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Colossally Abundant Numbers and Nicolas' Second Criterion

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Abstract

Robin's criterion states that the Riemann Hypothesis holds if and only if $\sigma(n) < e^{\gamma} n \log\log n$ for every $n > 5040$. Nicolas' second criterion, in the form given by Broughan, states that it holds if and only if $n/\varphi(n) < e^{\gamma}\log\log n + e^{\gamma}(2+\tau)/\sqrt{\log n}$ for every $n \geq N_{120569}$, where N_k is the k-th primorial and $\tau = 2 + \gamma - \log(4\pi)$; for N_k this inequality reads $c(p_k) < E(p_k)$. Let N be a colossally abundant number, let $q = p_k$ be the largest prime not exceeding the number $\xi(N)$ associated with N by Nicolas, and write $\Delta(N) = \log(\sigma(N)/N) - \gamma - \log\log\log N$. For $\log q \geq 31$ we prove unconditionally that $1.317/(\sqrt{q}\log^2 q) \leq \Delta(N) - c(q) + E(q) + (2\sqrt{2} - 2 - \tau)/(\sqrt{q}\log q) \leq 3.973/(\sqrt{q}\log^2 q)$. Consequently, the second inequality for N_k implies $\Delta(N) < -0.654/(\sqrt{q}\log q)$. We also prove that the primorials are the extremal cases of the second inequality, and that if it holds for every N_k with $e^{31} \leq p_k \leq X$, then Robin's inequality holds for every $n > 5040$ with $\log n \leq X(1 - \log^{-3} X) - \log X$. A computation over the primes up to $X = 100{,}000{,}008{,}765{,}439$ verifies this hypothesis, so Robin's inequality holds unconditionally for $5040 < n \leq 10^{10^{13.6377}}$. To the author's knowledge, this is the largest range for which Robin's inequality has been verified: it extends the range $5040 < n \leq 10^{10^{13.11485}}$ of Morrill and Platt.

Keywords: Riemann Hypothesis, Robin's inequality, Nicolas' inequality, Nicolas' second criterion, Colossally abundant numbers, Superabundant numbers, Primorial, Chebyshev function
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1. Introduction

The Riemann Hypothesis (RH) is the conjecture that the Riemann zeta function has its zeros only at the negative even integers and at complex numbers with real part $\frac{1}{2}$ [1]. It has been called the "Holy Grail of Mathematics" [2, 3], and numerous equivalent formulations exist [4]. One of particular interest here is Robin's criterion [5], which asserts that RH is equivalent to the inequality

$$\sigma(n) < e^{\gamma} \cdot n \cdot \log\log n \qquad \text{for every integer } n > 5040. \tag{1}$$

Here $\gamma \approx 0.57721$ is the Euler–Mascheroni constant, $\sigma(n) = \sum_{d|n} d$ is the sum-of-divisors function, and log denotes the natural logarithm. Grönwall's theorem [6] states that $\limsup_{n\to\infty} \sigma(n)/(n\log\log n) = e^{\gamma}$, so (1) asserts that this limiting value is never attained beyond 5040. A closely related elementary reformulation in terms of harmonic numbers is due to Lagarias [7], and the extremal integers relevant to both criteria have been studied computationally by Briggs [8] and structurally by Caveney, Nicolas and Sondow [9].

It is known that (1) holds for many classes of integers: odd integers exceeding 9 and squarefree integers exceeding 30 [10], sums of two squares exceeding 720 [11], and t-free integers for $t = 5, 7, 11$ and 20 [10, 12, 13, 14]. Akbary and Friggstad [15] showed that the least counterexample to (1), should one exist, must be a superabundant number, Nazardonyavi and Yakubovich [16] refined this to the smaller class of extremely abundant numbers, and the author proved that its largest prime factor exceeds $e^{31.018}$ [17]. Robin [5] proved that if RH is false, then (1) fails for infinitely many colossally abundant numbers.

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Nicolas' criterion concerns the primorials $q\# = \prod_{p\le q} p$. For a real number $x \ge 2$ put

$$c(x) = \sum_{p\le x} \log\frac{p}{p-1} - \gamma - \log\log\theta(x),$$

where $\theta(x) = \sum_{p\le x}\log p$ is the Chebyshev function, so that $c(q) > 0$ is the inequality $\frac{q\#}{\varphi(q\#)} > e^\gamma\cdot\log\log(q\#)$, with φ Euler's totient function. Nicolas [18] proved that $c(x) > 0$ for every $x \ge 2$ if RH is true, and that if RH is false, then $c(x) = \Omega_\pm(x^{-b})$ for some $0 < b < \frac{1}{2}$ (Proposition 2.7). Under RH, Ramanujan's asymptotic upper bound for $\sigma(n)/n$ [19] contains the term $-\frac{2(\sqrt{2}-1)}{\sqrt{\log n}}$, and Nicolas [20] gave an effective form of this bound, using the structure of colossally abundant numbers and his estimate [21] of $c(x)$ under RH.

Nicolas' second criterion also concerns the primorials. Let $N_k = p_1p_2\cdots p_k$ be the k-th primorial, where $p_1 = 2 < p_2 = 3 < \cdots$ are the primes, and put $\tau = 2 + \gamma - \log(4\pi) = 0.0461914\ldots$. From Nicolas' theorem [18] and a computation, Broughan [22, Theorem 5.34, Corollary 5.35] obtained that RH is equivalent to the inequality

$$\frac{n}{\varphi(n)} < e^\gamma\cdot\log\log n + \frac{e^\gamma\cdot(2+\tau)}{\sqrt{\log n}} \tag{2}$$

for every integer $n \ge N_{120569}$; the constant τ is denoted β in [22]. We call (2) Nicolas' second inequality. In Section 4 we prove that the primorials are the extremal cases of (2): if (2) holds for N_k, then it holds for every integer n with $N_k \le n < N_{k+1}$ (Lemma 4.2), so that RH is equivalent to (2) for the primorials N_k with $k \ge 120569$ (Corollary 4.3). For a prime $q = p_k$, the inequality (2) for N_k is the inequality $c(q) < E(q)$, where

$$E(q) = \log\left(1 + \frac{2+\tau}{\sqrt{\theta(q)}\cdot\log\theta(q)}\right) = \log\left(1 + \frac{2+\tau}{\sqrt{\log N_k}\cdot\log\log N_k}\right)$$

(Lemma 4.4).

In this paper we compare Robin's inequality at a colossally abundant number with Nicolas' second inequality at a primorial, without assuming RH. For an integer $n \ge 16$ put

$$\Delta(n) = \log\frac{\sigma(n)}{n} - \gamma - \log\log\log n,$$

so that Robin's inequality holds for $n > 5040$ if and only if $\Delta(n) < 0$. To each colossally abundant number N Nicolas associates a real number $\xi(N)$ [20, Definition 4.3], determined by the largest parameter of N (Section 3): every prime factor of N is at most $\xi(N)$, the colossally abundant number N_k of Proposition 3.2 is a multiple of N with $N_k/N \le \xi(N)$ whose prime factors are exactly the primes $p \le \xi(N)$, and, when $\log\xi(N) \ge 31$,

$$\xi(N)\cdot\left(1 - \frac{1}{\log^3\xi(N)}\right) < \log N \le \xi(N)\cdot\left(1 + \frac{0.0010488}{\log\xi(N)}\right)$$

(proof of Lemma 3.4). The notation is collected in Tables 1 and 2. Our main result is the following.

Theorem 1.1. *Let N be a colossally abundant number, and let $q = p_k$ be the largest prime with $q \le \xi(N)$. If $\log q \ge 31$, then*

$$\frac{1.317}{\sqrt{q}\cdot\log^2 q} \le \Delta(N) - c(q) + E(q) + \frac{2\sqrt{2}-2-\tau}{\sqrt{q}\cdot\log q} \le \frac{3.973}{\sqrt{q}\cdot\log^2 q}.$$

Here $2\sqrt{2}-2-\tau = 2(\sqrt{2}-1)-\tau = 0.7822357\ldots$, where $2(\sqrt{2}-1)$ is the constant of Ramanujan's bound [19]. Thus the sum of Robin's function $\Delta(N)$ and of the difference $E(q)-c(q)$, whose sign decides Nicolas' second inequality for N_k, equals $-\frac{2\sqrt{2}-2-\tau}{\sqrt{q}\log q}$ up to an explicit error of order $\frac{1}{\sqrt{q}\log^2 q}$, that is, smaller by a factor $\log q$. Combined with Lemma 4.2, Theorem 1.1 gives the following result.

Theorem 1.2. *Let N be a colossally abundant number, and let $q = p_k$ be the largest prime with $q \le \xi(N)$. Assume* $\log q \ge 31$.

(a) If

$$\Delta(N) < -\frac{2\sqrt{2}-2-\tau}{\sqrt{q}\cdot\log q} + \frac{1.317}{\sqrt{q}\cdot\log^2 q},$$

then $c(q) < E(q)$, and (2) holds for every integer n with $N_k \le n < N_{k+1}$.

(b) If (2) holds for N_k, then

$$\Delta(N) < -\frac{2\sqrt{2}-2-\tau}{\sqrt{q}\cdot\log q} + \frac{3.973}{\sqrt{q}\cdot\log^2 q} < -\frac{0.654}{\sqrt{q}\cdot\log q}.$$

(c) If the Riemann Hypothesis is true, then the conclusion of (b) holds for every colossally abundant number N with $\log q \ge 31$.

(d) If $\Delta(N) \ge 0$, then $c(q) > E(q) + \frac{0.654}{\sqrt{q}\cdot\log q}$; in particular, (2) does not hold for N_k.

Theorem 1.1 is deduced in Section 4 from the comparison between Robin's function at N and Nicolas' function at q proved in Section 3 (Theorem 3.1),

$$\frac{1.353}{\sqrt{q}\cdot\log^2 q} \le \Delta(N) - c(q) + \frac{2\sqrt{2}}{\sqrt{q}\cdot\log q} \le \frac{3.970}{\sqrt{q}\cdot\log^2 q}, \tag{3}$$

and from the bounds $-\frac{0.036}{\sqrt{q}\log^2 q} \le E(q) - \frac{2+\tau}{\sqrt{q}\log q} \le \frac{0.0022}{\sqrt{q}\log^2 q}$ (Lemma 4.5). Theorem 1.2 follows from Theorem 1.1, from Lemma 4.4, from Lemma 4.2 and, for (c), from Proposition 2.9. In Lemma 4.2, the left-hand side $\frac{n}{\varphi(n)}$ of (2) depends only on the set of primes dividing n, and an integer $n < N_{k+1}$ has at most k distinct prime factors, so $\frac{n}{\varphi(n)}$ is at most its value at N_k, whose prime factors are the first k primes; the right-hand side of (2) is an increasing function of n for $n \ge 3$ (Lemma 4.1), so on the interval $N_k \le n < N_{k+1}$ it is smallest at $n = N_k$.

The two bounds of Theorem 1.1 differ by $\frac{3.973-1.317}{\sqrt{q}\log^2 q}$. Consequently, Theorem 1.2 does not decide whether (2) holds for N_k when $\Delta(N)$ lies between $-\frac{2\sqrt{2}-2-\tau}{\sqrt{q}\log q} + \frac{1.317}{\sqrt{q}\log^2 q}$ and $-\frac{2\sqrt{2}-2-\tau}{\sqrt{q}\log q} + \frac{3.973}{\sqrt{q}\log^2 q}$. Remark 3.7 lists the terms of the proof that produce this interval, and Remark 4.6 shows that Theorems 1.3 and 1.4 below do not depend on its width.

Theorem 1.2 (b) transfers Nicolas' second inequality at primorials to Robin's inequality at colossally abundant numbers, and, through a result of Robin and Nicolas on the least counterexample (Corollary 6.6), to Robin's inequality for all integers in a corresponding range.

Theorem 1.3. *Let $X \ge e^{31}$, and assume that (2) holds for N_k for every k with $e^{31} \le p_k \le X$. Then:*

(i) every colossally abundant number N with $e^{31} \le q \le X$, where q is the largest prime with $q \le \xi(N)$, satisfies $\Delta(N) < -\frac{0.654}{\sqrt{q}\log q}$;

(ii) Robin's inequality (1) holds for every integer $n > 5040$ with

$$\log n \le X\cdot\left(1 - \frac{1}{\log^3 X}\right) - \log X.$$

The hypothesis of Theorem 1.3 concerns only the primorials N_k, that is, the partial sums $\sum_{p\le p_k} \log\frac{p}{p-1}$ and $\theta(p_k)$, whereas its conclusion (ii) concerns all integers n. We verified this hypothesis by computation (Section 7, Part 5). Put

$$X_0 = 100{,}000{,}008{,}765{,}439 \qquad \text{and} \qquad K_0 = \pi(X_0) = 3{,}204{,}942{,}022{,}702.$$

Theorem 1.4. *(i) The inequality (2) holds for N_k for every k with $120569 \le k \le K_0$. Consequently, it holds for every integer n with $N_{120569} \le n < N_{K_0+1}$, in particular for every integer $n \ge N_{120569}$ with $\log n \le 9.9999999338\cdot 10^{13}$.*

(ii) Every colossally abundant number N with $e^{31} \le q \le X_0$, where q is the largest prime with $q \le \xi(N)$, satisfies $\Delta(N) < -\frac{0.654}{\sqrt{q}\log q}$.

(iii) Robin's inequality (1) *holds for every integer* n *with* $5040 < n \le 10^{10^{13.6377}}$.

Part (i) of Theorem 1.4 is a computation; parts (ii) and (iii) follow from part (i) and Theorem 1.3. By Proposition 2.3, Robin's inequality was known for $5040 < n \le 10^{10^{13.11485}}$, and the verification of (2) in [22] covers the primorials N_k with $p_k \le 10^9$.

The hypothesis $\log q \ge 31$ of Theorems 1.1 and 3.1 covers every colossally abundant counterexample to (1). In [17, Theorem 4.2] the author proved that the largest prime factor p_k of the least counterexample satisfies $\log p_k > 31.018$, and Lemma 5.1 gives $\log p_k > 31.0319$ for the largest prime factor of every counterexample; since every prime factor of a colossally abundant number N is at most q, the prime q of a colossally abundant counterexample satisfies $\log q \ge 31$. Equivalently, Robin's inequality holds for every colossally abundant number $N > 5040$ with $\log q < 31$. The threshold 31 enters the proof through Lemma 2.11 (b) and through the numerical constants of Section 3, which are evaluated at $\log \xi = 31$; Remark 3.8 lists these uses.

