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FLAT LINEARITY OF UNIFORM GROUPS

OPUS 5.5 AND TEO 2.0

ABSTRACT. A uniform group is a quotient $\mathbb{Z}_K^{*M} \to \Gamma \to \mathbb{Z}_K^M$ on which S_M acts by permuting the generators, and a flat representation is a representation $\pi : \Gamma \to U_K$ such that each $\pi(g_i)$ has the K-th roots of unity as eigenvalues. These are the quasi-flat matrix models of the group duals $\widehat{\Gamma} \subset S_{KM}^+$. We show that $\mathbb{Z}_K^{*M}$ is flat linear, in the sense that its flat representations separate points, and that for finite groups, flat linearity and stationarity reduce to exact covering problems. We use this to compute a census, and to prove results on the Heisenberg and metabelian groups.

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INTRODUCTION

The quantum permutation group S_N^+ of Wang [29], which is a compact quantum group in the sense of Woronowicz [32], is defined via the magic matrices $u = (u_{ij})$, whose entries are projections, summing up to 1 on each row and each column. This quantum group has many interesting closed subgroups $G \subset S_N^+$, and a natural way of investigating them is via random matrix models, $\pi : C(G) \to M_K(C(X))$. Such a model is called inner faithful when its Hopf image, in the sense of [1], is the whole $C(G)$, and stationary when it recovers the Haar integration over G.

The simplest such models are the flat ones, where the standard coordinates $u_{ij} \in C(G)$ are mapped into projections of rank 1, as in [5], and their quasi-flat versions,

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where the coordinates u_{ij} are mapped into projections of rank ≤ 1. When G is quasi-transitive, with orbits of size K, there is a universal quasi-flat model, and the question is whether this model is inner faithful, or even stationary. See [2, 3, 4, 5].

In the group dual case, $G = \widehat{\Gamma}$, the situation is as follows:

(1) The work of Bichon [8] shows that the quasi-transitive examples of product type come from intermediate quotients $\mathbb{Z}_K^{*M} \to \Gamma \to \mathbb{Z}_K^M$, and the quasi-flat models become group representations $\pi : \Gamma \to U_K$, with each $\pi(g_i)$ having the K-th roots of unity as eigenvalues.

(2) Assuming that S_M acts on Γ by permuting the generators, we are led to the uniform groups from [2, 3], and to the questions whether the flat representations separate the points, which we call flat linearity, and whether they recover the Haar functional, which we call stationarity.

(3) At $K = 2$ the generators are involutions, Γ is a reflection group, and these questions are related to the work of Raum and Weber [22, 23], who showed that the strongly symmetric reflection groups classify the group-theoretical easy quantum groups.

(4) At $K \geq 3$, the uniform groups are expected to play a key role in the classification of the unitary easy quantum groups, in the hyperoctahedral case, complementary to the non-hyperoctahedral case and the noncrossing case, investigated by Mang-Weber and Tarrago-Weber, see [21, 26].

In quantum group terms, flat linearity means that the universal quasi-flat model is inner faithful, in the sense of [1], and stationarity means that this model is stationary, in the sense of [4]. Also, at $K = 2$, our results concern the matrix models of the group-theoretical easy quantum groups, at the level of their diagonal group duals.

The present paper is a systematic study of these questions, with an emphasis on explicit, computer-assisted group theory. Our first result concerns the free products, completing the inner faithfulness result from [4]:

Theorem A. *The free product $\mathbb{Z}_K^{*M}$ is flat linear. More precisely, the faithful flat representations of $\mathbb{Z}_K^{*M}$ form a dense G_δ subset of the space of all flat representations.*

In quantum group terms, this says that the universal quasi-flat model of the free product $\widehat{\mathbb{Z}}_K \,\widehat{*}\, \ldots \,\widehat{*}\, \widehat{\mathbb{Z}}_K \subset S_{KM}^+$, in the sense of Wang [28], is inner faithful, as stated in [4]. The proof uses a complexification argument, reducing the question to the existence of suitable elliptic elements of $PSL_2(\mathbb{R})$. In the finite case, our main observation is that everything becomes exact and computable:

Theorem B. *For a finite uniform group Γ, the flat representations correspond, up to unitary conjugacy, to the flat covers, namely the sets S of irreducible representations whose generator spectra exactly cover the K-th roots of unity. Moreover:*

(1) *Γ is flat linear precisely when the admissible irreducibles, which are those appearing in flat covers, have trivial common kernel.*

(2) *Γ is stationary precisely when the vector $(K \dim(\tau)/|\Gamma|)_{\tau \in \mathrm{Irr}(\Gamma)}$ is a convex combination of indicator vectors of flat covers.*

In particular, stationarity requires all irreducibles to have dimension $\leq K$.

This leads to an algorithm, which computes the character table numerically, but uses only integer data at the end, and decides stationarity by a rational linear program, with exact certificates. Running it on the basic families, we obtain:

Theorem C. *The following hold:*

(1) *If the diagonal kernel* $\Lambda_\Gamma = \ker(\Gamma \to \mathbb{Z}_K)$*, with* $g_i \to 1$*, is abelian, then* Γ *is stationary. In particular, the dihedral groups, the Heisenberg groups* H_K*, and more generally the uniform groups with* $M = 2$ *which are nilpotent of class* ≤ 2*, are all stationary.*

(2) *The groups* $\Gamma_{K,q} = (\mathbb{Z}_K * \mathbb{Z}_K)_{met}/qA$ *with* $K \geq 3$ *prime, and* $q \neq K$ *prime, are flat linear, but not stationary.*

(3) *In the case* $q|K$*, the groups* $\Gamma_{3,3}$ *and* $\Gamma_{4,2}$*, of order* 729 *and* 8192*, are not flat linear.*

Here $G_{met} = G/G''$ denotes the metabelianization of a group G, namely its largest metabelian quotient, and with $G = \mathbb{Z}_K * \mathbb{Z}_K$, the derived subgroup $A = G'/G''$ of G_{met} is free abelian of rank $(K-1)^2$, so that $\Gamma_{K,q}$ is obtained by killing the q-th powers in A. The first assertion comes from Thoma's construction [27], the point being that the representations induced from the characters of the diagonal kernel are automatically flat. The second assertion extends the main negative result of [3], concerning the case $K = 3$, $q = 2$, which appears here as a direct consequence of Theorem B. The third assertion, which is computer-assisted, shows that the hypothesis $q \nmid K$ is needed.

We have as well a systematic census, based on the SmallGroups library [7] of the GAP system, covering the uniform groups with $M = 2$ and $K \in \{3, 4, 5, 6, 7, 8, 9\}$, with $M = 3$ and $K \in \{2, 3\}$, and with $M = 4$ and $K = 2$, up to orders ranging between 112 and 2000, depending on the parameters. This census shows that none of the obvious conjectures that can be naively formulated hold:

Theorem D. *The following happen:*

(1) *The group* $A_4 \times \mathbb{Z}_3$ *has a uniform structure which is stationary, but whose diagonal kernel is not abelian.*

(2) *The groups* $\mathbb{Z}_3 \times SL_2(\mathbb{F}_3)$ *and* $\mathbb{Z}_5^2 \times A_5$ *have uniform structures whose irreducibles are all admissible, but which are not stationary.*

(3) *The group* $S_4 \times \mathbb{Z}_2^2$ *has a uniform structure, with* $K = 2$*,* $M = 3$*, which is not flat linear, and so does the extraspecial group* 2_-^{1+4} *of order* 32*, with* $K = 2$*,* $M = 4$*.*

(4) *The group* $\mathbb{Z}_3^2 \times ((\mathbb{Z}_6 \times S_3) \rtimes \mathbb{Z}_2)$ *of order* 648 *has a uniform structure, with* $K = 6$*, whose irreducibles are all admissible, which passes the dimension counting test for stationarity, but which is not stationary.*

At $K = 2$, the flat representations are given by configurations of points on the sphere S^2, via the Pauli matrices, the identifications of letters in the sense of Raum and Weber corresponding to collisions of points. For the strongly symmetric groups $\Gamma_N^{[s]} = \mathbb{Z}_2^{*N}/\langle (g_i g_j)^s \rangle$, behind the hyperoctahedral series from [22], we show that these are not flat linear, for $s \geq 4$ finite and even, and $N \geq 3$, and that at $s = 4$ their flat representations factor through the free nilpotent reflection group of class 2.

Regarding (4), all the other 730 admissible structures found are decided by dimension counting alone, and this example shows that stationarity depends as well on which irreducibles can appear together in a flat representation.

Finally, we discuss some notions going beyond flatness: a scale of decay properties between stationarity and inner faithfulness, flatness at height $m \geq 1$, and a second order flatness condition, which for the free products $\mathbb{Z}_K^{*M}$ is equivalent to the existence of M mutually unbiased bases in $\mathbb{C}^K$.

The paper is organized as follows: 1-2 contain generalities and the proof of Theorem A, 3-4 contain the finite group theory and a first computer census, 5-6 contain the stationarity and metabelian results, 7 discusses the reflection groups, in connection with the work of Raum and Weber, 8 contains the SmallGroups census, and 9 contains the material going beyond flatness.

1. Uniform groups, definitions

We use the following notion, from [2, 3], which is motivated by the work of Bichon [8] on the group dual subgroups of the quantum permutation groups S_N^+:

Definition 1.1. *A discrete group Γ is called uniform, with parameters $K, M \in \mathbb{N}$, when:*

(1) *Γ is generated by M elements, $\Gamma = \langle g_1, \ldots, g_M \rangle$.*
(2) *Γ appears as an intermediate quotient $\mathbb{Z}_K^{*M} \to \Gamma \to \mathbb{Z}_K^M$, with the standard generators of $\mathbb{Z}_K^{*M}$ mapping to $g_1, \ldots, g_M$, and then to the standard generators of $\mathbb{Z}_K^M$.*
(3) *We have an action $S_M \curvearrowright \Gamma$, given by $\sigma(g_i) = g_{\sigma(i)}$.*

Observe that (2) forces each generator g_i to have order exactly K. The motivation comes from the quantum permutation groups, as follows. Given a quotient $\mathbb{Z}_K^{*M} \to \Gamma$, with $N = KM$, Bichon's construction [8] produces an embedding $\widehat{\Gamma} \subset S_N^+$, the magic unitary being block-diagonal, with the i-th block being the Fourier transform of the i-th copy of $\mathbb{Z}_K$. Moreover, as explained in [3, 4], the quasi-transitive group duals $\widehat{\Gamma} \subset S_N^+$ which are of product type, with orbits of size K, are precisely those coming from intermediate quotients as in (2).

Regarding (3), this condition is there in order to avoid degenerate situations, such as the one exhibited in [4], where $\Gamma = (\mathbb{Z}_K * \mathbb{Z}_K) \times \mathbb{Z}_K$ has no inner faithful quasi-flat model, for the simple reason that a unitary with simple spectrum which commutes with $\pi(g_1)$ and $\pi(g_2)$ forces these 2 matrices to commute. Here are now the main examples:

Example 1.2. The following groups are uniform:

(1) The free products $\mathbb{Z}_K^{*M}$, and the abelian groups $\mathbb{Z}_K^M$.
(2) The dihedral groups D_n of order $2n$, with n even, with $K = M = 2$, generated by 2 reflections. Also, the infinite dihedral group $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$.
(3) The Heisenberg groups H_K of order K^3, generated by g_1, g_2 subject to the relations $g_1^K = g_2^K = 1$, with $z = [g_1, g_2]$ central and satisfying $z^K = 1$.
(4) The universal metabelian groups $(\mathbb{Z}_K^{*M})_{met} = \mathbb{Z}_K^{*M}/(\mathbb{Z}_K^{*M})''$, and their quotients by characteristic subgroups of the abelian group $A = (\mathbb{Z}_K^{*M})'/(\mathbb{Z}_K^{*M})''$.

