FURTHER CASES OF THE INDUCTIVE ALPERIN–MCKAY CONDITION

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Abstract. We verify the inductive Alperin–McKay condition for simple groups of types A, ${}^2A$, B, C, and F_4, for groups of type D in even defining characteristic, and for unipotent blocks of type D. The constructions at odd nondefining primes retain the actual defect normalizers, central characters, and block relations for all ordinary upper characters. These results supply further simple-group cases required by the reduction of the Alperin–McKay conjecture.

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1. Introduction

For a block B of a finite group G, let $\mathrm{Irr}_0(B)$ denote its ordinary irreducible characters of height zero. If D is a defect group of B, and b its Brauer correspondent in $N_G(D)$, the Alperin–McKay conjecture predicts

$$|\,\mathrm{Irr}_0(B)| = |\,\mathrm{Irr}_0(b)|.$$

Späth's reduction [34] asks for more than this equality on simple groups: the bijection on a universal cover must respect automorphisms, central characters, and blocks of ordinary characters above every matched pair. We use the character-triple formulation of this inductive condition, abbreviated iAM, recalled in Section 2.

Theorem 1.1. *Every nonabelian finite simple group of type A, ${}^2A$, B, C, or F_4 satisfies iAM for every prime. The same conclusion holds for simple groups of type $D_n^\epsilon(q)$, $n \geq 4$, when q is even. For arbitrary q, every unipotent block of the natural type-$D_n^\epsilon(q)$ group satisfies the iAM block condition on the prescribed central quotient.*

Here a unipotent block means the *whole block* containing a unipotent character; all its ordinary nonunipotent characters are included. In the last assertion, a block is considered on the prescribed central quotient of the natural simply connected group. At the extra universal cover of $\Omega_8^+(2)$, all central-character sectors are treated in Section 12. The theorem makes no assertion about nonunipotent type-D blocks in odd defining characteristic.

The prime two is a published case for all simple groups [29]. The defining-characteristic case, including exceptional covers, is [6, Lemma 7.4]. It therefore suffices to prove the theorem at odd nondefining primes. The additional assertions about natural-group source systems, actual defect normalizers, and ordinary regular fibres in the family sections concern these odd nondefining primes. The table separates the main previously published inputs from the additional arguments of this paper.

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Type	Published input	Argument retained here
$A, {}^2A$	$\ell \geq 5$: [28, Corollary 12.5]; maximal defect: [9]	Prime three, natural central covers, and all ordinary regular fibres
B, C	Odd q, $\ell \geq 5$: [7]; small even-field groups: [31]	Prime three and the remaining even-characteristic symplectic groups
D	Unipotent source classification and ordinary Jordan theory	Complete unipotent blocks; rank induction staying in characteristic two
F_4	Block classification [16, 17, 1]; $F_4(2)$: [30, Proposition 4]	Full ordinary inverses and automorphism actions, including the prime three

The proof has three common parts. An ordinary correction theorem upgrades a central character-triple comparison to a block comparison while retaining coefficients of arbitrary degree. A normal defect-zero source theorem passes from a source normalizer to the actual defect normalizer. Rational Levi assembly and ordinary Jordan transfer preserve the prescribed regular block anchors and actual field and graph returns. The family sections verify the hypotheses on concrete sources, including the inverse parametrization of the entire relevant height-zero set. Finite computations used at the small boundaries are described in Section A.

2. Ordinary character triples and block comparison

For a positive integer n, write $n_\ell = \ell^{v_\ell(n)}$ and $n_{\ell'} = n/n_\ell$, where $v_\ell(n)$ is the ℓ-adic valuation. For a finite group H, let $\mathrm{Lin}(H)$ be its group of linear ordinary characters and let $\mathrm{Lin}_{\ell'}(H)$ consist of those of order prime to ℓ. Write $\mathrm{dz}_\ell(H)$ for the irreducible ordinary characters θ with $\theta(1)_\ell = |H|_\ell$, and abbreviate this to $\mathrm{dz}(H)$ when ℓ is fixed. For $\theta \in \mathrm{Irr}(H)$ put

$$d_\ell^H(\theta) = v_\ell(|H|) - v_\ell(\theta(1)).$$

If $B = \mathrm{bl}(\theta)$ has defect group D, its height is $h_B(\theta) = v_\ell(|D|) - d_\ell^H(\theta)$. We write $\mathrm{Def}(B)$ for the set of defect groups of B, and $\mathrm{Irr}(H \mid \eta)$ for the irreducible characters of H lying above a specified character η of a normal subgroup. The groups $O_\ell(H)$ and $O_{\ell'}(H)$ are the largest normal ℓ- and ℓ'-subgroups of H, respectively. We write $O^\ell(H)$ for the intersection of the normal subgroups of H whose index is a power of ℓ.

Fix a splitting modular system $(\mathbb{K}, \mathcal{O}, k)$ for all finite groups under consideration, with residue characteristic ℓ. Reduction of an algebraic integer is denoted by a superscript $*$. For a union S of conjugacy classes of a finite group H, put $S^+ = \sum_{s \in S} s \in \mathcal{O}H$. For $\theta \in \mathrm{Irr}(H)$, its ordinary central character has value

$$\omega_\theta(S^+) = \frac{1}{\theta(1)} \sum_{s \in S} \theta(s).$$

This is an algebraic integer. Its reduction is the value of the central character of $\mathrm{bl}(\theta)$ on S^+. In particular, block induction can be tested by comparing an ambient class sum with its intersection with the subgroup.

2.1. The inductive condition and its quantifiers

Suppose $G \lhd Y = GH$, $M = G \cap H$, and $\chi \in \mathrm{Irr}(G)$, $\psi \in \mathrm{Irr}(M)$ are invariant under Y, H, respectively. An associated projective representation P of Y restricts to a representation of G affording χ; its factor set is inflated from Y/G. We use normalized finite factor sets, trivial if either argument belongs to the base group. The relation

$$(Y, G, \chi) \geq_c (H, M, \psi)$$

means that $C_Y(G) \leq H$ and associated systems P, Q have the same factor set on H and the same scalar matrices on the whole group $C_Y(G)$. For $G \leq V \leq Y$, the same irreducible

inverse-factor-set representation E of $V/G \simeq (V \cap H)/M$ gives the ordinary correspondence

$$\mathrm{tr}(E \otimes P|_V) \longmapsto \mathrm{tr}(E \otimes Q|_{V\cap H}). \tag{2.1}$$

Clifford theory says that this exhausts all ordinary characters above χ and ψ. We write $\geq_b$ if, in addition, a defect group D of $\mathrm{bl}(\psi)$ satisfies $C_Y(D) \leq H$ and every correspondence (2.1) satisfies

$$\mathrm{bl}(\mathrm{tr}(E \otimes Q|_{V\cap H}))^V = \mathrm{bl}(\mathrm{tr}(E \otimes P|_V)). \tag{2.2}$$

These are the ordinary character-triple relations of [24, 35, 36].

Let S be nonabelian simple, let G be its universal covering group, and let B be a block with noncentral defect D. Set $\Gamma = \mathrm{Aut}(G)_{B,D}$. The inductive Alperin–McKay condition requires a Γ-stable subgroup $N_G(D) \leq M < G$, its corresponding block b with $b^G = B$, and a Γ-equivariant bijection $\Omega : \mathrm{Irr}_0(B) \to \mathrm{Irr}_0(b)$ preserving all characters of $Z(G)$. For every χ, with $K_\chi = \ker\chi \cap Z(G)$, it further requires

$$\big((G/K_\chi) \rtimes \Gamma_\chi, G/K_\chi, \bar\chi\big) \geq_b \big((M/K_\chi) \rtimes \Gamma_\chi, M/K_\chi, \overline{\Omega(\chi)}\big). \tag{2.3}$$

This is [36, Definition 4.12]. The relation in (2.3) concerns every intermediate group and every ordinary upper character. The linear correction below is extended on the effective outer group; the relation (2.3) is imposed on Γ_χ.

We use restriction, transitivity, the induction theorem for character triples, and the block Butterfly theorem in [24, Section 3], [36, Theorem 4.6] and [27, Theorem 1.10]. In the last theorem the conjugation images of the two ambient groups must coincide, and the new local group is the inverse image of the old local action image. We verify those requirements whenever the ambient group is changed.

2.2. **An ordinary correction theorem**

Theorem 2.1. *Let $G \lhd Y = GH$, $M = G \cap H$, and let χ, ψ be as above. Put $B = \mathrm{bl}(\chi)$, $b = \mathrm{bl}(\psi)$. Assume*

$$b^G = B, \qquad D \in \mathrm{Def}(b), \qquad C_Y(D) \leq H.$$

Suppose associated projective systems P, Q give $(Y, G, \chi) \geq_c (H, M, \psi)$, with the same finite factor set. Write $C = C_Y(G)$ and $F = Y/(GC)$. Assume that every F-invariant linear character of order prime to ℓ of every normal subgroup of F extends to a linear character of F of order prime to ℓ. Then there is $\delta \in \mathrm{Lin}_{\ell'}(F)$ such that P and δQ give

$$(Y, G, \chi) \geq_b (H, M, \psi).$$

If G is perfect, b, B have the same actual defect group D, and $N_Y(D) \leq H$, the same assertion holds on each quotient by a Y-normal subgroup $K \leq Z(G) \cap \ker\chi \cap \ker\psi$. The lower matched pair is unchanged.

Proof. We first treat genuine representations, with characters $\widehat\chi \in \mathrm{Irr}(Y)$ and $\psi_0 \in \mathrm{Irr}(H)$ extending χ, ψ. Let $T = Y[B]$ be Dade's ramification group. Its cyclic membership test gives

$$V[B] = V \cap Y[B] \quad (G \leq V \leq Y). \tag{2.4}$$

In particular $T[B] = T$ and $V[B] = V$ when $V \leq T$. Moreover $C \leq T$: for $c \in C$, the element c is central in $G\langle c\rangle$ and acts in the restriction of $\widehat\chi$ as a root of unity, whose reduction is nonzero. This is the ramification test of [19, Notation 3.1].

Apply [19, Theorem C(b)(2)] to $(T, G, T \cap H, M)$. The chosen extension of χ restricts to T; the chosen extension of the prescribed ψ restricts to the local preimage of every Sylow ℓ-subgroup of T/G; and $C_T(D) \leq T \cap H$. The theorem gives an extension $\eta \in \mathrm{Irr}(T \cap H)$ of this same ψ satisfying

$$\mathrm{bl}(\eta|_{V\cap H})^V = \mathrm{bl}(\widehat\chi|_V) \quad (G \leq V \leq T). \tag{2.5}$$

There is a unique line $\lambda \in \mathrm{Lin}(T/G)$ with $\eta = (\psi_0|_{T\cap H})\lambda$. This is a tensor identity: after identifying the two representations on M, their matrix quotient centralizes that irreducible representation and is scalar. Write $\lambda = \lambda_\ell \lambda_{\ell'}$.

If V/G is cyclic of order prime to ℓ, with $V \le T$, [19, Lemma 3.7] gives a unique block of $V \cap H$ over b inducing $\mathrm{bl}(\widehat{\chi}|_V)$, and restriction from its entire ordinary character set to $\mathrm{Irr}(b)$ is bijective. Thus it contains exactly one extension of ψ. Two extensions satisfying (2.5) consequently differ by a line of ℓ-power order. For $h \in H$, the character η^h also satisfies (2.5), since $T \lhd Y$ and $\widehat{\chi}, \psi_0$ are ambient characters. Hence

$$\lambda_{\ell'}^h = \lambda_{\ell'}. \tag{2.6}$$

As $Y = GH$, this is invariance under the whole ambient group.

We next prove that this line kills the entire centralizer. The restrictions of the two genuine representations to C have the same scalar character ω. For $c \in C$, put $V = G\langle c\rangle$ and $I = V \cap H = M\langle c\rangle$. For $g \in G$ and every integer j one has exact set identities

$$\mathrm{Cl}_V(gc^j) = \mathrm{Cl}_G(g)c^j, \qquad \mathrm{Cl}_V(gc^j) \cap I = (\mathrm{Cl}_G(g) \cap M)c^j.$$

They remain true when $\langle c\rangle \cap G$ is nontrivial. Their ordinary central-character values are the corresponding base values multiplied by the same unit $\omega(c^j)$. Since $b^G = B$, their reductions agree. Induction of the covering local block is defined: its defect contains a conjugate of D, and the chosen centralizer lies in I. Therefore

$$\mathrm{bl}(\psi_0|_I)^V = \mathrm{bl}(\widehat{\chi}|_V).$$

For c of ℓ'-order, uniqueness in [19, Lemma 3.7] gives $\lambda(c) = 1$. For arbitrary c, its commuting primary parts lie in C, and $\lambda_{\ell'}$ also kills the ℓ-part. Thus $\lambda_{\ell'}|_C = 1$.

The line now defines an F-invariant ℓ'-line on $T/(GC) \lhd F$. Extend it to $\delta \in \mathrm{Lin}_{\ell'}(F)$ and put $\widehat{\psi} = \psi_0\delta|_H$. On $T \cap H$ this is $\eta\lambda_\ell^{-1}$. Tensoring with an ℓ-power line preserves every ℓ-block. Consequently

$$\mathrm{bl}(\widehat{\psi}|_{V\cap H})^V = \mathrm{bl}(\widehat{\chi}|_V) \quad (G \le V \le T). \tag{2.7}$$

The correction is trivial on G and C, so neither the lower pair nor the full centralizer scalar changes.

Fix $G \le V \le Y$, $I = V \cap H$, $A = V/G$, and any $\xi \in \mathrm{Irr}(A)$. For $x \in V$ write $a = xG$ and choose $h \in I$ above a. Put

$$L = G\langle h\rangle = G\langle x\rangle, \qquad S = \mathrm{Cl}_V(x) \cap hG.$$

The set S is a union of L-classes. Set

$$u_x = \omega_{\widehat{\chi}|_L}(S^+), \qquad v_x = \omega_{\widehat{\psi}|_{L\cap H}}((S \cap I)^+).$$

These are algebraic integers. If $h \in T$, equation (2.7) gives $u_x^* = v_x^*$. If $h \notin T$, Dade vanishing in [35, Lemma 4.2(b)] gives zero for each of these reduced class sums; its local block induction is defined by the same defect-centralizer containment. Thus $u_x^* = v_x^*$ in both cases. The full ambient class splits into fibers over $\mathrm{Cl}_A(a)$; conjugation in V identifies the global fibers, and conjugation in I, which surjects onto A, identifies their local intersections. It follows, in characteristic zero, that

$$\begin{aligned} \omega_{(\widehat{\chi}|_V)\xi}(\mathrm{Cl}_V(x)^+) &= u_x\,\omega_\xi(\mathrm{Cl}_A(a)^+), \\ \omega_{(\widehat{\psi}|_I)\xi}((\mathrm{Cl}_V(x) \cap I)^+) &= v_x\,\omega_\xi(\mathrm{Cl}_A(a)^+). \end{aligned} \tag{2.8}$$

All factors are algebraic integers. Reducing these products proves the block identity for every ξ, including $\ell \mid \xi(1)$. Gallagher's theorem gives every ordinary upper character, proving the genuine case.

For projective systems use the convention $P(y)P(y') = \alpha(y,y')P(yy')$. Let Z_0 be the finite cyclic group generated by the values of α and form

$$\widehat{Y} = \{(y,z) : y \in Y, z \in Z_0\}, \qquad (y,z)(y',z') = (yy', zz'\alpha(y,y')).$$

Embed the base as $G_0 = \{(g,1) : g \in G\}$ and put $D_0 = \{(d,1) : d \in D\}$. The genuine systems are $\widehat{P}(y,z) = zP(y)$ and $\widehat{Q}(h,z) = zQ(h)$. Direct multiplication shows

$$C_{\widehat{Y}}(G_0) = \pi^{-1}(C), \quad C_{\widehat{Y}}(D_0) = \pi^{-1}(C_Y(D)), \quad \widehat{Y}/(G_0 C_{\widehat{Y}}(G_0)) \simeq F.$$

The genuine argument therefore supplies a correction on the same F, trivial on Z_0. If E has factor set α^{-1}, then

$$\widehat{E}((v,z)G_0) = z^{-1}E(vG)$$

is a genuine irreducible coefficient on $\pi^{-1}(V)/G_0$. Its tensor products with the two corrected genuine systems kill Z_0 and are exactly the inflations of (2.1). The block identity descends by [35, Theorem 4.1(i)], or by [24, Proposition 2.4(b),(c)], first through the central ℓ'-part and then the central ℓ-part. Definedness downstairs follows from the original defect-centralizer condition.

Finally suppose the additional hypotheses concerning G, K, D hold. If yK centralizes G/K, the map $g \mapsto g^{-1}g^y$ takes values in $K \le Z(G)$ and is a homomorphism. Perfectness makes it trivial. Thus the full quotient centralizer has inverse image C, and the two systems descend with their common factor set and their entire scalar condition. Every central ℓ-subgroup is contained in D, so $DK = DK_{\ell'}$ has unique Sylow ℓ-subgroup D. Hence

$$\pi^{-1}N_{Y/K}(DK/K) = N_Y(D), \qquad C_{Y/K}(DK/K) \le H/K. \tag{2.9}$$

The quotient blocks have actual common defect DK/K, and their base block induction follows from [24, Proposition 2.4]. The effective quotient remains F. Apply the theorem just proved on Y/K itself. This constructs the block relation on the exact central quotient, completing the proof. □

Corollary 2.2 (Correction preserving a prescribed normal layer)**.** *Keep the genuine-representation hypotheses in the proof of Theorem 2.1, but do not assume its extension condition on the effective outer group. Suppose that $GC_Y(G) \le J \lhd Y$, and that every Y/J-invariant ℓ'-line of every normal subgroup of Y/J extends to an ℓ'-line of Y/J. In particular this holds if Y/J is abelian. Suppose the prescribed extensions $\widehat{\chi}, \psi_0$ satisfy*

$$\mathrm{bl}(\psi_0|_{V\cap H})^V = \mathrm{bl}(\widehat{\chi}|_V) \qquad (G \le V \le J).$$

Then a character $\delta \in \mathrm{Lin}_{\ell'}(Y/J)$ makes $\widehat{\chi}, \psi_0\delta$ give the full relation $(Y,G,\chi) \ge_b (H,M,\psi)$. In particular the correction fixes the specified extension on $J \cap H$, not only its block.

Proof. Set $T = Y[B]$ and choose the extension η of the prescribed ψ by [19, Theorem C(b)(2)], exactly as in (2.5). Write $\eta = (\psi_0|_{T\cap H})\lambda$. If x is an ℓ'-element of $T \cap J$, the two prescribed block identities on $G\langle x\rangle$ and [19, Lemma 3.7] give $\lambda(x) = 1$. Taking the commuting primary parts of an arbitrary element of $T \cap J$ therefore shows $\lambda_{\ell'}|_{T\cap J} = 1$. The resulting ℓ'-line of $T/(T \cap J) \simeq TJ/J$ is invariant by (2.6). Since TJ/J is normal in Y/J, the assumed extension property gives $\delta \in \mathrm{Lin}_{\ell'}(Y/J)$ with this restriction.

On $T \cap H$, the corrected extension is $\eta\lambda_\ell^{-1}$, so (2.7) holds. Since $C_Y(G) \le J$, its complete centralizer scalar is unchanged. The Dade vanishing argument and the integral products (2.8) now prove the block identity for every ordinary upper coefficient. No division by its degree is used. Finally $\delta|_J = 1$, which gives the asserted preservation of the specified normal layer. □

2.3. **The linear extension hypothesis**

Lemma 2.3. *Let $R \lhd F$ and let $\tau \in \mathrm{Lin}(R)$ be F-invariant. It extends linearly to F if and only if $\tau|_{R\cap F'} = 1$. In particular the extension hypothesis of Theorem 2.1 holds when F is abelian, and when F is a subgroup of $C_r \rtimes A$, where r is prime and A is abelian. If F' is an ℓ-group, every ℓ'-line of any subgroup extends to an ℓ'-line of F.*

Proof. Necessity is immediate. For sufficiency extend the induced line from RF'/F' to the finite abelian group F/F'. Such an extension is obtained by adjoining a generator at a time and choosing a root of its prescribed power value. Taking the ℓ'-part preserves a prescribed

ℓ' restriction. For a nonabelian subgroup $F \leq C_r \rtimes A$ one has $F' = C_r$, and some element acts nontrivially on C_r. If $R \cap C_r = 1$, then $[R, F] = 1$; if $C_r \leq R$, then $[R, F] = C_r$. Thus $R \cap F' = [R, F]$, which every invariant line kills. The last assertion follows because an ℓ'-line kills any ℓ-subgroup. □

For a perfect central cover of a simple group S, $\mathrm{Out}(G) \hookrightarrow \mathrm{Out}(S)$. Indeed an automorphism trivial on S differs from the identity by a homomorphism $G \to Z(G)$, which is trivial. Removing an inner lift proves the assertion for outer classes. Consequently the effective quotients in our applications are actual subgroups of $\mathrm{Out}(S)$. The same statement holds after the central quotients in Theorem 2.1.

2.4. **A centerless cyclic-outer endpoint**

Proposition 2.4. *Let G be centerless and perfect with cyclic $\mathrm{Out}(G)$. Let B have actual defect D, let $N_G(D) \leq M < G$ be $\mathrm{Aut}(G)_{B,D}$-stable, and let $b \in \mathrm{Bl}(M)$ have the same actual defect D with $b^G = B$. Every $\mathrm{Aut}(G)_{B,D}$-equivariant bijection $\Omega : \mathrm{Irr}_0(B) \to \mathrm{Irr}_0(b)$ satisfies the full inductive block-triple condition.*

Proof. For a matched pair χ, ψ put $Y = \mathrm{Aut}(G)_\chi$, $H = MN_Y(D)$. Identifying G with $\mathrm{Inn}(G)$, defect conjugacy gives $Y = GH$, $G \cap H = M$. The stability of M makes H a group. Equivariance and inner invariance give $H_\chi = H_\psi = H$. Both quotients by their bases are cyclic, so the two prescribed ordinary characters extend genuinely. To construct an extension, for a lift x of a cyclic generator rescale an intertwiner T so that $T^r = \rho(x^r)$; this keeps the actual return element.

Let $\Gamma = \mathrm{Aut}(G)_{B,D}$, $X = G \rtimes \Gamma_\chi$ and $X_M = M \rtimes \Gamma_\chi$. The map

$$\varepsilon(g, a) = \mathrm{inn}_g\, a$$

surjects onto Y, is injective on the base, and satisfies

$$\varepsilon^{-1}(H) = X_M \supseteq N_X(D), \qquad C_X(G) = \ker \varepsilon = \{(m, \mathrm{inn}_{m^{-1}}) : m \in N_G(D),\ \mathrm{inn}_{m^{-1}} \in \Gamma_\chi\}.$$

Pulling back the genuine extensions gives common factor set one and the identity scalar on this whole centralizer, which need not be central in X. Moreover $X/(GC_X(G)) \simeq Y/G$ is cyclic. Apply Theorem 2.1 directly in X. Its conclusion includes all coefficients in $\mathrm{Irr}(K)$ for every $K \leq \Gamma_\chi$; these groups can be noncyclic because they retain inner automorphisms. This is exactly (2.3). □

3. Normal defect-zero sources

This section gives the local passage from a source normalizer to the actual defect normalizer. It also fixes the ordinary upper characters and the central quotients needed when this passage is composed with a global correspondence.

Theorem 3.1. *Let $N \leq V$ be normal subgroups of a finite group A. Let $\kappa \in \mathrm{dz}(N)$, put $I = V_\kappa$, and suppose $Q = I/N$ has a normal ℓ-subgroup U with $C_Q(U) \leq U$. Then I has a unique block c_I over $\mathrm{bl}(\kappa)$. Let c_V be its Fong–Reynolds correspondent, assume that A stabilizes c_V, and fix an actual defect group $D \leq I$ of these blocks. There is a single $N_A(D)$-equivariant bijection*

$$\Omega_V : \mathrm{Irr}_0(c_V) \longrightarrow \mathrm{Irr}_0(b_V),$$

where b_V is the Brauer correspondent in $N_V(D)$, preserving the characters of $Z(V)$ and satisfying

$$(A_\chi, V, \chi) \geq_b (N_A(D)_{\Omega_V(\chi)}, N_V(D), \Omega_V(\chi)). \tag{3.1}$$

For every fixed A-stable subgroup $C \leq Z(V)$, the map can be chosen so that the corresponding relation also holds on the quotient by $K_\chi = \ker \chi \cap C$, with local group $N_V(D)/K_\chi = N_{V/K_\chi}(DK_\chi/K_\chi)$. All these relations include every ordinary upper coefficient.

3.1. **Invariant sources and their faithful scalar covers**

First suppose $V = I$ and κ is A-invariant. On the preimage W of a Sylow ℓ-subgroup of A/N, the unique block over κ has a defect group E with $W = NE$ and $N \cap E = 1$. This follows from covering Brauer pairs for a defect-zero block and an ℓ-group quotient. Hence $N_W(E) = C_N(E) \times E$. The Dade–Glauberman–Nagao correspondent $\kappa_E \in \mathrm{dz}(C_N(E))$ extends there as $\kappa_E \otimes 1_E$. The extension implication in [24, Theorem 5.5], with lower section $N \lhd NE$, therefore shows that κ extends to W.

The Clifford obstruction in $H^2(A/N, \mathbb{C}^\times)$ consequently has ℓ'-order, by restriction and corestriction. Represent it by a normalized finite cocycle α with values in a cyclic group Z of ℓ'-roots of unity, and form

$$\widehat{A} = A \times_\alpha Z, \qquad (a,z)(a',z') = (aa', zz'\alpha(aN, a'N)).$$

The embedded $N = \{(n,1) : n \in N\}$ is normal and disjoint from Z. The associated projective source becomes a genuine representation $\widehat{S}(a,z) = zS(a)$ of $\widehat{A}$. Its character $\widehat{\kappa}$ extends κ and has faithful scalar character μ on Z. Put

$$\widehat{I} = \pi^{-1}(I), \quad \widehat{Q} = \widehat{I}/N, \quad \overline{A} = \widehat{A}/N.$$

The preimage of U in $\widehat{Q}$ has the form $Z \times \widehat{U}$ with unique Sylow ℓ-subgroup $\widehat{U} \lhd \widehat{Q}$. Since $C_Q(U) = Z(U)$,

$$C_{\widehat{Q}}(\widehat{U}) = Z \times Z(\widehat{U}). \tag{3.2}$$

For each $\lambda \in \mathrm{Irr}(Z)$ its central idempotent e_λ is a block of the right side. It is also a block of $\widehat{Q}$: every block of $\widehat{Q}$ has a defect group containing its normal ℓ-subgroup $\widehat{U}$, and hence has nonzero Brauer image at $\widehat{U}$. If e_λ split into two or more such block idempotents, their nonzero orthogonal Brauer images would split the primitive idempotent e_λ of $kC_{\widehat{Q}}(\widehat{U})$, a contradiction. Its Brauer image at any $P \in \mathrm{Syl}_\ell(\widehat{Q})$ is $e_\lambda \neq 0$, so it has full defect. The same argument applies to $X = N_{\widehat{Q}}(P)$, using (3.2).

Take a $\widehat{A}$-stable lattice L_κ. Since κ has defect zero, kNe_κ is the full matrix algebra on the simple reduction $k \otimes_{\mathcal{O}} L_\kappa$, whose dimension is $\kappa(1)$. Thus the action map $\mathcal{O}Ne_\kappa \to \mathrm{End}_{\mathcal{O}}(L_\kappa)$ is surjective modulo the maximal ideal of $\mathcal{O}$, and Nakayama's lemma makes it surjective over $\mathcal{O}$. Both sides are free of rank $\kappa(1)^2$, so it is an isomorphism. The map

$$ge_\kappa \longmapsto \widehat{S}(\hat{g}) \otimes (\hat{g}N)e_{\mu^{-1}}$$

is independent of the lift $\hat{g}$, is multiplicative, and is an isomorphism on every N-coset. It gives the integral algebra isomorphism

$$\mathcal{O}Ie_\kappa \simeq M_{\kappa(1)}(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}\widehat{Q}e_{\mu^{-1}}. \tag{3.3}$$

Thus e_κ is the unique block c_I over κ, and its ordinary characters are the descents of

$$\widehat{\kappa}|_{\widehat{I}}\mathrm{Inf}(\gamma), \qquad \gamma \in \mathrm{Irr}(\widehat{Q} \mid \mu^{-1}). \tag{3.4}$$

An actual defect D of c_I has $D \cap N = 1$. Extend μ^{-1} trivially across a Sylow ℓ-subgroup of $\widehat{Q}$. Its induction to $\widehat{Q}$ has degree prime to ℓ, and all its irreducible constituents lie in the μ^{-1}-slice. At least one constituent has degree prime to ℓ. Thus the minimum ℓ-part of a degree in this full-defect scalar block is 1. The defect/degree formula and (3.4) therefore give

$$|D| = |I/N|_\ell, \qquad \chi_\gamma \in \mathrm{Irr}_0(c_I) \Longleftrightarrow \ell \nmid \gamma(1). \tag{3.5}$$

The preimage of D is $Z \times \widehat{D}$, with unique Sylow subgroup $\widehat{D}$, and $P = \widehat{D}N/N \in \mathrm{Syl}_\ell(\widehat{Q})$.

3.2. **Actual normalizers and the prescribed DGN extension**

Set

$$\widehat{J} = N_{\widehat{I}}(\widehat{D}), \quad \widehat{Y} = N_{\widehat{A}}(\widehat{D}), \quad N_0 = C_N(\widehat{D}), \quad M = N\widehat{D}.$$

Uniqueness of $\widehat{D}$ in its preimage gives $\widehat{J} = \pi^{-1}(N_I(D))$ and $\widehat{Y} = \pi^{-1}(N_A(D))$. Also $N_N(\widehat{D}) = N_0$. The unique block of M over κ has actual defect $\widehat{D}$: its Brauer image is nonzero there, while every defect intersects N trivially and has order at most $|M/N|$.

If xN normalizes P in $\overline{A}$, then x normalizes M and its unique block. Its two actual defects $\widehat{D}^x, \widehat{D}$ are M-conjugate. Correcting x by an element of N makes it normalize $\widehat{D}$ without changing its quotient image. Consequently the actual quotient maps give

$$\widehat{J}/N_0 = N_{\widehat{Q}}(P) = X, \qquad \widehat{Y}/N_0 = N_{\overline{A}}(P). \tag{3.6}$$

Put $A_0 = N\widehat{Y}$. The DGN setting of [24, Hypothesis 5.1] holds with $K = N$, $M = N\widehat{D}$, $G = A_0$, $D_0 = 1$, $D = \widehat{D}$, and $L = N_0$. Let $\varphi \in \mathrm{dz}(N_0)$ be the DGN correspondent. Apply [24, Proposition 5.12] to the already prescribed genuine extension $\widehat{\kappa}|_{A_0}$. It provides a genuine extension $\widehat{\varphi} \in \mathrm{Irr}(\widehat{Y})$ with the same scalars on $C_{A_0}(M)$ and, for every ℓ'-element h with $\widehat{D} \in \mathrm{Syl}_\ell(C_M(h))$,

$$\widehat{\kappa}(h)^* = e\widehat{\varphi}(h)^*, \qquad e = [\kappa|_{N_0}, \varphi].$$

The degree congruence in [24, Theorem 5.2(b)] gives $\ell \nmid e$ and $\kappa(1)^*_{\ell'} = e^*[N : N_0]^*_{\ell'}\varphi(1)^*_{\ell'}$. Dividing only by these units yields

$$\left(\frac{|N|_{\ell'}\widehat{\kappa}(h)}{\kappa(1)_{\ell'}}\right)^* = \left(\frac{|N_0|_{\ell'}\widehat{\varphi}(h)}{\varphi(1)_{\ell'}}\right)^*. \tag{3.7}$$

In particular $\widehat{\varphi}$ has the same faithful scalar μ on Z.

Using (3.6) and $\widehat{\varphi}$, the same integral untwisting gives

$$\mathcal{O}N_I(D)e_\varphi \simeq M_{\varphi(1)}(\mathcal{O}) \otimes \mathcal{O}Xe_{\mu^{-1}}.$$

There is therefore one local block b_0 over φ. Its defect order is $|X|_\ell = |D|$. Since $D \lhd N_I(D)$, the group D is an actual defect of b_0. Its height-zero characters are exactly the descents of $\widehat{\varphi}\,\mathrm{Inf}(\delta)$ for $\delta \in \mathrm{Irr}_{\ell'}(X \mid \mu^{-1})$. The next argument identifies b_0 as the Brauer correspondent.

3.3. **Block compatibility**

Theorem B of [13], applied to the finite group $\widehat{Q}$ and $P \in \mathrm{Syl}_\ell(\widehat{Q})$, gives one $\mathrm{Aut}(\widehat{Q})_P$-equivariant bijection

$$f : \mathrm{Irr}_{\ell'}(\widehat{Q}) \longrightarrow \mathrm{Irr}_{\ell'}(X)$$

with central character-triple systems. For $z \in Z$, the two scalar matrices in those systems have scalars $\gamma(z)/\gamma(1)$ and $f(\gamma)(z)/f(\gamma)(1)$. Their equality shows that f preserves the μ^{-1}-slice. For a fixed γ put

$$Y_\gamma = N_{\overline{A}}(P)_\gamma, \qquad \Gamma_0 = \varepsilon(Y_\gamma), \quad \varepsilon : \overline{A}_\gamma \longrightarrow \mathrm{Aut}(\widehat{Q}).$$

Sylow conjugacy gives $\overline{A}_\gamma = \widehat{Q}Y_\gamma$. Restrict the systems to $\widehat{Q} \rtimes \Gamma_0$ and $X \rtimes \Gamma_0$. Their global action image is $\mathrm{Inn}(\widehat{Q})\Gamma_0 = \varepsilon(\overline{A}_\gamma)$, and their local action image is $\mathrm{Inn}(X)\Gamma_0$. Moreover,

$$\varepsilon^{-1}(\mathrm{Inn}(X)\Gamma_0) = Y_\gamma.$$

Indeed, an element in the left side has the same action as some xy with $x \in X$ and $y \in Y_\gamma$; their quotient belongs to $C_{\overline{A}}(\widehat{Q})$, which normalizes P. The reverse inclusion is immediate. Theorem 2.8 of [13] therefore gives common-factor associated representations $R_\gamma, R_{f(\gamma)}$ on these actual quotient inertias. Let S_φ afford the genuine character $\widehat{\varphi}$ on $\widehat{Y}$, and put $\widehat{\chi} = \widehat{\kappa}|_{\widehat{I}}\mathrm{Inf}(\gamma)$. Tensor the associated representations with these genuine source systems:

$$T(\hat{a}) = \widehat{S}(\hat{a}) \otimes R_\gamma(\hat{a}N), \qquad T'(\hat{y}) = S_\varphi(\hat{y}) \otimes R_{f(\gamma)}(\hat{y}N_0). \tag{3.8}$$

They have the same factor set and full centralizer scalars, because $C_{\widehat{A}}(\widehat{I}) \le C_{A_0}(M)$.

