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Realizer Bounds and Repetition Constraints at the First Coefficient Crossing in the $3x + 1$ Problem

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Abstract

Terras's coefficient stopping time conjecture asks whether the shortcut Collatz orbit of every integer $n \geq 2$ lies below n when its coefficient $\rho_k = 3^d/2^k$ first falls below one. Here k counts shortcut steps and d counts their odd inputs. We prove that every eligible realizer descends when the first-crossing word has at most 75 maximal runs of odd inputs. Any counterexample requires at least 76 such runs, 10 439 860 591 steps and 6 586 818 670 odd inputs. The proof uses a lower bound on the seed from the length of its odd runs, rational approximations to $\log_2 3$, the verified range of Rozier and Terracol, and Rhin's estimate for a linear form in logarithms. We also prove descent for a family with arbitrarily many odd runs. Two further restrictions narrow the remaining words. An initial square UU in a failing word satisfies $|U| < 35 \log_2 k$. For a counterexample of shortest crossing length, the reciprocal-sum estimate of Nikzad, Frinkle and Bagherzadeh gives a lower bound on ρ_k whose value begins $0.9441733669\ldots$. The open inequality compares the least eligible starting value in each remaining parity class with an explicit threshold.

1 What is being asked

Write $T(n) = (3n + 1)/2$ at odd inputs and $T(n) = n/2$ at even inputs. Fix $n \geq 2$. For the parity word w of its first k input states $n, T(n), \ldots, T^{k-1}(n)$, let d count the odd inputs. Induction gives

$$2^k T^k(n) = 3^d n + C_w, \tag{1}$$

where the nonnegative integer C_w depends only on w. Write $\rho_k = 3^d/2^k$ with $\rho_0 = 1$. An odd step multiplies the coefficient by $3/2$ and an even step halves it. Let $\tau(n)$ be the first k with $\rho_k < 1$, and let $\chi(n)$ be the first k with $T^k(n) < n$. An empty defining set gives stopping time ∞. Equation (1) gives $\tau(n) \leq \chi(n)$, so any failure of Terras's conjecture $\tau(n) = \chi(n)$ has finite $k = \tau(n)$ and $T^k(n) \geq n$ [10, 2]. An even starting value descends at its first step, so a failure has odd n and $d \geq 1$.

A first-crossing word has $\rho_j \geq 1$ for every $j < k$ and $\rho_k < 1$. Its last step is even, so $\rho_k \in [1/2, 1)$. With $d \geq 1$, this also gives $2^{k-1} \leq 3^d < 2^k$ and $k = \lceil d \log_2 3 \rceil \leq 2d$. An *odd run* is a maximal block of consecutive odd input states.

2 The signed comparator

Dividing (1) by 2^k gives the form we use throughout.

Proposition 1. *With $M_w = C_w/3^d$,*

$$T^k(n) = \rho_k\,(n + M_w), \qquad \textit{so} \qquad T^k(n) < n \iff n\big(\rho_k^{-1} - 1\big) > M_w.$$

Moreover, if the odd runs have lengths $\ell_1, \dots, \ell_s$ and run i starts at time t_i,

$$M_w = \sum_{i=1}^{s} \frac{1-(2/3)^{\ell_i}}{\rho_{t_i}}. \tag{2}$$

Proof. The first display is (1). For (2), each odd step at time j contributes $2^j 3^{-d_{j+1}} = 1/(3\rho_j)$ to M_w, where d_{j+1} counts odd steps through time j. Summing a run of ℓ consecutive odd steps from t gives $\frac{1}{3\rho_t}\sum_{r<\ell}(2/3)^r = (1-(2/3)^\ell)/\rho_t$. □

If a first-crossing word contains $s \geq 1$ odd runs, every proper prefix has $\rho_t \geq 1$, so each run contributes less than one and each odd step contributes at most $1/3$. Thus

$$M_w < s, \qquad M_w \leq d/3. \tag{3}$$

The two bounds depend respectively on the number of runs and on the total number of odd inputs.