Indeed, in all cases the group Γ appears by quotienting $\mathbb{Z}_K^{*M}$ by a characteristic subgroup, contained in the derived subgroup, so (2) and (3) are satisfied. For the Heisenberg group, observe that the transposition of g_1, g_2 maps $z \to z^{-1}$. For the metabelian quotients, the structure of A was worked out in [3], as follows: at $M = 2$, we have $A \simeq \mathbb{Z}^{(K-1)^2}$, which can be identified with $V \otimes V$, where $V = \mathbb{Z}^{K-1}$, in such a way that g_1, g_2 act by $m \otimes 1$ and $1 \otimes m$, where m is the companion matrix of the polynomial $1 + t + \ldots + t^{K-1}$.

At $K = 2$, an important class of examples is formed by the strongly symmetric reflection groups, whose defining relations are stable under all the identifications of the generators, and which are uniform as soon as they are nondegenerate, in the sense that their abelianization is $\mathbb{Z}_2^M$. These groups produce, via the work of Raum and Weber [22, 23], the group-theoretical easy quantum groups, and they form an uncountable class. We will come back to them in section 7.

In what follows we will be particularly interested in the following finite groups:

Definition 1.3. *Given $K \geq 2$ and a prime q, we set*

$$\Gamma_{K,q} = (\mathbb{Z}_K * \mathbb{Z}_K)_{met}/qA$$

which is an extension of $\mathbb{Z}_K^2$ by the elementary abelian group $A/qA \simeq \mathbb{Z}_q^{(K-1)^2}$.

For $K = 3$, $q = 2$, this is the group of order 144 investigated in [3]. In general, when q does not divide K, the extension splits by the Schur-Zassenhaus theorem, and so $\Gamma_{K,q} = A_q \rtimes \mathbb{Z}_K^2$, with $A_q = \mathbb{Z}_q^{(K-1)^2}$, the action being by $m \otimes 1$ and $1 \otimes m$. With these conventions, we have the following summary of the above discussion:

Theorem 1.4. *The groups $\mathbb{Z}_K^{*M}$, $\mathbb{Z}_K^M$, D_n with $n \in 2\mathbb{N} \cup \{\infty\}$, H_K and $\Gamma_{K,q}$ are all uniform. Moreover, D_n with n even is a quotient of D_∞, $H_2 = D_4$, and when $q \nmid K$ we have $\Gamma_{K,q} = A_q \rtimes \mathbb{Z}_K^2$, with $A_q = \mathbb{Z}_q^{(K-1)^2}$.*

Proof. This follows from the above discussion, the only assertion which remains to be checked being the formula $H_2 = D_4$. But this comes from the fact that the dihedral group of order 8 is generated by 2 reflections whose commutator is the central rotation by π. □

2. Flat representations, models

Let us first explain the terminology. A real matrix $A \in M_N(\mathbb{R})$ can be thought of as a landscape built over the $N \times N$ grid, with height A_{ij} above the point (i, j), and A is called flat when this landscape is flat, that is, when all its entries are equal. Given a closed subgroup $G \subset S_N^+$ and a matrix model $\pi : C(G) \to M_K(C(X))$, mapping $u_{ij} \to P_{ij}$, this applies to the matrices of ranks

$$r_{ij}^x = rank(P_{ij}^x)$$

which are bistochastic, with sums K. When $K = N$, the model is called flat when all these matrices are flat, $r_{ij}^x = 1$, as in [5]. In the quasi-transitive case, with orbits of size K, the matrices r^x cannot be flat, because $r_{ij}^x = 0$ when i, j are in different orbits, and the model is called quasi-flat when each r^x is block-flat, with all entries equal to 1

inside the diagonal $K\times K$ blocks, which amounts in saying that we have $rank(P_{ij}^x)\leq 1$, for any i,j,x. See [3, 4].

In the group dual case, this flatness condition becomes a spectral condition. Indeed, let $w=e^{2\pi i/K}$, and consider the subgroup $\widehat{\Gamma}\subset S_{KM}^+$ coming from a quotient $\mathbb{Z}_K^{*M}\to\Gamma$. Bichon's construction [8] produces a block-diagonal magic matrix $u=diag(u^{(1)},\ldots,u^{(M)})$, the entries of the i-th block being the spectral projections of the generator $g_i\in C^*(\Gamma)$:

$$u_{ab}^{(i)}=\frac{1}{K}\sum_{m=0}^{K-1}w^{-(b-a)m}g_i^m$$

Thus, for the model coming from a representation $\pi:\Gamma\to U_K$, the rank of $\pi(u_{ab}^{(i)})$ is the multiplicity of w^{b-a} as an eigenvalue of $\pi(g_i)$. We conclude that the rank matrix is block-flat precisely when each $\pi(g_i)$ has all the K-th roots of unity as eigenvalues, each with multiplicity 1. With $W=diag(w,w^2,\ldots,w^K)\in U_K$, this leads to:

Definition 2.1. *Let $\Gamma=\langle g_1,\ldots,g_M\rangle$ be uniform.*

(1) *A representation $\pi:\Gamma\to U_K$ is called flat when each $\pi(g_i)$ is conjugate to W, that is, has all the K-th roots of unity as eigenvalues, with multiplicity 1.*
(2) *The space of flat representations is denoted X_Γ, and the universal flat model is $\pi_\Gamma:C^*(\Gamma)\to M_K(C(X_\Gamma))$, given by $\pi_\Gamma(g)(\pi)=\pi(g)$.*

Here X_Γ is a compact real algebraic subset of U_K^M, namely the set of the tuples $(V_1,\ldots,V_M)$ of conjugates of W satisfying the relations of Γ. This is a particular case of a representation variety, in the sense of [20, 25]. As explained in [3, 4], via the Fourier transform over $\mathbb{Z}_K$, the model π_Γ is the universal quasi-flat model of the corresponding group dual, $\widehat{\Gamma}\subset S_{KM}^+$. We use the following notions:

Definition 2.2. *A uniform group Γ is called:*

(1) *Flat linear, when its flat representations separate the points, in the sense that for any $g\neq 1$ there exists $\pi\in X_\Gamma$ with $\pi(g)\neq 1$.*
(2) *Stationary, when there is a probability measure μ on X_Γ such that*

$$\int_{X_\Gamma}tr(\pi(g))\,d\mu(\pi)=\delta_{g,1}$$

for any $g\in\Gamma$, where tr is the normalized trace of $M_K(\mathbb{C})$.

The terminology in (1) is inspired by the notion of inner linearity for discrete quantum groups: a flat linear group is one which embeds, in a parametric sense, into the unitary group U_K, via its flat representations. The flat linearity of Γ is precisely the inner faithfulness of π_Γ, in the sense of [1], and the stationarity of Γ is precisely the stationarity of π_Γ, in the sense of [4]. This latter condition forces the kernel of π_Γ to be contained in the kernel of the left regular representation, so the reduced group algebra $C_{red}^*(\Gamma)$ is a quotient of the image of π_Γ, which is contained in $M_K(C(X_\Gamma))$. Thus $C_{red}^*(\Gamma)$ is subhomogeneous, which by Thoma's theorem [27] requires Γ to be virtually abelian. See [2]. For the convenience of the reader, here is a dictionary between the group theory notions used in this paper, and the quantum group notions behind them:

Group side	Quantum group side
uniform group $\mathbb{Z}_K^{*M} \to \Gamma \to \mathbb{Z}_K^M$	product type group dual $\widehat{\Gamma} \subset S_{KM}^+$, with S_M symmetry
flat representation $\pi : \Gamma \to U_K$	quasi-flat model, $rank(P_{ij}) \leq 1$
flat linear group	inner faithful universal quasi-flat model
stationary group	stationary model, $\int tr \circ \pi = h$
flat representation at height m	quasi-flat model at height m, in M_{mK}
strongly symmetric, with $K = 2$	group-theoretical easy quantum group

The last line refers to the work of Raum and Weber [22, 23], to be discussed in section 7, and the height m notion will be discussed in section 9. At the elementary level, we have:

Proposition 2.3. *If Γ is stationary, then Γ is flat linear.*

Proof. If $g \in \Gamma$ satisfies $\pi(g) = 1$ for any $\pi \in X_\Gamma$, then the integral in the stationarity formula equals 1, and so $g = 1$. □

Let us discuss now the free products. For $\Gamma = \mathbb{Z}_K^{*M}$ there are no relations, and the space of flat representations is simply $X_\Gamma = C_W^M$, where $C_W = \{UWU^*|U \in U_K\} \simeq U_K/\mathbb{T}^K$ is the conjugacy class of W. The following statement was proved in [4], modulo the fact that a certain noncommutative polynomial does not vanish identically on U_K^M, and we provide here a complete proof, along with a stronger, generic version of the result:

Theorem 2.4. *The free product $\mathbb{Z}_K^{*M}$ is flat linear. More precisely, the set of faithful flat representations is a dense G_δ subset of $X_\Gamma = C_W^M$.*

Proof. At $M = 1$ the representation $g_1 \to W$ is faithful, so assume $M \geq 2$. Any $\gamma \neq 1$ in $\Gamma = \mathbb{Z}_K^{*M}$ can be written as a reduced word $\gamma = g_{i_1}^{k_1} \ldots g_{i_n}^{k_n}$, with $n \geq 1$, $i_p \neq i_{p+1}$ and $1 \leq k_p \leq K - 1$. Consider the following holomorphic map, defined on the connected complex manifold $GL_K(\mathbb{C})^M$:

$$F_\gamma(A_1, \ldots, A_M) = \prod_{p=1}^{n} A_{i_p} W^{k_p} A_{i_p}^{-1} - 1$$

Our first claim is that F_γ does not vanish identically on U_K^M. Indeed, assume that it does. Since the complexification of the Lie algebra of U_K is the full matrix algebra $M_K(\mathbb{C})$, a holomorphic function on $GL_K(\mathbb{C})^M$ which vanishes on U_K^M has all its derivatives vanishing at the points of U_K^M, and so vanishes identically, by the identity theorem. Thus we would have $F_\gamma = 0$ on $GL_K(\mathbb{C})^M$, and we will show that this is not the case.

We use the fact that $\mathbb{Z}_K^{*M}$ has a faithful representation $\sigma : \mathbb{Z}_K^{*M} \to PSL_2(\mathbb{R})$, mapping each generator g_i to an elliptic element, of rotation angle $2\pi/K$. Indeed, when $M(K - 1) > K$, the orbifold consisting of a sphere with M cone points of order K and one puncture is hyperbolic, its fundamental group is $\mathbb{Z}_K^{*M}$, and its uniformization produces such a representation, by the Poincaré polygon theorem. See [6, 17]. In the remaining

case $K = M = 2$ we can use the infinite dihedral group, generated inside $PSL_2(\mathbb{R})$ by the half-turns around 2 distinct points.

Now since an elliptic element of rotation angle $2\pi/K$ is conjugate in $PGL_2(\mathbb{C})$ to the class of $diag(w, w^2)$, we can find matrices $B_1, \ldots, B_M \in GL_2(\mathbb{C})$ such that $\sigma(g_i)$ is the class of $B_i\, diag(w, w^2)B_i^{-1}$. Consider then $A_i = B_i \oplus 1_{K-2}$. Since $W = diag(w, w^2) \oplus diag(w^3, \ldots, w^K)$, the upper left 2×2 block of the product appearing in $F_\gamma(A_1, \ldots, A_M)$ is:

$$\prod_{p=1}^{n} B_{i_p} diag(w, w^2)^{k_p} B_{i_p}^{-1}$$

The class of this matrix in $PGL_2(\mathbb{C})$ is $\sigma(\gamma) \neq 1$, and so this matrix is not scalar. Thus $F_\gamma(A_1, \ldots, A_M) \neq 0$, which proves our claim.

Now back to X_Γ, consider the closed subset $Z_\gamma = \{\pi \in X_\Gamma | \pi(\gamma) = 1\}$. Since $X_\Gamma = C_W^M$ is a connected real analytic manifold, and $\pi \to \pi(\gamma) - 1$ is a real analytic function on it, which is not identically zero by the above, Z_γ has empty interior. Since Γ is countable, the Baire category theorem shows that the complement of $\bigcup_{\gamma\neq 1} Z_\gamma$, which is the set of faithful flat representations, is a dense G_δ. □

Alternatively, one can use here Dekker's faithful representations of free products of cyclic groups by rotations [12], lifted to U_2, and completed with $w^2, \ldots, w^{K-1}$.

