A MIDPOINT LOCALLY UNIFORMLY ROTUND BANACH SPACE WITH THE DAUGAVET PROPERTY

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Abstract. We construct a separable Banach space with the Daugavet and Schur properties whose norm is midpoint locally uniformly rotund and Gâteaux smooth. The construction combines the ergodic orbit norm of Dantas, Kaasik and Perreau with a Bourgain–Rosenthal enlargement in the form developed by Kadets and Werner.

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1. Introduction

Recall that a normed space X has the *Daugavet property* if

$$\|\operatorname{Id}+T\| = 1 + \|T\|$$

for every rank-one operator $T\colon X \to X$. We write $X \in \mathrm{DPr}$. The spaces $C[0,1]$ and $L_1[0,1]$, with their usual norms, are standard examples; see [3].

Dantas, Kaasik and Perreau [2] constructed an equivalent norm on $L_1[0,1]$ that is weakly midpoint locally uniformly rotund and Gâteaux smooth and has the Daugavet property. They also proved in [2, Remark 3.16] that this norm is not midpoint locally uniformly rotund.

Definition 1.1. A norm on a Banach space X is *midpoint locally uniformly rotund* (*MLUR*) if, for every $x \in S_X$ and every sequence $(h_n) \subset X$,

$$\|x + h_n\| \longrightarrow 1 \quad \text{and} \quad \|x - h_n\| \longrightarrow 1 \quad \Longrightarrow \quad \|h_n\| \longrightarrow 0.$$

The space X has the *Schur property* if every weakly convergent sequence in X converges in norm. A norm is *Gâteaux smooth* if it is Gâteaux differentiable at every nonzero point.

In [2, Remark 3.16], the authors noted that the existence of an MLUR Banach space with the Daugavet property was unknown. We prove the following theorem.

Theorem 1.2. *There exist a closed subspace $E \subset L_1(\mathbb{T}^{\mathbb{N}})$ and an equivalent norm N on E such that (E, N) has the Daugavet and Schur properties, and N is midpoint locally uniformly rotund and Gâteaux smooth.*

The proof combines two constructions. The first is the orbit norm of Dantas, Kaasik and Perreau [2], defined using an ergodic measure-preserving transformation and a weighted ℓ_2-sum along its orbits. The second is the finite-dimensional enlargement method of Bourgain and Rosenthal [1], in the form used by Kadets and Werner [4] to construct a space with the Daugavet and Schur properties. We choose the transformation to be a product rotation whose coordinate speeds tend to zero. We will use the following standard geometric characterization of the Daugavet property; see [4, Lemmas 1.1 and 1.2].

Proposition 1.3. *A Banach space X has the Daugavet property if and only if, for every $x \in S_X$ and every $\varepsilon > 0$,*

$$B_X = \overline{\operatorname{conv}}\,\{y \in S_X : \|x + y\| > 2 - \varepsilon\}.$$

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2. An orbit norm and finite-dimensional enlargements

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, let

$$\Omega = \mathbb{T}^{\mathbb{N}},$$

and let μ be product Haar probability measure. A function in $L_1(\Omega)$ is said to *depend on finitely many coordinates* if it has a representative measurable with respect to the σ-algebra generated by finitely many coordinate maps. A subspace $G \subset L_1(\Omega)$ is said to *depend on finitely many coordinates* if there is a finite set $I \subset \mathbb{N}$ such that every $g \in G$ has a representative measurable with respect to $\sigma(\omega \mapsto \omega_r : r \in I)$.

Enumerate the primes as $(p_r)_{r\geq 1}$, and put

$$\alpha_r = \frac{2^{-r}}{\sqrt{p_r}}, \qquad T(\omega)_r = \omega_r + \alpha_r \pmod 1. \tag{2.1}$$

We state the following lemma without proof.

