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AXIOMATIC FOUNDATIONS OF ASTROPHYSICAL SYSTEMS: MODEL-INDEPENDENT OBSTRUCTIONS BETWEEN EQUATIONS OF STATE, OBSERVABLES, AND FIELD EQUATIONS

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Abstract. Continuing the axiomatic program for astrophysics initiated in previous works, we introduce a gravity-model-independent and scale-independent definition of an *astrophysical system* and of *clusters of astrophysical systems* over arbitrary piecewise smooth spatial manifolds. A cluster is modeled as a subspace of piecewise differentiable sections of matter and gravitational bundles (modulo coincidence on dense open sets), endowed with macroscopic observable functionals and coupled through a parametric field operator $\mathcal{D}_{\text{grav}}$ encompassing Newtonian gravity, General Relativity, and modified gravity theories ($f(R)$, Rastall, scalar-tensor, and MOND-type models). Restricting local equations of state to bounded extended-norm balls (which form affine Banach spaces on fixed partitions) and applying Implicit Function Theorems in locally convex spaces, we prove four universal obstruction theorems: (i) a direct multi-observable obstruction theorem; (ii) a gravity-coupled structural obstruction theorem via locally density-solvable field equations and reference evaluation operators; (iii) a causal lifting theorem for general acoustic cones; and (iv) an *inverse obstruction theorem*, showing that combining causality-constrained equations of state with empirical observable relations induces *a priori* bounds on the parameter space of the gravitational theory. We also prove that finite-dimensional gravity-coupled clusters are closed and nowhere dense in any Hausdorff locally convex topology, and apply our results to compact stars in Rastall and perturbative $f(R)$ gravity, Bowers–Liang anisotropic stars, and galaxies obeying the Baryonic Tully–Fisher Relation.

1. Introduction

The search for an axiomatic formulation of astrophysics dates back to the foundational treatises of Eddington [1] and Chandrasekhar [2], who sought to deduce the macroscopic regularities of stars from general mathematical principles governing thermodynamic equilibrium and gravitation. In two previous works [3, 4], this axiomatization problem was revisited from the perspective of infinite-dimensional functional analysis and differential topology. Specifically, in [3] we introduced the notion of a *cluster of stellar systems* of degree (k, l) as a subspace of piecewise differentiable radial profiles $(p, \rho) \in C^k_{\text{pw}}(I) \times C^l_{\text{pw}}(I)$ endowed with extended norms in the sense of Beer and Vanderwerff [5], and proved that empirical mass-radius relations $g(M, R, \delta) = 0$ combined with mass, radius, or causal bounds impose strict *a priori* obstructions on the admissible equations of state $f(p, \rho, \epsilon) = 0$. In [4], we specialized to clusters of Tolman–Oppenheimer–Volkoff (TOV) systems, formulating their classification problem in locally convex spaces and constructing a fifteen-dimensional space of extended analytic TOV solutions via pseudo-asymptotic limits and topological phase transitions of cavity and singularity submanifolds.

Despite their generality with respect to the equation of state and the mass-radius relation, the frameworks of [3, 4] retain three structural limitations when confronted with modern theoretical and observational astrophysics:

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(1) **Dependence on spherical symmetry and 1D intervals:** Defining a system solely as a pair of scalar functions (p, ρ) on a one-dimensional radial interval $I \subset \mathbb{R}$ restricts the formalism to isotropic, spherically symmetric configurations. Realistic astrophysical objects often exhibit anisotropy (distinct radial and tangential pressures $p_r \neq p_t$ [6, 7]), rotation (axisymmetric domains), multi-component fluids, or non-trivial spatial topologies.
(2) **Restriction to the classical continuity equation:** In [3], the coupling between macroscopic observables (M, R) and microscopic matter density ρ was mediated solely by a scalar ordinary differential continuity equation $M'(R) = F(R, \rho)$, motivated by the classical spherical mass relation $M'(r) = 4\pi r^2 \rho(r)$. However, in modified theories of gravity—such as $f(R)$ gravity [9, 11], scalar-tensor and Horndeski theories [12], Rastall gravity [13, 14], or Eddington-inspired Born–Infeld (EiBI) gravity [15]—as well as in full hydrostatic balance, the effective mass and pressure gradients depend on higher-order curvature derivatives, non-minimal matter-geometry couplings, pressure anisotropy, or auxiliary gravitational fields Φ, modifying both the continuity equation and the hydrostatic operator.
(3) **Restriction to (M, R) observables:** Contemporary multi-messenger astrophysics constrains compact objects and galaxies not merely through mass and radius, but through higher-multipole and dynamical observables: moment of inertia I, dimensionless tidal deformability Λ_{tidal}, quadrupole moment Q (the universal I–Love–Q and compactness–Love relations [16, 17]), surface redshift Z, and, at galactic scales, asymptotic flat rotation velocities V_{flat} and disk/halo scale radii R_d (such as the Baryonic Tully–Fisher Relation [18]).

The purpose of the present paper is to overcome all three limitations simultaneously. We formulate a definition of *astrophysical system* and *astrophysical cluster* that is completely independent of the underlying gravitational model, spatial dimension, and symmetry assumptions, and we prove universal structural and obstruction theorems based on it.

More precisely, in Section 2, we define an astrophysical system of degree (k, l, m) over a piecewise smooth spatial manifold Σ as a triple $\mathcal{S} = (p, \rho, \Phi)$ of piecewise differentiable sections of matter and gravitational bundles, belonging to an extended normed space $C^{k,l,m}_{\text{pw}}(\Sigma)$ topologized by canonical extended norms (whose bounded component $C^{k,l,m}_{\text{pw},b}(\Sigma)$ and fixed-partition strata carry canonical locally convex and affine Banach structures). A cluster of astrophysical systems $\mathcal{A} \subset C^{k,l,m}_{\text{pw}}(\Sigma)$ is equipped with:

- a finite tuple of macroscopic *observable functionals* $\mathcal{O} = (\mathcal{O}_1, \dots, \mathcal{O}_N) : \mathcal{A} \to \mathbb{R}^N$;
- a parametric *gravity-matter coupling operator* $\mathcal{D}_{\text{grav}}(\mathcal{S}, \alpha) = 0$, where $\alpha \in A_{\text{grav}}$ ranges over an ordered Banach or locally convex space of gravitational theory parameters (encoding, e.g., Newton's constant G, post-Newtonian parameters, Rastall or $f(R)$ curvature couplings, or MOND acceleration scales), together with a reference evaluation/trace operator ev_Γ extracting the characteristic density $\rho_\Gamma := \text{ev}_\Gamma(\rho)$;
- *state functions* $f(p, \rho, \epsilon) = 0$ and *observable relations* $g(\mathcal{O}, \delta) = 0$.

In Section 3, by noting that open extended-norm balls around a reference configuration carry a canonical affine Banach structure and applying the Implicit Function Theorem for locally convex spaces [20, 21] together with partial orderings induced by positive cones, we establish four main obstruction results:

- **Theorem 3.1 (Direct Multi-Observable Obstruction):** Bounds on the macroscopic observables $\mathcal{O} = (\mathcal{O}_1, \dots, \mathcal{O}_N)$ in a cluster endowed with a locally monotone observable relation $g(\mathcal{O}, \epsilon) = 0$ induce *a priori* bounds on every monotonically locally integrable (MLI) state function $f(p, \rho, \epsilon)$ and, when f is fully MLI, on the generalized acoustic propagation speeds $v^2 = \partial_\rho \xi$.
- **Theorem 3.2 (Gravity-Coupled Structural Obstruction):** When the state function and observable relation depend on independent parameters ϵ and δ, any *locally density-solvable*

gravity-matter coupling operator $\mathcal{D}_{\mathrm{grav}}(\mathcal{S}, \alpha) = 0$ bridges the two: bounds on the observables $\mathcal{O}$, their boundary jets, and the gravitational parameters α induce bounds on the reference matter density $\rho_\Gamma = \mathrm{ev}_\Gamma(\rho)$ and hence on any MLI state function and its speed of sound.

- **Theorem 3.3 (Causal Lifting in General Gravity):** Conversely, imposing an invertible causal cone condition $v^2(\rho_\Gamma, \epsilon) \preceq v_0^2(\rho_\Gamma)$ on the generalized acoustic operator lifts, via the gravity-matter coupling $\mathcal{D}_{\mathrm{grav}}(\mathcal{S}, \alpha) = 0$, to upper bounds on the admissible state functions in terms of the observable parameters δ and gravitational parameters α.
- **Theorem 3.6 (Inverse Obstruction Theorem for Gravity Models):** By exploiting the symmetry between matter parameters ϵ, observable parameters δ, and gravitational parameters α in our axiomatic setup, we prove that if the equation of state is constrained within a known physical regime (e.g., from nuclear physics or relativistic causality) and the cluster obeys an empirical observable relation $g(\mathcal{O}, \delta) = 0$, then any locally α-invertible gravity coupling $\mathcal{D}_{\mathrm{grav}}(\mathcal{S}, \alpha) = 0$ suffers *a priori* upper or lower bounds on the gravitational parameter space A_{grav} (depending on the co- or contra-monotonicity of the gravity solver). This turns empirical astrophysical scaling relations into rigorous mathematical falsifiability tests for modified gravity models.

In Section 4, we study the global linear and topological structure of gravity-coupled clusters $\mathcal{A}_\Sigma^{k,l,m}(\alpha) = \mathrm{span}(\mathcal{D}_{\mathrm{grav}}(\cdot, \alpha)^{-1}(0)) \subset C_{\mathrm{pw}}^{k,l,m}(\Sigma)$, generalizing the topological rigidity argument of [4] to arbitrary gravity models (Theorem 4.2) and comparing the linear span with the nonlinear equilibrium manifold $\mathcal{M}_{\mathrm{sol}}(\alpha)$. In Section 5, we illustrate the power and versatility of the general formalism with concrete applications across different gravity regimes and spatial scales: (1) compact stars in Rastall and perturbative $f(R) = R + \alpha R^2$ gravity, deriving explicit analytic and numerical bounds on the gravity parameter α from (M, R) and 3-observable $(M, R, \Lambda_{\mathrm{tidal}})$ tidal deformability relations; (2) anisotropic relativistic fluid spheres coupled via the Bowers–Liang hydrostatic equation with distinct radial and tangential sound speeds; and (3) galactic-scale astrophysical systems obeying the Baryonic Tully–Fisher Relation coupled to halo dispersion bounds. We conclude in Section 6 by outlining the sequel papers in this series.

2. General Astrophysical Systems and Clusters

2.1. **Piecewise Differentiable Section Spaces and Extended Norms.** To liberate the formalism of [3, 4] from one-dimensional intervals while preserving its analytical advantages, let Σ be a d-dimensional smooth manifold (possibly with corners or piecewise smooth boundary $\partial\Sigma$), which represents the spatial domain (or symmetry quotient) of an astrophysical configuration. For instance, $\Sigma = I \subset [0, \infty)$ for spherically symmetric radial profiles, $\Sigma = I \times [0, \pi]$ for axisymmetric rotating or tidally deformed configurations, or $\Sigma \subset \mathbb{R}^3$ for general three-dimensional configurations.

Let $\pi : E \to \Sigma$ be a finite-dimensional normed vector bundle over Σ equipped with a Riemannian metric on Σ and a compatible bundle connection ∇, so that higher-order covariant derivatives $\nabla^j u$ of smooth sections $u \in \Gamma^\infty(\Sigma; E)$ and their pointwise norms $|\nabla^j u(x)|_E$ are canonically defined.

Definition 2.1 (Piecewise C^k Sections and Extended Norm Topology)**.** A section $u : \Sigma \to E$ is said to be *piecewise C^k-differentiable* if there exists a finite family of pairwise disjoint open subsets $\{\Sigma_a\}_{a=1}^{m_\mathcal{P}}$ of Σ such that $\overline{\bigcup_{a=1}^{m_\mathcal{P}} \Sigma_a} = \Sigma$ and the restriction $u|_{\Sigma_a}$ belongs to $C^k(\Sigma_a; E)$ for each $a = 1, \ldots, m_\mathcal{P}$.

To ensure that the resulting function space is Hausdorff, two such sections u_1, u_2 are identified $(u_1 \sim u_2)$ whenever they coincide on a dense open subset of Σ on which both are C^k. For each equivalence class u, its intrinsic *singular/discontinuity locus* $\mathrm{Sing}(u) \subset \Sigma$ is defined as the complement of the largest open subset of Σ on which u admits a C^k representative (so that $\mathrm{Sing}(u) \subset \Sigma \backslash \bigcup_{a=1}^{m_\mathcal{P}} \Sigma_a$ is closed and nowhere dense). The quotient vector space of equivalence classes, denoted by $C_{\mathrm{pw}}^k(\Sigma; E)$, is endowed with the canonical *extended norm* (or generalized norm in the sense of Beer and Vanderwerff

[5])

$$\|u\|_k := \max_{0\le j\le k} \sup_{x\in\Sigma\setminus \mathrm{Sing}(u)} |\nabla^j u(x)|_E \in [0,\infty]. \tag{2.1}$$

Because the union of two nowhere dense closed sets is again nowhere dense, the supremum in (2.1) is independent of the choice of representative, and $\|u\|_k = 0$ if and only if $u(x) = 0$ on $\Sigma \setminus \mathrm{Sing}(u)$, i.e., $u = 0$ in $C^k_{\mathrm{pw}}(\Sigma; E)$. Thus $\|\cdot\|_k$ satisfies all axioms of a norm on $C^k_{\mathrm{pw}}(\Sigma; E)$ except that it may take the value $+\infty$. As established in [5], $\|\cdot\|_k$ turns $C^k_{\mathrm{pw}}(\Sigma; E)$ into an *extended normed linear space* whose canonical topology τ_k, generated by the open balls $B_r(u_0) = \{u \in C^k_{\mathrm{pw}}(\Sigma; E) : \|u - u_0\|_k < r\}$ for $r > 0$, makes $\left(C^k_{\mathrm{pw}}(\Sigma; E), \tau_k\right)$ a Hausdorff topological abelian group with a base of convex open neighborhoods; its connected component of the origin is the clopen subspace of bounded sections $C^k_{\mathrm{pw},b}(\Sigma; E) := \{u \in C^k_{\mathrm{pw}}(\Sigma; E) : \|u\|_k < \infty\}$, on which τ_k restricts to a genuine normed (hence locally convex Hausdorff) topological vector space topology, while each coset $u_0 + C^k_{\mathrm{pw},b}(\Sigma; E)$ is a clopen affine normed manifold. When Σ is compact (or when restricting to quasi-isometric metrics, fiber norms $|\cdot|_E$, and compatible connections ∇ of uniformly bounded geometry with uniformly bounded difference tensor up to order $k-1$ on non-compact Σ), any two such choices yield equivalent extended norms and therefore induce the exact same topology τ_k and bounded subspace $C^k_{\mathrm{pw},b}(\Sigma; E)$. Notice that while the full space $C^k_{\mathrm{pw}}(\Sigma; E)$ is neither Banach nor Fréchet, for any fixed open partition $\mathcal{P} = \{\Sigma_a\}_{a=1}^{m_{\mathcal{P}}}$, the *$\mathcal{P}$-regular stratum* $C^k_{\mathcal{P}}(\Sigma; E) := \left\{u \in C^k_{\mathrm{pw}}(\Sigma; E) : \mathrm{Sing}(u) \subset \Sigma \setminus \bigcup_{a=1}^{m_{\mathcal{P}}} \Sigma_a\right\} \cong C^k(\bigcup_{a=1}^{m_{\mathcal{P}}} \Sigma_a; E)$ is a subspace in which every open extended-norm ball $B_{r,\mathcal{P}}(u_0) := B_r(u_0) \cap C^k_{\mathcal{P}}(\Sigma; E)$ is isometric to an open ball in the affine Banach space $u_0 + C^k_b(\bigcup_{a=1}^{m_{\mathcal{P}}} \Sigma_a; E)$.

2.2. **Abstract Astrophysical Systems, Gravity Couplings, and Observables.** We now decompose the field content of an astrophysical configuration into thermodynamic/matter variables and gravitational/geometric potentials. Let W_p, W_ρ, and W_{grav} be finite-dimensional vector bundles over Σ, representing respectively:

- **Stress/Pressure variables (W_p):** e.g., isotropic pressure p ($W_p = \Sigma \times \mathbb{R}$), anisotropic pressures (p_r, p_t) ($W_p = \Sigma \times \mathbb{R}^2$), or full spatial stress tensors paired with electromagnetic/entropic stresses;
- **Density variables (W_ρ):** e.g., total mass-energy density ρ, baryon number density n_B, or multi-species densities $(\rho_{\mathrm{baryon}}, \rho_{\mathrm{DM}})$;
- **Gravitational/Geometric variables (W_{grav}):** e.g., enclosed mass function $M(r)$ and metric potentials $(\nu(r), \lambda(r))$ in General Relativity, tidal perturbation fields $H_2(r)$, the Newtonian potential Φ_N, or additional scalar/tensor gravitational fields $(\phi, \mathcal{R}, \dots)$ in modified gravity theories.

