THE INDUCTIVE ALPERIN–MCKAY CONDITION FOR $G_2(q)$ AND ${}^3D_4(q)$

QIYUE TANG

Abstract. We verify the inductive Alperin–McKay condition for $G_2(q)$ ($q \geq 3$) and ${}^3D_4(q)$ ($q \geq 2$) at every odd prime.

hexagon:2611.00450v1

1. Introduction

The Alperin–McKay conjecture relates the ordinary irreducible characters in a block of a finite group to characters of a local subgroup. McKay's 1972 paper [20] initiated the study of the number of irreducible characters of odd degree. Alperin formulated the blockwise conjecture in his account of the main problems of block theory [1]. Let ℓ be a prime, let B be an ℓ-block of a finite group G, and let D be a defect group of B. For a positive integer n, write n_ℓ for its ℓ-part. A character $\chi \in \mathrm{Irr}(B)$ has height zero when $\chi(1)_\ell = |G : D|_\ell$; denote the set of such characters by $\mathrm{Irr}_0(B)$. If b is the Brauer correspondent of B in $N_G(D)$, the conjecture asserts that

$$|\mathrm{Irr}_0(B)| = |\mathrm{Irr}_0(b)|.$$

Thus the local subgroup is determined by the defect group of the individual block.

For the ordinary McKay conjecture, Isaacs–Malle–Navarro reduced the character-counting equality to conditions on finite simple groups in 2007 [13]. Malle–Späth proved the odd-degree case in 2016 [19]. Cabanes–Späth subsequently completed the proof for all primes [10]; their theorem also gives a bijection equivariant under the automorphisms stabilizing the Sylow subgroup. In the blockwise problem, the correspondence must retain the individual block and its defect group.

Späth's 2013 reduction theorem [24] reduces the Alperin–McKay conjecture to the inductive Alperin–McKay condition (iAM) for the universal covering groups of nonabelian finite simple groups. The inductive condition requires an automorphism-equivariant bijection, preservation of central characters, and a block compatibility condition for characters in intermediate overgroups. Späth expressed these requirements in the language of block character triples in [25, Section 4]. For a simple group with trivial Schur multiplier and cyclic outer automorphism group, Brough–Ruhstorfer's criterion [8, Lemma 7.1] reduces the verification to an equivariant bijection for each block. Ruhstorfer's verification for the remaining quasi-isolated two-blocks of exceptional groups completed the Alperin–McKay conjecture at the prime two [22]. The present paper treats odd primes for the two families in its title.

The ordinary character theory of the two families considered here has been studied in detail. In 1987 Deriziotis–Michler computed the character table and blocks of ${}^3D_4(q)$ and verified the ordinary Alperin–McKay equality [11]. An proved that equality for $G_2(q)$ in 1994 [2] and studied Dade's conjecture for this family in 1996 [3]. His later work treats invariant forms of the counting conjectures and records the action of outer automorphisms on ordinary character correspondences [4, 6]. For nondefining primes $\ell \geq 5$, the 1997 invariant calculation already supplies ordinary correspondences for the noncyclic $G_2(q)$-blocks, retaining their outer-automorphism stabilizers [4, (3C)(1)]. For the triality groups, An's proof of Dade's invariant conjecture [5] contains defect-preserving, automorphism-equivariant correspondences for the

Date: November 14, 2026.

2020 *Mathematics Subject Classification.* Primary 20C20; Secondary 20C15, 20C33.

Key words and phrases. inductive Alperin–McKay condition, finite simple groups, height-zero characters, triality.

noncyclic abelian-defect blocks, as well as the height-zero characters of the full-defect three-blocks. These are inputs to the verification below.

Several general results supply the remaining ingredients. Cabanes–Späth developed equivariant Jordan decomposition and character extension methods for reductive groups with connected centre [9]. Malle related characters of blocks with abelian Sylow subgroups to characters of torus normalizers [18]. Kessar–Malle's results identify the blocks containing constituents of Lusztig induction and control the variation of the semisimple parameter within an ℓ-series [15, 16]. Rossi's equivariant parametrization of generalised Harish–Chandra series [21] gives the global parameters used here. Schulte determined the relevant local structure and automorphism actions in proving the inductive blockwise Alperin weight condition for $G_2(q)$ and ${}^3D_4(q)$ [23]. We use her structural and block-theoretic results together with ordinary character correspondences.

Theorem 1.1. *Let q be a prime power. Let $S = G_2(q)$ with $q \geq 3$, or let $S = {}^3D_4(q)$ with $q \geq 2$. Then S satisfies the inductive Alperin–McKay condition for every odd prime.*

The verification separates the toral blocks from the nonabelian-defect three-blocks. In type G_2, the two ordinary character sets of a toral block are parametrized by the same torus characters and relative Weyl-group characters. We identify their individual blocks using maximal Brauer pairs, prove that the parameters exhaust each block, and choose the parametrizations simultaneously under automorphisms. This includes the exceptional graph-field automorphism in characteristic three. When $3 \nmid q$, the principal three-block is the unique block of maximal defect; Cabanes–Späth's equivariant McKay theorem [10] and the cyclic-outer criterion apply to it. The groups $G_2(3)$ and $G_2(4)$ use Breuer's verification for their exceptional covering groups [7].

For ${}^3D_4(q)$, we use An's ordinary correspondences for abelian-defect blocks. For each full-defect three-block we compute the Sylow normalizer and match its nine height-zero characters with An's global list. If $q = p^f$, the calculation retains the complete outer group of order $3f$, including the inner return of the chosen field representative in the nonsplit torus model.

2. The cyclic-outer criterion

For a finite group H, write $\mathrm{Irr}(H)$ for its ordinary irreducible characters and $\mathrm{Irr}_{\ell'}(H)$ for those of degree prime to ℓ. Write $\mathrm{Bl}(H)$ for its ℓ-blocks and $B_0(H)$ for its principal block; $\mathrm{Syl}_\ell(H)$ is the set of Sylow ℓ-subgroups of H. The symbol C_n denotes a cyclic group of order n, and a subscript on an acting group denotes a stabilizer. For $h \in H$, $\mathrm{Int}(h)$ is conjugation by h; $\mathrm{Inn}(H)$ is the group of inner automorphisms and $\mathrm{Out}(H) = \mathrm{Aut}(H)/\mathrm{Inn}(H)$. Write $O_\ell(H)$ for the largest normal ℓ-subgroup of H; if T is abelian, $O_{\ell'}(T)$ denotes its subgroup of elements of order prime to ℓ. For $\chi \in \mathrm{Irr}(H)$, its defect is $v_\ell(|H|) - v_\ell(\chi(1))$, where $v_\ell(n)$ is the ℓ-adic valuation of n. For $a \in \mathrm{Aut}(H)$, put $\chi^a(h) = \chi(a^{-1}(h))$. All induced character actions use this convention. Induction and inflation are denoted by Ind and Inf, with the groups specified whenever they are not determined by the formula. If B is a block with defect group D, the relevant automorphism group is

$$\Gamma = \mathrm{Aut}(G)_{B,D}.$$

Equivariance for a fixed D is understood with respect to this stabilizer. Equivalently, one may use the block stabilizer on the conjugacy class of maximal Brauer pairs, transporting the local group by inner conjugation.

