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EKELAND–HOFER CAPACITIES AND CONVEX DUALITY: ORDINARY AND REAL SPECTRAL INVARIANTS

Abstract. We identify the rational Ekeland–Hofer capacities of convex bodies with Clarke's homogeneous dual-action selectors by a direct minimax argument. Convex reduction and a common nonresonant quadratic tail give the cohomological real analogue for bodies invariant under a linear anti-symplectic involution. In the real case, the mod-two index of half-period translation requires linking in the presence of fixed Fourier modes. The real selectors are full actions of generalized brake characteristics and converge to twice the least brake action. Convex approximation includes nonsmooth bodies, and Fourier decomposition computes both sequences on diagonal ellipsoids.

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1. Introduction

Fix $n \geq 1$, write $\mathbb{N} = \{1, 2, \ldots\}$, and identify $\mathbb{R}^{2n} = \mathbb{C}^n$. We use the Euclidean scalar product, $J_0 = i$, and

$$\omega_0(v, w) = \langle J_0 v, w \rangle, \qquad \lambda_0(z)[v] = \tfrac{1}{2}\langle J_0 z, v \rangle.$$

Thus Hamiltonian trajectories satisfy $\dot{x} = J_0 \nabla H(x)$. A convex body is a compact convex set with nonempty interior. Smooth boundaries are C^∞;

positive curvature means positive definite second fundamental form. Denote the closed Euclidean ball by $B^{2n}(R)$. For $0 \in \operatorname{int} C$, define

$$\begin{aligned} j_C(z) &= \inf\{r > 0 : z \in rC\}, \qquad h_C = j_C^2, \\ h_C^*(p) &= \sup_{z \in \mathbb{R}^{2n}} \big(\langle p, z\rangle - h_C(z)\big). \end{aligned} \tag{1.1}$$

Write $L^p = L^p(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n})$, with norm $\|\cdot\|_p$, and $H^s = H^s(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n})$; the subscript 0 denotes mean-zero loops. All function spaces have their strong topologies. The mean-zero primitive $\Pi : L^2_0 \to H^1_0$ is

$$(\Pi u)(t) = \int_0^t u(s)\, ds - \int_0^1 \int_0^\tau u(s)\, ds\, d\tau. \tag{1.2}$$

Set

$$\begin{aligned} \mathcal{A}(u) &= \tfrac{1}{2} \int_0^1 \langle J_0 \Pi u, u\rangle\, dt, \\ \mathcal{H}_C(u) &= \int_0^1 h_C^*(-J_0 u)\, dt, \end{aligned} \tag{1.3}$$

and define the homogeneous dual quotient by

$$\begin{aligned} \Lambda &= \{u \in L^2_0 : \|u\|_2 = 1,\ \mathcal{A}(u) > 0\}, \\ \Psi_C(u) &= \frac{\mathcal{H}_C(u)}{\mathcal{A}(u)}, \qquad u \in \Lambda. \end{aligned} \tag{1.4}$$

The circle $S^1 = \mathbb{R}/\mathbb{Z}$ acts by $(\theta \cdot u)(t) = u(t + \theta)$. For its rational Borel index $\operatorname{ind}_{\mathrm{FR}}$, defined in Section 2, Clarke's selectors are

$$s_k(C) = \inf\{a > 0 : \operatorname{ind}_{\mathrm{FR}}\{\Psi_C < a\} \geq k\}, \qquad k \geq 1. \tag{1.5}$$

Section 2 defines c_k^{EH} by Hamiltonian minimax.

Theorem A. *For every convex body $C \subset \mathbb{R}^{2n}$, $p \in \operatorname{int} C$, and $k \geq 1$,*

$$s_k(C - p) = c_k^{\mathrm{EH}}(C). \tag{1.6}$$

Consequently $s_k(C) := s_k(C - p)$ is independent of p, also for nonsmooth bodies and repeated selector values.

Equality (1.6) is recorded in [Mat25, remark following Theorem A], using [GR24, Theorem 1.2]. For smooth positively curved C, the inequality $c_k^{\mathrm{EH}}(C) \leq s_k(C)$ is proved in [BBLM23, equation (3.3), Proposition 3.1]. For the reverse bound, convex minimization over the negative and constant modes yields a compact graph deformation. Spectrality and continuity along a family with one nonresonant quadratic tail then give the comparison. The real proof reuses this analysis, with the mod-two linking developed in Section 7.

Let $\rho : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear anti-symplectic involution, so $\rho^2 = \mathrm{id}$ and $\rho^* \omega_0 = -\omega_0$, and let $C = \rho C$. On brake loops $x(-t) = \rho x(t)$, half-period translation

$$(\sigma x)(t) = x(t + 1/2)$$

defines a $\mathbb{Z}_2$-action. Its Borel index $\mathrm{ind}_{\mathbb{Z}_2}$ uses $\mathbb{F}_2[w]$, $|w| = 1$, whereas $\mathrm{ind}_{\mathrm{FR}}$ uses $\mathbb{Q}[e]$, $|e| = 2$. The real dual selectors $s_k(\cdot, \rho)$ are the corresponding strict sublevel thresholds of the same quotient on anti-brake velocities; see (7.1). The cohomological real EH quantities $c_k^{\mathrm{EH}}(\cdot, \rho)$ are defined independently in Section 8.

Theorem B. *For every ρ-invariant convex body C, every $p \in \mathrm{Fix}(\rho) \cap \operatorname{int} C$, and every $k \geq 1$,*

$$c_k^{\mathrm{EH}}(C, \rho) = s_k(C - p, \rho). \tag{1.7}$$

The values are positive, finite, and independent of the fixed interior center and adapted coordinates. They are monotone under inclusions, homogeneous of degree two, and satisfy

$$c_k^{\mathrm{EH}}(SC, S\rho S^{-1}) = c_k^{\mathrm{EH}}(C, \rho) \quad \textit{for every linear symplectic map } S.$$

Thus $s_k(C, \rho)$ is defined for uncentered invariant bodies. The first real capacity and its brake-action representation are due to Jin–Lu [JL20, Theorems 1.3 and 1.11]. Shi–Lu use the half-period action with a genus index [SL22, Section 7]; no equality between that index and $\mathrm{ind}_{\mathbb{Z}_2}$ is assumed here. The even positive Fourier modes are fixed by σ. Their presence changes linking and bounds the entire sequence.

For $\boldsymbol{a} = (a_1, \dots, a_n)$ with $0 < a_1 \leq \cdots \leq a_n$, set

$$E(\boldsymbol{a}) = \left\{ z \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\pi |z_j|^2}{a_j} \leq 1 \right\}, \qquad \rho_0(q, p) = (q, -p). \tag{1.8}$$

Theorem C. *For every ρ-invariant convex body C, with $\beta_\rho(C) = s_1(C, \rho)$,*

$$\begin{gathered} 0 < \beta_\rho(C) \leq s_k(C, \rho) \leq 2\beta_\rho(C) \qquad (k \geq 1), \\ \lim_{k \to \infty} s_k(C, \rho) = 2\beta_\rho(C). \end{gathered} \tag{1.9}$$

Every selector is the full action $\int_y \lambda_0$ of a generalized brake closed characteristic on ∂C, a classical characteristic when C is smooth with positive curvature. After translation by a fixed interior center, the generalized equation is (6.6). On diagonal ellipsoids,

$$s_k(E(\boldsymbol{a}), \rho_0) = \begin{cases} \min\{a_k, 2a_1\}, & 1 \leq k \leq n, \\ 2a_1, & k > n. \end{cases} \tag{1.10}$$

2. Ordinary minimax and dual-action foundations

In the ordinary theory, we use singular cohomology with coefficients in $\mathbb{Q}$. Let ES^1 be the unit sphere of $\ell^2(\mathbb{C})$ with scalar circle action, and put $BS^1 = ES^1/S^1$. For an S^1-space X, let $p_X : X \times_{S^1} ES^1 \to BS^1$ be its Borel projection. The universal first Chern class $e \in H^2(BS^1; \mathbb{Q})$ gives $H^*(BS^1; \mathbb{Q}) = \mathbb{Q}[e]$. Define

$$\mathrm{ind}_{\mathrm{FR}}(X) = \sup\{j \geq 1 : p_X^*(e^{j-1}) \neq 0\}, \qquad \mathrm{ind}_{\mathrm{FR}}(\varnothing) = 0. \tag{2.1}$$

An equivariant map $X \to Y$ implies $\mathrm{ind}_{\mathrm{FR}}(X) \le \mathrm{ind}_{\mathrm{FR}}(Y)$. In particular, no freeness assumption is required.

Lemma 2.1. *If $V = \bigoplus_{j=1}^{d} \mathbb{C}_{(m_j)}$, where $\theta \cdot z = e^{2\pi i m_j \theta} z$ and $m_j \in \mathbb{Z} \setminus \{0\}$, then its unit sphere satisfies $\mathrm{ind}_{\mathrm{FR}}(S(V)) = d$.*

Proof sketch. The equivariant Euler class is $(\prod_j m_j)e^d$. The sphere-bundle Gysin sequence gives exactly the asserted height of e; see [BBLM23, proof of Lemma A.2]. The weights are invertible over $\mathbb{Q}$. □

Let

$$\begin{aligned} E &= H^{1/2}(\mathbb{R}/\mathbb{Z}, \mathbb{C}^n) = E^- \oplus E^0 \oplus E^+, \\ x(t) &= \sum_{m \in \mathbb{Z}} e^{2\pi i m t} x_m, \\ \|x\|_E^2 &= |x_0|^2 + 2\pi \sum_{m \neq 0} |m| |x_m|^2. \end{aligned} \tag{2.2}$$

The summands are the negative, constant, and positive Fourier modes; P^-, P^0, P^+ denote their projections and $x^\pm = P^\pm x$. These projections also act on L^2; $\langle \cdot, \cdot \rangle_E$ is the real Hilbert scalar product for the displayed norm. The inclusion $i : E \hookrightarrow L^2$ is compact; write i^* for its Hilbert adjoint. Moreover, $\|x\|_p \le C_p \|x\|_E$ for every finite p. EH minimax sets, neighborhoods, and deformations use the E topology; H^1 records primitive and critical-loop regularity. The map

$$\Pi : L_0^2 \longrightarrow H_0^1 \hookrightarrow E$$

is bounded into H^1 and compact into E; differentiation recovers u. Define the free action on all of E by

$$\begin{aligned} \mathfrak{a}(x) &= \pi \sum_{m \in \mathbb{Z}} m |x_m|^2 \\ &= \tfrac{1}{2}(\|x^+\|_E^2 - \|x^-\|_E^2), \qquad x \in E. \end{aligned} \tag{2.3}$$

For $x \in H^1$,

$$\mathfrak{a}(x) = \int_x \lambda_0 = \tfrac{1}{2} \int_0^1 \langle J_0 x, \dot{x} \rangle \, dt.$$

In particular, $\mathcal{A}(u) = \mathfrak{a}(\Pi u)$ for $u \in L_0^2$.

Put $S^+ = \{x \in E^+ : \|x\|_E = 1\}$. The original EH group Γ consists of equivariant homeomorphisms

$$\Theta(x) = e^{\theta_+(x)} x^+ + x^0 + e^{\theta_-(x)} x^- + K(x), \tag{2.4}$$

where $\theta_\pm$ are continuous invariant functions bounded on bounded sets and K is continuous, equivariant, and compact on bounded sets. For some $\rho > 0$, all three vanish when $\mathfrak{a}(x) \le 0$ or $\|x\|_E \ge \rho$. Use the neighborhood pseudoindex

$$\mathrm{ind}_{\mathrm{EH}}(V) = \inf_{\Theta \in \Gamma} \inf_{\substack{U \supset V \cap \Theta(S^+) \\ U \text{ open, invariant in } E}} \mathrm{ind}_{\mathrm{FR}}(U). \tag{2.5}$$

These are the original deformations [EH90, Section II], with the singular-cohomology convention of [BBLM23, Section 3.1, Remark A.1]. For invariant open V, the inner infimum equals

$$\operatorname{ind}_{\mathrm{FR}}(V \cap \Theta(S^+)). \tag{2.6}$$

Indeed, for $Y = \Theta^{-1}V \cap S^+$, the open set $\{x : x^+ \neq 0,\ x^+/\|x^+\|_E \in Y\}$ retracts equivariantly to Y. The Borel spaces of these open Hilbert-manifold slices are metrizable and locally contractible: their bundles over BS^1 are locally products. Thus singular and Alexander–Spanier cohomology agree naturally there, preserving $p_X^* e^j$.

Let $\mathcal{F}(C)$ consist of smooth $H : \mathbb{R}^{2n} \to [0,\infty)$, zero near C, with $H(z) = b|z|^2$ outside a compact set, where $b > \pi$ and $b \notin \pi\mathbb{N}$. Set

$$\Phi_H(x) = \mathfrak{a}(x) - \int_0^1 H(x(t))\,dt. \tag{2.7}$$

$$\begin{aligned} c_k^{\mathrm{EH}}(H) &= \inf\{a \in \mathbb{R} : \operatorname{ind}_{\mathrm{EH}}\{\Phi_H < a\} \geq k\}, \\ c_k^{\mathrm{EH}}(C) &= \inf_{H \in \mathcal{F}(C)} c_k^{\mathrm{EH}}(H). \end{aligned} \tag{2.8}$$

An empty infimum is $+\infty$. The open-slice comparison identifies this threshold with the original minimax:

$$c_k^{\mathrm{EH}}(H) = \inf_{\operatorname{ind}_{\mathrm{EH}}(V) \geq k} \sup_{x \in V} \Phi_H(x).$$

Indeed, $\sup_V \Phi_H = a$ implies $V \subset \{\Phi_H < a + \varepsilon\}$; conversely, use the sublevels themselves.