Remark 2 (Testing all eligible realizers of a word). A length-k parity word specifies one residue class modulo 2^k. Indeed, the two lifts of a given length-j class modulo 2^{j+1} have starting values that differ by 2^j. Their images after j steps differ by 3^{d_j}, an odd integer, so exactly one lift has the prescribed next parity. For a first-crossing word, let $r(w)$ be the least integer ≥ 2 in its class. Every eligible realizer descends if and only if $r(w)(\rho_k^{-1} - 1) > M_w$, since the left side increases with the seed. If a seed n fails, its endpoint satisfies $0 \leq T^k(n) - n < s$ by Proposition 1 and (3).

Example 3. The exact odd-to-odd valuation word $(1,2,1,3,1)$ from [5, App. A.6(b)] has eight shortcut input bits 11011001 and an odd terminal state. The eight input bits give the class 187 mod 256. Terminal oddness adds a ninth bit and selects 187 mod 512. Indeed, $T^8(187) = 179$, while the other lift, 443, has $T^8(443) = 179 + 3^5 = 422$. The first coefficient crossing already occurs at step 7. Its prefix 1101100 gives the class 59 mod 128, and

$$59 - T^7(59) = \frac{(2^7 - 3^4)\cdot 59 - 85}{2^7} = 21.$$

Thus $T^7(59) = 38$ and every eligible seed in $59 + 128\mathbb{Z}$ descends at step 7. Within a period of 512 there are four representatives of this crossing class, of which one realizes the full valuation word.

3 Words with at most 75 odd runs

Lemma 4 (run-length realizer bound)**.** *If an odd run starts at x and lasts ℓ steps, the endpoint plus one equals $(3/2)^\ell(x+1)$, so $2^\ell \mid x+1$. At each odd step $x+1$ is multiplied by $3/2$, and at each even step it decreases. Writing D_i for the number of odd steps before run i and $L = \log_2(n+1)$, we get $2^{\ell_i} \leq (n+1)(3/2)^{D_i}$, hence $D_{i+1} \leq L + \alpha D_i$ with $\alpha = \log_2 3$, and therefore*

$$d \leq G_s L, \qquad G_s = \sum_{j=0}^{s-1} \alpha^j.$$

Theorem 5. *Every eligible realizer of a first-crossing word with at most 75 odd runs descends at that crossing. Any counterexample to Terras's conjecture has at least 76 odd runs, crossing time $k \geq 10\,439\,860\,591$ and odd-input count $d \geq 6\,586\,818\,670$.*

Proof. Rozier and Terracol [9, Cor. 5.4] verify $\chi(n) = \tau(n)$ for $2 \le n \le B$ with $B = 28 \cdot 10^{18}$, so a failure has $n > B$. Put $\Lambda = k \ln 2 - d \ln 3 > 0$. By Proposition 1 a failure gives $(e^{\Lambda} - 1)n \le M_w$, hence with (3)

$$0 < \Lambda < s/n < s/B, \qquad 0 < \Lambda/d < 1/(3n) < 1/(3B). \tag{4}$$

The two large integers. Take the convergents of $\alpha = \log_2 3$ of indices 20 and 19, respectively:

$$(a, b) = (9\,809\,721\,694,\ 6\,189\,245\,291), \qquad (u, v) = (630\,138\,897,\ 397\,573\,379).$$

One certifies $a/b < \alpha$ and $(u/v) \ln 2 - \ln 3 > 1/(3B)$. With the second half of (4) this forces $a/b < \alpha < k/d < u/v$. The determinant $bu - av = 1$ makes $bk - ad$ and $ud - vk$ positive integers, and expanding gives $(k, d) = (ud - vk)(a, b) + (bk - ad)(u, v)$. Hence $k \ge a + u = 10\,439\,860\,591$ and $d \ge b + v = 6\,586\,818\,670$; this pair is the convergent of index 21, the mediant of the two used, and it is the pair that occurs in Eliahou's work on nontrivial cycle lengths [1].