We use the block-character-triple formulation of iAM in [25, Definition 4.12]. In that formulation, a matched pair of characters must have compatible projective representations with a common factor set and matching centralizer scalars. The induced correspondences must preserve block induction for every ordinary character above the matched pair in each intermediate group.

Proposition 2.1 (Brough–Ruhstorfer)**.** *Let S be a nonabelian finite simple group with trivial Schur multiplier and cyclic outer automorphism group. Let B be an ℓ-block of S, with nontrivial*

defect group D, and let b be its Brauer correspondent in $N_S(D)$. If there is an $\mathrm{Aut}(S)_{B,D}$-equivariant bijection

$$\mathrm{Irr}_0(B) \longrightarrow \mathrm{Irr}_0(b),$$

then B satisfies the inductive Alperin–McKay condition.

Proof. Apply [8, Lemma 7.1(i)]. Its ambient group is the universal covering group; here this is S itself. The central-character requirement holds since $Z(S) = 1$. Moreover, $N_S(D) < S$: otherwise D would be a nontrivial normal ℓ-subgroup of the nonabelian simple group S. The lemma gives the block-character-triple condition, including the ordinary characters above each matched pair. □

The cited lemma applies to an arbitrary prime ℓ.

Lemma 2.2. *Let S be as in Proposition 2.1, and suppose that S has a unique ℓ-block B of maximal defect. Then B satisfies the inductive Alperin–McKay condition.*

Proof. Let $P \in \mathrm{Syl}_\ell(S)$. A character of degree prime to ℓ belongs to a block of maximal defect and has height zero there. Hence $\mathrm{Irr}_{\ell'}(S) = \mathrm{Irr}_0(B)$. Brauer's first main theorem gives a bijection between the blocks of S with defect group P and the blocks of $N_S(P)$ with that defect group. Consequently its Brauer correspondent b is the unique maximal-defect block of $N_S(P)$, and $\mathrm{Irr}_{\ell'}(N_S(P)) = \mathrm{Irr}_0(b)$. Cabanes–Späth's equivariant McKay theorem [10, Theorem B] gives an $\mathrm{Aut}(S)_P$-equivariant bijection between these two sets. The uniqueness of B also gives $\mathrm{Aut}(S)_{B,P} = \mathrm{Aut}(S)_P$. Thus this bijection has the block and equivariance properties in Proposition 2.1. □

We also use the known iAM results in defining characteristic, including exceptional covering groups [8, Lemma 7.4], and for cyclic-defect blocks at odd primes [17, Theorem 1.1]. Blocks of central defect are excluded from the verification in [25, Definition 4.12]; for the centreless groups below these are the defect-zero blocks.

3. Toral ordinary parameters

We use bold letters for algebraic groups over $\overline{\mathbb{F}}_p$ and F for a Frobenius endomorphism; the corresponding finite group is the group of F-fixed points. A superscript $*$ denotes a dual reductive group or dual torus. The standard field Frobenius raising root parameters to their p^eth powers is denoted by F_{p^e}. If $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ and $T = \mathbf{T}^F$, then $R_T^G(\theta)$ denotes its Deligne–Lusztig virtual character for $\theta \in \mathrm{Irr}(T)$. For any virtual character V, define

$$\mathrm{Irr}(V) = \{\chi \in \mathrm{Irr}(G) : \langle V, \chi\rangle \neq 0\}.$$

Here $\langle V, \chi\rangle = |G|^{-1}\sum_{g\in G} V(g)\overline{\chi(g)}$ is the ordinary character inner product. For semisimple $s \in \mathbf{G}^{*F}$, write $\mathcal{E}(G, s)$ for the rational Lusztig series and

$$\mathcal{E}_\ell(G, s) = \bigcup_{t\in C_{\mathbf{G}^*}(s)^F \text{ an } \ell\text{-element}} \mathcal{E}(G, st) \qquad (s \text{ of } \ell'\text{-order}).$$

Jordan decomposition labels characters in $\mathcal{E}(G, s)$ by unipotent characters of $C_{\mathbf{G}^*}(s)^F$ when this algebraic centralizer is connected.

A maximal Brauer pair (D, b_D) for a block B consists of a defect group D and a block b_D of $C_G(D)$ associated with B. The maximal pairs of B form one G-orbit. We use the usual containment of Brauer pairs and Brauer's first main theorem to identify corresponding blocks.

Lemma 3.1. *Suppose $T = A \times K$ is an abelian normal subgroup of $N = T \rtimes W$, where A is an ℓ-group and K is an ℓ'-group. Let $\lambda \in \mathrm{Irr}(K)$, inflated to T, and put $E = W_\lambda$. Assume that E is an ℓ'-group acting faithfully on A. Then N has a unique block b covering the N-orbit of the torus block containing λ. Its defect group is A.*

Let $\mathcal{H} \le \mathrm{Aut}(N)$ stabilize T and λ, and suppose that $\Lambda(\theta)$ extends each $\theta \in \mathrm{Irr}(T)$ to N_θ with $\Lambda(\theta^a) = \Lambda(\theta)^a$ for $a \in \mathcal{H}\,\mathrm{Inn}(N)$. Then the characters of b are

$$\psi_{\nu,\eta} = \mathrm{Ind}_{N_{\lambda\nu}}^{N}(\Lambda(\lambda\nu)\,\mathrm{Inf}\,\eta), \qquad \nu \in \mathrm{Irr}(A)/E, \quad \eta \in \mathrm{Irr}(E_\nu). \tag{3.1}$$

Here ν is inflated from A to T, $\mathrm{Inf}\,\eta$ is inflated through $N_{\lambda\nu}/T = E_\nu$, and $\mathrm{Irr}(A)/E$ means a set of E-orbit representatives. Changing a representative transports η by the same element of E. All these characters have height zero, and the parametrization is $\mathcal{H}$-equivariant.

Proof. The inertia group of the torus block is $I = T \rtimes E$, since its unique character of ℓ'-order is λ. The extension $\widehat{\lambda}(te) = \lambda(t)$ allows us to remove λ by tensoring and then to factor out K. The resulting group is $A \rtimes E$. Every defect group contains the normal ℓ-subgroup A; since A is Sylow, every block has defect group A. Faithfulness gives $C_{A\rtimes E}(A) = A$. This centralizer has a single block b_0, so (A, b_0) is the unique maximal Brauer pair and there is one block. Fong–Reynolds correspondence gives the assertion about b.

