SeqImpar and SeqPar: Analytic Characterizations of the Zeros of the Riemann Zeta and Dirichlet Beta Functions and a Criterion Equivalent to the Riemann Hypothesis

Felipe Martins
martins@lifequery.org

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Abstract

We study two rational sequences, called SeqImpar and SeqPar, originally defined by recurrences and converging to the common limit 5. The ratio of their convergence errors tends to 1/2. Using this ratio, we construct two Dirichlet series weighted by the von Mangoldt function and associated with the Riemann zeta function and the Dirichlet beta function. We prove that these series admit meromorphic continuations whose poles in the critical strip correspond exactly to the zeros of the respective functions, with residues equal to minus one-half of their multiplicities. We then prove that an estimate for the partial sum weighted by SeqImpar and SeqPar is equivalent to the Riemann Hypothesis, and state a parallel criterion for the Dirichlet beta function. The paper rigorously distinguishes proved identities, equivalences, and open problems: the coincidence of the number 1/2 with the critical line does not, by itself, constitute a proof of the Riemann Hypothesis.

Introduction

The Riemann Hypothesis (RH) states that all nontrivial zeros of the Riemann zeta function $\zeta(s)$ have real part equal to 1/2. The Dirichlet beta function $\beta(s) = L(s, \chi_4)$ has an analogous conjecture, which is a case of the Generalized Riemann Hypothesis (GRH). The relationship between the locations of these zeros and fluctuations in the distribution of prime numbers is classically established through the von Mangoldt function and explicit formulas [1, 2, 3, 4].

The aim of this paper is to show how two rational sequences, called SeqImpar and SeqPar, induce weights that preserve analytic information about the zeros of these L-functions. The central result is a rigorous reformulation of RH in terms of Dirichlet series and partial sums built from the sequences. We emphasize from the outset that introducing a weight with limit 1/2 does not prove that the zeros lie on the critical line: the limit of the weight and the real part of the complex parameter belong to different mathematical objects.

Conventions. We write $\mathrm{Re}(s)$ for the real part of s, $O(\cdot)$ for Landau's notation, and $\Lambda(n)$ for the von Mangoldt function. The identities for the sequences below are valid for the stated integer indices. The convergence of a Dirichlet series outside its region of absolute convergence is understood in the usual sense, with its terms summed in the natural order of the integers.

Closed Forms and Asymptotic Ratio

Let I_n be the nth term of SeqImpar, starting with $I_1 = 1$, $I_2 = 4/3$, $I_3 = 3$, and having increments $4/[(n-2)(n-1)]$ for $n \geq 4$. Let P_n be the nth term of SeqPar, starting with $P_1 = 1$, $P_2 = 2$, and having increments $2/[n(n-1)] + 6/[n(n+1)]$ for $n \geq 3$.

Proposition 1 (Closed-form expressions)**.** *For every integer $n \geq 3$,*

$$I_n = 5 - \frac{4}{n-1}, \qquad P_n = 5 - \frac{2}{n} - \frac{6}{n+1}. \tag{1}$$

Proof. Since $1/[(k-2)(k-1)] = 1/(k-2) - 1/(k-1)$, the first recurrence telescopes:

$$I_n = 3 + 4\sum_{k=4}^{n}\left(\frac{1}{k-2} - \frac{1}{k-1}\right) = 5 - \frac{4}{n-1}. \tag{2}$$