Lemma 2.1. *The transformation T is invertible, measure preserving and ergodic. Moreover, $\alpha_r \to 0$.*

Fix

$$a_k = \frac{3}{5}\,4^{-|k|} \quad (k \in \mathbb{Z}), \qquad C = \sum_{k\in\mathbb{Z}} \sqrt{a_k} = 3\sqrt{\frac{3}{5}}. \tag{2.2}$$

In particular, $a_k > 0$, $\sum_k a_k = 1$, and $C < \infty$. For $f \in L_1(\Omega)$, define

$$\mathcal{J}f(\omega) = \left(\sqrt{a_k} f(T^k\omega)\right)_{k\in\mathbb{Z}}, \qquad N(f) = \int_\Omega \|\mathcal{J}f(\omega)\|_2 \, d\mu. \tag{2.3}$$

This is the orbit norm introduced in [2, Section 3.1]. In particular,

$$\|f\|_1 \leq N(f) \leq C\,\|f\|_1 \qquad (f \in L_1(\Omega)).$$

Since T is ergodic, [2, Theorem 3.4] shows that N is strictly convex and Gâteaux smooth away from zero. For a subspace $G \subset L_1(\Omega)$, write

$$B_1(G) = \{g \in G : \|g\|_1 \leq 1\}, \qquad S_N(G) = \{g \in G : N(g) = 1\}.$$

We also write $B_N(G) = \{g \in G : N(g) \leq 1\}$. We use the following metric for convergence in measure:

$$d(f,g) = \inf\{\varepsilon > 0 : \mu(|f-g| \geq \varepsilon) \leq \varepsilon\}. \tag{2.4}$$

Notice that

$$d(f,g) \leq \sqrt{\|f-g\|_1}, \qquad d(tf,tg) \leq \max\{1,|t|\}\, d(f,g). \tag{2.5}$$

The following lemma is a reformulation of [4, Lemma 2.4 and its proof]. The statement there is formulated in terms of functions v_{ij}; its proof constructs them as products $v_{ij} = u_i q_{ij}$. We state explicitly the properties of the factors q_{ij} needed below.

Here a nonnegative multiplier means simply a measurable function $q : \Omega \to [0,\infty)$ belonging to $L_1(\Omega)$, used through pointwise multiplication:

$$(gq)(\omega) = g(\omega)q(\omega).$$

Let $I \subset \mathbb{N}$ be a finite set of coordinates supporting $G \subset L_1(\Omega)$. Saying that each block $(q_{ij})_{j=1}^s$ depends on one fresh coordinate means that there are distinct indices $r_1, \ldots, r_m \in \mathbb{N} \setminus I$ and nonnegative functions $Q_{ij} \in L_1(\mathbb{T})$ such that

$$q_{ij}(\omega) = Q_{ij}(\omega_{r_i}) \qquad \text{for almost every } \omega \in \Omega.$$

Consequently, each block is independent of the coordinates supporting G.

Lemma 2.2. *Let $I \subset \mathbb{N}$ be finite, let G be a finite-dimensional subspace of $L_1(\Omega)$ depending only on the coordinates in I, let $u_1, \dots, u_m$ belong to its L_1-unit sphere, and let $0 < \beta < 1$. There are an integer $s \geq 1$ and nonnegative functions $Q_{ij} \in L_1(\mathbb{T})$, $1 \leq i \leq m$, $1 \leq j \leq s$, such that, for every choice of pairwise distinct indices $r_1, \dots, r_m \in \mathbb{N} \setminus I$, the multipliers*

$$q_{ij}(\omega) = Q_{ij}(\omega_{r_i})$$

satisfy

$$\mathbb{E}q_{ij} = 1, \qquad \mathbb{E}\min(q_{ij}, 1) \leq \frac{\beta}{2}, \qquad \mathbb{E}\left|\frac{1}{s}\sum_j q_{ij} - 1\right| \leq \beta, \tag{2.6}$$

and, for

$$F = G + \operatorname{span}\{u_i q_{ij} : 1 \leq i \leq m,\ 1 \leq j \leq s\},$$

one has

$$\sup_{f \in B_1(F)} \operatorname{dist}_d(f, B_1(G)) \leq \beta. \tag{2.7}$$

The blocks $(q_{ij})_{j=1}^s$, $1 \leq i \leq m$, are mutually independent and independent of the coordinates in I.