Clifford theory over T gives exactly (3.1), with

$$\psi_{\nu,\eta}(1) = [W : E_\nu]\eta(1).$$

Since E_ν is an ℓ'-group, this degree has ℓ-part $|W|_\ell = |N : A|_\ell$. Thus every character has height zero. The equivariance follows by applying an automorphism to induction and to the chosen extension map. □

Lemma 3.2. *Let $\mathbf{C}$ be a connected reductive group with Frobenius F, and let $\mathbf{T}$ be an F-stable maximal torus such that F acts trivially on $N_{\mathbf{C}}(\mathbf{T})/\mathbf{T}$. If an F-stable connected closed algebraic subgroup $\mathbf{J}$ contains $\mathbf{T}$, then every $\mathbf{C}^F$-conjugate of $\mathbf{T}$ contained in $\mathbf{J}$ is $\mathbf{J}^F$-conjugate to $\mathbf{T}$.*

Proof. Write the other torus as ${}^g\mathbf{T}$ with $g \in \mathbf{C}^F$. Since maximal tori in the connected group $\mathbf{J}$ are conjugate, choose $h \in \mathbf{J}$ with ${}^h\mathbf{T} = {}^g\mathbf{T}$. Then $n = g^{-1}h \in N_{\mathbf{C}}(\mathbf{T})$, and triviality of the Weyl-group action gives $h^{-1}F(h) = n^{-1}F(n) \in \mathbf{T}$. Put $z = h^{-1}F(h)$. Lang's theorem on $\mathbf{T}$ supplies $t \in \mathbf{T}$ with $tF(t)^{-1} = z$. Hence $F(ht) = hzF(t) = ht$, and $ht \in \mathbf{J}^F$ has the required conjugation action. □

4. The groups $G_2(q)$

Throughout this section $q = p^f \ge 5$ and $G = G_2(q)$. The algebraic group of type G_2 has trivial centre and is both simply connected and adjoint. The finite group G is simple, has trivial Schur multiplier, and has cyclic outer automorphism group. In characteristic three its generator is the exceptional graph-field automorphism γ with $\gamma^2 = F_3$; its order on G is $2f$. These descriptions are recalled in [23, Section 4.1].

4.1. The blocks requiring a construction

We may assume that $\ell \ne p$ is odd and that the defect group of B is noncyclic. Denote the dth cyclotomic polynomial by Φ_d. If $\ell \ge 5$, the order polynomial

$$q^6\Phi_1(q)^2\Phi_2(q)^2\Phi_3(q)\Phi_6(q)$$

satisfies condition $(*)$ of [18]: exactly one cyclotomic factor whose index is the order of q modulo ℓ contributes to its ℓ-part. Theorem 2.6 of that paper describes B by a central-defect e-cuspidal pair. A noncyclic defect group must be the ℓ-part of the full split or full nonsplit torus:

$$T = T_\epsilon \simeq C_{q-\epsilon}^2, \qquad A = O_\ell(T) \simeq C_{\ell^a}^2, \qquad \epsilon \in \{1, -1\}, \quad \ell \mid q - \epsilon, \quad a = v_\ell(q - \epsilon). \tag{4.1}$$

Indeed the Φ_3 and Φ_6 tori are cyclic, and a proper Levi with nontrivial derived subgroup has connected central rank one. For $\ell = 3$, Proposition 4.28 of [23] says that the principal block is the unique maximal-defect block, all other blocks have abelian defect, and their noncyclic defect

groups are precisely of the form (4.1). The root calculation in Section 4.3 proves, in every case under consideration,

$$C_G(A) = T, \qquad N_G(A) = N_G(T). \tag{4.2}$$

The principal three-block will be treated in Section 4.6.

Fix a maximal Brauer pair (A, b_λ) of B. The block b_λ of T has a unique character λ of ℓ'-order. Put $K = O_{\ell'}(T)$. Let $(\mathbf{T}^*, s)$ be dual to $(\mathbf{T}, \lambda)$, where s is an ℓ'-element of the dual torus. Put $T^* = (\mathbf{T}^*)^F$ and $A^* = O_\ell(T^*)$; torus duality identifies A^* with $\mathrm{Irr}(A)$.

4.2. **The local ordinary characters**

The torus normalizer admits an actual splitting

$$N := N_G(T) = T \rtimes W, \qquad W \simeq S_3 \times C_2 \simeq D_{12}.$$

Here D_{12} denotes the dihedral group of order twelve. One model uses the long-root SL_3 subgroup: let P_π be the permutation matrix of $\pi \in S_3$ and put $c(\pi) = \mathrm{sgn}(\pi)P_\pi$. Then $c(\pi)c(\rho) = c(\pi\rho)$ and $\det c(\pi) = \mathrm{sgn}(\pi)^4 = 1$. The inverse-transpose involution v fixes every $c(\pi)$. The permutation subgroup intersects the diagonal torus trivially, and the Weyl action of v is inversion. Thus $W = c(S_3) \times \langle v \rangle$ is a complement. The construction is valid for both the Frobenius F_q and the twisted Frobenius $\mathrm{Int}(v)F_q$, and ordinary field maps fix the displayed complement. Compare [23, Lemma 4.4 and Section 4.2.2].

Set $E = W_\lambda$. This is an ℓ'-group: for $\ell \geq 5$ this follows from $|W| = 12$, and at three it is the inertial quotient $N_G(A, b_\lambda)/(AC_G(A)) = N_G(A, b_\lambda)/T$ of the maximal abelian Brauer pair, hence has order prime to three. Its action on A is faithful. Indeed the rank-two torus action is generated by

$$r = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \qquad u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad -I.$$

The twelve resulting matrices have entries in $\{0, 1, -1\}$ and remain distinct modulo every odd prime.

For ordinary field actions the complement gives an equivariant extension map for all $\theta \in \mathrm{Irr}(T)$ by assigning value one on W_θ. This extension is also compatible with conjugation by T: for $t \in T$ and $w \in W_\theta$, $\theta(t^{-1}w(t)) = \theta(t)^{-1}\theta(w(t)) = 1$. Conjugation by W transports the same formula, giving equivariance under $\mathrm{Inn}(N)$. When $p = 3$, use the extension map of [14, Lemma 3.1], transported from its rational torus model by Lang–Steinberg conjugacy. It is defined on all torus characters and is equivariant under the graph-field normalizer. Its parameter is $e = 1$ or 2 here. The hypotheses $f \geq 2$ and $\ell \geq 5$ hold because $q \geq 5$ and $\ell \neq p = 3$ is odd.

Lemma 3.1 therefore parametrizes the entire ordinary character set of a single block b of N by

$$\psi_{\nu,\eta} = \mathrm{Ind}_{N_{\lambda\nu}}^N(\Lambda(\lambda\nu)\,\mathrm{Inf}\,\eta), \qquad \nu \in \mathrm{Irr}(A)/E, \quad \eta \in \mathrm{Irr}(E_\nu). \tag{4.3}$$

The maximal pair of b is the pair fixed above, so b is the Brauer correspondent of B in the actual normalizer $N_G(A)$.

4.3. **Block membership**

Use torus coordinates $h(z_1, z_2, z_3)$ with $z_1z_2z_3 = 1$. The short-root values are $z_i^{\pm 1}$, and the long-root values are z_i/z_j for $i \neq j$. For ζ of order ℓ^a take

$$a_1 = h(\zeta, \zeta^{-1}, 1), \qquad a_2 = h(\zeta, 1, \zeta^{-1}).$$

The short-root values on a_1 are $\zeta, \zeta^{-1}, 1$ and their inverses; the long-root values are $\zeta^{\pm 1}, \zeta^{\pm 2}$. Thus only the pair indexed by z_3 centralizes a_1, and it has values $\zeta^{\pm 1}$ on a_2. Consequently

$$C^\circ_{\mathbf{G}}(a_1, a_2) = \mathbf{T}.$$

This also holds when $\ell = 3$ and $a = 1$. Since a_1, a_2 generate A, every element of $N_G(A)$ normalizes $\mathbf{T}$. The faithful Weyl action on A therefore gives $C_G(A) = T$. As $A = O_\ell(T)$ is characteristic in T, we obtain

$$N_G(A) = N_G(\mathbf{T}) = N_G(T),$$

which proves (4.2). For the generator chain a_1, a_2, [15, Proposition 2.16(1)] gives the nonvanishing Lusztig-restriction pairings required in Proposition 2.12 of that paper, since the source Levi is a torus. The terminal group is $\mathbf{T}$, so Proposition 2.13(1) gives the terminal single-block condition. Proposition 2.12 therefore places every constituent of $R_T^G(\lambda)$ in the block with Brauer pair (A, b_λ), namely B.