The remaining block condition is the normalized trace test of [24, Theorem 4.4], for every ℓ'-element $h \in \widehat{Y}$ in the matched inertia such that

$$\widehat{D} \in \mathrm{Syl}_\ell(C_{\widehat{I}}(h)). \tag{3.9}$$

We must compare $\left(|\widehat{I}|_{\ell'} \operatorname{tr} T(h)/\widehat{\chi}(1)_{\ell'}\right)^*$ and its local counterpart.

An automorphism α of Q fixing U pointwise has $g^{-1}\alpha(g) \in C_Q(U) = Z(U)$. Put $f_\alpha(g) = g^{-1}\alpha(g)$. Since α fixes $Z(U)$ pointwise, $f_{\alpha\beta}(g) = f_\alpha(g)f_\beta(g)$. Thus $\alpha \mapsto f_\alpha$ embeds this automorphism group into the finite abelian ℓ-group $\prod_{g\in Q} Z(U)$, with componentwise multiplication. The ℓ'-automorphism induced by h fixes the Sylow image of $\widehat{D}$, hence U, and is therefore trivial. Its action on $\widehat{Q}$ is of the form

$$\tau_h(g) = g^{-1}g^{hN} \in Z,$$

a homomorphism trivial on ZP.

If $\tau_h \neq 1$, faithfulness of μ and the intertwining identity imply $\operatorname{tr} R_\gamma(hN) = 0$: conjugating this matrix by a suitable lower matrix multiplies it by a nontrivial μ^{-1}-value. The same is true locally. Indeed $K_h = \ker \tau_h \lhd \widehat{Q}$ contains P, so the Frattini argument gives $\widehat{Q} = K_h X$ and $\tau_h(X) = \tau_h(\widehat{Q})$. Thus both tensor traces vanish in characteristic zero.

If $\tau_h = 1$, the two central-triple quotient matrices are the same scalar s_h. Their degrees cancel in the normalized traces because the lower coefficients have ℓ'-degree. The remaining expressions are

$$|\widehat{Q}|_{\ell'} s_h \frac{|N|_{\ell'}\widehat{\kappa}(h)}{\kappa(1)_{\ell'}}, \qquad |X|_{\ell'} s_h \frac{|N_0|_{\ell'}\widehat{\varphi}(h)}{\varphi(1)_{\ell'}}.$$

Equation (3.9) permits the use of (3.7), and $[\widehat{Q} : X] \equiv 1 \pmod{\ell}$ is the number of Sylow ℓ-subgroups. This proves the trace test. The conclusion of [24, Theorem 4.4] is the full block relation, with every ordinary upper inverse-factor coefficient, without a restriction on its degree.

We descend through the faithful source cover directly. If $a \in C_A(I)$, a lift $\hat{a}$ centralizes the embedded N. Its source matrix is scalar. For $g \in \widehat{I}$ the commutator $[g, \hat{a}]$ lies in Z and has trivial source matrix, so faithfulness gives $[g, \hat{a}] = 1$. Hence

$$\pi^{-1}(C_A(I)) = C_{\widehat{A}}(\widehat{I}).$$

The tensor systems kill Z and descend with all centralizer scalars. A downstairs ℓ'-test element centralizing D lifts to an ℓ'-element centralizing the unique $\widehat{D}$. No larger ℓ-subgroup can centralize the lift, since it would inject into the downstairs centralizer. The two normalized order factors upstairs are both multiplied by the unit $|Z|$. Thus the same trace test holds downstairs. In particular $b_0^I = c_I$, and the actual common defect D identifies b_0 as the Brauer correspondent. This proves the invariant-source case, with a single equivariant map since the source extensions and f were fixed on the whole ambient.

3.4. **Moving sources and all upper inductions**

Return to general V. Apply the invariant-source construction once in A_κ, where N, I are normal. Put $Y = N_A(D)$, $M_V = N_V(D)$, $J_I = N_I(D)$, $Y_\kappa = N_{A_\kappa}(D)$ and $N_0 = C_N(D)$. If φ is the D-DGN correspondent of κ, then

$$Y_\varphi = Y_\kappa, \qquad (M_V)_\varphi = J_I. \tag{3.10}$$

For if $\varphi^y = \varphi$, the unique blocks of ND over κ^y and κ have the same Brauer correspondent in $N_0 \times D$. They are equal by Brauer's first main theorem. Their covered D-invariant defect-zero blocks of N are then equal, and their unique ordinary characters give $\kappa^y = \kappa$. The converse is naturality of DGN.

Also $Y = Y_\kappa M_V$. To see this, for $y \in Y$ first choose $v \in V$ with $\kappa^y = \kappa^v$. The source blocks have actual defects D and D^v. Correct v inside its source inertia so that $vi \in N_V(D)$ while $\kappa^{vi} = \kappa^y$. This proves the product assertion with an actual defect correction.

Let Ω_I be the invariant-source map and define

$$\Omega_V(\mathrm{Ind}_I^V \tau) = \mathrm{Ind}_{J_I}^{M_V} \Omega_I(\tau). \tag{3.11}$$

Both inductions are Fong–Reynolds bijections, by (3.10), and preserve the actual defect D. Their local block induces c_V by transitivity, hence is the required b_V. Equivariance under Y_κ and inner invariance under M_V give equivariance under Y.

Fix $\chi = \mathrm{Ind}_I^V \tau$ and put $G_1 = (A_\chi)_\kappa$, $H = N_{A_\chi}(D)$, $H_1 = H \cap G_1$. Then $A_\chi = VG_1 = VH$ and $H = M_V H_1$. For every $V \le L \le A_\chi$, ordinary Clifford theory gives the two full-domain bijections

$$\mathrm{Irr}(L \cap G_1 \mid \tau) \xrightarrow{\mathrm{Ind}} \mathrm{Irr}(L \mid \chi), \qquad \mathrm{Irr}(L \cap H_1 \mid \Omega_I(\tau)) \xrightarrow{\mathrm{Ind}} \mathrm{Irr}(L \cap H \mid \Omega_V(\chi)).$$

All lower heights are preserved; $N_V(D) = M_V$ is the actual local group. These are exactly the hypotheses of [24, Theorem 3.14], which proves (3.1) for every ordinary upper character. Its central part preserves $Z(V)$.

3.5. **Constructing the strong map on the quotient first**

We record the quotient argument, since a central kernel need not lie in the source group N. A group Q with $C_Q(O_\ell(Q)) \le O_\ell(Q)$ retains this property after quotienting by $W \le Z(Q)$. Indeed $Z(Q)$ is an ℓ-group. With $R = O_\ell(Q)$, the preimage L of $C_{Q/W}(R/W)$ acts trivially on R/W and fixes W. Its action embeds, modulo $C_Q(R)$, into $\mathrm{Hom}(R/W, W)$ by $rW \mapsto r^{-1}r^x$. Thus L is a normal ℓ-group and lies in R.

Now let $C \le Z(V)$ be A-stable and $K = \ker\chi \cap C$. The image of C in $Q = I/N$ is central and hence an ℓ-group, so $C_{\ell'} \le N$. The defect-zero block of N implies $O_\ell(N) = 1$. Consequently

$$K \cap N = K_{\ell'}, \qquad K_\ell \le O_\ell(V) \le D, \qquad K \cap N \le \ker\kappa.$$

Thus κ descends to a defect-zero character of $\bar{N} = NK/K$, its inertia is I/K, and its quotient is $Q/(NK/N)$, to which the preceding paragraph applies. The actual quotient block has defect $\bar{D} = DK/K$ and the heights are unchanged. Since $DK = D \times K_{\ell'}$, one has, for every relevant ambient B stabilizing K,

$$N_{B/K}(DK/K) = N_B(D)/K. \tag{3.12}$$

The quotient of the local block is the Brauer correspondent: its defect is $\bar{D}$ and its block induction follows from [24, Propositions 2.4–2.5] and (3.12).

For one representative of every $N_A(D)$-orbit of possible K, run the already proved theorem in A_K/K. Restrict its map to characters faithful on C/K, and inflate. These are exactly the characters whose central kernel on C is K. Transport by $N_A(D)$. Equivariance under the stabilizer makes this independent of the transporter, and the distinct kernels give disjoint sets. The resulting single map has the required quotient relation for each matched pair. Inflation by [24, Lemma 3.12] gives the upstairs relation as well. This proves Theorem 3.1.

3.6. **Composition with a global correspondence**

Apply Theorem 3.1 once in $V \rtimes \Gamma$, where $\Gamma \le \mathrm{Aut}(V)$ preserves the normal source group, the block and D. In toral applications this preservation follows from the intrinsic description of the torus as the centralizer of a characteristic subgroup of D. A witness ambient B with the same actual action is treated by the block Butterfly theorem: if $\varepsilon : B \to \mathrm{Aut}(V)$ is its action map, defect conjugacy gives

$$\varepsilon(B) = \mathrm{Inn}(V)\varepsilon(N_B(D)), \qquad \varepsilon^{-1}(\varepsilon(N_B(D))) = N_B(D).$$

The second equality holds because the indicated automorphisms preserve the actual subgroup D.

Suppose a global construction reaches a subgroup V containing $N_G(D)$ and gives a block relation with ambient $H = VN_A(D)$. The theorem then reaches $N_V(D) = N_G(D)$ in that same ambient. Transitivity composes the two block relations. For a strong global relation, perform this composition on its exact central quotient first, using (3.12), and inflate only afterwards.

4. Ordinary Jordan parameters and actual relative groups

Let $\mathbf{G}$ be connected reductive, let F be a Frobenius map defining an $\mathbb{F}_q$-structure, and write $G = \mathbf{G}^F$. Here q is a power of the defining prime p, and the block prime is $\ell \neq p$. A star denotes a fixed dual root datum, including its Frobenius. For a semisimple element $s \in \mathbf{G}^{*F^*}$, $\mathcal{E}(G, s)$ denotes the rational Lusztig series and

$$\mathcal{E}_\ell(G, s) = \bigcup_{t \in C_{\mathbf{G}^*}(s)_\ell^{F^*}} \mathcal{E}(G, st) \qquad (s \text{ of } \ell'\text{-order}).$$

The union is over rational conjugacy classes of ℓ-elements. An e-split Levi is the centralizer of an F-stable Φ_e-torus; in the applications e is the order of q modulo the odd prime ℓ. All dualities, rational conjugations and Jordan maps below are fixed together.

4.1. The ordinary inverse required in a block construction

A block construction in a semisimple s-series uses the following three different statements. A block theorem identifies the block attached to a marked e-cuspidal pair. An ordinary Jordan theorem then identifies its intersection with each $\mathcal{E}(G, st)$. Finally unipotent e-Harish–Chandra theory labels that intersection by ordinary characters of the marked relative group.

We use the individual-block intersection assertion and its inverse in [15, Proposition 2.3.6], with its stated hypotheses on the prime and the actual isogeny type. For the exceptional bad-prime series, the source arrows are the individual-block arrows in [16, 17, 18]. The exceptional cases in which two source blocks share a union of parameters are separated in the family proofs below. For unipotent ordinary e-series we use [25, Theorem 3.4]. Its statement applies to connected reductive groups in nondefining characteristic, including the exceptional types.

More explicitly, an inverse starts with an arbitrary $\chi \in \mathrm{Irr}_0(B)$, determines its rational semisimple parameter st, recovers its marked unipotent source $(\mathbf{M}^*, \kappa)$ in $C_{\mathbf{G}^*}(s)$, and then recovers an ordinary coefficient η in the actual t-stabilizer of the marked relative group. The height formula must be applied at this stage. In particular, an upper character in a regular embedding may have positive height while its natural-group constituents have height zero. Each family proof records these cases. The map to a local group is defined only after this inverse, its block membership, and its equivariance have been established.

4.2. Relative normalizers and Jordan parameters

Lemma 4.1. *Let $\mathbf{X}$ have connected centre, let $\mathbf{L}$ be an F-stable e-split Levi, and fix dual data. Let $X = \mathbf{X}^F$ and $L = \mathbf{L}^F$. Let $s \in \mathbf{L}^{*F^*}$ be semisimple, take $\lambda \in \mathcal{E}(L, s)$, and put*

$$\mathbf{M}^* = C_{\mathbf{L}^*}(s), \qquad M^* = (\mathbf{M}^*)^{F^*}, \qquad \kappa = J_{L,s}(\lambda).$$

Use the Jordan maps characterized in [32]. Suppose

$$\mathbf{L}^* = C_{\mathbf{X}^*}(Z^\circ(\mathbf{M}^*)_{\Phi_e}). \tag{4.1}$$

Then, with a fixed opposite convention if necessary, there is an isomorphism of the actual finite groups

$$\frac{N_X(\mathbf{L}, \lambda)}{L} \simeq \frac{N_{C_{\mathbf{X}^*}(s)^{F^*}}(\mathbf{M}^*, \kappa)}{M^*}. \tag{4.2}$$

It commutes with every pair of dual field or diagram isotypies preserving the data. If s has ℓ'-order and $t \in Z(\mathbf{L}^*)^{F^*}$ has ℓ-power order, the ordinary inertia of $\lambda\theta_t$ corresponds under this map to the t-stabilizer on the right, where θ_t is the ordinary line dual to t.

Proof. The connected-centre hypothesis passes to $\mathbf{L}$. To see this, let X be the character lattice of a maximal torus and choose simple roots for which $\mathbf{L}$ is standard. Connectedness of $Z(\mathbf{X})$ says that the torsion subgroup of $\mathsf{X}/\mathbb{Z}\Phi(\mathbf{X})$ has p-power order. The Levi simple roots form a subset of the ambient simple roots, so $\mathbb{Z}\Phi(\mathbf{X})/\mathbb{Z}\Phi(\mathbf{L})$ is free abelian. In the exact sequence

$$0 \longrightarrow \mathbb{Z}\Phi(\mathbf{X})/\mathbb{Z}\Phi(\mathbf{L}) \longrightarrow \mathsf{X}/\mathbb{Z}\Phi(\mathbf{L}) \longrightarrow \mathsf{X}/\mathbb{Z}\Phi(\mathbf{X}) \longrightarrow 0,$$

the torsion subgroup of the middle term therefore injects into the torsion subgroup of the last term. It too has p-power order, proving that $Z(\mathbf{L})$ is connected. For each dual group the simply connected covering kernel of the derived group consequently has p-power order. It has trivial geometric points, so the covering is bijective on geometric points. The connectedness theorem for semisimple centralizers in simply connected derived groups therefore gives connected semisimple centralizers in the dual groups used here.

Relative algebraic normalizers on the two dual root data are identified by their Weyl actions. This identification commutes with Frobenius and the specified isotypies. Connected Lang on $\mathbf{L}$ and $\mathbf{L}^*$ identifies their fixed relative quotients with the quotients of their actual finite normalizers. Thus an actual coset nL has a rational dual coset n^*L^*.

If n fixes λ, disjointness of rational Lusztig series provides $h \in L^*$ such that $c = n^*h$ centralizes s. The isotypy equivariance of the fixed Jordan map, [32, Theorem 6.2], says exactly that c fixes the mark κ. It normalizes $C_{\mathbf{L}^*}(s) = \mathbf{M}^*$. The ambiguity in h is an element of M^*, so cM^* is well defined. Composing two rational normalizers composes their dual cosets and correcting conjugations; the remaining correction lies in M^*. This proves that the map is a homomorphism. If $c \in M^*$, then $n^* \in L^*$ and $n \in L$, proving injectivity.

Conversely an element c in the right numerator preserves $Z^\circ(\mathbf{M}^*)_{\Phi_e}$, hence normalizes $\mathbf{L}^*$ by (4.1). Take its rational dual coset nL. The Jordan images of λ^n and λ agree by the same isotypy theorem. Injectivity of $J_{L,s}$ gives $\lambda^n = \lambda$. This proves surjectivity. Transporting the correcting h proves equivariance, since its ambiguity remains exactly M^*.

The central-twist property of this Jordan map gives

$$C_{\mathbf{L}^*}(st) = \mathbf{M}^*, \qquad J_{L,st}(\lambda\theta_t) = \kappa.$$

The coprime orders of s, t give $C_{\mathbf{X}^*}(st) = C_{\mathbf{X}^*}(s) \cap C_{\mathbf{X}^*}(t)$. Repeating the preceding construction for st identifies its marked relative group with

$$\frac{N_{C_{\mathbf{X}^*}(s)^{F^*} \cap C_{\mathbf{X}^*}(t)^{F^*}}(\mathbf{M}^*, \kappa)}{M^*},$$

which is precisely the t-stabilizer in (4.2). This works for every ℓ-power order of t. □

The lemma also verifies that $\mathbf{M}^*$ is e-split in $C_{\mathbf{X}^*}(st)$: centralizing its source e-torus in $\mathbf{X}^*$ gives $\mathbf{L}^*$, and intersecting with $C(st)$ gives $\mathbf{M}^*$. Thus [25, Theorem 3.4] labels the complete unipotent ordinary series by $\mathrm{Irr}(W_{s,t})$, with $W_{s,t}$ the actual group in the proof. A source extension on its full semilinear stabilizer makes the coefficient actions equivariant. The family proofs supply this extension and the block membership. The argument applies for every $q \geq 2$.

4.3. **Three ordinary extension constructions**

We record the elementary operator constructions repeatedly used below. Let ρ be an irreducible representation of a normal subgroup N and suppose I/N is cyclic, generated by aN of order m. If $a^m = h \in N$, an intertwiner A for a has $A^m = c\rho(h)$ for a scalar c. Rescaling A by an mth root of c^{-1} gives an extension to the actual group I. This retains the return h.

If equal marked factors are permuted, use the permutation of tensor factors. For associated systems with factor sets α_i, the resulting factor set is

$$\alpha((x_i)\sigma, (y_i)\tau) = \prod_i \alpha_i(x_i, y_{\sigma^{-1}(i)}). \tag{4.3}$$

Matching local systems give exactly the same factor. A central product is formed only on the matching scalar slice of its actual kernel. Both operations preserve all ordinary inverse-factor coefficients.

Finally suppose a canonical virtual representation is a signed sum of induced modules whose defining ordinary linear characters are fixed by a pinned automorphism group. Those induced modules have actual semidirect-product actions. For an invariant constituent of virtual multiplicity one, let α be its Clifford obstruction and let the multiplicity spaces have dimensions m_i and

signs ϵ_i. Each multiplicity space is projective with factor α^{-1}. Taking determinants gives a coboundary for α^{m_i}, and $\sum_i \epsilon_i m_i = 1$ gives a coboundary for α. Thus the constituent extends. When a central lift occurs in a return relation, retain its scalar in every module before taking determinants; this fixes that same scalar in the extension. The family proofs specify the actual inducing characters and the multiplicity-one input when using this argument, including the disconnected orthogonal and half-spin cases.

All extensions in this section are ordinary extensions over $\mathbb{K}$. The integral block comparison is supplied by Theorem 2.1, Theorem 3.1, or the next section.

5. Rational Levi groups and ordinary Jordan transfer

5.1. The finite assembly theorem

Theorem 5.1. *Let $\mathbf{L}$ be an F-stable Levi of a simply connected semisimple group, put $L = \mathbf{L}^F$ and $H = [\mathbf{L}, \mathbf{L}]^F$, and let $c \in \mathrm{Bl}(L)$ have actual defect D. Let X be a finite ambient group normalizing L, whose action is induced by regular conjugations, algebraic isotypies, field automorphisms and diagram automorphisms preserving $\mathbf{L}, c$ and the algebraic derived subgroup. For every natural Frobenius factor block of H covered by c, assume a height-zero bijection to its actual defect normalizer, equivariant for the induced algebraic action, preserving its whole centre and having full ordinary block-triple systems in the induced algebraic automorphism action. Central-defect factors use the identity. Then there is a single $N_X(D)$-equivariant bijection*

$$\phi_c : \mathrm{Irr}_0(c) \longrightarrow \mathrm{Irr}_0(c_D), \qquad c_D \in \mathrm{Bl}(N_L(D)),$$

where c_D is the Brauer correspondent. It has full ordinary block-triple systems in the actual ambient and preserves all central scalars. It also supplies all ordinary regular fibres, covariance under every quotient line, genuine extension transfer, and the block equality at each specified regular Clifford inertia.

Proof. The quotient L/H is the full finite group of the connected quotient torus, hence is abelian. It includes the rational central-isogeny cosets. If F permutes simple derived components in a cycle of length e, projection to one component identifies its fixed points with the natural group for F^e. Retain that extension field and its actual diagram return.

Choose a covered H-block c_0 whose Fong–Reynolds block c_1 in $J = L_{c_0}$ has defect D; this is [24, Proposition 2.5(f)]. Then $D_0 = D \cap H$ is a defect of c_0. Decompose c_0 and D_0 into their actual factor blocks and factor defects. Inflate any strong factor systems from their specified central quotients by [24, Lemma 3.12] before inserting the natural group in $\mathbf{L}$.

Choose systems on representatives of the actual factor orbits and transport them. For a fixed product character, choose the constituent representatives so that constituents in the same factor-automorphism orbit are equal. Its stabilizer in the product action is then a product of wreath products of factor stabilizers with symmetric groups permuting equal constituents. Tensor the factor matrices and use the permutation matrices of (4.3). By [36, Proposition 2.18, Theorems 2.21 and 4.6], these systems give the full ordinary block relation. Restrict them to the actual algebraic action. Put $X_0 = X_{c_0}$ and $M_0 = N_H(D_0)$, and let e_0 be the Brauer correspondent of c_0 in M_0. For $\vartheta \in \mathrm{Irr}_0(c_0)$ set

$$Y_\vartheta = N_{X_0}(D_0)_\vartheta, \qquad \Gamma_\vartheta = \varepsilon(Y_\vartheta), \quad \varepsilon : X_{0,\vartheta} \longrightarrow \mathrm{Aut}(H).$$

Defect conjugacy gives $X_{0,\vartheta} = HY_\vartheta$. Since $C_{X_0}(H)$ centralizes D_0,

$$\varepsilon^{-1}(\mathrm{Inn}(M_0)\Gamma_\vartheta) = Y_\vartheta.$$

To check this equality, choose an element of $M_0 Y_\vartheta$ inducing any prescribed action in $\mathrm{Inn}(M_0)\Gamma_\vartheta$; every other lift differs from it by an element of $C_{X_0}(H)$. The block Butterfly theorem thus yields the factor map to M_0 in X_0 and retains the entire centralizer, including the regular torus scalars.

Defect conjugacy gives $X_0 = HN_{X_0}(D_0)$. Let C_1 be the unique block of $V_1 = N_J(D_0)$ of defect D inducing c_1. It exists since $N_J(D) \le V_1$. Apply [27, Proposition 1.12] with

$$N = H, \quad \mathcal{H} = N_{X_0}(D_0), \quad M = M_0, \quad J = L_{c_0}, \quad e = c_0, \quad E = e_0, \quad c = c_1, \quad C = C_1.$$

The resulting map on c_1 has target C_1 and its full ordinary upper-block relation. If c_0 moves, Brauer injectivity at D_0 identifies the inertia of its local block in $V = N_L(D_0)$ with V_1. Induce on both sides. At every upper subgroup these are the ordinary Clifford bijections: a global transporter of c_0 is corrected inside its block inertia by a defect conjugator to preserve D_0. Thus both quotient-surjection hypotheses of [24, Theorem 3.14] hold. The common defect is D, so the lower height differences vanish. We have a single equivariant map

$$\phi_1 : \mathrm{Irr}_0(c) \longrightarrow \mathrm{Irr}_0(C), \quad C \in \mathrm{Bl}(V), \quad C^L = c, \quad D \in \mathrm{Def}(C). \tag{5.1}$$

It remains to pass from $V = N_L(D_0)$ to $N_L(D)$. The argument must allow an ℓ-part in L/H. For $\psi \in \mathrm{Irr}_0(C)$, choose a linear constituent $\nu \in \mathrm{Irr}(D_0)$ whose Clifford correspondent τ in $R = V_\nu$ has a block C_R of actual defect D. Existence and conjugacy of valid choices under $N_V(D)$ are [24, Proposition 2.5(a),(f)]. Put

$$B_0 = R \cap H, \qquad K/B_0 = (R/B_0)_{\ell'}, \qquad Z = \ker \nu.$$

Put $U = (N_X(D_0))_{\nu, C_R}$. Then $R \lhd U$ and $Z \lhd U$. The abelian quotient makes $K \lhd U$, R/K is an ℓ-group, and $D \cap K = D_0$. Work modulo Z first. Bars denote these quotients. Faithfulness of $\bar\nu$ makes $\bar D_0$ central in $\bar R$. Retain the specified block $\bar C = \mathrm{bl}(\bar\tau)$, of actual defect $\bar D = D/Z$. Here Z need not be central in R: the defect assertion is [24, Proposition 2.5(c)], applied with its normal subgroup Z, its extending character 1_R, and its height-zero product τ. Retain the preimage of $(U/Z)_{\bar C}$ as the quotient-block inertia. It contains the character inertia, and can depend on the specified block $\bar C$ dominated by C_R. Every upper character above τ kills Z and therefore descends to the quotient.

In the specified block $\bar C$, choose a covered $\bar K$-block a whose Fong–Reynolds block C_a in $R_a = (\bar R)_a$ has the actual defect group $\bar D$. Let $\tau_a \in \mathrm{Irr}(C_a)$ be the Fong–Reynolds correspondent of $\bar\tau$, and choose $\bar\kappa$ as an irreducible constituent of $\tau_a|_{\bar K}$. Proposition 2.5(a),(e) of [24] gives height zero for both τ_a and $\bar\kappa$.

The block a has central defect $\bar D_0$, $R_a/\bar K$ is an ℓ-group, and the Fong–Reynolds block is its unique covering block. Therefore [24, Hypothesis 5.1] gives

$$R_a = \bar K \bar D, \qquad \bar K \cap \bar D = \bar D_0. \tag{5.2}$$

Clifford theory writes

$$\tau_a(1) = [R_a : (R_a)_{\bar\kappa}]\, m\, \bar\kappa(1),$$

where m is the degree of an irreducible projective representation of $(R_a)_{\bar\kappa}/\bar K$. The orbit index is an ℓ-power. The same is true of m: the factor-set class is annihilated by the order of this ℓ-group, so it has a representative realized by a finite central ℓ-extension. Both height-zero degrees have ℓ-part $|\bar K : \bar D_0|_\ell$, so the orbit index and m are both one. Thus $\bar\kappa$ is R_a-invariant and τ_a genuinely extends it. Write $\bar U = (U/Z)_{\bar C}$. In the actual inertia $\bar U_{a,\bar\kappa}$, the subgroups $\bar K$ and R_a are normal and $\bar\kappa$ is invariant. Apply [24, Theorem 5.13] with

$$(G, K, M, D_0, D) = (\bar U_{a,\bar\kappa}, \bar K, R_a, \bar D_0, \bar D).$$

It gives the central-defect DGN bijection

$$\mathrm{Irr}_0(R_a \mid \bar\kappa) \longrightarrow \mathrm{Irr}_0(N_{R_a}(\bar D) \mid \bar\kappa^{\bar D}) \tag{5.3}$$

with its full ordinary block relation. Its prescribed extension is the one in [24, Proposition 5.12]; the whole centralizer scalar is retained.

The inertia of the DGN block $a^{\bar D}$ in $N_{\bar R}(\bar D)$ is precisely $N_{R_a}(\bar D)$. Indeed, if a and a^x have the same local block, their unique covers in $\bar K \bar D$ have the same Brauer correspondent at $\bar D$. Brauer's first main theorem makes the covers equal and then $a = a^x$. Naturality proves the converse. The

character version follows from the natural bijection in [24, Theorem 5.2]. Thus the source-block inertia and the source-character inertia both match, inside the previously specified quotient-block inertia.

Induce (5.3) through these inertias. Clifford uniqueness gives the global surjection, and Proposition 2.5(f), followed by a defect-conjugacy correction, gives the local surjection at every upper subgroup. Theorem 3.14 of [24] gives the full ordinary relation on $\bar{C}$. Since $Z \leq D$, the normalizers satisfy

$$N_{R/Z}(D/Z) = N_R(D)/Z.$$

Inflate a resulting local character to $\eta \in \mathrm{Irr}(N_R(D))$. Proposition 2.4(a) of [24] gives $\mathrm{bl}(\eta)^R = C_R$. A defect of $\mathrm{bl}(\eta)$ contains the normal ℓ-subgroup D, and block induction places it inside a conjugate of a defect of C_R, whose order is $|D|$. Its defect is therefore exactly D. This identifies the actual Brauer block, proves height zero, and supplies the common-defect hypothesis for inflation of the relation by [24, Lemma 3.12].

Take the disjoint union over the dominated quotient blocks. For the inverse retain a local character's quotient block, take its inverse Brauer block, and invert the DGN and Clifford maps. Brauer uniqueness and the same defect calculation put its inflation in C_R. Choose representatives of the data $(\nu, \bar{C}, a, \bar{\kappa})$ under the full normalizer ambient once, construct on their stabilizers, and transport. Write ψ_ν for the Clifford correspondent of ψ over ν in V_ν. For a representative ν_i, let Θ_i be the disjoint union of the specified quotient-block maps, inflated to $N_{V_{\nu_i}}(D)$. If h carries a valid ν to ν_i, the final formula is

$$\Omega(\psi) = \left(\mathrm{Ind}_{N_{V_{\nu_i}}(D)}^{N_V(D)} \Theta_i((\psi_\nu)^h)\right)^{h^{-1}}.$$

Two valid choices are $N_V(D)$-conjugate; equivariance on the representative's stabilizer and inner invariance after induction make this independent of the choice. The same calculation proves full ambient equivariance. A second application of Theorem 3.14 handles the ν-induction at every upper subgroup. Since $N_V(D) = N_L(D)$, composition with (5.1) proves the asserted lower map and block relation.

Finally let $\widetilde{L}$ be the actual regular overgroup, $M = N_L(D)$, and $\widetilde{M} = N_{\widetilde{L}}(D)$. For a matched pair θ, θ', actual stabilizer equality and defect conjugacy give

$$I = \widetilde{L}_\theta = LJ, \qquad J = \widetilde{M}_{\theta'}, \qquad I/L = J/M.$$

Use the same irreducible inverse-factor coefficient E in $P \otimes E$ and $P' \otimes E$, and Clifford-induce over moving regular orbits. This produces

$$\widetilde{\phi}_c : \mathrm{Irr}(\widetilde{L} \mid \mathrm{Irr}_0(c)) \longrightarrow \mathrm{Irr}(\widetilde{M} \mid \mathrm{Irr}_0(c_D)).$$

Tensoring by every ordinary quotient line commutes with this formula. Whole-centre scalars agree because the same E and the same centralizer scalar are used. A genuine extension is characterized by a one-dimensional inverse-factor coefficient and is therefore preserved. At the specified Clifford inertia the block relation gives $\mathrm{bl}(\theta_0')^I = \mathrm{bl}(\theta_0)$, before induction to the top regular group. This proves all the final assertions. □

5.2. **Transfer from a canonical proper Jordan Levi**

Say that $s \in \mathbf{G}^{*F^*}$ is strictly quasi-isolated if no proper F^*-stable Levi contains

$$C(s) = C^\circ_{\mathbf{G}^*}(s) C_{\mathbf{G}^{*F^*}}(s).$$

For a non-strictly-quasi-isolated s, choose a minimal such Levi $\mathbf{L}^*$. Its finite BDR source is L itself: a rational normalizer fixing the s-series can be corrected by an element of L^* to centralize s, and then belongs to L^* by the defining containment.