For an integer $N \geq 0$, let $V_N \subset E^+$ be the sum of modes $1, \dots, N$ ($V_0 = \{0\}$), and P_{V_N} its L^2-orthogonal projection.

Proposition 2.2. *The body capacities are monotone, translation invariant, and homogeneous:*

$$\begin{aligned} C \subset D \implies c_k^{\mathrm{EH}}(C) \leq c_k^{\mathrm{EH}}(D), \\ c_k^{\mathrm{EH}}(rC + p) = r^2 c_k^{\mathrm{EH}}(C) \qquad (r > 0,\ p \in \mathbb{R}^{2n}). \end{aligned} \tag{2.9}$$

If $H \in \mathcal{F}(C)$ and $0 \in \operatorname{int} C$, then $c_k^{\mathrm{EH}}(H) > 0$; it is finite if its outer slope satisfies $b > \pi\lceil k/n \rceil$. Every finite positive $c_k^{\mathrm{EH}}(H)$ is a critical value of Φ_H. For admissible H, G,

$$H \leq G \quad \implies \quad c_k^{\mathrm{EH}}(H) \geq c_k^{\mathrm{EH}}(G).$$

If their difference is bounded and their selectors are finite, then

$$|c_k^{\mathrm{EH}}(H) - c_k^{\mathrm{EH}}(G)| \leq \|H - G\|_\infty. \tag{2.10}$$

Thus a family with a common quadratic tail, continuous in the uniform norm of differences, cannot have its finite kth selector cross a positive level which is never critical.

Proof sketch. These are the EH minimax properties [EH90, Section II, pp. 558–559]. We recall the estimates relevant to our conventions. If $H = 0$ on $B^{2n}(r)$, quadratic growth gives $K > 0$ such that

$$H(z) \le K|z|^4/r^2,$$
$$\Phi_H\big|_{\rho S^+} \ge \rho^2/2 - KC_4^4\rho^4/r^2 > 0$$

for small $\rho > 0$. For any $a > 0$, choose smooth cutoffs $\beta : \mathbb{R} \to [0,1]$ and $\gamma : [0,\infty) \to [0,1]$ with

$$\beta = 1 \text{ on } [\tfrac{1}{2}\min\{1,a^2\}, \tfrac{1}{2}\max\{1,a^2\}], \qquad \beta = 0 \text{ on } (-\infty, 0],$$
$$\gamma = 1 \text{ on } [0, \max\{1,a^2\}], \qquad \gamma \text{ of compact support.}$$

The time-$\log a$ flow of $\dot{x} = \beta(\mathfrak{a}(x))\gamma(\|x\|_E^2)x^+$ belongs to Γ and maps S^+ to aS^+. Indeed, its trajectories from S^+ are $x(t) = e^t v$ between times 0 and $\log a$; only the positive component is rescaled, and the cutoff fixes the required regions. The EH linking estimate [EH90, Proposition 1] gives

$$\operatorname{ind}_{\mathrm{EH}}(E^- \oplus E^0 \oplus V_N) \ge nN. \tag{2.11}$$

Its finite-dimensional proof uses the weighted Euler class and piercing; the singular argument uses [BBLM23, Proposition A.4]. Since $H \ge b|z|^2 - A_0$, the functional is bounded above by A_0 on this cone when $b > \pi N$, proving finiteness.

For compactness, put $L = P^+ - P^-$ and $q(z) = \nabla H(z) - 2bz$. Then

$$\nabla\Phi_H = (L - 2bi^*i)x - i^*q(x).$$

The last term has relatively compact image since q is bounded. The linear term has Fourier eigenvalues

$$\begin{cases} -2b, & m = 0, \\ 1 - b/(\pi m), & m > 0, \\ -1 - b/(\pi|m|), & m < 0. \end{cases}$$

Nonresonance bounds their absolute values away from zero. Thus every sequence with gradient tending to zero has a convergent subsequence. For a critical point $x \in E$, the Fourier equations give $2\pi m x_m = (\nabla H(x))_m$. Since $\nabla H(x) \in L^2$, they imply $x \in H^1$ and $\dot{x} = J_0\nabla H(x)$, hence x is smooth. This regularity identifies the E-critical points with the periodic orbits used below. Fix a positive noncritical value c and choose $\varepsilon > 0$ so that $[c - 2\varepsilon, c + 2\varepsilon] \subset (0,\infty)$ contains no critical value. Put $Y_\pm = \{\Phi_H < c \pm \varepsilon\}$. Choose $\chi \in C_c^\infty((c-2\varepsilon, c+2\varepsilon); [0,1])$ equal to one on $[c-\varepsilon, c+\varepsilon]$, and define

$$Z(x) = \frac{\chi(\Phi_H(x))}{\|\nabla\Phi_H(x)\|_E^2}\nabla\Phi_H(x)$$

on the band, with $Z = 0$ outside it. The compactness proved above bounds $\|\nabla\Phi_H\|_E$ away from zero on the band, so Z is bounded and locally Lipschitz.

For fixed $\Theta \in \Gamma$, choose

$$R > \sup_{v \in S^+} \|\Theta(v)\|_E + 3\varepsilon \|Z\|_\infty$$

and a smooth $\zeta : [0,\infty) \to [0,1]$, equal to one on $[0, R^2]$ and zero outside a bounded interval. Let φ_t be the flow of

$$\dot{x} = a_H(x)\big(Lx - i^*\nabla H(x)\big),$$
$$a_H(x) = \frac{\chi(\Phi_H(x))\zeta(\|x\|_E^2)}{\|\nabla \Phi_H(x)\|_E^2},$$

where $a_H = 0$ outside the band. With $\beta_t(x) = \int_0^t a_H(\varphi_s(x))\, ds$, variation of constants gives

$$\varphi_t(x) = e^{\beta_t(x)}x^+ + x^0 + e^{-\beta_t(x)}x^- + K_t(x).$$

Here β_t is continuous and invariant, and K_t is compact on bounded sets: its integral expression is a uniform limit, on bounded sets, of Riemann sums with relatively compact image. The norm cutoff fixes all sufficiently large points. Since $H \geq 0$, the flow also fixes $\{\mathfrak{a} \leq 0\}$. Thus $g = \varphi_{3\varepsilon} \in \Gamma$ and

$$Y_+ \cap g\Theta(S^+) \subset g\big(Y_- \cap \Theta(S^+)\big).$$

Transporting an open index witness for Y_- by g excludes an index threshold at c. Sublevel inclusions prove Hamiltonian monotonicity and (2.10), and continuity then gives no crossing.

Monotonicity is immediate from the admissible families; translation and dilation are the EH invariances [EH90, Section II, p. 559]. For the exact tail convention, allowing $b|z-q|^2$ gives the same body infimum for every center q: any such H admits a majorant with tail $b'|z-q'|^2$, $b' > b$, by interpolation outside a sufficiently large ball. The reverse comparison interchanges q, q'. Translation is therefore compatible with the centered definition; $r^2 H(z/r)$ proves scaling. □

Lemma 2.3. *For centered convex bodies, the quotient Ψ_C is continuous, and s_k is monotone and homogeneous of degree two. These assertions include nonsmooth bodies. If $rB^{2n}(1) \subset C \subset RB^{2n}(1)$, then*

$$\pi r^2 \lceil k/n \rceil \leq s_k(C) \leq \pi R^2 \lceil k/n \rceil. \tag{2.12}$$

Proof. For the support function $\sigma_C(p) = \max_{z \in C}\langle p, z\rangle$,

$$h_C^*(p) = \sup_{t \geq 0}(t\sigma_C(p) - t^2) = \tfrac{1}{4}\sigma_C(p)^2. \tag{2.13}$$

Boundedness of Π gives continuity of $\mathcal{A}$. Also

$$\tfrac{1}{4}r^2\|u\|_2^2 \leq \mathcal{H}_C(u) \leq \tfrac{1}{4}R^2\|u\|_2^2,$$

and $|\sigma_C(p) - \sigma_C(q)| \leq R|p-q|$ gives

$$|\mathcal{H}_C(u) - \mathcal{H}_C(v)| \leq \tfrac{1}{4}R^2(\|u\|_2 + \|v\|_2)\|u - v\|_2.$$

Moreover $C \subset D$ implies $\Psi_C \le \Psi_D$, and $\Psi_{rC} = r^2 \Psi_C$. Writing u_m for the Fourier coefficients of u gives

$$\mathcal{A}(u) = \frac{1}{4\pi} \sum_{m \neq 0} \frac{|u_m|^2}{m}.$$

On the L^2-unit sphere of V_N, $\Psi_C \le \pi R^2 N$ and the index is nN. Taking strict levels above $\pi R^2 N$ proves the upper bound. If $P_{V_N} u = 0$ and $\|u\|_2 = 1$, then $\mathcal{A}(u) \le 1/(4\pi(N+1))$ and $\mathcal{H}_C(u) \ge r^2/4$. Hence L^2-normalized projection to V_N is defined on $\{\Psi_C < a\}$ for $a \le \pi r^2 (N+1)$; its index is at most nN. Take $N = \lceil k/n \rceil - 1$, with an empty sublevel when $N = 0$. □

For a smooth positively curved body D with $0 \in \operatorname{int} D$, let $\mathcal{S}(D)$ be the periods of $\dot{z} = J_0 \nabla h_D(z)$ on ∂D. Euler's identity gives $\lambda_0(J_0 \nabla h_D) = 1$ there, so periods equal actions.

Lemma 2.4. *$\mathcal{S}(D)$ is closed and has measure zero in $(0, \infty)$.*

Proof sketch. Closedness follows from compactness of ∂D and continuous dependence of its nonvanishing flow. For nullness, fix $0 < a < b$ and choose smooth G equal to $h_D^2/2$ near $\{a \le h_D \le b\}$ and quadratic at infinity. Let z be a characteristic of period $T \in [a, b]$. Its rescaling satisfies

$$x(t) = \sqrt{T}\, z(Tt),$$
$$\dot{x} = J_0 \nabla G(x), \qquad \Phi_G(x) = T^2/2.$$

Let P_N project onto modes $|m| \le N$, $Q_N = 1 - P_N$. For $v \in P_N E$ and $w \in Q_N E$, set

$$\mathcal{T}_v(w) = (P^+ - P^-) Q_N i^* \nabla G(v + w).$$

The Fourier tail estimates give

$$\|\mathcal{T}_v(w) - \mathcal{T}_v(w')\|_E \le \frac{\|D^2 G\|_\infty}{2\pi(N+1)} \|w - w'\|_E.$$

Choose N so that this coefficient is less than one. The high-mode critical equation $w = \mathcal{T}_v(w)$ then has a unique solution $w(v)$. Every derivative of G of order at least two is bounded; Hölder's inequality and $E \hookrightarrow L^p$ for all finite p make $x \mapsto \nabla G(x)$ smooth from E to L^2. The implicit function theorem gives a smooth map $v \mapsto w(v)$. Thus every critical value of Φ_G is a critical value of the smooth finite-dimensional function $v \mapsto \Phi_G(v + w(v))$. Sard's theorem, followed by the change of variables $T \mapsto T^2/2$, proves nullness on $[a, b]$. □

3. Homogeneous duality and the graph deformation

Assume that C has smooth boundary and positive curvature and that $0 \in \operatorname{int} C$. Put $h = h_C$. Its Hessian is

$$D^2 h = 2\, dj_C \otimes dj_C + 2 j_C D^2 j_C.$$

Since $D^2 j_C$ annihilates the radial direction and is positive on $\ker dj_C$, this Hessian is positive definite away from zero. Compactness of the unit sphere and degree-zero homogeneity give uniform Hessian bounds away from zero. Thus ∇h is globally Lipschitz, and there are $\mu, L, m_0, M_0 > 0$ such that

$$\begin{aligned} \langle \nabla h(z) - \nabla h(z'), z - z' \rangle &\geq \mu |z - z'|^2, \\ |\nabla h(z) - \nabla h(z')| &\leq L |z - z'|. \end{aligned} \tag{3.1}$$

$$m_0 |z|^2 \leq h(z) \leq M_0 |z|^2. \tag{3.2}$$

Write $q_\pm(v) = \frac{1}{2}\|v\|_E^2$ on $E^\pm$. For $\tau > 0$, put $F_\tau(x) = \tau \int_0^1 h(x(t))\, dt$ for $x \in L^2$, and define, for $v \in E^+$,

$$K_\tau(v) = \min_{w \in E^-,\ c \in E^0} \big(F_\tau(v + w + c) + q_-(w) \big). \tag{3.3}$$

The minimized expression is strictly convex and weakly lower semicontinuous, with the coercive bound

$$F_\tau(v + w + c) + q_-(w) \geq \tau m_0 (\|v\|_2^2 + \|w\|_2^2 + |c|^2) + q_-(w).$$

The direct method gives a unique minimizer $(w_\tau(v), c_\tau(v))$. Define

$$\begin{aligned} x_\tau(v) &= v + w_\tau(v) + c_\tau(v), \\ f_\tau(v) &= q_+(v) - K_\tau(v). \end{aligned} \tag{3.4}$$

The minimizing pair is equivariant and positively homogeneous, and

$$f_\tau(v) = \max_{w \in E^-,\ c \in E^0} \big(\mathfrak{a}(v + w + c) - F_\tau(v + w + c) \big). \tag{3.5}$$

