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CONSTRUCTIONS AND EXPLICIT POWER SAVINGS FOR POINT–UNIT-CIRCLE INCIDENCES

YUDA CHEN AND YIFEI WANG

ABSTRACT. Let $\mathcal{U}(n,m)$ be the maximum number of incidences between n distinct points and m distinct unit circles in the plane. We give two constructions and an explicit upper bound for this function. Writing $s=\min(n,m)$ and $M=\max(n,m)$, the first construction gives a four-piece monomial lower bound for all cardinalities, with balanced exponent 1.041169920. The second gives balanced configurations along an unbounded sequence with exponent 1.0431722305539701, and further unbalanced configurations. The upper bound is $\mathcal{U}(n,m)\ll M+s^{2/3-2\delta}M^{2/3+\delta}$, where $\delta=2^{-2^{664}}/514560000000>2^{-2^{665}}$. The constructions use two windows in a relative-norm-one number-field lattice, strengthened arithmetic estimates, and interpolation between local profiles. The upper bound uses finite incidence regularization, prediction, arithmetic heights, and a six-by-five rigidity obstruction. The resulting bounds for the unit-distance problem follow as corollaries.

1. INTRODUCTION

For a finite point set $P\subset\mathbb{R}^2$ and a finite family $\mathcal{C}$ of circles of radius one, write

$$I(P,\mathcal{C})=\#\{(p,C)\in P\times\mathcal{C}:p\in C\},\qquad \mathcal{U}(n,m)=\max_{|P|=n,\ |\mathcal{C}|=m} I(P,\mathcal{C}).$$

The points are distinct, and the circles are distinct. If Q is the set of circle centers, then

$$I(P,\mathcal{C})=I(P,Q):=\#\{(p,q)\in P\times Q:|p-q|=1\}.$$

The sets P and Q may intersect. Interchanging them gives $\mathcal{U}(n,m)=\mathcal{U}(m,n)$. We use $s=\min(n,m)$ and $M=\max(n,m)$ throughout.

The main question is how large $\mathcal{U}(n,m)$ can be as the two cardinalities vary independently. We study it from two directions: explicit configurations give lower bounds, and restrictions on all configurations give upper bounds. The lower-bound results below also distinguish configurations available at every sufficiently large scale from configurations along an unbounded sequence of scales.

For comparison with the classical problem, we use the following normalization.

Let $u(N)$ denote the maximum number of unordered unit-distance pairs among N distinct planar points. The incidence formulation gives

$$2u(N)\le\mathcal{U}(N,N)\le 2u(2N). \tag{1.1}$$

Indeed, using a point set as its own set of centers counts every pair twice. Conversely, the union of the points and centers has at most $2N$ elements, and each unordered pair contributes at most two incidences.

1.1. Lower bounds: constructions.

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Bounds along an unbounded sequence. Erdős's lattice construction [10] gives, for some $c > 0$,

$$u(N) \geq N \exp(c \log N / \log \log N)$$

along an unbounded sequence. The extra factor grows more slowly than every positive power of N. The later number-field constructions produce a fixed power gain, and Table 1 records the principal steps relevant to our construction.

The lower-bound exponent changed in May 2026. The construction of OpenAI [27] gives $u(N) \geq N^{1+\eta}$ for some fixed $\eta > 0$ and arbitrarily large N. Its central idea is to use an ideal lattice in a number field of increasing degree and bounded root discriminant. An embedding into $\mathbb{C}$ turns suitable algebraic norm equations into Euclidean unit distances. Alon, Bloom, Gowers, Litt, Sawin, Shankar, Tsimerman, Wang and Wood [2] gave a streamlined account, including an explicit positive value of η. Sawin [36] sharpened the arithmetic and geometric estimates and obtained $1 + \eta = 1.014114$.

The passage from many small split primes to many elements of one ideal class has a useful antecedent in the work of Ellenberg and Venkatesh [8] on class-group torsion. In the present setting, choosing one class turns many ideal identities into elements with a prescribed relative norm. Those elements become the unit directions of the geometric construction.

The subsequent improvements came from several parts of the argument. Emmerich [9] studied the optimization of Sawin's integer parameters and the verification of explicit lower-bound certificates. Naslund's earlier manuscript [25] reached the sequence exponent 1.0358324. His mixed-signature construction [24] then reached 1.043171. Its tower is built over $\mathbb{Q}(\sqrt{241})$. The relative norm-one group, the local conditions at selected primes and a relative zeta estimate determine the final exponent. Our construction uses this arithmetic framework, with a stronger finite quotient, a sharper arithmetic estimate and separate local profiles for points and circle centers.

TABLE 1. Sequence lower bounds for $u(N)$ and, by (1.1), for $\mathcal{U}(N, N)$. Each power bound holds along an unbounded sequence of cardinalities.

Construction	Lower bound
Erdős [10]	$N \exp(c \log N / \log \log N)$
OpenAI; Alon et al. [2, 27]	$N^{1+\eta}$, for a fixed $\eta > 0$
Sawin [36]	$N^{1.014114}$
Naslund's earlier manuscript [25]	$N^{1.0358324}$
Naslund [24]	$N^{1.043171}$
Theorem 1.2	$N^{1.0431722305539701}$

Bounds at every sufficiently large scale. The distinction between a sequence of sizes and every sufficiently large size matters here. The degree of a field is discrete, and passing to a larger field can make the resulting configuration much larger. Adding isolated points fills a gap in cardinality but can lose the exponent one is trying to preserve. Tseng [43] treated this issue directly and obtained the exponent 1.03158935 for every sufficiently large N. Theorem 1.1 uses two phases of local profiles and interpolates between them within one field. This produces the exponent β_* at every scale and, more generally, the four terms of $\mathcal{L}(s, M)$. Theorem 1.2 instead chooses the strongest balanced profile and retains the larger exponent α_* along a sequence. The two constructions therefore optimize different questions.

TABLE 2. Lower bounds valid for every sufficiently large N. The displayed estimates apply to both $u(N)$ and $\mathcal{U}(N,N)$, with absolute positive implicit constants.

Construction	Lower bound at every large scale
Tseng [43]	$\gg N^{1.03158935}$
Theorem 1.1	$\gg N^{1.041169920}$

Two-parameter lower bounds. Theorems 1.1 and 1.2 give the incidence versions of these two conclusions. In the first, a piecewise monomial bound describes all pairs (n,m). In the second, selecting particular scales gives a stronger balanced exponent and further unbalanced configurations. Set

$$\alpha_* = 1.0431722305539701, \qquad \beta_* = 1.041169920, \tag{1.2}$$

$$r_* = \frac{10603071}{10000000}, \quad a_* = \frac{39597848504393}{50767750000000}, \quad b_* = \frac{1073769}{4061420},$$

$$q_* = \frac{60307177}{1060307100}, \quad R_* = \frac{1060307100}{999999923}. \tag{1.3}$$

Terminating decimals denote exact rational numbers.

Theorem 1.1 (Construction at every scale)**.** *There is an absolute constant $c>0$ such that, for all positive integers n,m,*

$$\mathcal{U}(n,m) \geq c\big(n+m+\mathcal{L}(s,M)\big), \tag{1.4}$$

where

$$\mathcal{L}(s,M) = \min\left\{ s^{0.65015512} M^{0.3910148},\ s^{0.7456363} M^{0.2974058},\ s^{a_*} M^{b_*},\ sM^{q_*} \right\}. \tag{1.5}$$

The active terms change at $M = s^{1.02}, s^{1.04}, s^{r_*}$. In particular, $\mathcal{U}(N,N) \gg N^{\beta_*}$. For every fixed $1 \leq r < R_* = 1.060307181643653\ldots$, the bound at $(n,m) = (N, \lceil N^r \rceil)$ is superlinear in the larger cardinality.

The points and circle centers in this theorem come from two windows in one S-integer lattice. The window functions control the two cardinalities separately. Mixing local functions at two scales covers every sufficiently large integer size; elementary constructions cover the remaining sizes.

Theorem 1.2 (Sequence constructions)**.** *There are point sets P_j and families $\mathcal{C}_j$ of distinct unit circles, with $|P_j| = |\mathcal{C}_j| = N_j \to \infty$, such that*

$$\frac{I(P_j, \mathcal{C}_j)}{N_j^{\alpha_*}} \longrightarrow \infty. \tag{1.6}$$

The circle centers can be chosen to be exactly P_j. There is also a sequence $s_j \to \infty$ such that

$$\mathcal{U}(s_j, \lceil s_j^{1.0629395} \rceil) \gg s_j^{1.062939573}. \tag{1.7}$$

For every fixed $r \geq 1$ and every $\epsilon > 0$, there is an unbounded sequence of integers s for which

$$\mathcal{U}(s, \lceil s^r \rceil) \gg_{r,\epsilon} s^{1+r(\alpha_*-1)-\epsilon}.$$

Arithmetic mechanism of the constructions. The field tower supplies many algebraic directions that become unit vectors in the plane. Golod–Shafarevich theory [11, 12] keeps the tower infinite, while prescribed local conditions control ramification and the degrees of selected primes. Cutting towers in this way was developed by Hajir, Maire and Ramakrishna [14]. We use the class field theory and pro-2 group conventions of [23, 26]; the relation between the group filtration and the associated graded algebra is the one studied by Jennings and Quillen [16, 32].

Choose a quadratic extension K/F in the tower and a complex embedding ϕ above a real place of F. The nontrivial involution ι of K/F acts there as complex conjugation. Thus

$$N_{K/F}(\gamma-\eta)=1 \quad\Longrightarrow\quad |\phi(\gamma)-\phi(\eta)|=1.$$

We choose finite subsets Γ_L,Γ_R of ideal cosets by intersecting a common translate of a lattice with two windows. The points and circles are then given explicitly by

$$\begin{aligned} P&=\{(\Re\phi(\gamma),\Im\phi(\gamma)):\gamma\in\Gamma_L\},\\ C_\eta:\ &(X-\Re\phi(\eta))^2+(Y-\Im\phi(\eta))^2=1, \qquad \eta\in\Gamma_R. \end{aligned} \tag{1.8}$$

The embedding is injective, so points and centers remain distinct. Ideal-class selection and a mass formula count the useful norm-one differences. Poisson summation controls the sizes of the two windows. The class-number and Fourier-analytic normalizations are those of [22, 42].

Three logarithmic quantities organize the transfer: the size of the point window, the size of the center window, and the number of incidences. Keeping all three quantities separate is what makes the unequal-cardinality estimates possible. At a complex place, the two coordinates of a relative norm-one element can have different absolute values. Averaging over the relative units accounts for these values, and the regulator cancels against the class-number formula. The finite places contribute independent choices of valuation shells. Mixing the local functions changes the three quantities in controlled proportions without forming a Cartesian product of planar configurations.

The arithmetic loss is governed by a relative zeta value. Character decompositions [37] reduce its estimation to L-functions, with functional equations coming from Hecke theory [15]. The limiting prime inequalities of Tsfasman and Vlăduţ [44] allow a comparison near $s=1$; explicit prime estimates [35] control the tails. The analytic evaluation follows the approximate-functional-equation approach of [7], with explicit error bounds at each stage. Together with the improved signature bound, this gives

$$\theta\ge\frac{1}{2}-2^{-35}, \qquad C=0.04226263259973019243081.$$

The balanced profile and the two-window profiles use this same arithmetic input.

1.2. **Upper bounds: incidence estimates.** An upper bound must hold for every configuration. The classical exponent is $4/3$ in the balanced problem; an improvement of its constant and an improvement of its exponent are different results. Table 3 separates these two kinds of progress.

The basic geometric estimate is

$$\mathcal{U}(n,m)\ll n^{2/3}m^{2/3}+n+m. \tag{1.9}$$

It has the same form as the point–line theorem of Szemerédi and Trotter [41]. Spencer, Szemerédi and Trotter [39] established the corresponding $4/3$ exponent for unit distances, and Székely [40] gave a short proof using crossing numbers. The more general point–curve theorem of Pach and Sharir [30] explains the role of two geometric properties: two distinct unit circles meet at at most two points, and two distinct points belong to at most two unit circles. We recall a crossing-number proof of (1.9) in Lemma 10.2.

The incidence viewpoint also connects the problem with arrangements of curves. Clarkson, Edelsbrunner, Guibas, Sharir and Welzl [6] developed decomposition methods that bound the complexity of curve arrangements and yield incidence estimates. Cutting curves into arcs with at most one intersection per pair gives another route; Sharir and Zahl [38] extended this approach to algebraic curves of bounded degree. Pach and Agarwal [29] give a broader account of these links between arrangements, repeated distances and geometric graphs. For unit circles, the common radius provides the additional structure used in the arithmetic stage below.

For unit distances, Ágoston and Pálvölgyi [1] sharpened the constant to $u(N)\le 1.94N^{4/3}$. OpenAI [28] subsequently proved $u(N)\ll N^{4/3-\delta}$ for an absolute $\delta>0$, improving the exponent. Theorem 1.3 gives a concrete value of the exponent difference and a bound for unequal cardinalities.

Polynomial partitioning supplies the passage to the two-parameter problem. Guth and Katz [13] used this method in their distinct-distance theorem, and Kaplan, Matoušek and Sharir [17] developed direct applications to classical incidence estimates.

TABLE 3. Upper bounds for $u(N)$. By (1.1), the same exponents give upper bounds for $\mathcal{U}(N,N)$. The constant 1.94 refers to $u(N)$.

Source	Upper bound	Exponent difference
Spencer–Szemerédi–Trotter [39]	$\ll N^{4/3}$	0
Ágoston–Pálvölgyi [1]	$\leq 1.94N^{4/3}$	0
OpenAI [28]	$\ll N^{4/3-\delta}$	Some $\delta > 0$
Theorem 1.3	$\ll N^{4/3-\delta_{\mathrm{up}}}$	$\dfrac{2^{-2^{664}}}{514560000000}$

An explicit two-parameter upper bound. Writing $M = s^r$ makes the role of the imbalance transparent: the gain over the classical nonlinear term is a factor $s^{(2-r)\delta_{\mathrm{up}}}$. It is positive for $1 \leq r < 2$; at the endpoint the linear term takes over. Define

$$\delta_{\mathrm{up}} = \frac{2^{-2^{664}}}{514560000000}, \qquad \delta_{\mathrm{simple}} = 2^{-2^{665}}. \tag{1.10}$$

Theorem 1.3 (Explicit incidence upper bound)**.** *For all positive integers n, m,*

$$\mathcal{U}(n,m) \ll M + s^{2/3-2\delta_{\mathrm{up}}} M^{2/3+\delta_{\mathrm{up}}}. \tag{1.11}$$

In particular,

$$\mathcal{U}(N,N) \ll N^{4/3-\delta_{\mathrm{up}}}.$$

The exponent difference satisfies $\delta_{\mathrm{up}} > \delta_{\mathrm{simple}} > 0$. The same bounds hold with δ_{simple} in place of δ_{up}.

For $M = s^r$, $1 \leq r \leq 2$, the nontrivial term in (1.11) is

$$s^{\frac{2}{3}(1+r)-(2-r)\delta_{\mathrm{up}}}.$$

Thus the saving persists throughout $1 \leq r < 2$. At the far end the elementary geometry gives an exact answer.

Proposition 1.4 (The terminal range)**.** *For positive integers s, M with $M \geq s(s-1)$,*

$$\mathcal{U}(s,M) = M + s(s-1).$$

Mechanism of the upper-bound proof. The upper-bound proof starts with an incidence graph having too many edges. Regularization produces a graph with controlled degrees and expansion. A finite prediction argument then reduces the variation of certain conditional distributions. In the arithmetic stage, an edge is represented by an algebraic ratio: these ratios have large height, but suitable products around rectangles have small height. This is the probabilistic and arithmetic strategy of [28].

Two quantitative principles support the geometric and algebraic steps. The Borsuk–Ulam theorem supplies simultaneous polynomial bisectors; Matoušek [20] explains this topological mechanism and its combinatorial applications. Effective forms of the Nullstellensatz, including Kollár's degree bounds [18], control polynomial identities certifying that a system has no common zero. Our argument also needs bounds on the heights of the coefficients in these identities.

To obtain a numerical exponent difference, we keep the parameters finite throughout the argument. Heights are normalized as in Bombieri and Gubler [5]. Effective algebraic estimates come from the arithmetic Nullstellensatz of Krick, Pardo and Sombra [19], the ideal-membership bounds

of Aschenbrenner [3], and real algebraic geometry [4]. These estimates bound the exceptional polynomial conditions before the size of the configuration tends to infinity.

The final obstruction is a six-by-five grid. The grid reductions of [33, 34] suggest the small configuration on which to concentrate. We use the full logarithmic-Jacobian rank pattern and prove the required arbitrary-flow statement. Independent radicals and rigidity then force a height bound incompatible with the selected edge heights. The resulting parameters are $K = 2^{68}$, $S_0 = 2^{649}$ and $a = 2^{-2^{664}}$, yielding (1.10). Finally, polynomial partitioning [13, 17] transfers the balanced estimate to arbitrary (n, m).

1.3. **Consequences for unit distances.** The two lower-bound theorems and the upper-bound theorem now transfer to $u(N)$ through (1.1). The quantifiers on the two lower bounds remain separate.

Corollary 1.5. *For every sufficiently large integer N,*

$$N^{\beta_*} \ll u(N) \ll N^{4/3-\delta_{\mathrm{up}}}.$$

There is a sequence $N_j \to \infty$ such that

$$\frac{u(N_j)}{N_j^{\alpha_*}} \longrightarrow \infty.$$

Consequently, for every $\epsilon > 0$, the sequence lower bound may be written as $u(N_j) \geq N_j^{1.0431722305539701-\epsilon}$ for all sufficiently large j.

Proof. For the lower bound at every scale, apply Theorem 1.1 with $s = M = \lfloor N/2 \rfloor$. The union of the points and centers has at most N elements and at least half as many unordered unit pairs as cross-incidences. Add distinct points to obtain exactly N elements. The upper bound follows from the left inequality in (1.1) and Theorem 1.3. For the sequence, use the coincident point and center sets in Theorem 1.2. □

1.4. **Organization and conventions.** The construction proof occupies Part 1. Sections 3 and 4 establish the field and arithmetic inputs; Section 5 proves the two-window transfer. The balanced margin in Section 6 proves (1.6), and Section 7 proves the all-scale envelope and (1.7). The elementary transfers in Section 2 give the other sequence statements and the terminal range. Part 2 proves Theorem 1.3; its final parameter calculation is in Section 26.

All logarithms are natural unless a base is indicated. The construction proof uses exact finite algebra and outward-rounded interval arithmetic. The defining coefficients, parameter tables and finite recurrences are given in Appendix A. The numerical sections specify their algorithms, error bounds and intermediate values. Standard results from class field theory, height theory, and real algebraic geometry are stated with the normalizations needed in their applications.

1.5. **AI model disclosure.** We acknowledge the use of GPT-6 Astra in the discovery and exploration of the theorems presented in this work.

Contents

1. Introduction 1

Part 1. Incidence constructions 8
2. Elementary geometric transfers and an exact terminal range 8
3. The arithmetic field family 16
4. The arithmetic constant 33
5. A two-window transfer to planar incidences 44
6. A certified balanced exponent 53
7. Asymmetric profiles and two independent cardinality bounds 62
8. An arbitrarily small twenty-circle configuration 74

Part 2. Explicit incidence upper bounds 80
9. The finite quantitative framework 80
10. A quantitative incidence piece and a prediction lemma 81
11. Heights, finite tests, and coordinate states 89
12. Scores on common levels and a quantitative lower bound for rare mass 94
13. Weighted extraction of a graph of ratios 103
14. A finite algebraic obstruction 106
15. A finite selection theorem for polynomial patterns 113
16. Transfer to unequal numbers of points and unit circles 116
17. Completion and the explicit exponent 119
18. The final parameter regime 121
19. A stronger finite incidence normalization 121
20. Uniform prediction and linear support transfer 124
21. Sharper joint-exception and averaged-template estimates 126
22. The improved explicit parameter chain 128
23. An eight-by-four finite algebraic obstruction 130
24. A six-by-five obstruction for arbitrary flows 132
25. A stronger exceptional set and the final algebraic threshold 140
26. Completion of the refined bound 142
Appendix A. Explicit local profiles 143
References 177

Part 1. Incidence constructions

2. Elementary geometric transfers and an exact terminal range

For finite sets $A, B \subset \mathbb{R}^2$, write

$$I(A,B) = \#\{(a,b) \in A \times B : |a-b| = 1\}, \qquad \mathcal{U}(n,m) = \max_{|A|=n,\ |B|=m} I(A,B).$$

Each element $b \in B$ represents the unit circle with centre b. The points within each set are distinct; the two sets are allowed to intersect. Thus this definition is exactly the point–unit-circle incidence problem, rather than a weighted version of it. Interchanging the two sets gives

$$\mathcal{U}(n,m) = \mathcal{U}(m,n). \tag{2.1}$$

Throughout this section, $n, m \geq 1$, $s = \min(n,m)$ and $M = \max(n,m)$. A *balanced seed* is an actual pair (A,B) with $|A| = |B| = N$ and $I(A,B) = E$. If the starting construction is a single set U with $e(U)$ unordered unit-distance pairs, take $A = B = U$; then $E = 2e(U)$. No disjointness between the two sides is needed for the arguments below.

2.1. Deterministic rectangular selection.

Lemma 2.1 (Two degree selections)**.** *Let (A,B) be a balanced seed with parameters (N,E). For integers $0 \leq a, b \leq N$, put $G_{N,E}(a,b) = 0$ if $ab = 0$, and otherwise put*

$$G_{N,E}(a,b) = \max\left\{ \left\lceil \frac{b}{N} \left\lceil \frac{aE}{N} \right\rceil \right\rceil, \left\lceil \frac{a}{N} \left\lceil \frac{bE}{N} \right\rceil \right\rceil \right\}. \tag{2.2}$$

There are subsets $A' \subseteq A$, $B' \subseteq B$ with $|A'| = a$, $|B'| = b$ and

$$I(A',B') \geq G_{N,E}(a,b) \geq \left\lceil \frac{abE}{N^2} \right\rceil. \tag{2.3}$$

They can be obtained by two degree rankings, performed in each of the two possible orders, and retaining the better selection.

Proof. The sum of the a largest degrees on the A side is at least aE/N. It is an integer, so retaining these a vertices leaves at least $\lceil aE/N \rceil$ edges. In the resulting bipartite graph, the sum of the b largest degrees on the B side is at least b/N times the total number of edges. This proves the first term in (2.2). Reversing the two rankings proves the second. Either term is an integer at least abE/N^2. Ties may be resolved by any fixed ordering of the seed vertices. □

The selected coordinates and circle equations are unchanged: a selected point is still a seed point, and a selected circle still has equation $|x-b| = 1$ for its original centre b. Thus Lemma 2.1 is a deterministic geometric construction, not only an averaging statement. Repeating the two rankings whenever the edge count strictly increases is optional; it can only improve the certified count.

2.2. Exact sizes, separated copies, and bridges.

Lemma 2.2 (Padding and joining)**.** *Suppose a unit-distance configuration has two nonempty sides of sizes $a \leq n$ and $b \leq m$, with F edges. It can be enlarged to sizes exactly (n,m) with at least*

$$F + (n-a) + (m-b) \tag{2.4}$$

edges. More generally, let finitely many nonempty configurations have parameters (a_i, b_i, F_i), where $\sum_i a_i \leq n$ and $\sum_i b_i \leq m$. They can be placed, joined and padded to give

$$\mathcal{U}(n,m) \geq n + m - 1 + \sum_i (F_i - a_i - b_i + 1). \tag{2.5}$$

For an empty list of configurations, the right-hand side is $n+m-1$ and is still attainable.

Proof. To add a point on the first side, put it on the unit circle about any existing point on the second side, avoiding the finitely many already used coordinates. It contributes at least one new edge. Apply the same procedure with the sides reversed for each remaining point on the second side. This proves (2.4). In particular, starting with a single unit-distance pair gives $n + m - 1$ edges.

To join a new configuration to those already placed, choose a point a on its first side and an existing point b on the second side. Translate the whole new configuration so that a becomes $b + u$, where $|u| = 1$. A collision between any new and old coordinate excludes at most one choice of u. There are only finitely many such exclusions, so a permitted u exists. This translation preserves every internal edge and guarantees a new bridge. Other cross-configuration unit pairs only increase the count. If there are $k \geq 1$ configurations, this procedure gives at least $\sum_i F_i + k - 1$ edges on the two sums of their vertex counts. Padding then gives (2.5). □

Configurations may instead be kept completely separated when bridges are unnecessary. For example, after enclosing a seed in a disc of radius R, translating successive copies by a fixed vector of length greater than $2R + 2$ keeps all inter-copy distances greater than one.

Proposition 2.3 (Integer transfer from one seed)**.** *Fix a balanced seed (N, E) and use $G = G_{N,E}$ from (2.2). For every integer $1 \leq k \leq s$, set*

$$a_k = \min\{N, \lfloor s/k \rfloor\}, \qquad b_k = \min\{N, \lfloor M/k \rfloor\}.$$

Then

$$\mathcal{U}(s, M) \geq kG(a_k, b_k), \tag{2.6}$$

$$\mathcal{U}(s, M) \geq s + M - 1 + k\big(G(a_k, b_k) - a_k - b_k + 1\big). \tag{2.7}$$

One may take the maximum of these bounds, the bound $s + M - 1$, and the bounds obtained from any other available seeds.

For a useful allocation without equal-sized remainder blocks, write

$$q = \lfloor s/N \rfloor, \qquad v = s - qN, \qquad t = \min\{N, M - qN\}.$$

Then

$$\mathcal{U}(s, M) \geq qE + G(v, t). \tag{2.8}$$

Writing $x_+ = \max\{x, 0\}$, the stronger bridge-and-padding guarantee

$$\mathcal{U}(s, M) \geq s + M - 1 + q(E - 2N + 1)_+ + H(v, t) \tag{2.9}$$

also holds, where $H(0, t) = 0$ and $H(v, t) = (G(v, t) - v - t + 1)_+$ for $v > 0$.

Proof. For (2.6), select an (a_k, b_k) rectangle by Lemma 2.1, take k separated copies, and add any missing vertices. Their sizes do not exceed (s, M) by the definitions of a_k, b_k. Joining these copies and using Lemma 2.2 proves (2.7). For (2.8), use q full copies and, when $v > 0$, one (v, t) selected rectangle. For (2.9), retain the full copies only when $E - 2N + 1 > 0$, and retain the remainder only when its analogous surplus is positive; then apply (2.5). Omitting a block frees vertices and does not violate either size budget. □

These are finite formulas for a given initial configuration and given integer targets. They do not assert that a seed is available at every requested size. They also distinguish the surplus in (2.5) from a gain that would require counting an extra incidence for an isolated padding point.

Proposition 2.4 (A common-root construction for diagonal seeds)**.** *Suppose the seed has $A = B = U$, $|U| = N \geq 2$, and $I(U, U) = E$. If*

$$k = \left\lfloor \frac{s - 1}{N - 1} \right\rfloor \geq 1,$$

then

$$\mathcal{U}(s, M) \geq s + M - 2 + k(E - 2N + 2). \tag{2.10}$$

Proof. Choose a root $u_0 \in U$. Take k rigid copies of U with their roots at the same point. Choose the rotations successively so that every non-root point of a new copy avoids all previously used coordinates; only finitely many angles are excluded at each stage. The resulting set V has exactly $1 + k(N-1)$ points. Different copies share only the root, so they share no unordered unit-distance edge. Hence $I(V,V) \geq kE$. Pad the two copies of V to sizes s and M using (2.4). The number of added vertices is $s + M - 2(1 + k(N-1))$, which gives (2.10). □

Both copies of the root, as a point and as a circle centre, are shared in this construction. Counting one as new would give incorrect sizes. The proposition makes no optimality claim about other gluing patterns.

2.3. **The exponent transferred from a balanced sequence.**

Theorem 2.5 (A specified rectangular sequence)**.** *Let $1 < \alpha < 2$. Suppose that, for every $\eta > 0$, there is an unbounded sequence of actual balanced seeds (N_j, E_j) satisfying $E_j \geq N_j^{\alpha-\eta}$. For every fixed $r \geq 1$ and every $\varepsilon > 0$, there is an unbounded sequence s_j for which*

$$\mathcal{U}(s_j, \lceil s_j^r \rceil) \geq s_j^{\max\{r,\, 1+r(\alpha-1)\}-\varepsilon}. \tag{2.11}$$

The sizes and the construction can be chosen as follows:

$$s_j = \lfloor N_j^{1/r} \rfloor, \qquad M_j = \lceil s_j^r \rceil, \tag{2.12}$$

followed by the two degree selections of Lemma 2.1.

Proof. Choose $0 < \eta < \min\{\alpha - 1, \varepsilon/(2r)\}$ and use its seed sequence. The integers in (2.12) satisfy $1 \leq s_j \leq M_j \leq N_j$ for all sufficiently large j, and $M_j/N_j \to 1$. Lemma 2.1 therefore gives

$$\mathcal{U}(s_j, M_j) \geq \frac{s_j M_j}{N_j^2} E_j \geq \frac{s_j M_j}{N_j^2} N_j^{\alpha-\eta} = s_j^{1+r(\alpha-1-\eta)+o(1)}.$$

For sufficiently large j, this is at least $s_j^{1+r(\alpha-1)-\varepsilon}$. Independently, $\mathcal{U}(s_j, M_j) \geq M_j \geq s_j^r$, by the elementary padding construction. Taking the better construction proves (2.11). □

With the certified balanced exponent used in this paper,

$$\alpha_0 = 1.0431722305539701, \tag{2.13}$$

the transferred exponent $1 + r(\alpha_0 - 1)$ exceeds the linear exponent r precisely when

$$1 \leq r < \frac{1}{2 - \alpha_0} = 1.045120168887829545\ldots. \tag{2.14}$$

For instance, at $r = 1.04$ the exponent before the arbitrary ε loss is 1.044899119776128904. Theorem 2.5 retains the sequence quantifier of its hypothesis. Its sequence may depend on r and ε; it is not an assertion for every sufficiently large s. The all-size constructions elsewhere in the paper require their separate uniform counting and scale-coverage arguments.

2.4. **An elementary construction at large imbalance.**

Theorem 2.6 (Exact terminal range)**.** *For $s \geq 2$, put $S = s(s-1)$. Then*

$$\mathcal{U}(s, M) \geq M + \min\{M, S\}, \qquad M \geq s, \tag{2.15}$$

and

$$\mathcal{U}(s, M) = M + s(s-1), \qquad M \geq s(s-1). \tag{2.16}$$

For $s = 1$ one has $\mathcal{U}(1, M) = M$.

Proof. Choose s distinct points $a_1, \dots, a_s$ in a disc of radius $1/4$, with no three lying on a common unit circle. This choice can be made recursively with rational coordinates: after finitely many points have been selected, exclude the unit circles through each pair of old points, as well as the old points themselves. This is a finite union of circles and points, so its complement in the disc contains a rational point.

Every pair a_i, a_j has distance strictly between zero and two, and therefore lies on exactly two unit circles. Explicitly, writing $d_{ij} = |a_i - a_j|$ and letting J denote rotation through $\pi/2$, their centres are

$$b_{ij}^{\pm} = \frac{a_i + a_j}{2} \pm \frac{\sqrt{1 - d_{ij}^2/4}}{d_{ij}} J(a_i - a_j).$$

The no-three condition makes all $2\binom{s}{2} = S$ circles distinct, and each contains exactly two of the chosen points. If $M \le S$, select any M of these circles, obtaining $2M$ incidences. If $M \ge S$, select all of them and then add $M - S$ distinct unit circles through a_1 and through none of $a_2, \dots, a_s$. Their centres can be chosen on the unit circle about a_1, avoiding finitely many forbidden centres. This gives $2S + (M - S) = M + S$ incidences. Thus (2.15) and the lower half of (2.16) hold.

For the matching upper bound, consider arbitrary s points and M distinct unit circles, and let d_C be the number of selected points on a circle C. Two distinct points lie on at most two unit circles, so

$$\sum_C \binom{d_C}{2} \le 2\binom{s}{2} = S.$$

The integer inequality $d \le 1 + \binom{d}{2}$ holds for every $d \ge 0$. Consequently

$$I \le M + \sum_C \binom{d_C}{2} \le M + S,$$

which proves the upper half of (2.16). For $s = 1$, at most one incidence is possible per circle, and any number of distinct unit circles can be chosen through the single point. □

The bounds from the terminal construction and the seed constructions can always be combined by taking their maximum. None of the elementary transfers above purports to determine the extremal order throughout the intermediate imbalance range.

2.5. **Continuous triple spectra and a sharper exact interval.** Call a set of s circle centres *admissible* if its diameter is less than two, no four of the corresponding unit circles concur, and every intersection has multiplicity two or three. Distinct centres then give exactly two transverse intersections per pair of circles. A *continuous triple spectrum up to T* means that, for each integer $0 \le t \le T$, an admissible s-centre configuration with exactly t triple intersections is available. Different values of t may use different configurations.

Theorem 2.7 (Incidences from a continuous spectrum)**.** *Let $s \ge 2$, $S = s(s-1)$, and suppose a continuous triple spectrum up to T is available with exactly s centres. For $s \le M \le S$,*

$$\mathcal{U}(s, M) \ge 2M + \min\left\{T, M, \left\lfloor \frac{S - M}{2} \right\rfloor\right\}. \tag{2.17}$$

If, in addition, $3T < S$, then

$$\mathcal{U}(s, M) = \left\lfloor \frac{3M + S}{2} \right\rfloor \quad \text{for} \quad \max\{s, S - 2T - 1\} \le M \le S. \tag{2.18}$$

Above S, the exact value is given by (2.16).

Proof. In a configuration with t triple intersections, let D_t be its number of double intersections. Counting the two intersections of every circle pair gives

$$D_t + 3t = 2\binom{s}{2} = S. \tag{2.19}$$

Choose $t = \min\{T, M, \lfloor (S-M)/2 \rfloor\}$. Then $t \le M$ and $M - t \le S - 3t = D_t$. Select all t triple intersections and any $M - t$ double intersections. These are M distinct points incident to the s unit circles, with exactly $2M + t$ incidences. Equation (2.1) gives (2.17) in the stated order of the two cardinalities.

For an arbitrary configuration of s unit circles and M selected points, let d_p be the number of circles through p. The inequalities

$$\sum_p \binom{d_p}{2} \le S, \qquad 2d - 3 \le \binom{d}{2} \quad (d \in \mathbb{Z}_{\ge 0})$$

are universal; the second follows from $\binom{d}{2} - 2d + 3 = (d-2)(d-3)/2 \ge 0$. Consequently $\mathcal{U}(s, M) \le \lfloor (3M + S)/2 \rfloor$. If $M \ge S - 2T - 1$, then $\lfloor (S-M)/2 \rfloor \le T$. The additional hypothesis $S \ge 3T + 1$ implies $M \ge S - 2T - 1 \ge T$, so the minimum in (2.17) is $\lfloor (S-M)/2 \rfloor$. Its resulting value is $\lfloor (3M + S)/2 \rfloor$, proving (2.18). □

The strict inequality $3T < S$ records the availability of a double intersection at the odd endpoint $M = S - 2T - 1$. It holds for every finite value used below and for the general construction at sufficiently large s. Thus no parity endpoint is supplied by deleting a nonexistent double intersection.

2.6. **Products of small configurations.** A triple of centres has a common unit-circle intersection precisely when the three centres lie on a unit circle. In the absence of fourfold intersections, such triples are in bijection with triple intersection points. We therefore regard the triple pattern as a 3-uniform hypergraph on the centres.

The geometric input is the ordered 20-centre configuration of Proposition 8.1. It has arbitrarily small diameter, exactly 60 triple intersections and no fourfold intersections. Its ordered prefixes have triple counts

$$\begin{aligned} (\tau_0, \dots, \tau_{10}) &= (0, 0, 0, 1, 2, 4, 8, 11, 14, 17, 20), \\ (\tau_{11}, \dots, \tau_{20}) &= (24, 28, 32, 36, 40, 44, 48, 52, 56, 60). \end{aligned} \tag{2.20}$$

In particular, $0 \le \tau_{a+1} - \tau_a \le 4$. The geometric realization, including the exclusion of unintended contacts, is essential here: a hypergraph with these numerical counts alone would not suffice.

Lemma 2.8 (Generic products and unions)**.** *Let $A, B \subset \mathbb{C}$ have diameter less than one and no four points on a unit circle. For all but finitely many angles ϕ, the sumset*

$$A + e^{i\phi}B = \{a + e^{i\phi}b : a \in A,\ b \in B\}$$

has $|A||B|$ elements. Its unit-circle triples are exactly those inherited from a single coordinate fibre. In particular it has no four points on a unit circle, and its number of triples is

$$|B|\, t(A) + |A|\, t(B). \tag{2.21}$$

Finite configurations of arbitrarily small diameter may also be placed by small generic translations so that their union has no new mixed triple and no repeated centre. These unions add both the numbers of centres and the numbers of triples. Generic singleton padding is allowed.

Proof. A collision between two distinct pairs (a, b) excludes at most one angle, unless both coordinates coincide. Consider a triple of distinct pairs. If one projection is constant, the triple is inherited from the other factor. Otherwise we show that its unit-circle condition excludes only finitely many angles.

Suppose first that both projections have three distinct values. After subtracting the first point, write its two differences as $x = \alpha + e^{i\phi}\beta$ and $y = \gamma + e^{i\phi}\delta$. A necessary condition for circumradius one is

$$|x|^2|y|^2|x-y|^2 - 4\det(x,y)^2 = 0.$$

The coefficient of $e^{3i\phi}$ in the first term is $\overline{\alpha}\beta\,\overline{\gamma}\delta\,\overline{\alpha-\gamma}(\beta-\delta)$, which is nonzero. The determinant-square term has frequencies of absolute value at most two. Thus this is a nonzero trigonometric polynomial.

If, say, the A projection has two values, relabel so that $a_1 = a_2 \neq a_3$. Necessarily $b_1 \neq b_2$. The possible centres of a unit circle through the first two sumset points are $a_1 + e^{i\phi}c_\pm$, where $c_\pm$ are the two unit-circle centres through b_1, b_2. A third contact requires

$$|a_3 - a_1 + e^{i\phi}(b_3 - c_\pm)| = 1.$$

For either sign this cannot hold identically: an identity would force $b_3 = c_\pm$ and $|a_3 - a_1| = 1$, contrary to the diameter assumption. Interchanging the two factors treats the other case. There are finitely many triples, so all mixed triples are excluded by avoiding finitely many angles.

If four points of the sumset lay on a unit circle, either they would all lie in one coordinate fibre, contradicting the hypothesis on that factor, or some three would form a mixed triple. This proves the no-four assertion and (2.21).

For the union assertion, add one translated configuration to those already placed. A mixed triple has two points in one of the two parts. The two points determine at most two unit circles, and requiring the third to lie on one of them excludes a finite union of circles in the translation plane. Repeated centres exclude finitely many additional translations. These sets have empty interior, so a translation can be chosen in an arbitrarily small open disc while avoiding all of them. No mixed fourfold intersection can then occur. The same argument applies to a singleton. □

For each fixed product dimension d, choose the primitive factors with diameter less than $1/(4d)$ and apply the lemma successively. The resulting product has diameter less than one, so every pair of its associated unit circles crosses twice. Its triple hypergraph has one edge for each copy of a core edge in a coordinate fibre, and

$$|H_{20}^d| = 20^d, \qquad t(H_{20}^d) = 60d\,20^{d-1} = 3d\,20^d. \tag{2.22}$$

Every induced subfamily inherits a valid geometric realization.

2.7. **A continuous spectrum for every sufficiently large size.** Order $[20]^d$ lexicographically, using the order in (2.20), and let $f_d(h)$ be the number of fibre triples in its first h vertices. Put $f_0(0) = f_0(1) = 0$ and $F_j = 3j20^j$ for $j \geq 0$. If $L = 20^{d-1}$ and $h = aL + b$, where $0 \leq b < L$, then

$$f_d(h) = aF_{d-1} + f_{d-1}(b) + \tau_a L + (\tau_{a+1} - \tau_a)b. \tag{2.23}$$

At $h = 20^d$ this means $f_d(20^d) = F_d$, with the last term omitted. Indeed, the first two terms count edges in full and partial suffix layers. A first-coordinate fibre contributes τ_{a+1} if its suffix is in the partial layer, and τ_a otherwise. In particular,

$$0 \leq f_d(h+1) - f_d(h) \leq 4d. \tag{2.24}$$

Each newly added vertex contributes at most four completed triples in each coordinate fibre.

To describe an optional efficient reservoir, let $c(v)$ be the smallest number of vertices in a subset of the core inducing exactly v edges; set $c(v) = +\infty$ when no such subset exists. This is a finite definition using the explicitly specified 20-vertex core. Define

$$c_*(0) = 0, \qquad c_*(t) = \min_{\substack{1 \leq v \leq \min\{60,t\} \\ c(v) < \infty}} \big(c(v) + c_*(t-v)\big) \quad (t \geq 1). \tag{2.25}$$

By Lemma 2.8, $c_*(t)$ centres suffice for exactly t triples: take the indicated generic union of core subsets. Conversely, every such union can be split off at its last positive-triple component, proving the recurrence for this specified reservoir family. The three-vertex prefix has one triple, so $c_*(t) \le 3t$.

For $s \ge 2$ put

$$d = \max\{1, \lceil \log_{20} s \rceil\}, \qquad R_d = \max_{0 \le v < 4d} c_*(v),$$
$$q = \max\{0, s - R_d + 1\}, \quad T_0(s) = f_d(q).$$

Proposition 2.9 (Prefix and reservoir interpolation)**.** *Every integer from zero to $T_0(s)$ is attained by an admissible configuration of exactly s centres. In particular,*

$$T_0(s) \ge \frac{3}{\log 20} s \log s - O(s). \tag{2.26}$$

A stronger explicitly computable endpoint is obtained by adjoining to $[0, T_0(s)]$ all integers

$$f_d(h) + v, \qquad q \le h \le s, \quad 0 \le v \le 3(s-h), \quad c_*(v) \le s - h, \tag{2.27}$$

and taking the first missing nonnegative integer minus one. All of these are constructive lower spectra, without an optimality claim for arbitrary subfamilies or planar configurations.

Proof. The case $q = 0$ asserts only the attainable value zero. Otherwise, for $t < T_0(s)$ take the largest $h < q$ with $f_d(h) \le t$. Then (2.24) gives $0 \le t - f_d(h) \le 4d - 1$, so the h-vertex prefix together with a reservoir uses at most

$$h + c_*(t - f_d(h)) \le q - 1 + R_d = s$$

centres and has exactly t triples. At $t = T_0(s)$ use the q-vertex prefix itself. Lemma 2.8 permits their union and padding to exactly s centres, without changing the triple count. The same argument directly realizes every value in (2.27).

For completeness, the logarithmic lower estimate can be proved directly from the recurrence. Write $g(x) = 3x \log_{20} x$, with $g(0) = 0$. Induction gives $f_d(h) \ge g(h) - 6h$. For a nonzero proper prefix, discard leading zero digits and write $h = aL + b$ with $a \ge 1$, $b < L$. The first two terms of (2.23) are at least $ag(L) + g(b) - 6b$. The difference $g(h) - ag(L) - g(b)$ is three times h times the base-20 entropy of the partition into a blocks of size L and one of size b; it is at most $3h$. Since $b \le h/2$, the desired estimate follows. Full prefixes satisfy it by (2.22). Finally $R_d \le 3(4d-1)$ and (2.24) imply $f_d(s) - f_d(q) = O(d^2)$ for sufficiently large s. Thus $T_0(s) \ge g(s) - 6s - O(d^2)$, proving (2.26). □

The finite values of $c(v)$ can be evaluated by enumerating the 2^{20} subsets of the explicit core, and (2.25) then involves only integer additions and comparisons. As $R_d = O(d)$, the extension (2.27) examines only $O(d)$ prefix sizes and $O(d)$ reservoir counts per prefix. The interval $[0, T_0(s)]$ may be stored implicitly; no array of length comparable to $s \log s$ is needed for this general construction.

Combining Proposition 2.9 with Theorem 2.7, and truncating the endpoint at $\lfloor (S-1)/3 \rfloor$ if necessary for small s, gives an exact interval below S of width at least

$$\frac{6}{\log 20} s \log s - O(s). \tag{2.28}$$

The construction also gives the additive term $T_0(s)$ in (2.17) over its middle range; its consequence is not limited to the terminal interval.

2.8. **Finite examples from the prefix construction.** The prefix construction also gives short numerical examples. The following table specifies the reservoirs used here. For an integer $w \ge 0$, let

$$B(w) = \{j \in \{0, \dots, 19\} : \lfloor w/2^j \rfloor \text{ is odd}\}.$$

Thus w is simply a compact notation for a subset of the twenty ordered centres. The table gives a subset with exactly v triples; its size is denoted by d_v.

(2.29)

v	w_v	d_v	v	w_v	d_v
0	0	0	8	63	6
1	7	3	9	3940	7
2	15	4	10	495	8
3	79	5	11	127	7
4	31	5	12	383	8
5	467	6	13	895	9
6	95	6	14	255	8
7	351	7	15	200164	9

The entries follow directly from the sixty triples in Section 8: for each row,

$$|B(w_v)| = d_v, \qquad \sum_{(e,g)\in\mathcal{T}_{20}} \prod_{j\in g} \left(\lfloor w_v/2^j \rfloor - 2\lfloor w_v/2^{j+1} \rfloor \right) = v.$$

Here $\mathcal{T}_{20}$ is the signed triple list defined in Section 8.4. These expressions use only the displayed integers and that list; minimality of the reservoir sizes is not needed. In particular, $c_*(v) \le 8$ for $0 \le v < 12$, and $c_*(v) \le 9$ for $0 \le v < 16$.

For $s = 500, 1000$ take $d = 3$ and $q = s - 7$; for $s = 10000$ take $d = 4$ and $q = s - 8$. The proof of Proposition 2.9 then supplies every count in $[0, f_d(q)]$. A few further prefixes give

(2.30)

s	q	$f_d(q)$	$T(s)$	$s(s-1) - 2T(s) - 1$
500	493	2738	2754	243991
1000	993	5944	5958	987083
10000	9992	85520	85546	99818907

For clarity, the short interval calculation is displayed in full below. Each row combines the prefix of size h with one of the reservoirs $B(w_v)$. The last column records all counts in the indicated range, and $h + d_v \le s$ in every case. Generic padding makes the number of centres exactly s.

s	h	$f_d(h)$	v	$f_d(h) + v$
500	493	2738	$0, \dots, 5$	$2738, \dots, 2743$
	494	2744	$0, \dots, 5$	$2744, \dots, 2749$
	495	2750	$0, \dots, 4$	$2750, \dots, 2754$
1000	993	5944	$0, \dots, 7$	$5944, \dots, 5951$
	994	5952	$0, \dots, 6$	$5952, \dots, 5958$
10000	9992	85520	$0, \dots, 9$	$85520, \dots, 85529$
	9993	85530	$0, \dots, 9$	$85530, \dots, 85539$
	9994	85540	$0, \dots, 6$	$85540, \dots, 85546$

Together with the initial interval, these overlapping ranges prove continuous coverage up to every endpoint in (2.30). For example,

$$\mathcal{U}(500, M) = \left\lfloor \frac{3M + 249500}{2} \right\rfloor \qquad (243991 \le M \le 249500).$$

The corresponding exact intervals start at 987083 for $s = 1000$ and at 99818907 for $s = 10000$. They follow from (2.18), and join $\mathcal{U}(s, M) = M + s(s-1)$ at the terminal range.

One can obtain larger individual triple counts by another explicit subfamily. In the cube of the core, set

$$F = \{(x, y, 0) : 0 \le x, y < 20\} \cup \{(x, 0, z) : 1 \le z \le r_x\},$$
$$(r_x) = (\underbrace{6, \dots, 6}_{15}, 5, 5, 0, 0, 0).$$

There are 500 vertices. Its three coordinate directions contribute

$$20 \cdot 60 + 5\tau_{17} + \tau_{15} = 1480, \qquad 20 \cdot 60 = 1200, \qquad 15\tau_7 + 2\tau_6 = 181,$$

respectively, giving 2861 triples. The product $F \times H_{20}$ has 10000 vertices and $20 \cdot 2861 + 500 \cdot 60 = 87220$ triples. These two configurations follow from the same generic-product lemma.

3. The arithmetic field family

This section gives the field input, including the strengthened signature bound, separately from the analytic estimate for its relative zeta value. The tower, its local specifications, and the argument for its infinitude originate in Naslund's construction [24]. We give the arithmetic data and the proofs needed here; in particular, a finite quotient calculation is not used as a substitute for proving that the underlying arithmetic tower is infinite. The standard results used below are Kummer theory, local duality and local class field theory, the Brauer–Hasse–Noether description of the Brauer group, pro-2 Nakayama, the filtered Fox exact sequence, and the Jennings basis theorem. Their specific applications and the finite calculations are stated explicitly.

3.1. **Base field and local data.** Put $B = \mathbb{Q}(\sqrt{241})$, and distinguish its two real embeddings $v_1(\sqrt{241}) > 0$ and $v_2(\sqrt{241}) < 0$. The following eight elements will also fix the coordinates of every elementary Galois image:

$$\begin{aligned} &\alpha_0 = -1, \qquad \alpha_1 = -71011068 + 4574225\sqrt{241}, \\ &\alpha_2 = (-6101 - 393\sqrt{241})/2, \quad \alpha_3 = (6101 - 393\sqrt{241})/2, \\ &\alpha_4 = 31 - 2\sqrt{241}, \quad \alpha_5 = 31 + 2\sqrt{241}, \\ &\alpha_6 = 326 - 21\sqrt{241}, \quad \alpha_7 = 326 + 21\sqrt{241}. \end{aligned} \tag{3.1}$$

Their respective norms are $1, -1, -2, -2, -3, -3, -5, -5$. Let $\mathfrak{p}_j = (\alpha_{1+j})$, $\mathfrak{q}_j = (\alpha_{3+j})$, and $\mathfrak{r}_j = (\alpha_{5+j})$ for $j = 1, 2$, and set

$$S = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{r}_1, \mathfrak{r}_2\}.$$

We additionally distinguish

$$\mathfrak{t}_1 = (-14127 + 910\sqrt{241}), \quad \mathfrak{t}_2 = (-14127 - 910\sqrt{241}), \quad \mathfrak{t}_3 = (621 + 40\sqrt{241}).$$

The first two primes have norm 29, the third norm 41. The conjugate prime $\mathfrak{t}_4 = (621 - 40\sqrt{241})$ is not capped.

Lemma 3.1. *The ring of integers is $\mathbb{Z}[(1+\sqrt{241})/2]$, its discriminant is 241, and its class number is one. The group of square classes with even valuations outside S is an eight-dimensional $\mathbb{F}_2$-space, with basis the classes of* (3.1).

Proof. The formula for the integers and discriminant is the quadratic-field formula. The displayed norms and $\alpha_2\alpha_3 = 2$, $\alpha_4\alpha_5 = -3$, $\alpha_6\alpha_7 = -5$ show that the primes over $2, 3, 5$ are principal. The prime 7 is inert, since $241 \equiv 3 \pmod 7$ is a nonsquare. Every ideal class has an integral ideal of norm at most $\sqrt{241}/2 < 8$ by Minkowski's theorem. Its prime factors are therefore among the principal primes just listed; this proves class number one. Consequently an element having even valuations outside S, modulo a square, is a unit times a product of the six specified prime generators. Their valuation vectors make those six square classes independent. The unit theorem gives two unit square classes as a vector-space dimension; -1 and the norm--1 unit α_1 are independent, as their two real sign vectors show. This proves the assertion without needing a claim that α_1 is the smallest fundamental unit. □

Let B_S be the maximal pro-2 extension of B unramified at all finite primes outside S, without a condition at infinity, and let $G_S = \mathrm{Gal}(B_S/B)$. Kummer theory and Lemma 3.1 identify its maximal elementary quotient with

$$\mathrm{Gal}(E/B) = \mathbb{F}_2^8, \qquad E = B(\sqrt{\alpha_0}, \dots, \sqrt{\alpha_7}).$$

For a Galois element g its elementary image is the string $a_0 \cdots a_7$ defined by $g\sqrt{\alpha_i} = (-1)^{a_i}\sqrt{\alpha_i}$. Throughout, $[g,h] = g^{-1}h^{-1}gh$.

Here is the complete local-vector input. At 3 and 5, t denotes inertia and f a Frobenius lift fixing the square root of the chosen uniformizer; F_j denotes the Frobenius at $\mathfrak{t}_j$.

(3.2)

ι_1	10111010	ι_2	11000101
x_1	00100000	x_2	00010000
y_1	11110010	y_2	10000001
z_1	10000100	z_2	11111000
t_{q1}	00001000	f_{q1}	10100111
t_{q2}	00000100	f_{q2}	11101011
t_{r1}	00000010	f_{r1}	01100101
t_{r2}	00000001	f_{r2}	01011010
F_1	00100111	F_2	00011011
F_3	00111000		

For reproducibility, these vectors can be calculated directly from (3.1), using the roots of 241 congruent respectively to $7, 25 \pmod{32}$ at the two dyadic places, $2, 1 \pmod 3$, $1, 4 \pmod 5$, $3, 26 \pmod{29}$, and $6 \pmod{41}$. At an odd prime, record the parity of the valuation and whether the unit part is a quadratic residue. At 2, record the valuation modulo two and the odd part modulo eight, in the square-class basis $-1, 2, 5$. The dyadic generators x, y, z act on $(\sqrt{2}, \sqrt{-1}, \sqrt{5})$ by $(-,+,+), (+,-,+), (+,-,-)$. Thus they evaluate a class $(-1)^a 2^b 5^c$ as $b, a, a+c$, respectively. These rules give every entry of (3.2); for example at $\mathfrak{p}_1$, $\alpha_1 \equiv 3 \equiv -5 \pmod 8$. The real rows are simply the signs of (3.1).

3.2. **The prescribed dyadic extension.** Let $i = \sqrt{-1}$, $k = \mathbb{Q}_2(i)$, $\pi_5 = 1 + 2i$, and let $U_4/\mathbb{Q}_2$ be unramified of degree four. Define

$$M_8 = \mathbb{Q}_2(i, \sqrt{5}, \sqrt{\pi_5}), \quad M_{16} = M_8(\sqrt{2}), \quad L_2 = M_{16}U_4.$$

The finite group needed at both dyadic places is

$$D = \langle x, y, z \mid x^2 = y^2 = z^4 = [y,z]^2 = [x,y] = [x,z] = [[y,z],z] = 1 \rangle. \tag{3.3}$$

Lemma 3.2. *The extension $L_2/\mathbb{Q}_2$ is Galois with group D, degree 32, ramification index 8, residue degree 4, and normalized different exponent $18/8 = 9/4$. Its Zassenhaus layers have dimensions $3, 2$ in degrees one and two, with all later layers zero. Therefore*

$$h_D(t) := h_{\mathbb{F}_2[D]}(t) = (1+t)^3(1+t^2)^2.$$

Proof. The identity $\pi_5\overline{\pi_5} = 5$ supplies both conjugate radicals. There are automorphisms y_0, z_0 of M_8 which both send i to $-i$ and $\sqrt{\pi_5}$ to $\sqrt{5}/\sqrt{\pi_5}$, and send $\sqrt{5}$ respectively to $\sqrt{5}$ and $-\sqrt{5}$. They satisfy $y_0^2 = 1$, $z_0^4 = 1$, $y_0 z_0 y_0 = z_0^{-1}$, with $z_0^2 \neq 1$. Over k, the extension by $\sqrt{5}$ is unramified whereas that by $\sqrt{\pi_5}$ is ramified, so $[M_8 : \mathbb{Q}_2] = 8$. Its three quadratic subfields have square classes $-1, 5, -5$; hence $\sqrt{2} \notin M_8$ and $[M_{16} : \mathbb{Q}_2] = 16$. The largest unramified subextension of M_{16} is $\mathbb{Q}_2(\sqrt{5})$: its Galois group has elementary abelianization, so every cyclic quotient has order at most two. Consequently $M_{16} \cap U_4 = \mathbb{Q}_2(\sqrt{5})$ and $[L_2 : \mathbb{Q}_2] = 32$. Lift y_0 trivially on U_4, lift z_0 by the Frobenius on U_4, and take x to negate only $\sqrt{2}$. These elements satisfy (3.3). Writing $w = [y,z]$, its normal forms are $x^a y^b z^c w^e$, with $a, b, e \in \{0,1\}$ and $0 \le c < 4$; their actions on the radicals and U_4 distinguish all 32 forms. The inertia group is $\langle x, y, w \rangle$ of order eight.

Here is also a direct different calculation. In k, put $\varpi = 1 + i$. The conductor exponents of the quadratic characters with square classes

$$1, \quad 5, \quad \pi_5, \quad 5\pi_5, \quad 2, \quad 10, \quad 2\pi_5, \quad 10\pi_5$$

are $0, 0, 2, 2, 4, 4, 4, 4$. To check the nonzero entries, use the Eisenstein polynomial $X^2 + 2(1 + \varpi)X/\varpi + 2/\varpi$ for the extension by $\sqrt{\pi_5}$, and $X^2 + 2X + 1 - u$ for $u = 2 - i$ and $u = i$, corresponding to $2\pi_5$ and 2 modulo squares. The derivative valuations give $2, 4, 4$; an unramified quadratic twist by 5 preserves these conductors. The conductor–discriminant formula gives relative discriminant exponent 20 for M_{16}/k. Since the different exponent of $k/\mathbb{Q}_2$ is two, $\operatorname{ord}_2 \mathfrak{d}_{M_{16}/\mathbb{Q}_2} = 8 \cdot 2 + 20 = 36$. Its residue degree is two, so its different exponent is 18; passage to the unramified compositum does not change this. Finally D_2D is the central elementary group generated by z^2, w, and $D_3D = 1$. Jennings's basis theorem yields the displayed Hilbert polynomial. □

We use the following standard local presentation statements. At a real place the maximal pro-2 group is C_2. At an odd prime p, it is generated by t, f with relation $ftf^{-1}t^{-p}$, whose quadratic initial over $\mathbb{F}_2$ is $[t, f] + ((p-1)/2)t^{[2]}$. The maximal pro-2 group of $\mathbb{Q}_2$ is a rank-three Demushkin group with one relation. In the dyadic coordinates above its quadratic initial is

$$y^{[2]} + [x, y] + [x, z]. \tag{3.4}$$

For the last assertion, local duality identifies the relation pairing with the Hilbert symbol. In the dual square-class basis $2, -5, 5$, that symbol has indicator matrix $\left(\begin{smallmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)$; off-diagonal entries give brackets and diagonal entries restricted squares. This proves (3.4), including its diagonal term, rather than choosing a convenient but unspecified dyadic lift. These local presentation and duality statements are the standard local Galois theorems used here [23, 26].

3.3. **The global presentation and the tower group.** Let $\mathfrak{F}$ be the free pro-2 group on eight generators mapping minimally onto G_S. For a local place choose lifts of its generators to $\mathfrak{F}$, and denote its lifted defining relation by ρ_ν. Let $\Sigma = S \cup \{v_1, v_2\}$.

Lemma 3.3. *The group G_S has relation rank seven. Any seven of the eight local relations ρ_ν normally generate the kernel of $\mathfrak{F} \longrightarrow G_S$.*

Proof. There are eight generators by Kummer theory. To bound the relation rank, let $X = \operatorname{Spec} \mathcal{O}_B[1/30]$. The Kummer exact sequence, $\operatorname{Pic}(X) = 0$, and the Brauer–Hasse–Noether theorem [23, 31] give

$$H^2(X, \mu_2) = \operatorname{Br}(X)[2] = \ker\left(\bigoplus_{\nu \in \Sigma} \operatorname{Br}(B_\nu)[2] \longrightarrow \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\right).$$

Each local term has dimension one and the sum map is onto, so the right side has dimension seven. The Hochschild–Serre sequence for finite Galois 2-covers of X [21] injects $H^2(G_S, \mathbb{F}_2)$ into this space: the intervening direct limit of H^1 vanishes, since a double cover of a finite 2-cover becomes trivial on a further 2-cover (take its Galois closure, whose group embeds in a wreath product of 2-groups). Thus the relation rank is at most seven.

For completeness, the seven local quadratic initials are independent. Under the relation–cup pairing their classes are the local Hilbert forms on the span of (3.1). The eight forms sum to zero by Hilbert reciprocity. Evaluate them on

$$(-1, \alpha_2),\ (-1, \alpha_1),\ (-1, \alpha_7),\ (-1, \alpha_4),\ (-1, \alpha_5),\ (\alpha_2, \alpha_6),\ (\alpha_1, \alpha_7).$$

Their nontrivial local supports, in that order, are

$$\{\mathfrak{p}_1, v_1\},\ \{\mathfrak{p}_1, v_2\},\ \{\mathfrak{p}_2, v_2\},\ \{\mathfrak{q}_1, v_1\},\ \{\mathfrak{q}_2, v_2\},\ \{\mathfrak{r}_1, v_1\},\ \{\mathfrak{r}_2, v_2\}.$$

These are obtained by the elementary Hilbert formulas. For odd p,

$$(p^a u, p^b v)_p = (-1)^{ab(p-1)/2}(u/p)^b (v/p)^a.$$

For odd u, v at 2, the exponent is $((u-1)/2)((v-1)/2) + a(v^2-1)/8 + b(u^2-1)/8$ modulo two. Together with the residue rules following (3.2), this is a finite direct check of the supports. They form a spanning tree on the eight places, so the sole dependence among the eight forms is their sum. In particular, omitting $\mathfrak{p}_1$ leaves seven independent quadratic initials. Their images are independent in the relation module $R/(R^2[R,\mathfrak{F}])$. Its dimension is at most seven by the cohomological bound. Pro-2 Nakayama now proves normal generation; the same argument applies to any omitted place. □

Define G as the quotient of G_S obtained by imposing the local quotient D at each dyadic place, $C_2 \times C_2$ at each place over $3, 5$, and $F_1^4 = F_2^4 = F_3^4 = 1$. Equivalently, quotient by the closed normal subgroup generated by the kernels of those local maps and the three fourth powers. Let B_G be its fixed field in B_S. All those kernels lie in the Frattini subgroup, so $E \subset B_G$ and $G/D_2G = \mathbb{F}_2^8$ with coordinates (3.2).

An explicit normal presentation of G over $\mathfrak{F}$ has thirty relations: the seven ρ_ν with $\nu \neq \mathfrak{p}_1$; for each $j = 1, 2$ the six words

$$x_j^2,\ [x_j, y_j],\ [x_j, z_j],\ [[y_j, z_j], z_j],\ z_j^4,\ [y_j, z_j]^2;$$

the eight squares t_ν^2, f_ν^2 at $3, 5$; and the three F_k^4. To justify the dyadic assertion without replacing an unknown relation, let N_D be the kernel of the free three-generator presentation (3.3). Its seven displayed relators span $N_D/(N_D^2[N_D, \mathfrak{F}_3])$. The true local arithmetic relation r belongs to N_D. In its expression in this relation module, the coefficient of y^2 is one: the four quadratic initials $x^2, y^2, [x, y], [x, z]$ are independent, whereas the other three relators start in degree at least three, and r has initial (3.4). Replacing y^2 by r therefore still spans the relation module and normally generates N_D by pro-2 Nakayama. In the local arithmetic group $r = 1$, leaving exactly the six additional relations stated above. At odd places the original tame relation together with the two squares makes the two involutions commute and therefore presents $C_2 \times C_2$. The omitted dyadic arithmetic relation already follows from the other seven global relations, by Lemma 3.3. We do not replace its true higher coefficients by a guessed word.

3.4. **Finite restricted-Lie calculations.** We specify the finite calculations sufficiently to determine all their defining matrices. Work in $\mathbb{F}_2\langle X_0, \dots, X_7\rangle$, with $[a, b] = ab + ba$ and $a^{[2]} = a^2$, and let L_i be the degree-i part of its free restricted Lie algebra. Identify every vector in (3.2) with the corresponding linear polynomial. Bases are obtained by row reduction of the following lists:

$$L_2 = \langle X_i^2, [X_i, X_j]\rangle, \quad L_3 = [L_2, L_1], \quad L_4 = [L_3, L_1] + [L_2, L_2] + \langle u^2 : u \in L_2\rangle.$$

For the final square list it suffices to square a basis, since the cross terms are already in $[L_2, L_2]$.

Let $q_1, \dots, q_{21}$ be, in order, the following quadratic initials: the squares of ι_1, ι_2; the four expressions $[t_\nu, f_\nu] + t_\nu^2$ at 3 and $[t_\nu, f_\nu]$ at 5; $y_2^2 + [x_2, y_2] + [x_2, z_2]$; the six expressions $x_j^2, [x_j, y_j], [x_j, z_j]$; and the eight t_ν^2, f_ν^2. Put

$$\begin{aligned} R_2 &= \langle q_i\rangle, \\ R_3 &= \langle [q_i, X_a] : 1 \le i \le 21, 0 \le a < 8\rangle + \langle \omega_1, \omega_2\rangle, \quad \omega_j = [[y_j, z_j], z_j], \\ I_4 &= [R_3, L_1] + [R_2, L_2] + \langle q_i^2 : 1 \le i \le 21\rangle \\ &\qquad + \langle z_1^4, z_2^4, [y_1, z_1]^2, [y_2, z_2]^2, F_1^4, F_2^4, F_3^4\rangle. \end{aligned} \tag{3.5}$$

Algorithm 3.4 (Row reduction for the restricted-Lie spaces)**.** (i) In each degree $r \le 4$, order the words $X_{i_1} \cdots X_{i_r}$ lexicographically. Write a polynomial as its coefficient vector in this order, with coefficients in $\mathbb{F}_2$. There are at most $8^4 = 4096$ columns.

(ii) Form the rows in the displayed spanning lists. In a product, multiply words by juxtaposition and collect equal words; every coefficient is reduced modulo two.

(iii) Apply Gaussian elimination, beginning with the last nonzero coefficient of each row. The surviving independent rows give a basis; their number is the required dimension or rank.

(iv) To find the relations among an ordered list of rows, append the corresponding identity matrix before elimination and perform the same row operations on both matrices. A zero row in the first matrix gives its linear relation in the second.

The resulting dimensions are

(3.6)

$\dim L_1$	$\dim L_2$	$\dim L_3$	$\dim L_4$	$\dim R_2$	$\dim R_3$
8	36	168	1044	21	142

$\dim I_4 = 963.$

The 170-row list defining R_3 has 28 dependencies; the coefficients on its last two rows are zero in every dependency. In addition,

$$\operatorname{rank}(\operatorname{ad}\iota_1 : L_1 \to L_2/R_2) = 7, \qquad \operatorname{rank}(\operatorname{ad}\iota_1 : L_2 \to L_3/R_3) = 8. \tag{3.7}$$

These equalities follow by exact finite linear algebra on the displayed lists, using the elimination algorithm above. The same elimination returns a row-dependency vector for each relation.

Lemma 3.5. *The group $H = G/D_4G$ has order 2^{49}; its conjugacy class of ι_1 has size 2^{15}. The prescribed local groups embed in G, and already embed in G/D_3G; the odd local elementary groups and real involutions embed in G/D_2G.*

Proof. The 21 quadratic initials are independent. Modulo $D_4\mathfrak{F}$, the subgroup $D_2\mathfrak{F}$ is elementary abelian and its third layer is central. Normal generation therefore says that the entire third relation space is exactly the list R_3 in (3.5): a product of the 21 quadratic relations can enter degree three only when every quadratic coefficient vanishes. Hence the first three dimensions of G are $8, 36-21, 168-142 = 8, 15, 26$.

The local injection assertions reduce to additional small tests on the same data: x_j, y_j, z_j are independent, and $z_j^2, [y_j, z_j]$ are independent modulo R_2; each odd pair t_ν, f_ν is independent; and $F_k^2 \notin R_2$. These tests can be performed by adding the specified vectors to the matrices and repeating the row reduction. They give ranks $3, 2, 2, 1$, respectively. Since $D_3D = 1$, the two graded injections imply the injection of the entire dyadic group. The capped Frobenius order is therefore exactly four, as it is at most four by definition.

In H, the map $h \mapsto [h, \iota_1]$ on the abelian group D_2H is a homomorphism into D_3H; its rank is eight by (3.7). Its kernel has order $2^{15+26-8} = 2^{33}$. The degree-one kernel in (3.7) is the line through ι_1, and ι_1 itself centralizes ι_1. Thus its full centralizer has order 2^{34} and its class has size 2^{49-34}. □

3.5. **Why the tower is infinite.** We include the filtered relation-module argument because the ordinary count of thirty defining relations does not by itself prove infinitude. For a finite 2-group Δ, write $\Lambda = \mathbb{F}_2[\Delta]$ with augmentation ideal $\mathfrak{I}$, and let $h_\Lambda(t)$ be its augmentation Hilbert polynomial. If Δ were a finite quotient equal to G, its local subgroups are embedded by Lemma 3.5.

The filtered Fox exact sequence for a free presentation with eight generators identifies the module $\mathcal{M}$ of relation rows with the kernel of $\Lambda^8(-1) \to \mathfrak{I}$, $(b_i) \mapsto \sum b_i(g_i - 1)$. This map is strict and onto: the augmentation ideal is generated by $g_i - 1$, and its rth power by products of r such elements. Consequently

$$h_{\mathcal{M}} = 1 - (1 - 8t)h_\Lambda. \tag{3.8}$$

The relevant Fox identities are the product rule and the fundamental identity $r - 1 = \sum_i (\partial_i r)(g_i - 1)$; the chain rule sends the local relation rows to the global ones. Equivalently, exactness follows from

the cellular chain complex of the covering of an eight-petal graph corresponding to the presentation kernel. These are the standard Fox statements used in (3.8).

For a local group Γ with k generators put

$$\psi_\Gamma(t) = kt - 1 + h_{\mathbb{F}_2[\Gamma]}(t)^{-1}.$$

Its global relation-row module has Hilbert polynomial at most $\psi_\Gamma h_\Lambda$ for $0 < t < 1$. Here is a proof of the filtered claim, which is needed rather than mere unfiltered freeness. The first two graded injections of Γ into Δ, and $D_3\Gamma = 1$, allow a homogeneous Jennings basis for Γ to be extended to one for Δ. Place the Γ basis last. The resulting ordered monomials give

$$\Lambda = \bigoplus_j \eta_j \mathbb{F}_2[\Gamma], \qquad \mathfrak{I}^r = \bigoplus_j \eta_j \mathfrak{I}_\Gamma^{r-w_j}, \qquad \sum_j t^{w_j} = h_\Lambda / h_{\mathbb{F}_2[\Gamma]}.$$

The local strict augmentation sequence gives local kernel polynomial $kth_\Gamma - h_\Gamma + 1$. Inducing it using this decomposition gives $\psi_\Gamma h_\Lambda$. The Fox chain-rule map to the global rows preserves the filtration and cannot increase the Hilbert polynomial when evaluated at $0 < t < 1$. This proves the asserted upper bound.

One overlap among the local modules must be subtracted. The omitted dyadic relation at $\mathfrak{p}_1$ follows from the other global local relations, so its Fox row belongs both to the dyadic module there and to the sum of the ten other local modules. The cyclic module generated by that row has Hilbert polynomial

$$s_D(t)h_\Lambda(t), \qquad s_D(t) = t^2\left(1 - \frac{t^7}{h_D(t)}\right). \tag{3.9}$$

To verify (3.9), work first in $\mathbb{F}_2[D]$. Writing $X = x - 1, Y = y - 1, Z = z - 1, W = [y, z] - 1$, the graded monomial basis is $X^a Y^b Z^c W^e$, with $a, b, e \le 1, c \le 3$, of weight $a + b + c + 2e$. The only joint right-annihilator of X, Y, Z in the graded algebra is its top-degree line $\mathbb{F}_2 XYZ^3W$ of degree seven: multiplication by X first forces $a = 1$, multiplication by Z forces $c = 3$, and $ZY = YZ + W$ then forces $b = e = 1$. The initial Fox row of (3.4) is $(Y + Z, Y + X, X)$. Its three entries are an invertible linear transform of (X, Y, Z), so right multiplication by this row is injective in every graded degree below seven. The top line is the sum of all group elements and kills every augmentation element. Higher terms in the actual dyadic relator therefore cannot affect either the kernel or these filtration jumps. Its local cyclic row module has Hilbert polynomial $t^2(h_D - t^7)$. The Jennings decomposition above induces this module to Λ and proves (3.9).