Similarly, using $1/[k(k-1)] = 1/(k-1) - 1/k$ and $1/[k(k+1)] = 1/k - 1/(k+1)$,

$$P_n = 2 + 2\sum_{k=3}^{n}\frac{1}{k(k-1)} + 6\sum_{k=3}^{n}\frac{1}{k(k+1)} \tag{3}$$

$$= 2 + 2\left(\frac{1}{2} - \frac{1}{n}\right) + 6\left(\frac{1}{3} - \frac{1}{n+1}\right) \tag{4}$$

$$= 5 - \frac{2}{n} - \frac{6}{n+1}. \tag{5}$$

In particular, both sequences converge to 5. □

Definition 1 (SeqImpar–SeqPar weight)**.** For $n \geq 3$, define the errors $E_I(n) = 5 - I_n$ and $E_P(n) = 5 - P_n$ and their ratio

$$R(n) = \frac{E_I(n)}{E_P(n)} = \frac{5 - I_n}{5 - P_n}. \tag{6}$$

By Proposition 1,

$$R(n) = \frac{2n(n+1)}{(n-1)(4n+1)}. \tag{7}$$

Consequently,

$$R(n) - \frac{1}{2} = \frac{7n+1}{2(n-1)(4n+1)} \tag{8}$$

and

$$R(n) = \frac{1}{2} + \frac{7}{8n} + O(n^{-2}). \tag{9}$$

Thus $R(n) > 1/2$ and $R(n) \to 1/2$. It is crucial to note that this is the ratio of the convergence errors: $I_n/P_n \to 1$, not $1/2$.

The combination of the errors also exhibits cancellation:

$$E_I(n) - \frac{1}{2}E_P(n) = \frac{7n+1}{n(n^2-1)} = O(n^{-2}). \tag{10}$$

The coefficient $1/2$ is the unique constant coefficient that eliminates the order-$1/n$ term in $E_I(n) - cE_P(n)$. This property is algebraic and, by itself, contains no information about the primality of n.

Exact Relationship with Prime Gaps

If $p < q$ are consecutive primes and $g = q - p$ is the gap between them, Proposition 1 gives

$$I_q - I_p = \frac{4g}{(p-1)(q-1)}, \tag{11}$$

$$P_q - P_p = g\left(\frac{2}{pq} + \frac{6}{(p+1)(q+1)}\right). \tag{12}$$

These identities encode a known gap g as increments of the two sequences. They are not, by themselves, independent formulas for locating the next prime or controlling g. The genuinely arithmetic information in the analytic constructions below will be introduced through the function $\Lambda(n)$.

Dirichlet Transform Associated with the Zeta Function

The von Mangoldt function is defined by

$$\Lambda(n) = \begin{cases} \log p, & n = p^k,\ p \text{ prime},\ k \geq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

For $\mathrm{Re}(s) > 1$, the classical identities [1, 2] hold:

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \qquad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{14}$$

Definition 2. We introduce the weighted series

$$F_\zeta(s) = \sum_{n=3}^{\infty} \frac{R(n)[\Lambda(n) - 1]}{n^s}, \qquad \mathrm{Re}(s) > 1. \tag{15}$$

Subtracting 1 is essential: it makes it possible to cancel the principal pole at $s = 1$ and measure the fluctuation of $\psi(x)$ around x.

Theorem 1 (Meromorphic continuation and zeta poles)**.** *Let*

$$c(n) = R(n) - \frac{1}{2} = \frac{7n + 1}{2(n - 1)(4n + 1)}. \tag{16}$$

Then F_ζ has a meromorphic continuation to the half-plane $\mathrm{Re}(s) > 0$, given by

$$F_\zeta(s) = -\frac{1}{2}\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{2}\zeta(s) + \frac{1}{2} + \frac{1 - \log 2}{2^{s+1}} + H_\zeta(s), \tag{17}$$

where

$$H_\zeta(s) = \sum_{n=3}^{\infty} \frac{c(n)[\Lambda(n) - 1]}{n^s} \tag{18}$$

is holomorphic for $\mathrm{Re}(s) > 0$. For every nontrivial zero ρ of $\zeta(s)$ with multiplicity m_ρ,

$$\operatorname*{Res}_{s=\rho} F_\zeta(s) = -\frac{m_\rho}{2}. \tag{19}$$

Conversely, the poles of F_ζ in the strip $0 < \mathrm{Re}(s) < 1$ are precisely the nontrivial zeros of ζ, with the stated multiplicity relation. The point $s = 1$ is a removable singularity of F_ζ.