Lemma 3.1. *The map $W_\tau : E^+ \to H^1 \cap (E^- \oplus E^0)$, $W_\tau = w_\tau + c_\tau$, is Lipschitz from the L^2 norm on E^+ to the H^1 norm and compact into E on E-bounded sets. The full graph point $x_\tau(v)$ is only required to lie in E.*

Proof. Fix $v_1, v_2 \in E^+$ and put $p_i = \tau \nabla h(x_\tau(v_i))$ for $i = 1, 2$. Let Δ denote the difference between the corresponding quantities at v_1 and v_2. The Fourier Euler equations are

$$\begin{aligned} (p_i)_0 &= 0, \\ (p_i)_m &= 2\pi m\, (w_\tau(v_i))_m \qquad (m < 0). \end{aligned} \tag{3.6}$$

Pairing with Δw and using $(\Delta p)_0 = 0$ gives

$$\langle \Delta p, \Delta x \rangle_{L^2} = \langle \Delta p, \Delta v \rangle_{L^2} - \|\Delta w\|_E^2.$$

Strong monotonicity and the Lipschitz bound therefore yield

$$\|\Delta w\|_E^2 + \tau\mu \|\Delta x\|_2^2 \leq \tau L \|\Delta x\|_2 \|\Delta v\|_2,$$

hence

$$\|\Delta w\|_E^2 + \frac{\tau\mu}{2} \|\Delta x\|_2^2 \leq \frac{\tau L^2}{2\mu} \|\Delta v\|_2^2. \tag{3.7}$$

The Fourier equations also give

$$\begin{aligned} |\Delta c| &\leq \|\Delta x\|_2, \\ \|\partial_t \Delta w\|_2 &= \|P^- \Delta p\|_2 \leq \tau L \|\Delta x\|_2. \end{aligned} \tag{3.8}$$

Together with (3.7), these prove the H^1 estimate. Compactness follows from $E \hookrightarrow L^2$. □

Lemma 3.2. *For every $\tau > 0$,*

$$\begin{aligned} &\operatorname{ind}_{\mathrm{FR}}(\{v \in E^+ : f_\tau(v) < 0\}) \\ &\qquad = \operatorname{ind}_{\mathrm{FR}}(\{u \in \Lambda : \Psi_C(u) < \tau\}). \end{aligned} \tag{3.9}$$

Intersecting the left-hand set with S^+ preserves its index.

Proof. The conjugate of F_τ on L^2 is $F_\tau^*(p) = \tau^{-1} \int h^*(p)$. For $v \in E^+$, define

$$\begin{aligned} p(v) &= \tau \nabla h(x_\tau(v)), \\ U(v) &= J_0 p(v), \qquad r(v) = P^+ \Pi U(v). \end{aligned} \tag{3.10}$$

Equation (3.6) gives $U(v) \in L^2_0$ and

$$\Pi U(v) = r(v) + w_\tau(v).$$

The maps $U : E^+ \to L^2_0$ and $r : E^+ \to E^+$ are continuous, by Lemma 3.1 and the Lipschitz continuity of ∇h. Abbreviate $r = r(v)$ and $w = w_\tau(v)$. Fenchel equality and Euler's identity give

$$\begin{aligned} F_\tau^*(p(v)) &= F_\tau(x_\tau(v)), \\ \langle r, v \rangle_E &= \langle p(v), x_\tau(v) \rangle_{L^2} + 2q_-(w) \\ &= 2K_\tau(v). \end{aligned}$$

Therefore

$$\begin{aligned} \tau^{-1} \mathcal{H}_C(U(v)) - \mathcal{A}(U(v)) &= K_\tau(v) - q_+(r) \\ &= f_\tau(v) - q_+(r - v) \le f_\tau(v). \end{aligned} \tag{3.11}$$

Conversely, for $u \in L^2_0$, put $z = \Pi u$ and $y = P^+ z$. Fenchel's inequality at $(x_\tau(y), -J_0 u)$ gives

$$\begin{aligned} \tau^{-1} \mathcal{H}_C(u) - \mathcal{A}(u) &\ge \langle -J_0 u, x_\tau(y) \rangle_{L^2} - F_\tau(x_\tau(y)) \\ &\qquad - q_+(y) + q_-(z^-) \\ &= f_\tau(y) + q_-(z^- - w_\tau(y)) \ge f_\tau(y). \end{aligned} \tag{3.12}$$

The pairing uses only $x_\tau(y) \in L^2$ and $z \in H^1$ and does not differentiate the graph point. The first inequality gives $\mathcal{A}(U(v)) > 0$ when $f_\tau(v) < 0$; the second gives $f_\tau(P^+ \Pi u) < 0$ when $\Psi_C(u) < \tau$. Hence the continuous equivariant maps

$$\begin{aligned} \{v \in E^+ : f_\tau(v) < 0\} &\longrightarrow \{u \in \Lambda : \Psi_C(u) < \tau\}, \\ v &\longmapsto \frac{U(v)}{\|U(v)\|_2}, \\ \{u \in \Lambda : \Psi_C(u) < \tau\} &\longrightarrow \{v \in E^+ : f_\tau(v) < 0\}, \\ u &\longmapsto P^+ \Pi u \end{aligned}$$

are well-defined. Index monotonicity in both directions proves the equality; positive radial normalization proves the sphere assertion. □

Lemma 3.3. *For $0 < \tau < \tau'$, put $X_{\tau'} = \{v \in S^+ : f_{\tau'}(v) < 0\}$ and*

$$\alpha = \frac{(\tau' - \tau)m_0}{2(\tau M_0 + (\tau' - \tau)m_0)} > 0.$$

Then $f_\tau \geq \alpha$ on $S^+ \setminus X_{\tau'}$.

Proof. Using the $K_{\tau'}$ minimizer as a competitor for K_τ gives

$$\begin{aligned} K_{\tau'}(v) - K_\tau(v) &\geq (\tau' - \tau) \int_0^1 h(x_{\tau'}(v)(t))\, dt \\ &\geq (\tau' - \tau)m_0 \|v\|_2^2. \end{aligned}$$

On the stated complement $K_{\tau'} \leq 1/2$, while the competitor $(w, c) = (0, 0)$ gives $K_\tau(v) \leq \tau M_0 \|v\|_2^2$. Consequently, with $t = \|v\|_2^2$,

$$\begin{aligned} f_\tau(v) &\geq \max\{(\tau' - \tau)m_0 t, \tfrac{1}{2} - \tau M_0 t\} \\ &\geq \alpha. \end{aligned}$$

□

Proposition 3.4. *Fix $0 < \tau < \tau'$ and $a > 0$. There are $\Theta \in \Gamma$ and $B_0 < \infty$ with $\|\Theta(v)\|_E \leq B_0$ on S^+ such that, for every $H \in \mathcal{F}(C)$,*

$$\int_0^1 (H(x(t)) - \tau h(x(t)))\, dt \leq 1 \quad (\|x\|_E \leq B_0) \tag{3.13}$$

implies

$$\mathrm{ind}_{\mathrm{EH}}(\{\Phi_H < a\}) \leq \mathrm{ind}_{\mathrm{FR}}(\{\Psi_C < \tau'\}). \tag{3.14}$$

Proof. Choose $R \geq 1$ with $R^2\alpha > a + 1$, put $W = W_\tau$, and take $\|W(v)\|_E \leq C_0\|v\|_E$. Set $B_0 = R(1 + C_0)$. For $v \in S^+ \setminus X_{\tau'}$ and $0 \leq t \leq 1$,

$$\begin{aligned} \mathfrak{a}(Rv + tRW(v)) &= R^2(q_+(v) - t^2 q_-(w_\tau(v))) \\ &\geq R^2 f_\tau(v) \geq R^2\alpha. \end{aligned} \tag{3.15}$$

These segments lie in the ball of radius B_0. Choose smooth $[0, 1]$-valued cutoffs $\chi = 0$ on $(-\infty, R^2\alpha/2]$, $\chi = 1$ on $[R^2\alpha, \infty)$, and $\zeta = 1$ on $[0, B_0^2]$ with bounded support. The equivariant field

$$\mathcal{V}(x) = \chi(\mathfrak{a}(x))\zeta(\|x\|_E^2)W(x^+) \tag{3.16}$$

is globally Lipschitz since W is Lipschitz and the norm cutoff bounds its support. Its flow φ_t has constant positive component, so its time-one map is

$$\begin{aligned} T(x) &= x + \eta(x)W(x^+), \\ \eta(x) &= \int_0^1 \chi(\mathfrak{a}(\varphi_t(x)))\zeta(\|\varphi_t(x)\|_E^2)\, dt \in [0, 1]. \end{aligned}$$

Here η is continuous and invariant. The added term is compact on bounded sets, and T fixes $\{\mathfrak{a} \leq 0\}$ and all sufficiently large points. Thus $T \in \Gamma$, and uniqueness of the flow along the displayed segments gives

$$T(Rv) = Rx_\tau(v) \qquad (v \in S^+ \setminus X_{\tau'}). \tag{3.17}$$

Let $D_R \in \Gamma$ be the radial deformation constructed in the proof of Proposition 2.2, so $D_R(v) = Rv$ on S^+. Put $\Theta = T \circ D_R$. Then $\|\Theta(v)\|_E \le B_0$ on S^+ and, outside $X_{\tau'}$,

$$\Phi_H(\Theta(v)) \ge R^2 f_\tau(v) - 1 > a.$$

Hence $\Theta^{-1}(\{\Phi_H < a\}) \cap S^+ \subset X_{\tau'}$. The open-slice formula (2.6) and Lemma 3.2 prove the assertion. □

3.1. **A cap with fixed quadratic slope.** Fix

$$\begin{gathered} 0 < c < \tau < M, \qquad \delta > 0, \\ [c/(1+2\delta), c] \cap \mathcal{S}(C) = \varnothing. \end{gathered} \tag{3.18}$$

Choose a smooth convex profile $f : \mathbb{R} \to [0, \infty)$ with

$$\begin{gathered} f = 0 \text{ on } (-\infty, 1+\delta], \\ 0 < f' \le 1 \text{ on } (1+\delta, \infty), \\ f(r) = r - \kappa \text{ for } r \ge 1 + 2\delta, \\ 1 + \delta < \kappa < 1 + 2\delta, \end{gathered} \tag{3.19}$$

and set $A_s = sf(h)$, $s \in [\tau, M]$.

Lemma 3.5. *Given $B, \zeta > 0$ and*

$$b > \max\{\pi, M \max_{|z|=1} h(z)\}, \qquad b \notin \pi\mathbb{N},$$

there is a family $\widehat{H}_s \in \mathcal{F}(C)$, $s \in [\tau, M]$, equal to $b|z|^2$ outside one fixed ball and continuous in the uniform norm of differences. Each member satisfies $\widehat{H}_s \ge A_s$ and $\Phi_{\widehat{H}_s}(x) \ne c$ for every one-periodic orbit x. Uniformly for $s \in [\tau, M]$,

$$0 \le \int_0^1 (\widehat{H}_s - A_s)(x(t))\, dt < \zeta \qquad (\|x\|_E \le B). \tag{3.20}$$

Proof. Choose $\alpha, \beta > 0$ with $\alpha|z|^2 \le h(z) \le \beta|z|^2$ and $b > M\beta$, and $\varepsilon > 0$ with $\tau\kappa - \varepsilon > c$. An even mollification ϑ of $|\cdot|$ may be chosen so that

$$\begin{gathered} \vartheta(t) = |t| \quad (|t| \ge \varepsilon), \qquad |\vartheta'| \le 1, \\ 0 \le \vartheta - t\vartheta' \le \varepsilon, \\ 0 \le \vartheta - |t| \le \varepsilon. \end{gathered}$$

Put $\mathrm{mx}(a, d) = (a + d + \vartheta(a - d))/2$. Choose $R \ge 1$ so large that

$$\begin{gathered} \alpha R^2 > 1 + 2\delta, \\ \tau(\alpha R^2 - \kappa) > M f(1 + 2\delta), \\ \frac{(b + \varepsilon/2) C_4^4 B^4}{R^2} < \zeta, \end{gathered} \tag{3.21}$$

where $\|x\|_4 \le C_4 \|x\|_E$. For $L > \max\{bR^2 + \varepsilon, c + \varepsilon\}$, define

$$F_s(z) = \mathrm{mx}(A_s(z), b|z|^2 - L).$$

Then $F_s = A_s$ for $|z| \le R$, while $b > M\beta$ gives a common $R_2 > R$ with

$$F_s(z) = b|z|^2 - L \qquad (|z| \ge R_2). \tag{3.22}$$

For $\mathcal{D}F = \langle \nabla F, z\rangle/2 - F$, on $\{h > 1 + 2\delta\}$ the chain rule gives

$$\begin{aligned} \mathcal{D}F_s &= w_1 s\kappa + w_2 L + \tfrac{1}{2}(t\vartheta'(t) - \vartheta(t)) \\ &\ge \min\{s\kappa, L\} - \varepsilon/2 > c, \end{aligned} \tag{3.23}$$

where $t = A_s - b|z|^2 + L$ and w_1, w_2 are the nonnegative partial derivatives of mx, with $w_1 + w_2 = 1$.

