Points from Processes:

Vakarelov's Dynamic Mereotopology and Whitehead's Point-Free Program

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Abstract

Alfred North Whitehead (1861–1947) held that neither points of space nor instants of time should be primitive: both ought to be definable from relations between material things. Region-based theories of space have largely carried out the spatial half of this program, but the temporal half has remained informal. This paper offers an expository and philosophical account of Dimiter Vakarelov's *dynamic mereotopology*, developed in five papers between 2010 and 2016, which aims at a theory of space and time that is point-free in this double sense. We argue that the program is best read as a *staged realization* of Whitehead's demand. Each stage enriches the primitive spatio-temporal relations between changing regions just enough to recover a further layer of temporal structure: first moments without order, then an ordered time, and finally nameable, spatially non-trivial epochs. The staging culminates in a structural result Whitehead anticipated but never formalized: points of space and points of time turn out to be different aggregations of one and the same stock of abstract atoms. We also argue that the representation theorems on which the program rests deliver a reconstruction, rather than a reduction, of points, and that reconstruction is what Whitehead's program requires.

Keywords: Whitehead; mereotopology; point-free theory of space and time; contact algebra; epochal theory of time; representation theorems.

> *How an actual entity becomes constitutes what that actual entity is … Its "being" is constituted by its "becoming."*
>
> — A. N. Whitehead, *Process and Reality*

1 Introduction

In a lecture of 1915, *Space, Time, and Relativity*, published two years later, Whitehead set out in a few sentences a program that he would spend much of the next decade trying to carry out:

> Our space concepts are concepts of relations between material things in space. Thus there is no such entity as a self-subsistent point. A point is merely the name for some peculiarity of the relations between matter which is, in common language, said to be in space. [. . .] Similar explanations apply to time. Before the theories of space and time have been carried to a satisfactory conclusion on the relational basis, a long and careful scrutiny of the definitions of points of space and instants of time will have to be undertaken, and

> many ways of effecting these definitions will have to be tried and compared. This is an unwritten chapter of mathematics.
>
> (Whitehead, 1917, ch. VIII, pp. 194–195)

The complaint is directed against the Euclidean tradition, in which points, lines and planes are undefined primitives. Whitehead's objection is not that these notions are useless, but that they are too abstract to stand at the foundation of a theory of the physical world: nothing in experience is a point. What we encounter are extended things standing in relations to one another, touching, overlapping, containing and preceding. If points and instants are to have a place in the theory, they must be *constructed* from those relations.

Two features of the program deserve emphasis from the start. The first is that it is *doubly* point-free. Influenced by relativity theory, Whitehead insisted that space and time cannot be treated separately: an adequate theory must be an integrated theory of space and time, in which neither spatial points nor temporal instants are primitive (Whitehead, 1919, 1920). The second is that the two halves of the program fared very differently. For space, *Process and Reality* contains something close to a mathematical blueprint: a primitive notion of region, a primitive relation of connection between regions, and a sequence of definitions leading to points, lines and planes (Whitehead, 1929/1978, Part IV, chs. II–III). That blueprint, together with independent contributions by de Laguna and Tarski, grew into what is now called the *region-based theory of space* (de Laguna, 1922; Tarski, 1956; Pratt-Hartmann, 2007; Vakarelov, 2007). For time, by contrast, Whitehead left no comparable blueprint. His later *epochal theory of time* introduces a rich vocabulary of durations, epochs, contemporaries and the present epoch, but presents it informally and in his own idiosyncratic philosophical terminology (Whitehead, 1925, 1929/1978). Any formalization of it is bound to be partial, and its fidelity to what Whitehead meant will always be open to dispute.

This paper is about the most sustained attempt we know of to supply the missing temporal half within the framework that succeeded for space. Between 2010 and 2016 Dimiter Vakarelov published five papers under the title *Dynamic Mereotopology*. The numbering of the series is confusing, so we fix short names here: DM-I (Vakarelov, 2010) and DM-II (Vakarelov, 2012) are two independent papers, while the third instalment appeared in three parts, which we call DM-III.1, DM-III.2 and DM-III.3 (Vakarelov, 2014, 2016a,b). Across these papers Vakarelov extends contact algebras, the standard algebraic framework of region-based theories of space, to algebras of *dynamic regions*, regions that change over time. He then shows how both space points and time points can be recovered from the resulting algebras.

Our aim is expository and philosophical. We present the main constructions of the five papers for a mixed readership of logicians and philosophers, keeping the formal apparatus to what is needed to follow the argument, and we assess what the program achieves in relation to Whitehead's demand. The thesis we defend is the following.

> Vakarelov's dynamic mereotopology is best understood as a *staged realization* of Whitehead's demand for a doubly point-free theory of space and time. Each stage enriches the primitive spatio-temporal relations just enough to recover a further layer of temporal structure: first moments of time without order; then time order; finally epochs that can be named and are spatially non-trivial. The staging culminates in a structural insight that Whitehead anticipated but never formalized: spatial points and temporal points are different aggregations of the same underlying atoms.

Two clarifications are needed. First, "staged" is meant literally. Each stage of the program

achieves something definite, leaves something definite inexpressible, and the next stage introduces new primitives precisely to repair that gap. Vakarelov is unusually candid about these gaps, and his own self-criticism, especially the diagnosis in DM-III.3 of what went wrong in DM-II, supplies much of the evidence for our reading. Second, "realization" needs care. The technical core of each stage is a *representation theorem*: every abstract algebra satisfying the axioms is isomorphic to an algebra built from explicit points and instants. One might object that this smuggles points back in, since the theorem is proved by exhibiting a point-based model. We argue in Section 7 that this objection misreads both the theorems and Whitehead. What the theorems establish is a *reconstruction*: points and instants are defined, within the point-free theory, as certain collections of regions, and the definitions are shown to recover all the structure the point-based picture contains. That, rather than an elimination of all talk of points, is what Whitehead asked for.

The plan of the paper is as follows. Section 2 reviews the static theory: contact algebras, their topological models, and the representation theory in which points are recovered from regions. Section 3 introduces the "snapshot" picture of dynamic regions that underlies all three stages. Sections 4–6 take the stages in turn. Section 7 offers a philosophical assessment, and Section 8 concludes with open problems. An appendix collects the axioms.

2 The static background: from mereology to mereotopology

Every construction in the dynamic theory is modeled on a construction in the static one. This section therefore sets out the static theory in some detail, following the presentation in DM-III.1, which Vakarelov wrote as the foundation for the later parts. Readers familiar with contact algebras may skim it, but should note three points that matter later: the treatment of the zero region as a predicate of existence (§2.1), the distinction between two kinds of abstract point (§2.3), and the role of the Efremovich axiom in producing a third kind (§2.4).

2.1 Mereology as Boolean algebra

Mereology is the theory of parts and wholes (Simons, 1987). Its basic relations are *part-of* and *overlap*. Following an observation going back to Tarski, the structure of these relations can be identified, for most purposes, with that of a Boolean algebra $(B, 0, 1, \leq, +, \cdot, ^*)$ (Simons, 1987; Vakarelov, 2014). The elements of B are *regions*. The ordering $a \leq b$ is read "a is part of b." The sum $a + b$ is the region composed of a and b, the product $a \cdot b$ their common part, and a^* the complement of a, the rest of the world. Two regions *overlap*, $a \; O \; b$, when they have a non-zero common part, that is, when $a \cdot b \neq 0$.

The zero element needs a word of interpretation, because it becomes important in the dynamic theory. Classical mereology usually has no null individual. In the Boolean setting there is a zero, and Vakarelov treats it as the *non-existing region*. He then uses it to define a predicate of existence: a region *a exists* when $a \neq 0$ (Vakarelov, 2014, §2.1). This is a predicate, not a quantifier, and it does real work later on. A dynamic region may exist at some moments of its history and fail to exist at others, and almost every temporal relation in the dynamic theory is defined in terms of when regions exist.

2.2 Contact algebras

Mereology alone cannot express the difference between two regions that touch and two that are separated by a gap: in both cases they fail to overlap. Whitehead's primitive of *connection*, now usually called *contact*, supplies exactly this. Adding contact to mereology yields *mereotopology*, so called because contact is a topological notion. Its standard algebraic form is the contact algebra.