3. The finite group case

In this section we assume that Γ is a finite uniform group. We denote by $\mathrm{Irr}(\Gamma)$ the set of its irreducible representations, up to equivalence, and for $\tau \in \mathrm{Irr}(\Gamma)$ we denote by χ_τ its character. The point is that everything reduces to integer data, as follows:

Definition 3.1. *Given a finite uniform group Γ, and $\tau \in \mathrm{Irr}(\Gamma)$:*

(1) *The spectral data of τ is the matrix $e_\tau \in M_{M\times K}(\mathbb{N})$ given by*

$$(e_\tau)_{ij} = \text{multiplicity of } w^j \text{ as eigenvalue of } \tau(g_i)$$

and τ is called admissible when e_τ has entries $0, 1$.

(2) *A flat cover is a set $S \subset \mathrm{Irr}(\Gamma)$ of admissible irreducibles such that $\sum_{\tau\in S} e_\tau$ is the all-one matrix. We denote by $\mathcal{F}_\Gamma$ the set of flat covers.*

Observe that the spectral data can be computed from the character, as follows, with χ_τ being evaluated on the powers of the generators:

$$(e_\tau)_{ij} = \frac{1}{K}\sum_{m=0}^{K-1} \chi_\tau(g_i^m) w^{-jm}$$

Geometrically, it is convenient to think of the spectral data e_τ as a subset of the $M \times K$ grid, formed by the pairs (i, j) such that w^j is an eigenvalue of $\tau(g_i)$. When τ is admissible, this subset has exactly $\dim(\tau)$ points on each row, so that $\dim(\tau) \leq K$, and a flat cover is a set of admissible irreducibles whose subsets partition the grid. Thus, the flat covers are the solutions of an exact covering problem, in the sense of combinatorics, and this will be the key to the computations below. With these notions in hand, we have:

Theorem 3.2. *Let Γ be a finite uniform group.*

(1) *The flat representations of Γ are, up to conjugation, the representations $\pi_S = \bigoplus_{\tau\in S}\tau$, with $S \in \mathcal{F}_\Gamma$, and we have a decomposition*

$$X_\Gamma = \bigsqcup_{S\in\mathcal{F}_\Gamma} O_S \quad , \quad O_S \simeq U_K/\mathbb{T}^{|S|}$$

into connected components, given by the conjugation orbits O_S of the representations π_S.

(2) *Γ is flat linear precisely when its admissible irreducibles have trivial common kernel, these being exactly the irreducibles $\tau \in \bigcup_{S\in\mathcal{F}_\Gamma} S$.*

(3) *Γ is stationary precisely when there are weights $c_S \geq 0$ satisfying*

$$\sum_{S\ni\tau} c_S = \frac{K\dim(\tau)}{|\Gamma|} \quad , \quad \forall \tau \in {\rm Irr}(\Gamma)$$

and in this case $\sum_S c_S = 1$, and the stationarity measure can be taken to be $\mu = \sum_S c_S\mu_S$, with μ_S being the Haar measure on O_S.

Proof. (1) Given a flat representation π, decompose it as $\pi = \bigoplus_\tau m_\tau\tau$. The spectrum of $\pi(g_i)$, counted with multiplicities, is then the sum of the spectra of the $\tau(g_i)$, taken with multiplicities m_τ. Since this spectrum consists of the K-th roots of unity, each with multiplicity 1, we must have $m_\tau \in \{0, 1\}$, each τ with $m_\tau = 1$ must be admissible, and the set $S = \{\tau | m_\tau = 1\}$ must be a flat cover. Conversely, any flat cover S produces a flat representation π_S, whose dimension is automatically K. Since π_S is multiplicity-free, the stabilizer of π_S under the conjugation action of U_K is the unitary group of its commutant, which is $\mathbb{T}^{|S|}$, and this gives the orbit description. Finally, distinct flat covers produce representations with distinct characters, and so disjoint closed orbits.

(2) Any admissible irreducible τ belongs to a flat cover. Indeed, in each row of the $M \times K$ grid, τ occupies $\dim(\tau)$ slots, and the remaining $K-\dim(\tau)$ slots of the M rows can be matched, in an arbitrary way, into $K - \dim(\tau)$ characters, all the elements of $\mathbb{Z}_K^M$ being characters, because $\Gamma^{ab} = \mathbb{Z}_K^M$. Thus the irreducibles appearing in flat covers are exactly the admissible ones, and since $\pi_S(g) = 1$ precisely when $\tau(g) = 1$ for any $\tau \in S$, this gives the result.

(3) Let μ be a probability measure on X_Γ, and set $c_S = \mu(O_S)$. Since the trace is invariant under conjugation, the function $\pi \to tr(\pi(g))$ is constant on each O_S, and we obtain:

$$\begin{aligned}\int_{X_\Gamma} tr(\pi(g))\,d\mu(\pi) &= \sum_S c_S\,tr(\pi_S(g))\\ &= \frac{1}{K}\sum_S c_S\sum_{\tau\in S}\chi_\tau(g)\\ &= \frac{1}{K}\sum_\tau\left(\sum_{S\ni\tau} c_S\right)\chi_\tau(g)\end{aligned}$$

On the other hand, the regular character gives $\delta_{g,1} = \frac{1}{|\Gamma|}\sum_\tau \dim(\tau)\chi_\tau(g)$. Since the irreducible characters are linearly independent, μ produces the stationarity precisely

when the equations in the statement are satisfied. Conversely, given weights $c_S \geq 0$ satisfying these equations, the measure $\mu = \sum_S c_S \mu_S$ produces the stationarity. Finally, multiplying the equations by $\dim(\tau)$ and summing gives $K \sum_S c_S = K$, because each flat cover has total dimension K, and because $\sum_\tau \dim(\tau)^2 = |\Gamma|$. □

As a first consequence, we have the following obstruction to stationarity:

Corollary 3.3. *If a finite uniform group Γ is stationary, then all its irreducible representations τ are admissible. In particular, we must have $\dim(\tau) \leq K$, and each matrix $\tau(g_i)$ must have simple spectrum.*

Proof. All the right-hand sides of the equations in Theorem 3.2 (3) are strictly positive, so each τ must belong to a flat cover, and so must be admissible. □

As an illustration, let us discuss the abelian case. Here the irreducibles are the characters χ_a, with $a \in \mathbb{Z}_K^M$, given by $\chi_a(g_i) = w^{a_i}$, they are all admissible, and the flat covers are the sets $S \subset \mathbb{Z}_K^M$ of size K whose coordinate projections $S \to \mathbb{Z}_K$ are all bijective. We have:

Theorem 3.4. *The abelian group $\mathbb{Z}_K^M$ is stationary, and therefore flat linear. Moreover, it has $(K!)^{M-1}$ flat covers, and the stationarity measure can be taken to be the uniform measure on the K^{M-1} translates of the diagonal $\{(t, \ldots, t) | t \in \mathbb{Z}_K\}$.*

Proof. A flat cover S is determined by the permutations $\sigma_2, \ldots, \sigma_M \in S_K$ such that $S = \{(t, \sigma_2(t), \ldots, \sigma_M(t)) | t \in \mathbb{Z}_K\}$, which gives the count $(K!)^{M-1}$. The translates $b + \Delta$ of the diagonal Δ are flat covers, which partition $\mathbb{Z}_K^M$ into K^{M-1} classes. Thus, the weights $c_{b+\Delta} = K^{1-M}$ satisfy the equations in Theorem 3.2 (3), whose right-hand side is K/K^M. □

4. Computer census, first round

Theorem 3.2 converts the flat linearity and stationarity questions, for a finite uniform group Γ, into a finite amount of exact integer computations. We have implemented this in the Python program `flatlin.py`, which works as follows:

(1) The group Γ is given by M permutations $g_1, \ldots, g_M$ of order K. The program computes the elements, the multiplication table, the conjugacy classes and the power maps, and checks the S_M symmetry, by checking that the transposition (12) and the M-cycle of the generators extend to automorphisms.

(2) The character table is computed numerically, by Burnside's algorithm [11, 13], as common eigenvectors of the class multiplication matrices, and is checked against the orthogonality relations. Only integer quantities are extracted from it: the dimensions, the spectral data e_τ, via the formula after Definition 3.1, and the kernels, all with rounding errors checked to be $< 10^{-6}$.

(3) The flat covers are enumerated by an exact cover search, in the style of Knuth's Algorithm X [19], and flat linearity is decided by the kernel criterion in Theorem 3.2 (2).

(4) Stationarity is decided by the linear program in Theorem 3.2 (3). In the feasible case, a vertex solution is computed in floating point and then recomputed

exactly, in rational arithmetic, from its support. In the infeasible case, the program produces either an irreducible belonging to no flat cover, or a Farkas certificate, namely a rational vector y with $\langle y, 1_S\rangle \leq 0$ for any flat cover S, and $\langle y, \lambda\rangle > 0$, where λ is the right-hand side vector, all checked in exact arithmetic.

The program `examples.py` runs this on the basic families of uniform groups from section 1, realized as permutation groups: $\mathbb{Z}_K^M$ acting on itself, D_n acting on the n-gon, H_K acting linearly on $\mathbb{Z}_K^3$, and $\Gamma_{3,q} = A_q \rtimes \mathbb{Z}_3^2$ acting affinely on $A_q = \mathbb{Z}_q^4$, with $g_1 : v \to (m\otimes 1)v$ and $g_2 : v \to (1\otimes m)v + a$, for a suitable vector a. Its output is recorded in `examples.log`. For the groups $\Gamma_{K,q}$ in general, including the case $q|K$, where the Schur-Zassenhaus theorem does not apply, we use instead the normal form $\gamma = c^v g_1^x g_2^y$, where $c_{ab} = [g_1^a, g_2^b]$ with $1 \leq a, b \leq K-1$ is the standard free basis of the derived subgroup of $\mathbb{Z}_K * \mathbb{Z}_K$, the multiplication being computed from the conjugation action of g_1, g_2 on the elements c_{ab}. This is implemented in the program `metab.py`, which recovers $\Gamma_{3,2}$, and computes $\Gamma_{3,3}$. For larger orders, the GAP program `dempsey.g` builds $\Gamma_{K,q}$ from its presentation, with generators g_1, g_2, and relations $g_1^K = g_2^K = 1$, $c_{ab}^q = 1$ and $[c_{ab}, c_{a'b'}] = 1$, and records the same data as the census programs from section 8 below, which is then analyzed in Python. The results are summarized below:

Γ	$\lvert\Gamma\rvert$	K	M	irreducibles	$\lvert\mathcal{F}_\Gamma\rvert$	Λ_Γ	flat	stat.
$\mathbb{Z}_2^2, \mathbb{Z}_2^3, \mathbb{Z}_2^4$	4, 8, 16	2	2, 3, 4	$1^{\lvert\Gamma\rvert}$	2, 4, 8	yes	yes	yes
$\mathbb{Z}_3^2, \mathbb{Z}_3^3$	9, 27	3	2, 3	$1^{\lvert\Gamma\rvert}$	6, 36	yes	yes	yes
$\mathbb{Z}_4^2, \mathbb{Z}_5^2$	16, 25	4, 5	2	$1^{\lvert\Gamma\rvert}$	24, 120	yes	yes	yes
D_n, $4 \leq n \leq 24$	$2n$	2	2	$1^4\, 2^{n/2-1}$	$n/2+1$	yes	yes	yes
H_3	27	3	2	$1^9\, 3^2$	8	yes	yes	yes
H_4	64	4	2	$1^{16}\, 2^4\, 4^2$	36	yes	yes	yes
H_5	125	5	2	$1^{25}\, 5^4$	124	yes	yes	yes
H_6	216	6	2	$1^{36}\, 2^9\, 3^8\, 6^2$	1036	yes	yes	yes
H_7	343	7	2	$1^{49}\, 7^6$	5046	yes	yes	yes
H_8	512	8	2	$1^{64}\, 2^{16}\, 4^8\, 8^4$	54068	yes	yes	yes
H_9	729	9	2	$1^{81}\, 3^{18}\, 9^6$	376326	yes	yes	yes
$\Gamma_{3,2}$	144	3	2	$1^9\, 3^6\, 9^1$	12	no	yes	no
$\Gamma_{3,5}$	5625	3	2	$1^9\, 3^{48}\, 9^{64}$	54	no	yes	no
$\Gamma_{3,7}$	21609	3	2	$1^9\, 3^{96}\, 9^{256}$	102	no	yes	no
$\Gamma_{3,11}$	131769	3	2	$1^9\, 3^{240}\, 9^{1600}$	246	no	yes	no
$\Gamma_{3,3}$	729	3	2	$1^9\, 3^{26}\, 9^6$	20	no	no	no
$\Gamma_{4,2}$	8192	4	2	$1^{16}\, 2^4\, 4^{30}\, 8^{24}\, 16^{24}$	52	no	no	no