Definition 2.2 (Astrophysical System and Cluster)**.** Given non-negative integers (k, l, m), an *astrophysical system of degree* (k, l, m) over Σ is a triple $\mathcal{S} = (p, \rho, \Phi)$ in the product extended normed space

$$C^{k,l,m}_{\mathrm{pw}}(\Sigma) := C^k_{\mathrm{pw}}(\Sigma; W_p) \times C^l_{\mathrm{pw}}(\Sigma; W_\rho) \times C^m_{\mathrm{pw}}(\Sigma; W_{\mathrm{grav}}). \tag{2.2}$$

A *cluster of astrophysical systems* (or *astrophysical cluster*) of degree (k, l, m) is a vector subspace $\mathcal{A} \subset C^{k,l,m}_{\mathrm{pw}}(\Sigma)$ endowed with the induced extended-norm topology $\tau_{k,l,m}$ (which restricts to a normed locally convex vector topology on its bounded subspace $\mathcal{A} \cap C^{k,l,m}_{\mathrm{pw},b}(\Sigma)$). When the gravitational potential Φ is left implicit or projected out, we denote by $\mathcal{A}_{\mathrm{mat}} \subset C^k_{\mathrm{pw}}(\Sigma; W_p) \times C^l_{\mathrm{pw}}(\Sigma; W_\rho)$ the underlying matter cluster.

Macroscopic astrophysics characterizes each system $\mathcal{S} \in \mathcal{A}$ by a finite set of measurable quantities (mass, radius, moment of inertia, tidal deformability, luminosity, flat rotation velocity, etc.).

Definition 2.3 (Macroscopic Observable Functionals and Bounded Subclusters)**.** An *observable structure* on an astrophysical cluster $\mathcal{A}$ is an everywhere-defined, piecewise continuous map

$$\mathcal{O} = (\mathcal{O}_1, \dots, \mathcal{O}_N) \in C^0_{\mathrm{pw}}(\mathcal{A}; \mathbb{R}^N), \tag{2.3}$$

(i.e., a point-defined map $\mathcal{O} : \mathcal{A} \to \mathbb{R}^N$ that is continuous on a finite family of pairwise disjoint open subsets whose union is dense in $\mathcal{A}$, without passing to equivalence classes modulo nowhere dense subsets of $\mathcal{A}$ so that pointwise evaluation on nowhere dense equilibrium loci remains unambiguous), assigning to each system $\mathcal{S} = (p, \rho, \Phi) \in \mathcal{A}$ its tuple of macroscopic observables $\mathcal{O}(\mathcal{S}) \in \mathbb{R}^N$. Given a subset $K \subset \mathbb{R}^N$ defined by componentwise upper bounds $\mathcal{O}_i(\mathcal{S}) \le \mathcal{O}_{i,0}$ (resp. lower bounds $\mathcal{O}_i(\mathcal{S}) \ge \mathcal{O}_{i,0}$), we say that a subcluster $\mathcal{B} \subset \mathcal{A}$ is *$\mathcal{O}$-bounded from above (resp. below) by* $\mathcal{O}_0 = (\mathcal{O}_{1,0}, \dots, \mathcal{O}_{N,0})$ if $\mathcal{O}(\mathcal{S}) \le \mathcal{O}_0$ (resp. $\mathcal{O}(\mathcal{S}) \ge \mathcal{O}_0$) in the componentwise order of $\mathbb{R}^N$ for all $\mathcal{S} \in \mathcal{B}$.

Example 2.4 (Recovering Classical Stellar Clusters)**.** When $\Sigma = I \subset [0, \infty)$, $W_p = W_\rho = W_{\mathrm{grav}} = I \times \mathbb{R}$, $\Phi = M$ is the enclosed mass function, and $\mathcal{O} = (M, R) : \mathcal{A} \to \mathbb{R}^2$ assigns the total mass $M(\mathcal{S}) = M(R)$ and stellar radius $R(\mathcal{S})$, Definitions 2.2 and 2.3 recover precisely the clusters of stellar systems $\mathrm{Stellar}^{k,l,m}_I$ introduced in [3, 4] (when k later denotes a polytropic stiffness constant in Section 5, we write k_p for the pressure differentiability degree to avoid notation clash).

Remark 2.5 (Representative-Independence and Continuity of Free-Boundary Observables). For free-boundary observables such as the stellar radius $R(\mathcal{S})$ in Example 2.4, because elements of $C^k_{\mathrm{pw}}(I; \mathbb{R})$ are equivalence classes modulo coincidence on dense open sets, $R(\mathcal{S})$ is defined in terms of the equivalence class $p \in C^k_{\mathrm{pw}}(I; \mathbb{R})$ via the interior continuous trace $\lim_{r \to R^-} p(r) = 0$ at the boundary of the connected regular interior $\Sigma_{\mathrm{int}} = (0, R(\mathcal{S}))$ where $p > 0$ (and set to 0 on the trivial zero configuration). On any open stratum of $\mathcal{A}$ where the inward boundary pressure derivative is transversal ($\lim_{r \to R^-} p'(r) < 0$ with $k \ge 1$), the Implicit Function Theorem guarantees that $R(\mathcal{S})$—and hence the evaluated total mass $M(\mathcal{S}) = \lim_{r \to R(\mathcal{S})^-} M(r)$—is completely independent of the choice of representative and continuous with respect to the extended-norm topology $\tau_{k,l,m}$.

Next, we formalize how a theory of gravity couples the matter variables (p, ρ) to the gravitational potentials Φ and macroscopic observables $\mathcal{O}$. In any metric or non-metric theory of gravity, equilibrium configurations satisfy a system of field equations (e.g., modified Einstein equations $G_{\mu\nu} + \alpha H_{\mu\nu} = 8\pi G T_{\mu\nu}$ together with divergence laws $\nabla^\mu T_{\mu\nu} = Q_\nu$). To rigorously link equivalence classes of sections $\rho \in C^l_{\mathrm{pw}}(\Sigma; W_\rho)$ with finite-dimensional observable tuples $\mathcal{O}(\mathcal{S}) \in \mathbb{R}^N$, we restrict to clusters whose density sections admit a regular interior approach domain $\Sigma_{\mathrm{int}} \subset \Sigma \setminus \mathrm{Sing}(\rho)$ with reference locus $\Gamma \subset \overline{\Sigma}_{\mathrm{int}}$, and introduce a continuous *reference evaluation (interior trace or averaging) operator*

$$\mathrm{ev}_\Gamma : \mathcal{A}_\rho \subset C^l_{\mathrm{pw}}(\Sigma; W_\rho) \to W_{\rho,\Gamma}, \qquad \rho \mapsto \rho_\Gamma := \mathrm{ev}_\Gamma(\rho), \tag{2.4}$$

taking values in an ordered Banach space $W_{\rho,\Gamma}$ (typically $W_{\rho,\Gamma} = \mathbb{R}^q$). Because $\Sigma_{\mathrm{int}} \setminus (\mathrm{Sing}(\rho_1) \cup \mathrm{Sing}(\rho_2))$ is dense in Σ_{int}, the interior continuous trace $\mathrm{ev}_\Gamma(\rho) := \lim_{x \to \Gamma,\, x \in \Sigma_{\mathrm{int}}} \rho(x)$ (as well as any integral average over Σ_{int}) is completely independent of the choice of representative in the quotient space $C^l_{\mathrm{pw}}(\Sigma; W_\rho)$, satisfies $|\mathrm{ev}_\Gamma(\rho)|_{W_{\rho,\Gamma}} \le \|\rho\|_0 \le \|\rho\|_l$, and is order-preserving whenever the fiber cone $W_{\rho,+}$ is closed. In concrete astrophysical problems, $\mathrm{ev}_\Gamma(\rho)$ represents either: (a) the interior boundary trace $\rho(R) := \lim_{r \to R^-} \rho(r)$ at a characteristic surface or fiducial radius $r = R$; (b) the central density $\rho_c = \rho(0^+) = \rho(R)/\rho_{\mathrm{rel}}$ linked to the surface via a relative density factor $\rho_{\mathrm{rel}} \in (0, 1]$ [3]; or (c) the volume-averaged density $\bar{\rho} = \mathrm{Vol}(\Sigma_{\mathrm{int}})^{-1} \int_{\Sigma_{\mathrm{int}}} \rho \, dV$.

Definition 2.6 (Gravity-Matter Coupling Operator and Local Density-Solvability)**.** Let A_{grav} be an ordered topological vector space of *gravitational theory parameters* (e.g., couplings $\alpha \in A_{\mathrm{grav}}$ parametrizing deviations from Newtonian or Einsteinian gravity). A *gravity-matter coupling operator*

on a cluster $(\mathcal{A}, \mathcal{O})$ is a map

$$\mathcal{D}_{\text{grav}} : \mathcal{A} \times A_{\text{grav}} \to \mathcal{V}, \tag{2.5}$$

where $\mathcal{V}$ is a locally convex space, such that a system $\mathcal{S} = (p, \rho, \Phi) \in \mathcal{A}$ is in gravitational equilibrium for the theory parameter $\alpha \in A_{\text{grav}}$ if and only if

$$\mathcal{D}_{\text{grav}}(\mathcal{S}, \alpha) = 0. \tag{2.6}$$

Suppose that along an observable relation, one subset of dependent observables $\mathcal{O}_{\text{dep}}$ (such as total mass M or rotation velocity V) is expressed as a piecewise C^s-differentiable function $\mathcal{O}_{\text{dep}} = \mu(\mathcal{O}_{\text{ind}}, \delta)$ of the remaining independent observables $\mathcal{O}_{\text{ind}}$ (such as radius R or tidal deformability Λ_{tidal}) and observable parameters $\delta \in F$, so that the s-jet $J^s\mathcal{O}(\delta) := (\mathcal{O}_{\text{ind}}, \mathcal{O}_{\text{dep}}, \partial\mathcal{O}_{\text{dep}}, \dots, \partial^s\mathcal{O}_{\text{dep}})$ is well-defined, and that the cluster is *s-th order boundary jet-compatible* at Γ. Specifically, writing $\mathcal{O}_{\text{ind}} = (\mathcal{O}_{\text{geom}}, \mathcal{O}_{\text{aux}})$ for the geometric coordinate $\mathcal{O}_{\text{geom}}$ parametrizing the reference boundary locus Γ together with any auxiliary macroscopic observables $\mathcal{O}_{\text{aux}}$, s-th order boundary jet-compatibility means that the internal boundary derivatives $\partial_r^j\Phi|_\Gamma$ $(1 \le j \le s)$ are determined by the macroscopic geometric jet derivatives $\partial^j_{\mathcal{O}_{\text{geom}}}\mu(\mathcal{O}_{\text{geom}}, \mathcal{O}_{\text{aux}}; \delta)$ via fixed non-zero jet homology factors $\kappa_{\text{hom},j}$ as $\partial_r^j\Phi|_\Gamma = \kappa_{\text{hom},j}^{-1}\partial^j_{\mathcal{O}_{\text{geom}}}\mu$. For instance, when $\mathcal{O}_{\text{geom}} = R$, $\mathcal{O}_{\text{aux}} = \Lambda_{\text{tidal}}$, and $\mathcal{O}_{\text{dep}}(R) = m(R; R)$ where $m(r; R)$ is the internal enclosed mass profile of the star of radius R (whose first total derivative obeys the Leibniz rule $\partial_R M(R) = \partial_r m(r; R)|_{r=R} + \partial_R m(r; R)|_{r=R}$), this reduces for $j = 1$ with $\kappa_{\text{hom}} := |\kappa_{\text{hom},1}| > 0$ to $\partial_r m(R; R) = \kappa_{\text{hom}}^{-1}|\partial_R\mu|$, and for $1 \le j \le s$ to a single positive constant $\kappa_{\text{hom},j} = \kappa_{\text{hom}} > 0$ on positive-slope homologous/envelope families where $m(r; R) = \kappa_{\text{hom}}^{-1}M(r)$ has $\partial_R m(r; R) \equiv 0$ (with $\kappa_{\text{hom}} = 1$ for strictly nested profiles). Given a reference evaluation operator ev_Γ as in (2.4) and a continuous reference projection/trace operator $\text{ev}_\Gamma^{\mathcal{V}} : \mathcal{V} \to \mathcal{V}_\Gamma$ into a Banach space $\mathcal{V}_\Gamma$, we say that $\mathcal{D}_{\text{grav}}$ induces a *locally density-solvable structural equation of order s* if, along s-th order boundary jet-compatible clusters, the evaluated equilibrium equation $\text{ev}_\Gamma^{\mathcal{V}}\big(\mathcal{D}_{\text{grav}}(\mathcal{S}, \alpha)\big) = 0$ factors through a reduced structural relation

$$\mathcal{K}\big(J^s\mathcal{O}(\delta), \rho_\Gamma, \alpha\big) = 0 \tag{2.7}$$

that can be locally solved for the reference density $\rho_\Gamma = \text{ev}_\Gamma(\rho) \in W_{\rho,\Gamma}$ in terms of $(J^s\mathcal{O}(\delta), \alpha)$: that is, there exists a local map Ψ_{grav} such that

$$\rho_\Gamma = \Psi_{\text{grav}}\big(J^s\mathcal{O}(\delta), \alpha\big). \tag{2.8}$$

If Ψ_{grav} is monotone with respect to the components of $J^s\mathcal{O}(\delta)$ and α (with respect to fixed ordering cones), we say that $\mathcal{D}_{\text{grav}}$ is *monotonically density-solvable*. More generally, on non-homologous equilibrium sequences where the ratio between the internal boundary jet $\partial_r^j\Phi|_\Gamma$ and the macroscopic observable jet $\partial^j_{\mathcal{O}_{\text{geom}}}\mu$ varies across the cluster away from critical turning points (including variations induced by $\mathcal{O}_{\text{aux}}(R)$ along a 1-parameter sequence), all obstruction inequalities in Section 3 remain valid by replacing $\kappa_{\text{hom},j}$ with its uniform upper or lower envelope bound $\kappa_{\text{hom},j} \in [\kappa_{\min,j}, \kappa_{\max,j}] \subset (0, \infty)$ over the subcluster.

Remark 2.7 (Physical Realizations and Envelope Bounds for Boundary Jet-Compatibility). In astrophysical applications, s-th order boundary jet-compatibility arises in two distinct ways:

(a) *Homologous, Strictly Nested, or Crust-Transition Families:* When a cluster describes stars sharing a common universal envelope or crust profile $m(r; R) = \kappa_{\text{hom}}^{-1}M(r)$ evaluated at a fiducial transition locus Γ $(r = R_\Gamma)$, the partial derivative with respect to the sequence parameter vanishes $(\partial_R m(r; R) \equiv 0)$, so that $\partial_r^j m(R_\Gamma; R_\Gamma) = \kappa_{\text{hom}}^{-1}M^{(j)}(R_\Gamma)$ holds with a constant factor $\kappa_{\text{hom},j} = \kappa_{\text{hom}} > 0$ for all $1 \le j \le s$ (with $\kappa_{\text{hom}} = 1$ for strictly nested profiles).

(b) *General Non-Homologous Equilibrium Sequences:* Along a single-EoS relativistic sequence parametrized by central density ρ_c (or radius R) on any compact radius interval $[R_1, R_2]$

disjoint from critical turning or inflection points where $\partial_r^j m(r;R)|_{r=R} = 0$ or $\frac{d^j M(R)}{dR^j} = 0$, the smooth jet ratio functions

$$\kappa_{\mathrm{hom},j}(R) := \left(\frac{\partial^j m(r;R)}{\partial r^j}\bigg|_{r=R} \right)^{-1} \frac{d^j M(R)}{dR^j} \qquad (1 \le j \le s) \tag{2.9}$$

are continuous and non-vanishing on $[R_1, R_2]$, and hence attain finite, strictly positive extremal envelope bounds $0 < \kappa_{\min,j} \le |\kappa_{\mathrm{hom},j}(R)| \le \kappa_{\max,j} < \infty$. Substituting $\kappa_{\min,j}$ or $\kappa_{\max,j}$ according to the monotonicity of Ψ_{grav} and Θ_{grav} yields rigorous, parameter-free bounds across non-homologous sequences.

Remark 2.8 (Independence of the Gravity Model). Definition 2.6 is completely independent of the specific gravitational action.

(i) In Newtonian gravity and standard General Relativity for spherical stars, $\mathcal{O} = (M, R)$, $\rho_\Gamma = \rho(R)/\rho_{\mathrm{rel}}$, and the continuity component of $\mathcal{D}_{\mathrm{grav}}$ is $m'(R) - 4\pi R^2 \rho(R) = 0$, which is of order $s = 1$ with $\Psi_{\mathrm{grav}}(R, M, M') = |M'|/(4\pi\kappa_{\mathrm{hom}}\rho_{\mathrm{rel}}R^2)$.

(ii) In Rastall gravity [13, 14], where $\nabla^\mu T_{\mu\nu} = \lambda \nabla_\nu \mathcal{R}$ ($\mathcal{R}$ being the Ricci scalar) and $\alpha = \kappa\lambda$ is the dimensionless Rastall parameter, the effective energy density is $\rho_{\mathrm{eff}} = \frac{(1-3\alpha)\rho - 3\alpha p}{1-4\alpha}$, modifying both the mass continuity equation and hydrostatic balance and yielding an α-dependent solver $\rho_\Gamma = \Psi_{\mathrm{grav}}(R, M, M', \alpha)$ (see Section 5.1).

(iii) In perturbative $f(R) = R + \alpha R^2$ (Starobinsky) gravity [10, 11], the modified continuity equation includes $\mathcal{O}(\alpha)$ higher-derivative terms involving $M''(R)$ and $M'''(R)$, giving a locally density-solvable structural equation of order $s = 3$.