Subadditivity for the sum of local modules, subtracting their indicated common submodule, now gives from (3.8) $1 \le h_\Lambda(t)P(t)$, where

$$\begin{aligned} P(t) &= 1 - 8t + 2\psi_{C_2}(t) + 4\psi_{C_2^2}(t) + 2\psi_D(t) + 3\psi_{C_4}(t) - s_D(t), \\ \psi_{C_2} &= \frac{t^2}{1+t}, \quad \psi_{C_2^2} = 2t - 1 + \frac{1}{(1+t)^2}, \\ \psi_{C_4} &= \frac{t^4}{(1+t)(1+t^2)}, \\ \psi_D &= 3t - 1 + \frac{1}{(1+t)^3(1+t^2)^2}. \end{aligned} \tag{3.10}$$

Direct rational substitution gives

$$P(34/117) = -187433948535241/88772225460489675 < 0.$$

This contradicts $h_\Lambda(34/117) > 0$. Thus G is infinite.

3.6. **The degree-four enclosure and the stronger finite quotient.** The unknown higher coefficients of arithmetic local lifts must be accounted for. In particular, using I_4 alone as the degree-four relation space would not be justified. We now describe the containing space T used to control every possible lift.

Algorithm 3.6 (The degree-four containing space)**.** (i) For $v = \sum v_i X_i$, take the group word $\prod_{i=0}^{7} f_i^{v_i}$ in increasing order. Use the literal squares for the two real relations, the words $ftf^{-1}t^{-p}$ at odd places, and the listed squares and commutators for the quadratic quotient relations. For the remaining dyadic initial, use

$$W = y_2^2[x_2, y_2][x_2, z_2]$$

when finding the cubic coefficients.

(ii) Expand each resulting $\rho_i - 1$ through degree four, writing its first homogeneous terms as $q_i + t_i + \cdots$. Products are products of words in the free associative algebra. To invert $1 + a$, use $1 + a + a^2 + a^3 + a^4$ and discard terms of degree above four.

(iii) Apply Algorithm 3.4 to the first 168 rows of R_3. For each of their 28 relations $k = (k_{ia})$, form

$$c(k) = \sum_{i,a} k_{ia}\big([t_i, X_a] + X_a[q_i, X_a]\big). \tag{3.11}$$

(iv) In turn, add each basis element U of L_2 to the quadratic coefficient of each of the sixteen local generators other than F_1, F_2, F_3. Recompute (3.11) and retain its change. The required changes in the quadratic expressions are

$$D_v(v^2)[U] = [v, U], \qquad D_v[v, w][U] = [U, w], \qquad D_w[v, w][U] = [v, U].$$

(v) Let T be the span of I_4, all the vectors $c(k)$, and all these changes. Apply Gaussian elimination to this list and to the images of $[L_3, \iota_j]$ modulo I_4 and T.

The calculation gives The same Gaussian elimination, now entirely on degree-four words, gives

$$\begin{gathered}\dim T = 996, \qquad \dim(T/I_4) \text{ before variations} = 2,\\ \operatorname{rank}([L_3, \iota_j] \bmod I_4) = \operatorname{rank}([L_3, \iota_j] \bmod T) = 18 \qquad (j = 1, 2).\end{gathered} \tag{3.12}$$

There are precisely $16 \cdot 36 = 576$ perturbation directions. The definition of T and the row-reduction rule specify a reproducible finite proof of (3.12); no floating-point arithmetic or unlisted arithmetic relator coefficients enter it.

Lemma 3.7. *If R_4 is the true degree-four relation space of the arithmetic group G, then $I_4 \subseteq R_4 \subseteq T$. In particular* $\operatorname{ad}\iota_1 : \operatorname{gr}_3 G \to \operatorname{gr}_4 G$ *has rank* 18*.*

Proof. The first inclusion follows from squares, brackets, and the seven explicit quartic relators. Work modulo $D_5\mathfrak{F}$ and I_4. The quadratic relations commute and square to one; their commutators with the eight free generators, and the two cubic relations, are central involutions. Indeed, the possible obstructions are $[R_2, L_2]$, $[R_3, L_1]$, and squares of R_2, all included in I_4. This accounts also for arbitrary conjugating words, not just the displayed conjugations. Independence of the q_i forces their coefficients to vanish in any normal-relation product of degree four. The remaining coefficients give a dependency among the 170 third degree rows. Every such dependency has zero cubic-relation coordinates. For a relation with initial terms $1 + q + t + \cdots$, direct multiplication in our commutator convention gives degree-four coefficient $[t, X] + X[q, X]$ for its commutator with $1 + X$. Thus every surviving correction has the form (3.11).

Any true lift of a local generator differs from its canonical lift by an element of $D_2\mathfrak{F}$. The cubic part of a quadratic relator depends affinely on the sixteen possible quadratic lift coefficients: two quadratic perturbations first interact in degree four, and cubic lift coefficients cancel because the

exponent sum of each generator in each quadratic relation is even. The derivative rules above therefore exhaust the changes in (3.11).

There remains the dyadic arithmetic relation, whose full word has not been specified. Both it and W belong to the true normal relation subgroup, and their quadratic initials agree. Their quotient is therefore in that subgroup intersected with $D_3\mathfrak{F}$; their cubic coefficients differ by an element of R_3. Replacing one by the other in (3.11) changes it only by $[R_3, L_1]$, already in I_4. This justifies the substitution for this purpose without claiming a replacement of the full presentation. We have now accounted for every possible degree-four relation, proving the second inclusion. The equal endpoint ranks in (3.12) give rank 18 modulo the intermediate space R_4 as well. □

Proposition 3.8. *There is a quotient Q of G of order 2^{67}, mapping onto $H = G/D_4G$, in which the conjugacy class of ι_1 has at least 2^{33} elements.*

Proof. The group D_4G/D_5G is central elementary abelian in G/D_5G. Choose eighteen vectors in $[L_3, \iota_1]$ independent modulo T. Extend them to a basis of L_4/T, and project onto their eighteen coordinates. This linear map vanishes on $R_4 \subseteq T$ and is an isomorphism on the specified adjoint image. Quotient the central fourth layer by its kernel. The result Q has order $|H|2^{18} = 2^{67}$.

Let $C_0 = C_H(\iota_1)$ and let $C = C_Q(\iota_1)$. Although $Q \to H$ is onto, its restriction $C \to C_0$ need not be onto. In fact $D_3H \subset C_0$, and for $u \in D_3H$ the commutator of a lift with ι_1 is the rank-eighteen adjoint map just retained. Changing the lift by the central kernel leaves this commutator unchanged. Hence $[C_0 : \operatorname{image} C] \geq 2^{18}$. Because the entire kernel is central, $|C| = 2^{18}|\operatorname{image} C|$, and thus

$$[Q : C] = [H : C_0]\,[C_0 : \operatorname{image} C] \geq 2^{15}2^{18} = 2^{33}.$$

This argument explicitly uses the obstruction to lifting a centralizer; it does not assume centralizer surjectivity. □

3.7. **Fields, signature, discriminant, and prime densities.**

Theorem 3.9. *For every power of two $d \geq 2^{67}$ there is a finite Galois extension K/B inside B_G with $[K : B] = d$ whose Galois group maps onto Q. For any such field choose the complex conjugation ι above v_1 and set $F = K^{\langle\iota\rangle}$. Then $[F : \mathbb{Q}] = d$, K/F is quadratic and unramified at every finite place, K is totally imaginary, and the signature (b, c) of F satisfies*

$$b \geq 1, \qquad \theta_* := \frac{17179869183}{34359738368} = \frac{1}{2} - 2^{-35} \leq \frac{c}{d} < \frac{1}{2}. \tag{3.13}$$

Both root discriminants equal

$$\lambda = 2^{9/4}\sqrt{3615}, \qquad \ell = \log\lambda = \tfrac{9}{4}\log 2 + \tfrac{1}{2}\log 3615. \tag{3.14}$$

Every prime of F above a selected prime of B splits in K. Their absolute types and counts are

(3.15)

selected rational prime or branch	(e, f)	$N\mathfrak{p}$	$\#\{\mathfrak{p} \subset F\}/d$
2	$(8, 4)$	2^4	$1/32$
3	$(2, 2)$	3^2	$1/4$
5	$(2, 2)$	5^2	$1/4$
29 *(both branches)*	$(1, 4)$	29^4	$1/4$
41 *(the branch $\mathfrak{t}_3$ only)*	$(1, 4)$	41^4	$1/8$

Proof. Let $V = \ker(G \to Q)$. Since G is infinite and finitely generated, it has open normal subgroups $V' \subset V$ of arbitrarily large index. For a desired dyadic $d \geq 2^{67}$ choose $[G : V'] \geq d$. A normal subgroup of a finite 2-group has ambient-normal subgroups of every intermediate power-of-two order: its intersection with the center contains an involution, and induction applies after quotienting by that central subgroup. Apply this to V/V' to get an open normal $U \subset V$ with $[G : U] = d$. Its fixed field is the required K. In particular, this is a statement for every sufficiently large dyadic degree, not just a sparse sequence chosen after the analytic estimates.

Since $i = \sqrt{\alpha_0} \in K$, the field K is totally imaginary. Let $J = \mathrm{Gal}(K/B)$ and let N_ι be the size of the conjugacy class of ι. Its image in Q gives $N_\iota \geq 2^{33}$. An embedding of F over v_1 is real exactly when its corresponding element of J centralizes ι; quotienting by $\langle\iota\rangle$ gives $b = |C_J(\iota)|/2$. There are no real embeddings over v_2, because the two real involutions have different elementary images in (3.2), and conjugation preserves elementary images. Since $d = |J|$,

$$\frac{b}{d} = \frac{1}{2N_\iota}, \qquad \frac{c}{d} = \frac{1}{2} - \frac{1}{4N_\iota},$$

which proves (3.13) and also $b \geq 1$.

The elementary image of neither real involution lies in the span of the local vectors of any selected finite place. This is a direct test on (3.2). Since the local decomposition groups are generated by these vectors' lifts and the local groups embed, no conjugate of ι belongs to any such decomposition group. Its intersection with $\mathrm{Gal}(K/F) = \langle\iota\rangle$ is therefore trivial. This proves splitting in K/F and preserves the prescribed local types. Outside S, K/B is unramified by its definition; at S, splitting in K/F also kills inertia. Thus K/F is unramified at every finite prime and $|\Delta_K| = |\Delta_F|^2$.

For the common root discriminant, the quadratic base contributes $241^{1/2}$. Each dyadic branch has relative different exponent divided by ramification index $9/4$, so the two branches together contribute $2^{9/4}$ to the absolute root discriminant. At $3, 5$ the ramification is tame of index two, hence the two branches contribute $3^{1/2}, 5^{1/2}$. The capped primes are unramified and no other prime ramifies. This proves (3.14).

For a rational prime having both degree-one branches in B, the sum of ef over primes of F is d, giving $d/(ef)$ selected primes. For the single selected branch above 41 it is $d/2$, giving $d/8$. The norms are p^f. This proves (3.15); no uniform type for the unselected branch $\mathfrak{t}_4$ is asserted. □

3.8. **The exact analytic input needed later.** For the fields just constructed write $L_F(s) = \zeta_K(s)/\zeta_F(s)$, with its value at one defined as the ratio of residues. The analytic class-number formula gives

$$L_F(1) = 2^c \pi^{d-c} \lambda^{-d/2} \frac{h_K \operatorname{Reg}_K w_F}{h_F \operatorname{Reg}_F w_K} > 0, \qquad w_F = 2.$$

Let $h_{\mathrm{rel}} = h_K/h_F$, let h_{cap} be the order of the kernel of $\mathrm{Cl}(F) \to \mathrm{Cl}(K)$, and let Reg^1 be the covolume of the relative-unit logarithms $\frac{1}{2}(\log|\phi_j^+\varepsilon| - \log|\phi_j^-\varepsilon|)$. The normalization used in the geometric transfer is

$$\frac{w_K}{h_{\mathrm{rel}} h_{\mathrm{cap}} \operatorname{Reg}^1} = \frac{2^{d-1}\pi^{b+c}}{\lambda^{d/2} L_F(1)}. \tag{3.16}$$

Its full unit/cohomology and regulator proof is given in Proposition 5.1; the factors of two in (3.16) correspond to the logarithmic normalization just stated.

Proposition 3.10 (Quantifier required of the arithmetic bound)**.** *The numerical arithmetic result used in this paper is the following uniform statement: there is d_0 such that* every *admissible K/B in the above tower with $[K : B] = d \geq d_0$ satisfies*

$$\frac{1}{d}\log L_F(1) < C, \qquad C = \frac{4226263259973019243081}{10^{23}}. \tag{3.17}$$

The proof and its signed-kernel, L-value, and remainder certificates are given in the arithmetic section. This statement applies simultaneously to every dyadic degree of Theorem 3.9 above d_0 and gives

$$\frac{w_K}{h_{\mathrm{rel}} h_{\mathrm{cap}} \operatorname{Reg}^1} > 2^{d-1}\pi^{(1-c/d)d}\lambda^{-d/2}e^{-dC}.$$

The uniform quantifier is essential: a bound along a selected sequence would not suffice for choosing the degree from a prescribed pair of cardinalities in the later all-scale construction. The arithmetic proof obtains the displayed eventual bound by ruling out every infinite sequence of

violating fields, using a convergent subsequence of normalized prime counts and a strict limiting inequality. This compactness argument does not require a nested tower and does not produce an effective numerical threshold. The stronger quotient Q only restricts the original family; therefore a uniform estimate for the original family remains valid for all fields used here.

3.9. **Explicit comparison fields and residue-degree detectors.** We now construct the comparison field used in the analytic estimate. The choice of forms and primitive radicals follows Naslund [24]; the rational witness checks and the proof organisation below are explicit. This also specifies the fields underlying every one of its 576 auxiliary L-functions. The construction uses the group G, its Zassenhaus filtration, and the eight elements $\alpha_0, \dots, \alpha_7$ of Section 3; put $E_0 = E = B(\sqrt{\alpha_0}, \dots, \sqrt{\alpha_7})$. In particular $[E_0 : \mathbb{Q}] = 512$. Every field K used here retains the quotient G/D_4G and therefore contains the fixed field of D_3G.

Write $V = \mathbb{F}_2^8$, and interpret a string $r_0 \cdots r_7$ in the order of the Kummer basis in (3.1). Set $\alpha_r = \prod_{i=0}^{7} \alpha_i^{r_i}$ and $\chi_r(g) = (-1)^{r \cdot \overline{g}}$. The 36 coordinates on the free degree-two restricted Lie algebra are, in order, the eight squares $e_i^{[2]}$ and the 28 brackets $[e_i, e_j]$, $i < j$. A linear functional ψ is determined by the quadratic function $q_\psi(v) = \psi(v^{[2]})$: indeed

$$(v + w)^{[2]} = v^{[2]} + w^{[2]} + [v, w].$$

Its polar form is $\psi([v, w])$. It descends to L_2/R_2 precisely when it kills the explicit relation space in (3.5).

For two independent rows r_1, r_2 and $\varrho \in \mathbb{F}_2^2 \setminus \{0\}$, define

$$\phi(v) = (r_1 \cdot v, r_2 \cdot v), \qquad q_\psi(v) = \mathbf{1}_{\{\phi(v) = \varrho\}}. \tag{3.18}$$

The three possible indicator functions on $\mathbb{F}_2^2$ are x_1x_2, $x_1 + x_1x_2$, and $x_2 + x_1x_2$. Consequently their polar form is $\det(\phi(v), \phi(w))$, of rank two, with radical $\ker \phi$. For direct verification the 36 entries of ψ are

$$\big(\mathbf{1}_{\{\phi(e_i) = \varrho\}}\big)_{i=0}^{7}, \qquad \big(r_{1i}r_{2j} + r_{2i}r_{1j}\big)_{i<j}.$$

Thus every linear-algebra test below is a specified computation over $\mathbb{F}_2$, not an inference from a numerical approximation.

TABLE 4. The seven comparison forms and their quadratic subfields.

o	$r_{1,o}$	$r_{2,o}$	ϱ_o	c_o
10	00010011	01001100	$(1,1)$	$213033204 - 13722675\sqrt{241}$
23	01011111	10001011	$(0,1)$	$155 - 10\sqrt{241}$
24	01011111	10011000	$(0,1)$	$(-378557 + 24385\sqrt{241})/2$
19	01001100	10000010	$(1,1)$	$-326 + 21\sqrt{241}$
20	01001100	10010001	$(1,1)$	$(47 - 3\sqrt{241})/2$
17	01000110	10001000	$(1,1)$	$-31 + 2\sqrt{241}$
7	00001010	11000100	$(1,1)$	$-3433342 + 221161\sqrt{241}$

The integer labels in Table 4 are just identifiers; the rows and (3.18), not the labels, define the forms. All seven functionals annihilate R_2. They obey

$$\psi_{24} = \psi_{10} + \psi_{23}, \quad \psi_{20} = \psi_{10} + \psi_{19}, \quad \psi_7 = \psi_{17} + \psi_{19}.$$

The space

$$W = \langle \psi_{10}, \psi_{23}, \psi_{19}, \psi_{17} \rangle \subset (L_2/R_2)^* \tag{3.19}$$

has dimension four. Besides the seven displayed forms, its nonzero elements are the following eight sums, all of polar rank four:

$$(a, b) = (19, 23), (19, 24), (7, 10), (17, 10), (17, 24), (7, 24), (7, 23), (17, 23). \tag{3.20}$$

For every pair the four rows $r_{1,a}, r_{2,a}, r_{1,b}, r_{2,b}$ are independent, and c_a/c_b is not a square in B. These assertions can be checked by forming the 16 sums of the four 36-bit vectors, row-reducing their alternating 8 by 8 polar matrices, and applying the ordinary quadratic-field square test to c_a/c_b. These finite eliminations and square tests give all the stated identities.

Exact radicals and their finite verification. For each form used below let

$$a_o = \alpha_{r_{1,o}}, \quad b_o = \alpha_{r_{2,o}}, \quad F_{0,o} = B(A_o, B_o), \qquad A_o^2 = a_o, \quad B_o^2 = b_o.$$

The rows are independent, so this field has degree eight over $\mathbb{Q}$. The coefficients $c_{o,0}, \dots, c_{o,7}$ are the rational numbers listed in Section 3.10. They define

$$\beta_o = c_{o,0} + c_{o,1}\sqrt{241} + c_{o,2}A_o + c_{o,3}\sqrt{241}A_o + c_{o,4}B_o + c_{o,5}\sqrt{241}B_o + c_{o,6}A_oB_o + c_{o,7}\sqrt{241}A_oB_o. \tag{3.21}$$

The eighteen rows have identifiers

$$0, 3, 4, 5, 6, 7, 9, 10, 12, 13, 14, 15, 17, 19, 20, 22, 23, 24.$$

The fifteen census rows occur in the order

$$(o_1, \dots, o_{15}) = (0, 5, 9, 17, 22, 14, 3, 7, 4, 15, 6, 20, 12, 24, 13). \tag{3.22}$$

Their row data are listed in Section 3.10 and are decoded by (3.18). The seven comparison rows are given by Table 4. Four rows occur in both lists.

For $\sigma = (\epsilon_a, \epsilon_b) \in \mathbb{F}_2^2$, its action is $A_o \mapsto (-1)^{\epsilon_a} A_o$, $B_o \mapsto (-1)^{\epsilon_b} B_o$, and $\sqrt{241} \mapsto \sqrt{241}$. For each of the three nonidentity σ, the square-root recursion below gives an element $u_{o,\sigma}$ in the same quadratic basis. These elements satisfy the exact identities

$$u_{o,\sigma}^2 = \frac{\sigma(\beta_o)}{\beta_o}, \qquad u_{o,\sigma}\sigma(u_{o,\sigma}) = \begin{cases} -1, & \sigma = \varrho_o, \\ 1, & \sigma \neq \varrho_o. \end{cases} \tag{3.23}$$

There is an especially short exact check of the required ramification condition. All coefficients of (3.21) have denominators supported on $2, 3, 5$, and all basis elements are integral outside these primes. The absolute norms are positive and are as follows; a row (h_2, h_3, h_5) means $\mathrm{N}_{F_{0,o}/\mathbb{Q}}(\beta_o) = 2^{h_2}3^{h_3}5^{h_5}$.

o	0	3	4	5	6	7	9	10	12
(h_2, h_3, h_5)	$(2,0,4)$	$(0,2,0)$	$(0,2,2)$	$(2,0,4)$	$(12,6,2)$	$(0,2,0)$	$(6,0,0)$	$(12,4,0)$	$(4,0,0)$

o	13	14	15	17	19	20	22	23	24
(h_2, h_3, h_5)	$(2,2,0)$	$(2,4,2)$	$(6,4,0)$	$(0,2,0)$	$(8,4,4)$	$(4,0,0)$	$(0,2,0)$	$(6,2,0)$	$(2,2,0)$

Nonnegative valuations outside $2, 3, 5$, together with these norms, imply that every β_o is a unit outside the primes above $2, 3, 5$. Thus no computation of a maximal order is needed for this step.

Algorithm 3.11 (Verification in the quadratic fields)**.** (i) In a quadratic extension $T(\sqrt{d})$ write an element as $x + y\sqrt{d}$, with $x, y \in T$. Multiply and take norms by

$$(x + y\sqrt{d})(x' + y'\sqrt{d}) = (xx' + dyy') + (xy' + yx')\sqrt{d}, \qquad N(x + y\sqrt{d}) = x^2 - dy^2.$$

Apply these formulas successively in $\mathbb{Q} \subset B \subset B(A_o) \subset F_{0,o}$.

(ii) To take a square root of $x + y\sqrt{d}$ when $y \neq 0$, try both square roots n of $x^2 - dy^2$ in T. For each, try a nonzero square root r of $(x + n)/2$ in T. The corresponding candidate is $r + (y/(2r))\sqrt{d}$. When $y = 0$, try a square root of x in T, and then $\sqrt{d}$ times a square root of x/d in T. Repeat in the smaller quadratic fields; in $\mathbb{Q}$, check whether numerator and denominator are nonnegative perfect squares.

(iii) For every listed β_o and each of the three nonidentity σ, apply this rule to $\sigma(\beta_o)/\beta_o$. Square the candidate u and calculate $u\sigma(u)$ to check (3.23). Successive norms check the displayed powers of $2, 3, 5$. Apply the same square test to the quotients c_a/c_b.

(iv) Form the quadratic forms and the four comparison sums, and check their ranks and identities by Gaussian elimination over $\mathbb{F}_2$.

Expanding $(r + s\sqrt{d})^2$ shows that the alternatives in the square root step exhaust all possibilities. The procedure gives all 54 identities (3.23), the eighteen norm statements, and the square-class assertions.

Lemma 3.12. *For every one of the eighteen rows, the field $M_o = F_{0,o}(\sqrt{\beta_o})$ is a Galois extension of B with dihedral group of order eight, lies in the fixed field of D_3G, and therefore lies in every admissible K. On D_2G, restriction to its central subgroup is $g \mapsto z^{\psi_o([g]_2)}$, where z is its central involution.*

Proof. The first identity in (3.23) makes M_o/B normal. The second implies that β_o is not a square: for a square $\beta_o = b^2$ the quotient root $\sigma(b)/b$ would give sign $+1$ for every σ. Any lift of σ to M_o has square z precisely for $\sigma = \varrho_o$. Of the groups of order eight with a central kernel of order two and quotient C_2^2, exactly the dihedral group has lifts of order four above exactly one of the three nonzero quotient classes. Consequently squares and commutators with elementary images v, w map to $z^{q_{\psi_o}(v)}$ and $z^{\psi_o([v,w])}$.

Since $F_{0,o} \subset E_0$ and β_o is a unit away from $2, 3, 5$, M_o is unramified outside the allowed set S: at a prime of odd residue characteristic a quadratic extension defined by a unit is unramified. Hence M_o lies in the maximal S-ramified pro-2 extension. It remains to check the extra relations defining G, rather than assume that every S-ramified extension belongs to our tower. The extra degree-two square and bracket relations map to the identity because $\psi_o(R_2) = 0$. The triple commutators, fourth powers and squares of commutators in the presentation of Section 3.3 vanish in every dihedral group of order eight. All its extra relations therefore hold. The quotient map factors through G, and kills D_3G because this dihedral group is of class two with central Frattini subgroup of order two. Its action on D_2G/D_3G is the stated functional, as squares and commutators span this layer. □

Let N_W be the inverse image of $W^\perp \subset L_2/R_2$ in D_2G, and define

$$E_* = (B_G)^{N_W}. \tag{3.24}$$

Since $[G, D_2G] \subset D_3G \subset N_W$, this is Galois over B and

$$1 \longrightarrow \mathbb{F}_2^4 \longrightarrow \mathrm{Gal}(E_*/B) \longrightarrow \mathbb{F}_2^8 \longrightarrow 1$$

is central. Its degree over $\mathbb{Q}$ is $2 \cdot 2^{12} = 8192$. It contains E_0 and each of the seven M_o: by Lemma 3.12, N_W lies in their kernels. Conversely E_0 and those seven fields generate E_*, since the common kernel on D_2G/D_3G is exactly $W^\perp$. The inclusion $D_4G \subset D_3G \subset N_W$ proves $E_* \subset K$ for *every* member of the field family, without passing to a subsequence.

The 576 factors, with explicit radicals for each twist. Let $H = \mathrm{Gal}(E_*/B)$. For a nontrivial central character $\omega \in W$, quotient out its kernel in the central subgroup. In this quotient the center is the inverse image of the radical of the polar form of ω, and its index is $2^{r(\omega)}$. An irreducible character nontrivial on the remaining central involution vanishes off this center: there a group element is conjugate to its product with that involution. On the center its absolute value is its degree. Character orthogonality therefore gives degree $2^{r(\omega)/2}$. The sum of their squared degrees is 256, so there are $256/2^{r(\omega)}$ of them.

Write ρ_o for the two-dimensional dihedral representation of M_o/B. For a rank-two character the complete list is $\rho_o \otimes \chi_w$, with w running over a complement of $\langle r_{1,o}, r_{2,o} \rangle$ in V. For a rank-four sum in (3.20) it is $\rho_a \otimes \rho_b \otimes \chi_w$, with w running over a complement of the four rows. These representations are irreducible: their dimension is the required smallest irreducible dimension for the central character. Two twists are isomorphic exactly when their difference annihilates the polar radical. This proves both the count and the lack of duplication. A deterministic convention for the complements is to scan $e_0, \dots, e_7$ and append a vector whenever it increases the rank of the rows

already chosen. All sums of the appended vectors are the twist representatives. Thus there are $7 \cdot 64 = 448$ quartic and $8 \cdot 16 = 128$ octic factors. The regular-representation identity proves (4.12) Euler factor by Euler factor, including ramified primes.

The following construction identifies every factor as a quadratic Hecke L-function and permits its conductor and Euler polynomials to be recomputed. For all seven rows choose $\tau_o = (1, 0)$; it differs from ϱ_o. Its fixed field in $F_{0,o}$ is $B(\sqrt{b_o}) = B(\sqrt{c_o})$. The last equality is an exact square-class test; the displayed c_o are small representatives of b_o modulo squares. Choose the recursively obtained square root u_o for τ_o, changing its sign if needed so that $u_o \neq -1$, and put

$$\gamma_o = (1 + u_o)^2 \beta_o \in B(\sqrt{c_o}). \tag{3.25}$$

Indeed $u_o \tau_o(u_o) = 1$ implies $\tau_o(\gamma_o) = \gamma_o$. The desired fields are exactly

$$A_o = B(\sqrt{c_o}), \qquad A'_{o,w} = A_o(\sqrt{\gamma_o \alpha_w}), \tag{3.26}$$

$$A_{ab} = B(\sqrt{c_a}, \sqrt{c_b}), \qquad A'_{ab,w} = A_{ab}(\sqrt{\gamma_a \gamma_b \alpha_w}). \tag{3.27}$$

Here $[A_o : \mathbb{Q}] = 4$ and $[A_{ab} : \mathbb{Q}] = 8$. The two quotients of Dedekind zeta functions associated with these quadratic extensions are, respectively, the two sorts of factors just listed. To prove this, lift τ_o to the involution sending $\sqrt{\beta_o}$ to $u_o\sqrt{\beta_o}$. It fixes $(1 + u_o)\sqrt{\beta_o}$, so the corresponding quadratic character over A_o induces ρ_o. Multiplication by α_w twists by χ_w. For a pair, c_a/c_b nonsquare means that the two index-two subgroups are distinct and their product is H. The elementary tensor-induction formula then induces their product character from their intersection, which is exactly the extension in (3.27). These quadratic characters are nontrivial because their induced representations are irreducible of dimension two or four.

For either extension A'/A the analytic conductor in the chosen Dedekind-zeta convention is $Q = |\Delta_{A'}|/|\Delta_A|$. Its archimedean factor is the quotient of the two Dedekind gamma factors; its root number is $+1$. At a rational prime p, if $D_T(p, X) = \prod_{\mathfrak{q}|p \text{ in } T}(1 - X^{f(\mathfrak{q}/p)})$, then its Euler denominator is $D_{A'}(p, X)/D_A(p, X)$. These definitions determine the conductors, signatures and every Euler factor by exact arithmetic in the displayed fields. The factors are quadratic Hecke L-functions, so their analytic continuation and functional equations follow from Hecke theory.

Algebraic detectors and the two strengthened floors. For an unramified prime $\mathfrak{p}$ of B, compute its elementary Frobenius vector v from the quadratic-residue signs of the eight α_i in its residue field. If $v \neq 0$, its residue degree in E_*/B is four if $\phi_o(v) = \varrho_o$ for any row of Table 4, and is two otherwise. This follows because a Frobenius g has g^2 in the central subgroup, and $\psi_o(g^2) = q_{\psi_o}(v)$.

If $v = 0$, the prime splits in E_0 and hence in every $F_{0,o}$. Its second-layer Frobenius $\bar{g}_2 \in L_2/R_2$ is independent of the choice of a prime above it, since $[G, D_2G] \subset D_3G$. Using the census order (3.22), form the fifteen-bit pattern

$$p_i = \mathbf{1}_{\{\beta_{o_i} \text{ is a nonsquare modulo } \mathfrak{p}\}} = \psi_{o_i}(\bar{g}_2). \tag{3.28}$$

For this test, embed F_{0,o_i} into the residue field by choosing square roots of a_{o_i} and b_{o_i}, then evaluate (3.21). All choices give the same quadratic-residue sign by (3.23). The evaluated element is nonzero by the unit property. The fifteen census functionals kill R_2 and have rank fifteen, so they form its full dual. In particular a nonzero pattern proves $\bar{g}_2 \neq 0$, which forces even residue degree in every K.

The four comparison-basis values are determined by

$$\psi_{17}(\bar{g}_2) = p_4, \qquad \psi_{19}(\bar{g}_2) = p_4 + p_8,$$
$$\psi_{10}(\bar{g}_2) = p_4 + p_8 + p_{12}, \qquad \psi_{23}(\bar{g}_2) = p_4 + p_8 + p_{12} + p_{14}.$$

Thus $f_{E_*}(\mathfrak{p}) = 1$ exactly when all four sums vanish; otherwise it is two. With bit $i - 1$ representing p_i, these are the masks $0x8, 0x88, 0x888, 0x2888$.

Define $\Sigma_2 = \{v : v^{[2]} \in R_2\}$. It has exactly 21 elements, found simply by testing all 256 vectors against the fifteen census functionals. If $v \notin \Sigma_2$, a Frobenius has nontrivial square already in G/D_3G.

Its order in every retained quotient is therefore a multiple of four. Let $\mathcal{P}_4$ consist of the free primes with $\mathrm{N}\mathfrak{p} \le 10^6$ and $v \notin \Sigma_2$. Let $\mathcal{P}_2$ consist of the degree-one free primes within the census cutoff, with $v = 0$, nonzero pattern, and all four comparison masks zero. These two sets are disjoint, and the prescribed floors

$$f_{\mathfrak{p}}^0 = \begin{cases} 4, & \mathfrak{p} \in \mathcal{P}_4, \\ 2, & \mathfrak{p} \in \mathcal{P}_2, \\ f_{E_*}(\mathfrak{p}), & \text{otherwise} \end{cases} \tag{3.29}$$

divide the actual residue degree for every field K.

Algorithm 3.13 (Residue-degree floors)**.** Let $\mathfrak{p}$ be a free prime ideal of B.

(i) Reduce the eight α_i in its residue field and test whether each is a square. These signs give $v \in \mathbb{F}_2^8$.

(ii) If $v \neq 0$, the seven comparison forms determine $f_{E_*}(\mathfrak{p})$ as above. Test $v \in \Sigma_2$ using the fifteen quadratic forms. If $N\mathfrak{p} \le 10^6$ and $v \notin \Sigma_2$, place $\mathfrak{p}$ in $\mathcal{P}_4$.

(iii) If $v = 0$, choose square roots of a_o, b_o in the residue field and evaluate the fifteen β_o. Their square signs give the pattern $(p_1, \dots, p_{15})$. The four displayed sums determine whether $f_{E_*}(\mathfrak{p})$ is one or two. If the pattern is nonzero, all four sums vanish, the degree over $\mathbb{Q}$ is one, and $N\mathfrak{p} \le 1125899906842000$, place $\mathfrak{p}$ in $\mathcal{P}_2$.

(iv) Assign the floor in (3.29). For a split rational prime, carry out these steps at each of its two prime ideals, using the two residue values of $\sqrt{241}$. For an inert prime use $\mathbb{F}_{p^2}$, and at 241 use $\mathbb{F}_{241}$.

The finite lower-norm check required in (4.9) can also be performed directly from these formulas. Enumerate prime ideals of B of norm at most $X_A = 12000000$, omitting the selected primes, compute (3.29), and test $(\mathrm{N}\mathfrak{p})^{f_{\mathfrak{p}}^0} \ge 41^4$. For primes of degree two use $\mathbb{F}_{p^2}$ instead of $\mathbb{F}_p$; the test is unchanged. The ramified rational prime 241 gives the single prime ideal with residue field $\mathbb{F}_{241}$ and is included. There are thirteen degree-two vector-zero primes in this range, over

$$409, 769, 1129, 1249, 1489, 1609, 1801, 2281, 2521, 2689, 3001, 3049, 3361;$$

all have a nonzero comparison mask, hence residue degree two in E_*. Every free prime of norm less than 1681 has floor four. The remaining small vector-zero degree-one primes are classified by the same exact radical evaluations. For norms greater than $X_A > 41^4$ the required inequality is automatic. The exact enumeration can be shortened: test all degree-one free primes below 1681, and all vector-zero degree-one primes up to 41^4. For larger norms a nonzero elementary vector already gives a sufficient floor of at least two. For split rational primes the vector-zero screen is exactly $p \equiv 1, 49 \pmod{120}$ together with the four residue tests for $\alpha_1, \alpha_2, \alpha_4, \alpha_6$; this follows from $\alpha_0 = -1$, $\alpha_2\alpha_3 = 2$, $\alpha_4\alpha_5 = -3$, $\alpha_6\alpha_7 = -5$ and $\mathrm{N}(\alpha_1) = -1$. The enumeration gives 248 small degree-one free primes, finds 658 degree-one vector-zero primes up to 41^4, all detected, and independently verifies the thirteen inert primes above. The least vector-zero norm is 65809. The same residue tests define the full finite prime sums used in Section 4.

3.10. **Rational data for the comparison fields.** Put $D = \sqrt{241}$. For each row o in the next table set $a_o = \alpha_{r_{1,o}}$, $b_o = \alpha_{r_{2,o}}$, $A_o^2 = a_o$ and $B_o^2 = b_o$, with the convention for α_r in Section 3.9. The element defining the quadratic extension is

$$\beta_o = c_{o,0} + c_{o,1}D + (c_{o,2} + c_{o,3}D)A_o + (c_{o,4} + c_{o,5}D)B_o + (c_{o,6} + c_{o,7}D)A_oB_o.$$

All the entries below are rational numbers. The label o is used only to match the field, its quadratic form, and its prime detector.

o	$r_{1,o}$	$r_{2,o}$	ϱ_o
0	00000100	00010000	$(1,1)$
3	00000101	01001000	$(1,1)$
4	00000111	01000001	$(0,1)$
5	00001000	10100000	$(1,1)$
6	00001010	11000001	$(1,1)$
7	00001010	11000100	$(1,1)$
9	00010001	11000001	$(0,1)$
10	00010011	01001100	$(1,1)$
12	00100010	11001100	$(1,1)$
13	00100100	01001011	$(1,1)$
14	00110011	01010001	$(0,1)$
15	01000010	10100010	$(1,0)$
17	01000110	10001000	$(1,1)$
19	01001100	10000010	$(1,1)$
20	01001100	10010001	$(1,1)$
22	01001101	10000100	$(0,1)$
23	01011111	10001011	$(0,1)$
24	01011111	10011000	$(0,1)$

o	$c_{o,0}$	$c_{o,1}$	$c_{o,2}$	$c_{o,3}$
0	-8	$-\frac{1}{2}$	$\frac{5}{2}$	0
3	$-\frac{65217}{2}$	$-\frac{4201}{2}$	$-\frac{323}{2}$	$-\frac{21}{2}$
4	$-\frac{74249}{2}$	$\frac{4783}{2}$	0	0
5	-8	$\frac{1}{2}$	$\frac{5}{2}$	0
6	$\frac{112303}{2}$	-3617	293	-17
7	$-\frac{65217}{2}$	$\frac{4201}{2}$	$-\frac{323}{2}$	$\frac{21}{2}$
9	$-\frac{31}{2}$	1	0	0
12	$-\frac{426347}{4}$	$\frac{27459}{4}$	231045	-14883
13	$\frac{10995005}{4}$	$\frac{708183}{4}$	0	0
14	$\frac{6753}{4}$	$-\frac{435}{4}$	0	0
15	$-\frac{3}{2}$	0	83000	$\frac{10693}{2}$
17	-39	$\frac{5}{2}$	$-\frac{1925}{2}$	-62
19	$-\frac{7886559}{4}$	$\frac{508001}{4}$	$-\frac{144775}{4}$	$-\frac{9375}{4}$
20	$-\frac{279699}{4}$	$\frac{18017}{4}$	$-\frac{41}{4}$	$-\frac{1}{4}$
22	$-\frac{15}{4}$	$-\frac{1}{4}$	0	0
24	$-\frac{91332223}{4}$	$\frac{5883225}{4}$	0	0

o	$c_{o,4}$	$c_{o,5}$	$c_{o,6}$	$c_{o,7}$
0	0	0	0	0
3	0	0	0	0
4	$-\frac{699}{2}$	$-\frac{49}{2}$	0	0
5	0	0	0	0
6	$-\frac{1661}{2}$	-65	-47767	-3077
7	0	0	0	0
9	$\frac{31}{2}$	1	0	0
12	$-\frac{236491}{4}$	$-\frac{15209}{4}$	-4341	-293
13	$-\frac{266484672787}{4}$	$-\frac{17165786781}{4}$	0	0
14	0	0	$-\frac{761}{40}$	$-\frac{49}{40}$
15	0	0	0	0
17	0	0	0	0
19	$\frac{92871}{2}$	$-\frac{6079}{2}$	-414500	-26700
20	-4797	309	15	1
22	0	0	$\frac{2251}{12}$	$\frac{145}{12}$
24	0	0	$-\frac{1321960925}{24}$	$-\frac{85154989}{24}$

The remaining two coefficient vectors are as follows:

$$\begin{aligned}
(c_{10,0}, c_{10,1}) &= \left(-\frac{2892966757041403369}{4}, -\frac{186352370650554249}{4}\right), \\
(c_{10,2}, c_{10,3}) &= (17869643568757778063, 1151084931623631393), \\
(c_{10,4}, c_{10,5}) &= \left(-\frac{7940871298299206214 81}{2}, -25575824330978012432\right), \\
(c_{10,6}, c_{10,7}) &= \left(\frac{39240143878073548164507}{4}, \frac{2527679869998307278733}{4}\right), \\
(c_{23,0}, \ldots, c_{23,7}) &= (C_0, -C_1/2, C_2/2, -C_3/2, C_4/2, -C_5, C_6/10, -C_7/2).
\end{aligned}$$

The positive integers C_j are given by their base-10^8 digits. A row $(d_0, \ldots, d_k)$ denotes $\sum_{i=0}^{k} d_i 10^{8(k-i)}$; leading zeroes inside a digit are only typographical. Within each block labelled j, read the digits from left to right and then from top to bottom.

j	Base-10^8 digits of C_j
0	1, 74510253, 60717598, 75908937, 16911066 39456000, 90234466, 55894358, 87867505, 86638758 17982229, 76772239, 12695036, 95162104, 21143576 27456833, 18370036, 34063252, 02922730, 86767929 09860055, 94974870, 20362575
1	22482387, 24719038, 17215355, 51947364, 98058428 56713531, 00247991, 74208515, 23159475, 64610967 25772018, 46869563, 15956800, 44527910, 70294152 90055354, 13384992, 95080980, 51277486, 24137160 73881828, 57274345
2	91, 59029286, 29453336, 12211730, 48598714 32070187, 25846677, 83448236, 86981542, 07191611 70993215, 14394504, 18874875, 27262368, 50555749 35646452, 64442163, 96813687, 18619874, 70246292 81871762, 89008821
3	5, 89984940, 61658962, 38887151, 81555788

j	Base-10^8 digits of C_j
	82412309, 92506419, 51981627, 39424367, 11802506 89500430, 98702276, 76479766, 26671906, 79133801 64568107, 79889859, 76753457, 48667474, 40359780 36314435, 22186127
4	19815307, 97740989, 17041772, 62334033, 89720818 97955614, 95504304, 79925327, 69857602, 62420292 04820133, 45180980, 41479110, 28140821, 90999693 94939042, 16228504, 98817114, 66783624, 56635009 10103644, 83583425
5	638208, 09689109, 13499645, 91657229, 94433351 90643553, 31608707, 34432674, 77108328, 18070756 02404113, 34330615, 13264175, 09610688, 14906045 01313072, 48188240, 73197257, 80643329, 93753240 95985823, 70173255
6	25, 99975966, 03372215, 40103718, 02733514 09760623, 78821630, 87248335, 90659344, 63093977 86652900, 77092944, 26508322, 62532508, 39111414 99761832, 50753640, 98407725, 18653042, 99398517 17387267, 63638587
7	33495834, 94007047, 33540442, 31150225, 90196915 32201182, 79844008, 86119143, 50511082, 26120257 23259866, 66480608, 17335618, 95066745, 20312236 13669879, 60461526, 77155044, 41490265, 17835603 27026885

The positive sign of every norm and the three exponents of $2, 3, 5$ are listed in Section 3.9. They follow by successively applying

$$(x + y\sqrt{d})(x' + y'\sqrt{d}) = (xx' + dyy') + (xy' + yx')\sqrt{d}, \qquad N(x + y\sqrt{d}) = x^2 - dy^2$$

with $d = b_o, a_o, 241$. The same formulas give a short recursive check of the Galois identities. For $x + y\sqrt{d}$, $y \neq 0$, first take either square root n of $x^2 - dy^2$, then a nonzero square root r of $(x + n)/2$; the candidate square root is $r + (y/(2r))\sqrt{d}$. When $y = 0$, use a square root of x in the smaller field or $\sqrt{d}$ times a square root of x/d. At the rational level, numerator and denominator must both be squares. Trying the two signs at each step exhausts all possibilities. Applied to $\sigma(\beta_o)/\beta_o$, this gives the three identities (3.23) for every row. Changing the sign of a returned square root does not change $u\sigma(u)$.

3.11. **Intrinsic coefficients and prime sums.** For each quadratic extension A'/A in (3.26)–(3.27), define

$$D_T(p, X) = \prod_{\mathfrak{q}|p \text{ in } T} (1 - X^{f(\mathfrak{q}/p)}), \qquad P_p(X) = \frac{D_{A'}(p, X)}{D_A(p, X)}.$$

The quotient P_p is a polynomial with integer coefficients. Its reciprocal coefficients are uniquely specified by

$$a_1 = 1, \qquad a_{p^k} = - \sum_{j=1}^{\min(k, \deg P_p)} [X^j]P_p(X)\, a_{p^{k-j}}, \qquad a_n = \prod_{p^k \| n} a_{p^k}.$$

Thus the rational radicals above, ordinary ideal factorisation and this recursion determine each coefficient of all 576 factors. Their conductors are $|\Delta_{A'}|/|\Delta_A|$, and their gamma factors follow from the signatures. The analogous definitions apply to the 255 quadratic factors of ζ_{E_0}/ζ_B.

The prime correction can equally be defined without bins. Let $X_C = 1125899906842000$ and let $\mathcal{P}_2$ be the set in (3.29), with the stated cutoff X_C. For $s > 1$ put

$$\mathcal{C}_2(s) = \sum_{\mathfrak{p}\in\mathcal{P}_2} \frac{1}{2}\operatorname{arctanh}((N\mathfrak{p})^{-s}), \qquad \mathcal{C}_2 = \sum_i c_i \mathcal{C}_2(\sigma_i).$$

Every summation condition is an exact residue-field test specified by (3.28). If a norm N is assigned an interval $[L, U]$ and $\tau(x) = \sum_i c_i \operatorname{arctanh}(x^{-\sigma_i})/2$ is decreasing there, then $\tau(N) \geq \tau(U)$. Consequently replacing any enlarged-bin lower bound by the exact sum $\mathcal{C}_2$ preserves the direction of the relative-zeta upper bound. This comparison applies to the signed kernel as a whole.

For a factor with gamma data (a, μ_j, κ), conductor Q, $t = \kappa/\sqrt{Q}$ and cutoff N, its intrinsic finite value is

$$F_N(s) = \frac{t^s}{\prod_j \Gamma(s/a + \mu_j)} \sum_{n=1}^{N} a_n \left\{ I_{s/a}((tn)^a) + I_{(1-s)/a}((tn)^a) \right\}.$$

The series for I_b and their remainder bounds in Section 4.7, together with the tail estimate (4.28), give two-sided rational enclosures for this expression and for $L(s)$. This formulation specifies the finite sum through its coefficients and kernel, independently of a choice of floating-point accumulation or stored moments.

4. The arithmetic constant

This section proves the arithmetic input

$$C_{\mathrm{ar}} = 0.04226263259973019243081$$

used in the incidence construction. It bounds a *relative zeta value*. It is neither a bound for an individual Dedekind zeta value at its pole nor a direct bound for an unnormalised class number or regulator. All decimal constants defining the witness are exact rational numbers. Numerical optimisation is used only to find the witness; none of its termination messages is used in the proof.

4.1. **Field input and the quantity being bounded.** We use the following data from the tower construction of Theorem 3.9, following the arithmetic framework of [24]. The base is $B = \mathbb{Q}(\sqrt{241})$. The fields K/B under consideration are finite Galois extensions in the fixed pro-2 tower, of unbounded degree, containing the fixed comparison field E_* of degree 8192 over $\mathbb{Q}$ constructed in Section 3.9. They are totally imaginary. If ι is the specified complex conjugation, then $F = K^{\langle\iota\rangle}$, $[F : \mathbb{Q}] = d$, and $[K : \mathbb{Q}] = 2d$. The quadratic extension K/F is unramified at finite places. Both fields have root discriminant

$$\lambda = 2^{9/4}\sqrt{3615}, \qquad \ell = \log\lambda = \tfrac{9}{4}\log 2 + \tfrac{1}{2}(\log 3 + \log 5 + \log 241). \tag{4.1}$$

Write (r_1, r_2) for the signature of F and $\theta = r_2/d$; thus $r_1/d = 1 - 2\theta$ and $0 \leq \theta \leq 1/2$.

There are nine selected primes of B: the two primes over each of $2, 3, 5$, two specified primes of norm 29, and one specified prime of norm 41. Their types (e, f) in K/B are respectively $(8, 4)$, $(2, 2)$, $(2, 2)$, $(1, 4)$ and $(1, 4)$. Every other prime of B is unramified in K/B. The residue-degree floors used below divide the actual residue degrees in every field of the family. These field-theoretic inputs are established in Section 3.

Put $L_F(s) = \zeta_K(s)/\zeta_F(s)$. The class-number formula gives the exact identity

$$L_F(1) = 2^{r_2}\pi^{d-r_2}\lambda^{-d/2}\frac{h_K \operatorname{Reg}_K w_F}{h_F \operatorname{Reg}_F w_K} > 0, \tag{4.2}$$

where w_A is the number of roots of unity in A.

Proposition 4.1. *For all sufficiently large fields in this family,*

$$\frac{1}{d}\log L_F(1) < C_{\mathrm{ar}}. \tag{4.3}$$

The proof uses the relative zeta estimate below.

4.2. **Reduction to a positive prime-density problem.** Use $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. The completed relative function is

$$\mathcal{L}_F(s) = \lambda^{ds/2}\Gamma_{\mathbb{R}}(s+1)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}L_F(s).$$

It is the completed L-function of the nontrivial quadratic Hecke character of K/F [15, 42], is entire of order at most one, and satisfies $\mathcal{L}_F(s) = \mathcal{L}_F(1-s)$. Its zeros lie in $0 \le \Re s \le 1$. For completeness, these facts imply monotonicity on the real interval $[1,\infty)$ without any assertion about zeros on the critical line. Indeed, set $\Xi(z) = \mathcal{L}_F(1/2+z)$ and group its zeros in pairs $\{z_0, -z_0\}$. The paired Hadamard product yields

$$\frac{\Xi'(x)}{\Xi(x)} = \frac{m}{x} + \sum_{\{z_0,-z_0\}} \Re\frac{2x}{x^2 - z_0^2} \ge 0 \qquad (x \ge 1/2),$$

because, if $z_0 = a+iy$, then $|a| \le 1/2$ and the summand equals $2x(x^2-a^2+y^2)/((x^2-a^2+y^2)^2+4a^2y^2)$. The paired series converges locally absolutely away from zeros; evenness removes the constant exponential factor. Positivity at $s=1$ follows from (4.2).

At each prime of F, the square of the Euler factor of $L_F(s)$ is at most the corresponding factor of $\zeta_K(s)$, for real $s > 1$. Consequently, for $0 < \varepsilon \le 1$,

$$\frac{1}{d}\log L_F(1) \le \frac{\log \zeta_K(1+\varepsilon)}{2d} + \Upsilon(\varepsilon,\theta), \tag{4.4}$$

$$\Upsilon(\varepsilon,\theta) = \frac{\varepsilon}{2}(\ell - \log\pi) - \varepsilon\theta\log 2 + (1-2\theta)\log\Gamma(1+\varepsilon/2) + \theta\log\Gamma(1+\varepsilon). \tag{4.5}$$

The last expression tends uniformly to zero for $0 \le \theta \le 1/2$.

For a prime power q and a sequence of fields of increasing degree, pass to a subsequence on which $\beta_q = \lim N_q(K)/(2d)$ exists for every q. This is possible by a diagonal argument, since $N_{p^f}(K)/(2d) \le 1/f$. Put

$$g_s(q) = -\log(1-q^{-s}), \qquad w(q) = 2\log q \sum_{m\ge 1}\frac{1}{q^m+1}, \qquad Z(s) = \sum_q \beta_q g_s(q). \tag{4.6}$$

For fixed $s > 1$, dominated convergence identifies $Z(s)$ with the limit of $(2d)^{-1}\log\zeta_K(s)$. The unconditional basic inequality of Tsfasman–Vlăduţ [44], applied with $\log|\Delta_K|^{1/2} = d\ell$ and K totally imaginary, is

$$\sum_q \beta_q w(q) \le \frac{\ell - \gamma - \log(4\pi)}{2} = 2\kappa_\infty. \tag{4.7}$$

Here γ is Euler's constant. The convergence of $Z(1)$ follows, for example, from $g_1(q) \le 3w(q)/(2\log 2)$. First passing to the field limit in (4.4) for a fixed ε, and then letting $\varepsilon \downarrow 0$, shows that every subsequential lower bound for $d^{-1}\log L_F(1)$ is at most $Z(1)$. This order of limits is essential.

The selected local types give

$$Z_{\mathrm{sel}}(s) = \tfrac{1}{32}g_s(16) + \tfrac{1}{4}g_s(9) + \tfrac{1}{4}g_s(25) + \tfrac{1}{4}g_s(29^4) + \tfrac{1}{8}g_s(41^4). \tag{4.8}$$

Define T_{sel} by replacing each g_s in this expression by w, and let $B_r = 2\kappa_\infty - T_{\mathrm{sel}}$.

For a free prime $\mathfrak{p}$ of B, let $f_{\mathfrak{p}}^0 \mid f_K(\mathfrak{p})$ be its certified floor and suppose $(\mathrm{N}\mathfrak{p})^{f_{\mathfrak{p}}^0} \ge q_0 = 41^4$. The associated floor sum is

$$Y_{\mathrm{fl}}(s) = \sum_{\mathfrak{p}\ \text{free}} \frac{g_s((\mathrm{N}\mathfrak{p})^{f_{\mathfrak{p}}^0})}{2f_{\mathfrak{p}}^0}.$$

Lemma 4.2 (Signed comparison)**.** *Let* $\mathsf{K}(q) = \sum_i c_i g_{\sigma_i}(q)$*, where* $1 < \sigma_i < 1.01$*,* $c_i \in \mathbb{R}$*, and* $b \geq 0$*. Suppose, for all* $q \geq q_0$*,*

$$\mathsf{K}(q) \geq g_1(q) - bw(q), \qquad \mathsf{K}(q) \geq 0, \qquad \mathsf{K}'(q) \leq 0. \tag{4.9}$$

Then the eventual relative-zeta ceiling is every rational number strictly larger than

$$Z_{\mathrm{sel}}(1) + \sum_i c_i Y_{\mathrm{fl}}(\sigma_i) + bB_r. \tag{4.10}$$

Proof. A free prime of norm N and actual residue degree f contributes $\mathsf{K}(N^f)/(2f)$. Nonnegativity and monotonicity give $\mathsf{K}(N^f)/(2f) \leq \mathsf{K}(N^{f^0})/(2f^0)$. Summing, then applying Fatou's lemma to the nonnegative kernel, gives $\sum_q \beta_q^{\mathrm{free}} \mathsf{K}(q) \leq \sum_i c_i Y_{\mathrm{fl}}(\sigma_i)$. Interchanging the finite signed sum with the floor sum is valid because $|\mathsf{K}| \leq \sum_i |c_i| g_{\sigma_i}$, whose floor sums converge. Apply the first inequality in (4.9) and the remaining budget $\sum \beta_q^{\mathrm{free}} w(q) \leq B_r$ from (4.7). This bounds $Z(1)$ by (4.10). A hypothetical sequence of exceptions of increasing degree contradicts the preceding limit argument. □

4.3. **The rational kernel and its global verification.** The witness has 36 nonzero coefficients, selected from a grid of 71 abscissae. Table 6 specifies all of them exactly by $\sigma_i = 1 + 1/n_i$, $c_i = A_i/10^{45}$. Also

$$b = \frac{1122702740701354531103966031006813729945 22}{10^{45}}. \tag{4.11}$$

In particular $\sum_i |c_i| < 3000000$. This last number is a property of the witness, not a restriction on all admissible kernels.

Here are full interval-check rules for (4.9). Set $x = \log q$ and

$$\Phi_1 = e^x(\mathsf{K} - g_1 + bw)(e^x), \quad \Phi_2 = e^x \mathsf{K}(e^x), \quad \Phi_3 = e^x \frac{d}{dx}\mathsf{K}(e^x).$$

Their required signs are respectively $+, +, -$. Use

$$e^x g_s(e^x) = e^{-(s-1)x} \sum_{k\geq 0} \frac{e^{-ksx}}{k+1}, \quad e^x \frac{d}{dx} g_s(e^x) = -se^{-(s-1)x} \sum_{k\geq 0} e^{-ksx},$$

and $e^x w(e^x) = 2x \sum_{m\geq 1} e^{-(m-1)x}/(1 + e^{-mx})$. Retain $k = 0, \ldots, 9$ in the first two series. For this particular witness the common lower endpoint satisfies $x > 14.85$, so $e^{-sx} < 3.56 \cdot 10^{-7}$. The omitted series and their Taylor coefficients through order three, after summing all signed coefficients, are bounded in absolute value by

$$2(3000000 \cdot 1.01 + 1)\frac{10.11^3}{6}(3.56 \cdot 10^{-7})^{10} < 3.413 \cdot 10^{-56} < 10^{-55}.$$

The geometric ratio of successive derivative majorants is less than $2 \cdot 10^{-6}$; the factor 2 above covers its sum. Thus a radius 10^{-55} on each of the four Taylor coefficients is sufficient. Only $m \leq 6$ is retained in w, which gives a pointwise lower function for Φ_1 because $b \geq 0$; its derivatives are then evaluated exactly as derivatives of this finite lower function.

Algorithm 4.3 (Verification of the kernel signs)**.**

(i) Put $x_* = 10^{-6}\lfloor 10^6 \log q_0 \rfloor$ and $X = 2000000$. Partition $[x_*, X]$ into intervals of length $1/20$ below 60; thereafter use length $x/100$ at a left endpoint x, shortening the last interval if necessary.

(ii) On an interval with midpoint a and half-length h, enclose each of Φ_1, Φ_2, Φ_3 by

$$\Phi(a) + \Phi'(a)t + \tfrac{1}{2}\Phi''(a)t^2 + \tfrac{1}{6}\Phi'''([a-h, a+h])t^3, \qquad |t| \leq h.$$

Use the finite series and remainder bounds above for the coefficients.

(iii) If these intervals establish the required signs $+, +, -$, retain the lower bounds for $\Phi_1, \Phi_2, -\Phi_3$. Otherwise divide the interval into two equal parts and repeat the preceding step.

(iv) Take the smallest of the retained lower bounds. Check the three displayed inequalities below to cover $x \geq X$.

The exact rational coefficients establish the signs on all 4428 resulting cells, with lower bounds

$$\Phi_1 > 2.9510 \cdot 10^{-11}, \quad \Phi_2 > 1.4846 \cdot 10^{-17}, \quad -\Phi_3 > 1.4846 \cdot 10^{-17}.$$

For $x \geq X$ the first abscissa is smallest and $c_1 > 0$. The following three strictly positive scalar checks cover the entire remaining ray:

$$\begin{aligned} c_1 &> \sum_{j>1} |c_j|(1 + e^{-X})e^{-(\sigma_j - \sigma_1)X}, \\ c_1\sigma_1 &> \sum_{j>1} |c_j|\sigma_j(1 + 2e^{-X})e^{-(\sigma_j - \sigma_1)X}, \\ 2bX &> 1 + 3e^{-X}. \end{aligned}$$

The first two bound Φ_2 and $-\Phi_3$ after multiplication by $e^{(\sigma_1 - 1)x}$; the third makes $g_1 - bw < 0$. This proves the global kernel conditions, including the infinite tail.

4.4. **Comparison-field factorisation and prime corrections.** Use the exact forms, radicals, twist characters and residue-degree detectors of Section 3.9; in particular the integer family labels below refer to Table 4 and the pairs in (3.20). Let $E_0 \subset E_*$ be the genus field of degree 512. The relevant class-two quotient over B has order 4096, elementary abelian quotient of dimension eight, and a four-dimensional central character space. Its fifteen nonzero central characters have seven polar forms of rank two and eight of rank four. For polar rank r, an irreducible representation with this central character has dimension $2^{r/2}$ and there are $256/2^r$ such representations. One way to see the dimension formula is to pass to the quotient in which the central character has order two: the character vanishes off the inverse image of the radical, and character orthogonality gives $(\dim \pi)^2 = 2^r$. The degree check is $256 + 7 \cdot 64 \cdot 4 + 8 \cdot 16 \cdot 16 = 4096$.

The regular representation and Artin formalism therefore give the *factor-by-factor* identity

$$\frac{\log \zeta_{E_*}(s)}{8192} = \frac{1}{16}\frac{\log \zeta_{E_0}(s)}{512} + \frac{1}{4096}\sum_{\nu=1}^{448} \log L_{4,\nu}(s) + \frac{1}{2048}\sum_{\nu=1}^{128} \log L_{8,\nu}(s). \tag{4.12}$$

The seven degree-four families are indexed by $10, 23, 24, 19, 20, 17, 7$, each with 64 twists. The eight degree-eight families are $(19, 23), (19, 24), (7, 10), (17, 10), (17, 24), (7, 24), (7, 23), (17, 23)$, each with 16 twists. Every factor here is a quotient $\zeta_{A'}/\zeta_A$ for a specified quadratic extension A'/A; hence it is a quadratic Hecke L-function with root number $+1$, not an assumed instance of Artin holomorphy. The defining radicals and their exact finite verification are given in Section 3.9. Exact local arithmetic in these fields determines their conductors, signatures and all Euler factors.

The floor of a free prime starts with its residue degree in E_*/B. For the certified set $\mathcal{P}_4$ it is raised from 2 to 4 when needed, and for the detected set $\mathcal{P}_2$, split in E_*, it is raised from 1 to 2. These sets are disjoint. Write $Y_*(s)$ for the left side of (4.12), and define

$$T_s(N) = \tfrac{1}{2}g_s(N) - \tfrac{1}{4}g_s(N^2) = \tfrac{1}{2}\operatorname{arctanh}(N^{-s}), \tag{4.13}$$

$$\Delta_4(s) = \sum_{\substack{\mathfrak{p} \in \mathcal{P}_4 \\ f_{E_*}(\mathfrak{p}) = 2}} \left(\tfrac{1}{4}g_s((\mathrm{N}\mathfrak{p})^2) - \tfrac{1}{8}g_s((\mathrm{N}\mathfrak{p})^4)\right). \tag{4.14}$$

Let Δ_{sel} be the difference between the selected-prime contribution in E_* and the actual selected-prime contribution in K. It is evaluated from the bad Euler factors at 2, together with $g_s(N^{f_{E_*}})/(2f_{E_*}) -$

$g_s(N^4)/8$ at the three capped primes. The primes over $3, 5$ give zero difference because their types agree. Termwise comparison of Euler products proves

$$Y_{\mathrm{fl}}(s) = Y_*(s) - \Delta_{\mathrm{sel}}(s) - Z_{\mathrm{sel}}(s) - \Delta_4(s) - \sum_{\mathfrak{p}\in\mathcal{P}_2} T_s(\mathrm{N}\mathfrak{p}). \tag{4.15}$$

In particular the signed coefficients cannot be applied to one-sided upper estimates for $Y_*(\sigma_i)$ separately.

The prime sum is defined directly by its residue tests and the endpoint $X_C = 1125899906842000$. Put

$$\mathcal{C}_2(s) = \sum_{\mathfrak{p}\in\mathcal{P}_2} T_s(N\mathfrak{p}), \qquad \tau(N) = \sum_i c_i T_{\sigma_i}(N).$$

Exact enumeration therefore determines every term, without a choice of binning. Grouping norms into intervals can accelerate the evaluation. Indeed

$$-2\frac{d}{dx}\tau(e^x) = \sum_i \frac{c_i\sigma_i e^{-\sigma_i x}}{1-e^{-2\sigma_i x}},$$

and directed interval bisection verifies positivity after multiplication by e^x on $16 \le x \le 36$; the finite check uses 327600 cells. If n norms belong to an interval $[L, U]$ in this range, their total contribution is at least $n\tau(U)$. Thus an exact prime sum is at least any such lower endpoint estimate. The signed kernel is combined before this comparison, so its direction is unaffected by negative coefficients.

4.5. **Finite evaluation of the arithmetic constant.** The number to be evaluated is the exact real scalar

$$\Phi = Z_{\mathrm{sel}}(1) + bB_r + \sum_i c_i \left\{ Y_*(\sigma_i) - \Delta_{\mathrm{sel}}(\sigma_i) - Z_{\mathrm{sel}}(\sigma_i) - \Delta_4(\sigma_i) - \mathcal{C}_2(\sigma_i) \right\}, \tag{4.16}$$

where

$$\mathcal{C}_2(s) = \frac{1}{2}\sum_{\mathfrak{p}\in\mathcal{P}_2} \operatorname{arctanh}((N\mathfrak{p})^{-s}).$$

The set $\mathcal{P}_2$ consists of the degree-one free primes of norm at most 1125899906842000 satisfying the residue tests in Section 3.9. The exact field coefficients are listed in Section 3.10. Thus every operand in (4.16) is determined by rational data and finite residue-field calculations, together with Dedekind zeta values at the specified rational arguments.

Here is a direct verification algorithm using only finite Euler products. It also gives an independent specification of the numerical inequality, while the factorisation (4.12) and the approximate functional equations below make its evaluation much faster. For an integer $P \ge 2$ put

$$Y_{*,P}(s) = \frac{1}{8192}\sum_{p\le P}\sum_{\mathfrak{q}\mid p \text{ in } E_*} -\log(1-p^{-f(\mathfrak{q}/p)s}).$$

Every local factor includes all prime ideals over p, including those at ramified primes. The field can be constructed directly as

$$E_* = E_0(\sqrt{\beta_{10}}, \sqrt{\beta_{23}}, \sqrt{\beta_{19}}, \sqrt{\beta_{17}}).$$

Its degree and its identification with the comparison field follow from Section 3.9.

Lemma 4.4. *For $s > 1$ and $P \ge 2$,*

$$0 \le Y_*(s) - Y_{*,P}(s) \le R_P(s) := \frac{P^{1-s}}{(s-1)(1-(P+1)^{-s})}.$$

Consequently, replacing Y_* by $Y_{*,P}$ in (4.16) changes the scalar by at most

$$\sum_i |c_i| R_P(\sigma_i) < \frac{300000000000}{P^{1/50000}}.$$

Proof. For each rational prime p, the number of prime ideals over p is at most 8192, and every residue degree is at least one. The normalised local logarithm is therefore at most $-\log(1-p^{-s})$. Hence

$$0 \le Y_*(s) - Y_{*,P}(s) \le \sum_{n>P} \frac{n^{-s}}{1-n^{-s}} \le \frac{1}{1-(P+1)^{-s}} \int_P^\infty x^{-s}\,dx.$$

The final bound uses $\sigma_i \ge 1 + 1/50000$, $\sum_i |c_i| < 3000000$ and $1-(P+1)^{-\sigma_i} > 1/2$. □

Algorithm 4.5 (An upper bound for the arithmetic scalar)**.** Fix a rational error allowance $\eta > 0$.

(i) Construct the eighteen listed quadratic extensions and verify their identities by Algorithm 3.11. Form E_* from the four displayed radicals.

(ii) Use Algorithm 3.13 to enumerate $\mathcal{P}_2$ and $\mathcal{P}_4$. For each rational prime that splits in B, include both prime ideals. Compute $\mathcal{C}_2$, adding the signed kernel at each norm before adding the contributions from different norms. Compute the selected-prime corrections by their stated local factors.

(iii) Enclose the required zeta values. One choice is to factor primes in E_* through a bound P with $300000000000P^{-1/50000} < \eta$ and use Lemma 4.4. The shorter choice is Algorithm 4.6 applied to the factors in (4.12). In either case retain the proved remainder intervals.

(iv) Evaluate the finite expressions with rational upper and lower bounds. Increase the number of terms and the arithmetic precision until the propagated interval width is less than η. Add the remainder intervals with the sign of each c_i, obtaining $\Phi \in [L, U]$.

(v) Verify the rational inequality $U < C_{\mathrm{ar}}$. The choice $\eta = 2^{-120}$ is smaller than the strict gap below. For the direct Euler-product calculation, $P = 2^{8000000}$ is a permissible fixed choice, since $300000000000P^{-1/50000} < 2^{-121}$.

For clarity, the elementary functions in this algorithm can be enclosed by elementary convergent series. For $x > 0$, put $z = (x-1)/(x+1)$ and use

$$\log x = 2\sum_{k=0}^{m} \frac{z^{2k+1}}{2k+1} + \varepsilon_m, \qquad |\varepsilon_m| \le \frac{2|z|^{2m+3}}{(2m+3)(1-|z|^2)}.$$

Range reduction by powers of two speeds convergence. Exponentials are enclosed by their Taylor series with the usual positive remainder majorant; rational powers are evaluated as $\exp(r\log x)$. The alternating arctangent series and $\pi = 16\arctan(1/5) - 4\arctan(1/239)$ give rational intervals for π. For Euler's constant one may use $H_n - \log(n+1) < \gamma < H_n - \log n$, or the sharper Euler–Maclaurin expansion. Each of these intervals converges to its stated value. These formulas also enclose the elementary local corrections and the signed kernel. For the prime correction one may use $\operatorname{arctanh} x = \sum_{k\ge0} x^{2k+1}/(2k+1)$ with its positive geometric remainder.

The finite checks and the resulting upper certificate are summarised below. The last line is an upper bound for the exact scalar (4.16); intermediate summation arrangements do not enter its definition.

Quantity	Result
Radical rows and square-ratio identities	18 and 54
Dimensions of R_2, the census dual, and W	$21, 15, 4$
Comparison field degree	8192
Degree-four and degree-eight factors	448 and 128
Kernel coefficients and total variation	36, $\sum_i \lvert c_i\rvert < 3000000$
Arithmetic scalar	$\Phi \le 2^{-363}V$

Here V is the integer obtained by concatenating the following three strings in order:

$$794044738564735584308537459113721899,$$
$$127207977731047345153771189872170342,$$
$$983861833465461068215627680053171377.$$

Equivalently, this is the exact-rational certificate

(4.17)
$$\Phi \le 2^{-363}V < 0.042262632599730192430809745728704125 < C_{\mathrm{ar}} = 0.042262632599730192430 81.$$

The gap between C_{ar} and $2^{-363}V$ is greater than $2.5427 \cdot 10^{-25}$. All these comparisons are rational comparisons. Together with Lemma 4.2, this proves Proposition 4.1.

4.6. **Evaluation by short Dirichlet sums.** The factorisation (4.12) reduces the comparison-field calculation to 448 degree-four and 128 degree-eight factors, together with the genus-field term. For a factor with conductor Q, write

$$G(s) = C_G \kappa^{-s} \prod_{j=1}^{m} \Gamma(s/a + \mu_j).$$

The gamma parameters are

Type	a	(μ_j)	κ
Degree eight	1	$(0,0,0,0)$	$(2\pi)^4$
Pure degree four	1	$(0,0)$	$(2\pi)^2$
Mixed $(3,1)$	2	$(0,0,0,1/2)$	π^2
Mixed $(1,3)$	2	$(0,1/2,1/2,1/2)$	π^2

Put $t = \kappa/\sqrt{Q}$, $x_n = (tn)^a$ and

$$p_s = \frac{t^s}{\prod_j \Gamma(s/a + \mu_j)}, \quad H(x) = \frac{1}{2\pi i} \int_{(u)} \prod_j \Gamma(z + \mu_j) x^{-z}\, dz, \quad I_b(x) = \int_1^\infty v^{b-1} H(xv)\, dv.$$

The functional equation with root number $+1$ gives

$$L(s) = p_s \sum_{n \ge 1} a_n \left\{ I_{s/a}(x_n) + I_{(1-s)/a}(x_n) \right\}. \tag{4.18}$$

Indeed the Mellin transform of $\sum_n a_n H(x_n y)$ is $\Lambda(az)/C_G$. Split its inverse transform at $y = 1$ and use $z \mapsto 1/a - z$. This yields (4.18), with no extra scaling factor. The coefficients are generated exactly from the Euler polynomials in Section 3.11.

Algorithm 4.6 (Enclosing the short Dirichlet sums)**.**

(i) For each factor, determine its conductor, signature and Euler polynomials from the displayed quadratic fields. Use the coefficient recurrence in Section 3.11 to obtain the exact integers a_n through a chosen bound N.

(ii) Evaluate the finite part of (4.18). For each retained argument, enclose both kernels by the residue series and remainder bounds of Section 4.7.

(iii) For a factor of degree D, use

$$\sum_{n \ge 1} |a_n| n^{-2} \le \zeta(2)^D < 2^D.$$

Take $\beta = 2$ and $\eta = 4$ in (4.28), multiply by the positive prefactor, and add the bounds for $b = s/a, (1-s)/a$. If the omitted tail is too large, double N and repeat.

(iv) Increase the number of residue terms and the precision until the finite part has the desired interval width. Once the lower endpoint of the complete L-value interval is positive, apply log to both endpoints. Combine the resulting intervals with the coefficients c_i and the weights in (4.12).

(v) Apply the degree-two kernel formulas below to the genus-field factors, and the Euler–Maclaurin formula to the remaining Hurwitz zeta values. Refine each enclosure until the total error allowance is met.

The coefficient bound follows from $|a_n| \le d_D(n)$. The choices in step (iii) satisfy $a(b+\eta) > 2$ at every abscissa used here, so the tail bound tends to zero as N increases. The kernel remainders also tend to zero. Thus the procedure gives enclosures of arbitrarily small width. Positivity of the real L-values follows from their expression as quotients of Dedekind zeta functions for $s > 1$.

The kernels may also be evaluated on short intervals. They are completely monotone, since H is a positive multiplicative convolution of $x^{\mu_j}e^{-x}$. About the right endpoint ξ of a cell $[\xi', \xi]$, the degree-D Taylor remainder is at most

$$\frac{|I_b^{(D+1)}(\xi')|}{(D+1)!}(\xi - \xi')^{D+1}. \tag{4.19}$$

The identity $xI_b'(x) + bI_b(x) = -H(x)$ supplies its derivatives. Exact integer moments can then replace term-by-term summation within a cell; this changes neither the finite sum nor its remainder bound.