Here the notation d^m in the irreducibles column means that there are m irreducibles of dimension d, the column Λ_Γ indicates whether the diagonal kernel, introduced in section 5 below, is abelian, and the dihedral groups were computed for the values $n = 4, 6, 8, 10, 12, 16, 20, 24$. For the dihedral groups we recover the stationarity result from [3], and the number of flat covers is $n/2+1$, corresponding to the $n/2-1$ irreducibles

of dimension 2, which are flat covers by themselves, and to the 2 ways of pairing the 4 characters. Regarding $\Gamma_{3,2}$, the failure of stationarity was established in [3] by a direct computation, and appears here as an instance of Corollary 3.3, because of the irreducible of dimension $9 > 3$. Also, the irreducibles of $\Gamma_{3,q}$ with $q = 2, 5, 7, 11$ are in agreement with the formula in section 6 below, and the computation of $\Gamma_{3,3}$ was done both with `metab.py` and with `dempsey.g`, with identical results. Summarizing, we have:

Theorem 4.1. *The groups $\mathbb{Z}_K^M$ and D_n, as well as H_K with $K \leq 9$, are stationary, and so flat linear. The groups $\Gamma_{3,q}$ with $q = 2, 5, 7, 11$ are flat linear, but not stationary, and the groups $\Gamma_{3,3}$ and $\Gamma_{4,2}$ are neither flat linear, nor stationary.*

Proof. This follows from the above computations, all certified in exact arithmetic as explained above, along with Theorem 3.4 for the groups $\mathbb{Z}_K^M$ with arbitrary K, M, and with [3] for the dihedral groups D_n with n arbitrary. □

5. Diagonal kernels, stationarity

In this section we prove a general stationarity result, which explains most of the positive results in the census, and in particular covers all the Heisenberg groups. The key object is the following normal subgroup:

Definition 5.1. *The diagonal kernel of a uniform group $\Gamma = \langle g_1, \ldots, g_M \rangle$ is*

$$\Lambda_\Gamma = \ker(\Gamma \to \mathbb{Z}_K)$$

where the morphism $\Gamma \to \mathbb{Z}_K$ is the composition of the quotient map $\Gamma \to \mathbb{Z}_K^M$ with the sum map $\mathbb{Z}_K^M \to \mathbb{Z}_K$, so that $g_i \to 1$ for any i.

Observe that Λ_Γ is a normal subgroup of index K, which contains the derived subgroup Γ', and which is invariant under the action of S_M. Moreover, we have $g_i \in g_1\Lambda_\Gamma$ for any i, and the cosets of Λ_Γ are represented by the powers $1, g_1, \ldots, g_1^{K-1}$. Our main result here is:

Theorem 5.2. *Let Γ be a uniform group, finite or not, whose diagonal kernel $\Lambda = \Lambda_\Gamma$ is abelian. Then for any character $\chi \in \widehat{\Lambda}$ the induced representation*

$$\pi_\chi = \text{Ind}_\Lambda^\Gamma(\chi) : \Gamma \to U_K$$

is flat, and the push-forward of the Haar measure of the compact abelian group $\widehat{\Lambda}$ via the map $\chi \to \pi_\chi$ is a stationarity measure. In particular, Γ is stationary, and so flat linear.

Proof. We use the coset representatives $x_j = g_1^j$, with $j \in \mathbb{Z}_K$, and the corresponding basis $\{e_j\}$ of the space of π_χ, so that $\pi_\chi(g)e_j = \chi(x_{j'}^{-1} g x_j)e_{j'}$, where j' is such that $gx_j \in x_{j'}\Lambda$. For $g = g_i$ we have $j' = j + 1$, because the image of $g_i g_1^j$ in $\mathbb{Z}_K$ is $j + 1$. Thus $\pi_\chi(g_i) = PD_i$, where P is the cyclic shift $e_j \to e_{j+1}$, and D_i is diagonal, with entries $d_{ij} = \chi(x_{j+1}^{-1} g_i x_j)$. Now observe that we have:

$$(PD_i)^K = \left(\prod_j d_{ij}\right) 1$$

Since $g_i^K = 1$, we conclude that $\prod_j d_{ij} = 1$. But this allows us to find a diagonal unitary E with $E_{j+1}/E_j = d_{ij}$ for any j, the compatibility condition around the cycle being precisely $\prod_j d_{ij} = 1$, and we have then $PD_i = EPE^{-1}$. Thus $\pi_\chi(g_i)$ is conjugate to P, whose eigenvalues are the K-th roots of unity, each with multiplicity 1, and so π_χ is flat.

Regarding now the stationarity, since Λ is normal, the character formula of Frobenius gives, for any $g \in \Gamma$:

$$Tr(\pi_\chi(g)) = \delta_{g\in\Lambda} \sum_{j\in\mathbb{Z}_K} \chi(x_j^{-1} g x_j)$$

By integrating over $\chi \in \widehat{\Lambda}$, and using $\int_{\widehat{\Lambda}} \chi(h) d\chi = \delta_{h,1}$ for $h \in \Lambda$, we obtain $K\delta_{g,1}$. Thus, after normalizing the trace, the push-forward measure is a stationarity measure, as claimed. Finally, the map $\chi \to \pi_\chi$ is continuous, so the push-forward is well-defined. □

In the case $M = 2$, the abelianity condition can be made explicit, as follows:

Proposition 5.3. *For a uniform group $\Gamma = \langle g_1, g_2\rangle$, we have $\Lambda_\Gamma = \Gamma'\langle h\rangle$, with $h = g_1 g_2^{-1}$, and so Λ_Γ is abelian precisely when Γ' is abelian, and is centralized by h.*

Proof. The subgroup $\Gamma'\langle h\rangle$ is contained in Λ_Γ, it is normal, because it contains Γ', and its image in $\mathbb{Z}_K^2$ is the subgroup generated by $(1, -1)$, which is the kernel of the sum map. Thus $\Gamma'\langle h\rangle = \Lambda_\Gamma$, and the second assertion follows. □

We have as well the following general principle, which allows to combine examples:

Proposition 5.4. *Assume that $\Gamma_1 = \langle x_1, \ldots, x_M\rangle$ and $\Gamma_2 = \langle y_1, \ldots, y_M\rangle$ are uniform, with parameters (K_1, M) and (K_2, M), where K_1, K_2 are coprime. Then $\Gamma = \Gamma_1 \times \Gamma_2$, with generators $g_i = (x_i, y_i)$, is uniform, with parameters (K_1K_2, M). Moreover, if Γ_1 and Γ_2 are stationary, so is Γ.*

Proof. The uniformity of Γ follows from the Chinese remainder theorem, $\mathbb{Z}_{K_1} \times \mathbb{Z}_{K_2} = \mathbb{Z}_{K_1K_2}$, with the S_M action being the diagonal one. Given flat representations π_1, π_2 of Γ_1, Γ_2, the tensor product $\pi_1 \otimes \pi_2$ is a representation of Γ on $\mathbb{C}^{K_1} \otimes \mathbb{C}^{K_2}$, and the eigenvalues of $\pi_1(x_i) \otimes \pi_2(y_i)$ are the products $\alpha\beta$, with $\alpha^{K_1} = 1$ and $\beta^{K_2} = 1$, which are the K_1K_2-th roots of unity, each appearing once, again by coprimality. Thus $\pi_1 \otimes \pi_2$ is flat. Now if μ_1, μ_2 are stationarity measures, the push-forward of $\mu_1 \times \mu_2$ under $(\pi_1, \pi_2) \to \pi_1 \otimes \pi_2$ is a stationarity measure for Γ, because $tr(A \otimes B) = tr(A)tr(B)$, and $\delta_{(x,y),1} = \delta_{x,1}\delta_{y,1}$. □

For the Heisenberg groups with K prime, the whole space of flat representations can be described explicitly, as follows:

Proposition 5.5. *For K prime, the Heisenberg group H_K has $K! + K - 1$ flat covers: the $K!$ sets $L_\sigma = \{\chi_{a,\sigma(a)} | a \in \mathbb{Z}_K\}$ with $\sigma \in S_K$, where $\chi_{a,b}$ is the character $g_1 \to w^a$, $g_2 \to w^b$, and the $K - 1$ singletons formed by the irreducibles τ_l of dimension K, with $\tau_l(z) = w^{-l}$. Moreover, a stationarity measure is given by the weights $c_{L_\sigma} = \frac{1}{K \cdot K!}$ and $c_{\{\tau_l\}} = \frac{1}{K}$.*

Proof. For K prime, the irreducibles of H_K are the K^2 characters $\chi_{a,b}$, factorizing through $\mathbb{Z}_K^2$, and the $K-1$ representations τ_l of dimension K, with $l=1,\ldots,K-1$, given by $\tau_l(g_1)=P$ and $\tau_l(g_2)=D^l$, where P is the cyclic shift and $D=diag(w^j)$. Both P and D^l have all the K-th roots of unity as eigenvalues, so each τ_l is a flat cover by itself. On the other hand, a flat cover containing a character must consist of K characters, and the condition on the spectra of g_1,g_2 says precisely that the corresponding set of pairs (a,b) is the graph of a permutation. Regarding the weights, the right-hand sides in Theorem 3.2 (3) are $1/K^2$ for the characters, and $1/K$ for the representations τ_l. Since each $\chi_{a,b}$ belongs to exactly $(K-1)!$ of the sets L_σ, the weights in the statement give $(K-1)!/(K\cdot K!)=1/K^2$, as desired. □

Observe that, for $K>2$, the measure produced by Theorem 5.2 for H_K is different, being supported by the flat covers L_σ with σ a translation of $\mathbb{Z}_K$, along with the singletons $\{\tau_l\}$. In general, there are many stationarity measures, the corresponding weights c_S in Theorem 3.2 (3) forming a convex polytope. For instance, for the abelian group $\mathbb{Z}_K^2$ the flat covers are the graphs of the permutations of $\mathbb{Z}_K$, and the weights c_S correspond to the decompositions of the matrix J/K, which is the barycenter of the Birkhoff polytope, as convex combinations of permutation matrices. Summarizing, we have obtained:

Theorem 5.6. *The following uniform groups are stationary, and so flat linear:*

(1) *The abelian groups $\mathbb{Z}_K^M$.*
(2) *The dihedral groups D_n, with $n\in 2\mathbb{N}\cup\{\infty\}$.*
(3) *The uniform groups with $M=2$ which are nilpotent of class ≤ 2.*
(4) *In particular, the Heisenberg groups H_K, for any $K\geq 2$.*

Proof. All these assertions follow from Theorem 5.2. Indeed, for $\mathbb{Z}_K^M$ the diagonal kernel is abelian, for D_n it is the subgroup of rotations, and in the class ≤ 2 case with $M=2$ the derived subgroup Γ' is central, so Proposition 5.3 applies. Finally, H_K has class ≤ 2, its commutator z being central. □

6. Metabelian groups, obstructions

We discuss now the metabelian groups $\Gamma_{K,q}=(\mathbb{Z}_K*\mathbb{Z}_K)_{met}/qA$, with q prime, $q\nmid K$, which extend the order 144 example $\Gamma_{3,2}$ from [3]. We use the description $\Gamma_{K,q}=A_q\rtimes\mathbb{Z}_K^2$, with $A_q=\mathbb{Z}_q^{(K-1)^2}$, and with g_1,g_2 acting by $m\otimes 1$ and $1\otimes m$.