Let $t \in A^*$ be dual to $\nu \in \mathrm{Irr}(A)$. Both $C_{\mathbf{G}^*}(s)$ and $C_{\mathbf{G}^*}(st)$ are connected, since $\mathbf{G}^*$ is simply connected. Frobenius acts trivially on the Weyl group of the full split or full nonsplit torus: the two torus types correspond to the central Weyl elements 1 and -1. Lemma 3.2, applied with ambient group $\mathbf{G}^*$ and subgroup $C_{\mathbf{G}^*}(st)$, therefore verifies the rational-conjugacy hypothesis of [16, Lemma 2.8]. Jordan decomposition identifies constituents of $R_T^G(\lambda\nu)$ with the unipotent torus series in $C_{\mathbf{G}^*}(st)^F$. That lemma now places every such constituent in the same block as a constituent of $R_T^G(\lambda)$, hence in B.

4.4. **Parametrization and exhaustion**

Put $e = e_\ell(q)$, the multiplicative order of q modulo ℓ. Thus $e = 1$ for $\epsilon = 1$ and $e = 2$ for $\epsilon = -1$. An e-split Levi subgroup is the centralizer of a torus whose order polynomial is a power of Φ_e. Here $\mathbf{T}$ is itself such a torus and is its own centralizer. It has no proper Levi subgroup, so every $(\mathbf{T}, \theta)$ is an e-cuspidal pair. Its relative Weyl group is $N_G(T, \theta)/T$; for $\theta = \lambda\nu$ this is E_ν.

The equivariant Jordan decomposition and relative-group identifications of [21, Theorem 3.2 and Corollary 3.3], together with its Theorem 3.4 and Lemmas 3.5–3.6, give bijections

$$I_{T,\lambda\nu} : \mathrm{Irr}(E_\nu) \longrightarrow \mathrm{Irr}(R_T^G(\lambda\nu)). \tag{4.4}$$

The algebraic group $\mathbf{G}$ has connected centre, and its derived group has type G_2, as required in [21, Hypothesis 3.1]. The automorphism group used there consists of restrictions of bijective algebraic morphisms commuting with F; it includes the exceptional graph-field morphism in characteristic three. The stated results apply at $\ell = 3$ as well. The maps will be chosen simultaneously in Section 4.5.

The packets indexed by $\mathrm{Irr}(A)/E$ are distinct. If st and st' are rationally conjugate, their ℓ'-parts show that a conjugator centralizes s. The two copies of $\mathbf{T}^*$ in $C_{\mathbf{G}^*}(st')$ can be identified by a rational element of that centralizer, by Lemma 3.2. Correcting the conjugator by this element puts it in the normalizer of the marked pair $(\mathbf{T}^*, s)$. Thus t and t' differ precisely by the action of E. For a fixed t the actual relative group is E_ν, as in the local formula. Rossi's relative-group identification is equivariant for this action.

For $\ell \geq 5$, [18, Theorem 2.9 and Proposition 2.7(c)] give

$$|\,\mathrm{Irr}(B)| = |\,\mathrm{Irr}(A \rtimes E)|.$$

Its extension hypothesis holds by the local construction above. The disjoint packets (4.4) already lie in B, and their total size is the right-hand side by Clifford theory. They therefore exhaust $\mathrm{Irr}(B)$.

For $\ell = 3$, since E is a $3'$-group, $\lambda \neq 1$: otherwise $E = W$, whose order is divisible by three. Hence $s \neq 1$. By [15, Theorem 2.3(a)], $\mathcal{E}_3(G, s)$ is a union of blocks. The already anchored series $R_T^G(\lambda)$ has a constituent in B, so $\mathrm{Irr}(B) \subseteq \mathcal{E}_3(G, s)$. The connected centralizer of this nonidentity $3'$-element has root type

$$T, \qquad A_1T, \qquad A_1^2.$$

Indeed the long-root A_2 centralizer would require $z_1 = z_2 = z_3$ and $z_1^3 = 1$, giving a parameter of order three; the full G_2 centralizer would force $s = 1$. The root conditions $z_i = 1$ and $z_i = z_j$ leave only the proper closed root systems A_1 and A_1^2 in the other non-toral cases.

Frobenius acts on the root subsystem by 1 or -1, so it preserves each A_1 component. For each such component,

$$|\operatorname{SL}_2(q)|_3 = (q-\epsilon)_3,$$

since 3 divides exactly one of $q-1$ and $q+1$. The central isogenies in these rank-one products have two-power kernels, which do not alter the three-part. Together with the central torus this gives

$$|C_{\mathbf{G}^*}(s)^F|_3 = (q-\epsilon)_3^2 = |A^*|.$$

Thus A^* is a Sylow three-subgroup of this centralizer. Start with any ordinary character in $\mathcal{E}_3(G,s)$ and conjugate the three-part of its semisimple label into A^*. Since s is a power of st, the centralizer of st is a connected subgroup of $C_{\mathbf{G}^*}(s)$ containing $\mathbf{T}^*$, again with only rank-one root components. Every unipotent character of such a centralizer belongs to its full ϵ-torus series: for a rank-one factor the two virtual characters from the split and nonsplit tori are $1+\mathrm{St}$ and $1-\mathrm{St}$, respectively, where St denotes the Steinberg character; for two factors one takes their products. The Jordan inverse thus places the arbitrary character in one of the packets (4.4). The block anchor already proved that all these packets lie in B. This proves exhaustion at three, including all nonunipotent ordinary characters in the block.

4.5. **Degrees and the full automorphism action**

Write $\chi_{\nu,\eta} = I_{T,\lambda\nu}(\eta)$. The degree formula in [21, Lemma 3.6] gives

$$\chi_{\nu,\eta}(1)_\ell = |G : N_G(T,\lambda\nu)|_\ell\, \eta(1)_\ell = |G:T|_\ell.$$

Since $|T:A|$ is prime to ℓ, this is $|G:A|_\ell$. The local degree in (4.3) has ℓ-part $|N:A|_\ell$. Thus all the characters just parametrized have height zero in their respective blocks, and

$$\chi_{\nu,\eta} \longmapsto \psi_{\nu,\eta} \tag{4.5}$$

is a bijection $\operatorname{Irr}_0(B) \longrightarrow \operatorname{Irr}_0(b)$.