The genus-field factorisation is

$$\zeta_{E_0}(s) = \zeta_B(s) \prod_{0 \ne w \in \mathbb{F}_2^8} \frac{\zeta_{B(\sqrt{\alpha_w})}(s)}{\zeta_B(s)}, \qquad \zeta_B(s) = \zeta(s)\, 241^{-s} \sum_{a=1}^{240} \left(\frac{a}{241}\right) \zeta(s, a/241).$$

The 255 quotients are quadratic Hecke factors over B. Their exact coefficients come from the same local definition, and their degree-two kernel formulas and explicit tails are given below. The remaining Hurwitz zeta values have elementary Euler–Maclaurin enclosures. For $x > 0$, let $y = N + x$. Then

$$\begin{aligned} \zeta(s, x) = {} & \sum_{n=0}^{N-1} (n+x)^{-s} + \frac{y^{1-s}}{s-1} + \frac{1}{2}y^{-s} \\ & + \sum_{k=1}^{M} \frac{B_{2k}}{(2k)!}(s)_{2k-1} y^{-s-2k+1} + R_{M,N}, \\ |R_{M,N}| \le {} & \frac{4(s)_{2M}}{6^{2M}(s+2M-1)} y^{-s-2M+1}. \end{aligned}$$

Here $(s)_j = s(s+1)\cdots(s+j-1)$ and the Bernoulli numbers are exact rationals. The remainder follows by integrating the periodic Bernoulli polynomial and using $|B_{2M}(\{u\})| \le 2(2M)!\zeta(2M)/(2\pi)^{2M}$, $\zeta(2M) < 2$ and $\pi > 3$. Increasing N with M fixed makes this bound tend to zero. These formulas specify the entire genus-field calculation, including its pole contribution at $s = 1$ evaluated at the arguments $s > 1$ used here.

4.7. **Explicit kernel series and their error bounds.** This subsection supplies the kernel expansions and remainder estimates used in Section 4.6. It follows the Mellin construction of Naslund [24]. All truncation bounds are inequalities between exact real numbers, so they can be used directly in rational interval evaluation.

Residues for the degree-four and degree-eight kernels. Use the notation H, I_b of Section 4.6; in particular $I_b(x) = \int_1^\infty v^{b-1} H(xv)\, dv$. Group the shifts μ_j into the distinct values $\alpha \in \mathcal{A} \subseteq \{0, 1/2\}$, with multiplicities m_α. Put $m = \sum m_\alpha$. For $n \geq 0$ define the function analytic at $e = 0$

$$F_{\alpha,n}(e) = \left\{ \frac{(-1)^n}{n!} \Gamma(1+e) \prod_{j=1}^{n} (1 - e/j)^{-1} \right\}^{m_\alpha} \prod_{\beta \in \mathcal{A} \setminus \{\alpha\}} \Gamma(-n + \beta - \alpha + e)^{m_\beta}. \tag{4.20}$$

Here an empty product is one, and $[e^j]$ means Taylor coefficient at zero. At the pole $z = -n - \alpha$, the product of gamma factors is $e^{-m_\alpha} F_{\alpha,n}(e)$. Taking residues yields

$$H(x) = \sum_{\alpha \in \mathcal{A}} \sum_{n \geq 0} [e^{m_\alpha - 1}] F_{\alpha,n}(e) x^{n+\alpha-e}, \tag{4.21}$$

$$x^b I_b(x) = \prod_j \Gamma(b + \mu_j) - \sum_{\alpha \in \mathcal{A}} \sum_{n \geq 0} [e^{m_\alpha - 1}] \frac{F_{\alpha,n}(e) x^{n+\alpha+b-e}}{n + \alpha + b - e}. \tag{4.22}$$

The second formula is used when none of $n + \alpha + b$ is zero, as is the case at all the rational abscissae here. The additional term comes from the pole of $1/z$ in the Mellin integral for $x^b I_b$. When $b < 0$, its initial contour lies to the right of the positive pole $z = -b$ as well. Thus no pole is accidentally lost near $b = 0$. These formulas follow either by shifting the inverse Mellin contour, or first on a compact interval of positive x and then by analytic continuation. The reflection formula and Stirling's estimate make the left-contour remainder tend to zero; the factorial bounds below also prove locally uniform absolute convergence of the residue series.

Here are explicit bounds for the omitted residues. Choose $r = 1/2$ if $|\mathcal{A}| = 1$, and $r = 1/4$ otherwise. For all sufficiently large n (take $n > k + 3$ for a derivative of order k), put

$$\begin{aligned} C_{\alpha,n}(x) = r^{1-m_\alpha} e^{r|\log x|} x^{n+\alpha} & \left\{ \frac{\Gamma(1-r)}{n!} \prod_{j=1}^{n} (1 - r/j)^{-1} \right\}^{m_\alpha} \\ & \times \prod_{\beta \neq \alpha} \left\{ \frac{\pi}{(19/20)\cos(\pi r)\Gamma(n + 1 - (\beta - \alpha) - r)} \right\}^{m_\beta}. \end{aligned} \tag{4.23}$$

Then $C_{\alpha,n}(x)$ bounds the absolute value of the corresponding term in (4.21). To check this, on $|e| = r$ use $|x^{-e}| \leq e^{r|\log x|}$ and $|\Gamma(1+e)| \leq \Gamma(1-r)$. For the other shift class the reflection formula gives

$$|\Gamma(-n + \beta - \alpha + e)| \leq \frac{\pi}{\cos(\pi r)|\Gamma(n + 1 - (\beta - \alpha) - e)|}.$$

If $\sigma \geq 1$ and $|y| \leq 1/4$, the gamma product formula implies

$$\frac{|\Gamma(\sigma + iy)|}{\Gamma(\sigma)} = \prod_{j \geq 0} \left(1 + \frac{y^2}{(\sigma + j)^2} \right)^{-1/2} \geq \left(\frac{\pi |y|}{\sinh(\pi |y|)} \right)^{1/2} > \frac{19}{20}.$$

At zero the ratio on the right is interpreted as one. The last rational bound follows, for instance, by using $\pi/4 < 11/14$ and the positive power series for $\sinh x/x$ to verify $\sinh(11/14)/(11/14) < 400/361$. The relevant real parts are at least $3/2$, where Γ is increasing. This proves the cross-shift factor in (4.23). Cauchy's coefficient estimate supplies the factor r^{1-m_α}.

For $H^{(k)}(x)/k!$, multiply $C_{\alpha,n}(x)$ by

$$D_{\alpha,n,k}(x) = \frac{\prod_{j=0}^{k-1} (|n + \alpha - j| + r)}{k! x^k}. \tag{4.24}$$

This follows by differentiating $x^{n+\alpha-e}$ before taking its coefficient. For $I_b(x)$ itself, the omitted term is at most $C_{\alpha,n}(x)/(n + \alpha + b - r)$ whenever the denominator is positive; the external factor x^b in (4.22) cancels.

There is a finite geometric bound for each infinite residue tail. The ratio of two consecutive C bounds is exactly

$$q_{\alpha,n}(x) = \frac{x}{(n+1-r)^{m_\alpha}} \prod_{\beta\neq\alpha} (n+1-(\beta-\alpha)-r)^{-m_\beta}. \tag{4.25}$$

It decreases with n. For the derivative bound include the ratio $D_{\alpha,n+1,k}/D_{\alpha,n,k}$; this also decreases once $n > k$. Choose $n_0 > k+3$ so that the appropriate ratio at n_0 is at most $1/2$. The tail beginning at n_0 is bounded by its first majorant divided by one minus that ratio. For I_b, the extra factor $(n+\alpha+b-r)^{-1}$ decreases and can be held at its value at n_0 while using (4.25). Sum over the one or two shift classes. This is a completely specified omitted-pole bound. The choice of n_0 may be enlarged arbitrarily; the inequalities, not a floating estimate of an optimal cutoff, decide whether it is acceptable.

Only the first m_α Taylor coefficients in e are needed for each retained residue in (4.21)–(4.22); the products, gamma functions and exponential x^{-e} are evaluated as outward power-series balls. If $h_j = H^{(j)}(c)/j!$ and $p_j = I_b^{(j)}(c)/j!$, the differential identity in Section 4.6 gives the finite recurrence

$$p_{j+1} = -\frac{h_j + (j+b)p_j}{c(j+1)}. \tag{4.26}$$

It carries both interval rounding and the explicit residue-tail bounds. The complete-monotonicity bound (4.19) then bounds the Taylor remainder on every cell.

The Dirichlet tail has an explicit bound. If $\eta > 0$ and $v_j = b + \mu_j + \eta > 0$, set $U = \max_j v_j$ and let $m_0 = 2$ for $m = 2$, $m_0 = 4$ for $m = 4$. Mellin inversion and the gamma product inequality give

$$0 < I_b(x) \le \frac{U}{m_0\eta} \prod_j \Gamma(v_j)\, x^{-b-\eta}. \tag{4.27}$$

Indeed the contour integrand contains $1/(\eta+iy)$, bounded by $1/\eta$, and each gamma factor is bounded by $\Gamma(v_j)(1+y^2/U^2)^{-1/2}$. The integrals of the product, divided by 2π, are $U/2$ and $U/4$ for two and four factors respectively. Consequently, if $\sum |a_n| n^{-\beta} \le E_\beta$ and $a(b+\eta) > \beta$, the omitted terms after N satisfy

$$\sum_{n>N} |a_n| I_b((tn)^a) \le E_\beta N^\beta \frac{U}{m_0\eta} \prod_j \Gamma(v_j)\, (tN)^{-a(b+\eta)}. \tag{4.28}$$

This is Rankin's inequality applied term by term. Multiplying by the positive AFE prefactor gives the omitted-tail interval used in Section 4.6.

Degree-two kernels for the genus-field factors. The degree-two factors admit the following shorter kernels. Define, for real μ,

$$H_\mu(x) = \int_1^\infty u^{\mu-1} e^{-xu}\, du, \qquad I_\mu^{\mathrm{B}}(x) = \int_1^\infty u^\mu K_0(xu)\, du, \tag{4.29}$$

$$V_1(x) = e^{-x}(1+1/x)/x, \qquad V_2(x) = \sqrt{\pi/2}\, e^{-x}(1+1/x)/x^{3/2}. \tag{4.30}$$

Here $K_0(x) = \int_0^\infty e^{-x\cosh v}\, dv$ and $K_1(x) = \int_0^\infty e^{-x\cosh v}\cosh v\, dv$. These definitions give $K_0' = -K_1$ and, by integration by parts, $x^2K_0'' + xK_0' - x^2K_0 = 0$. They also give $K_0(x) \le \sqrt{\pi/(2x)}e^{-x}$, by $\cosh v \ge 1 + v^2/2$. It follows directly that

$$0 < H_\mu(x) \le V_1(x) \quad (\mu \le 2), \qquad 0 < I_\mu^{\mathrm{B}}(x) \le V_2(x) \quad (\mu \le 3/2).$$

For the first inequality bound $u^{\mu-1}$ by u; for the second use the displayed bound for K_0 and bound $u^{\mu-1/2}$ by u. Both kernels are Laplace transforms of positive measures, as is seen for I_μ^{B} by substituting the integral for K_0 and applying Tonelli's theorem. Thus they are completely monotone and holomorphic on the right half-plane, with $|J(z)| \le J(\operatorname{Re} z)$ there.

Let a quadratic Hecke factor over B have conductor Q, real coefficients $|a_n| \le d_2(n)$, positive functional equation, and gamma factor $\Gamma_{\mathbb{R}}(s+\nu_1)\Gamma_{\mathbb{R}}(s+\nu_2)$, where $\nu_i \in \{0,1\}$. Put $t = 2\pi/\sqrt{Q}$. For the mixed type $\nu_1 \ne \nu_2$, the exact formula is

$$L(\sigma) = \frac{t^\sigma}{\Gamma(\sigma)} \sum_{n\ge 1} a_n \{H_\sigma(tn) + H_{1-\sigma}(tn)\}. \tag{4.31}$$

For the pure type $\nu_1 = \nu_2 = \nu$, it is

$$L(\sigma) = \frac{4(t/2)^{\sigma+\nu}}{\Gamma((\sigma+\nu)/2)^2} \sum_{n\ge 1} a_n n^\nu \{I^{\mathrm{B}}_{\sigma+\nu-1}(tn) + I^{\mathrm{B}}_{\nu-\sigma}(tn)\}. \tag{4.32}$$

In particular the factor n for odd pure parity is essential. Here is a derivation fixing all normalisations. The Mellin transforms of e^{-x} and $K_0(x)$ are $\Gamma(w)$ and $2^{w-2}\Gamma(w/2)^2$, respectively; the second follows by integrating $e^{-x\cosh v}$ first in x, followed by the beta integral and gamma duplication formula. For $\Theta(u) = \sum_n a_n n^\nu k(tnu)$, use $k = e^{-x}, \nu = 0$ in the mixed case, and $k = K_0$ in the pure case. Its Mellin transform is a constant multiple of the completed $L(w-\nu)$ and is invariant under $w \mapsto 1 + 2\nu - w$. Mellin inversion gives $\Theta(u) = u^{-1-2\nu}\Theta(1/u)$. Splitting its Mellin integral at one now gives (4.31) and (4.32). The contour shifts are legitimate because the factor is entire, has polynomial growth in fixed vertical strips after the gamma factors are removed, and Stirling's estimate gives exponential decay on the contours. These properties hold for the nontrivial quadratic Hecke factors specified above.

For either kernel write J for the kernel and V for its envelope. At a positive center c, Cauchy's estimate on $|z-c| = c/2$ gives

$$\frac{|J^{(j)}(c)|}{j!} \le V(c/2)(2/c)^j.$$

Thus, on $|x-c| \le c/32$, truncation after degree D has absolute error

$$R_D(c) = V(c/2)\frac{16^{-D-1}}{1-1/16}. \tag{4.33}$$

For $x \le c$ the error is nonnegative, since complete monotonicity makes every Taylor term about c evaluated to its left nonnegative. For a chosen rational $A \ge 64$, use the exact grid $c_i = A(31/32)^i$ until $c_J < \min(t, 2^{-80})$. Start each kernel at $c_0 = A$ with the enclosure $[0, V(A)]$. The top endpoint A, Taylor degree and arithmetic precision can be increased to meet the prescribed error budget. If $p_j = J^{(j)}(c)/j!$ and k_j are the Taylor coefficients of e^{-x} or $K_0(x)$, respectively, the recurrences are

$$\begin{aligned} c(j+1)p_{j+1} &= -k_j - (j+\mu)p_j && (J = H_\mu),\\ c(j+1)p_{j+1} &= -k_j - (j+\mu+1)p_j && (J = I^{\mathrm{B}}_\mu). \end{aligned} \tag{4.34}$$

In the first case $k_j = (-1)^j e^{-c}/j!$. In the second case $k_0 = K_0(c), k_1 = -K_1(c)$, and the Bessel differential equation gives

$$c^2(j+1)(j+2)k_{j+2} = -c(j+1)(2j+1)k_{j+1} - (j^2-c^2)k_j + 2ck_{j-1} + k_{j-2}, \tag{4.35}$$

where $k_{-1} = k_{-2} = 0$. Evaluate the Taylor polynomial at $-c_i/32$, then add $[0, R_D(c_i)]$, to enclose the kernel at c_{i+1}. Induction proves every grid enclosure. Exponentials are evaluated by the series given above. The defining integrals for K_0 and K_1 can be evaluated on $[0, V]$ by interval quadrature; their positive tails are bounded by

$$\int_V^\infty e^{-x\cosh v}\,dv \le \frac{2e^{-xe^V/2}}{xe^V}, \qquad \int_V^\infty e^{-x\cosh v}\cosh v\,dv \le \frac{2e^{-xe^V/2}}{x}.$$

These bounds follow from $e^v/2 \le \cosh v \le e^v$ for $v \ge 0$. Increase V and refine the finite interval partition until the desired width is obtained.

For completeness the finite sum is obtained from exact integer moments. On a nonempty bin $c_{i+1} < tn \le c_i$, let n_i be the floor of the average of its smallest and largest integer indices and set

$$M_{ij} = \sum_{n \text{ in bin } i} a_n n^\nu (n - n_i)^j, \qquad A_i = \sum_{n \text{ in bin } i} |a_n| n^\nu.$$

Here $\nu = 0$ for the mixed case. If p_{ik} are the kernel coefficients at c_i, translation of the polynomial gives the finite sum

$$\sum_i \sum_{j=0}^{D} M_{ij} t^j \sum_{k=j}^{D} \binom{k}{j} p_{ik} (tn_i - c_i)^{k-j}, \tag{4.36}$$

with error at most $\sum_i A_i R_D(c_i)$ for each of the two kernels. The bin endpoints are certified integer floors of interval-enclosed c_i/t; if the floor is unresolved, precision must be raised. The checks that the bins cover exactly $1, \dots, N$ and that $tN < A$ and $t > c_J$ ensure no term is omitted or assigned twice. The choices $D = 20$ and $N = 8\lfloor\sqrt{Q}\rfloor + 1$ are convenient initial values. Increase the cutoff, top endpoint and degree until the combined interval and tail errors meet the prescribed tolerance.

Finally, for any $\beta > 1$ and $X = t(N + 1)$, the omitted Dirichlet tails satisfy the explicit bounds

$$\sum_{n>N} |a_n| H_\mu(tn) \le \zeta(\beta)^2 (N+1)^\beta V_1(X), \qquad X \ge \beta - 1, \tag{4.37}$$

$$\sum_{n>N} |a_n| n^\nu I_\mu^{\mathrm{B}}(tn) \le \zeta(\beta)^2 (N+1)^{\beta+\nu} V_2(X), \qquad X \ge \beta + \nu - 3/2. \tag{4.38}$$

The required exponents are $\mu = \sigma, 1 - \sigma$ in the first line, and $\mu = \sigma + \nu - 1, \nu - \sigma$ in the second. For $1 < \sigma < 1.01$ their envelope conditions are satisfied. To prove the bounds, note that $x^\beta V_1(x)$ decreases for $x \ge \beta - 1$, and $x^{\beta+\nu} V_2(x)$ decreases for $x \ge \beta + \nu - 3/2$, as direct logarithmic differentiation shows. Multiply by $|a_n| n^{-\beta}$ and sum, using $\sum d_2(n) n^{-\beta} = \zeta(\beta)^2$. One may minimise the upper endpoint over $\beta = 101/100, 21/20, 11/10, 9/8, 6/5, 5/4, 4/3, 3/2, 2$, or simply use $\beta = 2$. Every candidate satisfying the displayed conditions gives a valid upper bound. Multiply these one-sided errors and the interpolation errors by the positive prefactor in the appropriate AFE, and add the errors for its two kernels. This proves the two-sided genus-field enclosures. All coefficients are the exact coefficients defined in Section 3.11.

5. A two-window transfer to planar incidences

This section proves the geometric part of the construction. In particular, the two sides of the incidence graph are allowed to have different local profiles, different counting powers, and different cardinalities. All averages below concern the *absolute number of cross edges*; no symmetry of the endpoint distributions is assumed.

5.1. **Number-field coordinates and the relative-unit mass.** Let K/F be a quadratic extension, with nontrivial automorphism ι, such that K is totally imaginary, F has a real place, and K/F is unramified at every finite place. Write

$$[F : \mathbb{Q}] = d = b + 2c, \qquad \theta = c/d, \qquad \mathrm{rd}(F) = \mathrm{rd}(K) = \lambda, \qquad \ell = \log \lambda.$$

Here (b, c) is the signature of F; in particular $b \ge 1$. The fields constructed in the arithmetic part satisfy these hypotheses. Put

$$L_F(1) = \frac{\operatorname{res}_{s=1} \zeta_K(s)}{\operatorname{res}_{s=1} \zeta_F(s)}.$$

The arithmetic estimate used in this section is

$$\log L_F(1) \le Cd. \tag{5.1}$$

Neither its numerical value nor the existence of the field family is deduced from a successful geometric computation.

For each real place v of F choose an extension $\phi_v : K \to \mathbb{C}$. For each complex place choose extensions ϕ_j^+ and $\phi_j^- = \phi_j^+ \circ \iota$ inducing the same embedding of F. These choices identify $K \otimes_{\mathbb{Q}} \mathbb{R}$ with $\mathbb{C}^d$; write x_∞ for the image of $x \in K$. There are b single, *compact* coordinates and c pairs of coordinates. Define

$$l_j(x) = \tfrac{1}{2}\big(\log|\phi_j^+(x)| - \log|\phi_j^-(x)|\big), \qquad l(x) = (l_j(x))_{j=1}^c.$$

If $N_{K/F}x = x\iota(x) = 1$, then

$$|\phi_v(x)| = 1, \qquad (|\phi_j^+(x)|, |\phi_j^-(x)|) = (e^{l_j(x)}, e^{-l_j(x)}). \tag{5.2}$$

Indeed, complex conjugation over a real place is ι, and the product of the two embeddings over a complex place is the corresponding embedding of $N_{K/F}x$. For $h \in \mathbb{R}^c$, let W_h fix the compact coordinates and send (z_j, w_j) to $(e^{-h_j}z_j, e^{h_j}w_j)$. It has real determinant one.

Let $\mathcal{U}^1 = \{u \in \mathcal{O}_K^\times : N_{K/F}u = 1\}$, let w_K be the number of roots of unity of K, and set

$$h_{\mathrm{cap}} = \big|\ker\big(\mathrm{Cl}(F) \to \mathrm{Cl}(K)\big)\big|, \quad h_{\mathrm{rel}} = h_K/h_F, \quad q_{\mathrm{cl}} = |\mathrm{Cl}(K)/\operatorname{im}\mathrm{Cl}(F)| = h_{\mathrm{cap}}h_{\mathrm{rel}}.$$

Lemma 5.1 (Relative-unit mass)**.** *The image $l(\mathcal{U}^1)$ is a full lattice in $\mathbb{R}^c$, and its kernel has order w_K. If R^1 denotes the covolume of this lattice ($R^1 = 1$ when $c = 0$), then*

$$\frac{w_K}{q_{\mathrm{cl}}R^1} = \frac{2^{d-1}\pi^{b+c}}{\lambda^{d/2}L_F(1)}. \tag{5.3}$$

Proof. We give the normalization details because an omitted power of two would change the final exponent. Write $U_K = \mathcal{O}_K^\times$, $U_F = \mathcal{O}_F^\times$, and $u_N = [U_F : N_{K/F}U_K]$. The norm image contains U_F^2, so this index is finite. The rank form of Dirichlet's unit theorem [22] gives

$$\operatorname{rank} U_K = d - 1, \quad \operatorname{rank} U_F = b + c - 1, \quad \operatorname{rank}\mathcal{U}^1 = c.$$

Under the ordinary logarithmic embedding of K, a norm-one unit has coordinates $(0, (2l_j, -2l_j)_j)$. Thus its l-image is discrete and has rank c. Its kernel consists of algebraic integers all of whose conjugates have modulus one; these are roots of unity. One can see the last assertion directly: the monic integral polynomials of their powers have bounded coefficients and bounded degree, so only finitely many powers are possible. Every root of unity of K has norm one, since its norm is a root of unity of F, hence ±1, and is positive at a real place. This proves the lattice and kernel assertions. The same positivity shows that -1 is not a norm from $K^\times$.

We next compute the capitulation factor. For the involution on U_K, the degree-zero Tate quotient has order u_N, and

$$H^1(\langle\iota\rangle, U_K) = \mathcal{U}^1/\{u/\iota(u) : u \in U_K\}.$$

The Herbrand quotient of this module is 2^{b-1}. For completeness, its rational plus and minus ranks are $b + c - 1$ and c, respectively. A trivial infinite cyclic module has quotient 2, a sign module has quotient $1/2$, and a finite module has quotient 1 (by counting the kernels and images of $1 - \iota$ and $1 + \iota$). Passing to a finite-index sum of plus and minus lattices therefore gives 2^{b-1}. It follows that $|H^1| = u_N/2^{b-1}$.

There is an isomorphism from the capitulation kernel to this H^1: if $\mathfrak{a}\mathcal{O}_K = x\mathcal{O}_K$, send $[\mathfrak{a}]$ to the class of $x/\iota(x)$. It is well defined. If this class is trivial, then $x/u \in F$ for a unit $u \in U_K$, and contraction of ideals shows that $\mathfrak{a}$ is principal. Conversely, Hilbert's theorem 90 [23] writes any norm-one unit as $x/\iota(x)$. Its principal ideal is then ι-invariant. Because no finite prime ramifies, every invariant fractional ideal of K is extended from F: the exponents at a split pair agree, and an inert prime has ramification index one. This proves surjectivity, and hence

$$h_{\mathrm{cap}} = u_N/2^{b-1}. \tag{5.4}$$

Use ordinary regulators, with logarithm $2\log|\cdot|$ at complex places. Delete the coordinate of one real place of F in both unit logarithmic lattices. The norm map keeps each remaining compact

coordinate and adds the two coordinates in each pair. The change $(x_j, y_j) \mapsto (x_j, x_j + y_j)$ has determinant one. Its kernel lattice is $2l(\mathcal{U}^1)$, of covolume $2^c R^1$. Its image has index $u_N/2$ in the logarithmic lattice of U_F, since logarithms kill $\{\pm 1\}$ and -1 is not a norm. Taking covolumes gives

$$R_K/R_F = 2^{c-1} u_N R^1. \tag{5.5}$$

The analytic class-number formula [23], with $w_F = 2$, now gives

$$L_F(1) = \frac{2^{c+1}\pi^{b+c} h_{mrel}}{w_K \lambda^{d/2}} \frac{R_K}{R_F}.$$

Substitution of (5.4) and (5.5), together with $d = b + 2c$, proves (5.3). Only the standard unit theorem, Hilbert 90 and analytic class-number formula are used here; the displayed calculation fixes their conventions for the present construction. □

5.2. **Finite coordinates, periods, and uniform point counts.** Let $\mathcal{P}$ be a finite set of primes of F that split in K. Choose $\mathfrak{P}_\mathfrak{p}$ over each $\mathfrak{p}$, so that $\mathfrak{p}\mathcal{O}_K = \mathfrak{P}_\mathfrak{p}\iota\mathfrak{P}_\mathfrak{p}$. Write $Q_\mathfrak{p}$ for the residue cardinality, $\mathcal{O}_\mathfrak{p}$ for the local valuation ring, and $\varpi_\mathfrak{p}$ for a uniformizer. Identify $K_{\mathfrak{P}_\mathfrak{p}}$ with $F_\mathfrak{p}$. Put

$$\mathbb{A}_\mathcal{P} = \prod_{\mathfrak{p}\in\mathcal{P}} F_\mathfrak{p}^2, \qquad \mathcal{E} = \mathbb{C}^d \times \mathbb{A}_\mathcal{P}.$$

Finite additive Haar measures give $\mathcal{O}_\mathfrak{p}$ mass one; at infinity we use ordinary real Lebesgue measure. Let $\mathcal{O}_\mathcal{P}$ be the ring of elements of K integral away from the chosen split pairs, embedded by

$$\gamma \longmapsto (\gamma_\infty, \gamma_f), \qquad \gamma_f = (\gamma_\mathfrak{p}, (\iota\gamma)_\mathfrak{p})_\mathfrak{p}.$$

For integers $a_\mathfrak{p}, b_\mathfrak{p}$ define a rectangular group

$$\mathcal{C}_f = \prod_\mathfrak{p} (\varpi_\mathfrak{p}^{a_\mathfrak{p}}\mathcal{O}_\mathfrak{p} \times \varpi_\mathfrak{p}^{b_\mathfrak{p}}\mathcal{O}_\mathfrak{p}), \qquad H = d^{-1}\log|\mathcal{C}_f|.$$

Lemma 5.2 (The period lattice)**.** *Set*

$$\mathfrak{a}_{\mathcal{C}_f} = \prod_\mathfrak{p} \mathfrak{P}_\mathfrak{p}^{a_\mathfrak{p}} (\iota\mathfrak{P}_\mathfrak{p})^{b_\mathfrak{p}}, \qquad \Delta_0 = (\lambda/2)^d.$$

The elements of $\mathcal{O}_\mathcal{P}$ with finite part in $\mathcal{C}_f$ are precisely $\mathfrak{a}_{\mathcal{C}_f}$, and $N\mathfrak{a}_{\mathcal{C}_f} = |\mathcal{C}_f|^{-1}$. Every coset of $\mathcal{C}_f$ in $\mathbb{A}_\mathcal{P}$ contains such a finite part. The group

$$\Lambda_h = \{(W_h\gamma_\infty, \gamma_f) : \gamma \in \mathcal{O}_\mathcal{P}\}$$

is a lattice in $\mathcal{E}$ of covolume Δ_0. A fundamental domain is $W_h P_{\mathcal{C}_f} \times \mathcal{C}_f$, where $P_{\mathcal{C}_f}$ is an archimedean fundamental parallelepiped of $\mathfrak{a}_{\mathcal{C}_f}$. Every nonzero vector in the Euclidean dual of $W_h(\mathfrak{a}_{\mathcal{C}_f})_\infty$ satisfies

$$\prod_{w=1}^{d} |\xi_w| \geq \mu^d, \qquad \mu = 2e^{H/2-\ell}. \tag{5.6}$$

Proof. The valuation conditions give the first assertion and the norm formula. For the coset assertion, enclose a given coset in a larger rectangular group $\widetilde{\mathcal{C}}_f$. The natural map

$$\mathfrak{a}_{\widetilde{\mathcal{C}}_f}/\mathfrak{a}_{\mathcal{C}_f} \longrightarrow \widetilde{\mathcal{C}}_f/\mathcal{C}_f$$

is injective by those valuation conditions. Both groups have order $|\widetilde{\mathcal{C}}_f|/|\mathcal{C}_f|$, so it is surjective. First reducing the finite coordinates by this assertion and then reducing the infinite coordinates modulo $\mathfrak{a}_{\mathcal{C}_f}$ proves the fundamental-domain assertion, with uniqueness off the chosen half-open boundaries. The standard ideal covolume formula is

$$\mathrm{covol}(\mathfrak{a}_\infty) = 2^{-d}|\Delta_K|^{1/2} N\mathfrak{a} = \Delta_0 N\mathfrak{a}.$$

Thus the product domain has volume Δ_0. Discreteness follows locally from the discreteness of the ideal lattice. The map W_h has determinant one and hence preserves this covolume.

For the last assertion, use the real inner product $\langle x, y\rangle = \operatorname{Re}\sum_w x_w\overline{y_w}$. If $\mathfrak{a}^\vee$ is the trace-dual ideal, the Euclidean dual of $\mathfrak{a}_\infty$ is $\overline{(2\mathfrak{a}^\vee)_\infty}$, since $\operatorname{Re}\sum_w \phi_w(xy) = \frac{1}{2}\operatorname{Tr}_{K/\mathbb{Q}}(xy)$. The discriminant formula gives $N\mathfrak{a}^\vee = \lambda^{-2d}(N\mathfrak{a})^{-1}$. For nonzero $y \in \mathfrak{a}^\vee$ its principal ideal is contained in $\mathfrak{a}^\vee$, so $|N_{K/\mathbb{Q}}y| \ge N\mathfrak{a}^\vee$. Consequently the product of the moduli of the corresponding dual vector is at least $2^d\lambda^{-d}(N\mathfrak{a})^{-1/2}$. Applying this to $\mathfrak{a}_{\mathcal{C}_f}$ gives (5.6). Finally the dual of $W_h\mathfrak{a}_\infty$ is W_{-h} applied to the old dual, and this map preserves the product of the moduli. □

Lemma 5.3 (Uniform Poisson bound)**.** *Put $\|\xi\|_* = \sum_w |\xi_w|$. Suppose a Euclidean lattice $\Lambda \subset \mathbb{C}^d$ obeys* (5.6)*, and a nonnegative, smooth, integrable function Ψ has bounded derivatives of every order and satisfies*

$$|\widehat{\Psi}(\xi)| \le M_F^d e^{-\sigma\|\xi\|_*}\int \Psi, \qquad M_F \ge 1.$$

If

$$\sigma\mu \ge \log M_F + 2\log 5 + 2, \tag{5.7}$$

then, for every translate y,

$$\sum_{x\in\Lambda+y}\Psi(x) \le \frac{2}{\operatorname{covol}\Lambda}\int\Psi.$$

Consequently, if Ψ_f is nonnegative, constant on $\mathcal{C}_f$-cosets and supported on finitely many such cosets, then for every h and $z_0 \in \mathcal{E}$,

$$\sum_{x\in\Lambda_h+z_0}\Psi_\infty(x_\infty)\Psi_f(x_f) \le \frac{2}{\Delta_0}\int\Psi_\infty\int\Psi_f, \tag{5.8}$$

provided Ψ_∞ satisfies the preceding Fourier bound and (5.7) *uses $H = d^{-1}\log|\mathcal{C}_f|$.*

Proof. Arithmetic–geometric mean gives $\|\xi\|_* \ge d\mu$ for nonzero dual vectors; the same bound separates distinct dual vectors. Disjoint balls of radius $d\mu/2$ show that the shell $jd\mu \le \|\xi\|_* < (j+1)d\mu$ contains at most $(2j+3)^{2d}$ points. Since $(2j+3) \le 5^j$ for $j \ge 1$, condition (5.7) gives

$$M_F^d\sum_{\xi\in\Lambda^*\setminus\{0\}} e^{-\sigma\|\xi\|_*} \le \sum_{j\ge1} e^{-2dj} \le 1.$$

Poisson summation now proves the assertion for a Schwartz function. For the stated hypotheses apply it to $\Psi(x)e^{-\varepsilon|x|^2}$. The Fourier transform is convolved with a Gaussian probability density; the exponential bound is multiplied by $m_\varepsilon = \int e^{\sigma\|u\|_*}G_\varepsilon(u)\,du \to 1$. The zero-frequency term is $\int\Psi e^{-\varepsilon|x|^2}$. Monotone convergence as $\varepsilon\downarrow 0$ gives the claimed pointwise bound for every translate.

To prove (5.8), partition by the finite $\mathcal{C}_f$-coset. By Lemma 5.2, the infinite coordinates in each nonempty part are a translate of $W_h(\mathfrak{a}_{\mathcal{C}_f})_\infty$. Its covolume is $\Delta_0/|\mathcal{C}_f|$, so the first assertion bounds each part by $2|\mathcal{C}_f|\int\Psi_\infty/\Delta_0$. The sum of the constant finite weights is $(\int\Psi_f)/|\mathcal{C}_f|$. □

5.3. **Bilinear shell resources.** For a local field with residue size Q, put $\mathcal{B}_i = \varpi^{-i}\mathcal{O}$, $S_0 = \mathcal{O}$, $S_i = \mathcal{B}_i \setminus \mathcal{B}_{i-1}$ for $i \ge 1$, and $m_0 = 1$, $m_i = (Q-1)Q^{i-1}$. Fix $k \ge 0$ and two finite nonnegative matrices w^L, w^R. The corresponding edge profiles are

$$f_a(x,y) = \sum_{i,j} w^a_{ij}\mathbf{1}_{S_i\times\varpi^{-k}S_j}(x,y), \qquad a \in \{L,R\}.$$

They are invariant under separate multiplication by local units and under the common additive period $\mathcal{O}\times\varpi^{-k}\mathcal{O}$. The latter follows because adding an integral element preserves every shell; finite support also makes the profiles compactly supported.

Lemma 5.4 (Exact bilinear shell formulas)**.** *Define the symmetric matrix D by*

$$D_{ii'} = m_{\min(i,i')}\ (i \neq i'), \quad D_{00} = 0, \quad D_{ii} = im_i - Q^{i-1}\ (i \geq 1),$$

and set

$$G_{(i,j),(i',j')} = (k+1)m_i m_j \mathbf{1}_{i=i',j=j'} + m_j D_{ii'}\mathbf{1}_{j=j'} + m_i D_{jj'}\mathbf{1}_{i=i'}.$$

For $s(n,\omega) = (\varpi^n\omega, \varpi^{-n}\omega^{-1})$, with Haar probability measure on $\mathcal{O}^\times$, the cross overlap is

$$I = \sum_{n\in\mathbb{Z}} \int_{\mathcal{O}^\times} \int f_L(x) f_R(x + s(n,\omega))\, dx\, d\omega = Q^k (w^L)^T G w^R. \tag{5.9}$$

If $I > 0$, the expectations of $-\log f_L$ at the left endpoint and $-\log f_R$ at the right endpoint under this normalized edge law are

$$E_L = -\frac{\sum_{i,j} w^L_{ij}(Gw^R)_{ij} \log w^L_{ij}}{(w^L)^T G w^R}, \qquad E_R = -\frac{\sum_{i,j} w^R_{ij}(Gw^L)_{ij} \log w^R_{ij}}{(w^L)^T G w^R}.$$

Terms with zero weight are omitted. For counting profiles $\psi_a = f_a^{v_a}$, $v_a > 0$, the exact local resources are

$$\begin{aligned} A_a &= \int \psi_a = Q^k \sum_{i,j} m_i m_j (w^a_{ij})^{v_a}, \\ X^{\mathrm{loc}}_a &= \log A_a + v_a E_a, \\ Z^{\mathrm{loc}} &= k \log Q + \log((w^L)^T G w^R) + E_L + E_R. \end{aligned} \tag{5.10}$$

The period contribution is $k \log Q$. It must not be added a second time to either resource in (5.10)*.*

Proof. Two nonarchimedean balls have intersection volume equal to the smaller volume when the displacement belongs to the larger ball, and otherwise have empty intersection. Therefore the cross overlap between the rectangles $\mathcal{B}_i \times \mathcal{B}_{j+k}$ and $\mathcal{B}_{i'} \times \mathcal{B}_{j'+k}$ is

$$Q^{k+\min(i,i')+\min(j,j')}(k + 1 + \max(i,i') + \max(j,j')).$$

Indeed the possible valuations are exactly the integers from $-\max(i,i')$ to $k + \max(j,j')$, and unit factors do not affect this test. Replace each shell indicator by the difference of its two ball indicators, omitting the lower ball when the index is zero. The double difference of $Q^{\min(i,i')}$ is $m_i \mathbf{1}_{i=i'}$, while that of $\max(i,i')Q^{\min(i,i')}$ is $D_{ii'}$; direct subtraction gives the displayed diagonal and off-diagonal values. Applying these differences in both coordinates gives $Q^k G$. Bilinear expansion proves (5.9). This argument does not require either matrix to be symmetric or to factor as a product of two vectors.

The contribution of a fixed left shell to the overlap is $Q^k w^L_{ij}(Gw^R)_{ij}$. Dividing by I gives its marginal probability, and hence E_L. Symmetry of G gives the right marginal and E_R. Finally the shell rectangles are disjoint and have volumes $Q^k m_i m_j$, which gives A_a and the remaining formulas. All G-entries are nonnegative, since $im_i - Q^{i-1} = Q^{i-1}(iQ - i - 1) \geq 0$. □

5.4. **The four energies and the two windows.** At each compact place, each paired complex place, and each selected finite place choose two edge profiles $f_{L,\tau}, f_{R,\tau}$ and two counting profiles $\psi_{L,\tau}, \psi_{R,\tau}$. Here τ indexes blocks, not individual real coordinates. At a compact block the shift is 1; at a pair block it is (e^u, e^{-u}), integrated over $u \in \mathbb{R}$; at a finite block it is $s(n,\omega)$ as above, summed over n and integrated over the unit group. Let $I_\tau > 0$ be the cross integral of $f_{L,\tau} f_{R,\tau}$ with these shifts.

We state precisely the hypotheses on the local profiles. Both edge and counting profiles are invariant under separate coordinate rotations at infinity and separate local-unit multiplications at finite places. Finite profiles have a common rectangular additive period and compact support. A counting profile is positive exactly where its edge profile is positive. Each infinite counting energy $-\log \psi_{a,\tau}$ is bounded below and coercive. The infinite counting profiles are smooth, integrable, have

bounded derivatives, and satisfy Fourier bounds with common constants (M_F, σ), with prefactor M_F on a compact block and M_F^2 on a paired block. Finally the following four local energies have finite means and variances under the *cross-edge* law:

$$-\log \psi_{L,\tau}(x), \quad -\log \psi_{R,\tau}(x+e), \quad -\log f_{L,\tau}(x), \quad -\log f_{R,\tau}(x+e).$$

The analytic part verifies these conditions for the profiles used here. Counting powers may differ between sides and between blocks. No restriction such as $\psi_L = f_L^p$ with one common p is imposed.

Write $f_a = \prod_\tau f_{a,\tau}$, $\psi_a = \prod_\tau \psi_{a,\tau}$, $A_a = \int_{\mathcal{E}} \psi_a$, and $I = \prod_\tau I_\tau$. Let $e(u, n, \omega)$ denote the product shift: its compact coordinates are 1, its paired coordinates are (e^{u_j}, e^{-u_j}), and its finite coordinates are $(\varpi_{\mathfrak{p}}^{n_{\mathfrak{p}}}\omega_{\mathfrak{p}}, \varpi_{\mathfrak{p}}^{-n_{\mathfrak{p}}}\omega_{\mathfrak{p}}^{-1})$. Tonelli's theorem shows that

$$\frac{f_L(x) f_R(x + e(u, n, \omega))}{I}\, dx\, du\, d\omega\, d\#n \tag{5.11}$$

is a probability measure, the product of the local edge laws. Define

$$\begin{aligned} W_a &= -\log \psi_a, \quad V_a = -\log f_a, \\ \mu_L &= \mathbb{E} W_L(x), \quad \mu_R = \mathbb{E} W_R(x+e), \\ \nu_L &= \mathbb{E} V_L(x), \quad \nu_R = \mathbb{E} V_R(x+e). \end{aligned}$$

These are total means, not means divided by d.

Lemma 5.5 (Four-energy truncation)**.** *Suppose each of the four total energies has variance at most Vd. For $\eta > 0$ and $d \geq 8V/\eta^2$, define*

$$\Omega_a = \{x : f_a(x) > 0,\ W_a(x) \leq \mu_a + \eta d\}.$$

Then Ω_L, Ω_R are bounded Borel sets, invariant under the common finite period and the rotations and units specified above. Put

$$\Phi(u, n) = |\Omega_L \cap (\Omega_R - e(u, n, 1))|, \qquad \mathcal{W}(n) = \int_{\mathbb{R}^c} \Phi(u, n)\, du.$$

Only finitely many n have nonzero $\mathcal{W}(n)$, and

$$\sum_n \mathcal{W}(n) \geq \tfrac{1}{2} I \exp(\nu_L + \nu_R - 2\eta d). \tag{5.12}$$

Also $\mathbf{1}_{\Omega_a} \leq e^{\mu_a + \eta d} \psi_a$.

Proof. Coercivity and lower boundedness of the infinite count energies, together with finite support and bounded energies at the finite places, make each sublevel set bounded. Measurability and all invariances follow from those of its factors. Let Y be the event in (5.11) where the two count energies are at most their means plus ηd, and the two edge energies are at least their means minus ηd. Chebyshev and the union bound give $\mathbb{P}(Y) \geq 1 - 4V/(\eta^2 d) \geq 1/2$. Independence between these four energies is neither asserted nor needed. On Y the endpoints lie in the appropriate two windows, and

$$f_L(x) f_R(x+e) \leq e^{-\nu_L - \nu_R + 2\eta d}.$$

Multiplying by this upper bound and integrating the event proves (5.12). The unit integral can be removed from the unweighted overlap: multiplication of the two finite coordinates by (ω, ω^{-1}) preserves both windows and measure. Coordinate rotations similarly remove arbitrary archimedean phases.

For finiteness, the support of the difference of two bounded windows has bounded coordinates. Thus both ϖ^n and ϖ^{-n} must lie in fixed local balls, restricting n to a finite interval at each prime. Likewise e^{u_j} and e^{-u_j} are bounded whenever the overlap is nonempty; hence the relevant u lie in a bounded box. The last inequality is immediate from the definition of W_a. □

For a fixed finite catalog of profiles and $O(d)$ finite places, the variance hypothesis holds with one constant V: each total energy is a sum of independent block contributions, and the maximum of finitely many local variances is finite. This statement remains uniform under arbitrary mixtures, including proportions tending to zero. It does not require each block population to tend to infinity.

5.5. **Class selection, common translations, and projection.**

Lemma 5.6 (Weighted class selection)**.** *For nonnegative finitely supported weights $\mathcal{W}(n)$ with positive sum, there exist a subset $\mathcal{N}'$ of valuation indices, an index $n_0 \in \mathcal{N}'$, and elements $\beta_n \in \mathcal{O}_{\mathcal{P}}$ such that*

$$\sum_{n\in\mathcal{N}'} \mathcal{W}(n) \ge q_{\mathrm{cl}}^{-1} \sum_{n} \mathcal{W}(n), \tag{5.13}$$

$$N_{K/F}\beta_n = 1, \qquad (\beta_n) = \prod_{\mathfrak{p}} \mathfrak{P}_{\mathfrak{p}}^{n_{\mathfrak{p}}-n_{0,\mathfrak{p}}} (\iota\mathfrak{P}_{\mathfrak{p}})^{-n_{\mathfrak{p}}+n_{0,\mathfrak{p}}}.$$

The cosets $\beta_n\mathcal{U}^1$ are pairwise disjoint.

Proof. Map n to the class of $\mathfrak{A}(n) = \prod_{\mathfrak{p}} \mathfrak{P}_{\mathfrak{p}}^{n_{\mathfrak{p}}}$ in $\mathrm{Cl}(K)/\operatorname{im}\mathrm{Cl}(F)$ and choose a fiber of largest total weight. It has positive weight, so choose n_0 in it with positive weight. For n in this fiber there are $x_n \in K^\times$ and a fractional ideal $\mathfrak{a}_n$ of F with $\mathfrak{A}(n)\mathfrak{A}(n_0)^{-1} = x_n\mathfrak{a}_n\mathcal{O}_K$. Put $\beta_n = x_n/\iota(x_n)$. Dividing the ideal identity by its ι-image cancels the extended ideal and gives the stated ideal of β_n, while its relative norm is one. Its valuations vanish away from the selected primes, so it belongs to $\mathcal{O}_{\mathcal{P}}$. Distinct n give distinct ideals. Two norm-one generators of the same ideal differ by a norm-one unit, which proves the coset assertion. □

Lemma 5.7 (Absolute cross-edge average)**.** *Let Ω_L, Ω_R be as in Lemma 5.5. Suppose that each side has a fixed finite cardinality cap for every common translate, uniformly under reciprocal finite-coordinate rescaling of the two windows. There exist finite sets of field elements Γ_L, Γ_R within those respective caps whose cross norm-one edges number at least*

$$\frac{w_K}{q_{\mathrm{cl}} R^1 \Delta_0} \sum_{n} \mathcal{W}(n). \tag{5.14}$$

Proof. Apply Lemma 5.6. Let α_0 act only on the finite coordinates, with components

$$(\varpi_{\mathfrak{p}}^{n_{0,\mathfrak{p}}}, \varpi_{\mathfrak{p}}^{-n_{0,\mathfrak{p}}}).$$

Set $\Omega'_a = \alpha_0^{-1}\Omega_a$, $\mathcal{C}'_f = \alpha_0^{-1}\mathcal{C}_f$. This reciprocal scaling preserves Haar measure, the period measure, and all infinite coordinates. It is not a change of the planar radius. For $n \in \mathcal{N}'$ and $\beta \in \beta_n\mathcal{U}^1$, its finite coordinates after multiplication by α_0 have the moduli prescribed by n and inverse unit parts. Thus rotation and unit invariance give

$$|\Omega'_L \cap (\Omega'_R - (W_h\beta_\infty, \beta_f))| = \Phi(l(\beta) - h, n). \tag{5.15}$$

For fixed h, average a single common translation z_0 uniformly over $W_h P_{\mathcal{C}'_f} \times \mathcal{C}'_f$, of volume Δ_0. Let $\Gamma_a(h, z_0)$ consist of the $\gamma \in \mathcal{O}_{\mathcal{P}}$ whose point $(W_h\gamma_\infty, \gamma_f) + z_0$ lies in Ω'_a. Let $E(h, z_0)$ count pairs (γ, β) with $\gamma \in \Gamma_L(h, z_0)$, $\beta \in \bigcup_{n\in\mathcal{N}'} \beta_n\mathcal{U}^1$, and $\gamma + \beta \in \Gamma_R(h, z_0)$. Tiling by the lattice and Tonelli give the exact conditional average

$$\mathbb{E}_{z_0} E(h, z_0) = \Delta_0^{-1} \sum_{n\in\mathcal{N}'} \sum_{\beta\in\beta_n\mathcal{U}^1} \Phi(l(\beta) - h, n).$$

Now average h uniformly over a fundamental parallelepiped Π^1 of $l(\mathcal{U}^1)$. Each point of $l(\beta_n) + l(\mathcal{U}^1)$ has exactly w_K preimages in $\beta_n\mathcal{U}^1$. The translates of Π^1 tile $\mathbb{R}^c$, and therefore

$$\frac{1}{R^1} \int_{\Pi^1} \sum_{\beta\in\beta_n\mathcal{U}^1} \Phi(l(\beta) - h, n)\, dh = \frac{w_K}{R^1} \int_{\mathbb{R}^c} \Phi(u, n)\, du.$$

This is an unfolding identity, not an equidistribution assertion about the arguments or finite-unit parts of algebraic units. Combining it with (5.13) proves that the average of E is at least (5.14).

The functions are measurable countable sums of indicators. No edge is counted twice: the unit cosets are disjoint, and the ordered pair of field elements determines its difference β. Hence E is bounded by the product of the two uniform cardinality caps. It is integrable, and some common choice of h, z_0 attains at least its average. Both cardinality caps hold for this choice because they hold for every choice. The same translation must be used on the two sides. □

Theorem 5.8 (Two-window geometric transfer)**.** *Assume* (5.1) *and the local-profile hypotheses above. Suppose the common period satisfies* (5.7)*, and the four energy variances are at most Vd. Define*

$$X_L = \log 2 - \ell + \frac{\log A_L + \mu_L}{d}, \qquad X_R = \log 2 - \ell + \frac{\log A_R + \mu_R}{d}, \tag{5.16}$$
$$Z = 2\log 2 - \tfrac{3}{2}\ell - C + (1-\theta)\log\pi + \frac{\log I + \nu_L + \nu_R}{d}.$$

For every $\eta > 0$ with $d \geq 8V/\eta^2$, there exist finite sets of distinct planar points P, Q such that

$$|P| \leq 2e^{d(X_L+\eta)}, \quad |Q| \leq 2e^{d(X_R+\eta)}, \quad \#\{(p,q) \in P \times Q : |p-q| = 1\} \geq \tfrac{1}{4}e^{d(Z-2\eta)}. \tag{5.17}$$

Putting a unit circle at every center in Q gives the same number of point–unit-circle incidences. The sets P and Q may overlap, but each set separately has no repeated point, and the circles are distinct.

Proof. Take the two windows of Lemma 5.5 and perform the common finite rescaling in Lemma 5.7. The rescaled counting majorant is $\psi_a \circ \alpha_0$ in the finite coordinates. Its integral is unchanged, and it is periodic under $\mathcal{C}'_f$, whose measure is unchanged. Its infinite part has Fourier bound $M_F^{b+2c}e^{-\sigma\|\xi\|_*}$ times its mass, by multiplication of the block Fourier bounds. Lemma 5.3 and the pointwise majorization in Lemma 5.5 therefore give, simultaneously for both sides and for every h, z_0,

$$|\Gamma_a(h, z_0)| \leq \frac{2A_a e^{\mu_a + \eta d}}{\Delta_0} = 2e^{d(X_a+\eta)}.$$

These are absolute caps, not bounds holding only for some favorable translate on each side.

Lemmas 5.7, 5.5 and 5.1 now give a common translate with at least

$$\frac{w_K I e^{\nu_L+\nu_R-2\eta d}}{2q_{\mathrm{cl}}R^1\Delta_0} = \frac{2^{2d-2}\pi^{b+c}}{\lambda^{3d/2}L_F(1)} I e^{\nu_L+\nu_R-2\eta d} \geq \tfrac{1}{4}e^{d(Z-2\eta)}$$

cross norm-one edges. This calculation accounts for both factors of two: one from the four-energy event and one from the class/unit mass identity. There is no additional division by two for ordered bipartite incidences.

Choose a real place v_0 of F and project both sets to its compact coordinate. Equivalently, discard their common translated coordinate and take $P = \{\phi_{v_0}(\gamma) : \gamma \in \Gamma_L\}$ and $Q = \{\phi_{v_0}(\gamma) : \gamma \in \Gamma_R\}$. The field embedding is injective, so neither cardinality decreases. For every counted difference β,

$$|\phi_{v_0}(\beta)|^2 = v_0(\beta\iota(\beta)) = 1.$$

Distinct ordered pairs remain distinct ordered planar pairs. Assigning the circle of radius one centered at each element of Q proves the incidence assertion. A coordinate common to P and Q causes no problem: its distance from its own center is zero and was never counted. □

Corollary 5.9 (Strict resource inequalities and exact sizes)**.** *Consider a family with degrees tending to infinity and a fixed finite catalog of local profiles. Suppose the preceding hypotheses hold uniformly, and, for some $\kappa > 0$, prescribed L, r, p satisfy*

$$X_L \leq L - \kappa, \qquad X_R \leq rL - \kappa, \qquad Z \geq pL + \kappa.$$

For all sufficiently large degrees there are exactly $\lceil e^{dL} \rceil$ points and $\lceil e^{drL} \rceil$ unit circles with at least e^{dpL} incidences. The threshold is uniform if L, r, p range over a compact set and the same κ, Fourier margin and variance bound work throughout.

Proof. Choose $0 < \eta < \kappa/4$ in Theorem 5.8. For large d, the strict exponential gaps absorb the constants 2 and $1/4$ in (5.17). Add distinct points and circle centers to reach the prescribed cardinalities; this cannot remove an incidence. The proof uses only the stated uniform constants. If one sets $L = \log s/d$ for an integer s, the same argument gives exactly s points and $\lceil s^r \rceil$ circles. The separate all-size argument must supply an admissible mixture at each such L; a mere unbounded sequence of field degrees does not by itself supply every cardinality. □

Corollary 5.10 (Balanced positive-margin criterion)**.** *Let $0 < \delta < 1$ and $p_0 = 2/(1+\delta)$. At every block take equal left and right edge profiles f, with counting profile f^{p_0}. Suppose the hypotheses of Theorem 5.8 hold uniformly along a field family with $d \to \infty$. Write*

$$J_{\mathbb{R}} = \log I_{\mathbb{R}} - (1+\delta)\log A_{\mathbb{R}}, \quad J_{\mathbb{C}} = \log I_{\mathbb{C}} - (1+\delta)\log A_{\mathbb{C}}, \quad F_{\mathrm{fin}} = \frac{1}{d}\sum_{\mathfrak{p}\in\mathcal{P}} (\log I_{\mathfrak{p}} - (1+\delta)\log A_{\mathfrak{p}}),$$

when the compact blocks share a profile and the paired blocks share a profile. If the margin

$$\mathcal{M} = F_{\mathrm{fin}} - (\tfrac{1}{2} - \delta)\ell - C + (1-\delta)\log 2 + (1-\theta)\log\pi + (1-2\theta)J_{\mathbb{R}} + \theta J_{\mathbb{C}} \tag{5.18}$$

has a positive uniform lower bound, there are finite planar sets U_j such that

$$|U_j| \longrightarrow \infty, \qquad \frac{u(U_j)}{|U_j|^{1+\delta}} \longrightarrow \infty,$$

where $u(U)$ counts unordered unit distances. Different profiles within either archimedean type are handled by replacing the corresponding terms by their normalized sums.

Proof. Every profile is even. For each fixed local displacement e, the map $x \mapsto -x - e$ preserves the symmetric density $f(x)f(x+e)$ and exchanges the two endpoint energies. Thus $\nu_L = \nu_R = \nu$, $\mu_L = \mu_R = p_0\nu$, and the two windows, their rescalings, and their selected field sets coincide. Write $X = X_L = X_R$. Substituting the resource formulas and using $p_0(1+\delta) = 2$ gives

$$Z - (1+\delta)X = \mathcal{M};$$

all energy means cancel exactly. There is no extra finite-period term: the factors $Q_{\mathfrak{p}}^{k_{\mathfrak{p}}}$ are already in both $I_{\mathfrak{p}}$ and $A_{\mathfrak{p}}$. Theorem 5.8, with its now identical projected sets U, gives

$$\frac{u(U)}{|U|^{1+\delta}} \geq 2^{-4-\delta}\exp(d[\mathcal{M} - (3+\delta)\eta]).$$

Here we used $u(U) \geq E/2$ for the counted ordered edges and $|U| \leq 2e^{d(X+\eta)}$. Choose $4\eta < \inf \mathcal{M}$. The right side diverges. Since $u(U) \leq |U|^2/2$ and $\delta < 1$, the cardinalities also tend to infinity. This gives the sequence of cardinalities used in Theorem 1.2. □

Algorithm 5.11 (Coordinates from two lattice windows)**.** Fix the field data and the common choices h, z_0 in the construction.

(i) List the finitely many $\mathcal{C}'_f$-cosets meeting the finite support. For each coset choose the representative γ_r supplied by Lemma 5.2; its fiber is $\gamma_r + \mathfrak{a}_{\mathcal{C}'_f}$.

(ii) Choose a $\mathbb{Z}$-basis of $\mathfrak{a}_{\mathcal{C}'_f}$. The bounded infinite window bounds every coefficient in this basis. List the lattice elements with integer coefficients in these bounds.

(iii) Select the elements satisfying the left window inequalities and, separately, those satisfying the right window inequalities. Project both sets by the embedding in Theorem 5.8. Use the first set as points and the second as centres of circles of radius one.

The selected elements lie in the number field, so the resulting point coordinates and circle centres are real algebraic numbers.

Lemma 5.12 (Effective extraction of coordinates)**.** *Given integers $N, M, E \geq 0$, there is a terminating algorithm which decides whether N distinct points and M distinct unit-circle centers can have at least E incidences, and, if so, returns such a configuration with real algebraic coordinates.*

Algorithm 5.13 (Extraction of an incidence configuration)**.** Fix integers $N, M, E \geq 0$.

(i) If $E > NM$, there is no such configuration. If $E = 0$, choose N distinct rational points and M distinct rational centres.

(ii) For $1 \leq E \leq NM$, list the bipartite graphs with exactly E edges on the labelled sides $\{1, \dots, N\}$ and $\{1, \dots, M\}$. For each graph, introduce real coordinates (x_i, y_i) and (u_j, v_j) and impose

$$(x_i - u_j)^2 + (y_i - v_j)^2 = 1 \quad \text{for each edge } (i, j).$$

Also impose

$$(x_i - x_k)^2 + (y_i - y_k)^2 > 0 \quad (i \neq k),$$
$$(u_j - u_l)^2 + (v_j - v_l)^2 > 0 \quad (j \neq l).$$

(iii) Apply effective quantifier elimination over real closed fields to each of these finitely many systems. If one is nonempty, choose a real algebraic point in it by algebraic sample-point extraction [4]. Its coordinates give the desired points and centres. If all the systems are empty, no required configuration exists.

Proof of Lemma 5.12. Each system in Algorithm 5.13 has integer coefficients. Effective quantifier elimination and algebraic sample-point extraction terminate for every such system [4]; there are only finitely many graphs to consider. Any configuration with at least E incidences contains one of their edge sets. Conversely, a solution realizes all E edges of its graph. Extra incidences and points common to the two sides are allowed. This proves termination and correctness. □

Whenever the construction guarantees a specified integer target E, Lemma 5.12 gives an alternative finite procedure for obtaining algebraic coordinates attaining that target.

6. A certified balanced exponent

This section proves a lower bound along an unbounded sequence of cardinalities. The assertions for every sufficiently large cardinality are separate. All terminating decimals occurring below denote exact rational numbers.

Theorem 6.1. *Put*

$$\alpha_* = \frac{10431722305539701}{10^{16}} = 1.0431722305539701.$$

For every $\varepsilon > 0$ there are arbitrarily large integers N and sets of N points and N distinct unit circles in the plane with at least $N^{\alpha_* - \varepsilon}$ incidences.

The proof uses the fields of Theorem 3.9, the arithmetic bound of Proposition 4.1, and the balanced transfer criterion of Corollary 5.10. The analytic and finite numerical inequalities needed by that criterion are proved here by explicit finite expressions with rigorous remainder bounds.

6.1. **The fixed data and the margin.** Write

$$\ell = \frac{9}{4}\log 2 + \frac{1}{2}\log 3615, \qquad \theta_0 = \frac{17179869183}{34359738368}, \qquad C_* = \frac{4226263259973019243081}{10^{23}}. \tag{6.1}$$

Here ℓ is the logarithmic root discriminant from (3.14), θ_0 is the lower bound for the proportion of complex places of the underlying field, and C_* is an upper bound for the normalized logarithm of the relative zeta value.

Let $m = 96$, $s = 57/50$, and

$$a = \frac{76524029038425969}{10^{22}} = 0.000007652402903842596 9. \tag{6.2}$$

Appendix A.5 specifies the polynomial P by an integer Chebyshev coefficient table. The rational change of basis given there produces its symmetric Bernstein matrix $B = (\beta_{ij})_{0\le i,j\le m}$:

$$P(t,u) = \sum_{i,j=0}^{m} \beta_{ij} b_i^m(t) b_j^m(u), \qquad b_i^m(t) = \binom{m}{i} t^i (1-t)^{m-i}. \tag{6.3}$$

The polynomial has degree at most 96 in each variable. Its Bernstein matrix may have negative entries; positivity follows from subdivision on the square.

For $0 < \delta < 1$ put $p = 2/(1+\delta)$. The complex-place profile is

$$t = (1 + a|z|^2)^{-1}, \quad u = (1 + a|w|^2)^{-1}, \qquad g(z,w) = t^s u^s P(t,u), \qquad (z,w) \in \mathbb{C}^2. \tag{6.4}$$

Its mass, overlap, and local functional are

$$A_{\mathbb{C}}(p) = \int_{\mathbb{C}^2} g(z,w)^p \, dz\, dw, \tag{6.5}$$

$$I_{\mathbb{C}} = \int_{\mathbb{R}} \int_{\mathbb{C}^2} g(z,w) g(z + e^v, w + e^{-v}) \, dz\, dw\, dv, \tag{6.6}$$

$$J_{\mathbb{C}}(\delta) = \log I_{\mathbb{C}} - (1+\delta)\log A_{\mathbb{C}}(p). \tag{6.7}$$

Lebesgue measure is used on each copy of $\mathbb{C} = \mathbb{R}^2$.

At a real place use the radial profile on $\mathbb{C}$

$$f(z) = \exp(-a_{\mathbb{R}}|z|^2/p), \qquad a_{\mathbb{R}} = 2p\delta = \frac{4\delta}{1+\delta}.$$

Completing a square gives

$$A_{\mathbb{R}} = \frac{\pi}{a_{\mathbb{R}}}, \qquad I_{\mathbb{R}} = \frac{\pi p}{2a_{\mathbb{R}}} e^{-a_{\mathbb{R}}/(2p)},$$

and hence

$$J_{\mathbb{R}}(\delta) = \log(p/2) + \delta\left(\log\frac{2p\delta}{\pi} - 1\right), \qquad J_{\mathbb{R}}''(\delta) = \frac{1}{\delta(1+\delta)}. \tag{6.8}$$

The five finite-place choices and their densities per degree are

rational prime	Q	ν_Q	k	weight-matrix size
2	16	1/32	7	64×64
3	9	1/4	9	64×64
5	25	1/4	6	64×64
29	707281	1/4	1	12×12
41	2825761	1/8	1	12×12

At 41, the local ramification index and residue degree are $(1, 4)$. Selecting only one of the two primes of B gives the density 1/8.

The finite recurrence in Appendix A.4 specifies nonnegative rational weight matrices w. These matrices are fixed as δ varies. Here is an explicit formula for their contribution, in the notation of Lemma 5.4. Set

$$\mu_0 = 1, \qquad \mu_i = (Q-1)Q^{i-1} \quad (i \ge 1),$$

and let $D_{ij} = \mu_{\min(i,j)}$ for $i \ne j$, $D_{00} = 0$, and $D_{ii} = i\mu_i - Q^{i-1}$ for $i \ge 1$. On pairs of indices define

$$G_{(i,j),(a,b)} = (k+1)\mu_i\mu_j \mathbf{1}_{i=a}\mathbf{1}_{j=b} + \mu_j D_{ia}\mathbf{1}_{j=b} + \mu_i D_{jb}\mathbf{1}_{i=a}.$$

With zero weights beyond the stated matrix sizes, put

$$A_Q(p) = Q^k \sum_{i,j} \mu_i\mu_j w_{ij}^p, \qquad I_Q = Q^k \sum_{i,j,a,b} w_{ij} G_{(i,j),(a,b)} w_{ab}, \tag{6.9}$$

$$\mathcal{F}(\delta) = \sum_{\text{five types}} \nu_Q\big(\log I_Q - (1+\delta)\log A_Q(p)\big), \qquad H = \sum_{\text{five types}} \nu_Q k \log Q. \tag{6.10}$$

Thus the factor Q^k has already been included; there is no additional term $-\delta H$ in the following margin:

$$\begin{aligned}\Phi(\delta;\theta,C) &= \mathcal{F}(\delta) - (\tfrac{1}{2}-\delta)\ell - C + (1-\delta)\log 2 + (1-\theta)\log\pi \\ &\quad + (1-2\theta)J_{\mathbb{R}}(\delta) + \theta J_{\mathbb{C}}(\delta).\end{aligned} \tag{6.11}$$

The balanced transfer criterion applies when this margin is positive, the profiles are admissible, and the Fourier separation condition holds. The remaining subsections establish these statements with exact data.

6.2. **Exact subdivision, positivity, and a Fourier tube.** We describe the subdivision calculation explicitly. Let $P(t,u) = \sum d_{ij} t^i u^j$, and write $D = (d_{ij})$. The change of basis matrices are

$$T_{ri} = \begin{cases} \binom{m}{i}\binom{m-i}{r-i}(-1)^{r-i}, & r \ge i, \\ 0, & r < i, \end{cases} \qquad V_{ir} = \begin{cases} \binom{i}{r}/\binom{m}{r}, & r \le i, \\ 0, & r > i. \end{cases}$$

Then $D = TBT^{\mathsf{T}}$. For a rational interval $[l, l+h]$, let $U_{rj} = \binom{j}{r} l^{j-r} h^r$ for $j \ge r$, and $U_{rj} = 0$ otherwise. On a rectangle $[l, l+h] \times [l', l'+h']$, the Bernstein coefficient matrix in the two rescaled variables is

$$B_{\square} = (VUT)B(VU'T)^{\mathsf{T}}. \tag{6.12}$$

Every operation in this formula is rational. Since Bernstein basis functions are nonnegative and sum to one, the smallest and largest entries of $B_{\square}$ bound P on that rectangle.

Lemma 6.2. *The polynomial in* (6.3) *is positive on* $[0,1]^2$. *For every*

$$\delta \in [\delta_0, \delta_1] = [0.04317218, 0.04317228] \tag{6.13}$$

the profile (6.4) *satisfies*

$$|\widehat{g^p}(\xi_1,\xi_2)| \le K_{\mathbb{C}} A_{\mathbb{C}} e^{-\sigma_{\mathbb{C}}(|\xi_1|+|\xi_2|)}, \qquad K_{\mathbb{C}} < 4, \quad \sigma_{\mathbb{C}} > 1.$$

The Fourier transform has phase $e^{-2\pi i x\cdot\xi}$.

Proof. Apply (6.12) to the uniform 8×8 grid. The exact minimum L of all cell coefficients and their maximum U satisfy the exact rational inequalities

$$\frac{8305722}{10^7} < L \le P \le U < \frac{264213}{10^3}.$$

Here L and U denote the exact extrema of the rational matrices in (6.12), so this is a finite rational comparison. The positive lower bound applies to the polynomial on the whole square.

Let $\rho = 9/20000$. Complexify a scaled real vector $x \in \mathbb{R}^2$ to $x + iy$, where $|y| \le \rho$, and put $Z = 1 + (x_1 + iy_1)^2 + (x_2 + iy_2)^2$. If $r = |x|$, then

$$\operatorname{Re} Z \ge (1-\rho^2)(1+r^2), \qquad \left|Z^{-1} - (1+r^2)^{-1}\right| \le \frac{2r\rho + \rho^2}{(1-\rho^2)(1+r^2)^2} \le \omega := \frac{(13/20)\rho + \rho^2}{1-\rho^2}.$$

The last step follows from $\max_{r\ge 0} 2r/(1+r^2)^2 = 3\sqrt{3}/8 < 13/20$.

For each real grid cell $\square$, with minimum coefficient $L_\square > 0$, form the rational number

$$q_\square = \frac{1}{L_\square} \sum_{r+k>0} \binom{m}{r}\binom{m}{k} 8^{r+k} \max_{i,j} |(\Delta_1^r \Delta_2^k B_\square)_{ij}| \omega^{r+k}, \qquad q_0 = \max_\square q_\square. \tag{6.14}$$

The finite sum has $0 \le r, k \le m$. The Bernstein derivative identity bounds $|\partial_t^r \partial_u^k P|/(r!k!)$ by the coefficient in this sum without ω^{r+k}. Taylor's identity for a polynomial is exact also at complex arguments. Therefore, at a real point in $\square$, the complexified polynomial differs from its real value by at most $q_\square L_\square \le q_0 P$. Derivatives need only be bounded on the cell containing the real Taylor centre; the complex point need not stay in that cell.

Exact rational evaluation of (6.14) gives

$$q_0 < 0.976994 < 1.$$

Consequently P stays in the open right half-plane throughout the specified tube, and its principal real powers are holomorphic there. Pointwise in that tube,

$$|g(x+iy, x'+iy')^p| \le (1-\rho^2)^{-2sp}(1+q_0)^p g(x,x')^p.$$

Here the variables have first been scaled by $\sqrt{a}$. At the largest p, namely $p_0 = 2/(1+\delta_0)$, outward evaluation gives

$$\log K_{\mathbb{C}} = -2sp_0 \log(1-\rho^2) + p_0 \log(1+q_0) < 1.306742 < \log 4, \qquad \sigma_{\mathbb{C}} = \frac{2\pi\rho}{\sqrt{a}} > 1.022100 > 1. \tag{6.15}$$

Both upper factors decrease as p decreases. Shift each of the two Fourier integration contours in the direction of its frequency by $\rho/\sqrt{a}$. The bound just proved supplies an integrable majorant because $sp > 1$. Cauchy's theorem, followed by passage from finite rectangles to the full plane, gives the asserted Fourier bound. $\square$

The real-place Gaussian has Fourier transform $A_{\mathbb{R}} \exp(-\pi^2|\xi|^2/a_{\mathbb{R}})$. Since $a_{\mathbb{R}} \le 1$ on (6.13), $r - \pi^2 r^2/a_{\mathbb{R}} \le a_{\mathbb{R}}/(4\pi^2) < \log 2$. Thus both profiles are admissible with the common constants

$$M = 2, \qquad \sigma = 1. \tag{6.16}$$

Here the convention in the transfer criterion assigns prefactor M to a real-place block and M^2 to a complex-place block; thus the complex bound $K_{\mathbb{C}} < 4$ has exactly the required normalization. The remaining admissibility conditions also follow directly: the profiles are smooth with bounded derivatives; their energies are bounded below and coercive; their masses are finite because $sp > 1$. Finiteness of the overlap follows from the reduction below. Finally, $\log^2(1+x) \le C_\eta (1+x)^\eta$ and any $0 < \eta < s-1$ reduce the endpoint second moments to the same overlap integrals with slightly smaller positive Student exponents. This verifies the second-moment condition.

6.3. **The overlap and its absolute error.**

Lemma 6.3. *For $A, B > 0$ define*

$$Q_{A,B}(y) = \int_{\mathbb{R}} (1 + ye^{2v})^{-A} (1 + ye^{-2v})^{-B}\, dv.$$

For $0 < y < 1$,

$$Q_{A,B}(y) = \sum_{n=0}^{\infty} \frac{(A)_n (B)_n}{(n!)^2} y^{2n} \left[\log(1/y) + \psi(n+1) - \frac{\psi(A+n) + \psi(B+n)}{2} \right]. \tag{6.17}$$

If $A, B \geq 1$, $ABy^2 < 1$, and $2\log(1/y) \geq H_{\lceil A \rceil - 1} + H_{\lceil B \rceil - 1}$, then the terms are nonnegative and the remainder after $n \leq N$ is at most

$$\log(1/y) \frac{(ABy^2)^{N+1}}{1 - ABy^2}. \tag{6.18}$$

Proof. The substitutions $x = ye^{2v}$ and $t = x/(x + y^2)$ give

$$2Q_{A,B}(y) = \int_0^1 t^{B-1}(1-t)^{A-1}(1 - (1-y^2)t)^{-A}\, dt.$$

One can obtain (6.17) without assigning a numerical value to a special function: differentiation under this beta integral shows that $F(x) = Q_{A,B}(\sqrt{x})$ satisfies

$$x(1-x)F'' + [1 - (A+B+1)x]F' - ABF = 0.$$

The logarithmic Frobenius recurrence at $x = 0$ gives exactly the coefficients in (6.17). To fix the two constants, split the original integral at $v = 0$ and compare the decaying factor on each half with 1. The error is $O(y)$ and the constant term is $-(\psi(A) + \psi(B) + 2\gamma)/2$, using

$$\int_0^1 \frac{t^{A-1} - 1}{1-t}\, dt = -\psi(A) - \gamma.$$

Thus $F(x) = -\frac{1}{2}\log x - (\psi(A) + \psi(B) + 2\gamma)/2 + o(1)$, which fixes the logarithmic and regular solutions and proves the identity. The coefficients grow polynomially, so the series is absolutely convergent for $x < 1$.