The relation of Theorem 3.1, and hence of Theorem 1.1, to earlier work is the following. The comparison of $\sigma(N)/N$ with the Mertens product $\prod_{p \le \xi(N)} \frac{p}{p-1}$ at colossally abundant numbers goes back to Robin [5, Section 4], and the decomposition used here is that of Nicolas' proof of [20, Proposition 5.1], which gives, under RH and for $\xi(N) \ge 10^9 + 7$, an upper and a lower bound for $\sigma(N)/N$ in terms of $\log N$. In that proof RH is used in two ways. First, the Mertens product is evaluated by the estimate of $c(x)$ under RH [21, (2.16)], [20, (1.11)] (Proposition 2.8). Second, the numbers $\log N$, $\theta(\xi(N))$ and $\xi(N)$ are compared in [20, Lemma 4.16], which is stated under RH and whose proof uses the bounds $|\psi(x) - x| \le \frac{\sqrt{x}\log^2 x}{8\pi}$ and $|\theta(x) - x| \le \frac{\sqrt{x}\log^2 x}{8\pi}$ valid under RH [20, (2.7), (2.8)]; the comparison of $\log\log\theta(\xi(N))$ with $\log\log\log N$ in [20, Lemma 4.17] is deduced from that lemma. The estimates of the sums over the primes $p \le \xi(N)$ according to their exponent in [20, (5.5)–(5.8)] rest on bounds that do not assume RH. In Theorem 3.1 the Mertens product is not evaluated; it is kept inside $c(q)$. The comparison of $\log\log\log N$ with $\log\log\theta(\xi(N))$ is made directly in Lemma 3.4, by the mean value theorem applied to the difference $\log N - \theta(\xi(N))$, in which $\theta(\xi_2(N))$ is estimated by the bounds of Büthe [23] and Dusart [24], which give $|\theta(t) - t| \le \frac{0.022}{\log \xi(N)} \cdot t$ for every $t \ge \xi_2(N)$ when $\log \xi(N) \ge 31$ (Lemma 2.11), and $\theta(\xi(N))$ by the bound of Dusart. The sum $\sum_{\xi_2 < p \le \xi} \frac{1}{p^2}$ is estimated by Stieltjes integration (Lemmas 2.12 and 3.3), as in [20, Lemma 2.1], and the primes with exponent at least 2 are treated as in [20, (5.5), (5.6)] (Lemma 3.5). By Proposition 2.1, a counterexample to (1) exists only if RH is false, so an estimate proved under RH does not apply to a counterexample; Theorem 3.1 applies to every colossally abundant number with $\log q \ge 31$, whether RH holds or not, and it is used in this way in Theorem 1.2 and in Corollaries 5.2 and 5.3. Under RH and for $\xi(N) \ge 10^9 + 7$, the estimates of [20] give a narrower interval than Theorem 3.1; Remark 3.6 compares the constants.

Theorem 3.1 also has the following consequences for counterexamples to (1) (Section 5).

(1) Every colossally abundant counterexample to (1) yields a prime q with $\log q > 31.03$ and $c(q) > \frac{2.70}{\sqrt{q}\log q}$ (Corollary 5.2), whereas under RH $c(q) < \frac{2.0462}{\sqrt{q}\log q}$ for every prime $q \ge 10^9$ (Proposition 2.8).

(2) Conversely, every prime q with $\log q \ge 31$ and $c(q) > \frac{2\sqrt{2}}{\sqrt{q}\log q}\left(1 - \frac{0.4787}{\log q}\right)$ yields a colossally abundant counterexample to (1), namely $N_{F(q,1)}$ (Corollary 5.3).

(3) Under RH, every colossally abundant number N with $\log q \ge 31$ satisfies $-\frac{0.709}{\sqrt{q}\log^2 q} \le \Delta(N) + \frac{2\sqrt{2}-2}{\sqrt{q}\log q} - \frac{S_1(q)}{\log q} \le \frac{3.915}{\sqrt{q}\log^2 q}$, where S_1 is the sum over the zeros of the Riemann zeta function of Proposition 2.8, and $\Delta(N) < -\frac{0.654}{\sqrt{q}\log q}$ (Corollary 5.4); the constant $2\sqrt{2} - 2 = 2(\sqrt{2} - 1)$ is the constant of Ramanujan's bound.

For superabundant counterexamples, which need not be colossally abundant, we prove the following (Section 6).

(4) Every superabundant number n with largest prime factor p_k satisfies $\log n \ge \psi(p_k)$ (Lemma 6.1, following [16]), and every superabundant counterexample satisfies $c(p_k) > \frac{0.9991}{\sqrt{p_k}\log p_k}$ (Theorem 6.4).

(5) If n is the least counterexample, then the least colossally abundant number $N \ge n$ is a counterexample, and $\log N - \log \xi(N) \le \log n \le \log N$ (Corollary 6.6).

The status of these statements with respect to the literature is as follows. The equivalence of RH with (2) for $n \ge N_{120569}$ is due to Broughan [22]. The first claim of item (4) is due to Nazardonyavi and Yakubovich [16], and item

(5) is obtained by combining two results of Robin and Nicolas (Proposition 3.2 (f) and (g)) with item (1). Theorems 1.1, 1.2, 1.3, 1.4 and 3.1, Lemma 4.2, Corollary 4.3, items (1), (2) and (3), and the second claim of item (4) are proved in this paper.

The proofs use the description of colossally abundant numbers by Alaoglu and Erdős [25], Robin [5] and Nicolas [20], explicit estimates for the Chebyshev function of Rosser and Schoenfeld [26], Büthe [23] and Dusart [24], the estimate of Axler [27] for the Mertens product, Broughan's form of Nicolas' second criterion [22], and the numerical verification of (1) for all $5040 < n \le 10^{10^{13.11485}}$ due to Morrill and Platt [14]. Section 2 collects these results, Section 3 proves Theorem 3.1, Section 4 treats Nicolas' second criterion and proves Theorems 1.1 and 1.2, Section 5 derives the consequences of Theorem 3.1 for counterexamples, Section 6 treats superabundant counterexamples and proves Theorems 1.3 and 1.4, and Section 7 describes the computer verification of the numerical constants.

2. Background

Table 1 collects the notation used throughout the paper; the notation for colossally abundant numbers is collected in Table 2 of Section 3.

Table 1: Notation used throughout the paper.

Symbol	Meaning
γ	Euler–Mascheroni constant
$\sigma(n)$, $I(n)$	sum of the divisors of n, and $I(n) = \frac{\sigma(n)}{n}$
$\varphi(n)$	Euler's totient function
$\theta(x)$, $\psi(x)$	$\sum_{p\le x} \log p$ and $\sum_{p^j\le x} \log p$
p_k, $p\#$, N_k	k-th prime, $\prod_{q\le p} q$, and $N_k = p_k\#$, with $\log N_k = \theta(p_k)$
$\Delta(n)$	$\log I(n) - \gamma - \log\log\log n$; Robin's inequality holds for $n > 5040$ if and only if $\Delta(n) < 0$
$c(x)$	$\sum_{p\le x} \log \frac{p}{p-1} - \gamma - \log\log\theta(x)$; Nicolas' inequality at x is $c(x) > 0$
τ	$2 + \gamma - \log(4\pi) = 0.0461914\ldots$, denoted β in [22]
$S_1(x)$	$\sum_\rho \frac{x^{\rho-1}}{\rho(1-\rho)}$, over the nontrivial zeros ρ of the Riemann zeta function
$\xi(N)$, q	real number associated by Nicolas with a colossally abundant number N (Section 3), and the largest prime $q \le \xi(N)$
$G(x)$	$e^\gamma \log x + \frac{e^\gamma(2+\tau)}{\sqrt{x}}$; (2) is $\frac{n}{\varphi(n)} < G(\log n)$ (Section 4)
$E(q)$	$\log\left(1 + \frac{2+\tau}{\sqrt{\theta(q)}\log\theta(q)}\right)$; (2) for N_k is $c(p_k) < E(p_k)$ (Lemma 4.4)

We write $I(n) = \frac{\sigma(n)}{n}$ for the abundancy index, and we say that *Robin's inequality holds for n* if $I(n) < e^\gamma \cdot \log\log n$. An integer $n > 5040$ for which Robin's inequality does not hold is called a *counterexample*. We write $\psi(x) = \sum_{p^j\le x} \log p = \sum_{j\ge1} \theta(x^{1/j})$ for the second Chebyshev function, so that $\psi(x) = \sum_{q\le x} \left\lfloor \frac{\log x}{\log q} \right\rfloor \log q$ and $\psi(x) \ge \theta(x) + \theta(\sqrt{x})$. For a prime p we write $p\# = \prod_{q\le p} q$, so that $\log(p\#) = \theta(p)$, and $p_1 = 2 < p_2 = 3 < \cdots$ denotes the sequence of primes. Since $\sum_{p\le x} \log \frac{p}{p-1}$ and $\theta(x)$ change only at primes, $c(x) = c(q)$ when q is the largest prime with $q \le x$.

2.1. Robin's Criterion and Counterexamples

Proposition 2.1. *Robin's inequality holds for all natural numbers $n > 5040$ if and only if the Riemann Hypothesis is true [5, Theorem 1, p. 188].*

Proposition 2.2. *For $n > 1$, $I(n) < \prod_{q\mid n} \frac{q}{q-1} = \frac{n'}{\varphi(n')}$, where n' is the product of the distinct primes dividing n [10, (2.7), p. 362].*

Proposition 2.3. *Robin's inequality holds for every integer n with $5040 < n \le 10^{10^{13.11485}}$ [14, Theorem 5].*

A natural number n is *superabundant* if $I(m) < I(n)$ for every $m < n$, and *colossally abundant* if there is $\varepsilon > 0$ with $\frac{\sigma(m)}{m^{1+\varepsilon}} \le \frac{\sigma(n)}{n^{1+\varepsilon}}$ for every $m \ge 1$ [25]; such a number ε is called a *parameter* of n. Every colossally abundant number is superabundant [25].

Proposition 2.4. *If n is superabundant with largest prime factor p_k, then $n = \prod_{i=1}^{k} p_i^{a_i}$ with $a_1 \ge a_2 \ge \cdots \ge a_k \ge 1$ [25, Theorem 1, p. 450].*

Proposition 2.5. *If $n > 5040$ is the smallest integer such that Robin's inequality does not hold for n, then n is superabundant [15, Theorem 3, p. 273].*

Proposition 2.6. *If the Riemann Hypothesis is false, then Robin's inequality fails for infinitely many colossally abundant numbers [5, Section 4, Proposition 1].*

2.2. Nicolas' Criterion

Proposition 2.7. *If the Riemann Hypothesis is true, then $c(x) > 0$ for every $x \ge 2$. If the Riemann Hypothesis is false, then there is b with $0 < b < \frac{1}{2}$ such that $c(x) = \Omega_{\pm}(x^{-b})$; more precisely, this holds for every b with $1 - \Theta < b < \frac{1}{2}$, where Θ is the supremum of the real parts of the zeros of the Riemann zeta function [18, Theorem 3 (b), (c)], see also [21].*

In [18, 21] the statement is given for $\log f(x) = -c(x)$, where $f(x) = e^{\gamma} \cdot \log\theta(x) \cdot \prod_{p \le x}\left(1 - \frac{1}{p}\right)$. Put

$$\tau = 2 + \gamma - \log(4\pi) = 0.0461914\ldots, \qquad S_1(x) = \sum_{\rho} \frac{x^{\rho-1}}{\rho(1-\rho)},$$

where ρ runs over the nontrivial zeros of the Riemann zeta function [20, (1.3), (1.5)].

Proposition 2.8. *Assume the Riemann Hypothesis. Then $|S_1(x)| \le \frac{\tau}{\sqrt{x}}$ for $x > 1$ [20, (1.4)], and for $x \ge 10^9$ [21, (2.16)], [20, (1.11)],*

$$\frac{0.055}{\sqrt{x}\log^2 x} \le -c(x) + \frac{2}{\sqrt{x}\log x} + \frac{S_1(x)}{\log x} \le \frac{2.062}{\sqrt{x}\log^2 x}.$$

Consequently $c(x) < \frac{2+\tau}{\sqrt{x}\log x} < \frac{2.0462}{\sqrt{x}\log x}$ for every $x \ge 10^9$.

Proof. The left inequality gives $c(x) \le \frac{2}{\sqrt{x}\log x} + \frac{S_1(x)}{\log x} - \frac{0.055}{\sqrt{x}\log^2 x} < \frac{2+\tau}{\sqrt{x}\log x}$, and $2 + \tau < 2.0462$. □

For $k \ge 1$ we write $N_k = p_1 p_2 \cdots p_k = p_k\#$ for the k-th primorial, so that $\log N_k = \theta(p_k)$.

Proposition 2.9. *The Riemann Hypothesis is true if and only if (2) holds for every integer $n \ge N_{120569}$ [22, Theorem 5.34 (2), Corollary 5.35].*

2.3. Explicit Estimates for Primes

Proposition 2.10. *The following hold.*

(a) $\theta(x) < x$ *for* $0 < x \le 10^{19}$, *and* $\theta(x) > x - 1.94\sqrt{x}$ *for* $1427 < x \le 10^{19}$ *[23, Theorem 2], see also [20, (2.1), (2.2)].*

(b) $|\theta(x) - x| < \frac{x}{\log^3 x}$ *for* $x \ge 89{,}967{,}803$ *[24, Theorem 4.2].*

(c) For every $x \ge 89{,}693$ *there is a prime* p *with* $x < p \le x \cdot \left(1 + \frac{1}{\log^3 x}\right)$ *[24, Proposition 5.4].*

(d) $\theta(x) \ge 0.945 \cdot x$ *for* $x \ge 853$ *[26, Theorem 10], and* $\pi(x) < \frac{1.26 \cdot x}{\log x}$ *for* $x > 1$ *[26, (3.6)], where* $\pi(x)$ *is the number of primes* $p \le x$.

(e) $\sum_{p \ge x} \frac{1}{p^3} \le \frac{0.086}{x^2}$ *for* $x \ge 1418$ *[20, (2.15)].*

(f) For $x \geq 46{,}909{,}038$ [27, Proposition 6.1],

$$\prod_{p \leq x} \frac{p}{p-1} < e^{\gamma} \cdot \log x \cdot (1 - \varepsilon(\log x))^{-1}, \qquad \varepsilon(P) = \frac{1}{20P^3} + \frac{3}{16P^4}.$$

Lemma 2.11. *The following hold.*

(a) $\theta(t) < 1.000012 \cdot t$ for every $t > 0$.

(b) Let $\lambda \geq 31$. Then $|\theta(t) - t| \leq \frac{0.022}{\lambda} \cdot t$ for every $t \geq 1.39838 \cdot e^{\lambda/2}$.

Proof. (a) For $t \leq 10^{19}$ this is Proposition 2.10 (a); for $t > 10^{19}$, Proposition 2.10 (b) gives $\theta(t) < t \cdot \left(1 + \frac{1}{\log^3(10^{19})}\right) < 1.000012 \cdot t$.