Lemma 5.2. *Let $\mathbf{X}$ be connected reductive, let F_0 be a Steinberg endomorphism, and put $F = F_0^r$. Suppose $s \in \mathbf{X}^F$ is semisimple and its $\mathbf{X}^F$-conjugacy class is F_0-stable. Let $\mathbf{L}$ be the minimal Levi containing $C^\circ_{\mathbf{X}}(s)C_{\mathbf{X}^F}(s)$. There is $x \in \mathbf{X}^F$ such that $\mathbf{K} = x\mathbf{L}x^{-1}$ is F_0-stable and the $\mathbf{K}^F$-conjugacy class of xsx^{-1} is F_0-stable.*

Proof. The minimal Levi is unique: it is the centralizer of the connected centre of $C^\circ_{\mathbf{X}}(s)C_{\mathbf{X}^F}(s)$. Indeed this group contains a maximal torus of $\mathbf{X}$, so its connected centre is a torus. If a Levi contains it, the connected centre of that Levi is contained in this torus; taking centralizers gives the asserted minimality. Choose $g \in \mathbf{X}^F$ with $F_0(s) = gsg^{-1}$. Then $F_0(\mathbf{L}) = g\mathbf{L}g^{-1}$. By Lang's theorem choose $h \in \mathbf{X}$ with $F_0(h) = hg^{-1}$. Consequently $h\mathbf{L}h^{-1}$ is F_0-stable and hsh^{-1} is fixed by F_0. Put

$$a = h^{-1}F(h) = g^{-1}F_0(g^{-1})\cdots F_0^{r-1}(g^{-1}).$$

Since $g \in \mathbf{X}^F$ and F, F_0 commute, $a \in \mathbf{X}^F$. The F-fixedness of hsh^{-1} gives $a \in C_{\mathbf{X}}(s)$, and hence $a \in C_{\mathbf{X}^F}(s) \le \mathbf{L}$. Apply Lang's theorem in the connected F-stable group $\mathbf{L}$ to choose $l \in \mathbf{L}$ satisfying $l^{-1}aF(l) = 1$. Then $x = hl$ belongs to $\mathbf{X}^F$ and $x\mathbf{L}x^{-1} = h\mathbf{L}h^{-1} = \mathbf{K}$. Finally set

$$u = F_0(x)gx^{-1} = h(g^{-1}F_0(l)g)l^{-1}h^{-1}.$$

The middle factors lie in $\mathbf{L}$, so $u \in \mathbf{K}$; $x, g \in \mathbf{X}^F$ also give $u \in \mathbf{X}^F$. Direct multiplication now gives

$$F_0(xsx^{-1}) = u(xsx^{-1})u^{-1}, \qquad u \in \mathbf{K}^F,$$

as required. Every return in this construction is the actual r-fold field return. □

Proposition 5.3. *Let $\mathbf{G}$ be simply connected with its regular embedding and standard field and diagram automorphisms, in the scope of* [27, Theorems 2.1, 2.11, 2.12]*; in particular, $G = \mathbf{G}^F$ is the universal covering group of its nonabelian simple central quotient, as assumed in Section 2 of that paper. Let B be in the ℓ-series of an ℓ'-element s and let $\mathbf{L}^*$ be the canonical proper Levi above. Suppose its source block c has the systems of Theorem 5.1, including the full ordinary regular fibres and the specified Clifford-inertia block equalities. Then the maps of the global and local Brauer complexes give*

$$\Omega_B = R_D\phi_c R^{-1} : \mathrm{Irr}_0(B) \longrightarrow \mathrm{Irr}_0(b), \qquad b \in \mathrm{Bl}(N_G(D)). \tag{5.4}$$

This is equivariant under the full required automorphism stabilizer and has the strong relation (2.3). Ordinary extension transfer in this statement uses $C(s) \le \mathbf{L}^*$ and the full field and diagram inertia, including its ℓ-part.

Proof. The splendid complex

$$\mathcal{C} = G\Gamma_c(Y^{\mathbf{G}}_{\mathbf{U}}, \mathcal{O})^{\mathrm{red}}e_s^L$$

has concentrated cohomology giving the Morita equivalence under $C(s) \le \mathbf{L}^*$, by [26, Theorem 2.28] and [3, Theorem 7.7]. Fix a maximal c-Brauer pair with subgroup $D \le L$. Its nonzero corner in $\mathrm{Br}_{\Delta D}(\mathcal{C})$ gives the corresponding B-pair with this same subgroup. Equality of defect orders makes it maximal, and the corresponding centralizer block is unique. This identifies the subgroup D in the subpair correspondence of [26, Theorem 1.3, Proposition 1.4]. Normalizer induction of that same Brauer complex gives the local Morita map R_D between $N_L(D)$ and $N_G(D)$. These maps preserve heights. Thus (5.4) and its inverse $R\phi_c^{-1}R_D^{-1}$ have the stated domains and blocks.

For the geometric field generator, use Lemma 5.2 in the dual group. It gives both the stable minimal Levi and its stable rational source series. The subsequent commuting diagram operator construction in [26, Proposition 4.8] therefore applies to this choice. More explicitly, write σ for the diagram operation commuting with F_0. Its rational correction gives $y \in \mathbf{G}^F$ such that $y\sigma$ stabilizes the chosen Levi and its source idempotent, with $b = F_0(y)y^{-1} \in \mathbf{L}^F$. Choose $l \in \mathbf{L}$ by Lang's theorem with $F_0(l)l^{-1} = b$, and put $x = l^{-1}y$. Then $F_0(x) = x$, so $x \in \mathbf{G}^{F_0} \le \mathbf{G}^F$ and $l = yx^{-1} \in \mathbf{L}^F$. Thus $x\sigma$ still stabilizes the Levi and source idempotent and commutes with F_0. This supplies the required commuting operators.

The containment in [26, Theorem 5.8] can be weakened for the ordinary extension statement over $\mathbb{K}$ as follows. This is an argument from its proof, rather than an application of its stronger stated hypothesis. Its proof uses the diagonal field action of Proposition 5.7 and the fact that the underlying bimodule is Morita. The parabolic independence immediately preceding Theorem 2.27, and its restriction-of-scalars realization in Section 5.3, require only $C^{\circ}_{\mathbf{G}^*}(s) \leq \mathbf{L}^*$. Lemma 5.6 and the transport of the natural field action then give Proposition 5.7 on this same bimodule. The Morita property follows from Theorem 2.28 under $C(s) \leq \mathbf{L}^*$. Rouquier's cyclic module-extension argument and Theorem 1.7 in that proof therefore apply with this weaker containment. Locally, Proposition 5.10, the diagonal version of Theorem 5.2, and the actual local equivalence of Theorem 3.10 give the same argument for Theorem 5.11. For this use take coefficients in $\mathbb{K}$. Every finite order is invertible there, including the order-three diagram operation at $\ell = 3$ and field orders divisible by ℓ. The resulting ordinary functors commute with restriction, induction and the chosen corrected automorphisms, and transfer genuine extensions. The integral block anchor below is constructed separately; this argument makes no claim of an integral extension by a diagram or field operator whose order is divisible by ℓ.

For the regular Clifford inertia put

$$I = \widetilde{G}_{\chi}, \qquad I_L = I \cap \widetilde{L} = \widetilde{L}_{\theta}, \qquad J = N_I(D), \qquad J_L = N_{I_L}(D), \quad \theta = R^{-1}(\chi).$$

Equivariance and $\widetilde{G} = G\widetilde{L}$ give $I = GI_L$ and $I/G \cong I_L/L$. The complex $\mathcal{C}$ carries the integral diagonal $\widetilde{L}$-action. Form

$$\mathcal{C}_I = \mathrm{Ind}_{(G\times L^{\mathrm{opp}})\Delta I_L}^{I\times I_L^{\mathrm{opp}}}(\mathcal{C}c).$$

As in [27, Lemma 2.14], [3, Lemma 7.4] and [26, Lemma 1.9] give its splendid Rickard equivalence; cohomology concentration and [26, Theorem 1.7] give the corresponding Morita bimodule. Denote its ordinary correspondence by R_I. The integral local complex in [27, Lemma 2.14] reduces to the specified block corner of $\mathrm{Br}_{\Delta D}(\mathcal{C}_I)$ followed by normalizer induction. Its concentrated cohomology gives the local Morita correspondence $R_{I,D}$ between J_L and J.

Both correspondences commute with induction to the top regular groups, by [26, Remark 1.8] and the local calculation in [27, Lemma 2.14]. Consequently, for the chosen regular characters and their Clifford correspondents χ_0, χ_0', the characters

$$\theta_0 = R_I^{-1}(\chi_0), \qquad \theta_0' = R_{I,D}^{-1}(\chi_0')$$

are the specified Clifford correspondents in the assembly theorem. That theorem gives $\mathrm{bl}(\theta_0')^{I_L} = \mathrm{bl}(\theta_0)$. The Harris–Knorr compatibility of the integral complex and its Brauer construction, [26, Remark 1.21], therefore gives

$$\mathrm{bl}(\chi_0')^I = \mathrm{bl}(\chi_0). \tag{5.5}$$

For the genuine extensions used in Theorem 2.1 of [27], Lemma 2.2 of that paper gives the corresponding equality at every $G \leq I_0 \leq I$.

The same Brauer pairs and actual defect conjugacy give

$$N_G(D)N_{\widetilde{L}\mathcal{A}}(D, c_D) = N_{\widetilde{G}\mathcal{A}}(D, b),$$

where $\mathcal{A}$ is the corrected geometric stabilizer. The ordinary fibres in the assembly theorem are covariant under all quotient lines and preserve the entire regular centre. They preserve genuine extension obstruction by using the same coefficient. These are exactly hypotheses (i)–(iii) of [27, Theorem 2.12].

The passage back to the standard automorphism ambient is part of that theorem. The good representatives of Theorem 2.11 give stabilizer factorization and genuine standard extensions. Lemmas 2.3–2.4 transfer them to the corrected geometric generators and back. In split D_4, Lemmas 2.7–2.8 apply separately to every required Sylow two and Sylow three inertia; vanishing of both restricted Clifford obstructions gives the full extension by restriction and corestriction. The comparison lines and whole-centre scalars are those of Lemmas 2.9–2.10, and (5.5) is the

required block anchor. These are the five conditions of Theorem 2.1, including its nonabelian outer quotient case.

That theorem constructs the systems first on G/K_χ, $K_\chi = \ker\chi \cap Z(G)$, and proves the full ordinary block relation there. The block Butterfly theorem realizes the actual semidirect automorphism inertia. For a perfect natural group no additional action kernel appears on this quotient: an automorphism acting identically modulo K_χ defines a homomorphism $G \to K_\chi$, which is trivial. Its full character inertia image is generated by inner automorphisms and the stabilizer of the actual block and defect, by defect conjugacy. Finally D is the unique Sylow ℓ-subgroup of DK_χ, since the central ℓ-part lies in D. Thus the local quotient satisfies

$$N_{G/K_\chi}(DK_\chi/K_\chi) = N_G(D)/K_\chi.$$

This proves the strong assertion. Natural small groups are handled directly in the family proofs before this perfect group passage is used. □

6. Ordinary source systems for classical groups

Throughout this section p is the defining characteristic, $q = p^f$, and $\ell \neq p$ is odd. A statement about a natural group refers to the fixed points of the specified simply connected algebraic group. In the classical sections, a rational Levi L has algebraic Levi $\mathbf{L}$, with $L = \mathbf{L}^F$; its finite dual is $L^* = (\mathbf{L}^*)^{F^*}$. Centres of algebraic groups are taken before fixed points when they parametrize dual torus lines. For $\epsilon \in \{1, -1\}$, write $\mathrm{GL}_n^\epsilon(q) = \mathrm{GL}_n(q), \mathrm{GU}_n(q)$, respectively, and use the analogous convention for $\mathrm{SL}_n^\epsilon(q)$. We retain the action on a finite factor induced by its actual embedding: in particular, an automorphism returning from a cycle of factors can have a nontrivial inner power.

The phrase *full source system* means an equivariant height-zero bijection with the prescribed strong block character-triple relation. Its associated systems give maps on all ordinary characters above a paired character at every intermediate group, and have the same scalar on the entire ambient centralizer.

We first record several constructions used more than once. They also specify precisely the ordinary information carried by the source maps.

6.1. Independent coordinates and ordinary packets

Lemma 6.1. *Let*

$$A = \prod_i (C_{\ell^{a_i}})^{w_i}, \qquad D = A \rtimes P,$$

where P is an ℓ-group acting faithfully by permutations of coordinates of equal order. Then A is the unique maximal abelian normal subgroup of D. In particular A is characteristic in D.

Proof. Let $U \lhd D$ be abelian and let $x \in U$, with permutation image π. For every $a \in A$, the commutator $[x, a]$ belongs to $U \cap A$; hence x centralizes it. Thus $(\pi - 1)^2 A = 0$. If $\pi \neq 1$, choose a coordinate i moved by π. Its cycle has odd ℓ-power length at least three, and

$$(\pi - 1)^2 e_i = e_{\pi^2 i} - 2e_{\pi i} + e_i \neq 0.$$

The three positions are distinct and the two coefficients 1 remain nonzero even at $\ell = 3$. Therefore $\pi = 1$, so $U \leq A$. □

Proposition 6.2. *Let $\mathbf{G}$ have connected centre and a simple classical derived group, and put $X = \mathbf{G}^F$. Let $\mathbf{L}$ be an F-stable e-split Levi, put $L = \mathbf{L}^F$, and let (L, λ) be an e-cuspidal source of an ℓ-block B, with semisimple ℓ'-parameter $s \in (\mathbf{L}^*)^{F^*}$. Put*

$$A = Z(L)_\ell, \quad N = N_X(L), \quad W_\lambda = N_\lambda/L.$$

Suppose that $C_X(A) = L$, that λ is the canonical central-defect character of its block c, and that the corresponding dual unipotent source $\mathbf{M}^* = C_{\mathbf{L}^*}(s)$ satisfies

$$(Z(\mathbf{M}^*)^{F^*})_\ell = (Z(\mathbf{L}^*)^{F^*})_\ell. \tag{6.1}$$

Assume the actual source decomposition

$$L = A \times K, \qquad K = O^\ell(L), \qquad \lambda = 1_A \otimes \kappa, \qquad \kappa \in \mathrm{dz}(K).$$

Central duality is assumed to identify $(Z(\mathbf{L}^*)^{F^*})_\ell$ with $\mathrm{Irr}(A)$, by restriction of its linear characters to A. Suppose also that the source characters have genuine equivariant extensions to their actual normalizer inertias. For $t \in (Z(\mathbf{L}^*)^{F^*})_\ell$, let $\widehat{t}$ be the ordinary linear character of L dual to t, put $\theta_t = \lambda\widehat{t}$ and $W_t = N_{\theta_t}/L$, and write $\Lambda(\theta_t)$ for its chosen genuine extension to N_{θ_t}. There are ordinary packet maps

$$I_t : \mathrm{Irr}(W_t) \longrightarrow \mathcal{E}(X, (L, \theta_t)) = \mathrm{Irr}(B) \cap \mathcal{E}(X, st), \tag{6.2}$$

and the map

$$I_t(\eta) \longmapsto \mathrm{Ind}_{N_{\theta_t}}^N(\Lambda(\theta_t)\eta) \tag{6.3}$$

is a bijection on the height-zero sets of the specified global and local blocks. Its parameter set is

$$\{(t, \eta) : [W_\lambda : W_t]_\ell\, \eta(1)_\ell = 1\}/N. \tag{6.4}$$

The actual defect order is $|A|\,|W_\lambda|_\ell$. These statements apply separately to every regular covering source block. The resulting maps preserve every regular linear twist and the complete ordinary central character.

Proof. The hypotheses on classical groups in [15, Assumptions 2.1.2 and 2.2.3] allow every odd non-defining prime. Apply its height-preserving block Jordan correspondence, Theorem 1.4(B.1),(B.3), to the unipotent block dual to $C_{\mathbf{G}^*}(s)$. The height-zero exhaustion theorem [8, Theorem 5.2] places its ℓ-element parameter in $(Z(\mathbf{M}^*)^{F^*})_\ell$. Condition (6.1) therefore places every height-zero character of B in one of the displayed series. No rational ${}^3D_4$ factor occurs in this use of the theorem.

For a fixed t, the centralizer of t is a Levi subgroup. Indeed, an odd-order semisimple element has no nontrivial self-inverse eigenvalue; reciprocal eigenvalue packets give general linear or unitary factors and the 1-space gives the residual classical factor. Let $\mathbf{H}^* = C_{\mathbf{G}^*}(t)$, and choose the dual Levi $\mathbf{H}$ containing $\mathbf{L}$. Since $t \in Z(\mathbf{L}^*)$ and $\mathbf{L}$ is e-split,

$$C_{\mathbf{H}^*}(Z^\circ(\mathbf{L}^*)_{\Phi_e}) = C_{\mathbf{G}^*}(Z^\circ(\mathbf{L}^*)_{\Phi_e}) = \mathbf{L}^*.$$

Thus $(\mathbf{L}, \lambda)$ is the same e-cuspidal pair inside $\mathbf{H}$. In [15, Proposition 2.3.6], take $\mathbf{L}_0 = \mathbf{L}$ and $s_0 = t$. Its auxiliary dual Levi is $\mathbf{L}^*$, since

$$C_{\mathbf{G}^*}(Z^\circ(\mathbf{L}_0^*)_{\Phi_e}) = \mathbf{L}^*.$$

Its induced source character is therefore $\lambda\widehat{t}$, and its equation (2.3.6.3) gives

$$\mathrm{Irr}(B) \cap \mathcal{E}(X, st) = \mathcal{E}(X, (L, \lambda\widehat{t})),$$

with the original embedded source and its specified block. The ordinary parameterization and its equivariance are [25, Theorems 3.2 and 3.4, Lemmas 3.5–3.7]. Hypothesis 3.1 holds for this ambient group with simple classical derived group; its statements already include the Levi subgroups appearing here. Products of natural Frobenius factors, their common centres and their rational central-isogeny cosets are treated by Theorem 5.1 after constructing the individual factor systems. The degree formulas are

$$I_t(\eta)(1)_\ell = \frac{|X|_\ell}{|A|\,|W_t|_\ell}\eta(1)_\ell, \qquad \mathrm{Ind}_{N_{\theta_t}}^N(\Lambda(\theta_t)\eta)(1)_\ell = \frac{|N|_\ell}{|A|\,|W_t|_\ell}\eta(1)_\ell. \tag{6.5}$$

The inverse locally is ordinary Clifford correspondence, followed by division by the chosen genuine extension. Uniqueness of the marked cuspidal support proves that fusion on both sides is exactly the same actual normalizer action.

We verify the block and defect before using (6.5) as a height comparison. The source theorem [15, Proposition 2.1.6] gives the actual pair (A, c) of B. The displayed decomposition makes K characteristic in L, hence normal in N, and $N_\kappa = N_\lambda$. In $Q = N_\kappa/K$, the image of A is a normal ℓ-subgroup and is self-centralizing: an element centralizing it modulo K has commutators in $A \cap K = 1$, so belongs to $C_N(A) = L$. Theorem 3.1, with defect-zero source κ on K, therefore gives one covering block in N for this source orbit and an actual defect of order $|A|\,|W_\lambda|_\ell$. In all applications below its ℓ-part is represented by the specified permutation matrices, or their odd-order Spin lifts. Choose its maximal pair (D, c_D) above (A, c). Since

$$C_X(D) \le C_X(A) = L \le N,$$

the centralizers and Brauer maps along a normal-step chain from A to D are the same locally and globally. Thus (D, c_D) is a B-pair. On the other hand the minimum in (6.5) is attained at $t = 1, \eta = 1$. Every displayed character belongs to B, and every height-zero character of B has already been exhausted. The block degree-minimum formula gives defect order $|A|\,|W_\lambda|_\ell$. The constructed pair is therefore maximal in X as well. This proves the asserted induced-block identity and (6.4).

Repeat the argument on each regular covering block, with the whole regular central ℓ-torus included in A and the same source decomposition verified there. Its ordinary source restricts to the given source orbit by [15, Proposition 2.1.5] and source uniqueness. Conversely every such covering source produces a covering block. Induction modules have the same action of every regular central element as their source, so preserve its scalar in characteristic zero. Central dual multiplication is precisely regular linear twisting; [25, Lemma 3.7] makes the two constructions covariant for all these twists. For types B, C, D, the following lemma identifies their union with the entire ordinary regular fibre. The full fibre in type A is constructed in Section 7. □

Lemma 6.3. *Let $G \le X$ be a regular embedding of a natural group of type B, C, or D, and let ℓ be odd and nondefining. Let B be an ℓ-block of G. Then*

$$\mathrm{Irr}(X \mid \mathrm{Irr}_0(B)) = \bigcup_{B_X \text{ covering } B} \mathrm{Irr}_0(B_X).$$

Proof. Put $Z = Z(X)$ and $X_0 = GZ$. The quotient X/X_0 is a two-group. Choose a block B_X covering B. Let $\widetilde{\chi} \in \mathrm{Irr}(B_X)$, and choose a constituent $\chi \in \mathrm{Irr}(B)$ of its restriction to G. A constituent on X_0 is obtained from χ and the prescribed scalar character of Z by descent through their central intersection, and has degree $\chi(1)$. Clifford theory across the two-group X/X_0 consequently gives $\widetilde{\chi}(1)_\ell = \chi(1)_\ell$.

If D is a defect group of B, the block on X_0 containing that constituent has defect of order

$$|D|\,|Z|_\ell/|Z \cap G|_\ell.$$

Indeed, on the central product this is the image of $D \times Z_\ell$. Passing from X_0 to X does not change its defect order: every ℓ-subgroup of X lies in X_0, and the normal-subgroup defect theorem identifies its intersection with the defect of the covered block. Hence, for a defect group D_X of B_X,

$$\frac{|X|_\ell}{|D_X|} = \frac{|G|_\ell}{|D|}.$$

The degree definition of height now gives $h_{B_X}(\widetilde{\chi}) = h_B(\chi)$. Every constituent over G belongs to a conjugate of B, and conjugation preserves height, proving the equality. □

6.2. **Fourier restriction, twisting, and actual returns**

Lemma 6.4. *Let $N \lhd R$, with $\Delta = R/N$ abelian of ℓ'-order. Suppose a block idempotent e of $\mathcal{O}R$ has free orbit under* $\mathrm{Irr}(\Delta)$. *Then*

$$E = \sum_{\delta \in \mathrm{Irr}(\Delta)} \delta(e) \in Z(\mathcal{O}N)$$

is a block idempotent, and

$$\mathcal{O}NE \xrightarrow{\sim} \mathcal{O}Re, \quad a \mapsto ae, \qquad x \mapsto |\Delta| \operatorname{pr}_1(x) \tag{6.6}$$

are inverse algebra maps, where pr_1 *projects onto the identity component of the* Δ*-grading. Restriction is consequently an irreducible bijection on the entire ordinary block. Every lower character has a unique extension in the specified upper block.*

Proof. Use the Δ-grading of $\mathcal{O}R$. Fourier orthogonality places E in the identity component and gives $\operatorname{pr}_1(e) = E/|\Delta|$. For $x \in \mathcal{O}Re$,

$$|\Delta| \operatorname{pr}_1(x) = \sum_{\delta} \delta(x).$$

Multiplication by e kills every summand except x, because the idempotents $\delta(e)$ are orthogonal. Conversely $|\Delta| \operatorname{pr}_1(ae) = aE = a$. This proves (6.6). The upper algebra is indecomposable, so the lower central idempotent is primitive. The equivalence preserves the underlying ordinary representation on N, proving the character assertion. Uniqueness also gives covariance for every automorphism stabilizing the chosen upper block. □

Lemma 6.5. *Suppose*

$$(Y, N, \chi) \geq_b (H, M, \psi), \qquad M = N \cap H,$$

and β *is a* Y*-invariant linear character of* N*. Then the same relation holds for* $(\beta\chi, \beta|_M\psi)$*.*

Proof. Put $Q = Y/N$, choose $t : Q \to Y$, and set

$$a(x, y) = \beta(t(x)t(y)t(xy)^{-1}).$$

Invariance makes this a normalized cocycle with values in a finite cyclic group C. Form $\widehat{Q} = Q \times C$ with multiplication $(x, z)(y, w) = (xy, zw\, a(x, y))$, and let $\widehat{Y} = Y \times_Q \widehat{Q}$. Embed the original N by $n \mapsto (n, (1, 1))$. For $g = nt(x)$, the formula

$$\widehat{\beta}(g, (x, z)) = \beta(n)z^{-1}$$

is a genuine linear character of $\widehat{Y}$, because the cocycle in the numerator and denominator cancels in a product. Its restriction is β. The action images of Y and $\widehat{Y}$ on this same N are equal. Apply the block Butterfly theorem, multiply the two associated systems by $\widehat{\beta}$, and apply Butterfly back. Every intermediate ordinary coefficient receives the same linear twist; block induction commutes with it, and the full centralizer receives the same scalar on both sides. □

An invariant irreducible character extends over a cyclic quotient with its actual return: if yN has order r, choose an intertwiner U, compare U^r with $\rho(y^r)$, and rescale U by an r-th root of the scalar discrepancy. This works in characteristic zero even when $\ell \mid r$. With an induced-block anchor and equal ordinary centralizer scalars, Theorem 2.1 then gives every upper block relation. For an abelian quotient it applies to the genuine joint extensions constructed below.

7. Type A and twisted type A

Throughout this section, ℓ is an odd prime different from the characteristic of the defining field. Write $\mathrm{GL}_n^+(q) = \mathrm{GL}_n(q)$, $\mathrm{GL}_n^-(q) = \mathrm{GU}_n(q)$, and use the analogous convention for $\mathrm{SL}_n^\epsilon(q)$.

Theorem 7.1. *The inductive Alperin–McKay condition holds for every* ℓ*-block of a nonabelian finite simple group of type* A *or* ${}^2A$*. For the natural groups* $\mathrm{SL}_n^\epsilon(q)$*, the constructions also supply the following data in their actual regular, field and graph ambients: a height-zero bijection to the Brauer correspondent in the actual defect normalizer, the associated block character-triple relations on the prescribed central quotients, and the full ordinary maps on every regular fibre above these lower characters. These maps preserve all quotient linear twists and the complete ordinary central character. They are compatible with central products and permutations of equal factors, with the actual return automorphism on each factor cycle.*

The iAM assertion for $\ell \geq 5$ is already contained in [28, Corollary 12.5]; maximal-defect blocks are covered by [9, Theorem 1.1]. The purpose of the argument below is to include the nonmaximal-defect blocks at three and to retain the natural central-cover data needed in Levi subgroups. We do not infer these data merely from the iAM assertion for a simple quotient.

An ordinary fibre in this section means

$$\mathrm{Irr}(\widetilde{X} \mid \mathrm{Irr}_0(B)),$$

not just $\mathrm{Irr}_0(\widetilde{B})$. Its characters may have positive height. On an intermediate inertia group, if $\mathcal{P}$ is an associated projective representation, every ordinary coefficient of factor set α^{-1} is retained in $\mathcal{P} \otimes \mathcal{R}$. This is the meaning of a full ordinary system; no restriction on $\mathcal{R}(1)_\ell$ is imposed. The common block-triple, normal-source and Levi-assembly results apply to these systems.

7.1. **Ordinary source packets and exact normalizers**

Lemma 7.2. *Let*

$$A = \prod_i (C_{\ell^{a_i}})^{w_i}, \qquad D = A \rtimes P,$$

where P is an ℓ-group acting faithfully by permutations of coordinates of equal order. Then A is the unique maximal abelian normal subgroup of D. In particular A is characteristic in D.

Proof. Let $U \lhd D$ be abelian and let $x \in U$, with permutation image π. For every $a \in A$, the commutator $[x, a]$ belongs to $U \cap A$; hence x centralizes it. Thus $(\pi - 1)^2 A = 0$. If $\pi \neq 1$, choose a coordinate i moved by π. Its cycle has odd ℓ-power length at least three, and

$$(\pi - 1)^2 e_i = e_{\pi^2 i} - 2e_{\pi i} + e_i \neq 0.$$

The three positions are distinct and the two coefficients 1 remain nonzero even at $\ell = 3$. Therefore $\pi = 1$, so $U \leq A$. □

Proposition 7.3. *Let $\mathbf{G}$ have connected centre and simple derived group of type A, and put $X = \mathbf{G}^F$. Let $\mathbf{L}$ be an F-stable e-split Levi, put $L = \mathbf{L}^F$, and let (L, λ) be an e-cuspidal source of an ℓ-block B, with semisimple ℓ'-parameter $s \in (\mathbf{L}^*)^{F^*}$. Put*

$$A = Z(L)_\ell, \quad N = N_X(L), \quad W_\lambda = N_\lambda / L.$$

Suppose that $C_X(A) = L$, that λ is the canonical central-defect character of its block c, and that the corresponding dual unipotent source $\mathbf{M}^* = C_{\mathbf{L}^*}(s)$ satisfies

$$(Z(\mathbf{M}^*)^{F^*})_\ell = (Z(\mathbf{L}^*)^{F^*})_\ell. \tag{7.1}$$

We require the actual source decomposition

$$L = A \times K, \qquad K = O^\ell(L), \qquad \lambda = 1_A \otimes \kappa, \quad \kappa \in \mathrm{dz}_\ell(K).$$

In particular K is characteristic in L, and every ordinary ℓ-power-order line of L kills K. Central defect alone is not used as a substitute for this decomposition. We also require that duality identifies $(Z(\mathbf{L}^*)^{F^*})_\ell$ with $\mathrm{Irr}(A)$, by restriction of $\widehat{t}$ to A. This holds for the torus sources used below by the perfect pairing on each rational cyclic torus. Suppose also that the source characters have genuine equivariant extensions to their actual normalizer inertias. For $t \in (Z(\mathbf{L}^*)^{F^*})_\ell$, let $\widehat{t}$ be the ordinary linear character of L dual to t, put $\theta_t = \lambda\widehat{t}$ and $W_t = N_{\theta_t}/L$, and write $\Lambda(\theta_t)$ for its chosen genuine extension to N_{θ_t}. There are ordinary packet maps

$$I_t : \mathrm{Irr}(W_t) \longrightarrow \mathcal{E}(X, (L, \theta_t)) = \mathrm{Irr}(B) \cap \mathcal{E}(X, st), \tag{7.2}$$

and the map

$$I_t(\eta) \longmapsto \mathrm{Ind}_{N_{\theta_t}}^N(\Lambda(\theta_t)\eta) \tag{7.3}$$

is a bijection on the height-zero sets of the specified global and local blocks. Its parameter set is

$$\{(t, \eta) : [W_\lambda : W_t]_\ell \, \eta(1)_\ell = 1\} / N. \tag{7.4}$$

The actual defect order is $|A|\,|W_\lambda|_\ell$. These statements apply separately to every regular covering source block. The resulting maps preserve every regular linear twist and the complete ordinary central character.

Proof. The hypotheses on type A groups in [15, Assumptions 2.1.2 and 2.2.3] allow every odd non-defining prime. Apply its height-preserving block Jordan correspondence, Theorem 1.4(B.1),(B.3), to the unipotent block dual to $C_{\mathbf{G}^*}(s)$. The height-zero exhaustion theorem [8, Theorem 5.2] places its ℓ-element parameter in $(Z(\mathbf{M}^*)^{F^*})_\ell$. Condition (7.1) therefore places every height-zero character of B in one of the displayed series. No rational ${}^3D_4$ factor occurs in this use of the theorem.

For a fixed t, the centralizer of t is a Levi subgroup. In type A, a semisimple centralizer is the block-diagonal Levi on the distinct eigenspaces. The determinant and central-isogeny relations are those of the specified dual group; they are retained. Frobenius permutes the eigenspaces, giving the actual rational linear or unitary factors and their rational central multiplication. Let $\mathbf{H}^* = C_{\mathbf{G}^*}(t)$, and choose the dual Levi $\mathbf{H}$ containing $\mathbf{L}$. Since $t \in Z(\mathbf{L}^*)$ and $\mathbf{L}$ is e-split,

$$C_{\mathbf{H}^*}(Z^\circ(\mathbf{L}^*)_{\Phi_e}) = C_{\mathbf{G}^*}(Z^\circ(\mathbf{L}^*)_{\Phi_e}) = \mathbf{L}^*.$$

Thus $(\mathbf{L}, \lambda)$ is the same e-cuspidal pair inside $\mathbf{H}$. In [15, Proposition 2.3.6], take $\mathbf{L}_0 = \mathbf{L}$ and $s_0 = t$. Its auxiliary dual Levi is $\mathbf{L}^*$, since

$$C_{\mathbf{G}^*}(Z^\circ(\mathbf{L}_0^*)_{\Phi_e}) = \mathbf{L}^*.$$

Its induced source character is therefore $\lambda\widehat{t}$, and its equation (2.3.6.3) gives

$$\mathrm{Irr}(B) \cap \mathcal{E}(X, st) = \mathcal{E}(X, (L, \lambda\widehat{t})),$$

with the original embedded source and its specified block. The ordinary parameterization and its equivariance are [25, Theorems 3.2 and 3.4, Lemmas 3.5–3.7]. Hypothesis 3.1 holds for these connected-centre groups with simple derived group of type A. A torus is handled directly by ordinary duality; products and their actual rational central multiplication are handled by Theorem 5.1. The degree formulas are

$$I_t(\eta)(1)_\ell = \frac{|X|_\ell}{|A|\,|W_t|_\ell}\eta(1)_\ell, \qquad \mathrm{Ind}_{N_{\theta_t}}^{N}(\Lambda(\theta_t)\eta)(1)_\ell = \frac{|N|_\ell}{|A|\,|W_t|_\ell}\eta(1)_\ell. \tag{7.5}$$

The inverse locally is ordinary Clifford correspondence, followed by division by the chosen genuine extension. Uniqueness of the marked cuspidal support proves that fusion on both sides is exactly the same actual normalizer action.

We verify the block and defect before using (7.5) as a height comparison. The source theorem [15, Proposition 2.1.6] gives the actual pair (A, c) of B. In the source inertia, the core block has defect zero, and the toral part has normal subgroup A. Theorem 3.1 gives one covering block for this source orbit and an actual defect of order $|A|\,|W_\lambda|_\ell$. In all applications below its ℓ-part is represented by the specified permutation matrices, in their natural linear or unitary realizations. Choose its maximal pair (D, c_D) above (A, c). Since

$$C_X(D) \le C_X(A) = L \le N,$$

the centralizers and Brauer maps along a normal-step chain from A to D are the same locally and globally. Thus (D, c_D) is a B-pair. On the other hand the minimum in (7.5) is attained at $t = 1, \eta = 1$. Every displayed character belongs to B, and every height-zero character of B has already been exhausted. The block degree-minimum formula gives defect order $|A|\,|W_\lambda|_\ell$. The constructed pair is therefore maximal in X as well. This proves the asserted induced-block identity and (7.4).