Definition 2.1 (Dimov and Vakarelov, 2006; Vakarelov, 2014)**.** A *contact algebra* is a pair (B, C) where B is a non-degenerate Boolean algebra and C is a binary relation on B satisfying:

(C1) if $a \ C \ b$ then $a \neq 0$ and $b \neq 0$;

(C2) if $a \ C \ b$, $a \leq a'$ and $b \leq b'$, then $a' \ C \ b'$;

(C3) if $a \ C \ (b + c)$ then $a \ C \ b$ or $a \ C \ c$;

(C4) if $a \ C \ b$ then $b \ C \ a$;

(C5) if $a \cdot b \neq 0$ then $a \ C \ b$.

The axioms are intuitive. Only existing regions touch (C1). Enlarging regions preserves contact (C2). If a region touches a sum, it touches one of the summands (C3). Contact is symmetric (C4). Overlapping regions touch (C5). Two consequences show how much latitude the axioms leave. By (C5), overlap is the *smallest* contact relation on any Boolean algebra. By (C1), the relation that holds between any two existing regions is the *largest*. Every contact relation lies between these two extremes.

Contact allows Whitehead's own mereotopological relations to be defined (Whitehead, 1929/1978, Part IV, ch. II; 1929 ed., pp. 453–454). Two regions are in *external contact* when they touch without overlapping. A region is a *tangential part* of another when it is part of it and both are externally connected to some third region. It is a *non-tangential part* when it is a part that is not tangential, which amounts to $a \ \overline{C} \ b^*$: a does not even touch the outside of b. Vakarelov's illustrations are worth repeating for a non-specialist audience. A glass standing on a table is in external contact with the table. A nail driven into the table, with its head still visible, is a tangential part of the table. The same nail driven so deep that its head disappears is a non-tangential part (Vakarelov, 2014, §2.1).

One caveat is important for anyone comparing the formal theory with *Process and Reality*. Whitehead's connection relation satisfies a further axiom of *extensionality*: if $a \not\leq b$, then some region touches a but not b. With this axiom part-of becomes definable from contact, since $a \leq b$ holds exactly when everything touching a touches b. Contact algebras do not assume extensionality, so their contact is weaker than Whitehead's (Vakarelov, 2014, §2.1). The dynamic theory inherits this choice. Its primitives are therefore a Boolean algebra *plus* contact, not contact alone, and this is the first of several respects in which the formal program is a disciplined approximation to Whitehead rather than a transcription of him.

Standard models. The intended models of contact algebras are topological. Let X be a topological space with closure and interior operators Cl and Int. A set $a \subseteq X$ is *regular closed* if $a = Cl(Int(a))$, that is, if it is the closure of its own interior. Intuitively, it is a solid body with no lower-dimensional fins or hairs. The regular closed sets form a Boolean algebra $RC(X)$, in which sum is union, product is the closure of the interior of the intersection, and complement is the closure of the set-theoretic complement. Setting $a \ C \ b$ iff $a \cap b \neq \emptyset$ makes

$RC(X)$ a contact algebra. Here two regions touch exactly when they share a point, and the familiar geometrical solids (balls, cubes, pyramids) are all regular closed (Vakarelov, 2014, §2.2). Contact algebras of this form, and their subalgebras, are the *standard* contact algebras.

2.3 Recovering points from regions

The central question for Whitehead's program is whether the standard models are merely one interpretation among many, or whether they capture the abstract theory completely. If every contact algebra is isomorphic to a standard one, then the point-based picture contains nothing the point-free theory lacks. Points can then be *defined* from regions, by whatever construction the proof of the representation theorem uses. DM-III.1 explains in a particularly instructive way that there are two candidate constructions, and that only the second gives the right answer.

First attempt: ultrafilters. The obvious candidates for abstract points come from the Stone representation of Boolean algebras. Here the points are *ultrafilters*: maximal consistent collections of regions, thought of as all the regions containing a given point. Contact can be transferred to ultrafilters by declaring two of them *adjacent*, URV, when every region in U touches every region in V. Each region a is then represented by the set of ultrafilters containing it. Two regions turn out to be in contact iff some ultrafilter containing the first is adjacent to some ultrafilter containing the second (Vakarelov, 2014, Thm. 2.9); see also Düntsch and Vakarelov (2007). The resulting models are "discrete" in Galton's sense: they resemble the squares of a chessboard, where two regions touch when some square of one is adjacent to some square of the other (Galton, 1999).

This is a genuine representation theorem, but Vakarelov stresses that it is *not the intended one*. In it, contact is not the sharing of a point. Two regions can touch without containing a common ultrafilter. Whitehead's notion of connection, on which touching just is having a common boundary point, is therefore not captured. Ultrafilters are too few, or too fine, to serve as the points at which regions meet (Vakarelov, 2014, §2.3).

Second attempt: clans. The solution adopted by Dimov and Vakarelov (2006) is to add more points. A *clan* is a collection Γ of regions that behaves like "all the regions containing a given point" in a weaker sense than an ultrafilter. It contains 1 but not 0, it is closed upward, if it contains $a + b$ it contains a or b, and any two of its members are in contact. Every ultrafilter is a clan, but there are many more. Every clan is the union of a set of pairwise adjacent ultrafilters (a *clique*), and every such union is a clan (Vakarelov, 2014, Lemma 2.11). Taking clans as points, and representing a region a by the set $g(a)$ of clans containing it, gives the result that justifies the program.

Theorem 2.2 (Topological representation; Dimov and Vakarelov, 2006)**.** *Every contact algebra (B, C) embeds into the standard contact algebra $RC(X)$, where X is the space of clans of (B, C) with the sets $g(a)$ as a base of closed sets. Under this embedding, $a\ C\ b$ iff $g(a) \cap g(b) \neq \emptyset$: two regions are in contact exactly when they share a point.*

In Dimov and Vakarelov (2006) the clan space is moreover shown to be compact, T_0 and semiregular, but the essential point for us is the last clause. Defined as clans, points are exactly what Whitehead needed. They are constructed from regions and contact alone, and contact between regions becomes the sharing of a constructed point.

Vakarelov draws a striking moral from the fact that this theorem uses two kinds of points. Ultrafilters may be regarded as *atoms*. Clans that are not ultrafilters are unions of adjacent ultrafilters, and so may be regarded as *molecules* composed of atoms. Moreover, the boundary points of a region are never ultrafilters, only genuine molecules (Vakarelov, 2014, §2.4). In Vakarelov's words, these facts "throw a new light on the ancient atomistic view of space." We flag this remark because our thesis is, in effect, that the dynamic program extends it. In the static theory, spatial points are molecules built from ultrafilter atoms. In the completed dynamic theory, temporal points turn out to be molecules built from the *same* atoms, assembled in a different way.

2.4 Precontact relations, the Efremovich axiom, and clusters

Two further pieces of static theory are needed later.

Precontact. A relation satisfying (C1)–(C3) and the mirror image of (C3), but not necessarily symmetry (C4) or reflexivity on existing regions (C5), is a *precontact* relation (Düntsch and Vakarelov, 2007; Vakarelov, 2014). The ultrafilter construction works for precontact relations too. The canonical relation R on ultrafilters is then neither symmetric nor reflexive in general, and the correspondence is exact: R is symmetric iff (C4) holds and reflexive iff (C5) holds (Vakarelov, 2014, Lemma 2.8). This matters because the temporal relation of *precedence* in the dynamic theory is a precontact relation. "a exists before b exists" is neither symmetric nor reflexive, yet it distributes over sums exactly as contact does.

The Efremovich axiom and clusters. The third ingredient is an axiom borrowed from the theory of proximity spaces:

$$\text{(CE)} \qquad \text{if } a \mathrel{\overline{C}} b \text{ then there is } c \text{ with } a \mathrel{\overline{C}} c \text{ and } c^* \mathrel{\overline{C}} b.$$

Intuitively, if a and b are apart, some region c separates them: a is apart from c, and b is apart from everything outside c. The canonical adjacency relation R on ultrafilters is transitive exactly when (CE) holds (Vakarelov, 2014, Lemma 2.8). In a contact algebra satisfying (CE), then, adjacency is an *equivalence relation*, and the ultrafilters fall into disjoint classes. The unions of these classes are the *clusters*, and they coincide with the maximal clans (Vakarelov, 2014, Lemma 2.16), an algebraic version of a classical result of Leader on proximity spaces. Two features of clusters will matter. Distinct clusters are always separated by some region, which belongs to one cluster while its complement belongs to the other (Vakarelov, 2014, Lemma 2.18). And, by themselves, clusters are too coarse to tell regions apart, so a representation theorem cannot be built on clusters alone (Vakarelov, 2014, §2.6).

In the static theory, clusters are a technical curiosity. In the dynamic theory they become central. Applied to the relation of *time contact* rather than spatial contact, they turn out to be the time points, Vakarelov's formal counterpart of Whitehead's epochs. Their coarseness then becomes an advantage. It is exactly what one expects of an epoch, which gathers together everything happening at once without distinguishing one region from another.