(b) Let $t \geq 1.39838 \cdot e^{\lambda/2}$. If $t \leq 10^{19}$, Proposition 2.10 (a) gives $|\theta(t) - t| < \frac{1.94}{\sqrt{t}} \cdot t$, and $\sqrt{t} \geq \sqrt{1.39838} \cdot e^{\lambda/4} > 1.18253 \cdot e^{\lambda/4}$ since $\sqrt{1.39838} > 1.18253$, so $\lambda \cdot \frac{|\theta(t)-t|}{t} < \frac{1.94 \cdot \lambda \cdot e^{-\lambda/4}}{1.18253}$; the right-hand side is decreasing for $\lambda \geq 4$, and is less than 0.022 at $\lambda = 31$. If $t > 10^{19}$, Proposition 2.10 (b) gives $\lambda \cdot \frac{|\theta(t)-t|}{t} < \frac{\lambda}{\log^3 t}$ with $\log t > \max\left(\log(10^{19}), \frac{\lambda}{2}\right)$. If $\lambda \leq 2\log(10^{19})$, this is at most $\frac{2\log(10^{19})}{\log^3(10^{19})} < 0.022$, and otherwise it is at most $\frac{8}{\lambda^2} < \frac{8}{31^2} < 0.022$. □

The constant 1.39838 of Lemma 2.11 (b) is chosen so that the lemma applies to every $t \geq \xi_2$: by the lower bound of Robin for ξ_2 (Proposition 3.2 (d)), $\xi_2 \geq \sqrt{2} \cdot \left(1 - \frac{\log 2}{2\lambda}\right) \cdot e^{\lambda/2}$ with $\lambda = \log \xi$, and $\sqrt{2} \cdot \left(1 - \frac{\log 2}{2 \cdot 31}\right) = 1.39840\ldots > 1.39838$ (see (4)).

Lemma 2.12. *Let $y > 1$ and $0 \leq \delta < 1$ be such that $|\theta(t) - t| \leq \delta t$ for every $t \geq y$. Put $A_j = \frac{1}{y \log^j y}$. Then*

$$(1 - 3\delta) \cdot A_1 - A_2 \leq \sum_{p > y} \frac{1}{p^2} \leq (1 + 3\delta) \cdot A_1 - (1 + \delta) \cdot A_2 + 2(1 + \delta) \cdot A_3.$$

Proof. Put $J_j = \int_y^{\infty} \frac{dt}{t^2 \log^j t}$. Since $\frac{d}{dt}\left(-\frac{1}{t \log^j t}\right) = \frac{1}{t^2 \log^j t} + \frac{j}{t^2 \log^{j+1} t}$, we have $J_j = A_j - jJ_{j+1}$, hence $J_1 = A_1 - J_2$, $J_2 = A_2 - 2J_3$, and $0 < J_j \leq A_j$. By Stieltjes integration,

$$\sum_{p > y} \frac{1}{p^2} = \int_y^{\infty} \frac{d\theta(t)}{t^2 \log t} = -\frac{\theta(y)}{y^2 \log y} + \int_y^{\infty} \theta(t) \cdot \frac{2\log t + 1}{t^3 \log^2 t}\, dt,$$

and $\int_y^{\infty} \frac{2\log t + 1}{t^2 \log^2 t}\, dt = 2J_1 + J_2 = 2A_1 - J_2$. Using $(1 - \delta)t \leq \theta(t) \leq (1 + \delta)t$ for $t \geq y$, the sum is at least $-(1 + \delta)A_1 + (1 - \delta)(2A_1 - A_2) \geq (1 - 3\delta)A_1 - A_2$, and at most $-(1 - \delta)A_1 + (1 + \delta)(2A_1 - A_2 + 2A_3)$, using $J_2 \geq A_2 - 2A_3$. □

3. Colossally Abundant Numbers

In this section we prove the following comparison between Robin's function at a colossally abundant number N and Nicolas' function at the largest prime $q \leq \xi(N)$, from which Theorem 1.1 is deduced in Section 4.

Theorem 3.1. *Let N be a colossally abundant number, and let q be the largest prime with $q \leq \xi(N)$. If $\log q \geq 31$, then*

$$\frac{1.353}{\sqrt{q} \cdot \log^2 q} \leq \Delta(N) - c(q) + \frac{2\sqrt{2}}{\sqrt{q} \cdot \log q} \leq \frac{3.970}{\sqrt{q} \cdot \log^2 q}.$$

Thus Robin's function at N and Nicolas' function at q differ by $\frac{2\sqrt{2}}{\sqrt{q}\log q}$, up to an explicit error of order $\frac{1}{\sqrt{q}\log^2 q}$. Theorem 3.1 gives $\Delta(N) \geq 0$ when $c(q)\sqrt{q}\log q \geq 2\sqrt{2}\left(1 - \frac{0.4787}{\log q}\right)$ and $\Delta(N) < 0$ when $c(q)\sqrt{q}\log q < 2\sqrt{2}\left(1 - \frac{1.4035}{\log q}\right)$, and it does not determine the sign of $\Delta(N)$ when $c(q)\sqrt{q}\log q$ lies between these two numbers.

3.1. The Structure of Colossally Abundant Numbers

We recall the description given by Nicolas [20, Section 4], which goes back to Alaoglu and Erdős [25] and Robin [5]. For real $t > 1$ and an integer $j \geq 1$ put

$$F(t, j) = \frac{\log\left(1 + \frac{1}{t+t^2+\cdots+t^j}\right)}{\log t},$$

which is decreasing in t. Let N be a colossally abundant number and κ its largest parameter. The number κ determines a sequence of real numbers $\xi = \xi_1, \xi_2, \xi_3, \ldots$, which are the thresholds at which the exponent of a prime changes (Proposition 3.2 (a)): $\xi = \xi(N) > 1$ is defined by $F(\xi, 1) = \kappa$, and $\xi_j = \xi_j(N)$ for $j \geq 2$ by $F(\xi_j, j) = \kappa$, with $\xi_1 = \xi$; K is the largest j with $\xi_j \geq 2$ [20, Definition 4.3]. Since $\xi_1 > \xi_2 > \cdots$, the primes $p \leq \xi_j$ are the primes whose exponent in N_κ (defined below) is at least j; in particular, the primes dividing N_κ are the primes $p \leq \xi$. Put $N_\kappa = \prod_{j=1}^{K} \prod_{\xi_{j+1} < p \leq \xi_j} p^j$. Throughout, the letter ξ without index denotes $\xi_1 = \xi(N)$, and the other terms of the sequence always carry their index $j \geq 2$. The numbers N and N_κ have the same largest parameter κ, hence the same numbers ξ, ξ_j and K; they need not be equal, and by Proposition 3.2 (a) and (b) below, N divides N_κ and $\frac{N_\kappa}{N} \leq \xi$. The exponents of the primes are described for N_κ, and the results are stated for N.

The thresholds ξ_j arise as follows. A colossally abundant number with parameter ε maximizes $\frac{\sigma(n)}{n^{1+\varepsilon}}$ over the positive integers n. For a prime p and $j \geq 1$,

$$\frac{\sigma(p^j)}{p^{j(1+\varepsilon)}} \Big/ \frac{\sigma(p^{j-1})}{p^{(j-1)(1+\varepsilon)}} = \left(1 + \frac{1}{p + p^2 + \cdots + p^j}\right) \cdot p^{-\varepsilon},$$

and this ratio is at least 1 if and only if $F(p, j) \geq \varepsilon$. Since $F(t, j)$ is decreasing in t, for $\varepsilon = \kappa$ this holds if and only if $p \leq \xi_j$: raising the exponent of p from $j - 1$ to j does not decrease $\frac{\sigma(n)}{n^{1+\kappa}}$ exactly when $p \leq \xi_j$, and N_κ is the number in which every prime has the largest exponent allowed by this rule. In particular $\xi = \xi_1$ is the threshold for a prime to divide N_κ, which is why the largest prime $q \leq \xi$ and the primorial $q\#$ appear in Theorems 1.1 and 3.1. Since N and N_κ both have the parameter κ, both maximize $\frac{\sigma(n)}{n^{1+\kappa}}$, so $\frac{\sigma(N)}{N^{1+\kappa}} = \frac{\sigma(N_\kappa)}{N_\kappa^{1+\kappa}}$, that is, $I(N) = I(N_\kappa) \cdot \left(\frac{N_\kappa}{N}\right)^{-\kappa}$. With $1 \leq \frac{N_\kappa}{N} \leq \xi$ and $\xi^{-\kappa} = \left(1 + \frac{1}{\xi}\right)^{-1} = \frac{\xi}{\xi+1}$, which follows from $\kappa = F(\xi, 1) = \frac{\log(1+1/\xi)}{\log \xi}$, this gives the inequality $\frac{\xi}{\xi+1} \cdot I(N_\kappa) \leq I(N) \leq I(N_\kappa)$ of Proposition 3.2 (b).

For example, let $\kappa = F(7, 1) = \frac{\log(8/7)}{\log 7} = 0.0686\ldots$ Then $\xi = 7$. Since $F(3, 2) = 0.0729\ldots \geq \kappa > F(5, 2) = 0.0204\ldots$, we have $3 \leq \xi_2 < 5$; since $F(2, 3) = 0.0995\ldots \geq \kappa > F(3, 3) = 0.0230\ldots$, we have $2 \leq \xi_3 < 3$; and since $F(2, 4) = 0.0473\ldots < \kappa$, we have $\xi_4 < 2$ and $K = 3$. Hence $N_\kappa = 2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$, with $q = 7$ and $q\# = 210$.

Proposition 3.2. *With this notation, the following hold.*

(a) N_κ *is colossally abundant with largest parameter* κ*, and* N *divides* N_κ*; every prime dividing* N *is at most* ξ*, and a prime* $p \leq \xi$ *has exponent* j *in* N_κ *if* $\xi_{j+1} < p \leq \xi_j$ *[20, (4.11), Proposition 4.2, Lemma 4.4], [25, Theorem 10].*

(b) $\sum_{j=1}^{K} \theta(\xi_j) - \log \xi \leq \log N \leq \sum_{j=1}^{K} \theta(\xi_j)$*, and* $\frac{\xi}{\xi+1} \cdot I(N_\kappa) \leq I(N) \leq I(N_\kappa)$*, where*

$$I(N_\kappa) = \prod_{j=1}^{K} \prod_{\xi_{j+1} < p \leq \xi_j} \frac{1 - p^{-(j+1)}}{1 - p^{-1}}$$

[20, Proposition 4.5].

(c) $\xi^{1/j} \leq \xi_j \leq (j\xi)^{1/j}$ *for every* $j \geq 2$ *[20, (4.26)].*

(d) $\sqrt{2\xi} \cdot \left(1 - \frac{\log 2}{2 \log \xi}\right) \leq \xi_2$ *for* $\xi \geq 1530$ *[5, p. 190], [20, (4.28)], and* $\xi_2 \leq \sqrt{2\xi} \cdot \left(1 - \frac{0.323}{\log \xi}\right)$ *for* $\xi \geq 10^9$ *[20, (4.29)].*

(e) If $\xi > 10^9 + 7$*, then* $\sum_{j=3}^{K} \xi_j \leq 1.9769 \cdot \xi^{1/3}$ *[20, Lemmas 4.8–4.10].*

(f) For every integer $m \geq 2$ *there are two consecutive colossally abundant numbers* $N' < N$ *with* $\frac{N}{\xi(N)} \leq N' < m \leq N$ *[20, Proposition 4.6].*

(g) If $3 \le N' < N$ are consecutive colossally abundant numbers and $N' \le m \le N$, then

$$\frac{I(m)}{\log\log m} \le \max\left\{\frac{I(N')}{\log\log N'}, \frac{I(N)}{\log\log N}\right\}$$

[5, Proposition 1], see also [16].

In (e), Nicolas states [20, Lemma 4.10] for $N \ge N^{(0)}$, where $N^{(0)}$ is the colossally abundant number with $\xi(N^{(0)}) = 10^9 + 7$. If $\xi > 10^9 + 7$, then $\kappa < F(10^9 + 7, 1)$, and since the parameters of colossally abundant numbers decrease as the numbers increase [20, (4.4)], we have $N > N^{(0)}$, so (e) applies.

3.2. Three Estimates

Throughout this subsection, N is a colossally abundant number with $\lambda = \log\xi \ge 31$, where $\xi = \xi(N)$, and we put

$$M_\xi = \frac{\sqrt{2}}{\sqrt{\xi}\cdot\lambda}, \qquad L(t) = \log\log t, \qquad S_2 = \sum_{\xi_2 < p \le \xi} \frac{1}{p^2}.$$

The proof of Theorem 3.1 rests on three estimates, each with main term M_ξ or a multiple of it. Lemma 3.3 estimates S_2: by Lemma 2.12, S_2 has the main term $\frac{1}{\xi_2\log\xi_2}$, and $\sqrt{2\xi}\cdot\frac{\lambda}{2} = \frac{1}{M_\xi}$. Lemma 3.4 estimates $L(\log N) - L(\theta(\xi))$: by the mean value theorem this difference has the main term $\frac{\log N - \theta(\xi)}{\xi\lambda}$, where $\log N - \theta(\xi)$ has the main term $\theta(\xi_2)$, which is $\sqrt{2\xi}$ up to a factor $1 + O\left(\frac{1}{\lambda}\right)$, and $\frac{\sqrt{2\xi}}{\xi\lambda} = M_\xi$. Lemma 3.5 shows that $\log I(N)$ is $\sum_{p\le q}\log\frac{p}{p-1} - S_2$ up to a term of smaller order. The three lemmas give $\Delta(N) - c(q) = -2M_\xi$ up to terms of the form $\frac{C\cdot M_\xi}{\lambda}$, and the decimal constants in their statements are the values of C obtained for $\lambda \ge 31$; these constants are verified in Section 7 (Table 5).

By Proposition 3.2 (d), and since $1 - \frac{\log 2}{2\cdot 31} > 0.98882$ and $\sqrt{2}\cdot 0.98882 > 1.39838$,

$$\xi_2^- := \sqrt{2\xi}\cdot\left(1 - \frac{\log 2}{2\lambda}\right) \le \xi_2 \le \xi_2^+ := \sqrt{2\xi}\cdot\left(1 - \frac{0.323}{\lambda}\right), \qquad \xi_2 \ge 1.39838\cdot e^{\lambda/2}, \tag{4}$$

so Lemma 2.11 (b) applies to every $t \ge \xi_2$ with $\delta = \frac{0.022}{\lambda}$. Table 2 collects the notation of this section. Here ξ_2^- and ξ_2^+ are the lower and the upper bound for ξ_2 of Proposition 3.2 (d); they are used only through (4). For $\ell = \log\xi_2$ we have $\ell \le \log\sqrt{2\xi} = \frac{1}{2}(\lambda + \log 2)$ by $\xi_2 \le \sqrt{2\xi}$. By the lower bound of (4), $\ell \ge \frac{1}{2}(\lambda + \log 2) + \log\left(1 - \frac{\log 2}{2\lambda}\right)$, and since $\lambda \ge 31$ and $-\log\left(1 - \frac{\log 2}{2\cdot 31}\right) < 0.011306$, also $\ell \ge \frac{1}{2}(\lambda + \log 2) - 0.011306 = \frac{\lambda}{2}\cdot\left(1 + \frac{\log 2 - 0.022612}{\lambda}\right)$ with $\log 2 - 0.022612 > 0.670535$, so that

$$\frac{\lambda}{2}\cdot\left(1 + \frac{0.670535}{\lambda}\right) \le \ell = \log\xi_2 \le \frac{\lambda}{2}\cdot\left(1 + \frac{\log 2}{\lambda}\right). \tag{5}$$

Lemma 3.3. $M_\xi\cdot\left(1 - \frac{2.43616}{\lambda}\right) \le S_2 \le M_\xi\cdot\left(1 - \frac{1.920374}{\lambda}\right)$.