We now make the simultaneous choice of the global maps. Set $\Gamma = \operatorname{Aut}(G)_{B,A}$ and $H = \Gamma_{b_\lambda}$. An automorphism stabilizing A also stabilizes $T = C_G(A)$, and H fixes λ, the unique character of ℓ'-order in b_λ. The maximal Brauer pairs of B with first component A form one N-orbit. Therefore

$$\Gamma = \operatorname{Inn}(N)H. \tag{4.6}$$

Here $\operatorname{Inn}(N)$ denotes the automorphisms of G induced by elements of N.

Let $\Theta = \{\lambda\nu : \nu \in \operatorname{Irr}(A)\}$. For one θ in each H-orbit on Θ, choose the H_θ-equivariant map supplied by [21, Lemma 3.5]. If $h \in H$, define the map at θ^h by

$$I_{T,\theta^h}(\eta^h) = I_{T,\theta}(\eta)^h.$$

Two choices of h differ by an element of H_θ; the stabilizer equivariance makes this definition independent of that choice. Rossi's relative-group isomorphism transports η by the conjugation action on N_θ/T.

First use all pairs (ν,η) with $\nu \in \operatorname{Irr}(A)$ and $\eta \in \operatorname{Irr}(E_\nu)$. The group H contains the automorphisms induced by N_λ, so the transported global maps respect the $E = N_\lambda/T$-identifications on these pairs. The equivariant extension map gives the same identifications in the local Clifford parametrization. Consequently (4.5) is an H-equivariant bijection on the quotient parameter set. Conjugation by N acts trivially on both $\operatorname{Irr}(G)$ and $\operatorname{Irr}(N)$; (4.6) proves Γ-equivariance. In the nonsplit model an ordinary field representative may satisfy $F_p^f = \operatorname{Int}(v)$; this actual return is retained by the complement construction. When $p=3$, the whole graph-field group is retained by Johansson's extension map. Proposition 2.1 proves iAM for every noncyclic abelian-defect block considered here.

4.6. **The principal three-block and small groups**

If $3 \nmid q$, [23, Proposition 4.28(i)] states that the principal three-block of G is its unique maximal-defect three-block. Lemma 2.2 proves iAM for this block.

The defining-characteristic and cyclic-defect cases were already covered in Section 2. For $q = 3, 4$, the universal covering groups have exceptional multipliers. We use the computations recorded in [7]: both $G_2(3)$ and $G_2(4)$ are explicitly listed for the inductive Alperin–McKay condition, with all required Schur covers and prime divisors. This supplies the two exceptional-cover cases. This completes the G_2 part of Theorem 1.1.

5. The triality groups

Theorem 5.1. *For every prime power q, the simple group ${}^3D_4(q)$ satisfies the inductive Alperin–McKay condition at every odd prime.*

Write $G = {}^3D_4(q)$, where $q = p^f$. This group is centreless and is its own universal covering group, including when $q = 2$. Moreover $\mathrm{Out}(G)$ is cyclic of order $3f$; the usual field automorphism represents a generator [23, Section 5.1 and Proposition 5.4]. Consequently, by [8, Lemma 7.1(i)], it suffices to construct, for each block B with defect group D, an $\mathrm{Aut}(G)_{B,D}$-equivariant bijection

$$\mathrm{Irr}_0(B) \longrightarrow \mathrm{Irr}_0(b), \qquad b \in \mathrm{Bl}(N_G(D)), \quad b^G = B.$$

The central-character condition is automatic. This cited criterion supplies the required block character-triple condition. The central-defect convention was recalled in Section 2. The prime p is covered by [8, Lemma 7.4], and nontrivial cyclic-defect blocks at odd primes by [17, Theorem 1.1].

5.1. **Blocks with abelian defect**

An constructs his maps using the full outer generator α of order $3f$, as specified in [5, (2.1)]. Here and below the numbered parts of (3C) refer to the author's report linked in the bibliography.

Lemma 5.2. *Let $\ell \neq p$ be odd, and suppose that B has noncyclic abelian defect group D. There is an $\mathrm{Aut}(G)_{B,D}$-equivariant, defect-preserving bijection $\mathrm{Irr}(B) \longrightarrow \mathrm{Irr}(b)$ for its Brauer correspondent b.*

Proof. For $\ell \geq 5$, the noncyclic defect groups are the ℓ-parts of the tori

$$T_\epsilon \simeq C_{q^3-\epsilon} \times C_{q-\epsilon}, \qquad T_{2,\epsilon} \simeq C^2_{q^2+\epsilon q+1}, \qquad \epsilon \in \{1, -1\}.$$

Their centralizers are the indicated tori, and their normalizers are the corresponding torus normalizers. The same two forms describe the noncyclic abelian three-defect groups [5, Section 2 and (3C)]; see also [23, Proposition 5.26].

For $\ell \geq 5$, and also for $\ell = 3$ with $D = O_3(T_{2,\epsilon})$, the map in [5, (3C)(3), equations (3.5)–(3.6)] has domain $\mathrm{Irr}(b)$ with b the block of $N_G(D)$ inducing B. It is a bijection on the whole ordinary block, preserves defects, and commutes with the outer generator. For $\ell = 3$ and $D = O_3(T_\epsilon)$, use the map called Ψ_2 in [5, (3C)(4)]. Its domain is again the block of $N_G(D) = N_G(T_\epsilon)$.

The formulas use a semisimple torus label and an ordinary coefficient in its inertia quotient. They commute with the normalizer action and with every power of α that carries the chosen block to itself. Inner automorphisms of $N_G(D)$ act trivially on its irreducible characters. The Frattini argument, applied to the block and a maximal Brauer pair, therefore gives equivariance for the full stabilizer $\mathrm{Aut}(G)_{B,D}$. Restricting the defect-preserving correspondence to height zero gives the required Alperin–McKay bijection. □

In these correspondences the semisimple label is paired with a unipotent label in its centralizer. A toral centralizer contributes one label; a rank-one centralizer contributes 1 and the Steinberg character; a centralizer of type A_2^ϵ contributes the labels of degrees $1, q(q+\epsilon), q^3$. Here A_2^1 and A_2^{-1} denote the split and unitary types, respectively. An's formulas also specify the four

product labels for an involution centralizer and the unipotent labels at the identity. The block membership is part of [5, (3C)(3)–(4)].