2.5 The template

We can now state the methodological template that every stage of the dynamic program follows. It has three steps, each modeled on the static theory just described.

1. *A concrete model.* Begin with an intuitive point-based structure, in the static case $RC(X)$, in which the intended relations between regions are defined from points.
2. *Abstraction.* Select properties of these relations that are true in the concrete model, and take them as the axioms of an abstract, point-free theory.
3. *Representation.* Prove that every abstract model is isomorphic to (a subalgebra of) a concrete one. The proof constructs points from regions (clans, in the static case) and thereby shows how points can be defined.

The third step is where Whitehead's program is actually carried out, and the choice of construction in it is philosophically loaded. As the contrast between ultrafilters and clans shows, the same algebra can be represented using different notions of point, and only some of them vindicate the intended reading of the primitives. The dynamic program repeats this lesson on a larger scale. Its history is, to a large extent, a search for the right notion of *time* point.

3 The snapshot picture

All three stages of the program share a single concrete model, which Vakarelov calls the *snapshot* or *cinematographic* construction (Vakarelov, 2016a, §1; Vakarelov, 2020, §3). This section presents it and then sets it against the abstract algebras that each stage axiomatizes. Much of the philosophical interest of the program lies in the relation between these two layers: what the snapshot model takes as given, the abstract algebra must construct, and the comparison shows exactly where the point-free theory has work to do.

3.1 Snapshots and dynamic regions

The construction starts from a *time structure* $(T, \prec)$: a non-empty set T of time points with a binary relation $m \prec n$, read "m is before n." Nothing is assumed about $\prec$ in advance. Familiar conditions such as transitivity, linearity, density, seriality or directedness may be imposed or not, and different selections give different pictures of time: some subsets are typical of classical time, others of the time order of special relativity (Vakarelov, 2016a, §1.1).

To each time point m we then attach a contact algebra (B_m, C_m), thought of as a *snapshot*: the static configuration of regions at m, with its own relation of contact. The cinematic image is Vakarelov's, and so is the caution that goes with it. A snapshot is not what an observer happens to see at m. Light from a distant star may reach us long after the star has ceased to exist. It is rather the actual spatial configuration at m, however that is known (Vakarelov, 2020, Remark 3.2).

A changing region is identified with its history: a sequence $a = \langle a_m \rangle_{m \in T}$ assigning to each moment the static region $a_m \in B_m$ that a occupies then. Vakarelov calls such sequences *dynamic regions*. Sums, products and complements are computed moment by moment, so that, for instance, $(a + b)_m = a_m +_m b_m$, and part-of holds when it holds at every moment. A *dynamic model of space* is any Boolean algebra $B(T)$ of dynamic regions closed under these operations, that is, any subalgebra of the product $\prod_{m \in T} B_m$. It is *full* if it is the whole product, and *rich* if it contains every region that, at each moment, is either the whole of space or nothing (Vakarelov, 2016a, Def. 1.2).

Two features of this definition carry philosophical weight. First, the predicate of existence from Section 2.1 becomes temporal: *a exists at m* iff $a_m \neq 0_m$. A non-zero dynamic region

may exist at some moments and not at others, and may even exist intermittently. Vakarelov mentions viruses (Vakarelov, 2020, §3.2). Second, distinct dynamic regions may coincide at a moment. Vakarelov's example is political: before the Second World War there was one Germany, for a time afterwards two, and now again one (Vakarelov, 2020, §3.2). Identity of dynamic regions is identity of whole histories, not of stages.

Finally, the model contains points of both kinds explicitly. Time points are the elements of T. Space points can be supplied in two ways. By the representation theorem for contact algebras (Theorem 2.2), each snapshot (B_m, C_m) has its own space X_m, whose points are the clans of B_m, so that each moment has a *local space*. Alternatively, by a joint embedding theorem, all the snapshots can be placed inside the regular closed sets of a single *absolute space* X (Vakarelov, 2010, Thm. 1.2; Vakarelov, 2014, Thm. 2.13). The snapshot model is therefore point-based twice over, which is exactly what the abstract theory must eventually reconstruct.

3.2 Spatio-temporal relations as quantifications over time

In a dynamic model of space, relations between dynamic regions are defined by quantifying over moments. The relations studied in the three stages are the following (Vakarelov, 2010, 2012; Vakarelov, 2016a, §1.2):

$$
\begin{aligned}
a\ C^{\forall}\ b &\iff (\forall m \in T)\,(a_m\ C_m\ b_m) && \text{stable contact}\\
a\ C^{s}\ b &\iff (\exists m \in T)\,(a_m\ C_m\ b_m) && \text{space contact}\\
a\ C^{t}\ b &\iff (\exists m \in T)\,(a_m \neq 0_m \text{ and } b_m \neq 0_m) && \text{time contact}\\
a\ \mathcal{B}\ b &\iff (\exists m, n \in T)\,(m \prec n,\ a_m \neq 0_m \text{ and } b_n \neq 0_n) && \text{precedence}
\end{aligned}
$$

Space contact is the relation called *unstable contact*, $C^{\exists}$, in the first stage. Stable contact means touching at every moment, space contact touching at some moment, time contact coexisting at some moment, and precedence existing at some moment before a moment at which the other exists. Figure 1 shows how the first three are read off a film strip, row by row.

Three observations will matter later. First, only precedence uses the time order; the other three would be unchanged if the frames of the film were shuffled. This is why the first stage of the program could get by with an unordered set of moments. Second, precedence is obtained from time contact by replacing "the same moment" with "an earlier and a later moment," which Vakarelov notes explicitly (Vakarelov, 2020, §3.3). Time contact is the degenerate, simultaneous case of precedence. Third, space contact requires the two regions to touch *at the same moment*. A region that occupies at one time a place that another occupies at a later time is not in contact with it, so space contact is genuinely spatio-temporal rather than merely spatial.

3.3 From the model to the axioms

The abstract theory is obtained by selecting properties of these relations that hold in the snapshot model and taking them as axioms. For the relations of the second and third stages, the basic facts are these (Vakarelov, 2016a, Lemma 1.4). Space contact and time contact are both contact relations in the sense of Definition 2.1. Space contact implies time contact, since regions that touch at some moment certainly coexist at it. Precedence is a precontact relation. And, in rich models, time contact satisfies the Efremovich axiom.

The proof of the last fact is worth seeing, because it shows how much the axioms depend on what the model contains. Suppose a and b are never contemporaries. Let c be the region

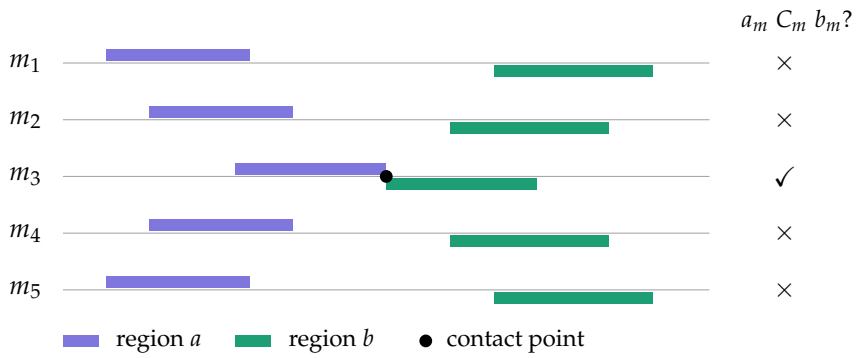


Figure 1: A film strip for two dynamic regions a and b across five moments; each row is a snapshot, drawn here in one spatial dimension. The regions touch only at m_3. Hence $a\ C^{\exists}\ b$ (equivalently $a\ C^{s}\ b$) holds, witnessed by m_3, while $a\ C^{\forall}\ b$ fails, witnessed by any other row; $a\ C^{t}\ b$ holds because both regions exist together at every moment. Reordering the rows changes none of these verdicts, but it can change verdicts of precedence, which is the only relation in the list that depends on $\prec$.

that is the whole of space at exactly those moments at which a does not exist, and nothing at the others. Then a never coexists with c, and b, which exists only when a does not, never coexists with the complement of c. The argument needs the region c to be present in the model, and richness guarantees exactly that. Rich models, in other words, contain regions of the form "the whole world, during such-and-such moments." These regions are the formal ancestors of the *universal time representatives* of the third stage, which are the whole world during a single moment.