2.3. **State Functions, Observable Relations, and Causal Cones in Locally Convex Spaces.** To handle multi-parameter equations of state, multi-observable empirical relations, and modified gravity couplings within a unified differential calculus on locally convex spaces, we work with ordered Banach spaces. Recall that a topological vector space X equipped with a closed convex cone $X_+ \subset X$ (satisfying $X_+ + X_+ \subset X_+$, $\mathbb{R}_{\ge 0} X_+ \subset X_+$, and $X_+ \cap (-X_+) = \{0\}$) carries a canonical partial order: $x_1 \preceq_X x_2$ if and only if $x_2 - x_1 \in X_+$. A map $\phi : U \subset X \to Y$ between ordered topological vector spaces is *co-monotone* (or *monotone non-decreasing*) if $x_1 \preceq_X x_2$ implies $\phi(x_1) \preceq_Y \phi(x_2)$, and *contra-monotone* (or *antitone / monotone non-increasing*) if $x_1 \preceq_X x_2$ implies $\phi(x_2) \preceq_Y \phi(x_1)$ (which is equivalent to being co-monotone with respect to the opposite cone $-Y_+$). When no ambiguity arises, "monotone" without qualification refers to co-monotonicity; when an argument is contra-monotone, upper bounds on the input invert to lower bounds on the output.

On the section spaces $C^k_{\mathrm{pw}}(\Sigma; W_p)$ and $C^l_{\mathrm{pw}}(\Sigma; W_\rho)$, fixing closed convex cones $W_{p,+} \subset W_p$ and $W_{\rho,+} \subset W_\rho$ in the fibers induces canonical pointwise positive cones: $\rho_1 \preceq \rho_2$ if and only if $\rho_2(x) - \rho_1(x) \in W_{\rho,+}|_x$ on a dense open subset of Σ, which by the closedness of $W_{\rho,+}$ immediately implies $\rho_{1,\Gamma} \preceq \rho_{2,\Gamma}$ under the interior limit or integral averaging map ev_Γ.

Let us also make precise how Implicit Function Theorems apply to state functions on $\mathcal{A}_{\mathrm{mat}} \times E$ (or $\mathcal{A} \times E$ for quasi-local equations of state that depend parametrically on Φ). As noted in [3], the classical Banach IFT [19] and the Nash–Moser theorem for Fréchet spaces cannot be applied globally to $C^{k,l,m}_{\mathrm{pw}}(\Sigma)$ because the latter is neither Banach nor Fréchet (and even its bounded connected component $C^{k,l,m}_{\mathrm{pw},b}(\Sigma)$ is an *incomplete* normed space, since uniform limits of piecewise differentiable sections with unfixed finite partitions may accumulate infinitely many jump discontinuities). However, Hiltunen [20] and Glöckner [21] established Implicit Function Theorems for maps $F : P \times Z \to Y$ where P is an arbitrary locally convex (or topological) vector space—which need not be complete—and only the solved variable space Z and target space Y must be complete Banach spaces. In our setting:

(a) On the $\mathcal{P}$-regular stratum $C^k_{\mathcal{P}}(\Sigma; W_p)$ associated with any fixed open partition $\mathcal{P} = \{\Sigma_a\}_{a=1}^{m_\mathcal{P}}$, the open extended-norm ball $B_{r,\mathcal{P}}(p_0) = \{p \in C^k_{\mathcal{P}}(\Sigma; W_p) : \|p - p_0\|_k < r\}$ around

$p_0 \in C^k_{\mathcal{P}}(\Sigma; W_p)$ is isometric to an open ball in the complete affine Banach space $p_0 + C^k_b(\bigcup_{a=1}^{m_{\mathcal{P}}} \Sigma_a; W_p)$;

(b) Whereas the classical Banach IFT [19] suffices when both p and ρ are restricted to a fixed partition $\mathcal{P}$ and a Banach parameter space E, the Hiltunen–Glöckner IFT [20, 21] applies more generally whenever the solved variable p and target Y_f lie in complete Banach spaces while the parameter variables (ρ, Φ, ϵ) vary in an arbitrary locally convex or incomplete normed space P. Specifically, for reference-locus relations evaluated via ev_Γ into the complete finite-dimensional Banach fiber $Y_f = W_{p,\Gamma}$ (for arbitrary degrees (k, l, m)), (ρ, Φ) may vary freely over an open extended-norm ball $B_{r_\rho}(\rho_0) \times B_{r_\Phi}(\Phi_0)$ of the incomplete normed space $C^l_{\mathrm{pw},b}(\Sigma; W_\rho) \times C^m_{\mathrm{pw},b}(\Sigma; W_{\mathrm{grav}})$ with unfixed partition; similarly, for Nemytskii (superposition) operators $f(p, \rho, \epsilon)(x) = f_0(p(x), \rho(x), \epsilon)$ induced by a C^{k+2} fiber map $f_0 : W_p \times W_\rho \times E \to W_p$ (with f_0 and its fiber derivatives up to order $k+2$ uniformly bounded on bounded fiber subsets of $W_p \times W_\rho$ so as to ensure continuous Fréchet differentiability on C^k_b, preserving the singular locus $\Sigma \setminus \bigcup_{a=1}^{m_{\mathcal{P}}} \Sigma_a$) when $l \geq k$, evaluated around a bounded reference configuration (p_0, ρ_0) on $B_{r_p,\mathcal{P}}(p_0) \times B_{r_\rho,\mathcal{P}}(\rho_0) \times W$ (and possibly depending parametrically on auxiliary fields Φ or parameters in an incomplete normed or locally convex space P), f maps $(p - p_0, \rho - \rho_0, \epsilon)$ from an open subset of a product of locally convex and Banach spaces into the complete Banach space $Y_f = C^k_b(\bigcup_{a=1}^{m_{\mathcal{P}}} \Sigma_a; W_p)$.

Consequently, whenever the partial Fréchet derivative $\partial_p f$ is an invertible continuous linear operator on the bounded Banach component around p_0, the Hiltunen–Glöckner IFT rigorously guarantees unique local solvability $p = \xi(\rho, \epsilon)$ and differentiability with respect to ρ and ϵ. Throughout Section 3, whenever local implicit solvers $(\xi, \eta, \Psi_{\mathrm{grav}}, \Upsilon, \Theta_{\mathrm{grav}})$ are evaluated along an $\mathcal{O}$-bounded subcluster up to extremal bound tuples $(\mathcal{O}_0, \{M_\beta(\delta)\}, \rho_{\mathrm{caus}}(\epsilon_0), \alpha_0)$, we implicitly restrict to subclusters contained within a single order-convex (box-adapted) Hiltunen–Glöckner solvability chart containing the order interval between the subcluster evaluations and the extremal bound tuple.

Definition 2.9 (State Functions and Generalized Speed of Sound)**.** Let $\mathcal{A}$ be an astrophysical cluster and let (E, E_+) be an ordered Banach space of thermodynamic/microphysical parameters. A *state function* on $\mathcal{A}$ is a map

$$f : \mathcal{A}_{\mathrm{mat}} \times E \to Y_f \tag{2.10}$$

(or $f : \mathcal{A} \times E \to Y_f$ for *quasi-local state functions* depending parametrically on gravitational potentials Φ or macroscopic observables $\mathcal{O}$, in which case partial derivatives with respect to p and ρ are taken at fixed Φ), where $(Y_f, Y_{f,+})$ is an ordered Banach space (such as $W_{p,\Gamma} = \mathbb{R}^r$ under ev_Γ or $C^k_b(\bigcup_{a=1}^{m_{\mathcal{P}}} \Sigma_a; W_p)$).

(1) We say that f is *locally integrable at* $p_0 \in C^k_{\mathcal{P}}(\Sigma; W_p)$ if the equation of state $f(p, \rho, \epsilon) = 0$ can be locally solved for p: there exist open neighborhoods $U \subset B_{r,\mathcal{P}}(p_0) \subset C^k_{\mathcal{P}}(\Sigma; W_p)$ of p_0, $V \subset C^l_{\mathcal{P}}(\Sigma; W_\rho)$ (or $U \subset W_{p,\Gamma}$ and $V \subset C^l_{\mathrm{pw},b}(\Sigma; W_\rho)$ factoring through $V_\Gamma \subset W_{\rho,\Gamma}$ under ev_Γ), and $W \subset E$, together with a map $\xi : V \times W \to U$, such that

$$f\big(\xi(\rho, \epsilon), \rho, \epsilon\big) = 0 \quad \text{for all } (\rho, \epsilon) \in V \times W. \tag{2.11}$$

(2) If the implicit pressure map $\xi(\rho, \epsilon)$ is monotone in both variables (ρ, ϵ) with respect to the positive pressure cone $W_{p,+}$ and $f(p, \rho, \epsilon)$ is monotone in (ρ, ϵ) with respect to $Y_{f,+}$, we say that f is a *monotonically locally integrable (MLI)* state function. Note that by implicit differentiation ($\partial_\epsilon \xi = -(\partial_p f)^{-1} \partial_\epsilon f$ and $\partial_\rho \xi = -(\partial_p f)^{-1} \partial_\rho f$), whenever f is C^1 and $(\partial_p f)^{-1} : Y_{f,+} \to -W_{p,+}$ is cone-inverting—as in canonical state functions $f(p, \rho, \epsilon) = p - \xi(\rho, \epsilon)$ with $\partial_p f = \mathrm{id} > 0$ and $Y_{f,+} := -W_{p,+}$—monotonicity of $\xi(\rho, \epsilon)$ in (ρ, ϵ) follows automatically from that of $f(p, \rho, \epsilon)$, with $f(p, \rho, \epsilon_1) \preceq_{Y_f} f(p, \rho, \epsilon_2) \iff f(p, \rho, \epsilon_1) \succeq_{W_p} f(p, \rho, \epsilon_2) \iff \xi(\rho, \epsilon_1) \preceq_{W_p} \xi(\rho, \epsilon_2)$. Throughout what follows, an upper (resp. lower) bound

on an MLI state function f refers equivalently to the physical pressure upper (resp. lower) bound $p = \xi(\rho, \epsilon) \preceq_{W_p} \xi(\rho, \epsilon_0)$ in $W_{p,+}$ and to $f(p, \rho, \epsilon) \preceq_{Y_f} f(p, \rho, \epsilon_0)$ in $Y_{f,+}$.

(3) When ξ is differentiable with respect to ρ, we define the *generalized squared speed of sound* (or *acoustic operator*) near p_0 as the partial derivative

$$v^2(\rho, \epsilon) := \partial_\rho \xi(\rho, \epsilon), \tag{2.12}$$

ordered by the positive operator cone mapping the positive density cone into the positive pressure cone (which for a pointwise Nemytskii equation of state $\xi(\rho, \epsilon)(x) = \xi_0(\rho(x), \epsilon)$ corresponds to the pointwise fiber bundle cone $\mathrm{Hom}_+(W_\rho, W_p)$ on a dense open subset of Σ, and when evaluated at the reference locus Γ gives a finite-dimensional matrix/vector of squared acoustic phase speeds $v^2(\rho_\Gamma, \epsilon) \in \mathrm{Hom}(W_{\rho,\Gamma}, W_{p,\Gamma})$ ordered by $\mathrm{Hom}_+(W_{\rho,\Gamma}, W_{p,\Gamma})$). If f is MLI and $v^2(\rho, \epsilon)$ is also monotone in (ρ, ϵ), we say that f is *fully monotonically locally integrable (fully MLI)* at p_0.

Definition 2.10 (Causal Cone Condition)**.** Let f be a fully MLI state function at p_0 on an astrophysical cluster $\mathcal{A}$. A *causal condition* for f near p_0 is an order upper bound

$$v^2(\rho_\Gamma, \epsilon) \preceq v_0^2(\rho_\Gamma), \tag{2.13}$$

where $v_0^2(\rho_\Gamma)$ is a reference threshold function (e.g., the squared speed of light $c^2 = 1$ in relativistic theories, or the maximum stable Alfvén/acoustic phase speed in magneto-hydrodynamic or anisotropic media). We say that the causal condition is *locally monotone invertible for* ϵ if the map $\epsilon \mapsto v^2(\rho_\Gamma, \epsilon)$ can be locally inverted to write $\epsilon = \Upsilon(v^2, \rho_\Gamma)$ monotonically in v^2, and the composite threshold map $\rho_\Gamma \mapsto \Upsilon\big(v_0^2(\rho_\Gamma), \rho_\Gamma\big)$ is monotone (either co-monotone or contra-monotone, with respect to fixed cones). Similarly, we say that (2.13) is *locally monotone invertible for* ρ_Γ if (2.13) is equivalent to an upper bound $\rho_\Gamma \preceq \rho_{\mathrm{caus}}(\epsilon)$ for a monotone map $\epsilon \mapsto \rho_{\mathrm{caus}}(\epsilon)$.

Definition 2.11 (Observable Relation Function)**.** Let $(\mathcal{A}, \mathcal{O})$ be an astrophysical cluster with N macroscopic observables $\mathcal{O} = (\mathcal{O}_1, \dots, \mathcal{O}_N) \in C^0_{\mathrm{pw}}(\mathcal{A}; \mathbb{R}^N)$, and let (F, F_+) be an ordered Banach space of empirical/macroscopic parameters. An *observable relation function* is a map

$$g : \mathbb{R}^N \times F \to Y_g \tag{2.14}$$

(inducing by post-composition a map on $\big(C^0_{\mathrm{pw}}(\mathcal{A}; \mathbb{R})\big)^N \times F$) that defines the macroscopic scaling relation $g(\mathcal{O}_1, \dots, \mathcal{O}_N, \delta) = 0$ on the observable values $\mathcal{O}(\mathcal{S}) \in \mathbb{R}^N$.

(1) We say that g is *parameter-solvable* (or a *locally monotone observable function with respect to* δ) if $g(\mathcal{O}, \delta) = 0$ can be locally solved for δ as $\delta = \eta(\mathcal{O}_1, \dots, \mathcal{O}_N)$, where η is monotone in each observable $\mathcal{O}_i$.

(2) Given a partition of the observables into a dependent observable $\mathcal{O}_{\mathrm{dep}} = \mathcal{O}_1$ (or more generally a dependent tuple $\mathcal{O}_{\mathrm{dep}} \in \mathbb{R}^{N_{\mathrm{dep}}}$, such as total mass M) and independent observables $\mathcal{O}_{\mathrm{ind}} = \mathcal{O}' = (\mathcal{O}_2, \dots, \mathcal{O}_N)$ (such as radius R, tidal deformability Λ_{tidal}, etc.), we say that g is *observable-solvable for* $\mathcal{O}_1$ if $g(\mathcal{O}_1, \mathcal{O}', \delta) = 0$ can be locally solved as $\mathcal{O}_1 = \mu(\mathcal{O}', \delta)$ with μ piecewise C^s-differentiable in $\mathcal{O}'$.

3. Gravity-Independent and Inverse Obstruction Theorems

We are now ready to state and prove four general obstruction theorems that hold across any astrophysical cluster $\mathcal{A}$, regardless of the underlying gravitational action or spatial geometry.

3.1. Direct Multi-Observable and Gravity-Coupled Obstructions.

Our first theorem extends Theorem III.1 of [3] from (M, R) stellar clusters to arbitrary N-observable astrophysical systems.

Theorem 3.1 (Direct Multi-Observable Obstruction Theorem)**.** *Let $\mathcal{C} = (\mathcal{A}, \mathcal{O}_1, \dots, \mathcal{O}_N)$ be a cluster of astrophysical systems of degree (k, l, m) over Σ, endowed with a parameter-solvable, locally monotone observable relation $g(\mathcal{O}_1, \dots, \mathcal{O}_N, \epsilon) = 0$, where ϵ takes values in an ordered Banach*

space (E, E_+). Then any simultaneous upper (resp. lower) bounds $\mathcal{O}_i \le \mathcal{O}_{i,0}$ (resp. $\mathcal{O}_i \ge \mathcal{O}_{i,0}$) on the observables $i = 1, \dots, N$ induce an upper (resp. lower) bound in $W_{p,+}$ on the implicit pressure operator $p = \xi(\rho, \epsilon)$ and in $Y_{f,+}$ on every MLI state function $f(p, \rho, \epsilon)$ of $\mathcal{C}$ depending on ϵ (and possibly on additional fixed parameters). Furthermore, if f is fully MLI at some p_0, then these bounds descend to the generalized squared speed of sound $v^2(\rho, \epsilon)$ near p_0.

Proof. We establish the statement for upper bounds with co-monotone η; the proof for lower bounds or contra-monotone arguments is completely dual by reversing the corresponding order cones. By Definition 2.11(1), the observable relation $g(\mathcal{O}_1, \dots, \mathcal{O}_N, \epsilon) = 0$ can be locally solved for the parameter ϵ in a neighborhood of a reference value, yielding

$$\epsilon = \eta(\mathcal{O}_1, \dots, \mathcal{O}_N). \tag{3.1}$$

Suppose the cluster (or the $\mathcal{O}$-bounded subcluster under consideration) satisfies the macroscopic observable bounds $\mathcal{O}_i \le \mathcal{O}_{i,0}$ for each $i = 1, \dots, N$. Since η is locally monotone non-decreasing in each of its N arguments with respect to the order cone E_+, repeated application of monotonicity yields the chain of inequalities

$$\epsilon = \eta(\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_N) \preceq_E \eta(\mathcal{O}_{1,0}, \mathcal{O}_2, \dots, \mathcal{O}_N) \preceq_E \cdots \preceq_E \eta(\mathcal{O}_{1,0}, \mathcal{O}_{2,0}, \dots, \mathcal{O}_{N,0}) =: \epsilon_0. \tag{3.2}$$

Thus the microphysical parameter ϵ is bounded from above in (E, E_+) by ϵ_0.