Repeat the argument on each regular covering block, with the whole regular central ℓ-torus included in A. Its ordinary source restricts to the given source orbit by [15, Proposition 2.1.5] and source uniqueness. Conversely every such covering source produces a covering block. Induction modules have the same action of every regular central element as their source, so preserve its

scalar in characteristic zero. Central dual multiplication is precisely regular linear twisting; [25, Lemma 3.7] makes the two constructions covariant for all these twists. The full fibre for special linear and unitary groups is constructed below; in particular, restriction to a special group is not presumed to preserve height in a regular extension of ℓ-divisible index. □

Lemma 7.4. *Let $N \lhd R$, with $\Delta = R/N$ abelian of ℓ'-order. Suppose a block idempotent e of $\mathcal{O}R$ has free orbit under* $\mathrm{Irr}(\Delta)$. *Then*

$$E = \sum_{\delta \in \mathrm{Irr}(\Delta)} \delta(e) \in Z(\mathcal{O}N)$$

is a block idempotent, and

$$\mathcal{O}NE \xrightarrow{\sim} \mathcal{O}Re, \quad a \mapsto ae, \qquad x \mapsto |\Delta| \operatorname{pr}_1(x) \tag{7.6}$$

are inverse algebra maps, where pr_1 *projects onto the identity component of the Δ-grading. Restriction is consequently an irreducible bijection on the entire ordinary block. Every lower character has a unique extension in the specified upper block.*

Proof. Use the Δ-grading of $\mathcal{O}R$. Fourier orthogonality places E in the identity component and gives $\operatorname{pr}_1(e) = E/|\Delta|$. For $x \in \mathcal{O}Re$,

$$|\Delta| \operatorname{pr}_1(x) = \sum_{\delta} \delta(x).$$

Multiplication by e kills every summand except x, because the idempotents $\delta(e)$ are orthogonal. Conversely $|\Delta| \operatorname{pr}_1(ae) = aE = a$. This proves (7.6). The upper algebra is indecomposable, so the lower central idempotent is primitive. The equivalence preserves the underlying ordinary representation on N, proving the character assertion. Uniqueness also gives covariance for every automorphism stabilizing the chosen upper block. □

Lemma 7.5. *Suppose*

$$(Y, N, \chi) \geq_b (H, M, \psi), \qquad M = N \cap H,$$

and β is a Y-invariant linear character of N. Then the same relation holds for $(\beta\chi, \beta|_M \psi)$.

Proof. Put $Q = Y/N$, choose $t : Q \to Y$, and set

$$a(x, y) = \beta(t(x)t(y)t(xy)^{-1}).$$

Invariance makes this a normalized cocycle with values in a finite cyclic group C. Form $\widehat{Q} = Q \times C$ with multiplication $(x, z)(y, w) = (xy, zw\, a(x, y))$, and let $\widehat{Y} = Y \times_Q \widehat{Q}$. Embed the original N by $n \mapsto (n, (1, 1))$. For $g = nt(x)$, the formula

$$\widehat{\beta}(g, (x, z)) = \beta(n) z^{-1}$$

is a genuine linear character of $\widehat{Y}$, because the cocycle in the numerator and denominator cancels in a product. Its restriction is β. The action images of Y and $\widehat{Y}$ on this same N are equal. Apply the block Butterfly theorem, multiply the two associated systems by $\widehat{\beta}$, and apply Butterfly back. Every intermediate ordinary coefficient receives the same linear twist; block induction commutes with it, and the full centralizer receives the same scalar on both sides. □

An invariant irreducible character extends over a cyclic quotient with its actual return: if yN has order r, choose an intertwiner U, compare U^r with $\rho(y^r)$, and rescale U by an r-th root of the scalar discrepancy. This works in characteristic zero even when $\ell \mid r$. With an induced-block anchor and equal ordinary centralizer scalars, Theorem 2.1 then gives every upper block relation. For an abelian quotient it applies to the genuine joint extensions constructed below.

7.2. **Full general linear and unitary factors at three**

Proposition 7.6. *Let $R = \mathrm{GL}_n^\epsilon(Q)$, $3 \nmid Q$, and let B be a unipotent 3-block or a central $3'$-linear twist of such a block. Its source normalizer supports a height-zero correspondence with full ordinary block-triple and regular source data, preserving every central scalar and every ordinary regular twist. The assertion holds in the full field–graph ambient, and in a product ambient with the actual return of each factor cycle. These are relations with base R and all of its ordinary upper characters. The prescribed strong central quotients of the natural special groups are supplied by their descent below; this proposition does not assert a separate strong relation on every central quotient of R.*

Proof. For $n = 1$ the source normalizer is the torus itself. For $R = \mathrm{GL}_2(2)$ or $\mathrm{GU}_2(2)$, the source torus contains the normal Sylow three-subgroup and its normalizer is R. Choose the identity correspondence and identical associated systems in these cases. This matches the whole actual ambient centralizer, including any inner graph lift. We may exclude these identity cases in the following construction. Put $h = \mathrm{ord}_3(\epsilon Q)$ and write $n = r + hw$, where the block has h-core κ of size r. Write χ_κ for its corresponding unipotent character. Its source is

$$L = \mathrm{GL}_r^\epsilon(Q) \times T_h^w, \qquad |T_h| = Q^h - \epsilon^h, \qquad \lambda = \chi_\kappa \otimes 1. \tag{7.7}$$

The core character has defect zero by the hook formula. For $h = 1$ the core is empty; for $h = 2$ its hook lengths are odd and the denominator has no factor 3. The source has central defect $A = (T_h)_3^w$ and its entire ordinary block is $\{\chi_\kappa \otimes \theta : \theta \in \mathrm{Irr}(A)\}$. More precisely,

$$L = A \times K, \qquad K = \mathrm{GL}_r^\epsilon(Q) \times (T_h)_{3'}^w = O^3(L).$$

For $h = 1$, the core factor is absent. For $h = 2$, the abelianization of the core factor has order dividing $Q - \epsilon$, which is prime to three; the small core factors have the same conclusion directly. Thus the displayed equality with $O^3(L)$ follows, and the core character on K has defect zero. This verifies the actual splitting hypothesis of Proposition 7.3. The integer h is the partition hook parameter. The algebraic split parameter is $\mathrm{ord}_3(Q)$, which need not equal h for unitary groups.

The natural representation gives $C_R(A) = L$ and the actual monomial normalizer with quotient $C_h \wr S_w$. Permutations of equal summands give a genuine Sylow 3-subgroup P of the relative quotient and $D = A \rtimes P$. Lemma 7.2 applies. The source pair and the argument of Proposition 7.3 prove the actual global and local block anchor. They also give the complete parameter set

$$(t, \eta), \quad t \in A^*, \quad \eta \in \mathrm{Irr}((C_h \wr S_w)_t), \quad [C_h \wr S_w : (C_h \wr S_w)_t]_3 \eta(1)_3 = 1, \tag{7.8}$$

modulo the actual monomial action. For $h = 1$, the signed hook formula gives this directly. If t has eigenvalue multiplicities m_c, then $C_R(t) = \prod_c \mathrm{GL}_{m_c}^\epsilon(Q)$. For a partition α of m, write χ_α for the unipotent character of $\mathrm{GL}_m^\epsilon(Q)$, η_α for the Specht character of S_m, and $h(b)$ for the hook length of a box b of α. Since $\epsilon Q \equiv 1 \pmod 3$, the lifting-the-exponent identity gives, for every $j \geq 1$,

$$v_3(Q^j - \epsilon^j) = v_3((\epsilon Q)^j - 1) = v_3(Q - \epsilon) + v_3(j).$$

The signed hook numerator and denominator each contain m factors, and the remaining power of Q is a three-adic unit. Therefore

$$v_3(\chi_\alpha(1)) = v_3(m!) - \sum_{b \in \alpha} v_3(h(b)) = v_3(\eta_\alpha(1)).$$

Conversely a degree prime to 3 has a semisimple centralizer containing a Sylow 3-subgroup, and its parameter is therefore in the diagonal A.

For $h = 2$, choose one member of each nonzero inverse colour pair. Then the little group is

$$(C_2 \wr S_{m_0}) \times \prod_{[c] \neq [0]} S_{m_{[c]}}.$$

Extend the 3-primary torus line trivially across the actual splitting. Its extension is unique among 3-primary lines because this little group's abelianization is a 2-group. In general a cyclic phase stabilizer of order $a \mid h$ is handled by assigning its actual lift v the unique 3-primary a-th root of $\theta(v^a)$. Tensor permutations on equal colour spaces then give the extensions for all ordinary coefficients. This is also the ordinary extension-map statement of [8, Corollary 4.7(b)]. Restriction to L, followed by division by this extension, recovers each local coefficient. The global inverse is (7.2).

We specify the full field action in the nonmatching linear case. If $Q = p^f \equiv -1 \pmod 3$, then $p \equiv -1 \pmod 3$ and f is odd. On a quadratic summand $K = \mathbb{F}_{Q^2}$, use a basis of $\mathbb{F}_{p^2}/\mathbb{F}_p$, put $\tau : x \mapsto x^Q$ and $\phi : x \mapsto x^p$. Then

$$\phi^f = \tau, \qquad \alpha = \tau\phi, \qquad \alpha^f = 1.$$

Thus α is the actual coefficient-field generator. The trace form $\mathrm{Tr}_{K/\mathbb{F}_Q}(xy)$ has a symmetric prime-field Gram matrix H; the graph map $\gamma_H(g) = H^{-1}g^{-T}H$ inverts the torus, fixes τ and the summand permutations, and commutes with α. For odd p, with $u^2 = d$ nonsquare, one can take

$$M_{a+bu} = \begin{pmatrix} a & db \\ b & a \end{pmatrix}, \quad H = \begin{pmatrix} 2 & 0 \\ 0 & 2d \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For $p = 2$, $u^2 + u + 1 = 0$ gives

$$M_{a+bu} = \begin{pmatrix} a & b \\ b & a+b \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Multiplication verifies $M_c^T H = H M_c$, $\tau^T H \tau = H$, and $\tau M_c \tau = M_{c^Q}$. The full quotient $C_f \times C_2$ is cyclic, and the cyclic extension construction therefore retains its entire 3-part and its quadratic return.

For the matching linear case the actual field–graph quotient is abelian. Local extensions are genuine tensor-permutation extensions: order equal-colour positions and match the j-th position under each colour automorphism. These matching permutations multiply exactly. Use the same Specht space for each equal partition label and induce the resulting representation from its colour inertia. Its restriction is the original irreducible local character. Globally [10, Remark 4.7 and its preceding proof] gives ordinary extension to the full field–graph inertia. For its pinned graph representative v_0, one has $v_0^2 = (-1)^{n+1}I$. A unipotent 3-block kills that central involution. Multiplying its graph operator by $\rho(v_0)^{-1}$ therefore gives the transpose-inverse operator with square 1, commuting with the prime-field operators. In a unitary factor the prime-field action already includes the graph return; its complete finite quotient is cyclic, and the cyclic extension construction applies to that quotient.

The ordinary source and global induction modules give the same scalar on $Z(R)$, including its 3-part. Their block anchor was proved before (7.8). Theorem 2.1, on the cyclic or abelian effective quotient just described, proves the full block relation for every ordinary coefficient. A central $3'$-parameter is a genuine linear twist. Its block inertia fixes that line; apply Lemma 7.5. Here twisting is performed on the entire global and local associated systems, including the extension on the normalizer, not just on the source character. This retains the value of a determinant line on a permutation lift and proves covariance for its 2-part as well. Finally tensor the systems and their actual return operators around factor cycles, and use the product, wreath and Butterfly construction of Theorem 5.1. For the endpoint, write $L = A \times K$, where $K = \mathrm{GL}_r^{\epsilon}(Q) \times (T_h)_{3'}^{w}$, and take the defect-zero character $\chi_\kappa \otimes 1_{(T_h)_{3'}^{w}}$ of K. The phase group C_h acts faithfully on A, and the independent coordinates give $C_{N_R(L)/K}(A) = A$. Since A is characteristic in D, $N_R(D) \le N_R(L)$. The normal-source theorem therefore refines the constructed system to $N_R(D)$, with the same ordinary coefficients and return operators. □

7.3. **Matching sources for special linear and unitary groups**

In natural dimension two, every odd nondefining Sylow subgroup is cyclic. We use [20, Theorem 1.1 and Definition 7.2(iii)]; the proof of that theorem chooses the actual subgroup $N_G(D)$. At the exceptional dimension-two parameters, the kernel from the universal cover to the natural group has order prime to the odd nondefining block prime. The prescribed quotients by the kernels of the ordinary central characters coincide, and the actual defect normalizers commute with this central quotient. Thus the same strong system applies to the natural perfect group.

There is an actual ambient kernel in rank one which must be retained. For $j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, inverse transpose on $\mathrm{GL}_2(q)$ has the form

$$\gamma(g) = \det(g)^{-1} j g j^{-1}.$$

Consequently $j^{-1}\gamma$ centralizes $\mathrm{SL}_2(q)$, while it can still act on the regular centre. Instead of applying a criterion which assumes that this centralizer is just the regular centre, pull the cyclic-defect strong system back along the actual conjugation action, using the block Butterfly theorem [27, Theorem 1.10]. The new local group is the full inverse image of the old local action image. It therefore contains the entire ambient kernel, with its same ordinary scalar on both sides, and has lower group $N_G(D)$. The unitary rank-one realization uses the same argument after identifying its derived group with $\mathrm{SL}_2(q)$. Perform this pullback first on $G/(\ker\chi \cap Z(G))$. Perfection prevents an additional action kernel on that quotient.

We define the maps on every regular fibre by the associated systems of this very pullback, using the same ordinary inverse-factor-set coefficient globally and locally. Thus their specified block anchors, quotient-line covariance and whole-centre scalars are inherited from one system; no separately chosen regular extension is substituted. The nonperfect leaves use the identity systems described at the end of the induction. We may consequently take $n \geq 3$ below.

Let $G = \mathrm{SL}_n^{\epsilon}(q)$ and first suppose $\ell \mid q - \epsilon$. There is one unipotent ℓ-block, the principal block. Put $a = v_\ell(q - \epsilon)$, let T be the diagonal split or unitary torus, and put $M_0 = N_G(T)$. The maximal-defect correspondence of [9, Theorem 1.1 and Sections 4–6] gives the required ordinary system at M_0, with the actual Sylow-normalizer exception treated below. We refine that correspondence to the actual defect normalizer; this refinement is needed later for rational type-A factors.

Write the toral Sylow subgroup additively as

$$A = \{(x_1, \dots, x_n) \in (\mathbb{Z}/\ell^a)^n : \textstyle\sum_i x_i = 0\}, \qquad Z_\ell = \{(c, \dots, c) : nc = 0\}. \tag{7.9}$$

The action of S_n on A/Z_ℓ is faithful except when $(n, \ell, a) = (3, 3, 1)$. For $n \geq 5$, a permutation acting trivially makes

$$e_{\sigma i} - e_{\sigma j} - e_i + e_j$$

a constant vector. Its support has size at most four, so it is zero; varying i, j gives $\sigma = 1$. For $n = 4$, $Z_\ell = 1$, and the same difference argument applies. For $n = 3$ only $\ell = 3$ needs attention; the second difference $(-1, 2, -1)$ is constant precisely at $a = 1$. For $n = 2$, the nonidentity permutation acts by inversion and is nontrivial at odd ℓ.

Assume for the moment that G is perfect and we are outside that exception. A principal height-zero character has ℓ'-degree. Its determinant is trivial, so its scalar on every central ℓ-element is 1. Its central ℓ'-character is also trivial because its block is principal. Thus it kills $Z(G)$, as does its paired local character. Work first in $S = G/Z(G)$. The faithfulness just proved gives, with bars denoting images,

$$C_{\overline{M}_0}(\overline{A}) = \overline{T}, \qquad C_{\overline{M}_0/O_{\ell'}(\overline{T})}(\overline{A}) = \overline{A}. \tag{7.10}$$

Apply Theorem 3.1 with defect-zero source $1_{O_{\ell'}(\overline{T})}$, in the full block/defect automorphism ambient. It refines the specified principal local block to the actual Sylow normalizer. If $P \in \mathrm{Syl}_\ell(G)$, then

P is the unique Sylow subgroup of $PZ(G)$, and hence

$$N_{G/Z(G)}(PZ(G)/Z(G)) = N_G(P)/Z(G). \tag{7.11}$$

Composition on this same quotient, followed by ordinary inflation, supplies the strong actual-normalizer system. For the exceptional triple $(3,3,1)$, the direct Sylow-normalizer correspondence is [9, Proposition 6.3], except for the small matrix group $\mathrm{SL}_3(4)$, which is constructed below.

7.4. **A natural-cover criterion and the small fields**

We use the following form of the regular-embedding criterion. The point of stating it is that a natural group need not be the universal cover of its simple quotient.

Lemma 7.7 (Criterion on an actual perfect central cover)**.** *Let G be perfect, let $G \lhd \widetilde{G}$ with $\widetilde{G}/G$ abelian, let an abelian group E act on $\widetilde{G}$, and suppose that $C_{\widetilde{G}E}(G) = Z(\widetilde{G})$. Let $D \le G$, and let $N_G(D) \le M \le G$ be stable under the relevant D-stabilizing automorphisms. Put $\widetilde{M} = MN_{\widetilde{G}}(D)$. Suppose $\chi \in \mathrm{Irr}_0(G)$ is paired with $\psi \in \mathrm{Irr}_0(M)$, their blocks have the same actual defect group D, the local block induces the global block, and the following statements hold.*

(i) *The global and local inertia groups factor as in* [9, Theorem 4.1(iii),(iv)]*; χ and ψ extend to their respective field–graph inertia groups.*

(ii) *Their inertia groups have the same image modulo the lower normal group:*

$$(\widetilde{G}E)_\chi = G\big(\widetilde{M}N_{GE}(D)\big)_\psi.$$

(iii) *There are extensions $\widetilde{\chi}$ to $\widetilde{G}_\chi$ and $\widetilde{\psi}$ to $\widetilde{M}_\psi$ lying over the same linear character of $Z(\widetilde{G})$. For every $a \in N_{GE}(D)_\psi$, including the actual elements ge with $g \in G$, $e \in E$, there is a line $\nu_a \in \mathrm{Irr}(\widetilde{G}_\chi/G)$ such that*

$$\widetilde{\chi}^{,a} = \nu_a\widetilde{\chi}, \qquad \widetilde{\psi}^{,a} = \nu_a|_{\widetilde{M}_\psi}\widetilde{\psi}.$$

(iv) *Every regular intermediate group has the prescribed block anchor:*

$$\mathrm{bl}(\widetilde{\chi}|_J) = \mathrm{bl}(\widetilde{\psi}|_{J\cap\widetilde{M}})^{,J} \qquad (G \le J \le \widetilde{G}_\chi).$$

Then, with $Z = \ker\chi \cap Z(G)$, these data give the block character-triple relation on G/Z and M/Z, for their actual ambient inertia groups. The relation includes every ordinary coefficient over every intermediate subgroup. Universality of the central cover is not needed.

Proof. Conditions (1)–(3) are the projective construction in [9, Proposition 4.2] and [27, Theorem 2.1(i)–(iv)]. The displayed common lines make the global and local cross factors identical. Condition (4) is exactly the replacement for equation (4.2) in that construction; no block is subsequently chosen by an arbitrary extension.

Let ρ afford χ and let $\mathcal{P}$ be its projective system on the automorphism inertia. Write

$$\rho(g) = s(g)\mathcal{P}(\mathrm{ad}\, g).$$

Choose the same inner section on the local group. In a finite scalar extension containing the factor-set values and the scalars $s(g)$, the map

$$g \longmapsto (\mathrm{ad}\, g, s(g)) \tag{7.12}$$

is a homomorphism with kernel $\ker\chi \cap Z(G)$. The multiplication law follows from the actual matrices $\rho(g)$; conjugation by an automorphism proves normality of its image, since the scalar kernel acts faithfully. Any automorphism trivial on this central quotient has central deviation $g \mapsto g^{-1}g^a$, a homomorphism from perfect G to its centre, and is therefore trivial. The complete ambient centralizer is consequently the scalar one already matched. If C is the added scalar group, the corresponding block on the central product $(G/Z)C$ has actual defect $C_\ell\overline{D}$. Its intersection with the embedded G/Z is $\overline{D}$, so its normalizer is the normalizer of $\overline{D}$; their centralizers are also equal because C is central.

Let Y be this finite scalar extension and let J be the preimage of the action of $\widetilde{G}_{\chi}$. Then J contains the embedded G/Z and the whole scalar centralizer, is normal in Y, and Y/J is a quotient of a subgroup of the abelian group E. The prescribed genuine systems on the regular layer lift the same matrices as $\widetilde{\chi}, \widetilde{\psi}$, multiplied by the same scalar. The class-sum comparison in the last paragraph of [9, Proposition 4.2] therefore transports (4) to every subgroup between the embedded G/Z and J. That comparison uses these matrix identities and central block descent; its use of the covering group has already been replaced by the embedding (7.12).

Apply Corollary 2.2 in Y. It corrects the local genuine system by an ℓ'-line trivial on J, and hence retains the specified regular extensions. Its integral class-sum argument proves the block identity for all ordinary coefficients, even when their degrees are divisible by ℓ. Inverse-factor-set coefficients are the scalar-killing slice of these ordinary coefficients, as in the proof of Theorem 2.1. Finally the block Butterfly theorem transports this system to the actual ambient inertia, whose conjugation image is the same; its local group is the complete inverse image of the actual local action image. Together with [24, Lemma 3.12], this gives the asserted strong relation on the actual central quotient. □

Lemma 7.8 (Proper Levi transfer on a natural type-A cover)**.** *Let $G = \mathrm{SL}_n^{\epsilon}(q)$, $n \geq 3$, be perfect with nonabelian simple central quotient, let $\widetilde{G} = \mathrm{GL}_n^{\epsilon}(q)$, and let E be its actual abelian standard field–graph automorphism group. Let B have semisimple ℓ'-label s, and suppose that the canonical Levi $\mathbf{L}^*$ containing $C^{\circ}_{\mathbf{G}^*}(s)C_{\mathbf{G}^{*F^*}}(s)$ is proper. Let c be the corresponding block of the dual finite Levi L, with its actual defect group D. Assume that c, and its conjugate source blocks, have the systems of Theorem 5.1 in the actual regular and corrected geometric automorphism ambients. In particular assume the whole ordinary regular fibres, all quotient-line covariance, the whole regular-centre scalars, extension transfer at every character inertia, and the specified regular Clifford-inertia block equalities. Then B has the full natural-group system with endpoint $N_G(D)$, including its prescribed strong central quotients and its whole ordinary regular fibres. The finite group G need not be a universal covering group.*

Proof. We separate the geometric construction from the finite-cover criterion. Theorem 2.28 of [26] gives the integral splendid complex $\mathcal{C}$ for G, L; its Brauer construction at ΔD and normalizer induction give the local complex for $N_G(D), N_L(D)$. These constructions use the specified simply connected algebraic group, and impose no condition on its finite Schur multiplier. The corresponding Brauer pairs have the same actual subgroup D, by [26, Theorem 1.3, Proposition 1.4]. Write R, R_D for their ordinary bijections and set

$$\Omega = R_D \phi_c R^{-1}.$$

The top regular complexes, composed with the assumed entire source fibre map, likewise define

$$\widetilde{\Omega} : \mathrm{Irr}(\widetilde{G} \mid \mathrm{Irr}_0(B)) \longrightarrow \mathrm{Irr}(N_{\widetilde{G}}(D) \mid \mathrm{Irr}_0(b_D)).$$

Restriction and induction commute with these functors by [26, Remark 1.8]. Thus these are exactly the displayed fibres, including regular characters of positive height. The inverse is obtained by reversing all three maps.

Lemma 5.2 makes the canonical Levi and its rational source series stable under the required field generator. The corrected diagram generator of [26, Proposition 4.8] commutes with it. The ordinary extension construction from Proposition 5.3 uses Theorem 2.27 and Lemma 5.6 of that paper for parabolic independence, Proposition 5.7 for the full field action, and Proposition 5.10 and Theorem 5.2 for its local version. The cyclic module-extension argument is over $\mathbb{K}$. Its Morita hypothesis is Theorem 2.28 under the displayed rational total-centralizer containment. Consequently it transfers genuine extensions in both directions for the full corrected geometric inertia $\mathcal{A}$, including its ℓ-part. This is the geometric portion of the earlier proof, before its use of the universal-cover criterion; it remains valid for the present natural group.

Here is the passage to the actual standard ambient. By [10, Theorem 4.1], choose a regular conjugate $\chi \in \mathrm{Irr}_0(B)$, transporting B, c, D simultaneously and renaming them, with

$$(\widetilde{G}E)_\chi = \widetilde{G}_\chi E_\chi, \qquad \chi \text{ extends to } GE_\chi.$$

The extension comparisons in [27, Lemmas 2.3–2.4] apply on these actual groups. Their proofs compare an extension on the cyclic field preimage and its invariance under the standard graph lift and the corrected lift $\mathrm{ad}(x)\gamma$, with x fixed by the field generator. Locally the two lifts are $y\,\mathrm{ad}(x)\gamma$ and $yx\gamma$, after the actual defect conjugator y is inserted. These are statements about these group actions and their return elements, and use neither a universal central extension nor a presentation of the Schur multiplier. They transfer the displayed extension to the geometric ambient. Transfer through R^{-1}, the assumed source system, and R_D, followed by the local comparison, therefore proves that $\psi = \Omega(\chi)$ extends to $N_{GE}(D)_\psi$.

Equivariance and defect conjugacy give

$$(\widetilde{G}E)_\chi = G\big(N_{\widetilde{G}}(D)N_{GE}(D)\big)_\psi.$$

They also give the local inertia factorization. Indeed, write an element of its right-hand stabilizer as te, where $t \in \widetilde{G}_\chi$, $e \in E_\chi$. Correct e by an element of G carrying D^e back to D; equivariance makes this corrected element fix ψ. The remaining factor lies in $N_{\widetilde{G}}(D)_\psi$. This proves (1)–(2) of Lemma 7.7.

Put $I = \widetilde{G}_\chi$ and $J = N_I(D)$. Both I/G and $J/N_G(D)$ are the same cyclic group. Choose an extension χ_0 of χ to I, and let ψ_0 be the Clifford correspondent of $\widetilde{\Omega}(\mathrm{Ind}_I^{\widetilde{G}}\,\chi_0)$. The restriction compatibility makes ψ_0 an extension of ψ. The isotypy covariance of the geometric maps is [26, Lemma 2.23]; their quotient-line covariance is [27, Lemma 2.9]. Combined with the assumed source covariance, these make $\widetilde{\Omega}$ equivariant for the actual standard defect stabilizer as well. Indeed, a standard lift and its corrected geometric lift differ by an actual G-inner action; after both lifts are corrected to preserve D, their local difference lies in $N_G(D)$. These inner actions fix the respective ordinary characters.

Explicitly, let $a \in N_{GE}(D)_\psi$. It fixes χ and normalizes I, J. There is a unique $\nu_a \in \mathrm{Irr}(I/G)$ with $\chi_0^a = \nu_a\chi_0$. Extend this line to $\lambda_a \in \mathrm{Irr}(\widetilde{G}/G)$, and write $\widetilde{M} = N_{\widetilde{G}}(D)$. Actual equivariance, quotient-line covariance, and Clifford induction give

$$\begin{aligned}\mathrm{Ind}_J^{\widetilde{M}}(\psi_0^a) &= \widetilde{\Omega}\big((\mathrm{Ind}_I^{\widetilde{G}}\,\chi_0)^a\big)\\ &= \lambda_a|_{\widetilde{M}}\,\widetilde{\Omega}(\mathrm{Ind}_I^{\widetilde{G}}\,\chi_0)\\ &= \mathrm{Ind}_J^{\widetilde{M}}(\nu_a|_J\,\psi_0).\end{aligned}$$

The two inducing characters lie over the same ψ, so Clifford uniqueness gives $\psi_0^a = \nu_a|_J\psi_0$. This verifies the common-line assertion for every actual ge in the required inertia, including its defect correction. The diagonal action on the complexes and [27, Lemma 2.10] preserve the whole $Z(\widetilde{G})$-character. This proves (3). The corrected geometric ambient has only been used to transport extensions and maps. Its centralizer of G may be larger, as in [27, Remark 2.5]; the criterion is applied to the standard ambient $\widetilde{G}E$ after the two extension comparisons.

The block anchor is integral and independent of that ordinary field extension. Set $I_L = I \cap \widetilde{L}$, so that $I = GI_L$ and $I/G = I_L/L$, and form

$$\mathcal{C}_I = \mathrm{Ind}_{(G\times L^{\mathrm{opp}})\Delta I_L}^{I\times I_L^{\mathrm{opp}}}(\mathcal{C}c).$$

The diagonal regular action exists over $\mathcal{O}$. [3, Lemma 7.4] and [26, Lemma 1.9, Theorem 1.7] give its Rickard equivalence and concentrated Morita cohomology, also when $\ell \mid |I : G|$. Its Brauer complex gives the local correspondence between $N_{I_L}(D)$ and J. Commutation with induction identifies the preimages of χ_0, ψ_0 as the specified source Clifford correspondents. The assumed

source anchor and Harris–Knörr compatibility [26, Remark 1.21] therefore give

$$\mathrm{bl}(\psi_0)^I = \mathrm{bl}(\chi_0).$$

The normal-subgroup argument of [27, Lemma 2.2] gives this equality after restriction to every $G \le I_0 \le I$, proving (4). Its only relevant quotient assumption is that I/G is abelian.

Finally [10, Section 3.2, equation (3.1)] gives $C_{\widetilde{GE}}(G) = Z(\widetilde{G})$ for these perfect natural groups of dimension at least three. All hypotheses of Lemma 7.7 are now verified on the actual group. That lemma constructs the relation on $G/(\ker\chi \cap Z(G))$, including every ordinary coefficient at every intermediate group. Its anchored correction fixes χ_0, ψ_0 on the regular layer. Since quotient-line covariance determines the entire cyclic regular fibre from such a pair, it also retains the particular map $\widetilde{\Omega}$. Transporting the construction on orbit representatives gives the single equivariant natural system asserted. □

For $G = \mathrm{SL}_n(4)$ or $\mathrm{SU}_n(2)$, $n \ge 4$, let

$$R_0 = \ker(\mathbb{F}_3^n \xrightarrow{\sum} \mathbb{F}_3), \qquad P = R_0 \rtimes P_n, \quad P_n \in \mathrm{Syl}_3(S_n).$$

The group order gives $|P| = |G|_3$. If an abelian subgroup $U \le P$ has nontrivial permutation image H, write its orbit sizes as $s_1, \dots, s_o$. An abelian transitive permutation quotient is regular, so

$$|H| \le \prod_i s_i, \qquad \log_3 |H| \le (n-o)/2.$$

The dimension of R_0^H is $o-1$ if there is a fixed coordinate, and o otherwise. In the first case $\log_3 |U| \le (n+o)/2 - 1 < n-1$; in the second, $o \le n/3$, so $\log_3 |U| \le 2n/3 < n-1$. Thus R_0 is the unique largest abelian subgroup of P. Its distinct weights give the actual normalizer

$$M_0 = (C_3 \wr S_n) \cap G,$$

which contains $N_G(P)$. The regular quotient is C_3.

For every perfect natural group of dimension at least three used here, [10, Section 3.2, equation (3.1)] identifies the actual regular field–graph ambient modulo its centre with its conjugation image on G. Hence its centralizer of G is exactly the regular centre. In the unitary case its field–graph group is the actual cyclic group, with the graph return already a field power. This verifies the centralizer hypothesis of the natural-cover criterion, including $\mathrm{SU}_4(2)$, $\mathrm{SU}_6(2)$, and the Hessian case below.

The natural-group GGGR and regular correspondence theorems [10, Theorem 4.1, Theorems 4.4–4.6, and Theorem 6.1] provide the global extension, splitting and regular maps. In characteristic two the graph-pinning correction has square one and is prime-field fixed. There is also a direct local splitting argument. On the monomial group, entrywise squaring ϕ and inverse transpose γ coincide. Hence $\delta = \phi\gamma$ centralizes the entire local group, while ϕ inverts its diagonal quotient C_3. On a three-element constituent orbit, an invariant involution has a fixed constituent. On a regular-extension torsor it acts as $t \mapsto a - t$, which has a unique fixed point. Choose that constituent and extension and extend across the cyclic involution. For $\mathrm{SL}_n(4)$, let the additional local centralizing element δ act identically. For $\mathrm{SU}_n(2)$, field and graph are already the same actual automorphism of the group, so the actual field–graph group is cyclic; no second independent involution is introduced. This proves the full local inertia assertion with the actual relations.

Both $C_G(P)$ and $C_{M_0}(R_0) = R_0$ are 3-groups. The Sylow normalizer therefore has one block, and G has one full-defect block. The local monomial group also has one 3-block. Its regular 3-extension has a unique covering block, the principal block. Consequently the regular block anchor holds even for upper characters of positive height. The natural-cover criterion applies, and (7.10)–(7.11) gives the actual endpoint. This includes the natural groups $\mathrm{SU}_4(2)$ and $\mathrm{SU}_6(2)$.

7.5. **The Hessian endpoint for** $\mathrm{SL}_3(4)$

Let $\omega^2+\omega+1=0$ in $\mathbb{F}_4$, and on $\mathbb{F}_4^3$ put

$$d=\mathrm{diag}(1,\omega,\omega^2),\quad p(e_i)=e_{i+1},\quad E=\langle d,p\rangle,\quad F=(\omega^{ij})_{0\le i,j\le 2},\quad u=\mathrm{diag}(1,1,\omega).$$

Indices are read modulo three. The commutator is a nontrivial scalar, so E is extraspecial of order 27, exponent 3. Its distinct d-eigenspaces and transitive p-action make it absolutely irreducible. Let the bar denote entrywise squaring in $\mathbb{F}_4$. Direct multiplication gives

$$FdF^{-1}=p^2,\quad FpF^{-1}=d,\quad upu^{-1}=\omega^2dp,\quad F^2=3J,\quad \overline{F}^{T}F=3I,\quad \overline{u}^{T}u=I, \tag{7.13}$$

where J exchanges the last two basis vectors. The actions of F,u generate $\mathrm{SL}_2(3)$ on $E/Z(E)$. A normalizer fixes the scalar commutator, so has symplectic image; its kernel is E, because an automorphism trivial on both the centre and central quotient is inner. Thus

$$N_{\mathrm{GL}_3(4)}(E)=\langle E,F,u\rangle=\mathrm{GU}_3(2),\qquad N_{\mathrm{SL}_3(4)}(E)=\mathrm{SU}_3(2)=E:Q_8. \tag{7.14}$$

In characteristic two, (7.13) is unitary, and the two matrix groups have orders 648 and 216, respectively.