To restore the exact quadratic tail, take a smooth nonincreasing $\chi : [0,\infty) \to [0,1]$, equal to 1 on $[0,1]$ and 0 on $[2,\infty)$. Choose

$$T > R_2^2, \qquad (L/T)\|\chi'\|_\infty < \tfrac{1}{2}\operatorname{dist}(b, \pi\mathbb{Z}).$$

Set

$$\widehat{H}_s(z) = F_s(z) + L\big(1 - \chi(|z|^2/T)\big). \tag{3.24}$$

For $|z| \ge R_2$ this equals $q(|z|^2)$, where

$$\begin{aligned} q(t) &= bt - L\chi(t/T), \\ q'(t) &= b - (L/T)\chi'(t/T) \notin \pi\mathbb{Z}. \end{aligned}$$

A trajectory meeting this region stays on its radial sphere by uniqueness of the Hamiltonian flow, and hence cannot be one-periodic. Thus every nonconstant one-periodic orbit lies in $\{|z| < R_2\}$, where $\widehat{H}_s = F_s$. If it meets $F_s > A_s$, its energy exceeds $Mf(1+2\delta)$ by (3.21). It cannot enter $\{h \le 1 + 2\delta\}$, where $F_s = A_s \le Mf(1+2\delta)$; (3.23) therefore makes its action $\int_0^1 \mathcal{D}F_s(x(t))\,dt > c$. Otherwise it is an A_s-orbit (the nonnegative difference $F_s - A_s$ has zero gradient at its zeros). Its level $h = r$ is constant. For $r \ge 1 + 2\delta$ its action is $s\kappa > c$; for $1 + \delta < r < 1 + 2\delta$ its characteristic period and action satisfy

$$\begin{aligned} P &= sf'(r) \in \mathcal{S}(C), \\ a &= s(rf'(r) - f(r)), \\ (1+\delta)P \le a &\le (1+2\delta)P. \end{aligned}$$

Here convexity gives $f(r) \le (r - 1 - \delta)f'(r)$. Thus $a = c$ contradicts (3.18); constant orbits have action $-\widehat{H}_s \le 0$.

The family vanishes on $\{h < 1+\delta\}$ and has the asserted common quadratic tail. The smooth maximum is 1-Lipschitz in its first argument, so the compactly supported differences depend Lipschitz continuously on s. Finally,

$$\begin{aligned} 0 \le \widehat{H}_s - A_s &\le (b + \varepsilon/2)|z|^2 \mathbf{1}_{\{|z|>R\}}, \\ \int_0^1 (\widehat{H}_s - A_s)(x(t))\,dt &\le \frac{(b+\varepsilon/2)C_4^4 B^4}{R^2} < \zeta. \end{aligned}$$

The pointwise bound uses $\mathrm{mx} \le \max + \varepsilon/2$ before the radial correction and $\widehat{H}_s \le b|z|^2$ during it. This proves (3.20). □

4. Comparison and convex approximation

Lemma 4.1. *Let $0 \in \operatorname{int} C$ and let $v \in H^1$ satisfy $\mathfrak{a}(v) > 0$ and $\mathcal{H}_C(\dot v) \le b\mathfrak{a}(v)$ for some $b > 0$. For $r > 0$ and $y \in E^- \oplus E^0$,*

$$\mathfrak{a}(y + rv) \le b \int_0^1 h_C(y(t) + rv(t))\, dt. \tag{4.1}$$

Proof. Put $x = y + rv$. The assertion is immediate if $\mathfrak{a}(x) \le 0$. Otherwise set $B(v,x) = \int_0^1 \langle -J_0\dot v, x\rangle\, dt$. Since $\mathfrak{a}(x - rv) = \mathfrak{a}(y) \le 0$,

$$\begin{aligned} rB(v,x) &= \mathfrak{a}(x) - \mathfrak{a}(y) + r^2\mathfrak{a}(v) \\ &\ge \mathfrak{a}(x) + r^2\mathfrak{a}(v) \ge 2r\sqrt{\mathfrak{a}(x)\mathfrak{a}(v)} > 0. \end{aligned}$$

Fenchel's inequality for $t > 0$ gives

$$tB(v,x) - t^2\mathcal{H}_C(\dot v) \le \int_0^1 h_C(x(t))\, dt.$$

Its maximum over positive t is attained at $t = B(v,x)/(2\mathcal{H}_C(\dot v))$. Hence

$$\int_0^1 h_C(x(t))\, dt \ge \frac{B(v,x)^2}{4\mathcal{H}_C(\dot v)} \ge \frac{\mathfrak{a}(x)}{b}.$$

Only positive scalings of $\dot v$ were used. □

Proposition 4.2. *If D is smooth and positively curved, with $0 \in \operatorname{int} D$, then $c_k^{\mathrm{EH}}(D) = s_k(D)$ for every $k \ge 1$.*

Proof. Write $h = h_D$ and $s = s_k(D) \in (0,\infty)$. The upper inequality is proved in [BBLM23, equation (3.3), Proposition 3.1]. For its proof, fix $b > s \ge s_1(D) = \min \mathcal{S}(D)$. By [BBLM23, Lemma 3.5], choose a compact invariant set $V_b \subset E$ of mean-zero H^1 primitives and put $W_b = \{rv : r > 0,\ v \in V_b\}$. These sets satisfy

$$\begin{gathered} \mathfrak{a}(v) > 0, \qquad \mathcal{H}_D(\dot v) \le b\mathfrak{a}(v) \quad (v \in V_b), \\ \operatorname{ind}_{\mathrm{FR}}(W_b) \ge \operatorname{ind}_{\mathrm{FR}}\{\Psi_D \le b\} \ge k. \end{gathered}$$

Put $Z_b = E^- + E^0 + W_b$. Lemma 4.1 gives $\mathfrak{a}(x) \le b\int_0^1 h(x(t))\, dt$ on Z_b. For every $\varepsilon > 0$, the majorant in [BBLM23, Lemma 3.3] satisfies $H_b \in \mathcal{F}(D)$ and $H_b \ge bh - b - \varepsilon$. Together with the linking statement for this carrier [BBLM23, Lemma 3.6], this yields

$$\sup_{Z_b} \Phi_{H_b} \le b + \varepsilon, \qquad \operatorname{ind}_{\mathrm{EH}}(Z_b) \ge k.$$

Every strict level above $b+\varepsilon$ therefore qualifies in the minimax, so $c_k^{\mathrm{EH}}(D) \le c_k^{\mathrm{EH}}(H_b) \le b + \varepsilon$. Let $\varepsilon \downarrow 0$ and $b \downarrow s$.

For the lower inequality, fix $H \in \mathcal{F}(D)$ and $c \in (0,s) \setminus \mathcal{S}(D)$. Choose

$$c < a < \tau < \tau' < s$$

and fix the graph and radius B_0 supplied by Proposition 3.4 for these parameters. Since H vanishes near D and $\mathcal{S}(D)$ is closed, choose $\delta > 0$ so that

$$H = 0 \text{ on } \{h \le 1 + 2\delta\},$$
$$[c/(1+2\delta), c] \cap \mathcal{S}(D) = \varnothing.$$

For the profile f in (3.19), the quotient $H/f(h)$ is bounded where defined: H vanishes near its denominator's zero set, and both functions have quadratic growth. Choose $M > \tau$ with $H \le Mf(h)$. Apply Lemma 3.5 with error $\zeta = 1$, radius B_0, and common slope

$$b_\infty > \max\{\pi\lceil k/n\rceil, M \max_{|z|=1} h(z)\},$$
$$b_\infty \notin \pi\mathbb{N}.$$

For $\|x\|_E \le B_0$, (3.20) and $f(h) \le h$ give

$$\int_0^1 (\widehat{H}_\tau - \tau h)(x(t))\,dt \le \int_0^1 (\widehat{H}_\tau - A_\tau)(x(t))\,dt < 1.$$

Proposition 3.4 therefore gives

$$\operatorname{ind}_{\mathrm{EH}}\{\Phi_{\widehat{H}_\tau} < a\} \le \operatorname{ind}_{\mathrm{FR}}\{\Psi_D < \tau'\} < k,$$
$$c_k^{\mathrm{EH}}(\widehat{H}_\tau) \ge a > c.$$

The choice of b_∞ and Proposition 2.2 make $\nu \mapsto c_k^{\mathrm{EH}}(\widehat{H}_\nu)$ finite-valued and continuous on $[\tau, M]$. Spectrality and the exclusion of c in Lemma 3.5 give $c_k^{\mathrm{EH}}(\widehat{H}_\nu) > c$ throughout this interval. At the endpoint, Hamiltonian monotonicity gives

$$H \le Mf(h) \le \widehat{H}_M,$$
$$c_k^{\mathrm{EH}}(H) \ge c_k^{\mathrm{EH}}(\widehat{H}_M) > c.$$

Take the infimum over H, then let $c \uparrow s$ through the complement of $\mathcal{S}(D)$, using Lemma 2.4. □

Lemma 4.3. *Every convex body C with $0 \in \operatorname{int} C$ admits smooth positively curved approximants D_ε and $\eta_\varepsilon \to 0$, $0 < \eta_\varepsilon < 1$, such that*

$$\sqrt{1-\eta_\varepsilon}\,C \subset D_\varepsilon \subset \sqrt{1+\eta_\varepsilon}\,C. \tag{4.2}$$

The approximants may preserve every orthogonal symmetry of C.

Proof. Choose $R > 0$ with $C \subset B^{2n}(R)$ and a radial nonnegative smooth unit-mass mollifier ρ_ε supported in $B^{2n}(\varepsilon)$. Put

$$g_\varepsilon = h_C * \rho_\varepsilon + \varepsilon|z|^2, \qquad D_\varepsilon = \{g_\varepsilon \le 1\}. \tag{4.3}$$

Then

$$D^2 g_\varepsilon \ge 2\varepsilon I,$$
$$g_\varepsilon(z) \ge \frac{(\max\{|z| - \varepsilon, 0\})^2}{R^2},$$

Thus $D_\varepsilon \subset B^{2n}(R+\varepsilon)$, and $g_\varepsilon \to h_C$ locally uniformly. For small ε, $g_\varepsilon(0) < 1$ and on its unit level

$$\langle \nabla g_\varepsilon(z), z\rangle \geq 1 - g_\varepsilon(0) > 0.$$

Thus D_ε is smooth and positively curved. For small ε, both C and D_ε lie in $B^{2n}(2R)$. Set

$$\eta_\varepsilon = \varepsilon + \sup_{B^{2n}(2R)} |g_\varepsilon - h_C| \longrightarrow 0.$$

Then $\{h_C \leq 1-\eta_\varepsilon\} \subset D_\varepsilon \subset \{h_C \leq 1+\eta_\varepsilon\}$, giving (4.2). Radial convolution and $|z|^2$ preserve every orthogonal symmetry of h_C. □

Proof of Theorem A. First suppose $0 \in \operatorname{int} C$. Both values on C are finite and positive: for s_k use (2.12), and for c_k^{EH} compare with inner and outer smooth balls and apply Proposition 4.2. Take the approximants from Lemma 4.3. For $I_k = s_k$ and $I_k = c_k^{\mathrm{EH}}$, monotonicity and scaling give

$$(1-\eta_\varepsilon) I_k(C) \leq I_k(D_\varepsilon) \leq (1+\eta_\varepsilon) I_k(C).$$

Since $s_k(D_\varepsilon) = c_k^{\mathrm{EH}}(D_\varepsilon)$ by Proposition 4.2, letting $\varepsilon \downarrow 0$ proves $s_k(C) = c_k^{\mathrm{EH}}(C)$. For arbitrary $p \in \operatorname{int} C$, translation invariance gives $s_k(C-p) = c_k^{\mathrm{EH}}(C-p) = c_k^{\mathrm{EH}}(C)$. □

Corollary 4.4. *If $rB^{2n}(1) \subset C$ and the Hausdorff distance between convex bodies C, D is $\delta < r$, then, for $I_k = s_k$ or $I_k = c_k^{\mathrm{EH}}$,*

$$(1-\delta/r)^2 I_k(C) \leq I_k(D) \leq (1+\delta/r)^2 I_k(C). \tag{4.4}$$

In particular both sequences are termwise Hausdorff continuous.

Proof. The support functions satisfy $|\sigma_D(v) - \sigma_C(v)| \leq \delta|v|$ and $\sigma_C(v) \geq r|v|$. Hence $(1-\delta/r)C \subset D \subset (1+\delta/r)C$. Apply monotonicity, scaling, and translation invariance. □

5. Ordinary ellipsoid computations

For the ellipsoid $E(\boldsymbol{a})$ defined in the introduction, let $\mu_k(\boldsymbol{a})$ be the kth element of $\{ma_j : m \geq 1,\ 1 \leq j \leq n\}$, counted with multiplicity. Then [EH90, Proposition 4]

$$s_k(E(\boldsymbol{a})) = c_k^{\mathrm{EH}}(E(\boldsymbol{a})) = \mu_k(\boldsymbol{a}). \tag{5.1}$$

Indeed, for $u_j(t) = \sum_{m\neq 0} u_{jm} e^{2\pi i m t}$,

$$\begin{aligned} 4\pi \mathcal{H}_{E(\boldsymbol{a})}(u) &= \sum_{j,m\neq 0} a_j |u_{jm}|^2, \\ 4\pi \mathcal{A}(u) &= \sum_{j,m\neq 0} \frac{|u_{jm}|^2}{m}. \end{aligned} \tag{5.2}$$

For $a > 0$, define

$$q_a(u) = \sum_{j,m\neq 0} (a_j - a/m)|u_{jm}|^2. \tag{5.3}$$

Since $q_a = 4\pi(\mathcal{H}_{E(\boldsymbol{a})} - a\mathcal{A})$ and $\mathcal{H}_{E(\boldsymbol{a})}(u) > 0$ for $u \neq 0$, the strict sublevel is

$$\{\Psi_{E(\boldsymbol{a})} < a\} = \{u \in L^2_0 : \|u\|_2 = 1,\ q_a(u) < 0\}.$$

Let V_a consist of modes $m > 0$ with $ma_j < a$. Writing $u = v + w \in V_a \oplus V_a^{\perp}$, the equivariant retraction

$$u \longmapsto \frac{v + (1-t)w}{\|v + (1-t)w\|_2}, \qquad 0 \le t \le 1, \tag{5.4}$$

stays in that sublevel because $q_a(v) < 0 \le q_a(w)$ and

$$q_a(v) + (1-t)^2 q_a(w) \le q_a(u) < 0.$$

Lemma 2.1 therefore gives

$$\begin{aligned}\operatorname{ind}_{\mathrm{FR}}\{\Psi_{E(\boldsymbol{a})} < a\} &= \dim_{\mathbb{C}} V_a \\ &= \#\{(j,m) : m \ge 1,\ ma_j < a\},\end{aligned}$$

including zero when $V_a = 0$. Taking the threshold proves (5.1).