For $A \geq 1$, the recurrence and monotonicity of ψ give

$$0 \leq \psi(A+n) - \psi(n+1) \leq H_{\lceil A \rceil - 1}.$$

Also $(A)_n/n! \leq A^n$. These two inequalities bound the bracket between 0 and $\log(1/y)$ and give the geometric remainder (6.18). □

The elementary Laplace calculation

$$\int_{\mathbb{C}} (1 + |z|^2)^{-\alpha} (1 + |z+h|^2)^{-\alpha'}\, dz = \frac{\pi}{\alpha + \alpha' - 1} \mathbb{E}(1 + T(1-T)|h|^2)^{-(\alpha+\alpha'-1)} \tag{6.19}$$

holds for $\alpha, \alpha' > 0$, $\alpha + \alpha' > 1$, with $T \sim \mathrm{Beta}(\alpha, \alpha')$. Indeed, insert the two Gamma integrals, complete the square in z, and put the two Laplace variables equal to RT and $R(1-T)$; the Jacobian is R. Tonelli's theorem applies to the nonnegative integrand.

Set $\eta = 2s - 1$. Applying (6.19) termwise to the finite monomial expansion $D = (d_{ij})$ yields

$$Z := \frac{a^2 I_{\mathbb{C}}}{\pi^2} = \sum_{i,j,k,l} \frac{d_{ij} d_{kl}}{(\eta + i + k)(\eta + j + l)} \mathbb{E} Q_{\eta+i+k, \eta+j+l}(y), \tag{6.20}$$

where the independent variables and their product parameter are

$$T \sim \mathrm{Beta}(s+i, s+k), \qquad T' \sim \mathrm{Beta}(s+j, s+l), \qquad y = a\sqrt{T(1-T)T'(1-T')} \leq a/4.$$

Every expectation is finite: $Q_{A,B}(y)$ is bounded by $\log(1/y)$ plus a constant, and logarithms are integrable under beta laws. Hence the finite signed sum is legitimate.

Here is the fully rational coefficient calculation for the truncated sum. Define $D_x(j) = \sum_{h=0}^{j-1}(x+h)^{-1}$, with $D_x(0) = 0$, and

$$\begin{aligned}
K_n(i,k) &= \frac{(s+i)_n(s+k)_n}{n!(\eta+i+k+n)_{n+1}},\\
r_n(i,k) &= -\eta^{-1} + \frac{H_n}{2} - \frac{D_\eta(i+k+n)}{2} + D_\eta(i+k+2n+1) - \frac{D_s(i+n)+D_s(k+n)}{2},\\
R_n(i,k) &= K_n(i,k)r_n(i,k), \qquad c_s = -2\psi(s) + \psi(2s) + \eta^{-1} - \gamma,\\
G_n &= \sum_{i,j,k,l} d_{ij}d_{kl}K_n(i,k)K_n(j,l),\\
\widetilde{G}_n &= \sum_{i,j,k,l} d_{ij}d_{kl}\{R_n(i,k)K_n(j,l) + K_n(i,k)R_n(j,l)\}.
\end{aligned}$$

The finite expression is

$$Z_N = \sum_{n=0}^{N} a^{2n}\{G_n[\log(1/a) + c_s] + \widetilde{G}_n\}. \tag{6.21}$$

To verify it, multiply the beta densities by $(T(1-T))^n$. Their normalizing ratios are $(s+i)_n(s+k)_n/(2s+i+k)_{2n}$, and the tilted logarithmic moments are $\psi(s+i+n)+\psi(s+k+n)-2\psi(2s+i+k+2n)$. Substitution in (6.17), followed by $\psi(x+j) = \psi(x) + D_x(j)$, gives (6.21).

Put

$$\rho_0 = \frac{(\eta+2m)^2a^2}{16}, \qquad S_D = \sum_{i,j}|d_{ij}|.$$

For the present data, $\rho_0 < 1$, $\log(4/a) > 1/2$, and $\log(4/a) > H_{\lceil\eta+2m\rceil-1}$. The function $y^{2N+2}\log(1/y)$ is increasing on $(0, a/4]$. Consequently (6.18) and (6.20) give the two-sided, signed-coefficient bound

$$|Z - Z_N| \le E_N := \frac{\log(4/a)S_D^2}{\eta^2}\frac{\rho_0^{N+1}}{1-\rho_0}. \tag{6.22}$$

The absolute values in S_D are essential. Discarding positive cross-remainders alone would prove only a lower bound and would not justify the upper root endpoint below.

At $N = 45$, exact rational evaluation of the coefficients and outward evaluation of the displayed logarithm and digamma values give

$$\begin{aligned}
&E_{45} < 1.439\cdot 10^{-39},\\
&Z > 20687.627426656863755481552062778868241323 45,\\
&Z < 20687.627426656863755481552062778868241323 47.
\end{aligned} \tag{6.23}$$

6.4. **Mass integration with a proved remainder.** Polar coordinates in each complex variable give, for $q = sp - 1 > 0$,

$$A_{\mathbb{C}} = \pi^2 a^{-2}q^{-2}\mathcal{A}(p), \qquad \mathcal{A}(p) = \mathbb{E}P(T,U)^p, \quad T,U \text{ independent with density } qt^{q-1}. \tag{6.24}$$

The partition consists of all $i/64$, $0 \le i \le 64$, together with

$$1/512, 1/256, 1/128, \quad 127/128, 255/256, 511/512.$$

There are 70 intervals and 2485 rectangles after symmetry is used. For each rectangle, (6.12) supplies positive rational bounds $l_\square \le P \le u_\square$. Write

$$c_\square = (l_\square + u_\square)/2, \quad r_\square = (u_\square - l_\square)/(u_\square + l_\square) < 1.$$

For $1 < p < 2$ and $N \geq 2$, the ratio of consecutive absolute binomial coefficients beyond N is less than 1. Thus

$$\left| P^p - c_\square^p \sum_{k=0}^{N} \binom{p}{k} (P/c_\square - 1)^k \right| \leq c_\square^p \left| \binom{p}{N+1} \right| \frac{r_\square^{N+1}}{1 - r_\square}. \tag{6.25}$$

Multiply this by the beta probability of the rectangle and, for an off-diagonal rectangle, by 2. Choose the least $N \geq 2$ for which this error is less than $10^{-8}/2485$. At both anchors the required orders are between 3 and 7; the sum of errors is less than $2.607 \cdot 10^{-9}$.

No numerical quadrature is needed for the finite part. Re-expand it as

$$\sum_{j=0}^{N} c_\square^{p-j} H_{N,j} P^j, \qquad H_{N,j} = \sum_{k=j}^{N} \binom{p}{k} \binom{k}{j} (-1)^{k-j}. \tag{6.26}$$

At a rational anchor p every $H_{N,j}$ is rational. Form each P^j exactly over the rationals. Its monomial integral over $[l, r]$ uses only

$$\mu_k(l, r) = \int_l^r q t^{q+k-1}\, dt = \frac{q(r^{q+k} - l^{q+k})}{q+k}. \tag{6.27}$$

If $P^j = \sum c_{ab}^{(j)} t^a u^b$, its rectangle integral is $\sum c_{ab}^{(j)} \mu_a(l, r) \mu_b(l', r')$. This is a finite matrix contraction, enclosed outward. One may clear denominators and use integer multiplication; under Kronecker substitution the base is chosen larger than either resulting variable degree, so no two monomials alias. Sharing the exact powers between the two anchors changes neither expression nor error bound.

Using 5000-bit outward interval arithmetic for these finite expressions and adding both signs of (6.25) gives

δ	$\mathcal{A}(p)$ lower bound	$\mathcal{A}(p)$ upper bound
0.04317218	1822.463794230170096512406	1822.463794235383458823703
0.04317228	1822.461874029453779304104	1822.461874034667151066175

(6.28)

These are bounds on the normalized mass $\mathcal{A}$, not on $A_\mathbb{C}$ itself. From (6.24) and $Z = a^2 I_\mathbb{C}/\pi^2$,

$$J_\mathbb{C}(\delta) = \log Z + 2(1+\delta)\log q - 2\delta \log \pi + 2\delta \log a - (1+\delta)\log \mathcal{A}(p). \tag{6.29}$$

6.5. **Finite evaluation of the balanced profile.** We enclose each exact expression between rational lower and upper bounds. The following steps give the numerical inequalities used below.

Algorithm 6.4 (Checking the balanced profile)**.** (1) Form B from the integer coefficient table in Appendix A.5 and form the finite weights from the recurrence in Appendix A.4. Check symmetry, nonnegativity of the finite weights, and the five place densities in the displayed table.

(2) Compute (6.12) by rational matrix arithmetic on the 8-grid and the stated 70-interval grid. Extract the minimum and maximum entries; all minima must be positive. Compute the finite differences in (6.14) and check (6.15).

(3) Convert B to D exactly, compute (6.21) through 45, and add the absolute error (6.22).

(4) For each anchor choose the cell orders by (6.25), form the exact powers through order 7, and integrate by (6.26)–(6.27).

(5) Evaluate (6.9), (6.10), (6.8), and (6.29) with outward rounding. Use the rational constant C_* from (6.1) to obtain lower and upper bounds for the margin Φ.

Each transcendental evaluation can itself be reduced to explicit series bounds. For example, logarithms follow from $\log x = 2\sum_{j\geq 0} z^{2j+1}/(2j+1)$, $z = (x-1)/(x+1)$, after rational range reduction, with tail bounded by $2|z|^{2N+1}/((2N+1)(1-z^2))$ after $j < N$. Exponentials have the usual Taylor tail bounded by a geometric majorant once the successive term ratio is less than 1; $x^r = e^{r \log x}$.

One may enclose π using $\pi = 16\arctan(1/5) - 4\arctan(1/239)$ and the alternating arctangent series. For positive arguments, the digamma recurrence and the Euler–Maclaurin expansion

$$\psi(X) = \log X - \frac{1}{2X} - \sum_{j=1}^{n-1} \frac{B_{2j}}{2jX^{2j}} + R_n(X), \qquad |R_n(X)| \le \frac{|B_{2n}|}{2nX^{2n}}$$

give arbitrary precision after shifting X by a positive integer. The corresponding harmonic-number expansion encloses γ. The stated remainder bound follows by expanding the positive-integrand Binet integral; thus it is valid for every real $X > 0$. These series and their remainder bounds reduce every numerical comparison to finitely many rational inequalities.

6.6. **Curvature, arithmetic binding, and an exact root bound.** We need more than two sign tests, because the final decimal lies much closer to the root than the anchor spacing. The requisite curvature bound is independent of numerical differentiation.

Lemma 6.5. *For a fixed positive profile h with finite logarithmic second moment, let $L(p) = \log \int h^p$ and $J(\delta) = \log I - (1+\delta)L(2/(1+\delta))$, with I independent of δ. Then*

$$J''(\delta) = -\frac{p^2}{1+\delta} \operatorname{Var}_p(\log h) \le 0,$$

where the probability density is $h^p / \int h^p$.

Proof. Differentiation under the integral gives $L' = \mathbb{E}_p \log h$ and $L'' = \operatorname{Var}_p(\log h)$. Since $p' = -p/(1+\delta)$, one has $J' = -L + pL'$ and the displayed second derivative. The moment bounds already proved justify these derivatives uniformly on the compact anchor interval. □

The finite-place variances use the probabilities proportional to $\mu_i \mu_j w_{ij}^p$, ignoring zero weights. Their contribution K_f to $-\mathcal{F}''$ is enclosed by the finite sums

$$K_f = \sum \frac{\nu_Q p^2}{1+\delta} \left\{ \frac{\sum \mu_i \mu_j w_{ij}^p (\log w_{ij})^2}{\sum \mu_i \mu_j w_{ij}^p} - \left(\frac{\sum \mu_i \mu_j w_{ij}^p \log w_{ij}}{\sum \mu_i \mu_j w_{ij}^p} \right)^2 \right\}.$$

For the complex-place variance, put $X = -s(\log t + \log u) \ge 0$ and $Y = \log(P/L) \ge 0$. Then $\log g - \log L = -X + Y$, and $(-X+Y)^2 \le X^2 + Y^2$. On a mass rectangle replace P^p in this numerator by $u_\square^p$ and Y by $\log(u_\square/L)$, and bound its denominator below by the sum of $l_\square^p$ times the rectangle probabilities. The required one-dimensional logarithmic moments have primitives

$$t^q(\log t - 1/q), \qquad t^q\{(\log t)^2 - 2\log t/q + 2/q^2\},$$

with limiting value 0 at $t = 0$. The mixed term in X^2 factors into the two first moments. Thus this variance bound is again a finite sum with explicit interval endpoints.

Uniformly on (6.13), these calculations give

$$K_f < 2, \qquad K_{\mathbb{C}} := \frac{p^2}{1+\delta} \operatorname{Var}_p(\log g) < 100. \tag{6.30}$$

More precisely, the enclosing interval for K_f is

$$[1.5538279509314, 1.5538298808880].$$

The explicit upper-bounding expression for $K_{\mathbb{C}}$ just described lies in

$$[96.937586626011038, 96.939887401158926];$$

the upper endpoint therefore bounds $K_{\mathbb{C}}$. Writing $\Phi(\delta) = \Phi(\delta; \theta_0, C_*)$, equations (6.8) and Lemma 6.5 imply

$$-52 < \Phi''(\delta) \le \frac{1-2\theta_0}{\delta_0(1+\delta_0)} < 10^{-8}. \tag{6.31}$$

Here are outward rational anchor enclosures using the final C_*:

(6.32)

	lower endpoint	upper endpoint
$\Phi(\delta_0)$	0.00000120512546272971262 0	0.00000120512695478631685 3
$\Phi(\delta_1)$	−0.00000117871394528733953 4	−0.00000117871245322631717 8

The displayed endpoints enclose the finite expressions above and suffice for the root comparison below. Further checks give

$$15.602654682637374 < H < 15.602654682637375, \qquad 2e^{H/2-\ell} > 10, \tag{6.33}$$

and, throughout the anchor interval,

$$-\log\pi - 2J_{\mathbb{R}}(\delta) + J_{\mathbb{C}}(\delta) > 0.6247407 > 0. \tag{6.34}$$

For the last inequality, $J_{\mathbb{C}}$ is concave by Lemma 6.5, so its minimum on the interval is bounded below by the smaller enclosed endpoint value; the elementary Gaussian term is enclosed on the whole interval. It follows that increasing θ from θ_0 can only improve the margin.

The anchor bounds use the fixed coefficient table and finite recurrences in the appendices, together with the arithmetic constant C_* of Proposition 4.1. The constant enters (6.11) through the single term $-C_*$.

Lemma 6.6. *The function Φ has a unique zero δ_* in $[\delta_0, \delta_1]$, and*

$$1.04317223055397014 < 1 + \delta_* < 1.04317223055403600.$$

In particular, $1 + \delta_* > \alpha_* + 10^{-17}$.

Proof. Put $h = \delta_1 - \delta_0 = 10^{-7}$ and let a_i^-, a_i^+ be the two endpoints in row i of (6.32). Their signs give existence. By (6.31),

$$\Phi'(\delta) \le \frac{a_1^+ - a_0^-}{h} + 52h < -20,$$

so the zero is unique. A function with second derivative between $-K$ and k differs from its endpoint chord by an amount between $-kh^2/8$ and $Kh^2/8$. This follows by subtracting the corresponding quadratic with zero endpoint values and using convexity or concavity. The zero of the line through $a_0 > 0 > a_1$ is $\delta_0 + ha_0/(a_0 - a_1)$, increasing in both endpoint values. The inverse-slope bound above therefore gives

$$\delta_0 + \frac{ha_0^-}{a_0^- - a_1^-} - \frac{10^{-8}h^2}{160} \le \delta_* \le \delta_0 + \frac{ha_0^+}{a_0^+ - a_1^+} + \frac{52h^2}{160}.$$

Substituting the four rational endpoints in (6.32) and clearing denominators yields the displayed bounds. More precisely, one plus the lower endpoint is greater than $1.0431722305539701490282 6$, while one plus the upper endpoint is less than $1.0431722305540359896051 7$. □

Proof of Theorem 6.1. Take $\delta = \alpha_* - 1 + 10^{-17}$. Lemma 6.6 and strict decrease give $\Phi(\delta; \theta_0, C_*) > 0$. Theorem 3.9 and Proposition 4.1 supply arbitrarily large admissible field degrees with $\theta \ge \theta_0$ and the required arithmetic bound. Equation (6.34) preserves the positive margin for their actual signatures. The exact finite weights, the admissibility proof, and (6.33) verify every remaining hypothesis of Corollary 5.10. Its concentration and lattice selection argument, with loss $10^{-17}/2$, produces an unbounded sequence with incidence count at least $N^{\alpha_* + 5\cdot 10^{-18}}$. Hence the quotient by N^{α_*} tends to infinity, proving (1.6) as well as the stated ε form. Replacing each point of such a unit-distance configuration by the unit circle centred there gives the required balanced incidence configuration; the factor between ordered distances and unordered edges does not change the exponent. Adding isolated points or circles, if necessary, makes both cardinalities exactly N. □

This proves the sequence construction from explicit profiles and finite expressions with stated remainder bounds.

7. Asymmetric profiles and two independent cardinality bounds

All logarithms in this section are natural. We write $U(n,m)$ for the largest number of unit-distance pairs between sets $P,Q\subset\mathbb{C}$ with $|P|=n$ and $|Q|=m$. Equivalently, Q is the set of centres of m distinct unit circles and P is a set of n distinct points. An element of $P\cap Q$ is permitted. A pair is ordered by its two sides, so that no factor $1/2$ is introduced when passing from this bipartite graph to incidences. In particular, $U(n,m)=U(m,n)$.

This section uses the fields of Theorem 3.9, the relative-unit mass identity 3.16, and the arithmetic input stated in 3.10. The numerical values used below are

$$\begin{aligned}\ell&=\frac{9}{4}\log 2+\frac{1}{2}\log 3615, \qquad C=\frac{4226263259973019243081}{10^{23}},\\ \theta_*&=\frac{17179869183}{34359738368}, \qquad (\sigma,M_F)=(30,4).\end{aligned}\tag{7.1}$$

The degrees are $d=[F:\mathbb{Q}]=2^k$, for every sufficiently large integer k. Preserving the specified finite quotient requires $k\ge 67$; this is not an effective threshold for all the subsequent asymptotic estimates. The integer 96 occurring in the balanced polynomial certificate is a polynomial degree, not a factor in the field degrees.

7.1. **Local masses, overlaps and true endpoint energies.** At a complex-place pair, choose $a,b>0$, $s,t>0$, $s+t>1$, counting powers $p_A,p_B>0$, two symmetric polynomials P_A,P_B with rational coefficients which are strictly positive on $[0,1]^2$, and $\lambda_B\ge 0$. Put

$$T_a(z)=\frac{1}{1+a|z|^2}, \qquad \phi(x,y)=(1-x)(1-y),$$

and define the two edge densities

$$g_A(z,w)=T_a(z)^sT_a(w)^sP_A(T_a(z),T_a(w)),\tag{7.2}$$

$$g_B(z,w)=T_b(z)^tT_b(w)^tP_B(T_b(z),T_b(w))e^{\lambda_B\phi(T_b(z),T_b(w))}.\tag{7.3}$$

The counting densities are $g_A^{p_A}$ and $g_B^{p_B}$. All examples have $sp_A>1$ and $tp_B>1$, so their counting masses are finite. The condition $s+t>1$ gives finiteness of the overlap, also in an open neighbourhood of the chosen exponents. Positivity and boundedness of the polynomial and exponential factors reduce these assertions to the corresponding product-Student assertions proved below.

Let $x=(z,w)$, $e(u)=(e^u,e^{-u})$, and set

$$I=\int_{\mathbb{R}}\int_{\mathbb{C}^2} g_A(x)g_B(x+e(u))\,dx\,du, \qquad A_A=\int_{\mathbb{C}^2}g_A^{p_A}, \qquad A_B=\int_{\mathbb{C}^2}g_B^{p_B}.\tag{7.4}$$

All endpoint expectations in this section refer to the probability measure $I^{-1}g_A(x)g_B(x+e(u))\,dx\,du$. In particular, put

$$E_A=\mathbb{E}\{\log(1+a|z|^2)+\log(1+a|w|^2)\},\tag{7.5}$$

$$E_B=\mathbb{E}\{\log(1+b|z+e^u|^2)+\log(1+b|w+e^{-u}|^2)\},\tag{7.6}$$

$$v_A=\mathbb{E}[-\log g_A(x)]=sE_A-\mathbb{E}\log P_A,\tag{7.7}$$

$$v_B=\mathbb{E}[-\log g_B(x+e(u))]=tE_B-\mathbb{E}\log P_B-\lambda_B\mathbb{E}\phi_B.\tag{7.8}$$

The complex-pair resources are

$$y_A=\log A_A+p_Av_A, \qquad y_B=\log A_B+p_Bv_B, \qquad z=\log I+v_A+v_B.\tag{7.9}$$

Different counting powers and different endpoint energies are essential. For example, replacing E_A,E_B by expectations under the counting mass laws would not give the cardinality bounds in Theorem 5.8. Neither equality of the endpoint energies nor an Euler–Lagrange stationarity assumption is used.

At each real place of F, which gives one complex coordinate of K, use the edge density $f(z) = e^{-\alpha|z|^2}$ on both sides and the counting density $h(z) = e^{-|z|^2/1000}$, where $\alpha = 2/3999$. The edge law has $\mathbb{E}|z|^2 = 1/(2\alpha) + 1/4 = 1000$. Direct Gaussian integration gives

$$y_R = \log(1000\pi) + 1, \qquad z_R = \log(3999\pi/4) + 1. \tag{7.10}$$

Indeed $\int f(z)f(z+1)\,dz = \pi e^{-\alpha/2}/(2\alpha)$ and the sum of the two edge energies is 2000α.

For clarity, the finite-place formula used in the computation is recorded here as well. For a shell pair with residue cardinality Q, common period parameter k, shell measures $m_0 = 1$ and $m_i = Q^i - Q^{i-1}$, let G be the Gram matrix of Lemma 5.4. For nonnegative rational matrices w_A, w_B define

$$J = w_A^{\mathsf{T}} G w_B, \qquad I_f = Q^k J, \tag{7.11}$$

$$e_A = -J^{-1} \sum_{i,j} (w_A)_{ij} (Gw_B)_{ij} \log(w_A)_{ij}, \qquad e_B = -J^{-1} \sum_{i,j} (w_B)_{ij} (Gw_A)_{ij} \log(w_B)_{ij}, \tag{7.12}$$

$$A_{f,A} = Q^k \sum_{i,j} m_i m_j (w_A)_{ij}^{q_A}, \qquad A_{f,B} = Q^k \sum_{i,j} m_i m_j (w_B)_{ij}^{q_B}, \tag{7.13}$$

$$X_{f,A} = \log A_{f,A} + q_A e_A, \qquad X_{f,B} = \log A_{f,B} + q_B e_B, \tag{7.14}$$

$$Z_f = \log I_f + e_A + e_B, \qquad H_f = k \log Q. \tag{7.15}$$

Here matrices are vectorized in the quadratic form, $q_A, q_B > 0$, and $0 \log 0 = 0$. The cross-Gram numerator and the two marginal numerators are computed with exact integer arithmetic after clearing rational denominators. Thus (7.15) is valid also when the matrices do not solve any optimization problem.

The selected finite-place densities per degree are

rational prime	2	3	5	29	41
Q	16	9	25	29^4	41^4
number of selected places/d	1/32	1/4	1/4	1/4	1/8.

(7.16)

At 41 only the places above the designated prime of $\mathbb{Q}(\sqrt{241})$ are selected. Their actual absolute local type is $(1, 4)$, and their density is $1/8$ as displayed in (7.16).

For one complete local-profile specification, let $X_{f,A}, X_{f,B}, Z_f, H$ now denote the sums of the local quantities with the weights in (7.16). A signature-safe triple is

$$X_A = \log 2 - \ell + X_{f,A} + \max_{\vartheta \in \{\theta_*, 1/2\}} \{(1 - 2\vartheta) y_R + \vartheta y_A\}, \tag{7.17}$$

$$X_B = \log 2 - \ell + X_{f,B} + \max_{\vartheta \in \{\theta_*, 1/2\}} \{(1 - 2\vartheta) y_R + \vartheta y_B\}, \tag{7.18}$$

$$Z = 2 \log 2 - \frac{3}{2}\ell - C + Z_f + \min_{\vartheta \in \{\theta_*, 1/2\}} \{(1 - \vartheta) \log \pi + (1 - 2\vartheta) z_R + \vartheta z\}. \tag{7.19}$$

These are affine signature expressions before taking maxima or minima. For every profile in Appendix A $y_A, y_B > 2y_R$ and $z - 2z_R - \log \pi > 0$, both checked with intervals. Consequently the point caps use $y_A/2, y_B/2$, while the edge lower bound uses θ_*. Thus the point bounds and the edge bound use the indicated signature endpoints. The term $-C$ occurs once, and only once, in Z; changing a certified arithmetic upper bound therefore changes Z affinely without requiring another local integral evaluation.

By Theorem 5.8, an admissible specification with these resources gives, for the field of degree d, a common-translate configuration with

$$|P| \le e^{dX_A + o(d)}, \qquad |Q| \le e^{dX_B + o(d)}, \qquad I(P, Q) \ge e^{dZ - o(d)}. \tag{7.20}$$

There is no division of the edge lower bound by a random endpoint count. The geometric Fourier condition is

$$30 \cdot 2e^{H/2-\ell} > \log 4 + 2\log 5 + 2. \tag{7.21}$$

Since H is the normalized logarithm of the *whole* finite period, this condition is imposed on a complete mixture, not separately on every individual finite-place factor.

7.2. **Exact analytic evaluation of the asymmetric profiles.** We use Γ for Euler's gamma function, $\psi = \Gamma'/\Gamma$, $(x)_n = \Gamma(x+n)/\Gamma(x)$, and $H_m = \sum_{j=1}^{m} j^{-1}$ with $H_0 = 0$. We give formulas and explicit remainder bounds, so the numerical assertions below are finite checks rather than appeals to numerical quadrature. The same formulas cover the product, finite-factor, exponential and general-polynomial profiles in all the stated witnesses.

Lemma 7.1 (One-coordinate Feynman integral)**.** *Let $s, t > 0$, $\nu = s + t - 1 > 0$, and $a, b > 0$. For $h \geq 0$ put*

$$L(x) = ax + b(1-x), \quad D(x) = \frac{abx(1-x)}{L(x)}, \quad w(x) = \frac{x^{s-1}(1-x)^{t-1}}{L(x)}.$$

Then

$$\int_{\mathbb{C}} (1 + a|z|^2)^{-s}(1 + b|z+h|^2)^{-t}\, dz = \frac{\pi\Gamma(\nu)}{\Gamma(s)\Gamma(t)} \int_0^1 w(x)(1 + D(x)h^2)^{-\nu}\, dx.$$

Moreover $D(x) \leq D_* := ab/(\sqrt{a} + \sqrt{b})^2$ and $D(x) \geq \min(a, b)x(1-x)$.

Proof. Insert the beta identity for the product of the two negative powers. Completing the square leaves $L(x)|z + b(1-x)h/L(x)|^2 + 1 + abx(1-x)h^2/L(x)$. The planar radial integral is $\pi L(x)^{-1}(1 + D(x)h^2)^{1-s-t}/(s+t-1)$. The beta prefactor gives the displayed formula. Maximizing $abx(1-x)/(ax + b(1-x))$ gives D_*; bounding $L(x)$ by $\max(a, b)$ gives the lower bound. All interchanges are justified by nonnegativity. □

For two possibly different coordinate exponent pairs (s_i, t_i), $i = 1, 2$, put $\nu_i = s_i + t_i - 1 > 0$ and use the corresponding functions w_i, D_i. Integration in the hyperbolic variable reduces to

$$Q_{\nu_1,\nu_2}(y) = \frac{1}{2}\int_0^\infty (1+yx)^{-\nu_1}(1+y/x)^{-\nu_2}\frac{dx}{x} \tag{7.22}$$

$$= \sum_{n=0}^{\infty} \frac{(\nu_1)_n(\nu_2)_n}{(n!)^2} y^{2n}\left[-\log y + \psi(n+1) - \frac{\psi(\nu_1+n) + \psi(\nu_2+n)}{2}\right], \quad 0 < y < 1. \tag{7.23}$$

Here is a derivation of the logarithmic series. Set $r = y^2$ and first replace yx by x in the integral. For $0 < \operatorname{Re} z < \min(\nu_1, \nu_2)$ its Mellin transform in r is, by two beta integrals,

$$\int_0^\infty r^{z-1} Q_{\nu_1,\nu_2}(\sqrt{r})\, dr = \frac{\Gamma(z)^2\Gamma(\nu_1 - z)\Gamma(\nu_2 - z)}{2\Gamma(\nu_1)\Gamma(\nu_2)}.$$

Mellin inversion therefore writes Q as $1/(4\pi i\Gamma(\nu_1)\Gamma(\nu_2))$ times the vertical integral of $\Gamma(z)^2\Gamma(\nu_1 - z)\Gamma(\nu_2 - z)r^{-z}$. Move the line to $\operatorname{Re} z = -N - 1/2$. The crossed double poles are $z = -n$, and

$$\Gamma(z) = \frac{(-1)^n}{n!}\left(\frac{1}{z+n} + \psi(n+1) + O(z+n)\right).$$

The residue, multiplied by the factor $1/2$ from the Mellin transform, is precisely the nth term of (7.23). The horizontal integrals tend to zero by the exponential decay of gamma on vertical lines. On the new vertical line, the reflection formula replaces $\Gamma(z)^2$ by $\pi^2/(\sin^2(\pi z)\Gamma(1-z)^2)$; the remaining gamma ratios grow only polynomially in $N + |\operatorname{Im} z|$. Thus the remaining integral is bounded by a polynomial in N times $r^{N+1/2}$, and tends to zero for $0 < r < 1$. This proves the series. The explicit tail bound below permits its subsequent integration against the beta densities.

Here is an explicit scalar version convenient for finite verification. Define

$$A_{i,n} = \int_0^1 w_i(x) D_i(x)^n \, dx, \qquad B_{i,n} = \int_0^1 w_i(x) D_i(x)^n \log D_i(x) \, dx, \tag{7.24}$$

$$K_{i,n} = \frac{\Gamma(\nu_i + n)}{n!\Gamma(s_i)\Gamma(t_i)} A_{i,n}, \qquad F_{i,n} = \frac{\Gamma(\nu_i + n)}{n!\Gamma(s_i)\Gamma(t_i)} \{[\psi(n+1) - \psi(\nu_i + n)]A_{i,n} - B_{i,n}\}. \tag{7.25}$$

The product-Student overlap is

$$\mathcal{J}(s_1, t_1; s_2, t_2) = \frac{\pi^2}{2} \sum_{n=0}^{\infty} (K_{1,n} F_{2,n} + F_{1,n} K_{2,n}). \tag{7.26}$$

When $a < b$, set $q = 1 - a/b$. For a fixed coordinate pair (s, t),

$$A_n = \frac{a^n}{b} \sum_{j=0}^{\infty} u_j, \tag{7.27}$$

$$u_j = \frac{(n+1)_j}{j!} q^j B(s + n + j, t + n), \tag{7.28}$$

$$[\psi(n+1) - \psi(\nu + n)]A_n - B_n = \frac{a^n}{b} \sum_{j=0}^{\infty} u_j h_j, \tag{7.29}$$

$$h_j = h_\infty + [\psi(s + t + 2n + j) - \psi(n + 1 + j)] + [\psi(s + t + 2n + j) - \psi(s + n + j)], \tag{7.30}$$

$$h_\infty = 2\psi(n+1) - \psi(\nu + n) - \psi(t + n) - \log a. \tag{7.31}$$

These identities follow by expanding $(1 - qx)^{-n-1}$ in the integral for A_n, and differentiating with respect to the real variable n. The ratios satisfy

$$\frac{u_{j+1}}{u_j} = q \frac{n + j + 1}{j + 1} \frac{s + n + j}{s + t + 2n + j}.$$

After retaining $j = 0, \dots, J$, every remaining ratio is at most $q_J = q(1 + n/(J + 2))$. If $q_J < 1$, the omitted unscaled sum $\sum u_j$ is at most $u_{J+1}/(1 - q_J)$. The absolute tail of the unscaled sum $\sum u_j h_j$ is at most

$$\frac{u_{J+1}}{1 - q_J} \left\{ |h_\infty| + (\nu + n)(x_1^{-1} + x_1^{-2}) + (t + n)(x_2^{-1} + x_2^{-2}) \right\}, \quad x_1 = J + n + 2, \quad x_2 = J + s + n + 1. \tag{7.32}$$

Indeed $\psi'(x) = \sum_{j \ge 0} (x + j)^{-2} \le x^{-2} + x^{-1}$, and each of the two digamma differences in h_j decreases as j increases. In particular, this bound remains valid if some h_j is negative. Multiply these two unscaled tail bounds by a^n/b to bound the tails in A_n and its logarithmic combination, and then by $\Gamma(\nu + n)/(n!\Gamma(s)\Gamma(t))$ to bound K_n, F_n. If $b < a$ the two endpoint parameters are exchanged. If $a = b$, $q = 0$ and the inner sum consists exactly of its $j = 0$ term.

An elementary absolute outer tail will be useful when polynomial coefficients have both signs. Assume $D_{i,*} < 1$ for each coordinate, as holds for all the small-scale profiles used here. Put $a_i^- = \min(a_i, b_i)$, $c_i = \max(1, \nu_i)$, and

$$L_i = -\log a_i^- + 2\psi(s_i + t_i) - \psi(s_i) - \psi(t_i) + |\psi(1) - \psi(\nu_i)|, \qquad \rho = c_1 c_2 D_{1,*} D_{2,*} < 1.$$

Then the tail after $n = N$ in (7.26) is in absolute value at most

$$\frac{\pi^2 (L_1 + L_2)}{2\nu_1 \nu_2 a_1^- a_2^-} \frac{\rho^{N+1}}{1 - \rho}. \tag{7.33}$$

To prove this, use $(\nu)_n/n! \le \max(1, \nu)^n$, $D \le D_*$, $L(x) \ge a^-$, and $-\log D(x) \le -\log a^- - \log x - \log(1 - x)$. The last two logarithms have beta expectations $2\psi(s + t) - \psi(s) - \psi(t)$. Also

$|\psi(n+1)-\psi(n+\nu)| \le |\psi(1)-\psi(\nu)|$. These give separate bounds on $K_{i,n}$ and $F_{i,n}$, and a geometric sum proves (7.33). A tighter version replaces the final absolute digamma term by its positive part when the bracket is certified nonnegative. The sufficient check $-\log D_* > H_{\lceil\nu\rceil-1}$ is explicitly performed. This also explains why $0 < \nu < 1$ is permitted; it is not silently excluded.

For polynomial factors, write $P_A(x,y) = \sum A_{ij}x^i y^j$ and $P_B(x,y) = \sum B_{kl}x^k y^l$. Their overlap is the finite signed sum

$$\sum_{i,j,k,l} A_{ij}B_{kl}\mathcal{J}(s+i,t+k;s+j,t+l). \tag{7.34}$$

Finite summation is exact; a common absolute error for a scalar overlap is multiplied by $\|A\|_1\|B\|_1$. No favourable cancellation is used to shrink a remainder. Equivalently, the same finite sum can be grouped as $A^{\mathsf{T}}KB$ and $A^{\mathsf{T}}FB$. All shifted gamma and digamma values may be obtained from $\Gamma(x+1) = x\Gamma(x)$ and $\psi(x+1) = \psi(x) + 1/x$.

For the exponential factor let $E_L(v) = \sum_{j=0}^{L} \lambda_B^j v^j/j!$. On $0 \le v \le 1$,

$$0 \le e^{\lambda_B v} - E_L(v) \le R_L := e^{\lambda_B}\lambda_B^{L+1}/(L+1)!. \tag{7.35}$$

Since the unmodified polynomial densities are positive, if I_0 is their overlap and I_L is the finite polynomial contraction with $E_L(\phi)$, then $I_L \le I \le I_L + R_L I_0$. This inequality is applied also at the perturbed exponent values needed for the next lemma.

Lemma 7.2 (Rigorous endpoint expectations)**.** *Holding the polynomial coefficients, scales and λ_B fixed,*

$$E_A = -\partial_s \log I, \qquad E_B = -\partial_t \log I.$$

For any sufficiently small exact rational $h > 0$, $-\partial_s \log I$ is enclosed by

$$-\frac{\log I(s+h,t) - \log I(s,t)}{h} \le E_A \le -\frac{\log I(s,t) - \log I(s-h,t)}{h},$$

and likewise for E_B.

Proof. The overlap is finite in an open neighbourhood, so differentiation under the integral is justified by exponential moment domination of the logarithmic energies. Its logarithm is convex separately in s,t by Hölder's inequality. The derivative identity and the two secant bounds follow. There is no factor $1/2$: the two exponents are varied separately. □

To evaluate $\mathbb{E}\log P$, suppose exact Bernstein subdivision proves $0 < L \le P \le U$ on the square. If $L = U$, P is constant and no approximation is needed. Otherwise put $\zeta = (\sqrt{U} - \sqrt{L})/(\sqrt{U} + \sqrt{L})$ and $Y = (L + U - 2P)/(U - L)$. Then

$$\log P = 2\log\frac{\sqrt{U}+\sqrt{L}}{2} - 2\sum_{j=1}^{N}\frac{\zeta^j}{j}T_j(Y) + \mathcal{R}_N, \qquad |\mathcal{R}_N| \le \frac{2\zeta^{N+1}}{(N+1)(1-\zeta)}. \tag{7.36}$$

The infinite series is the real logarithm of the factorization into $1 - \zeta e^{i\theta}$ and its conjugate when $Y = \cos\theta$. The error bound follows by summing the geometric tail and using $|T_j(Y)| \le 1$. Multiply the displayed polynomial by the *full* P_A or P_B in (7.34) before division by I. The normalized expectation error is at most the same uniform bound, rather than that bound multiplied by an inverse minimum of P. If the exponential is truncated, add $R_L I_0 \max(|\log L|, |\log U|)/I$. Similarly, multiplying the right density by ϕ evaluates $\mathbb{E}\phi_B$, with exponential-tail error at most $R_L I_0/I$. Equations (7.8) now give the two true edge energies.

7.3. **Counting masses and Fourier admissibility.** For $g = T_a(z)^s T_a(w)^s P(T_a(z), T_a(w)) e^{\lambda\phi}$ and $V = sp > 1$, polar coordinates followed by $x = T_a(z), y = T_a(w)$ give

$$\int_{\mathbb{C}^2} g^p = \frac{\pi^2}{a^2(V-1)^2} \mathbb{E}_{\mu_V \otimes \mu_V}[P(x,y)^p e^{p\lambda(1-x)(1-y)}], \qquad d\mu_V(x) = (V-1)x^{V-2}\,dx. \tag{7.37}$$

Thus all polynomial integrals are explicit moments. On a cell $[a_0, b_0] \times [c_0, d_0]$,

$$\int_{a_0}^{b_0} x^j \, d\mu_V(x) = \frac{V-1}{V-1+j}(b_0^{V-1+j} - a_0^{V-1+j}),$$

with the analogous formula in y. If Bernstein coefficients on the cell give $0 < L_C \le P \le U_C$, put $c_C = (L_C + U_C)/2$ and $r_C = (U_C - L_C)/(U_C + L_C) < 1$. For an integer $N > p$,

$$P^p = c_C^p \sum_{j=0}^{N} \binom{p}{j} (P/c_C - 1)^j + \mathcal{E}_C, \qquad |\mathcal{E}_C| \le c_C^p \frac{|\binom{p}{N+1}| r_C^{N+1}}{1 - r_C}. \tag{7.38}$$

After the first omitted term the ratio of absolute binomial terms is $r_C(j-p)/(j+1) < r_C$, proving the bound. All polynomial powers and coefficient convolutions are rational operations. Integrating the finite polynomial and adding the error multiplied by the cell mass gives a rigorous counting-mass enclosure. For the exponential use (7.35) with $p\lambda$; the product remainder is bounded by $e^{p\lambda}$ times the polynomial error plus its uniform exponential error times $\max P^p$. This proves the finite counting procedure for the general-polynomial sequence witness as well as for the all-size rows.

The special factor $P = \prod_\ell (1 - q_\ell \phi)$, $q_\ell < 1$, has a shorter certificate. For one factor expand

$$(1 - q\phi)^p = \sum_{h \ge 0} \binom{p}{h} (-q)^h \phi^h, \qquad \mathbb{E}\phi^h = \left[\frac{h!}{(V)_h}\right]^2.$$

For $|q| < 1$ a geometric tail is immediate once $h > p$. For arbitrary $q < 1$, including the negative values below -1 in our fourth-degree examples, use instead

$$1 - q\phi = (1 - q/2)\left[1 - \frac{q}{2-q}(2\phi - 1)\right], \qquad \left|\frac{q}{2-q}\right| < 1. \tag{7.39}$$

The same uniform binomial remainder now applies on $|2\phi - 1| \le 1$. Convolving the finite expansions proves the factorized counting certificates without any restriction $q > -1$. Explicitly, if factor i is bounded by $B_i = \max(1, (1-q_i)^p)$ and its approximation has uniform error ε_i, the total product error is at most $\prod_i (B_i + \varepsilon_i) - \prod_i B_i$, by expanding the product of the true factors plus their errors. For logarithmic expectations one may sum the individual log-factor expansions and their uniform remainders.

We next justify the complex contour shifts used for the count densities. For a shift of a real two-vector by ih, with $|h| \le \sigma/(2\pi)$, put $\rho = a\sigma^2/(4\pi^2) < 1$. If $R = a|z|^2$ and $R_c = a\sum (z_j + ih_j)^2$, then

$$|1 + R_c| \ge (1 - \rho)(1 + R), \qquad |T_c - T| \le \frac{2\sqrt{\rho}\, T^{3/2}\sqrt{1-T} + \rho T^2}{1 - \rho}. \tag{7.40}$$

The first follows from $\mathrm{Re}(1 + R_c) = 1 + R - a|h|^2$; the second follows by subtracting the two reciprocals.

For completeness we describe an exact finite Fourier certificate for any polynomial used in (7.3). Express P in a Bernstein basis on each rational cell $C = [a_0, b_0] \times [c_0, d_0]$. Its minimum and maximum Bernstein coefficients bound P there. Finite differences of the coefficients give upper bounds

$$D_{ij,C} \ge \sup_C \frac{|\partial_x^i \partial_y^j P|}{i!\,j!}.$$

This follows by differentiating the Bernstein basis; for bidegree (m,n) the factor is $\binom{m}{i}\binom{n}{j}$ divided by the side lengths to the powers i,j. Let δ_x,δ_y be cellwise upper bounds from (7.40), and set

$$R_C = L_C^{-1}\sum_{i+j>0} D_{ij,C}\delta_x^i\delta_y^j, \tag{7.41}$$

$$D_C = (1-c_0)\delta_x + (1-a_0)\delta_y + \delta_x\delta_y. \tag{7.42}$$

The polynomial Taylor expansion is finite, so these bounds hold even though the displaced arguments are complex. If every $R_C<1$, then $|P(T_c,U_c)-P(T,U)|<P(T,U)$, proving zero-freeness and a holomorphic logarithm throughout the required tube. A sufficient per-coordinate Fourier multiplier bound is

$$p\left[-s\log(1-\rho)+\frac{1}{2}\log(1+R_C)+\frac{\lambda}{2}D_C\right]<\log M_F \quad \text{on every cell.} \tag{7.43}$$

The factor $1/2$ appears because one complex-place pair contains two complex coordinates. Bounds for the product of the two coordinates give M_F^2, not M_F. The compact Gaussian count density satisfies the corresponding bound if $\sigma^2/(4000\pi^2)<\log M_F$.

For the finite-factor profiles the following simpler bound is useful. Write $c=1-q>0$ and

$$1-q\phi(T,U)=\frac{1+R+S+cRS}{(1+R)(1+S)}.$$

Require $1-2\rho+c\rho^2>0$ and $1-c\rho>0$. On writing $R_c=U-\rho, S_c=V-\rho$, the numerator is $A+B(U+V)+cUV$ with $A,B>0$ and $\operatorname{Re}U,\operatorname{Re}V\ge 0$. The real part of $(A+BU)/(B+cU)$ is positive, since its numerator after multiplication by the conjugate denominator is $AB+(Ac+B^2)\operatorname{Re}U+Bc|U|^2$. Hence the numerator cannot vanish. Moreover $|R_c|\le R+\rho$, giving a per-coordinate logarithmic cost at most

$$p\{\log(1+\max(1,c)\rho)-\log(1-\rho)\}.$$

Alternatively let $\delta=((3\sqrt{3}/8)\sqrt{\rho}+\rho)/(1-\rho)$ and $D=2\delta+\delta^2$. Since $|\phi_c-\phi|\le D$, the cost is at most

$$\frac{p}{2}\log\left(1+\frac{|q|D}{\min(1,1-q)}\right).$$

The smaller of the two bounds may be used, retaining the zero-free test. Sum these costs over the factors, add $-sp\log(1-\rho)$ for the base Student density, and add $p\lambda D/2$ for an exponential factor. These inequalities give the stated Fourier bounds.

All these contour bounds dominate the displaced count density by a fixed multiple of its integrable real counterpart, uniformly throughout the tube. For each complex coordinate block $\mathbb{C}\simeq\mathbb{R}^2$ with $\xi_j\neq 0$, rotate its real plane so that the first axis is parallel to ξ_j, and shift this one real variable by $-i\sigma/(2\pi)$. Equivalently the real two-vector is shifted by ih_j, where $h_j=-\sigma\xi_j/(2\pi|\xi_j|)$; it is not necessary or correct to shift both real scalar coordinates by that length. For $\xi_j=0$ no shift is needed. Cauchy's theorem applies to each truncated rectangle because the chosen logarithms are holomorphic in the tube. To remove its vertical sides, integrate their absolute values over the other real variables. The uniform domination bounds them by a fixed constant times an integrable real slice function at the two rectangle endpoints. There is a sequence of endpoints tending to infinity on which the sum of these two slice functions tends to zero. Along that sequence the vertical integrals vanish; dominated convergence handles the horizontal integrals. Fubini's theorem permits the shifts to be made successively in all blocks. The Fourier exponential supplies $e^{-\sigma|\xi_j|}$ in block j. Consequently

$$|\widehat{\Psi}(\xi)|\le M_F^d e^{-\sigma\sum_j|\xi_j|}\int\Psi.$$

The real count densities are smooth and have bounded derivatives, as is also immediate from their positive polynomial factors on the compact T-square and their Student decay. Together with (7.21), these are exactly the uniform lattice-count hypotheses of Theorem 5.8.

7.4. **Frozen witnesses and numerical inequalities.** A complete primitive consists of one archimedean specification and one finite shell specification for each column of (7.16). A phase is a finite list of primitives with nonnegative rational weights summing exactly to one. A phase resource is the weighted sum of the primitive resources (7.19); its H is the same weighted sum of the finite period logarithms. Independent mixtures of archimedean and finite choices are represented by their Cartesian product, so this definition imposes no artificial coupling of those choices.

Appendix A gives the thirteen archimedean choices $A_1, \dots, A_{13}$ and every exact phase proportion. The finite-place choices $F_1, \dots, F_{27}$ are defined by the parameter table and fixed recurrence in Appendix A.3. Together these specify all local functions in the four all-scale rows, the row with $r = 1.0603071$, and the sequence row with $r = 1.0629395$. All entries outside the stated finite matrix sizes are zero.

Algorithm 7.3 (Evaluating an asymmetric phase)**.** Fix one of the phases listed in Appendix A.

(1) For each pair (A_i, F_j) in that phase, take the archimedean parameters from A_i and construct the five finite-place matrices from F_j by Algorithm A.1.

(2) Compute the finite Gram sums in (7.15). The matrices are rational, so the Gram sums are exact. Bound the logarithms and powers by the series described in Section 6.5.

(3) Calculate the archimedean overlap from (7.31)–(7.33). At each truncation add the displayed upper bound for the omitted terms. For an exponential factor, also include the error in (7.35).

(4) Calculate the two endpoint expectations using the secant bounds of Lemma 7.2 and the polynomial-logarithm formula (7.36). Calculate the counting masses by the beta-moment formulas in Section 7.3.

(5) Average the local quantities with the exact phase proportions. Insert ℓ, C, θ_* in (7.19) to obtain intervals for X_A, X_B, Z.

(6) For phase number j, bound the three differences $jA - X_A$, $jrA - X_B$, and $Z - jpA$ from below. Finally check the Fourier inequality (7.21) for the whole phase.

The tables below give the resulting intermediate values and lower bounds for the three differences.

The resource enclosures for the explicitly generated shell matrices at the final arithmetic constant are as follows. Every coordinate in the next table has absolute error less than 10^{-14}. The two triples in each row correspond to the low and high phases.

(7.44)

r	X_A	X_B	Z
1	16.78868158843516031	16.78868158843516311	17.47987129259717701
	33.57736376332853005	33.57736376332851334	34.95974183376258108
1.02	15.35493718282736379	15.66203593439365147	16.10717960172946562
	30.70987476114307411	31.32407226427460953	32.21435864299422203
1.04	13.95715205046627547	14.51543816483972122	14.72393608412320808
	27.91430490980325812	29.03087713854989154	29.44787119439221925
1.06	12.77577752622962604	13.54232418328576750	13.54520419129381841
	25.55155514383284223	27.08464845794506936	27.09040812622815656

For $r = 1.0603071$, each coordinate below has absolute error less than 10^{-10}:

(7.45) $$(12.7596514059164,\ 13.5291490151536,\ 13.5291510087870),$$

(7.46) $$(25.5193034072574,\ 27.0582986256356,\ 27.0583012571363).$$

The corresponding low/high values of H, each with absolute error less than 10^{-8}, are

$$\begin{array}{r|rr} 1 & 10.37313535827153 & 20.54980773330726 \\ 1.02 & 9.13486236690564 & 18.48399167887957 \\ 1.04 & 7.74967445154824 & 17.04332318075847 \\ 1.06 & 6.94792967872249 & 15.60265468263737 \\ 1.0603071 & 6.94792967872249 & 15.60265468263737. \end{array} \tag{7.47}$$

Substituting the exact A, r, p values of Table 7 into the displayed intervals verifies all three strict inequalities. The logarithmic counting and edge exponents are normalized by d.

The one-phase witness has

$$r_s = 1.0629395, \qquad p_s = 1.062939573, \qquad A_s = 17.884565041128177, \tag{7.48}$$

and its resource triple is enclosed by

$$(17.8845647411425,\ 19.0102103225485,\ 19.0102119817337) + [-3.1 \cdot 10^{-9}, 3.1 \cdot 10^{-9}]^3.$$

Its three resource slacks exceed $5.23 \cdot 10^{-8}$, and its Fourier gap exceeds 45.23154.

It uses the general-polynomial profile (7.3), with

$$\begin{aligned} s &= 1.2643704818394643, & t &= 0.3569370831364012, \\ \log a &= -6.960105659675447, & \log b &= -6.8719742698834, \\ \lambda_B &= 0.7958371960711847, & p_A &= \frac{2621439999847412109375}{2093974328671816122368}, \\ & & p_B &= \frac{13107199999237060546875}{325646890540637329817 6}. \end{aligned}$$

The exact polynomial coefficient matrices are listed as profile A_{13} in Appendix A. Its two finite specifications have period vectors $(5, 7, 5, 1, 0)$ and $(5, 7, 4, 1, 0)$, and exact proportions $192596700618323/10^{15}$ and $807403299381677/10^{15}$. The polynomial counting and cell-Fourier bounds above apply directly to these matrices.

Algorithm 7.3 gives all the inequalities in Table 7 directly from the parameter tables and the stated series bounds.

7.5. **Dyadic degree interpolation and all integer sizes.**

Theorem 7.4 (Two phases give all sufficiently large sizes)**.** *Suppose two finite mixtures of admissible primitives from one fixed finite list have a common Fourier pair (σ, M_F) and satisfy, for some $A > 0, r \ge 1, p > 0$,*

$$X_{A,j} < jA, \qquad X_{B,j} < jrA, \qquad Z_j > jpA, \qquad j = 1, 2. \tag{7.49}$$

Suppose also that their whole-period Fourier inequalities are strict. Then, for all sufficiently large integers N, there are exactly N points and $\lceil N^r \rceil$ distinct unit circles with $\Omega(N^p)$ incidences.

Proof. Let $\delta > 0$ be smaller than every slack in (7.49). For $0 \le t \le 1$, assign proportions $1 - t, t$ to the low and high phase profiles, respectively. Since all logarithmic local resources are additive in the numbers of places, the mixed resources satisfy

$$X_A \le (1 + t)A - \delta, \quad X_B \le r(1 + t)A - \delta, \quad Z \ge p(1 + t)A + \delta.$$

The whole-period condition is preserved: it is equivalent to the linear inequality $H > 2\ell + 2\log((\log M_F + 2\log 5 + 2)/(2\sigma))$.

Choose a sufficiently large available dyadic degree d such that $Ad \le \log N < 2Ad$, and put $t = \log N/(Ad) - 1$. Allocate the actual complex-place pairs and the selected finite places according to these proportions, rounding to integers and correcting the last part to keep the total fixed. The number of profile types is fixed, so this changes the sum of local logarithmic quantities by $O(1)$, uniformly in t. There is no need for every part to contain a growing number of places. The four

total energies in Theorem 5.8 are sums of independent variables, each drawn from a fixed finite family with finite second moment. Consequently their variances are bounded by Kd with one K for all t. Joint concentration therefore holds uniformly. The arithmetic estimate is uniform over this field family as well.

Choose the energy-band tolerance small compared with δ, and then choose d large. The uniform $o(d)$ terms and the $O(1)$ allocation errors are absorbed by the strict reserve. Formula (7.20) gives $|P| \le N$, $|Q| \le \lceil N^r \rceil$, and at least cN^p cross unit distances, with $c > 0$ independent of N. Append distinct points and centres far from the finite existing configuration until the prescribed cardinalities are exact. Existing edges remain. Making one unit circle per centre completes the construction. □

A fixed one-phase witness gives a different quantifier. Taking $N_j = \lceil e^{A_s d_j} \rceil$ and $M_j = \lceil N_j^{r_s} \rceil$, its strict slacks give exact sets of those sizes by the same padding argument. Thus (7.48) proves

$$U(N_j, \lceil N_j^{1.0629395} \rceil) = \Omega(N_j^{1.062939573}), \qquad N_j \longrightarrow \infty. \tag{7.50}$$

This one-phase construction gives the second sequence in Theorem 1.2; the two-phase hypothesis of Theorem 7.4 gives the all-scale conclusion.

7.6. **Physical mixing of different ratios and the exact envelope.** The next interpolation is at the level of local functions in the same number field; it would not follow from a list of unrelated incidence lower bounds.

Theorem 7.5 (Uniform interpolation in the size ratio)**.** *Let finitely many two-phase witnesses satisfy Theorem 7.4, with parameters (A_i, r_i, p_i), one common finite primitive list, and common strict Fourier bounds. For any $\beta_i \ge 0$ with $\sum_i \beta_i = 1$ put*

$$A = \left(\sum_i \frac{\beta_i}{A_i} \right)^{-1}, \qquad \lambda_i = \frac{\beta_i / A_i}{\sum_j \beta_j / A_j}, \qquad r = \sum_i \beta_i r_i, \qquad p = \sum_i \beta_i p_i. \tag{7.51}$$

Then $U(N, \lceil N^r \rceil) \ge cN^p$ for all sufficiently large N, with $c > 0$ and the size threshold uniform over the entire mixture simplex. In particular, on a closed interval between two successive upper-concave-hull vertices (r_i, p_i) and (r_{i+1}, p_{i+1}),

$$U(N, M) \ge cN^{a_i} M^{b_i}, \qquad b_i = \frac{p_{i+1} - p_i}{r_{i+1} - r_i}, \quad a_i = p_i - b_i r_i, \quad N^{r_i} \le M \le N^{r_{i+1}}. \tag{7.52}$$

Proof. At both the low and high ends mix witness i with physical proportion λ_i. The identity $\lambda_i A_i = A\beta_i$ gives, for $j = 1, 2$,

$$\sum_i \lambda_i X_{A,i,j} < jA, \qquad \sum_i \lambda_i X_{B,i,j} < jrA, \qquad \sum_i \lambda_i Z_{i,j} > jpA.$$

If δ is the minimum of all original positive raw slacks, each mixed slack is at least $\sum_i \lambda_i \delta = \delta$. The Fourier threshold is linear in H and hence also survives mixing. Next perform the low/high interpolation of Theorem 7.4. The quantities A and the primitive variances remain in fixed compact ranges, the number of possible primitives is finite, and every operation has the same positive reserve. The proof of that theorem is therefore uniform in β, including boundary points where some parts vanish. It applies even if r depends on N.

For two adjacent vertices choose $\beta_{i+1} = (r - r_i)/(r_{i+1} - r_i)$, $\beta_i = 1 - \beta_{i+1}$. Then $p = a_i + b_i r$. Substituting $r = \log M / \log N$ gives $N^p = N^{a_i} M^{b_i}$. Exact ceilings or floors change neither the uniform constant nor the argument, since the strict exponential reserves absorb the bounded cardinality corrections. □

The upper concave hull of the five rows of Table 7 has exactly four vertices, at

$$r = 1, \quad 1.02, \quad 1.04, \quad 1.0603071.$$

The $r = 1.06$ row is below the chord joining the last two vertices: that chord takes the exact value

$$\frac{53825287754393}{50767750000000} > 1.060225408 \quad \text{at } r = 1.06.$$

All these comparisons use rational arithmetic on the terminating witness targets. The three segment pairs (a_i, b_i) are exactly

$$\left(\frac{8126939}{12500000}, \frac{977537}{2500000}\right), \quad \left(\frac{7456363}{10000000}, \frac{1487029}{5000000}\right), \quad \left(\frac{39597848504393}{50767750000000}, \frac{1073769}{4061420}\right). \tag{7.53}$$

Their slopes are strictly decreasing.

To extend past the last ratio, let $(r_*, p_*) = (1.0603071, 1.060307177)$. For $M \geq s^{r_*}$ construct an all-size seed with $x = \lceil M^{1/r_*} \rceil$ left vertices and $\lceil x^{r_*} \rceil$ right vertices. Retain M right vertices of largest degree, then retain s left vertices of largest degree. Each step keeps at least the corresponding fraction of all current edges, since the mean of the largest degrees is at least the overall mean. As $\lceil x^{r_*} \rceil = O(M)$, this yields

$$U(s, M) \geq csM^{(p_*-1)/r_*}, \qquad \frac{p_* - 1}{r_*} = \frac{60307177}{1060307100}. \tag{7.54}$$

This is an actual deletion operation in a planar configuration; it does not tensor or formally multiply incidence graphs.

Corollary 7.6 (A global four-monomial lower bound)**.** *Let $s = \min(n, m)$ and $M = \max(n, m)$. There is an absolute $c > 0$ such that for all positive integers n, m,*

$$U(n, m) \geq c\Big[n + m + \min\Big\{s^{8126939/12500000} M^{977537/2500000}, \tag{7.55}$$

$$s^{7456363/10000000} M^{1487029/5000000}, \tag{7.56}$$

$$s^{39597848504393/50767750000000} M^{1073769/4061420}, \tag{7.57}$$

$$sM^{60307177/1060307100}\Big\}\Big]. \tag{7.58}$$

The nonlinear term is strictly larger in power order than $M = s^r$ for

$$1 \leq r < \frac{1060307100}{999999923} = 1.060307181643\ldots.$$

This is a constructive lower bound, not a matching upper bound for arbitrary unit-circle incidences.

Proof. The first three lines in logarithmic coordinates describe the concave polygonal chain through the four hull vertices. Appending the line $1 + ((p_* - 1)/r_*)r$ preserves strict decrease of slopes. Such a concave polygonal function is the minimum of its extended affine pieces, giving the four monomials by Theorem 7.5 and (7.54). The tail line meets $p = r$ at $r = r_*/(1 + r_* - p_*)$, the stated rational number. One point and arbitrarily many distinct unit circles through it give $U(n, m) \geq M$ after padding. Taking the better of the two constructions and decreasing the constant gives their sum. For bounded s, the nonlinear minimum is at most $sM^{(p_*-1)/r_*} = O(M)$, so a further constant adjustment covers all remaining small sizes. □

7.7. **A fully specified N versus $N^{1.04}$ recipe.** Use the $r = 1.04$ row, with $A = 13.957152859335954$ and $p = 1.054938332$. For an arbitrary sufficiently large integer N choose

$$d = 2^{\lfloor \log_2(\log N/A) \rfloor}, \qquad t = \frac{\log N}{Ad} - 1. \tag{7.59}$$

Assign low and high phases with proportions $1 - t, t$. Their archimedean parameters are as follows; all displayed decimals are exact inputs.

	low phase	high phase
s	1.3325274080489524	1.2128575878994745
t in the profile exponent	0.53875810170025862	0.30695909643943042
$\log a$	−7.2898212608618529	−10.185246038705765
$\log b$	−7.1737906334714499	−10.105937104613458
p_A	1.3678970737119656	1.2281456888253364
p_B	3.1221335967204542	4.5512896290149945

Here the symbol t for an endpoint exponent is distinct from the interpolation proportion in (7.59). Both phases use $\lambda_B = 0$ and $P = \prod_{q\in\mathbf{q}}(1 - q\phi)$. The exact factor vectors are

$$\begin{aligned}
\mathbf{q}_{A,\text{low}} &= (.88036964413350938, -1.2599682068584213,\\
&\qquad .21830494292067165, .13926299502740697),\\
\mathbf{q}_{B,\text{low}} &= (.72924190769296271, -.24792205322667021,\\
&\qquad -.26836424936085079, -.28090971334204096),\\
\mathbf{q}_{A,\text{high}} &= (.85224580275921558, -2.0550667048347195,\\
&\qquad .28466325079340971, .18710522486521236),\\
\mathbf{q}_{B,\text{high}} &= (.63561360296940816, -.30103276870604978,\\
&\qquad -.29459829242192803, -.19660203638447701).
\end{aligned}$$

The three low-phase finite specifications have exact proportions

$$\left(\frac{222724515886789}{500000000000000}, \frac{313697877428409}{10^{15}}, \frac{240853090798013}{10^{15}}\right)$$

and period vectors $(5,7,4,0,0)$, $(5,7,5,0,0)$, $(5,7,4,0,0)$, in the prime order of (7.16). The high-phase proportions are

$$\left(\frac{16116346494661}{250000000000000}, \frac{6607013992789}{50000000000000}, \frac{100424291770697}{125000000000000}\right),$$

with period vector $(8,10,7,1,1)$ for all three specifications. These are the finite choices F_{12}, F_{13}, F_{14} in the low phase and F_{15}, F_{16}, F_{17} in the high phase. Their counting powers and finite recurrences are given in Appendix A.3. The archimedean choices are A_6 and A_7, respectively, with the exact phase mixtures displayed in Appendix A.

Choose the field K/F of degree d; allocate its complex-place pairs and each selected finite-place type with the above proportions, multiplied by $1 - t$ or t. Use the two counting/edge-energy windows of Theorem 5.8 on the *same* S-integer lattice, the same relative-unit parameter and the same translate. A common translate with enough norm-one differences exists by that theorem, while the two point-count caps hold uniformly for every translate. Project both sets to a compact complex coordinate, and take the right-hand images as unit circle centres. Padding gives exactly $(N, \lceil N^{1.04}\rceil)$, with $\Omega(N^{1.054938332})$ incidences.

For example, at the symbolic scales $N_k = \lceil \exp(20.935729289003931\, 2^k)\rceil$ the two phases have proportions $1/2 + o(1)$ and $1/2 + o(1)$. This is only an illustrative subsequence; formula (7.59) covers every sufficiently large N. For an intermediate ratio, such as $r = 1.03$, first apply (7.51) to the 1.02 and 1.04 witnesses, then apply (7.59) to their common A. This gives $p(1.03) = 1.051964274$ and the monomial $N^{0.7456363}M^{0.2974058}$ on the whole closed range $N^{1.02} \le M \le N^{1.04}$.

The logarithmic formulas above determine the allocation from (n, m). The condition $k \ge 67$ gives the required field quotient; the asymptotic threshold absorbs concentration and rounding errors. The field and common-translate construction in Section 5 then specifies the coordinates.

Finally, the balanced degree-96 construction has a direct, weaker asymmetric consequence requiring no new local analysis. If its sets U_j have N_j vertices and at least $N_j^{\alpha-\varepsilon}$ unordered unit pairs, fix $r \ge 1$, put $n_j = \lceil N_j^{1/r}\rceil$, and choose the n_j vertices of largest unit-distance degree as the left side.

Keep all N_j vertices as centres and pad to $m_j = \lceil n_j^r \rceil$. Since the sum of the degrees is twice the number of unordered edges, this gives

$$U(n_j, m_j) \geq \frac{2n_j}{N_j} u(U_j) = \Omega(n_j^{1+r(\alpha-1)-\varepsilon}),$$

after renaming ε. This gives the stated sequence of asymmetric configurations. The all-scale bounds use the independent left and right profiles constructed above.

8. An arbitrarily small twenty-circle configuration

The construction in this section supplies the geometric primitive used in the exact-tail results. A zero in a limiting determinant will only be used to exclude additional contacts. Every contact that is claimed to exist is separately forced by an exact distance identity.

Proposition 8.1. *For every $\eta > 0$ there are twenty distinct centres, of diameter less than η, whose unit circles have exactly sixty triple points and no point on four circles. Every pair of circles meets in two distinct points. The centres admit an order such that the number of triple points among the first h circles, for $0 \leq h \leq 20$, is*

$$(\tau_0, \dots, \tau_{20}) = (0, 0, 0, 1, 2, 4, 8, 11, 14, 17, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60). \tag{8.1}$$

In particular every prefix increment is at most 4.

We identify the plane with $\mathbb{C}$. All subscripts of centres in this appendix start at 0. Opposite centres are always taken relative to the common origin of the construction.

8.1. A nonsingular ten-centre seed.

Let T_0 be the real root of

$$H(T) = T^3 + 54T + 108 = 0, \qquad A_0 = -T_0^2/18 + 2T_0/3 + 4. \tag{8.2}$$

The derivative $H'(T) > 0$ and $H(-2) < 0 < H(-1)$ show that the root is unique and $-2 < T_0 < -1$. The polynomial is irreducible over $\mathbb{Q}$: a rational root of a monic integral cubic would be an integer, and the unique real root lies strictly between consecutive integers. Consequently an element of $\mathbb{Q}[T]/(H)$ vanishes at T_0 if and only if its three reduced rational coefficients vanish.

Initially let A, T vary near A_0, T_0. For a small positive real parameter ϵ, put

$$a = (a_1, a_2, a_3, a_4) = (0, 3, A, T), \qquad v_j = e^{i\epsilon a_j}, \qquad W = \frac{1}{2}\sum_{j=1}^{4} v_j.$$

The first six centres are

$$d_0 = v_1 + v_2 - W, \quad d_1 = v_1 + v_3 - W, \quad d_2 = v_1 + v_4 - W, \quad d_3 = -d_0,\ d_4 = -d_1,\ d_5 = -d_2.$$

Write their first two nonzero coefficients as

$$d_j = i\epsilon b_j + \epsilon^2 c_j + O(\epsilon^3). \tag{8.3}$$

If $B = (3 + A + T)/2$ and $C = -(9 + A^2 + T^2)/4$, then

$$(b_j, c_j) = (a_{j+2} - B, -a_{j+2}^2/2 - C), \quad 0 \leq j \leq 2,$$

and the other three pairs of coefficients are their negatives.

These six circles have eight exact, prescribed triple points. The four points $v_j - W$ are incident to the three pair-sum centres containing v_j. The four points $v_i + v_j + v_k - W$, with i, j, k distinct, are incident to the three pair-sum centres from that triple. In each case the difference between the point and an incident centre is one of the unit vectors v_l.

For $j = 0, 1, 2$, choose the intersection of the two circles with centres $d_j, -d_j$ that tends to e_j, where $(e_0, e_1, e_2) = (1, -1, -1)$. For positive ϵ it has the exact expression

$$q_j = -e_j \operatorname{sign}(b_j) \frac{id_j}{|d_j|} \sqrt{1 - |d_j|^2}. \tag{8.4}$$

The sign is constant in a neighbourhood of the chosen algebraic parameters. Seek a seventh centre $w = \epsilon^2 X + i\epsilon Y$ with $|w - q_j| = 1$ for all three j. Writing $u_j = -c_j/b_j$, the equations after division by ϵ^2 have limits

$$Y^2 - b_j^2 - 2e_j(X + u_j Y) = 0. \tag{8.5}$$

The last two equations determine

$$Y = \frac{b_1^2 - b_2^2}{2(u_1 - u_2)}, \qquad X = \frac{b_1^2 - Y^2}{2} - u_1 Y. \tag{8.6}$$

Let

$$F(A, T) = Y^2 - b_0^2 - 2(X + u_0 Y).$$

Add $d_6 = w, d_7 = -w$, with coefficient pairs $(Y, X), (-Y, -X)$.

We next prescribe three contacts for a ninth centre. If two distinct centres have coefficient pairs $(b_i, c_i), (b_j, c_j)$, their unit-circle intersection branch tending to $e \in \{-1, 1\}$ has expansion

$$p_e(i, j) = e + i\epsilon u + \epsilon^2 v + O(\epsilon^3), \quad u = \frac{b_i + b_j}{2} - e\frac{c_i - c_j}{b_i - b_j}, \quad v = c_i - \frac{e}{2}(u - b_i)^2. \tag{8.7}$$

This follows by expanding the two exact equations $|p - d_i|^2 = |p - d_j|^2 = 1$. Choose

$$r_0 = p_{-1}(0, 6), \qquad r_1 = -p_1(1, 6), \qquad r_2 = p_1(6, 7). \tag{8.8}$$

Denote their coefficient triples by (e_l, u_l, v_l) and put $h_l = u_l^2 + 2e_l v_l$. Seek $z = \epsilon^2 U + i\epsilon V$ with $|z - r_l| = 1$. The divided limiting equations are

$$V^2 - 2u_l V - 2e_l U + h_l = 0 \qquad (0 \le l \le 2). \tag{8.9}$$

Subtract the equation with $l = 0$ from the other two and solve

$$2(e_l - e_0)U + 2(u_l - u_0)V = h_l - h_0 \qquad (l = 1, 2). \tag{8.10}$$

Let $G(A, T)$ be the left-hand side of (8.9) for $l = 0$ after these substitutions. Set $d_8 = z, d_9 = -z$.

All expressions in (8.6)–(8.10) are rational functions of A, T. Substitution of (8.2) and reduction by $T^3 = -54T - 108$ gives the exact identities

$$\begin{aligned} F(A_0, T_0) = G(A_0, T_0) = 0, \\ \det \frac{\partial(F, G)}{\partial(A, T)}(A_0, T_0) = -\frac{5427}{176} + \frac{1053}{352}T_0 - \frac{441}{352}T_0^2 < 0. \end{aligned} \tag{8.11}$$

The two elimination blocks have determinants

$$4(u_2 - u_1) = 6 - 3T_0 > 0, \qquad \det \begin{pmatrix} 2(e_1 - e_0) & 2(u_1 - u_0) \\ 2(e_2 - e_0) & 2(u_2 - u_0) \end{pmatrix} = 40 + \frac{8}{3}T_0 + \frac{4}{9}T_0^2 > 0. \tag{8.12}$$

These equalities, including the rational derivatives in (8.11), follow by rational substitution and reduction as detailed in Section 8.4. No decimal approximation to the root is used in that calculation.

We justify that this limiting calculation gives actual circles. For a pair of centres with distinct b-coefficients, their difference has the form $\epsilon h(\epsilon)$ with $h(0) \neq 0$. Replace the quotient of that difference by its norm by $h(\epsilon)/|h(\epsilon)|$, choosing the sign that gives the desired branch. The ordinary formula for intersections of equal circles then shows that the selected branch is real analytic at $\epsilon = 0$. Conjugating ϵ to $-\epsilon$ conjugates all centres and branches, so the squared-distance equations are even in ϵ. Their constant terms vanish; division by ϵ^2 is therefore analytic. The six resulting equations in X, Y, U, V, A, T have nonsingular Jacobian at zero: elimination of the two blocks in (8.12) leaves exactly the determinant in (8.11). The real analytic implicit-function theorem supplies exact parameter functions for all sufficiently small ϵ. They are even by uniqueness. Thus the coefficient pairs computed above remain the leading pairs of actual centres, with the prescribed six contacts for $w, -w$ and six for $z, -z$.