Lemma 6.1. *Let K be prime, and $q\neq K$ prime. The dual group $\widehat{A}_q$, with the dual action of $\mathbb{Z}_K^2$, has the following properties:*

(1) *The elements $g_1^ag_2^b$ with exactly one of a,b nonzero have no nontrivial fixed points.*
(2) *The elements $g_1^ag_2^b$ with a,b both nonzero have a fixed point subgroup of order q^{K-1}.*

In particular, for $K\geq 3$ there exist characters of A_q having trivial stabilizer in $\mathbb{Z}_K^2$.

Proof. Since $q\neq K$, the polynomial $1+t+\ldots+t^{K-1}$ is separable over $\mathbb{F}_q$, so over the algebraic closure, m is diagonalizable, with eigenvalues the primitive K-th roots

of unity $\zeta, \zeta^2, \dots, \zeta^{K-1}$. Thus $m^a \otimes m^b$ is diagonalizable, with eigenvalues ζ^{ai+bj}, with $1 \leq i, j \leq K-1$. The fixed space is spanned by the eigenvectors with $ai + bj \equiv 0$, and the number of such pairs (i, j) is 0 when exactly one of a, b is zero, and $K-1$ when both are nonzero. Since the dimension of a fixed space does not change under field extensions, and since the dual action is by the inverse transposes, which have the same fixed space dimensions, this gives (1) and (2).

Finally, a character with nontrivial stabilizer is fixed by one of the $K-1$ subgroups $\langle g_1 g_2^c \rangle$ with $c \neq 0$, so the number of such characters is at most $(K-1)q^{K-1}$, which for $K \geq 3$ is smaller than $|\widehat{A}_q| = q^{(K-1)^2}$. □

With this, we can now prove our main negative result:

Theorem 6.2. *For $K \geq 3$ prime and $q \neq K$ prime, the group $\Gamma_{K,q}$ is flat linear, but not stationary. More precisely, $\Gamma_{K,q}$ has irreducibles of dimension K^2.*

Proof. Since $\Gamma_{K,q} = A_q \rtimes H$ with A_q and $H = \mathbb{Z}_K^2$ abelian, its irreducibles are obtained by the method of little groups [24]: for each orbit of H on $\widehat{A}_q$, with a chosen point χ having stabilizer H_χ, and each character ν of H_χ, we obtain an irreducible representation induced from $\chi\nu$, of dimension $[H : H_\chi]$. By Lemma 6.1, there are characters with $H_\chi = \{1\}$, producing irreducibles of dimension $K^2 > K$. Thus $\Gamma_{K,q}$ is not stationary, by Corollary 3.3.

Regarding flat linearity, this is [3, Theorem 5.6], which states that for K prime, a metabelian uniform group with $M = 2$ whose derived subgroup has trivial K-primary component is flat linear. Here the derived subgroup is A_q, which is a q-group, so this result applies. □

Observe that this is in agreement with Proposition 5.3, because $h = g_1 g_2^{-1}$ acts on A_q by $m \otimes m^{-1}$, which is nontrivial for $K \geq 3$, so the diagonal kernel of $\Gamma_{K,q}$ is not abelian. Also, for $K = 2$ the situation is different, because $(K-1)^2 = K - 1 = 1$, and the element $g_1 g_2$ acts trivially on $A_q = \mathbb{Z}_q$, so all the orbits have size ≤ 2. In fact, we have here $\Gamma_{2,q} = D_{2q}$, which is stationary. For $K = 3$ the irreducible dimensions can be computed explicitly, from Lemma 6.1, and we obtain the following formula, which is in agreement with the computations in section 4, at $q = 2, 5, 7, 11$:

$$\mathrm{Irr}(\Gamma_{3,q}) \ : \ 1^9 \ 3^{2(q^2-1)} \ 9^{(q^2-1)^2/9}$$

Indeed, the trivial character of A_q produces 9 characters, the $2(q^2-1)$ nontrivial characters fixed by $g_1 g_2$ or $g_1 g_2^2$ form $2(q^2-1)/3$ orbits of size 3, each producing 3 irreducibles of dimension 3, and the remaining $q^4 - 2q^2 + 1 = (q^2-1)^2$ characters form orbits of size 9.

In the case $q|K$, the result from [3] used above does not apply, because the derived subgroup A/qA is then a q-group with $q|K$, and the computations in section 4 show that flat linearity can indeed fail. For $\Gamma_{3,3}$, of order 729, there are 20 flat covers, and the 23 admissible irreducibles, out of 41, have a common kernel of order 3. For $\Gamma_{4,2}$, of order 8192, the 52 flat covers produce a common kernel of order 8. Observe that for K prime the infinite group $(\mathbb{Z}_K * \mathbb{Z}_K)_{met}$ is flat linear, by [3, Theorem 5.6], its derived subgroup $\mathbb{Z}^{(K-1)^2}$ being torsion-free. Thus $\Gamma_{3,3}$ is a quotient of a flat linear group which is not

flat linear, as is the case for the quotients $\Gamma_N^{[s]}$ of $\mathbb{Z}_2^{*N}$ in section 7 below. Summarizing, we have:

Theorem 6.3. *For the metabelian groups $\Gamma_{K,q} = (\mathbb{Z}_K * \mathbb{Z}_K)_{met}/qA$ with $K \geq 3$, we have:*

(1) *For K prime and $q \neq K$ prime, $\Gamma_{K,q}$ is flat linear, but not stationary.*
(2) *For $(K,q) = (3,3)$ and $(K,q) = (4,2)$, $\Gamma_{K,q}$ is neither flat linear, nor stationary.*

Proof. (1) is Theorem 6.2, and (2) follows from the computations in section 4, the failure of stationarity being clear from the irreducibles of dimension $9 > 3$, respectively $8, 16 > 4$, via Corollary 3.3. □

7. Reflection groups, geometry

In this section we discuss in more detail the case $K = 2$, where the generators are involutions, Γ is a reflection group, and the flat representations have a simple geometric interpretation, in terms of configurations of points on the sphere. The reflection groups are central objects in the work of Raum and Weber on the easy quantum groups [22, 23]. They showed in [22] that the group-theoretical easy quantum groups $H_N \subset G \subset H_N^{[\infty]}$ are precisely the semi-direct products $G = \widehat{\Gamma} \rtimes S_N$ coming from the strongly symmetric reflection groups Γ, and this was a key ingredient of their classification of the orthogonal easy quantum groups [23]. The flat representations below concern the diagonal group duals $\widehat{\Gamma}$ of these quantum groups, and the results might be of interest in connection with this work.

Let $\sigma_1, \sigma_2, \sigma_3$ be the Pauli matrices, namely:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For $p \in S^2$ we set $\sigma_p = p_1\sigma_1 + p_2\sigma_2 + p_3\sigma_3$, which is a self-adjoint unitary with eigenvalues ± 1. Also, following [22], let us call a reflection group $\mathbb{Z}_2^{*N} \to \Gamma$ strongly symmetric when its kernel is invariant under any identification of letters, that is, under the endomorphisms of $\mathbb{Z}_2^{*N}$ given by $g_i \to g_{\varphi(i)}$, with $\varphi : \{1, \ldots, N\} \to \{1, \ldots, N\}$ arbitrary. We have:

Proposition 7.1. *Let $\Gamma = \mathbb{Z}_2^{*N}/R$ be a uniform reflection group, with $K = 2$, $M = N$.*

(1) *The flat representations of Γ are, up to conjugation, given by $g_i \to \sigma_{p_i}$, with $(p_1, \ldots, p_N) \in (S^2)^N$ being such that the relations in R are satisfied.*
(2) *The matrix $\sigma_p\sigma_q$ has eigenvalues $e^{\pm i\theta}$, where $\theta \in [0, \pi]$ is the angle between p, q, so that $(\sigma_p\sigma_q)^s = 1$ precisely when $\theta \in \frac{2\pi}{s}\mathbb{Z}$.*
(3) *If Γ is strongly symmetric, the set $X_\Gamma \subset (S^2)^N$ of point configurations as in (1) is stable under all the maps $(p_i) \to (p_{\varphi(i)})$, with φ arbitrary.*

Proof. (1) A flat representation maps each g_i to a self-adjoint unitary $V_i \in U_2$ with eigenvalues ± 1, which is of the form $V_i = \sigma_{p_i}$, with $p_i \in S^2$ being the Bloch vector of its $+1$ eigenspace. (2) This follows from $\sigma_p\sigma_q = \langle p, q\rangle 1 + i\sigma_{p\times q}$, which has trace $2\cos\theta$ and determinant 1. (3) If $w \in \ker(\mathbb{Z}_2^{*N} \to \Gamma)$, then $\varphi(w)$ is in the kernel too, so the representation $g_i \to \sigma_{p_{\varphi(i)}}$ kills w. □

Thus, the identifications of letters correspond to collisions of points on the sphere. As a basic family of examples, consider the following groups, with $s \in 2\mathbb{N} \cup \{\infty\}$:

$$\Gamma_N^{[s]} = \mathbb{Z}_2^{*N} \Big/ \langle (g_i g_j)^s = 1 \rangle$$

These groups are uniform and strongly symmetric, and are those behind the hyperoctahedral series of easy quantum groups, whose operator algebraic properties were studied in [22]. We have $\Gamma_N^{[2]} = \mathbb{Z}_2^N$, $\Gamma_N^{[\infty]} = \mathbb{Z}_2^{*N}$, and $\Gamma_2^{[s]} = D_s$. By Proposition 7.1, their flat representations correspond to the configurations of N points on the sphere whose pairwise angles are multiples of $2\pi/s$. We have the following result, which is computer-assisted, via the program `bloch.py`:

Proposition 7.2. *For $N = 3$ and $s = 4, 6$, every flat representation of $\Gamma_3^{[s]}$ has finite image, of order at most 16, respectively 48. For $s = 8, 10, 12$ there are flat representations of $\Gamma_3^{[s]}$ with infinite image.*

Proof. At $s = 4$ this follows from the proof of Theorem 7.3 below, the image being contained in the Pauli group, of order 16. At $s = 6$, distinct lines $\mathbb{R}p_i$ satisfy $|\langle p_i, p_j \rangle| = 1/2$. For 3 distinct lines, after orienting 2 of the inner products positively, comparing Gram matrices shows that, up to a rotation, the lines are those through the normalized roots $e_1 - e_j$ with $j = 2, 3, 4$, inside the sum-zero hyperplane $V \subset \mathbb{R}^4$, if the product of the 3 inner products is positive, and those through $e_1 - e_2, e_1 - e_3, e_2 - e_3$ otherwise, and the configurations having at most 2 distinct lines fit into these models too. Now the conjugation by σ_p acts on the Bloch sphere by the half-turn $-r_p$, where r_p is the reflection perpendicular to p, and for these roots r_p is a transposition of coordinates. Thus the image of our group in $PU(2) = SO(3)$ is contained in $\{sgn(t)t | t \in S_4\}$, acting on V, which has 24 elements. Moreover, a word whose image is scalar must satisfy $sgn(t)t = 1$, and so $t = 1$, because the action of S_4 on V does not contain -1, so this word has even length, determinant 1, and is ± 1. Thus the image has at most $2 \cdot 24 = 48$ elements.