For $a_1 = \cdots = a_n = \pi R^2$ this yields

$$s_k(B^{2n}(R)) = c_k^{\mathrm{EH}}(B^{2n}(R)) = \pi R^2 \lceil k/n \rceil.$$

For example, $E(a, 2a)$ has sequence $a, 2a, 2a, 3a, 4a, 4a, 5a, \ldots$. A pure mth mode has stabilizer $\mathbb{Z}/m\mathbb{Z}$, allowed by the rational index.

6. Real loops and the half-period action

Let ρ be a linear anti-symplectic involution, and let $C = \rho C$ be a convex body. Set $L = \operatorname{Fix}(\rho)$. The eigenspaces $L_{\pm} = \ker(\rho \mp I)$ are complementary Lagrangians. Dual bases for the nondegenerate pairing $\omega_0 : L_+ \times L_- \to \mathbb{R}$ give a symplectic map A such that

$$A\rho_0 = \rho A. \tag{6.1}$$

The adapted structures

$$J = AJ_0A^{-1}, \qquad g(v,w) = \omega_0(v, Jw)$$

satisfy $\rho J = -J\rho$ and make ρ an isometry. Work in these adapted coordinates, with $\rho = \rho_0$, $J = J_0$, and $L = \mathbb{R}^n \times \{0\}$; the Hamiltonian tails use this adapted norm. Convexity and ρ-invariance give $(z + \rho z)/2 \in L \cap \operatorname{int} C$ for every $z \in \operatorname{int} C$. Until centering independence is established, assume $0 \in \operatorname{int} C$ and put $h = h_C$.

The gauge and its Legendre transform in (1.1) and (2.13) are ρ-invariant. Define the closed real spaces E_ρ, V_ρ and the positive-action unit slice Λ_ρ by

$$\begin{aligned}E_\rho &= \{x \in E : x(-t) = \rho x(t)\},\\ V_\rho &= \{u \in L^2_0 : u(-t) = -\rho u(t)\},\\ \Lambda_\rho &= \Lambda \cap V_\rho.\end{aligned} \tag{6.2}$$

They carry the induced strong topologies. The primitive (1.2) maps V_ρ isomorphically to $H_0^1 \cap E_\rho$, compactly into E_ρ. The Fourier splitting (2.2) restricts to

$$\begin{aligned} x(t) &= \sum_{m\in\mathbb{Z}} e^{2\pi m J_0 t} x_m, \qquad x_m \in L, \\ E_\rho &= E_\rho^- \oplus E_\rho^0 \oplus E_\rho^+, \qquad E_\rho^0 = L. \end{aligned} \tag{6.3}$$

The norm, projections, embeddings, and action are inherited from E. Restrict $\mathcal{A}$ and $\mathcal{H}_C$ to V_ρ, and Ψ_C to Λ_ρ, where its index sublevels are taken. The continuity and quadratic energy estimates of Lemma 2.3 apply to these restrictions.

Half-period translation

$$(\sigma x)(t) = x(t+\tfrac{1}{2}), \qquad (\sigma u)(t) = u(t+\tfrac{1}{2}) \tag{6.4}$$

preserves the real spaces and functionals and acts on mode m by $(-1)^m$. The first mode shows that a translation by θ preserves every brake loop only if $\sin(2\pi\theta) = 0$. Thus the residual translation group is $\mathbb{Z}_2 = \{0, 1/2\}$; brake reflection itself acts trivially on E_ρ. For an absolutely continuous brake loop, $x(0), x(1/2) \in L$ and $\rho^*\lambda_0 = -\lambda_0$ give

$$\int_x \lambda_0 = 2\int_0^{1/2} \lambda_0(x)[\dot{x}]\, dt.$$

Brake characteristic actions use this full-period normalization.

Proposition 6.1. *For every centered invariant convex body with $B^{2n}(r) \subset C \subset B^{2n}(R)$, $0 < r \le R$,*

$$\begin{aligned} \beta_\rho(C) := \inf_{\Lambda_\rho} \Psi_C = \min_{\substack{u\in V_\rho \\ \mathcal{A}(u)=1}} \mathcal{H}_C(u), \\ \pi r^2 \le \beta_\rho(C) \le \pi R^2. \end{aligned} \tag{6.5}$$

This is the least generalized brake action on ∂C, equivalently the least $T > 0$ admitting an absolutely continuous $y : \mathbb{R}/\mathbb{Z} \to \partial C$ with

$$\begin{aligned} y(-t) &= \rho y(t), \\ \dot{y}(t) &= TJp(t), \qquad p(t) \in \partial h(y(t)) \quad a.e. \end{aligned} \tag{6.6}$$

Its full action is T. For smooth C, these are Hamiltonian trajectories.

Proof. Jin–Lu's dual formula [JL20, Proposition 7.1, $p = 2$] applies to the mean-zero H^1 brake loops $z = \Pi u$ with $\mathfrak{a}(z) = 1$; their $H_C = j_C^2$ and action (1.8) agree with ours. It gives attainment and the brake minimum without boundary regularity. For the normalization, their proof, Steps 2–4, yields, at a minimizer of value s, a constant $c \in L$ such that

$$q = s\Pi u + c \in \partial h^*(-Ju), \qquad h(q) \equiv s. \tag{6.7}$$

Thus $y = q/\sqrt{s}$ satisfies (6.6) with $T = s$. Conversely, (6.6) gives $\mathfrak{a}(y) = T$ and $\mathcal{H}_C(\dot{y}/\sqrt{T}) = T$ by Euler and Fenchel identities. Positive homogeneity

identifies the two infima in (6.5). The bounds follow from the dual estimates in Lemma 2.3 and the first positive brake mode. □

Remark 6.2. This minimum equals Jin–Lu's real EH capacity [JL20, Theorems 1.3, 1.11 and Remark 1.12].

Proposition 6.3. *For symplectic A and $\widehat{\rho} = A\rho A^{-1}$, the map*

$$\Lambda_\rho \longrightarrow \Lambda_{\widehat{\rho}}, \qquad u \longmapsto \frac{Au}{\|Au\|_2}$$

is an equivariant homeomorphism preserving the quotient for C and AC. Moreover, $\beta_\rho(C-p)$ is independent of $p \in L \cap \operatorname{int} C$.

Proof. In the original Euclidean coordinates,

$$h^*_{AC}(q) = h^*_C(A^T q), \qquad -J_0 A = A^{-T}(-J_0),$$
$$\Pi(Au) = A\Pi u,$$
$$\mathcal{H}_{AC}(Au) = \mathcal{H}_C(u), \qquad \mathcal{A}(Au) = \mathcal{A}(u).$$

Homogeneity proves the first assertion; the inverse uses A^{-1}. The second follows from the first-capacity identification above and its fixed-translation invariance [JL20, Propositions 1.7(i), 1.10]. □

7. The mod-two index and linking

All spaces are metrizable and all actions continuous. We use singular cohomology over $\mathbb{F}_2$. Take $E\mathbb{Z}_2 = S(\ell^2(\mathbb{R}))$ with the antipodal action and put $B\mathbb{Z}_2 = E\mathbb{Z}_2/\mathbb{Z}_2$. For a $\mathbb{Z}_2$-space X, write $X_{\mathbb{Z}_2} = E\mathbb{Z}_2 \times_{\mathbb{Z}_2} X$ and $H^*_{\mathbb{Z}_2}(X) = H^*(X_{\mathbb{Z}_2};\mathbb{F}_2)$. The canonical projection is

$$p_X : X_{\mathbb{Z}_2} \longrightarrow B\mathbb{Z}_2, \qquad [e,x] \longmapsto [e].$$

Let $w \in H^1(B\mathbb{Z}_2;\mathbb{F}_2)$ be the nonzero class, so that $H^*(B\mathbb{Z}_2;\mathbb{F}_2) = \mathbb{F}_2[w]$. The cohomological index [FR78] is

$$\operatorname{ind}_{\mathbb{Z}_2}(X) = \sup\{j \geq 1 : p_X^*(w^{j-1}) \neq 0\}, \qquad \operatorname{ind}_{\mathbb{Z}_2}(\varnothing) = 0.$$

Below, w also denotes its pullback to each Borel space. In relative cohomology, multiplication by w means cup product with this absolute class.

An equivariant map $f : X \to Y$ induces $f_{\mathbb{Z}_2} : X_{\mathbb{Z}_2} \to Y_{\mathbb{Z}_2}$ with $p_Y \circ f_{\mathbb{Z}_2} = p_X$. Hence $\operatorname{ind}_{\mathbb{Z}_2}(X) \leq \operatorname{ind}_{\mathbb{Z}_2}(Y)$, and equivariant homotopy equivalences preserve the index. A fixed point gives a section of p_X, so $X^{\mathbb{Z}_2} \neq \varnothing$ implies $\operatorname{ind}_{\mathbb{Z}_2}(X) = \infty$.

For a centered invariant convex body define

$$s_k(C,\rho) = \inf\{a > 0 : \operatorname{ind}_{\mathbb{Z}_2}(\{u \in \Lambda_\rho : \Psi_C(u) < a\}) \geq k\}, \qquad k \geq 1. \quad (7.1)$$

Every nonempty space has index at least one, so $s_1(C,\rho) = \beta_\rho(C)$. The support-function identity (2.13) gives $C \subset D \Rightarrow \mathcal{H}_C \leq \mathcal{H}_D$ and $\mathcal{H}_{rC} = r^2\mathcal{H}_C$. Sublevel inclusions therefore give monotonicity and quadratic homogeneity of every $s_k(\cdot,\rho)$.

Lemma 7.1. *An invariant open cover $X = U \cup V$ satisfies* $\operatorname{ind}_{\mathbb{Z}_2}(X) \leq \operatorname{ind}_{\mathbb{Z}_2}(U) + \operatorname{ind}_{\mathbb{Z}_2}(V)$.

Proof. For finite $a = \operatorname{ind}_{\mathbb{Z}_2}(U)$ and $b = \operatorname{ind}_{\mathbb{Z}_2}(V)$, the pairs (X, U) and (X, V) give relative lifts of w^a and w^b. Their cup product gives

$$w^{a+b} = 0 \quad \text{in } H^*_{\mathbb{Z}_2}(X), \qquad H^*_{\mathbb{Z}_2}(X, U \cup V) = 0.$$

The relative product applies because the Borel cover is open. □

Lemma 7.2 (Compact detection)**.** *If $\operatorname{ind}_{\mathbb{Z}_2}(X) \geq k$ for a finite $k \geq 1$, some compact invariant $K \subset X$ satisfies $\operatorname{ind}_{\mathbb{Z}_2}(K) \geq k$.*

Proof. Over a field, the nonzero class w^{k-1} pairs with a finite singular cycle. Its support $Z \subset X_{\mathbb{Z}_2}$ is compact. The free diagonal action makes $E\mathbb{Z}_2 \times X \to X_{\mathbb{Z}_2}$ a two-sheeted covering, hence a proper map. Project the compact inverse image of Z to X. The resulting invariant compact set K supports the same cycle in $K_{\mathbb{Z}_2}$ and the same nonzero pairing. □

A cone is an invariant subset Y of a normed real representation containing zero and closed under nonnegative scalar multiplication. Write $Y_* = Y \setminus \{0\}$; $\mathbb{R}_{\mathrm{sgn}}$ denotes the sign line. The next three lemmas adapt the suspension and relative-class argument of [BBLM23, Lemmas A.2–A.3, Proposition A.4] to the degree-one class w, allowing fixed directions in Y.