Proof. Since $c(n) = O(1/n)$ and $|\Lambda(n) - 1| \leq \log n + 1$ for $n \geq 3$, each compact subset of $\mathrm{Re}(s) > 0$ admits a convergent majorant for $H_\zeta(s)$ of the form

$$\sum_{n=3}^{\infty} O\left(\frac{\log n + 1}{n^{1+\sigma_0}}\right), \qquad \sigma_0 > 0. \tag{20}$$

The normal convergence criterion shows that H_ζ is holomorphic in the stated half-plane. Separating $R(n) = 1/2 + c(n)$ and subtracting the $n = 1, 2$ terms from the classical series, we find

$$\sum_{n=3}^{\infty} \frac{\Lambda(n) - 1}{n^s} = -\frac{\zeta'(s)}{\zeta(s)} - \zeta(s) + 1 + \frac{1 - \log 2}{2^s}. \tag{21}$$

This identity proves the asserted formula. If $\zeta(s)$ has a zero of order m_ρ at ρ, then ζ'/ζ has a simple pole with residue m_ρ, whereas ζ and H_ζ are holomorphic at ρ; hence the residue of F_ζ is $-m_\rho/2$. Near $s = 1$, the residue of $-\zeta'/(2\zeta)$ is $+1/2$, and that of $-\zeta/2$ is $-1/2$; they cancel. □

Interpretation. The coefficient $1/2$ appears in the residues of the poles associated with zeros because it is the limit of the weight $R(n)$. This identity identifies zeros as singularities, but does not prove that their real parts equal $1/2$.

Criterion Equivalent to the Riemann Hypothesis

We define the Chebyshev function

$$\psi(x) = \sum_{n \le x} \Lambda(n) \tag{22}$$

and the sum weighted by the sequences

$$A_\zeta(x) = \sum_{3 \le n \le x} R(n)[\Lambda(n) - 1], \qquad x \ge 3. \tag{23}$$

Lemma 1 (Comparison of errors)**.** *Uniformly for $x \ge 3$,*

$$A_\zeta(x) = \frac{1}{2}[\psi(x) - x] + O(\log^2 x). \tag{24}$$

Proof. Using $R(n) = 1/2 + c(n)$,

$$A_\zeta(x) = \frac{1}{2} \sum_{3 \le n \le x} [\Lambda(n) - 1] + \sum_{3 \le n \le x} c(n)[\Lambda(n) - 1]. \tag{25}$$

The first term is $[\psi(x) - x]/2 + O(1)$, since removing the indices 1 and 2 and using $\lfloor x \rfloor$ change the sum by only a constant. The second term is bounded in absolute value by

$$O\left(\sum_{3 \le n \le x} \frac{\log n + 1}{n} \right) = O(\log^2 x). \tag{26}$$

This completes the proof. □

The classical theorem of von Koch, in one of its usual forms, states [1, 2, 3, 5] that

$$\mathrm{RH} \quad \Longleftrightarrow \quad \psi(x) - x = O\left(\sqrt{x} \log^2 x\right). \tag{27}$$

Theorem 2 (SeqImpar–SeqPar criterion equivalent to RH)**.** *The following statements are equivalent:*

1. *All nontrivial zeros of $\zeta(s)$ have real part $1/2$.*
2. *As $x \to \infty$, the weighted sum satisfies*

$$\sum_{3 \le n \le x} \frac{5 - I_n}{5 - P_n} [\Lambda(n) - 1] = O\left(\sqrt{x} \log^2 x\right). \tag{28}$$

3. *The original Dirichlet series*

$$\sum_{n=3}^{\infty} \frac{(5 - I_n)[\Lambda(n) - 1]}{(5 - P_n) n^s} \tag{29}$$

converges at every point of the half-plane $\mathrm{Re}(s) > 1/2$.