The same pattern recurs throughout. The correspondence between properties of the time order and conditions on time contact and precedence (Section 5) is proved for rich models, and its proofs construct regions of this kind. Vakarelov is candid that choosing axioms is not a matter of reading them off the model. It is an iterative process: one guesses the steps of a canonical construction, checks which properties each step needs, verifies them in the model and adds them to the list, a process he describes as a long and not always successful experiment (Vakarelov, 2020, §1.4). This is worth bearing in mind when the axioms look unmotivated. Many of them are there because a particular step of the representation proof requires them.

3.4 Two layers: what the contrast reveals

The snapshot model and the abstract algebra describe the same subject matter, and the representation theorems tie them together. But they distribute the work very differently. Table 1 sets the two side by side, using the mature theory of the third stage for the right-hand column.

Read row by row, the table brings out five contrasts.

An inversion of priority. In the snapshot model, time comes first. The time structure is fixed before anything else, snapshots are indexed by it, and regions are functions on it. In the abstract algebra the order is reversed. Regions and their relations come first, and time points, their order and the snapshots themselves are all constructed from them. This reversal is Whitehead's program in miniature. The snapshot model is the picture a point-based theorist

	Snapshot model	**Abstract algebra (third stage)**
Dynamic region	derived: a function $\langle a_m \rangle$ on T	primitive: an element of the algebra
Time point	given: an element of T	constructed: a t-cluster containing a universal time representative
Time order	given: the relation $\prec$	constructed: induced on t-clusters by precedence $\mathcal{B}$
Space point	given: a clan of some B_m, or a point of X	constructed: an s-clan inside a t-cluster
Snapshot at a time	given: the contact algebra (B_m, C_m)	constructed: a quotient of the algebra determined by the cluster's s-clans
"a exists at m"	$a_m \neq 0_m$	$a \cdot c \neq 0$, where c names the epoch
"a touches b at m"	$a_m \; C_m \; b_m$	$(a \cdot c) \; C^s \; (b \cdot c)$, where c names the epoch

Table 1: What is given and what is constructed in the two layers of the theory.

starts from; the algebra is the picture Whitehead asked for, in which instants are "the name for some peculiarity of the relations" between things. The representation theorem is what licenses moving from one to the other.

Talking about a moment requires a name for it. The snapshot model can say directly what is the case at a given moment, because moments are among its objects. The abstract algebra cannot, at least not without help, because its quantifiers range over regions only. It can say that two regions coexist at *some* moment, or touch at *some* moment, but not *which*. The last two rows of Table 1 show how the third stage overcomes this. A universal time representative c is a region that, in the standard model, is the whole world during one moment and nothing at any other. Multiplying by it isolates a single moment: a exists at the epoch named by c iff $a \cdot c \neq 0$, and a touches b then iff $(a \cdot c) \; C^s \; (b \cdot c)$ (Vakarelov, 2016a, Lemma 3.4, §3.1). This is the formal content of Vakarelov's motivating examples: we pick out an epoch by naming something that existed then and only then, as in "the epoch of Leonardo" (Vakarelov, 2016a, §3). Without such names, the algebra can describe the pattern of coexistence and succession only up to an existential quantifier over moments, and that is the root of most of the expressive limitations of the first two stages.

Richness as hidden structure. The axioms are verified in rich models, and richness is not something the algebra can say about itself. Vakarelov makes this point explicitly about irreflexivity of the time order. In a rich standard model there is a condition on time contact and precedence that holds exactly when the time order is irreflexive, because richness supplies the special regions the proof needs. In an abstract algebra without time representatives, however, he can prove only one direction of the correspondence, and conjectures that the other fails (Vakarelov, 2020, Remark 4.10). Richness is a feature of the model that the abstract language must find some way to encode. The time representatives of the third stage are, in effect, that encoding: they postulate inside the algebra the special regions that richness provides in the model.

What the algebra cannot see. The representation theorems embed an abstract algebra into a standard model; they do not set up a one-to-one correspondence between algebras and

models. Vakarelov compares the situation with the representation of Boolean algebras as algebras of sets (Vakarelov, 2020, §4.4). Some of what is in a snapshot model is invisible to its algebra of dynamic regions. In particular, a static region that occurs in the snapshot B_m but is not the stage at m of any dynamic region in the model plays no role at all. The algebra recovers each snapshot only up to its *essential subalgebra*, the part realized by stages of dynamic regions (Vakarelov, 2016a, §4.2). The philosophical moral is pleasingly Whiteheadian. A static region has no standing in the theory except as a stage of something that persists or happens; there are no snapshots of nothing in particular.

Exact recovery where it matters. Within these limits the reconstruction of time is exact. For a rich standard model, each moment i determines a t-cluster $\hat{\imath}$, consisting of the regions that exist at i. Distinct moments determine distinct t-clusters, and two regions are contemporaries iff they belong to a common one (Vakarelov, 2016a, Lemmas 4.1–4.2). The time order is recovered too: $i \prec j$ iff every region existing at i precedes every region existing at j (Vakarelov, 2016a, Lemma 4.4). Since each such $\hat{\imath}$ contains the universal time representative of i, these are exactly the time points of the third-stage construction. In the favorable case, then, nothing temporal is lost in passing from the model to the algebra and back. What is lost is only spatial material that no dynamic region instantiates.

The observer. Finally, the snapshot model presupposes a single clock: one time structure, along which one sequence of snapshots is taken. Vakarelov describes the camera as a fixed observer (Vakarelov, 2020, §3.2). The abstract algebra mentions no observer and no clock. Yet the single time line has not disappeared. It is encoded in the axioms, and in Section 7.3 we argue that it enters through one axiom in particular, the Efremovich axiom for time contact.

One last observation completes the picture. Any static contact algebra can be read as a dynamic one, by taking its contact as space contact, the largest contact ("both regions exist") as time contact, and precedence equal to time contact. The construction then finds exactly one time point, and the time order on it is trivial. In Vakarelov's phrase, the clock is not ticking; time is stopped, or degenerate (Vakarelov, 2020, Lemma 4.2 and §4.2). Static mereotopology is thus not an alternative to the dynamic theory but its limiting case. The stages that follow can be read as successive attempts to make the clock tick in more and more articulate ways, while still extracting the clock from the regions rather than building it in.

4 Stage one: time without order

The first stage, DM-I (Vakarelov, 2010), is deliberately experimental. Vakarelov later describes its purpose as testing whether the strategy of Section 2.5 could work at all for space and time together, and for that reason it uses only relations that do not presuppose that time flows (Vakarelov, 2020, §1.4). The time structure of the intended models is a bare set T of moments, with no before or after.

4.1 Stable and unstable contact

The two primitive relations are *stable contact*, $a\ C^{\forall}\ b$, "a and b are always in contact," and *unstable contact*, $a\ C^{\exists}\ b$, "a and b are sometimes in contact." In the standard models they are the universally and existentially quantified versions of the contact relations of the snapshots

(Section 3.2). Vakarelov's illustration is a human body. Considered as a dynamic region, the body is always in contact with its head, while its two hands are in contact only sometimes (Vakarelov, 2010, Introduction).

The axioms are economical. A *dynamic contact algebra* in the sense of DM-I is a Boolean algebra with two relations such that $C^{\exists}$ is a contact relation (Definition 2.1) and $C^{\forall}$ satisfies three further conditions (Vakarelov, 2010, Def. 2.1):

($C^{\forall}1$) $1\ C^{\forall}\ 1$;

($C^{\forall}2$) not $0\ C^{\forall}\ 0$;

($C^{\forall}3$) if $p\ C^{\forall}\ q$ and not $a\ C^{\forall}\ b$, then $(p \cdot a^*)\ C^{\exists}\ (q \cdot a^*)$ or $(p \cdot b^*)\ C^{\exists}\ (q \cdot b^*)$.

The first two say that the whole of space always touches itself and that the non-existing region does not. The third does the real work, and it has a clear intuitive reading in the snapshot model. Suppose p and q are always in contact, while a and b are not. Then there is a moment at which a and b fail to touch. At that moment p and q still touch, at some point, and that point cannot lie in both a and b, since otherwise a and b would touch there. So at that moment either the parts of p and q outside a touch, or the parts outside b do. The axiom is the abstract trace of a static fact about contact, applied at the moment that witnesses the failure of stable contact (Vakarelov, 2010, Lemma 1.1(vi)). From these axioms it follows that stable contact implies unstable contact, and that stable contact is symmetric and monotone (Vakarelov, 2010, Lemma 2.1).