Excluding at most 75 runs. Suppose now that $s \le 75$, so that $\Lambda < 75/B$. For $p/q < \alpha$ put $e = q \ln 3 - p \ln 2 > 0$. Since $k/d > \alpha$, $q\Lambda + de = (qk - pd) \ln 2 \ge \ln 2$. Two pairs suffice, with certified enclosures $e_{\mathrm{I}} \in (5.22, 5.24) \cdot 10^{-18}$ at $(p, q) = (9\,881\,527\,843\,552\,324,\ 6\,234\,549\,927\,241\,963)$ and $e_{\mathrm{II}} \in (1.24, 1.25) \cdot 10^{-19}$ at $(p, q) = (6\,724\,555\,128\,221\,608\,268,\ 4\,242\,721\,909\,926\,539\,673)$. Pair I gives $d > (0.69 - q \cdot 75/B)/(5.24 \cdot 10^{-18}) > 1.26 \cdot 10^{17}$. If $d < 2^{61} < 2.31 \cdot 10^{18}$, pair II gives $\Lambda > (0.69 - 2.31 \cdot 10^{18} \cdot 1.25 \cdot 10^{-19})/(4.25 \cdot 10^{18}) > 9 \cdot 10^{-20}$, hence $n < 75/(9 \cdot 10^{-20})$ and $n + 1 < 2^{70}$; Lemma 4 with $G_{75} < 1.72 \cdot 10^{15}$ then gives $d < 70 \cdot 1.72 \cdot 10^{15} = 1.204 \cdot 10^{17}$, contradicting $d > 1.26 \cdot 10^{17}$.

The tail. For $d \ge 2^{61}$, Rhin's effective estimate [8] (in the form used by [9, Prop. 6.3]) gives $\Lambda \ge k^{-13.3}$, so $n < 75(2d)^{13.3}$ and $\log_2(n + 1) < 7 + 14 \log_2(2d)$. With $2^j \le d < 2^{j+1}$, Lemma 4 forces $2^j \le d < 1.72 \cdot 10^{15}\,[\,7 + 14(j + 2)\,]$. At $j = 61$ the upper bound is $1.52908 \cdot 10^{18} < 2^{61}$. It stays below 2^j by induction, since the bracket increases by 14 while 2^j doubles. This contradiction excludes the remaining values of d. □

Remark 6 (Dependencies of Theorem 5). The Rozier–Terracol verification supplies the seed bound $n > B$. Combined with (4), the Farey interval then gives the two integer bounds. In the proof of Theorem 5, Rhin's estimate enters only in the case $d \ge 2^{61}$. Rozier and Terracol [9, Rem. 5.5] credit Simons with proposing the classification by local maxima. Priority for the stated first-crossing bounds remains open.

4 A family with arbitrarily many runs

Proposition 7. *Repeat the block* 110 *exactly* $h \ge 1$ *times and append zeros up to the first coefficient crossing. Every eligible realizer descends at that crossing.*

Proof. One block acts by $H(x) + 5 = 9(x + 5)/8$, so -5 is its fixed point. The block's full coefficient is $9/8$ and its two nonempty proper-prefix coefficients are $3/2$ and $9/4$. Consequently the first crossing occurs among the appended zeros at $k = \lceil h \log_2 9 \rceil \le 4h$. Iteration gives $H^h(n) + 5 = 9^h(n + 5)/8^h$. Integrality implies $8^h \mid n + 5$, hence $n \ge 8^h - 5$. The normalized memory is $5(1 - (8/9)^h) < 5$ and the appended zeros leave it unchanged.