5.2. **The normalizer of a Sylow three-subgroup**

It remains to consider $\ell = 3 \neq p$ and nonabelian defect. The block classification shows that such a defect group is a Sylow three-subgroup [5, (3C)]. Choose $\epsilon \in \{1, -1\}$ with $q \equiv \epsilon \pmod 3$, and set

$$a = v_3(q - \epsilon), \qquad m = q^2 + \epsilon q + 1 = 3c, \qquad (c, 6) = 1.$$

In $T = T_\epsilon$, put

$$A = O_3(T) \simeq C_{3^{a+1}} \times C_{3^a}.$$

Write $N_{\mathrm{alg}}(T) = N_{\mathbf{G}}(\mathbf{T})^F$ for the rational points of the algebraic torus normalizer. The root construction below first gives

$$N_{\mathrm{alg}}(T) = T \rtimes K, \qquad K = \langle \sigma, \tau \rangle \times \langle \iota \rangle \simeq S_3 \times C_2.$$

Take the simply connected algebraic group $\mathbf{G}$ of type D_4 with simple roots r_1, r_2, r_3, r_4, where r_2 is the central node. Pinned triality ρ cycles r_1, r_3, r_4 and fixes r_2. Write $h_i = h_{r_i}(-1)$ and $n_i = n_{r_i}(1)$ for the Chevalley torus elements and root representatives. Put $r_{12} = r_1 + 2r_2 + r_3 + r_4$, $n_{12} = n_{r_{12}}(1)$, and $n_0 = n_1 n_3 n_4 n_{12}$. The representatives for the two tori are

$$u = h_1 h_3 h_4 n_2, \qquad v = h_2 n_1 n_3 n_4, \qquad \sigma = (uv)^2, \quad \tau = u, \quad \iota = (uv)^3.$$

Lemma A.1 proves $u^2 = v^2 = 1$, $(uv)^6 = 1$ and $(uv)^3 = n_0$. The images of u, v generate the triality-fixed Weyl group of order twelve, so they give a complement. All representatives are fixed by the prime-field map and by ρ. They are rational for both ρF_q and $\mathrm{Int}(n_0)\rho F_q$. These are tori 1 and 7 in [12, Table 3]; their normalizers also occur in [5, (2.6)].

Set $Q = \epsilon q$. In coroot coordinates the finite torus consists of

$$(t_1, t_2, t_1^Q, t_1^{Q^2}), \qquad t_1^{Q^3-1} = t_2^{Q-1} = 1. \tag{5.1}$$

Negative exponents have their usual multiplicative meaning. The root reflections act by

$$u(t_1, t_2) = (t_1, t_1^m / t_2), \qquad v(t_1, t_2) = (t_2 / t_1, t_2), \qquad m = Q^2 + Q + 1.$$

Using $m \equiv 3 \pmod{Q-1}$ and $m(m-3) = (Q^3 - 1)(Q + 2)$ gives

$$\sigma(t_1, t_2) = (t_1^{1-m} t_2, t_1^{-m} t_2). \tag{5.2}$$

Also $\tau = u$ and $\iota = n_0$ acts by inversion.

The action on the unequal cyclic coordinates $x \in \mathbb{Z}/3^{a+1}\mathbb{Z}$ and $y \in \mathbb{Z}/3^a\mathbb{Z}$ is

$$\sigma(x, y) = (-2x + 3y, -x + y), \qquad \tau(x, y) = (x, x - y), \qquad \iota(x, y) = (-x, -y). \tag{5.3}$$

To obtain these formulas from (5.1)–(5.2), put $t_1 = \zeta^x$, $t_2 = \zeta^{3y}$ with $|\zeta| = 3^{a+1}$. If $q - \epsilon = 3^a d$, then

$$\frac{m}{3} = 1 + \epsilon 3^a d + 3^{2a-1} d^2 \equiv 1 \pmod{3^a}.$$

Thus $m \equiv 3 \pmod{3^{a+1}}$ and (5.3) follows. The equations $\sigma(x, y) = (x, y)$ give $x \equiv 0 \pmod{3^a}$ and $y = 0$. The image of $\sigma - 1$ is generated by $(-3, -1)$ and $(3, 0)$. Hence

$$C_A(\sigma) = \{(3^a j, 0) : j \in \mathbb{F}_3\}, \qquad [A, \sigma] = 3\mathbb{Z}/3^{a+1}\mathbb{Z} \times \mathbb{Z}/3^a\mathbb{Z}. \tag{5.4}$$

The action of σ has order three, and τ is an involution conjugating it to its inverse; together their image is S_3. They both fix x modulo three, whereas ι negates it. Therefore ι is outside that image, and K acts faithfully on A.

Put $S = A \rtimes \langle \sigma \rangle$. The order of G gives $|S| = 3^{2a+2} = |G|_3$. If an abelian subgroup $U \leq S$ is not contained in A, choose an element of $U \setminus A$. It acts on A by σ or σ^2, so $U \cap A \leq C_A(\sigma)$. Also $U/(U \cap A)$ embeds in $S/A \simeq C_3$. Thus $|U| \leq 9$. Since $|A| \geq 27$, A is the unique abelian

subgroup of its order and is characteristic in S. For a root $r = \sum_i b_i r_i$, restriction to A has dual coordinates

$$(2b_1 - mb_2 + 2Qb_3 + 2Q^2 b_4,\ -b_1 + 2b_2 - b_3 - b_4)$$

modulo 3^{a+1} and 3^a, respectively. Modulo three, the invertible target change $(x, y) \mapsto (2x, x+2y)$ changes this map into

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

For the twelve positive roots, listed in Appendix A, the images are

$$(1,0), (0,1), (1,0), (1,0), (1,1), (1,1),$$
$$(1,1), (2,1), (2,1), (2,1), (0,1), (0,2).$$

All are nonzero; the negative roots have their negatives. Thus no root centralizes A, and $C^\circ_{\mathbf{G}}(A) = \mathbf{T}$. Every element of $N_G(A)$ normalizes this identity component, so $N_G(A) \le N_{\mathrm{alg}}(T)$. The faithful action of K on A then gives $C_G(A) = T$. Since $A = O_3(T)$ is characteristic in the finite torus T,

$$N_G(T) \le N_G(A) \le N_{\mathrm{alg}}(T) \le N_G(T).$$

Thus the algebraic and finite torus normalizers coincide, including at $q = 2$. The characteristic property of A in S now puts $N_G(S)$ in this group.

The fixed-point equations in (5.2) give $t_1^m = 1$ and $t_2 = 1$. Hence

$$C_T(\sigma) = \{(t_1, 1) : t_1^m = 1\} \simeq C_m.$$

Since $C_G(A) = T$, this is also $C_G(S)$, in agreement with [23, Proposition 5.19(ii)]. Set $C = O_{3'}(C_T(\sigma)) \simeq C_c$. An element of T normalizes S precisely when its σ-commutator belongs to A, equivalently when its $3'$-part belongs to C. Therefore

$$N := N_G(S) = (A \times C) \rtimes K, \qquad C_G(S) = C_A(\sigma) \times C. \tag{5.5}$$

On $C_T(\sigma)$, the element u fixes t_1 and v replaces it by t_1^{-1}. Thus $S_3 = \langle \sigma, u \rangle$ fixes C, and $\iota = (uv)^3$ inverts it. For $q = 2$, these formulas give $|S| = 81$ and $c = 1$.

5.3. **Nine characters in each full-defect block**

The $N_G(S)$-conjugacy classes of maximal Brauer pairs with first component S are indexed by $\lambda \in \mathrm{Irr}(C)$ up to inversion. Write $B_{[\lambda]}$ for the corresponding block of G and $b_{[\lambda]}$ for its Brauer correspondent in N. The principal block corresponds to $\lambda = 1$.