Proof. We use the construction in the proof of [4, Lemma 2.4] with error tolerance β. Choose its auxiliary parameter $\delta > 0$ sufficiently small, also requiring $\delta \leq \beta$, and let $h \in L_1[0,1]$ satisfy all conclusions of [4, Lemma 2.2] for this parameter. In particular,

$$h \geq 0, \qquad \int_0^1 h(t)\,dt = 1, \qquad \int_0^1 |h(t) - 1|\,dt \geq 2 - \delta.$$

Let λ denote normalized Haar measure on $\mathbb{T}$. Since $(\mathbb{T}, \lambda)$ is an atomless standard probability space, it supports a sequence of independent random variables $Q_j : \mathbb{T} \to [0, \infty)$, each having the same distribution as h. By [4, Lemma 2.2(c)], there is an integer $s \geq 1$ such that

$$\int_{\mathbb{T}} \left|\frac{1}{s}\sum_{j=1}^s Q_j(t) - 1\right| d\lambda(t) \leq \delta.$$

Set $Q_{ij} = Q_j$ for all i, j, and fix any pairwise distinct indices $r_1, \dots, r_m \in \mathbb{N} \setminus I$. Define

$$q_{ij}(\omega) := Q_{ij}(\omega_{r_i}).$$

These are nonnegative integrable functions. For each i, the entire block $(q_{ij})_{j=1}^s$ depends only on coordinate r_i. The product structure of μ makes these blocks mutually independent and independent of the coordinates in I.

We now verify the asserted properties. Since q_{ij} has the same distribution as h,

$$\mathbb{E}q_{ij} = 1, \qquad \mathbb{E}|q_{ij} - 1| \geq 2 - \delta.$$

For every $x \geq 0$, one has

$$|x - 1| = x + 1 - 2\min(x, 1).$$

Consequently,

$$\mathbb{E}\min(q_{ij}, 1) = \frac{\mathbb{E}q_{ij} + 1 - \mathbb{E}|q_{ij} - 1|}{2} \leq \frac{\delta}{2} \leq \frac{\beta}{2}.$$

Moreover, the definition of the block gives

$$\mathbb{E}\left|\frac{1}{s}\sum_{j=1}^s q_{ij} - 1\right| = \int_{\mathbb{T}} \left|\frac{1}{s}\sum_{j=1}^s Q_j(t) - 1\right| d\lambda(t) \leq \delta \leq \beta.$$

This proves (2.6).

For every $g \in G$, independence and nonnegativity yield

$$\|gq_{ij}\|_1 = \mathbb{E}(|g|q_{ij}) = (\mathbb{E}|g|)(\mathbb{E}q_{ij}) = \|g\|_1.$$

In particular, the products $v_{ij} := u_i q_{ij}$ belong to $L_1(\Omega)$, and $\|v_{ij}\|_1 = 1$. Thus

$$F = G + \operatorname{span}\{v_{ij} : 1 \le i \le m,\ 1 \le j \le s\}$$

is a well-defined finite-dimensional subspace, depending only on the coordinates in $I \cup \{r_1, \dots, r_m\}$.

The above construction is the one used in the proof of [4, Lemma 2.4], carried out on the fresh coordinates $r_1, \dots, r_m$. Their particular indices do not affect the argument. Thus part (c) of that lemma gives

$$\sup_{f \in B_1(F)} \operatorname{dist}_d(f, B_1(G)) \le \beta,$$

which is (2.7). □

The next estimate is the additional ingredient.

Lemma 2.3. *Let $y \in L_1(\Omega)$ depend on a finite coordinate set I, and let $q \in L_1(\mathbb{T})$ be nonnegative with $\int_{\mathbb{T}} q = 1$. For $r \notin I$, put $q_r(\omega) = q(\omega_r)$ and*

$$D_r(q) = \sum_{k \in \mathbb{Z}} \sqrt{a_k}\ \|q(\cdot + k\alpha_r) - q\|_{L_1(\mathbb{T})}.$$

Then $D_r(q) \to 0$ as $r \to \infty$, and

$$\|\mathcal{J}(yq_r) - q_r \mathcal{J} y\|_{L_1(\Omega;\ell_2)} \le \|y\|_1 D_r(q). \tag{2.8}$$

Consequently,

$$|N(yq_r) - N(y)| \le \|y\|_1 D_r(q). \tag{2.9}$$

Proof. For each fixed integer k, translation continuity in $L_1(\mathbb{T})$ and $\alpha_r \to 0$ give

$$\|q(\cdot + k\alpha_r) - q\|_1 \longrightarrow 0.$$

These quantities are at most $2\|q\|_1 = 2$, so the summability of $(\sqrt{a_k})_k$ proves $D_r(q) \to 0$.