Proof. The equivalence of (1) and (2) follows directly from Lemma 1 and von Koch's classical criterion: the correction $O(\log^2 x)$ is smaller than the scale $O(\sqrt{x} \log^2 x)$ at issue. To prove (2)⇒(3), we apply summation by parts to a Dirichlet series with partial sums $A_\zeta(x) = O(\sqrt{x} \log^2 x)$. The resulting integral converges for $\mathrm{Re}(s) > 1/2$, since for $\sigma = \mathrm{Re}(s) > 1/2$,

$$\int_3^{\infty} \frac{\sqrt{t} \log^2 t}{t^{\sigma+1}} \, dt < \infty. \tag{30}$$

For (3)⇒(1), we use the theorem on locally uniform convergence of Dirichlet series in their half-plane of convergence. The sum would be holomorphic for $\mathrm{Re}(s) > 1/2$. By Theorem 1, any zeta zero in this region would produce a genuine pole, a contradiction. Hence there are no zeros with real part greater than 1/2. The functional equation and the symmetry of nontrivial zeros under $\rho \mapsto 1 - \rho$ also exclude zeros with real part less than 1/2. Thus all lie on the critical line. □

This equivalence is a characterization of RH, not a proof that RH is true. To establish RH one would need to prove (2) or (3) by methods independent of the hypothesis itself.

Corresponding Transform for the Dirichlet Beta Function

Define the primitive real character modulo 4 by

$$\chi_4(n) = \begin{cases} 0, & n \text{ even}, \\ 1, & n \equiv 1 \pmod 4, \\ -1, & n \equiv 3 \pmod 4. \end{cases} \tag{31}$$

For $\mathrm{Re}(s) > 1$,

$$\beta(s) = L(s, \chi_4) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^s} \tag{32}$$

and [2, 4]

$$-\frac{\beta'(s)}{\beta(s)} = \sum_{n=1}^{\infty} \frac{\chi_4(n)\Lambda(n)}{n^s}. \tag{33}$$

Definition 3. We introduce

$$F_\beta(s) = \sum_{n=3}^{\infty} \frac{R(n)\chi_4(n)\Lambda(n)}{n^s}, \qquad \mathrm{Re}(s) > 1. \tag{34}$$

Theorem 3 (Meromorphic continuation and beta poles)**.** *The function F_β has a meromorphic continuation to the half-plane $\mathrm{Re}(s) > 0$, given by*

$$F_\beta(s) = -\frac{1}{2}\frac{\beta'(s)}{\beta(s)} + H_\beta(s), \tag{35}$$

where

$$H_\beta(s) = \sum_{n=3}^{\infty} \frac{c(n)\chi_4(n)\Lambda(n)}{n^s} \tag{36}$$

is holomorphic for $\mathrm{Re}(s) > 0$. For every nontrivial zero ρ_β of β with multiplicity m_β,

$$\operatorname*{Res}_{s=\rho_\beta} F_\beta(s) = -\frac{m_\beta}{2}. \tag{37}$$

Proof. The same normal convergence argument used in Theorem 1 shows that H_β is holomorphic for $\mathrm{Re}(s) > 0$. Separating $R(n) = 1/2 + c(n)$ and using the identity for $-\beta'/\beta$ gives the stated continuation. The terms $n = 1$ and $n = 2$ do not contribute because $\Lambda(1) = 0$ and $\chi_4(2) = 0$. The residue at the zeros follows from the behavior of the logarithmic derivative of a holomorphic function. □

Corollary 1 (Analytic criterion for beta)**.** *All nontrivial zeros of $\beta(s)$ lie on $\mathrm{Re}(s) = 1/2$ if and only if the series defining $F_\beta(s)$ converges for every $\mathrm{Re}(s) > 1/2$.*

Justification. The convergence condition implies the absence of poles of F_β to the right of $1/2$, hence the absence of zeros of β in that region. The symmetry of the zeros of the L-function $L(s, \chi_4)$ completes one direction. Conversely, under GRH for χ_4, the classical estimate for the sum $\sum_{n\le x} \chi_4(n)\Lambda(n)$ is $O(\sqrt{x}\log^2 x)$, with fixed modulus 4. Since $R(n) - 1/2 = O(1/n)$, the weighted partial sum has the same order, and summation by parts guarantees convergence.