8.2. **Excluding all unprescribed contacts.** A third circle is incident to the branch in (8.7) only if its coefficient pair lies on the line through the first two transformed pairs

$$(b_j, c_j - eb_j^2/2).$$

Indeed the coefficient of ϵ^2 in $|p - d_j|^2 - 1$ is $2e(v - c_j) + (u - b_j)^2$. Consequently every actual triple with branch sign e satisfies the necessary limiting equation

$$\det \begin{pmatrix} 1 & b_i & c_i - eb_i^2/2 \\ 1 & b_j & c_j - eb_j^2/2 \\ 1 & b_k & c_k - eb_k^2/2 \end{pmatrix} = 0. \tag{8.13}$$

If the determinant is nonzero, the corresponding actual contact is absent for every sufficiently small positive ϵ.

Here are all the limiting triples for the first ten centres. An entry ijk denotes the unordered set $\{i, j, k\}$.

$e = -1$	012	037	045	068	135
	146	234	256	478	679
$e = 1$	015	024	036	123	147
	169	257	345	379	678

(8.14)

The exact rational enumeration in Section 8.4 gives precisely this list, and no collinear group of four transformed pairs. Every listed triple is an exact contact already prescribed: eight are the original unit-vector contacts, six come from $w, -w$, and six from $z, -z$. Since all other signed determinants are nonzero, these are all the actual triple points for sufficiently small ϵ, and there are no fourfold points. The exact check also gives $b_i \neq b_j$ for all distinct indices; in particular all centres are distinct.

8.3. **An exact four-contact identity and five additions.** Let $d, -d, z, -z$ be four centres of sufficiently small norm. Let n_1 be perpendicular to $z - d$, and n_2 perpendicular to $z + d$, with positive real parts and with lengths

$$|n_1|^2 = 1 - |z - d|^2/4, \qquad |n_2|^2 = 1 - |z + d|^2/4.$$

Their selected orientations give $n_1, n_2 \longrightarrow 1$ as $\epsilon \longrightarrow 0$. Put $D = n_1 - n_2$. The circle centred at D contains the following four existing pair-intersection points:

$$\frac{d+z}{2} + n_1, \quad -\frac{d+z}{2} + n_1, \quad \frac{d-z}{2} - n_2, \quad -\frac{d-z}{2} - n_2. \tag{8.15}$$

For example, subtracting the first point from D gives $-(d+z)/2 - n_2$, of norm 1 by perpendicularity. The other three identities are identical applications of the two displayed length relations. The reflected centre $-D$ gives the reflected contacts. If all eight cross-pair intersections of $\{d, -d\}$ and $\{z, -z\}$ are ordinary, adding $D, -D$ therefore forces eight distinct new triple points. This assertion is an exact norm identity, not an inference from a limiting determinant.

For coefficient pairs (b, c) of d and (V, U) of z, set

$$R_1 = \frac{U - c}{V - b}, \qquad R_2 = \frac{U + c}{V + b}.$$

Expansion of the two oriented normals gives the coefficient pair of D:

$$b_D = -R_1 + R_2, \qquad c_D = -\frac{(V-b)^2}{8} - \frac{R_1^2}{2} + \frac{(V+b)^2}{8} + \frac{R_2^2}{2}. \tag{8.16}$$

Thus $D = O(\epsilon)$, and both the construction and its exact algebraic check can be iterated. Perform the following five additions:

parent indices (i, j)	new indices $D, -D$	triple points after addition
$(2, 8)$	$10, 11$	28
$(0, 10)$	$12, 13$	36
$(1, 10)$	$14, 15$	44
$(6, 10)$	$16, 17$	52
$(0, 14)$	$18, 19$	60

At each step the rational check establishes three assertions. The four cross-pairs of parents belong to no old triple, for either sign; all the new b-coefficients remain distinct; and the complete signed inventory consists of the old inventory together with exactly the eight forced triples. It also excludes every fourfold group. The signed list of newly forced triples is particularly simple. If i', j' are the opposite indices and k, k' are the new opposite indices, it is

$$\begin{array}{l|llll} \text{centre } k & (+,i,j,k) & (+,i',j',k) & (-,i,j',k) & (-,i',j,k) \\ \text{centre } k' & (-,i',j',k') & (-,i,j,k') & (+,i',j,k') & (+,i,j',k'). \end{array}$$

This list follows directly from (8.15) and reflection. Section 8.4 excludes every signed triple outside this forced list at each stage.

Proof of Proposition 8.1. Choose a positive ϵ small enough for the seed implicit functions and all five additions. There are finitely many nonzero limiting determinants, so a single sufficiently small choice excludes every unprescribed triple and fourfold point. All sixty prescribed triples hold by the exact identities above. Distinct leading b-coefficients give distinct centres, and all centres have norm $O(\epsilon)$. Taking ϵ smaller makes their diameter less than $\min\{\eta, 2\}$. Hence every pair of unit circles crosses in two distinct points. Deleting centres cannot create a new triple; the prefix counts are therefore obtained by counting which of the sixty forced triples lie in the initial segment. The exact enumeration below gives (8.1). For example the full arrangement has $20 \cdot 19 - 3 \cdot 60 = 200$ ordinary double points, since a triple point uses three of the $20 \cdot 19$ pairwise intersection occurrences. □

8.4. **A short modular verification.** All nonvanishing statements needed for the twenty-centre construction can be checked in a small residue ring. This avoids large rational coefficients while keeping every input in the paper. Set

$$q = 43550027, \qquad t = 10001.$$

Then $H(t) = 103 \cdot 223 \cdot q$. Reduce the coefficient pairs (b_j, c_j) from the preceding formulas modulo q, sending T_0 to t. The ten pairs below determine all twenty: the opposite indices are $(0, 3), (1, 4), (2, 5), (6, 7), (8, 9), (10, 11), (12, 13), (14, 15), (16, 17), (18, 19)$, and opposite coefficient pairs have opposite residues.

(8.17)

j	b_j	c_j	j	b_j	c_j
0	33013073	8157915	10	15899737	14658938
1	10526953	20077072	12	18988445	3023102
2	33023071	1697946	14	18737466	18438607
6	33023077	23427946	16	42220276	15928316
8	22138069	35882666	18	12631489	31859312

The denominator checks are equally short. The two elimination quantities in (8.12) reduce to 43520030 and 5768908. Each is relatively prime to q, as are $2, 3$, and the product B_{20} defined below. The remaining divisors in the coefficient formulas are powers of 2 and 3, individual b-coefficients, and differences or sums of two b-coefficients. An individual b-coefficient is half its difference from its opposite, and a sum is a difference from an opposite coefficient. The formulas therefore reduce unambiguously modulo q.

Define $\mathcal{T}_{10}$ to be the twenty signed triples in (8.14). At each of the five additions, append the eight signed triples displayed after (8.16); let $\mathcal{T}_n$ denote the resulting list at size $n = 10, 12, 14, 16, 18, 20$. For $i < j < k$, put

$$\Delta_e(i,j,k) = (b_j - b_i)(c_k - c_i - e(b_k^2 - b_i^2)/2) - (b_k - b_i)(c_j - c_i - e(b_j^2 - b_i^2)/2).$$

This is the determinant in (8.13). Finally put

$$B_n = \prod_{0 \le i < j < n} (b_j - b_i),$$

$$D_n^e = \prod_{\substack{0 \le i < j < k < n \\ (e,\{i,j,k\}) \notin \mathcal{T}_n}} \Delta_e(i,j,k).$$

Algorithm 8.2 (Finite verification of the signed contacts)**.** (i) Complete the twenty coefficient pairs in (8.17) by assigning opposite residues to opposite indices. Work in $\mathbb{Z}/q\mathbb{Z}$.

(ii) For each $n = 10, 12, 14, 16, 18, 20$, form $\mathcal{T}_n$ from the initial list and the prescribed additions. Multiply the differences $b_j - b_i$ to obtain B_n. For each sign e, multiply $\Delta_e(i,j,k)$ over the triples outside $\mathcal{T}_n$ to obtain D_n^e.

(iii) For each sign, count the triples of $\mathcal{T}_{20}$ containing each pair of indices. Check the four cross-pairs used in each addition against the preceding list. Finally, count the triples contained in each initial segment $\{0, \dots, h-1\}$ to obtain τ_h.

The products in the second step are as follows. The last column is the number of factors in either D_n^e.

(8.18)

n	B_n	$D_n^{-1} = D_n^{+1}$	factors
10	5626121	5376484	110
12	15711630	15793488	206
14	5343041	33688737	346
16	37739968	31118037	538
18	35855152	9075633	790
20	3964820	26158359	1110

Every residue is nonzero. Thus all b_j are distinct, and every signed triple outside the prescribed list has a nonzero determinant in $\mathbb{Q}[T_0]$. Indeed, a zero element would reduce to zero under any homomorphism for which its denominators are invertible. This argument does not require primality of q.

For each sign, no pair of indices belongs to two triples in $\mathcal{T}_{20}$, as is immediate from the initial list and the five lists of additions. A collinear group of four transformed pairs would contain two triples sharing a pair, so the displayed inventory also excludes every fourfold point. The same lists show that each pair of opposite parents used in an addition has all four cross-pairs ordinary before the addition. The exact norm identities then give all the prescribed contacts.

The identities $F = G = 0$, the two elimination determinants, and the Jacobian in (8.11) are obtained by substituting $A_0 = 4 + 2T_0/3 - T_0^2/18$ in (8.6)–(8.10) and reducing $T_0^3 = -54T_0 - 108$. For differentiation one first differentiates these rational expressions in the independent variables A, T, and only then makes the substitution. This gives the exact coefficients in (8.11). Thus the modular products are used only for the nonvanishing inventory; existence of every contact follows from the preceding identities and the implicit-function theorem.

Finally,

$$\tau_h = \#\{(e,g) \in \mathcal{T}_{20} : \max g < h\} \qquad (0 \le h \le 20)$$

gives precisely (8.1), by direct counting in the sixty-entry list. This completes the finite verification using only the displayed formulas and integer tables.

TABLE 6. Exact signed kernel: $\sigma = 1 + 1/n$, $c = A/10^{45}$.

n	A
50000	594647068573300982896594075703454333754041927
46739	-45166657905922747968921708062913488702245586149
43691	12708410204807445262504127880643351463400663051 1
40842	-13729317328298619947543072076223382838779566 4061
33361	21077691202522637452539862373147726405237410427 7
29151	-42695323172719884698182459590605536934610785290 3
25472	46070176428788620994739947181982089155367991550 0
19450	-89918568425296636497102465101343989614464018899 4
16996	13878310442849069099797415734727225415799472363 18
13882	-15023796415865987008206399036351300946105358672 25
11340	16529056113284965531387677077167253396853147494 16
8659	-31665000436559772900190129783791657895352420259 77
7566	36725376316445235887645626936554790820753339953 30
5777	-30434544061074113069907407139094300083268666194 76
4411	50020467550150315475623888361592625174773064588 81
3854	-38186335568851657070411177792013847763677 4
3368	-16200194533448661474438768629192832987384268242585
2943	22697787623332014355304850430278768962645583046129
2248	-29880027710674424722863821958732498498273331827071
1836	49468850188032721736372616322944656457907169703407
1499	-60882223293988663534267525953997994183154575396766
1145	10123359690359591893046419676181982380638652038994 6
935	-17810883546701279576882451771951449809044478718110 7
764	24633364205372950039694240271437708851901538848002 3
624	-28130763342890777026750897335796624221082004032858 8
510	24542985768498900044694296537000058370917169543425 3
389	-24602036260525659234062267722990808630736347569531 8
318	28625104901718107315835484735319340538449012393774 8
260	-26206872342807104072195609401993952541299657409506 4
212	22235174164214326049466326146666830127132030942066 1
173	-23284304980967865085521037253931882953879807166045 9
151	21894034078199719947280907831626251222059467997254 4
132	-13343541919426787322679188990739800992907127464881 3
115	86573284906192695497138732121014432151352869446345
108	-50032097264347342407127955807531256085215217123001
101	82005189514505043992780199180415521840064153964 50

r	p	A	certified slack	Fourier gap
1	1.041169920	16.788682174888262	$7.98\,10^{-8}$	30.91608
1.02	1.048990216	15.354937578311516	$6.91\,10^{-8}$	13.59661
1.04	1.054938332	13.957152859335954	$8.05\,10^{-8}$	3.50131
1.06	1.060225408	12.77577761760308	$5.19\,10^{-8}$	0.16350
1.0603071	1.060307177	12.759652001254373	$7.12\,10^{-8}$	0.16350

TABLE 7. Two-phase witnesses. At phase $j = 1, 2$ the three resource slacks are $jA - X_A$, $jrA - X_B$, and $Z - jpA$. The last column bounds $30 \cdot 2e^{H_{\min}/2-\ell} - \log 4 - 2\log 5 - 2$ from below.

Part 2. Explicit incidence upper bounds

9. The finite quantitative framework

For the upper bound, a unit circle is represented by its center. We use the incidence notation of Section 1 and the unit-distance count from Corollary 1.5. The following fixed parameters establish a finite framework. Sections 18–26 then sharpen the estimates and select the parameters giving Theorem 1.3.

Define the following absolute numbers:

$$\begin{gathered} C_{\mathrm{alg}} = 10^{100000}, \qquad K = 10^{100004}, \qquad k = (2^{1024}K)^{1024}, \\ S_0 = k^{100}, \qquad a = 2^{-2^{200}k^{100}}, \qquad \varepsilon = a^2, \qquad b = a/10^6, \qquad \delta_1 = \frac{a^2}{3}, \qquad \delta_0 = 2^{-2^{10^{11}}}. \end{gathered} \tag{9.1}$$

These are specified real numbers, not parameters to be selected after an existence argument. In particular $0 < \delta_0 < \delta_1 < 1/3$.

Proposition 9.1 (An auxiliary parameter choice)**.** *There is an absolute finite constant C such that, for every positive integer N,*

$$u(N) \le CN^{4/3-\delta_1}, \qquad \mathcal{U}(N,N) \le CN^{4/3-\delta_1}.$$

Both inequalities also hold with the simpler exponent difference δ_0 in place of δ_1.

Proposition 9.2 (Transfer for the auxiliary parameters)**.** *Set $s = \min(n,m)$ and $M = \max(n,m)$. For $\delta = \delta_1$, and also for $\delta = \delta_0$, one has*

$$\mathcal{U}(n,m) \ll M + s^{2/3-2\delta}M^{2/3+\delta}. \tag{9.2}$$

More explicitly, the two regimes may be written as

$$\mathcal{U}(n,m) \ll \begin{cases} s^{2/3-2\delta}M^{2/3+\delta}, & s \le M \le s^2, \\ M, & M \ge s^2. \end{cases}$$

In the second regime the elementary inequality $\mathcal{U}(n,m) \le M + s(s-1)$ holds without an unspecified constant.

The principal earlier input is the qualitative power-saving method of [28]. Its incidence regularization, prediction, height decomposition, and radical obstruction supply the architecture of this argument. We give the needed probability and geometric arguments below, with finite parameter choices. The additional finite algebraic bridge uses critical values and effective elimination; finite polynomial zero-pattern selection replaces the passage to a negligible-height field. The quantitative estimates below make the exponent difference explicit.

Background results. We use the product formula, extension of absolute values, real-closed-field transfer, plane Bézout, polynomial ideals and derivations in characteristic zero, and Borsuk–Ulam for polynomial bisection. The quantitative algebraic inputs are the effective Nullstellensatz of Krick–Pardo–Sombra [19] and Hermann's ideal-membership bound in the form given by Aschenbrenner [3]. Their required numerical forms are stated where used.

Conventions. All logarithms are natural unless a base is displayed. $\mathbb{E}$ and Pr refer to the indicated finite probability laws; zero-probability conditional laws can be assigned arbitrarily. The absolute logarithmic height is denoted by h. A weighted sum over places is written $\sum_v^*$. In complex coordinates $z = x + iy$, $w = x - iy$, the physical real plane has $w = \overline{z}$. The involution on algebraic curves therefore combines conjugation of coefficients with interchange of z, w. The algebraic quadratic form used over extension fields is bilinear, not Hermitian.

Throughout the finite framework, K, k, a, b, ε are fixed numbers from (9.1); only the scale t tends to infinity. Assertions for sufficiently large t have thresholds depending only on these fixed

numbers and on fixed polynomial families. Such thresholds affect the final constant C, which is not numerically optimized or evaluated. They do not alter either displayed exponent difference.

Lemma 9.3. *One has* $\mathcal{U}(n,m) = \mathcal{U}(m,n)$ *and* $2u(N) \le \mathcal{U}(N,N) \le 2u(2N)$.

Proof. Replace each circle by its center. An incidence becomes an ordered cross-pair at unit distance. Swapping the two vertex sets proves the first identity. Taking the same N-point set on both sides counts each unordered unit pair twice, proving the lower bound. For two arbitrary N-point sets, their union has at most $2N$ points and each unordered unit pair gives at most two cross-incidences. This proves the upper bound, also when the two sets overlap. □

10. A quantitative incidence piece and a prediction lemma

This section supplies the finite probabilistic input to the argument. Its structure follows the incidence and information reductions in [28]; all parameter estimates used here are proved below with fixed, rather than slowly varying, exponents. All logarithms in the paper are natural unless a base is displayed. An assertion for sufficiently large t in this section has a threshold depending only on the fixed parameter a.

10.1. **Elementary incidence estimates and a cutting.**

Lemma 10.1 (Algebraic replacement)**.** *Let* $x_1, \dots, x_N$ *be distinct points of* $\mathbb{R}^2$*, and let* F *be any specified collection of pairs at distance one. For any prescribed neighbourhoods of the* x_i*, there are distinct points* x_i' *with real algebraic coordinates in these neighbourhoods such that* $|x_i' - x_j'| = 1$ *for every* $(i,j) \in F$*.*

Proof. Choose rational open rectangles containing the original points and contained in their prescribed neighbourhoods. For $2N$ real variables impose membership in these rectangles, the equations

$$(X_{i,1} - X_{j,1})^2 + (X_{i,2} - X_{j,2})^2 = 1 \quad ((i,j) \in F),$$

and the inequalities

$$(X_{i,1} - X_{j,1})^2 + (X_{i,2} - X_{j,2})^2 > 0 \quad (i \ne j).$$

This finite system over $\mathbb{Q}$ has a real solution. The transfer theorem for real closed fields says that an existential formula over $\mathbb{Q}$ which holds in $\mathbb{R}$ holds in the real closure of $\mathbb{Q}$. Applying that theorem gives the required solution. Additional unit pairs cause no difficulty, because the graph F may be retained as a subgraph. □

Lemma 10.2 (A point–unit-circle bound)**.** *For* m *distinct points and* l *distinct unit circles,*

$$I \le 16m^{2/3}l^{2/3} + 16m + l.$$

Only the existence of an absolute constant in this inequality will be needed later.

Proof. We first recall a short proof of the crossing estimate needed here. For a simple graph with v vertices and $e \ge 4v$ edges, take a drawing with the minimum number c of crossings. Such a drawing can be chosen with no crossings of adjacent edges: a crossing of two edges with a common endpoint is removed by interchanging their initial portions and then separating the coincident portions locally. It can also be chosen with no three edge interiors meeting at one point. Removing one edge at each crossing leaves a planar simple graph, so $c \ge e - 3v$ by the planar Euler bound. Independently retain each vertex with probability $p = 4v/e$. Apply the same planar bound to the induced drawing and take expectations. The four endpoints of every crossing are distinct, whence

$$p^4 c \ge p^2 e - 3pv, \qquad c \ge \frac{e^3}{64v^2}.$$

Discard circles incident to at most one of the specified points; their total contribution is at most l. On every remaining circle draw the cyclic arcs between successive incident points. A circle with two

incident points contributes its two complementary arcs. The resulting multigraph has m vertices and E edges, where E equals the number of incidences on the retained circles. A given pair of points lies on at most two unit circles, and each of these contributes at most two arcs with that pair of endpoints. Thus edge multiplicity is at most four. Retain one arc for each unordered pair of endpoints. The resulting simple graph has $e \geq E/4$ edges and a drawing with at most $2\binom{l}{2} \leq l^2$ crossings. Multiple circle intersections may be separated by local perturbations, with at most one crossing for each pair of arcs at each former intersection. Intersections which are graph vertices do not contribute crossings. If $e < 4m$, then $E < 16m$. Otherwise the crossing estimate gives $e^3 \leq 64m^2l^2$, and consequently $E \leq 16m^{2/3}l^{2/3}$. Adding back the discarded incidences proves the assertion. Empty point or circle sets are immediate. □

Lemma 10.3 (A finite cutting)**.** *Let P be a finite point set and let $\mathcal{C}$ consist of $l \geq 1$ distinct unit circles. For $R \geq 1$ there is a multiset of $s = O(R\log(2+l))$ sampled circles such that, after deleting points on the sampled circles, the remaining points can be assigned to $O(1+s^2)$ open regions, each meeting at most l/R circles of $\mathcal{C}$. Each assigned point belongs to its assigned region. All constants here are absolute.*

Proof. Fix a horizontal direction satisfying the following finite list of genericity conditions. A point of P and a circle-intersection point have different horizontal coordinates unless they are the same point. Also a point of P has an extremal horizontal coordinate of a circle only if it is that extremal point. For the second condition the excluded directions satisfy $u \cdot (p-c) = \pm 1$, where u is a unit direction and c is a circle centre; there are finitely many such directions. The first condition also excludes only finitely many.

Sample $s = \lceil CR\log(2+l)\rceil$ circles independently and uniformly. Repetitions may be ignored when forming their arrangement. Write each sampled circle as its upper and lower semicircle graphs. Their endpoints and intersections are the events of a vertical sweep. Between events, the gaps between consecutive graphs are open vertical regions. Continue a gap across an event precisely when the same two boundary graphs remain consecutive and its closed vertical cross section contains no event point. Otherwise terminate it and begin a new gap on the other side. This procedure produces open regions bounded by at most two semicircle graphs and two vertical event lines, with unbounded boundaries allowed. They avoid all sampled circles. The direction conditions ensure that an input point on an event line which is not in one of these open regions is itself an event point, and hence is already deleted.

There are $O(1+s^2)$ regions. Indeed extrema create $O(s)$ events. If $c_x \geq 2$ sampled circles meet at an intersection x, only $O(c_x)$ new gaps can start there. Since $c_x \leq 2\binom{c_x}{2}$ and each circle pair intersects at most twice, the sum of these contributions is $O(s^2)$. This argument includes multiple intersections and coincidences of extrema and intersections.

Every possible region has a description involving at most six original circles: two boundary graphs and at most two circles for each vertical endpoint. Semicircle signs and choices of events contribute only an absolute factor, so there are at most $C_1(1+l)^6$ candidate regions. A fixed candidate meeting more than l/R circles is avoided by all samples with probability at most $\exp(-s/R)$. An actual region necessarily avoids every sampled circle. A union bound therefore shows that the probability of any actual region violating the conflict bound is at most $C_1(1+l)^6\exp(-s/R)$. Choose the absolute C so this is less than $1/10$. Some sample has the required property. □

10.2. **Fixed-parameter regularization.**

Proposition 10.4. *Fix $0 < a < 10^{-12}$ and put*

$$\varepsilon = a^2, \qquad b = a/10^6, \qquad \gamma = \frac{27}{50}, \qquad \eta = \gamma - \frac{1}{2} = \frac{1}{25}.$$

Suppose $N = t^3$ distinct real points determine at least $t^{4-\varepsilon}$ ordered unit pairs. For all sufficiently large t, there are finite sets P, Q of distinct real algebraic points and a nonempty incidence graph

$E \subseteq P \times Q$, with $|p-q|=1$ on every edge, satisfying

$$t^{1+2a-18b} \le |P| \le t^{1+2a+\varepsilon}, \qquad t^{2+a-18b} \le |Q| \le t^{2+a}, \tag{10.1}$$

$$t^{2+2a-10b} \le |E| \le t^{2+2a+2\varepsilon}, \tag{10.2}$$

$$t^{1-11b} \le d(p) \le t^{1+b}, \qquad t^{a-11b} \le d(q) \le t^{a+b}. \tag{10.3}$$

Let ω be the uniform edge law, and write $\mu(p) = d(p)/|E|$ and $\kappa(q) = d(q)/|E|$ for its marginals. Then

$$\mu(p) \ge \frac{\gamma}{|P|}, \qquad \kappa(q) \ge \frac{\gamma}{|Q|}. \tag{10.4}$$

For every $F \subseteq P$, $G \subseteq Q$, with counting fractions $u = |F|/|P|$ and $v = |G|/|Q|$,

$$\omega(F,G) \le \sqrt{uv}\,((uv)^{\eta} + \xi), \qquad \xi = t^{-a/100}. \tag{10.5}$$

In particular $\max_p \mu(p) \le t^{-1+11b}$.

Proof. Apply Lemma 10.1 to the original configuration. Regard one copy as points and the other as centres, retaining the ordered unit pairs under consideration. Use Lemma 10.3 with $R = t^{1-a}$. Its sample size is $O(t^{1-a}\log t)$. Incidences with sampled circles number at most sN. Every other circle contains at most $2s$ deleted points, so deletion loses at most $3sN$ incidences. Since $\varepsilon < a$, this is negligible compared with $t^{4-\varepsilon}$.

Each open region meets at most t^{2+a} circles. Subdivide its assigned point set into groups of size at most $2t^{1+2a}$. The total number of groups is $O(t^{2-2a}\log^2 t)$, and hence at most $\frac{1}{2}t^{2-2a+\varepsilon}$ for large t. At least $\frac{1}{2}t^{4-\varepsilon}$ incidences survive deletion. One group, with its intersecting circles, has at least $t^{2+2a-2\varepsilon}$ retained incidences. Denote its sets temporarily by P_0, Q_0 and its edge count by e_0. Lemma 10.2 also gives, after absorbing absolute factors into t^{ε},

$$|P_0| \le t^{1+2a+\varepsilon}, \quad |Q_0| \le t^{2+a}, \quad t^{2+2a-2\varepsilon} \le e_0 \le t^{2+2a+2\varepsilon}.$$

The edges may be a subgraph of all available incidences; all upper incidence estimates remain valid for such a subgraph.

Delete point vertices whose degree exceeds t^{1+b} and centre vertices whose degree exceeds t^{a+b}, measuring these degrees before deletion. There are at most $t^{1+2a-b+2\varepsilon}$ excessive point vertices and $t^{2+a-b+2\varepsilon}$ excessive centres. Applying the incidence bound to either excessive set against the full opposite set bounds its lost edges by an absolute constant times

$$t^{2+2a-2b/3+3\varepsilon} + t^{2+a+\varepsilon} + t^{1+2a+\varepsilon}.$$

Here the harmless third ε includes the fractional errors in the mixed term. Since $\varepsilon = a^2 < b/10^6$ for the present range of a, both deletions lose $o(e_0)$ edges. Relabel the trimmed graph as (P_0, Q_0, E_0); it has $e_0 \ge t^{2+2a-3\varepsilon}$ and the required degree upper bounds.

For subsets with old counting fractions u, v, let w be their fraction of edges of E_0. The incidence bound and the degree bounds imply

$$w \le T\big((uv)^{2/3} + t^{-a}v + t^{-1}u\big), \qquad w \le T\min(u,v), \qquad T \le t^{2b}.$$

For example before increasing T one can take $T \le t^{b+5\varepsilon}$. The linear terms are the two vertex terms in the incidence bound divided by e_0. For nonnegative A, B, C, D, $\min(A+B+C, D) \le A + \min(B,D) + \min(C,D)$. Apply this and $\min(x,y) \le \sqrt{xy}$ to obtain

$$w \le t^{2b}\sqrt{uv}\,\big((uv)^{1/6} + 2t^{-a/2}\big). \tag{10.6}$$

Set $\tau = t^{-a/8}$. Among induced rectangles with both old fractions at least τ, choose one maximizing $w/(uv)^{\gamma}$. This is a maximum over a nonempty finite family, and is at least one. Write U, V, W for the fractions of the chosen rectangle. Since $UV \ge t^{-a/4}$, equation (10.6) gives

$$1 \le t^{2b}(UV)^{19/150} + 2t^{2b-49a/100}.$$

If $UV < t^{-16b}$, both terms on the right tend to zero: the first is at most $t^{-2b/75}$. Thus $UV \ge t^{-16b}$ for large t, and

$$U, V \ge t^{-16b}, \qquad W \ge (UV)^{\gamma} \ge t^{-9b}. \tag{10.7}$$

Let P, Q, E be the chosen induced subgraph. These estimates and the trimmed degree upper bounds give (10.1)–(10.2). For instance the old point count is at least $e_0/t^{1+b} \ge t^{1+2a-b-3\varepsilon}$; multiply by U to get the stated lower bound. The centre calculation is identical.

If $F \subseteq P$ and $G \subseteq Q$ have new fractions u, v, the maximization yields $\omega(F, G) \le (uv)^{\gamma}$ whenever $Uu, Vv \ge \tau$. In the other case $uv \le t^{-a/8+16b}$. Applying (10.6) to old fractions Uu, Vv and dividing by W then bounds $\omega(F, G)/\sqrt{uv}$ by

$$t^{-a/48+(41/3)b} + 2t^{-a/2+11b} \le t^{-a/100}$$

for sufficiently large t. This proves (10.5); empty subsets satisfy it as well.

Deleting a single vertex from the maximizing rectangle still leaves a feasible rectangle, by (10.7). Optimality after deleting p implies

$$1 - \mu(p) \le (1 - |P|^{-1})^{\gamma} \le 1 - \gamma|P|^{-1}.$$

This proves the first half of (10.4); the other is the same argument. Multiply by $|E|$ and use the side-size bounds to obtain the degree lower bounds in (10.3). Finally $\mu(p) \le t^{1+b}/t^{2+2a-10b} \le t^{-1+11b}$. □

10.3. **Consequences for labels, small sets and wedge information.** For later use, (10.5) has the following marginal version. If $h = \mu(F)$ and $j = \kappa(G)$, then

$$\omega(F, G) \le 2\sqrt{hj}\,((hj)^{1/25} + \xi). \tag{10.8}$$

Indeed $u \le h/\gamma$, $v \le j/\gamma$, and $\gamma^{-1-2/25} < 2$, $\gamma^{-1} < 2$.

Lemma 10.5 (Matching labels)**.** *Assume $\xi \le 2^{-112}$, and assign labels from any finite set to the vertices of P and Q. Let s be the probability that an edge has different endpoint labels. If $s \le 2^{-112}$, there is a label whose two marginal exceptional masses have sum at most $2s$. For independent points $p, p' \sim \mu$, at every $s \in [0, 1]$,*

$$\Pr(\ell(p) \ne \ell(p')) \le 2^{112} s.$$

The same statement holds for two independent centres.

Proof. Let u_i, v_i be the counting fractions of label i. Summing (10.5) over equal-label rectangles and using $\sum_i \sqrt{u_i v_i} \le 1$ gives

$$1 - s \le \max_i (u_i v_i)^{1/25} + \xi.$$

Choose i_0 attaining the maximum. Bernoulli's inequality yields $u_{i_0} v_{i_0} \ge (1 - s - \xi)^{25} \ge 1 - 25(s + \xi)$. Both complementary counting fractions are therefore at most $25(s + \xi)$. Applying (10.5) to a complementary class against the whole opposite side bounds each complementary marginal mass by $10\sqrt{s + \xi} < 2^{-50}$. Call these marginal masses h, j, and put $J = \omega(\ell(p) \ne i_0, \ell(q) \ne i_0)$. By (10.8) and $2\sqrt{hj} \le h + j$,

$$J \le (h + j)\big((hj)^{1/25} + \xi\big) \le \frac{h + j}{4}.$$

The event that exactly one endpoint has label i_0 is contained in the mismatch event. Hence $s \ge h + j - 2J \ge (h + j)/2$. Two independent point labels disagree with probability at most $2h \le 4s$. If $s > 2^{-112}$, the bound $1 \le 2^{112} s$ suffices. This proves the asserted global bound, including $s = 0$. □

The *wedge law* chooses q with law κ and, conditional on q, chooses two neighbours p, r independently and uniformly. Thus

$$\Omega(p, q, r) = \frac{\mathbf{1}_{pq \in E} \mathbf{1}_{rq \in E}}{|E| d(q)}. \tag{10.9}$$

Its two point marginals are μ, and each point–centre marginal is ω.

Lemma 10.6 (Finite small-set estimates)**.** *Let $0 < \alpha \le 1$. If $F \subseteq P$ has mass $h \le \alpha$ and $G \subseteq Q$ has arbitrary mass j, then*

$$\omega(F, G) \le 2(\alpha^{1/25} + \xi)\sqrt{hj}.$$

If also $j \le \alpha$, then

$$\omega(F, G) \le (\alpha^{2/25} + \xi)(h + j).$$

For every $0 < d \le 1$, the wedge law satisfies

$$\Pr(p \in F, r \in F) \le h\left[d + 2d^{-1/2}\left\{(\alpha^2/d)^{1/25} + \xi\right\}\right]. \tag{10.10}$$

Proof. The first two assertions follow immediately from (10.8). For the last one put $g(q) = \Pr(p \in F \mid q)$. Then $\mathbb{E}_\kappa g = h$, and the required probability is $\mathbb{E}_\kappa g^2$. The set $G_d = \{q : g(q) > d\}$ has mass at most h/d. Splitting according to G_d gives

$$\mathbb{E}_\kappa g^2 \le dh + \mathbb{E}_\kappa[g\mathbf{1}_{G_d}] = dh + \omega(F, G_d).$$

Apply (10.8) with $hj \le h^2/d \le \alpha^2/d$ and $\sqrt{hj} \le h/\sqrt{d}$. The case $h = 0$ is immediate. □

Lemma 10.7 (Variation of an edge bit)**.** *Give every edge a bit. Let V_P and V_Q be its disagreement probabilities in independent two-edge stars based at a point and at a centre, respectively. If f is the smaller of the two global bit probabilities, then, for $\xi \le 2^{-112}$,*

$$f \le 2^{112}(V_P + V_Q).$$

Proof. Label each vertex by a conditional majority bit, breaking ties by a fixed rule. If the conditional bit parameter is ρ, the chance of disagreement with this label is $\min(\rho, 1 - \rho)$, at most $2\rho(1 - \rho)$. Thus the two endpoint labels differ on an edge with probability at most $v = V_P + V_Q$. If $v \le 2^{-112}$, the local part of Lemma 10.5 gives a common label whose total marginal exceptional mass is at most $2v$. The edge bit differs from that label with probability at most $3v$, which bounds f. If $v > 2^{-112}$, use $f \le 1/2 \le 2^{112}v$. □

Lemma 10.8 (Information in a wedge)**.** *Let $\pi(p, r) = \sum_q \Omega(p, q, r)$, and let*

$$D = \sum_{p,r} \pi(p, r) \log \frac{\pi(p, r)}{\mu(p)\mu(r)}.$$

For the graph of Proposition 10.4,

$$\pi(p, r) \le \frac{2}{|E| d_{Q,\min}} \quad (p \ne r), \qquad \Pr(p = r) = \frac{|Q|}{|E|}, \tag{10.11}$$

$$0 \le D \le 2a \log t \quad \text{for sufficiently large } t. \tag{10.12}$$

Proof. Two distinct real points have at most two common unit-circle centres. Summing (10.9) therefore gives the off-diagonal estimate. On the diagonal,

$$\sum_p \pi(p, p) = \sum_q \sum_{p : pq \in E} \frac{1}{|E| d(q)} = \frac{|Q|}{|E|}.$$

Put $e_* = 30b$. The bounds of Proposition 10.4 imply

$$|P| \le t^{1+2a+e_*}, \quad |E| d_{Q,\min} \ge t^{2+3a-2e_*}, \quad |Q|/|E| \le t^{-a+2e_*}.$$

For a finite random variable X let $\mathcal{H}(X) = -\sum_x \Pr(X = x) \log \Pr(X = x)$. Dropping the nonnegative diagonal contribution to $\mathcal{H}(p, r)$ and using the off-diagonal atom bound gives

$$\mathcal{H}(p, r) \ge (1 - t^{-a+2e_*})((2 + 3a - 2e_*) \log t - \log 2).$$

On the other hand $\mathcal{H}(p), \mathcal{H}(r) \le \log|P|$. Expanding the definition gives $D = \mathcal{H}(p) + \mathcal{H}(r) - \mathcal{H}(p,r)$. Hence

$$\frac{D}{\log t} \le a + 4e_* + (2+3a)t^{-a+2e_*} + \frac{\log 2}{\log t} \le 2a$$

for sufficiently large t, since $4e_* = 120b < a/1000$. Nonnegativity is the log-sum inequality, also proved below. □

10.4. **A finite prediction principle.** We give the probability argument in a form independent of geometry. Let ω be any probability law on finite sets $P \times Q$, with positive marginals μ, κ, and set

$$\Omega(p,q,r) = \kappa(q)\omega(p \mid q)\omega(r \mid q), \qquad \pi(p,r) = \sum_q \Omega(p,q,r), \qquad D = D_{\mathrm{KL}}(\pi \| \mu \otimes \mu).$$

For finite laws u, v write

$$D_{\mathrm{KL}}(u\|v) = \sum_x u(x) \log \frac{u(x)}{v(x)}, \qquad \mathrm{TV}(u,v) = \tfrac{1}{2}\sum_x |u(x) - v(x)|.$$

Terms of weight zero are omitted; a positive weight divided by zero gives infinite relative entropy.

Lemma 10.9 (Elementary information inequalities)**.** *Relative entropy is nonnegative, obeys the chain rule*

$$D_{\mathrm{KL}}(u_{XY}\|v_{XY}) = D_{\mathrm{KL}}(u_X\|v_X) + \sum_x u_X(x) D_{\mathrm{KL}}(u_{Y|x}\|v_{Y|x}),$$

and cannot increase under a deterministic map. Moreover

$$2\,\mathrm{TV}(u,v)^2 \le D_{\mathrm{KL}}(u\|v), \qquad |\mathbb{E}_u F - \mathbb{E}_v F| \le 2\|F\|_\infty \mathrm{TV}(u,v).$$

Proof. The inequality $x\log x - x + 1 \ge 0$ implies nonnegativity after weighting by v; the infinite cases are immediate. Splitting $\log(u_{XY}/v_{XY})$ into the marginal log ratio and the conditional log ratio proves the chain rule by summation. Applying that identity to a variable and its deterministic image proves data processing. For the total variation bound let $A = \{x : u(x) > v(x)\}$ and set $p = u(A)$, $q = v(A)$. Then $\mathrm{TV}(u,v) = p - q$. Data processing gives a lower bound by

$$B(p,q) = p\log(p/q) + (1-p)\log((1-p)/(1-q)).$$

For fixed $0 < q < 1$, $B(q,q) = \partial_p B(q,q) = 0$ and $\partial_p^2 B(p,q) = 1/(p(1-p)) \ge 4$. Integrating twice gives $B(p,q) \ge 2(p-q)^2$. Endpoints follow by continuity or by the infinite-entropy convention. The bounded-expectation estimate follows by applying the triangle inequality to $\sum_x (u(x) - v(x))F(x)$. □

A random choice T, independent of all vertices, specifies a labelling $c_T : P \to [M]$. We call this choice a test. Thus a test partitions the points into at most M classes, and a history records their labels in a sequence of tests. The test space may be infinite; all maps are measurable. For a history map $\mathscr{H}$ define

$$\mu_h(x) = \frac{\mu(x)\mathbf{1}_{\mathscr{H}(x)=h}}{\mu(\mathscr{H} = h)} \quad \text{when } \mu(\mathscr{H} = h) > 0.$$

Histories of zero mass are irrelevant.

Lemma 10.10 (Prediction from a strict prefix)**.** *Suppose each test also specifies vector maps $f_T : [M] \to \mathbb{R}^d$ and $g_T : Q \to \mathbb{R}^d$ with norms at most B and B' respectively. Let $H \ge 1$ be an integer, take H independent tests of the same law, and choose a step uniformly from $\{1, \dots, H\}$. Use only its strict prefix as the point history. For a fresh independent test put $m_{T,h} = \sum_x \mu_h(x) f_T(c_T(x))$. The average, over prefixes and the fresh test, of*

$$\left| \mathbb{E}_{(p,q)\sim\omega} \left\langle f_T(c_T(p)) - m_{T,\mathscr{H}(p)}, g_T(q) \right\rangle \right|$$

is at most $4BB'(D/H)^{1/4}$. In particular one may fix a history, before the fresh test is drawn, with at most M^{H-1} values for which the same bound holds after averaging only the fresh test.

Proof. Freeze all H tests temporarily. Apply the map recording the complete sequence of labels to both coordinates of (p, r). The actual law is its image under π; the reference law is its image under $\mu \otimes \mu$. Their relative entropy is at most D. At step j let $\mathscr{H}_j$ be the strict prefix. Given prefix values (h, h'), denote the actual law of the next two labels by $A_{j,h,h'}$. The reference conditional law is

$$B_{j,h,h'} = (c_{T_j})_* \mu_h \otimes (c_{T_j})_* \mu_{h'}.$$

Every actual prefix of positive probability also has positive reference probability. The chain rule in Lemma 10.9 gives

$$\sum_{j=1}^{H} \mathbb{E}_\pi D_{\mathrm{KL}}(A_{j,\mathscr{H}_j(p),\mathscr{H}_j(r)} \| B_{j,\mathscr{H}_j(p),\mathscr{H}_j(r)}) \le D.$$

Consequently Pinsker and Cauchy–Schwarz give

$$\frac{1}{H} \sum_{j=1}^{H} \mathbb{E}_\pi \operatorname{TV}(A_{j,\mathscr{H}_j(p),\mathscr{H}_j(r)}, B_{j,\mathscr{H}_j(p),\mathscr{H}_j(r)}) \le \sqrt{\frac{D}{2H}}. \tag{10.13}$$

For each step define the residual

$$U_j(p) = f_{T_j}(c_{T_j}(p)) - \sum_x \mu_{\mathscr{H}_j(p)}(x) f_{T_j}(c_{T_j}(x)).$$

Its norm is at most $2B$. Under the reference conditional product, the two residuals have mean zero and are independent, so their inner product has expectation zero. Its absolute value is bounded by $4B^2$. The bounded-expectation estimate and (10.13) therefore yield

$$\frac{1}{H} \sum_j \mathbb{E}_\pi \langle U_j(p), U_j(r) \rangle \le 8B^2 \sqrt{\frac{D}{2H}}.$$

Each summand is nonnegative. More precisely, conditional independence of the two endpoints of a wedge proves the identity

$$\mathbb{E}_\pi \langle U_j(p), U_j(r) \rangle = \mathbb{E}_{q \sim \kappa} \left\| \mathbb{E}_{p|q} U_j(p) \right\|^2.$$

It follows that

$$\left| \mathbb{E}_\omega \langle U_j(p), g_{T_j}(q) \rangle \right| \le B' \left(\mathbb{E}_\pi \langle U_j(p), U_j(r) \rangle \right)^{1/2}.$$

Average over j and use Cauchy–Schwarz again. The result is at most $B'(8B^2\sqrt{D/(2H)})^{1/2} \le 4BB'(D/H)^{1/4}$. Now average over the previously frozen tests. The residual at step j depends only on the strict prefix and the current test; the latter is independent and identically distributed with a fresh test. Thus it may be replaced by such a fresh test without changing the law. Finally choose one prefix whose conditional average discrepancy is no greater than the overall average. Its number of values is at most M^{H-1}. □

Corollary 10.11 (Acceptance of finite labels)**.** *Suppose a test assigns a set of accepted labels $A_T(q) \subseteq [M]$ to each centre, and let $s_0 = \mathbb{E}_T \Pr_\omega(c_T(p) \notin A_T(q))$. There is a fixed history with at most M^{H-1} values such that a prediction $\widehat{p} \sim \mu_{\mathscr{H}(p)}$ satisfies*

$$\mathbb{E}_T \Pr(c_T(\widehat{p}) \notin A_T(q)) \le s_0 + 4\sqrt{M}(D/H)^{1/4}.$$

The probability on the left first samples (p, q) with law ω and then samples the prediction conditional only on $\mathscr{H}(p)$. The test is independent of this entire experiment.

Proof. Use $f_T(c) = e_c \in \mathbb{R}^M$ and let $g_T(q)$ be the indicator vector of $A_T(q)$. Their norms are at most 1 and $\sqrt{M}$. The difference between actual and predicted acceptance probabilities is exactly the inner product discrepancy in Lemma 10.10. □

Lemma 10.12 (The prediction coupling)**.** *For any fixed history and integer $r \geq 1$, define the joint law*

$$\Pr(p, \widehat{p}, q_1, \ldots, q_r) = \mu(p)\mu_{\mathscr{H}(p)}(\widehat{p}) \prod_{j=1}^{r} \omega(q_j \mid p).$$

Both point marginals are μ; every (p, q_j) has law ω; every q_j has law κ; and

$$\Pr(\widehat{p} = p) \leq |\mathscr{H}(P)| \max_p \mu(p).$$

With a fresh independent test, the probability of any of the r acceptance failures in Corollary 10.11 is bounded by r times that corollary's bound.

Proof. All marginal claims follow by summation. In particular, if $\mathscr{H}(x) = h$, then

$$\sum_{p:\mathscr{H}(p)=h} \mu(p)\mu_h(x) = \mu(x).$$

The collision probability equals

$$\sum_{h:\mu(\mathscr{H}=h)>0} \frac{\sum_{p:\mathscr{H}(p)=h} \mu(p)^2}{\mu(\mathscr{H} = h)} \leq |\mathscr{H}(P)| \max_p \mu(p).$$

Every pair $(\widehat{p}, q_j)$ has the required prediction law, so the last assertion is a union bound; independence of failures is not asserted or needed. □

Corollary 10.13 (Prediction of a strict cross)**.** *A test now assigns each vertex a state $y_T(x) = (A_T(x), B_T(x))$ in a finite product of label sets. For a state $y = (u, v)$ put*

$$\mathcal{R}_y = \{(u', v') : \textit{exactly one of } u' = u,\ v' = v \textit{ holds}\}.$$

For integers $H \geq 1$, $L \geq 2$, there is a fixed history with at most L^{2H} values such that, writing $\nu_{h,T} = (y_T)_* \mu_h$,

$$\mathbb{E}_T \mathbb{E}_\omega[1 - \nu_{\mathscr{H}(p),T}(\mathcal{R}_{y_T(q)})] \leq \mathbb{E}_T \Pr_\omega(y_T(p) \notin \mathcal{R}_{y_T(q)}) + \frac{4}{L} + 18(D/H)^{1/4}. \tag{10.14}$$

If the tests instead have a finite measure of total mass W, and $m(T)$ bounds their actual failure probability, the integrated bound is

$$\int \mathbb{E}_\omega[1 - \nu_{\mathscr{H}(p),T}(\mathcal{R}_{y_T(q)})]\, dT \leq \int m(T)\, dT + W\left(\frac{4}{L} + 18(D/H)^{1/4}\right). \tag{10.15}$$

Proof. For each attained first-coordinate label u, choose $h_1(u)$ independently and uniformly from $[L]$. Independently make the same choices $h_2(v)$ for the second-coordinate labels. Use these two random maps for both points and centres. Only finitely many labels occur; ordering their first occurrences among a fixed ordering of the vertices makes these choices measurable even when the test space is infinite. The new label of a state (u, v) is the pair $(a, b) = (h_1(u), h_2(v))$. Set

$$f(a, b) = (e_a, e_b, e_{(a,b)}), \qquad g(q) = (e_{a_q}, e_{b_q}, -2e_{(a_q,b_q)}).$$

Their norms are $\sqrt{3}$ and $\sqrt{6}$. The inner product is

$$\mathbf{1}_{a=a_q} + \mathbf{1}_{b=b_q} - 2\mathbf{1}_{(a,b)=(a_q,b_q)},$$

which is the indicator of the strict cross after relabelling. Lemma 10.10 therefore gives error at most $4\sqrt{18}(D/H)^{1/4}$. For two fixed original states, these random maps preserve equalities and create either of the two previously false coordinate equalities with probability at most $1/L$. The probability that the original and relabelled strict-cross indicators differ is at most $2/L$. This applies to both the actual and the predicted pair: the predicted point uses previous random maps only, and the current random maps are fresh. The two comparisons therefore add $4/L$. Average the new random maps and test before choosing the history, and use $4\sqrt{18} < 18$. The set of labels has size

L^2, so the history count is at most $(L^2)^{H-1} \le L^{2H}$. For $W > 0$, apply the probability statement to the normalized test measure and multiply by W. For $W = 0$, use the empty history. □

11. Heights, finite tests, and coordinate states

This section supplies the arithmetic part of the argument, including the quantitative treatment of the bounded-height alternative. The underlying height and prediction strategy is that of [28]; all constants needed below are specified here. Write ω for the uniform law on the incidence edges of the preceding section, and μ, κ for its point and centre marginals. We use the following conclusions already proved there:

$$\begin{aligned} D \le 2a \log t, \qquad \max_p \mu(p) \le t^{-1+11b}, \\ d_{P,\min} \ge t^{1-11b}, \qquad d_{Q,\min} \ge t^{a-11b}, \qquad \xi \le t^{-a/100}. \end{aligned} \tag{11.1}$$

Here D is the mutual information of the two point endpoints of a wedge, and $b = a/10^6$. The matching-label inequality is available with $C_0 = 2^{112}$: for every finite labelling of the two vertex sets, the probability that two independent μ-points have different labels is at most C_0 times the edge mismatch probability. We also use the finite-label acceptance and strict-cross prediction lemmas proved in the preceding section. Throughout, sufficiently large t means a threshold depending only on the fixed numerical parameters, never on the configuration.

For a point or centre $x = (x_1, x_2)$, set

$$z_x = x_1 + ix_2, \qquad w_x = x_1 - ix_2.$$

The algebraic specialization already made ensures that all these coordinates are algebraic. Choose a finite normal number field F containing them and i. On an incidence edge $e = (p, q)$ put

$$d_e = z_p - z_q, \qquad (z_p - z_q)(w_p - w_q) = 1, \qquad w_p - w_q = d_e^{-1}. \tag{11.2}$$

In particular every d_e is nonzero. Distinct original real points have distinct z-coordinates and distinct w-coordinates. Their nonzero differences remain nonzero under every field embedding, although the two embedded coordinates need not remain complex conjugates.

11.1. **Normalizations and elementary height estimates.** We recall the place normalizations explicitly. If $d = [F : \mathbb{Q}]$, include each embedding $\sigma : F \longrightarrow \mathbb{C}$ as an archimedean entry of weight $1/d$, with absolute value $|u|_\sigma = |\sigma(u)|$. Thus conjugate embeddings are included separately. At a finite place v above the rational prime ℓ, use the absolute value extending $|\ell|_\ell = \ell^{-1}$ and the weight $w_v = [F_v : \mathbb{Q}_\ell]/d$. Write $w_\sigma = 1/d$ at infinity. The standard product formula, with these normalizations, states

$$\sum_v w_v \log |u|_v = 0 \qquad (u \in F^\times), \qquad \sum_{v|\infty} w_v = 1. \tag{11.3}$$

We use $\sum_v^* g(v) = \sum_v w_v g(v)$. Expectations denoted by $\mathbb{E}_E$ or $\mathbb{E}_\omega$ both use the uniform law on incidence edges. Only finitely many finite places contribute for any fixed $u \ne 0$. These statements follow from the prime factorization of a principal fractional ideal and the norm formula; they are the usual number-field product formula, rather than an additional geometric hypothesis [5].

Define $\log^+ r = \max(0, \log r)$ for $r > 0$, and set $\log^+ 0 = 0$. The absolute logarithmic height is

$$\mathrm{h}(u) = \sum_v w_v \log^+ |u|_v, \qquad \mathrm{h}(0) = 0.$$

Lemma 11.1 (Height identities and embedding tails)**.** *This height is unchanged by enlarging F. For $u \ne 0$,*

$$\mathrm{h}(u^{-1}) = \mathrm{h}(u), \qquad \sum_v w_v \big|\log |u|_v\big| = 2\,\mathrm{h}(u). \tag{11.4}$$

For arbitrary u, v,

$$\mathrm{h}(uv) \le \mathrm{h}(u) + \mathrm{h}(v), \qquad \mathrm{h}(u+v) \le \mathrm{h}(u) + \mathrm{h}(v) + \log 2. \tag{11.5}$$

Suppose that X is an F-valued random variable and that σ is an independent uniformly random embedding of F into $\mathbb{C}$. For $B > 1$ and $0 < r < 1$,

$$\Pr(|\sigma(X)| > B) \le \frac{\mathbb{E}\,\mathrm{h}(X)}{\log B}, \qquad \Pr(X \ne 0,\ |\sigma(X)| < r) \le \frac{\mathbb{E}\,\mathrm{h}(X)}{\log(1/r)}. \tag{11.6}$$

Proof. For a finite extension F'/F and a finite place v, the identity $\sum_{v'|v}[F'_{v'} : F_v] = [F' : F]$ shows that the sum of the new weights above v is its old weight. Each embedding of F has $[F' : F]$ extensions, so the assertion also holds at infinity. A common overfield compares any two choices of F.

The positive and negative parts of the logarithm in (11.3) have equal weighted sums. This proves (11.4). The product inequality is pointwise. For sums, the ultrametric triangle inequality contributes no extra constant at finite places, whereas the ordinary triangle inequality contributes at most $\log 2$ at infinity. The archimedean weights sum to one, giving (11.5). Finally, the average over embeddings of $\log^+ |\sigma(X)|$ is at most $\mathrm{h}(X)$. On $X \ne 0$, the average negative logarithm is bounded by $\mathrm{h}(X^{-1}) = \mathrm{h}(X)$. Markov's inequality gives both bounds in (11.6). □

11.2. **The level space and its arithmetic scale.** For an edge and a place put $n_e(v) = -\log |d_e|_v$. Choose a finite closed interval $I_v = [\ell_v, u_v]$ containing 0 and all $n_e(v)$. Enlarge it, if necessary, so that every pair of distinct point vertices satisfies

$$\ell_v \le -\log |z_p - z_{p'}|_v, \qquad u_v \ge \log |w_p - w_{p'}|_v. \tag{11.7}$$

There are only finitely many relevant algebraic numbers, so $I_v = [0, 0]$ is possible at all but finitely many places. The level space and its measure are

$$\mathcal{L} = \coprod_v \{v\} \times I_v, \qquad \int_{\mathcal{L}} g = \sum_v w_v \int_{I_v} g(v, s)\, ds.$$

Every integral without an indicated domain in the subsequent level argument is with respect to this measure. Define

$$\begin{aligned} Y_e(v, s) &= \mathbf{1}_{\{n_e(v) > s\}}, \qquad \pi(v, s) = \Pr_\omega(Y_e = 1), \\ f(v, s) &= \min\{\pi(v, s), 1 - \pi(v, s)\}, \qquad S = \int f. \end{aligned} \tag{11.8}$$

Further enlarging the intervals does not change S: outside the range of the edge thresholds all edge bits coincide.

For independent uniform edges e, e', the threshold representation of the distance between two real numbers and (11.4) give

$$\begin{aligned} 2\mathbb{E}\,\mathrm{h}(d_e/d_{e'}) &= \mathbb{E} \sum_v w_v |n_e(v) - n_{e'}(v)| \\ &= 2 \int \pi(1 - \pi) \le 2S. \end{aligned} \tag{11.9}$$

All terms are nonnegative, so the exchanges of integration and expectation are justified by Tonelli's theorem.

11.3. **Transferring edge heights to point heights.**

Lemma 11.2 (A quantitative height transfer)**.** *Let U assign an element of F to every point and centre vertex. The matching-label inequality with constant C_0 implies*

$$\mathbb{E}_{p,p' \sim \mu}\, \mathrm{h}(U_p - U_{p'}) \le C_0 \mathbb{E}_{(p,q) \sim \omega}\, \mathrm{h}(U_p - U_q) + 2C_0 + 1. \tag{11.10}$$

The two point variables on the left are independent.

Proof. At a finite place and a level $s \geq 0$, label vertices by the equivalence classes of $|U_x - U_{x'}|_v \leq e^s$. The ultrametric inequality makes this an equivalence relation. Its mismatch indicator integrates over $s \geq 0$ to $\log^+ |U_x - U_{x'}|_v$.

At an infinite place, identify the embedded coordinates with complex numbers and use square cells of side e^s. Shift the grid by a uniformly random fractional vector in $[0,1)^2$. Two coordinates at Euclidean distance A have different cells with probability at most

$$\min\{1, \sqrt{2}Ae^{-s}\}.$$

Indeed, in each real coordinate the mismatch probability is at most the coordinate separation divided by the side length; summing the two coordinate separations costs at most $\sqrt{2}A$. Direct integration gives

$$\int_0^\infty \min\{1, \sqrt{2}Ae^{-s}\}\, ds \leq \log^+ A + 2.$$

Conversely, two points at distance greater than $\sqrt{2}e^s$ must occupy different cells for every shift. Consequently their positive logarithmic distance is at most the integral of their mismatch indicator plus $\log\sqrt{2}$.

At each fixed place, level, and shift, apply the matching-label inequality to the finite set of labels that occurs. Integrate it, average the archimedean shifts, and then sum the place weights. Finite places cause no additive error. The total weight at infinity is one, so the edge error is at most $2C_0$, and the point error is less than one. This is (11.10). □

11.4. **Excluding a prescribed bounded scale.** For the rest of this section the numerical parameters are

$$K = 10^{100004}, \qquad k = (2^{1024}K)^{1024}, \qquad S_0 = k^{100}, \qquad a = 2^{-2^{200}S_0}, \qquad b = a/10^6. \tag{11.11}$$

The proof below also works for any fixed $K \geq 10$ with these definitions.

Proposition 11.3 (The bounded-scale alternative)**.** *For every sufficiently large t, the incidence piece under consideration satisfies $S > S_0$.*

Proof. Assume $S \leq S_0$. By (11.9), some edge value d_0 satisfies $\mathbb{E}_\omega \,\mathrm{h}(d_e/d_0) \leq S_0$. Initially rescale the two coordinates to $z/d_0, d_0 w$, and write $D_e = d_e/d_0$ for the first rescaled jump. Both rescaled edge jumps, D_e and D_e^{-1}, have mean height at most S_0. Apply Lemma 11.2 to both coordinates and average over one of the two point variables. A point $p_0 \in P$ can be chosen such that, after putting

$$Z_x = (z_x - z_{p_0})/d_0, \qquad W_x = d_0(w_x - w_{p_0}),$$

one has

$$A_0 := \mathbb{E}_\mu(\mathrm{h}(Z_p) + \mathrm{h}(W_p)) \leq 2C_0S_0 + 4C_0 + 2.$$

The identities $Z_q = Z_p - D_{pq}$ and $W_q = W_p - D_{pq}^{-1}$ on an edge and (11.5) give

$$B_0 := \mathbb{E}_\kappa(\mathrm{h}(Z_q) + \mathrm{h}(W_q)) \leq A_0 + 2S_0 + 2\log 2.$$

Also $C_0' := \mathbb{E}_\omega(\mathrm{h}(D_e) + \mathrm{h}(D_e^{-1})) \leq 2S_0$. Thus, with the fixed numerical bound

$$C_h = 2^{117}S_0, \qquad A_0 + B_0 + C_0' \leq 16C_0(S_0+1) \leq C_h. \tag{11.12}$$

This normalization preserves the exact identity $(Z_p - Z_q)(W_p - W_q) = 1$ on every edge.

Given any finite history map $\mathcal{H}$ on P, sample a point $p \sim \mu$, a prediction $\widehat{p}$ independently from μ conditional on $\mathcal{H}(\widehat{p}) = \mathcal{H}(p)$, and three independent neighbours q_1, q_2, q_3 of p conditional on p. Choose a fresh uniform embedding $\sigma : F \longrightarrow \mathbb{C}$, independently of this entire experiment. Both point marginals are μ, every centre marginal is κ, and each (p, q_j) has law ω. These facts hold for every history map, including one that will be selected below. Put $D_j = D_{pq_j}$.

Choose fixed geometric tolerances

$$\log B = 2^{10}C_h, \qquad \log(1/r) = 2^{12}C_h, \qquad \tau = \frac{r^4}{256B^5}. \tag{11.13}$$

In particular $B \geq 2$, $0 < r < 1$, and $0 < \tau < 1$. The total expected height of the sixteen quantities

$$Z_p, W_p, Z_{\widehat{p}}, W_{\widehat{p}}, \quad (Z_{q_j}, W_{q_j}, D_j, D_j^{-1})_{j=1}^{3}$$

is $2A_0 + 3B_0 + 3C_0' \leq 3C_h$. By (11.6), the probability that any of their embedded absolute values exceeds B is at most $3/1024 < 1/100$. The sum of the expected heights of $Z_{\widehat{p}} - Z_p$ and of the three differences $D_j - D_l$, $j < l$, is at most

$$2A_0 + 3C_0' + 4\log 2 \leq 12C_h.$$

The probability that any of these differences is nonzero but its embedded absolute value is less than r is consequently at most $12/4096 < 1/100$. Equal differences are not counted in this event.

For each embedding, divide $[-B-1, B+1]$ into half-open intervals of length at most τ. Label a point by the four intervals containing the real and imaginary parts of Z_p, W_p; give all points outside this box one additional label. An embedding is a test, and the number M of possible labels can be chosen to satisfy

$$M \leq 2(6B/\tau)^4, \qquad \log M \leq 2^{18}C_h \leq 2^{135}S_0. \tag{11.14}$$

For instance, the number of one-dimensional intervals is at most $6B/\tau$; the additional label is covered by the factor two. Substitution of (11.13) proves the logarithmic estimate. At a centre q, accept a label if it occurs at an actual neighbour of q. Every actual edge is accepted.

Take the maximum history length

$$H_b = \left\lceil 2^{40}M^2(D+1) \right\rceil.$$

The finite-label acceptance lemma gives a fixed strict-prefix history $\mathcal{H}$ with at most M^{H_b} values such that the acceptance failure of the prediction at a fresh embedding is at most

$$4\sqrt{M}(D/H_b)^{1/4} \leq 2^{-8} = 1/256. \tag{11.15}$$

The factor $\sqrt{M}$ is retained here; dropping it would invalidate this numerical parameter choice.

For large enough t, (11.1) implies $H_b \leq 2^{42}aM^2\log t$. The choice of a makes

$$2^{42}aM^2\log M < \tfrac{1}{2}. \tag{11.16}$$

To verify this without an unspecified small constant, let $A = 2^{136}S_0$. Equation (11.14) gives $\log_2 M \leq A$ and $\log_2\log M \leq \log_2 A$. The logarithm to base two of the left side of (11.16) is at most

$$42 - 2^{200}S_0 + 2A + \log_2 A < -1,$$

since $S_0 \geq 1$ and $\log_2 S_0 \leq S_0$. It follows that $M^{H_b} \leq t^{1/2}$.

The prediction collision probability is

$$\Pr(\widehat{p} = p) = \sum_h \frac{\sum_{p:\mathcal{H}(p)=h}\mu(p)^2}{\mu(\mathcal{H}=h)} \leq |\mathcal{H}(P)| \max_p \mu(p) \leq t^{-1/2+11b}.$$

Also $\Pr(q_j = q_l) \leq d_{P,\min}^{-1}$ for $j \neq l$. Use the selected history in the earlier experiment. Each of the three prediction–centre comparisons has failure probability at most $1/256$; no independence of those failures is needed. The two geometric error events cost less than $2/100$ in total, and the point and centre collision probabilities tend to zero. Therefore an outcome exists where all three tests accept, all sixteen absolute values are at most B, and

$$|Z_{\widehat{p}} - Z_p| \geq r, \qquad |D_j - D_l| \geq r \quad (j < l). \tag{11.17}$$

Here and in the rest of this proof the embedding symbols are suppressed. The embedding is chosen only after the entire sampling experiment, so all height tail estimates used above retain their stated independence.

Acceptance at q_j supplies an actual neighbour with the same label as $\widehat{p}$. This is a label of a point inside the box, and both complex coordinate errors are at most $\sqrt{2}\tau$. Comparing its exact unit equation with the predicted coordinates gives

$$|(Z_{\widehat{p}} - Z_{q_j})(W_{\widehat{p}} - W_{q_j}) - 1| \le 8B\tau.$$

Set $\Delta Z = Z_{\widehat{p}} - Z_p$ and $\Delta W = W_{\widehat{p}} - W_p$. Subtraction of the original unit equation yields

$$\Delta Z D_j^{-1} + \Delta W D_j + \Delta Z \Delta W = E_j, \qquad |E_j| \le 8B\tau.$$

Subtract the $j = 1$ equation from the other two. The resulting two-by-two system for $(\Delta Z, \Delta W)$ has determinant, up to sign,

$$\frac{(D_1 - D_2)(D_1 - D_3)(D_2 - D_3)}{D_1 D_2 D_3}.$$

Its absolute value is at least r^3/B^3. Every matrix entry is at most $2B$ in absolute value, and both right-hand entries are at most $16B\tau$. Cramer's rule therefore gives

$$|\Delta Z| \le \frac{64B^5\tau}{r^3} = r/4,$$

contradicting (11.17). The assumption $S \le S_0$ is impossible for sufficiently large t. □

11.5. **Coordinate states with a bounded consistency defect.** We now use the original z, w coordinates, not the temporary normalization from the preceding proof. At a finite place and level (v, s), let A_x be the class of z_x under $|z_x - z_{x'}|_v \le e^{-s}$, and let B_x be the class of w_x under $|w_x - w_{x'}|_v \le e^s$. At an infinite place use square-grid cells of side lengths e^{-s} and e^s, respectively. For each of these two grids choose a fractional shift in $[0, 1)^2$ and use that same fractional shift, scaled by the side length, for every s. The state of a vertex is the ordered pair

$$y_x = (A_x, B_x).$$

For a state $y = (A, B)$ its strict cross is

$$R_y = \{(A', B') : \text{exactly one of } A' = A,\ B' = B \text{ holds}\}.$$

An edge is called consistent if $Y_e = 1$ and exactly its A-labels agree, or $Y_e = 0$ and exactly its B-labels agree. Define

$$m(v, s) = \Pr_\omega(e \text{ is inconsistent at } (v, s)), \qquad m_e = \int \mathbf{1}_{\{e \text{ is inconsistent}\}}.$$

Lemma 11.4 (Uniform consistency budget)**.** *The shifts can be fixed so that*

$$\int m = \mathbb{E}_\omega m_e \le 4. \tag{11.18}$$

At a finite place every edge is consistent except at its single threshold. Equality of A-labels implies $|z_x - z_{x'}|_v \le c_v e^{-s}$, and equality of B-labels implies $|w_x - w_{x'}|_v \le c_v e^s$, where $c_v = 1$ at finite places and $c_v = \sqrt{2}$ at infinity. All the states and indicators may be chosen measurably.

Proof. At a finite place, $n_e(v) > s$ says that the first jump has absolute value less than e^{-s} and its inverse has absolute value greater than e^s. Thus exactly the A-labels agree. When $n_e(v) < s$ the roles reverse. The exceptional equality is a null set in the level variable.

At infinity put $r = |s - n_e(v)|$. The jump that should match, divided by its grid side, has size e^{-r}, so its mismatch probability is at most $\sqrt{2}e^{-r}$. The other jump, divided by its grid side, has size e^r and cannot match if $r \ge \log\sqrt{2}$. The expected inconsistency indicator is therefore at most

$$\mathbf{1}_{\{|s - n_e(v)| \le \log\sqrt{2}\}} + \sqrt{2}e^{-|s - n_e(v)|}.$$

Its integral over the whole real line is $\log 2+2\sqrt{2}<4$. Restriction to I_v can only decrease it. Average over edges and sum the archimedean place weights, whose sum is one. Averaging the finitely many grid shifts now fixes shifts with total inconsistency at most four. The claimed distance implications are the diameter bounds of a cell, or the definitions of the nonarchimedean classes. At finite places label a class by the first vertex it contains in a fixed ordering; at infinite places use integer grid indices. This makes the state comparisons measurable functions of s. □

11.6. **A fixed quantitative choice of the later histories.**

Lemma 11.5 (Compatibility of the large-scale parameters)**.** *Let*

$$H=\lceil 2ak^{400}\log t\rceil,\qquad L=\lceil k^{100}\rceil,\qquad q_*=k^{-1/400}.$$

For all sufficiently large t,

$$(D/H)^{1/4}\le k^{-100},\quad L^{-1}\le k^{-100},\quad S^{-1}<k^{-100},\quad d_{Q,\min}^{-1}\le k^{-100},\quad \xi\le k^{-100},\tag{11.19}$$

$$L^{2Hk}\le t^{1/2},\qquad \frac{L^{2Hk}}{d_{P,\min}}\le k^{-100},\qquad L^{2H}\max_p\mu(p)\le k^{-100-1/16}.\tag{11.20}$$

In the last inequality, the right side can be replaced by any other fixed positive constant by increasing the threshold for t.

Proof. The bounds involving $D/H, L, S$ follow from their definitions and Proposition 11.3. The bounds involving ξ and the minimum centre degree follow from (11.1), since $a-11b>0$. The integer k makes $L=k^{100}$. For large enough t we have $1\le (a/2)k^{400}\log t$, so $H\le (5a/2)k^{400}\log t$. Consequently

$$\frac{\log L^{2Hk}}{\log t}\le 500ak^{401}\log k<\tfrac{1}{2}.$$

For the final numerical inequality, take logarithms to base two: the negative term $-2^{200}k^{100}$ from a exceeds in absolute value $\log_2 500+401\log_2 k+\log_2\log k+1$. This holds already for $k\ge 2$, using $\log_2 k\le k$; our $k\ge 2^{1048576}$ is much larger. Thus the first inequality in (11.20) holds. Dividing by $d_{P,\min}\ge t^{1-11b}$ proves the second for large t. Since $L^{2H}\le L^{2Hk}\le t^{1/2}$, the last follows from $\max_p\mu(p)\le t^{-1+11b}$. The relevant power of t is strictly negative, which proves the final assertion as well. □

The strict-cross prediction lemma can therefore be applied to any finite-measure family of the coordinate-state tests. Its history has at most L^{2H} values, and a k-tuple of histories has at most L^{2Hk} possible values. The next section selects the test family and keeps track of its integrated error; no asymptotic choice of k or of a remains.

12. Scores on common levels and a quantitative lower bound for rare mass

This section gives finite-parameter versions of the level and score arguments. The organization follows the method introduced in [28], but every estimate used below is stated with a numerical error. In particular, no passage to a subsequence, and no unspecified rate in a little-oh term, is used.

Write ω for the uniform probability measure on the retained incidence edges, and write μ and κ for its point and center marginals. All integrals over levels use the weighted measure of the preceding section. For a level, $Y_e\in\{0,1\}$ is its edge bit, $y_x=(A_x,B_x)$ its vertex state, and m its probability of inconsistency. An edge is consistent when exactly its A labels agree if $Y_e=1$, and exactly its B labels agree if $Y_e=0$. Put

$$\alpha=k^{-1/16},\qquad \vartheta=k^{-1/400},$$
$$\mathcal{C}=\{f\ge\alpha\},\quad \mathcal{B}=\{f<\alpha,\ m>k^{-1}\},\quad \mathcal{G}=\{f<\alpha,\ m\le k^{-1}\}.$$

The letters $\mathcal{C}, \mathcal{B}, \mathcal{G}$ denote common, damaged rare, and good rare levels, respectively. Denote their first two widths by $W_c = \int_{\mathcal{C}} 1$ and $W_b = \int_{\mathcal{B}} 1$. We have

$$S \geq k^{100}, \qquad \int m \leq 4, \qquad W_c \leq k^{1/16} S, \qquad W_b \leq 4k. \tag{12.1}$$

We shall also use the marginal expansion estimate

$$\Pr_{\omega}(F, G) \leq 2\sqrt{\mu(F)\kappa(G)}\big((\mu(F)\kappa(G))^{1/25} + \xi\big), \qquad \xi \leq k^{-100}. \tag{12.2}$$

All these statements hold for sufficiently large t with the fixed numerical parameters chosen earlier.

12.1. **Rare exceptions and the history map.** At a level of $\mathcal{G}$, let r be the minority bit. The coordinate which agrees on consistent majority-bit edges is B when $r = 1$, and A when $r = 0$. Its mismatch probability is at most $f + m$. The quantitative matching-label lemma, with its local constant two, supplies one label whose point and center exception probabilities have sum at most $2(f + m)$. Select such a label by a fixed order. Let I_x indicate that the selected coordinate of x is not this label, and define

$$h_P = \mathbb{E}_{\mu} I_p, \quad h_Q = \mathbb{E}_{\kappa} I_q, \quad H_x^* = \int_{\mathcal{G}} I_x, \qquad \mathcal{A} = \mathbb{E}_{\mu} H_p^*, \quad \mathcal{B}_* = \mathbb{E}_{\kappa} H_q^*.$$

The label selection is measurable: at every level only the labels on the finite vertex set are relevant, and ties are settled by their first vertex.