We give the complete regular ordinary fibre. The three unipotent characters of $\mathrm{GL}_3(4)$, of degrees $1,20,64$, have three determinant twists each. The semisimple parameter with eigenvalues $1,\omega,\omega^2$ has degree 105 and is determinant-self-twisted; its restriction splits into three degree-35 characters. These are all the lower characters of degree prime to 3. Indeed a split multiplicity pattern $2+1$ has centralizer 3-part 27, compared with $|\mathrm{GL}_3(4)|_3=81$, and has trivial scalar stabilizer. A quadratic or cubic field factor has centralizer 3-part at most 9; even division of a restriction degree by three leaves a factor three. This exhausts the semisimple patterns in dimension three. Hence the lower global degrees are

$$1,\ 20,\ 64,\ 35,\ 35,\ 35. \tag{7.15}$$

Locally the trivial character of E gives the four lines and the degree-two character of Q_8. The eight nonzero characters of $E/Z(E)$ form a free Q_8-orbit and give one degree-eight character. The nonlinear E-characters have degree three and contribute no further height-zero characters. Thus the local degrees are

$$1,\ 1,\ 1,\ 1,\ 2,\ 8. \tag{7.16}$$

The regular diagonal C_3 fixes three characters on each side and cycles the other three. The field and graph involutions induce the same transposition on that orbit; their product centralizes the entire actual local group in (7.14). Choose invariant regular extensions on the three fixed torsors, using $t\mapsto a-t$, pair them and all determinant twists, and pair the regular character of degree 105 with the induced local character of degree three. This pairs all ten regular ordinary characters, including the upper character of positive height. All have trivial scalar on the central C_3, globally by perfection and (7.15), locally by the construction through $E/Z(E)$.

The centralizer $C_G(E)=Z(E)$ and the normal Sylow E of the local group show that the local block is unique. Brauer correspondence identifies the unique global full-defect block, and the regular 3-extensions have unique principal covering blocks. The natural-cover criterion therefore proves the full strong relation at the exact group (7.14). The same realization proves that $\mathrm{SU}_3(2)$ has normal Sylow three-subgroup and has the identity source system. The group $\mathrm{SL}_2(2)=\mathrm{SU}_2(2)=S_3$ is handled in the same way.

7.6. **Nonmatching unipotent sources**

Now let $e=\mathrm{ord}_\ell(\epsilon q)>1$. Write $n=r+ew$, with e-core κ of size r, and work in $\widetilde{G}=\mathrm{GL}_n^\epsilon(q)$. If $w=0$, the block of $\widetilde{G}$ has defect zero. Since $\ell\nmid|\widetilde{G}:G|=q-\epsilon$, every covered block of G also has defect zero; its identity correspondence gives the required character-triple system. Assume henceforth that $w>0$. The source is

$$\widetilde{L}=\mathrm{GL}_r^\epsilon(q)\times T_e^w,\qquad |T_e|=q^e-\epsilon^e,\qquad \widetilde{\lambda}=\kappa\otimes 1.$$

For a unitary group, $T_e = \mathrm{GU}_1(q^e)$ if e is odd and $T_e = \mathrm{GL}_1(q^e)$ if e is even. Its determinant map onto $\mathrm{GU}_1(q)$ is surjective. For odd e, the exponent on a norm-one element is the product

$$\frac{q^e+1}{q+1}\frac{q^e-1}{q-1};$$

the second factor is 1 modulo $q+1$. For $e = 2d$, the action on a space and its dual has determinant

$$\mathrm{Norm}_{\mathbb{F}_{q^e}/\mathbb{F}_{q^2}}(x)^{1-q} = x^{-(q^e-1)/(q+1)}.$$

This proves surjectivity in both cases. In the linear case it is the ordinary norm map. Since $\ell \nmid q-\epsilon$, surjectivity already holds on the ℓ'-part.

Consequently, for $L = \widetilde{L} \cap G$,

$$L = A \times K_0, \qquad A = (T_e)^w_\ell, \qquad K_0 = (\mathrm{GL}^\epsilon_r(q) \times (T_e)^w_{\ell'}) \cap G. \tag{7.17}$$

There is no augmentation relation on A. The e-core character has defect zero; no nontrivial determinant twist can fix its product with the trivial active-torus line, because the latter determinant is onto. Cyclic Clifford theory therefore makes its restriction κ_0 to K_0 irreducible and defect zero. It is invariant under the actual relative normalizer and the source field–graph action. Distinct independent coordinate weights give

$$C_G(A) = L, \quad N_G(A) = N_G(L) = M, \quad C_{M/K_0}(A) = A. \tag{7.18}$$

The relative phase group is C_e, acting faithfully on each coordinate; its order is prime to ℓ. Genuine odd-order block permutations give $D = A \rtimes P$, $P \in \mathrm{Syl}_\ell(S_w)$. They have determinant one and are isometries in the unitary model. The actual source pair and the degree calculation of Proposition 7.3 prove that D is a global defect. Lemma 7.2 gives $N_G(D) \le M$.

We verify the whole ordinary regular restriction. The determinant group $\widetilde{G}/G$ has ℓ'-order. A nontrivial determinant line changes the central ℓ'-semisimple label of a regular unipotent block. Lemma 7.4 therefore identifies that whole block algebra with its lower block algebra. The same argument applies locally, since the active torus detects every determinant line. Moreover, if the quotient has order m, its Fourier homogeneous central unit u, normalized by $u^m = E$, gives

$$\mathcal{O}\widetilde{G}E \simeq \mathcal{O}GE \otimes \mathcal{O}C_m.$$

In particular every covering block in every intermediate regular group is regular-invariant. For $\ell \ge 5$, these are the additional block hypotheses of [8, Corollary 6.1]; its condition $\ell \nmid 3q(q-\epsilon)$ is satisfied. For the strengthened conclusion we use the inputs of that proof. The bijection in [8, Theorem 5.6] preserves the whole ordinary central character, all determinant lines, and the specified induced block. Since the determinant quotient has ℓ'-order and Fourier restriction is irreducible, its height-zero domain is exactly the union of all regular fibres over the lower height-zero set. Thus no additional upper characters are omitted. The actual local inertia factorization and extension are [8, Theorem 1.2(b), Corollary 4.7]; the global statements are [10, Theorem 4.1]. Here the automorphism group is the actual image E_0 in [8, Theorem 1.2], so the return relation of a rational torus, including a unitary graph return, is already imposed. The regular block invariance just proved and Harris–Knörr give every intermediate block anchor exactly as in [9, Proposition 4.2, equation (4.2)]. The twist-equivariant upper bijection supplies the common cross-factor lines of Lemma 7.7. Its other hypotheses are the common actual defect established above, the cyclic determinant quotient, and the actual ambient centralizer identified in [10, Section 3.2, equation (3.1)]. Consequently that lemma supplies the strong natural-cover system and every ordinary intermediate coefficient. This argument uses neither universality of the natural cover nor the simple-group conclusion of [8, Corollary 6.1] as a substitute for these data.

For $\ell = 3$, $e = 2$. Use Proposition 7.6 instead. In particular, (7.2) supplies the same-core ordinary inverse for every t, and the full regular ambient uses the actual quadratic return described there. In the unitary nonmatching case the geometric parameter is $\mathrm{ord}_3(q) = 1$; the

algebraic source is a one-split Levi with ordinary cuspidal 2-core. The source therefore satisfies the algebraic cuspidality hypothesis. Fourier restriction supplies the full regular fibre and every diagonal intermediate block anchor, by the same Harris–Knörr argument. The natural-cover criterion gives the strong lower map to M also at three.

Finally apply Theorem 3.1 to (7.17)–(7.18), with κ_0, once in the full automorphism stabilizer. It refines M to $N_M(D) = N_G(D)$. Here $Z(G)$ is an ℓ'-group, lies in K_0, and is killed by the unipotent source; perform the construction on the prescribed central quotient first. Composition gives the required exact endpoint and all ordinary upper maps.

7.7. **All blocks and strict dimension induction**

We prove the type-A assertion simultaneously for every field and sign by induction on n. For a non-strictly-quasi-isolated parameter, choose the proper rational total-centralizer Levi. Its derived factors have natural dimension less than n. Apply Theorem 5.1, including its actual central multiplication and moving block inertias, then Lemma 7.8. Thus the proper-Levi step uses the actual natural-cover criterion even when the natural group is not universal. For universal covers it also verifies the all-ordinary source hypotheses of [27, Theorem 2.12].

We give the additional strictly quasi-isolated step. Let $s \neq 1$ be a strictly quasi-isolated semisimple ℓ'-element of the projective dual. Its rational component group has order m dividing $\gcd(n, q-\epsilon)$, and m is prime to ℓ. Indeed, for a lift s_0, choose an ℓ'-integer k with $s_0^k = zI$. A scalar component $s_0 \mapsto cs_0$ satisfies $c^k = 1$, which proves the prime assertion. Choose a prime $r \mid m$. [28, Lemmas 6.1–6.2] gives a proper Levi L' and an actual group N', with

$$N'/L' \simeq C_r.$$

Its derived natural factors have dimension $n/r < n$. The original Jordan Levi L lies in L', and the actual defect D lies in L, because the component extension has ℓ'-order. The induction hypothesis and Theorem 5.1 give the complete source system on L'.

There are three covering situations. If a covered block c_1 is stable and has one cover, [27, Proposition 1.12] gives the selected block of N'. If it has r covers, retain all their ordinary colours and the particular cover selected by the Jordan equivalence; the source subgroup is the one specified in [28, Definition 10.5]. If c_1 moves, its inertia is L'; use Fong–Reynolds globally and locally. Brauer's first main theorem identifies the actual local block inertia, and [24, Theorem 3.14] induces the associated systems at every intermediate ordinary group. The block Frattini factorizations are obtained in these inertias by correcting conjugate defect groups.

The hypotheses on the local geometric module are as follows. In the untwisted nonmatching case, $\ell \nmid 2|H^1(F, Z(\mathbf{G}))|$; use [28, Theorem 9.2]. There are two rational Levi possibilities. If L' is one-split, its defining complex is the permutation module and its Brauer quotient is the local module of [28, Lemma 8.3]. If L' is not one-split, [28, Lemma 6.11] gives the abelian ambient quotient; Lemmas 8.2–8.3 then give the same local extension. Untwistedness of G alone does not imply that this rational Levi is one-split. In the matching case the source on the original L is a central ℓ'-twist of a principal source, so D is Sylow in this original L. The actual equality

$$N_{\widetilde{N}'}(D)\,\widetilde{L}' = \widetilde{N}'$$

then follows from the normalizer factorization and Sylow Frattini in L, and verifies the generation condition of Lemma 8.2. For the twisted nonmatching case, [28, Theorem 10.6] applies with $Q = D$; the extra generation condition of Lemma 8.2 is explicitly an untwisted condition.

We spell out the upper maps used here. On an inertia subgroup J of a source character θ, choose the associated projective systems $\mathcal{P}_\theta$ and $\mathcal{P}_{\phi(\theta)}$ with their common factor set α. For every irreducible ordinary projective coefficient $\mathcal{R}$ of factor set α^{-1}, set

$$\operatorname{tr}(\mathcal{P}_\theta \otimes \mathcal{R}) \longmapsto \operatorname{tr}(\mathcal{P}_{\phi(\theta)} \otimes \mathcal{R}),$$

and then apply ordinary Clifford induction to J. Restriction to the lower group recovers the source orbit and division by the associated system recovers $\mathcal{R}$; hence this map and its inverse are

defined on the whole ordinary fibre. Do this on all covered block colours before choosing the particular Jordan cover. In the multiple-cover case, this also supplies the characters of $\widetilde{N}'$ above $T = L'$, not only those of $\widetilde{T}$. In the moving case, the same construction is made in the block inertia and is then induced on both sides. The block-triple relation on that inertia, followed by [24, Theorem 3.14], preserves the chosen induced block. Thus the domain is exactly

$$\mathrm{Irr}(\widetilde{N}' \mid \mathrm{Irr}_0(T, \widehat{c})),$$

as required by the published [28, Theorem 10.6(i)].

Multiplying $\mathcal{R}$ by a quotient line proves covariance for every ordinary line. The matched scalar on the full regular centre proves the central-character clause. A genuine extension is precisely an appropriate degree-one coefficient, and is therefore transported as a genuine extension. Finally the common block-triple relation supplies the block equality for every coefficient and each Clifford correspondent; in the matching or twisted case it in particular gives the top regular-block equality in [28, Theorem 10.6(iii)]. These are the three actual hypotheses of the cited reduction, including upper characters of positive height. If R and R_D denote the actual global and local ordinary Jordan maps, the result is

$$\Omega = R_D \phi R^{-1}, \qquad \Omega^{-1} = R \phi^{-1} R_D^{-1}. \tag{7.19}$$

The same-complex Brauer comparison supplies its prescribed regular block anchor. The strong quotient follows from [28, Theorem 9.1] for universal covers and from the natural-cover construction above for the exceptional natural parameters.

We finish the small natural boundaries of this induction. The nonperfect leaves are $\mathrm{SL}_2(2) = \mathrm{SU}_2(2) = S_3$, $\mathrm{SU}_3(2) = 3^{1+2} : Q_8$, and $\mathrm{SL}_2(3) = \mathrm{SU}_2(3)$. For the first two, three is the only odd nondefining divisor of the order, and their Sylow three-subgroups are normal and therefore characteristic. Each centralizer of that subgroup is a three-group, so each group has one three-block, of full defect. Its actual defect normalizer is the whole group. Globally and locally take the identical associated projective system in every actual factor ambient; the same ordinary coefficients then give the identity on every regular fibre, with every central scalar and return operator retained. This supplies full natural-group data. The last group has order 24, so no odd nondefining prime divides its order. At every such prime all its blocks have defect zero, and the same identity construction applies with $D = 1$. Primes not dividing the orders of the first two groups are handled identically. When a subsequent regular extension has a larger defect, the second stage of Theorem 5.1 passes from the normalizer of the natural factor defect to the actual enlarged defect normalizer; it is not identified with the whole regular group. The exceptional natural type-A parameters are

$$\mathrm{SL}_2(4),\ \mathrm{SL}_3(2),\ \mathrm{SL}_2(9),\ \mathrm{SL}_4(2),\ \mathrm{SU}_4(2),\ \mathrm{SL}_3(4),\ \mathrm{SU}_6(2),\ \mathrm{SU}_4(3).$$

At $\ell \geq 5$ their relevant natural Sylow subgroups are cyclic, as follows from their group orders. The kernel from their universal cover has only prime divisors $2, 3$. For its ℓ'-kernel Y, $\mathcal{O}Ue_Y \simeq \mathcal{O}(U/Y)$, with commuting Brauer maps. The unique Sylow subgroup above a lower defect gives the actual quotient normalizer, so the prescribed strong central quotient in the cyclic theorem [20, Theorem 1.1 and Definition 7.2(iii)] supplies the natural system at the actual defect normalizer. At three the matching cases were proved above. $\mathrm{SL}_4(2) = \mathrm{GL}_4(2)$ is the nonmatching case with weight at most two; $\mathrm{SL}_2(4)$ and $\mathrm{SL}_3(2)$ use their cyclic universal-cover systems with central 2-kernel. For $\mathrm{SU}_4(3)$ and $\mathrm{SL}_2(9)$, three is defining. Finally, at the remaining exceptional matching parameters the strictly quasi-isolated component order divides $\gcd(n, q - \epsilon) \in \{1, 3\}$ and is prime to three, so is 1; their nonunipotent blocks enter the proper Levi step through Lemma 7.8. This includes the noncyclic source in $\mathrm{SU}_6(2)$ at three: an order-five label has the four-dimensional unitary torus T_4, of order $2^4 - 1 = 15$, and a two-dimensional unitary residual space. Its actual finite Levi is

$$L = \{(x, g) \in T_4 \times \mathrm{GU}_2(2) : \det(x) \det(g) = 1\}.$$

The torus determinant is onto C_3, and $\mathrm{GU}_2(2) \simeq C_3 \times S_3$; eliminating the torus 3-part identifies L with $C_5 \times \mathrm{GU}_2(2)$. Central Jordan twisting, [15, Proposition 1.3.2(ii)], identifies the source block as a nontrivial order-five line on C_5 tensored with the unique three-block of $\mathrm{GU}_2(2)$. The latter uniqueness follows from $\mathrm{GU}_2(2) = C_3 \times S_3$ and the normal self-centralizing Sylow three-subgroup of S_3. Thus the source has normal defect $D = C_3^2$ and $N_L(D) = L$. It has the identity full ordinary system in every actual source-block inertia, since this Sylow subgroup is characteristic in L; the lower and upper matrices and all specified blocks are literally the same on both sides. The smaller natural factor $\mathrm{SU}_2(2) = S_3$ has the identity system proved above; Levi assembly retains the determinant 3-quotient and its full ordinary fibres. The new transfer lemma supplies the global ordinary extensions, every regular-inertia block anchor, and the prescribed quotient for this block. Thus this boundary is not reduced to a cyclic-defect claim. This terminates the induction and proves the assertion for every natural group. For the eight simple groups with exceptional Schur multipliers, the all-prime iAM input is recorded explicitly in [9, Section 6.1]. Thus the natural-group argument is not used to discard an extra Schur kernel. Together with that input, the construction proves Theorem 7.1.

8. Types B and C

Write $G = \mathbf{G}^F$, where $\mathbf{G}$ is simply connected of type B_n or C_n, and put $q = p^f$. In this section ℓ is odd and $\ell \neq p$. The notation $\widetilde{G}$ always denotes a regular embedding of this natural group. In particular, an exceptional covering group is not silently identified with G. In even characteristic the natural groups of types B_n and C_n are identified by their standard exceptional isogeny, which commutes with the defining Frobenius; we use the type-C description for this finite group.

The case q odd and $\ell \geq 5$ is [7, Theorem 9.12]. The proof below treats the remaining case $\ell = 3$, and also type C in even characteristic. The two normalizers used below are distinguished throughout: first $N_G(A)$, where A is the characteristic abelian subgroup of a defect group, and finally the actual normalizer $N_G(D)$. Defect-zero blocks use the identity correspondence and will be omitted from the source-normalizer arguments.

8.1. Central defect, ordinary packets, and integral returns

The global use of Proposition 6.2 is in the connected-centre regular group, separately for each covering source block. Its derived algebraic group is simple. The natural group's lower correspondence is then obtained by the block criterion; we do not apply a connected-centre statement directly to G. Here is the source information used in that construction. If $(\mathbf{L}, \lambda)$ is the relevant e-cuspidal source, $A = Z(L)_\ell$, $C_G(A) = L$, and

$$Z(C_{L^*}(s))_\ell = Z(L^*)_\ell, \tag{8.1}$$

and the actual decomposition $L = A \times O^\ell(L)$ and full central duality below hold, its source block c has central defect A. For every $t \in Z(L^*)_\ell$ put $\theta_t = \lambda\widehat{t}$ and $W_t = N_G(L, \theta_t)/L$. Clifford theory gives the local parameters (t, η), $\eta \in \mathrm{Irr}(W_t)$. On each regular covering source, the packet proposition gives the corresponding global parametrization with its regular versions of these parameters. Height zero on both sides is precisely

$$[W_1 : W_t]_\ell \eta(1)_\ell = 1. \tag{8.2}$$

The conclusion is used separately for each prescribed regular covering source, with its own Brauer pair.

Here are the hypotheses that make this input applicable at three. The classical groups have good prime 3, the Mackey formula in [15, Assumption 2.2.3] holds for their Levi subgroups, and no rational factor of type ${}^3D_4$ occurs. Thus [17, Theorem A(e)] applies. The ordinary block identification, including the series st with t of 3-power order, is [15, Proposition 2.3.6, especially (2.3.6.2)–(2.3.6.3)]; it is not an inference from the cardinalities of the two character sets. The

equivariant choice of the ordinary series bijections and its covariance under central twisting are the inputs from [25, Theorems 3.2 and 3.4, Lemmas 3.5–3.7] already recorded in Proposition 6.2.

The following observation removes an unnecessary assertion about the order of a scalar factor set in the type-B argument.

Lemma 8.1. *Let $K \lhd K \rtimes P$, let P be a finite ℓ-group, and let $\kappa \in \mathrm{dz}_\ell(K)$ be P-invariant. Suppose that κ has an ordinary extension to $K \rtimes P$. Then the crossed product of $\mathcal{O}Kb_\kappa$ by the actual action of P is a matrix algebra over $\mathcal{O}P$. The same conclusion with $\mathcal{O}[A \rtimes P]$ holds after adjoining a central ℓ-group A on which P acts.*

Proof. Choose a splitting complete discrete valuation ring $\mathcal{O}$ with fraction field $\mathcal{K}$. The defect-zero algebra is $\mathcal{O}Kb_\kappa \simeq M_d(\mathcal{O})$. Each automorphism induced by $x \in P$ is inner, so choose a matrix $U_x \in \mathrm{GL}_d(\mathcal{O})$ implementing it, with $U_1 = 1$. Write $U_xU_y = \alpha(x,y)U_{xy}$, where $\alpha(x,y) \in \mathcal{O}^\times$. The assumed ordinary extension says that this factor set is a coboundary in $\mathcal{K}^\times$: there are $\beta(x) \in \mathcal{K}^\times$, $\beta(1) = 1$, for which $\alpha(x,y) = \beta(x)\beta(y)\beta(xy)^{-1}$. Applying the valuation gives $v\beta(xy) = v\beta(x) + v\beta(y)$. A homomorphism from the finite group P to $\mathbb{Z}$ is zero. Thus every $\beta(x)$ is a unit. Rescaling U_x by $\beta(x)^{-1}$ untwists the crossed product over $\mathcal{O}$. The matrices act trivially on A, and the assertion with A follows by retaining the given action of P on its group algebra. □

This lemma is only an integral block and defect calculation. The equivariant ordinary correspondence below continues to use the extension maps provided by the cited normalizer results; no equivariance of an arbitrarily selected coboundary is assumed.

8.2. **The terminal symplectic sources**

Let $G = \mathrm{Sp}_{2n}(q)$ and $e = \mathrm{ord}_\ell(q)$, $d_0 = e/\gcd(e,2)$, $\epsilon = (-1)^{e+1}$. For q odd we need the quasi-isolated 3-blocks; for q even the terminal blocks are unipotent. Their sources have

$$L = \mathrm{Sp}_{2r}(q) \times T^w, \qquad |T| = q^{d_0} - \epsilon, \qquad A = T_\ell^w. \tag{8.3}$$

At 3 one has $d_0 = 1$. The two eigenspaces of a quasi-isolated dual involution give orthogonal factors of types B and D. An e-cuspidal residual symbol has no removable e-hook or e-cohook. A residual split or nonsplit SO_2 with nontrivial Φ_e-part is therefore excluded. All other residual centres, component groups, and central-isogeny discrepancies have order a power of two. This proves (8.1). For q even the dual centre is connected and the only quasi-isolated terminal label is 1; the same equality follows from the unipotent e-cuspidal degree formula.

The decomposition in (8.3) gives

$$L = A \times K, \qquad K = \mathrm{Sp}_{2r}(q) \times (T_{\ell'})^w = O^\ell(L), \qquad \lambda = 1_A \otimes \kappa, \quad \kappa \in \mathrm{dz}_\ell(K).$$

The residual group is generated by elements of defining-prime power order, so has no nontrivial ℓ-group quotient. Central defect gives the stated defect-zero degree on K. For a regular covering source, the derived centres and the central-isogeny kernel and cokernel have 2-power order; hence the same canonical splitting exists at every odd ℓ. The duality on each cyclic torus is perfect on its ℓ-part, and these 2-power isogenies preserve it. Thus the dual central ℓ-group parametrizes all of $\mathrm{Irr}(A)$, also for every regular covering source. These are the additional decomposition and duality hypotheses of the packet proposition.

The individual coordinates of A have different kernels on the natural module. Since ℓ is odd, a nontrivial coordinate character is different from its inverse. Consequently

$$C_G(A) = L, \qquad N_G(A) = N_G(L). \tag{8.4}$$

The ℓ-part of the relative source group is a group of permutations of equal coordinates. These permutations have genuine representatives in G. If P is its Sylow ℓ-subgroup, a maximal source pair therefore has

$$D = A \rtimes P. \tag{8.5}$$

The defect-zero residual character is fixed, and tensor permutation operators provide its ordinary extension across P. Lemma 8.1 verifies the integral defect calculation.

The full coordinate module proves that A is the unique maximal abelian normal subgroup of D. Indeed, if an abelian normal subgroup has an element inducing a nonidentity permutation x, normality and commutativity imply $(x-1)^2A = 0$. A nontrivial cycle of an odd-order permutation has length at least three; on three consecutive coordinate basis vectors its second difference is nonzero, including in characteristic three. Thus every abelian normal subgroup is contained in A. In particular A is characteristic in D and $N_G(D) \le N_G(A)$.

The source Brauer pair (A, c) belongs to $B = b_G(\mathbf{L}, \lambda)$ by [15, Proposition 2.1.6]. Its extension to a maximal pair is the one in (8.5); the self-centralizing pair construction in the proof of [17, Proposition 3.10] requires only good $\ell \ge 3$ here. It does not require the additional large-prime assumption used for the final Cabanes assertion in the older source quoted there: the preceding coordinate argument proves that assertion directly. Uniqueness of inclusion of Brauer pairs and [17, Lemma 3.9] identify the same source orbit globally and locally. This also identifies the block used in (8.2).

For completeness, no unproved ordinary correspondence for strict type D is being used inside this calculation. Only unipotent residual symbols enter the Jordan source. A degenerate symbol with a one-hook or one-cohook core has equal rows consisting of consecutive initial entries and hence rank zero. Thus in the $e = 1, 2$ core used here no positive-rank degenerate mark is discarded. The ordinary series correspondence is the connected-centre classical one stated above.

8.3. **The complete local characters in type** C

Put $N = N_G(L)$ and $\widetilde{N} = N_{\widetilde{G}}(L)$. The source characters in c are all the $\theta_t = \lambda\widehat{t}$ in the packet, not merely the canonical character λ. Tensoring by a central line preserves e-cuspidality, so each θ_t is in the domain of [5, Theorems 1.1 and 1.2]. Those theorems have no restriction $\ell \ge 5$ and allow both odd and even defining characteristic.

Theorem 1.2(a) gives an equivariant extension map for $\mathrm{Irr}(L)$; Theorem 1.2(b) gives invariance of the relative characters over the regular relative inertia subgroup. The required multiplicity-one restriction follows from [7, Remark 4.6(b)], whose hypothesis excludes type D_{2m} and therefore holds here. Theorem 1.1 applies to each complete local character

$$\psi_{t,\eta} = \mathrm{Ind}_{N_{\theta_t}}^{N}(\Lambda(\theta_t)\eta). \tag{8.6}$$

It supplies a regular conjugate with the diagonal–field stabilizer factorization and a genuine extension to the full field inertia. In particular, the ℓ-part of a field automorphism group has not been removed.

We spell out why all these choices are compatible with the prescribed block. The distinguished constituent of multiplicity one and unique degree in [15, 2.4.1], together with [15, Proposition 2.4.2(a)], gives

$$(N\widetilde{L})_\lambda = N_\lambda \widetilde{L}_\lambda. \tag{8.7}$$

For odd q, the global diagonal–field factorization is [11, Theorem 3.1]. For even q there is no diagonal outer action, so this factorization is immediate; the field inertia quotient is cyclic. The distinguished-degree argument in [7, Corollary 4.3], with the good-prime block uniqueness just proved, therefore chooses λ with $(\widehat{N}\widetilde{L})_\lambda = \widehat{N}_\lambda\widetilde{L}_\lambda$. Since $N \cap \widetilde{L} = L$, equation (8.7) and block uniqueness imply

$$\widetilde{L}_B = \widetilde{L}_\lambda. \tag{8.8}$$

Every element of this group centralizes A, and consequently fixes every θ_t in c. The proof of [7, Proposition 4.7] now adjusts η by a $\widetilde{L}_{\theta_t}$-conjugation. This conjugation remains in $\widetilde{L}_B$ by (8.8). Thus the preferred complete character can be selected inside the required block orbit. Its inertia quotient over N is cyclic, so this preferred character also has a genuine extension to the full field inertia. Thus changing to the block-preserving representative does not discard the extension requirement.

Use this same extension map to construct its regular companion $\widetilde{\Lambda}$ as in the proof of [12, Theorem 4.2]. A regular character above θ_t and $\Lambda(\theta_t)$ have a common extension on their common inertia; induction gives $\widetilde{\Lambda}$. This construction is equivariant and commutes with multiplication by every member of $\mathrm{Irr}(\widetilde{N}/N)$. It is not a second independent choice of local extensions.

For each specified covering source block $\widetilde{c}$, match the regular global and local characters with equal $(\widetilde{\theta}, \widetilde{\eta})$ parameters. Their restrictions to $Z(\widetilde{G})$ agree, and all quotient-line twists commute with this matching. The common Brauer pair above (A, c) identifies its local covering block $\widetilde{b}$ and gives

$$\widetilde{b}^{\widetilde{G}} = \widetilde{B}. \tag{8.9}$$

The assertion is made separately for each covering source; it is stronger than preservation of a union of covering blocks. Lemma 6.3 identifies these sets with the entire ordinary fibres above lower height-zero characters, because $\widetilde{G}/GZ(\widetilde{G})$ is a 2-group.

Apply the block criterion to the relevant regular and field orbit of blocks, transporting their maximal Brauer pairs to the fixed subgroup D. Since A is characteristic in D, the same transporters act on $N_G(A)$ and its corresponding blocks. These facts verify every part of [8, Theorem 2.4]: cyclic regular quotients give the regular inertia extensions in (i); the common parameter map, central scalars, (8.9), and all linear twists give (ii); [11, Theorem 3.1] and cyclic field inertia give (iii); the complete-character assertion of [5, Theorem 1.1] and the block-preserving choice above give (iv); and the relevant outer group is abelian, giving (v). Proposition 2.5 of the same paper gives the block-triple relation on the central quotient belonging to the particular character. Theorem 2.1 then gives the full ordinary systems used in the subsequent Levi assembly.

For q even and $n \geq 4$, the natural group is centreless and the field group is cyclic, so the same argument applies without a diagonal obstruction. For $n = 2, 3$, use instead the actual normalizer bijections in [31, Theorem 5.7]; this retains the exceptional graph–field automorphism when $n = 2$. The exceptional universal covers $6.A_6$ and $2.\mathrm{Sp}_6(2)$ are covered explicitly by [31, Theorem 5.8]. The trivial central sector gives the natural $\mathrm{Sp}_6(2)$ system. For the centreless natural groups in these ranks the outer group is cyclic, including the exceptional graph–field generator in rank two. Proposition 2.4 therefore supplies the full ordinary relation from these actual-normalizer bijections. A regular embedding in characteristic two is the product with its regular centre, so adjoining the same central line on both sides gives every ordinary regular fibre. An actual factor $\mathrm{Sp}_4(2)$ occurring inside a Levi uses its inherited algebraic action, not the full automorphism group of its derived simple quotient.

8.4. **Isolated type B at three**

Let $G = \mathrm{Spin}_{2n+1}(q)$, with q odd and $3 \nmid q$. In [7, Table 1 and Lemma 5.1], $d_0 = \mathrm{ord}_3(q^2) = 1$. Consequently only the following two source shapes occur:

dual connected centralizer	derived source	coordinate multiplicity
$C_k(q)C_{n-k}(q)$	B_m	1
$C_{n/2}(q^2)$	$B_m A_1^r$	2.

The third line of that table requires even d_0 and is absent. In the second line, an A_1 factor has roots $\pm(e_{2i} - e_{2i-1})$ and central coordinate $z_{2i} = z_{2i-1}$. This is why the actual weight multiplicity is two. Since $3 \nmid 2$, it causes no loss of a toral coordinate or of the central-defect equality (8.1).

All derived centres and rational central-isogeny discrepancies in these sources are 2-groups. The central 3-group $A = Z(L)_3$ maps isomorphically to the 3-part of the abelianization. Its canonical retraction gives $L = A \times K$, and $\lambda = 1_A \otimes \kappa$ with $\kappa \in \mathrm{dz}_3(K)$. Here $K = O^3(L)$, since the derived factors are generated by defining-prime elements and the remaining torus in K has order prime to three. The cyclic torus dualities, unchanged by the 2-power isogenies, identify the whole dual central three-group with $\mathrm{Irr}(A)$. This also verifies the packet hypotheses in each connected-centre regular covering source. The coordinate argument proves (8.4); it also remains

valid through the Spin cover, since the commutator of an odd-order element with a centralizing lift cannot be a nontrivial central involution.

We use the actual normalizer and the actual extension maps in [7, Sections 6–9]. Their applicability at 3 can be checked before invoking any of that paper's final theorems. The construction of V', H' and its Weyl image is a root-group calculation. Proposition 8.4 and its following observation give $H' \cap L \le Z(L)$. Lemmas 9.3–9.6 identify the action as

$$\left(\mathrm{SL}_2(\epsilon q^{d_0}) \rtimes \langle c_1' H' \rangle\right) \wr S_r \ \times B_m(q), \tag{8.10}$$

with the full Frobenius action on the factors; the residual $B_m(q)$ is fixed by V'. Proposition 9.2 extends the central characters on H', and Propositions 9.7–9.8 extend the source characters by these actual tensor and wreath-product actions. They use $A(\infty)$ for the ordinary characters of SL_2 and B_m, the latter supplied by [12, Theorem 3.10]. A rank-one residual factor uses the type-A input and a rank-zero factor uses the identity. None of these steps uses $\ell \ge 5$.