For $h \le 63$, $k \le 252$ is below the crossing-length bound in Theorem 5. For $h \ge 64$, Rhin's estimate gives $n\Lambda \ge (8^h - 5)/(4h)^{13.3} > 5$. Indeed, $8^{64} - 5 > 5 \cdot 256^{14}$, and $((h + 1)/h)^{14} \le (65/64)^{14} < 8$, so $8^h - 5 > 5(4h)^{14}$ persists by induction. A failure requires $n\Lambda < n(e^{\Lambda} - 1) \le M_w < 5$, a contradiction. □

The prefix $(110)^h$ and the completed word generally have different least seeds. Write $m = k - 3h$ and $n = 8^h t - 5$. All these n form the binary prefix class, since $2^{3h} = 8^h$ is its exact modulus. Its least positive representative is $8^h - 5$. To append m zeros we also need $2^m \mid 9^h t - 5$, the block endpoint. At $h = 12$ this gives $k = 39$, $m = 3$ and $t \equiv 5 \pmod 8$. The prefix and completed-word least seeds are therefore 68 719 476 731 and 343 597 383 675, respectively.

5 A finite-word repetition bound

Proposition 8. *Let a first-crossing word from n begin with UU, where $|U| = \ell \geq 1$. Then $(4/3)^\ell < n+1$. If the crossing fails to give descent, then*

$$\ell < \frac{\log_2(n+1)}{\log_2(4/3)} < \frac{14.3}{\log_2(4/3)} \log_2 k < 35 \log_2 k.$$

Proof. Set $x = T^\ell(n)$. Both n and x start with U, so the parity-class congruence gives $x \equiv n \pmod{2^\ell}$. Since $\ell < k$, $\rho_\ell \geq 1$. Equality would give $3^{d_\ell} = 2^\ell$, impossible for $\ell \geq 1$. Thus $x \geq \rho_\ell n > n$, so $x - n$ is a positive multiple of 2^ℓ. Equivalently $2^\ell(x-n)$ is a positive multiple of $2^{2\ell}$. At each step $T(y) + 1 \leq (3/2)(y+1)$, whence

$$2^\ell \leq x - n < x + 1 \leq (3/2)^\ell (n+1).$$

Dividing gives the first assertion, independently of whether the seed descends.

For a failure set $\Lambda = k \ln 2 - d \ln 3 > 0$. Then $n\Lambda < n(e^\Lambda - 1) \leq M_w \leq d/3$ by Proposition 1 and (3). Since $k \geq 2$ and $d \leq k$, Rhin's bound [9, Prop. 6.3] gives $\Lambda \geq k^{-13.3}$. It follows that $n < (d/3)k^{13.3} \leq k^{14.3}/3$ and hence $n + 1 < k^{14.3}$. Taking logarithms completes the proof, since $14.3/\log_2(4/3) < 35$. □

The restriction concerns repetition inside a single finite first-crossing prefix. The periodic-approximation results in [3] concern infinite parity words.

6 A restriction from reciprocal sums

Write $x_j = T^j(n)$ and form the product over the odd input values of the first k steps,

$$Q = \prod_{\substack{0 \leq j < k \\ x_j \text{ odd}}} \left(1 + \frac{1}{3x_j}\right).$$

Multiplying the single-step factors gives $x_k/n = \rho_k Q$. Hence Proposition 1 gives $Q = 1 + M_w/n$.

Let $D = 28\,019\,077\,177\,231\,758\,495$. For any finite shortcut path with distinct inputs all exceeding D, let R be the sum of the reciprocals of its odd inputs. Nikzad, Frinkle and Bagherzadeh prove $R < B_0$ in [7, Theorem 1A.1], where

$$B_0 = \frac{45}{256} + 10^{-6} + 10^{-15}.$$

Their proof gives a second rational upper bound by retaining the coefficient in (153) before the rounding in (154). Set

$$K = \frac{6\,719\,267}{13\,693\,680}, \qquad c = \frac{93\,657\,291\,203\,213}{10^{15}}, \qquad B_1 = \frac{15}{4} Kc + 10^{-6} + 10^{-15}.$$

Indeed, for each short gap bracket of total length m, (153) bounds the sum over its admissible starting values by $K2^{-m}$. The one-step balance bound (142) contributes the factor $15/4$. Equation (164) bounds the weighted bracket count by the exact dyadic $\sum_{m=61}^{1000} \widehat{C}_m 2^{-m} = 0.09365729120321290556\ldots < c$. Adding the long-shell allowance from (139) and the boundary allowance from (149), as in (151)–(154), gives $R < B_1 < B_0$. These equation numbers refer to [7]. The unpublished verification [6] independently reconstructs the high-path estimate and reports an exact reproduction of the finite count in (164).