The inertia subgroup of λ in N is $(A \times C) \rtimes K_\lambda$, where $K_\lambda = K$ for $\lambda = 1$, and $K_\lambda = S_3$ otherwise. The character λ extends to this subgroup by being trivial on A and K_λ. Tensoring by its inverse and then factoring out C identifies its covering characters with $\mathrm{Irr}(A \rtimes K_\lambda)$. This latter group has a single three-block. Indeed every defect group contains the normal three-subgroup A, so each block idempotent has nonzero Brauer image at A. Faithfulness gives $C_{A \rtimes K_\lambda}(A) = A$. The group algebra of A over a field of characteristic three is local and has a single nonzero central idempotent. The Brauer images of distinct block idempotents are orthogonal, so there is exactly one block. Fong–Reynolds correspondence gives one block of N above $[\lambda]$. Its maximal Brauer pair is (S, b_λ), with b_λ the block of $C_G(S) = C_3 \times C$ labelled by λ; hence it is $b_{[\lambda]}$.

By (5.4), the σ-invariant characters of A are exactly

$$\nu_j(x, y) = \exp(2\pi i j x/3), \qquad j \in \mathbb{F}_3.$$

For a local irreducible character to have degree prime to three, its A-constituent must be one of these three characters. The element τ fixes all three and ι sends j to $-j$. For $(\lambda, j) \ne (1, 0)$, the local parameters are thus

$$(\lambda, j, \eta), \qquad \eta \in \mathrm{Irr}(S_3), \qquad (\lambda, j, \eta) \sim (\lambda^{-1}, -j, \eta). \tag{5.6}$$

Extend $\lambda\nu_j$ trivially over S_3, writing the extension as $\widehat{\lambda\nu_j}$. The corresponding character is

$$\psi_{\lambda,j,\eta} = \mathrm{Ind}_{(A\times C)\rtimes S_3}^{N}(\widehat{\lambda\nu_j}\ \mathrm{Inf}\,\eta),$$

where η is inflated through the quotient by $A \times C$. Its degree is $2\eta(1)$. At $(\lambda, j) = (1, 0)$ use instead the six inflated characters of $K \simeq D_{12}$. Clifford restriction to $A \times C$ recovers the parameters, so this list is both injective and exhaustive. It gives nine height-zero characters in each $b_{[\lambda]}$.

Let $T^* = (\mathbf{T}^*)^F$ be the finite torus dual to T, with the dual Weyl action. Choose ω of order $q^3 - \epsilon = m(q - \epsilon)$ and write $t_1 = \omega^X$, $t_2 = \omega^{mY}$, where X is taken modulo $q^3 - \epsilon$ and Y modulo $q - \epsilon$. Equation (5.2) gives

$$\sigma(X, Y) = ((1 - m)X + mY, -X + Y).$$

Thus the image of $\sigma - 1$ is generated by $(-m, -1)$ and $(m, 0)$, equivalently by $(m, 0)$ and $(0, 1)$. The quotient is cyclic of order m, with coordinate X modulo m. Torus duality therefore gives

$$C_{T^*}(\sigma) \simeq \mathrm{Irr}(T/[T, \sigma]) \simeq C_m.$$

Use the scalar correspondence in [5, (3.1) and (3C)(9)] to choose the $3'$-element $s_\lambda \in C_{T^*}(\sigma)$ whose associated linear character on C is λ. This is the correspondence used for the individual blocks, and it commutes with the field action and inversion. Choose a generator z of $O_3(C_{T^*}(\sigma))$; its powers are labelled by $j \in \mathbb{F}_3$. The individual block descriptions in [5, (3C)(6),(9)] give, for $(\lambda, j) \neq (1, 0)$, the three characters with semisimple label $s_\lambda z^j$ and unipotent labels of degrees

$$1, \quad q(q + \epsilon), \quad q^3.$$

Their centralizer is of type A_2^ϵ with its central torus. The rational fusion is precisely $(s_\lambda, z^j) \sim (s_\lambda^{-1}, z^{-j})$. Pair the trivial, sign and degree-two characters of S_3 with 1, the Steinberg character, and the middle unipotent character, respectively.

For $\lambda \neq 1$, the group H_ϵ used in [5, (3C)(9)] has the form

$$H_\epsilon = C \times H_0, \qquad H_0 \simeq \mathrm{SL}_3^\epsilon(q) \cdot 3.$$

Here $\mathrm{SL}_3^+(q) = \mathrm{SL}_3(q)$ and $\mathrm{SL}_3^-(q) = \mathrm{SU}_3(q)$. The block used there is $B_0(H_0) \otimes \lambda$, with $S \in \mathrm{Syl}_3(H_0)$ and $C_{H_\epsilon}(S) = C_G(S) = C_3 \times C$. The centralizer of S in H_0 is C_3. The maximal Brauer pair of its principal block uses the principal block of C_3, which is its unique block. Consequently $B_0(H_0) \otimes \lambda$ has maximal pair (S, b_λ) in H_ϵ. The block induction to G in An's construction preserves this pair. Uniqueness of the block containing the pair identifies that induced block with $B_{[\lambda]}$.

At $(1, 0)$, the principal block has six unipotent characters of degree prime to three. Together with the three characters at $z \neq 1$, these are exactly $\mathrm{Irr}_0(B_{[1]})$ [5, (3C)(6), equation (3.7)]. An also shows there that all nine height-zero characters of the principal block are fixed by the full outer generator. Pair the six unipotent characters with the six characters of K in any fixed order. The preceding descriptions give a bijection

$$\Omega_{[\lambda]} : \mathrm{Irr}_0(B_{[\lambda]}) \longrightarrow \mathrm{Irr}_0(b_{[\lambda]}).$$

5.4. **The entire outer action**

The prime-field root representatives defining K are fixed by the prime-field map $a_0 = F_p$. With the character convention of Section 2, its action in the rational torus models on both sets of parameters is

$$(\lambda, j, \eta) \longmapsto (\lambda^{p^{-1}}, p^{-1}j, \eta).$$

Here the inverse exponents are taken modulo c and modulo three, respectively. The three A_2^ϵ unipotent labels are distinguished by their degrees and hence are fixed, including after a correcting inverse-transpose automorphism. At $(1, 0)$, the six local characters and their six global partners are fixed. For the global action one may also use [5, (3.1)–(3.3)].

Recall $n_0 = (uv)^3$ from Lemma A.1. It has order two, is fixed by ρ and F_p, and centralizes K. The two rational models have Frobenius

$$F_+ = \rho F_q, \qquad F_- = \mathrm{Int}(n_0)\rho F_q.$$

Thus the prime-field representative $a_0 = F_p$ satisfies

$$a_0^{3f} = 1 \quad (\epsilon = 1), \qquad a_0^{3f} = \mathrm{Int}(n_0) \quad (\epsilon = -1).$$

Its outer image is the full generator of order $3f$. For the nonsplit model, Lang's theorem gives $h \in \mathbf{G}$ with $F_+(h) = hn_0$. Conjugation by h identifies $\mathbf{G}^{F_-}$ with $\mathbf{G}^{F_+}$ and transports F_p to $\mathrm{Int}(hF_p(h)^{-1})F_p$. Since n_0 is fixed by F_p, we have $F_+(hF_p(h)^{-1}) = hF_p(h)^{-1}$. The transported map thus has the outer class of the standard generator. The element n_0 normalizes S, inverts T, and centralizes K. Its return therefore acts on (λ, j) by simultaneous inversion, exactly the identification in (5.6). The congruence $q^3 \equiv \epsilon \pmod m$ checks the same return on labels. The element a_0 stabilizes S and maps to a generator of $\mathrm{Out}(G)$. Identifying G with its inner automorphism group therefore gives

$$N_{\mathrm{Aut}(G)}(S) = N_G(S)\langle a_0\rangle.$$

An element stabilizing $B_{[\lambda]}$ has the form $\mathrm{Int}(n)a_0^d$ with $n \in N_G(S)$ and $[\lambda^{p^{-d}}] = [\lambda]$. Inner automorphisms of $N_G(S)$ fix the characters on both sides, and the displayed parameter action commutes with $\Omega_{[\lambda]}$. Hence this bijection is $\mathrm{Aut}(G)_{B_{[\lambda]},S}$-equivariant.