Now let $f(p, \rho, \epsilon)$ be any MLI state function on $\mathcal{C}$ depending monotonically on ϵ. From $\epsilon \preceq_E \epsilon_0$, it follows immediately that for every (p, ρ) in the local solvability domain, the implicit pressure solution $p = \xi(\rho, \epsilon)$ and the state function f satisfy

$$\xi(\rho, \epsilon) \preceq_{W_p} \xi(\rho, \epsilon_0) \quad \text{and} \quad f(p, \rho, \epsilon) \preceq_{Y_f} f(p, \rho, \epsilon_0) \tag{3.3}$$

(where for canonical state functions $f(p, \rho, \epsilon) = p - \xi(\rho, \epsilon)$ with $Y_{f,+} = -W_{p,+}$, the latter is equivalent to $f(p, \rho, \epsilon) \succeq_{W_p} f(p, \rho, \epsilon_0)$). Finally, if f is fully MLI at p_0, then by Definition 2.9(3) the generalized squared speed of sound $v^2(\rho, \epsilon) = \partial_\rho \xi(\rho, \epsilon)$ is well-defined near p_0 and depends monotonically on ϵ, which implies

$$v^2(\rho, \epsilon) \preceq v^2(\rho, \epsilon_0), \tag{3.4}$$

completing the proof. □

Theorem 3.1 applies directly whenever the state function f and the observable relation g share a common parameter $\epsilon \in E$—for instance, in effective halo equations of state parametrized directly by empirical scaling constants (see Corollary 5.3(1)) or in homologous polytropic families where the macroscopic mass-radius normalization is an explicit monotone function of the polytropic stiffness. In more general astrophysical settings, however, the empirical observable relation $g(\mathcal{O}, \delta) = 0$ depends on macroscopic fitting parameters $\delta \in F$, whereas the microphysical equation of state $f(p, \rho, \epsilon) = 0$ depends on independent thermodynamic parameters $\epsilon \in E$. The bridge between the two is then provided by the gravity-matter coupling operator $\mathcal{D}_{\text{grav}}(\mathcal{S}, \alpha) = 0$ and the reference evaluation map ev_Γ.

Theorem 3.2 (Gravity-Coupled Structural Obstruction Theorem)**.** *Let $\mathcal{C} = (\mathcal{A}, \mathcal{O}_1, \mathcal{O}')$ be an astrophysical cluster of degree (k, l, m) over Σ, where $\mathcal{O}_1$ is piecewise C^s-differentiable with respect to the remaining observables $\mathcal{O}' = (\mathcal{O}_2, \dots, \mathcal{O}_N)$. Assume that $\mathcal{C}$ is endowed with:*

(i) an observable-solvable relation $g(\mathcal{O}_1, \mathcal{O}', \delta) = 0$ allowing one to write $\mathcal{O}_1 = \mu(\mathcal{O}', \delta)$;

(ii) a monotonically density-solvable gravity-matter coupling operator $\mathcal{D}_{\text{grav}}(\mathcal{S}, \alpha) = 0$ of order s in the sense of Definition 2.6, with reference density solver $\rho_\Gamma = \Psi_{\text{grav}}(J^s\mathcal{O}(\delta), \alpha)$.

Then any upper (resp. lower) bounds on the independent observables $\mathcal{O}' \le \mathcal{O}'_0$ (or $\mathcal{O}' \ge \mathcal{O}'_{\min}$ for components in which Ψ_{grav} is contra-monotone), on the active geometric observable jets $\partial^\beta_{\mathcal{O}'}\mu(\mathcal{O}', \delta) \le M_\beta(\delta)$ entering Ψ_{grav} (for multi-indices $\beta \in \mathbb{Z}^{N-1}_{\ge 0}$ with $0 \le |\beta| \le s$), and on the gravitational parameter $\alpha \preceq_{A_{\text{grav}}} \alpha_0$ induce an upper (resp. lower) bound depending on (δ, α_0) on the reference

density ρ_Γ and hence on the implicit pressure map $\xi(\rho_\Gamma, \epsilon)$ in $W_{p,+}$ and on every MLI state function $f(p_\Gamma, \rho_\Gamma, \epsilon)$ in $Y_{f,+}$, which descends to the generalized speed of sound $v^2(\rho_\Gamma, \epsilon)$ whenever f is fully MLI at p_0.

Proof. Since $g(\mathcal{O}_1, \mathcal{O}', \delta) = 0$ is observable-solvable for $\mathcal{O}_1$, we locally express $\mathcal{O}_1 = \mu(\mathcal{O}', \delta)$, so that the s-jet of the observables along the cluster takes the form

$$J^s\mathcal{O}(\delta) = \Big(\mathcal{O}',\, \mu(\mathcal{O}', \delta),\, \partial_{\mathcal{O}'}\mu(\mathcal{O}', \delta),\, \ldots,\, \partial^s_{\mathcal{O}'}\mu(\mathcal{O}', \delta)\Big). \tag{3.5}$$

By the hypothesis that the gravity-matter coupling $\mathcal{D}_{\mathrm{grav}}(\mathcal{S}, \alpha) = 0$ is monotonically density-solvable of order s, the reduced structural equation $\mathcal{K}(J^s\mathcal{O}(\delta), \rho_\Gamma, \alpha) = 0$ is locally solved for the reference density $\rho_\Gamma = \mathrm{ev}_\Gamma(\rho)$ as

$$\rho_\Gamma(\mathcal{O}', \delta, \alpha) = \Psi_{\mathrm{grav}}\big(J^s\mathcal{O}(\delta), \alpha\big). \tag{3.6}$$

Imposing the bounds $\mathcal{O}' \le \mathcal{O}'_0$ (or $\mathcal{O}' \ge \mathcal{O}'_{\min}$ for contra-monotone components), $\partial^\beta_{\mathcal{O}'}\mu(\mathcal{O}', \delta) \le M_\beta(\delta)$ for the active jet multi-indices $0 \le |\beta| \le s$, and $\alpha \preceq_{A_{\mathrm{grav}}} \alpha_0$, the monotonicity of Ψ_{grav} yields an explicit upper bound on the reference density:

$$\rho_\Gamma(\mathcal{O}', \delta, \alpha) \preceq \Psi_{\mathrm{grav}}\big(\mathcal{O}'_0, \{M_\beta(\delta)\}_{|\beta|\le s}, \alpha_0\big) =: \rho_{\Gamma,0}(\delta, \alpha_0). \tag{3.7}$$

Now let $f(p_\Gamma, \rho_\Gamma, \epsilon)$ be any MLI state function evaluated at the reference locus Γ. Since the local pressure solution $p_\Gamma = \xi(\rho_\Gamma, \epsilon)$ (and f itself) depends monotonically on ρ_Γ, substituting (3.7) provides the upper bound

$$\xi(\rho_\Gamma, \epsilon) \preceq_{W_p} \xi\big(\rho_{\Gamma,0}(\delta, \alpha_0), \epsilon\big) \quad \text{and} \quad f(p_\Gamma, \rho_\Gamma, \epsilon) \preceq_{Y_f} f\big(p_\Gamma, \rho_{\Gamma,0}(\delta, \alpha_0), \epsilon\big). \tag{3.8}$$

When f is fully MLI at p_0, the generalized squared speed of sound $v^2(\rho_\Gamma, \epsilon) = \partial_\rho\xi(\rho_\Gamma, \epsilon)$ is monotone in ρ_Γ, so that $v^2(\rho_\Gamma, \epsilon) \preceq v^2\big(\rho_{\Gamma,0}(\delta, \alpha_0), \epsilon\big)$ near p_0. □

3.2. **Causal Lifting and the Inverse Obstruction Theorem for Gravity Models.** In Theorems 3.1 and 3.2, bounds on observables and gravity couplings propagate forward to constrain the state function and the speed of sound. Conversely, physical principles such as relativistic causality or thermodynamic stability impose *a priori* bounds on the speed of sound $v^2(\rho_\Gamma, \epsilon) \preceq v_0^2(\rho_\Gamma)$. Our third theorem shows that in any gravity model with a density-solvable coupling operator $\mathcal{D}_{\mathrm{grav}}$, causal bounds lift back to constrain the equation of state.

Theorem 3.3 (Causal Lifting Theorem in General Gravity)**.** *Let $\mathcal{C} = (\mathcal{A}, \mathcal{O}_1, \mathcal{O}')$ be an astrophysical cluster of degree (k, l, m) over Σ, with $\mathcal{O}_1$ piecewise C^s-differentiable in $\mathcal{O}'$, endowed with:*

(i) a fully MLI state function $f(p, \rho, \epsilon)$ at p_0;
(ii) an observable-solvable relation $g(\mathcal{O}_1, \mathcal{O}', \delta) = 0$ with $\mathcal{O}_1 = \mu(\mathcal{O}', \delta)$;
(iii) a monotonically density-solvable gravity-matter coupling operator $\mathcal{D}_{\mathrm{grav}}(\mathcal{S}, \alpha) = 0$ of order s.

Then any causal condition $v^2(\rho_\Gamma, \epsilon) \preceq v_0^2(\rho_\Gamma)$ on f that is locally monotone invertible for ϵ, together with bounds $\mathcal{O}_i \le \mathcal{O}_{i,*}$ (or $\mathcal{O}_i \ge \mathcal{O}_{i,*}$ for components $\mathcal{O}_i$ of $\mathcal{O}'$ in which the composite threshold map $\widetilde{\epsilon}$ in (3.11) is contra-monotone), active geometric jet bounds $\partial^\beta_{\mathcal{O}'}\mu(\mathcal{O}', \delta) \le M_\beta(\delta)$ (for the multi-indices $\beta \in \mathbb{Z}^{N-1}_{\ge 0}$ with $0 \le |\beta| \le s$ entering Ψ_{grav}, which hold automatically at $\mathcal{O}' = \mathcal{O}'_*$ whenever the active jet components have the matching monotonicity in $\mathcal{O}'$), and $\alpha \preceq \alpha_0$ on the gravitational parameters, induces an upper bound on the microphysical parameter $\epsilon \preceq \epsilon_0(\delta, \alpha_0)$ and consequently an upper bound on the implicit pressure map $\xi(\rho, \epsilon) \preceq_{W_p} \xi(\rho, \epsilon_0(\delta, \alpha_0))$ and on the state function $f(p, \rho, \epsilon) \preceq_{Y_f} f(p, \rho, \epsilon_0(\delta, \alpha_0))$ depending only on $(\mathcal{O}'_*, \delta, \alpha_0)$.

Proof. Using the observable relation $\mathcal{O}_1 = \mu(\mathcal{O}', \delta)$ and the gravity-matter coupling solver (3.6), the reference density along the cluster is given by $\rho_\Gamma(\mathcal{O}', \delta, \alpha) = \Psi_{\mathrm{grav}}(J^s\mathcal{O}(\delta), \alpha)$. Substituting this expression into the generalized squared speed of sound $v^2(\rho_\Gamma, \epsilon)$ transforms it into a composite map of the observables, observable parameters, gravitational parameters, and microphysical parameters:

$$\widetilde{v}^2(\mathcal{O}', \delta, \alpha, \epsilon) := v^2\big(\Psi_{\mathrm{grav}}(J^s\mathcal{O}(\delta), \alpha), \epsilon\big). \tag{3.9}$$

The causal condition (2.13) then reads

$$v^2\big(\rho_\Gamma(\mathcal{O}',\delta,\alpha),\epsilon\big) \preceq v_0^2\big(\rho_\Gamma(\mathcal{O}',\delta,\alpha)\big). \tag{3.10}$$

By the local monotone invertibility of the causal condition for ϵ (Definition 2.10), we can invert this inequality for the state parameter ϵ to obtain

$$\epsilon \preceq_E \Upsilon\Big(v_0^2\big(\rho_\Gamma(\mathcal{O}',\delta,\alpha)\big),\ \rho_\Gamma(\mathcal{O}',\delta,\alpha)\Big) =: \widetilde{\epsilon}(\mathcal{O}',\delta,\alpha). \tag{3.11}$$

Applying the observable, jet, and gravity parameter bounds $\mathcal{O}_i \le \mathcal{O}_{i,*}$ (or $\mathcal{O}_i \ge \mathcal{O}_{i,*}$ when $\widetilde{\epsilon}$ is contra-monotone in $\mathcal{O}_i$), $\partial^\beta_{\mathcal{O}'}\mu \le M_\beta(\delta)$, and $\alpha \preceq_{A_{\text{grav}}} \alpha_0$, the monotonicity of the composite threshold map $\widetilde{\epsilon}$ yields the parameter bound

$$\epsilon \preceq_E \widetilde{\epsilon}(\mathcal{O}'_*,\delta,\alpha_0) =: \epsilon_0(\delta,\alpha_0). \tag{3.12}$$

Since $f(p,\rho,\epsilon)$ is MLI, this upper bound on ϵ immediately induces the desired upper bound on the pressure solution $\xi(\rho,\epsilon) \preceq_{W_p} \xi\big(\rho,\epsilon_0(\delta,\alpha_0)\big)$ and on $f(p,\rho,\epsilon) \preceq_{Y_f} f\big(p,\rho,\epsilon_0(\delta,\alpha_0)\big)$ in terms of $(\mathcal{O}'_*,\delta,\alpha_0)$. □

Remark 3.4 (Reference-Locus vs. Global Spatial Profile Bounds). Notice that in Theorem 3.2, the gravity coupling bounds the local reference density $\rho_\Gamma = \text{ev}_\Gamma(\rho)$, which directly constrains the reference-locus state function $f(p_\Gamma,\rho_\Gamma,\epsilon)$ (and lifts to the full spatial profile $\xi(\rho(x),\epsilon)$ on Σ whenever the cluster satisfies a dominant envelope condition $\rho(x) \preceq C(x)\rho_\Gamma$). By contrast, in Theorem 3.3, the causal condition at Γ constrains the spatially homogeneous microphysical parameter $\epsilon \preceq_E \epsilon_0(\delta,\alpha_0)$ in E, which automatically bounds the full Nemytskii operator $\xi(\rho,\epsilon) \preceq_{W_p} \xi(\rho,\epsilon_0(\delta,\alpha_0))$ across all of Σ.

We now formulate a fourth theorem that has no counterpart in [3]: the *Inverse Obstruction Theorem*. Because our axiomatic framework places the gravitational parameter $\alpha \in A_{\text{grav}}$ on the same footing as the thermodynamic parameter $\epsilon \in E$ and the macroscopic observable parameter $\delta \in F$, we can invert the causal-structural chain with respect to α. Physically, this addresses a central problem in relativistic astrophysics and cosmology: *if the equation of state of matter is known within a trusted microphysical regime (e.g., from nuclear laboratory data or chiral effective field theory) and we observe an empirical scaling relation $g(\mathcal{O},\delta)=0$, what theories of gravity $\alpha \in A_{\text{grav}}$ are compatible with causality and equilibrium?*

Definition 3.5 (Gravity-Invertible Coupling Operator)**.** Let $\mathcal{D}_{\text{grav}} : \mathcal{A} \times A_{\text{grav}} \to \mathcal{V}$ be a gravity-matter coupling operator with reduced structural equation $\mathcal{K}(J^s\mathcal{O}(\delta),\rho_\Gamma,\alpha)=0$, where A_{grav} is an ordered Banach space. We say that $\mathcal{D}_{\text{grav}}$ is *locally monotonically α-invertible* if $\mathcal{K}(J^s\mathcal{O}(\delta),\rho_\Gamma,\alpha)=0$ can be locally solved both for $\rho_\Gamma = \Psi_{\text{grav}}(J^s\mathcal{O}(\delta),\alpha)$ and for the gravitational parameter α as

$$\alpha = \Theta_{\text{grav}}\big(J^s\mathcal{O}(\delta),\rho_\Gamma\big), \tag{3.13}$$

where Θ_{grav} is monotone (either co-monotone or contra-monotone) in each argument. (By the Implicit Function Theorem for locally convex spaces [20, 21], when A_{grav} is Banach, this holds in a neighborhood of any regular point where $\partial_\alpha\mathcal{K}$ and $\partial_{\rho_\Gamma}\mathcal{K}$ are invertible continuous linear operators.)