Proposition 9. *If a counterexample exists, let n have the least crossing length k among all counterexamples. For $i = 0, 1$ it satisfies*

$$0 < \Lambda = -\ln \rho_k < B_i/3, \qquad \rho_k > \exp(-B_i/3).$$

Proof. We use the external height record $H(y) \le 2456$ through D, reported in [7, eq. (34)]. Here H counts ordinary Collatz operations to first arrival at 1. For $2 \le y \le D$, this gives $\tau(y) \le \chi(y) \le H(y) \le 2456$, since shortcut steps combine ordinary operations. The lower bound on k in Theorem 5 therefore gives $n > D$. For $0 \le j < k$, the prefix coefficient is at least one and its affine correction is nonnegative, so $x_j \ge n > D$.

If two of $x_0, \dots, x_{k-1}$ coincided, the intervening segment would contain a cycle of length $L < k$. Let m be its minimum. One full turn from m has positive affine correction, since any positive cycle has an odd input. The return to m therefore forces its coefficient below one. Thus $\tau(m) \le L$, while every value on the cycle is at least m, giving a failure of shorter crossing length. This proves that the k inputs are distinct. The high-path estimate applies even when the endpoint repeats an input. Finally $x_k \ge n$ gives $1 \le \rho_k Q$, and

$$\Lambda \le \ln Q < \frac{1}{3} \sum_{\substack{0 \le j < k \\ x_j \text{ odd}}} \frac{1}{x_j} < B_i/3.$$

□

In the coordinate $-\log_2 \rho_k$, the nominal first-crossing interval has length one. For each bound the surviving interval has length $B_i/(3 \ln 2)$. The values below are rounded for display.

	$\exp(-B_i/3)$	surviving width	excluded width
B_0	0.9430895072	8.45334%	91.54666%
B_1	0.9441733669	8.28763%	91.71237%

These percentages measure intervals of log coefficients. The restriction concerns a counterexample of minimal crossing length.

7 The open question

A shortest counterexample would have at least 76 odd runs, obey the coefficient bound in Proposition 9, and have no initial square whose period exceeds the bound in Proposition 8. For each remaining first-crossing word w, its coefficient and its additive correction are known exactly. The open estimate for its least eligible realizer $r(w)$ is

$$r(w) > \frac{C_w}{2^k - 3^d} = \frac{M_w}{\rho_k^{-1} - 1}.$$

Equality already gives a counterexample. A lower bound for $r(w)$ or an upper bound for the right side can certify a word if it preserves this strict inequality. The rational-shadow estimates of [4] supply bounds for individual blocks. Applying one to this question requires transporting it back to the starting seed and comparing it with the threshold of the actual first-crossing prefix.

Reproducibility

From the companion repository at commit `a93b1e8`, run

```
python3 scripts/verify_crossing.py
```

The script checks the numerical inequalities with integers and rationals, encloses $\ln 2$ and $\ln 3$ by a 96-term series with a geometric tail, recomputes the finite bracket sum in (164) of [7], and encloses each exponential cutoff by alternating Taylor bounds. The SHA-256 digest of the verifier begins `f852c88311071a0d`. The orbit and residue-class examples are finite checks. The proofs for arbitrary word lengths are in the text.

The finite seed verification of Rozier–Terracol, Rhin's estimate, the high-path accounting of Nikzad–Frinkle–Bagherzadeh and the height record cited above are external inputs. The general case of Terras's conjecture remains open.

AI assistance and author responsibility

The author directed this research and used Claude (Anthropic) and ChatGPT (OpenAI) for proof exploration and revision, cross-checking the reciprocal-sum estimates against the source paper, developing and testing the exact-arithmetic verifier, and drafting the manuscript. The author takes responsibility for the mathematical claims, their interpretation, and any errors.

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