Proof. Put $\ell = \log\xi_2$ and $A_j = \frac{1}{\xi_2\ell^j}$. Since $t \mapsto t\log t$ is increasing, the bounds (4) on ξ_2 and (5) on ℓ give

$$A_1 \ge \frac{M_\xi}{\left(1 - \frac{0.323}{\lambda}\right)\left(1 + \frac{\log 2}{\lambda}\right)} \ge \frac{M_\xi}{1 + \frac{0.370148}{\lambda}} \ge M_\xi\cdot\left(1 - \frac{0.370148}{\lambda}\right)$$

and, since $\left(1 - \frac{\log 2}{2\lambda}\right)\left(1 + \frac{0.670535}{\lambda}\right) \ge 1 + \frac{0.316464}{\lambda}$ for $\lambda \ge 31$,

$$A_1 \le \frac{M_\xi}{1 + \frac{0.316464}{\lambda}} \le M_\xi\cdot\left(1 - \frac{0.316464}{\lambda} + \frac{0.316464^2}{\lambda^2}\right) \le M_\xi\cdot\left(1 - \frac{0.313233}{\lambda}\right).$$

Table 2: Notation of Section 3.

Symbol	Meaning
κ	largest parameter of the colossally abundant number N
$\xi = \xi_1, \xi_j$	real numbers with $F(\xi_j, j) = \kappa$; a prime p with $\xi_{j+1} < p \le \xi_j$ has exponent j in N_κ
K	largest j with $\xi_j \ge 2$
N_κ	colossally abundant number $\prod_{j=1}^{K} \prod_{\xi_{j+1}<p\le\xi_j} p^j$, a multiple of N
q, u	largest prime with $q \le \xi$, and $u = \log q$
λ	$\log \xi$
M_ξ, M_q	$\frac{\sqrt{2}}{\sqrt{\xi}\cdot\lambda}$ and $\frac{\sqrt{2}}{\sqrt{q}\cdot u}$
$L(t)$	$\log\log t$
S_2	$\sum_{\xi_2<p\le\xi} \frac{1}{p^2}$
D	$\log N - \theta(\xi)$
A_j	$\frac{1}{y\log^j y}$ (Lemma 2.12), with $y = \xi_2$ and $y = \xi$ in the proof of Lemma 3.3

Since $\frac{\lambda}{2} \le \ell \le \frac{\lambda}{2}\left(1 + \frac{\log 2}{\lambda}\right)$, we get $A_2 = \frac{A_1}{\ell} \le \frac{2M_\xi}{\lambda}$, $A_3 = \frac{A_2}{\ell} \le \frac{4M_\xi}{\lambda^2}$ and

$$A_2 \ge \frac{2M_\xi}{\lambda} \cdot \frac{1 - \frac{0.370148}{\lambda}}{1 + \frac{\log 2}{\lambda}} \ge \frac{2M_\xi}{\lambda} \cdot \left(1 - \frac{1.063296}{31}\right) > \frac{1.9314 \cdot M_\xi}{\lambda}.$$

By Lemma 2.12 with $y = \xi_2$ and $3\delta = \frac{0.066}{\lambda}$,

$$\sum_{p>\xi_2} \frac{1}{p^2} \ge \left(1 - \frac{0.066}{\lambda}\right)\left(1 - \frac{0.370148}{\lambda}\right) M_\xi - \frac{2M_\xi}{\lambda} \ge M_\xi \cdot \left(1 - \frac{2.436148}{\lambda}\right)$$

and

$$\sum_{p>\xi_2} \frac{1}{p^2} \le \left(1 + \frac{0.066}{\lambda}\right)\left(1 - \frac{0.313233}{\lambda}\right) M_\xi - \frac{1.9314 \cdot M_\xi}{\lambda} + \frac{8.006 \cdot M_\xi}{\lambda^2} \le M_\xi \cdot \left(1 - \frac{1.920374}{\lambda}\right),$$

where we used $2(1+\delta)\cdot 4 < 8.006$ and $\frac{8.006}{\lambda} \le \frac{8.006}{31} < 0.258259$. By Lemma 2.12 with $y = \xi$, using $-(1+\delta)A_2(\xi) + 2(1+\delta)A_3(\xi) \le 0$,

$$0 \le \sum_{p>\xi} \frac{1}{p^2} \le \frac{1.0022}{\xi\lambda} = \frac{1.0022}{\sqrt{2}\cdot\sqrt{\xi}} \cdot M_\xi \le \frac{1.0022 \cdot 31 \cdot e^{-31/2}}{\sqrt{2}} \cdot \frac{M_\xi}{\lambda} < \frac{4.08 \cdot 10^{-6} \cdot M_\xi}{\lambda},$$

because $\lambda e^{-\lambda/2}$ is decreasing for $\lambda \ge 2$. Since $S_2 = \sum_{p>\xi_2} \frac{1}{p^2} - \sum_{p>\xi} \frac{1}{p^2}$, the claim follows. □

Lemma 3.4. $M_\xi \cdot \left(1 - \frac{0.369659}{\lambda}\right) \le L(\log N) - L(\theta(\xi)) \le M_\xi \cdot \left(1 - \frac{0.05272}{\lambda}\right)$.

Proof. Put $D = \log N - \theta(\xi)$. By Proposition 3.2 (c), $\xi_3 \ge \xi^{1/3} \ge e^{31/3} > 853$, so $K \ge 3$, and by Proposition 2.10 (d), $\theta(\xi_3) \ge 0.945 \cdot \xi^{1/3} > \lambda$, because $0.945 \cdot e^{s/3} > s$ for $s \ge 31$. Proposition 3.2 (b) and (e) and Lemma 2.11 then give

$$\theta(\xi_2) \le D \le \theta(\xi_2) + 1.000012 \cdot 1.9769 \cdot \xi^{1/3}.$$

By (4) and Lemma 2.11 (b),

$$\theta(\xi_2) \ge \left(1 - \frac{0.022}{\lambda}\right)\left(1 - \frac{\log 2}{2\lambda}\right)\sqrt{2\xi} \ge \left(1 - \frac{0.368574}{\lambda}\right)\sqrt{2\xi}$$

and

$$\theta(\xi_2) \le \left(1 + \frac{0.022}{\lambda}\right)\left(1 - \frac{0.323}{\lambda}\right)\sqrt{2\xi} \le \left(1 - \frac{0.301}{\lambda}\right)\sqrt{2\xi}.$$

Moreover

$$\frac{1.000012 \cdot 1.9769 \cdot \xi^{1/3}}{\sqrt{2\xi}} \le 1.397905 \cdot e^{-\lambda/6} \le \frac{1.397905 \cdot 31 \cdot e^{-31/6}}{\lambda} < \frac{0.24717}{\lambda},$$

because $\lambda e^{-\lambda/6}$ is decreasing for $\lambda \ge 6$. Hence

$$\left(1 - \frac{0.368574}{\lambda}\right)\sqrt{2\xi} \le D \le \left(1 - \frac{0.0538}{\lambda}\right)\sqrt{2\xi}.$$

By Proposition 2.10 (b), $\xi \cdot (1 - \lambda^{-3}) < \theta(\xi) < \xi \cdot (1 + \lambda^{-3})$. Hence $\log N = \theta(\xi) + D \le \xi \cdot \left(1 + \lambda^{-3} + \sqrt{2}e^{-\lambda/2}\right) \le \xi \cdot \left(1 + \frac{0.0010488}{\lambda}\right)$, so that $\log\log N \le \lambda + \frac{0.0010488}{\lambda} \le \lambda \cdot \left(1 + \frac{0.0000339}{\lambda}\right)$ and

$$\log N \cdot \log\log N \le \xi\lambda \cdot \left(1 + \frac{0.001085}{\lambda}\right).$$

Also $\log\theta(\xi) \ge \lambda + \log(1 - \lambda^{-3}) \ge \lambda - 1.0001\lambda^{-3}$, so that

$$\theta(\xi) \cdot \log\theta(\xi) \ge \xi\lambda \cdot (1 - \lambda^{-3})(1 - 1.0001\lambda^{-4}) \ge \xi\lambda \cdot \left(1 - \frac{0.00108}{\lambda}\right).$$

Since $L'(t) = \frac{1}{t\log t}$ is decreasing for $t > 1$, the mean value theorem gives

$$\frac{D}{\log N \cdot \log\log N} \le L(\log N) - L(\theta(\xi)) \le \frac{D}{\theta(\xi) \cdot \log\theta(\xi)}.$$

With $\frac{\sqrt{2\xi}}{\xi\lambda} = M_\xi$, the lower bound is at least $M_\xi \cdot \frac{1-0.368574/\lambda}{1+0.001085/\lambda} \ge M_\xi \cdot \left(1 - \frac{0.369659}{\lambda}\right)$, and the upper bound is at most $M_\xi \cdot \frac{1-0.0538/\lambda}{1-0.00108/\lambda} \le M_\xi \cdot \left(1 - \frac{0.05272}{\lambda}\right)$. □

Lemma 3.5. *Let q be the largest prime with $q \le \xi$. Then*

$$\sum_{p\le q} \log\frac{p}{p-1} - S_2 - \frac{1.0156 \cdot M_\xi}{\lambda} \le \log I(N) \le \sum_{p\le q} \log\frac{p}{p-1} - S_2.$$

Proof. For $p \le \xi$ let v_p be the exponent of p in N_κ. By Proposition 3.2 (a) and (b), the primes $p \le \xi$ are the primes $p \le q$, and

$$\log I(N_\kappa) = \sum_{p\le q} \log\frac{p}{p-1} + \sum_{p\le q} \log\left(1 - p^{-(v_p+1)}\right).$$

Every term of the second sum is negative, and $v_p = 1$ for $\xi_2 < p \le \xi$; since $\log(1-u) \le -u$, the second sum is at most $-S_2$, and the upper bound follows from $I(N) \le I(N_\kappa)$.

For the lower bound we split the second sum into three parts. For $\xi_2 < p \le \xi$, $-\log(1 - p^{-2}) \le \frac{p^{-2}}{1-\xi_2^{-2}}$, so this part is at least $-S_2 - 2\xi_2^{-2}S_2$. For $\xi_3 < p \le \xi_2$, $v_p = 2$, and by Proposition 2.10 (e), since $\xi_3 > 1418$, this part is at least $-\frac{1}{1-\xi_3^{-3}} \cdot \frac{0.086}{\xi_3^2} \ge -0.0861 \cdot \xi^{-2/3}$. For $p \le \xi_3$, if $\xi_{j+1} < p \le \xi_j$ with $j \ge 3$, then $p^{j+1} > \xi_{j+1}^{j+1} \ge \xi$ by Proposition 3.2 (c), so $-\log(1 - p^{-(j+1)}) \le \frac{1}{\xi-1}$; by Proposition 2.10 (d) and Proposition 3.2 (c), the number of such primes is at most $\pi\left((3\xi)^{1/3}\right) < \frac{3\cdot 1.26\cdot 3^{1/3}\cdot\xi^{1/3}}{\lambda} < \frac{5.4518\cdot\xi^{1/3}}{\lambda}$, so this part is at least $-\frac{5.4518}{31} \cdot \frac{\xi^{1/3}}{\xi-1} \ge -0.17587 \cdot \xi^{-2/3}$. Together with $I(N) \ge \frac{\xi}{\xi+1} I(N_\kappa)$ and $\log\left(1 + \frac{1}{\xi}\right) \le \frac{1}{\xi} \le e^{-31/3} \cdot \xi^{-2/3}$,

$$\log I(N) \ge \sum_{p\le q} \log\frac{p}{p-1} - S_2 - 2\xi_2^{-2}S_2 - 0.26201 \cdot \xi^{-2/3}.$$

Finally $\xi^{-2/3} = \frac{\lambda^2 e^{-\lambda/6}}{\sqrt{2}} \cdot \frac{M_\xi}{\lambda}$, and $\lambda^2 e^{-\lambda/6}$ is decreasing for $\lambda \ge 12$, so $0.26201 \cdot \xi^{-2/3} \le \frac{0.26201\cdot 31^2\cdot e^{-31/6}}{\sqrt{2}} \cdot \frac{M_\xi}{\lambda} < \frac{1.0155\cdot M_\xi}{\lambda}$, while $2\xi_2^{-2}S_2 \le 2\xi_2^{-2}M_\xi < \frac{10^{-4}\cdot M_\xi}{\lambda}$ by Lemma 3.3 and (4). □

Table 3 summarizes the proof of Lemma 3.5: it lists the four terms of $\log I(N) - \sum_{p\le q} \log\frac{p}{p-1}$, according to the exponent v_p of the prime p in N_κ, with the bounds obtained for each of them.

Table 3: The terms of $\log I(N) - \sum_{p\le q} \log \frac{p}{p-1}$ in the proof of Lemma 3.5.

Primes	**Term**	**Upper bound**	**Lower bound**
$\xi_2 < p \le \xi$ $(v_p = 1)$	$\sum \log\left(1 - p^{-2}\right)$	$-S_2$	$-S_2 - 2\xi_2^{-2} S_2$
$\xi_3 < p \le \xi_2$ $(v_p = 2)$	$\sum \log\left(1 - p^{-3}\right)$	0	$-0.0861 \cdot \xi^{-2/3}$
$p \le \xi_3$ $(v_p \ge 3)$	$\sum \log\left(1 - p^{-(v_p+1)}\right)$	0	$-0.17587 \cdot \xi^{-2/3}$
all $p \le \xi$	$\log I(N) - \log I(N_\kappa)$	0	$-e^{-31/3} \cdot \xi^{-2/3}$
Total		$-S_2$	$-S_2 - \frac{1.0156 \cdot M_\xi}{\lambda}$

3.3. *Proof of Theorem 3.1*

The proof of Theorem 3.1 writes $\Delta(N) - c(q)$ as the difference of $\log I(N) - \sum_{p\le q} \log \frac{p}{p-1}$, which is $-S_2$ up to a smaller term (Lemma 3.5) with $S_2 = M_\xi \cdot \left(1 + O\left(\frac{1}{\lambda}\right)\right)$ (Lemma 3.3), and of $L(\log N) - L(\theta(\xi)) = M_\xi \cdot \left(1 + O\left(\frac{1}{\lambda}\right)\right)$ (Lemma 3.4); the main term of $\Delta(N) - c(q)$ is therefore $-2M_\xi$, and the error terms are collected and expressed in terms of q.