At $s = 8, 10, 12$, consider 3 unit vectors with common pairwise angle θ, equal respectively to $\pi/4$, $2\pi/5$, $\pi/6$, which is a multiple of $2\pi/s$, and with inner products $c = \cos\theta$. These vectors exist, because their Gram matrix G has eigenvalues $1-c, 1-c, 1+2c > 0$, and define a flat representation, by Proposition 7.1. With $B = \sigma_{p_1}\sigma_{p_2}\sigma_{p_3}$, the Pauli multiplication rule gives $Tr(B) = 2i \det(p_1, p_2, p_3)$, and so:

$$T = |Tr(B)|^2 = 4 \det G = 4(1 - 3c^2 + 2c^3)$$

If B had finite order, its eigenvalues would be roots of unity, with product $\det B = -1$, so we would have $T = 2 + u + u^{-1}$ for a root of unity u, and all the algebraic conjugates of T would belong to $[0, 4]$. But we have $T = -2 + 2\sqrt{2}$, respectively $T = (-5 + 5\sqrt{5})/2$, respectively $T = -5 + 3\sqrt{3}$, whose algebraic conjugates $-2 - 2\sqrt{2}$, $(-5 - 5\sqrt{5})/2$, $-5 - 3\sqrt{3}$ are negative. Thus B has infinite order. All this is confirmed numerically by the program `bloch.py`, for all the realizable angle patterns. □

At $s = 4$ the situation can be fully elucidated, as follows:

Theorem 7.3. *The quotient of $\Gamma_N^{[4]}$ by the common kernel of its flat representations is the free nilpotent reflection group of class 2, namely*

$$\Phi_N = \mathbb{Z}_2^{*N} \big/ \gamma_3(\mathbb{Z}_2^{*N})$$

where γ_3 denotes the third term of the lower central series, which has order $2^{N(N+1)/2}$. In particular, $\Gamma_N^{[4]}$ is not flat linear for $N \geq 3$.

Proof. In a flat representation of $\Gamma_N^{[4]}$, the points p_i are pairwise equal, opposite or orthogonal, by Proposition 7.1. Thus they belong to $\{\pm e_1, \pm e_2, \pm e_3\}$ for some orthonormal frame, and up to conjugation the representation takes values in the Pauli group, which has order 16:

$$P = \{\pm 1, \pm i\} \cdot \{1, \sigma_1, \sigma_2, \sigma_3\}$$

Since P is nilpotent of class 2, any flat representation factorizes through Φ_N. On the other hand, Φ_N is itself a quotient of $\Gamma_N^{[4]}$, because in Φ_N we have, for any 2 generators a, b, the following formulae:

$$(ab)^2 = [a,b] \quad , \quad [a,b]^2 = 1 \quad , \quad [a,b] \in Z(\Phi_N)$$

Now recall that any element of Φ_N can be written as follows, with $\varepsilon_i, c_{ij} \in \{0,1\}$:

$$x = g_1^{\varepsilon_1} \ldots g_N^{\varepsilon_N} \prod_{i<j} [g_i, g_j]^{c_{ij}}$$

Thus $|\Phi_N| \leq 2^{N+N(N-1)/2} = 2^{N(N+1)/2}$, and we will show that the Pauli representations separate these elements, which proves both that we have equality, and the first assertion. There are 2 cases, as follows.

Case 1: $\varepsilon \neq 0$. We use a representation $g_i \to s_i \sigma_3$, with signs $s_i = \pm 1$, under which:

$$x \to \Big(\prod_i s_i^{\varepsilon_i} \Big) \sigma_3^{|\varepsilon|}$$

Thus, a suitable choice of the signs s_i makes the image of x different from 1.

Case 2: $\varepsilon = 0$, $(c_{ij}) \neq 0$. Consider the graph E on $\{1, \ldots, N\}$ having an edge ij when $c_{ij} = 1$, and a representation $g_i \to \sigma_{f(i)}$, with $f : \{1, \ldots, N\} \to \{1,2,3\}$. Since $[\sigma_a, \sigma_b] = -1$ for $a \neq b$, we obtain in this way:

$$x \to (-1)^{n_f} \quad , \quad n_f = \#\Big\{ ij \in E \Big| f(i) \neq f(j) \Big\}$$

If E has a vertex v of odd degree, we take $f(v) = 1$ and $f = 3$ elsewhere, and we get $n_f = \deg(v)$, which is odd. Otherwise, we pick an edge $ij \in E$, and take $f(i) = 1$, $f(j) = 2$, and $f = 3$ elsewhere, and we get $n_f = \deg(i) + \deg(j) - 1$, which is odd.

Finally, for $N \geq 3$ the group $\Gamma_N^{[4]}$ maps onto the Coxeter group of the hyperbolic triangle $(4,4,4)$, which is infinite, so it is not isomorphic to the finite group Φ_N. □

The orders $8, 64, 1024$ of Φ_2, Φ_3, Φ_4 were also checked directly with the program `flatpart.py`, which computes the image of the direct sum of all the Pauli representations. We can now formulate our main result regarding the groups $\Gamma_N^{[s]}$, which covers as well the case $s \geq 8$, where some flat representations have infinite image:

Theorem 7.4. *The reflection groups $\Gamma_N^{[s]}$ behave as follows:*

(1) *The group $\Gamma_N^{[2]} = \mathbb{Z}_2^N$ is stationary.*
(2) *The group $\Gamma_N^{[\infty]} = \mathbb{Z}_2^{*N}$ is flat linear, and for $N \geq 3$ it is not stationary.*
(3) *For $s \geq 4$ finite and even, and $N \geq 3$, the group $\Gamma_N^{[s]}$ is not flat linear, the nontrivial element $w = [(g_1g_2)^{s/2}, g_3]$ being trivial in all its flat representations.*
(4) *For $s = 4, 6$ and $N \geq 3$, the flat representations of $\Gamma_N^{[s]}$ all factor through a finite quotient, while $\Gamma_N^{[s]}$ is infinite.*
(5) *For $N = 2$ and any even s, the group $\Gamma_2^{[s]} = D_s$ is stationary.*

Proof. (1) and (5) follow from Theorem 5.6, and (2) follows from Theorem 2.4, along with the fact that $\mathbb{Z}_2^{*N}$ is not amenable for $N \geq 3$.

(3) In a flat representation $g_i \to \sigma_{p_i}$, the matrix $U = \sigma_{p_1}\sigma_{p_2}$ has determinant 1, and satisfies $U^s = 1$, so its eigenvalues are $e^{\pm i\theta}$ with $s\theta \in 2\pi\mathbb{Z}$, and so $U^{s/2} = \pm 1$. Thus $(g_1g_2)^{s/2}$ is mapped to a scalar, and w is mapped to 1. On the other hand, consider a hyperbolic triangle with angles π/s, map g_1, g_2, g_3 to the reflections in its sides, and map the other generators g_i to the image of g_1, which gives a morphism, because s is even. The image of $(g_1g_2)^{s/2}$ is then the half-turn around the vertex v where the first 2 sides meet, and since this half-turn is an involution, the image of w is the product of the half-turns around v and around its reflection $v' \neq v$ by the third side, which is a nontrivial translation. Thus $w \neq 1$, as claimed.

(4) For $s = 4$ this follows from Theorem 7.3. For $s = 6$, consider a flat representation $g_i \to \sigma_{p_i}$. The lines $\mathbb{R}p_i$, when distinct, are pairwise at angle $\pi/3$, and there are at most 3 distinct such lines in $\mathbb{R}^3$, because the Gram matrix of 4 such lines would be of the form $1+S/2$, with S being a 4×4 symmetric matrix with zero diagonal and off-diagonal entries ± 1, whose smallest eigenvalue is always -3, $-\sqrt{5}$ or -1, and never -2, so this Gram matrix cannot be positive of rank ≤ 3. Thus the image of the representation is contained in the group generated by -1 and by 3 matrices σ_{p_i}, which is finite by Proposition 7.2. Since there are finitely many such configurations up to rotation, the kernels of the flat representations have a finite index intersection. Finally, $\Gamma_N^{[s]}$ is infinite by (3), or by mapping it onto the Coxeter group of the hyperbolic triangle (s, s, s). □

For $s \geq 8$ the flat representations can have infinite image, by Proposition 7.2, and it would be interesting to compute the flat quotients of the groups $\Gamma_N^{[s]}$, namely their quotients by the common kernel of the flat representations, which are proper quotients by Theorem 7.4 (3), and to know for which values of s these quotients are finite, as is the case for $s = 4, 6$, by Theorem 7.4 (4). More generally, it would be interesting to understand the flat representations of the strongly symmetric reflection groups. By the Raum-Weber correspondence [22], this would provide information on the quasi-flat models of all the group-theoretical easy quantum groups, at the level of their diagonal group duals. Observe that the strongly symmetric reflection groups form an uncountable class, as shown in [22].

8. The SmallGroups census

For a finite uniform group Γ we have the following implications, coming from Theorem 5.2 and Corollary 3.3, where "admissible" means that all the irreducible representations of Γ are admissible, in the sense of Definition 3.1:

$$\Lambda_\Gamma \text{ abelian} \implies \Gamma \text{ stationary} \implies \Gamma \text{ admissible}$$

The natural questions are whether these implications can be reversed, and whether all the finite uniform groups are flat linear. In order to investigate these questions, we have used the SmallGroups library [7] of the GAP system, version 4.16.1, as follows:

(1) The GAP programs produced from our template `template.g`, namely `manul.g`, `leopard.g`, `ali.g`, `cheetah.g`, `frazier.g`, `liston.g` and `canelo.g`, run, for given parameters K, M, over all the groups G of order n in a given range, with $K^M | n$, except for the orders having more than 2500 groups, such as $n = 256, 576, 1152, 1728$, and keep the groups with $G^{ab} = \mathbb{Z}_K^M$.

(2) For each of these groups, the programs compute the epimorphisms $\mathbb{Z}_K^{*M} \to G$ up to automorphisms, via the GQuotients algorithm [16], and for each such epimorphism they record whether the S_M symmetry holds, whether Λ_Γ is abelian, and the character degrees, spectral data, kernels and class sizes, all computed exactly, with cyclotomic arithmetic.

(3) The cases $K = 8, 9$ were done by `liston.g`, and the case $K = 4$ with $1000 < n \leq 2000$ by `canelo.g`, the 2 programs running in parallel, on 2 different computers.

(4) The Python program `census.py` reads this data, and computes the flat covers, the flat linearity and the stationarity, exactly as in section 4. For the data from `liston.g` and `canelo.g`, we have used its variant `census_sym.py`, which only analyzes the symmetric epimorphisms. Also, the program `explore.py` searches for uniform groups among random permutation groups, and was used in order to find the first examples below.

Among these programs, `frazier.g` was particularly useful, punching like a heavyweight champion, with a few decisive left hooks: in a short run, it covered the small orders at $K = 6$ and the case $M = 4$, producing the group 2_-^{1+4} and the group of order 648 discussed below, which disproves the natural conjecture that stationarity is decided by dimension counting.

K	M	epimorphisms	uniform	Λ_Γ abelian	admissible	stationary	flat linear
2	3	3939	20	8	9	9	14
2	4	4187	4	2	2	2	2
3	2	558	260	131	150	148	239
3	3	7224	13	6	6	6	7
4	2	1458	348	291	311	308	317
5	2	128	42	29	42	32	42
6	2	813	170	120	150	141	160
7	2	50	10	10	10	10	10
8	2	84	24	24	24	24	24
9	2	97	27	22	27	25	27

The ranges of orders covered are as follows: $n \leq 2000$ for $K = 3, 4, 5, 7, 8, 9$ with $M = 2$, $n \leq 1368$ for $K = 6$ with $M = 2$, $n \leq 248$ for $K = 2$ with $M = 3$, $n \leq 729$ for $K = 3$ with $M = 3$, and $n \leq 128$ for $K = 2$ with $M = 4$, the last order being only partially covered for $K = 3$ with $M = 3$, for $K = 2$ with $M = 4$, and for $K = 6$ with $M = 2$. In total, these programs find 18538 epimorphisms, 918 of which are symmetric, and so correspond to uniform groups, and their properties are summarized in the above table, where the columns after the third one count uniform groups only. Observe that at $M = 3, 4$ the symmetric epimorphisms are quite rare, the S_M symmetry being a strong condition. Also, at $K = 9$ there are 4 admissible uniform structures having more than 400000 flat covers, whose stationarity was decided by hand, as explained at the end of this section. As the table shows, the answer to all the above questions is negative. In order to explain this, let us start with the following simple sufficient condition for stationarity:

Proposition 8.1. *Assume that all the irreducible representations of a finite uniform group Γ have dimension 1 or K, and that those of dimension K are admissible. Then Γ is stationary.*

Proof. The characters of Γ factorize through $\Gamma^{ab} = \mathbb{Z}_K^M$, so they are the characters χ_a with $a \in \mathbb{Z}_K^M$, and as in the proof of Theorem 3.4, the K^{M-1} translates of the diagonal are flat covers by characters, partitioning the set of characters. On the other hand, each admissible irreducible τ of dimension K forms a flat cover by itself. With weights $K/|\Gamma|$ on the translates of the diagonal, and $K^2/|\Gamma|$ on the singletons $\{\tau\}$, the equations in Theorem 3.2 (3) are satisfied. □

The census contains 62 uniform groups which are stationary without having an abelian diagonal kernel. Those with K prime, namely $K = 2, 3, 5$, have character degrees 1 and K only, so they are explained by this result, while for K composite other examples appear, such as the group $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes ((\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4) = \text{SmallGroup}(288, 429)$ at $K = 4$, with irreducibles $1^{16}\, 2^4\, 4^{16}$. The simplest example is $\Gamma = A_4 \times \mathbb{Z}_3 = \text{SmallGroup}(36, 11)$, with $K = 3$, which has irreducibles $1^9\, 3^3$, and 2 uniform structures, both stationary. Here $\Lambda_\Gamma = V_4\langle h \rangle$, with $h = g_1 g_2^{-1}$, by Proposition 5.3, and the diagonal kernel is abelian for one of the 2 structures only.