In the second source shape, [7, Lemma 6.1 and Remark 6.2] give $\widetilde{L}_\lambda = LZ(\widetilde{G})$ and $W(\theta) = W(\widetilde{\theta})$ for each $\theta \in \mathrm{Irr}(c)$. Thus the relative-character requirement of Assumption 4.5(ii) is the identity, for both central characters of the Spin kernel. We extend Proposition 9.8 to this specified source block using that equality. No assertion that Remark 6.2 applies to every block of L is needed. For the toral source, the construction in [7, Remark 9.11] uses [12, Section 5.D] and gives the same two requirements of Assumption 4.5.

The relative Sylow 3-subgroup consists of permutations: the remaining coordinate factors have order two because $d_0 = 1$. Its inverse image in the finite Tits group has a normal 2-kernel, so Schur–Zassenhaus gives a genuine subgroup P. By (8.10) it permutes equal A_1 factors and coordinates, and fixes the B_m factor. The extension map just constructed gives an ordinary extension of κ across this actual P. Lemma 8.1 therefore proves (8.5) integrally, without an unproved claim that an arbitrary intertwiner factor set has 2-power values. The coordinate proof again makes A characteristic in D.

The rest of the block-criterion verification is now the one given above, with [12, Theorem 3.10] replacing [11, Theorem 3.1]. In particular we use the same distinguished source constituent, nonfusion, regular extension map, every ordinary regular fibre, and the individual covering-block equality (8.9). This proves the strong iAM relation relative to $N_G(A)$ for each isolated type-B block at 3.

8.5. **Terminal blocks at the actual defect normalizer**

We complete the terminal construction before using it in a Levi subgroup. In both source shapes already proved, put

$$L = C_G(A) = A \times K, \qquad K = O^\ell(L), \qquad M = N_G(A), \qquad I = M_\kappa.$$

The group A is characteristic in the actual defect D, and K is characteristic in $C_G(A)$. Thus both are preserved by the full block-and-defect stabilizer. In I/K, the image of A is a normal self-centralizing ℓ-subgroup: if an element centralizes it modulo K, its commutators with A lie in $A \cap K = 1$, so it belongs to $C_G(A) = L$. The source character κ has defect zero. These are all the hypotheses of Theorem 3.1, with normal source group K and base group M.

The unique source block in I and its Fong–Reynolds block in M are the blocks identified by the actual source Brauer pair above. Applying that theorem in $M\,\mathrm{Aut}(G)_{B,D}$ gives the full block relation from this block of M to its Brauer correspondent in

$$N_M(D) = N_G(D).$$

The equality follows from the characteristic property of A. The incoming global correspondence and this outgoing relation therefore have exactly the same middle group and block. Since $Z(G)$ has order at most two and is contained in K, the quotient clause of the source theorem applies to $Z(G) \cap \ker \chi$. Inflate both relations to the same middle triple and compose. This proves the strong relation on that prescribed central quotient. Applying the same ordinary coefficients in

the regular ambient also gives every ordinary regular fibre, every quotient-line twist, and the specified covering-block equality at its Clifford inertia. The terminal natural-group systems now have the actual normalizer endpoint required by Levi assembly.

8.6. **Full linear and unitary factors at all odd primes**

Lemma 8.2. *Let $R = \mathrm{GL}_m^\epsilon(Q)$ and let $\ell \nmid Q$ be odd. Each block of R has a height-zero correspondence to its actual defect normalizer with full ordinary systems in its actual algebraic, regular, field and graph ambient. These systems preserve all central scalars, quotient-line twists, and specified regular Clifford-inertia blocks, including in products with actual Frobenius returns.*

Proof. For $m = 1$ use the identity system on the torus. For $m \geq 2$, realize the algebraic group with fixed group R as the Levi

$$g \longmapsto \mathrm{diag}(g, \det(g)^{-1})$$

of type A_m. In the unitary case take the orthogonal sum of the m-dimensional Hermitian space and a nondegenerate line; this gives an F-stable algebraic Levi with fixed group $\mathrm{GU}_m(Q)$. Its derived fixed group is the natural $H = \mathrm{SL}_m^\epsilon(Q)$. Theorem 7.1 supplies the full natural systems for every covered H-block, including the nonperfect small groups and their identity systems. Theorem 5.1 therefore applies. It keeps the full determinant quotient R/H, including its ℓ-part, and explicitly passes from $N_R(D \cap H)$ to $N_R(D)$. Its last paragraph supplies the stated full ordinary fibres and extension transfer. Tensor permutations and the actual return on a Frobenius cycle are included in that theorem. In particular this argument uses more than an iAM statement for the simple quotient of H. □

8.7. **Actual rational Levi groups and the nonisolated blocks**

We induct on the semisimple rank, simultaneously over all field sizes in the remaining range. Rank one is type A. The even-characteristic ranks two and three use the published actual-normalizer systems cited above; the nonperfect natural factor $\mathrm{Sp}_4(2)$ uses Section 8.8. The terminal systems at larger rank have just been constructed independently of this induction.

For type C, place the residual $+1$ and -1 eigenspaces of the semisimple dual label in one symplectic summand. All other Frobenius and inverse orbits give actual linear or unitary factors. The resulting finite Levi is

$$L = \mathrm{Sp}_{2r}(q) \times \prod_i \mathrm{GL}_{a_i}(q^{d_i}) \times \prod_j \mathrm{GU}_{b_j}(q^{e_j}). \tag{8.11}$$

It has no determinant constraint. The dual Levi contains the full centralizer, so its series normalizer is the Levi itself. The symplectic factor has smaller rank, and the other factors use Lemma 8.2 at the actual prime ℓ. Similitudes act on each linear or unitary factor by inner automorphisms; in the unitary case use surjectivity of the finite-field norm. Factor cycles keep their extension fields and contragredient returns. Theorem 5.1 therefore gives the system at the actual $N_L(D)$ for every specified source block.

For type B, let $\mathbf{L}^*$ be the minimal Levi containing $C^\circ_{\mathbf{G}^*}(s)$. The connected inverse image of a linear packet in Spin is retained. In a split packet it is

$$\mathbf{H} = \{(g, z) : z^2 \det(g) = 1\}. \tag{8.12}$$

For a rational orbit use the product determinant and the actual Frobenius. Its derived fixed group is the natural special linear or unitary group, and the kernel μ_2 of the orthogonal projection is disjoint from this derived group. If K_1 is the full finite linear or unitary group and δ its quadratic determinant character, then

$$H^F/\mu_2 = \ker \delta, \qquad \widehat{H}/\mu_2 = K_1, \qquad \widehat{H} = \{x : F(x)x^{-1} \in \mu_2\}. \tag{8.13}$$

Thus $F(x) = z(x)x$, $z(x) \in \mu_2$, is the actual return. The product of these connected packets and the residual Spin factor maps to $\mathbf{L}$ with a central 2-kernel Z_0. Its rational image L_0 has quotient

$$L/L_0 = H^1(F, Z_0). \tag{8.14}$$

Apply Theorem 5.1 directly to $[\mathbf{L}, \mathbf{L}]^F$, using the natural type-A factors and the smaller-rank Spin factor. It retains the whole finite quotient $L/[\mathbf{L}, \mathbf{L}]^F$, hence all the cosets in (8.14). Only tensor characters agreeing on each central intersection descend; all surviving central characters, including a faithful μ_2-character, remain in the ordinary systems. This gives each specified L-block its actual-normalizer system.

We also include the series-normalizer coset explicitly. Write $R = N'$ for the finite BDR source normalizer and let $c \in \mathrm{Bl}(R)$ be the source block. In type B, $L \lhd R$ and $|R : L| \leq 2$, by the construction in [28, Theorem 4.2]. Choose a covered L-block c_0 with the prescribed defect D. This is the same actual subgroup on both layers because ℓ is odd. In the ambient stabilizer of c_0, apply [27, Proposition 1.12] with base L, intermediate layer R_{c_0}, and local ambient the stabilizer of D. The local base is $N_L(D)$ and the resulting local layer is $N_{R_{c_0}}(D)$. Its block is the specified Brauer block of the Fong–Reynolds correspondent of c. If c_0 moves, Brauer's first main theorem identifies its local block inertia, and [24, Theorem 3.14] induces both sides to R and $N_R(D)$. A block transporter is corrected within its inertia by a defect conjugator; this gives both quotient-surjection hypotheses at every upper subgroup. The defects agree, so the height differences agree. Consequently we have the full ordinary block relation for

$$\phi_c : \mathrm{Irr}_0(c) \longrightarrow \mathrm{Irr}_0(c_D), \qquad c_D \in \mathrm{Bl}(N_R(D)).$$

Using the same ordinary coefficients and inducing over moving regular orbits gives all regular fibres and quotient-line covariance. The upper relations and their specified-block restrictions are [24, Proposition 3.9(b),(d)]; genuine extensions and all centralizer scalars are preserved. For type C take $R = L$ and use the preceding assembly map.

For type B apply [28, Theorem 9.2 and Remark 9.5] with $Q = D$, which is a characteristic subgroup of itself. For type C apply Proposition 5.3 in its full-centralizer case. All three transfer inputs are now explicit: the entire ordinary regular fibres and their central scalars; extension transfer in the actual field inertia and equality of fibres over each lower character; and the specified regular Clifford-inertia block equality. For the last input use the same global complex and its Brauer construction, as in [27, Lemma 2.14] and [28, Lemma 9.4]. The numerical hypothesis in type B is $\ell \nmid 2|H^1(F, Z(\mathbf{G}))|$, since the latter group has order at most two.

The full field action uses the two branches of the proof of published [28, Theorem 4.2]. When its field norm lies in L, that proof gives the full field extension. In the other branch the actual quotient over L is cyclic; the corresponding local quotient embeds in it. Over the ordinary coefficient field $\mathbb{K}$, the cyclic module extension retains the entire field order and its genuine inner return, globally and locally. This is the local version used in Remark 9.5. The integral block comparison at the regular Clifford inertia is the separate complex construction just specified. Thus the general integral statement over the ℓ'-part of the field group is not used as a statement over its ℓ-part.

Let R_G and R_D denote the ordinary maps of the global and local Brauer complexes. The resulting correspondence is

$$\Omega_B = R_D \phi_c R_G^{-1} : \mathrm{Irr}_0(B) \longrightarrow \mathrm{Irr}_0(b_D), \qquad b_D \in \mathrm{Bl}(N_G(D)). \tag{8.15}$$

The inverse is $R_G \phi_c^{-1} R_D^{-1}$. The source and local complexes carry the same actual defect subgroup. For an arbitrary regular ordinary character above a lower height-zero character, use the regular global map, the full source fibre, and the regular local map. This also supplies the strengthened natural system needed at the next rank. The strong central quotient is the one in the transfer criterion, with local group the quotient of this actual $N_G(D)$. This proves the remaining odd nondefining cases of types B/C, and completes their contribution to the main theorem.

8.8. **The nonperfect natural boundary** $\mathrm{Sp}_4(2)$

The natural factor $\mathrm{Sp}_4(2) \cong S_6$ requires its actual algebraic automorphism action, although its derived simple subgroup is A_6. We give this step separately. At an odd nondefining prime the only possibilities are $3, 5$. The proof of [31, Theorem 5.8] constructs the iAM system for $U = 6.A_6$ with the actual endpoint $N_U(P)$, for every noncentral defect group P; the same proof shows that these are Sylow subgroups. On the trivial central-character sector, its prescribed strong quotient gives the corresponding system on A_6, with endpoint $N_{A_6}(D)$.

Put $N = A_6$ and $J = S_6$. Since $|J : N| = 2$ is prime to ℓ, a block of J and any block of N it covers have the same actual defect group after a single conjugation; in particular $D \le N$. Use the full automorphism system on N, and take its local ambient subgroup to be the stabilizer of D. Applying [27, Proposition 1.12], whose normal-subgroup argument is [24, Proposition 4.7(b)], to this intermediate group J gives a bijection for each specified covering block of S_6, with local group

$$N_{S_6}(D).$$

In the case of a moving covered block, apply the proposition first in its inertia and then induce on both sides by [24, Theorem 3.14]. All ordinary inverse-factor-set coefficients are retained; the upper character-triple and specified-block compatibilities are [24, Proposition 3.9(b),(d)].

Finally the restriction $\mathrm{Aut}(S_6) \longrightarrow \mathrm{Aut}(A_6)$ is injective. Indeed, if α fixes A_6 pointwise, then $x^{-1}\alpha(x) \in C_{S_6}(A_6) = 1$ for every $x \in S_6$, by comparing its conjugation on A_6. Its image preserves the given S_6; equivalently, S_6/A_6 is one of the order-two subgroups of $\mathrm{Out}(A_6) = V_4$. Thus the construction respects the actual inherited field and exceptional graph action. This proves the natural small-factor system needed in the Levi induction, including its ordinary regular fibres. The simple group $\mathrm{Sp}_4(2)'$ and the extra cover of $\mathrm{Sp}_6(2)$ themselves are already covered by [31, Theorem 5.8].

9. Unipotent blocks of type D

Let $G = \mathrm{Spin}^{\epsilon}_{2n}(q)$, $n \ge 4$. We first construct all its unipotent blocks at odd nondefining primes. In this section a unipotent block means the entire block in the blockwise series $\mathcal{E}_\ell(G, 1)$, including its ordinary nonunipotent characters.

Theorem 9.1. *Let $n \ge 4$, $\epsilon \in \{+, -\}$, and let ℓ be an odd prime not dividing q. Every unipotent ℓ-block of $\mathrm{Spin}^{\epsilon}_{2n}(q)$ has a full ordinary source system at its actual defect normalizer, on the prescribed central quotient. This includes the full diagram action in split D_4. The comparison uses the entire height-zero set, including its ordinary nonunipotent characters.*

9.1. **Actual signed-cycle sources**

Work initially with $K = \mathbf{G}^F_{\mathrm{ad}}$, $S = [K, K]$, and $\Delta = K/S$. The group Δ is an abelian two-group of order at most four. Let (L, κ) be the marked unipotent e-cuspidal source of a block B of K, where $e = \mathrm{ord}_\ell(q)$, and put

$$k = e/\gcd(e, 2), \quad m = 2k, \quad A = Z(L)_\ell, \quad K_0 = O^\ell(L), \quad M = N_K(A).$$

The unipotent source is the marked e-cuspidal pair supplied by [15, Assumption 2.1.2 and Propositions 2.1.6–2.1.7]. These results apply to all classical groups at every odd nondefining prime, including three. In the signed-cycle model its toral part consists of w cycles of length k: positive cycles if e is odd and negative cycles if e is even. The residual orthogonal factor carries an e-cuspidal unipotent character. That character has central defect by [15, Assumption 2.1.2 and Proposition 2.1.11]. The centre of its residual derived group and the central multiplication kernel have two-power order. The same holds for the cokernel on rational points. Thus the odd part of the central toral factor splits off, giving the actual finite groups

$$L = A \times K_0, \qquad \kappa = 1_A \otimes \kappa_0, \quad \kappa_0 \in \mathrm{dz}_\ell(K_0), \qquad C_K(A) = L, \quad N_K(A) = N_K(L). \tag{9.1}$$

Here A is a full product of w independent cyclic ℓ-coordinates. More explicitly, the residual factors and the prime-to-ℓ torus have no abelian quotient of ℓ-power order, while A embeds in the abelianization and is its entire ℓ-part. Its projection therefore gives $L = A \times O^{\ell}(L)$. This also includes a residual one-dimensional torus of order prime to ℓ.

For completeness the centralizer assertion concerns all components. Every root outside the residual core is nontrivial on at least one independent coordinate of A; a coefficient two is nonzero since ℓ is odd. Hence the connected algebraic centralizer is the displayed Levi. An element of the full centralizer can be multiplied by an element of that connected centralizer to normalize its maximal torus. Its Weyl element fixes every coordinate of A. The signed-coordinate action is faithful, so that Weyl element belongs to the residual Weyl group. This removes every extra component. Finally A is characteristic in L, which proves both normalizer equalities. The Brauer-pair assertion in Proposition 2.1.6 of the cited reference supplies the specified pair $(A, \mathrm{bl}(\kappa))$.

For $r \mid m$, let $G(m, r, w)$ be the subgroup of $C_m \wr S_w$ whose product of diagonal entries belongs to $C_{m/r}$; the group with $w = 0$ is trivial. The inertia M_κ can be proper in M. Its relative group is B_w or D_w for $e = 1, 2$, and

$$W = M_\kappa/L = G(m, 1, w) \quad \text{or} \quad G(m, 2, w) \tag{9.2}$$

for $e > 2$. All permutations of equal signed cycles and all their rotations have actual normalizer lifts. The only restriction is the total sign: an odd sign change on the moving support must be compensated by the graph involution on the residual core. If that core involution preserves κ, the full group $G(m, 1, w)$ occurs; otherwise only the even phase product occurs. The index-two possibility therefore includes the empty residual core and a degenerate symbol. A normalizer can exchange the two degenerate marks; one works in the actual source inertia and then induces over both marks. Ordinary extension of the unipotent core to its actual stabilizer, with rational permutations, is [37, Theorem 3.3 and its tensor lemmas]. It applies to every invariant ordinary unipotent source character.

We make the splitting needed for every torus colour in the full ambient explicit. Suppose $A \le R$ are both normal in T, $R/A = W$, and W has a normal ℓ'-subgroup E with $A^E = 1$. In the preimage of E, choose an ℓ'-complement E_1. Conjugacy of complements gives

$$R = A\, N_R(E_1), \qquad A \cap N_R(E_1) = 1.$$

Thus $W_0 = N_R(E_1)$ is an actual complement. Every other complement is A-conjugate: first conjugate its E-part to E_1, then its whole normalizer is $N_R(E_1)$. It follows that

$$T = A \rtimes Q, \qquad Q = N_T(W_0), \quad W_0 \lhd Q. \tag{9.3}$$

In the application, work in the actual adjoint ambient generated by K and its field and graph automorphisms. Let $\widehat{T}$ be the stabilizer there of the embedded source pair and its torus A, and take $R = M_\kappa/K_0$ and $T = \widehat{T}/K_0$, identifying A with its image. Here $K_0 = O^{\ell}(L)$ is characteristic in L, and M_κ is normal in $\widehat{T}$, so both normality requirements hold. For (9.2), take the sign or phase subgroup. Its fixed space is zero except for the trivial group D_1 on one coordinate. For that boundary case the defect is cyclic and the relative coefficient group is trivial. There is also a direct extension argument, needed for the ordinary source system: the obstruction for a line of the normal ℓ-group A has ℓ-power order. Its inertia quotient in T/A has cyclic Sylow ℓ-subgroups, since the field group is cyclic and the possible graph part has order two; the split rank-four triality case has already been excluded for this source by Lemma 9.3. Restriction to a cyclic Sylow ℓ-subgroup kills that obstruction, and restriction–corestriction kills its entire ℓ-part. Thus every colour extends to its actual inertia in this boundary case as well. If $A = 1$, the block has defect zero and uses the identity comparison. In the split-complement cases, for any ordinary $\theta \in \mathrm{Irr}(A)$,

$$\widehat{\theta}(au) = \theta(a), \qquad u \in Q_\theta,$$

is therefore a genuine extension in the actual ambient, with all of its field and graph returns.

9.2. **All cyclotomic coefficients**

Lemma 9.2 (Root coefficients and scalar returns)**.** *Let ℓ be odd, $a \geq 1$, and let $V = (\mathbb{Z}/\ell^a)^w$. Let W be $W(B_w)$ or $W(D_w)$, in its signed-coordinate action, with $w \geq 2$ in the D-case, and let R be a finite group of automorphisms of the corresponding root system containing W. Interpret half-integral D_4 root matrices over $\mathbb{Z}_\ell$. Put $U = \langle pI \rangle \leq \mathrm{Aut}(V)$, where $\ell \nmid p$. For $\theta \in V$, every $\eta \in \mathrm{Irr}(W_\theta)$ extends to its inertia in $(RU)_\theta$. The same conclusion holds for $W = G(m, r, w)$, $r \in \{1, 2\}$ with $r \mid m$, $\ell \nmid m$, $m \mid \ell - 1$, with $R = G(m, 1, w)$.*

Proof. First consider the real root systems. Let Φ_θ consist of the roots whose pairing with θ vanishes modulo ℓ^a. The explicit signed-coordinate description gives

$$W_\theta = W(\Phi_\theta) = W(B_z) \times \prod_j S_{b_j} \quad \text{or} \quad W(D_z) \times \prod_j S_{b_j}.$$

Here z is the number of zero coordinates; the b_j are the multiplicities of nonzero coordinates modulo sign, with conjugate signed permutation realizations on those supports. At $z = 0, 1$ the evident empty or trivial factors are used. Since ℓ is odd, a nonzero coordinate cannot equal its negative, at any valuation. These equalities follow directly by writing a stabilizing signed permutation coordinate by coordinate.

It is essential to allow a root element to return θ up to a field scalar. Set

$$R^* = \{r \in R : r\theta \in U\theta\}.$$

This is a subgroup. Every $r \in R^*$ preserves Φ_θ, since multiplying a root pairing by a unit does not change its vanishing. Thus R^* acts on this root system. The kernel of this action intersects $W(\Phi_\theta)$ trivially and centralizes it.

We recall the elementary extension argument for root automorphisms. For a finite reduced root system Ψ, $\mathrm{Aut}(\Psi) = W(\Psi) \rtimes \mathcal{D}$, where $\mathcal{D}$ permutes its simple roots. For an irreducible component the character inertia in its diagram group is cyclic or a subgroup of S_3. Its Schur multiplier is trivial, so each invariant ordinary irreducible character extends to that inertia. For repeated isomorphic components take these extensions on the tensor factors and let permutations act by permutation of tensor coordinates. This gives an extension of every $\eta \in \mathrm{Irr}(W(\Psi))$ to its full inertia in $\mathrm{Aut}(\Psi)$. Restriction to the action image of R^*, followed by inflation, gives an extension to R^*_η.

For the complex monomial groups, write each nonzero coordinate as a phase multiple of a chosen representative of its C_m-orbit. No nonidentity phase fixes a nonzero coordinate: if $(\zeta - 1)x = 0$, then $\zeta - 1$ is an ℓ-adic unit. Consequently

$$W_\theta = G(m, r, z) \times \prod_j S_{b_j}.$$

The group $R^* = \{r \in G(m, 1, w) : r\theta \in U\theta\}$ preserves the zero support and permutes nonzero colour supports. After the chosen diagonal change of coordinates, θ is constant on each nonzero support. An element of R^* has a uniform phase on each such support: two different phases cannot give the same multiple of its nonzero coordinate, by the unit calculation above. Forgetting these uniform phases gives an actual homomorphism

$$R^* \longrightarrow G(m, 1, z) \times \prod_{d \geq 1} (S_d \wr S_{c_d}),$$

where c_d is the number of nonzero colour supports of size d. Its kernel intersects W_θ trivially and centralizes W_θ; its restriction identifies the latter with $G(m, r, z) \times \prod_j S_{b_j}$. Use ordinary Specht modules and their tensor permutation operators on the nonzero factors. On the zero factor an irreducible of $G(m, 2, z)$ extends to its inertia in $G(m, 1, z)$, whose quotient has order at most two. For $r = 1$ the zero-factor representation is already defined on the full group. Restrict the tensor extension to the image of R^*_η and inflate. This treats the actual subgroup even when its

zero action and its permutations of nonzero supports are coupled; no independent motion of these factors is assumed.

In either case let E afford this extension. The subgroup $R_\eta^* \cap U$ is central, so its matrices are the values of a linear character ω. A character of a subgroup of a finite cyclic group extends to that cyclic group; choose $\nu \in \mathrm{Irr}(U)$ with $\nu|_{R_\eta^* \cap U} = \omega$. Then

$$E^+(ru) = E(r)\nu(u)$$

is well-defined on $R_\eta^* U$. Its restriction to $(RU)_{\theta,\eta}$ is the required extension. In the real root case the scalar intersection is contained in $\{I, -I\}$. Indeed orthogonality gives $u^2 = 1$ modulo ℓ^a, and the torsion-free congruence kernel in $\mathrm{GL}_w(\mathbb{Z}_\ell)$, for odd ℓ, then identifies the root matrix with I or $-I$. The monomial case allows all scalar phases in C_m. The construction imposes these actual intersections, rather than replacing a field return by an independent cyclic generator. □

Lemma 9.3 (Image of an actual unipotent source)**.** *For the source in (9.1)–(9.2), its actual ambient action on A has the following form. For $e = 1, 2$ its root part preserves the relative root system B_w or D_w, including the full D_4 root automorphism group when triality occurs. For $e > 2$ its image is contained in $G(m, 1, w)\langle pI\rangle$, except for the split D_4, $e = 4$ source in Lemma 9.4. The kernel of this action centralizes the relative Weyl group.*

Proof. Use signed cycles for the rational torus in the orthogonal coordinate model. A moving coordinate is a signed cycle of length $k = e/\gcd(e, 2)$. A root normalizer commuting with its rational return can permute cycles of the same length, rotate them, and change their signs; on the cyclotomic coordinate these operations give $C_{2k} \wr S_w$. The even-sign condition selects its subgroup $G(2k, 2, w)$ when the source mark requires it. Allowing the ambient graph involution removes that condition, but introduces no further operation. A field map raises torus coordinates to the p-th power, hence contributes the scalar pI; a correction taking the source back contributes one of the preceding root operators. All of these descriptions are on the actual signed cycle, so their return elements are retained.

The only extra root automorphisms occur in split D_4. For $e = 1, 2$, the only possible residual cuspidal ranks at most four are $0, 1, 4$. Here is the small-rank check, including the rational sign. In the symbol description of [15, proof of Proposition 5.2(a.3)], a 1-cuspidal symbol has no hook of length one. After a common shift its rows are an empty row and a consecutive row of even length $2d$; its rank is d^2, and the residual sign is $(-1)^d$. For $e = 2$, absence of a cohook of length one gives alternating consecutive rows; the allowed even total length gives rank d^2 and positive residual sign. Thus, in split rank four, the rank-one residual core would have negative total sign: for $e = 1$ the core is negative, while for $e = 2$ the three moving negative cycles have negative product. It does not occur in the split form. The remaining sources have residual rank four (and no moving coordinate), or residual rank zero (the full four-dimensional torus). Hence triality acts on the full relative root system D_4 whenever this argument is needed.

For $e = 3, 6$, there is one cyclotomic coordinate. The rational cyclotomic plane is a one-dimensional vector space over $\mathbb{Q}(\zeta_3)$. The finite root centralizer acts by units of finite order in this field and therefore has image contained in μ_6. This is the allowed scalar subgroup of $G(6, 1, 1)$. The remaining nonzero case is $e = 4$: these are all possibilities because the order polynomial of D_4 has cyclotomic indices $1, 2, 3, 4, 6$. Its full root centralizer is computed in Lemma 9.4. The accompanying root calculation independently gives orders $36, 96, 36$ for the full root centralizers at $e = 3, 4, 6$, respectively, and cyclotomic image C_6 in the two rank-one cases. Finally, after (9.3), a kernel Q_A has $Q_A \cap W_0 = 1$. Both are normalized by Q, and $[Q_A, W_0] \le Q_A \cap W_0$, proving the last assertion. □

Lemma 9.4. *For the D_4, $e = 4$ source, every ordinary little-group coefficient extends to its full root and triality inertia at every ℓ-adic valuation.*

Proof. Identify $\mathbb{R}^4$ with $\mathbb{C}^2$ by $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$, and put

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad H = \frac{1+i}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then

$$W = \langle iI, X, Z \rangle, \qquad R = \langle P, H \rangle, \qquad |W| = 16, \quad |R| = 96, \quad R/W \simeq S_3. \tag{9.4}$$

Here $W = \{i^a X^b Z^c : 0 \le a < 4,\ 0 \le b, c < 2\}$. The identities $P^2 = Z$, $H^2 = iI$, $HZH^{-1} = X$, and $(HP)^3 = -I$ show $W \le R$ and give its quotient of order six. The images act as the full permutation group on the three Pauli axes. The realifications preserve the 24 roots $\pm e_j \pm e_k$ and commute with

$$v = \operatorname{diag}(J, J), \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The even signed monomial centralizer of v is W. Since the full root automorphism quotient over $W(D_4)$ has order six, (9.4) is the entire actual root centralizer.

Exactly six nonidentity W-elements have eigenvalue 1. Their fixed lines are

$$(1,0),\ (0,1),\ (1,1),\ (1,-1),\ (1,i),\ (1,-i). \tag{9.5}$$

The squared norms of pairwise determinants are $1, 2, 4$. For every other nonidentity element, the normal form $i^a X^b Z^c$ gives nonzero norm of $\det(w - I)$ equal to a power of two. These are units at every possible $e = 4$ prime, which is at least five. Taking a Teichmüller fourth root i, the same calculation holds over $\mathbb{Z}/\ell^a$. Divide a nonzero vector by its largest common power of ℓ and reduce modulo ℓ: two distinct lines in (9.5) cannot fix it. Thus a nonzero colour has stabilizer 1 or C_2, uniformly at all valuations. The centralizer of a reflection in R is diagonal on its two distinct complex eigenspaces, hence abelian. Its lines extend all ordinary C_2-coefficients.

For the field action, use the saturated inertia $R^* = \{r \in R : r\theta \in \langle pI \rangle \theta\}$. If $W_\theta = C_2$, this group normalizes W_θ and therefore centralizes its unique nonidentity reflection. Consequently it lies in the abelian centralizer just described; every ordinary character of W_θ extends to R^*. If $W_\theta = 1$, use the trivial representation. Thus the argument covers a root operator which fixes the colour only after a field scalar has been applied.

For zero colour, W has eight lines and two irreducibles of degree two. The two degree-two representations are the displayed natural representation and its complex conjugate; both extend to R. The R-invariant lines are 1 and the determinant, which extend as those same characters of R. The other six form two orbits of length three; their inertia quotient over W has order two, so each extends. This proves every zero-colour case. The scalar intersection with the field action is imposed by the central-product construction of Lemma 9.2. □

9.3. **Ordinary inverse and the diagonal block anchor**

Tensor the genuine source extension with the colour and coefficient extensions. Put $I_\theta = (M_\kappa)_\theta$, so $I_\theta/L = W_\theta$; the colour extension is inflated from the quotient by K_0. The local ordinary characters are

$$\operatorname{Ind}_{I_\theta}^{M_\kappa}(\Lambda(\kappa\theta)\eta), \qquad \eta \in \operatorname{Irr}(W_\theta), \tag{9.6}$$

over all source colours, modulo actual source inertia. This is ordinary Clifford theory on the actual split quotient (9.3). If the source moves, Fong–Reynolds induces its unique covering block and all characters to M. The unique source correspondent also shows that induction of its full inertia extension remains a genuine extension. The actual defect can be chosen as $D = A \rtimes P$, where $P \in \operatorname{Syl}_\ell(W)$. Here is an identification before any height-zero count is used. The unipotent defect description, restated in the proof of [8, Theorem 5.2], says that a Sylow ℓ-subgroup of $C^\circ_{\mathbf{G}_{\mathrm{ad}}}([\mathbf{L}, \mathbf{L}])^F$ is a defect of B. On the moving signed-cycle support its order is $|A|\,|W|_\ell$: the classical order formula and the lifting-the-exponent identity give the toral contribution $|A|$ and the permutation contribution $(w!)_\ell$; the possible sign index has order two. Odd permutation cycles have genuine lifts: the Tits kernel is a two-group, so Schur–Zassenhaus lifts the permutation Sylow

subgroup. Together with A these form a Sylow subgroup of that centralizer. They fix the marked core, so this same D is a defect of the unique source block in M_κ, and of its Fong–Reynolds block in M. The common Brauer pair then identifies its induced block with B. The source normalizer has not replaced the defect normalizer: the permutation-coordinate argument of Lemma 6.1 makes A characteristic in this D, and hence $N_K(D) \le M$. The local degree formula gives height zero exactly when

$$[W : W_\theta]_\ell \eta(1)_\ell = 1. \tag{9.7}$$

We now give both ordinary parameter inverses. Start with an arbitrary $\chi \in \mathrm{Irr}_0(B)$. By [8, Theorem 5.2] its dual semisimple ℓ-parameter has a representative $t \in (Z(\mathbf{L}^*)^{F^*})_\ell$, dual to a colour θ of A. The centralizer of this odd-order element is a rational Levi: in the orthogonal eigenvalue description its nonunit reciprocal eigenvalue pairs give general linear or unitary factors and the unit eigenspace gives the residual D-factor. There is no minus-one eigenvalue. Lifting across the central isogeny of two-power degree changes none of this odd-order centralizer. Denote by $\mathbf{G}(t)$ the dual rational Levi in $\mathbf{G}_{\mathrm{ad}}$, chosen to contain $\mathbf{L}$.

Proposition 2.3.6 of [15], applied with its $s = 1$, $s_0 = t$, has all its hypotheses here: the algebraic centre is connected, its Assumption 2.1.2 permits the prime, and its Assumption 2.2.3 is the classical Mackey formula. Since t is central in $\mathbf{L}^*$, the source in that proposition is exactly $(L, \kappa\widehat{t})$. Its formula (2.3.6.3) gives

$$\mathrm{Irr}(B) \cap \mathcal{E}(K, t) = \mathcal{E}\big(K, (L, \kappa\widehat{t})\big) = \varepsilon\, R^{\mathbf{G}_{\mathrm{ad}}}_{\mathbf{G}(t)}\big(\widehat{t}\,\mathcal{E}(G(t), (L, \kappa))\big).$$

The sign ε is the usual Jordan sign, chosen to make the corresponding characters ordinary irreducibles. Apply [25, Theorem 3.4] to this unipotent e-Harish–Chandra series, and use the normalized Jordan map of [32, Theorems 2.1 and 6.2]. To identify the relative group, a normalizer of $(L, \kappa\widehat{t})$ must fix the dual central parameter t. After correction inside L^*, its dual representative lies in $C_{\mathbf{G}^*}(t)$. Rational normalizer duality, with the same marked unipotent character, then gives the actual relative group

$$N_{G(t)}(L, \kappa)/L \simeq N_K(L, \kappa\widehat{t})/L = W_\theta.$$

Thus χ determines exactly one $\eta \in \mathrm{Irr}(W_\theta)$ for the chosen marked parameter. Its degree satisfies

$$\chi(\theta, \eta)(1)_\ell = \frac{|K|_\ell}{|A|\,|W_\theta|_\ell}\eta(1)_\ell.$$

This proves that a height-zero χ gives precisely a pair in (9.7). Conversely, the same formula shows that every pair in (9.7) produces a height-zero character in this specific block, by the displayed block identity from Proposition 2.3.6.