The k-th coordinate of the difference in (2.8) is

$$\sqrt{a_k}\, y(T^k\omega)\big(q(\omega_r + k\alpha_r) - q(\omega_r)\big).$$

Using $\|\cdot\|_2 \le \|\cdot\|_1$ and Tonelli's theorem, we obtain

$$\begin{aligned}
&\|\mathcal{J}(yq_r) - q_r \mathcal{J} y\|_{L_1(\Omega;\ell_2)} \\
&= \int_\Omega \left(\sum_{k \in \mathbb{Z}} a_k\, |y(T^k\omega)|^2 |q(\omega_r + k\alpha_r) - q(\omega_r)|^2 \right)^{1/2} d\mu(\omega) \\
&\le \int_\Omega \sum_{k \in \mathbb{Z}} \sqrt{a_k}\, |y(T^k\omega)|\, |q(\omega_r + k\alpha_r) - q(\omega_r)|\, d\mu(\omega) \\
&= \sum_{k \in \mathbb{Z}} \sqrt{a_k} \int_\Omega |y(T^k\omega)|\, |q(\omega_r + k\alpha_r) - q(\omega_r)|\, d\mu(\omega).
\end{aligned}$$

Since $y \circ T^k$ depends only on the coordinates in I and $r \notin I$, the product measure structure gives

$$\begin{aligned}
&\int_\Omega |y(T^k\omega)|\, |q(\omega_r + k\alpha_r) - q(\omega_r)|\, d\mu(\omega) \\
&= \left(\int_\Omega |y(T^k\omega)|\, d\mu(\omega) \right) \left(\int_{\mathbb{T}} |q(t + k\alpha_r) - q(t)|\, dt \right) \\
&= \|y\|_1\ \|q(\cdot + k\alpha_r) - q\|_{L_1(\mathbb{T})},
\end{aligned}$$

where the last equality uses that T preserves measure. Consequently,

$$\|\mathcal{J}(yq_r) - q_r \mathcal{J} y\|_{L_1(\Omega;\ell_2)} \le \|y\|_1 \sum_{k \in \mathbb{Z}} \sqrt{a_k}\ \|q(\cdot + k\alpha_r) - q\|_{L_1(\mathbb{T})} = \|y\|_1 D_r(q),$$

which proves (2.8).

Finally, $\|\mathcal{J}y\|_2$ depends only on the coordinates in I. Independence, $q_r \geq 0$, and $\mathbb{E}q_r = 1$ therefore give

$$\|q_r\mathcal{J}y\|_{L_1(\Omega;\ell_2)} = \int_\Omega q_r \|\mathcal{J}y\|_2 \, d\mu = (\mathbb{E}q_r) \int_\Omega \|\mathcal{J}y\|_2 \, d\mu = N(y).$$

The reverse triangle inequality now yields

$$\begin{aligned} |N(yq_r) - N(y)| &= \Big| \|\mathcal{J}(yq_r)\|_{L_1(\Omega;\ell_2)} - \|q_r\mathcal{J}y\|_{L_1(\Omega;\ell_2)} \Big| \\ &\leq \|\mathcal{J}(yq_r) - q_r\mathcal{J}y\|_{L_1(\Omega;\ell_2)} \\ &\leq \|y\|_1 \, D_r(q), \end{aligned}$$

which is (2.9). □

Here C is the constant defined in (2.2).

Proposition 2.4. *Let $G \subset L_1(\Omega)$ be finite dimensional and depend on finitely many coordinates. Given $y_1, \ldots, y_m \in S_N(G)$ and $0 < \beta, \eta < 1$, there are a finite-dimensional space $F \supset G$, depending on finitely many coordinates, an integer $s \geq 1$, and $w_{ij} \in S_N(F)$ such that*

$$N(x + w_{ij}) \geq 2 - 2\beta - 2\eta \qquad (x \in S_N(G),\ i \leq m,\ j \leq s), \tag{2.10}$$

$$N\left(y_i - \frac{1}{s}\sum_{j=1}^{s} w_{ij}\right) \leq C\beta + \eta \qquad (i \leq m), \tag{2.11}$$

$$\sup_{f \in B_1(F)} \operatorname{dist}_d(f, B_1(G)) \leq \beta. \tag{2.12}$$

Proof. Let $I \subset \mathbb{N}$ be a finite coordinate set supporting G. Apply Lemma 2.2 with tolerance β to

$$u_i = \frac{y_i}{\|y_i\|_1}, \qquad i = 1, \ldots, m.$$

This gives an integer s and fixed profiles Q_{ij}. By Lemma 2.3, $D_r(Q_{ij}) \to 0$ for each of these finitely many profiles. Choose pairwise distinct $r_1, \ldots, r_m \in \mathbb{N} \setminus I$ sufficiently large that

$$D_{r_i}(Q_{ij}) < \eta \qquad (1 \leq i \leq m,\ 1 \leq j \leq s). \tag{2.13}$$