Proof of Theorem 3.1. Put $\xi = \xi(N)$, $\lambda = \log \xi$ and $u = \log q \ge 31$, so that $\lambda \ge u \ge 31$. Since $q \le \xi$ and the primes $p \le \xi$ are the primes $p \le q$, we have $\theta(\xi) = \theta(q)$ and

$$\Delta(N) - c(q) = \left(\log I(N) - \sum_{p\le q} \log \frac{p}{p-1}\right) - \left(L(\log N) - L(\theta(\xi))\right).$$

By Lemmas 3.3, 3.4 and 3.5,

$$-M_\xi \cdot \left(2 - \frac{1.973094 - 1.0156}{\lambda}\right) \le \Delta(N) - c(q) \le -M_\xi \cdot \left(2 - \frac{0.369659 + 2.43616}{\lambda}\right),$$

that is, $-M_\xi \cdot \left(2 - \frac{0.957494}{\lambda}\right) \le \Delta(N) - c(q) \le -M_\xi \cdot \left(2 - \frac{2.805819}{\lambda}\right)$. Put $M_q = \frac{\sqrt{2}}{\sqrt{q}\cdot u}$. The prime following q exceeds ξ, so by Proposition 2.10 (c), $\xi < q \cdot (1 + u^{-3})$ and $\lambda < u + u^{-3}$. Hence $M_\xi \le M_q$, $\frac{1}{\lambda} \ge \frac{1-u^{-4}}{u}$, and

$$\frac{M_\xi}{M_q} \ge \frac{1}{(1+u^{-3})^{1/2}(1+u^{-4})} \ge 1 - \frac{u^{-2}/2 + u^{-3}}{u} \ge 1 - \frac{0.000554}{u}.$$

Therefore

$$\Delta(N) - c(q) \ge -M_q \cdot \left(2 - \frac{0.957494 \cdot (1 - 31^{-4})}{u}\right) \ge -M_q \cdot \left(2 - \frac{0.9574}{u}\right)$$

and

$$\Delta(N) - c(q) \le -M_q \cdot \left(1 - \frac{0.000554}{u}\right)\left(2 - \frac{2.805819}{u}\right) \le -M_q \cdot \left(2 - \frac{2.8070}{u}\right).$$

Since $2M_q = \frac{2\sqrt{2}}{\sqrt{q}\log q}$, $0.9574 \cdot \sqrt{2} > 1.353$ and $2.8070 \cdot \sqrt{2} < 3.970$, the theorem follows. □

In the notation of the proof, Theorem 3.1 is the pair of inequalities

$$c(q) - \frac{2\sqrt{2}}{\sqrt{q}\log q}\left(1 - \frac{0.4787}{\log q}\right) \le \Delta(N) \le c(q) - \frac{2\sqrt{2}}{\sqrt{q}\log q}\left(1 - \frac{1.4035}{\log q}\right). \tag{6}$$

Remark 3.6. The decomposition of $\Delta(N) - c(q)$ used in the proof of Theorem 3.1 follows Nicolas' proof of [20, Proposition 5.1], in which $\log(\sigma(N)/N)$ is written as $U = U_1 + U_2 + U_3 - U_4 - U_5$ [20, (5.4)], with $-U_5 = \sum_{p\le\xi} \log \frac{p}{p-1}$ the Mertens product; a comparison of $\sigma(N)/N$ with this product for colossally abundant numbers N also underlies Robin's proof of Proposition 2.6 [5, Section 4]. In [20, Proposition 5.1] the Mertens product is evaluated under RH by [21,

(2.16)], that is, by Proposition 2.8. In Theorem 3.1 it is kept in the form $c(q)+\gamma+\log\log\theta(q)$, so that no hypothesis on the zeros of the Riemann zeta function is used, and the remaining terms are bounded from above and from below with explicit constants for every colossally abundant number with $\log q \ge 31$. The estimate of the Mertens product that does not assume RH, Proposition 2.10 (f), is an upper bound with the factor $(1-\varepsilon(\log x))^{-1}$, where $\varepsilon(\log x) > \frac{1}{20\log^3 x}$, whereas the terms compared in Theorem 3.1 are of order $\frac{1}{\sqrt{q}\log q}$; in this paper it is used only in Lemmas 5.1 and 6.3. Under RH, [20] contains a comparison of the same form as Theorem 3.1, with $\xi=\xi(N)$ in place of q: for $\xi \ge 10^9+7$, the estimates [20, (5.5)–(5.8), (4.68), (4.69)] give, for the number $W = U - \sum_{p\le\xi}\log\frac{p}{p-1} - \log\log\log N + \log\log\theta(\xi)$,

$$\frac{3.0187}{\sqrt{\xi}\log^2\xi} - \frac{0.448}{\xi^{2/3}} \le W + \frac{2\sqrt{2}}{\sqrt{\xi}\log\xi} \le \frac{3.844}{\sqrt{\xi}\log^2\xi} + \frac{0.0015}{\xi^{2/3}},$$

where U is the sum of [20, (5.4)], which satisfies $\frac{\xi}{\xi+1}\cdot U \le \log\frac{\sigma(N)}{N} \le U$ [20, (5.3)], so that W is $\Delta(N)-c(q)$ with $\log\frac{\sigma(N)}{N}$ replaced by U. Here $3.0187 = 2.5627 + 0.456$ and $3.844 = 3.353 + 0.491$ are the sums of the constants of [20, (5.8), (4.69)] and of [20, (5.7), (4.68)], and $0.448 = 0.251 + 0.087 + 0.11$ is the sum of the constants of [20, (5.5), (5.6), (4.69)]; with [20, (1.11)] they give the constants $0.9567 = 3.0187 - 2.062$ and $3.789 = 3.844 - 0.055$ of [20, (5.11), (5.12)]. The interval from 3.0187 to 3.844, of length 0.8253, is contained in the interval from 1.353 to 3.970 of Theorem 3.1, of length 2.617. Thus, under RH and for $\xi \ge 10^9+7$, the estimates of [20] are more precise than Theorem 3.1; Theorem 3.1 holds without RH, and its constants include, at the threshold $\log q \ge 31$, the terms of order $\xi^{-2/3}$ (Remark 3.7).

Remark 3.7. The two bounds of Theorem 3.1 differ by $\frac{3.970-1.353}{\sqrt{q}\log^2 q}$, and those of Theorem 1.1 by $\frac{3.973-1.317}{\sqrt{q}\log^2 q}$, the additional $\frac{0.036+0.0022}{\sqrt{q}\log^2 q}$ coming from the bounds for $E(q)$ of Lemma 4.5. Consequently, Corollaries 5.2 and 5.3 below do not decide whether Robin's inequality holds at a colossally abundant number N when $c(q)\sqrt{q}\log q$ lies between $2\sqrt{2}\left(1-\frac{1.4035}{\log q}\right)$ and $2\sqrt{2}\left(1-\frac{0.4787}{\log q}\right)$. In the proof, the two sides differ by $(2.805819 - 0.957494)\cdot\frac{M_\xi}{\lambda} = 1.848325\cdot\frac{M_\xi}{\lambda}$; of this, 1.0156 comes from Lemma 3.5 (the primes $p\le\xi_2$, which have exponent at least 2), $2.43616 - 1.920374 = 0.515786$ from Lemma 3.3, and $0.369659 - 0.05272 = 0.316939$ from Lemma 3.4. Two of these contributions are bounds, evaluated at the threshold $\lambda = 31$, of quantities that tend to 0 as $\lambda\to\infty$. The term $\frac{1.0156\cdot M_\xi}{\lambda}$ of Lemma 3.5 bounds a quantity which is at most $\left(\frac{0.26201\cdot\lambda^2 e^{-\lambda/6}}{\sqrt{2}} + 10^{-4}\right)\cdot\frac{M_\xi}{\lambda}$ (Table 3), and in the proof of Lemma 3.4 the term $\frac{0.24717}{\lambda}$, which is part of the contribution 0.316939, bounds $1.397905\cdot e^{-\lambda/6}$, which comes from the primes with exponent at least 3 in $\log N$; the functions $\lambda^2 e^{-\lambda/6}$ and $\lambda e^{-\lambda/6}$ are decreasing for $\lambda\ge 12$ and tend to 0. The rest of the difference comes from the relative error $\frac{0.022}{\lambda}$ of $\theta(t)$ for $t\ge\xi_2$ (Lemma 2.11 (b)) and from the two bounds for ξ_2 of Proposition 3.2 (d), whose constants are $\frac{\log 2}{2} = 0.3465\ldots$ and 0.323, the latter being stated for $\xi\ge 10^9$.

Remark 3.8. The hypothesis $\lambda = \log\xi \ge 31$ is used in the proof of Theorem 3.1 in the following places: in Lemma 2.11 (b), whose constant 0.022 is obtained from the bound of Büthe at $\lambda = 31$; in (4), through the upper bound for ξ_2 of Proposition 3.2 (d), which requires $\xi\ge 10^9$; in Proposition 3.2 (e), which requires $\xi > 10^9+7$; in the proofs of Lemmas 3.4 and 3.5, through $\xi^{1/3} > 1418$; in the proof of Theorem 3.1, through Proposition 2.10 (c), which requires $q\ge 89{,}693$; and in every decimal constant of Lemmas 3.3, 3.4 and 3.5 and of Theorem 3.1, each of which is the value at $\lambda = 31$ of a function of λ that is monotone in the direction used in the proofs (Section 7). Among these, the conditions that do not come from a numerical evaluation at $\lambda = 31$ require $\xi > 10^9+7$, that is, $\lambda > \log(10^9+7) = 20.7232\ldots$. For a threshold λ_0 with $\log(10^9+7) < \lambda_0 < 31$, the constant of Lemma 2.11 (b) and the constants of Lemmas 3.3, 3.4 and 3.5 would have to be recomputed at λ_0; this paper states Theorems 3.1 and 1.1 only for $\lambda\ge 31$, the range that contains the prime q of every colossally abundant counterexample (Corollary 5.2).

4. Nicolas' Second Criterion

In this section we prove that the primorials are the extremal cases of the inequality (2) of Proposition 2.9, and we prove Theorems 1.1 and 1.2. For real $x > 1$ put

$$G(x) = e^\gamma\cdot\log x + \frac{e^\gamma\cdot(2+\tau)}{\sqrt{x}},$$

so that (2) is the inequality $\frac{n}{\varphi(n)} < G(\log n)$.

Lemma 4.1. *The function G is increasing on $[\log 3, \infty)$.*

Proof. We have $G'(x) = e^{\gamma} \cdot x^{-3/2} \cdot \left(\sqrt{x} - \frac{2+\tau}{2}\right)$, and $\left(\frac{2+\tau}{2}\right)^2 < 1.0468 < \log 3 \le x$ for $x \ge \log 3$, so $G'(x) > 0$. □

Lemma 4.2. *Let $k \ge 2$, and let n be an integer with $N_k \le n < N_{k+1}$. Then*

$$\frac{n}{\varphi(n)} \le \frac{N_k}{\varphi(N_k)} \qquad \text{and} \qquad G(\log n) \ge G(\log N_k),$$

so that

$$G(\log n) - \frac{n}{\varphi(n)} \ge G(\log N_k) - \frac{N_k}{\varphi(N_k)}.$$

In particular, if (2) holds for N_k, then it holds for every integer n with $N_k \le n < N_{k+1}$, and if (2) holds for N_k and for N_{k+1}, then it holds for every integer n with $N_k \le n \le N_{k+1}$.

Proof. Let $q_1 < q_2 < \cdots < q_r$ be the distinct primes dividing n. Since $q_i \ge p_i$ for $1 \le i \le r$, we have $n \ge q_1 \cdots q_r \ge p_1 \cdots p_r = N_r$, and $n < N_{k+1}$ gives $r \le k$. Since $t \mapsto \frac{t}{t-1}$ is decreasing for $t > 1$ and $\frac{p}{p-1} > 1$ for every prime p,

$$\frac{n}{\varphi(n)} = \prod_{i=1}^{r} \frac{q_i}{q_i - 1} \le \prod_{i=1}^{r} \frac{p_i}{p_i - 1} \le \prod_{i=1}^{k} \frac{p_i}{p_i - 1} = \frac{N_k}{\varphi(N_k)}.$$

Since $k \ge 2$, we have $\log n \ge \log N_k \ge \log 6 > \log 3$, and Lemma 4.1 gives $G(\log n) \ge G(\log N_k)$. The last claims follow, the case $n = N_{k+1}$ being the hypothesis for N_{k+1}. □

Corollary 4.3. *Let $K \ge 2$. Then (2) holds for every integer $n \ge N_K$ if and only if it holds for N_k for every $k \ge K$. Consequently, the Riemann Hypothesis is true if and only if (2) holds for N_k for every $k \ge 120569$.*

Proof. Every integer $n \ge N_K$ satisfies $N_k \le n < N_{k+1}$ for exactly one $k \ge K$, so the first claim follows from Lemma 4.2. The second claim follows from the first with $K = 120569$ and Proposition 2.9. □

Lemma 4.4. *Let $q = p_k$ be a prime with $q \ge 3$. Then (2) holds for N_k if and only if $c(q) < E(q)$, where*

$$E(q) = \log\left(1 + \frac{2+\tau}{\sqrt{\theta(q)} \cdot \log \theta(q)}\right).$$

Proof. We have $\log N_k = \theta(q) \ge \log 6 > 1$ and $\frac{N_k}{\varphi(N_k)} = \prod_{p \le q} \frac{p}{p-1}$, so (2) for $n = N_k$ is the inequality

$$\prod_{p \le q} \frac{p}{p-1} < e^{\gamma} \cdot \log \theta(q) + \frac{e^{\gamma} \cdot (2+\tau)}{\sqrt{\theta(q)}}.$$

Dividing by $e^{\gamma} \cdot \log \theta(q) > 0$, it becomes

$$\frac{\prod_{p \le q} \frac{p}{p-1}}{e^{\gamma} \cdot \log \theta(q)} < 1 + \frac{2+\tau}{\sqrt{\theta(q)} \cdot \log \theta(q)},$$

and, since the logarithm is increasing and the logarithm of the left-hand side is $\sum_{p \le q} \log \frac{p}{p-1} - \gamma - \log\log \theta(q) = c(q)$, this is the inequality $c(q) < E(q)$. □

Lemma 4.5. *Let q be a prime with $u = \log q \ge 31$. Then*

$$\frac{2+\tau}{\sqrt{q} \cdot u} - \frac{0.036}{\sqrt{q} \cdot u^2} \le E(q) \le \frac{2+\tau}{\sqrt{q} \cdot u} + \frac{0.0022}{\sqrt{q} \cdot u^2}.$$

Proof. We first prove the lower bound. By Proposition 2.10 (a) if $q \le 10^{19}$, and by Proposition 2.10 (b) if $q > 10^{19}$, we have $\theta(q) \le q \cdot (1 + u^{-3})$. Hence $\sqrt{\theta(q)} \le \sqrt{q} \cdot \left(1 + \frac{u^{-3}}{2}\right)$ and $\log \theta(q) \le u + \log(1 + u^{-3}) \le u \cdot (1 + u^{-4})$. Put $y_0 = \frac{2+\tau}{\sqrt{q} \cdot u}$ and $y = \frac{2+\tau}{\sqrt{\theta(q)} \cdot \log \theta(q)}$. Then

$$y \ge \frac{y_0}{\left(1 + \frac{u^{-3}}{2}\right)(1 + u^{-4})} \ge y_0 \cdot \left(1 - \frac{u^{-3}}{2} - u^{-4}\right) =: y_1,$$

and $0 < y_1 \le y_0$. Since $\log(1 + t) \ge t - \frac{t^2}{2}$ for $t \ge 0$,

$$E(q) = \log(1 + y) \ge \log(1 + y_1) \ge y_1 - \frac{y_1^2}{2} \ge y_0 - y_0 \cdot \left(\frac{u^{-3}}{2} + u^{-4}\right) - \frac{y_0^2}{2}.$$

Here $y_0 \cdot \left(\frac{u^{-3}}{2} + u^{-4}\right) = \frac{2+\tau}{\sqrt{q} \cdot u^2} \cdot \left(\frac{1}{2u} + \frac{1}{u^2}\right)$ and $\frac{y_0^2}{2} = \frac{(2+\tau)^2}{2\sqrt{q}} \cdot \frac{1}{\sqrt{q} \cdot u^2}$ with $\sqrt{q} \ge e^{31/2}$. By Proposition 2.8, $2 + \tau < 2.0462$, and

$$2.0462 \cdot \left(\frac{1}{62} + \frac{1}{31^2}\right) + \frac{2.0462^2}{2} \cdot e^{-31/2} < 0.036,$$

which proves the lower bound.