Lemma 7.3 (Sign stabilization)**.** *For every cone Y and integer $q \geq 0$,*

$$\operatorname{ind}_{\mathbb{Z}_2}\big((Y \oplus \mathbb{R}^q_{\mathrm{sgn}})_*\big) = \operatorname{ind}_{\mathbb{Z}_2}(Y_*) + q, \qquad \infty + q = \infty.$$

Proof. For $q = 1$, the empty and infinite-index cases follow from the two-point sign sphere and monotonicity. Otherwise put $k = \operatorname{ind}_{\mathbb{Z}_2}(Y_*) \in \mathbb{N}$ and $X = (Y \oplus \mathbb{R}_{\mathrm{sgn}})_*$. The open cover

$$X = (Y_* \oplus \mathbb{R}_{\mathrm{sgn}}) \cup (Y \oplus (\mathbb{R}_{\mathrm{sgn}})_*)$$

gives $k \leq \operatorname{ind}_{\mathbb{Z}_2}(X) \leq k + 1$. The sign line bundle over $X_{\mathbb{Z}_2}$ has Euler class w and sphere bundle $E\mathbb{Z}_2 \times X$. Its mod-two Gysin sequence contains

$$H^{k-1}(E\mathbb{Z}_2 \times X) \xrightarrow{\mathrm{tr}} H^{k-1}_{\mathbb{Z}_2}(X) \xrightarrow{\smile w} H^k_{\mathbb{Z}_2}(X).$$

Here tr is the transfer. If $\operatorname{ind}_{\mathbb{Z}_2}(X) = k$, lift w^{k-1} to the first group. The inclusion $Y_* \to X$ is nullhomotopic by

$$(y, t) \longmapsto ((1 - t)y, t), \qquad 0 \leq t \leq 1.$$

For $k > 1$, the induced map $E\mathbb{Z}_2 \times Y_* \to E\mathbb{Z}_2 \times X$ has zero pullback in degree $k - 1$. Naturality of transfer therefore forces $w^{k-1} = 0$ on $(Y_*)_{\mathbb{Z}_2}$. For $k = 1$, X is path connected: the two sign rays connect through any nonzero point of Y. The degree-zero transfer of its connected two-sheeted covering is multiplication by $2 = 0$ in $\mathbb{F}_2$, so exactness forces $w \neq 0$ on $X_{\mathbb{Z}_2}$. Both cases contradict $\operatorname{ind}_{\mathbb{Z}_2}(X) = k$. Induction proves the formula. □

Lemma 7.4 (Relative cone class)**.** *Let Y be a cone and $T = \mathbb{R}^d$ a trivial representation, $d \geq 1$. Choose a generator $\mu \in H^{d-1}(T_*; \mathbb{F}_2)$ for $d > 1$, and a nonconstant $\mu \in H^0(T_*; \mathbb{F}_2)$ for $d = 1$. Under $(T_*)_{\mathbb{Z}_2} = B\mathbb{Z}_2 \times T_*$, we also denote by μ its pullback from T_*. Let*

$$\delta : H^*_{\mathbb{Z}_2}(T_*) \longrightarrow H^{*+1}_{\mathbb{Z}_2}((Y \oplus T)_*, T_*)$$

be the connecting map. Then, for every $j \geq 0$,

$$w^j \delta\mu \neq 0 \quad \Longleftrightarrow \quad w^j \neq 0 \text{ on } (Y_*)_{\mathbb{Z}_2}.$$

Proof. Apply Mayer–Vietoris to

$$(Y \oplus T)_* = (Y_* \oplus T) \cup (Y \oplus T_*).$$

Since Y contracts equivariantly to zero and $T_* \simeq S^{d-1}$, the Künneth formula identifies the restriction-difference map as

$$H^*_{\mathbb{Z}_2}(Y_*) \oplus H^*_{\mathbb{Z}_2}(T_*) \longrightarrow H^*_{\mathbb{Z}_2}(Y_*) \otimes H^*(T_*),$$
$$(\eta, w^j\mu) \longmapsto \eta \otimes 1 + w^j|_{Y_*} \otimes \mu.$$

The classes $1, \mu$ are independent, including when $d = 1$. Thus $w^j\mu$ extends from T_* to $(Y \oplus T)_*$ exactly when $w^j|_{Y_*} = 0$. Exactness for the pair proves the assertion; δ commutes with multiplication by w. □

Lemma 7.5 (Piercing)**.** *Let Y be a cone in a finite-dimensional representation, $T = \mathbb{R}^d$ trivial with $d \geq 1$, and B an invariant compact neighborhood of zero in the ambient space of $Y \oplus T$. Suppose*

$$\psi : Y \oplus T \longrightarrow \mathbb{R}^q_{\mathrm{sgn}} \oplus T \quad \text{is continuous and equivariant,}$$
$$\psi(0, t) = (0, t).$$

Every invariant relatively open neighborhood $U \subset (Y \oplus T)_*$ of $\psi^{-1}(0) \cap \partial B$ satisfies

$$\mathrm{ind}_{\mathbb{Z}_2}(U) + q \geq \mathrm{ind}_{\mathbb{Z}_2}(Y_*).$$

Proof. Put $X = (Y \oplus T)_*$ and suppose $l = \mathrm{ind}_{\mathbb{Z}_2}(U) < \infty$, $l + q < \mathrm{ind}_{\mathbb{Z}_2}(Y_*)$. Set

$$V = X \setminus \psi^{-1}(0), \qquad W = U \cup V,$$
$$X' = (X \cap B) \cup W, \qquad X'' = (X \setminus B) \cup W.$$

Since $\psi|_{T_*} = \mathrm{id}$, we have $T_* \subset V \subset W$. As W contains $X \cap \partial B$, the sets X', X'' form an open cover of X with intersection W. Choose invariant balls about zero, the smaller contained in $\operatorname{int} B$ and the larger containing B. Radial shrinking into the smaller ball and expansion beyond the larger give homotopy right inverses to the inclusions, as pairs relative to T_*. Both restrictions from $H^*_{\mathbb{Z}_2}(X, T_*)$ are therefore injective. Mayer–Vietoris also gives an injection

$$H^*_{\mathbb{Z}_2}(X, T_*) \longrightarrow H^*_{\mathbb{Z}_2}(W, W \cap T_*).$$

Indeed, if $\alpha|_W = 0$, the matching pair $(\alpha|_{X'}, 0)$ extends to β on (X, T_*); injectivity on X'' gives $\beta = 0$, and injectivity on X' gives $\alpha = 0$.

For $\eta = \delta\mu$, Lemma 7.4 yields

$$w^{l+q}\eta \neq 0 \quad \text{in } H^*_{\mathbb{Z}_2}(W, W \cap T_*).$$

On the target pair $((\mathbb{R}^q_{\mathrm{sgn}} \oplus T)_*, T_*)$, $w^q \delta\mu = 0$ by sign stabilization and the relative cone lemma (also for $q = 0$, when the pair is identical). Pullback by ψ therefore gives $w^q\eta = 0$ on $(V, V \cap T_*)$. The triple $W \cap T_* \subset V \subset W$ and the pair (W, U) give the respective lifts

$$w^q\eta \in H^*_{\mathbb{Z}_2}(W, V), \qquad w^l \in H^*_{\mathbb{Z}_2}(W, U).$$

Their relative product lies in $H^*_{\mathbb{Z}_2}(W, U \cup V) = 0$, contradicting the displayed nonvanishing. □

On the brake space put

$$S^+_\rho = \{v \in E^+_\rho : \|v\|_E = 1\}.$$

Let Γ_ρ consist of the halfshift-equivariant homeomorphisms

$$\Theta(x) = e^{\theta_+(x)}x^+ + x^0 + e^{\theta_-(x)}x^- + K(x) \qquad (x \in E_\rho), \tag{7.2}$$

with precisely the boundedness, compactness, and vanishing conditions of (2.4), now on E_ρ. The coefficient bounds, compactness of K, and cutoff outside a ball ensure that these maps and their inverses preserve bounded sets. They preserve $\{\mathfrak{a} > 0\}$ because they fix its complement. Solving the formula for the inverse and composing maps preserve its form and conditions. Hence Γ_ρ is a group. Define

$$\mathrm{ind}_{\mathrm{EH},\rho}(A) = \inf_{\Theta \in \Gamma_\rho} \inf_{\substack{O \supset A \cap \Theta(S^+_\rho) \\ O \text{ open, invariant in } E_\rho}} \mathrm{ind}_{\mathbb{Z}_2}(O). \tag{7.3}$$

Take the Fourier truncations $F_N \subset E_\rho$ with $|m| \leq N$ and orthogonal projections P_N.

Proposition 7.6 (Compact-cone linking)**.** *Let $V \subset \{\mathfrak{a} > 0\}$ be compact and invariant, and put $\mathcal{W} = \{rv : r > 0,\ v \in V\}$. Then*

$$\mathrm{ind}_{\mathrm{EH},\rho}(E^-_\rho + E^0_\rho + \mathcal{W}) \geq \mathrm{ind}_{\mathbb{Z}_2}(\mathcal{W}).$$

Positive fixed directions are allowed.

Proof. The empty case is immediate. By inversion in Γ_ρ, it suffices to prove $\mathrm{ind}_{\mathbb{Z}_2}(O) \geq \mathrm{ind}_{\mathbb{Z}_2}(\mathcal{W})$ whenever

$$O \supset \Theta(E^-_\rho + E^0_\rho + \mathcal{W}) \cap S^+_\rho, \qquad O \text{ open and invariant.}$$

Write $F^\pm_N = F_N \cap E^\pm_\rho$, $P^\pm_N = P_N P^\pm$. Compactness and positive action give $\min_{v \in V} \|P^+ v\| > 0$. Uniform convergence of P^+_N on V gives

$$\inf_{v \in V} \|P^+_N v\| \geq \varepsilon > 0 \quad \text{for all sufficiently large } N.$$

Decompose the negative truncation and define

$$\begin{aligned} F_N^- &= N_N \oplus T_N^-, \\ T_N &= T_N^- \oplus E_\rho^0, \\ W_N &= P_N^+(\mathcal{W}) \cup \{0\}, \\ Y_N &= N_N \oplus W_N. \end{aligned}$$

Here N_N consists of negative odd modes, and T_N is the trivial representation of negative even modes and constants. Positive even modes remain in W_N. The full negative space has infinite index, but $q_N = \dim N_N$ is finite. Monotonicity and sign stabilization give

$$\begin{aligned} \operatorname{ind}_{\mathbb{Z}_2}((Y_N)_*) &= q_N + \operatorname{ind}_{\mathbb{Z}_2}((W_N)_*) \\ &\geq q_N + \operatorname{ind}_{\mathbb{Z}_2}(\mathcal{W}). \end{aligned}$$

We claim, for large N,

$$P_N\Theta(Y_N \oplus T_N) \cap S_\rho^+ \subset O.$$

Otherwise choose $x_N \in Y_N \oplus T_N$ with $P_N\Theta(x_N) \in S_\rho^+ \setminus O$. The identity of Θ outside a ball bounds x_N, and

$$x_N^- + x_N^0 = -e^{-\theta_-(x_N)} P_N^- K(x_N) - P^0 K(x_N)$$

is precompact. Write $x_N^+ = r_N P_N^+ v_N$, with $v_N \in V$, $r_N \geq 0$. Since $\|x_N^+\| = r_N \|P_N^+ v_N\| \geq \varepsilon r_N$, the sequence (r_N) is bounded. Passing to a subsequence, we obtain

$$\begin{gathered} r_N \to r, \qquad v_N \to v, \qquad x_N \to x, \\ x^+ = rP^+ v, \qquad \Theta(x) \in S_\rho^+ \setminus O. \end{gathered}$$

If $r = 0$, then $x \in E_\rho^- \oplus E_\rho^0$ and $\Theta(x) = x \notin S_\rho^+$. If $r > 0$, then $x \in E_\rho^- + E_\rho^0 + \mathcal{W}$. Both cases contradict the choice of O.

Fix such an N and put

$$\begin{gathered} \phi = P_N\Theta|_{F_N}, \qquad B = \phi^{-1}\{z \in F_N : \|z\| \leq 1\}, \\ \psi = (P_N^- + P^0)\phi|_{Y_N \oplus T_N} : Y_N \oplus T_N \longrightarrow N_N \oplus T_N. \end{gathered}$$

Since ϕ is the identity outside a ball and on $F_N^- \oplus E_\rho^0$, B is a compact invariant neighborhood of zero and $\psi|_{T_N} = \mathrm{id}$. For $Z = \psi^{-1}(0) \cap \partial B$,

$$\phi(Z) \subset S_\rho^+ \cap O.$$

Choose an invariant relatively open neighborhood $U \subset (Y_N \oplus T_N)_*$ of Z with $\phi(U) \subset O$. Piercing gives

$$\begin{aligned} \operatorname{ind}_{\mathbb{Z}_2}(O) + q_N &\geq \operatorname{ind}_{\mathbb{Z}_2}(U) + q_N \\ &\geq \operatorname{ind}_{\mathbb{Z}_2}((Y_N)_*) \\ &\geq \operatorname{ind}_{\mathbb{Z}_2}(\mathcal{W}) + q_N. \end{aligned}$$

Since $q_N < \infty$, cancellation is valid, including when $\operatorname{ind}_{\mathbb{Z}_2}(\mathcal{W}) = \infty$. □

Lemma 7.7. *The halfshift fixed sector has attained minimum*

$$\min_{\Lambda_\rho \cap \mathrm{Fix}(\sigma)} \Psi_C = 2\beta_\rho(C). \tag{7.4}$$

Thus $\{\Psi_C < a\}$ *is free for* $a \le 2\beta_\rho(C)$ *and has infinite* $\mathbb{F}_2$ *Borel index for* $a > 2\beta_\rho(C)$.