Let q_{ij} be the corresponding multipliers from Lemma 2.2, and put

$$v_{ij} = y_i q_{ij}, \qquad F = G + \operatorname{span}\{v_{ij} : 1 \leq i \leq m,\ 1 \leq j \leq s\}.$$

Since each y_i is a nonzero scalar multiple of u_i, this is the same space F as in Lemma 2.2. Hence (2.12) follows directly from (2.7).

Norm equivalence gives $\|y_i\|_1 \leq N(y_i) = 1$. Consequently, (2.8) and (2.13) imply

$$\|\mathcal{J}v_{ij} - q_{ij}\mathcal{J}y_i\|_{L_1(\Omega;\ell_2)} \leq \|y_i\|_1 \, D_{r_i}(Q_{ij}) \leq \eta.$$

We repeat the argument from Lemma 2.3. Since T acts coordinatewise, $\mathcal{J}y_i$ still depends only on I. Independence and $\mathbb{E}q_{ij} = 1$ therefore give

$$\|q_{ij}\mathcal{J}y_i\|_{L_1(\Omega;\ell_2)} = \int_\Omega q_{ij} \|\mathcal{J}y_i\|_2 \, d\mu = (\mathbb{E}q_{ij})N(y_i) = 1.$$

The reverse triangle inequality now yields

$$|N(v_{ij}) - 1| \leq \|\mathcal{J}v_{ij} - q_{ij}\mathcal{J}y_i\|_{L_1(\Omega;\ell_2)} \leq \eta. \tag{2.14}$$

As $N(v_{ij}) \geq 1 - \eta > 0$, define

$$w_{ij} = \frac{v_{ij}}{N(v_{ij})} \in S_N(F).$$

The cost of this normalization is exactly

$$N(w_{ij} - v_{ij}) = \left| \frac{1}{N(v_{ij})} - 1 \right| N(v_{ij}) = |1 - N(v_{ij})| \leq \eta.$$

Now fix any $x \in S_N(G)$ and any pair (i,j), and write

$$A = \mathcal{J}x, \qquad B = \mathcal{J}y_i, \qquad q = q_{ij}.$$

Both A and B depend only on I, so the pair (A,B) is independent of q. At each point, the reverse triangle inequality in ℓ_2 gives

$$\begin{aligned}\|A + qB\|_2 &\geq \big|\|A\|_2 - q\,\|B\|_2\big| \\ &= \|A\|_2 + q\,\|B\|_2 - 2\min\{\|A\|_2\,, q\,\|B\|_2\}.\end{aligned}$$

The last term measures the possible cancellation. To control it, observe that for $a, b, t \geq 0$,

$$\min\{a, tb\} \leq (a+b)\min\{1,t\}.$$

Hence independence and (2.6) imply

$$\begin{aligned}\int_\Omega \min\{\|A\|_2\,, q\,\|B\|_2\}\,d\mu &\leq \int_\Omega (\|A\|_2 + \|B\|_2)\min\{1,q\}\,d\mu \\ &= \big(N(x) + N(y_i)\big)\mathbb{E}\min\{1,q\} \\ &\leq (1+1)\frac{\beta}{2} = \beta.\end{aligned}$$

Also,

$$\int_\Omega \|A\|_2\; d\mu = 1, \qquad \int_\Omega q\,\|B\|_2\; d\mu = (\mathbb{E}q)N(y_i) = 1.$$

Integrating the pointwise lower bound therefore gives

$$\int_\Omega \|\mathcal{J}x + q_{ij}\mathcal{J}y_i\|_2\; d\mu \geq 2 - 2\beta.$$