For the upper bound, put $a = u^{-3}$ and $b = \frac{u^{-4}}{1-u^{-3}}$. Since $q \ge e^{31} > 89{,}967{,}803$, Proposition 2.10 (b) gives $\theta(q) > q \cdot (1-a)$. Hence $\sqrt{\theta(q)} > \sqrt{q} \cdot \sqrt{1-a} \ge \sqrt{q} \cdot (1-a)$, and, since $\log(1-a) \ge -\frac{a}{1-a}$, $\log \theta(q) > u - \frac{u^{-3}}{1-u^{-3}} = u \cdot (1-b)$. Since $\log(1 + t) \le t$ for $t \ge 0$ and $(1 - a)(1 - b) \ge 1 - a - b$,

$$E(q) \le y < \frac{y_0}{(1-a)(1-b)} \le \frac{y_0}{1-a-b} = y_0 \cdot \left(1 + \frac{a+b}{1-a-b}\right).$$

Here $\frac{a+b}{1-a-b} = u^{-3} \cdot \frac{1 + \frac{u^{-1}}{1-u^{-3}}}{1-a-b}$, and the last factor is decreasing in u and is less than 1.0323 at $u = 31$. Hence $E(q) < y_0 + \frac{(2+\tau) \cdot 1.0323}{u^2} \cdot \frac{1}{\sqrt{q} \cdot u^2}$, and $2.0462 \cdot \frac{1.0323}{31^2} < 0.0022$. □

Proof of Theorem 1.1. By Theorem 3.1,

$$\frac{1.353}{\sqrt{q} \cdot \log^2 q} \le \Delta(N) - c(q) + \frac{2\sqrt{2}}{\sqrt{q} \cdot \log q} \le \frac{3.970}{\sqrt{q} \cdot \log^2 q}.$$

We have

$$\Delta(N) - c(q) + E(q) + \frac{2\sqrt{2} - 2 - \tau}{\sqrt{q} \log q} = \Delta(N) - c(q) + \frac{2\sqrt{2}}{\sqrt{q} \log q} + \left(E(q) - \frac{2+\tau}{\sqrt{q} \log q}\right),$$

and by Lemma 4.5 the last term lies between $-\frac{0.036}{\sqrt{q} \log^2 q}$ and $\frac{0.0022}{\sqrt{q} \log^2 q}$. Since $1.353 - 0.036 = 1.317$ and $3.970 + 0.0022 \le 3.973$, the theorem follows. □

Proof of Theorem 1.2. Put $u = \log q$ and $V = \Delta(N) - c(q) + E(q)$. By Theorem 1.1,

$$-\frac{2\sqrt{2} - 2 - \tau}{\sqrt{q} \cdot u} + \frac{1.317}{\sqrt{q} \cdot u^2} \le V \le -\frac{2\sqrt{2} - 2 - \tau}{\sqrt{q} \cdot u} + \frac{3.973}{\sqrt{q} \cdot u^2},$$

and $\Delta(N) = V - (E(q) - c(q))$. Since $q \ge 3$, Lemma 4.4 shows that (2) holds for N_k if and only if $E(q) - c(q) > 0$, and $q \ge 3$ gives $k \ge 2$.

(a) If $\Delta(N)$ is less than the lower bound for V, then $E(q) - c(q) = V - \Delta(N) > 0$. Hence (2) holds for N_k, and by Lemma 4.2 for every integer n with $N_k \le n < N_{k+1}$.

(b) If (2) holds for N_k, then $E(q) - c(q) > 0$, so $\Delta(N) < V$, and $\Delta(N)$ is less than the upper bound for V. Since $u \ge 31$, this upper bound is at most $-\frac{1}{\sqrt{q} \cdot u}\left(2\sqrt{2} - 2 - \tau - \frac{3.973}{31}\right)$, and $2\sqrt{2} - 2 - \tau - \frac{3.973}{31} > 0.654$.

(c) By Proposition 2.10 (b) at $x = 10^8$, $\pi(10^8) \ge \frac{\theta(10^8)}{\log 10^8} > \frac{10^8 \cdot (1 - \log^{-3} 10^8)}{\log 10^8} > 5 \cdot 10^6 > 120569$, so $p_{120569} < 10^8 < e^{31} \le q = p_k$ and $k > 120569$. If RH is true, Proposition 2.9 shows that (2) holds for N_k, and (b) applies.

(d) If $\Delta(N) \ge 0$, then $c(q) - E(q) = \Delta(N) - V \ge -V \ge \frac{1}{\sqrt{q} \cdot u}\left(2\sqrt{2} - 2 - \tau - \frac{3.973}{u}\right) > \frac{0.654}{\sqrt{q} \cdot u}$, as in (b). Hence $c(q) > E(q)$, and (2) does not hold for N_k by Lemma 4.4. □

The number $2\sqrt{2}-2-\tau = 0.7822\ldots$ in Theorems 1.1 and 1.2 is the constant discussed after Corollary 5.4. Theorem 1.2 (c) gives, under RH, the bound $\Delta(N) < -\frac{0.654}{\sqrt{q}\log q}$ of Corollary 5.4 from Proposition 2.9 instead of Proposition 2.8, and Theorem 1.2 (d) is the analogue of Corollary 5.2 for $E(q)$; since $E(q) \ge \frac{2+\tau}{\sqrt{q}\log q} - \frac{0.036}{\sqrt{q}\log^2 q}$, it gives $c(q) > \frac{2.70}{\sqrt{q}\log q} - \frac{0.036}{\sqrt{q}\log^2 q}$.

Remark 4.6. Let N, $q = p_k$ and $u = \log q \ge 31$ be as in Theorem 1.1. Since $\Delta(N) = V - (E(q) - c(q))$ with $V = \Delta(N) - c(q) + E(q)$, the upper bound of Theorem 1.1 gives $\Delta(N) < 0$ whenever

$$c(q) < E(q) + \frac{2\sqrt{2}-2-\tau}{\sqrt{q}\cdot u} - \frac{3.973}{\sqrt{q}\cdot u^2},$$

in particular whenever $c(q) \le E(q) + \frac{0.654}{\sqrt{q}\cdot u}$. Hence Robin's inequality holds for N also for the values $E(q) \le c(q) \le E(q) + \frac{0.654}{\sqrt{q}\cdot u}$, for which (2) does not hold for N_k (Lemma 4.4): the inequalities $\Delta(N) < 0$ and $c(q) < E(q)$ differ by the term $\frac{2\sqrt{2}-2-\tau}{\sqrt{q}\cdot u}$ of Theorem 1.1, which does not depend on the constants 1.317 and 3.973. Theorems 1.3 and 1.4 use only Theorem 1.2 (b), that is, only the upper bound of Theorem 1.1; the range of Theorem 1.3 (ii) is $\log n \le X(1-\log^{-3} X) - \log X$ and depends on X only, and not on the constants 1.317 and 3.973. The terms of the proof of Theorem 3.1 that produce the difference $3.970-1.353$ are listed in Remark 3.7. The constant 0.654 of Theorem 1.2 (b) and (d) depends only on the upper bound of Theorem 1.1, that is, on the upper bound 3.970 of Theorem 3.1 and on the upper bound 0.0022 of Lemma 4.5: it is a lower bound for the value at $u = 31$ of $2\sqrt{2}-2-\tau-\frac{3.973}{u}$, which is $0.65407\ldots$, and for $u \ge 31$ this function increases to $2\sqrt{2}-2-\tau = 0.78223\ldots$ as $u \to \infty$. Thus Theorem 1.2 (b) and (d) hold with 0.654 replaced by $2\sqrt{2}-2-\tau-\frac{3.973}{\log q}$.

5. Counterexamples to Robin's Inequality

In this section we derive from Theorem 3.1 consequences for counterexamples to (1), and an explicit form under RH of Ramanujan's bound at colossally abundant numbers.

Lemma 5.1. *Let n be a counterexample with largest prime factor p_k. Then $\log p_k > 31.0319$ and $\log\log n < 1.000002 \cdot \log p_k$.*

Proof. Put $L = \log\log n$ and $P = \log p_k$. By Proposition 2.3, $n > 10^{10^{13.11485}}$, so $L > 13.11485 \cdot \log 10 + \log\log 10 > 31.032$. By Proposition 2.2, $e^\gamma L \le I(n) < \prod_{p\le p_k} \frac{p}{p-1}$. If $p_k < 46{,}909{,}038$, then by Proposition 2.10 (f) at $x = 46{,}909{,}038$, this product is less than $e^\gamma \cdot 17.6638 \cdot 1.000011 < e^\gamma \cdot 31.032$, a contradiction. Hence $p_k \ge 46{,}909{,}038$, and Proposition 2.10 (f) gives $L < P \cdot (1-\varepsilon(P))^{-1}$. Since ε is decreasing and $P \ge \log(46{,}909{,}038)$, $(1-\varepsilon(P))^{-1} < 1.000011$, so $P > \frac{31.032}{1.000011} > 31.0316$; then $(1-\varepsilon(P))^{-1} \le (1-\varepsilon(31.0316))^{-1} < 1.000002$, so $L < 1.000002 \cdot P$ and $P > \frac{31.032}{1.000002} > 31.0319$. □

For the least counterexample, the author proved $\log p_k > 31.018$ [17, Theorem 4.2]; Lemma 5.1 applies to every counterexample.

Corollary 5.2. *Let N be a colossally abundant counterexample, and q the largest prime with $q \le \xi(N)$. Then $\log q > 31.0319$ and*

$$c(q) > \frac{2\sqrt{2}}{\sqrt{q}\log q}\left(1 - \frac{1.4035}{\log q}\right) > \frac{2.70}{\sqrt{q}\log q}, \qquad \textit{that is,} \qquad \frac{q\#}{\varphi(q\#)} > e^\gamma \cdot \log\theta(q) \cdot \exp\left(\frac{2.70}{\sqrt{q}\log q}\right).$$

Proof. The largest prime factor p_k of N is at most $\xi(N)$ by Proposition 3.2 (a), so $p_k \le q$ and $\log q > 31.0319$ by Lemma 5.1. Since Robin's inequality fails for N, $\Delta(N) \ge 0$, and the upper bound in (6) gives the first inequality. The function $u \mapsto 2\sqrt{2}\left(1 - \frac{1.4035}{u}\right)$ is increasing, and its value at $u = 31.0319$ exceeds 2.70. □

Corollary 5.3. *Let q be a prime with $\log q \geq 31$ and*

$$c(q) > \frac{2\sqrt{2}}{\sqrt{q}\log q}\left(1 - \frac{0.4787}{\log q}\right).$$

Then every colossally abundant number N such that q is the largest prime with $q \leq \xi(N)$ is a counterexample. In particular, the colossally abundant number $N_{F(q,1)}$ of Proposition 3.2 (a), for which $\xi = q$, is a counterexample.

Proof. The lower bound in (6) gives $\Delta(N) > 0$, that is, $I(N) > e^{\gamma}\log\log N$, and $N > 5040$ because q divides N_κ and $\log N \geq \theta(q) - \log q$ by Proposition 3.2 (b). For $\kappa = F(q,1)$, the number N_κ is colossally abundant with largest parameter κ by Proposition 3.2 (a), and $\xi(N_\kappa) = q$ because $F(\cdot,1)$ is decreasing. □

By Corollaries 5.2 and 5.3, for colossally abundant numbers N with $\log q \geq 31$, Robin's inequality fails at N when $c(q)\sqrt{q}\log q > 2\sqrt{2}\left(1 - \frac{0.4787}{\log q}\right)$ and holds at N when $c(q)\sqrt{q}\log q < 2\sqrt{2}\left(1 - \frac{1.4035}{\log q}\right)$.

The next corollary is an explicit inequality, for colossally abundant numbers with $\log q \geq 31$, of the type of Ramanujan's upper bound for $\sigma(n)/n$ under RH, of which [20, Theorem 1.1] gives an effective form for every $n \geq 2$.

Corollary 5.4. *Assume the Riemann Hypothesis. Let N be a colossally abundant number, and q the largest prime with $q \leq \xi(N)$. If $u = \log q \geq 31$, then*

$$-\frac{0.709}{\sqrt{q}\cdot u^2} \leq \Delta(N) + \frac{2\sqrt{2}-2}{\sqrt{q}\cdot u} - \frac{S_1(q)}{u} \leq \frac{3.915}{\sqrt{q}\cdot u^2},$$

and

$$\Delta(N) < -\frac{0.654}{\sqrt{q}\log q}, \qquad \textit{that is,} \qquad \frac{\sigma(N)}{N} < e^{\gamma}\cdot\log\log N\cdot\exp\left(-\frac{0.654}{\sqrt{q}\log q}\right).$$

Proof. Since $q > 10^9$, Proposition 2.8 at $x = q$ gives

$$\frac{2}{\sqrt{q}\cdot u} + \frac{S_1(q)}{u} - \frac{2.062}{\sqrt{q}\cdot u^2} \leq c(q) \leq \frac{2}{\sqrt{q}\cdot u} + \frac{S_1(q)}{u} - \frac{0.055}{\sqrt{q}\cdot u^2}.$$

Theorem 3.1 gives $c(q) - \frac{2\sqrt{2}}{\sqrt{q}\cdot u} + \frac{1.353}{\sqrt{q}\cdot u^2} \leq \Delta(N) \leq c(q) - \frac{2\sqrt{2}}{\sqrt{q}\cdot u} + \frac{3.970}{\sqrt{q}\cdot u^2}$, and the first claim follows from $1.353 - 2.062 = -0.709$ and $3.970 - 0.055 = 3.915$. For the second claim, (6) and $c(q) < \frac{2+\tau}{\sqrt{q}\cdot u}$ (Proposition 2.8) give

$$\Delta(N)\sqrt{q}\log q < 2 + \tau - 2\sqrt{2} + \frac{2\sqrt{2}\cdot 1.4035}{\log q} \leq 2 + \tau - 2\sqrt{2} + \frac{2\sqrt{2}\cdot 1.4035}{31} < -0.654.$$

□

The constants $-0.709 = 1.353 - 2.062$ and $3.915 = 3.970 - 0.055$ of the first claim of Corollary 5.4 correspond to the constants $0.9567 = 3.0187 - 2.062$ and $3.789 = 3.844 - 0.055$ of [20, (5.11), (5.12)], which Nicolas obtains under RH for $\xi(N) \geq 10^9 + 7$ (Remark 3.6). The number $2\sqrt{2} - 2 - \tau = 2(\sqrt{2} - 1) - \tau = 0.7822\ldots$ appearing in the proof is the constant obtained from Ramanujan's bound [19] together with $|S_1(x)| \leq \frac{\tau}{\sqrt{x}}$, as in [20, (1.8)]; here $\sqrt{q}\log q$ plays the role of $\sqrt{\log N}\cdot\log\log N$, since $\log N = q\cdot(1 + o(1))$ by the proof of Lemma 3.4 and Proposition 2.10 (c).