Theorem 3.6 (Inverse Obstruction Theorem: Constraining Gravity Models from EoS and Observables)**.** *Let $\mathcal{C} = (\mathcal{A},\mathcal{O}_1,\mathcal{O}')$ be an astrophysical cluster of degree (k,l,m) over Σ, endowed with:*

- *(i) a fully MLI state function $f(p,\rho,\epsilon)$ satisfying a causal condition $v^2(\rho_\Gamma,\epsilon) \preceq v_0^2(\rho_\Gamma)$ that is locally monotonically invertible for the reference density ρ_Γ, in the sense that causality implies an upper bound $\rho_\Gamma \preceq \rho_{\text{caus}}(\epsilon)$;*
- *(ii) an observable-solvable relation $g(\mathcal{O}_1,\mathcal{O}',\delta)=0$ with $\mathcal{O}_1 = \mu(\mathcal{O}',\delta)$ piecewise C^s and monotone active geometric jet components $\partial^\beta_{\mathcal{O}'}\mu(\mathcal{O}',\delta)$ $(0 \le |\beta| \le s)$ entering Θ_{grav};*
- *(iii) a locally monotonically α-invertible gravity-matter coupling operator $\mathcal{D}_{\text{grav}}(\mathcal{S},\alpha)=0$ of order s with gravity solver $\alpha = \Theta_{\text{grav}}(J^s\mathcal{O}(\delta),\rho_\Gamma)$.*

Suppose the microphysical equation of state parameter ϵ is constrained so that $\rho_{\mathrm{caus}}(\epsilon) \preceq \rho_{\mathrm{caus}}(\epsilon_0)$ (for instance, $\epsilon \preceq_E \epsilon_0$ when ρ_{caus} is co-monotone, or $\epsilon \succeq_E \epsilon_0$ when ρ_{caus} is contra-monotone, as in polytropes where $\rho_{\mathrm{caus}}(k,\gamma) = (k\gamma)^{-1/(\gamma-1)}$), and each independent observable $\mathcal{O}_i$ ($i = 2,\dots,N$) in $\mathcal{O}'$ is bounded by an extremal value $\mathcal{O}_{i,*}$ matching the monotonicity of the composite map $\mathcal{O}_i \mapsto \Theta_{\mathrm{grav}}(J^s\mathcal{O}(\delta), \rho_\Gamma)$ (namely, $\mathcal{O}_i \le \mathcal{O}_{i,*}$ in Case (a) when co-monotone and $\mathcal{O}_i \ge \mathcal{O}_{i,*}$ when contra-monotone, and conversely $\mathcal{O}_i \le \mathcal{O}_{i,*}$ in Case (b) when contra-monotone and $\mathcal{O}_i \ge \mathcal{O}_{i,*}$ when co-monotone). Writing $\mathcal{O}'_* = (\mathcal{O}_{2,*},\dots,\mathcal{O}_{N,*})$, we have:

(a) If $\Theta_{\mathrm{grav}}(J^s\mathcal{O}(\delta), \rho_\Gamma)$ is co-monotone *(non-decreasing) in ρ_Γ, the gravitational parameter space A_{grav} obeys the* a priori *upper bound*

$$\alpha \preceq_{A_{\mathrm{grav}}} \Theta_{\mathrm{grav}}\Big(J^s\mathcal{O}(\delta)\big|_{\mathcal{O}'=\mathcal{O}'_*},\, \rho_{\mathrm{caus}}(\epsilon_0)\Big) =: \alpha_{\max}(\mathcal{O}'_*, \delta, \epsilon_0). \tag{3.14}$$

(b) If $\Theta_{\mathrm{grav}}(J^s\mathcal{O}(\delta), \rho_\Gamma)$ is contra-monotone *(non-increasing) in ρ_Γ, the gravitational parameter space A_{grav} obeys the* a priori *lower bound*

$$\alpha \succeq_{A_{\mathrm{grav}}} \Theta_{\mathrm{grav}}\Big(J^s\mathcal{O}(\delta)\big|_{\mathcal{O}'=\mathcal{O}'_*},\, \rho_{\mathrm{caus}}(\epsilon_0)\Big) =: \alpha_{\min}(\mathcal{O}'_*, \delta, \epsilon_0). \tag{3.15}$$

In particular, any gravitational model whose coupling parameter α violates (3.14) (resp. (3.15)) cannot simultaneously accommodate the empirical observable relation $g(\mathcal{O}, \delta) = 0$ and a causal equation of state in the given microphysical regime.

Proof. Since the causal condition $v^2(\rho_\Gamma, \epsilon) \preceq v_0^2(\rho_\Gamma)$ is locally monotonically invertible with respect to ρ_Γ, solving the causal inequality for ρ_Γ yields the maximum causally admissible reference density for a given microphysical parameter ϵ:

$$\rho_\Gamma \preceq \rho_{\mathrm{caus}}(\epsilon). \tag{3.16}$$

Under the microphysical constraint guaranteeing $\rho_{\mathrm{caus}}(\epsilon) \preceq \rho_{\mathrm{caus}}(\epsilon_0)$, equation (3.16) implies the uniform causal density ceiling

$$\rho_\Gamma \preceq \rho_{\mathrm{caus}}(\epsilon_0). \tag{3.17}$$

On the other hand, by Definition 3.5, the reduced structural equation $\mathcal{K}\big(J^s\mathcal{O}(\delta), \rho_\Gamma, \alpha\big) = 0$ can be solved locally for the gravitational theory parameter α as

$$\alpha = \Theta_{\mathrm{grav}}\big(J^s\mathcal{O}(\delta), \rho_\Gamma\big), \tag{3.18}$$

where $J^s\mathcal{O}(\delta)$ is determined by the empirical observable relation $\mathcal{O}_1 = \mu(\mathcal{O}', \delta)$. In Case (a), where Θ_{grav} is co-monotone in ρ_Γ, substituting (3.17) gives $\alpha \preceq_{A_{\mathrm{grav}}} \Theta_{\mathrm{grav}}\big(J^s\mathcal{O}(\delta), \rho_{\mathrm{caus}}(\epsilon_0)\big)$, and evaluating at the observable bounds $\mathcal{O}_i \le \mathcal{O}_{i,*}$ (for co-monotone components of $\mathcal{O}'$) and $\mathcal{O}_i \ge \mathcal{O}_{i,*}$ (for contra-monotone components of $\mathcal{O}'$) preserves the upper bound and yields (3.14). In Case (b), where Θ_{grav} is contra-monotone in ρ_Γ, the upper bound (3.17) on ρ_Γ reverses the inequality to $\alpha \succeq_{A_{\mathrm{grav}}} \Theta_{\mathrm{grav}}\big(J^s\mathcal{O}(\delta), \rho_{\mathrm{caus}}(\epsilon_0)\big)$, and evaluating at the observable bounds $\mathcal{O}_i \le \mathcal{O}_{i,*}$ (for contra-monotone components of $\mathcal{O}'$) and $\mathcal{O}_i \ge \mathcal{O}_{i,*}$ (for co-monotone components of $\mathcal{O}'$) preserves the lower bound to produce (3.15). □

Remark 3.7 (Multi-Scale Intersection Falsifiability Test). When N_{cl} distinct astrophysical clusters $\{\mathcal{C}^{(a)}\}_{a=1}^{N_{\mathrm{cl}}}$ across different physical scales or density regimes (e.g., neutron stars, white dwarfs, and rotation-supported galaxies) are governed by the same fundamental gravitational theory parameter $\alpha \in A_{\mathrm{grav}}$, Theorem 3.6 constrains α simultaneously to the intersection of the admissible order intervals $\bigcap_{a=1}^{N_{\mathrm{cl}}} \big[\alpha_{\min}^{(a)}, \alpha_{\max}^{(a)}\big] \subset A_{\mathrm{grav}}$. Whenever this intersection is empty, the modified gravity family is falsified across scales without numerical integration of the interior field equations.

4. Topological Rigidity of Gravity-Coupled Astrophysical Clusters

In [4], we formulated the classification problem for TOV clusters and observed (Remark 1 of [4]) that topological connectedness prevents a finite-dimensional TOV cluster from being invariant under arbitrary small perturbations in a Hausdorff locally convex topology. Here we show that this topological rigidity is a universal feature of any gravity-coupled astrophysical cluster, independent of the specific gravitational field equations.

Definition 4.1 (Gravity-Generated Astrophysical Cluster)**.** Let $\mathcal{D}_{\text{grav}} : C^{k,l,m}_{\text{pw}}(\Sigma) \times A_{\text{grav}} \to \mathcal{V}$ be a gravity-matter coupling operator. For each fixed gravitational parameter $\alpha \in A_{\text{grav}}$, the *nonlinear equilibrium solution locus* is $\mathcal{M}_{\text{sol}}(\alpha) := \{\mathcal{S} \in C^{k,l,m}_{\text{pw}}(\Sigma) : \mathcal{D}_{\text{grav}}(\mathcal{S}, \alpha) = 0\}$, and the *$\alpha$-gravity astrophysical cluster* of degree (k, l, m) over Σ, denoted by $\mathcal{A}^{k,l,m}_{\Sigma}(\alpha)$, is the vector subspace of $C^{k,l,m}_{\text{pw}}(\Sigma)$ generated by (i.e., the linear span of) $\mathcal{M}_{\text{sol}}(\alpha)$:

$$\mathcal{A}^{k,l,m}_{\Sigma}(\alpha) := \text{span}\big(\mathcal{M}_{\text{sol}}(\alpha)\big) = \text{span}\Big\{\mathcal{S} \in C^{k,l,m}_{\text{pw}}(\Sigma) : \mathcal{D}_{\text{grav}}(\mathcal{S}, \alpha) = 0\Big\}. \tag{4.1}$$

We say that $\mathcal{D}_{\text{grav}}(\cdot, \alpha)$ is *non-degenerate and selective* on $C^{k,l,m}_{\text{pw}}(\Sigma)$ if $\mathcal{M}_{\text{sol}}(\alpha)$ contains at least one non-zero astrophysical system $\mathcal{S}_* \neq 0$, and there exists at least one bounded test configuration $\mathcal{S}_{\text{test}} \in C^{k,l,m}_{\text{pw},b}(\Sigma)$ that does not belong to $\mathcal{A}^{k,l,m}_{\Sigma}(\alpha)$.

We say that a subset X of a topological vector space or topological group W is *invariant under arbitrary small perturbations* if for every $x \in X$ there exists an open neighborhood $U \subset W$ of 0 such that $x + U \subset X$; equivalently, X is an open subset of W.

Theorem 4.2 (Universal Topological Rigidity of Gravity-Coupled Clusters)**.** *Let Σ be a smooth manifold (possibly with corners or piecewise smooth boundary $\partial\Sigma$) and let $\mathcal{D}_{\text{grav}}(\cdot, \alpha)$ be any non-degenerate and selective gravity-matter coupling operator on $C^{k,l,m}_{\text{pw}}(\Sigma)$. Let τ be either any locally convex vector topology on $C^{k,l,m}_{\text{pw}}(\Sigma)$ or the canonical extended-norm topology $\tau_{k,l,m}$ of Definition 2.1 (whose connected component of the origin is the locally convex normed vector space $W_0 = C^{k,l,m}_{\text{pw},b}(\Sigma)$). Then the following three conditions cannot hold simultaneously:*

(1) The topology τ is Hausdorff;
(2) The gravity-generated cluster $\mathcal{A}^{k,l,m}_{\Sigma}(\alpha)$ is finite-dimensional;
(3) The cluster $\mathcal{A}^{k,l,m}_{\Sigma}(\alpha)$ is invariant under arbitrary small τ-perturbations.

In fact, whenever τ is Hausdorff, every vector subspace $\mathcal{A} \subset C^{k,l,m}_{\text{pw}}(\Sigma)$ that does not contain the entire connected component of the origin W_0 (where $W_0 = C^{k,l,m}_{\text{pw}}(\Sigma)$ for a vector topology and $W_0 = C^{k,l,m}_{\text{pw},b}(\Sigma)$ for $\tau_{k,l,m}$) has empty τ-interior, and every finite-dimensional gravity-generated cluster $\mathcal{A}^{k,l,m}_{\Sigma}(\alpha)$ is closed and nowhere dense (so that its underlying equilibrium subset $\mathcal{M}_{\text{sol}}(\alpha)$ is likewise nowhere dense, and is also closed whenever $\mathcal{D}_{\text{grav}}(\cdot, \alpha)$ is τ-continuous into a Hausdorff space $\mathcal{V}$).

Proof. Let W_0 denote the connected component of the origin 0 in $\big(C^{k,l,m}_{\text{pw}}(\Sigma), \tau\big)$, so that $W_0 = C^{k,l,m}_{\text{pw}}(\Sigma)$ when τ is a topological vector space topology and $W_0 = C^{k,l,m}_{\text{pw},b}(\Sigma)$ when $\tau = \tau_{k,l,m}$ is the canonical extended-norm topology. In either case, $(W_0, \tau|_{W_0})$ is a locally convex topological vector space over $\mathbb{R}$, hence path-connected (and contractible via the linear homotopy $H(t, \mathcal{S}) = t\mathcal{S}$ on W_0), and W_0 is open in $\big(C^{k,l,m}_{\text{pw}}(\Sigma), \tau\big)$.

Suppose that condition (3) holds, so that $\mathcal{A}^{k,l,m}_{\Sigma}(\alpha)$ is an open subset of $\big(C^{k,l,m}_{\text{pw}}(\Sigma), \tau\big)$. Then $\mathcal{A}_0 := \mathcal{A}^{k,l,m}_{\Sigma}(\alpha) \cap W_0$ is a non-empty (since $0 \in \mathcal{A}_0$) open vector subspace of W_0. If (1) and (2) also hold, then $\mathcal{A}_0$ is a finite-dimensional vector subspace of the Hausdorff topological vector space W_0, and therefore $\mathcal{A}_0$ is automatically closed in W_0 [22, 23]. Since W_0 is connected and $\mathcal{A}_0 \neq \emptyset$ is

both open and closed in W_0, we must have $\mathcal{A}_0 = W_0$. However, by the hypothesis that $\mathcal{D}_{\mathrm{grav}}(\cdot,\alpha)$ is selective, there exists $\mathcal{S}_{\mathrm{test}} \in C^{k,l,m}_{\mathrm{pw},b}(\Sigma) \subset W_0$ with $\mathcal{S}_{\mathrm{test}} \notin \mathcal{A}^{k,l,m}_{\Sigma}(\alpha)$, so $\mathcal{A}_0 \subsetneq W_0$ (and moreover $\dim W_0 = \infty$ whereas $\dim \mathcal{A}_0 < \infty$), yielding a contradiction.

More generally, if $\mathcal{A}$ is any vector subspace of $C^{k,l,m}_{\mathrm{pw}}(\Sigma)$ with non-empty τ-interior, then by translation invariance $\mathcal{A}$ contains an open neighborhood $U \subset W_0$ of 0; since U is absorbing in the topological vector space W_0 ($W_0 = \bigcup_{n=1}^{\infty} nU$), linear invariance of $\mathcal{A}$ implies $W_0 \subset \mathcal{A}$. When τ is Hausdorff and $\dim \mathcal{A}^{k,l,m}_{\Sigma}(\alpha) < \infty$, $\mathcal{A}^{k,l,m}_{\Sigma}(\alpha)$ is closed in $\left(C^{k,l,m}_{\mathrm{pw}}(\Sigma), \tau\right)$ (as its intersection with each clopen affine coset $\mathcal{S}_0 + W_0$ is a finite-dimensional affine subspace of a Hausdorff topological vector space, hence closed) and has empty interior, hence is nowhere dense; since $\mathcal{M}_{\mathrm{sol}}(\alpha) \subset \mathcal{A}^{k,l,m}_{\Sigma}(\alpha)$, $\mathcal{M}_{\mathrm{sol}}(\alpha)$ is likewise nowhere dense (and closed whenever $\mathcal{D}_{\mathrm{grav}}(\cdot,\alpha)$ is continuous and $\{0\} \subset \mathcal{V}$ is closed). □

Remark 4.3 (Linear Clusters vs. Nonlinear Solution Manifolds). Because gravitational field equations $\mathcal{D}_{\mathrm{grav}}(\mathcal{S},\alpha) = 0$ are generally nonlinear, a linear combination of two equilibrium solutions in $\mathcal{M}_{\mathrm{sol}}(\alpha)$ need not belong to $\mathcal{M}_{\mathrm{sol}}(\alpha)$. Notice that all four obstruction theorems of Section 3 (Theorems 3.1–3.6) hold directly on the nonlinear equilibrium locus $\mathcal{M}_{\mathrm{sol}}(\alpha)$ (or on any $\mathcal{O}$-bounded subset thereof), without requiring $\mathcal{D}_{\mathrm{grav}}$ to be linear. As explained in [4], the reason for defining the ambient cluster $\mathcal{A}^{k,l,m}_{\Sigma}(\alpha)$ as the linear span $\mathrm{span}(\mathcal{M}_{\mathrm{sol}}(\alpha))$ is functional-analytic: to classify exact or approximate solutions via series expansions, Schauder bases, or finite-dimensional subspaces of $C^{k,l,m}_{\mathrm{pw}}(\Sigma)$, one seeks minimal vector subspaces containing $\mathcal{M}_{\mathrm{sol}}(\alpha)$. Theorem 4.2 shows that in any gravity theory, neither $\mathcal{M}_{\mathrm{sol}}(\alpha)$ nor its linear hull $\mathcal{A}^{k,l,m}_{\Sigma}(\alpha)$ can be open in $C^{k,l,m}_{\mathrm{pw}}(\Sigma)$; physical perturbations of an equilibrium system $\mathcal{S}_0 \in \mathcal{M}_{\mathrm{sol}}(\alpha)$ must instead be constrained to the *tangent cone* $\mathcal{T}_{\mathcal{S}_0}\mathcal{M}_{\mathrm{sol}}(\alpha) = \ker\left(D_{\mathcal{S}}\mathcal{D}_{\mathrm{grav}}(\mathcal{S}_0,\alpha)\right)$ of the nonlinear operator. In Paper II of this series, we will construct explicit linearly independent families in $\mathcal{A}^{k,l,m}_{\Sigma}(\alpha)$ when $\mathcal{D}_{\mathrm{grav}}$ reduces to generalized Riccati or Abel differential operators.