Proof. Define

$$D_2 : V_\rho \longrightarrow V_\rho \cap \mathrm{Fix}(\sigma), \qquad (D_2 v)(t) = v(2t).$$

Fixed velocities are $1/2$-periodic, so the inverse is $e \mapsto (t \mapsto e(t/2))$. Moreover,

$$\begin{aligned} \|D_2 v\|_2 &= \|v\|_2, \\ \mathcal{H}_C(D_2 v) &= \mathcal{H}_C(v), \\ \mathcal{A}(D_2 v) &= \tfrac{1}{2}\mathcal{A}(v). \end{aligned}$$

Apply Proposition 6.1. A fixed point gives a section of the Borel projection, so all powers of its universal class survive. □

8. Real Hamiltonian minimax and comparison

For an invariant convex body C, put

$$\mathcal{F}_\rho(C) = \{H \in \mathcal{F}(C) : H \circ \rho = H\}.$$

Its quadratic tails use the adapted norm of Section 6. Restrict Φ_H from (2.7) to E_ρ and define

$$\begin{aligned} c_k^{\mathrm{EH}}(H,\rho) &= \inf\{a \in \mathbb{R} : \mathrm{ind}_{\mathrm{EH},\rho}(\{x \in E_\rho : \Phi_H(x) < a\}) \ge k\}, \\ c_k^{\mathrm{EH}}(C,\rho) &= \inf_{H \in \mathcal{F}_\rho(C)} c_k^{\mathrm{EH}}(H,\rho). \end{aligned} \tag{8.1}$$

Empty infima are $+\infty$. The normalized positive-projection retraction used in (2.6) restricts to E_ρ. Hence, for invariant open $V \subset E_\rho$,

$$\mathrm{ind}_{\mathrm{EH},\rho}(V) = \inf_{\Theta \in \Gamma_\rho} \mathrm{ind}_{\mathbb{Z}_2}(\Theta^{-1}(V) \cap S_\rho^+). \tag{8.2}$$

As in the ordinary minimax, strict sublevels give

$$c_k^{\mathrm{EH}}(H,\rho) = \inf_{\mathrm{ind}_{\mathrm{EH},\rho}(V) \ge k} \sup_{x \in V} \Phi_H(x).$$

Proposition 8.1. *If* $0 \in \operatorname{int} C$ *and* $H \in \mathcal{F}_\rho(C)$, *then* $c_k^{\mathrm{EH}}(H,\rho) > 0$; *all these values are finite if the quadratic tail has slope* $b > 2\pi$. *Every finite value is a critical value of* $\Phi_H|_{E_\rho}$, *attained at a one-periodic brake orbit of* H. *For admissible* ρ*-invariant Hamiltonians* H, G *with finite* k*th selectors and bounded difference,*

$$|c_k^{\mathrm{EH}}(H,\rho) - c_k^{\mathrm{EH}}(G,\rho)| \le \|H - G\|_\infty.$$

The Hamiltonian selectors decrease under increasing H. *The body selectors are monotone and homogeneous of degree two.*

Proof. The reflection $\mathcal{R}x(t) = \rho x(-t)$ is an isometric involution of E preserving Φ_H and each Fourier summand. Hence

$$\nabla(\Phi_H|_{E_\rho})(x) = \nabla\Phi_H(x) \in E_\rho \qquad (x \in E_\rho).$$

The small-sphere estimate, compact nonlinear term, nonresonant linear inverse, and slice deformation of Proposition 2.2 therefore restrict to E_ρ. Its Fourier regularity argument gives $x \in H^1$ and $\dot{x} = J_0\nabla H(x)$ at every critical point.

For finiteness let $\mathcal{V}_{\rho,2} = \{t \mapsto e^{4\pi J_0 t}v : v \in L\} \subset E_\rho^+$ be the second Fourier mode. Its sphere is fixed, so Proposition 7.6 gives

$$\operatorname{ind}_{\mathrm{EH},\rho}(E_\rho^- \oplus L \oplus \mathcal{V}_{\rho,2}) = \infty.$$

Since $H(z) \geq b|z|^2 - M_0$,

$$\Phi_H(x) \leq M_0 + \sum_{m\leq 0}(\pi m - b)|x_m|^2 + (2\pi - b)|x_2|^2 \leq M_0$$

on this cone when $b > 2\pi$. All cutoffs and normalized gradient flows in Proposition 2.2 preserve E_ρ and commute with σ, hence belong to Γ_ρ; the slice proof uses only index monotonicity and is valid for $\operatorname{ind}_{\mathbb{Z}_2}$. Sublevel inclusions give the remaining Hamiltonian claims. Dilation conjugates Γ_ρ to itself. The radial flow in Proposition 2.2 restricts to E_ρ and maps S_ρ^+ to rS_ρ^+ at time $\log r$. Thus $H_r(z) = r^2H(z/r)$ and $\Phi_{H_r}(rx) = r^2\Phi_H(x)$ give homogeneity; inclusion of Hamiltonian classes gives body monotonicity. □

Lemma 8.2. *For $p \in L$, $c_k^{\mathrm{EH}}(C + p, \rho) = c_k^{\mathrm{EH}}(C, \rho)$.*

Proof. The equivariant translation $T_px = x + p$ preserves $\mathfrak{a}$ and conjugates Γ_ρ to itself. For $v \in S_\rho^+$ and $0 \leq t \leq 1$,

$$\mathfrak{a}(v + tp) = \tfrac{1}{2}, \qquad \|v + tp\|_E^2 = 1 + t^2|p|^2.$$

Choose smooth $\chi : \mathbb{R} \to [0,1]$ and $\zeta : [0,\infty) \to [0,1]$ with $\chi = 0$ on $(-\infty, 0]$, $\chi = 1$ near $1/2$, and $\zeta = 1$ on $[0, 1+|p|^2]$ with compact support. The time-one map of $\dot{x} = \chi(\mathfrak{a}(x))\zeta(\|x\|_E^2)p$ agrees with T_p on S_ρ^+. It belongs to Γ_ρ: its bounded displacement lies in L, and it fixes $\{\mathfrak{a} \leq 0\}$ and all sufficiently large points. Hence T_p preserves the pseudoindex.

The tail-center comparison in Proposition 2.2 uses invariant radial annuli when both centers lie in L: for a tail $b|z-p|^2$ and $q \in L$, choose nonresonant $b' > b$; then $b'|z-q|^2$ dominates outside a large ball. Interpolation gives an invariant majorant with the new tail. Interchanging the centers proves cofinality. Apply this to $\Phi_{H(\cdot - p)}(T_px) = \Phi_H(x)$. □

8.1. **Restriction of the convex reduction.** Throughout this subsection, assume $C = \rho C$, $0 \in \operatorname{int} C$, and that ∂C is smooth with positive curvature.

For $v \in E_\rho^+$ define

$$\begin{aligned} K_{\tau,\rho}(v) &= \min_{w\in E_\rho^-,\ c\in L} \{F_\tau(v+w+c) + q_-(w)\}, \\ W_{\tau,\rho}(v) &= w_{\tau,\rho}(v) + c_{\tau,\rho}(v), \\ x_{\tau,\rho}(v) &= v + W_{\tau,\rho}(v), \\ f_{\tau,\rho}(v) &= q_+(v) - K_{\tau,\rho}(v). \end{aligned} \tag{8.3}$$

Here F_τ and $q_\pm$ are those of Section 3. The involution $\mathcal{R}$ preserves the Fourier splitting and the minimized functional. Uniqueness of its ordinary minimizer therefore gives

$$K_{\tau,\rho} = K_\tau|_{E_\rho^+}, \qquad W_{\tau,\rho} = W_\tau|_{E_\rho^+}, \qquad f_{\tau,\rho} = f_\tau|_{E_\rho^+}.$$

In particular Lemma 3.1, including (3.8), applies: the correction is Lipschitz from the L^2 norm into $H^1 \cap E_\rho$ and compact into E_ρ. The full graph point is only required to lie in $H^{1/2}$.

Proposition 8.3. *For every $\tau > 0$, the sublevels $\{f_{\tau,\rho} < 0\} \subset E_\rho^+$ and $\{\Psi_C < \tau\} \subset \Lambda_\rho$ admit continuous equivariant maps in both directions. Their indices satisfy, including infinite values,*

$$\begin{aligned} \operatorname{ind}_{\mathbb{Z}_2}\{v \in E_\rho^+ : f_{\tau,\rho}(v) < 0\} &= \operatorname{ind}_{\mathbb{Z}_2}\{v \in S_\rho^+ : f_{\tau,\rho}(v) < 0\} \\ &= \operatorname{ind}_{\mathbb{Z}_2}\{u \in \Lambda_\rho : \Psi_C(u) < \tau\}. \end{aligned}$$

Proof. For $p(v) = \tau\nabla h(x_{\tau,\rho}(v))$, the maps of Lemma 3.2 restrict to

$$v \longmapsto \frac{J_0 p(v)}{\|J_0 p(v)\|_2}, \qquad u \longmapsto P^+\Pi u.$$

Indeed $p(v)(-t) = \rho p(v)(t)$, hence $J_0 p(v)$ is anti-brake. Its constant coefficient vanishes: $p(v)_0 \in L$, and variation in the constant directions L gives $p(v)_0 = 0$. The negative Euler equations (3.6) remain valid. Thus (3.11) and (3.12) restrict without change and prove nonvanishing, positivity of action, and the strict sublevel inclusions. Neither argument differentiates a graph point. Index monotonicity in both directions and positive radial normalization prove the claim. □

Proposition 8.4. *For $0 < \tau < \tau'$ and $a > 0$, there are $\Theta \in \Gamma_\rho$ and $B_0 < \infty$ with $\sup_{v\in S_\rho^+} \|\Theta(v)\|_E \le B_0$ such that, for every $H \in \mathcal{F}_\rho(C)$,*

$$\int_0^1 (H(x(t)) - \tau h(x(t)))\, dt \le 1 \quad (x \in E_\rho,\ \|x\|_E \le B_0)$$

implies

$$\operatorname{ind}_{\mathrm{EH},\rho}\{\Phi_H < a\} \le \operatorname{ind}_{\mathbb{Z}_2}\{u \in \Lambda_\rho : \Psi_C(u) < \tau'\}. \tag{8.4}$$

Proof. Lemma 3.3 applies to $f_{\tau,\rho}$. Use the graph field (3.16) with $W_{\tau,\rho}$ and the radial flow from Proposition 3.4 on E_ρ. They preserve the brake space and commute with σ; their compact corrections and action/norm cutoffs satisfy (7.2). The positive-path estimate (3.15), exact graph identity (3.17),

and radius B_0 are unchanged. The resulting inverse sphere slice lies in $\{v \in S^+_\rho : f_{\tau',\rho}(v) < 0\}$. Apply (8.2) and Proposition 8.3. □

8.2. **Equality and invariant approximation.**

Proof of Theorem B. First let C be centered, smooth, and positively curved. Proposition 6.1 and Lemma 7.7 give $0 < s_k(C,\rho) < \infty$. Fix $b > s_k(C,\rho)$. Compact detection supplies invariant compact $K \subset \{\Psi_C < b\} \subset \Lambda_\rho$ with $\mathrm{ind}_{\mathbb{Z}_2}(K) \geq k$. The primitive maps K homeomorphically to $V = \Pi K$; put $\mathcal{W} = \{rv : r > 0,\ v \in V\}$. Then

$$\mathrm{ind}_{\mathbb{Z}_2}(\mathcal{W}) \geq k, \qquad \mathcal{H}_C(\dot{v}) < b\mathfrak{a}(v) \quad (v \in V).$$

The positive-ray estimate of Lemma 4.1 restricts to $E^-_\rho + L + \mathcal{W}$. In particular (4.1) holds there without central symmetry. For $\varepsilon > 0$, choose $b < M < b + \varepsilon$ and a profile (3.19) with $M\kappa < b + \varepsilon$. Then $A = Mf(h) \geq bh - b - \varepsilon$. For a nonresonant $B > \max\{2\pi, M \max_{|z|=1} h(z)\}$, an invariant annular cutoff from A to its majorant $B|z|^2$ gives $H \in \mathcal{F}_\rho(C)$ with $H \geq A$. Consequently

$$\sup_{E^-_\rho + L + \mathcal{W}} \Phi_H \leq b + \varepsilon.$$

Proposition 7.6 and the minimax give $c^{\mathrm{EH}}_k(C,\rho) \leq b + \varepsilon$. Let $\varepsilon \downarrow 0$ and $b \downarrow s_k(C,\rho)$.

For the lower comparison put $s = s_k(C,\rho)$ and follow the lower proof of Proposition 4.2, with $H \in \mathcal{F}_\rho(C)$ and the graph from Proposition 8.4. Thus choose $c \notin \mathcal{S}(C)$ and $0 < c < a < \tau < \tau' < s$; construct the graph and radius B_0 before the profile (3.19) and $M > \tau$, with $H \leq Mf(h)$ and the spectrum gap (3.18). These choices use only the same vanishing and quadratic-growth hypotheses. Apply Lemma 3.5 with radius B_0, error 1, and

$$b_\infty > \max\{2\pi, M \max_{|z|=1} h(z)\}, \qquad b_\infty \notin \pi\mathbb{N}.$$

Its smooth maximum and radial correction preserve ρ, so $\widehat{H}_\nu \in \mathcal{F}_\rho(C)$ for every $\nu \in [\tau, M]$. The lemma excludes c as a critical value of $\Phi_{\widehat{H}_\nu}$ on E, hence on E_ρ by Proposition 8.1. Since $f(h) \leq h$, the initial cap satisfies

$$\mathrm{ind}_{\mathrm{EH},\rho}\{\Phi_{\widehat{H}_\tau} < a\} \leq \mathrm{ind}_{\mathbb{Z}_2}\{\Psi_C < \tau'\} < k,$$
$$c^{\mathrm{EH}}_k(\widehat{H}_\tau, \rho) \geq a > c.$$

Proposition 8.1 makes the selectors finite and continuous along this common-tail family. None can cross c. At the other endpoint $H \leq Mf(h) \leq \widehat{H}_M$, hence

$$c^{\mathrm{EH}}_k(H,\rho) \geq c^{\mathrm{EH}}_k(\widehat{H}_M, \rho) > c.$$

Take the infimum over H and let $c \uparrow s_k(C,\rho)$ through the complement of the closed null spectrum from Lemma 2.4.