Lemma 12.1. *The exception means and the integrated probabilities of joint exceptions satisfy*

$$h_P + h_Q \leq 2(f + m) \leq 4\alpha \qquad \text{on } \mathcal{G}, \qquad \mathcal{A} + \mathcal{B}_* \leq 3S, \tag{12.3}$$

$$J_e := \int_{\mathcal{G}} \Pr_{\omega}(I_p = I_q = 1) \leq 9\vartheta^2 S, \tag{12.4}$$

$$J_w := \int_{\mathcal{G}} \Pr(I_p = I_{p'} = 1) \leq 21\vartheta S. \tag{12.5}$$

In the last line the law samples a center with marginal κ and two independent neighbors of that center. Moreover $f \leq h_P + h_Q + m$ on $\mathcal{G}$.

Proof. The local matching lemma applies because $\alpha + k^{-1} \leq 2^{-112}$. Integrating its conclusion gives $\mathcal{A} + \mathcal{B}_* \leq 2(S + 4) \leq 3S$. If both endpoints have the selected coordinate label and the edge is consistent, its bit is the majority bit. Every minority edge therefore has an exceptional endpoint or is inconsistent, proving the last assertion.

For the edge estimate, write $h = h_P$ and $u = h_Q$. Since $2\sqrt{hu} \leq h + u$, (12.2) gives

$$\Pr_{\omega}(I_p = I_q = 1) \leq (h + u)\big((hu)^{1/25} + \xi\big) \leq 3\vartheta^2(h + u).$$

Here $h + u \leq 4\alpha$ implies $hu \leq 4\alpha^2$, and $\alpha^{2/25} = \vartheta^2$. Integration proves (12.4).

For the wedge estimate, fix a point subset F of marginal mass $h \leq 4\alpha$, and put $g(q) = \Pr(p \in F \mid q)$. Its wedge self-intersection probability is $\mathbb{E}_{\kappa} g^2$. For $0 < d \leq 1$, the set $G_d = \{q : g(q) > d\}$ has marginal mass at most h/d. Consequently

$$\mathbb{E}_{\kappa} g^2 \leq dh + \Pr_{\omega}(F, G_d) \leq h\left[d + 2d^{-1/2}\left(\left(\frac{(4\alpha)^2}{d}\right)^{1/25} + \xi\right)\right].$$

Choose $d = (4\alpha)^{1/25}$. Then $d \leq 2\vartheta$, and the first nonconstant term inside the second summand is $2(4\alpha)^{73/1250} \leq 2d$; the term containing ξ is at most ϑ. The bracket is therefore at most 7ϑ. Apply this estimate to the point exceptions at each good rare level and integrate to obtain $J_w \leq 7\vartheta\mathcal{A} \leq 21\vartheta S$. □

For a state $y = (a, b)$ define its strict cross by

$$R_y = \{(u, v) : (u = a,\ v \neq b) \text{ or } (u \neq a,\ v = b)\}.$$

The prediction lemma applied to the normalized measure on $\mathcal{C}$ provides one fixed history map H with at most L^{2H} values. At a level, let ν_h be the distribution of the true state of a μ-point conditional on its history being h. We use true states, not the auxiliary relabelled states used in constructing the history. Put

$$\mathcal{R} = \int_{\mathcal{C}} \mathbb{E}_\omega[1 - \nu_{\mathsf{H}(p)}(R_{y_q})]. \tag{12.6}$$

Consistency implies strict-cross membership. The prediction bound thus gives

$$\mathcal{R} \leq 4 + W_c(4/L + 18(D/H)^{1/4}) \leq 26k^{-99}S. \tag{12.7}$$

When $W_c = 0$, take the empty history and set $\mathcal{R} = 0$. Throughout, the history is fixed before the fresh level is sampled.

For each center q, sample k independent neighbors and record their histories in an ordered template $\theta = (h_1, \ldots, h_k)$. Write $a(q, \theta) = \Pr(\theta \mid q)$ and M for the number of possible templates. The joint edge–template measure is

$$\omega(p, q, \theta) = \frac{\mathbf{1}_{(p,q)\in E}}{|E|} a(q, \theta), \qquad M \leq L^{2Hk}. \tag{12.8}$$

In this law the original point and the template are independent conditional on q. The parameter choice ensures

$$d_{Q,\min}^{-1} \leq k^{-100}, \qquad d_{P,\min}^{-1} \leq k^{-100}, \qquad \frac{M}{d_{P,\min}} \leq k^{-100}, \qquad |\mathsf{H}(P)| \max_p \mu(p) \leq k^{-100-1/16}. \tag{12.9}$$

12.2. **Three pair laws and an upper score bound.** We use the following three experiments, each fixed across the level space:

(1) Q: sample q with law κ and then two independent neighbors p, p' of q;
(2) P: sample p with law μ and then two independent neighbors q, q' of p;
(3) P^θ: sample (p, θ) with its marginal in (12.8), and then sample q, q' independently conditional on (p, θ).

For any one of these laws write x for the base, y_1, y_2 for the opposite endpoints, and Y_1, Y_2 for the two bits. The base includes θ in the third law. At a level let ρ be the probability of bit one conditional on this full base, and define

$$C_i = \int_{\mathcal{C}} \mathbb{E}_{\text{base}}[2\rho(1 - \rho)], \qquad i \in \{Q, P, P^\theta\}.$$

Lemma 12.2. *Every edge slot, after its base is averaged, has law ω. In the P^θ experiment, for every fixed p, the probability of $q = q'$, averaged over the conditional template law, is at most $M/d(p)$.*

Proof. Summing (12.8) over θ gives the original edge law; conditioning and then resampling one slot preserves that marginal. For the collision claim put $a_p(\theta) = \Pr(\theta \mid p)$. Whenever this is positive,

$$\Pr(q \mid p, \theta) = \frac{\mathbf{1}_{(p,q)\in E} a(q, \theta)}{d(p) a_p(\theta)} \leq \frac{1}{d(p) a_p(\theta)}.$$

The sum of the squares of a probability vector is at most its largest entry. Multiplication by $a_p(\theta)$ and summation over at most M templates proves the bound. This is a pointwise assertion in p, so it can be multiplied by an arbitrary nonnegative function of p before averaging. □

Define the score constants

$$d_* := \frac{1}{100}, \qquad \ell_* := \frac{1}{20000} = \frac{d_*^2}{2}, \qquad c(\rho) = \begin{cases} 1+\ell_*, & |\rho - \frac{1}{2}| \le d_*, \\ 1-\ell_*, & |\rho - \frac{1}{2}| > d_*. \end{cases}$$

If $Y_1 = Y_2 = 1$, let N mean $B_{y_1} = B_{y_2}$; if $Y_1 = Y_2 = 0$, let N mean $A_{y_1} = A_{y_2}$. Set N false when the bits differ. Thus N tests the coordinate not shared with the base on a consistent edge. The score is

$$\mathrm{sc} = \begin{cases} c(\rho)\mathbf{1}_{\{Y_1=Y_2,\, N\}}, & \text{on } \mathcal{C}, \\ 2I_x\mathbf{1}_{\{Y_1=Y_2=r,\, N\}}, & \text{on } \mathcal{G}, \\ 0, & \text{on } \mathcal{B} \end{cases} - \mathbf{1}_{\{Y_1\neq Y_2\}}. \tag{12.10}$$

Let D_i denote its expected integral in experiment i. Superscripts c and nc will denote its contributions on $\mathcal{C}$ and on the complement of $\mathcal{C}$.

Lemma 12.3 (An interval calculation)**.** *Let $b \le b'$, $T \ge 0$, $I_- = [b-T, b]$, and $I_+ = [b', b'+T]$. For any interval J and $u = |J \cap I_-| + |J \cap I_+|$,*

$$[(1-\ell_*)2T + 2\ell_* u - 2T]_+ \le \ell_* |J| \mathbf{1}_{\{b,b' \in \operatorname{int} J\}}.$$

If a real random variable X satisfies $|\Pr(X > s) - 1/2| \le d_*$ throughout J, then $\Pr(X \in \operatorname{int} J) \le 2d_*$.

Proof. The positive part on the left is $2\ell_*(u-T)_+$. Positivity requires both interval intersections to have positive length, which places b, b' in the interior of J. Also $u \le 2T$ and $u \le |J|$, whence $2(u-T) \le u \le |J|$. For the probability assertion, if $s < s'$ belong to the interior of J, then $\Pr(s < X \le s') = \Pr(X > s) - \Pr(X > s') \le 2d_*$. Exhaustion of the interior by such intervals gives the result, including possible atoms at its endpoints. □

Proposition 12.4 (Upper and rare lower budgets)**.** *Put $v_* = 10d_*^2\ell_* = 1/20000000$. For all three experiments,*

$$D_i \le v_* C_i + \vartheta S. \tag{12.11}$$

If x is the base side and y the opposite side, then

$$D_i^{nc} \ge 2\mathbb{E}H_x^* - 2\mathbb{E}H_y^* - 40\vartheta^2 S, \tag{12.12}$$

where the expectations use the original marginals, also for P^θ.

Proof. We first prove the upper bound for distinct opposite endpoints. Their two unit-edge jumps, oriented from point to center in both cases, are nonzero and distinct; call them d, d'. At a place v, set

$$b = \min(-\log|d|_v, -\log|d'|_v), \quad b' = \max(-\log|d|_v, -\log|d'|_v),$$
$$u = -\log|d-d'|_v, \quad \beta = u - b.$$

The two opposite-endpoint coordinate differences are, up to sign, $d-d'$ and $1/d - 1/d'$. Consequently

$$-\log|(d-d')(1/d - 1/d')|_v = 2\beta - (b'-b). \tag{12.13}$$

Let $c_v = \log 2$ at the archimedean places and zero otherwise. The triangle inequality gives $\beta \ge -c_v$; put $T_v = \beta + c_v \ge 0$. A positive score with both bits one must lie in $[b - T_v, b]$, and a positive score with both bits zero must lie in $[b', b' + T_v]$. Indeed the matching nonshared coordinate bounds its difference by the corresponding grid scale, with factor at most $\sqrt{2}$ at infinity; substitution of the definitions gives these two inclusions. The integrated negative score at this place is exactly $b' - b$.

There are two possibilities. If a rare positive score with minority bit one occurs at a threshold s_0, then $s_0 < b$ and $\Pr(Y = 1) < \alpha$ at s_0. The monotonicity of this probability in the threshold puts every common level, and every rare level of minority bit zero, below s_0. They therefore cannot produce a positive score in the right interval. The assertion with the two bits reversed follows in the same way. Thus, if any rare positive score occurs, all positive scores are supported on only one of the two intervals, and their integral is at most $2T_v$.

Otherwise all positive scores are common. For the fixed base define

$$J_v = \{s : (v,s) \in \mathcal{C},\ |\rho(v,s) - 1/2| \le d_*\}.$$

This is an interval: both the global and the base-conditional bit probabilities are decreasing functions of the threshold, since the sampling laws are fixed across levels. Extending the coefficient $1-\ell_*$ to both full intervals and its additional coefficient $2\ell_*$ to J_v can only increase the positive score. Lemma 12.3 bounds the excess over $2T_v$ by

$$\ell_* |J_v| \mathbf{1}_{\{-\log|d|_v, -\log|d'|_v \in \operatorname{int} J_v\}}.$$

This nonnegative expression is also an upper allowance in the previous, rare-positive case. Subtract the negative score, use (12.13), and sum with the place weights. The product formula for $(d - d')(1/d - 1/d') \neq 0$ cancels the logarithmic expression. The remaining unconditional allowance is $2\sum_v w_v c_v = 2\log 2$.

For independent slots at the fixed base, the expected indicator in the last display is at most $4d_*^2$, by Lemma 12.3. We may drop the distinctness restriction here because this allowance is nonnegative. On J_v, the variation $2\rho(1-\rho)$ is at least $2(1/4 - d_*^2)$. Averaging and integrating therefore bounds the total contribution of distinct pairs by

$$2\log 2 + \frac{4d_*^2\ell_*}{2(1/4 - d_*^2)} C_i \le 2\log 2 + v_* C_i.$$

The argument also includes places whose chosen level interval has width zero: their positive and negative scores vanish and the same placewise upper inequality still holds.

For equal opposite endpoints the score integral is at most $2(W_c + H_x^*)$. Ordinary collision probabilities are $1/d(x)$, and the pointwise template collision bound is Lemma 12.2. Thus every diagonal contribution is at most $2k^{-100}(W_c + 3S)$. Together with $S \ge k^{100}$ and (12.1), this is enough to absorb the constant $2\log 2$ and all diagonal terms into ϑS. This proves (12.11).

For the lower bound, work at a good rare level and let M_1, M_2 indicate inconsistency of the two slots. If the base is exceptional, both opposite endpoints are normal, and both edges are consistent, then both bits are the minority bit and the required nonshared coordinate matches. Hence the positive score is at least

$$2I_x - 2I_x I_{y_1} - 2I_x I_{y_2} - 2M_1 - 2M_2.$$

If both opposite endpoints are normal and both edges consistent, their bits agree, whether or not the base is exceptional. The negative score is therefore at least $-I_{y_1} - I_{y_2} - M_1 - M_2$. The single-slot marginals are the original edge marginals, so integration gives a lower bound

$$2\mathbb{E}H_x^* - 2\mathbb{E}H_y^* - 4J_e - 6\int m.$$

On damaged levels the score is at least minus one. By (12.4) and (12.1), the full error is at most $36\vartheta^2 S + 24 + 4k \le 40\vartheta^2 S$, proving (12.12). □

12.3. **Candidate centers and transfer to the actual point.** At a fixed level set

$$\widetilde{\nu}_q = \mathbb{E}_{p|q}\nu_{\mathsf{H}(p)}, \qquad \mathcal{Y}_\theta = \{y \text{ a center state} : \nu_{h_j}(R_y) \ge 1 - k^{-2} \text{ for every } 1 \le j \le k\}.$$

A union bound followed by Markov's inequality shows that

$$\int_{\mathcal{C}} \Pr(y_q \notin \mathcal{Y}_\theta) \le k^3 \mathcal{R} \le 26k^{-96} S. \tag{12.14}$$

Lemma 12.5 (Sampling strict crosses)**.** *If $X_1, \dots, X_k$ are independent states of law ν, then, for $0 < \eta < 1$ and $k\eta \ge 8$,*

$$\Pr(\exists y : \nu(R_y) < 1 - \eta,\ X_1, \dots, X_k \in R_y) \le 2(2k+1)^2 2^{-k\eta/2}.$$

In particular,

$$\Pr(\exists y \in \mathcal{Y}_\theta : \widetilde{\nu}_q(R_y) < 1 - k^{-1/4} \mid q) \le 4(2k+1)^2 2^{-k^{3/4}/2}. \tag{12.15}$$

Proof. Draw k further independent states. If the first k states admit a cross as in the event, select one in a fixed order. For that cross, the number of misses among the new states has mean $k\beta$ and variance at most $k\beta$, where $\beta > \eta$. Chebyshev's inequality bounds the probability of fewer than $k\eta/2$ misses by $4/(k\beta) \le 1/2$. It remains to bound twice the probability of a cross which contains one half of $2k$ samples and misses at least $k\eta/2$ members of the other half.

Condition on the $2k$ indexed samples and split their indices uniformly into two groups of size k. A strict cross produces at most $(2k+1)^2$ different subsets of these indices: each of its two coordinate labels can be one of the at most $2k$ labels observed in that coordinate, or a label not observed. If a pattern has m misses, the probability that all misses belong to the second group is zero for $m > k$, and otherwise at most

$$\frac{k(k-1)\cdots(k-m+1)}{(2k)(2k-1)\cdots(2k-m+1)} \le 2^{-m}.$$

The union bound proves the first assertion. This argument allows repeated states, since the sampling indices remain distinct.

For the second assertion, after selecting the histories in θ, draw one synthetic state independently from each ν_{h_j}. Before the histories are revealed these are independent of law $\widetilde{\nu}_q$. If the template admits a violating candidate, choose one using only the template. Conditional on this choice, all synthetic states are in its cross with probability at least $1 - k \cdot k^{-2}$. Apply the finite framework with $\eta = k^{-1/4}$ and divide by $1 - 1/k \ge 1/2$. □

Lemma 12.6 (Restricted-support transfer)**.** *Fix a level. Under the law* (12.8)*, let $\mathcal{U}(q,\theta)$ be a selected row, a selected column, or a selected set of at most two states. On an event G determined by (q,θ) suppose that $\widetilde{\nu}_q(\mathcal{U}(q,\theta)) \ge 1 - 3\eta^2$, where $0 < \eta < 1/2$. Then*

$$\Pr(G,\ y_p \notin \mathcal{U}(q,\theta)) \le 8\eta.$$

Proof. The conditional history law of the original point, given (q,θ), is the ordinary history law of a neighbor of q. Consequently, on G,

$$\mathbb{E}[\nu_{\mathsf{H}(p)}(\mathcal{U}(q,\theta)^c) \mid q,\theta] \le 3\eta^2.$$

The event that this conditional measure is greater than η has probability at most 3η.

For each history h, define exceptional states using only h and the level. If there is a row of ν_h-mass at least $1-\eta$, that row is unique; include its complement as the first exceptional set. Use the analogous column complement as the second set. Each has mass at most η, with an empty set used when the respective concentrated support does not exist. If there is a set of at most two states of mass at least $1-\eta$, put $M_h = \{x : \nu_h(x) > \eta\}$. Every such concentrated two-state support contains M_h, and one such support shows that $\nu_h(M_h^c) \le 3\eta$. Use M_h^c as the third exceptional set; again use the empty set if no such support exists. The union has ν_h-mass at most 5η.

The original point has marginal μ, and conditional on its history its state has exactly law ν_h. Its probability of belonging to this union is therefore at most 5η, under the full edge–template measure. Outside this event and the preceding event of probability at most 3η, the actual state belongs to whichever permitted concentrated support has been selected. The assertion follows. No assertion about the point law after conditioning on the selected support is required. □

Two distinct states $(a,b),(a,b')$ in the same row have intersecting crosses contained in that row, namely

$$R_{(a,b)} \cap R_{(a,b')} = \{(a,v) : v \notin \{b,b'\}\}. \tag{12.16}$$

A third state outside the row reduces this intersection to at most one state. The column statements are identical with the coordinates exchanged. When both coordinates differ,

$$R_{(a,b)} \cap R_{(a',b')} = \{(a,b'),(a',b)\}. \tag{12.17}$$

Three states with pairwise distinct first and pairwise distinct second coordinates have empty common cross. These identities prove the following exhaustive alternatives for the candidate set: it is empty; it is a singleton; it has a repeated coordinate, with either all candidates in that row or column or a third outside it; or it has pairwise distinct coordinates, with either two candidates or at least three.

Select the two or three witnesses used in these alternatives by a fixed order. On the complement of the failure event in (12.15), their common cross has $\widetilde{\nu}_q$-mass at least $1 - 3k^{-1/4}$. The containing row, column, or at most two states furnished by (12.16)–(12.17) is therefore a permitted support in Lemma 12.6, with $\eta = k^{-1/8}$. Its transfer cost, integrated over common levels, is at most $8k^{-1/8}W_c \le 8\vartheta^{25}S$. Empty common crosses cannot have the asserted mass and yield no good edge.

Declare an edge with its template bad at a common level if it is inconsistent, if its center state is not a candidate, if the event in (12.15) occurs, or if the selected support fails to contain the actual point. By (12.14), the two lemmas, and (12.1), its integrated probability satisfies

$$B_{\mathrm{bad}} \le 4 + 26k^{-96}S + 36k^2 2^{-k^{3/4}/2}W_c + 8\vartheta^{25}S \le 10\vartheta^2 S. \tag{12.18}$$

For example, the exponential term times W_c/S is at most ϑ^2 for $k \ge 2^{1048576}$: the logarithm of its polynomial prefactor is smaller than $10\log_2 k$, whereas $k^{3/4}/2$ exceeds this and any required fixed multiple of $\log_2 k$ by that threshold. The remaining terms are bounded directly by powers of k.

12.4. **Suppression of common center variation.** The following elementary inequality is useful for every $0 \le \rho \le 1$:

$$c(\rho)\big(\rho^2 + (1-\rho)^2\big) - 2\rho(1-\rho) \ge \ell_*/2. \tag{12.19}$$

To verify it, subtract $2\rho(1-\rho)$ from the bracket with coefficient one, obtaining $(2\rho-1)^2$. In the central window the extra term is at least $\ell_*/2$. Outside the window the result is at least $4d_*^2 - \ell_* > \ell_*/2$.

Proposition 12.7. *We have*

$$C_Q + C_{P^\theta} \le 2^{19}\vartheta S, \qquad \mathcal{B}_* \le \mathcal{A} + 2^{20}\vartheta S. \tag{12.20}$$

Proof. Fix a level and a positive-mass template slice. Let b_θ be the single-edge bad probability in this slice. In the center experiment, sample its center using the conditional law of q given θ; the two points still have their ordinary independent law conditional on q, because the template was independent of the original point given q. In the point–template experiment sample the point conditional on θ, then its two centers independently conditional on (p,θ). Both experiments have the same conditional single-edge law. Thus a pair has a bad slot with probability at most $2b_\theta$ in each experiment.

Let V_Q, V_{P^θ} be the average full-law variations in this slice, and let d_Q, d_{P^θ} be its common expected scores. We claim

$$d_Q + d_{P^\theta} \ge \frac{\ell_*}{4}(V_Q + V_{P^\theta}) - 16b_\theta. \tag{12.21}$$

Each variation is at most $1/2$, and each score is at least minus its variation. We check all the candidate alternatives explicitly.

If there are no candidates, every edge is bad, so $b_\theta = 1$, and the bound follows from $d_Q + d_{P^\theta} \ge -1$. For a single candidate state, two good edges of a point–template star have the same opposite state and hence the same bit and the required coordinate match. Its mean score is at least $1 - \ell_* - 4b_\theta$. The other score is at least $-1/2$. Their sum is more than sufficient for (12.21).

If all candidates belong to one row or one column and at least two are distinct, support transfer puts all good point states in the same row or column. Every good edge therefore has the same bit. A bit difference requires a bad slot, so $V_Q + V_{P^\theta} \le 4b_\theta$. The general lower bound by minus variation proves the claim. If there is instead a third candidate outside the shared row or column, the selected intersection has at most one state. If it is empty all edges are bad; otherwise all good points have the same state, and the center-star score is at least $1 - \ell_* - 4b_\theta$. The other score is at least $-1/2$, as in the singleton case.

Finally suppose distinct candidates share no coordinate. Three candidates have empty common cross and no good edges. With exactly two candidates, good point states are their opposite corners. At either kind of base, the bit of a good edge determines its opposite state. Thus equal bits on a good pair imply the required coordinate match. The full-law expected score is at least the left side of (12.19) minus $4b_\theta$, after averaging the bases. Each orientation is at least $\ell_*/2 - 4b_\theta$, which proves (12.21). The penalty $4b_\theta$ used here follows by bounding the missing positive reward by $(1 + \ell_*)$ times the probability $2b_\theta$ of a bad slot.

All these comparisons use the full conditional ρ in $c(\rho)$. In particular, no deletion and renormalization of the bit law is made near the discontinuities of c. Multiply (12.21) by the actual probability of its template, sum the slices, and integrate. By (12.18),

$$D^c_Q + D^c_{P^\theta} \ge \frac{\ell_*}{4}(C_Q + C_{P^\theta}) - 160\vartheta^2 S.$$

Adding the two rare lower budgets cancels their main exception means. The upper budgets then give

$$g_*(C_Q + C_{P^\theta}) \le (2\vartheta + 240\vartheta^2)S, \qquad g_* := \ell_*(1/4 - 10d_*^2) = 0.00001245 > 10^{-5}.$$

Since ϑ is smaller than $1/240$, the first assertion of (12.20) follows. The common center score alone is at least $-C_Q$. Combining this with both center budgets gives

$$2\mathcal{B}_* - 2\mathcal{A} \le (1 + v_*)C_Q + \vartheta S + 40\vartheta^2 S.$$

The first assertion now implies the second, with the displayed constant. □

12.5. **A second score and positive rare point mass.** For a state measure ν, let ν_A, ν_B be its coordinate marginals. Define

$$\tau(\nu) = \|\nu_A\|_2^2 + \|\nu_B\|_2^2 - 1, \qquad \mathcal{T} = \int_{\mathcal{C}} \mathbb{E}_\mu \tau(\nu_{\mathsf{H}(p)}).$$

The squared norms are sums of squares of the atom probabilities.

Lemma 12.8. *The two-prediction score satisfies*

$$\mathcal{T} \le 2\mathcal{A} + 2\vartheta S. \tag{12.22}$$

Proof. Choose a history according to its μ-marginal, and then choose two independent points p', p'' from the conditional point law of that history. This coupling is fixed across the levels, and both its marginals are μ. The integrand of $\mathcal{T}$ is the mean of

$$F(p', p'') = \mathbf{1}_{\{A_{p'} = A_{p''}\}} + \mathbf{1}_{\{B_{p'} = B_{p''}\}} - 1.$$

For a distinct pair put $Z = z_{p'} - z_{p''}$ and $W = w_{p'} - w_{p''}$. Both are nonzero. At a place let the level interval be $[l, u]$. Its endpoint convention gives $l \le -\log|Z|_v$ and $u \ge \log|W|_v$. An A match can occur only below $-\log|Z|_v + c'_v$, and a B match only above $\log|W|_v - c'_v$, where $c'_v = \log\sqrt{2}$ at infinity and zero otherwise. Bounding the lengths of these two threshold ranges gives

$$\int_l^u F(p', p'')\, ds \le -\log|Z|_v - \log|W|_v + 2c'_v.$$

The product formula yields an all-level integral at most $\log 2$. On a good rare level where both points are normal their dominant labels match, so $F \ge 0$. On every good rare level $F \ge -I_{p'} - I_{p''}$, and on damaged levels $F \ge -1$. Thus a distinct pair has common-level integral at most $\log 2 + H^*_{p'} + H^*_{p''} + W_b$.

For a diagonal pair the common-level integral is W_c. Its probability is

$$\sum_{p \in P} \frac{\mu(p)^2}{\mu(\mathsf{H}^{-1}(\mathsf{H}(p)))} \le |\mathsf{H}(P)| \max_p \mu(p).$$

By (12.9) its contribution is at most $k^{-100} S \le \vartheta S$. The two preserved marginals bound the mean of the distinct-pair height terms by $2\mathcal{A}$, while $\log 2 + 4k \le \vartheta S$. This proves (12.22). □

Proposition 12.9 (A numerical rare-mass lower bound)**.** *The good rare point mass satisfies*

$$\mathcal{A} \ge 2^{-132} S. \tag{12.23}$$

Furthermore

$$C_P \le 2^{18} \mathcal{A} + 2^{38} \vartheta S. \tag{12.24}$$

Proof. At a common level call an ordinary edge bad when it is inconsistent or when $\nu_{\mathsf{H}(p)}(R_{y_q}) < 1 - \zeta$, where $\zeta = k^{-2}$. Its integrated bad probability is at most $4 + k^2 \mathcal{R} \le \vartheta^2 S$. Fix a point p and write b_p for its conditional bad probability, $\nu = \nu_{\mathsf{H}(p)}$, and ρ_p for its full conditional bit probability. Let $\mathcal{Y}_p$ be the states of its good neighbors. For every $y \in \mathcal{Y}_p$, $\nu(R_y) \ge 1 - \zeta$, and consistency places the actual point state in R_y. We prove

$$d_P(p) + \tau(\nu) \ge \frac{\ell_*}{4} 2\rho_p(1 - \rho_p) - 16(b_p + \zeta), \tag{12.25}$$

where $d_P(p)$ is the conditional common score.

If $\mathcal{Y}_p$ is empty then $b_p = 1$, while $d_P(p) \ge -1/2$ and $\tau(\nu) \ge -1$, which suffices. If it is a singleton (a, b), set $r = \nu_A(a)$, $s = \nu_B(b)$. Then $r + s \ge 1 - \zeta$, and hence $\tau(\nu) \ge r^2 + s^2 - 1 \ge -1/2 - \zeta$. Two good neighbors have the same state and bit, giving $d_P(p) \ge 1 - \ell_* - 4b_p$. This proves the claim in that case.

If two states share a row, their common cross has ν-mass at least $1 - 2\zeta$ and lies in that row, so $\tau(\nu) \ge (1 - 2\zeta)^2 - 1 \ge -4\zeta$. If all good neighbor states belong to this row, consistency puts the actual point in it and makes every good bit one. Therefore $2\rho_p(1 - \rho_p) \le 2b_p$ and $d_P(p) \ge -2b_p$, proving (12.25). Columns work identically with bit zero. If a third state is outside the shared row or column, the three crosses intersect in at most one state and their intersection has mass at least $1 - 3\zeta$. It must therefore be a singleton. Both coordinate marginals have an atom of at least that mass, and $\tau(\nu) \ge 2(1 - 3\zeta)^2 - 1 \ge 1 - 12\zeta$. The lower bound $d_P(p) \ge -1/2$ suffices.

In the remaining nontrivial case, the distinct neighbor states share neither coordinate. There cannot be three, because the actual point belongs to all their crosses. For two, their opposite corners carry ν-mass at least $1 - 2\zeta$. Each coordinate marginal consequently has at least this mass on two labels, and $\tau(\nu) \ge (1 - 2\zeta)^2 - 1 \ge -4\zeta$. The actual point is an opposite corner. Its good-edge bit determines the neighbor state, so (12.19) gives $d_P(p) \ge \ell_*/2 - 4b_p$. This finishes the proof of (12.25); the coefficient sixteen bounds each of the displayed penalties without changing the full bit law.

Average and integrate (12.25). Since $\zeta W_c \le \vartheta^2 S$, it follows that

$$D^c_P + \mathcal{T} \ge \frac{\ell_*}{4} C_P - 32\vartheta^2 S.$$

Combining this with the rare lower budget, the upper budget, and (12.22) gives

$$g_* C_P \le 2\mathcal{B}_* + 3\vartheta S + 72\vartheta^2 S \le 2\mathcal{B}_* + 4\vartheta S.$$

Use (12.20), $2/g_* < 2^{18}$, and $(2^{21} + 4)/g_* < 2^{38}$ to obtain (12.24). In particular,

$$C_P + C_Q \le 2^{18} \mathcal{A} + 2^{39} \vartheta S. \tag{12.26}$$

It remains to compare the edge minority probability to ordinary star variation. At any level give each point and center its conditional majority-bit label. If V_P, V_Q are the two ordinary star variations, the bit differs from these labels with probabilities at most V_P, V_Q, because $\min(\rho, 1-\rho) \le 2\rho(1-\rho)$. Thus endpoint labels disagree with probability at most $v = V_P + V_Q$. When $v \le 2^{-112}$, the local matching-label bound supplies a common label with total marginal exceptions at most $2v$; the edge bit differs from this label with probability at most $3v$. When $v > 2^{-112}$, the trivial bound $f \le 1/2$ is enough. Uniformly,

$$f \le 2^{112}(V_P + V_Q). \tag{12.27}$$

Integrating (12.27) over common levels and using (12.26) bounds their minority mass by $2^{130}\mathcal{A} + 2^{151}\vartheta S$. On good rare levels Lemma 12.1 bounds it by $\mathcal{A} + \mathcal{B}_* + 4$, and damaged levels contribute at most $2k$. Consequently

$$S \le (2^{130}+2)\mathcal{A} + (2^{151}+2^{20})\vartheta S + 4 + 2k \le 2^{131}\mathcal{A} + 2^{152}\vartheta S.$$

Since $\vartheta \le 2^{-154}$, this implies $\mathcal{A} \ge 2^{-132}S$, as asserted. □

13. Weighted extraction of a graph of ratios

We retain the probability laws and notation of the preceding sections. In particular, μ, κ are the two edge marginals, $\mathcal{C}, \mathcal{B}, \mathcal{G}$ are the common, damaged, and good rare levels, and I_x is the exception indicator on $\mathcal{G}$. Write

$$A = \mathbb{E}_\mu H_p^*, \quad B = \mathbb{E}_\kappa H_q^*, \quad \vartheta = k^{-1/400}, \quad R_0 = k^{1/8}, \quad c_0 = 2^{-133}.$$

The quantitative conclusions already proved are

$$A + B \le 3S, \quad A \ge 2^{-132}S, \quad C_Q \le 2^{19}\vartheta S, \quad W_c \le k^{1/16}S, \quad W_b \le 4k, \quad \mathbb{E}_E m_e \le 4. \tag{13.1}$$

Here C_Q is the integral, over common levels, of the probability that the two bits in an ordinary center star differ. We prove the extraction rather than assuming that a small exceptional probability controls lost height.

Lemma 13.1 (A pointwise inequality)**.** *If q, q' are distinct neighbors of a point p, then*

$$2H_p^* \le 3(H_q^* + H_{q'}^* + m_{pq} + m_{pq'}) + W_c + W_b + 2\log 2. \tag{13.2}$$

Proof. At a good rare level set $X = I_p$, $U = I_q$, $V = I_{q'}$ and let M, M' record the two inconsistencies. The rare positive score P_+ equals two when p is exceptional, both bits are minority bits, and the two other endpoints share the nonmatched coordinate; otherwise it is zero. Let $N = |Y_{pq} - Y_{pq'}|$. When $U = V = M = M' = 0$, both bits are the same function of X, and $P_+ = 2X$. Therefore, without restrictions,

$$P_+ \ge 2X - 2(U + V + M + M'), \qquad N \le U + V + M + M',$$

and $P_+ - N \ge 2X - 3(U + V + M + M')$.

We also need a placewise upper bound for this score. Put $d = z_p - z_q$, $d' = z_p - z_{q'}$, and, at a place v, let

$$b = \min(-\log|d|_v, -\log|d'|_v), \quad b' = \max(-\log|d|_v, -\log|d'|_v), \quad u = -\log|d - d'|_v.$$

Let $c_v = \log 2$ at an archimedean entry and zero otherwise, and put $T_v = u - b + c_v \ge 0$. Coordinate agreement permits positive scores only in $[b - T_v, b]$ or $[b', b' + T_v]$. Since the fraction of edges with bit one decreases with the threshold, a rare positive score can occur on only one of these two intervals. Its integrated positive contribution is at most $2T_v$. The all-level negative contribution is $b' - b$. Moreover,

$$-\log|(d - d')(d^{-1} - d'^{-1})|_v = 2(u - b) - (b' - b).$$

The element inside the absolute value is nonzero. Summing over places annuls its logarithm by the product formula and leaves at most $2\log 2$. Outside $\mathcal{G}$, the negative score has absolute value at

most one, and its total cost is at most $W_c + W_b$. Integrating the lower bound and comparing with this upper bound proves (13.2). □

Lemma 13.2 (A uniform tail bound)**.** *For all sufficiently large t, with the fixed parameters of the paper,*

$$\mathbb{E}_\mu\big[H_p^*\mathbf{1}_{H_p^*>R_0S}\big] \le 2^{13}\vartheta^4 S. \tag{13.3}$$

Proof. Denote the expectation on the left by A_T. On the event $H_p^* > R_0S$, the last three terms of (13.2) have sum at most H_p^*: their ratio to R_0S is at most $k^{-1/16} + (4k + 2\log 2)/(R_0S) < 1$. After absorption, cap either neighbor height at H_p^*. If a cap changes a term, that term alone makes the resulting inequality true. Thus

$$H_p^* \le 3\big(\min(H_p^*, H_q^*) + \min(H_p^*, H_{q'}^*) + m_{pq} + m_{pq'}\big).$$

Sampling two neighbors independently and retaining distinct neighbors on the left gives

$$(1 - d_{P,\min}^{-1})A_T \le 6\mathbb{E}_E[\min(H_p^*, H_q^*)\mathbf{1}_{H_p^*>R_0S}] + 24. \tag{13.4}$$

For $u \ge 0$ put $F_u = \{p : H_p^* > \max(u, R_0S)\}$ and $G_u = \{q : H_q^* > u\}$. The first marginal mass is at most $3/R_0$. The marginal expansion inequality consequently gives, uniformly in u,

$$\Pr_E(F_u, G_u) \le \theta\sqrt{\mu(F_u)\kappa(G_u)}, \qquad \theta = 2(3/R_0)^{1/25} + 2\xi \le 4\vartheta^2.$$

Layer cake and Cauchy–Schwarz imply

$$\mathbb{E}_E[\min(H_p^*, H_q^*)\mathbf{1}_{H_p^*>R_0S}] \le \theta\sqrt{A_T B} \le \theta\sqrt{3A_T S}.$$

For $d_{P,\min} \ge 2$, (13.4) and Young's inequality yield

$$\tfrac{1}{2}A_T \le 6\theta\sqrt{3A_T S} + 24 \le \tfrac{1}{4}A_T + 108\theta^2 S + 24.$$

Hence $A_T/S \le 432\theta^2 + 96/S \le 2^{13}\vartheta^4$, since $S \ge k^{100}$. □

For each point define a profile on the set of places by

$$a_p(v) = \int_{\{s:(v,s)\in\mathcal{G}\}} \sigma(v,s)I_p(v,s)\,ds,$$

where $\sigma = 1$ when the minority bit is one and $\sigma = -1$ otherwise. For a place profile b use the weighted norm $\|b\|_1 = \sum_v^* |b(v)|$. Let $n_{pq}(v) = -\log|z_p - z_q|_v$.

Lemma 13.3 (Wedge approximation)**.** *For an ordinary wedge (p, q, r) there are nonnegative errors E_1, E_2 such that*

$$\|(n_{pq} - n_{rq}) - (a_p - a_r)\|_1 \le E_1, \tag{13.5}$$

$$\|n_{pq} - n_{rq}\|_1 \ge H_p^* + H_r^* - E_2, \tag{13.6}$$

$$\mathbb{E}(E_1 + E_2) \le 2^{20}\vartheta S. \tag{13.7}$$

Proof. At a good rare level let b be the majority bit and M_{pq} the inconsistency indicator. The identity $Y_{pq} = b + \sigma(I_p + I_q)$ has error at most $2(I_pI_q + M_{pq})$. Subtract its two versions in the wedge; the center term cancels. Threshold integration gives (13.5) with

$$E_1 = \int_{\mathcal{C}} |Y_{pq} - Y_{rq}| + W_b + 2\int_{\mathcal{G}} (I_pI_q + I_rI_q + M_{pq} + M_{rq}).$$

Pointwise one also has

$$|Y_{pq} - Y_{rq}| \ge I_p + I_r - 2(I_pI_r + I_pI_q + I_rI_q + M_{pq} + M_{rq}).$$

If no deducted term occurs, exactly one exceptional point forces opposite bits; whenever a deducted term occurs the displayed lower bound is nonpositive or follows immediately. Integrating proves (13.6) with

$$E_2 = 2\int_{\mathcal{G}} (I_p I_r + I_p I_q + I_r I_q + M_{pq} + M_{rq}).$$

The preceding joint-exception estimates bound the edge and wedge integrals by $9\vartheta^2 S$ and $21\vartheta S$, respectively. Thus

$$\mathbb{E}(E_1 + E_2) \le C_Q + W_b + 72\vartheta^2 S + 42\vartheta S + 32 \le 2^{20}\vartheta S.$$

□

Lemma 13.4 (Selecting height before probability)**.** *Put $\theta = 1/(4K)$ and $B_k = 1 + \lceil \log_2(2R_0/c_0) \rceil$. There is $J \ge K$ and a set of ordinary wedges of probability at least $c_0/(R_0 B_k)$ such that*

$$J \le H_p^* + H_r^* \le 2J, \qquad E_1 + E_2 \le \theta(H_p^* + H_r^*). \tag{13.8}$$

All these wedges have $p \ne r$.

Proof. Set $W = H_p^* + H_r^*$ and $Z = E_1 + E_2$. Then $\mathbb{E}W = 2A \ge 4c_0 S$. Remove $Z > \theta W$. Its lost W-mass is at most

$$\mathbb{E}Z/\theta \le 2^{22} K\vartheta S \le c_0 S/4.$$

The inequality uses $\vartheta \le 2^{-157}/K$, which follows from the chosen k. Removing $W < c_0 S$ loses at most $c_0 S$. Finally, on $W > 2R_0 S$, one summand exceeds $R_0 S$ and is at least $W/2$. Thus

$$\mathbb{E}[W\mathbf{1}_{W>2R_0S}] \le 4\mathbb{E}_\mu[H_p^*\mathbf{1}_{H_p^*>R_0S}] \le 2^{15}\vartheta^4 S \le c_0 S/4.$$

At least $2c_0 S$ weighted mass remains, and every remaining weight is at most $2R_0 S$. The remaining probability is at least c_0/R_0. Divide $[c_0 S, 2R_0 S]$ into at most B_k dyadic intervals, assigning each endpoint to one interval. One interval has the stated probability. Its lower endpoint is $J \ge c_0 S \ge K$. If $p = r$, (13.6) would imply $0 \ge W - E_2 \ge (1-\theta)W > 0$, a contradiction. □

Lemma 13.5 (Deleting all popular unit circles)**.** *Call any real unit circle popular if it contains at least $t^{9/10}$ points of P, whether or not its center belongs to Q. The wedge probability that p, r share such a circle tends to zero as $t \to \infty$ with K fixed.*

Proof. Let $v = |P| \le t^{1+2a+\varepsilon}$. Pair counting bounds the number M_c of popular circles by $O(v^2 t^{-9/5}) = O(t^{1/5+4a+2\varepsilon})$. At most $M_c(M_c - 1)$ points lie on two of them, since each pair of circles intersects in at most two points. By the marginal atom bound their total point probability is

$$O(t^{-3/5+8a+4\varepsilon+11b}) = o(1).$$

Centers whose own unit circles are popular have probability at most

$$M_c \max_q \kappa(q) = O(t^{-9/5+3a+2\varepsilon+11b}) = o(1).$$

Outside these events, fix p, q. There is at most one popular circle through p, and it differs from the circle centered at q. At most two neighbors r of q lie on it. The remaining probability is at most $2/d_{Q,\min} \le 2t^{-a+11b}$, which tends to zero. All exponents are strictly negative for the fixed numerical parameters. □

Proposition 13.6 (Finite organized pair graph)**.** *For all sufficiently large t in a putative incidence counterexample, there are $J \ge K$ and a graph $G \subset P \times P$, using two copies of P, such that*

$$|P| \le t^{1+1/K}, \qquad |G| \ge t^{2-1/K}.$$

Each edge has different endpoints and a nonzero algebraic ratio λ_{pr} satisfying

$$(z_p - z_r)(w_p - w_r) = 2 - \lambda_{pr} - \lambda_{pr}^{-1}, \tag{13.9}$$

$$\mathrm{h}(\lambda_{pr}) \geq J/3, \tag{13.10}$$

$$\mathrm{h}\left(\frac{\lambda_{pr}\lambda_{p'r'}}{\lambda_{pr'}\lambda_{p'r}}\right) \leq J/K \quad \textit{on every complete rectangle.} \tag{13.11}$$

No edge has both endpoints on a popular unit circle.

Proof. The probability in Lemma 13.4 is a fixed positive number once K is fixed. Lemma 13.5 therefore removes less than half of it for sufficiently large t. For a fixed off-diagonal pair (p, r), there are at most two possible centers, so its original wedge probability is at most $2/(|E|d_{Q,\min})$. Thus the number of retained pairs is at least

$$\frac{c_0}{4R_0B_k}|E|d_{Q,\min} \geq c(K)t^{2+3a-21b} \geq t^{2-1/K}.$$

Also $|P| \leq t^{1+2a+\varepsilon} \leq t^{1+1/K}$. Select one retained center $q = q(p, r)$ for every pair, and define $\lambda_{pr} = (z_p - z_q)/(z_r - z_q)$. Writing the two jumps as d, d', the unit equations give

$$(z_p - z_r)(w_p - w_r) = (d - d')(d^{-1} - d'^{-1}) = 2 - d/d' - d'/d.$$

The product formula and (13.6) give $2\,\mathrm{h}(\lambda_{pr}) \geq (1 - \theta)J$, proving (13.10). Put $b_{pr}(v) = \log|\lambda_{pr}|_v + a_p(v) - a_r(v)$. Equation (13.5) and (13.8) give $\|b_{pr}\|_1 \leq 2\theta J$. On a rectangle, the unary profiles cancel. The norm of the resulting logarithmic defect is at most $8\theta J$, and its height is half that norm. Hence it is at most $4\theta J = J/K$. □

14. A finite algebraic obstruction

This section replaces the limiting height field by a finite algebraic argument. The geometric square-root obstruction follows the mechanism of [28]; we give its proof, including the field-enlargement step and the exceptional curve case. The additional ingredient is a critical-value construction with a uniform numerical height bound. All algebraic extensions in this section have characteristic zero.

For an ordinary coordinate pair $x = (x_1, x_2)$ put $z_x = x_1 + \mathrm{i}x_2$ and $w_x = x_1 - \mathrm{i}x_2$. We use the complexified quadratic form, not a Hermitian form:

$$L(p, r) = (p_1 - r_1)^2 + (p_2 - r_2)^2 = (z_p - z_r)(w_p - w_r).$$

There are always 20 left positions $p_1, \ldots, p_{20}$ and 5 right positions $r_1, \ldots, r_5$. A grid therefore has 50 position coordinates and 100 pairs. We abbreviate $L(p_i, r_j)$ to L_{ij}.

14.1. **A fixed set of exceptional position tuples.**

Definition 14.1. Let $\mathcal{Z} \subseteq \mathbb{A}^{50}_{\mathbb{C}}$ be the set of position tuples for which there are scalars R_{ij}, A_i, A'_j and vectors $H_i, U_j \in \mathbb{C}^2$ satisfying

$$R_{ij}^2 = L_{ij}(L_{ij} - 4), \tag{14.1}$$

$$2(p_i - r_j) \cdot (H_i - U_j) = R_{ij}(A_i - A'_j), \tag{14.2}$$

$$A_i - A'_j \neq 0 \qquad (1 \leq i \leq 20,\ 1 \leq j \leq 5). \tag{14.3}$$

Here the dot product is bilinear. Membership in $\mathcal{Z}$ means solvability in an algebraically closed overfield when the positions are themselves given over a smaller field.

Lemma 14.2 (Finite polynomial tests)**.** *There is a fixed finite list of polynomials with integer coefficients in the* 50 *position coordinates whose zero/nonzero pattern determines membership in $\mathcal{Z}$. The list is independent of every point configuration and of every rectangle-defect parameter introduced below.*

Proof. Introduce one further variable v and replace (14.3) by

$$v\prod_{i=1}^{20}\prod_{j=1}^{5}(A_i - A'_j) - 1 = 0.$$

We now have 201 equations in 176 auxiliary variables: 100 variables R_{ij}, 50 vector coordinates, 25 variables A_i, A'_j, and v. Their degree in the auxiliary variables is at most 101. Their coefficients are integer polynomials of degree at most 4 in the positions.

The finite list is obtained by linear algebra. We use the standard effective ideal-membership theorem: if polynomials of degree at most d in n variables over a field generate 1, then they have a representation of 1 with multiplier degrees at most $(2d)^{2^n}$. This is Hermann's bound in the form stated and proved in [3, Theorem 3.4]. Thus put

$$B = 202^{2^{176}}.$$

The required list is specified by the following finite procedure.

Algorithm 14.3 (Polynomial conditions for solvability)**.** (i) For each auxiliary equation, form an arbitrary polynomial multiplier of degree at most B, with one unknown coefficient for each monomial.

(ii) Set the sum of the multiplied equations equal to 1. Equate coefficients of all monomials of degree at most $B + 101$, obtaining

$$M(p_1, \dots, r_5)c = b,$$

where b is the coefficient vector of the constant polynomial 1.

(iii) Form all minors of M and $[M \mid b]$. These are the desired integer polynomials in the position coordinates.

For any specialization of the positions, the degree bound and Hilbert's Nullstellensatz identify nonsolvability of the auxiliary equations with solvability of this linear system. The latter is equivalent to $\operatorname{rank} M = \operatorname{rank}[M \mid b]$, which is determined by the vanishing or nonvanishing of the listed minors. Thus the list determines membership in $\mathcal{Z}$ itself. □

14.2. **Generic grids and independent square roots.** We call a plane curve *real-compatible* if its zero set is preserved by

$$(z, w) \longmapsto (\overline{w}, \overline{z}).$$

This is the real structure corresponding to ordinary real x_1, x_2. A curve containing more real points than its degree squared is real-compatible: its image under this involution has the same degree, and Bézout forces the two irreducible curves to coincide. The curves to which we apply the next proposition have nonconstant z- and w-projections. A curve with constant z or w contains at most one point of the original real plane, so it cannot occur after the large-set restrictions used in the grid-selection argument.

We use a generic point in the algebraic sense. A point of an irreducible variety V over an algebraically closed field B is generic over B if its coordinates have transcendence degree $\dim V$ over B. A jointly generic tuple is generic on the product of its specified varieties. Generic evaluation identifies the function field $B(V)$ with the field generated by the generic point: the kernel is a prime ideal of the same dimension as V, and so is its defining prime ideal. The same statement holds after enlarging the coefficient field provided genericity is established again over that enlarged field.

Lemma 14.4 (Independent quadratic radicals)**.** *Let F be a field of characteristic different from 2. Suppose $D_1, \dots, D_m \in F^\times$ have independent classes in $F^\times/(F^\times)^2$. If $R_j^2 = D_j$, then*

$$[F(R_1, \dots, R_m) : F] = 2^m,$$

and every independent choice of signs for the R_j is induced by an F-automorphism of this extension. In particular the single radicals $R_1, \dots, R_m$ are linearly independent over F.

Proof. We prove the degree and automorphism assertions by induction. They are immediate for one nonsquare. Assume them for the first $m-1$ radicals and write $E = F(R_1, \dots, R_{m-1})$. If D_m were a square in E, a root $u \in E$ would transform under each sign-change automorphism σ as $\sigma u = \pm u$. The signs define a character of the group $(\mathbb{Z}/2\mathbb{Z})^{m-1}$. The products $\prod_{j\in J} R_j$ form an F-basis of E by the induction hypothesis and have distinct sign characters. Expanding u in this basis shows that it is an F-multiple of one such product. Squaring contradicts independence of the classes of the D_j. Thus adjoining R_m doubles the degree. Each preceding automorphism extends by either sign on R_m, because it fixes D_m; the assertions follow. Finally, in a linear relation between the single radicals, changing just one sign and subtracting kills every other term. □

Proposition 14.5 (Generic avoidance)**.** *Let V_1, V_2 each be either $\mathbb{A}^2_{\mathbb{C}}$ or an irreducible real-compatible plane curve with nonconstant coordinate projections. Suppose they are not both the same real unit circle. A jointly generic point of $V_1^{20} \times V_2^5$ does not belong to $\mathcal{Z}$.*

Proof. Take a generic tuple over $B = \mathbb{C}$ in an algebraically closed extension Ω of B and suppose a solution of (14.1)–(14.3) exists in Ω, enlarging Ω if necessary. Put $d = \dim V_1$. Let K be the algebraic closure, inside Ω, of

$$B(r_1, \dots, r_5, U_1, \dots, U_5, A'_1, \dots, A'_5).$$

The last fifteen coordinates are arbitrary and may depend on all left positions. Adjoining them nevertheless reduces transcendence degree by at most fifteen. Joint genericity gives

$$\operatorname{trdeg}_K K(p_1, \dots, p_{20}) \geq 20d - 15.$$

If every p_i had transcendence degree at most $d-1$ over K, subadditivity would give an upper bound $20(d-1)$, a contradiction. Choose a surviving point $p = p_i$ generic on V_1 over K. Hence

$$F = K(V_1) = K(p).$$

This argument is why independence over the enlarged field is valid; it has not been assumed automatically.

Write $a_j = z_{r_j}$, $b_j = w_{r_j}$ and consider the functions

$$L_j(z, w) = (z - a_j)(w - b_j), \qquad D_j = L_j(L_j - 4) \in F.$$

We first prove the following stronger assertion: for every j there is an integer-valued valuation v_j of F with

$$v_j(D_j) = 1, \qquad v_j(D_l) = 0 \quad (l \neq j). \tag{14.4}$$

The nonvanishing assertions in its construction will be established using the original joint genericity over B, before any flow data are adjoined.

Plane case. If $V_1 = \mathbb{A}^2$, the polynomial

$$Q_j = (z - a_j)(w - b_j) - 4$$

is irreducible in $K[z, w]$. Indeed its two coefficients as a linear polynomial in w are coprime in $K[z]$, and it is linear over $K(z)$. The polynomials Q_j are pairwise distinct: their normalized coefficients recover a_j, b_j. None divides any L_l, a product of two linear polynomials. The exponent of Q_j in a rational function is therefore the valuation in (14.4).

A curve with a different anchor locus or a multiple projection. Let V_1 have irreducible equation $f(z, w) = 0$. Suppose either $V_1 \neq V_2$ or $\deg_w f \geq 2$. Fix j and put $a = a_j$. The coordinate a is transcendental over B. Since f is irreducible and has positive w-degree, characteristic zero implies that f and f_w are coprime in $B(z)[w]$. A Bézout identity in this polynomial ring, with

denominators cleared, shows that $f(a,w)$ has no repeated root; its degree is unchanged because its leading coefficient cannot vanish at the transcendental a.

If $V_1 \neq V_2$, genericity puts r_j outside V_1, so any root b of $f(a,w)$ differs from b_j. If $V_1 = V_2$ and $\deg_w f \geq 2$, choose a root other than b_j. Thus in both cases

$$s_j = (a,b) \in V_1, \qquad f_w(a,b) \neq 0, \qquad b - b_j \neq 0.$$

The element b is algebraic over $B(a)$ and belongs to K because K is algebraically closed and contains a. For $l \neq j$, the anchor r_l remains generic on V_2 over $B(s_j)$: s_j is algebraic over one coordinate of r_j, and the anchors were jointly generic. The two projections of V_2 are nonconstant, so neither coordinate of r_l equals its counterpart in s_j. Consequently $L_l(s_j) \neq 0$.

Also $L_l(s_j) \neq 4$. Otherwise genericity would force V_2 to be contained in the curve

$$(z-a)(w-b) = 4.$$

This is impossible for a plane. For a curve, irreducibility would make its equation proportional to this one. After normalizing the zw coefficient, the w coefficient would put a in B, contrary to its transcendence. These inequalities are assertions about the chosen elements of Ω and remain true after passing to K.

There is a unique formal solution

$$w(T) = b + c_1 T + c_2 T^2 + \cdots \in K[[T]], \qquad f(a+T, w(T)) = 0.$$

At the mth step the new coefficient occurs with the nonzero coefficient $f_w(a,b)$, which proves existence recursively. The substitutions $z \mapsto a+T$, $w \mapsto w(T)$ embed F in $K((T))$: $a+T$ is transcendental over K and the irreducible equation of V_1 is the only polynomial relation. Now

$$L_j(a+T, w(T)) = T\big(b - b_j + O(T)\big)$$

has a simple zero. The functions $L_j - 4$ and D_l for $l \neq j$ have nonzero constant terms. Order in T proves (14.4) in this case.

A common curve of projection degree one. The only remaining case has $V_1 = V_2$ and $\deg_w f = 1$. Real compatibility gives $\deg_z f = 1$ as well. Thus the common curve has one of the forms

$$w = ez + e', \quad e \neq 0, \qquad \text{or} \qquad (z-u)(w-v) = b, \quad b \neq 0. \tag{14.5}$$

To see this directly, write its equation as $c_{11}zw + c_{10}z + c_{01}w + c_{00} = 0$. If $c_{11} = 0$, nonconstant projections give the first form. Otherwise completion of the product gives the second form, with $b \neq 0$ by irreducibility. In the second form, real compatibility implies $v = \overline{u}$ and $b \in \mathbb{R}$. If $b = 1$, the real locus is the unit circle centred at u, excluded by the hypothesis. Hence

$$b \neq 1. \tag{14.6}$$

For a line in (14.5), $L_j = e(z-a_j)^2$. Choose c with $c^2 = 4/e$ and take the point s_j of the line with z-coordinate $a_j + c$. The function $L_j - 4$ has a simple zero there, and $L_j = 4$. For $l \neq j$, either equation $L_l(s_j) = 0$ or $L_l(s_j) = 4$ would make a_l algebraic over $B(s_j)$, contradicting the independence of the anchors. Order at this point of the rational function field $K(z)$ gives (14.4).

For the second curve in (14.5), put $\zeta = (z-u)/(a_j - u)$. Since a_j is generic, $a_j - u \neq 0$. Substitution gives

$$L_j = -b\frac{(\zeta-1)^2}{\zeta}.$$

Zeros of $L_j - 4$ correspond to roots of

$$b\zeta^2 + (4-2b)\zeta + b = 0.$$

Its discriminant is $16(1-b) \neq 0$; its roots are nonzero since $b \neq 0$. Either root gives a finite point s_j with $z_{s_j} \neq u$ at which $L_j - 4$ has a simple zero and $L_j = 4$. This point is algebraic over $B(a_j)$. As a

function of the independent coordinate a_l, for $l \neq j$, the value $L_l(s_j)$ is

$$-b\frac{(z_{s_j}-a_l)^2}{(z_{s_j}-u)(a_l-u)}.$$

It is nonconstant, since its numerator has degree two, its denominator has degree one, and $z_{s_j} \neq u$. It therefore cannot take the values 0 or 4 at the transcendental a_l over $B(s_j)$. Order at s_j again proves (14.4).

We have exhausted all cases. The valuations show that no nonempty product of the D_j is a square: a valuation indexed by one factor of that product has odd value on it. Thus the D_j have independent square classes in F, and Lemma 14.4 applies to the actual roots $R_j = R_{ij}$ for the surviving point $p_i = p$.

We finish with a linear system. Set $H = H_i$, $A = A_i$ and rewrite (14.2) as

$$2(p-r_j)\cdot H - R_j A = 2(p-r_j)\cdot U_j - R_j A'_j \qquad (1 \leq j \leq 5). \tag{14.7}$$

If the first three rows $p-r_1, p-r_2, p-r_3$ have rank two, the determinant of their three equations in H_1, H_2, A is an F-linear combination of R_1, R_2, R_3 with at least one nonzero coefficient. Indeed its coefficients, up to signs and factors of two, are the two-by-two minors of the position rows. The determinant is nonzero by Lemma 14.4. Cramer's rule places H_1, H_2, A in $F(R_1, R_2, R_3)$. Apply to the fourth equation the automorphism that changes only R_4 and subtract the original equation. Since $R_4 \neq 0$, this gives $A = A'_4$, contradicting (14.3).

Here is the complete rank check. If V_2 is a plane or a curve other than a line, three jointly generic anchors are noncollinear: after fixing two distinct anchors, containment of a generic third on their line would force its entire irreducible locus to be that line. The three position rows then have rank two for every p. If V_2 is a line, the first two rows already have rank two unless p is on that line. As p is generic on V_1 over K, the latter possibility forces $V_1 = V_2$ to be that same line. In this last case write $p - r_j = s_j v$ for a nonzero direction $v \in K^2$ and $s_j \in F^\times$. Only the unknown $h = v \cdot H$ is needed. The first two equations in h, A have determinant $2(s_2R_1 - s_1R_2) \neq 0$. Thus $h, A \in F(R_1, R_2)$, and changing the sign of R_3 in the third equation gives $A = A'_3$, again a contradiction. The surviving left point is distinct from the anchors by its genericity over K, which justifies $s_j \neq 0$. This proves generic avoidance. Notice that the flow variables have nowhere been required to be derivatives of the positions. □

14.3. **An effective height bound for finite nonexceptional grids.** We recall the precise effective algebraic input used in the height calculation. If $F_1, \ldots, F_s \in L[X_1, \ldots, X_n]$, where L is a number field, have degree at most d and no common zero over an algebraic closure, then there are $G_i \in L[X]$ with

$$1 = \sum_i G_i F_i, \qquad \deg G_i \leq 4nd^n. \tag{14.8}$$

This is the degree assertion of [19, Theorem 3.6]. Although that theorem is stated with algebraic-integer coefficients and a nonzero constant on the right, clearing denominators and dividing by that constant gives exactly (14.8). We do not use the height assertion of the quoted theorem.

For a nonzero polynomial with algebraic coefficients, its coefficient height means the absolute projective height of its coefficient vector. If $F(\alpha) = 0$ and $F \neq 0$, the local Cauchy root bound gives

$$\mathrm{h}(\alpha) \leq \mathrm{h}(F) + \log 2. \tag{14.9}$$

Indeed, after dividing by the leading coefficient, at a nonarchimedean place a root has absolute value at most the maximum of 1 and the absolute values of the remaining coefficients. At an archimedean place the same bound with a factor 2 follows from the geometric-series estimate. Sum these local inequalities; the leading coefficient disappears by the product formula.

Theorem 14.6 (Numerical critical-value bound)**.** *Suppose a grid has algebraic position coordinates and nonzero algebraic numbers λ_{ij} such that*

$$L_{ij} = 2 - \lambda_{ij} - \lambda_{ij}^{-1}.$$

For $2 \le i \le 20$, $2 \le j \le 5$ set

$$\chi_{ij} = \frac{\lambda_{ij}\lambda_{11}}{\lambda_{i1}\lambda_{1j}}.$$

If its position tuple is outside $\mathcal{Z}$, at least one pair $e = (i, j)$ satisfies

$$\mathrm{h}(\lambda_e) \le C_{\mathrm{alg}} \left(1 + \sum_{i=2}^{20} \sum_{j=2}^{5} \mathrm{h}(\chi_{ij}) \right), \qquad C_{\mathrm{alg}} = 10^{100000}. \tag{14.10}$$

Proof. Fix the actual values of the 76 numbers χ_{ij} and put $L = \mathbb{Q}(\chi_{ij} : i \ge 2, j \ge 2)$. Consider the following affine scheme in 150 variables over L: its variables are the 50 position coordinates and the 100 quantities λ_{ij}, and its 176 defining equations are

$$f_{ij} = \lambda_{ij}^2 + (L_{ij} - 2)\lambda_{ij} + 1 = 0, \qquad 1 \le i \le 20, \quad 1 \le j \le 5, \tag{14.11}$$

$$g_{ij} = \lambda_{ij}\lambda_{11} - \chi_{ij}\lambda_{i1}\lambda_{1j} = 0, \qquad 2 \le i \le 20, \quad 2 \le j \le 5. \tag{14.12}$$

Write these equations as $F_1 = \cdots = F_{176} = 0$ and the variables as $X_1, \ldots, X_{150}$. Their total degree is at most three. Every λ_{ij} is nonzero at a solution, by the constant term in (14.11).

Let x denote the actual solution. We use the literal kernel of the Jacobian matrix at x, so the argument does not presuppose smoothness or a radical presentation. For a vector in this kernel write $t_{ij} = d\lambda_{ij}/\lambda_{ij}$. Differentiating (14.12) gives

$$t_{ij} + t_{11} - t_{i1} - t_{1j} = 0.$$

Consequently $t_{ij} = A_i - A'_j$ on the full grid, with $A_i = t_{i1}$ and $A'_j = t_{11} - t_{1j}$. Differentiating the distance relation gives

$$dL_{ij} = (\lambda_{ij}^{-1} - \lambda_{ij})t_{ij}.$$

The position part of the tangent vector supplies H_i, U_j, so it satisfies (14.2) with $R_{ij} = \lambda_{ij}^{-1} - \lambda_{ij}$. The identity $R_{ij}^2 = L_{ij}(L_{ij} - 4)$ is immediate by expanding both sides. Thus a tangent vector for which all $d\lambda_{ij}$ are nonzero would put the actual position tuple in $\mathcal{Z}$.

A finite-dimensional vector space over an infinite field cannot be a union of finitely many proper linear subspaces. For example, the product of their nonzero defining linear forms is a nonzero polynomial and cannot vanish on the entire vector space. It follows that for some fixed edge e, the linear functional $d\lambda_e$ vanishes on the entire Jacobian kernel at x. By finite-dimensional linear algebra its row vector is in the row span of that Jacobian. Choose 176 multipliers $a_1, \ldots, a_{176}$ realizing this row-span relation.

Introduce these multipliers as new variables and define a closed affine algebraic set C_e by

$$F_l(X) = 0 \qquad (1 \le l \le 176), \tag{14.13}$$

$$\frac{\partial \lambda_e}{\partial X_v} = \sum_{l=1}^{176} a_l \frac{\partial F_l}{\partial X_v} \qquad (1 \le v \le 150). \tag{14.14}$$

The chosen point lifts to C_e. There are 326 variables and 326 equations, all of degree at most 3. The combined vector of input coefficients, together with 1, has height at most

$$H \le \sum_{i,j} \mathrm{h}(\chi_{ij}) + \log 4. \tag{14.15}$$

In fact every coefficient is 0, an integer of absolute value at most 2, or $\pm\chi_{ij}$; the factor 4 safely allows all displayed expansions.

The projection of C_e to its λ_e coordinate is finite. To prove this, take an irreducible component over $\overline{L}$ and its function field T. Equations (14.13) and (14.14) imply, for every $\overline{L}$-derivation $D : T \to T$,

$$D\lambda_e = \sum_l a_l DF_l = 0.$$

If λ_e were transcendental over $\overline{L}$, it could be included in a transcendence basis. Assigning it derivative 1 and the other basis elements derivative 0 gives a derivation of the corresponding rational-function field. It extends uniquely through the finite separable algebraic extension to T: for an algebraic element with minimal polynomial P, implicit differentiation defines its derivative since P' is nonzero. This contradicts the preceding identity. Thus λ_e is algebraic over the algebraically closed field $\overline{L}$, and is constant on that component. There are finitely many components, which proves the assertion. This reasoning applies at singular points as well; it asserts constancy on components of C_e, not on the original distance–rectangle fibre.

We next bound these finitely many critical values. Put $n = 326$ and

$$D_0 = 4n3^n.$$

Adjoin to the equations of C_e the equation $\lambda_e - T = 0$, where T is one new parameter, not an additional unknown for the degree calculation. For every rational specialization $T = t$ outside the finite projection, the resulting equations have no common zero. By (14.8) there is a representation of 1 with all multiplier degrees in the n unknowns at most D_0.

Expand those multipliers and equate monomial coefficients. This is a finite linear system

$$M(T)c = b$$

over $L[T]$. Its entries have degree at most one in T, and its number of rows is at most

$$R = \binom{n + D_0 + 3}{n} \leq (8n3^n)^n. \tag{14.16}$$

The system is compatible for all but finitely many rational t. Ranks of a univariate polynomial matrix and its augmentation are constant away from finitely many values, as follows by inspecting their nonzero maximal minors. Therefore it is compatible over $L(T)$.

Choose a nonzero maximal pivot minor $\Delta(T)$ of $M(T)$. Using its pivot columns and rows, Cramer's rule gives a solution whose only denominator is $\Delta(T)$; compatibility ensures that the other rows are satisfied as well. Clearing this denominator produces the polynomial identity

$$\sum_l G_l(X,T)F_l^{\mathrm{crit}}(X) + G_*(X,T)(\lambda_e - T) = \Delta(T), \tag{14.17}$$

where the F_l^{crit} are the 326 critical equations. At any point of C_e, substitute $T = \lambda_e$ into (14.17). It follows that $\Delta(\lambda_e) = 0$, and Δ is a nonzero polynomial. All steps were performed after fixing the actual χ_{ij}. In particular this is not a specialization of a generic eliminant whose coefficients might all disappear at the actual parameter.

We finish the arithmetic estimate explicitly. At a place v of L, let A_v be the maximum of 1 and the absolute values of all input coefficients. For a matrix entry linear in T, its archimedean coefficient ℓ^1 norm is at most $2A_v$ and its nonarchimedean Gauss norm is at most A_v. A determinant of order $r \leq R$ consequently has local coefficient norm at most $r!(2A_v)^r$ at infinity and A_v^r at a finite place. Summing with normalized place weights and using (14.15) gives

$$\deg \Delta \leq R, \qquad \mathrm{h}(\Delta) \leq R\big(H + \log(2R)\big).$$

Combining with (14.9) yields

$$\mathrm{h}(\lambda_e) \leq R\left(\sum_{i,j} \mathrm{h}(\chi_{ij}) + \log 4 + \log(2R)\right) + \log 2. \tag{14.18}$$

No number-field degree occurs in this estimate. For an entirely elementary rounding of (14.16), use $\log_{10}(8 \cdot 326) < 4$ and $\log_{10} 3 < 1/2$ to obtain

$$\log_{10} R < 326(4 + 326/2) = 54442 < 55000.$$

Thus the right side of (14.18) is less than $10^{55006}(1 + \sum_{i,j} \mathrm{h}(\chi_{ij}))$, which is certainly bounded by (14.10). This proves the theorem. □

Corollary 14.7 (A numerical contradiction on one grid)**.** *Put $K = 10^{100004}$. There is no algebraic grid outside $\mathcal{Z}$ for which*

$$J \geq K, \qquad \mathrm{h}(\lambda_{ij}) \geq J/3 \quad \textit{for all } i, j, \qquad \mathrm{h}(\chi_{ij}) \leq J/K \quad (i, j \geq 2),$$

and $L_{ij} = 2 - \lambda_{ij} - \lambda_{ij}^{-1}$.

Proof. There are 76 displayed defects. Theorem 14.6 would give, for some e,

$$\mathrm{h}(\lambda_e) \leq 10^{100000}\left(1 + \frac{76J}{K}\right) \leq \frac{77}{10000} J < J/3.$$

□

Remark 14.8. The requirement that the positions avoid $\mathcal{Z}$ is essential. The proof uses the selected generic grid rather than every complete distance grid with small rectangle defects. The separate grid-selection argument is needed to produce a nonexceptional grid from a large dense graph. It uses the finite Boolean pattern of Lemma 14.2, not merely a polynomial vanishing on the closure of $\mathcal{Z}$.

15. A finite selection theorem for polynomial patterns

The algebraic argument requires a grid outside a fixed constructible set. Avoiding its Zariski closure would be a stronger and generally incorrect requirement. We therefore preserve the entire finite pattern of polynomial equalities and inequalities. The important quantitative point is that the exponent in the following selection theorem does not depend on the degrees of the fixed polynomials. Those degrees affect only the starting value of the size parameter.

We identify the physical real plane with

$$\{(z, w) \in \mathbb{C}^2 : w = \overline{z}\}, \qquad z = x + iy, \quad w = x - iy.$$

The real structure on the complex plane of these coordinates is the antilinear involution $\iota(z, w) = (\overline{w}, \overline{z})$. A locus will mean either $\mathbb{C}^2$ or an irreducible complex algebraic curve. A polynomial is said to vanish identically on a product of loci when its restriction to that product is zero in its coordinate ring.

Lemma 15.1 (A blockwise zero count)**.** *Let $V_1, \dots, V_r$ be irreducible affine varieties over $\mathbb{C}$, and let $A_i \subset V_i$ be finite sets with $|A_i| \leq T$. Fix an integer $D \geq 1$. Suppose that, on each A_i, the restriction of a polynomial of total degree at most D either vanishes identically on V_i or has at most Z zeros. If a polynomial F of total degree at most D is not identically zero on $V_1 \times \cdots \times V_r$, then*

$$\#\{(a_1, \dots, a_r) \in A_1 \times \cdots \times A_r : F(a_1, \dots, a_r) = 0\} \leq rZT^{r-1}.$$

Proof. Induct on r. The assertion for $r = 1$ is a hypothesis. In the last block, choose a basis for the restrictions to V_r of the monomials of degree at most D. Expanding the restriction of F in this basis gives coefficient polynomials in the first $r - 1$ blocks. They are constant linear combinations of the original coefficient polynomials, so their degrees are still at most D. At least one coefficient polynomial G is not identically zero on $V_1 \times \cdots \times V_{r-1}$. This follows, for example, by extending the linearly independent basis elements from $\mathbb{C}$ to the fraction field of the coordinate ring of the earlier product. The tuples annihilating G contribute at most $(r - 1)ZT^{r-2}T$, by induction. At every other earlier tuple the last-block restriction is nonzero and contributes at most Z choices. Their contribution is at most ZT^{r-1}, proving the result. □

Lemma 15.2 (Counting complete ordered grids)**.** *If a bipartite graph $G \subset A \times B$ has e edges and $|A| = a > 0$, $|B| = b > 0$, then for positive integers r, s,*

$$\hom(K_{r,s}, G) \geq \frac{e^{rs}}{a^{rs-r} b^{rs-s}}.$$

Here a homomorphism is an ordered grid; repetitions are allowed.