In the other direction, we have the following obstruction, which only uses the dimensions of the irreducibles. Given a flat cover S, its type is the multiset of the dimensions of its elements, which is a partition of K, and we denote by n_d the number of irreducibles of dimension d:

Proposition 8.2. *If Γ is stationary, there are weights $c_T \geq 0$, indexed by the types T of flat covers, such that for any d we have*

$$\sum_T c_T\, m_d(T) = \frac{K d\, n_d}{|\Gamma|}$$

where $m_d(T)$ is the multiplicity of d in T. In particular, at $K = 3$ we must have $n_1 \geq 2n_2$.

Proof. The first assertion follows from Theorem 3.2 (3), by summing the equations over the irreducibles of each given dimension, and grouping the flat covers by type. At $K = 3$

the possible types are $(1,1,1)$, $(2,1)$, (3), and with a, b being the total weights of the first 2 types, the equations for $d = 2$ and $d = 1$ read $b = 6n_2/|\Gamma|$ and $3a + b = 3n_1/|\Gamma|$, so $a \geq 0$ gives $n_1 \geq 2n_2$. □

Now consider the group $\Gamma = \mathbb{Z}_3 \times SL_2(\mathbb{F}_3) = \text{SmallGroup}(72, 25)$, which has irreducibles $1^9\, 2^9\, 3^3$, and 2 uniform structures. The census shows that all the irreducibles are admissible, the point being that the elements of order 3 of $SL_2(\mathbb{F}_3)$ have distinct eigenvalues in the 2-dimensional representations. However, since $n_1 = 9 < 18 = 2n_2$, Proposition 8.2 shows that these structures are not stationary. The same phenomenon appears in a very natural way at $K = 5$, as follows:

Proposition 8.3. *Consider the group $\Gamma = A_5 \times \mathbb{Z}_5^2$, with generators $g_1 = (a, e_1)$ and $g_2 = (b, e_2)$, where $a = (12345)$, $b = (21345)$, and e_1, e_2 are the standard generators of $\mathbb{Z}_5^2$. Then Γ is uniform, with $K = 5$, $M = 2$, all its irreducibles are admissible, but Γ is not stationary.*

Proof. The 5-cycles a, b are not powers of each other, so they generate a subgroup of A_5 having at least 2 Sylow 5-subgroups, which must be A_5 itself. Since A_5 is perfect, $\Gamma^{ab} = \mathbb{Z}_5^2$, with g_1, g_2 mapping to a basis, and the subgroup generated by g_1, g_2 contains its derived subgroup, which is $A_5 \times 1$, so it is Γ. The conjugation by the transposition (12), which exchanges a, b, combined with the flip of e_1, e_2, gives the S_2 symmetry. The irreducibles of Γ are the products $\tau \otimes \chi$, with $\tau \in \text{Irr}(A_5)$, of dimensions $1, 3, 3, 4, 5$, and $\chi \in \widehat{\mathbb{Z}}_5^2$, so that $\text{Irr}(\Gamma) = 1^{25}\, 3^{50}\, 4^{25}\, 5^{25}$, and since the 5-cycles have simple spectrum in all the irreducibles of A_5, all these are admissible. Finally, a flat cover containing an irreducible of dimension 3 must be of type $(3,1,1)$. Thus, by summing the equations in Theorem 3.2 (3), the stationarity would require the characters to carry a total weight $\geq 2 \cdot 50 \cdot 15/1500 = 1$, while their total weight is $25 \cdot 5/1500 = 1/12$. □

The other admissible but non-stationary groups in the census are as follows, all failing because of the dimension obstruction in Proposition 8.2:

(1) At $K = 4$, the groups $\mathbb{Z}_4 \times (A_4 \rtimes \mathbb{Z}_4) = \text{SmallGroup}(192, 969)$ and $\mathbb{Z}_4 \times (H_3 \rtimes \mathbb{Z}_4) = \text{SmallGroup}(432, 275)$, with H_3 being the Heisenberg group of order 27.
(2) At $K = 5$, the group $\mathbb{Z}_5 \times (((\mathbb{Z}_2 \times Q_8) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_5) = \text{SmallGroup}(800, 1065)$, with Q_8 being the quaternion group.
(3) At $K = 6$, the groups $\mathbb{Z}_2 \times \mathbb{Z}_6 \times SL_2(\mathbb{F}_3) = \text{SmallGroup}(288, 981)$ and $\mathbb{Z}_3 \times (((\mathbb{Z}_2 \times Q_8) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_3) = \text{SmallGroup}(288, 985)$.
(4) Also at $K = 6$, the groups $\mathbb{Z}_3^2 \times S_3 \times S_3 = \text{SmallGroup}(324, 165)$ and $\mathbb{Z}_3^2 \times (H_3 \rtimes \mathbb{Z}_2^2) = \text{SmallGroup}(972, 781)$.

There is one more admissible but non-stationary group in the census, which is of a different nature, and which will be discussed at the end of this section.

Regarding flat linearity, the census finds 76 uniform groups which are not flat linear. At $M = 2$ the smallest one is $(\mathbb{Z}_2 \times ((\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2)) \rtimes \mathbb{Z}_4 = \text{SmallGroup}(128, 36)$, with $K = 4$, irreducibles $1^{16}\, 2^4\, 4^6$, and flat kernel of order 2, and with $K = 3$ the smallest one is $(\mathbb{Z}_9 \rtimes \mathbb{Z}_9) \rtimes \mathbb{Z}_3 = \text{SmallGroup}(243, 28)$, with irreducibles $1^9\, 3^8\, 9^2$, and flat kernel of order 3. At $M = 4$, the smallest one is the extraspecial group $2_-^{1+4} = \text{SmallGroup}(32, 50)$, with irreducibles $1^{16}\, 4^1$. Here the reason is simple: the irreducible

of dimension $4 > 2$ cannot appear in a flat representation, so the flat representations factorize through Γ^{ab}, and the flat kernel is the derived subgroup, of order 2. Finally, at $M = 3$ the smallest one is $\mathbb{Z}_2^2 \times S_4 = \text{SmallGroup}(96, 226)$, with irreducibles $1^8\, 2^4\, 3^8$, and flat kernel of order 4. In all these examples, the failure of flat linearity can be read on the character table alone, via Theorem 3.2 (2): the admissible irreducibles have a nontrivial common kernel. The last group above can be described explicitly, as follows:

Proposition 8.4. *Consider the group $\Gamma = S_4 \times \mathbb{Z}_2^2$, with generators $g_i = (t_i, f_i)$, where $t_1 = (12)$, $t_2 = (13)$, $t_3 = (14)$, and f_1, f_2, f_3 are the nonzero elements of $\mathbb{Z}_2^2$. Then Γ is uniform, with $K = 2$, $M = 3$, but it is not flat linear, the subgroup $V_4 \times 1$ being contained in the kernel of any flat representation.*

Proof. The vectors $(1, f_i) \in \mathbb{Z}_2 \times \mathbb{Z}_2^2 = \Gamma^{ab}$ form a basis, so $\Gamma^{ab} = \mathbb{Z}_2^3$ with the g_i mapping to a basis. The subgroup H generated by the g_i projects onto S_4, so its derived subgroup is $A_4 \times 1$, and since H^{ab} maps onto $\mathbb{Z}_2^3$, we get $|H| \geq 12 \cdot 8 = 96$, so $H = \Gamma$. The permutations of $\{2, 3, 4\}$ act on S_4 by conjugation, permuting the t_i, and $GL_2(\mathbb{F}_2) \simeq S_3$ acts on $\mathbb{Z}_2^2$ by permuting the f_i, so we have the S_3 symmetry. Finally, a flat representation has dimension 2, so its irreducible components have dimension ≤ 2, and those of $S_4 \times \mathbb{Z}_2^2$ are of the form $\sigma \otimes \chi$ with σ an irreducible of S_4 of dimension ≤ 2, which factorizes through $S_4/V_4 = S_3$. □

Despite these negative results, the census shows that, in most cases, stationarity is decided by dimension counting alone, once admissibility holds. Indeed, out of the 731 admissible uniform structures in the census, 730 are stationary precisely when the linear system in Proposition 8.2 has a solution $c_T \geq 0$, with T ranging over the types of flat covers. The remaining one is as follows:

Proposition 8.5. *The group $\Gamma = \mathbb{Z}_3^2 \times ((\mathbb{Z}_6 \times S_3) \rtimes \mathbb{Z}_2) = \text{SmallGroup}(648, 648)$ has a uniform structure with $K = 6$, $M = 2$, whose irreducibles $1^{36}\, 2^{81}\, 4^{18}$ are all admissible, for which the linear system in Proposition 8.2 has a solution, but which is not stationary.*

Proof. The census data shows that the irreducibles are all admissible, and that there are 4224 flat covers, of types (1^6), $(1^4, 2)$, $(1^2, 2^2)$, $(1^2, 4)$, (2^3), $(2, 4)$. The right-hand sides in Proposition 8.2 are $1/3$, $3/2$, $2/3$ for $d = 1, 2, 4$, and a solution is given by the weights $1/18$, $5/18$, $2/3$ on the types (1^6), (2^3), $(2, 4)$. Assume now that Γ is stationary, with weights c_S as in Theorem 3.2 (3), and let a, b be the total weights of the flat covers of type $(1^2, 4)$ and $(2, 4)$. Since each of the 18 irreducibles of dimension 4 has weight $6 \cdot 4/648 = 1/27$, and appears only in covers of these 2 types, once, we have $a + b = 18/27 = 2/3$. Since each cover of type $(1^2, 4)$ contains 2 characters, and the 36 characters have total weight $36 \cdot 6/648 = 1/3$, we have $2a \leq 1/3$, and so $b \geq 1/2$. On the other hand, the census data shows that only 9 of the 81 irreducibles of dimension 2 appear in covers of type $(2, 4)$, and since these have total weight $9 \cdot 12/648 = 1/6$, we obtain $b \leq 1/6$, a contradiction. This argument is confirmed by the exact Farkas certificate produced by `census.py`. □

Thus, stationarity depends not only on the dimensions of the irreducibles, but also on which irreducibles can appear together in a flat representation.

Finally, at $K = 9$ there are 4 admissible uniform structures having more than 400000 flat covers, which are too many for our enumeration, but which can be decided by hand, by using the symmetries of their spectral data, as follows:

Proposition 8.6. *At $K = 9$, $M = 2$, the 2 uniform structures on the group $\mathbb{Z}_9 \times (Q_8 \rtimes \mathbb{Z}_9) = \text{SmallGroup}(648, 90)$, with irreducibles $1^{81}\, 2^{81}\, 3^{27}$, are not stationary, while those on the groups $((\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_9 = \text{SmallGroup}(729, 26)$ and $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes ((\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_9) = \text{SmallGroup}(972, 173)$, with irreducibles $1^{81}\, 3^{72}$ and $1^{81}\, 3^{99}$, are stationary.*

Proof. The census data shows, as checked by the program `k9.py`, that in all cases the irreducibles are admissible, that the spectrum of each irreducible of dimension 3 is, on each generator, a residue class modulo 3, inside $\{1, \ldots, 9\}$, and that the spectrum of each irreducible of dimension 2, on the first generator, consists of 2 elements of a residue class modulo 3.