5. Concrete Applications Across Gravity Models and Scales

To demonstrate that our axiomatic framework and Theorems 3.1–3.6 yield explicit, quantitative physical results beyond classical GR/Newtonian spheres, we work out concrete applications across three regimes: (1) compact stars in modified gravity (Rastall and perturbative $f(R)$ gravity, including 3-observable tidal deformability relations); (2) anisotropic relativistic stars coupled via the Bowers–Liang hydrostatic equation; and (3) galactic rotation systems governed by the Baryonic Tully–Fisher Relation.

5.1. Compact Stars in Modified Gravity: Rastall, Perturbative $f(R)$, and Tidal Relations.

Consider a spherically symmetric static compact star over $\Sigma = (0, R_{\max}]$ in a modified theory of gravity parametrized by a coupling constant $\alpha \in \mathbb{R}$ (working in geometric units $G = c = 1$ throughout Subsections 5.1 and 5.2). In wide classes of modified gravity theories—including Rastall gravity [13, 14], perturbative Starobinsky gravity $f(R) = R + \alpha R^2$ [10, 11], and Eddington-inspired Born–Infeld gravity [15]—the internal enclosed mass profile $m(r;R)$ (whose boundary derivatives relate to the macroscopic mass-radius derivatives via $m'(R;R) = \kappa_{\mathrm{hom}}^{-1}|M'(R)|$ at order $s = 1$, and $m^{(j)}(R;R) = \kappa_{\mathrm{hom}}^{-1} M^{(j)}(R)$ for $1 \le j \le 3$ on $s = 3$ homologous/envelope clusters with positive slope $M'(R) > 0$, such as $b > 0$ in Eq. (5.4) below, where $\kappa_{\mathrm{hom}} = 1$ for strictly nested profiles) and the matter fields $(p(r), \rho(r))$ evaluated at a characteristic boundary or reference radius $r = R$ obey a modified continuity equation of the form

$$\mathcal{D}^{\mathrm{cont}}_{\mathrm{grav}}(p,\rho,M;\alpha) := \kappa_{\mathrm{hom}}^{-1}|M'(R)| - 4\pi R^2\Big[\rho(R) + \alpha\,\Phi_{\mathrm{mod}}\big(R, M(R), M'(R), M''(R), M'''(R)\big)\Big] = 0, \tag{5.1}$$

where Φ_{mod} encodes the effective gravitational stress-energy contribution due to non-minimal geometry-matter coupling or higher curvature terms. Specifically:

- In **Rastall gravity** [13, 14], the field equations $G_{\mu\nu}+\alpha g_{\mu\nu}\mathcal{R}=\kappa T_{\mu\nu}$ (where $\alpha=\kappa\lambda$ is the dimensionless Rastall parameter, $\alpha\neq 1/4$, with trace $(4\alpha-1)\mathcal{R}=\kappa T$) yield an effective mass continuity source $\rho_{\rm eff}(r)=\frac{(1-3\alpha)\rho(r)-3\alpha p(r)}{1-4\alpha}$. At the zero-pressure stellar surface $p(R)=0$, this takes the exact multiplicative form
$$\kappa_{\rm hom}^{-1}|M'(R)|=4\pi R^2\big(1+\chi(\alpha)\big)\rho(R),\qquad 1+\chi(\alpha):=\frac{1-3\alpha}{1-4\alpha}=1+\alpha+\mathcal{O}(\alpha^2),\tag{5.2}$$
which in the linearized coupling regime $|\alpha|\ll 1/4$ is written as $1+\alpha\chi$ with $\chi=1>0$.
- In **perturbative $f(R)=R+\alpha R^2$ gravity** [10, 11], we expand the $(0,0)$ component $(1+2\alpha\mathcal{R})G_0^0+\frac{1}{2}\alpha\mathcal{R}^2-2\alpha(\nabla_0\nabla^0-\Box)\mathcal{R}=-8\pi\rho$ of the modified field equations to first order in α at a zero-pressure reference boundary $r=R$. Using $G_0^0=-\frac{2m'(r)}{r^2}$ and $(\nabla_0\nabla^0-\Box)\mathcal{R}_0=-e^{-2\lambda}\big(\mathcal{R}_0''+(\frac{2}{r}-\lambda')\mathcal{R}_0'\big)$ with $e^{-2\lambda}=1-\frac{2m(r)}{r}$ and $-e^{-2\lambda}\lambda'=\frac{m-rm'}{r^2}$, the zeroth-order Ricci scalar $\mathcal{R}_0(r)=8\pi(\rho_0(r)-3p_0(r))$ satisfies $\mathcal{R}_0(R)=\frac{2m'(R)}{R^2}$, $\mathcal{R}_0'(r)=8\pi(1-3v_0^2(r))\rho_0'(r)=\zeta_{\rm crust}\big(\frac{2m'(r)}{r^2}\big)'$, and $\mathcal{R}_0''(R)=\zeta_{\rm crust}\big(\frac{2m'(r)}{r^2}\big)''\big|_{r=R}-24\pi(v_0^2)'(R)\rho_0'(R)$ near $r=R$, where $v_\Gamma^2:=dp_0/d\rho_0|_{r=R}$ is the boundary sound speed and $\zeta_{\rm crust}:=1-3v_\Gamma^2$. Suppose the boundary sound-speed gradient $(v_0^2)'(R)$ vanishes; this holds identically when Γ is placed at the base of a linear or bag-like outer crust ($p=v_\Gamma^2(\rho-\rho_{\rm surf})$, so $v_0^2(r)\equiv v_\Gamma^2$), in the soft outer crust regime $3v_\Gamma^2\ll 1$ ($\zeta_{\rm crust}\simeq 1$, while core causality is tracked via $\rho_{\rm rel}=\rho(R)/\rho_c\ll 1$), or at constant-density scaling $b=3$ where $\rho_0'(R)=0$. (For a single self-bound polytrope $p=k\rho^\gamma+k_0$ with $\gamma>1$ at $b\neq 3$, $(v_0^2)'(R)=(\gamma-1)\frac{v_\Gamma^2}{\rho_0(R)}\rho_0'(R)$ simply adds $-\frac{3(\gamma-1)v_\Gamma^2}{RM'}(1-\frac{2M}{R})(RM''-2M')^2$ inside the bracket of (5.3).) Evaluating on a 3rd-order boundary jet-compatible cluster $(m^{(j)}(R;R)=\kappa_{\rm hom}^{-1}M^{(j)}(R)$ for $j=1,2,3$, with $M'(R)>0)$ then yields the explicit $s=3$ jet reduction $\Phi_{\rm mod}=\Phi_{\rm mod}(R,M,M',M'',M''')$:
$$\Phi_{\rm mod}=\frac{1}{2\pi\kappa_{\rm hom}R^3}\Bigg[\zeta_{\rm crust}\left(1-\frac{2M}{R}\right)\left(R\,M'''-2M''+\frac{2M'}{R}\right)+\zeta_{\rm crust}\frac{\big(M-\kappa_{\rm hom}^{-1}RM'\big)(RM''-2M')}{R^2}-\frac{3\kappa_{\rm hom}^{-1}(M')^2}{2R}\Bigg].\tag{5.3}$$
Physically, this first-order perturbative reduction holds in the effective field theory (EFT) regime where $|\alpha|\mathcal{R}_0(R)=\frac{4|\alpha|m'(R)}{R^2}\ll 1$ (so the scalaron Compton wavelength $m_\phi^{-1}\sim\sqrt{6|\alpha|}$ is small compared to the local stellar curvature radius) and where the reference locus Γ is placed either just inside a C^3-smooth crust/envelope transition layer ($r=R_\Gamma<R$) or at a C^3-regularized stellar surface so that the curvature gradients $\mathcal{R}_0'(R)$ and $\mathcal{R}_0''(R)$ remain finite and free of distributional boundary jumps.

Let us apply Theorem 3.3 and Theorem 3.6 to a boundary jet-compatible cluster $\mathcal{C}$ of stars in modified gravity equipped with the empirical monomial mass-radius relation [24, 25]
$$g(M,R,a,b)=M-aR^b=0\quad(a>0,\ b\neq 0),\tag{5.4}$$
and the general affine/shifted polytropic equation of state [3]
$$f(p,\rho,k,k_0,\gamma)=p-k\rho^\gamma-k_0=0\quad(k>0,\ \gamma>1,\ k_0\leq 0).\tag{5.5}$$
Notice that (5.5) accommodates both self-bound compact stars (such as strange quark stars or MIT bag-like crusts with $k_0<0$, where $p(R)=0$ occurs at a non-vanishing interior surface trace $\rho_\Gamma=\rho(R):=\lim_{r\to R^-}\rho(r)=(-k_0/k)^{1/\gamma}>0$, corresponding to $\rho_{\rm rel}=1$) and standard gaseous or core-coupled polytropes ($k_0=0$, where the boundary density $\rho(R)$ is linked to the reference core/transition density $\rho_\Gamma=\rho_c$ via the relative density factor $\rho_{\rm rel}:=\rho(R)/\rho_\Gamma\in(0,1]$ [3]). Here ρ denotes the total mass-energy density, so that fixing the adiabatic index $\gamma>1$ and treating the

polytropic stiffness $\epsilon = k \in (0,\infty)$ as the active microphysical parameter, both $\xi(\rho_\Gamma, k) = k\rho_\Gamma^\gamma + k_0$ and the exact relativistic adiabatic squared speed of sound $v^2(\rho_\Gamma, k, \gamma) = dp/d\rho = k\gamma\rho_\Gamma^{\gamma-1}$ are strictly co-monotone in (ρ_Γ, k) for all $\rho_\Gamma > 0$ (with $Y_f = \mathbb{R}$ ordered by $Y_{f,+} = \mathbb{R}_{\le 0}$ as in Definition 2.9(2)), with relativistic causal condition $v^2(\rho_\Gamma, k, \gamma) \le 1$ and causal density ceiling $\rho_{\mathrm{caus}}(k,\gamma) = (k\gamma)^{-1/(\gamma-1)}$. (If a polytrope is instead prescribed in terms of rest-mass density ρ_B as $p = k\rho_B^\gamma + k_0$ with total energy density $\rho(\rho_B) = \rho_B + \frac{k}{\gamma-1}\rho_B^\gamma - k_0$ by the first law of thermodynamics, the relativistic squared sound speed is $v^2 = \frac{dp}{d\rho} = \frac{k\gamma\rho_B^{\gamma-1}}{1+\frac{k\gamma}{\gamma-1}\rho_B^{\gamma-1}}$, which is automatically subluminal for $1 < \gamma \le 2$ and for $\gamma > 2$ simply replaces the causal ceiling $(k\gamma)^{-1/(\gamma-1)}$ on ρ_B by $\left(\frac{\gamma-1}{k\gamma(\gamma-2)}\right)^{1/(\gamma-1)}$ in all bounds below.)

Corollary 5.1 (Causal and Inverse Gravity Bounds for Rastall-Type Polytropic Stars)**.** *Let $\mathcal{C}$ be a cluster of astrophysical systems of degree $(k_p, l, l+1)$ (writing k_p for the pressure differentiability degree to avoid notation clash with the polytropic constant k; note that no restriction $l \ge k_p$ is required when the state function is evaluated at Γ via ev_Γ, whereas $l \ge k_p$ guarantees full Nemytskii solvability in $C^{k_p}_{\mathrm{pw}}$ as in Section 2) satisfying the monomial mass-radius relation (5.4), the polytropic equation of state (5.5), the causal condition $v^2(\rho_\Gamma, k, \gamma) = k\gamma\rho_\Gamma^{\gamma-1} \le 1$ at $\rho_\Gamma = \rho(R)/\rho_{\mathrm{rel}}$, and the modified continuity coupling $\kappa_{\mathrm{hom}}^{-1}|M'(R)| = 4\pi R^2(1+\alpha\chi)\rho(R)$ with $\chi > 0$ and $1+\alpha\chi > 0$. Let $R_* > 0$ denote the upper radius bound $R \le R_* := R_0$ when $b \ge 3$, or the lower radius bound $R \ge R_* := R_{\min} > 0$ when $b < 3$ (so that $R^{b-3} \le R_*^{b-3}$ across the cluster). Then:*

*(1) **(Causal Lifting Bound on EoS for Fixed Gravity α):** For any fixed modified gravity parameter $\alpha > -1/\chi$, the polytropic constant k at $R = R_*$ must satisfy the upper bound*

$$k \le \frac{1}{\gamma}\left[\frac{4\pi\kappa_{\mathrm{hom}}\rho_{\mathrm{rel}}(1+\alpha\chi)}{a|b|}\right]^{\gamma-1} R_*^{-(\gamma-1)(b-3)}. \tag{5.6}$$

*(2) **(Inverse Obstruction Bound on the Modified Gravity Parameter α):** Conversely, if the microphysical equation of state parameters (k,γ) are constrained by $k \ge k_{\min} > 0$ from nuclear/atomic theory, then any modified gravity theory compatible with causality and the mass-radius relation (5.4) at radius R_* must satisfy the* a priori *lower bound on α:*

$$\alpha \ge \frac{1}{\chi}\left[\frac{a|b|}{4\pi\kappa_{\mathrm{hom}}\rho_{\mathrm{rel}}}(k_{\min}\gamma)^{\frac{1}{\gamma-1}} R_*^{b-3} - 1\right]. \tag{5.7}$$

In exact (non-linearized) Rastall gravity (5.2) with $\alpha < 1/4$, where $1+\chi(\alpha) = \frac{1-3\alpha}{1-4\alpha} > 3/4$, this inverse bound reads explicitly (whereas for $0 < \mathcal{Q}_* \le 3/4$ the inequality $\frac{1-3\alpha}{1-4\alpha} \ge \mathcal{Q}_*$ holds automatically for all $\alpha < 1/4$)

$$\frac{\mathcal{Q}_* - 1}{4\mathcal{Q}_* - 3} \le \alpha < \frac{1}{4} \quad (\text{for } \mathcal{Q}_* > 3/4), \qquad \text{where } \mathcal{Q}_* := \frac{a|b|}{4\pi\kappa_{\mathrm{hom}}\rho_{\mathrm{rel}}}(k_{\min}\gamma)^{\frac{1}{\gamma-1}} R_*^{b-3}. \tag{5.8}$$

Proof. Differentiating $M(R) = aR^b$ with respect to R gives $|M'(R)| = a|b|R^{b-1}$. Solving the modified continuity equation for the reference density $\rho_\Gamma = \rho(R)/\rho_{\mathrm{rel}}$ gives the explicit density solver of Definition 2.6:

$$\rho_\Gamma(R, a, b, \alpha) = \Psi_{\mathrm{grav}}(R, M', \alpha) = \frac{a|b|R^{b-3}}{4\pi\kappa_{\mathrm{hom}}\rho_{\mathrm{rel}}(1+\alpha\chi)}. \tag{5.9}$$

Substituting (5.9) into the causal condition $v^2 = k\gamma\rho_\Gamma^{\gamma-1} \le 1$ yields

$$k\gamma\left[\frac{a|b|R^{b-3}}{4\pi\kappa_{\mathrm{hom}}\rho_{\mathrm{rel}}(1+\alpha\chi)}\right]^{\gamma-1} \le 1. \tag{5.10}$$

Solving (5.10) for k at $R = R_*$ in accordance with Theorem 3.3 (where $b \ge 3$ corresponds to the co-monotone upper bound $R \le R_0$ and $b < 3$ corresponds to the contra-monotone lower bound

$R \geq R_{\min}$ in Theorem 3.3) immediately produces (5.6) (which reduces to Eq. (9) of [3] when $\alpha = 0$ and $\kappa_{\rm hom} = 1$). Conversely, solving (5.9) for the gravitational parameter α gives

$$\alpha = \Theta_{\rm grav}(R, M', \rho_\Gamma) = \frac{1}{\chi}\left[\frac{a|b|R^{b-3}}{4\pi\kappa_{\rm hom}\rho_{\rm rel}\rho_\Gamma} - 1\right], \tag{5.11}$$

which is contra-monotone (non-increasing) in ρ_Γ. Applying Case (b) of the Inverse Obstruction Theorem 3.6 with the causal density ceiling $\rho_\Gamma \leq \rho_{\rm caus}(k,\gamma) = (k\gamma)^{-1/(\gamma-1)} \leq (k_{\min}\gamma)^{-1/(\gamma-1)}$ at $R = R_*$ yields

$$1 + \alpha\chi \geq \frac{a|b|R_*^{b-3}}{4\pi\kappa_{\rm hom}\rho_{\rm rel}}(k_{\min}\gamma)^{\frac{1}{\gamma-1}} = \mathcal{Q}_*, \tag{5.12}$$

which rearranges to (5.7) in the linearized case and to (5.8) upon substituting $1 + \chi(\alpha) = \frac{1-3\alpha}{1-4\alpha}$. □