Proof. For an ordered r-tuple $\boldsymbol{p} \in A^r$, let $c(\boldsymbol{p})$ be its number of common neighbors. Then

$$\sum_{\boldsymbol{p} \in A^r} c(\boldsymbol{p}) = \sum_{q \in B} \deg(q)^r \geq b(e/b)^r.$$

Convexity, now applied to the s-th power, gives

$$\sum_{\boldsymbol{p} \in A^r} c(\boldsymbol{p})^s \geq a^r \left(\frac{e^r}{a^r b^{r-1}} \right)^s,$$

which is the claimed bound. □

Theorem 15.3 (Selection of a generic polynomial pattern)**.** *Fix a finite collection $F_1, \dots, F_L$ of complex polynomials in the coordinates of twenty left points and five right points. After discarding zero polynomials, put*

$$D = \max \left\{ 1, \sum_{\nu=1}^{L} \deg F_\nu \right\}.$$

Fix $0 < \varepsilon \leq 10^{-8}$. For all sufficiently large t, depending only on D and ε, the following holds. Let A, B be finite subsets of the physical real plane and $G \subset A \times B$, with

$$|A|, |B| \leq t^{1+\varepsilon}, \qquad |G| \geq t^{2-\varepsilon}.$$

There are loci V_A, V_B, each a plane or a curve of degree at most D, and a complete ordered 20-by-5 grid in G, such that its vertices lie on these loci and, for every ν,

$$F_\nu(\mathit{grid}) = 0 \quad \Longleftrightarrow \quad F_\nu|_{V_A^{20} \times V_B^5} \equiv 0. \tag{15.1}$$

Every curve locus is invariant under ι, and both its coordinate projections are nonconstant. Distinctness of all twenty-five vertices may be required by adding the pairwise differences of their z coordinates to the fixed polynomial collection.

If, additionally, $A, B \subset P$ and no edge of G has both endpoints on a real unit circle containing at least $t^{9/10}$ points of P, the loci V_A, V_B cannot both be that same unit circle.

Proof. We keep a subgraph, a locus for each side, and a parameter γ such that

$$e \geq t^{2-\gamma}, \qquad d_{\min} \geq t^{1-\gamma-2\varepsilon}, \qquad |A'|, |B'| \leq t^{1+\varepsilon}. \tag{15.2}$$

Initially take both loci to be planes. Repeatedly delete a vertex whose current degree is below $t^{1-3\varepsilon}$. The total deletion cost is at most $2t^{2-2\varepsilon}$. If $t^\varepsilon \geq 4$, at least half of the original edges survive, and their number is at least $t^{2-2\varepsilon}$. Thus (15.2) holds initially with $\gamma = 2\varepsilon$.

Suppose a side whose current locus is a plane has at least $t^{1-300\gamma}$ vertices on the zero set of a nonzero polynomial of degree at most D. One of its at most D irreducible factors contains at least $D^{-1} t^{1-300\gamma}$ such vertices. Restrict that side to this irreducible curve. The old minimum degree gives at least

$$D^{-1} t^{2-301\gamma-2\varepsilon} \geq t^{2-301\gamma-3\varepsilon}$$

remaining edges, provided $t^\varepsilon \geq D$. Prune again, now at degree threshold $t^{1-301\gamma-5\varepsilon}$. The number of deleted edges is at most $2t^{2-301\gamma-4\varepsilon}$, so at least $t^{2-301\gamma-4\varepsilon}$ edges remain for $t^\varepsilon \geq 4$. Reset γ to 305γ. The inequalities

$$301\gamma + 4\varepsilon \leq 305\gamma, \qquad 301\gamma + 5\varepsilon \leq 305\gamma + 2\varepsilon$$

show that (15.2) is preserved.

Each such step replaces one plane locus by a curve. There can therefore be at most two steps. After each step we repeat the concentration test on every side still assigned the plane, using the new value of γ. At termination,

$$2\varepsilon \le \gamma \le 2 \cdot 305^2 \varepsilon = 186050\varepsilon. \tag{15.3}$$

On a plane side, a nonzero polynomial of degree at most D has fewer than $t^{1-300\gamma}$ zeros among the retained vertices. On a curve side, a polynomial of degree at most D not vanishing identically on that curve has at most D^2 zeros, by the plane-curve Bézout bound. Thus the one-block zero bound is

$$Z = \max\{t^{1-300\gamma}, D^2\}.$$

Multiply together those F_ν which are not identically zero on $V_A^{20} \times V_B^5$; denote their product by F, with $F = 1$ if the collection is empty. The product of irreducible varieties over $\mathbb{C}$ is irreducible, so F is not identically zero, and its degree is at most D. Lemma 15.1 bounds the number of ordered vertex tuples on which F vanishes by

$$25 \max\{t^{1-300\gamma}, D^2\}\, t^{24(1+\varepsilon)}. \tag{15.4}$$

On the other hand, Lemma 15.2 gives at least

$$\frac{e^{100}}{|A'|^{80}|B'|^{95}} \ge t^{25-100\gamma-175\varepsilon} \tag{15.5}$$

complete ordered grids. The exponent in (15.5) exceeds the two exponents in (15.4), respectively, by

$$200\gamma - 199\varepsilon > 0, \qquad 1 - 100\gamma - 199\varepsilon \ge 1 - 18605199 \cdot 10^{-8} = 0.81394801 > 0.$$

For sufficiently large t, the fixed factors 25 and D^2 are also absorbed. A complete grid with $F \neq 0$ exists. The polynomials omitted from the product vanish on every tuple of the product loci, so this grid satisfies (15.1).

By the minimum-degree condition, each retained side contains at least $t^{1-\gamma-2\varepsilon}$ vertices. For sufficiently large t this exceeds D^2. A retained curve and its image under ι share all these physical real points; Bézout therefore makes the two irreducible curves equal. A curve with constant z, or constant w, contains at most one physical real point, so both projections are nonconstant. In particular, the coordinate differences added to enforce distinctness are not identically zero on the relevant product of loci. Condition (15.1) then makes these differences nonzero on the selected grid.

Finally,

$$\gamma + 2\varepsilon \le 0.00186052 < 1/10.$$

If both loci were the same real unit circle, that circle would contain more than $t^{9/10}$ points of P. Every retained edge would be forbidden by the last hypothesis. This proves the final assertion. □

Remark 15.4. All size thresholds in the proof are uniform in the polynomial coefficients. For example, it is enough to make

$$t^\varepsilon \ge \max\{100, D\}, \qquad t^{0.81394801} > 100D^2.$$

The exponent 10^{-8} in the theorem is independent of D. It is neither asserted nor needed that this threshold is smaller than the arithmetic parameter used elsewhere in the proof.

Corollary 15.5 (Avoidance of a constructible set)**.** *Let Z be a fixed constructible subset of the space of* 20*-by-*5 *position grids, described by a finite Boolean combination of polynomial equations over $\mathbb{Q}$. Suppose the generic point of $V_A^{20} \times V_B^5$ lies outside Z for every pair of loci allowed in Theorem 15.3, apart from two copies of the same real unit circle. Under the hypotheses of that theorem, including the popular-circle exclusion, the graph contains a complete grid outside Z.*

Proof. Use the polynomials in a Boolean description of Z as the fixed collection in Theorem 15.3, adding the coordinate differences for distinctness. Membership in Z depends solely on which of these polynomials vanish. The selected grid has the same pattern as the generic point of the product of its loci. The common unit circle is excluded by the theorem, so that generic point and the selected grid both lie outside Z. □

Theorem 15.6 (Explicit finite obstruction)**.** *Set $K = 10^{100004}$. There exists an absolute t_0 such that no data of the following kind exist when $t \geq t_0$: a finite algebraic point set P in the physical real plane, a graph $G \subset P \times P$, a real number $J \geq K$, and nonzero algebraic edge ratios λ_{pr}, satisfying*

$$|P| \leq t^{1+1/K}, \qquad |G| \geq t^{2-1/K}, \tag{15.6}$$

$$(z_p - z_r)(w_p - w_r) = 2 - \lambda_{pr} - \lambda_{pr}^{-1}, \qquad \mathrm{h}(\lambda_{pr}) \geq J/3, \tag{15.7}$$

$$\mathrm{h}\left(\frac{\lambda_{pr}\lambda_{p'r'}}{\lambda_{pr'}\lambda_{p'r}}\right) \leq J/K \qquad \text{on every complete rectangle.} \tag{15.8}$$

Moreover, no edge may have both endpoints on a real unit circle containing at least $t^{9/10}$ points of P.

Proof. Apply Corollary 15.5 with $\varepsilon = 10^{-8}$, the constructible set of Definition 14.1, its finite Boolean description from Lemma 14.2, and the generic avoidance statement of Proposition 14.5. This is permitted because $1/K < 10^{-8}$. The selected 20-by-5 grid is outside that set. Its seventy-six normalized rectangle defects χ_{ij}, $2 \leq i \leq 20$, $2 \leq j \leq 5$, all satisfy $\mathrm{h}(\chi_{ij}) \leq J/K$. Theorem 14.6 gives an edge for which

$$\mathrm{h}(\lambda_e) \leq 10^{100000}\left(1 + \sum_{i=2}^{20}\sum_{j=2}^{5}\mathrm{h}(\chi_{ij})\right) \leq 10^{100000} + \frac{76J}{10000}.$$

Since $J \geq K = 10^4 \cdot 10^{100000}$, the right side is at most $77J/10000 = 0.0077J < J/3$, contradicting (15.7). The starting threshold t_0 is the threshold for the fixed finite polynomial collection. It need not equal K. □

16. Transfer to unequal numbers of points and unit circles

We prove the transfer separately from the preceding quantitative argument. In particular, any later improvement in the balanced exponent can be inserted into the same formula. We include the partition construction used below. Its only topological input is the Borsuk–Ulam theorem in the following standard form: every continuous odd map $S^d \longrightarrow \mathbb{R}^d$ has a zero.

Lemma 16.1 (Polynomial bisection)**.** *Let $A_1, \ldots, A_s$ be finite subsets of $\mathbb{R}^2$. If $\binom{d+2}{2} - 1 \geq s$, there is a nonzero real polynomial f of degree at most d such that*

$$\#\{p \in A_i : f(p) > 0\} \leq |A_i|/2, \qquad \#\{p \in A_i : f(p) < 0\} \leq |A_i|/2$$

for every i. Points at which f vanishes are omitted from both inequalities.

Proof. For a positive ρ, replace each point of A_i by a smooth nonnegative bump of integral one supported in its radius-ρ disc. Let $\mu_{i,\rho}$ be the sum of these measures. The real space of polynomials of degree at most d has dimension $m = \binom{d+2}{2}$. On its coefficient unit sphere define

$$\Phi_\rho(f)_i = \int_{\mathbb{R}^2} \operatorname{sgn}(f(x))\, d\mu_{i,\rho}(x).$$

This map is odd and continuous. To see continuity, a nonzero polynomial has a planar zero set of Lebesgue measure zero, by Fubini applied to its one-variable fibers, and coefficient convergence gives pointwise sign convergence away from that zero set. Dominated convergence then applies. Pad the map with zero coordinates if necessary and apply Borsuk–Ulam on S^{m-1}. There is a unit-coefficient polynomial f_ρ for which all integrals vanish.

Choose $\rho \downarrow 0$ along a sequence and take a convergent subsequence of the coefficient vectors. Its limit f has coefficient norm one, so is nonzero. For each original point with $f(p) > 0$, all the mass of its bump eventually lies in the positive set of f_ρ: coefficient convergence is uniform on a fixed compact neighborhood, and f is positive in a sufficiently small neighborhood of p. The positive mass of every $\mu_{i,\rho}$ is exactly $|A_i|/2$, because its signed integral is zero and a polynomial zero set has measure zero. Thus there can be at most $|A_i|/2$ original points with $f(p) > 0$. The argument for negative values is identical. □

Lemma 16.2 (Planar partition into sign regions)**.** *For a finite $P \subset \mathbb{R}^2$, $|P| = M$, and an integer $D \geq 1$, there is a nonzero real polynomial f of degree at most D, and a partition of $\mathbb{R}^2 \setminus Z(f)$ into at most D^2 open sign regions, each containing at most $256M/D^2$ points of P. A unit circle not contained in $Z(f)$ meets at most $2D + 1$ of these regions. The sign regions need not be connected.*

Proof. For $D < 16$, take $f = 1$ and the single region $\mathbb{R}^2$. Suppose now that $D \geq 16$, and put

$$j = \lfloor 2 \log_2(D/8) \rfloor .$$

At step $\ell = 1, \ldots, j$, apply Lemma 16.1 to the at most $2^{\ell-1}$ point classes specified by the nonzero signs of the previously chosen polynomials. We may choose a nonzero bisecting polynomial f_ℓ of degree

$$d_\ell = \lceil 2^{\ell/2} \rceil,$$

since $\binom{d_\ell+2}{2} - 1 \geq d_\ell^2/2 \geq 2^{\ell-1}$. Each final sign class has at most $M/2^j$ points. Put $f = \prod_{\ell=1}^{j} f_\ell$. The nonempty sets

$$\{x : \operatorname{sgn}(f_\ell(x)) = \sigma_\ell,\ 1 \leq \ell \leq j\}, \qquad \sigma_\ell \in \{-1, 1\},$$

form the required open sign regions. The estimates

$$\deg f \leq \sum_{\ell=1}^{j} (2^{\ell/2} + 1) \leq 8\, 2^{j/2} \leq D, \qquad D^2/128 < 2^j \leq D^2$$

give both the number of regions and their point bound, with room in the stated constant 256.

A circle not contained in $Z(f)$ has at most $2D$ intersection points with that zero set. One can use the degree-two Bézout bound here, or substitute the rational parametrization of a circle into f: after multiplying by the denominator to the power D, one obtains a nonzero polynomial of degree at most $2D$, with the possible point at infinity counted projectively. On each connected arc left after removing these intersections, all the signs of the f_ℓ are constant. There are at most $2D + 1$ such arcs (including the case of no intersections), so the circle meets at most that many sign regions. □

Remark 16.3. The usual polynomial partition theorem describes connected components of a complement. The sign-region version above, sufficient here, avoids any separate bound on their number of connected components. The polynomial ham-sandwich construction is the standard mechanism behind polynomial partitioning; see [13, 17].

Theorem 16.4 (Balanced-to-unbalanced transfer)**.** *Let $1 < \beta \leq 4/3$, and suppose that, for every positive integer r,*

$$\mathcal{U}(r, r) \leq C r^\beta.$$

For positive integers n, m, set $s = \min\{n, m\}$ and $M = \max\{n, m\}$. Then

$$\mathcal{U}(n, m) \leq C'(M + s^{2\beta-2} M^{2-\beta}), \tag{16.1}$$

where C' depends only on C and β. More precisely,

$$\begin{cases} \mathcal{U}(n, m) = O_{C,\beta}(s^{2\beta-2} M^{2-\beta}), & s \leq M \leq s^2, \\ \mathcal{U}(n, m) \leq M + s(s-1) \leq 2M, & M \geq s^2. \end{cases}$$

Proof. Replacing a unit circle by its center interprets an incidence as a cross-pair of points at distance one. Exchanging the two point sets gives $\mathcal{U}(n,m) = \mathcal{U}(m,n)$. It is therefore enough to consider a set P of M points and a family $\mathcal{C}$ of s distinct unit circles.

First, for $d(p) = \#\{C \in \mathcal{C} : p \in C\}$, pair counting gives

$$\sum_{p \in P} \binom{d(p)}{2} \le 2\binom{s}{2} = s(s-1).$$

Since $d \le 1 + \binom{d}{2}$ for every nonnegative integer d,

$$I(P,\mathcal{C}) \le M + s(s-1). \tag{16.2}$$

This proves the stated bound for $M \ge s^2$ and deals with $s = 1$.

Assume $s \le M \le s^2$. Set

$$D = \lfloor M/s \rfloor, \qquad T = \left\lceil \frac{256M}{D^2} \right\rceil.$$

Then $M/(2s) \le D \le M/s$, and

$$T = O(s^2/M), \qquad sD \le M, \qquad D^2T = O(M), \qquad D^2 \le M. \tag{16.3}$$

Use Lemma 16.2. Let $\mathcal{C}_0$ consist of the circles contained in $Z(f)$, and put $\mathcal{C}_1 = \mathcal{C} \setminus \mathcal{C}_0$. For each sign region Ω_i, let b_i be the number of circles of $\mathcal{C}_1$ meeting it. The circle-crossing bound implies

$$\sum_i b_i \le (2D+1)s \le 3sD.$$

Each region contains at most T points. Split its b_i circles into batches containing at most T circles. Pad the points and each batch with distinct dummy points and distinct unit circles to obtain T points and T circles. Padding only adds incidences, so the balanced hypothesis bounds each batch by CT^β. Thus the incidences off $Z(f)$ are at most

$$\begin{aligned}\sum_i CT^\beta \left\lceil \frac{b_i}{T} \right\rceil &\le C\left(T^{\beta-1}\sum_i b_i + D^2T^\beta\right) \\ &\le O(C)MT^{\beta-1} \le O_{C,\beta}\big(s^{2\beta-2}M^{2-\beta}\big).\end{aligned}$$

An empty batch is omitted; counting an additional CT^β for each region only increases this upper bound. In particular, no balance between the point and circle counts of an individual region has been assumed.

For every circle of $\mathcal{C}_1$, at most $2D$ points lie on $Z(f)$, so these boundary incidences total at most $2sD \le 2M$. If $k_0 = |\mathcal{C}_0|$, then $k_0 \le D/2$: the distinct irreducible quadratic equations of these circles divide f. For completeness, a circle having infinitely many common zeros with f must be a component by Bézout, which proves this divisibility claim. Applying the same pair count as in (16.2) to $\mathcal{C}_0$ yields

$$I(P,\mathcal{C}_0) \le M + 2\binom{k_0}{2} \le M + D^2 \le 2M.$$

This includes the possibility that every point of P lies on $Z(f)$, as well as arbitrary concurrence of component circles. All boundary incidences have therefore been included.

Combining the bounds proves (16.1). In the middle range,

$$\frac{s^{2\beta-2}M^{2-\beta}}{M} = \left(\frac{s^2}{M}\right)^{\beta-1} \ge 1,$$

so the separate linear term can be absorbed there. □

Corollary 16.5 (The explicit unbalanced exponent)**.** *If $\mathcal{U}(r,r) = O(r^{4/3-\delta})$ for a fixed $0 < \delta < 1/3$, then*

$$\mathcal{U}(n,m) = O\left(M + s^{2/3-2\delta}M^{2/3+\delta}\right).$$

For $s \le M \le s^2$, the non-linear term is

$$(sM)^{2/3}\left(\frac{s^2}{M}\right)^{-\delta}.$$

In particular, writing $M = s^\alpha$ with $1 \le \alpha \le 2$ gives the exponent $\frac{2}{3}(1+\alpha) - (2-\alpha)\delta$ in s.

Proof. Substitute $\beta = 4/3 - \delta$ into Theorem 16.4; the displayed identities are immediate. □

17. Completion and the explicit exponent

Proof of Proposition 9.1. We first show that arbitrarily large point sets with at least $N^{4/3-\delta_1}$ ordered unit pairs cannot exist. Write $t = N^{1/3}$; the supposed number of ordered pairs is at least $t^{4-\varepsilon}$. The incidence regularization gives the graph, degree bounds, marginals, and expansion proved in the preceding sections. Algebraic specialization preserves its required edges and distinct vertices. The bounded-height argument gives $S > S_0$ for all sufficiently large t. The explicit history and score estimates then yield the hypotheses of Proposition 13.6.

Consequently, for every sufficiently large alleged counterexample there is a graph of ratios with

$$|P| \le t^{1+1/K}, \quad |G| \ge t^{2-1/K}, \quad J \ge K,$$

all edge heights at least $J/3$, all rectangle defect heights at most J/K, and no edge on a popular unit circle. The finite obstruction proved in Theorem 15.6 excludes precisely this data for $t \ge t_0(K)$. The two lower bounds on t can be imposed simultaneously. This contradiction proves a uniform bound for all sufficiently large N. Increasing its finite multiplicative constant handles the remaining positive integers. Since unordered pairs number half the ordered ones, this proves the claim for $u(N)$. Lemma 9.3 gives the bound for $\mathcal{U}(N,N)$.

It remains to justify the simpler numerical difference. Since $10 < 2^{10/3}$,

$$\log_2\left(2^{201}k^{100}\right) = 201 + 102400(1024 + 100004\log_2 10) < 35 \cdot 10^9 < 10^{11} - 1.$$

It follows that $2^{201}k^{100} + \log_2 3 < 2^{10^{11}}$, and therefore

$$\delta_1 = \frac{1}{3}\,2^{-2^{201}k^{100}} > 2^{-2^{10^{11}}} = \delta_0.$$

A bound with the larger difference implies the bound with the smaller one. This completes the proof. □

Proof of Proposition 9.2. Apply the balanced-to-unbalanced transfer proved in the preceding section with $\beta = 4/3 - \delta$. Its mixed exponents are $2\beta - 2 = 2/3 - 2\delta$ and $2 - \beta = 2/3 + \delta$. If $M \le s^2$, the mixed term is at least M, because their ratio is $(s^2/M)^{\beta-1} \ge 1$. The direct pair count proves the other regime. □

For illustration, if $M = s^\alpha$ with $1 \le \alpha \le 2$, the bound is

$$\mathcal{U}(s, s^\alpha) \ll s^{\frac{2}{3}(1+\alpha)-(2-\alpha)\delta}.$$

In particular,

M	Upper bound in terms of s
s	$O(s^{4/3-\delta})$
$s^{5/4}$	$O(s^{3/2-3\delta/4})$
$s^{3/2}$	$O(s^{5/3-\delta/2})$
$s^{7/4}$	$O(s^{11/6-\delta/4})$
s^2	$O(s^2)$

These formulas describe the upper bound in each cardinality range. For $M \geq s^2$, the order M is attained by placing M distinct points on one unit circle and adding the other distinct circles.

18. The final parameter regime

The finite framework supplies the geometric, arithmetic and probabilistic notation used below. In this parameter regime, $K, k, a, b, \varepsilon, S_0$ are given by (22.1). Each estimate in the following sections is proved with these choices.

Proposition 18.1 (The final quantitative estimate)**.** *Put*

$$\delta_{\mathrm{up}} = \frac{2^{-2^{664}}}{514560000000}, \qquad \delta_{\mathrm{simple}} = 2^{-2^{665}}.$$

There is an absolute finite constant C such that for every positive integer N,

$$u(N) \le CN^{4/3-\delta_{\mathrm{up}}}, \qquad \mathcal{U}(N,N) \le CN^{4/3-\delta_{\mathrm{up}}}.$$

For $s = \min(n,m)$ and $M = \max(n,m)$,

$$\mathcal{U}(n,m) \ll M + s^{2/3-2\delta_{\mathrm{up}}} M^{2/3+\delta_{\mathrm{up}}}.$$

All these conclusions also hold with δ_{simple} in place of δ_{up}.

The proof uses four improvements: a larger expansion exponent and a smaller matching-label constant; a second-moment small-set estimate; one prediction history controlling all subsequent bounded comparison vectors, which makes restricted-support transfer linear; and a rank-sensitive arithmetic height bound for a six-by-five grid. The final refinement keeps the logarithmic Jacobian and the full rigidity-rank pattern, so that only the actual geometric rank cases enter the height calculation. The intermediate eight-by-four calculation is included to make the algebraic reductions and their numerical consequences separately checkable. The final obstruction uses $K = 2^{68}$.

Relation to public results. The power saving of [28] asserts the existence of an absolute positive exponent difference without specifying its size. The public grid-refinement drafts [33, 34] contain eight-by-four and six-by-five reductions, respectively, but do not supply an explicit final power saving. We prove their required finite, arbitrary-flow versions below, including the line and conic cases, and combine them with explicit height and probability bounds.

The additional standard arithmetic inputs in the finite framework are the arbitrary-intersection and projection height inequalities of [19, Corollary 2.11 and Lemma 2.6]. Their precise numerical forms and treatment of singular or positive-dimensional fibers are given below. All new problem-specific arguments are included.

19. A stronger finite incidence normalization

We now replace the expansion exponent and the matching-label constant used in the finite framework. The change is quantitative: the cutting and the elementary incidence theorem remain the same. The estimates in this section are proved with the new parameters, so their use does not rely on substituting numbers into the old regularization statement.

Let $r_{\mathrm{ht}} \ge 17$ and define

$$\rho = \frac{r_{\mathrm{ht}} + 16}{6r_{\mathrm{ht}} + 98}, \qquad \gamma = \frac{1}{2} + \rho, \qquad R_{\mathrm{rect}} = 128(r_{\mathrm{ht}} + 17). \tag{19.1}$$

Thus

$$\frac{33}{200} \le \rho < \frac{1}{6}, \qquad \gamma \ge \frac{133}{200}, \qquad \frac{2}{3} - \gamma = \frac{1}{18r_{\mathrm{ht}} + 294}, \qquad R_{\mathrm{rect}}\left(\frac{2}{3} - \gamma\right) > 7.$$

Fix $0 < a < 10^{-12}$ and put

$$h_0 = \frac{a}{10^6}, \qquad b = \frac{h_0}{R_{\mathrm{rect}}}, \qquad \varepsilon = \frac{b}{10}. \tag{19.2}$$

All these parameters are fixed while t tends to infinity.

Proposition 19.1 (Optimized regularization)**.** *Suppose that $N = t^3$ distinct real points determine at least $t^{4-\varepsilon}$ ordered unit pairs. For sufficiently large t, there is a nonempty graph $E \subseteq P \times Q$ on real algebraic positions, with unit distance on every edge, such that*

$$t^{1+2a-2h_0} \le |P| \le t^{1+2a+\varepsilon}, \qquad t^{2+a-2h_0} \le |Q| \le t^{2+a}, \tag{19.3}$$

$$t^{2+2a-2h_0} \le |E| \le t^{2+2a+2\varepsilon}, \tag{19.4}$$

$$t^{1-3h_0} \le d(p) \le t^{1+b}, \qquad t^{a-3h_0} \le d(q) \le t^{a+b}. \tag{19.5}$$

For the uniform edge measure ω and its marginals μ, κ,

$$\mu(p) \ge \frac{\gamma}{|P|}, \qquad \kappa(q) \ge \frac{\gamma}{|Q|}, \qquad \max_p \mu(p) \le t^{-1+3h_0}, \qquad \max_q \kappa(q) \le t^{-2-a+3h_0}. \tag{19.6}$$

If $F \subseteq P, G \subseteq Q$ have counting fractions u, v, then

$$\omega(F,G) \le \sqrt{uv}((uv)^\rho + \xi), \qquad \xi = t^{-a/100}. \tag{19.7}$$

Writing $h = \mu(F), j = \kappa(G)$ instead gives

$$\omega(F,G) \le 2\sqrt{hj}((hj)^\rho + \xi). \tag{19.8}$$

The mutual information of the two point endpoints of an ordinary wedge satisfies $D \le 2a \log t$.

Proof. First make the real algebraic replacement of Lemma 10.1. Apply the cutting of Lemma 10.3 with $R = t^{1-a}$ to the t^3 circles centered at the points. The $O(t^{1-a}\log t)$ sampled circles and points on them remove at most $O(t^{4-a}\log t)$ incidences, which is $o(t^{4-\varepsilon})$. Each remaining region meets at most t^{2+a} circles. Splitting its assigned points into groups of size at most $2t^{1+2a}$ produces $O(t^{2-2a}\log^2 t)$ groups. One group therefore has at least $t^{2+2a-2\varepsilon}$ edges, for sufficiently large t. The incidence bound gives the initial estimates

$$|P_0| \le t^{1+2a+\varepsilon}, \quad |Q_0| \le t^{2+a}, \quad t^{2+2a-2\varepsilon} \le e_0 \le t^{2+2a+2\varepsilon}.$$

Remove point degrees exceeding t^{1+b} and center degrees exceeding t^{a+b}, with degrees measured before either deletion. The excessive vertex sets have at most $t^{1+2a-b+2\varepsilon}$ and $t^{2+a-b+2\varepsilon}$ members respectively. Applying Lemma 10.2 to each such set against the opposite full set bounds the loss by an absolute constant times

$$t^{2+2a-2b/3+3\varepsilon} + t^{2+a+\varepsilon} + t^{1+2a+\varepsilon}.$$

Relative to the lower bound for e_0, the first exponent has loss $-2b/3 + 5\varepsilon = -b/6 < 0$; both linear terms also have strictly negative relative exponent. Thus, after trimming and relabeling, $e_0 \ge t^{2+2a-3\varepsilon}$ and the stated degree upper bounds hold.

For old counting fractions u, v and the corresponding edge fraction w, the incidence and degree bounds give

$$w \le T((uv)^{2/3} + t^{-a}v + t^{-1}u), \qquad w \le T\min(u,v), \qquad T \le t^{b+5\varepsilon} \le t^{2b}.$$

Using $\min(x,y) \le \sqrt{xy}$ on the two linear terms yields

$$w \le t^{2b}\sqrt{uv}((uv)^{1/6} + 2t^{-a/2}). \tag{19.9}$$

Set $\tau = t^{-a/8}$. Among old induced rectangles with both fractions at least τ, maximize $w/(uv)^\gamma$. The full rectangle shows that the maximum is at least one. Let U, V, W be the fractions of a maximizing rectangle. Since $UV \ge t^{-a/4}$,

$$1 \le t^{2b}(UV)^{2/3-\gamma} + 2t^{2b-a/2+a\rho/4}.$$

If $UV < t^{-h_0}$, the first term is at most t^{-5b} by (19.1), and the second tends to zero because $\rho < 1/6$. Consequently

$$U, V \ge t^{-h_0}, \qquad W \ge (UV)^\gamma \ge t^{-h_0}. \tag{19.10}$$

Take this rectangle for P, Q, E. Its edge lower bound follows from $e_0 W \ge t^{2+2a-h_0-3\varepsilon} \ge t^{2+2a-2h_0}$. Before this restriction the point and center counts are at least $t^{1+2a-b-3\varepsilon}$ and $t^{2+a-b-3\varepsilon}$, by their degree upper bounds. Multiplying by U, V proves the two side lower bounds, since $b + 3\varepsilon \le h_0$.

For a new subrectangle with fractions u, v, maximality gives $\omega(F, G) \le (uv)^\gamma$ when $Uu, Vv \ge \tau$. Otherwise $uv \le t^{-a/8+h_0}$. Dividing (19.9), applied to Uu, Vv, by $W\sqrt{uv}$ and using $W \ge (UV)^\gamma$ gives

$$\begin{aligned}\frac{\omega(F,G)}{\sqrt{uv}} &\le t^{2b}(UV)^{2/3-\gamma}(uv)^{1/6} + 2t^{2b-a/2}(UV)^{-\rho} \\ &\le t^{-a/48+2b+h_0/6} + 2t^{-a/2+2b+h_0/6} \le t^{-a/100}\end{aligned}$$

for sufficiently large t. This proves (19.7), also for empty subsets.

Deleting a single vertex of the maximizing rectangle leaves a feasible rectangle for large t: the side fractions are at least t^{-h_0}, whereas $h_0 < a/8$, and both side cardinalities tend to infinity. Optimality thus gives $1 - \mu(p) \le (1 - |P|^{-1})^\gamma \le 1 - \gamma/|P|$, and likewise for κ. Combining these lower marginals with the edge and side-size estimates proves the degree lower bounds in (19.5). The upper marginal bounds follow by dividing the degree upper bounds by the edge lower bound. For (19.8), use $u \le h/\gamma$, $v \le j/\gamma$ and $\gamma^{-1-2\rho} < 2$, $\gamma^{-1} < 2$. For example the first strict inequality follows from $\gamma \ge 133/200$ and $1 + 2\rho < 4/3$.

Finally, the off-diagonal wedge atom is at most $2/(|E| d_{Q,\min})$, and its total diagonal probability is $|Q|/|E|$. Here

$$|E| d_{Q,\min} \ge t^{2+3a-5h_0}, \qquad |Q|/|E| \le t^{-a+2h_0}.$$

Repeating the entropy identity with these explicit bounds gives

$$\frac{D}{\log t} \le a + 5h_0 + 2\varepsilon + o(1) < 2a.$$

The $o(1)$ here is explicitly bounded by $(2 + 3a)t^{-a+2h_0} + \log 2/\log t$; it introduces no dependence on the configuration. This completes the proof. □

Lemma 19.2 (Matching labels with constant sixteen)**.** *Assume the counting expansion* (19.7) *and the lower marginals in* (19.6)*, with* $33/200 \le \rho < 1/6$*,* $\gamma = 1/2 + \rho$*, and* $\xi \le 2^{-20}$*. Assign labels from any finite set to both sides, and let* s *be the edge mismatch probability. Then:*

(1) if $s \le 1/16$*, some label has complementary marginal masses* h, j *satisfying* $h + j \le 8s$*;*

(2) if $s \le 1/128$*, some label has* $h + j \le 2s$*;*

(3) for two independent points, the probability of different labels is at most $16s$*, and the same holds for two independent centers.*

The threshold $1/128$ *in the second assertion is distinct from the threshold* $1/16$ *in the first.*

Proof. Write u_i, v_i for the counting fractions of label i. Summing the counting expansion over equal-label rectangles gives

$$1 - s \le \max_i (u_i v_i)^\rho + \xi,$$

because $\sum_i \sqrt{u_i v_i} \le 1$. Choose a maximizing label, with fractions u_0, v_0. Whenever $s + \xi < 1$, $u_0 v_0 \ge (1 - s - \xi)^{1/\rho}$. For the complementary counting fractions $u = 1 - u_0$, $v = 1 - v_0$,

$$\sqrt{uv} \le 1 - \sqrt{u_0 v_0} \le 1 - (1 - s - \xi)^{1/(2\rho)} \le \frac{s + \xi}{2\rho}. \tag{19.11}$$

The first inequality follows from $u + v \ge 2\sqrt{uv}$; the last is Bernoulli's inequality, as $1/(2\rho) > 1$. Let h, j be the corresponding complementary marginal masses and J their joint edge probability. Lower

marginals imply $h \ge \gamma u$, $j \ge \gamma v$, so

$$J \le \frac{h+j}{2\gamma}((uv)^{\rho}+\xi). \tag{19.12}$$

If $s \le 1/16$, then (19.11) gives $\sqrt{uv} < 19/100$. Hence

$$(uv)^{\rho}+\xi < (19/100)^{33/100} + 2^{-20} < 93/160 + 2^{-20} < 931/1600 \le 7\gamma/8.$$

For clarity the only nonrational comparison here is equivalent to the integer inequality $19^{33}160^{100} < 100^{33}93^{100}$. It follows that $J \le 7(h+j)/16$. The edges with exactly one endpoint carrying the selected label are mismatches, so $s \ge h + j - 2J \ge (h+j)/8$. This proves the first assertion.

For $s \le 1/128$ we may use the weaker hypotheses $\rho \ge 4/25$ and $\xi \le 2^{-14}$ in (19.11), to obtain

$$\sqrt{uv} \le \frac{3225}{131072} < \frac{1}{40}.$$

Therefore

$$(uv)^{\rho}+\xi < 40^{-8/25} + 2^{-14} < 5/16 + 2^{-14} < 33/100 \le \gamma/2.$$

The power comparison is equivalent to $5^{33} > 2^{76}$. Now (19.12) gives $J \le (h+j)/4$, and $s \ge (h+j)/2$. This proves the second assertion. Finally, for $s \le 1/16$ two independent point labels disagree with probability at most $2h \le 16s$, using the first assertion. For $s > 1/16$, use the bound $1 \le 16s$. Centers are treated identically. □

Lemma 19.3 (Bit variation with constant sixteen)**.** *Under the hypotheses of Lemma 19.2, assign a bit to each edge. If V_P, V_Q are the disagreement probabilities in independent two-edge stars based at a point and a center, and f is the smaller global bit probability, then*

$$f \le 16(V_P + V_Q).$$

Proof. Give each vertex its conditional majority-bit label, breaking ties by a fixed rule. If its conditional bit probability is π, disagreement with this label has probability $\min(\pi, 1-\pi) \le 2\pi(1-\pi)$. The probability of different endpoint labels on an edge is thus at most $v := V_P + V_Q$. If $v \le 1/16$, the first assertion of Lemma 19.2 gives a common label with total marginal exceptions at most $8v$. The edge bit differs from that label with probability at most $V_P + 8v \le 9v$, which bounds f. If $v > 1/16$, $f \le 1/2 \le 8v$. Both estimates imply the stated bound. □

Remark 19.4. Lemma 11.2 now applies with $C_0 = 16$. The rare-level exception construction still uses the stronger local conclusion with constant two, at its explicitly smaller threshold $1/128$. For the later extraction, the positive degree exponents remain $1-3h_0$ and $a-3h_0$, and $|E|d_{Q,\min} \ge t^{2+3a-5h_0}$. The popular-circle deletion proof has respective error bounds

$$O(t^{-3/5+8a+4\varepsilon+3h_0}), \quad O(t^{-9/5+3a+2\varepsilon+3h_0}), \quad 2t^{-a+3h_0},$$

all tending to zero. These are direct substitutions in the already proved counting argument, so it applies uniformly with the optimized parameters.

20. Uniform prediction and linear support transfer

20.1. **One history for all bounded comparison vectors.** The following strengthening makes the order of selection explicit. The history is chosen before any of the comparison vectors used later in the template argument.

Lemma 20.1 (Uniform prediction of bounded vectors)**.** *Use the finite incidence law ω, its marginals μ, κ, the wedge endpoint law π, and $D = D_{\mathrm{KL}}(\pi \| \mu \otimes \mu)$. A random test T, independent of the vertices, specifies a labelling $c_T : P \to [M]$ and a vector map $f_T : [M] \to \mathbb{R}^d$ satisfying $\|f_T(c)\| \le B$.*

Let $H \geq 1$ be an integer. There is a fixed strict prefix of fewer than H independent tests, giving a history map H with at most M^{H-1} values, such that, on putting

$$U_T(p) = f_T(c_T(p)) - \sum_x \mu_{\mathsf{H}(p)}(x) f_T(c_T(x)),$$

one has for a fresh independent test

$$\mathbb{E}_T \mathbb{E}_{q\sim\kappa} \left\| \mathbb{E}_{p|q} U_T(p) \right\|^2 \leq 8B^2 \sqrt{\frac{D}{2H}}. \tag{20.1}$$

The same history consequently satisfies, simultaneously for every measurable family $g_T : Q \to \mathbb{R}^d$ with $\|g_T(q)\| \leq B'$,

$$\mathbb{E}_T \left| \mathbb{E}_\omega \langle U_T(p), g_T(q) \rangle \right| \leq 4BB'(D/H)^{1/4}. \tag{20.2}$$

In particular, g may be defined after H has been fixed.

Proof. Temporarily freeze independent tests $T_1, \ldots, T_H$. Let H_j be the history of the first $j-1$ tests. The image of π under the two full sequences of labels has relative entropy at most D with respect to the corresponding image of $\mu \otimes \mu$. Given two prefix values (h, h'), the reference law of the next two labels is $(c_{T_j})_*\mu_h \otimes (c_{T_j})_*\mu_{h'}$. The entropy chain rule and Pinsker's inequality therefore give

$$\frac{1}{H} \sum_{j=1}^{H} \mathbb{E}_\pi \, \mathrm{TV}(A_{j,h,h'}, B_{j,h,h'}) \leq \sqrt{\frac{D}{2H}},$$

where A denotes the actual conditional law of the next labels, B the displayed reference law, and (h, h') the actual random prefixes. This is exactly the calculation for these sequences of labels in Lemma 10.10; it uses no comparison vector.

Let U_j be the residual defined above with T_j, H_j. Its norm is at most $2B$. Under the conditional reference product, the two residuals have zero means and are independent. Their inner product has absolute value at most $4B^2$, so the bounded-expectation estimate for total variation yields

$$\frac{1}{H} \sum_{j=1}^{H} \mathbb{E}_\pi \langle U_j(p), U_j(r) \rangle \leq 8B^2 \sqrt{\frac{D}{2H}}.$$

Conditional independence of the two point endpoints of a wedge gives the identity

$$\mathbb{E}_\pi \langle U_j(p), U_j(r) \rangle = \mathbb{E}_{q\sim\kappa} \left\| \mathbb{E}_{p|q} U_j(p) \right\|^2 \geq 0.$$

Average over the tests and over a uniformly chosen index j. Because the residual uses a strict prefix, T_j is independent of that prefix and may be replaced by a fresh test of the same law. One can therefore fix a prefix for which the fresh-test squared norm satisfies (20.1). This selects the history without reference to g.

For any subsequently specified g, conditional expectation and Cauchy–Schwarz give

$$\left| \mathbb{E}_\omega \langle U_T(p), g_T(q) \rangle \right| \leq B' \left(\mathbb{E}_\kappa \left\| \mathbb{E}_{p|q} U_T(p) \right\|^2 \right)^{1/2}.$$

Averaging over T and using (20.1) proves (20.2), since $\sqrt{8}\, 2^{-1/4} < 4$. The inequality holds for every g with this one fixed history, which proves the asserted simultaneous conclusion. □

Lemma 20.2 (Linear restricted-support transfer)**.** *Consider any finite-measure family of coordinate-state levels, of measure $W > 0$, sampled with its normalized measure. Include in each test two independent random maps h_1, h_2 from the coordinate-label sets to $[L]$, assigning each attained label an independent uniform value as in Corollary 10.13. Use the label $(h_1(A), h_2(B))$ and the vector*

$$f(a, b) = (e_a, e_b, e_{(a,b)}), \qquad \|f(a, b)\| = \sqrt{3}.$$

Fix the single history supplied by Lemma 20.1 for this vector-valued map and test law. Let ν_h be the true state law under μ_h at the current level, and put $\widetilde{\nu}_q = \mathbb{E}_{p|q}\nu_{\mathsf{H}(p)}$.

Conditional on q, sample a template θ independently of the original point p. At each level let $G = G(q,\theta)$ be an event and let $\mathcal{U}(q,\theta)$ be a selected row, a selected column, or a set of at most two true states. These selections may depend on the fixed history and the current level, but not on the new random maps. Suppose that on G

$$\widetilde{\nu}_q(\mathcal{U}(q,\theta)) \geq 1 - 3\eta.$$

Then the integral over this family of levels satisfies

$$\int \Pr(G,\ y_p \notin \mathcal{U}(q,\theta)) \leq W\left(3\eta + \frac{4}{L} + 4\sqrt{6}(D/H)^{1/4}\right). \tag{20.3}$$

The same fixed history simultaneously gives the strict-cross prediction bound and this support-transfer bound for every such selected support.

Proof. At a fixed fresh test, membership after relabelling in U has a representing vector $s(q,\theta)$ in the Euclidean space of these vectors. For a row or column use its single basis vector in the corresponding block. For at most two true states use the sum of the distinct basis vectors of their relabelled pairs in the third block. Counting each distinct relabelled pair only once makes this exactly an indicator and gives $\|s(q,\theta)\| \leq \sqrt{2}$. Set

$$g_T(q) = \mathbb{E}_{\theta|q}[\mathbf{1}_G s(q,\theta)].$$

Its norm is at most $\sqrt{2}$. This comparison vector may be defined after the history has been chosen, by Lemma 20.1. Because θ and p are conditionally independent given q, its inner product with the residual is the difference between the actual membership after relabelling probability restricted to G and the corresponding probability for $\widehat{p} \sim \mu_{\mathsf{H}(p)}$. Thus their difference, averaged over normalized levels and new random maps, has absolute value at most $4\sqrt{6}(D/H)^{1/4}$.

For a state outside a selected row or column, false membership after relabelling has probability at most $1/L$. For a state outside a selected set of at most two states, each of the two possible matches requires at least one collision between unequal coordinate labels. Its total probability is at most $2/L$. The selected support and all sampled vertices are independent of the new random maps conditional on the level and fixed history, so these estimates remain valid for their selections here. Actual and predicted probabilities after relabelling can each be replaced by true probabilities at cost at most $2/L$.

For the conditionally resampled point, averaging first over the original neighbor of q gives exactly $\widetilde{\nu}_q$. Hence

$$\Pr(G,\ y_{\widehat{p}} \notin \mathcal{U}(q,\theta)) = \mathbb{E}_{q,\theta}[\mathbf{1}_G \widetilde{\nu}_q(\mathcal{U}(q,\theta)^c)] \leq 3\eta.$$

Combining the three bounds and multiplying by W proves (20.3).

For completeness, a strict cross of a relabelled state (a,b) has comparison vector $(e_a, e_b, -2e_{(a,b)})$, of norm $\sqrt{6}$. Its inner product with the same vector-valued map is precisely the strict-cross indicator. Consequently the uniform comparison lemma gives that estimate with the same history; the usual error from random relabelling is unchanged. No separate history is selected for support transfer, and no conditional point law after conditioning on a selected support has been assumed. □

21. Sharper joint-exception and averaged-template estimates

The estimates in this section replace the initial parameter choices of the finite framework. All probability laws retain their earlier definitions; only the explicitly specified thresholds change.

Lemma 21.1 (An L^2 consequence of marginal expansion)**.** *Suppose, for some $0 < \rho < 1/2$ and $\gamma = 1/2 + \rho$, that*

$$\omega(F,G) \leq 2\sqrt{\mu(F)\kappa(G)}((\mu(F)\kappa(G))^{\rho} + \xi).$$

For a point set F of marginal mass h put $g(q) = \Pr(p \in F \mid q)$, and let $\kappa_{\min}$ be the smallest positive center mass. Then

$$\mathbb{E}_\kappa g^2 \le \frac{4}{\rho} h^{1+2\rho} + 8\xi^2 h(1 + \log(1/\kappa_{\min})).$$

Proof. Order the centers so that $g_1 \ge g_2 \ge \cdots$, and write $j_r = \sum_{i \le r} \kappa_i$. Apply expansion to the first r centers:

$$g_r \le 2h^\gamma j_r^{\gamma-1} + 2\xi\sqrt{h}\, j_r^{-1/2}.$$

Squaring and using $(u+v)^2 \le 2u^2 + 2v^2$ bounds the second moment by

$$8h^{2\gamma} \sum_r \kappa_r j_r^{2\gamma-2} + 8\xi^2 h \sum_r \kappa_r / j_r.$$

The first sum is a right-endpoint Riemann sum for a decreasing integrable function, and is at most $\int_0^1 x^{2\gamma-2}\,dx = 1/(2\rho)$. The first term of the second sum is one, and its remaining terms total at most $\int_{\kappa_1}^1 dx/x \le \log(1/\kappa_{\min})$. These facts prove the assertion, including zero-mass sets by continuity. □

In our incidence piece $\kappa_{\min} \ge \gamma/|Q|$ and $|Q| \le t^{2+a}$. Thus $\xi^2(1 + \log(1/\kappa_{\min})) \to 0$ uniformly as $t \to \infty$. This is a consequence of the explicit negative power of t in ξ; it introduces no loss in the final exponent.

Lemma 21.2 (Few-miss sampling)**.** *Let $X_1, \dots, X_k$ be independent label pairs with law ν. If $0 < \eta < 1$ and $k\eta \ge 8$, then*

$$\Pr\left(\exists y : \nu(R_y) < 1 - \eta, \quad \#\{j : X_j \notin R_y\} \le k\eta/8\right) \le 2(2k+1)^2 2^{-k\eta/16}.$$

Proof. On the event in question choose a witness using only the first sample. Draw a second independent sample of size k. Its number of misses has mean $k\beta$ and variance at most $k\beta$, with $\beta > \eta$. Chebyshev's inequality bounds the probability of fewer than $k\eta/2$ misses by $4/(k\beta) \le 1/2$. It suffices to bound twice the probability of a cross with at most $k\eta/8$ misses in the first half and at least $k\eta/2$ in the second half.

Condition on the $2k$ indexed observations and split their indices into uniform random halves. There are at most $(2k+1)^2$ strict-cross patterns. For one pattern let m be its total number of misses and X its misses in the first half. The event requires $m \ge k\eta/2$ and $X \le m/4$. Any fixed j indices all lie in the second half with probability $(k)_j/(2k)_j \le 2^{-j}$, interpreted as zero for $j > k$. Expanding the product defining 2^{m-X} consequently gives $\mathbb{E}2^{-X} \le (3/4)^m$. Therefore

$$\Pr(X \le m/4) \le [2^{1/4}(3/4)]^m \le 2^{-m/8} \le 2^{-k\eta/16}.$$

The middle inequality follows from $4(3/4)^8 = 26244/65536 < 1/2$. The union bound proves the lemma. □

Replace the earlier candidate set by an averaged template:

$$\mathcal{Y}_\theta = \left\{ y : \frac{1}{k} \sum_{j=1}^{k} \nu_{h_j}(R_y) \ge 1 - \eta/32 \right\}. \tag{21.1}$$

Then the two required candidate bounds are

$$\int_{\mathcal{C}} \Pr(y_q \notin \mathcal{Y}_\theta) \le 32\eta^{-1}\mathcal{R}, \tag{21.2}$$

$$\Pr\left(\exists y \in \mathcal{Y}_\theta : \widetilde{\nu}_q(R_y) < 1 - \eta \mid q\right) \le 4(2k+1)^2 2^{-k\eta/16}. \tag{21.3}$$

For the first, apply Markov to the average template miss probability. Its conditional expectation is $1 - \widetilde{\nu}_q(R_{y_q})$. For the second, select a violating candidate from the template and draw one independent synthetic state from each ν_{h_j}. Conditional on the template, the expected number of misses is at most $k\eta/32$. With probability at least $3/4$, hence at least $1/2$, it is at most $k\eta/8$. Before conditioning on

the template, the synthetic states are independent with common law $\widetilde{\nu}_q$. Lemma 21.2 and a factor two prove (21.3).

22. The improved explicit parameter chain

Let $K \geq 128$ be an integer for which the finite algebraic obstruction holds. Define

$$\begin{gathered} r_{\mathrm{ht}} = 49 + \lceil \log_2 K \rceil, \quad q = 2^{-r_{\mathrm{ht}}}, \quad \alpha = 2^{-(3r_{\mathrm{ht}}+49)}, \quad \rho = \frac{r_{\mathrm{ht}}+16}{6r_{\mathrm{ht}}+98}, \quad \gamma = \frac{1}{2} + \rho, \\ R_{\mathrm{rect}} = 128(r_{\mathrm{ht}}+17), \quad S_0 = 2^{64} q^{-5}, \quad a = 2^{-2^{15} S_0}, \quad b = \frac{a}{10^6 R_{\mathrm{rect}}}, \quad \varepsilon = b/10, \\ k = q^{-8}, \quad L = q^{-32}, \quad H = \lceil 2aq^{-128} \log t \rceil, \quad \eta = 2^{-53} q^4, \quad \zeta = 2^{-56} q^4. \end{gathered} \tag{22.1}$$

All parameter formulas are fixed before t tends to infinity; the history length H depends on t through the displayed formula. We use the strengthened regularization and matching-label estimates proved in the preceding optimization section: their global matching constant is $C_0 = 16$; their local constant is still two when mismatch is at most $1/128$. In particular $D \leq 2a \log t$, and all earlier positive degree exponents remain positive with the displayed R_{rect} and b.

22.1. **Heights and histories.** We retain the bounded-height experiment, but tighten its geometric tolerances. With $C_0 = 16$, its three height means satisfy

$$A_0 \leq 32S_0 + 66, \qquad B_0 \leq 34S_0 + 68, \qquad C_0' \leq 2S_0.$$

Since $S_0 \geq 336$, the total height of its sixteen quantities is at most $2A_0 + 3B_0 + 3C_0' \leq 173S_0$; the total height of its four nonzero differences is at most $2A_0 + 3C_0' + 4\log 2 \leq 71S_0$. Choose

$$\log B = 300S_0, \quad \log(1/r) = 220S_0, \quad \tau = r^4/(256B^5).$$

The two geometric failure probabilities total at most $173/300 + 71/220 < 0.9$. The three acceptance failures total at most $3/256$, and the collision probability tends to zero; their sum is therefore strictly less than one. The determinant contradiction is unchanged. The label count satisfies

$$\log M \leq 24 \log B + 16 \log(1/r) + 31 \leq 10721 S_0, \qquad \log_2 M < 15500 S_0.$$

The logarithm to base two of the history compatibility expression $2^{42} a M^2 \log M$ is at most

$$42 - 32768 S_0 + 31000 S_0 + \log_2(15500 S_0) < -1.$$

Thus the full bounded-scale proof applies and yields $S > S_0$. The large-scale histories obey

$$(D/H)^{1/4} \leq q^{32}, \qquad L^{-1} = q^{32}, \qquad \frac{\log L^{2Hk}}{\log t} \leq 160 a q^{-136} \log(1/q) < 1/2.$$

For the last inequality, take base-two logarithms and use $-\log_2 a = 2^{79+5r_{\mathrm{ht}}}$, which exceeds $136 r_{\mathrm{ht}} + \log_2(160 r_{\mathrm{ht}} \log 2) + 1$ for $r_{\mathrm{ht}} \geq 56$. All the collision probabilities used in the score arguments can therefore be made at most q^{100} by increasing the starting scale. The history is selected by the energy criterion in Lemma 20.1, so the same history also permits the linear support estimate of Lemma 20.2.

22.2. **Joint exceptions and bad edges.** Declare rare levels good when $f < \alpha$ and $m \leq \alpha$. Then

$$W_c \leq 2^{49} q^{-3} S, \qquad W_b \leq 2^{51} q^{-3}, \qquad \sigma := \alpha^{2\rho} = 2^{-16} q.$$

The local matching lemma gives $h_P + h_Q \leq 4\alpha$ and $\mathcal{A} + \mathcal{B}_* \leq 3S$. Taking t large enough that $\xi \leq \sigma$ and $8\xi^2(1 + \log(1/\kappa_{\min})) \leq \sigma$, expansion yields

$$J_e \leq 9\sigma S \leq qS/4096.$$

Lemma 21.1, $\rho \geq 1/7$, and $4^{2\rho} < 2$ give $\mathbb{E} g^2 \leq 57\sigma h$. After integration,

$$J_w \leq 171\sigma S \leq qS/256. \tag{22.2}$$

The history prediction error satisfies

$$\mathcal{R} \le 4 + 22q^{32}W_c \le 4 + 22 \cdot 2^{49}q^{29}S. \tag{22.3}$$

Use (21.1) with the displayed η. The actual-candidate failure divided by S is at most

$$32\eta^{-1}\mathcal{R}/S \le q/16 + 22 \cdot 2^{107}q^{25} \le q/16 + q/4096.$$

The integrated mixture failure is also at most $qS/4096$: its ratio to S is at most $36 \cdot 2^{49}q^{-19}2^{-2^{-64}q^{-4}}$, and $2^{4r_{\mathrm{ht}}-64} \ge 20r_{\mathrm{ht}} + 67$ for $r_{\mathrm{ht}} \ge 56$. For every good mixture, the containing row, column, or at most two states has predicted mass at least $1 - 3\eta$. Lemma 20.2 bounds the integrated support failure by

$$3\eta W_c + (4/L + 4\sqrt{6}(D/H)^{1/4})W_c \le \frac{3}{16}qS + 14 \cdot 2^{49}q^{29}S \le 3qS/16 + qS/4096.$$

Finally, inconsistency contributes at most $4 \le qS/4096$. Thus the combined bad-edge budget satisfies

$$B_{\mathrm{bad}} \le (q/4 + q/1024)S < 17qS/64. \tag{22.4}$$

22.3. **Scores and positive rare mass.** The upper score remains $D_i \le v_*C_i + qS$: its error is bounded by $2\log 2 + 2q^{100}(W_c + 3S)$. The rare lower score is now

$$D_i^{nc} \ge 2\mathbb{E}H_x^* - 2\mathbb{E}H_y^* - qS/512, \tag{22.5}$$

since its error is at most $4J_e + 24 + W_b$, and $24 + W_b \le qS/1024$. The two common-score inequalities consequently give

$$g_*(C_Q + C_{P^\theta}) \le (2 + 1/256)qS + 16B_{\mathrm{bad}} < \frac{13}{2}qS.$$

The exact value $g_* = 0.00001245$ satisfies $2^{19}g_* = 6.5273856 > 13/2$. Therefore, as before,

$$C_Q + C_{P^\theta} \le 2^{19}qS, \qquad \mathcal{B}_* \le \mathcal{A} + 2^{20}qS. \tag{22.6}$$

The two-prediction proof still gives $\mathcal{T} \le 2\mathcal{A} + 2qS$, because $W_b + \log 2 \le qS$ and the diagonal contribution is at most qS.

For the ordinary-point classification replace its threshold by the displayed ζ. Then

$$4 + \mathcal{R}/\zeta \le (q/64 + q/4096)S, \qquad \zeta W_c \le qS/128.$$

The integrated classification penalty is at most $97qS/256 < qS/2$. Combining with the upper, rare lower, and two-prediction budgets yields

$$g_*C_P \le 2\mathcal{B}_* + (3 + 1/512 + 1/2)qS \le 2\mathcal{B}_* + 4qS.$$

Hence

$$C_P \le 2^{18}\mathcal{A} + 2^{38}qS, \qquad C_P + C_Q \le 2^{18}\mathcal{A} + 2^{39}qS.$$

The improved bit-variation bound is $f \le 16(V_P + V_Q)$, and thus

$$S \le (2^{22} + 2)\mathcal{A} + (2^{43} + 2^{20})qS + 4 + W_b/2 \le 2^{23}\mathcal{A} + 2^{44}qS.$$

Since $q \le 2^{-56}$, this proves

$$\mathcal{A} \ge 2^{-24}S. \tag{22.7}$$

22.4. **Weighted extraction and its numerical endpoint.** In the extraction argument replace c_0 by 2^{-25} and R_0 by q^{-13}. Its small-set factor is at most $4q^2$: indeed

$$q^{13\rho-2} = 2^{-r_{\mathrm{ht}}(r_{\mathrm{ht}}+12)/(6r_{\mathrm{ht}}+98)}, \qquad r_{\mathrm{ht}} \geq 56.$$

This makes $2 \cdot 3^{\rho} q^{13\rho} + 2\xi \leq 4q^2$ for sufficiently large t. The absorption of common and damaged widths also holds, since $W_c/(R_0 S) \leq 2^{49} q^{10}$ and $(W_b + 2\log 2)/(R_0 S_0)$ is smaller still. The original tail calculation therefore gives $\mathbb{E}[H_p^* \mathbf{1}_{H_p^* > R_0 S}] \leq 2^{13} q^4 S$. The wedge approximation error is bounded by

$$C_Q + W_b + 8J_e + 2J_w + 32 \leq 2^{20} qS.$$

In the height-band selection the weighted error deletion consequently costs at most $2^{22} KqS \leq c_0 S/4$, because $q \leq 2^{-49}/K$. The tail deletion costs at most $2^{15} q^4 S \leq c_0 S/4$, and $c_0 S_0 \geq K$. All other steps of the organized graph extraction and finite polynomial selection are unchanged.

Thus, once the finite algebraic obstruction has been established with this K, the improved exponent difference is

$$\boxed{\delta = \frac{2^{-2^{79+5r_{\mathrm{ht}}}}}{3 \cdot 10^7 R_{\mathrm{rect}}}, \qquad r_{\mathrm{ht}} = 49 + \lceil \log_2 K \rceil, \quad R_{\mathrm{rect}} = 128(r_{\mathrm{ht}} + 17).} \tag{22.8}$$

The finite algebraic obstruction below supplies the value of K; together with the preceding incidence normalization, it fixes the exponent difference in this formula.

23. An eight-by-four finite algebraic obstruction

This section replaces the twenty-by-five algebraic grid and its Macaulay height estimate. The eight-by-four normalization has a public antecedent in the manuscript *A smaller generic grid in the unit-distance argument* [33]; the argument below is given independently because its application here permits arbitrary flow variables, not only derivatives. Its new quantitative use is through the finite, specialized height estimate proved below.

There are eight positions p_i and four positions r_j. Write $L_{ij} = (p_i - r_j) \cdot (p_i - r_j)$. Define the constructible set $Z_{8,4}$ by the existence of $R_{ij}, H_i, U_j, A_i, A'_j$ satisfying

$$R_{ij}^2 = L_{ij}(L_{ij} - 4), \qquad 2(p_i - r_j) \cdot (H_i - U_j) = R_{ij}(A_i - A'_j), \qquad A_i - A'_j \neq 0. \tag{23.1}$$

The positions comprise 24 coordinates. This set has a fixed finite Boolean description by integer polynomial tests: adjoin one inverse to $\prod_{i,j}(A_i - A'_j)$ and use effective ideal membership and matrix ranks as in the proof of Lemma 14.2. More explicitly, there are 69 auxiliary variables, 65 equations, and degree at most 33 in the auxiliary variables, so the multiplier bound $66^{2^{69}}$ suffices. The size of this description affects the eventual starting scale only.

Proposition 23.1. *Let V_1, V_2 be either the plane or irreducible real-compatible curves with non-constant z- and w-projections. If they are not the same real unit circle, a jointly generic point of $V_1^8 \times V_2^4$ lies outside $Z_{8,4}$.*

Proof. Suppose a solution of (23.1) exists. Let $J(x, y) = (-y, x)$. First replace all H_i, U_j by $H_i - U_1, U_j - U_1$, and all A_i, A'_j by $A_i - A'_1, A'_j - A'_1$. Thus $U_1 = 0$ and $A'_1 = 0$. These operations preserve all the equations and nonzero differences.

The transformation

$$H_i \longmapsto H_i + \tau J(p_i - r_1), \qquad U_j \longmapsto U_j + \tau J(r_j - r_1)$$

also preserves the equations: the added difference is $\tau J(p_i - r_j)$, whose scalar product with $p_i - r_j$ is zero. It preserves $U_1 = 0$. Since $r_2 \neq r_1$, one coordinate of $J(r_2 - r_1)$ is nonzero. Choose τ so that the corresponding coordinate of U_2 is zero. The choice of this coordinate is one of two finite possibilities.

Of the original twelve right-hand flow coordinates, only eight can now be nonzero. If all are zero, they generate no transcendence. Otherwise multiply every H_i, U_j, A_i, A'_j by the reciprocal of one nonzero remaining right-hand coordinate. All of (23.1) still holds, and that coordinate is now 1. In either case the normalized right-hand flow data have transcendence degree at most seven over the field generated by the right-hand positions.

Let B be the algebraically closed field defining the loci and let K be the algebraic closure after adjoining all four right-hand positions and the normalized right-hand flow data. For $d = \dim V_1$, joint genericity therefore gives

$$\operatorname{trdeg}_K K(p_1, \dots, p_8) \geq 8d - 7 > 8(d-1).$$

Some p_i remains generic on V_1 over K.

The private-valuation proof in Proposition 14.5 applies verbatim to these four anchors: it is proved one anchor at a time and is independent of the number of other anchors. It gives independent square classes for all four $D_j = L(p_i, r_j)(L(p_i, r_j) - 4)$ over $K(V_1)$. In the noncollinear case the first three equations determine H_i, A_i over the field with the first three radicals; flipping the fourth gives $A_i = A'_4$, a contradiction. When both loci are the same line, the first two equations determine the two effective unknowns, and flipping the third radical gives the same contradiction. The rank check and all private-valuation cases are exactly those already proved there. Hence four anchors suffice, and (23.1) is impossible. □

We use the following standard consequence of the arithmetic Bézout and projection inequalities of Krick–Pardo–Sombra. Here the coefficient height H includes the constant 1 in the joint coefficient vector. If a variety $V \subset \mathbb{A}^n_{\overline{\mathbb{Q}}}$ is defined by polynomials of degree at most d and a coordinate projection $\pi(V)$ is finite, then

$$h(\alpha) \leq d^n \left(\frac{n}{d} H + (5n+3)\log(n+1) \right) \qquad (\alpha \in \pi(V)). \tag{23.2}$$

Indeed, Corollary 2.11 of [19] bounds its arithmetic height by $nd^{n-1}H + 2nd^n \log(n+1)$, with $\deg V \leq d^n$. Lemma 2.6 of that paper adds at most $3(n+1)\log(n+1)d^n$ on projection. Arithmetic height is nonnegative and additive over components; for a point its value is at least the ordinary absolute logarithmic height. This proves (23.2). These quoted results explicitly permit non-equidimensional and singular varieties, arbitrary intersections, and projections with positive-dimensional fibers. No complete-intersection or generic-coefficient assumption is being made.

Theorem 23.2. *Suppose an algebraic real-position grid lies outside $Z_{8,4}$ and its nonzero algebraic ratios satisfy $L_{ij} = 2 - \lambda_{ij} - \lambda_{ij}^{-1}$. For $i \geq 2, j \geq 2$ put $\chi_{ij} = \lambda_{ij}\lambda_{11}/(\lambda_{i1}\lambda_{1j})$ and $\Sigma = \sum_{i=2}^{8} \sum_{j=2}^{4} h(\chi_{ij})$. Some edge e satisfies*

$$h(\lambda_e) \leq 4^{65} \left[\frac{65}{4}(2\Sigma + \log 8) + 328 \log 66 \right] < 1.4 \cdot 10^{39}(32.5\Sigma + 1500). \tag{23.3}$$

Proof. Translate and rotate the actual real grid so that $r_1 = (0,0)$ and $r_2 = (s,0)$ with $s \neq 0$. This is possible because the anchors are distinct real points. The transformation has algebraic coefficients. It preserves distances, the ratios, and membership in $Z_{8,4}$, since flows transform by the same orthogonal linear map. There are now 21 position variables: sixteen for the left positions, one for r_2, and four for r_3, r_4.

Set $\chi_{i1} = \chi_{1j} = 1$, $b_1 = 1$, and introduce eleven variables $a_1, \dots, a_8, b_2, b_3, b_4$. Put

$$\Lambda_{ij} = \chi_{ij} a_i b_j, \qquad F_{ij} = \Lambda_{ij}^2 + (L_{ij} - 2)\Lambda_{ij} + 1.$$

These are 32 polynomials of degree at most four in 32 variables X. The actual point is obtained by taking $a_i = \lambda_{i1}$ and $b_j = \lambda_{1j}/\lambda_{11}$. Every solution has all a_i, b_j and Λ_{ij} nonzero, as follows from the constant term of F_{ij}.

If a vector in the literal Jacobian kernel of all F_{ij} had every $d\Lambda_{ij}$ nonzero, logarithmic differentiation of the factorization would give $d\Lambda_{ij}/\Lambda_{ij} = A_i - A'_j$. Differentiating the distance identities would then give (23.1), with $R_{ij} = \Lambda_{ij}^{-1} - \Lambda_{ij}$ and the position components of the vector as flows. This is impossible at the actual position tuple. The finite-union-of-subspaces argument therefore gives an edge e such that $d\Lambda_e$ annihilates the whole Jacobian kernel. Its gradient is in the row span of the 32 gradients ∇F_{ij}.

Adjoin 32 multipliers c_{ij} and a coordinate y. Define C_e by

$$F_{ij}(X) = 0, \qquad \frac{\partial \Lambda_e}{\partial X_v} = \sum_{i,j} c_{ij} \frac{\partial F_{ij}}{\partial X_v} \quad (1 \le v \le 32), \qquad y - \Lambda_e(X) = 0.$$

This system has 65 variables and 65 equations, all of degree at most four; the actual point lifts to it. It is defined over the number field of the actual defects. Its combined coefficient height, together with 1, is at most $2\Sigma + \log 8$: all coefficients are bounded integers times $1, \chi_{ij}, \chi_{ij}^2$, with integer absolute value at most four. The factor eight is a harmless uniform allowance.

On the function field of every irreducible component of C_e, for every derivation over the algebraically closed coefficient field, the defining equations imply

$$Dy = D\Lambda_e = \sum_{i,j} c_{ij} DF_{ij} = 0.$$

Characteristic zero implies that y is constant on that component: if it were transcendental, extend it to a transcendence basis and take the derivation sending it to 1. Consequently the projection of C_e to y is finite. This applies without any smoothness assumption. Apply (23.2) with $n = 65, d = 4$. It gives the first inequality in (23.3). Finally $4^{65} < 1.4 \cdot 10^{39}$, $\log 8 < 2.1$, and $\log 66 < 4.2$ give the displayed strict bound. □

Corollary 23.3. *Take $K = 10^{43}$. No grid outside $Z_{8,4}$ can satisfy*

$$J \ge K, \qquad h(\lambda_{ij}) \ge J/3, \qquad h\left(\frac{\lambda_{ij}\lambda_{i'j'}}{\lambda_{ij'}\lambda_{i'j}}\right) \le J/K$$

on every complete rectangle, along with the distance identities.

Proof. There are 21 normalized defects, hence $\Sigma \le 21J/K$. Theorem 23.2 gives for some edge

$$\frac{h(\lambda_e)}{J} < \frac{1.4 \cdot 10^{39}}{K}(32.5 \cdot 21 + 1500) = 0.30555 < \frac{1}{3},$$

a contradiction. One may instead take the convenient power of two $K = 2^{143} > 10^{43}$. □

The finite polynomial-pattern selection proof requires no new geometric argument. One may apply its existing twenty-by-five statement to the tests for $Z_{8,4}$, interpreted as polynomials in the first eight left and first four right blocks. Projection of a jointly generic twenty-by-five product tuple is jointly generic on the required eight-by-four product. If the two loci are the same unit circle, the popular-circle exclusion applies exactly as before. Thus the new numerical contradiction replaces the old one in the finite obstruction theorem without altering its fixed exponent 10^{-8}.

24. A six-by-five obstruction for arbitrary flows

The joint linear-system and spanning-tree approach in this section has a public antecedent in PR 3 of *unit-distance-grid-refinement* [34]. We prove the needed geometric statements here, including all line cases for arbitrary flow variables. In particular the argument does not require the flows to be values of a derivation, and it does not import a rigidity theorem. This distinction is needed for a finite constructible bad set.

Take six positions p_i and five positions r_j. Let $Z_{6,5}$ be the set where there exist scalars R_{ij}, A_i, A'_j and vectors H_i, U_j such that

$$\begin{aligned} L_{ij} &= (p_i - r_j)\cdot(p_i - r_j), \qquad R_{ij}^2 = L_{ij}(L_{ij}-4), \\ 2(p_i - r_j)\cdot(H_i - U_j) &= R_{ij}(A_i - A'_j), \qquad A_i - A'_j \neq 0. \end{aligned} \tag{24.1}$$

As before this is a fixed constructible set with a finite Boolean description by integer polynomial tests. The same inverse-product and effective ideal-membership proof applies, now with 64 auxiliary variables, 61 equations, and auxiliary degree at most 31.

Theorem 24.1. *Let each of V_1, V_2 be the plane or an irreducible real-compatible curve with non-constant complex coordinate projections. Suppose they are not the same real unit circle. A jointly generic point of $V_1^6 \times V_2^5$ lies outside $Z_{6,5}$.*

24.1. **Independent radicals on the whole product.** Let B be an algebraically closed coefficient field for the two loci and define, at a jointly generic tuple,

$$F = B(p_1, \dots, p_6, r_1, \dots, r_5).$$

All thirty elements $D_{ij} = L_{ij}(L_{ij} - 4)$ have independent square classes in $F^\times/(F^\times)^2$. Indeed, fix i and take the algebraic closure K_i after adjoining every position except p_i. Joint genericity implies that p_i remains generic on V_1 over K_i. The private-valuation construction already proved in Proposition 14.5, applied over K_i, gives a valuation with value one on D_{ij} and zero on the other four radicands in that row. Every radicand in another row is a nonzero element of K_i and consequently has value zero. Thus every nonempty product of distinct D_{ij} has odd valuation somewhere and cannot be a square. By Lemma 14.4, all square-free products of the R_{ij} are linearly independent over F.

24.2. **An elementary complete-bipartite rigidity calculation.** For positions on a graph G, write $\mathrm{Rig}(G)$ for the matrix whose edge row at pr has $p - r$ in the two columns of p, $r - p$ in the two columns of r, and zero elsewhere. Thus its kernel consists of solutions of $(p - r)\cdot(H - U) = 0$. Its row rank is unchanged by an invertible affine change of positions $x \mapsto Mx + t$, provided the velocity vectors are changed by $H \mapsto M^{-\mathsf{T}}H$. This is only a rank calculation, not a claim that an arbitrary affine map preserves Euclidean distances.