Two further facts set the stage. Every static contact algebra becomes a DCA in this sense if stable and unstable contact are both taken to be its contact relation (Vakarelov, 2010, Lemma 2.3), a first instance of the static theory as the limiting case of the dynamic one. And, since all the axioms are universal sentences, every finite set of regions generates a finite subalgebra that is again a DCA (Vakarelov, 2010, Lemma 2.2, Cor. 2.1), a fact that will matter for the logic.

4.2 A vocabulary of "always" and "sometimes"

A large part of DM-I shows how much can be expressed with these two relations alone. The method is to take a familiar mereological or mereotopological relation, consider its "always" and "sometimes" versions in the snapshot model, and find a definition of each in the language of DCAs (Vakarelov, 2010, §2). A few examples convey the flavor.

- A region is *sometimes non-existent* iff it is not stably in contact with itself. In the snapshot model, $a\ C^{\forall}\ a$ fails exactly when some a_m is empty. The regions that *always exist* are the others, and the regions that *sometimes exist* are simply the non-zero ones.
- a is *sometimes part of* b iff $a \cdot b^*$ is sometimes non-existent: at some moment, nothing of a lies outside b.
- a and b *always overlap* iff $a \cdot b$ always exists, and they *sometimes overlap* iff $a \cdot b \neq 0$.
- a is *sometimes a non-tangential part* of b iff a is not stably in contact with b^*, and *always* a non-tangential part iff a is not even unstably in contact with b^*.

In the same way the stable versions of all eight RCC-8 relations of Egenhofer and Franzosa (1991) and Randell et al. (1992) become definable. Vakarelov conjectures, however, that their unstable versions are not definable, and that neither is Whitehead's relation of *contemporaneity*,

"a and b exist at some common moment" (Vakarelov, 2010, §2). That conjecture, if correct, is significant for our narrative. It means that the most basic temporal relation in Whitehead's later theory cannot be recovered from contact, however it is quantified over time. It has to be added as a new primitive, which is exactly what the second stage does.

4.3 Time moments as witness sets

The representation theory of DM-I follows the template of Section 2.5 with clans as the basic abstract points. Fix a DCA and consider the clans of the contact algebra $(B, C^{\exists})$. A *time moment* is a set α of clans satisfying a single condition (Vakarelov, 2010, Def. 3.1):

$$\text{(tm)} \qquad \text{whenever } p\ C^{\forall}\ q, \text{ some clan in } \alpha \text{ contains both } p \text{ and } q.$$

A time moment, in other words, is a *witness set*: a collection of candidate spatial points large enough that every stable contact is realized at it. The definition is tailored to recover the quantifier pattern of the standard model. Two regions are in unstable contact iff some time moment contains a clan to which both belong, and in stable contact iff every time moment does (Vakarelov, 2010, Lemmas 3.1 and 3.3). The "sometimes" and "always" of the snapshot model become existential and universal quantifiers over constructed moments.

Each time moment then yields a snapshot. The regions that belong to no clan in α form an ideal, the regions that are "absent at α." Factoring them out gives a Boolean algebra, in which two classes are in contact iff some clan in α contains representatives of both (Vakarelov, 2010, Lemma 1.4). Mapping each region to the vector of its classes at all time moments embeds the original algebra into a product of contact algebras. The resulting representation theorem has three forms (Vakarelov, 2010, Thm. 3.1). Every DCA is isomorphic to a subalgebra of a product of contact algebras. Each factor can be taken to be an algebra of regular closed sets of its own topological space, so that each moment has its own *local space*. And all of the factors can be embedded in the regular closed sets of one *absolute space*. Vakarelov's gloss is that each moment of time defines its own local space, while all local spaces can be embedded into one absolute space (Vakarelov, 2010, Remark 3.1). It is a neat formal echo of the old dispute between relative and absolute conceptions of space, with the twist that both are recovered from one and the same point-free algebra.

Moments without individuality. One feature of this construction is easy to overlook, and it explains much of what happens next. Condition (tm) is preserved under enlargement: any set of clans that contains a time moment is itself a time moment, and the set of all clans is one. The canonical model therefore contains a great many moments, most of them redundant, and nothing in the construction picks out a privileged collection of them. The moments of the first stage are adequate for representation, but they have no identity conditions of their own. They are not ordered, which was the intended limitation, but neither are they individuated: there is no sense in which the algebra determines *which* moments there are. This is the clearest sense in which stage one recovers time only in a minimal form. The later stages can be read as the search for time points that are not just adequate but *canonical*.

4.4 A decidable logic

DM-I also turns the algebraic theory into a logic. The language of *Dynamic Contact Logic* (DCL) has Boolean terms built from variables, and atomic formulas $a = b$, $a\ C^{\exists}\ b$ and $a\ C^{\forall}\ b$,

combined by the propositional connectives. It has no quantifiers. Since the axioms of DCAs are universal, they can be written as axiom schemes of such a language, and the resulting Hilbert-style system, with modus ponens as its only rule, is complete for several classes of models at once: all DCAs, the standard topological DCAs, and the standard "coherent" DCAs built over a single absolute space (Vakarelov, 2010, Thm. 4.1). Because every finite set of regions generates a finite subalgebra, the logic has the finite model property: a formula with n variables that is not a theorem fails in a DCA with at most 2^{2^n} elements. DCL is therefore decidable (Vakarelov, 2010, Cor. 4.1).

DCL extends BRCC-8, the constraint logic for the RCC-8 relations with Boolean region terms introduced by Wolter and Zakharyaschev (2000) and axiomatized in Balbiani et al. (2007). Vakarelov also notes its relation to an earlier modal approach of his own, in which stable and unstable contact are defined as $\Box(a\ C\ b)$ and $\Diamond(a\ C\ b)$ over an S5 modality (Vakarelov, 2008). The product models of DCAs imitate that modal semantics, and DCL is contained in the modal logic (Vakarelov, 2010, §5). The difference that matters for our purposes is that the modal logic assumes its possible worlds, whereas DCAs construct their moments.

4.5 What stage one achieves, and what it leaves open

Stage one establishes the basic claim of the program in its simplest form. Moments of time can be constructed from dynamic regions and two contact relations, alongside spatial points, in such a way that every abstract algebra is represented by a family of snapshots. It also shows that a substantial vocabulary of temporally qualified mereotopological relations is available without any primitive notion of time.

Its limits are equally clear, and Vakarelov states them himself (Vakarelov, 2010, §5). The moments it constructs are not ordered, so nothing that depends on before and after can be expressed. He gives two examples of what he would like to express. One is *stabilized contact*: from some moment on, two regions are always in contact. The other is a principle of *continuous change*: if a is part of b at one moment and disjoint from it at a later one, and both exist throughout, then at some intermediate moment a is in external contact with b. Contemporaneity, as we saw, is probably not definable. And, as we have added, the moments are not individuated. The second stage addresses the first two problems directly; the third addresses the last.

5 Stage two: ordering time

The second stage, DM-II (Vakarelov, 2012), gives the time of the standard models an order, and therefore needs a primitive relation sensitive to it. Its material is revisited and extended in the first two sections of DM-III.2 (Vakarelov, 2016a), and we draw on both.

5.1 Space contact, time contact and precedence

The primitive relations of the second stage are the three defined in Section 3.2. *Space contact* C^s is the unstable contact of the first stage under a new name. The renaming signals a new role: this is the relation responsible for the spatial points of the snapshots. *Time contact* C^t, coexistence at some moment, is Vakarelov's formalization of Whitehead's contemporaneity, the relation that the first stage probably could not define. *Precedence* $\mathcal{B}$, "a exists at some

moment before a moment at which b exists," has a less direct Whiteheadian pedigree. In DM-II Vakarelov notes that Whitehead used nothing like it, and in DM-III.2 that Whitehead mentioned such a relation without defining it (Vakarelov, 2012, Introduction; Vakarelov, 2016a, §1.2). Either way, he adopts it for two reasons: it is responsible for the time order, and, together with time contact, it characterizes a surprising range of properties of that order.

The basic axioms are those suggested by the snapshot model (Section 3.3): C^s and C^t are contact relations, space contact implies time contact, and $\mathcal{B}$ is a precontact relation (Vakarelov, 2012, Def. 3.1). The bridge axiom $a\ C^s\ b \Rightarrow a\ C^t\ b$ has the intuitive content that meeting somewhere in space requires coexisting at some time. Precedence is only a precontact relation because it is neither symmetric nor reflexive: a region need not exist before itself, and a existing before b does not imply the converse.

Stable contact disappears at this stage. Vakarelov explains that the canonical constructions of the first and second stages are not compatible, and leaves its characterization in the new setting as an open problem (Vakarelov, 2012, §6). It returns in the third stage, as a defined relation.