There is no extra fusion of labels in this inverse. If two pairs produce the same character, Jordan decomposition first conjugates their semisimple parameters. The unipotent e-Harish–Chandra partition in that centralizer then conjugates the marked cuspidal pairs. Adjusting the first conjugation by this centralizer conjugation gives an element of the actual marked-source normalizer. This is precisely Clifford equivalence on the local side of (9.6). The reverse implication follows from naturality of induction. The two degenerate markings are retained as characters of their actual core; if a normalizer interchanges them, the same Fong–Reynolds induction is used on both sides. Propositions 2.1.7–2.1.8 and the inverse in Proposition 2.3.6 of [15] keep these rational marked pairs, not merely their geometric types. The cited Jordan and series maps are equivariant, so the resulting height-zero bijection is equivariant in the whole source ambient.

We pass to the actual defect normalizer before descending the base group. Apply Theorem 3.1 with $N = K_0$, $V = M$, and the defect-zero source κ_0. The inertia quotient $M_{\kappa_0}/K_0 = A \rtimes W$ has $C_{A\rtimes W}(A) = A$, since the coordinate action of W is faithful. The actual ambient is the stabilizer of the source block in the normalizer of A. It normalizes K_0 and M, as required. The theorem gives an equivariant bijection from the source block in M to its Brauer correspondent b_K in

$N_K(D)$. Composing with the preceding ordinary packet map gives

$$\mathrm{Irr}_0(B) \longrightarrow \mathrm{Irr}_0(b_K), \qquad b_K^K = B.$$

The normal-source relation has matching Clifford factor sets on the entire ambient. Therefore a genuine extension of the source-normalizer character gives a genuine extension of the matched $N_K(D)$-character to its full inertia in the actual defect normalizer of that ambient. The source, colour, and coefficient constructions above provide precisely the former extension. Let $\mathcal{Y}$ be the full actual adjoint automorphism ambient stabilizing B. Since A is characteristic in D, every element of $N_{\mathcal{Y}}(D)$ normalizes $M = N_K(A)$ and K_0. It also fixes the source block of M: that block is the unique block of M with defect D inducing B, by Brauer correspondence through $N_K(D) \le M$. Thus it lies in the ambient used for the normal-source theorem. The target extension is consequently defined on the full required inertia in $N_{\mathcal{Y}}(D)$, including automorphisms which move a chosen core mark and return after an element of M.

For a nontrivial $\delta \in \mathrm{Irr}(\Delta)$, twisting the unipotent block B changes its dual central ℓ'-label from 1 to a nontrivial central 2-element. The $\mathrm{Irr}(\Delta)$-orbit is free. Lemma 6.4 proves that B restricts irreducibly and bijectively onto one block b of S, including all its ordinary characters. Every lower character has a canonical K-extension in B. The Fourier identity also shows that all blocks of K covering b are the displayed twists $B\delta$. Only B has unipotent central label. Thus every automorphism stabilizing b stabilizes B, so the preceding full ambient contains the required lower character inertia. The rational source L surjects onto Δ, because its simply connected preimage is connected and Lang's theorem gives the full connecting image in $H^1(F, Z(\mathbf{G}))$. At the actual endpoint one also has

$$K = S\,N_K(D), \qquad N_K(D)/N_S(D) \simeq \Delta.$$

Indeed $D \le S$, since it has odd order, and the unique block b of S under B is K-stable. The Frattini argument for its actual defect groups gives the first equality. For a nontrivial line δ of Δ, $b_K\delta = b_K$ would imply $B\delta = B$ by block induction. Consequently the orbit of b_K is free as well. Lemma 6.4 now applies to $N_S(D) \lhd N_K(D)$. It gives the canonical local diagonal extension at the actual defect normalizer, and the paired extensions have the specified induced-block anchor at K. Write these extensions as χ_K, ψ_K. Their restrictions are irreducible at every $S \le V \le K$, because they already restrict irreducibly to S and $N_S(D)$, respectively. Since K/S is abelian, [27, Lemma 2.2] now gives

$$\mathrm{bl}(\psi_K|_{N_V(D)})^V = \mathrm{bl}(\chi_K|_V) \qquad (S \le V \le K).$$

This is an anchor on every intermediate diagonal preimage, including the three order-two preimages when $\Delta = V_4$.

Except for the natural group $\Omega_8^+(2)$, the global canonical diagonal extension extends over its full field–graph inertia by [37, Theorem 3.21]: inflate to the regular group, extend there, and descend its already trivial central torus kernel. For $\Omega_8^+(2)$, its field–graph group is S_3. Each character inertia quotient is a subgroup of S_3, whose complex Schur multiplier is trivial; the same applies locally. Thus the needed genuine extensions exist in this case directly, without a universal-cover hypothesis on the natural group. The local genuine extension was constructed above. We explain the remaining block correction for split triality. If $S \le K \lhd Y \le \mathrm{Aut}(S)$ is the character inertia and $R = Y[b]$, the prescribed-character ramification theorem gives a correction line on R/S. Its ℓ'-part is invariant, by uniqueness on each cyclic ℓ'-layer [19, Lemma 3.7]. The canonical diagonal anchor gives the same block equality on every subgroup below K [27, Lemma 2.2]; the same uniqueness forces this line to be trivial on $(R \cap K)/S$. It therefore defines an invariant line on RK/K.

Outside split D_4, Y/K is abelian, so the descended line extends. This includes the twisted groups, whose field and graph operations can form a single cyclic group of order $2f$; they need not embed in $C_2 \times C_f$. In split D_4, Y/K is a subgroup of $S_3 \times C_f$. Every subgroup $F \le S_3 \times C_f$

has derived group 1 or C_3. For each $R_0 \lhd F$,

$$R_0 \cap F' = [R_0, F].$$

Indeed, if $F' = C_3$, either R_0 contains that subgroup, in which case a transposition acts by inversion, or their intersection is trivial. An invariant line on R_0 kills this intersection and extends through the abelianization of F. Taking its ℓ'-part gives the required extension of the correction line on Y/K. It leaves the entire diagonal anchor unchanged. Corollary 2.2, with its diagonal subgroup $J = K$, now supplies the full block relation at every intermediate upper group. Its line-extension hypothesis is imposed on Y/K, just verified above; no such property is being asserted for the larger quotient Y/S.

The comparison is now directly between S and $N_S(D)$. No assertion that an upper-base block-triple relation automatically restricts to a different base group is used in this descent. Unipotent natural blocks kill their whole central two-subgroup. The construction on S is therefore already the prescribed strong quotient for their natural covers; ordinary inflation is performed only afterwards. For the full ordinary regular fibres, inflate the canonical K-extension to a regular group $\widetilde{G}$. Every character above the fixed natural character is this extension multiplied by a line of $\widetilde{G}/G$, by Gallagher's theorem; the quotient is abelian. Use the same line, restricted to the actual local regular normalizer, on its canonical local extension. Frattini gives the identical regular quotient on both sides. The associated block systems therefore preserve every ordinary regular character, every quotient linear twist, and the full scalar on $Z(\widetilde{G})$. This is the natural source system at odd nondefining primes asserted in Theorem 9.1, with the entire split triality action.

10. Type D in even characteristic

Theorem 10.1. *At every odd prime, the inductive Alperin–McKay condition holds for the simple groups of type $D_n^\epsilon(q)$ with q even. The natural groups carry the full ordinary source systems used in rational Levi subgroups.*

Throughout this section q is even and ℓ is odd.

10.1. Characteristic two and actual rational factors

In characteristic two, the identity is the only quasi-isolated semisimple parameter of adjoint type D_n; this is [2, Theorem 4.6, Example 4.8]. In fact the ordinary orthogonal model gives the stronger fact needed for transfer. For a semisimple s, decompose the natural space into its unit eigenspace and the pairs of eigenspaces for $\lambda, \lambda^{-1} \neq 1$. In characteristic two there is no separate minus-one eigenspace. The centralizer is a product of general linear factors on the nonunit pairs and a connected orthogonal factor on the unit eigenspace; the latter must have trivial Dickson invariant because the full element does. It is precisely a Levi subgroup, with the indicated rational Frobenius action. Passing through the central isogeny, whose kernel has only trivial geometric points in this characteristic, preserves this description. Thus the whole algebraic $C_{\mathbf{G}^*}(s)$ is a proper F^*-stable Levi whenever $s \neq 1$. In particular the strictly quasi-isolated initial blocks are exactly the whole unipotent blocks already constructed.

We induct on the absolute rank n, simultaneously over all field sizes and both rational signs. If $\mathbf{L}$ is a proper Levi of D_n, its simply connected derived group is a product of type-A components and at most one residual D_r; the sum of their ranks is less than n. For a cycle $j, Fj, \dots, F^{a-1}j$ of components, the map

$$x \longmapsto xF(x)\cdots F^{a-1}(x), \qquad x \in \mathbf{H}_j^{F^a}, \tag{10.1}$$

is an isomorphism onto the fixed points of that component product; projection to the first factor is its inverse. A type-A return gives $\mathrm{SL}_m(q^a)$ or $\mathrm{SU}_m(q^a)$ with its actual inner correction.

On a residual D_r-support, an ambient signed permutation restricts to a signed permutation. Modulo $W(D_r)$ its graph image has order at most two. It cannot be the half-integral triality transformation on D_4. This proves that the proper-Levi rank induction of natural D_n^ϵ uses only $D_r^\pm$ at smaller rank. At top rank D_4, every proper Levi has only type-A components; triality can permute its actual A_1^3-factors. The notations $D_3 = A_3$, $D_2 = A_1^2$, and D_1 as a torus give the low-rank interpretations in (10.1).

Let $L_0 = [\mathbf{L}, \mathbf{L}]^F$, $\mathbf{T} = Z^\circ(\mathbf{L})$. The central multiplication has kernel $\mathcal{K}$, and its fixed-point sequence is

$$1 \longrightarrow \mathcal{K}^F \longrightarrow \mathbf{T}^F \times L_0 \longrightarrow L \longrightarrow H^1(F, \mathcal{K}) \longrightarrow 1. \tag{10.2}$$

The kernel need not have two-power order: in the even-characteristic D_4 Levi of derived type A_2,

$$L \simeq \mathrm{GL}_3(q) \times \mathbb{F}_q^\times, \quad L_0 \simeq \mathrm{SL}_3(q), \quad [L : \mathbf{T}^F L_0] = \gcd(3, q-1).$$

Its scalar μ_3 intersection persists although the ambient rational centre is trivial. For $l = zh \in L$ with geometric $z \in \mathbf{T}, h \in [\mathbf{L}, \mathbf{L}]$,

$$h^{-1} F(h) = z F(z)^{-1} \in \mathcal{K}.$$

This is the genuine rational diagonal action on the factors. Choose actual quotient representatives l_a in the relevant defect normalizer. Their products and operators have the form

$$l_a l_b = m_{a,b} l_{ab}, \qquad U_a U_b = \alpha(a,b)\rho(m_{a,b}) U_{ab}. \tag{10.3}$$

The lower factor systems have the same embedded return $m_{a,b}$ and factor set. Theorem 5.1 applies to (10.1)–(10.3), including moving covered blocks and the specified central-defect quotient blocks. It gives the source map at the actual $N_L(D)$ and all ordinary regular fibres. Every type-A factor is supplied by the natural-group, all-odd-prime assertion of Theorem 7.1, including its full ordinary regular fibres. For non-universal natural factors, its proper-Levi step is Lemma 7.8, together with the direct small-factor systems in Section 7; no universal-cover hypothesis is imposed on these inputs. The needed general linear and unitary rational factors are then furnished by Theorem 5.1; the prime-three-only linear-factor proposition is not used for other primes.

Every residual D_r is at strictly smaller rank.

10.2. **The actual Jordan functors and the standard ambient**

For a semisimple ℓ'-parameter $s \neq 1$, take $\mathbf{L}^* = C_{\mathbf{G}^*}(s)$, the proper rational Levi just described, and let $\mathbf{L}$ be its dual. Put $L = \mathbf{L}^F$. This satisfies the full algebraic centralizer containment of [26, Theorems 5.8 and 5.11]; no weakening of that hypothesis is used in this section. The BDR series normalizer is L, since a rational series normalizer can be corrected inside L^* to centralize s. Let

$$\mathcal{C} = G\Gamma_c(Y_{\mathbf{U}}^{\mathbf{G}}, \mathcal{O})^{\mathrm{red}} e_s^L.$$

Its block summand gives the global equivalence and, by [3, Theorem 7.7], the actual subpair correspondence $(Q, c_Q) \mapsto (Q, b_Q)$, with subgroup coordinate unchanged. Conjugate once to install the specified actual $D \leq L$. The Brauer construction at ΔD and normalizer induction give the local equivalence between $N_L(D)$ and $N_G(D)$. If their ordinary maps are R, R_D, set

$$\Omega_b = R_D \phi_c R^{-1}, \qquad \Omega_b^{-1} = R \phi_c^{-1} R_D^{-1}. \tag{10.4}$$

The same-complex construction and its Frattini equality give the required lower equivariance.

For extension transfer through an order-three diagram element at $\ell = 3$, use the ordinary coefficient field $\mathbb{K}$, of characteristic zero, in [26, Theorems 5.8 and 5.11]. All finite orders are invertible there. For the universal natural groups, Proposition 5.3 supplies the ordinary transfer and its separate integral block anchor; here the stronger algebraic containment holds. For $\Omega_8^+(2)$ only the preceding ordinary global and local maps and their natural equivariance are used at this point. Their construction by the same Brauer complexes does not require that the finite natural group be a universal cover; the final extension step is supplied directly below.

For the following standard-ambient extension argument, first assume that the natural group is the universal cover of its simple quotient. The sole extra-cover exception in this section, $\Omega_8^+(2)$, has the same lower Brauer-complex bijection and is treated directly below.

Let $\mathcal{B}$ be the standard field–graph group. It includes the whole S_3 in split D_4, and

$$C_{\widetilde{G}\mathcal{B}}(G) = Z(\widetilde{G}).$$

[37, Theorem A] gives good natural-group origins with

$$(\widetilde{G}\mathcal{B})_\chi = \widetilde{G}_\chi \mathcal{B}_\chi, \quad \chi \text{ extending to } G\mathcal{B}_\chi. \tag{10.5}$$

When a regular conjugation selects such an origin, transport the whole tuple (b, c, D, ϕ_c), both functors and both systems, then transport the relation back. The single equivariant map (10.4) is unchanged. Theorem 3.21 of the same reference supplies the corresponding regular-group extensions.

The corrected automorphism group $\mathcal{A}$ stabilizing the Jordan Levi can have a larger centralizer than the standard $\mathcal{B}$. Outside split D_4, [27, Lemmas 2.3–2.4] transfers extension between the commuting corrected generators and the standard ones, globally and locally. Transfer (10.5) backwards through R, across the source system, and forwards through R_D, then return to $\mathcal{B}$. The final criterion is applied in the standard ambient.

For split D_4, do this for each noncyclic Sylow two- and three-subgroup of the actual inertia. The corrected groups have generators

$$\mathrm{ad}(x)\gamma_r,\ F_r, \qquad r = 2, 3, \quad x \in G^{F_0},$$

as in [27, Section 2.3, Lemmas 2.7–2.8]. The ordinary functors apply to these commuting corrected pairs and their genuine inner returns. The source transfers their extensions, and the local functor returns them. [27, Proposition 2.13] then kills every Sylow obstruction by restriction and corestriction of the Clifford cocycle. All remaining prime Sylow subgroups are cyclic. This proves joint extension to the full standard inertia.

10.3. **The block correction and ordinary regular fibres**

Since q is even, the natural type-D group has trivial rational centre and has no diagonal outer automorphisms. Its effective outer inertia is cyclic, a subgroup of $C_2 \times C_f$, or, in split rank four, a subgroup of $S_3 \times C_f$. The preceding transfer gives genuine extensions of each matched global and local character to its actual automorphism inertia. The composite (10.4) has the specified induced-block anchor and the same actual defect on both sides.

The line-extension condition in Theorem 2.1 holds for every such outer inertia. The abelian cases are immediate. If $F \le S_3 \times C_f$ is nonabelian, then $F' = C_3$. For a normal subgroup $R_0 \lhd F$, either $R_0 \cap F' = 1$, or a lift of a transposition in F inverts $R_0 \cap F' = C_3$. Hence

$$R_0 \cap F' = [R_0, F].$$

Every invariant line of R_0 therefore extends through the abelianization of F. Taking its prime-to-ℓ part preserves a prescribed line of prime-to-ℓ order. Apply Theorem 2.1 in the actual automorphism inertia, whose centralizer of the centreless natural group is trivial. It supplies every intermediate block equality and every ordinary upper coefficient, including a nonlinear coefficient of S_3. The block Butterfly theorem then gives the required semidirect automorphism realization.

For the regular fibres no diagonal ambiguity remains in this characteristic. Indeed $\widetilde{G} = GZ(\widetilde{G})$ and $G \cap Z(\widetilde{G}) = 1$, by Lang's theorem and the trivial rational centre. Each regular character above χ is $\chi \otimes \nu$, for a line ν of the regular centre. Match it to $\psi \otimes \nu$. The common line preserves the entire central scalar, all quotient twists, and the specified block at every regular inertia. Equivalently these are the maps of the integral diagonal complex and its ΔD-Brauer construction, as in [27, Lemmas 2.14 and 2.2]. This proves the full ordinary system required for subsequent Levi assembly.

The natural group $\mathrm{Spin}_8^+(2)$ is treated directly in this last extension step. The lower maps (10.4) and their inverses are obtained from the same ordinary global and local Brauer complexes and their natural equivariance. This construction does not use the universal-cover hypothesis of the final criterion in [27]. The natural group has trivial centre and an extra universal kernel V_4. Its natural lower construction is (10.4). Here $\mathrm{Aut}(S) = S \rtimes S_3$. For a matched pair, put $Y = \mathrm{Aut}(S)_\chi$, $H = N_Y(D)$, $M = N_S(D)$. Then

$$Y = SH, \qquad H/M = Y/S \le S_3, \qquad C_Y(S) = 1.$$

Every subgroup of S_3 has trivial ordinary Schur multiplier, so both characters extend. For $S_3 = \langle r, t : r^3 = t^2 = 1,\ trt^{-1} = r^{-1}\rangle$, this follows directly: normalize projective operators $U^3 = V^2 = 1$; the remaining relation is $VUV^{-1} = cU^{-1}$, $c^3 = 1$. Replacing U by ζU, $\zeta^3 = 1$, changes c by ζ^2 and removes it. The cyclic cases require only one power normalization.

Apply the prescribed-character ramification theorem. If its normal outer ramification subgroup is C_3, an invariant ℓ'-correction is trivial when $\ell \neq 3$; at $\ell = 3$ any three-primary correction changes no block. The other normal subgroup possibilities are immediate. Theorem 2.1 therefore gives the full ordinary relation, including the nonlinear S_3-coefficient. The trivial-extra-kernel block of the universal cover is precisely this natural block, by $\mathcal{O}Ue_{V_4} \simeq \mathcal{O}S$; its actual defect normalizer quotient is exact. The other three central colours are the separate cases of Theorem 12.1.

In this characteristic the natural central quotient is the natural group itself. At the extra cover just discussed, the quotient by the prescribed central kernel is taken before ordinary inflation; its defect-normalizer quotient is the actual one. For all other natural type-D groups here the standard universal-cover hypotheses apply.

At rank four the quasi-isolated initial blocks in characteristic two are the unipotent blocks of Section 9, and every proper Levi has only type-A derived factors. At rank $n > 4$, any residual D_r-factor has $r < n$. Equations (10.1)–(10.4) therefore give an induction on rank which stays in characteristic two, regardless of the extension-field size. No nonunipotent quasi-isolated block of type D in odd characteristic is an initial case of this induction.

11. Groups of type F_4

Theorem 11.1. *The simple groups $F_4(q)$ satisfy the inductive Alperin–McKay condition for every prime. The condition is understood on their universal covering groups, including $2.F_4(2)$.*

The defining prime, the prime two, cyclic defect groups, and $F_4(2)$ are covered by the results recalled in the introduction and [30, Proposition 4]. We verify below the remaining ordinary character correspondences for $G = F_4(q)$, $q > 2$, and an odd prime $\ell \nmid q$. Here G is centreless and is its own universal cover. Its outer automorphism group is cyclic; in characteristic two we include the exceptional graph–field generator. We use the cyclic-outer criterion [6, Lemma 7.1(i)] only after reaching the Brauer correspondent in the actual group $N_G(D)$.

11.1. **Sources and ordinary parameters**

Fix a block B and an actual maximal Brauer pair (D, b_D). Whenever $A \le D$ is an abelian characteristic subgroup with $C_D(A) = A$, put

$$L = C_G(A), \qquad M = N_G(A), \qquad (A, b_A) \le (D, b_D).$$

In the cases used below, except for one explicitly separated row, $L = \mathbf{L}^F = A \times K$ with $K = O^\ell(L)$ and

$$\mathrm{Irr}(b_A) = \{\lambda\theta : \theta \in \mathrm{Irr}(A)\}, \qquad \lambda = 1_A \otimes \kappa, \qquad \kappa \in \mathrm{dz}(K). \tag{F1}$$

The character λ is intrinsic: it is the unique character of b_A trivial on A. Thus its inertia group $I = M_\lambda$ is also the inertia group of b_A. Since $C_I(A) = L$, the quotient I/K has the self-centralizing normal ℓ-subgroup AK/K.

We record two ordinary character facts used repeatedly. A genuine extension E of λ to I untwists its defect-zero source:

$$\mathcal{O}Ie_\kappa \simeq \mathrm{End}_{\mathcal{O}}(V_\kappa) \otimes_{\mathcal{O}} \mathcal{O}[I/K]. \tag{F2}$$

Here one takes an I-stable lattice affording E; the defect-zero algebra $\mathcal{O}Ke_\kappa$ is the corresponding full matrix algebra. Since AK/K is normal and self-centralizing, I/K has one block. Fong–Reynolds correspondence consequently gives one block of M over the orbit of b_A, namely the actual correspondent c of B. This assertion concerns the whole ordinary block.

If I/L is an ℓ'-group, each $\theta \in \mathrm{Irr}(A)$ has a unique extension of ℓ-power order to I_θ/K. Existence follows by taking a complement to A in I_θ/K; uniqueness follows because a quotient of two such extensions has both ℓ-power order and ℓ'-order. Write it as $\widehat{\theta}$. If E is invariant under the full stabilizer of the source, then

$$E_\theta = E|_{I_\theta}\widehat{\theta} \tag{F3}$$

is covariant under every return to the chosen source. Clifford theory now gives all ordinary local characters, with parameters (θ, η), $\eta \in \mathrm{Irr}(I_\theta/L)$, modulo I/L.

On the global side we use the equivariant Jordan decomposition and unipotent e-Harish–Chandra parameterization of [25, Theorems 3.2 and 3.4]. Their equivariance includes bijective algebraic endomorphisms commuting with F. For Theorem 3.2 the ambient group is the simple group $\mathbf{G}$ of type F_4, whose centre is connected; the theorem supplies its Jordan maps for every F-stable Levi subgroup. Theorem 3.4 itself applies to an arbitrary connected reductive group, so it also applies to the semisimple centralizers below at the bad prime three. If s is an ℓ'-element, set $\mathbf{C} = C_{\mathbf{G}^*}(s)$ and $C = \mathbf{C}^F$. All semisimple centralizers in $\mathbf{G}^*$ are connected. Let $(\mathbf{M}, \mu)$ be a marked unipotent e-cuspidal pair in $\mathbf{C}$, and suppose its central ℓ-group A^* is identified dually with $\mathrm{Irr}(A)$. For $t \in A^*$ put

$$C_t = C_{\mathbf{C}}(t)^F, \qquad W_t = N_{C_t}(\mathbf{M}, \mu)/\mathbf{M}^F.$$

The global parameter attached to $\eta \in \mathrm{Irr}(W_t)$ is

$$X(t, \eta) = J^{-1}_{G,st}(I^{C_t}_{(\mathbf{M},\mu)}(\eta)). \tag{F4}$$

The local parameter is

$$Y(t, \eta) = \mathrm{Ind}^M_{I_{\theta_t}}(E_{\theta_t} \mathrm{Inf}(\eta)), \tag{F5}$$

where θ_t is the line dual to t. A rational normalizer stabilizing the $\mathbf{L}^F$-class of s can be corrected inside $\mathbf{L}^{*F}$ to centralize s. This correction fixes every element of $Z(\mathbf{L}^*)^F_\ell$, and identifies the groups W_t with I_{θ_t}/L, simultaneously for all t. This is the normalizer identification in [25, Corollary 3.3]; it is also immediate from the proof of [18, Lemma 2.1].

11.2. **Good odd primes**

Assume $\ell \geq 5$. The non-defining part of the order polynomial is

$$\Phi_1^4\Phi_2^4\Phi_3^2\Phi_4^2\Phi_6^2\Phi_8\Phi_{12}.$$

Only $\Phi_e(q)$ can contribute to its ℓ-part, so condition $(*)$ in [23] holds. By Propositions 2.3–2.4 and Theorem 2.6 there, the block B has an actual maximal pair of the form (F1), with

$$D = A = Z(\mathbf{L})^F_\ell, \qquad N_G(D) = N_G(\mathbf{L}), \qquad \ell \nmid |N_G(\mathbf{L})/L|.$$

In particular, the local group already is the defect normalizer.

Let s label λ, put $\mathbf{M} = C_{\mathbf{L}^*}(s)$ and $\mu = J_{L,s}(\lambda)$. The dual source construction of [23, Theorem 2.9 and its proof] identifies

$$A^* = Z(\mathbf{M})^F_\ell = Z(\mathbf{L}^*)^F_\ell \quad \text{and} \quad W = N_C(\mathbf{M}, \mu)/\mathbf{M}^F = I/L, \tag{F6}$$

including their mutual action. The centre closure is equivalently [18, Proposition 2.2]. The equality of the embedded central ℓ-groups also follows directly from degrees: λ and μ have central ℓ-defect, so the Jordan degree formula gives $|Z(\mathbf{M})^F|_\ell = |Z(\mathbf{L}^*)^F|_\ell$, while $Z(\mathbf{L}^*) \leq Z(\mathbf{M})$. For

each $t \in A^*$ the same marked pair is e-cuspidal in C_t. The map of blocks in [18, Theorem 1, Propositions 3.6 and 4.1] sends its unipotent block to B. Thus every character in (F4) belongs to B.

There is no fusion missing from these parameters. Indeed, if two values of (F4) coincide, first conjugate their ℓ'-parts to s. Equality of the Jordan correspondents then identifies their marked e-Harish–Chandra pairs inside C_t. Correcting by that conjugacy leaves an element of $N_C(\mathbf{M}, \mu)$, and hence exactly the W-identification of (t, η). This proves injectivity on the parameter orbits. For completeness, the total number of these orbits is

$$|\operatorname{Irr}(A \rtimes W)| = |\operatorname{Irr}(B)| \tag{F7}$$

by [23, Theorems 2.8–2.9]. Consequently the injection is surjective onto the entire ordinary block. This use of Malle's ordinary isotypy is stronger than a count of a unipotent basic set. All its characters have height zero, and the local assertion follows from [23, Proposition 2.7].

The required source extensions exist by [1, Corollary 4.20, Lemma 4.21 and Theorem 7.1(b)]. For nonunipotent sources their invariance under the full field stabilizer is [1, Proposition 8.13]; the cases $l(B) = 1$ have $W = 1$ by [23, Proposition 3.6]. For unipotent sources use [37, Theorem 3.3, Lemmas 3.4–3.6 and Remark 3.7(a)]. The characteristic-two exceptional returns are verified below. Formula (F3) supplies all central ordinary twists, so (F4) and (F5) give a full-stabilizer equivariant bijection.

11.3. **The principal three-block**

Now let $\ell = 3$, $e = \operatorname{ord}_3(q)$, $\varepsilon = 1$ or -1 according as $e = 1$ or 2, and $a = v_3(q - \varepsilon)$. Tables 1–19 of [1] show that the principal block is the unique block of full defect 3^{4a+2}. The apparent full-defect $A_2\widetilde{A}_2$ centralizer does not provide another block in a $3'$-series: its isolated parameter has order three. Brauer's first main theorem gives the same uniqueness for the full-defect block of $N_G(P)$, $P \in \operatorname{Syl}_3(G)$. The equivariant McKay correspondence [13, Theorem B] therefore restricts to the two principal blocks. We use uniqueness of the full-defect block here, not an assertion that either group has only one block.

11.4. **All nonprincipal nonabelian three-blocks**

The exhaustion of the remaining nonabelian defect groups is [1, Proposition 6.18(b) and Section 6.20]: they belong to the rows $R_{29}, \dots, R_{34}$, with the existence conditions in its Tables 25–26. Thus a row absent at a small value of q introduces no additional case. In Table 26 the rows $R_{29}, \dots, R_{32}$ require $q \neq 2$, whereas R_{33}, R_{34} require $q \notin \{2, 4, 8\}$; a block row additionally requires an actual $3'$-parameter s of the indicated rational class type. The abelian subgroup in their torus-normalizer models is as follows; the intrinsic argument after the table proves its uniqueness.

D	A in the notation of AHL	A abstractly	$\lvert D\rvert$
R_{29}, R_{30}	R_{12}, R_{11}	$C_{3^{a+1}} \times C_{3^a}$	3^{2a+2}
R_{31}, R_{32}	R_{16}, R_{17}	$C_{3^a}^3$	3^{3a+1}
R_{33}, R_{34}	R_{18}	$C_{3^a}^4$	3^{4a+1}

(F8)

There is a small-field qualification to the names in the second column. Table 26 omits R_{11}, R_{12} from its list of radical subgroups at $q = 4, 8$, although R_{29}, R_{30} are present. For those two values we take A to be the three-part of the same mixed torus in the Sylow model of D; it is normal of index three and has the displayed abelian structure. We do not use the additional assertion in AHL Lemma 6.9 that this A is radical. Neither our source construction nor Theorem 3.1 requires that assertion. Away from these values, the indicated centralizers and subgroup names agree with AHL Lemmas 6.7–6.9. In the first and last rows $C_G(A)$ is respectively a mixed maximal torus of order $(q - \varepsilon)(q^3 - \varepsilon)$ and a pure ε-torus of order $(q - \varepsilon)^4$. In the middle row it is a rank-one Levi subgroup. These are actual rational groups; the rational central-isogeny coset in $(\mathrm{SL}_3^{\varepsilon} \circ_3 \mathrm{SL}_3^{\varepsilon}).3$ is retained.

Here is also an intrinsic check of the subgroup used in (F8). Write $D = \langle A, u\rangle$ with D/A of order three. Its action satisfies $(u-1)^2A \neq 1$, whence $D' = (u-1)A$ is noncentral and $C_D(D') = A$. An abelian normal subgroup not contained in A would generate D together with A, hence contain D' and centralize it, a contradiction. Thus $A = C_D(D')$ is characteristic and $C_D(A) = A$. This is the argument of AHL Lemma 6.9. It can be checked directly in the F_4 root lattice: for a mixed torus put $w = s_1s_2$ or s_3s_4, $N = 1 + w + w^2$, $K_0 = w + 2w^2$. A character x is trivial on its three-part precisely when

$$(N + 3^a vK_0)x \equiv 0 \pmod{3^{a+1}}, \qquad 3 \nmid v. \tag{F9}$$

Enumeration of the 48 roots and 1152 Weyl matrices gives no trivial root and only the identity pointwise stabilizer. For $a \geq 3$ the largest entries of $N\alpha$ and $N(h-1)$ are at most 12 and 16, so the congruence reduces to $Nx = 0$, $K_0x \equiv 0 \pmod 3$. The perpendicular A_2 has $(u-1)^2A \neq 1$; the two pure-torus order-three classes with two-dimensional fixed space both have size 32 and nonzero square difference modulo three. The rank-one annihilator has exactly the roots $\{\alpha, -\alpha\}$ and pointwise Weyl stabilizer $\{1, s_\alpha\}$. These finite checks certify the asserted centralizers and actions; the ordinary character exhaustion is proved separately below. There really are $3'$-parameters for both mixed rows at the two small fields. For example, use the simple roots $e_2 - e_3, e_3 - e_4, e_4, (e_1 - e_2 - e_3 - e_4)/2$, and represent a torus point by its values on these roots, as exponents modulo a prime r. For $w = s_1s_2$ or $w = s_3s_4$, one may take

q	r	w	y	rational type
4	7	s_1s_2	$(5,3,1,0)$	$(17,5)$
4	7	s_3s_4	$(0,1,4,1)$	$(13,5)$
8	19	s_1s_2	$(15,10,1,0)$	$(17,4)$
8	19	s_3s_4	$(0,10,11,1)$	$(13,4)$

Each column vector y satisfies $(\varepsilon qw - 1)^{\mathsf{T}}y = 0 \pmod r$. Exactly six roots annihilate it, forming the perpendicular A_2; its stabilizer among all 1152 Weyl matrices is precisely that A_2 Weyl group, of order six, and commutes with w. The long-A_2 twist s_1s_2 has a short-A_2 centralizer (type 17), while the short-A_2 twist s_3s_4 has a long-A_2 centralizer (type 13). This gives the rational types in the table, as in AHL Tables 23 and 25. The executable calculation `f4_small_field_parameters_round3.py` records the coordinates, matrices and assertions. Identifying their blocks and the dual actual defect groups still uses AHL Tables 13 and 17 and Proposition 6.18; parameter existence alone does not supply that identification.