The constant 2.70 of Corollary 5.2 is the value at $\log q = 31.0319$ of $2\sqrt{2}\left(1 - \frac{1.4035}{\log q}\right)$, which tends to $2\sqrt{2} = 2.8284\ldots$ as $\log q \to \infty$. Under RH the constant $2 + \tau$ of Proposition 2.8 cannot be replaced by a smaller one: by [22, Lemma 5.33, Theorem 5.34 (1) and its proof], under RH,

$$\limsup_{k\to\infty}\left(\frac{N_k}{\varphi(N_k)} - e^{\gamma}\log\log N_k\right)\sqrt{\log N_k} = e^{\gamma}(2 + \tau),$$

and since $\frac{N_k}{\varphi(N_k)} = e^{\gamma}\log\theta(p_k)\cdot e^{c(p_k)}$, $\log N_k = \theta(p_k) = p_k\cdot(1 + o(1))$ and $c(p_k) \to 0$ under RH (Proposition 2.8), this gives $\limsup_{q\to\infty} c(q)\sqrt{q}\log q = 2 + \tau$, the limit superior being taken over primes q. Thus, if RH is true, then $c(q) < \frac{2.0462}{\sqrt{q}\log q}$ for every prime $q \geq 10^9$ (Proposition 2.8), whereas the primes q of the colossally abundant counterexamples satisfy $c(q) > \frac{2.70}{\sqrt{q}\log q}$.

6. Superabundant Counterexamples

The results of Section 5 concern counterexamples that are colossally abundant, whereas Proposition 2.5 states that the least counterexample is superabundant. In this section a superabundant counterexample n is treated directly, in terms of its own largest prime factor p_k, without assuming that n is colossally abundant. Theorem 6.4 gives an explicit lower bound for $c(p_k)$; in particular, Nicolas' inequality $c(p_k) > 0$ holds at the largest prime factor of every superabundant counterexample, hence at the largest prime factor of the least counterexample. The constant 0.9991 of Theorem 6.4 is smaller than the constant 2.70 of Corollary 5.2, which is available only when the counterexample is colossally abundant, and it is smaller than $2\sqrt{2}\left(1-\frac{0.4787}{\log p_k}\right)$, so Theorem 6.4 and Corollary 5.3 do not imply that the colossally abundant number $N_{F(p_k,1)}$ is a counterexample. Corollary 6.6 relates the least counterexample n to a colossally abundant counterexample $N \geq n$, to which Corollary 5.2 applies with the prime q attached to N. All proofs of this section are given in full, and the numerical constants of Lemmas 6.2 and 6.3 and of Theorem 6.4, in particular 0.9991, are verified in Section 7 (Table 4).

Lemma 6.1. *Let $n = \prod_{i=1}^{k} p_i^{a_i}$ be a superabundant number with largest prime factor p_k. Then $a_i \geq \left\lfloor \frac{\log p_k}{\log p_i} \right\rfloor$ for every $1 \leq i \leq k$, and consequently $\log n \geq \psi(p_k)$.*

Proof. The first claim is due to Nazardonyavi and Yakubovich [16]; we give the proof for completeness. For a prime p and an integer $b \geq 0$ put $g(p,b) = \sum_{j=0}^{b} p^{-j}$. For $i = k$ the claim is $a_k \geq 1$. Let $i < k$, write $q = p_i$ and $a = a_i \geq 1$, and suppose that $a \leq \left\lfloor \frac{\log p_k}{\log q} \right\rfloor - 1$. Then $q^{a+1} \leq p_k$, and since $q^{a+1} \neq p_k$, we have $q^{a+1} < p_k$. Put $m = \frac{n \cdot q^{a+1}}{p_k}$, which is an integer with $m < n$, and write $b = a_k \geq 1$. Since I is multiplicative,

$$\frac{I(n)}{I(m)} = \frac{g(q,a)}{g(q,2a+1)} \cdot \frac{g(p_k,b)}{g(p_k,b-1)} = \frac{1 + p_k^{-b}/g(p_k,b-1)}{1+q^{-(a+1)}} \leq \frac{1+\frac{1}{p_k}}{1+\frac{1}{q^{a+1}}} < 1,$$

where we used $g(q,2a+1) = g(q,a)\cdot(1+q^{-(a+1)})$, $g(p_k,b-1) \geq 1$ and $q^{a+1} < p_k$. Thus $I(m) > I(n)$ with $m < n$, which contradicts the superabundance of n. Finally, by Proposition 2.4 the primes dividing n are exactly the primes $q \leq p_k$, so

$$\log n = \sum_{q \leq p_k} a_q \log q \geq \sum_{q \leq p_k} \left\lfloor \frac{\log p_k}{\log q} \right\rfloor \log q = \psi(p_k),$$

where a_q denotes the exponent of q in n. □

Lemma 6.2. *Let n be a counterexample, and n' the product of the distinct primes dividing n. Then $n' > 15$ and*

$$\log \frac{n'}{\varphi(n')} - \gamma - \log\log\log n' > \log\log\log n - \log\log\log n' \geq 0.$$

Proof. By Proposition 2.2, and since Robin's inequality fails for n, $\frac{n'}{\varphi(n')} > I(n) \geq e^{\gamma} \cdot \log\log n$. By Proposition 2.3, $\log\log n > 31.032$, so $\frac{n'}{\varphi(n')} > 55$. Since $\frac{m}{\varphi(m)} \leq 3$ for every $m \leq 15$, we get $n' > 15$, hence $\log\log n' > 0$. Taking logarithms gives $\log \frac{n'}{\varphi(n')} > \gamma + \log\log\log n$, and subtracting $\log\log\log n'$ gives the first inequality. The second follows from $n \geq n'$. □

For a superabundant counterexample with largest prime factor p_k, Proposition 2.4 gives $n' = p_k\#$, so Lemma 6.2 gives

$$c(p_k) > \log\log\log n - \log\log\theta(p_k) \geq 0. \tag{7}$$

Lemma 6.3. *Let n be a superabundant counterexample with largest prime factor p_k. Then $\log\log n < \log p_k + \frac{0.0562}{\log^2 p_k}$ and $\log n \cdot \log\log n < 1.0000603 \cdot p_k \cdot \log p_k$.*

Proof. Put $P = \log p_k$ and $L = \log\log n$. By Lemma 5.1, $P > 31.0319$, and by Propositions 2.2 and 2.10 (f), $e^{\gamma} L \leq I(n) < e^{\gamma} P \cdot (1-\varepsilon(P))^{-1}$. Hence

$$L - P < \frac{P \cdot \varepsilon(P)}{1-\varepsilon(P)} = \frac{1}{P^2} \cdot \frac{\frac{1}{20} + \frac{3}{16P}}{1-\varepsilon(P)} \leq \frac{1}{P^2} \cdot \frac{\frac{1}{20} + \frac{3}{16 \cdot 31.0319}}{1-\varepsilon(31.0319)} < \frac{0.0562}{P^2}.$$

Therefore $\log n < p_k \cdot e^{0.0562/P^2}$ and $L < P \cdot \left(1 + \frac{0.0562}{P^3}\right)$, so that

$$\log n \cdot \log\log n < p_k \cdot P \cdot e^{\frac{0.0562}{31.03^2}} \cdot \left(1 + \frac{0.0562}{31.03^3}\right) < 1.0000603 \cdot p_k \cdot P.$$

□

Theorem 6.4. *Let n be a superabundant counterexample with largest prime factor p_k. Then*

$$c(p_k) > \log\log\psi(p_k) - \log\log\theta(p_k) > \frac{0.9991}{\sqrt{p_k} \cdot \log p_k}.$$

Proof. By (7) and Lemma 6.1, $c(p_k) > \log\log\log n - \log\log\theta(p_k) \geq \log\log\psi(p_k) - \log\log\theta(p_k)$. The function $t \mapsto \log\log t$ has the decreasing derivative $\frac{1}{t\log t}$ for $t > 1$, and $\theta(p_k) < \theta(p_k) + \theta(\sqrt{p_k}) \leq \psi(p_k) \leq \log n$. By the mean value theorem and Lemma 6.3,

$$\log\log\psi(p_k) - \log\log\theta(p_k) \geq \frac{\psi(p_k) - \theta(p_k)}{\psi(p_k) \cdot \log\psi(p_k)} \geq \frac{\theta(\sqrt{p_k})}{\log n \cdot \log\log n} > \frac{\theta(\sqrt{p_k})}{1.0000603 \cdot p_k \cdot \log p_k}.$$

By Lemma 5.1, $\sqrt{p_k} > e^{15.515} > 5.47 \cdot 10^6$, so Proposition 2.10 (a) gives $\theta(\sqrt{p_k}) > \left(1 - \frac{1.94}{\sqrt{5.47 \cdot 10^6}}\right)\sqrt{p_k} > 0.99917 \cdot \sqrt{p_k}$ if $\sqrt{p_k} \leq 10^{19}$, and Proposition 2.10 (b) gives $\theta(\sqrt{p_k}) > (1 - 0.000012)\sqrt{p_k}$ otherwise. Since $\frac{0.99917}{1.0000603} > 0.9991$, the second inequality follows. □

Remark 6.5. The second inequality of Theorem 6.4 compares two functions of the prime p_k alone, and its constant is bounded by 1 in the following sense: for every $\eta > 0$ and every sufficiently large prime p,

$$\log\log\psi(p) - \log\log\theta(p) < \frac{1+\eta}{\sqrt{p} \cdot \log p}.$$

Indeed, $\psi(x) - \theta(x) = \theta(\sqrt{x}) + \sum_{j\geq 3} \theta(x^{1/j})$, where the terms with $j > \frac{\log x}{\log 2}$ vanish, so by Lemma 2.11 (a) the sum over $j \geq 3$ is at most $\frac{1.000012 \cdot x^{1/3} \log x}{\log 2}$, and Proposition 2.10 (b) gives $\theta(\sqrt{x}) = \sqrt{x} \cdot (1 + o(1))$ and $\theta(x) = x \cdot (1 + o(1))$ as $x \to \infty$; hence $\psi(x) - \theta(x) = \sqrt{x} \cdot (1 + o(1))$. Since $\frac{1}{t\log t}$ is decreasing, the mean value theorem gives $\log\log\psi(x) - \log\log\theta(x) \leq \frac{\psi(x)-\theta(x)}{\theta(x)\cdot\log\theta(x)} = \frac{1+o(1)}{\sqrt{x}\log x}$.

Corollary 6.6. *Suppose that the Riemann Hypothesis is false, and let n be the least counterexample. Let $N' < N$ be the consecutive colossally abundant numbers with $N' < n \leq N$. Then N is a counterexample,*

$$\log N - \log\xi(N) \leq \log N' < \log n \leq \log N,$$

and the largest prime $q \leq \xi(N)$ satisfies $c(q) > \frac{2.70}{\sqrt{q}\log q}$.

Proof. By Proposition 3.2 (f) applied to $m = n$, the consecutive colossally abundant numbers $N' < n \leq N$ satisfy $\frac{N}{\xi(N)} \leq N'$, which gives the displayed inequalities. The number 55440 is colossally abundant [20, Figure 1], and $55440 < n$ by Proposition 2.3, so $N' \geq 55440 > 5040$. Since $N' < n$, Robin's inequality holds for N', that is, $\frac{I(N')}{\log\log N'} < e^\gamma$, whereas $\frac{I(n)}{\log\log n} \geq e^\gamma$. By Proposition 3.2 (g), $\frac{I(N)}{\log\log N} \geq \frac{I(n)}{\log\log n} \geq e^\gamma$, and $N \geq n > 5040$, so N is a counterexample. The last claim is Corollary 5.2. □

Proof of Theorem 1.3. (i) Let $q = p_k$. Since $e^{31} \leq p_k \leq X$, (2) holds for N_k, and Theorem 1.2 (b) gives $\Delta(N) < -\frac{0.654}{\sqrt{q}\log q}$.

(ii) Suppose that Robin's inequality fails for some integer $m > 5040$ with $\log m \leq L$, where $L = X(1-\log^{-3} X) - \log X$. By Proposition 2.1, RH is false; let $n \leq m$ be the least counterexample, and let N be the colossally abundant number of Corollary 6.6. Then N is a counterexample and $\log N - \log\xi(N) \leq \log n \leq L$. Let q be the largest prime with $q \leq \xi(N)$. By Corollary 5.2, $\log q > 31.0319$, and since $\Delta(N) \geq 0$, part (i) gives $q > X$. Put $\xi = \xi(N)$ and $\lambda = \log\xi$, so that

$\xi \geq q > X$ and $\lambda > 31$. By the proof of Lemma 3.4, $\log N = \theta(\xi) + D$ with $D \geq \theta(\xi_2) > 0$, and by Proposition 2.10 (b), $\theta(\xi) > \xi \cdot (1 - \lambda^{-3})$. Hence

$$\log n \geq \log N - \log \xi > \xi \cdot (1 - \lambda^{-3}) - \lambda.$$

The function $t \mapsto t(1 - \log^{-3} t) - \log t$ is increasing for $t \geq e^{31}$, since its derivative is $1 - \log^{-3} t + 3 \log^{-4} t - \frac{1}{t} > 0$, so $\log n > X(1 - \log^{-3} X) - \log X = L$, a contradiction. □

Proof of Theorem 1.4. (i) By Lemma 4.4, (2) holds for N_k if and only if $c(p_k) < E(p_k)$, and the computation of Part 5 of Section 7 gives $c(p_k) < E(p_k)$ for every k with $120569 \leq k \leq K_0$. By Lemma 4.2, (2) then holds for every integer n with $N_k \leq n < N_{k+1}$ and $120569 \leq k \leq K_0$, that is, for $N_{120569} \leq n < N_{K_0+1}$. Since $p_{K_0+1} > X_0$, we have $\log N_{K_0+1} = \theta(p_{K_0+1}) > \theta(X_0)$, and the computation gives $\theta(X_0) > 9.9999999338 \cdot 10^{13}$.