We now account separately for the orbit error and the normalization error. By linearity of $\mathcal{J}$ and (2.14),

$$\begin{aligned}N(x + v_{ij}) &= \|\mathcal{J}x + \mathcal{J}v_{ij}\|_{L_1(\Omega;\ell_2)} \\ &\geq \|\mathcal{J}x + q_{ij}\mathcal{J}y_i\|_{L_1(\Omega;\ell_2)} - \|\mathcal{J}v_{ij} - q_{ij}\mathcal{J}y_i\|_{L_1(\Omega;\ell_2)} \\ &\geq 2 - 2\beta - \eta.\end{aligned}$$

It follows that

$$N(x + w_{ij}) \geq N(x + v_{ij}) - N(w_{ij} - v_{ij}) \geq 2 - 2\beta - 2\eta.$$

The multipliers and coordinates were chosen independently of x. Thus this bound holds for every $x \in S_N(G)$, which proves (2.10).

For each fixed i, write $\bar{q}_i = s^{-1}\sum_{j=1}^s q_{ij}$. Before normalization, the average of the new vectors is simply

$$\frac{1}{s}\sum_{j=1}^s v_{ij} = y_i\bar{q}_i.$$

The scalar function $\bar{q}_i$ depends only on r_i, so it is independent of y_i. By norm equivalence and (2.6), we obtain

$$\begin{aligned}N\left(y_i - \frac{1}{s}\sum_{j=1}^s v_{ij}\right) &= N\big(y_i(1-\bar{q}_i)\big) \\ &\leq C\int_\Omega |y_i|\,|1-\bar{q}_i|\,d\mu \\ &= C\,\|y_i\|_1\,\mathbb{E}|1-\bar{q}_i| \\ &\leq C\beta.\end{aligned}$$

Finally, averaging the normalization errors gives

$$N\left(y_i - \frac{1}{s}\sum_{j=1}^{s} w_{ij}\right) \le N\left(y_i - \frac{1}{s}\sum_{j=1}^{s} v_{ij}\right) + \frac{1}{s}\sum_{j=1}^{s} N(v_{ij} - w_{ij}) \le C\beta + \eta.$$

This proves (2.11) and completes the proof. □

3. A midpoint locally uniformly rotund Daugavet space

We first record how total boundedness in measure interacts with the orbit norm. The midpoint-defect estimate below is the one used in [2, Theorem 3.5]. Total boundedness in measure strengthens the conclusion to norm convergence, giving MLUR directly.

Lemma 3.1. *Let Y be a closed subspace of $L_1(\Omega)$ whose L_1-unit ball is totally bounded for convergence in measure. Then the restriction $N|_Y$ is MLUR, and Y has the Schur property.*

Proof. By (2.5), every L_1-bounded subset of Y is totally bounded for convergence in measure.

Let $x \in S_N(Y)$, and let $(h_n) \subset Y$ satisfy

$$N(x + h_n) \longrightarrow 1, \qquad N(x - h_n) \longrightarrow 1.$$

The sequence (h_n) is bounded. Put $U = \mathcal{J}x$, $V_n = \mathcal{J}h_n$, and

$$\Delta_n = \frac{\|U + V_n\|_2 + \|U - V_n\|_2}{2} - \|U\|_2 .$$

Then $\Delta_n \ge 0$, and

$$\int_\Omega \Delta_n \, d\mu = \frac{N(x + h_n) + N(x - h_n)}{2} - 1 \longrightarrow 0. \tag{3.1}$$

The triangle inequality also gives

$$\|V_n\|_2 \le \|U\|_2 + \Delta_n.$$

Using the coordinate indexed by zero, we obtain, for every measurable $A \subset \Omega$,

$$\sqrt{a_0} \int_A |h_n| \, d\mu \le \int_A \|U\|_2 \, d\mu + \int_\Omega \Delta_n \, d\mu. \tag{3.2}$$

The integral of $\|U\|_2$ is absolutely continuous, and the last term tends to zero. Treating the finitely many initial indices separately, (3.2) proves that (h_n) is uniformly integrable.

Take any subsequence of (h_n). Total boundedness in measure supplies a further subsequence converging in measure to a measurable function h; here we use completeness of the usual space of measurable functions for convergence in measure. Uniform integrability and Vitali's theorem give convergence in L_1, and hence in N, to h. Since Y is closed, $h \in Y$. Passing to the limit yields

$$N(x + h) = N(x - h) = N(x) = 1.$$

Strict convexity, established in [2], implies $h = 0$. Every subsequence therefore has a further subsequence converging to zero in N, so $N(h_n) \to 0$. This proves MLUR.