5.2 Correspondence: properties of time as properties of processes

The conceptual centerpiece of the second stage is a correspondence theory, in a sense closely analogous to correspondence theory in modal logic. In the snapshot model, properties of the time order $\prec$ are statements about time points. Vakarelov shows that each of a standard list of such properties is equivalent, over rich models, to a condition stated purely in terms of time contact and precedence between regions (Vakarelov, 2012, Lemma 2.4); (Vakarelov, 2016a, Lemma 2.1). Table 2 gives the list in the form of DM-III.2.

Property of the time order	**Condition on regions**
(RS) every moment has a later one	(rs) if $a \neq 0$ then $a\ \mathcal{B}\ 1$
(LS) every moment has an earlier one	(ls) if $a \neq 0$ then $1\ \mathcal{B}\ a$
(Up Dir) any two moments have a common later one	(up dir) if $a, b \neq 0$ then $a\ \mathcal{B}\ p$ or $b\ \mathcal{B}\ p^*$
(Down Dir) any two moments have a common earlier one	(down dir) if $a, b \neq 0$ then $p\ \mathcal{B}\ a$ or $p^*\ \mathcal{B}\ b$
(Dens) between two ordered moments lies a third	(dens) if $a\ \mathcal{B}\ b$ then $a\ \mathcal{B}\ p$ or $p^*\ \mathcal{B}\ b$
(Ref) every moment precedes itself	(ref) if $a\ C^t\ b$ then $a\ \mathcal{B}\ b$
(Irr) no moment precedes itself	(irr) if $a\ \mathcal{B}\ b$ then for some c, d: $c\ \mathcal{B}\ d$, $a\ C^t\ c$, $b\ C^t\ d$ and not $c\ C^t\ d$
(Lin) any two moments are ordered	(lin) if $a, b \neq 0$ then $a\ \mathcal{B}\ b$ or $b\ \mathcal{B}\ a$
(Tri) any two moments are equal or ordered	(tri) if $a\ C^t\ c$, $b\ C^t\ d$ and not $c\ C^t\ d$, then $a\ \mathcal{B}\ b$ or $b\ \mathcal{B}\ a$
(Tr) the order is transitive	(tr) if not $a\ \mathcal{B}\ b$ then for some c: not $a\ \mathcal{B}\ c$ and not $c^*\ \mathcal{B}\ b$

Table 2: Properties of the time order and their counterparts in terms of time contact and precedence (Vakarelov, 2016a, §2). In each row, the left-hand property holds of the time structure of a rich standard model iff the right-hand condition holds in its algebra of dynamic regions. Variables p range over all regions.

Three of the rows show how the correspondences work. Reflexivity of the time order corresponds to the requirement that contemporaries precede one another: if each moment counts as "before" itself, then any two regions that coexist at a moment exist before each other. Right seriality corresponds to the requirement that every existing region precedes the whole world: if every moment has a successor, then whatever exists at some moment exists before a moment at which something exists. Upward directedness is subtler. It says that any two existing regions a and b, together with *any* region p, satisfy: either p exists after a or the complement of p exists after b. The quantification over p is what does the work. If a and b have a common later moment k, then at k either p or its complement is non-empty, so one disjunct holds; and if they have no common later moment, a suitably chosen p refutes the condition. The trick of quantifying over a region and its complement, to simulate a quantifier over moments, recurs throughout the list.

The philosophical significance of the table is considerable. Properties such as density, linearity or directedness are normally formulated as properties of instants. The correspondence shows that, in rich models, they are equally well properties of how processes can coexist and succeed one another. The *shape* of time is encoded in the relations between the things in time, which is precisely what Whitehead's relationalism requires. Different selections from the list yield different conceptions of time within one framework. Reflexivity, transitivity and linearity, or irreflexivity, transitivity, trichotomy and density, are typical of classical time. Reflexivity, transitivity and directedness in both directions characterize the time order of special relativity (Vakarelov, 2016a, §1.1).

Two qualifications are needed. First, the correspondences are proved for *rich* models, and the proofs use the special regions that richness supplies (Section 3.3). Second, the conditions differ in logical form. Most are universal sentences, but transitivity is not: its counterpart (tr) asserts the existence of a separating region c, a temporal analog of the Efremovich axiom. In DM-II two rows were also missing. Vakarelov did not find counterparts for irreflexivity and trichotomy (Vakarelov, 2012, Note 1), and the versions in Table 2 first appear in DM-III.2. Even then, as we shall see, the correspondence for irreflexivity holds in rich standard models but could not be established for abstract algebras without further resources.

5.3 Time moments as cliques of ultrafilters

The representation theory of DM-II changes the basic abstract points from clans to ultrafilters. Each of the three relations induces a canonical relation on ultrafilters as in Section 2.3: UR^sV, UR^tV and $U \prec V$ hold when every region in U stands in space contact, time contact or precedence, respectively, with every region in V (Vakarelov, 2012, §3). Each property in Table 2 then has a counterpart as a property of these canonical relations, and again the correspondences are exact (Vakarelov, 2012, Lemma 3.3).

A *time moment* is now a non-empty set of ultrafilters any two of which are related by R^t, that is, a set of mutually contemporaneous atoms. One moment precedes another when every ultrafilter in the first precedes every ultrafilter in the second. The snapshot at a moment is obtained by identifying regions that belong to exactly the same ultrafilters of the moment, with contact defined through R^s (Vakarelov, 2012, Defs. 4.1, 4.4). The representation theorem states that every DCA embeds into the snapshot model built in this way, and that the model satisfies exactly the time conditions whose counterparts hold in the algebra (Vakarelov, 2012, Thm. 4.7).

There is, however, a catch. When the algebra satisfies one of the conditions (rs), (ls),

(up dir), (down dir) or (dens), the proof requires a time moment to contain *at most two* ultrafilters (Vakarelov, 2012, Def. 4.1, (tm3)). Vakarelov explains the condition as technical: the ultrafilter counterparts of these properties are not universal sentences, and the proof needs small moments to construct the required witnesses. The price is high. A snapshot built from one or two ultrafilters is a Boolean algebra with two or four elements, which is spatially trivial. Vakarelov's own later verdict is that contact algebras of this size "are of no interest," and that the language and axioms of the second stage were too weak to formalize changing regions with non-trivial snapshots (Vakarelov, 2016b, §4.2).

5.4 Logics, and a rule for transitivity

As in the first stage, the algebraic theory yields quantifier-free logics. For a set Ax of time conditions, the logic L_{Ax} adds to the axioms of DCAs the counterparts in Ax as further axiom schemes. Transitivity cannot be added in this way, because its counterpart is not universal. Instead Vakarelov adds a rule of inference that imitates it: from $\alpha \Rightarrow (a \mathcal{B} p \vee p^* \mathcal{B} b)$, where p is a variable not occurring in α, a or b, infer $\alpha \Rightarrow a \mathcal{B} b$ (Vakarelov, 2012, §5). The rule was studied earlier under the name of a "rule of normality" (Balbiani et al., 2007). Each logic L_{Ax} is strongly complete for the DCAs satisfying Ax, and, by the representation theorem, for the corresponding snapshot models. Logics without transitivity have the finite model property and are decidable; whether decidability survives the rule for transitivity is left open (Vakarelov, 2012, Thms. 5.1–5.2).

5.5 What stage two achieves, and what it leaves open

Stage two achieves something stage one could not. It recovers a time *order* from point-free primitives, and the correspondence theory shows that a wide range of properties of time can be stated without mentioning time points at all. Contemporaneity, which stage one probably could not express, is now primitive.

Its limits are of three kinds. The snapshots of its canonical models are spatially trivial whenever the algebra satisfies certain natural time conditions, so the dynamic and the spatial aspects of the theory have come apart. Irreflexivity and trichotomy are not captured in the abstract setting, although both are central to classical time and to temporal logic. And the algebra still has no means of referring to particular moments, so it cannot say what exists *now*, or what existed in the past. Vakarelov's diagnosis in DM-III.3 adds a fourth point that connects these failures. The time moments of the second stage are *minimal*: small cliques of mutually contemporaneous ultrafilters. Whitehead's durations, by contrast, are *complete*. Turning minimal moments into maximal ones is the step that defines the third stage.

6 Stage three: epochs

The third stage occupies the three parts of the third instalment. DM-III.1 supplies the static background (our Section 2), DM-III.2 the snapshot model, the time representatives and an analysis of standard models, and DM-III.3 the abstract definition and representation theorem (Vakarelov, 2014, 2016a,b). Two innovations define it: maximal time points, made possible by the Efremovich axiom for time contact, and names for time points, supplied by time representatives.