For the rank-one row, the kernel and rational cokernel of the central multiplication map have order at most two. Its derived finite group is $\mathrm{SL}_2(q)$, which is perfect for the present $q > 2$, $3 \nmid q$. Hence $L = A \times O^3(L)$. The other rows have the same decomposition by primary decomposition of their tori. Defect intersection for the actual subpair at A now gives (F1). In every row the canonical character λ is regular semisimple in L. This is immediate for a torus; in rank one its three-part forces the opposite torus of order $q + \varepsilon$ in the derived factor. The canonical character kills A, so its parameter has no nontrivial central three-part. The degree test is also the rank-one case of [1, Lemma 6.14(b)].

Recovering the precise global parameter. The label of this source is the actual rational $3'$-class s labeling B. To see this without losing a rational flag, take a toral pair $(\mathbf{S}, \zeta)$ for the source, with $A \leq \mathbf{S}$ and ζ of $3'$-order. Apply [3, Proposition 4.11 and Theorem 4.14] along a cyclic flag of A. Every intermediate identity component contains the same full-rank torus $\mathbf{S}$. The torus enlargement in the cyclic construction is therefore $\mathbf{S}$ itself. The flag generators centralize this torus, so its Lang commutator is trivial. On quotienting by a flag subgroup the kernel and cokernel on rational torus points are three-groups; ζ has a unique $3'$-order continuation. Thus i_A sends the source pair to the identical rational toral pair in $\mathbf{G}$. Since $\mathrm{Br}_A(e_s)b_A \neq 0$, the BDR identity

$$\mathrm{Br}_A(e_s) = \sum_{i_A(Y) = \mathcal{E}(s)} e_Y \tag{F10}$$

identifies the source parameter with s in its actual rational conjugacy class. The argument allows the disconnected intermediate groups in BDR Proposition 4.11.

By the proof of [1, Proposition 6.18(b)], a block in $\mathcal{E}_3(G,s)$ not corresponding to the principal block of C has abelian defect. Our block is therefore the row marked 1 in column 5 of the relevant table and has $|D| = |C|_3$. The tables determine this block uniquely by $(s,|D|)$, as stated in AHL Section 6.16. The semisimple character of $\mathcal{E}(G,s)$ has degree $|G|_3/|C|_3$ in three-part, so its block has this maximal defect and is B. Accordingly [18, Proposition 4.3] applies to this precise block. The inverse for every height-zero ordinary character. Put $\mathbf{T}^* = C_{\mathbf{L}^*}(s)$. The local rows give

$$A^* = (\mathbf{T}^{*F})_3 = Z(\mathbf{L}^*)_3^F, \qquad W = N_C(\mathbf{T}^*)/\mathbf{T}^{*F}, \qquad |C|_3 = 3|A^*|, \quad |W|_3 = 3. \tag{F11}$$

For $t \in A^*$ let $W_t = N_{C_t}(\mathbf{T}^*)/\mathbf{T}^{*F}$. These are precisely the local inertia quotients for the twists $\lambda\theta_t$. A Sylow subgroup P_C of this torus normalizer is Sylow in C, and $Z(P_C) \le A^*$ because A^* is self-centralizing in P_C.

Take an arbitrary $\chi \in \mathrm{Irr}_0(B)$, say $\chi \in \mathcal{E}(G,st)$, and write $\rho = J_{G,st}(\chi)$. The height formula gives

$$[C : C_t]_3\, \rho(1)_3 = 1. \tag{F12}$$

Thus t centralizes a Sylow three-subgroup of C. After conjugating that subgroup to P_C, the group $P_C\langle t\rangle$ is a three-subgroup, so Sylow maximality gives $t \in Z(P_C) \le A^*$. The torus $\mathbf{T}^*$ is the centralizer of a Sylow e-torus in C_t: in the mixed row its derived A_2 is split of sign ε and its remaining torus has order $\Phi_{3e}(q)$; in the other rows Frobenius acts as εq on the relevant root subsystem. Malle's prime-to-three unipotent criterion [21, Proposition 6.5 and Corollary 6.6] therefore places ρ in the toral e-Harish–Chandra series. Rossi's degree formula identifies its coefficient η and gives exactly

$$\mathcal{P}_B = \{(t,\eta) : t \in A^*,\ \eta \in \mathrm{Irr}(W_t),\ [W : W_t]_3\eta(1)_3 = 1\}/W. \tag{F13}$$

This is an inverse starting from an arbitrary ordinary character. Conversely every parameter in (F13) lies in the principal unipotent block of C_t and hence in B by [18, Theorem 1 and Proposition 4.3]. If two parameters give the same character, conjugate their $3'$-parts to s and restore the toral e-Harish–Chandra support inside C_t. The remaining conjugation belongs to $N_C(\mathbf{T}^*)$. Thus the quotient by W is exactly the global fusion relation.

Extensions for all ordinary parameters that occur. For the toral sources in characteristic two, $N_G(\mathbf{T}) = T \rtimes C_V(v)$, where V is the Tits group, v is the rational twist, and the Tits kernel is trivial. The extension $E_\xi(tw) = \xi(t)$ on its inertia is covariant under field returns; the relation $F_p^f = \mathrm{Int}(v^{-1})$ on this frame is retained. By [1, Proposition 5.13(a) and Section 8.21], no nonprincipal nonabelian three-block is stabilized by an odd power of the exceptional graph–field generator.

The rank-one sources in characteristic two also admit extensions for every ordinary twist. In the standard positive or negative torus frame, [1, Proposition 4.19(a),(b) and its proof] gives

$$L = Z \times K_0, \qquad M = (Z \rtimes V_0) \times K_0, \qquad K_0 \cong \mathrm{SL}_2(q), \quad V_0 \cong W(C_3).$$

Indeed the root lifts in V_0 centralize K_0 because the possible signs in their root actions are trivial in characteristic two; this is the explicit calculation $n^{-1}u_\alpha(x)n = u_\alpha(\pm x)$ in the proof of AHL Proposition 4.19(a). The Tits kernel is trivial, so these lifts give the stated complement, rather than merely its Weyl quotient. The negative longest lift centralizes these lifts, so the same complement survives the rational negative twist. More precisely, in the frame with Frobenius $F_v = \mathrm{Int}(\dot w_\varepsilon)F_q$, where $\dot w_\varepsilon = 1$ or the longest lift, F_p preserves Z, V_0 and K_0 and satisfies $F_p^f = \mathrm{Int}(\dot w_\varepsilon^{-1})$ on the fixed-point group. If $\lambda\theta = \zeta_\theta \otimes \nu$, put $E_\theta(zvk) = \zeta_\theta(z)\nu(k)$ on its inertia. Its inertia is $(Z \rtimes (V_0)_{\zeta_\theta}) \times K_0$, so this is a genuine extension for every ordinary twist. The formula is covariant under F_p and under conjugation by V_0; conjugation by L acts innerly on the corresponding inertia character. Since F_p preserves A, any return to this source is a product of a field power and an element of M. Consequently covariance holds for the complete field stabilizer, including its relation $F_p^f = \mathrm{Int}(\dot w_\varepsilon^{-1})$. This case does not require a weight-relevant subpair.

For odd q, in the pure-torus row put $H_2 = T \cap V$. All three-power twists have the same restriction $\delta = \lambda|_{H_2}$. The construction in [33, Lemmas 5.5–5.6 and 6.1] gives a line ξ_δ on V_δ extending δ, with value one on the specified generating root lifts. Thus

$$E_\theta(tv) = (\lambda\theta)(t)\xi_\delta(v), \qquad v \in V_{\lambda\theta}, \tag{F14}$$

is an extension. Field automorphisms fix V; the lift of the longest Weyl element is central in V. A return stabilizing the source has Tits part in V_δ, where the line is invariant under inner conjugation. Formula (F14) therefore respects the full field stabilizer.

In any characteristic, for a mixed-torus row $W \cong \mathfrak{S}_3$ is the full Weyl group of a regular A_2 Levi. Its connected root stabilizers are 1, C_2 and $\mathfrak{S}_3$. Condition (F13) forces $W_t = W$, so t is central in $\mathbf{C}$. Let $\mathbf{H}$ be the regular Levi in $\mathbf{G}$ dual to $\mathbf{C}$, with the source torus $\mathbf{T} \leq \mathbf{H}$. The dual normalizer identification gives $I = I_{\theta_t} = N_H(\mathbf{T})$, where $H = \mathbf{H}^F$. The linear character of H with central dual parameter st restricts to $\lambda\theta_t$ on T, so its restriction to I is the required extension. A return fixes the $3'$-part s and normalizes $\mathbf{H}$; the linear character of H is determined by its restriction to T and is therefore fixed whenever that twist is fixed. This proves covariance under the complete field return. This construction uses the regular A_2 Levi and its linear character, not maximal extendibility for an e-split Levi of $\mathbf{G}$; it therefore also covers $q = 4, 8$ with the intrinsic mixed-torus subgroup A specified above.

For the rank-one row with odd q, maximal extendibility is [1, Corollary 4.20 and Lemma 4.21]. The possible geometric and rational centralizer types, and their full relative groups, are

i	$\varepsilon = 1$	$\varepsilon = -1$	W
6	$(6, 2)$	$(6, 1)$	$W(B_3)$
8	$(8, 2)$	$(8, 3)$	$W(A_3)$
13	$(13, 2)$	$(13, 3)$	$W(A_2)$
10	$(10, 2)$	$(10, 1)$	$W(C_3)$
17	$(17, 3)$	$(17, 2)$	$W(A_2)$

(F15)

The full field stabilizer of each such source has odd order. In fact an even return degree forces $\varepsilon = 1$; AHL Table 23 then gives rational type 1 for $i = 6, 8, 10$, and type 1 or 5 for $i = 13, 17$. None occurs in the first column of (F15). This is the field-return argument used in the proof of AHL Proposition 8.13; it is applied here to the actual source, independently of whether A supports a weight. Each W_t is the Weyl group of a connected root subsystem, since Frobenius fixes the ambient Weyl group pointwise. Hence its group of linear characters is an elementary abelian two-group. The extensions of $\lambda\theta_t$ form a torsor for this group. The point stabilizer acts on that torsor through the odd-order outer stabilizer. Averaging in the two-group gives a fixed extension. Choosing on orbits and transporting supplies the covariant extensions needed in (F5).

These constructions give the full local inverse as well. A local character over $\lambda\theta$ has degree $[M : I_\theta]\lambda(1)m$, where m is the degree of its irreducible projective Clifford coefficient. Since $|W|_3 = 3$, height zero forces

$$m_3 = |W_\theta|_3/3.$$

Thus $|W_\theta|_3 = 3$ and $m_3 = 1$. These are exactly the twists treated above; their genuine extensions turn the coefficient into an ordinary character η. The parameters are exactly (F13).

The actual defect normalizer. Matching (F4) and (F5) first gives a bijection from $\mathrm{Irr}_0(B)$ to $\mathrm{Irr}_0(c)$ in M. Correct an automorphism fixing B, D by an element of $N_G(D)$ to fix the maximal Brauer pair. It then fixes the intrinsic source, and all preceding constructions commute with its action. The correcting element acts innerly on G and M.

Apply Theorem 3.1 to $K \lhd M$, with source κ. Its quotient I/K has the self-centralizing normal three-subgroup AK/K: if xK centralizes AK/K, then $[x, A] \leq A \cap K = 1$, so $x \in L = AK$. The identities $A = C_D(D')$ and $K = O^3(C_G(A))$ ensure invariance in the entire block-and-defect stabilizer. The actual group D is contained in I, since it stabilizes its subpair at A and hence the intrinsic character λ. Moreover $N_M(D) = N_G(D)$, since A is characteristic in D. Brauer

correspondence therefore supplies a block of M with the prescribed subpair at D. It lies over the M-orbit of b_A, so the uniqueness in (F2) identifies it with c, with this actual defect group. We obtain an equivariant bijection from $\mathrm{Irr}_0(c)$ to the actual Brauer correspondent in $N_G(D)$, with the ordinary upper character systems of that theorem. Compose the two bijections. Only now do we apply [6, Lemma 7.1(i)].

11.5. **The abelian three-blocks**

Cyclic defect is already known, so assume D noncyclic abelian. The source classification is [1, Corollary 6.19 and Lemmas 6.13–6.14]. Apart from its disconnected-centralizer row, $D = A$ is the three-part of the centre of the regular subgroup $\mathbf{L}$. This is an e-split Levi except for the mixed-torus rows with s of class type $(18,3),(19,8)$ when $e = 1$, and $(18,7),(19,9)$ when $e = 2$. In those exceptions $\mathbf{L}$ is an F-stable maximal torus; its dual torus is still the centralizer of a Sylow e-torus in $\mathbf{C}$. There are exactly two source forms: a regular semisimple source, and

$$A = C_{3^a}^2, \qquad [\mathbf{L}^*, \mathbf{L}^*] \text{ of type } B_2, \qquad \lambda = \widehat{s}\,\zeta_e, \qquad \zeta_e(1) = q(q-\varepsilon)^2/2, \tag{F16}$$

where $s \in Z(\mathbf{L}^*)^F$ is a $3'$-element. The BDR argument (F10) applies to a toral pair in the rational series of either source, so it recovers the same global s. In the regular semisimple cases the preceding toral inverse applies with $|C|_3 = |A^*|$ and $|W|_3 = 1$: among the noncyclic abelian rows of AHL Tables 1–19 these are precisely the principal-centralizer rows. The other nonprincipal-centralizer rows are either cyclic or have the B_2 source below. Thus the same maximal-defect argument identifies B as the semisimple block, and KM25 Proposition 4.3 gives the reverse block arrow. In (F16), AHL Section 6.16 and the B_2-marked table row identify the individual block from s and $|A| = 3^{2a}$. Its relative group satisfies $W \le D_8$.

We give the missing ordinary inverse in the B_2 case explicitly. Put $\mathbf{H} = [\mathbf{L}^*, \mathbf{L}^*]$, $R = C_{\mathbf{C}}^\circ(\mathbf{H})^F$ and $Q = C_{\mathbf{C}}(\mathbf{H})^F$. Use the usual Euclidean F_4 roots and put the B_2 subsystem on the e_3, e_4 plane. In odd characteristic its connected centralizer has roots $\pm e_1 \pm e_2$; in characteristic two it also has $\pm e_1, \pm e_2$, since the extra root commutators have coefficient ± 2. The cocharacter lattice on this plane is generated by $e_1 + e_2, e_1 - e_2$. Thus the respective derived centralizers are simply connected of type A_1^2 or B_2. Their displayed maximal torus is $Z(\mathbf{L}^*)$, with Frobenius εq. Centralizing s retains a root subsystem of this group. Its degrees are among two and four, and the other factors of those degrees are prime to three. Hence

$$|R|_3 = 3^{2a} = |A^*|, \qquad A^* = Z(\mathbf{L}^*)_3^F \in \mathrm{Syl}_3(R). \tag{F17}$$

This also controls the possibly disconnected centralizer Q. For $g \in Q$, the groups $(A^*)^g$ and A^* are Sylow subgroups of the normal subgroup R, so correct g by an element of R to normalize A^*. Since $C_{\mathbf{G}^*}(A^*) = \mathbf{L}^*$, the corrected element normalizes $\mathbf{L}^*$ and fixes its unipotent character ζ_e. Its image modulo $Z(\mathbf{L}^*)^F$ embeds in $W \le D_8$. As the three-part of this centre is already in R, Q/R is a $3'$-group. Consequently every three-element of Q lies in R and is R-conjugate into A^*.

Now take any ordinary $\chi \in \mathrm{Irr}(B)$ in the st-series, and let $\rho = J_{G,st}(\chi)$. The block containing ρ has a marked unipotent e-cuspidal pair $(\mathbf{M}_t, \mu_t)$ of quasi-central three-defect, as in the proof of KM25 Proposition 4.1(b). By [18, Theorem 1 and Proposition 4.1], its transported pair defines the particular block B. This implication must be combined with injectivity: [17, Theorem A(d)] gives a bijection on e-Jordan quasi-central cuspidal pairs for F_4 at three, and [18, Proposition 2.2] is a bijection on their dual unipotent marked pairs. Both pairs are in this scope. For the source, ζ_e is the unipotent e-cuspidal character in AHL Lemmas 5.7 and 6.14(a), and

$$\zeta_e(1)_3 = 3^{2a} = |\mathrm{Sp}_4(q)|_3.$$

Thus it has three-defect zero on the derived B_2 factor, and central, hence quasi-central, three-defect on $\mathbf{L}^{*F}$; for the transported pair, quasi-central defect is retained by KM25 Proposition 3.6. Thus that transported pair is C-conjugate to $(\mathbf{L}^*, \zeta_e)$, and the derived pair of $(\mathbf{M}_t, \mu_t)$ is C-conjugate to $(\mathbf{H}, \zeta_e|_{\mathbf{H}^F})$. The replacement in Proposition 3.6(b) of that paper requires an E_n component with $n \ge 6$, as its proof states, and cannot occur in F_4. After this conjugation, $t \in Q$.

By (F17) and the preceding paragraph we may assume $t \in A^*$. Both marked pairs are now e-split in $C_{\mathbf{C}}(t)$ and have the same derived pair. The rational-conjugacy argument at the start of the proof of KM25 Proposition 3.6 restores the entire marked pair inside this centralizer, while fixing t.

There are no omitted unipotent coefficients at this last step. If $st \neq 1$, the list of semisimple centralizers in [1, Table 25] shows that its centralizer has classical components and no rational ${}^3D_4$ component. In particular, the $A_2\widetilde{A}_2$ centralizer cannot contain this B_2 subsystem. For classical groups at three, the cuspidal block classification in [15, Assumption 2.1.2 and Proposition 2.1.7], together with the unipotent e-Harish–Chandra partition, says that the unipotent characters of this marked block are exactly its marked series. This cited assumption permits three for classical groups. If $st = 1$, then $s = t = 1$, and [1, Lemma 5.7] lists exactly the five unipotent characters of the B_2-source block. Thus in both cases $\rho = I^{C_t}_{(\mathbf{L}^*,\zeta_e)}(\eta)$ for a unique ordinary relative coefficient, up to the prescribed normalizer action. The converse follows from KM25 Theorem 1. The same source-restoring conjugation proves that its fusion is exactly the W-fusion of the parameters.

For all the connected-centralizer abelian rows, the source extension is supplied by AHL Theorem 7.1(b) and Proposition 8.13, or by the unipotent extension result of Späth when the source is unipotent. Here (D, b_D) is a maximal abelian pair and its inertial quotient is a $3'$-group, so it is relevant in AHL's sense. If $W = 1$ the extension is tautological; otherwise the tables give $l(B) > 1$, as required in its Hypothesis 8.2. This argument includes the non-e-split mixed tori just identified. Since W is a $3'$-group, (F3) supplies every ordinary twist. The degree formula gives height zero on both sides. This proves the full ordinary bijection for (F16); here $M = N_G(D)$ from the start.

The disconnected centralizer. The remaining row is $D = R_8 \cong C_3^2$, with

$$S = \mathbf{S}^F, \qquad |S| = \Phi_{3e}(q)^2, \qquad C_G(D) = S.3, \qquad N_G(D)/S = 3.\mathrm{SL}_2(3).$$

For a block with this defect, the $3'$-line λ_s on S has a free orbit under the group of order 72; this is precisely the defect-group case of [1, Lemma 6.13(a)]. The local source is $\mathrm{Ind}_S^{S.3}(\lambda_s)$. The sum of its three torus block idempotents is primitive by Fong–Reynolds correspondence, since the inertia of λ_s is S. Formula (F10) identifies its actual maximal pair and global parameter. The nine local characters are

$$\mathrm{Ind}_S^{N_G(D)}(\lambda_s\theta_t), \qquad t \in (\mathbf{S}^{*F})_3.$$

Globally $C_{\mathbf{G}^*}(s) = C_{\mathbf{G}^*}(st) = \mathbf{S}^*$, so there are exactly nine singleton Jordan series. They are in one block by [18, Lemma 2.7]. There is no additional fusion after fixing s, since its normalizer orbit is free. The inverse is restriction to S locally and the Jordan parameter globally. A return fixing s is unique modulo S, which proves full equivariance without an extension choice.

11.6. **Exceptional returns and the small covering group**

Let q be even. The source extensions for unipotent characters in [37, Theorem 3.3 and Remark 3.7(a)] use all bijective algebraic endomorphisms, and hence include the exceptional isogeny. For a nonunipotent block fixed by an odd graph–field power, AHL Proposition 5.13 and Section 8.21 leave toral sources or mixed sources of derived type A_1^2 or B_2. For the toral sources use the complement $C_V(v)$ and the extension already described: the exceptional endomorphism preserves the Tits representatives, and returning to the torus frame uses a Tits representative.

For the mixed sources the commuting root lifts in [1, Propositions 4.18–4.19 and the proof of Proposition 8.22] give a complement $V_0 \cong C_2^2$ or D_8 centralizing the derived source. In characteristic two there is no toral Tits kernel and the group has the product form $(Z \rtimes V_0) \times K_0$. The negative-torus twist centralizes V_0, and the exceptional endomorphism preserves the indicated mixed root subsystem and its Tits group. For an invariant source $\lambda = \zeta \otimes \nu$, the formula

$$E(zvk) = \zeta(z)\nu(k)$$

defines a genuine extension on its inertia and is invariant under the actual return. Here ν need not be unipotent: invariance of λ and preservation of the two factors suffice. The unique power-order extensions in (F3) supply all ordinary central twists. These observations complete the full automorphism action in the preceding good-prime and abelian three-block constructions.

For $q > 2$ every branch has now reached the actual Brauer correspondent in $N_G(D)$. The maps are equivariant under the block-and-defect stabilizer, $Z(G) = 1$, and $\mathrm{Out}(G)_B$ is cyclic. Thus [6, Lemma 7.1(i)] applies to each of these blocks, including the principal three-block.

For clarity about the exceptional Schur multiplier, the result used for $F_4(2)$ is the published iAM statement [30, Proposition 4], whose proof treats the double cover. Its full-defect three-block argument can be written with the following precise normalizer calculation. Put $X = 2.F_4(2).2$. The proof of [22, Theorem 4.1] places its Sylow-three normalizer inside

$$H = C_2 \times (\mathrm{PSL}_4(3).2^2).$$

If P is the upper-unitriangular Sylow three-subgroup, then $N_X(P) = N_H(P)$. A diagonal representative centralizing $P/\Phi(P)$ is scalar; a graph coset exchanges the first and third simple-root spaces and cannot centralize it. The remaining unitriangular centralizer is $Z(P)$. Hence

$$C_X(P) = Z(P) \times C_2.$$

There are exactly two full-defect blocks of $N_X(P)$, distinguished by their central C_2 character, and therefore exactly two such blocks of X. The central-character-preserving McKay map of [13, Theorem B] restricts to each colour. This supplies the upper-group Alperin–McKay assertion used in Sambale's proof. The other blocks are treated there by the cyclic-defect and explicit defect-two arguments. In particular one does not identify the nonsolvable overgroup H with a Sylow normalizer. The universal-cover case and hence Theorem 11.1 follow.

12. The extra cover of $\Omega_8^+(2)$

Theorem 12.1. *The additional central-character sectors of the universal cover of $\Omega_8^+(2)$ satisfy iAM at every odd prime.*

Proof. Write $S = \Omega_8^+(2)$, and let U be its universal cover. Its centre is V_4, and the outer group S_3 permutes its three nontrivial central characters [14, 22]. For each such character ν, first work on the exact quotient $G = U/K$, where $K = \ker \nu$. Thus $G = 2.S$, its central character is faithful, and its full outer colour stabilizer is C_2. It suffices to construct one equivariant system and transport it over this orbit of three colours.

Here is an actual group model for that stabilizer. Generate the Weyl group $Y = W(E_8)$ on the 240 roots. In doubled coordinates these roots are the 112 vectors with two entries ± 2, together with the 128 vectors with eight entries ± 1 and an even number of minus signs. The reflection in a root α is $v \mapsto v - (v \cdot \alpha)\alpha/4$. Let L be the lattice generated by the roots after dividing these coordinates by two. Its Gram determinant is one. On $L/2L$ the quadratic form $x \mapsto (x, x)/2 \pmod 2$ is nonsingular and has 136 zeros, including zero, so has plus type. Two roots have the same image modulo two precisely when they differ by sign: otherwise their difference has norm less than eight, whereas a nonzero vector in twice this even lattice has norm at least eight. The kernel of the Weyl action on the quotient is therefore $\{\pm I\}$; connectedness of the root system forces an element fixing every root line to be a common scalar. The central inversion belongs to Y: it is the product of reflections in eight mutually orthogonal roots. The computed orders

$$|Y| = 696729600, \qquad |Y'| = 348364800, \qquad |Z(Y')| = 2$$

identify the image with the full orthogonal group $O_8^+(2)$ and its derived image with S. The script also verifies that Y' is perfect. Since the determinant kernel has index two, it equals Y' and contains $-I$. Thus $G = Y'$ is the required perfect double cover.

We also need the action, not just these group orders. An element of $Z(Y)$ preserves every root line, so the same connectedness argument gives $Z(Y) = \{\pm I\} = Z(G)$. If an element outside G centralized G, it would, together with G, generate Y, and would therefore be central in Y, a contradiction. Hence

$$C_Y(G) = Z(G).$$

Conjugation by Y consequently has image of order $2|S|$ in $\mathrm{Aut}(G)$, exactly the full colour stabilizer. In particular its nontrivial outer action is the same action as that induced by the specified library inclusion $2.S < 2.S.2$; there is only one nonidentity element of $\mathrm{Out}(G)$.

For $\ell = 3, 5$, let $P \in \mathrm{Syl}_\ell(G)$, $M = N_G(P)$, and $H = N_Y(P)$. The following data are obtained in that permutation inclusion; the supplied GAP scripts construct the roots, the groups, the Sylow subgroups, the normalizers, and their class fusion. The ordinary global tables and their specified index-two fusion are from [4].

ℓ	$\lvert P\rvert$	$\lvert M\rvert$	$\lvert H\rvert$	$\lvert C_G(P)\rvert$	faithful local degrees	involution orbits
3	243	1944	3888	18	$1^4, 2^{10}, 4^4$	$1^{12}, 2^3$
5	25	800	1600	50	$1^8, 2^2, 8^6$	$1^8, 2^4$

In both rows $C_G(P) = Z(P) \times Z(G)$. The centralizer has exactly one block over the faithful character of $Z(G)$, because $Z(P)$ is an ℓ-group. This block is $N_G(P)$-stable. Brauer's first main theorem therefore gives exactly one faithful block b of M with defect P. The global calculation gives exactly one faithful block B of full defect, with

$$(k(B), k_0(B)) = (27, 18) \quad (\ell = 3), \qquad (k(B), k_0(B)) = (16, 16) \quad (\ell = 5).$$

Brauer correspondence preserves the central ℓ'-character, so $b^G = B$. Its global height-zero set has the same orbit data as the local set displayed above. Pair singleton orbits and two-element orbits, using the actual order-two action to transport the second member of each pair. Inner automorphisms of M act trivially on its characters, and $H/M \simeq Y/G$; thus this is an $\mathrm{Aut}(G)_{B,P}$-equivariant, central-character-preserving bijection $\Omega : \mathrm{Irr}_0(B) \longrightarrow \mathrm{Irr}_0(b)$.

For clarity, the passage from these finite data to the full inductive condition is a character-triple argument. For $\psi = \Omega(\chi)$, put $Y_\chi = \mathrm{Stab}_Y(\chi)$ and $H_\chi = N_{Y_\chi}(P)$. Then Y_χ is Y or G, according as χ is fixed or moved by the outer action, and equivariance gives $H_\chi = \mathrm{Stab}_H(\psi)$. Frattini gives $Y_\chi = GH_\chi$, with $G \cap H_\chi = M$. The quotients Y_χ/G and H_χ/M are cyclic, so χ and ψ have genuine extensions to those two groups. Their factor sets are both one, and their scalar characters coincide on the whole $C_{Y_\chi}(G) = Z(G)$. Moreover P is an actual defect group of both b and B, $b^G = B$, and $C_{Y_\chi}(P) \le H_\chi$. The effective quotient $Y_\chi/(GC_{Y_\chi}(G))$ is trivial or C_2, so it satisfies the linear-extension hypothesis of Theorem 2.1. That theorem supplies

$$(Y_\chi, G, \chi) \ge_b (H_\chi, M, \psi),$$

with every intermediate group and every ordinary upper coefficient included.

Here is the precise change to the ambient in (2.3). Put $\Gamma = \mathrm{Aut}(G)_{B,P}$, $X = G \rtimes \Gamma_\chi$, and let $\varepsilon : X \longrightarrow \mathrm{Aut}(G)$ be conjugation on G. The conjugation images of X and Y_χ both equal $\mathrm{Aut}(G)_\chi$: any character stabilizer can be corrected by an inner automorphism to stabilize P. The image of H_χ is precisely $\mathrm{Aut}(G)_{\chi,P}$. Consequently its actual inverse image in X is

$$\varepsilon^{-1}(\mathrm{Aut}(G)_{\chi,P}) = M \rtimes \Gamma_\chi = N_X(P).$$

The block Butterfly theorem [27, Theorem 1.10] therefore gives the required relation on these actual semidirect groups. This also retains the scalar condition on all of $C_X(G) = \ker \varepsilon$, which need not be central in X, and the ordinary coefficients for noncyclic subgroups of Γ_χ.

All other positive-defect faithful odd blocks in the ordinary tables have defect one. At seven this also follows from $|G|_7 = 7$. Their lifts to U still have cyclic defect, so they use [20, Theorem 1.1 and Definition 7.2(iii)] on the universal cover. The proof of that theorem uses $M = N_U(D)$ and the quotient by $\ker \chi \cap Z(U)$. Defect-zero blocks have no noncentral-defect requirement and use

the identity construction. The extra central kernel has order prime to ℓ; its trivial-character idempotent $e_K = |K|^{-1}\sum_{k\in K} k$ gives $\mathcal{O}Ue_K \simeq \mathcal{O}G$, with commuting Brauer maps, and

$$N_{U/K}(DK/K) = N_U(D)/K.$$

Here DK has the unique Sylow ℓ-subgroup D, which proves the normalizer equality. Every automorphism of G lifts to U; perfectness makes that lift unique, and the colour stabilizer fixes K. This identifies the block and defect stabilizers used above with the ones in the universal-cover condition. Since the central character on G is faithful, its inflation has $\ker\chi \cap Z(U) = K$. Thus the construction was made on the exact central quotient required in iAM. Transport over the three central colours completes the proof. □

The exceptional type-A universal covers are included in the published all-prime input [9, Section 6.1]; the natural groups needed for the Levi induction were treated separately in Section 7. The remaining small covers of types B/C and F_4 use the published results identified in their respective sections.

13. Completion of the proof

Proof of Theorem 1.1. The published prime-two and defining-characteristic results dispose of those primes. At odd nondefining primes, Theorem 7.1 gives types A and ${}^2A$, including the natural systems and the exceptional universal covers. Section 8, together with Section 8.8, gives types B/C; the remaining previously published range is included as stated there. Theorem 9.1 gives the whole unipotent blocks of type D, and Theorem 10.1 gives every block when the defining characteristic is two. Theorem 12.1 supplies the extra central colours of $\Omega_8^+(2)$. Finally Theorem 11.1 gives F_4, including its small universal cover.

At odd nondefining primes, the constructions above are made in the block stabilizer of an actual defect group. The central quotients in the character-triple condition are the quotients by $\ker\chi \cap Z(G)$; their local groups are the corresponding quotients of the actual defect normalizers. The common source and transfer results retain every ordinary inverse-factor-set coefficient, so the required upper block relations follow for every intermediate group. These are exactly the quantifiers in (2.3) for the newly constructed cases. The prime-two and defining-characteristic conclusions use the published iAM results cited in the introduction, with their own specified local subgroups; no additional regular-fibre assertion at those primes is needed here. □

Appendix A. Finite certificates

The **computational certificates on GitHub** include the complete scripts and their recorded outputs. The finite checks used in the proof have the following precise roles.

Check	Exact result	Use in the proof
Hessian boundary	All 262144 matrices over $\mathbb{F}_4$ are tested; $\lvert\mathrm{GL}_3(4)\rvert = 181440$, the full normalizer has order 648, and its determinant-one part has order 216	The natural $\mathrm{SL}_3(4)$ endpoint
Augmentation modules	The tested boundary cases isolate the order-three kernel at $(n, \ell, a) = (3, 3, 1)$	A check of the separate exceptional case; the general argument is algebraic
Quartic D_4	$\lvert W\rvert = 16$, $\lvert R\rvert = 96$; six reflection lines; abelian reflection centralizers of order 16; linear-character orbits $1, 1, 3, 3$	All ordinary relative coefficients and their actual triality action
F_4 root sources	All 48 roots and 1152 Weyl elements; mixed-torus valuation bounds, order-three classes, rank-one annihilators, and the B_2 complement	The stated actual centralizers and source actions
Extra $D_4(2)$ cover	Actual Weyl-group Sylow normalizers at $3, 5$; faithful global and local block characters and involution orbits	The two noncyclic extra-cover cases

The quartic calculation is reproduced both by exact Gaussian-rational matrix arithmetic and by an independent Wolfram Language evaluation. The determinant norms and unit arguments in the proof extend these finite identities to every admissible ℓ-adic valuation. The F_4 calculation similarly uses explicit coefficient bounds for the unbounded valuation parameter; it is not an extrapolation from finitely many field sizes. An independent reconstruction at (q=4,8), in both mixed-torus orientations, checks the source orders, the commutator subgroups, the root annihilators, and the pointwise Weyl stabilizers. These small-field checks do not assume that the torus subgroup is radical. A separate root-datum computation constructs rational semisimple parameters of orders (7) and (19), respectively, and verifies their root centralizers and rational relative groups in both orientations. Identifying their Brauer blocks uses the independent arguments and references in Section 11.

The GAP computations were executed with GAP 4.14.0 and CTblLib 1.3.9. They use the actual group inclusions for local class fusion and the specified library fusion for the global index-two inclusion. The archive records version information, source hashes, and commands. Character counts and orbit checks are used only at the indicated finite boundaries; the ordinary block and character-triple arguments are supplied in the text.

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