Proof of Theorem 5.1. For a nondefining odd prime, the cyclic-defect case was cited above. Lemma 5.2 covers all remaining abelian-defect blocks. The classification leaves only the full-defect three-blocks, for which $\Omega_{[\lambda]}$ is the required equivariant Alperin–McKay bijection at the actual defect normalizer. Apply [8, Lemma 7.1(i)] in each case. The defining-prime result completes the proof. □

6. Completion of the verification

Proof of Theorem 1.1. For $G_2(q)$ with $q \geq 5$, the nondefining odd-prime blocks of noncyclic defect consist of the toral blocks treated in Section 4 and the principal three-block. Their correspondences are to the actual defect normalizer and are equivariant for its full block stabilizer. Proposition 2.1 supplies iAM. The two smaller values $q = 3, 4$ use the separate universal-cover verification specified there.

For ${}^3D_4(q)$, Theorem 5.1 treats all nondefining odd-prime blocks of noncyclic defect, including $q = 2$ and the complete field-triality action. In both families the defining-prime and cyclic-defect results in Section 2 complete the list of blocks. □

Appendix A. Chevalley representatives in type D_4

Take an orthonormal basis $\varepsilon_1, \ldots, \varepsilon_4$ for the Euclidean root space and set

$$r_1 = \varepsilon_1 - \varepsilon_2, \quad r_2 = \varepsilon_2 - \varepsilon_3, \quad r_3 = \varepsilon_3 - \varepsilon_4, \quad r_4 = \varepsilon_3 + \varepsilon_4.$$

The remaining positive roots, in order, are

$$r_1 + r_2, \quad r_2 + r_3, \quad r_2 + r_4,$$
$$r_1 + r_2 + r_3, \quad r_1 + r_2 + r_4, \quad r_2 + r_3 + r_4,$$
$$r_1 + r_2 + r_3 + r_4, \quad r_{12} = r_1 + 2r_2 + r_3 + r_4.$$

Let $e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4$ be the hyperbolic basis of the eight-dimensional vector representation, with quadratic form

$$\mathcal{Q}\Big(\sum_i a_i e_i + \sum_i b_i f_i\Big) = \sum_i a_i b_i.$$

Its polar form satisfies $\langle e_i, f_j\rangle = \delta_{ij}$. Write E_{ij} for the matrix units in this ordered basis. The root matrices are

$$x_{\varepsilon_i-\varepsilon_j}(t) = I + t(E_{ij} - E_{4+j,4+i}) \quad (i \neq j),$$
$$x_{\varepsilon_i+\varepsilon_j}(t) = I + t(E_{i,4+j} - E_{j,4+i}) \quad (i < j),$$
$$x_{-\varepsilon_i-\varepsilon_j}(t) = I + t(E_{4+j,i} - E_{4+i,j}) \quad (i < j).$$

For each root r, put $n_r = x_r(1)x_{-r}(-1)x_r(1)$ and $h_r(-1) = n_r^2$. These are the images of the corresponding elements of the simply connected group.

Lemma A.1. *Let $u = h_1h_3h_4n_2$, $v = h_2n_1n_3n_4$ and $n_0 = n_1n_3n_4n_{12}$. Then*

$$u^2 = v^2 = n_0^2 = 1, \qquad (uv)^3 = n_0, \qquad [n_0, u] = [n_0, v] = 1.$$

The elements are fixed by pinned triality and by the prime-field map. The group $\langle u, v\rangle$ is a complement of order twelve for the two tori in Section 5.

Proof. Multiplication of the displayed root matrices gives the following signed images of the basis vectors:

	u	v	n_0
e_1	$-e_1$	e_2	$-f_1$
e_2	$-e_3$	e_1	$-f_2$
e_3	$-e_2$	f_3	$-f_3$
e_4	e_4	$-f_4$	$-f_4$
f_1	$-f_1$	f_2	$-e_1$
f_2	$-f_3$	f_1	$-e_2$
f_3	$-f_2$	e_3	$-e_3$
f_4	f_4	$-e_4$	$-e_4$

Applying the columns successively proves all the stated relations in the vector representation, over the integers and hence in every characteristic. The Weyl images of u and v are s_{r_2} and $s_{r_1}s_{r_3}s_{r_4}$. They generate the triality-fixed Weyl group D_{12}. The vector images therefore generate a group of order twelve.

To lift the relations to the simply connected group, observe that the three outer simple root subgroups commute and pinned triality permutes them cyclically. The highest root r_{12} is fixed by triality. On its integral root subgroup the pinned automorphism has a sign $\delta \in \{1, -1\}$; its order three gives $\delta^3 = 1$, hence $\delta = 1$. It therefore fixes n_{12}. Thus u, v, n_0 are all triality-fixed. A word in them whose vector image is the identity is a triality-fixed element of the central kernel. In odd characteristic, triality cyclically permutes the three nonidentity elements of the centre of Spin_8, corresponding to the three outer fundamental-weight classes modulo the root lattice; in characteristic two the centre has only the identity as a geometric point. The word is therefore the identity. This proves the relations in $\mathbf{G}$. They bound $|\langle u, v\rangle|$ by twelve; its Weyl image has order twelve, so the map to D_{12} is an isomorphism. This proves the complement assertion. The matrix representatives and the root words are defined over the prime field, completing the proof. □

Appendix B. Reproducible finite checks

The **computational supplement on GitHub** contains the Python source, recorded JSON output, and a runner for the exact matrix and root calculations in this paper. The block classification, ordinary character parametrizations, and inductive criterion are supplied by the proofs and cited theorems in the main text.

For G_2, the program generates the twelve integral matrices from Section 4, verifies their relations and entries, and checks their reductions modulo three and five. Since two distinct entries in $\{0, 1, -1\}$ have nonzero difference of absolute value at most two, the entry calculation proves faithful reduction for every odd prime.

For triality, the programs check (5.3) and (5.4) by exact enumeration for $1 \le a \le 4$, together with the twelve complement actions and the three invariant dual characters. They also check the twenty-four D_4 roots against the displayed restriction map, and evaluate the return congruences for the recorded prime powers. A separate root-lattice calculation verifies the Smith invariants of the fixed-torus matrix and checks the coordinate formulas at the recorded integer specializations. The additional standard-library program reconstructs the eight-dimensional Chevalley root matrices and verifies the complete signed-image table in Appendix A. It also checks the torus coordinate formulas at the recorded integer specializations and the dual-coordinate restriction map. The uniform arguments in the text explain their validity for all relevant prime powers.

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Zhili College, Tsinghua University, Beijing 100084, China
Email address: tangqy24@mails.tsinghua.edu.cn