(ii) and (iii) Since $e^{31} < 2.91 \cdot 10^{13} < X_0$ and $p_{120569} < e^{31}$ (proof of Theorem 1.2 (c)), part (i) gives the hypothesis of Theorem 1.3 with $X = X_0$. Part (ii) is Theorem 1.3 (i), and Theorem 1.3 (ii) gives Robin's inequality for every $n > 5040$ with $\log n \leq X_0(1 - \log^{-3} X_0) - \log X_0$. Theorem 1.3 (ii) is stated for every $n > 5040$: its proof starts from the least counterexample to (1), so no separate treatment of small n is needed, and the threshold N_{120569} of Proposition 2.9 enters only through $p_{120569} < e^{31}$, since the hypothesis of Theorem 1.3 concerns only the primes $p_k \geq e^{31}$. It remains to evaluate the bound. We have $\log X_0 = 32.23619\ldots$, hence $\log^{-3} X_0 = 2.98516\ldots \cdot 10^{-5}$ and $X_0 \cdot \log^{-3} X_0 = 2.98516\ldots \cdot 10^9$, so

$$X_0(1 - \log^{-3} X_0) - \log X_0 = 100{,}000{,}008{,}765{,}439 - 2.98516\ldots \cdot 10^9 - 32.23\ldots = 9.99970235968\ldots \cdot 10^{13}.$$

An integer n satisfies $n \leq 10^{10^a}$ if and only if $\log n \leq 10^a \cdot \log 10$, and $\frac{9.99970235968\ldots \cdot 10^{13}}{\log 10} = 4.34281555\ldots \cdot 10^{13} = 10^{13.637771\ldots}$. Since $10^{13.6377} \cdot \log 10 = 9.99805\ldots \cdot 10^{13} < 9.99970235968 \cdot 10^{13}$, Robin's inequality holds for every n with $5040 < n \leq 10^{10^{13.6377}}$. □

7. Experimental Verification

Every numerical inequality between explicit constants used in this paper was checked by Parts 1–4 of the Python script `verify_second_criterion.py`, run under Python 3.12, written with the help of Claude Opus 5.5 (Anthropic) and run by the author; these parts are self-contained and need only the library `mpmath`. They use the arbitrary-precision library `mpmath` with 60 significant decimal digits, and exact rational arithmetic (Python's `fractions` module) for the inequalities that involve only decimal constants, some of which hold with equality. For each inequality the script prints the computed value of its left-hand side. All 82 checks of Parts 1–4 passed: 14 in Part 1, 50 in Part 2, 7 in Part 3 and 11 in Part 4. We now describe the output.

7.1. Part 1: Counterexamples

Part 1 checks the constants of Lemmas 5.1, 6.2 and 6.3 and of Theorem 6.4. The computed values are listed in Table 4. The starting point is $\log\log\left(10^{10^{13.11485}}\right) = 31.0320905\ldots$, the bound for $\log\log n$ coming from Proposition 2.3. At $x = 46{,}909{,}038$, where Proposition 2.10 (f) begins to apply, $\log x = 17.6637\ldots$ and $(1 - \varepsilon(\log x))^{-1} = 1.0000109986\ldots$, which lies below 1.000011 by $1.4 \cdot 10^{-9}$; this is the smallest margin in Part 1. The two steps of Lemma 5.1 give $\frac{31.032}{1.000011} = 31.03165\ldots$ and $\frac{31.032}{1.000002} = 31.031937\ldots$. In Lemma 6.3, the constant 0.0562 exceeds the computed value $0.0560422\ldots$, and the factor 1.0000603 exceeds the computed value $1.0000602505\ldots$ by $4.9 \cdot 10^{-8}$.

7.2. Part 2: The Proof of Theorem 3.1

Part 2 checks the constants of Lemmas 2.11, 3.3, 3.4 and 3.5 and of the proof of Theorem 3.1, all at the threshold $\lambda = 31$, where $e^{31} = 29048849665247.4\ldots$; each quantity checked at $\lambda = 31$ is monotone in λ in the direction stated in the proofs. Table 5 lists the main values. In Lemma 2.11 (b), the relative error $\frac{0.022}{\lambda}$ for $\theta(t)$ on $t \geq \xi_2$ comes from Büthe's bound, with computed value $0.0219063\ldots$ at $\lambda = 31$, whereas Dusart's bound contributes at most 0.00105. In Lemma 3.4, the contribution $1.397905 \cdot 31 \cdot e^{-31/6} = 0.2471636\ldots$ of the primes with exponent at least 3 reduces the coefficient 0.301 coming from ξ_2 to 0.0538. In Lemma 3.5, the primes with exponent 2 and those with exponent at least 3 contribute $0.262002530\ldots \cdot \xi^{-2/3}$, which is $1.0154801\ldots \cdot \frac{M_\xi}{\lambda}$ at $\lambda = 31$. Twelve of the checks in Part 2 are identities or inequalities between the decimal constants of the proofs, such as $0.066 + 0.370148 + 2 = 2.436148$ and

Table 4: Values computed in Part 1 of the script.

Quantity	Computed value	Bound used
$\log\log\left(10^{10^{13.11485}}\right)$	$31.0320905521\ldots$	> 31.032
$\log(46{,}909{,}038)$	$17.6637209227\ldots$	< 17.6638
$(1-\varepsilon(\log 46{,}909{,}038))^{-1}$	$1.0000109986\ldots$	< 1.000011
$31.032/1.000011$	$31.0316586518\ldots$	> 31.0316
$(1-\varepsilon(31.0316))^{-1}$	$1.0000018754\ldots$	< 1.000002
$31.032/1.000002$	$31.0319379361\ldots$	> 31.0319
$\left(\frac{1}{20}+\frac{3}{16\cdot 31.0319}\right)/(1-\varepsilon(31.0319))$	$0.0560422746\ldots$	< 0.0562
$e^{0.0562/31.03^2}\left(1+\frac{0.0562}{31.03^3}\right)$	$1.0000602505\ldots$	< 1.0000603
$e^{15.515}$	$5471153.3376\ldots$	$> 5.47\cdot 10^6$
$1-1.94/\sqrt{5.47\cdot 10^6}$	$0.9991705159\ldots$	> 0.99917
$0.99917/1.0000603$	$0.9991097537\ldots$	> 0.9991
$e^{\gamma}\cdot 31.032$	$55.2702392751\ldots$	> 55
$\max_{m\le 15} m/\varphi(m)$	3	$= 3$

$0.313233 - 0.066 + 1.9314 - 0.258259 = 1.920374$; they are verified in exact rational arithmetic. The smallest positive margins in Part 2 are those of $\frac{1}{2\cdot 31^2}+\frac{1}{31^3} = 0.000553858\ldots < 0.000554$, of $2\left(\frac{\log 2}{2}-0.011306\right) = 0.6705351805\ldots \ge 0.670535$, and of $2\left(1-\frac{0.370148+\log 2}{31}\right) = 1.9314003\ldots > 1.9314$, with differences $1.4\cdot 10^{-7}$, $1.8\cdot 10^{-7}$ and $3.1\cdot 10^{-7}$. The final constants of Theorem 3.1 are $0.9574\cdot\sqrt{2} = 1.3539680\ldots > 1.353$ and $2.8070\cdot\sqrt{2} = 3.9696974\ldots < 3.970$.

Table 5: Values computed in Part 2 of the script, at $\lambda = 31$.

Quantity	Computed value	Bound used
$1-\frac{\log 2}{62}$	$0.9888202067\ldots$	> 0.98882
$1.94\cdot 31\cdot e^{-31/4}/1.18253$	$0.0219062995\ldots$	< 0.022
$-\log\left(1-\frac{\log 2}{62}\right)$	$0.0112427568\ldots$	< 0.011306
$0.670535-\frac{\log 2}{2}-\frac{\log 2}{2}\cdot\frac{0.670535}{31}$	$0.3164649670\ldots$	> 0.316464
$0.316464-0.316464^2/31$	$0.3132333721\ldots$	> 0.313233
$2\left(1-\frac{0.370148+\log 2}{31}\right)$	$1.9314003109\ldots$	> 1.9314
$1.0022\cdot 31\cdot e^{-31/2}/\sqrt{2}$	$4.0760229901\ldots\cdot 10^{-6}$	$< 4.08\cdot 10^{-6}$
$1.397905\cdot 31\cdot e^{-31/6}$	$0.2471636095\ldots$	< 0.24717
$\frac{1}{31^2}+\sqrt{2}\cdot 31\cdot e^{-31/2}$	$0.0010487169\ldots$	< 0.0010488
$0.0861+0.17587+e^{-31/3}$	$0.2620025305\ldots$	< 0.26201
$0.26201\cdot 31^2\cdot e^{-31/6}/\sqrt{2}$	$1.0154801390\ldots$	< 1.0155
$\frac{1}{2\cdot 31^2}+\frac{1}{31^3}$	$0.0005538585\ldots$	< 0.000554
$0.957494\cdot(1-31^{-4})$	$0.9574929632\ldots$	> 0.9574
$0.9574\cdot\sqrt{2}$	$1.3539680646\ldots$	> 1.353
$2.8070\cdot\sqrt{2}$	$3.9696974696\ldots$	< 3.970

7.3. Part 3: The Corollaries

Part 3 checks the constants of Proposition 2.8 and Corollaries 5.2 and 5.4. The script computes $\tau = 2+\gamma-\log(4\pi) = 0.0461914179\ldots$, so that $2+\tau = 2.0461914179\ldots < 2.0462$, and $2\sqrt{2}-2-\tau = 0.7822357068\ldots$, the constant discussed after Corollary 5.4. The constant of Corollary 5.2 is $2\sqrt{2}\left(1-\frac{1.4035}{31.0319}\right) = 2.7005040047\ldots > 2.70$, and the constant of Corollary 5.4 is $2+\tau-2\sqrt{2}+\frac{2\sqrt{2}\cdot 1.4035}{31} = -0.6541809497\ldots < -0.654$. The identities $1.353 - 2.062 =$

-0.709 and $3.970 - 0.055 = 3.915$ of Corollary 5.4 are verified in exact rational arithmetic. The constant 2.70 of Corollary 5.2 exceeds the bound 2.0462 of Proposition 2.8 by 0.6538. Finally, $e^{31.0319} = 2.999\ldots \cdot 10^{13} > 10^9$, so the primes q of Corollary 5.2 lie in the range of Proposition 2.8.

7.4. Part 4: Nicolas' Second Criterion

Part 4 checks the constants of Lemmas 4.1 and 4.5 and of Theorems 1.1, 1.2 and 1.3. The script computes $\left(\frac{2+\tau}{2}\right)^2 = 1.04672\ldots$, which lies below 1.0468, and $1.0468 < \log 3 = 1.09861\ldots$. The constant of Lemma 4.5 is $2.0462 \cdot \left(\frac{1}{62} + \frac{1}{31^2}\right) + \frac{2.0462^2}{2} \cdot e^{-31/2} = 0.03513\ldots < 0.036$, and the identity $1.353 - 0.036 = 1.317$ of Theorem 1.1 is verified in exact rational arithmetic. For the upper bound of Lemma 4.5, the script checks that $\frac{1+\frac{31^{-1}}{1-31^{-3}}}{1-31^{-3}-\frac{31^{-4}}{1-31^{-3}}} < 1.0323$ and $2.0462 \cdot \frac{1.0323}{31^2} < 0.0022$, and then $3.970 + 0.0022 \le 3.973$ (Theorem 1.1). For Theorem 1.2, it checks $2\sqrt{2} - 2 - \tau - \frac{3.973}{31} > 0.654$, $2 + \tau + 0.654 > 2.70$, and $\frac{10^8 \cdot (1-\log^{-3} 10^8)}{\log 10^8} > 5 \cdot 10^6$. For Theorem 1.3, it checks $10^{13.11485} \cdot \log 10 > 2.9996 \cdot 10^{13}$ and $X(1 - \log^{-3} X) - \log X > 10^{13.11485} \cdot \log 10$ for $X = 3.01 \cdot 10^{13}$.

7.5. Part 5: The Second Inequality at the Primorials

Part 5 of the script checks the inequality $c(q) < E(q)$ of Lemma 4.4 at every prime q with $3 \le q \le X_0$, where $X_0 = 100{,}000{,}008{,}765{,}439$ is the last integer of the sub-block of 2^{24} integers that contains 10^{14}. The integers up to X_0 are divided into 5,960,465 consecutive sub-blocks of 2^{24} integers. For each sub-block the primes are generated by a segmented sieve of Eratosthenes, and the sums $\sum \log \frac{p}{p-1}$ and $\sum \log p$ over its primes are computed in double precision; the prefix sums over all earlier sub-blocks are accumulated with `mpmath` at 40 significant digits. Within a sub-block the primes are taken in increasing order in batches of 64 consecutive primes $q_a < \cdots < q_b$. Since $\sum_{p \le q} \log \frac{p}{p-1}$ and $\theta(q)$ increase with q and $E(q)$ decreases, every prime q of the batch satisfies

$$E(q) - c(q) \ge E(q_b) - \sum_{p \le q_b} \log \frac{p}{p-1} + \gamma + \log\log\theta(q_a),$$

and the batch is accepted when the right-hand side exceeds 10^{-13}; otherwise $E(q) - c(q)$ is computed for each prime of the batch, a prime with $E(q) - c(q) \le 0$ is recorded as a failure, and a prime with $0 < E(q) - c(q) < 10^{-13}$ is recorded as unresolved. The term $\log \frac{p}{p-1}$ is computed as $-\log(1 - \frac{1}{p})$ for $p < 10^5$ and by its Taylor series up to p^{-4} for $p \ge 10^5$, and for $p \ge 10^{11}$ the term $\log p$ is computed as $\log B + \log(1 + \frac{p-B}{B})$, where B is the first odd number of the sub-block, with the Taylor series of $\log(1 + r)$ up to r^5. The computation uses floating-point arithmetic and not interval arithmetic.

The sub-blocks below $891264 \cdot 2^{24} = 14{,}952{,}928{,}641{,}024$ were processed under Python 3.12 with the sieve of the script, compiled with `numba`, and the remaining sub-blocks under Python 3.10 with the library `primesieve`, on 8 threads; the total running time was 20.31 hours. The output is the following.

- The number of primes $p \le X_0$ is $K_0 = 3{,}204{,}942{,}022{,}702$.
- There is no failure and no unresolved prime p_k with $120569 \le k \le K_0$.
- The largest k with $c(p_k) \ge E(p_k)$ is $k = 120568$.
- For every prime q with $e^{31} \le q \le X_0$, $(E(q) - c(q)) \cdot \sqrt{q} \cdot \log^2 q > 0.4169$.
- $\sum_{p \le X_0} \log \frac{p}{p-1} = 4.05030544560\ldots$ and $\theta(X_0) = 99{,}999{,}999{,}338{,}267.4\ldots$
- $X_0(1 - \log^{-3} X_0) - \log X_0 = 9.99970235968\ldots \cdot 10^{13} = 10^{13.637771\ldots} \cdot \log 10$.

The value $k = 120568$ agrees with the threshold N_{120569} of Proposition 2.9.

Statements and Declarations

Declaration of interests

The author declares no competing interests.

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Declaration of generative AI in scientific writing

The author used Claude Opus 5.5 to write the verification script `verify_second_criterion.py` of Section 7 and to generate parts of the text of this manuscript. The experiments were run by the author. After using this tool, the author reviewed and edited the code and the content as needed and takes full responsibility for the content of the publication.

Data availability

The Python script `verify_second_criterion.py` used in Section 7 is available in the preprint repository server Hexagon.

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