For the Schur assertion, let $(f_n) \subset Y$ be weakly null. Its image in $L_1(\Omega)$ is weakly null and therefore uniformly integrable by the Dunford–Pettis criterion. Every subsequence has a further subsequence converging in measure, and then in L_1, by uniform integrability. The limit must be zero because the sequence is weakly null in L_1. It follows that $\|f_n\|_1 \to 0$. □

Proof of Theorem 1.2. We complete the proof along the lines of the Kadets–Werner construction [4, Section 2], obtaining the desired space as the closure of an increasing union of finite-dimensional subspaces. For this, choose

$$\beta_n = 2^{-n-5} \qquad (n \ge 1).$$

We construct increasing finite-dimensional subspaces

$$E_1 \subset E_2 \subset \cdots \subset L_1(\Omega), \qquad E_1 = \operatorname{span}\{\mathbf{1}\},$$

each depending on finitely many coordinates. Suppose that E_n has been chosen. Take a finite β_n-net with respect to N,

$$\{y_{n,1}, \ldots, y_{n,m_n}\} \subset S_N(E_n),$$

and apply Proposition 2.4 with $\beta = \eta = \beta_n$. This gives E_{n+1}, an integer s_n, and $w_{n,i,j} \in S_N(E_{n+1})$ satisfying

$$N(x + w_{n,i,j}) \geq 2 - 4\beta_n \qquad (x \in S_N(E_n)), \tag{3.3}$$

$$N\left(y_{n,i} - \frac{1}{s_n}\sum_j w_{n,i,j}\right) \leq (C+1)\beta_n, \tag{3.4}$$

$$\sup_{f \in B_1(E_{n+1})} \operatorname{dist}_d(f, B_1(E_n)) \leq \beta_n. \tag{3.5}$$

In the last line one may choose an approximant attaining the infimum, since $B_1(E_n)$ is compact.

Set

$$E = \overline{\bigcup_{n\geq 1} E_n}^{\|\cdot\|_1} = \overline{\bigcup_{n\geq 1} E_n}^{N}. \tag{3.6}$$

(E, N) is a separable Banach space.

Fix $x, y \in S_N(E)$ and $\varepsilon, \delta > 0$. Approximate x and y by $x_0, y_0 \in S_N(E_{n_0})$ so that

$$N(x - x_0) < \varepsilon/4, \qquad N(y - y_0) < \delta/2.$$

Choose $n \geq n_0$ sufficiently large that

$$4\beta_n < \varepsilon/2, \qquad (C+2)\beta_n < \delta/2.$$

Take a net point $y_{n,i}$ with $N(y_0 - y_{n,i}) \leq \beta_n$. For every j, (3.3) gives

$$N(x + w_{n,i,j}) \geq N(x_0 + w_{n,i,j}) - N(x - x_0) > 2 - \varepsilon,$$

whereas (3.4) gives

$$N\left(y - \frac{1}{s_n}\sum_j w_{n,i,j}\right) \leq N(y - y_0) + (C+2)\beta_n < \delta.$$

Thus, for every $x \in S_N(E)$ and $\varepsilon > 0$, the closed convex hull of

$$\{w \in S_N(E) : N(x + w) > 2 - \varepsilon\}$$

contains $S_N(E)$, and hence $B_N(E)$. Proposition 1.3 proves the Daugavet property.

Now put $\sigma_n = \sum_{j=n}^{\infty} \beta_j$. Repeated use of (3.5) shows that, whenever $m > n$ and $f \in B_1(E_m)$, there exists $g \in B_1(E_n)$ such that

$$d(f, g) \leq \sum_{j=n}^{m-1} \beta_j \leq \sigma_n.$$

The union $\bigcup_m B_1(E_m)$ is L_1-dense in $B_1(E)$. Using (2.5), we conclude that

$$\sup_{f \in B_1(E)} \operatorname{dist}_d(f, B_1(E_n)) \leq \sigma_n.$$

Since $\sigma_n \to 0$ and each finite-dimensional ball $B_1(E_n)$ is compact in measure, $B_1(E)$ is totally bounded in measure. Lemma 3.1 now shows that $N|_E$ is MLUR and that E has the Schur property. Finally, the Gâteaux differentiability of $N|_E$ follows by restriction from that of N on $L_1(\Omega)$.

□

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