6.1 The Efremovich axiom and maximal time points

The first innovation is a single additional axiom on time contact, the Efremovich axiom of Section 2.4:

$$(C^tE) \qquad \text{if } a\ \overline{C^t}\ b \text{ then for some } c,\ a\ \overline{C^t}\ c \text{ and } c^*\ \overline{C^t}\ b.$$

If a and b never coexist, some region separates them: a never coexists with c, and b never coexists with anything outside c. In rich snapshot models the axiom holds, with c the region that is the whole world exactly when a does not exist (Section 3.3). Its effect on the abstract theory is decisive. By the static theory, it makes the canonical time-contact relation R^t on ultrafilters an *equivalence relation* (Vakarelov, 2014, Lemma 2.8). The atoms of the construction therefore fall into disjoint classes of mutually contemporaneous atoms, and the unions of these classes are the *t-clusters*: the maximal collections of regions closed under the clan conditions with respect to time contact (Vakarelov, 2016b, §2).

Time points can now be taken to be t-clusters. This inverts the choice of the second stage. Instead of the smallest admissible sets of contemporaneous atoms, the third stage uses the largest. Maximality is what makes the snapshots non-trivial: a t-cluster contains as many atoms as are contemporaneous with one another, so the snapshot at that time point has as many regions as the algebra can supply. It is also what makes time points canonical. Each atom lies in exactly one t-cluster, and two distinct t-clusters are always separated by some region that belongs to one while its complement belongs to the other (Vakarelov, 2014, Lemma 2.18). Unlike the moments of the first stage, which could be enlarged at will, t-clusters are individuated by the algebra itself.

6.2 Time representatives

The second innovation addresses the problem identified in Section 3.4: an algebra whose quantifiers range over regions cannot single out a time point. Vakarelov's motivation is drawn from ordinary language. We refer to periods by naming something that existed then and only then: "the epoch of Leonardo," "the epoch of the Renaissance," "the geological age of the dinosaurs," "the time of the First World War" (Vakarelov, 2016a, §3). In the snapshot model, a *time representative* is a region that exists at exactly one moment. It is a *universal* time representative if, at that moment, it is the whole of space (Vakarelov, 2016a, Def. 3.1). In rich models every moment has a universal time representative, and multiplying it by an ordinary region that exists at that moment, without filling it, yields a time representative that is not universal (Vakarelov, 2016a, Lemma 3.2).

In the abstract theory, time representatives and universal time representatives are given as two distinguished sets of regions, TR and UTR, together with a distinguished element NOW, the universal time representative of the present epoch. Two axioms say what membership in these sets amounts to (Vakarelov, 2016b, §1.1):

(TR1) $c \in TR$ iff $c \neq 0$ and any two regions contemporaneous with c are contemporaneous with each other;

(TR2) $c \in UTR$ iff $c \in TR$ and c is not contemporaneous with its own complement.

Both have a transparent reading in the snapshot model. If c exists at only one moment, everything that coexists with c exists at that moment, so any two such things coexist. If c is moreover the whole of space at that moment, its complement is empty then and full at every other moment, so c and its complement never coexist. Vakarelov notes that both sets

are definable from time contact, but includes them in the signature so that they are preserved by the embeddings of the representation theory (Vakarelov, 2016b, §1.1).

A further group of axioms ties time representatives to the three relations, and each is a fact about the snapshot model (Vakarelov, 2016a, Lemma 3.3). Any two contemporaries coexist with some universal time representative. Any space contact can be localized to the epoch of some universal time representative. Precedence passes through time representatives, in the sense that if c names an epoch, c precedes b and a coexists with c, then a precedes b. And every instance of precedence is witnessed by universal time representatives at both ends. Four more axioms, (UTR$\mathcal{B}$11)–(UTR$\mathcal{B}$22), say that the existence of an intermediate or common earlier or later epoch can be expressed either by quantifying over all regions or by quantifying over universal time representatives (Vakarelov, 2016a, Lemma 3.7). The first of them, for example, says that two epochs $c(i)$ and $c(j)$ have a common earlier epoch iff every region p either exists before $c(i)$ or has a complement that exists before $c(j)$. These axioms are the bridge between the region-level conditions of Table 2 and conditions on time points.

6.3 What names for epochs make expressible

With universal time representatives, the algebra can finally refer to time points, and its expressive power increases sharply. The basic facts are simple. If c is a universal time representative, then a coexists with c iff $a \cdot c \neq 0$, and two universal time representatives name the same time point iff they overlap (Vakarelov, 2016a, Lemma 3.4; Vakarelov, 2016b, Lemmas 1.4, 2.3). Precedence between universal time representatives is the time order between the epochs they name.

Translating the theory of time. These facts yield a translation τ of the first-order language of time structures into the language of DCAs. Variables for time points become variables restricted to universal time representatives, identity of time points becomes time contact between their representatives, and $m \prec n$ becomes $c(m)\ \mathcal{B}\ c(n)$ (Vakarelov, 2016a, §3). In rich standard models, a first-order sentence about time is true of the time structure iff its translation is true in the algebra (Vakarelov, 2016a, Lemma 3.5). And in abstract DCAs, each condition in Table 2 is equivalent to the translation of the corresponding property of time (Vakarelov, 2016b, Lemma 1.5). Irreflexivity and trichotomy, which resisted the second stage, are now handled like the rest. Irreflexivity, for instance, translates to the statement that no universal time representative precedes itself.

Past, present and future. With NOW, tensed notions become definable (Vakarelov, 2016a, §3.3.2). A region *exists now* iff it coexists with NOW. It *exists sometimes in the future* iff NOW precedes it, and *sometimes in the past* iff it precedes NOW. It *exists always in the future* iff it coexists with every time representative that NOW precedes, and *always* iff it coexists with every time representative whatever. Spatial relations can be tensed in the same way. *a is in contact with b now* iff $(a \cdot \mathrm{NOW})\ C^s\ (b \cdot \mathrm{NOW})$, and *$a$ will be in contact with b* iff, for some universal time representative c that NOW precedes, $(a \cdot c)\ C^s\ (b \cdot c)$.

Stable contact recovered. Among the tensed notions is the relation with which the program began. a is *always in contact with b* iff $(a \cdot c)\ C^s\ (b \cdot c)$ for every universal time representative c. Stable contact, dropped in the second stage, returns as a defined relation, and the third stage subsumes the first (Vakarelov, 2016b, §4.2).

Societies of contemporaries. Time representatives also allow time contact and precedence to be generalized from pairs of regions to sets. A non-empty set A of regions is a *society of contemporaries* iff there is a moment at which all its members exist, which in rich models holds iff some time representative coexists with every member of A. Precedence between two such sets is defined in the same way (Vakarelov, 2016a, Def. 3.8, Lemma 3.9). Vakarelov takes the name from Whitehead; we return in Section 7.3 to how far the notion matches Whitehead's.

6.4 Interpret first, then imitate

Before the abstract theory is set up, DM-III.2 carries out a step that makes the method of the third stage unusually transparent. It asks how the ingredients of a rich standard model, which are given in the model, could be *described* in terms of the algebra of dynamic regions alone (Vakarelov, 2016a, §4). The answers become the definitions of the abstract construction.

- *Time points as t-clusters.* For a moment i, the set $\hat{\imath}$ of regions that exist at i is a t-cluster; distinct moments give distinct t-clusters, and two regions are contemporaries iff some such t-cluster contains both (Vakarelov, 2016a, Lemmas 4.1–4.2). Vakarelov points out that the criterion for distinguishing moments is exactly the static criterion for distinguishing clusters, Lemma 2.18 of DM-III.1.
- *Atoms of a time point.* Each ultrafilter U of the snapshot at i yields an ultrafilter of the algebra, consisting of the regions whose stage at i lies in U, and $\hat{\imath}$ is the union of these. Two such ultrafilters are contemporaneous iff they come from the same moment (Vakarelov, 2016a, Lemma 4.3).
- *Time order between clusters.* $i \prec j$ iff every ultrafilter of $\hat{\imath}$ precedes every ultrafilter of $\hat{\jmath}$ (Vakarelov, 2016a, Lemma 4.4).
- *Snapshots as quotients.* The snapshot at i, restricted to its essential part, is isomorphic to the quotient of the algebra by the congruence "same stage at i." Each clan of the snapshot yields a clan of the algebra with respect to space contact, an *s-clan*, contained in $\hat{\imath}$ (Vakarelov, 2016a, Lemma 4.5).