Similarly, for the additive modified continuity equation (5.1) in perturbative $f(R) = R + \alpha R^2$ gravity on a 3rd-order boundary jet-compatible cluster with $b > 0$ and $\Phi_{\rm mod}$ given by (5.3) (in the vanishing boundary sound-speed gradient regime $(v_0^2)'(R) = 0$), solving (5.1) for $\rho_\Gamma = \rho(R)/\rho_{\rm rel}$ gives

$$\rho_\Gamma(R, a, b, \alpha) = \frac{abR^{b-3}}{4\pi\kappa_{\rm hom}\rho_{\rm rel}} - \frac{\alpha}{\rho_{\rm rel}}\,\Phi_{\rm mod}\big(R, aR^b, abR^{b-1}, ab(b-1)R^{b-2}, ab(b-1)(b-2)R^{b-3}\big). \tag{5.13}$$

Notice that in (5.3), for $M(R) = aR^b$ with $\kappa_{\rm hom} = 1$ and surface compactness $u_* := aR_*^{b-1} = M(R_*)/R_* \in (0, 1/2)$, one has $RM''' - 2M'' + \frac{2M'}{R} = ab(b-2)(b-3)R^{b-2}$ and $RM'' - 2M' = ab(b-3)R^{b-1}$, so that at $R = R_*$,

$$\Phi_{\rm mod}(R_*, a, b) = \frac{abR_*^{b-3}}{2\pi}\left[\zeta_{\rm crust}(1-2u_*)(b-2)(b-3) - u_*\left(\zeta_{\rm crust}(b-1)(b-3) + \frac{3}{2}b\right)\right]. \tag{5.14}$$

In particular, for canonical self-bound or nearly constant-density mass-radius scalings with $b \in [2,3]$ and $\zeta_{\rm crust} \in [0,1]$ (including $b = 3$ or an ultrarelativistic bag surface $v_\Gamma^2 = 1/3$ with $\zeta_{\rm crust} = 0$, where the derivative terms vanish identically and $\Phi_{\rm mod}(R_*, a, 3) = -\frac{27a^2R_*^2}{4\pi} < 0$), one strictly has $\Phi_{\rm mod}(R_*, a, b) < 0$ and $\Theta_{\rm grav}$ is *co-monotone* in ρ_Γ (Case (a) of Theorem 3.6), whereas for sufficiently steep positive-slope branches ($b > 3$ with $\zeta_{\rm crust}(1-2u_*)(b-2)(b-3) > u_*(\zeta_{\rm crust}(b-1)(b-3) + \frac{3}{2}b)$) one has $\Phi_{\rm mod}(R_*, a, b) > 0$ and $\Theta_{\rm grav}$ is *contra-monotone* in ρ_Γ (Case (b) of Theorem 3.6). Here, when bounding $\alpha = \Theta_{\rm grav}(R, a, b, \rho_\Gamma)$ uniformly over a radius interval $R \in [R_{\min}, R_0]$, $R_* \in \{R_{\min}, R_0\}$ denotes the extremal radius that maximizes (when $\Phi_{\rm mod} < 0$) or minimizes (when $\Phi_{\rm mod} > 0$) the composite threshold map $R \mapsto \Theta_{\rm grav}\big(R, a, b, (k_{\min}\gamma)^{-\frac{1}{\gamma-1}}\big)$ (for instance, at $b = 3$, where $|\Phi_{\rm mod}(R, a, 3)| = \frac{27a^2R^2}{4\pi}$, $\Theta_{\rm grav}(R, a, 3, \rho_{\rm caus}) \propto R^{-2}$ is contra-monotone in R whenever $(k_{\min}\gamma)^{-\frac{1}{\gamma-1}} > \frac{3a}{4\pi\kappa_{\rm hom}\rho_{\rm rel}}$, so $R_* = R_{\min}$, whereas $(k_{\min}\gamma)^{-\frac{1}{\gamma-1}} < \frac{3a}{4\pi\kappa_{\rm hom}\rho_{\rm rel}}$ yields $\alpha_{\max} < 0$, immediately ruling out all tachyon-free Starobinsky models $\alpha \geq 0$). Consequently, for $k \geq k_{\min}$, Theorem 3.6 yields both upper and lower gravity parameter obstructions:

$$\begin{cases} \alpha \leq \alpha_{\max} := \dfrac{\rho_{\rm rel}}{|\Phi_{\rm mod}(R_*, a, b)|}\left[(k_{\min}\gamma)^{-\frac{1}{\gamma-1}} - \dfrac{abR_*^{b-3}}{4\pi\kappa_{\rm hom}\rho_{\rm rel}}\right] & \text{if } \Phi_{\rm mod}(R_*, a, b) < 0, \\ \alpha \geq \alpha_{\min} := \dfrac{\rho_{\rm rel}}{\Phi_{\rm mod}(R_*, a, b)}\left[\dfrac{abR_*^{b-3}}{4\pi\kappa_{\rm hom}\rho_{\rm rel}} - (k_{\min}\gamma)^{-\frac{1}{\gamma-1}}\right] & \text{if } \Phi_{\rm mod}(R_*, a, b) > 0, \end{cases} \tag{5.15}$$

which are self-consistent within the tachyon-free perturbative EFT validity window

$$\alpha \geq 0 \qquad \text{and} \qquad |\alpha|\mathcal{R}_0(R_*) = \frac{4|\alpha|abR_*^{b-3}}{\kappa_{\rm hom}} \ll 1. \tag{5.16}$$

In accordance with Remark 3.7, intersecting the upper bound $\alpha \leq \alpha_{\max}$ from a self-bound/core scaling branch ($\Phi_{\rm mod} < 0$, e.g. $b \in [2,3]$) with the lower bound $\alpha \geq \alpha_{\min}$ from a steeper scaling

branch ($\Phi_{\rm mod} > 0$, $b > 3$) and the no-tachyon EFT domain $[0, \alpha_{\rm EFT}]$ traps the Starobinsky parameter α inside the compact interval $[\max\{0, \alpha_{\min}\}, \min\{\alpha_{\max}, \alpha_{\rm EFT}\}]$ and falsifies the model whenever this intersection is empty.

Example 5.2 (Multi-Messenger 3-Observable Cluster $(M, R, \Lambda_{\rm tidal})$ and Numerical Illustration)**.** To illustrate Theorem 3.6 with $N = 3$ observables on a cluster of neutron stars, take $\mathcal{O} = (M, R, \Lambda_{\rm tidal}) : \mathcal{A} \to \mathbb{R}^3$, where $\Lambda_{\rm tidal} = \frac{2}{3} k_2 (R/M)^5$ is the dimensionless tidal deformability (with k_2 the quadrupolar Love number). In the compactness range of canonical neutron stars, $\Lambda_{\rm tidal}$ obeys the empirical scaling relation [16, 17]

$$g(M, R, \Lambda_{\rm tidal}, \Lambda_0, q) := M - R \left(\frac{\Lambda_0}{\Lambda_{\rm tidal}} \right)^{1/q} = 0 \qquad (q \approx 6,\ \Lambda_0 > 0). \tag{5.17}$$

Here $\mathcal{O}_1 = M$ is solved in terms of the independent observables $\mathcal{O}' = (\mathcal{O}_{\rm geom}, \mathcal{O}_{\rm aux}) = (R, \Lambda_{\rm tidal})$ as $\mu(R, \Lambda_{\rm tidal}; \Lambda_0, q) = R(\Lambda_0/\Lambda_{\rm tidal})^{1/q}$, which is contra-monotone in $\Lambda_{\rm tidal}$. Since $\mu(R, \Lambda_{\rm tidal}; \Lambda_0, q)$ is homogeneous of degree 1 in R at fixed $\Lambda_{\rm tidal}$, its active geometric partial derivative is the compactness $\partial_R \mu = (\Lambda_0/\Lambda_{\rm tidal})^{1/q} = M/R = \frac{4\pi}{3} R^2 \bar{\rho}_{\rm eff}$ (which also serves as a lower bound for the total derivative $dM/dR = (\Lambda_0/\Lambda_{\rm tidal})^{1/q} \big(1 - \frac{R}{q\Lambda_{\rm tidal}} \frac{d\Lambda_{\rm tidal}}{dR}\big)$ along any 1-parameter equilibrium sequence with $d\Lambda_{\rm tidal}/dR \le 0$), whereas the bare internal boundary matter-mass gradient is $\partial_r m_{\rm mat}(R; R) = 4\pi R^2 \rho(R)$, related to the effective gravitational mass gradient in Rastall gravity via $\partial_r m(R; R) = 4\pi R^2 \rho_{\rm eff}(R) = (1 + \alpha\chi)\partial_r m_{\rm mat}(R; R) = (1 + \alpha\chi)\kappa_{\rm hom}^{-1} \partial_R \mu$. Here $\kappa_{\rm hom} := \frac{\partial_R \mu}{4\pi R^2 \rho(R)} = \frac{\bar{\rho}_{\rm eff}}{3\rho(R)}$ denotes the matter-referenced boundary jet homology factor, so that the combined homology-density factor entering the core reference density $\rho_\Gamma = \rho_c$ has the exact physical interpretation $\kappa_{\rm hom}\rho_{\rm rel} = \frac{\bar{\rho}_{\rm eff}}{3\rho(R)} \cdot \frac{\rho(R)}{\rho_c} = \frac{\bar{\rho}_{\rm eff}}{3\rho_c}$ (one-third of the mean-effective-to-central matter density ratio). For any subcluster satisfying the observational upper bounds $\Lambda_{\rm tidal} \le \Lambda_*$ and $R \le R_*$, one has $\partial_R \mu = (\Lambda_0/\Lambda_{\rm tidal})^{1/q} \ge (\Lambda_0/\Lambda_*)^{1/q}$, so that $\rho_\Gamma = \frac{(\Lambda_0/\Lambda_{\rm tidal})^{1/q}}{4\pi\kappa_{\rm hom}\rho_{\rm rel} R^2 (1+\alpha\chi)} \ge \frac{(\Lambda_0/\Lambda_*)^{1/q}}{4\pi\kappa_{\rm hom}\rho_{\rm rel} R_*^2 (1+\alpha\chi)}$ and the gravity solver $\Theta_{\rm grav}$ is contra-monotone in both R and $\Lambda_{\rm tidal}$ (matching Case (b) of Theorem 3.6). Using observational bounds from GW170817 [17] ($\Lambda_{1.4} \le \Lambda_* \approx 580$, $R \le R_* \approx 11.5$ km, compactness lower bound $M/R \ge (\Lambda_0/\Lambda_*)^{1/q} \approx 0.18$ in geometric units $G = c = 1$, where $\rho_{\rm geom} = (G/c^2)\rho_{\rm CGS}$ and $p_{\rm geom} = (G/c^4)p_{\rm CGS}$ have dimensions of km^{-2} and $k_{\rm geom} = p_{\rm geom}/\rho_{\rm geom}^2$ for $\gamma = 2$ has dimensions of km^2, related to $k_{\rm CGS} = p_{\rm CGS}/\rho_{\rm CGS}^2 = G k_{\rm geom}$ via $1\ \mathrm{km}^2 \approx 7.426 \times 10^{-19}\ \mathrm{cm}^2/(\mathrm{g/cm}^3) \approx 6.674 \times 10^2\ \mathrm{cm}^5\,\mathrm{g}^{-1}\,\mathrm{s}^{-2}$ in CGS units) and a stiff nuclear polytrope with $\gamma = 2$, $k \ge k_{\min}$, combined homology-density factor $\kappa_{\rm hom}\rho_{\rm rel} = \frac{\bar{\rho}_{\rm eff}}{3\rho_c} \approx 0.15$ (corresponding to $\bar{\rho}_{\rm eff}/\rho_c \approx 0.45$), and causal density ceiling $\rho_{\rm caus} \le (2k_{\min})^{-1}$, Corollary 5.1(2) gives $\mathcal{Q}_* = \frac{k_{\min}(\Lambda_0/\Lambda_*)^{1/q}}{2\pi\kappa_{\rm hom}\rho_{\rm rel} R_*^2} = \frac{k_{\min}}{k_{\rm crit}}$, where the critical threshold stiffness is $k_{\rm crit} := \frac{2\pi\kappa_{\rm hom}\rho_{\rm rel} R_*^2}{(\Lambda_0/\Lambda_*)^{1/q}} \approx 692\ \mathrm{km}^2$ ($\approx 4.62 \times 10^5\ \mathrm{cm}^5\,\mathrm{g}^{-1}\,\mathrm{s}^{-2}$ in CGS units). Consequently, for any stiff nuclear equation of state with $k_{\min} = 850\ \mathrm{km}^2$ ($\approx 5.67 \times 10^5\ \mathrm{cm}^5\,\mathrm{g}^{-1}\,\mathrm{s}^{-2}$, so $\mathcal{Q}_* \approx 1.23 > 1$), Eq. (5.8) forces $\alpha \ge \frac{\mathcal{Q}_* - 1}{4\mathcal{Q}_* - 3} \approx 0.12 > 0$, immediately ruling out all Rastall couplings $\alpha < 0.12$ across the subcluster $\Lambda_{\rm tidal} \le \Lambda_*$ without numerical integration of the interior TOV system.

5.2. **Anisotropic Relativistic Astrophysical Systems.** In dense nuclear matter inside neutron stars ($10^{15}\ \mathrm{g/cm}^3$), strong magnetic fields, pion condensation, or relativistic nuclear interactions induce pressure anisotropy, so that the radial pressure p_r differs from the tangential pressure p_t [6, 7, 8]. In our general framework (Definition 2.2), an anisotropic spherically symmetric system over $\Sigma = I \subset [0, \infty)$ is a section of the rank-2 pressure bundle $W_p = I \times \mathbb{R}^2$, where

$$p = (p_r, p_t) \in C^{k_p}_{\rm pw}(I; \mathbb{R}^2), \quad \rho \in C^{l}_{\rm pw}(I; \mathbb{R}), \quad M \in C^{m}_{\rm pw}(I; \mathbb{R}), \tag{5.18}$$

coupled via the full Bowers–Liang anisotropic TOV gravity-matter operator $\mathcal{D}^{\mathrm{BL}}_{\mathrm{grav}}(p_r, p_t, \rho, M) = 0$:

$$p_r'(r) + \frac{(\rho + p_r)(M + 4\pi r^3 p_r)}{r^2\left(1 - \frac{2M}{r}\right)} - \frac{2\Delta(r)}{r} = 0, \qquad M'(r) - 4\pi r^2 \rho(r) = 0, \tag{5.19}$$

where $\Delta(r) := p_t(r) - p_r(r)$ is the anisotropy factor.

To couple both the hydrostatic equation and the continuity equation in (5.19), consider a quasi-local vector-valued state function $f : \mathcal{A} \times E \to C^{k_p}_{\mathrm{pw}}(I; \mathbb{R}^2)$ (Definition 2.9) given by a radial polytrope paired with the Quasi-Local (Bowers–Liang / Horvat) anisotropy equation of state [6, 7]:

$$f_r(p_r, \rho, k_r, \gamma_r) = p_r - k_r \rho^{\gamma_r} - k_0 = 0, \qquad f_t(p_r, p_t, \rho, \sigma, u) = p_t - p_r - \sigma\, h(\rho, u) = 0, \tag{5.20}$$

where $\epsilon = (k_r, \gamma_r, \sigma) \in \mathbb{R}^3$ is the microphysical parameter vector, $\sigma \in \mathbb{R}$ is the dimensionless anisotropy parameter, and $u := M/R \in (0, 1/2)$ denotes the stellar compactness observable (we use u rather than $\mathcal{C}$ to avoid any notation clash with the cluster $\mathcal{C}$). In the standard Bowers–Liang / Horvat quasi-local models [6, 7], $h(\rho, u) = \frac{4\pi\rho^2 R^2}{3(1-2u)} > 0$ (or $h(\rho, u) = \frac{\rho u}{1-2u} > 0$, both having $\partial_\rho h(\rho, u) > 0$ and the physical dimension $[\text{length}]^{-2}$ of pressure in geometric units $G = c = 1$) near the zero-radial-pressure surface $p_r(R) = 0$. Moreover, for a self-bound surface ($k_0 < 0$), evaluating the hydrostatic component of (5.19) at $r = R$ (where $p_r(R) = 0$, $\rho(R) = (-k_0/k_r)^{1/\gamma_r} > 0$, and $|p_r'(R)| = -\partial_r p_r(R) \geq 0$ is the inward surface pressure gradient) relates $\rho(R)$ directly to the compactness $u = M/R$, the surface pressure slope $|p_r'(R)|$, and the anisotropy coupling σ:

$$|p_r'(R)| = \frac{\rho(R)\, u}{R(1 - 2u)} - \frac{2\sigma\, h\big(\rho(R), u\big)}{R} \tag{5.21}$$

(whereas if $k_0 = 0$ and the evaluation is instead performed at an interior reference locus $r = R_\Gamma < R$ where $\rho_\Gamma = \rho(R_\Gamma) > 0$ and $p_\Gamma := p_r(R_\Gamma) = k_r \rho_\Gamma^{\gamma_r} > 0$, the numerator $\rho(R)\, u$ in (5.21) and in the hydrostatic ceiling of (5.24) below is replaced by $(\rho_\Gamma + p_\Gamma)(u_\Gamma + 4\pi R_\Gamma^2 p_\Gamma)$). Here the local pressure solution map at fixed compactness u is $\xi(\rho, u, \epsilon) = \big(k_r\rho^{\gamma_r} + k_0,\ k_r\rho^{\gamma_r} + k_0 + \sigma h(\rho, u)\big)$, and the generalized squared speed of sound (Definition 2.9(3)) is the two-component acoustic vector

$$v^2(\rho, u, \epsilon) = \big(v_r^2(\rho, \epsilon),\ v_t^2(\rho, u, \epsilon)\big) = \Big(k_r\gamma_r\rho^{\gamma_r - 1},\ k_r\gamma_r\rho^{\gamma_r - 1} + \sigma\partial_\rho h(\rho, u)\Big) \in \mathbb{R}^2. \tag{5.22}$$