In the first case, a residue class modulo 3 can contain the spectrum of at most one irreducible of dimension 2, and its remaining element must then be covered by a character. Thus any flat cover contains at least as many characters as irreducibles of dimension 2. On the other hand, by summing the equations in Theorem 3.2 (3), stationarity would require the expected numbers of such irreducibles in a flat cover, with respect to the weights c_S, to be $9 \cdot 2 \cdot 81/648 = 9/4$ for the irreducibles of dimension 2, and $9 \cdot 81/648 = 9/8$ for the characters, a contradiction. Observe that this argument only uses the types of the flat covers, so the dimension counting test fails as well.

In the second case, the census data shows that the 9 pairs of residue classes, for the 2 generators, appear equally often among the irreducibles of dimension 3, namely $n_3/9$ times each, where $n_3 = 72, 99$ is their number. Consider the random flat cover obtained by choosing a uniform random bijection between the residue classes for the first generator and those for the second generator, and then, in each of the 3 resulting cells, an irreducible of dimension 3, uniformly. Each irreducible of dimension 3 is chosen with probability $(1/3) \cdot (9/n_3) = 3/n_3$. Also, by choosing uniformly one of the 9 translates of the diagonal, as in Theorem 3.4, each character is chosen with probability $1/9$. With total weights $81/|\Gamma|$ and $9n_3/|\Gamma|$ on these 2 constructions, which sum up to 1, because $|\Gamma| = 81 + 9n_3$, each character gets weight $9/|\Gamma|$, and each irreducible of dimension 3 gets weight $27/|\Gamma|$, as required by Theorem 3.2 (3). □

We can now summarize the main findings of the census, as follows:

Theorem 8.7. *For finite uniform groups, the following happen:*

(1) *Stationarity does not imply that Λ_Γ is abelian, as shown by $A_4 \times \mathbb{Z}_3$.*
(2) *Admissibility does not imply stationarity, as shown by $\mathbb{Z}_3 \times SL_2(\mathbb{F}_3)$ and $A_5 \times \mathbb{Z}_5^2$.*
(3) *Flat linearity can fail, as shown by $S_4 \times \mathbb{Z}_2^2$, with $K = 2$, $M = 3$, and by 2_-^{1+4}, with $K = 2$, $M = 4$.*
(4) *Admissibility and dimension counting do not imply stationarity, as shown by the group $\mathbb{Z}_3^2 \times ((\mathbb{Z}_6 \times S_3) \rtimes \mathbb{Z}_2)$, with $K = 6$.*

Proof. This follows from Propositions 8.1-8.6, and from the above discussion. □

9. Beyond flatness

In this final section we discuss some notions going beyond the flatness considered so far, both at the level of faithfulness, and of flatness itself. Let us start with faithfulness. Given a probability measure μ on the space X_Γ of flat representations, consider the positive definite function on Γ given by $\psi(g) = \int_{X_\Gamma} tr(\pi(g))\, d\mu(\pi)$. This function satisfies $\psi(1) = 1$ and $|\psi(g)| \leq 1$, and it is a convex combination of the normalized characters of the flat representations, with respect to the measure μ. In the quantum group picture, ψ is the random matrix trace of the model, and the inner faithfulness is related to the Cesàro limit of its convolution powers, see [30]. In the group dual case, $\Delta(g) = g \otimes g$, so the convolution powers are simply the pointwise powers, $\psi^{*r} = \psi^r$, and we obtain:

Proposition 9.1. *With the above notations, the following hold:*

1. *The measure μ is a stationarity measure precisely when $\psi = \delta_1$.*
2. *The flat representations in the support of μ separate the points of Γ precisely when $\psi(g) \neq 1$, for any $g \neq 1$.*
3. *If Γ is not amenable, then $\psi \notin \ell^p(\Gamma)$, for any $p < \infty$.*

Proof. (1) is clear. Regarding (2), since $|\psi(g)| \leq 1$, the Cesàro means of $\psi(g)^r$ converge to 1 if $\psi(g) = 1$, and to 0 otherwise. Moreover, since $Re\, tr(V) \leq 1$ for $V \in U_K$, with equality only at $V = 1$, by continuity $\psi(g) = 1$ happens precisely when $\pi(g) = 1$ for any π in the support of μ. Regarding (3), assume that $\psi \in \ell^p(\Gamma)$, and pick an integer r with $2r \geq p$. Since $|\psi| \leq 1$, we have $\psi^r \in \ell^2(\Gamma)$, and ψ^r is the average of the normalized traces of the representations $\pi_1 \otimes \ldots \otimes \pi_r$, with respect to $\mu^{\otimes r}$, which have size K^r. Thus we can assume $p = 2$, with K replaced by K^r. Now a positive definite function in $\ell^2(\Gamma)$ is a coefficient of the left regular representation λ, so its GNS representation is contained in λ, and so extends to $C^*_{red}(\Gamma)$. On the other hand, the image of this GNS representation is a quotient of the image of the model, which is contained in $M_K(C(X_\Gamma))$, and so is subhomogeneous. Thus $C^*_{red}(\Gamma)$ would have a nonzero finite dimensional representation, which forces Γ to be amenable. See [10]. □

In the spirit of Kesten's amenability criterion [18], statement (3) says that for a non-amenable group, the random matrix trace of a flat model cannot decay too fast, and in fact cannot belong to any of the spaces $\ell^p(\Gamma)$ with $p < \infty$. Thus, the decay of ψ provides a scale of notions between stationarity and inner faithfulness which is of a different nature, such as $\psi \in c_0(\Gamma)$, or $\sup_{g \neq 1} |\psi(g)| < 1$, and it would be interesting to investigate them for the free products $\mathbb{Z}_K^{*M}$, where for the Haar measure on $X_\Gamma = C_W^M$ the values $\psi(\gamma)$ are integrals of products of randomly rotated copies of W.

Let us discuss now flatness itself. Recall from section 2 that flatness means that the landscape of ranks associated to the model is flat, at height 1. The first natural generalization is as follows:

Definition 9.2. *A representation $\pi : \Gamma \to U_{mK}$ is called flat at height m when each $\pi(g_i)$ has the K-th roots of unity as eigenvalues, each with multiplicity m. We call Γ flat linear, or stationary, at height m, when the corresponding properties hold for the flat representations at height m.*

In quantum group terms, a family of representations which are flat at height m produces a matrix model $\pi : C(\widehat{\Gamma}) \to M_{mK}(C(X))$ whose rank matrices are block-flat, with all entries equal to m inside the diagonal $K \times K$ blocks, and the regular representation below corresponds to the canonical model of $\widehat{\Gamma}$ on $\ell^2(\Gamma)$. For a finite group, the flat representations at height m are the sums $\sum_\tau x_\tau \tau$ with $x \in \mathbb{N}^{\text{Irr}(\Gamma)}$ satisfying $\sum_\tau x_\tau e_\tau = m$, the all-$m$ matrix, and the analogue of Theorem 3.2 holds, with the stationarity equations becoming $\sum_x c_x x_\tau = mK \dim(\tau)/|\Gamma|$. These conditions are checked by the program `heights.py`. We have:

Proposition 9.3. *Any finite uniform group Γ is flat linear and stationary at height $|\Gamma|/K$, via its regular representation. Thus, we have minimal heights*

$$h_{lin}(\Gamma) \leq h_{stat}(\Gamma) \leq |\Gamma|/K$$

and these are respectively $2, 3$ for $S_4 \times \mathbb{Z}_2^2$, $1, 2$ for $\mathbb{Z}_3 \times SL_2(\mathbb{F}_3)$, and $1, 3$ for $\Gamma_{3,2}$.

Proof. Since each g_i has order K, it acts freely on Γ by left multiplication, so the restriction of the regular representation to $\langle g_i \rangle$ is a multiple of the regular representation of $\mathbb{Z}_K$, with multiplicity $|\Gamma|/K$. Thus the regular representation is flat at height $|\Gamma|/K$, and it is faithful and stationary, its normalized trace being δ_1. The numeric values come from `heights.py`. □

A second generalization concerns the joint behavior of the generators. For a flat representation π, let $P_a^{(i)}$ be the rank one spectral projections of $\pi(g_i)$. Flatness is a condition on the ranks of the $P_a^{(i)}$, and the next condition concerns the products:

Definition 9.4. *A flat representation $\pi : \Gamma \to U_K$ is called 2-flat when $tr(P_a^{(i)} P_b^{(j)}) = 1/K^2$, for any $i \neq j$ and any a, b.*

Since $tr(P_\xi P_\eta) = |\langle \xi, \eta \rangle|^2/K$ for unit vectors ξ, η, this means that the eigenbases of $\pi(g_i)$ and $\pi(g_j)$ are mutually unbiased, for any $i \neq j$, that is, that the matrices $\sqrt{K}\, U_i^* U_j$ are complex Hadamard matrices, where U_i is a unitary diagonalizing $\pi(g_i)$. This brings the mutually unbiased bases, and the Hadamard matrices, into the picture [14, 31]. Recall that a family of mutually unbiased bases in $\mathbb{C}^K$ has at most $K + 1$ elements, that this bound is attained when K is a prime power, via the construction of Wootters and Fields, and that for $K = 6$ it is widely believed that at most 3 such bases exist, despite many analytic and numerical studies. We have the following result, where the last assertion refers to the well-known open question of the existence of 4 mutually unbiased bases in $\mathbb{C}^6$:

Theorem 9.5. *The following hold:*

(1) *Any finite uniform group Γ is flat linear and stationary at height $|\Gamma|/K$.*
(2) *For K prime, the irreducibles of dimension K of the Heisenberg group H_K are 2-flat.*
(3) *The 2-flat representations of $\mathbb{Z}_K^{*M}$ correspond to the M-tuples of mutually unbiased bases of $\mathbb{C}^K$, formed by eigenvectors labeled by eigenvalues, and taken modulo phases. In particular, for $K \geq 2$, they exist for $M \leq K + 1$ when K is a prime power, they do not exist for $M > K + 1$, and their existence for $K = 6$, $M = 4$ is equivalent to the existence of 4 mutually unbiased bases in $\mathbb{C}^6$.*

Proof. (1) is Proposition 9.3. (2) follows from the fact that $\tau_l(g_1) = P$ is diagonalized by the Fourier matrix, and $\tau_l(g_2) = D^l$ by the identity, and these 2 bases are mutually unbiased. (3) follows from the fact that the flat representations of $\mathbb{Z}_K^{*M}$ are the tuples $(U_1WU_1^*, \ldots, U_MWU_M^*)$, with arbitrary $U_i \in U_K$, so that the 2-flatness condition is precisely the mutual unbiasedness of the columns of $U_1, \ldots, U_M$, together with the known results on mutually unbiased bases [14, 31]. □

Finally, let us mention a few questions in the infinite case. Here the space X_Γ is in general no longer a finite union of orbits, and the basic examples are the infinite dihedral group, covered by Theorem 5.2, and the universal metabelian groups $(\mathbb{Z}_K * \mathbb{Z}_K)_{met}$, which are extensions of $\mathbb{Z}_K^2$ by $\mathbb{Z}^{(K-1)^2}$. For K prime these groups are flat linear, by [3], and the question is open for K composite, and for their quotients whose derived subgroup has K-primary torsion, where the finite quotients $\Gamma_{3,3}$ and $\Gamma_{4,2}$ are not flat linear, by Theorem 6.3. Also, Theorem 2.4 suggests that the arguments in [4] for the amalgamated free products of cyclic groups, and for the free products of copies of $\mathbb{Z}_K^{M_i}$, can be completed in the same way, by complexification.

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