Now let C be any centered invariant convex body. Proposition 6.1 and Lemma 7.7 give $0 < s_k(C,\rho) < \infty$. Comparison with inner and outer balls, where smooth equality is already proved, gives $0 < c^{\mathrm{EH}}_k(C,\rho) < \infty$.

Lemma 4.3 supplies invariant smooth, positively curved approximations, since its radial mollifier preserves ρ_0. By (4.2), monotonicity and quadratic homogeneity, both selectors on the approximations converge to their values on C. Equality follows from the smooth case.

A fixed interior center exists by averaging. Lemma 8.2 makes the common value independent of this center. The normalized symplectic velocity map of Proposition 6.3 preserves every real dual sublevel and its index. Equality therefore transfers its covariance to the Hamiltonian construction and proves adapted-coordinate independence. □

For a bounded open invariant convex domain U, use the value on $\overline{U}$. It is also the supremum over invariant compact convex bodies contained in U: after a fixed interior translation, $(1-\varepsilon)\overline{U} \subset U$, and monotonicity and homogeneity squeeze the supremum to this value.

9. Bounds and the terminal level

Fix a centered invariant convex body and write $\beta = \beta_\rho(C)$.

Proposition 9.1. *The dual selectors satisfy (1.9). Strict sublevels below 2β have finite index; those above 2β have infinite index. If $B^{2n}(r) \subset C \subset B^{2n}(R)$, then*

$$\pi r^2 \le s_k(C,\rho) \le 2\pi R^2 \qquad (k \ge 1). \tag{9.1}$$

Proof. Choose $r, R > 0$ with $B^{2n}(r) \subset C \subset B^{2n}(R)$. $s_1(C,\rho) = \inf \Psi_C = \beta$, and Lemma 7.7 gives infinite index above 2β, hence $s_k(C,\rho) \le 2\beta$.

Fix $0 < b < 2\beta$ and $u \in \Lambda_\rho$. Write

$$e = \tfrac{1}{2}(u + \sigma u), \qquad o = \tfrac{1}{2}(u - \sigma u).$$

Convexity and shift invariance imply $\mathcal{H}_C(u) \ge \mathcal{H}_C(e)$. If $\mathcal{A}(e) > 0$, the compression $v(t) = e(t/2)$ satisfies $\mathcal{H}_C(v) = \mathcal{H}_C(e)$ and $\mathcal{A}(v) = 2\mathcal{A}(e)$. Using $\mathcal{H}_C(e) \ge 0$ when $\mathcal{A}(e) \le 0$, we obtain

$$\mathcal{H}_C(u) \ge \mathcal{H}_C(e) \ge 2\beta\mathcal{A}(e).$$

Let Q_N project onto the positive odd frequencies $m \le N$. If $Q_N u = 0$, the Fourier formula and $\|u\|_2 = 1$ give

$$\begin{aligned} \mathcal{A}(o) &\le \frac{1}{4\pi(N+1)}, \\ \mathcal{A}(u) &= \mathcal{A}(e) + \mathcal{A}(o) \\ &\le \frac{\mathcal{H}_C(u)}{2\beta} + \frac{1}{4\pi(N+1)}. \end{aligned}$$

For $u \in \{\Psi_C < b\}$, this and $\mathcal{H}_C(u) \ge r^2/4$ imply

$$\frac{r^2}{4}\left(1 - \frac{b}{2\beta}\right) < \frac{b}{4\pi(N+1)}.$$

Choose N for which this inequality fails. Then $Q_N u \ne 0$ on $\{\Psi_C < b\}$, and $u \mapsto Q_N u / \|Q_N u\|_2$ maps this sublevel equivariantly to the sign sphere

$S(\operatorname{im} Q_N)$. Lemma 7.3 gives finite index for this sphere, hence for the sublevel. Consequently $\liminf_k s_k(C,\rho) \geq b$ for every $b < 2\beta$, proving the limit. The bounds $\pi r^2 \leq \beta \leq \pi R^2$ give (9.1). □

10. Spectrality of the dual selectors

Assume $C = \rho C$, $0 \in \operatorname{int} C$, and that ∂C is smooth with positive curvature. Set

$$M = \{u \in V_\rho : \mathcal{A}(u) = 1\}, \qquad G = \mathcal{H}_C|_M, \qquad B = J\Pi.$$

The compact self-adjoint operator $B : V_\rho \to V_\rho$ satisfies $\mathcal{A}(u) = \frac{1}{2}\langle Bu, u\rangle_{L^2}$. Since $\langle Bu, u\rangle = 2$ on M, this is a closed smooth Hilbert hypersurface. The equivariant maps

$$\begin{aligned} \Lambda_\rho &\longrightarrow M, \quad u \longmapsto u/\sqrt{\mathcal{A}(u)}, \\ M &\longrightarrow \Lambda_\rho, \quad v \longmapsto v/\|v\|_2 \end{aligned} \tag{10.1}$$

are inverse and identify the sublevels of Ψ_C and G.

For $u \in V_\rho$, put $q_u = \nabla h^*(-Ju)$ and $\bar{q}_u = \int_0^1 q_u(t)\,dt \in L$. Then

$$g(u) := \nabla(\mathcal{H}_C|_{V_\rho})(u) = J(q_u - \bar{q}_u).$$

The inverse Hessian bounds for h^* and this orthogonal projection give positive constants μ_*, L_* such that

$$\begin{aligned} \langle g(u) - g(v), u - v\rangle_{L^2} &\geq \mu_* \|u - v\|_2^2, \\ \|g(u) - g(v)\|_2 &\leq L_* \|u - v\|_2. \end{aligned} \tag{10.2}$$

The constrained gradient is

$$\begin{aligned} \nabla_M G(u) &= g(u) - \ell(u) Bu, \\ \ell(u) &= \frac{\langle g(u), Bu\rangle_{L^2}}{\|Bu\|_2^2}. \end{aligned} \tag{10.3}$$

Lemma 10.1. *If $(u_j) \subset M$, $G(u_j)$ is bounded, and $\nabla_M G(u_j) \to 0$, then a subsequence converges strongly in V_ρ.*

Proof. The quadratic lower bound for $\mathcal{H}_C$ bounds $\|u_j\|_2$. Euler's identity gives

$$\ell(u_j) = G(u_j) - \tfrac{1}{2}\langle \nabla_M G(u_j), u_j\rangle_{L^2}.$$

The multipliers $\ell(u_j)$ are bounded. Compactness of B therefore allows a subsequence along which both $\ell(u_j)$ and Bu_j converge. Equation (10.3) then makes $g(u_j)$ Cauchy, and (10.2) gives

$$\mu_* \|u_i - u_j\|_2 \leq \|g(u_i) - g(u_j)\|_2.$$

The strong limit lies in M and is critical. □

Proposition 10.2. *Every finite positive $s_k(C,\rho)$ is a critical value of G and a full brake action on ∂C.*

Proof. If $c > 0$ is noncritical, Lemma 10.1 gives $\delta, \eta > 0$, $2\delta < c$, such that

$$\|\nabla_M G\|_2 \geq \eta \qquad \text{on } \{|G - c| \leq 2\delta\}.$$

Choose a smooth $[0,1]$-valued cutoff χ, supported in $(c-2\delta, c+2\delta)$ and equal to one on $[c-\delta, c+\delta]$. The tangent field

$$Z = -\chi(G)\frac{\nabla_M G}{\|\nabla_M G\|_2^2},$$

extended by zero, is bounded, locally Lipschitz, and equivariant. Indeed, $\|Bu\|_2 \geq 2/\|u\|_2$ on M controls the denominator in (10.3). Bounded speed and closedness of M give a global flow. Since $\dot{G} = -\chi(G)$, its time-2δ map sends

$$\{G < c + \delta\} \longrightarrow \{G < c - \delta\}.$$

Equivariance and inclusion give equal indices, even when infinite. At a selector $c = s_k(C, \rho)$ this contradicts

$$\operatorname{ind}_{\mathbb{Z}_2}\{G < c - \delta\} < k \leq \operatorname{ind}_{\mathbb{Z}_2}\{G < c + \delta\}.$$

For a critical $u \in M$ of value c, Euler's identity gives $g(u) = cBu$. The displayed projection formula yields, with $d = \bar{q}_u \in L$,

$$\begin{aligned} q := \nabla h^*(-Ju) &= c\Pi u + d, \\ -Ju &= \nabla h(q), \qquad \dot{q} = cu. \end{aligned}$$

Thus $q \in H^1$. The chain rule and Fenchel equality give

$$\frac{d}{dt}h(q) = \langle -Ju, cu \rangle = 0, \qquad h(q) = \int_0^1 h^*(-Ju)\, dt = c.$$

The loop $y = q/\sqrt{c}$ satisfies

$$\begin{aligned} h(y) &= 1, \qquad y(-t) = \rho y(t), \\ \dot{y} &= cJ\nabla h(y), \qquad \int_y \lambda_0 = c. \end{aligned}$$

□

Corollary 10.3. *Every selector of an invariant convex body is a generalized brake action on its boundary.*

Proof. Center C at a fixed interior point. Take invariant approximations with smooth boundary and positive curvature,

$$(1 - \varepsilon_j)C \subset C_j \subset (1 + \varepsilon_j)C, \qquad \varepsilon_j \downarrow 0.$$

Monotonicity and homogeneity give $s_j := s_k(C_j, \rho) \to s_k(C, \rho) = s$. Choose brake loops

$$y_j : \mathbb{R}/\mathbb{Z} \longrightarrow \partial C_j, \qquad \dot{y}_j = s_j J p_j, \qquad p_j = \nabla h_{C_j}(y_j).$$

For fixed $r, R > 0$, the inclusions $B^{2n}(r) \subset C_j \subset B^{2n}(R)$ imply

$$2 = \langle p_j, y_j \rangle = \sigma_{C_j}(p_j) \geq r|p_j|.$$

Hence (y_j) is bounded and equi-Lipschitz. A subsequence satisfies

$$y_j \to y \text{ uniformly}, \qquad p_j \rightharpoonup^* p \text{ in } L^\infty.$$

The limit $y : \mathbb{R}/\mathbb{Z} \to \partial C$ satisfies $y(-t) = \rho y(t)$ and $\dot{y} = sJp$ almost everywhere. For each $z \in \mathbb{R}^{2n}$, the subgradient inequality gives

$$h_{C_j}(z) \geq 1 + \langle p_j(t), z - y_j(t) \rangle.$$

Integrate against nonnegative L^1 test functions and use locally uniform gauge convergence. A countable dense set of z, followed by continuity, gives $p(t) \in \partial h_C(y(t))$ almost everywhere. Finally,

$$\int_y \lambda_0 = \frac{s}{2} \int_0^1 \langle y, p \rangle \, dt = s.$$

□

11. Real ellipsoids and symmetry examples

Proposition 11.1. *For $E(\boldsymbol{a})$ with $0 < a_1 \leq \cdots \leq a_n$,*

$$\operatorname{ind}_{\mathbb{Z}_2}\{u \in \Lambda_{\rho_0} : \Psi_{E(\boldsymbol{a})}(u) < b\} = \begin{cases} \#\{j : a_j < b\}, & 0 < b \leq 2a_1, \\ \infty, & b > 2a_1. \end{cases} \tag{11.1}$$

Its selectors therefore satisfy (1.10).

Proof. The Fourier formulas (5.2) restrict to the anti-brake coefficients $u_{jm} \in i\mathbb{R}$. The negative eigenspace of (5.3) is thus the real span V_b of the positive modes with $ma_j < b$. The normalized deformation (5.4) restricts to an equivariant retraction of the real strict sublevel onto $S(V_b)$; zero eigenvalues remain in its nonnegative complement, so the argument includes resonant endpoints. If $b \leq 2a_1$, only $m = 1$ occurs, giving $\#\{j : a_j < b\}$ sign lines. If $b > 2a_1$, the second mode in the first coordinate is fixed. Sign stabilization and the fixed-point index therefore give (11.1). □

For the ball, $a_j = \pi R^2$ and the sequence is

$$\underbrace{\pi R^2, \ldots, \pi R^2}_{n \text{ entries}}, 2\pi R^2, 2\pi R^2, \ldots.$$

More generally, the first $N = \#\{j : a_j < 2a_1\}$ entries have their axis multiplicities; all later entries equal $2a_1$. For arbitrary convex bodies, only convergence to 2β is asserted.

Proof of Theorem C. Apply Proposition 9.1, Proposition 10.2, Corollary 10.3, and Proposition 11.1. □

Remark 11.2. The triangle with vertices $(-1,-1)$, $(-1,1)$, $(2,0)$ is ρ_0-invariant and contains zero, but is not centrally symmetric. Half-period translation still preserves its dual action. The symplectic shear $A_s(q,p) = (q + sp, p)$ gives

$$A_s \rho_0 A_s^{-1}(q,p) = (q - 2sp, -p).$$

This involution is nonorthogonal for $s \neq 0$; the selectors of $A_s E(\boldsymbol{a})$ equal those of $E(\boldsymbol{a})$ by Theorem B.

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