Lemma 24.2. *If both vertex sets of a complete bipartite framework affinely span the plane, its infinitesimal motion space has dimension $3 + q$, where q is the dimension of the vector space of quadratic polynomials vanishing on all its vertices.*

Proof. Choose three affinely independent right vertices. For a left vertex p, write $c = p\cdot H$. The three edge equations are $c - r_j\cdot H = p\cdot U_j - r_j\cdot U_j$. Their coefficient matrix in the three unknowns (H_1, H_2, c) is invertible. Hence there are a matrix A, vectors b, c_0, and scalar d such that every left velocity has $H = Ap + b$ and $p\cdot H = c_0\cdot p + d$. In particular

$$Q(p) := p^{\mathsf{T}}Ap + (b - c_0)\cdot p - d = 0.$$

The remaining edge equations read

$$p\cdot(c_0 - A^{\mathsf{T}}r - U) + d - r\cdot b + r\cdot U = 0.$$

Since the left vertices affinely span the plane, they imply $U = c_0 - A^{\mathsf{T}}r$ and $Q(r) = 0$. Conversely these formulas produce every motion whenever Q vanishes at all vertices. The nine parameters in (A, b, c_0, d) determine the six coefficients of Q surjectively. The kernel is precisely $A = \omega J$, $b = c_0 = T$, $d = 0$, the three-dimensional space of rigid translations and rotations. The parameterization of motions is injective, as three left vertices determine A, b, the right velocities then determine c_0, and any one equality determines d. This proves the stated dimension. □

Lemma 24.3. *If neither locus is a line, four generic points of V_1 and three generic points of V_2 lie on a conic exactly when the two loci are the same irreducible conic. In that case the space of quadratics vanishing on these seven points has dimension one.*

Proof. The conic condition is the vanishing of the six-by-six minors of the seven-by-six quadratic evaluation matrix, hence a polynomial condition on the product. It suffices to exhibit a tuple violating it whenever the loci are not the same conic.

If one locus is the plane, first choose the three or four points on the other locus in general position. They impose three or four independent conditions on quadratics. Successive plane points can each be chosen off the zero set of a surviving nonzero quadratic, reducing the remaining dimension by one, until no quadratic remains. There are enough plane points on the prescribed side. If $V_1 = V_2$ is a curve of degree at least three, the six quadratic monomials are linearly independent when restricted to that curve; successive evaluation points therefore give rank six.

If $V_1 \neq V_2$ are curves, choose four points on V_1 off V_2, with no three collinear. These impose four independent conditions on conics. The pencil through them has base locus exactly the four chosen points: two distinct unions of opposite joining lines already have exactly those four common points. Choose a point on V_2 outside this base locus. One conic of the pencil passes through it. That conic cannot contain V_2: a nonlinear irreducible curve contained in a conic must be the conic itself, whereas the four first points were off V_2. Choose a second point of V_2 off the surviving conic; then no conic remains. A third point does not change this.

Finally, if the loci are the same irreducible conic, its defining quadratic vanishes on all points, and any independent quadratic meets it in at most four points by Bézout. Seven distinct points therefore leave precisely a one-dimensional quadratic space. □

Lemma 24.4. *At a jointly generic complete bipartite framework with at least three vertices on each side, the motion dimension is three when precisely one locus is a line, four when the two loci are different lines, and $v + 1$ when both loci are the same line and v is the total vertex count.*

Proof. For exactly one line, normalize it to $r_j = (t_j, 0)$ and write $U_j = (u_j, v_j)$, with distinct t_j. The other vertices $p_i = (x_i, y_i)$ affinely span the plane. Work modulo the span of the two vectors $\mathbf{1}, (t_j)_j$ in the vector space of right-indexed lists, using brackets for quotient classes. The edge equations give

$$x_i[u] + y_i[v] - [tu] = 0 \quad \text{for every } i.$$

Affine spanning forces $[u] = [v] = [tu] = 0$. Hence $u_j = A + Bt_j$ and $v_j = C + Dt_j$; since there are at least three distinct t_j, the condition $[tu] = 0$ forces $B = 0$. Thus the right velocities have just the three parameters A, C, D. Every left point is off the line and two right neighbors determine its velocity uniquely. The three rigid motions supply all these solutions, so the dimension is three. If the line is on the other side, interchange the sides.

For two different lines, an invertible affine change sends them either to $p_i = (s_i, 0), r_j = (t_j, 1)$ or to $p_i = (s_i, 0), r_j = (0, t_j)$. In the second case all generic s_i, t_j are nonzero. Write the velocities as (h_i, k_i) and (u_j, v_j). In the parallel case set

$$F_i = s_i h_i - k_i, \quad B_i = h_i, \quad C_j = u_j, \quad G_j = t_j u_j + v_j;$$

in the intersecting case set

$$F_i = s_i h_i, \quad B_i = k_i, \quad C_j = u_j, \quad G_j = t_j v_j.$$

Both changes of velocity variables are invertible, and the edge system becomes $F_i + G_j - t_j B_i - s_i C_j = 0$. Subtracting two rows with distinct s_i shows $C_j = D - Bt_j$; comparing any other row, using two distinct t_j, gives

$$B_i = A + Bs_i, \quad F_i = C + Ds_i, \quad C_j = D - Bt_j, \quad G_j = At_j - C.$$

These are exactly all solutions, with four free parameters.

For the same line, put it on the horizontal axis. All edge differences are nonzero horizontal vectors. The horizontal velocity is common on the connected graph, and the v vertical velocities are unrestricted. The dimension is $v + 1$, as asserted. □

24.3. **A tree whose removal preserves rigidity rank.** Let E be the thirty edges of $K_{6,5}$ and put

$$T = \{p_1r_4, p_2r_4, p_3r_5, p_4r_5, p_5r_1, p_5r_2, p_6r_3, p_6r_4, p_6r_5, p_5r_5\}.$$

This is a connected graph on eleven vertices with ten edges, hence a spanning tree. Its complement consists of the core $K_{4,3}$ on $p_1, \dots, p_4$ and r_1, r_2, r_3, followed by four attachments: r_5 to p_1, p_2; r_4 to p_3, p_4; p_6 to r_1, r_2; and p_5 to r_3, r_4.

Except when both loci are the same line, each attachment increases rigidity rank by two. The two new rows have an invertible two-by-two minor in the new vertex columns exactly when the three points are not collinear. Genericity gives this noncollinearity: a generic point cannot lie on the line through the other two unless its entire locus is that line, and the latter possibility with two generic points of the other locus forces both loci to be that same line.

If neither locus is a line and they are not the same conic, the core has rank $14 - 3 = 11$ by Lemmas 24.2–24.3. Its four attachments give rank 19, the maximum possible rank of the full framework. For a common irreducible conic, the core rank is $14 - 4 = 10$ and the full rank is $22 - 4 = 18$, again agreeing after the attachments. Exactly one line gives the ranks 11 and 19 by Lemma 24.4; two different lines give 10 and 18. Finally, for one common line both the full graph and the complement of T are connected, so both rigidity ranks equal ten. In all cases

$$\operatorname{rank} \operatorname{Rig}(E \setminus T) = \operatorname{rank} \operatorname{Rig}(E). \tag{24.2}$$

24.4. **The full flow system and its kernel.** The thirty equations in (24.1) form a matrix with 33 columns: 22 velocity columns and eleven scalar columns. Write its edge row as $a_e + R_e b_e$, where a_e is twice the rigidity row on the velocity columns, and b_e is the signed incidence row on the scalar columns. Put $r = \operatorname{rank} \operatorname{Rig}(E)$.

Choose ten scalar columns on which the tree rows b_e, $e \in T$, have nonzero determinant, and choose r independent rigidity rows from $E \setminus T$ and r velocity columns with nonzero determinant. The determinant of the resulting $(10 + r)$-row square minor is a linear combination over F of square-free products of the radicals. The coefficient of $\prod_{e \in T} R_e$ is, up to sign, the product of the two nonzero determinants just chosen. Independence of all the square-free radical products makes this minor nonzero. The full matrix therefore has rank at least $10 + r$.

Its kernel already contains every rigidity motion together with a common scalar value on all eleven vertices, a space of dimension $22 - r + 1 = 23 - r$. The rank is consequently at most $10 + r$. Equality holds, and every kernel vector has a constant scalar part. This remains true after any extension of the coefficient field, since matrix rank is unchanged. In particular no solution can have all $A_i - A'_j \neq 0$. This proves Theorem 24.1.

Remark 24.5 (A twenty-nine-edge version). One may delete the fixed core edge p_4r_3. At a generic core, all twelve edges are symmetric under permutations within the two parts. If deleting one row decreased the rank, every core row would be a coloop; then all twelve rows would be independent. This is impossible, as the ranks just computed are 11, 10, or 6. Deleting p_4r_3 thus leaves the core rank unchanged. The attachments and tree minor argument still apply. Consequently the generic-avoidance theorem also holds for the sparse graph $E \setminus \{p_4r_3\}$, with the nonzero flow constraints required only on its twenty-nine edges.

24.5. **The direct finite height bound.** For the full grid, translate and rotate actual algebraic real positions so that $r_1 = (0, 0)$ and $r_2 = (s, 0)$. There are 19 position variables. Set $b_1 = 1$ and $\chi_{i1} = \chi_{1j} = 1$, and write

$$\Lambda_{ij} = \chi_{ij} a_i b_j, \qquad F_{ij} = \Lambda_{ij}^2 + (L_{ij} - 2)\Lambda_{ij} + 1.$$

The six a_i and four b_j give ten further variables, for a total of 29 variables X. There are thirty distance equations of degree at most four. There are twenty defects, and their sum of heights is denoted by Σ.

Outside $Z_{6,5}$, the same literal-Jacobian-kernel argument as in Theorem 23.2 gives an edge e with $d\Lambda_e$ in the span of the gradients of the F_{ij}. Adjoin thirty multipliers and $y = \Lambda_e$. The critical-value variety has sixty variables and sixty equations of degree at most four, with coefficient height at most $2\Sigma + \log 8$. Its y-projection is finite, by the derivation argument already given. Thus

$$h(\lambda_e) \le 4^{60}\,[15(2\Sigma + \log 8) + 303 \log 61] < 1.33 \cdot 10^{36}(30\Sigma + 1280).$$

With $\Sigma \le 20J/K$ and $J \ge K$, $K = 10^{40}$ is already sufficient to contradict $h(\lambda_e) \ge J/3$.

For completeness the sparse version gives a better direct numerical bound without any rank-stratification argument. Delete F_{43} and its defect. There remain 29 variables, 29 equations, and 19 defects. Adjoining 29 multipliers and y gives 59 variables. Of the 59 equations, six have degree at most three, 52 have degree at most four, and one has degree at most two. The product and reciprocal sum of this upper degree list are

$$\Delta = 2 \cdot 3^6 \cdot 4^{52}, \qquad \sigma = \frac{31}{2}.$$

Using Corollary 2.11 of [19] with these individual degrees, followed by its Lemma 2.6, gives

$$h(\lambda_e) \le \Delta \left[\frac{31}{2}(2\Sigma + \log 8) + 298 \log 60\right]. \tag{24.3}$$

The harmless upper degree bounds can be substituted since each term in the product-times-reciprocal sum is increasing in each degree. Here $\Sigma \le 19J/K$. Since

$$3\Delta \left(31 \cdot 19 + \frac{31}{2}\log 8 + 298 \log 60\right) < 1.64 \cdot 10^{38} < 2^{127},$$

the choice $K = 2^{127}$ gives the required strict contradiction. The stronger rank-stratified bound below instead uses the full grid; this avoids any ambiguity when its distinguished edge is chosen as the base of a new ratio chart.

24.6. **A rank-sensitive logarithmic-Jacobian height estimate.** We use the arbitrary-intersection and projection estimates of Krick–Pardo–Sombra, Corollary 2.11 and Lemma 2.6, with their absolute number-field height conventions. We also record explicitly a useful individual-equation version of the intersection estimate.

Lemma 24.6 (Keeping the individual equation heights)**.** *Let $f_1, \dots, f_s$ be polynomials on $\mathbb{A}^n$, with degree bounds $d_i \ge 1$ and absolute affine coefficient heights at most h_i. Put $D = \prod_i d_i$, and let Z be their common zero set. Then*

$$\deg Z \le D, \qquad h_{\mathrm{KPS}}(Z) \le D\left[h_{\mathrm{KPS}}(\mathbb{A}^n) + \sum_{i=1}^{s} \frac{h_i}{d_i} + s\log(n+1)\right].$$

Degrees and heights here include all irreducible components, regardless of their dimensions; no proper-intersection hypothesis is imposed.

Proof. For a component Y on which f_i is not identically zero, the single-hypersurface arithmetic Bezout estimate used in the proof of KPS Corollary 2.11 gives

$$h_{\mathrm{KPS}}(Y \cap V(f_i)) \le d_i h_{\mathrm{KPS}}(Y) + (h_i + d_i \log(n+1)) \deg Y, \qquad \deg(Y \cap V(f_i)) \le d_i \deg Y.$$

If f_i vanishes identically on Y, the intersection retains Y. The same two inequalities still hold, since $d_i \ge 1$, $h_i \ge 0$, and $h_{\mathrm{KPS}}(Y) \ge 0$. Summing over components and deleting any duplicates or components contained in larger ones only decreases the degree and height sums. Thus these recurrences hold for an arbitrary intermediate zero set. Iterating and dividing the height recurrence at stage i by $\prod_{j \le i} d_j$ gives the assertion. □

The ambient height in this convention is the explicit Stoll number

$$s_n := h_{\mathrm{KPS}}(\mathbb{A}^n) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{1}{j} = \frac{1}{2}\left((n+1)\sum_{j=1}^{n}\frac{1}{j} - n\right).$$

It increases with n. We will use only $s_{29} < 45$ and $s_{50} < 90$. For completeness their exact rational values are

$$s_{29} = \frac{6975593267347}{155272637520} < 45, \qquad s_{50} = \frac{10904958651492685640759}{121531157029254772800} < 90.$$

Coordinates and a fixed coefficient tuple. The grid has six left vertices and five right vertices. Choose the critical edge before imposing a Euclidean gauge, and relabel its endpoints as $(1,1)$. Normalize the rectangle defects relative to it. All twenty new defects are complete rectangle defects from the hypothesis. The old and new normalized families are Laurent monomials in one another, so fixing either family fixes the other; their literal tangent conditions agree on the locus of nonzero ratios. Use ten ratio variables $a_1, \dots, a_6, b_2, \dots, b_5$, with $b_1 = 1$, so that

$$\Lambda_{ij} = \lambda_{ij} = \chi_{ij} a_i b_j, \qquad \chi_{i1} = \chi_{1j} = 1.$$

In particular the critical ratio is the coordinate a_1. An algebraic Euclidean isometry puts $r_1 = (0,0)$ and $r_2 = (s,0)$; the two centers are distinct real points, so this is permissible. The twelve left-point coordinates remain free. There are $N = 29$ variables X: nineteen position coordinates and ten ratio variables. The thirty equations are

$$F_{ij}(X) = \Lambda_{ij}^2 + (L_{ij} - 2)\Lambda_{ij} + 1 = 0.$$

Six equations F_{i1} have degree at most three and all others at most four. The gauged tangent vectors induce ungauged tangent vectors, so da_1 still annihilates the literal Jacobian kernel at the actual point. The actual values of all defects are fixed before any of the following varieties is formed.

A basis containing ten coefficient-free equations. Let c be the Jacobian rank at the actual point. The six rows of F_{i1} are independent, since

$$F_{i1} = a_i^2 + (|p_i|^2 - 2)a_i + 1, \qquad \nabla_{p_i} F_{i1} = 2a_i p_i \neq 0.$$

Here $a_i \neq 0$ and $p_i \neq r_1$, and the six rows have disjoint supports in the corresponding triples (p_i, a_i). Under the contradictory lower-height hypothesis, no λ_{ij} equals 1 or -1, since either number has height zero. Moreover, at a solution,

$$\frac{\partial F_{1j}}{\partial b_j} = a_1(\lambda_{1j} - \lambda_{1j}^{-1}) \neq 0 \qquad (2 \le j \le 5).$$

Each b_j is absent from the other nine equations in the base star $\{F_{i1}\}_{i=1}^{6} \cup \{F_{1j}\}_{j=2}^{5}$. These four columns, followed by the preceding six-row independence, show that all ten star gradients are independent. Thus $10 \le c \le 29$. Extend them to a basis of c rows, corresponding to $F_1, \dots, F_c$. Let $\mathcal{I}$ be the $c - 10$ selected nonstar indices and set

$$\Sigma = \sum_{(i,j)\in\mathcal{I}} h(\chi_{ij}).$$

Only these defects occur in the selected equations, and $\Sigma \le (c-10)J/K$ under the assumed rectangle-height bounds.

The logarithmic Jacobian. Let D_X be diagonal, with entry one in each position column and entry a_i or b_j in the corresponding ratio column. Divide row (i,j) of the ordinary Jacobian by Λ_{ij} and multiply its columns by D_X. The result B is a polynomial matrix:

$$B_{ij,x} = \frac{\partial L_{ij}}{\partial x} \qquad \text{in position columns,}$$
$$B_{ij,a_i} = B_{ij,b_j} = T_{ij} := 2\Lambda_{ij} + L_{ij} - 2 \quad \text{in ratio columns.}$$

with all other ratio entries zero and the absent b_1 column omitted. The position entries have degree at most one and the ratio entries degree at most two. On the nonzero-ratio locus this row and column scaling is invertible. The criticality of a_1 therefore says that the coordinate row e_{a_1} is in the span of the selected c rows of B at the actual point.

All common zeros of the selected equations are on this nonzero-ratio locus: $F_{i1} = 0$ forces every $a_i \neq 0$, and $F_{1j} = 0$ then forces every $b_j \neq 0$. Thus no inverse variables are needed. For any field derivation ∂, form the vector consisting of ∂x in the position columns and $\partial a_i/a_i, \partial b_j/b_j$ in the ratio columns. Its product with row B_i equals $\partial F_i/\Lambda_i$.

Ranks at most twenty-one. In $n = N + c \leq 50$ variables (X, A) impose

$$F_i(X) = 0 \quad (1 \leq i \leq c), \qquad e_{a_1} = \sum_{i=1}^{c} A_i B_i(X).$$

The nineteen position-column equations have degree at most two and the ten ratio-column equations degree at most three. Together with the six cubic and $c - 6$ quartic distance equations, their degree product is bounded by

$$\Delta_c = 2^{19} 3^{16} 4^{c-6}.$$

The individual coefficient heights give a sharper sum than their joint maximum. Each selected distance equation satisfies $h(F_{ij}) \leq 2h(\chi_{ij}) + \log 2$. The nineteen position-column equations have integer coefficients of absolute value at most two. In a ratio-column equation the coefficients involve only defects of selected incident edges, each to the first power, so its height is at most the sum of those defect heights plus $\log 2$. Each selected nonstar defect occurs in exactly two ratio columns. Consequently the sum of equation heights divided by their assigned degree bounds is at most

$$\frac{\Sigma}{2} + \frac{2\Sigma}{3} + \left(\frac{c}{4} + \frac{1}{2} + \frac{19}{2} + \frac{10}{3}\right) \log 2 = \frac{7}{6}\Sigma + \left(\frac{c}{4} + \frac{40}{3}\right) \log 2.$$

Lemma 24.6 bounds the height of the whole critical zero set by

$$\Delta_c \left[s_n + \frac{7}{6}\Sigma + \left(\frac{c}{4} + \frac{40}{3}\right) \log 2 + n \log(n+1) \right].$$

Choose an irreducible component W through the lifted actual point. On the nonempty open set where the selected rows have rank c, the multipliers are uniquely determined by X. Thus $\dim W \leq N - c$. Applying a derivation to the selected equations, the logarithmic-Jacobian identity gives $\partial a_1/a_1 = 0$, and hence a_1 is constant on W. By the KPS coordinate-projection bound,

$$h(a_1) \leq \Delta_c \left[\frac{7}{6}\Sigma + \left(\frac{c}{4} + \frac{40}{3}\right) \log 2 + s_n + (119 - 2c) \log(n+1) \right].$$

Using $s_n < 90$, $\log 2 < 0.7$, $\log 51 < 4$, $10 \leq c \leq 21$, and $\Sigma \leq 11J/K$, this yields

$$h(a_1) < 2^{49} 3^{16} \left(\frac{7}{6}\Sigma + 500\right) \leq 513 \cdot 2^{49} 3^{16} \frac{J}{K} \qquad (J \geq K). \tag{24.4}$$

Ranks at least twenty-two. Let V be the irreducible component through the actual point of the selected distance zero set. The selected gradients are independent there, so $\dim V = r = N - c$. Put $D_0 = 3^6 4^{c-6}$. The preceding individual-height lemma gives

$$\deg V \leq D_0, \qquad h_{\mathrm{KPS}}(V) \leq D_0 \left[s_{29} + \frac{1}{2}\Sigma + \left(\frac{c}{4} + \frac{1}{2}\right) \log 2 + c \log 30 \right].$$

Delete the a_1 column of the selected c-by-N matrix B and take all its c-by-c minors. At most nine remaining ratio columns have degree two, while all position columns have degree one, so every minor has degree at most $d = c + 9$. For $c = 29$ there are no such minors and V is already zero-dimensional.

Their joint affine coefficient height is at most $\Sigma + 6c$. Indeed row i involves only its own defect χ_i, with $\chi_i = 1$ on a star edge. At an infinite place, each entry in that row has coefficient ℓ^1 norm at most $12 \max(1, |\chi_i|_v)$: this follows by expanding $T_i = 2\Lambda_i + L_i - 2$, whose squared-distance

part has coefficient ℓ^1 norm at most eight. A determinant therefore has coefficient norm at most $c!12^c \prod_i \max(1, |\chi_i|_v)$. At finite places the bound is the same product without $c!12^c$. Summing over places gives

$$H_{\min} \le \Sigma + \log(c!) + c \log 12 < \Sigma + 6c,$$

because $c \le 29$ and $12c \le 348 < e^6$.

Intersect V with the zero set of these minors and choose a component W through the actual point. A full c-by-c minor of B is nonzero there, so it remains nonzero on a dense open subset of W. Deleting the target column lowers the rank on this subset; hence e_{a_1} is in the row span of B. The derivation argument above again shows that a_1 is constant on W. Components contained in the rank-drop locus are not being asserted to have this property.

Applying KPS Corollary 2.11 to the arbitrary intersection and then Lemma 2.6 to its chosen component gives, with $D = D_0 d^r$,

$$h(a_1) \le D\Big[\left(\frac{1}{2} + \frac{r}{c+9}\right)\Sigma + \left(\frac{c}{4} + \frac{1}{2}\right)\log 2 + s_{29} + (29 + 3(r+1))\log 30 + \frac{6cr}{c+9}\Big]. \tag{24.5}$$

The same formula applies when $r = 0$ by omitting the minor intersection. For $c \ge 22$, $D(c)$ decreases: for $c < 29$,

$$\frac{D(c+1)}{D(c)} = \frac{4}{c+9}\left(1 + \frac{1}{c+9}\right)^{28-c} \le \frac{4}{31} e^{6/31} < 1.$$

If $c = 22$, then $r = 7$, $\Sigma \le 12J/K$, and the bracket in (24.5), divided by J/K, is less than

$$\frac{1194}{31} + 6(0.7) + 45 + 53(3.41) < 270.$$

Here the fraction combines the defect term and $6cr/(c+9)$, and we used $J \ge K$, $s_{29} < 45$, and $\log 30 < 3.41$. Thus

$$h(a_1) < 270 \cdot 3^6 2^{32} 31^7 \frac{J}{K}.$$

For $c \ge 23$ use $r \le 6$, $\Sigma \le 19J/K$ and $D \le D(23) = 3^6 2^{64}$. The bracket divided by J/K is at most

$$0.7(19) + 7.75(0.7) + 45 + 50(3.41) + \frac{6 \cdot 29 \cdot 6}{32} < 300,$$

so in this case

$$h(a_1) < 300 \cdot 3^6 2^{64} \frac{J}{K}.$$

The explicit threshold. Take $K = 2^{86}$. All three rank cases give $h(a_1) < J/3$, as certified by the integer inequalities

$$\begin{aligned} 3 \cdot 513 \cdot 3^{16} &= 66248903619 < 137438953472 = 2^{37}, \\ 3 \cdot 270 \cdot 3^6 \cdot 31^7 &= 16245923506404390 < 18014398509481984 = 2^{54}, \\ 3 \cdot 300 \cdot 3^6 &= 656100 < 4194304 = 2^{22}. \end{aligned}$$

This contradicts the lower edge-height hypothesis. Every variety in the argument is defined after fixing the actual algebraic coefficient tuple. Nonproper intersections and positive-dimensional multiplier fibers have been included in the height bounds, and no constant depends on the degree of the number field.

25. A stronger exceptional set and the final algebraic threshold

The proof of the six-by-five obstruction establishes more than the nonexistence of a flow with every scalar difference nonzero. Its tree minor argument shows that every solution of the generic flow system has *all* scalar entries equal. Retaining this stronger conclusion removes the intermediate Jacobian ranks responsible for the larger thresholds above.

Definition 25.1. Let Z^+ be the set of six-by-five position tuples for which there exist $R_{ij}, H_i, U_j, A_i, A'_j$ satisfying

$$R_{ij}^2 = L_{ij}(L_{ij} - 4), \qquad 2(p_i - r_j) \cdot (H_i - U_j) = R_{ij}(A_i - A'_j),$$

with $A_{ij} := A_i - A'_j \neq 0$ for *at least one* edge (i, j).

This is a fixed constructible set with a finite integer-polynomial Boolean description. Indeed, for each of the thirty possible witness edges add $v(A_i - A'_j) - 1 = 0$, apply the finite ideal-membership and matrix-rank argument, and take the union of these thirty Boolean conditions. Each branch has 64 auxiliary variables and 61 equations of auxiliary degree at most two. There is no simultaneous product-nonvanishing requirement.

Proposition 25.2. *For any allowable pair of loci other than the same unit circle, a jointly generic point of $V_1^6 \times V_2^5$ lies outside Z^+.*

Proof. The full-radical independence and tree determinant proof of Theorem 24.1 gives

$$\ker M = \{(H, U, A, A') : \ (H, U) \in \ker \mathrm{Rig}, \ A_i = A'_j \text{ for all } i, j\}.$$

This equality of kernels, rather than just the exclusion of one kind of vector, was proved by equality of their dimensions. Matrix rank is unchanged by extending the field. It therefore rules out every witness in Definition 25.1. □

Proposition 25.3 (Finite selection with the rigidity rank retained)**.** *Fix $K \geq 10^9$. For sufficiently large t, suppose a graph $G \subset P \times P$ on a finite set of real algebraic points satisfies $|P| \leq t^{1+1/K}$ and $|G| \geq t^{2-1/K}$, and no edge has both endpoints on a real unit circle containing at least $t^{9/10}$ points of P. Then it contains a complete six-by-five grid outside Z^+. The position rigidity rank of this grid belongs to $\{19, 18, 10\}$; in the last case all its vertices are collinear. Its vertices may be taken distinct.*

Proof. Enlarge the finite polynomial family to contain all minors of the ordinary 30-by-22 position rigidity matrix and all collinearity determinants of triples of grid vertices, together with the coordinate-difference tests for distinctness. These are fixed integer polynomials. The existing finite zero-pattern selection applies to their union with the tests for Z^+. One may still use the original twenty-by-five statement of Theorem 15.3, with its fixed exponent 10^{-8}, letting these polynomials depend only on the first six left and five right blocks. The size hypotheses are stronger than those required there because $1/K < 10^{-8}$.

The selected actual grid has the same zero/nonzero pattern as a generic tuple of the selected loci. Write r_R for its position rigidity rank. The complete rank calculations in the preceding section show that

$$r_R \in \{19, 18, 10\}. \tag{25.1}$$

The rank is nineteen for the general nonconic case or for exactly one line; it is eighteen for a common irreducible conic or two different lines; and it is ten for one common line. In the last case all actual vertices are collinear, because their collinearity tests agree with those of the generic tuple. A common unit circle is excluded by the already proved popular-circle deletion: the selected edges cannot all have both endpoints on that circle. Proposition 25.2 and the identical Boolean pattern therefore place the actual grid outside Z^+ as claimed. □

The actual logarithmic-Jacobian rank. Fix the selected algebraic ratios and defects. Translate and rotate the actual real positions so that $r_1 = (0,0)$ and $r_2 = (s,0)$, $s \neq 0$. Membership in Z^+ and the rigidity rank are invariant under this Euclidean isometry. Use the nineteen position and ten ratio variables, and the thirty equations F_{ij}, from the logarithmic Jacobian construction above.

Let a vector be in the literal kernel of the actual logarithmic Jacobian B. Write its ratio coordinates as α_i, β_j, with $\beta_1 = 0$. Thus its logarithmic ratio velocities are $d\Lambda_{ij}/\Lambda_{ij} = \alpha_i + \beta_j$. At a solution, $2\Lambda_{ij} + L_{ij} - 2 = \Lambda_{ij} - \Lambda_{ij}^{-1}$, so the kernel equations supply a solution of the flow system with

$$A_i = \alpha_i, \qquad A'_j = -\beta_j, \qquad R_{ij} = \Lambda_{ij}^{-1} - \Lambda_{ij}.$$

Since the actual position tuple lies outside Z^+, all $A_i - A'_j$ are zero. Taking $j = 1$ gives $\alpha_i = 0$ for all i, and then $\beta_j = 0$ for all j. Thus the kernel of B consists exactly of the position rigidity motions satisfying the three coordinate gauge constraints, with all ratio velocities zero.

Before gauging, the position rigidity kernel has dimension $22 - r_R$. The two conditions at r_1 and the vertical condition at r_2 have rank three on the three-dimensional rigid-motion subspace: translations set the velocity at r_1 arbitrarily, and, after those are fixed, rotation gives vertical velocity ωs at r_2. Consequently the gauged rigidity kernel has dimension $19 - r_R$. It follows that the actual distance Jacobian has rank

$$c = 29 - (19 - r_R) = 10 + r_R \in \{29, 28, 20\}. \tag{25.2}$$

In particular every ratio coordinate, including the original a_1, is critical. No further change of base edge is necessary.

The noncollinear strata. When r_R is nineteen or eighteen, the actual rank is respectively 29 or 28. Choose a basis of the actual distance gradients containing the ten coefficient-free star rows. The selected nonstar height sum satisfies $\Sigma \leq (c-10)J/K$. The logarithmic-Jacobian high-rank argument already proved gives

$$\begin{aligned} h(a_1) \leq D\Bigg[&\left(\frac{1}{2} + \frac{r}{c+9}\right)\Sigma + \left(\frac{c}{4} + \frac{1}{2}\right)\log 2 + s_{29} \\ &+ (29 + 3(r+1))\log 30 + \frac{6cr}{c+9}\Bigg], \\ r = 29 - c, &\qquad D = 3^6 4^{c-6}(c+9)^r. \end{aligned} \tag{25.3}$$

For clarity, this argument is still the critical-minor construction: take the component of the selected equations which is smooth at the actual point, intersect with the maximal minors after deleting the target column, and retain a component meeting the full-row-rank open set. On that component the target coordinate is constant by derivations. It does *not* assume that the selected distance component remains inside a conic or keeps its actual position rigidity rank.

If $c = 28$, then $r = 1$, $\Sigma \leq 18J/K$, and the bracket in (25.3), divided by J/K, is less than

$$\left(\frac{1}{2} + \frac{1}{37}\right)18 + 7.5(0.7) + 45 + 35(3.41) + \frac{168}{37} < 184.$$

Here we used $J \geq K$, $\log 2 < 0.7$, $\log 30 < 3.41$, and $s_{29} < 45$. Therefore

$$h(a_1) < 184 \cdot 3^6 2^{44} \cdot 37\,\frac{J}{K}. \tag{25.4}$$

For $c = 29$, no critical minors are needed and $r = 0$. The bracket divided by J/K is less than

$$\frac{19}{2} + 7.75(0.7) + 45 + 32(3.41) < 170,$$

so

$$h(a_1) < 170 \cdot 3^6 2^{46}\,\frac{J}{K}. \tag{25.5}$$

The common-line stratum. When $r_R = 10$, all actual positions are on one line. After the same Euclidean normalization, that line is the horizontal axis. Set all vertical position coordinates identically to zero and retain ten horizontal coordinates: the six left coordinates and the four right coordinates other than $r_1 = 0$. Together with the ten ratio variables this gives 20 variables.

Every kernel vector of this reduced logarithmic Jacobian extends by zero vertical velocities to a kernel vector of the former one. Thus all its ratio velocities are zero. The remaining equations are the horizontal rigidity equations on a complete bipartite graph. Every edge has nonzero horizontal displacement: a zero displacement would make $L_{ij} = 0$ and $\Lambda_{ij} = 1$, contrary to the positive lower height. Hence every horizontal velocity is the same, and the velocity at $r_1 = 0$ forces this common value to be zero. The reduced Jacobian consequently has full rank twenty.

Its ten star rows are still independent. For F_{i1} the unique left-position derivative is $2a_i x_i \neq 0$, and the other four star pivots are the same nonzero b_j derivatives as before. Extend these rows to twenty independent equations. Six have degree at most three and fourteen have degree at most four; at most ten defects occur. The actual point is an isolated component of their zero set, because the Jacobian is invertible there.

Apply the individual-equation arithmetic height estimate in $\mathbb{A}^{20}$. The arithmetic height of that point dominates the ordinary height of its coordinate a_1, and is bounded by

$$3^6 4^{14} \left[s_{20} + \frac{1}{2}\Sigma + 5.5 \log 2 + 20 \log 21 \right].$$

Since $\Sigma \leq 10J/K$, $s_{20} < s_{29} < 45$, $\log 2 < 0.7$ and $\log 21 < 3.05$, this is less than

$$120 \cdot 3^6 2^{28} \, \frac{J}{K}. \tag{25.6}$$

No coefficient of a separately specified line is used: the line was made horizontal by the same algebraic Euclidean gauge, so the reduced equations have only the displayed defect coefficients.

Theorem 25.4 (Final explicit algebraic threshold)**.** *Take $K = 2^{68}$. A selected grid as above cannot simultaneously satisfy $J \geq K$, all edge heights at least $J/3$, and all rectangle defect heights at most J/K.*

Proof. The three cases are exhaustive by (25.1). Bounds (25.4), (25.5), and (25.6) respectively give $h(a_1) < J/3$, with the following exact integer certificates after cancellation of powers of two:

$$3 \cdot 184 \cdot 3^6 \cdot 37 = 14889096 < 16777216 = 2^{24},$$
$$3 \cdot 170 \cdot 3^6 = 371790 < 4194304 = 2^{22},$$
$$3 \cdot 120 \cdot 3^6 = 262440 < 1099511627776 = 2^{40}.$$

Each contradicts the lower height hypothesis for the edge $(1, 1)$. □

26. Completion of the refined bound

Proof of Proposition 18.1 and Theorem 1.3. Use the strengthened constructible bad set and the rigidity-rank selection in Proposition 25.3, followed by the height certificate in Theorem 25.4. They give the finite obstruction with $K = 2^{68}$. More explicitly, apply the polynomial-pattern selection theorem of the finite framework to the Boolean tests for Z^+ together with all minors of the position rigidity matrix. These are fixed polynomials in the first six left blocks and all five right blocks of the twenty-by-five selection. Projection to these eleven blocks preserves joint genericity. The popular-circle exclusion rules out two copies of the same unit-circle locus. The selected finite tuple therefore lies outside Z^+ and has the same rigidity rank as its generic product of loci. Its normalized distance–ratio Jacobian has rank 28 or 29, unless all positions lie on one real line. In the latter case the twenty-variable restriction has full rank. The corresponding three height estimates contradict the edge lower heights and rectangle upper heights.

For this value of K, the parameters in (22.1) have

$$r_{\mathrm{ht}} = 49 + 68 = 117, \qquad q = 2^{-117}, \qquad R_{\mathrm{rect}} = 128(117 + 17) = 17152,$$

and consequently

$$S_0 = 2^{64+5\cdot 117} = 2^{649}, \qquad a = 2^{-2^{15}S_0} = 2^{-2^{664}}, \qquad \frac{\varepsilon}{3} = \frac{a}{3 \cdot 10^7 R_{\mathrm{rect}}} = \frac{2^{-2^{664}}}{514560000000} = \delta_{\mathrm{up}}.$$

All numerical hypotheses of the improved parameter chain have now been supplied by proved statements.

If arbitrarily large $N = t^3$ admitted at least $N^{4/3-\delta_{\mathrm{up}}} = t^{4-\varepsilon}$ ordered unit pairs, the improved regularization would give the incidence piece. The bounded-height argument in the improved parameter chain forces $S > S_0$; its uniform history, exception estimates, scores, and weighted extraction produce a graph with at least $t^{2-1/K}$ edges on at most $t^{1+1/K}$ positions, edge heights at least $J/3$, rectangle-defect heights at most J/K, and $J \geq K$. All of its edges avoid popular unit circles. For sufficiently large t this contradicts the finite obstruction just established. Therefore the asserted upper bound holds for all large N, and an absolute multiplicative constant covers the finitely many smaller integers. Unordered unit pairs and balanced point-circle incidences are related by Lemma 9.3.

The balanced-to-unbalanced transfer in the finite framework gives the asserted two-parameter bound with the same δ_{up}. Finally $514560000000 < 2^{40}$, so

$$-\log_2 \delta_{\mathrm{up}} = 2^{664} + \log_2(514560000000) < 2^{664} + 40 < 2^{665}.$$

Thus $\delta_{\mathrm{up}} > \delta_{\mathrm{simple}}$, proving the simpler displayed bound as well. □

This also strictly improves the stronger baseline value δ_1, not only its simplified lower bound δ_0. Let k_{I} denote the parameter in (9.1). Since $k_{\mathrm{I}} > 2^{1024}$,

$$-\log_2 \delta_1 = 2^{201} k_{\mathrm{I}}^{100} + \log_2 3 > 2^{102601} > 2^{665} > -\log_2 \delta_{\mathrm{up}}.$$

Hence $\delta_{\mathrm{up}} > \delta_1$.

The resulting piecewise form is

$$\mathcal{U}(n,m) \ll \begin{cases} s^{2/3-2\delta_{\mathrm{up}}} M^{2/3+\delta_{\mathrm{up}}}, & s \leq M \leq s^2, \\ M, & M \geq s^2. \end{cases}$$

For $M = s^\alpha$ with $1 \leq \alpha \leq 2$, the first line is $O(s^{\frac{2}{3}(1+\alpha)-(2-\alpha)\delta_{\mathrm{up}}})$. For $M \geq s^2$, Proposition 1.4 gives the exact value.

Appendix A. Explicit local profiles

All numbers in this appendix are exact. A terminating decimal denotes its rational value. The exponential of a displayed logarithmic scale is taken as a real number, so no rounding of a scale is part of the definition.

A.1. **The asymmetric archimedean profiles.** We use thirteen archimedean choices. In the notation of (7.3), put

$$\phi(x,y) = (1-x)(1-y), \qquad P_A(x,y) = \prod_{q \in \boldsymbol{q}_A} (1 - q\phi(x,y)), \qquad P_B(x,y) = \prod_{q \in \boldsymbol{q}_B} (1 - q\phi(x,y)).$$

A one-entry vector gives a single factor. Unless specified otherwise, $\lambda_B = 0$. Profile A_{13} uses the two displayed coefficient matrices instead of these product formulas. In every row $a = \exp(\log a)$ and $b = \exp(\log b)$; the counting powers are p_A, p_B.

Profile A_1.

$$\begin{aligned} s &= 1.2043989997038962, & t &= 1.2043886824255563,\\ \log a &= -8.5510686631045161, & \log b &= -8.550651149265045,\\ p_A &= 1.9021620032341802, & p_B &= 1.9020662133751431, \end{aligned}$$

$$\begin{aligned} \boldsymbol{q}_A &= (0.76887061009101054, -0.10783506631385534, -0.22871734724239584, -0.07264260580682555),\\ \boldsymbol{q}_B &= (0.77117724356602846, -0.14696642002230928, -0.1534442371149321, -0.11105762495305768). \end{aligned}$$

Profile A_2.

$$\begin{aligned} s &= 1.2043886824255563, & t &= 1.2043989997038962,\\ \log a &= -8.550651149265045, & \log b &= -8.5510686631045161,\\ p_A &= 1.9020662133751431, & p_B &= 1.9021620032341802, \end{aligned}$$

$$\begin{aligned} \boldsymbol{q}_A &= (0.77117724356602846, -0.14696642002230928, -0.1534442371149321, -0.11105762495305768),\\ \boldsymbol{q}_B &= (0.76887061009101054, -0.10783506631385534, -0.22871734724239584, -0.07264260580682555). \end{aligned}$$

Profile A_3.

$$\begin{aligned} s &= 1.1245092973972035, & t &= 1.1247709010373244,\\ \log a &= -14.911457126049363, & \log b &= -14.912506062422008,\\ p_A &= 1.9407041990130223, & p_B &= 1.9405879030029978, \end{aligned}$$

$$\begin{aligned} \boldsymbol{q}_A &= (0.65417664278633447, -0.15486713748163783, -0.10075053127190532, -0.24170651732520421),\\ \boldsymbol{q}_B &= (0.65379078025747095, -0.21555353527088272, -0.10638723412874324, -0.17180448022262906). \end{aligned}$$

Profile A_4.

$$\begin{aligned} s &= 1.3594315662614589, & t &= 0.70699635212401812,\\ \log a &= -7.9465391424177376, & \log b &= -7.9447708032757562,\\ p_A &= 1.4740864760107992, & p_B &= 2.6806106845531392, \end{aligned}$$

$$\begin{aligned} \boldsymbol{q}_A &= (0.76872077289684793),\\ \boldsymbol{q}_B &= (0.50877380881889245). \end{aligned}$$

Profile A_5.

$$\begin{aligned} s &= 1.2255540979059307, & t &= 0.40460385905497714,\\ \log a &= -12.220791783262671, & \log b &= -12.285429182578341,\\ p_A &= 1.3037203671086752, & p_B &= 3.7748411700555744, \end{aligned}$$

$$\begin{aligned} \boldsymbol{q}_A &= (0.7215142797140347),\\ \boldsymbol{q}_B &= (0.35785243603442674). \end{aligned}$$

Profile A_6.

$$\begin{aligned} s &= 1.3325274080489524, & t &= 0.53875810170025862,\\ \log a &= -7.2898212608618529, & \log b &= -7.1737906334714499,\\ p_A &= 1.3678970737119656, & p_B &= 3.1221335967204542, \end{aligned}$$

$$\begin{aligned} \boldsymbol{q}_A &= (0.88036964413350938, -1.2599682068584213, 0.21830494292067165, 0.13926299502740697),\\ \boldsymbol{q}_B &= (0.72924190769296271, -0.24792205322667021, -0.26836424936085079, -0.28090971334204096). \end{aligned}$$

Profile A_7.

$$\begin{aligned} s &= 1.2128575878994745, & t &= 0.30695909643943042,\\ \log a &= -10.185246038705765, & \log b &= -10.105937104613458,\\ p_A &= 1.2281456888253364, & p_B &= 4.5512896290149945, \end{aligned}$$

$$\begin{aligned} \boldsymbol{q}_A &= (0.85224580275921558, -2.0550667048347195, 0.28466325079340971, 0.18710522486521236),\\ \boldsymbol{q}_B &= (0.63561360296940816, -0.30103276870604978, -0.29459829242192803, -0.19660203638447701). \end{aligned}$$

Profile A_8.

$$\begin{aligned} s &= 1.3305301253599091, & t &= 0.47060916132191954,\\ \log a &= -6.5625552610604423, & \log b &= -6.4379422235146553,\\ p_A &= 1.3173941176848287, & p_B &= 3.395835141488464, \end{aligned}$$

$$\begin{aligned} \boldsymbol{q}_A &= (0.87714813827272919, -1.5244741451477446, 0.47906719965973621),\\ \boldsymbol{q}_B &= (0.71467078140860318, -0.41308437981809476, -0.41307579827320773). \end{aligned}$$

Profile A_9.

$$\begin{aligned} s &= 1.2118504781572903, & t &= 0.27097683672088069,\\ \log a &= -9.1090438455467684, & \log b &= -9.3167870504534669,\\ p_A &= 1.1959224755575879, & p_B &= 5.0110364991429845, \end{aligned}$$

$$\begin{aligned} \boldsymbol{q}_A &= (0.84409554192296543),\\ \boldsymbol{q}_B &= (0.36503284547727038). \end{aligned}$$

Profile A_{10}.

$$\begin{aligned} s &= 1.3317397837887492, & t &= 0.4892702297723529,\\ \log a &= -6.7376369289230027, & \log b &= -6.612451176882594,\\ p_A &= 1.3310896776022887, & p_B &= 3.31273523536685, \end{aligned}$$

$$\begin{aligned} \boldsymbol{q}_A &= (0.90327587325239911, -1.1886509103101082, 0.29319880855158598, 0.00019948162526266087),\\ \boldsymbol{q}_B &= (0.7515748826447175). \end{aligned}$$

Here $\lambda_B = 0.79381059197794501$.

Profile A_{11}.

$$s = 1.216127878632149, \qquad t = 0.26791281537822215,$$
$$\log a = -8.9171753192348788, \qquad \log b = -8.8407237731055606,$$
$$p_A = 1.1946063213291391, \qquad p_B = 5.0308042073269617,$$

$$\boldsymbol{q}_A = (0.83140364256019872,\ 0.59452433678117078,\ -1.9884905298678195),$$
$$\boldsymbol{q}_B = (0.61188917676187526,\ -0.43810531231284111,\ -0.2919158644300448).$$

Profile A_{12}.

$$s = 1.2070837456977095, \qquad t = 0.27270146505219034,$$
$$\log a = -9.6653965853318624, \qquad \log b = -9.5865452608277746,$$
$$p_A = 1.2011561014509533, \qquad p_B = 4.9645738745963817,$$

$$\boldsymbol{q}_A = (0.87989553320102365,\ 0.3371138266507665,\ -1.9840895161416183,\ 0.00078042641678250225),$$
$$\boldsymbol{q}_B = (0.66490582684150423).$$

Here $\lambda_B = 0.79887701051583659$.

Profile A_{13}.

$$s = 1.2643704818394643, \qquad t = 0.3569370831364012,$$
$$\log a = -6.960105659675447, \qquad \log b = -6.8719742698834,$$
$$p_A = \frac{2621439999847412109375}{2093974328671816122368}, \qquad p_B = \frac{13107199999237060546875}{3256468905406373298176}.$$

Here $\lambda_B = 0.7958371960711847$. Writing $P_A(x,y) = \sum_{i,j=0}^{4} a_{ij} x^i y^j$ and $P_B(x,y) = \sum_{i,j=0}^{3} b_{ij} x^i y^j$, the symmetric coefficient matrices are as follows:

i	j	a_{ij}
0	0	0.08986490242942435
0	1	1.6882926023090434
0	2	−1.0295317237726267
0	3	−0.9419136560076167
0	4	1.1932878750417757
1	1	3.6568888170936775
1	2	−11.964534678648683
1	3	11.576781135858283
1	4	−4.957427876612321
2	2	27.567384358283995
2	3	−16.845106163973167
2	4	2.2717882081104745
3	3	−13.967139715783283
3	4	20.17737839990578
4	4	−18.68502660644571

i	j	b_{ij}
0	0	0.24276175946342138
0	1	1.003335297720344
0	2	−0.9926933584947837
0	3	0.7465963013110183
1	1	−1.5068996810513218

i	j	b_{ij}
1	2	3.6068291735844076
1	3	−3.10326479025343
2	2	−9.954310501105798
2	3	7.340174686016173
3	3	−4.983506197073761

A.2. **Phase mixtures.** Write $F_1, \dots, F_{27}$ for the finite-place choices in the finite parameter table. A pair (A_i, F_j) means that all archimedean places use A_i and all selected finite places use F_j. The following formulas give the exact proportions. A phase is the physical mixture described in the proof of the transfer theorem; every list of proportions sums to one. The first and second phases correspond to the low and high rows of Table 7.

Ratio $r = 1$. Phase 1 has the following nonzero proportions:

$$\lambda(A_1, F_1) = \frac{16958907852215594713173329267}{625000000000000000000000000000},$$
$$\lambda(A_1, F_2) = \frac{24339448698597155286826670733}{625000000000000000000000000000},$$
$$\lambda(A_2, F_1) = \frac{87063202412886552868266707 33}{625000000000000000000000000000},$$
$$\lambda(A_2, F_2) = \frac{12495323207898594713173329267}{625000000000000000000000000000}.$$

Phase 2 has the following nonzero proportions:

$$\lambda(A_3, F_3) = \frac{24934353685121}{200000000000000},$$
$$\lambda(A_3, F_4) = \frac{196019949657233}{500000000000000},$$
$$\lambda(A_3, F_5) = \frac{483288332259929}{1000000000000000}.$$

Ratio $r = 1.02$. Phase 1 has the following nonzero proportions:

$$\lambda(A_4, F_6) = \frac{1798339241187}{62500000000000},$$
$$\lambda(A_4, F_7) = \frac{96423893118511}{500000000000000},$$
$$\lambda(A_4, F_8) = \frac{389189392951993}{500000000000000}.$$

Phase 2 has the following nonzero proportions:

$$\lambda(A_5, F_9) = \frac{1571914426989}{2000000000000},$$
$$\lambda(A_5, F_{10}) = \frac{208994562167819}{1000000000000000},$$
$$\lambda(A_5, F_{11}) = \frac{5048224337681}{1000000000000000}.$$

Ratio $r = 1.04$. Phase 1 has the following nonzero proportions:

$$\lambda(A_6, F_{12}) = \frac{222724515886789}{500000000000000},$$
$$\lambda(A_6, F_{13}) = \frac{313697877428409}{1000000000000000},$$
$$\lambda(A_6, F_{14}) = \frac{240853090798013}{1000000000000000}.$$

Phase 2 has the following nonzero proportions:

$$\lambda(A_7, F_{15}) = \frac{16116346494661}{250000000000000},$$
$$\lambda(A_7, F_{16}) = \frac{6607013992789}{50000000000000},$$
$$\lambda(A_7, F_{17}) = \frac{100424291770697}{125000000000000}.$$

Ratio $r = 1.06$. Phase 1 has the following nonzero proportions:

$$\lambda(A_8, F_{18}) = \frac{3993394610163}{12500000000000},$$
$$\lambda(A_8, F_{19}) = \frac{275259845593321}{500000000000000},$$
$$\lambda(A_8, F_{20}) = \frac{65004370000159}{500000000000000}.$$

Phase 2 has the following nonzero proportions:

$$\lambda(A_9, F_{21}) = \frac{91393322620347}{1000000000000000},$$
$$\lambda(A_9, F_{22}) = \frac{90545392677349}{1000000000000000},$$
$$\lambda(A_9, F_{23}) = \frac{25564415146947}{31250000000000}.$$

Ratio $r = 1.0603071$. Phase 1 has the following nonzero proportions:

$$\lambda(A_8, F_{24}) = \frac{32983406912355683594422356 27}{50000000000000000000000000000},$$
$$\lambda(A_8, F_{20}) = \frac{32526359599855431640557764373}{50000000000000000000000000000},$$
$$\lambda(A_{10}, F_{24}) = \frac{13051042286592816405577643 73}{50000000000000000000000000000},$$
$$\lambda(A_{10}, F_{20}) = \frac{12870195480249718359442356 27}{50000000000000000000000000000}.$$

Phase 2 has the following nonzero proportions:

$$\lambda(A_{11}, F_{25}) = \frac{16105327022701024995863770053}{12500000000000000000000000000000},$$
$$\lambda(A_{11}, F_{23}) = \frac{98157173024743100004136229947}{12500000000000000000000000000000},$$
$$\lambda(A_{12}, F_{25}) = \frac{15134532157999750041362299 47}{12500000000000000000000000000000},$$
$$\lambda(A_{12}, F_{23}) = \frac{92240467367558999958637700 53}{12500000000000000000000000000000}.$$

Ratio $r = 1.0629395$. Phase 1 has the following nonzero proportions:

$$\lambda(A_{13}, F_{26}) = \frac{192596700618323}{1000000000000000},$$
$$\lambda(A_{13}, F_{27}) = \frac{807403299381677}{1000000000000000}.$$

A.3. **The finite-place profiles.** The shell matrices are specified by a finite recurrence. This gives their entries directly from the following parameter table. For $x > 0$, write $e(x) = \lfloor \log_2 x \rfloor$ and set

$$\mathcal{R}(x) = 2^{e(x)-96}\lfloor 2^{96-e(x)}x \rfloor, \qquad \mathcal{R}(0) = 0.$$

The number $\mathcal{R}(x)$ is rational and satisfies

$$0 \le x - \mathcal{R}(x) < 2^{-96}x \qquad (x > 0).$$

Thus the error is bounded relative to x, including when x is very small. For a residue cardinality Q, put $\mu_0 = 1$ and $\mu_i = (Q-1)Q^{i-1}$ for $i \ge 1$. Define the $n \times n$ matrix

$$K_{ia} = \begin{cases} \mu_{\min(i,a)}/\mu_i, & i \ne a, \\ i - 1/(Q-1), & i = a \ge 1, \\ 0, & i = a = 0, \end{cases} \qquad T_k(W) = (k+1)W + KW + WK^{\mathsf{T}}.$$

For $v > 1$ let

$$[\mathcal{P}_{k,v}(W)]_{ij} = \mathcal{R}\left(\left(\frac{T_k(W)_{ij}}{T_k(W)_{00}} \right)^{1/(v-1)} \right).$$

Algorithm A.1 (Constructing the finite-place matrices)**.** Fix Q, n, k, v_L, v_R from the parameter table.

(1) Form the matrix K above. Let E_{00} have entry 1 in position $(0,0)$ and entry 0 elsewhere, and set $L_0 = R_0 = E_{00}$.

(2) For each $j = 0, 1, \ldots, 511$, calculate

$$L_{j+1} = \mathcal{P}_{k,v_L}(R_j), \qquad R_{j+1} = \mathcal{P}_{k,v_R}(L_{j+1}).$$

In each application of $\mathcal{P}$, first form $T_k(W)$, divide its entries by $T_k(W)_{00}$, take the indicated positive real powers, and then apply $\mathcal{R}$ to each entry.

(3) Set $w_A = L_{512}$ and $w_B = R_{512}$. These rational matrices are the finite-place weights.

In every finite choice below, $n = 32$ at $Q = 16, 9, 25$, and $n = 8$ at $Q = 29^4, 41^4$. The period vector is ordered as $(16, 9, 25, 29^4, 41^4)$. The same two counting powers are used at all five places.

Choice	Period vector	v_L	v_R
F_1	$(6,7,5,0,0)$	1.9022689819024641	1.9019590028041919
F_2	$(6,7,5,1,0)$	1.9019716183498516	1.9022566182314828
F_3	$(10,13,9,1,1)$	1.9406729483332157	1.9401892938290983
F_4	$(11,13,9,1,1)$	1.9396621452441349	1.9415133796297677
F_5	$(11,13,9,1,1)$	1.9411333073005879	1.9401684191430082
F_6	$(6,7,5,1,0)$	1.5150990483651603	2.5549211308995434
F_7	$(6,7,5,1,0)$	1.4846999478560836	2.6463310741257056
F_8	$(6,7,5,0,0)$	1.4756691623397213	2.6753750220619135
F_9	$(9,11,8,1,1)$	1.3058560497722596	3.7576821673361782
F_{10}	$(9,11,8,1,1)$	1.3025106190128546	3.7837028164053099
F_{11}	$(9,11,8,1,1)$	1.3037447922823469	3.774609208029104
F_{12}	$(5,7,4,0,0)$	1.3728917907209091	3.0943782316699675

Choice	Period vector	v_L	v_R
F_{13}	$(5,7,5,0,0)$	1.3699424065203709	3.1115949392473694
F_{14}	$(5,7,4,0,0)$	1.3684277759573105	3.1192389794694209
F_{15}	$(8,10,7,1,1)$	1.2351336586754948	4.4451277217711773
F_{16}	$(8,10,7,1,1)$	1.2322835046253935	4.5088452416972107
F_{17}	$(8,10,7,1,1)$	1.228835306227319	4.5421194493594781
F_{18}	$(5,6,4,0,0)$	1.3373525036196185	3.2859074942080362
F_{19}	$(5,6,4,0,0)$	1.3214178043370663	3.375419719293832
F_{20}	$(5,6,4,0,0)$	1.3209948616341927	3.3752128297683086
F_{21}	$(7,9,6,1,1)$	1.2083788305014338	4.8126854566529449
F_{22}	$(7,9,6,1,1)$	1.19326286612058	5.0651385979551264
F_{23}	$(7,9,6,1,1)$	1.1967000740450726	4.9971089399496451
F_{24}	$(5,6,4,0,0)$	1.320295962717883	3.3767412090880851
F_{25}	$(7,9,6,1,1)$	1.2003100925483199	4.9285262048455696
F_{26}	$(5,7,5,1,0)$	1.2522867984705557	4.0212303103124363
F_{27}	$(5,7,4,1,0)$	1.2521690761030486	4.0223722356362588

The recurrence has nonnegative entries and preserves symmetry in the shell indices. Substitution in (7.15), followed by the five densities in (7.16), gives the following intermediate values. Each displayed number has absolute error less than 10^{-13}. Here $X_{f,A}, X_{f,B}, Z_f$ denote the complete weighted finite-place resources, including the terms $k \log Q$.

Choice	$X_{f,A}$	$X_{f,B}$	Z_f
F_1	8.75655867279179	8.75653190899294	9.80998821817131
F_2	12.12381511637379	12.12383970217037	13.35055616439864
F_3	20.87357373220748	20.87354800397013	22.45992559462880
F_4	20.95753542234338	20.95763312879784	22.54652329134110
F_5	20.96076454682457	20.96071354888252	22.54977460504412
F_6	12.18372151632316	12.29655447963901	13.46428342347475
F_7	12.19723738444273	12.32877451515930	13.48568608819373
F_8	8.83498016883761	8.97562085819919	9.95390224718216
F_9	19.64772635987888	19.96825713051418	21.30449246106980
F_{10}	19.61889322789329	19.94648271230000	21.27660951529022
F_{11}	19.63825352508882	19.96341232488492	21.29594604619528
F_{12}	8.22068639405639	8.50099747958038	9.35576499184704
F_{13}	8.83495059431152	9.09508618656232	9.99511903512623
F_{14}	8.24042867766068	8.52910213089902	9.37921474920714
F_{15}	18.20015023528243	18.77892344481823	19.87221182260070
F_{16}	18.47222574993064	19.07432174978183	20.15872231569113
F_{17}	18.32241478232529	18.93838228245181	20.00694235178725
F_{18}	7.88966666910419	8.25309877304806	9.03350114477615
F_{19}	7.79308079667491	8.19172596255829	8.94241331291362
F_{20}	7.76971901690643	8.16827188209009	8.91778247016684
F_{21}	17.18308072187814	17.96054470990158	18.86281575745856
F_{22}	17.09844631489943	17.99425409756351	18.79914495427625
F_{23}	17.04815117598609	17.91285807347277	18.74087730931423
F_{24}	7.74685058717637	8.14599529909959	8.89386783140356
F_{25}	17.00174429332277	17.83542657797112	18.68655383965078
F_{26}	12.50471278568225	13.07528224987849	13.93875913297122

Choice	$X_{f,A}$	$X_{f,B}$	Z_f
F_{27}	12.01985330510428	12.62605306785589	13.43986132053947

Algorithm A.2 (Determining the rational rounding)**.** The values $\mathcal{R}(0) = 0$ and $\mathcal{R}(1) = 1$ are immediate. For every other positive number x occurring in Algorithm A.1, proceed as follows.

(1) Enclose x between rational numbers, using the series bounds for log and exp in Section 6.5.
(2) Refine these bounds until they determine an integer e with $2^e \le x < 2^{e+1}$.
(3) Multiply the bounds by 2^{96-e} and refine them until they determine the integer $h = \lfloor 2^{96-e}x \rfloor$. Then $\mathcal{R}(x) = 2^{e-96}h$.

The finite calculations here use precision 224 in base 2 and determine every displayed integer e, h. Once the rational matrices are obtained, their Gram sums are exact; the remaining logarithms and powers are bounded by the same series.

A.4. **The balanced finite-place profiles.** Use Algorithm A.1 with the common power

$$v_L = v_R = v_0 = \frac{2}{1 + 0.04317220235645} = \frac{40000000000000}{20863444047129}.$$

The five parameter choices are

Q	16	9	25	29^4	41^4
n	64	64	64	12	12
k	7	9	6	1	1

For each choice take $w = R_{512}$. Equivalently, apply the one-matrix map $\mathcal{P}_{k,v_0}$ exactly 1024 times to E_{00}. The generation power v_0 is fixed; the exponent $p = 2/(1+\delta)$ in (6.9) varies with δ.

Substitution in (6.10) gives the following finite-place intermediate values, each with absolute error less than 10^{-34}:

δ	$\mathcal{F}(\delta)$
0.04317218	0.71826540946064369003155693342528624
0.04317228	0.71826381501041550320423926484819084
0.0431722305539701	0.71826460340275401512882615794339374

On the full interval $[0.04317218, 0.04317228]$, the finite-place curvature expression satisfies

$$1.5538279509314 < K_f < 1.5538298808880 < 2.$$

To obtain this enclosure, evaluate the finite moment sums in Section 6.6 with the interval $\delta \in [0.04317218, 0.04317228]$, retaining the fixed rational matrix w. The same logarithm and power series used for the recurrence enclose all terms. The period contribution is unchanged and obeys

$$15.602654682637374 < H < 15.602654682637375.$$

A.5. **The balanced polynomial.** Let $T_0(X) = 1$, $T_1(X) = X$, and $T_{j+1}(X) = 2XT_j(X) - T_{j-1}(X)$ be the Chebyshev polynomials. The balanced polynomial is

$$P(t,u) = 10^{-25} \sum_{i,j=0}^{96} c_{ij} T_i(2t-1) T_j(2u-1), \qquad c_{ji} = c_{ij}. \tag{A.1}$$

The integers c_{ij} for $i \le j$ are listed below and specify all the coefficients of P.

Algorithm A.3 (Changing to Bernstein coefficients)**.** (1) Starting with $T_0(2t-1) = 1$ and $T_1(2t-1) = 2t-1$, apply the Chebyshev recurrence to expand $T_i(2t-1)$ for $i = 2, \dots, 96$.
(2) Substitute these expansions and the integers c_{ij} in (A.1). Collect powers to obtain $P(t,u) = \sum d_{rk} t^r u^k$.

(3) For $0 \le i, j \le 96$, calculate

$$\beta_{ij} = \sum_{r \le i,\, k \le j} d_{rk} \frac{\binom{i}{r}}{\binom{96}{r}} \frac{\binom{j}{k}}{\binom{96}{k}}.$$

The resulting rational matrix is the Bernstein matrix used in Section 6.1.

i	j	c_{ij}	i	j	c_{ij}	i	j	c_{ij}
0	0	434636754068339169013769327	0	1	417204451912167217576633170	0	2	168236398817351803547172783
0	3	693743984333127248422738 26	0	4	259761121156953101728062 40	0	5	107080816624309059348380 34
0	6	367450242679507019679288 0	0	7	166239971850109579024515 8	0	8	460757986865318848211479
0	9	275221448618923589298634	0	10	47108164576348339376574	0	11	41507862955668047779052
0	12	15103940971610518866631	0	13	-11126743166377384884693	0	14	25466524515027958260335
0	15	-29628276388467589982713	0	16	35317646917060039692728	0	17	-38496769699901173253032
0	18	41366960406283092703940	0	19	-43147814723156518530556	0	20	44641825872811735790279
0	21	-45450081898249214659564	0	22	46136896933605698505443	0	23	-46317963481818161906316
0	24	46499961686100604221108	0	25	-46277881396366436249779	0	26	46136823688389846551364
0	27	-45654322003459180752675	0	28	45305252638660518163776	0	29	-44654624442312259008567
0	30	44172473585973545359297	0	31	-43414499851030288237461	0	32	42849044001908610306796
0	33	-42024372178031549707281	0	34	41409164348320556954791	0	35	-40545288289637214023842
0	36	39903181781718095854587	0	37	-39018829546825805696150	0	38	38365489357428639951009
0	39	-37473444436241065900711	0	40	36819631844074150799592	0	41	-35928584443625587981564
0	42	35281672515297135878043	0	43	-34397456419948094831832	0	44	33762450602793168445059
0	45	-32888886931772312092982	0	46	32269111676559459637461	0	47	-31408610977743859557294
0	48	30806158686080751697135	0	49	-29960152250059307513901	0	50	29376225873604484670729
0	51	-28545385509994294006477	0	52	27980643844307892851226	0	53	-27164987176530977465064
0	54	26619747104347749719584	0	55	-25818813032348640834160	0	56	25293116509406852991074
0	57	-24506093343904164105675	0	58	23999828958886944603836	0	59	-23225547939067964954999
0	60	22738658314771562028007	0	61	-21975486359310436587726	0	62	21508158274271065661932
0	63	-20754167963091101482104	0	64	20305717546698978669789	0	65	-19561680707874956456143
0	66	19126475853284806171651	0	67	-18396319194949646213543	0	68	17971422595954659391644
0	69	-17251580157643367486138	0	70	16839705767254353903595	0	71	-16125724616726804185990
0	72	15727500098412704463964	0	73	-15016719952991726661378	0	74	14631585737670452460402
0	75	-13921593752894640822193	0	76	13548769163913059695278	0	77	-12836887348830413946426
0	78	12475555388786690227543	0	79	-11758622395196147376806	0	80	11407944407037488833026
0	81	-10682086172236475848462	0	82	10341190394312187398104	0	83	-9601477808083759083546
0	84	926944600348732421051 7	0	85	-85093278073802780510 43	0	86	8185187919167694882770
0	87	-739547258046108541177 4	0	88	707822011485510907154 7	0	89	-624506114962501282991 0
0	90	5933806277836504206651	0	91	-503414761305507262612 8	0	92	472882850993075761505 3
0	93	-371793558868979426206 8	0	94	342275251769389691894 7	0	95	-218876059177807238051 9
0	96	19292889049971533536 90	1	1	335710238237256833611506122	1	2	12144770726802445192360076 4
1	3	43738880842450683251397697	1	4	15262739616106354123529894	1	5	52048994244431913577810 32
1	6	19517618292222328507771483	1	7	46308353344087388277456 2	1	8	37569734755309107783262 8
1	9	-10045727577292615632431 2	1	10	19385094063306696339466 1	1	11	-16083678426336121559903 8
1	12	16766766158332869331742 5	1	13	-16016899805553970426431 8	1	14	15685215379418386083726 2
1	15	-15169534781654544399833 0	1	16	14722352023989102178241 7	1	17	-14234448912041575232193 3
1	18	13791316479452101227046 5	1	19	-13327057578941066585198 8	1	20	12909565947699711585341 0
1	21	-12474474864752166948355 1	1	22	12087445152956336418986 9	1	23	-11683043264667089403277 2
1	24	11326587402038068894147 1	1	25	-10951878007493258348370 3	1	26	10624342147642948533902 0
1	27	-10277245516312980495314 2	1	28	99763286785210361109649	1	29	-96544123491237637689071
1	30	93776672888863228483018	1	31	-90784865786481234152434	1	32	88235468271413219211160
1	33	-85447950290928393345199	1	34	83094707044942074661011	1	35	-80490372139989311043112
1	36	78313469316800881600294	1	37	-75873309963000732596709	1	38	73855032587189438955997
1	39	-71562067528262499374756	1	40	69686675386086147588226	1	41	-67525786355395712323783
1	42	65779334968008902544891	1	43	-63737072449309265201592	1	44	62107221333951401591107
1	45	-60171611705494741307841	1	46	58647431347509406171763	1	47	-56807809637963752025333
1	48	55379584057631076600063	1	49	-53626456662850082306775	1	50	52285528077809177513712
1	51	-50610381326133321277613	1	52	49349096784556641003229	1	53	-47744188950080114205892
1	54	46555803668162970006335	1	55	-45014081459456221427073	1	56	43892604171513123297861
1	57	-42407615789972598461552	1	58	41347726588772500920638	1	59	-39913449260996247387903
1	60	38910493243777356877182	1	61	-37521076691769897447074	1	62	36571133185384827199548
1	63	-35220783606309554645566	1	64	34320166138835643568180	1	65	-33005438889819700421426

i	j	c_{ij}	i	j	c_{ij}	i	j	c_{ij}
1	66	32146646531883035828721	1	67	-30867513800187096498882	1	68	30045487126872580359531
1	69	-28795552988169510261898	1	70	28010104955757860270073	1	71	-26782884604071907170510
1	72	26031975699511596380627	1	73	-24822592658130890138622	1	74	24103709876887774302960
1	75	-22907361180491941213308	1	76	22218012265732809345686	1	77	-21029502255927711955646
1	78	20367265873140998193229	1	79	-19180754174335022531457	1	80	18543267392516558569126
1	81	-17351941869596135648742	1	82	16736869861486276788569	1	83	-15532429684657697645320
1	84	14937420611116185033721	1	85	-13709211651840083754187	1	86	13131840335632599000937
1	87	-11865310955571183226721	1	88	11303030726043047833482	1	89	-9976705663226596491436
1	90	942690593730614265937 2	1	91	-800562018783454545816 0	1	92	7466289300922730755844
1	93	-588281556852453633444 8	1	94	5356578700582286538902	1	95	-344410634251034125840 2
1	96	2963009181926239682492	2	2	43248132992862316374282747	2	3	14803632219555051086369330
2	4	5488730649172909141150283	2	5	16811822669703593917920 60	2	6	826973648431505335140720
2	7	85189644126558302837077	2	8	22467260249845892246989 9	2	9	-98911226575523381701866
2	10	132399845010890586972613	2	11	-10950868017567337375089 6	2	12	10853895208222925835511 7
2	13	-10044541074989833121423 7	2	14	95921203954226865322186	2	15	-90701904641530798266253
2	16	8644720009204670967491 9	2	17	-82272461757637278640734	2	18	78621275487260462263215
2	19	-75087918629024890571904	2	20	71959314626044001889740	2	21	-68904387668758359299896
2	22	66196411563524866475666	2	23	-63519488632954956241844	2	24	61151080264889055905417
2	25	-58780143860868223229603	2	26	56689512289635941713394	2	27	-54570236154546785315298
2	28	52709865492372113714167	2	29	-50800236192621228881088	2	30	49133018922295090280543
2	31	-47399967336044149501068	2	32	45896492545414051988858	2	33	-44313606265221746230943
2	34	42950253764715624453445	2	35	-41496165302559971012138	2	36	40253741047313871621826
2	37	-38910964523464745986240	2	38	37773705243056212460172	2	39	-36527780022289918736367
2	40	35482615126636367543476	2	41	-34321465044328708556568	2	42	33357460705910062922424
2	43	-32270906974384104857039	2	44	31378840652032745378830	2	45	-30358226491891413445273
2	46	29530255690094778899349	2	47	-28568159684671474719534	2	48	27797553990374424000717
2	49	-26887568269857315225595	2	50	26168502325187164146295	2	51	-25305032673615706823401
2	52	24632453770578508102575	2	53	-23810545406374633940892	2	54	23180050901468440077208
2	55	-22395279614385092107094	2	56	21802995059848280231302	2	57	-21051380792576505219161
2	58	20493891594962695879408	2	59	-19771811021246821678163	2	60	19246148208161717450494
2	61	-18550246256861915004216	2	62	18053905502125139901792	2	63	-17380993439360359108385
2	64	16911616701461522665173	2	65	-16259344610187152938215	2	66	15813273190276955701568
2	67	-15180711373610397534871	2	68	14755277713224853979965	2	69	-14139186192168153311922
2	70	13733613019264564514606	2	71	-13130868992187618109309	2	72	12743762220814346729690
2	73	-12151756597769094543721	2	74	11781531327296910678478	2	75	-11197684993992167195322
2	76	10842839259077221128714	2	77	-10264470432302562326753	2	78	992359651657437053275 8
2	79	-9347783488899368748696	2	80	901953104132042145399 3	2	81	-8442934878346828455101
2	82	81259704660947752508 33	2	83	-75445813785765994733 85	2	84	72375487079247945050 93
2	85	-6646283416549643540808	2	86	6347760601303947076221	2	87	-573975688620366255759 5
2	88	5448209974202604120298	2	89	-481345361387961798265 4	2	90	45272100999761049306 76
2	91	-384948060626634999836 4	2	92	35668879034494208725 01	2	93	-28155297756092565810 40
2	94	25363456193777831432 96	2	95	-16362535055428798492 34	2	96	13720281021067299577 60
3	3	42900253452877790383360 17	3	4	16324539784726949269305 75	3	5	28061024268630207668602 3
3	6	26609460124202207535799 6	3	7	-76953353816059793420668	3	8	11199889396075551277006 8
3	9	-88407459371592676133506	3	10	82934920117196599608028	3	11	-75423635353171226984722
3	12	69529457085899762391383	3	13	-64568617566464863008389	3	14	60189310244305531688695
3	15	-56373904555376008331544	3	16	53020476462276044876321	3	17	-49978203737808590780568
3	18	47303826043389528492849	3	19	-44815364307340707771590	3	20	42620217884497559896333
3	21	-40539479039115333240924	3	22	38699194488260209493219	3	23	-36927111421859966859323
3	24	35358598818473998745933	3	25	-33826262017383119402370	3	26	32471309310572752374786
3	27	-31129354631168832222481	3	28	29945749527998737038562	3	29	-28757816128352075377892
3	30	27714093139153364992557	3	31	-26652729084608099360710	3	32	25724882696595424607028
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Email address: Yuda Chen: math.yuda.chen@gmail.com

School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, China

Email address: Yifei Wang: yifeiwang.math@gmail.com

School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, China