Endowing $\mathbb{R}^2$ with the standard positive quadrant cone $\mathbb{R}^2_{\geq 0}$, the relativistic causal cone condition reads

$$(0, 0) \preceq \big(v_r^2, v_t^2\big) \preceq (1, 1), \tag{5.23}$$

which automatically guarantees $|v_t^2 - v_r^2| \leq 1$ (whereas imposing the stricter Abreu–Hernández–Núñez no-cracking stability cone $-1 \leq v_t^2 - v_r^2 \leq 0$ [8] additionally restricts the anisotropy parameter to $\sigma \leq 0$). Applying Theorem 3.3 (and inverting the pointwise acoustic cone $0 \leq v_t^2(\rho_\Gamma, u, \epsilon) \leq 1$ directly for σ at the reference locus Γ, which applies for both $\sigma \geq 0$ and $\sigma < 0$) to a cluster of self-bound anisotropic stars ($k_0 < 0$, $\rho_{\mathrm{rel}} = 1$) satisfying a mass-radius relation $g(M, R, \delta) = 0$ evaluated at a reference configuration $R = R_*$ with reference surface density $\rho_0(\delta) := \rho(R_*) = |\partial_R M(R_*, \delta)|/(4\pi\kappa_{\mathrm{hom}} R_*^2)$ and compactness $u_0 = M(R_*, \delta)/R_*$, combined with the inward hydrostatic balance condition $|p_r'(R_*)| \geq 0$ in (5.21), yields at $R = R_*$ an upper bound on the radial polytropic constant k_r and, on each fixed-k_r slice, two-sided bounds on the anisotropy parameter σ incorporating the full relativistic compactness factor $(1 - 2u_0)^{-1}$ (where, because the upper threshold maps $\frac{1 - k_r\gamma_r\rho_\Gamma^{\gamma_r - 1}}{\partial_\rho h(\rho_\Gamma, u)}$ and $\frac{\rho_\Gamma u}{2(1-2u)h(\rho_\Gamma, u)}$ are contra-monotone in (ρ_Γ, u) whenever $\partial_\rho h(\rho, u)$ and $h(\rho, u)/(\rho u)$ are co-monotone, the upper bounds in (5.24) hold either for a universal constant σ shared by a cluster containing $R = R_*$, or pointwise across any subcluster satisfying $\rho_\Gamma(R, \delta) \geq \rho_0(\delta)$ and $u(R, \delta) \geq u_0$):

$$k_r \leq \frac{1}{\gamma_r \rho_0(\delta)^{\gamma_r - 1}}, \qquad -\frac{k_r\gamma_r\rho_0(\delta)^{\gamma_r - 1}}{\partial_\rho h(\rho_0(\delta), u_0)} \leq \sigma \leq \min\left\{\frac{1 - k_r\gamma_r\rho_0(\delta)^{\gamma_r - 1}}{\partial_\rho h(\rho_0(\delta), u_0)},\ \frac{\rho_0(\delta)\, u_0}{2(1 - 2u_0)\, h(\rho_0(\delta), u_0)}\right\}. \tag{5.24}$$

Because the causal upper bound $\frac{1-k_r\gamma_r\rho_0(\delta)^{\gamma_r-1}}{\partial_\rho h(\rho_0(\delta),u_0)}$ in (5.24) is contra-monotone in k_r while the lower bound is co-monotone in $-k_r$, eliminating k_r over the admissible range $k_{r,\min} \le k_r \le (\gamma_r\rho_0(\delta)^{\gamma_r-1})^{-1}$ (with $k_{r,\min} \ge 0$) yields the k_r-independent envelope

$$-\frac{1}{\partial_\rho h(\rho_0(\delta),u_0)} \le \sigma \le \min\left\{\frac{1-k_{r,\min}\gamma_r\rho_0(\delta)^{\gamma_r-1}}{\partial_\rho h(\rho_0(\delta),u_0)},\ \frac{\rho_0(\delta)\,u_0}{2(1-2u_0)\,h(\rho_0(\delta),u_0)}\right\} \tag{5.25}$$

(note that for the Horvat ansatz $h(\rho,u)=\frac{\rho u}{1-2u}$ the hydrostatic ceiling simplifies to the universal constant $\sigma \le 1/2$, while for the Bowers–Liang ansatz $h(\rho,u)=\frac{4\pi\rho^2R^2}{3(1-2u)}$ it equals $\sigma \le \frac{3u_0}{8\pi\rho_0(\delta)R_*^2}$).

5.3. **Galactic-Scale Astrophysical Systems and the Baryonic Tully–Fisher Relation.** To illustrate that our definition of an astrophysical system applies equally well beyond stellar interiors, consider a cluster $\mathcal{A}_{\rm gal}$ of axisymmetric or spherical galactic systems over $\Sigma=(0,R_{\max}]$ equipped with the macroscopic observable tuple

$$\mathcal{O} = (M_b, V_{\rm flat}, R_d) : \mathcal{A}_{\rm gal} \to \mathbb{R}^3, \tag{5.26}$$

where M_b is the total baryonic mass, $V_{\rm flat}$ is the asymptotic outer rotation velocity, and R_d is the characteristic virial/outer halo radius. Observationally, disk and gas-rich galaxies obey the tight empirical *Baryonic Tully–Fisher Relation (BTFR)* [18]:

$$g(M_b, V_{\rm flat}, A_{\rm TF}, b) := M_b - A_{\rm TF}V_{\rm flat}^b = 0, \tag{5.27}$$

where empirically $b \approx 3.8$–4.0 and $A_{\rm TF} > 0$ is the normalization scale.

In a general modified gravity or effective dark-halo model (such as AQUAL/MOND [26] with acceleration scale a_0, or a barotropic dark/baryonic halo fluid with effective pressure $p_{\rm eff}$ and density ρ_b), the radial centripetal equilibrium and mass continuity at $r=R_d$ are governed by the gravity-matter coupling operator

$$\mathcal{D}^{\rm gal}_{\rm grav}(\mathcal{S};a_0) := \left(\frac{V_{\rm rot}(R_d)^2}{R_d} - \nu\!\left(\frac{GM_b(R_d)}{a_0R_d^2}\right)\frac{GM_b(R_d)}{R_d^2},\ M_b'(R_d) - 4\pi R_d^2\rho_b(R_d)\right) = (0,0), \tag{5.28}$$

where $V_{\rm rot}(R_d)$ is the circular velocity at $r=R_d$ (approaching $V_{\rm flat}$ in the outer halo) and $\nu(y)$ is the gravitational interpolation function ($\nu(y)\to 1$ in Newtonian gravity $a_0\to 0$, and $\nu(y)\sim y^{-1/2}$ in the deep-MOND regime $y \ll 1$).

Corollary 5.3 (Obstructions on Galactic Halo State Functions and Acceleration Scales)**.** *Let $\mathcal{C}_{\rm gal} = (\mathcal{A}_{\rm gal}, M_b, V_{\rm flat}, R_d)$ be a cluster of galactic systems with rotation velocity in $[V_{\min}, V_0]$ and halo radius $R_d \in [R_{d,\min}, R_{d,0}]$, obeying the Baryonic Tully–Fisher Relation (5.27) and the coupling (5.28).*

(1) ***(Direct Halo EoS Bound):*** *By Theorem 3.1, any MLI effective halo state function $f(p_{\rm eff},\rho_b,A_{\rm TF},b)$ depending monotonically on the BTFR parameters $(A_{\rm TF},b)$ is bounded from above on $\mathcal{C}_{\rm gal}$, and its effective sound/velocity-dispersion speed $v^2_{\rm disp}=\partial_{\rho_b}\xi$ is bounded from above.*

(2) ***(Direct and EoS-Coupled Inverse Bounds on a_0):*** *Suppose the outer halo gas obeys a barotropic dispersion equation of state $p_{\rm eff}=k_{\rm gal}\rho_\Gamma^{\gamma_{\rm gal}}$ ($\gamma_{\rm gal}>1$) subject to the virial/acoustic stability bound $v^2_{\rm disp} = k_{\rm gal}\gamma_{\rm gal}\rho_\Gamma^{\gamma_{\rm gal}-1} \le v^2_{\max}$ (corresponding to the constant threshold $v_0^2(\rho_\Gamma)\equiv v^2_{\max}$ in Definition 2.10). We obtain self-consistent kinematic and EoS-coupled bounds on a_0 in two regimes:*

(a) Asymptotic deep-MOND regime ($\nu(y)=y^{-1/2}$, $V_{\rm rot}(R_d)^4 = Ga_0M_b(R_d)$) with volume-averaged evaluation: *Combining (5.27) at $V_{\rm rot}(R_d)=V_{\rm flat}$ with centripetal balance gives the observable reduction $A_{\rm TF}Ga_0V_{\rm flat}^{b-4}=1$, which restricts $a_0 \le a_{0,\max}^{\rm TF} :=$*

$\max\{(GA_{\mathrm{TF}}V_{\min}^{b-4})^{-1},(GA_{\mathrm{TF}}V_0^{b-4})^{-1}\}$ *for any galaxy in* $[V_{\min},V_0]$ *(or the stricter uniform ceiling* $a_0\le\min_{V\in\{V_{\min},V_0\}}(GA_{\mathrm{TF}}V^{b-4})^{-1}$ *if* $A_{\mathrm{TF}}Ga_0V_{\mathrm{flat}}^{b-4}\le 1$ *is enforced simultaneously across all galaxies in* $[V_{\min},V_0]$*, reducing to* $b=4$*,* $a_0=(GA_{\mathrm{TF}})^{-1}$ *when equality holds with constant* a_0 *across* $[V_{\min},V_0]$*). Using the volume-averaged reference evaluation* $\rho_\Gamma=\bar\rho_b(R_d):=\frac{3M_b(R_d)}{4\pi R_d^3}=\frac{3V_{\mathrm{flat}}^4}{4\pi Ga_0R_d^3}$ *via the deep-MOND dynamical mass relation* $M_b(R_d)=\frac{V_{\mathrm{flat}}^4}{Ga_0}$ *(whereas substituting* $M_b(R_d)=A_{\mathrm{TF}}V_{\mathrm{flat}}^b$ *directly into* $\bar\rho_b(R_d)$ *yields a pure EoS-BTFR compatibility bound independent of* a_0*; alternatively, if* $V'_{\mathrm{rot}}(R_d)>0$*, one may use the local deep-MOND continuity density* $\rho_b(R_d)=\frac{V_{\mathrm{rot}}(R_d)^3V'_{\mathrm{rot}}(R_d)}{\pi Ga_0R_d^2}$*), Theorem 3.6(b) yields*

$$a_0\ge a_{0,\min}^{(1)}:=\frac{3V_{\min}^4}{4\pi GR_{d,0}^3}\left(\frac{k_{\mathrm{gal}}\gamma_{\mathrm{gal}}}{v_{\max}^2}\right)^{\frac{1}{\gamma_{\mathrm{gal}}-1}},\tag{5.29}$$

trapping a_0 *in* $\left[a_{0,\min}^{(1)},a_{0,\max}^{\mathrm{TF}}\right]$.

(b) Finite-R_d transition regime with local boundary evaluation: *For the standard interpolation function* $\mu_{\mathrm{MOND}}(x)=x/(1+x)$ *(where* $x=V_{\mathrm{rot}}^2/(a_0r)$*,* $y=x\mu_{\mathrm{MOND}}(x)=\frac{GM_b(r)}{a_0r^2}$*, and* $\nu(y)=\frac12+\sqrt{\frac14+\frac1y}$ *satisfies* $\nu(y)\mu_{\mathrm{MOND}}(x)=1$*), the exact enclosed mass profile* $GM_b(r)=\frac{rV_{\mathrm{rot}}(r)^4}{a_0r+V_{\mathrm{rot}}(r)^2}$ *evaluated at* $r=R_d$ *along a strictly flat outer rotation curve* ($V_{\mathrm{rot}}(r)\equiv V_{\mathrm{flat}}$*,* $V'_{\mathrm{rot}}(R_d)=0$*) simultaneously yields the tightened kinematic BTFR upper bound* $a_0=\frac{V_{\mathrm{flat}}^{4-b}}{GA_{\mathrm{TF}}}-\frac{V_{\mathrm{flat}}^2}{R_d}\le\widetilde{a}_{0,\max}^{\mathrm{TF}}:=a_{0,\max}^{\mathrm{TF}}-\frac{V_{\min}^2}{R_{d,0}}$ *(obtained by bounding the two terms separately over* $[V_{\min},V_0]\times[R_{d,\min},R_{d,0}]$*, and attained at* $(V_{\min},R_{d,0})$ *for* $b\ge4$*) and, upon differentiating with respect to* r *in the continuity component of* (5.28)*, the positive local boundary density solver* $\rho_\Gamma=\rho_b(R_d)=\frac{V_{\mathrm{flat}}^6}{4\pi GR_d^2(a_0R_d+V_{\mathrm{flat}}^2)^2}>0$*. Applying Theorem 3.6(b) (taking the positive root of* $(a_0R_d+V_{\mathrm{flat}}^2)^2$ *for* $a_0>0$*, which is strictly co-monotone in* $V_{\mathrm{flat}}\in[V_{\min},V_0]$ *and strictly contra-monotone in* $R_d\in[R_{d,\min},R_{d,0}]$ *whenever* $\left(k_{\mathrm{gal}}\gamma_{\mathrm{gal}}/v_{\max}^2\right)^{\frac{1}{2(\gamma_{\mathrm{gal}}-1)}}\ge\sqrt{4\pi G}\,R_{d,0}/V_{\min}$*) yields*

$$a_0\ge a_{0,\min}^{(2)}:=\frac{V_{\min}^3}{R_{d,0}}\left[\frac{1}{\sqrt{4\pi G}\,R_{d,0}}\left(\frac{k_{\mathrm{gal}}\gamma_{\mathrm{gal}}}{v_{\max}^2}\right)^{\frac{1}{2(\gamma_{\mathrm{gal}}-1)}}-\frac{1}{V_{\min}}\right],\tag{5.30}$$

trapping a_0 *in* $\left[a_{0,\min}^{(2)},\widetilde{a}_{0,\max}^{\mathrm{TF}}\right]$ *and ruling out the model whenever* $a_{0,\min}^{(2)}>\widetilde{a}_{0,\max}^{\mathrm{TF}}$.

6. Concluding Remarks

In this paper, we have formulated a general, gravity-model-independent definition of an *astrophysical system* and of *clusters of astrophysical systems* over arbitrary smooth manifolds Σ (possibly with corners or piecewise smooth boundary), unifying and extending the functional-analytic framework of [3, 4]. By decoupling the Hausdorff extended-norm structure of piecewise differentiable matter and gravitational sections $C_{\mathrm{pw}}^{k,l,m}(\Sigma)$ (and the locally convex and affine Banach structure of its bounded component and partition strata) from the specific choice of gravitational field equations $\mathcal{D}_{\mathrm{grav}}(\mathcal{S},\alpha)=0$ and macroscopic observables $\mathcal{O}=(\mathcal{O}_1,\dots,\mathcal{O}_N)$, we established four universal obstruction theorems:

- Direct propagation of multi-observable bounds to MLI equations of state and acoustic speeds (Theorem 3.1);
- Structural propagation across locally density-solvable gravity couplings via reference evaluation operators (Theorem 3.2);

- Causal lifting from acoustic cone bounds to state function bounds in arbitrary gravity theories (Theorem 3.3);
- An *Inverse Obstruction Theorem* (Theorem 3.6) that uses known microphysical EoS bounds and empirical observable scaling laws to place rigorous *a priori* upper or lower constraints on the parameter space A_{grav} of modified gravity theories.

In particular, intersecting the inverse bounds of Theorem 3.6 across multiple density regimes or astrophysical scales (Remark 3.7—such as combining positive- and negative-Φ_{mod} branches in perturbative $f(R)$ or Rastall compact star clusters in Section 5.1, combining kinematic BTFR and halo EoS bounds on a_0 in Corollary 5.3, or intersecting stellar and galactic constraints in unified modified-gravity theories sharing a common coupling $\alpha \in A_{\text{grav}}$) provides an analytical, integration-free consistency test capable of ruling out windows of modified gravity parameter space. Furthermore, we proved that finite-dimensional gravity-generated clusters $\mathcal{A}_{\Sigma}^{k,l,m}(\alpha)$ are topologically rigid (closed and nowhere dense) in any Hausdorff locally convex topology (Theorem 4.2).

In the next paper of this series, we will study the global solution space $\mathcal{A}_{\Sigma}^{k,l,m}(\alpha)$ when the hydrostatic component of $\mathcal{D}_{\text{grav}}(\mathcal{S}, \alpha) = 0$ is of generalized Riccati or Abel type, extending the pseudo-asymptotic limit and compactification theory of [4] to modified gravity and anisotropic systems.

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