Thus the correspondence between poles and zeros is rigorous for both ζ and β, but they are two distinct sets of zeros.

Joint Relationship and the Dedekind Zeta Function

The Dedekind zeta function of the Gaussian quadratic field satisfies the classical factorization [4, 6]

$$\zeta_{\mathbb{Q}(i)}(s) = \zeta(s)\beta(s). \tag{38}$$

Defining $F_G(s) = F_\zeta(s) + F_\beta(s)$, we obtain

$$F_G(s) = -\frac{1}{2}\frac{d}{ds}\log \zeta_{\mathbb{Q}(i)}(s) - \frac{1}{2}\zeta(s) \tag{39}$$

$$+ \frac{1}{2} + \frac{1-\log 2}{2^{s+1}} + H_\zeta(s) + H_\beta(s), \tag{40}$$

initially for $\mathrm{Re}(s) > 1$ and then by meromorphic continuation. The logarithmic derivative is to be understood as $\zeta'_{\mathbb{Q}(i)}/\zeta_{\mathbb{Q}(i)}$, avoiding any global choice of a logarithm branch.

This factorization provides a unified language for the two zero-location problems. At points where zeta and beta share a zero, the multiplicities and residues of F_G add. It is not claimed that the zeros of ζ and β coincide.

Limits of the Interpretation and Mathematical Originality

The appearance of the number $1/2$ in two different places calls for an essential caveat. The limit $\lim R(n) = 1/2$ is an algebraic fact about rational weights. The line $\mathrm{Re}(s) = 1/2$ is a geometric condition on the complex parameter of the zeros. The formulas established in this paper provide bridges between these contexts, but numerical equality does not imply conceptual identity.

More generally, if a weight $w(n) = c + O(1/n)$ with $c \neq 0$ is used instead of $R(n)$, analogous series will also have poles at the zeros of zeta and beta, now with residues equal to $-c$ times their multiplicity. Thus turning zeros into poles is an instance of a broader principle of holomorphic perturbations of Dirichlet series; it is not unique to these two sequences. What is specific to SeqImpar and SeqPar is the exact rational expression for the weight, its limit $1/2$, and its asymptotic coefficients.

Possible research problem. Determine whether the particular rational structure of $R(n)$ can yield new estimates for the weighted partial sums that are not merely reformulations of classical results. This problem remains open in the present work. In particular, no unconditional $O(\sqrt{x}\log^2 x)$ estimate for $A_\zeta(x)$ has been derived.

Conclusions

The contribution of this paper is a proved, self-contained construction: starting from the closed forms of SeqImpar and SeqPar, we define the weight $R(n) = (5 - I_n)/(5 - P_n)$, prove that $R(n) = 1/2 + O(1/n)$, and use it in Dirichlet series involving the von Mangoldt function. The resulting transforms identify the nontrivial zeros of the Riemann zeta and Dirichlet beta functions as poles, with residue $-m/2$ at a zero of multiplicity m.

The result most directly related to the Riemann Hypothesis is the rigorous equivalence

$$\boxed{\text{RH} \quad \Longleftrightarrow \quad \sum_{3\le n\le x} \frac{5-I_n}{5-P_n}[\Lambda(n)-1] = O\Big(\sqrt{x}\log^2 x\Big)\,.} \tag{41}$$

There is also a parallel series-convergence criterion for the Dirichlet beta function. These formulations make the relationship between the sequences and zero locations mathematically precise, without claiming that they have already proved RH or GRH. The future challenge is to identify, if one exists, an independent argument forcing the critical cancellation estimates.

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