In Vakarelov's words, these constructions "will be imitated" for abstract DCAs (Vakarelov, 2016a, Remark 4.6). The method is worth emphasizing because it answers a natural worry about the abstract definitions, namely that they are reverse-engineered to make the representation theorem come out. They are, but in a principled way. Each abstract definition is the description, in the language of regions, of a feature of the intended model, and the representation theorem verifies that the description is faithful.

6.5 The canonical construction

The abstract theory of DM-III.3 carries out the imitation. A dynamic contact algebra is now a Boolean algebra with space contact, time contact, precedence, the sets TR and UTR, and NOW, satisfying all the axioms described so far: the basic axioms of the second stage, the Efremovich axiom for time contact, and the axioms for time representatives. Optionally it also satisfies some of the time axioms of Table 2 (Vakarelov, 2016b, §1.1). The representation proceeds in three steps.

Step 1: time. Ultrafilters carry the canonical relations R^s, R^t and $\prec$. Since space contact implies time contact, R^s is included in R^t, and by the Efremovich axiom R^t is an equivalence relation. Clans with respect to space contact are *s-clans*, clans with respect to time contact are *t-clans*, and every s-clan is a t-clan. The maximal t-clans are the t-clusters. A t-cluster containing a universal time representative is a *UTR-cluster*, and these are the time points (Vakarelov, 2016b, §2, Def. 3.1). Each universal time representative lies in exactly one UTR-cluster, and a region belongs to the UTR-cluster of c iff it coexists with c, iff $a \cdot c \neq 0$ (Vakarelov, 2016b, Lemmas 2.3–2.4). One t-cluster precedes another iff every ultrafilter in the first precedes every ultrafilter in the second. For UTR-clusters this holds iff their representatives stand in the precedence relation (Vakarelov, 2016b, Def. 2.5, Lemma 2.6). The present epoch, *now*, is the unique UTR-cluster containing NOW. The result is the canonical time structure, consisting of the UTR-clusters, their order, and now, and each time axiom holds in the algebra iff the corresponding property holds of this structure (Vakarelov, 2016b, Lemma 3.2).

Step 2: space at a time. For each time point Γ, the s-clans contained in Γ determine a contact algebra, by the factor construction of the static theory applied to space contact (Vakarelov, 2016b, Def. 3.3). These are the canonical coordinate contact algebras, the reconstructed snapshots, and their spatial points are the s-clans inside Γ.

Step 3: the embedding. The snapshot construction, applied to the canonical time structure and the canonical snapshots, gives a full standard DCA. Each region a is mapped to the vector of its classes, one coordinate per time point. This map preserves the Boolean operations, space contact, time contact, precedence, and membership in *TR* and *UTR*, and it sends NOW to the universal time representative of now (Vakarelov, 2016b, Lemma 3.6). Since the time axioms are preserved as well, the result is the main theorem of the series.

Theorem 6.1 (Vakarelov, 2016b, Thm. 3.7)**.** *Every dynamic contact algebra embeds isomorphically into a full standard dynamic contact algebra, which satisfies exactly the same time axioms.*

The construction makes visible three kinds of abstract point, as summarized in Figure 2. Ultrafilters are the atoms. UTR-clusters, the equivalence classes of mutually contemporaneous atoms that contain a name, are the time points. And the s-clans inside a time point, cliques of atoms related by spatial adjacency, are the spatial points of that time point's snapshot. Vakarelov himself highlights this, noting that the three kinds of point had not been visible in the first two stages, and that both time points and space points are "structured aggregates" of the same atoms (Vakarelov, 2016b, §4.2).

6.6 What stage three achieves, and what it leaves open

Stage three resolves each of the difficulties left by stage two. Its snapshots are as large as the algebra allows. All the time conditions in Table 2, including irreflexivity and trichotomy, are handled in the abstract setting. Past, present and future are expressible. Stable contact is recovered. And its time points are canonical, individuated by the algebra itself. It also gives the clearest formal expression of the atomism Vakarelov had noticed in the static theory: spatial points and temporal points are built from one stock of atoms, assembled in two different ways.

Vakarelov is again explicit about the costs (Vakarelov, 2016b, §4.2). The most important concerns universal time representatives. Ordinary time representatives are well motivated

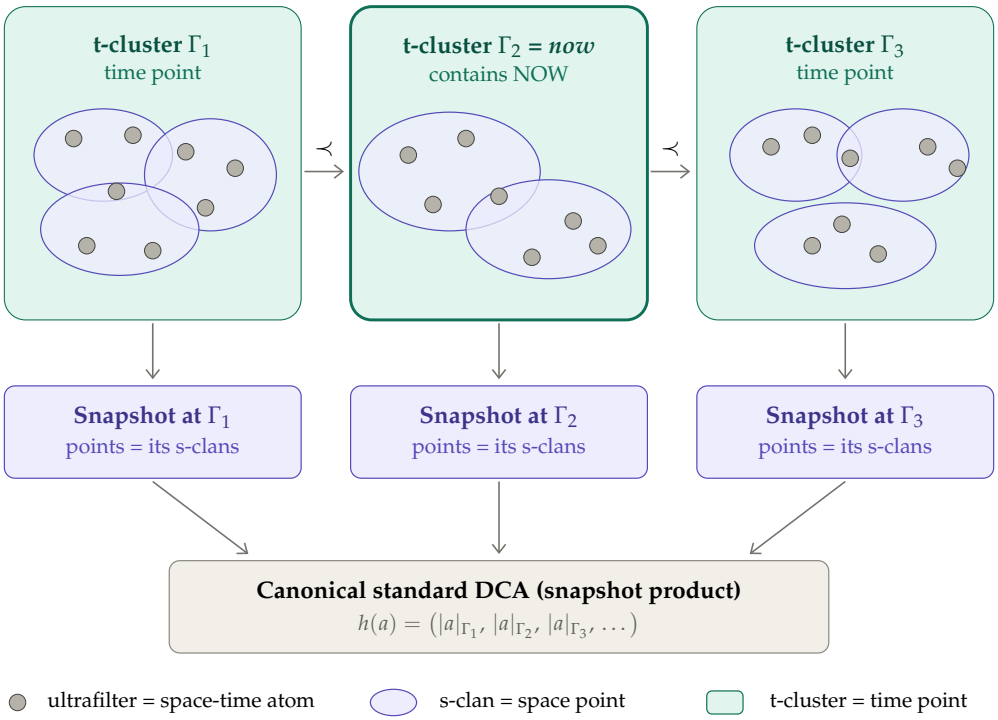


Figure 2: The three kinds of abstract points in a dynamic contact algebra (DM-III.3). Ultrafilters act as space-time atoms. They fall into disjoint t-clusters, which serve as time points and are ordered by $\prec$, inherited from the precedence relation $\mathcal{B}$; the cluster containing NOW is *now*. Within each t-cluster, s-clans (which may overlap, since spatial contact is not transitive) serve as the space points of that moment's coordinate contact algebra. The original algebra embeds into the product of these snapshots by sending each region a to its vector of equivalence classes.

by language, but a universal one must, in the intended model, occupy the whole of space during its epoch, and that is a strong and rather abstract requirement. It is needed because the representation theory works with UTR-clusters, and Vakarelov poses as an open problem either to develop the theory with ordinary time representatives alone or to find a better intuitive motivation for universal ones. The construction also yields only one time line, the time of a single observer, whereas Whitehead assumed many. Vakarelov leaves a model with many time lines to future research. And the language is open to extension, for instance by a relation of temporal *nearness* that would allow "the near future" and "the remote past" to be expressed.

Vakarelov's later survey adds a different way of simplifying the theory (Vakarelov, 2020, §§4–6). It drops time representatives and instead strengthens the interaction between time contact and precedence, by two "compositional" axioms that make the canonical precedence relation compatible with contemporaneity of atoms. With these axioms, *all* t-clusters can serve as time points, and the representation theorem goes through for every time condition except irreflexivity, which again needs the expressive resources that time representatives provide (Vakarelov, 2020, Remark 4.10). The survey also develops topological models and a duality theory for this version, generalizing Stone duality. That work lies beyond the five papers we are discussing, but it confirms the pattern we have traced: each refinement trades expressive resources against the generality and naturalness of the time points.

7 Assessment: what has been realized?

7.1 Reconstruction, not reduction

Every stage of the program rests on a representation theorem, and every representation theorem is proved by exhibiting a point-based model. This invites an objection. If the point-free theory is justified by showing that its algebras are isomorphic to algebras built from points and instants, then the points and instants were there all along, and the theory is "point-free" only in its choice of vocabulary. We think this objection misreads both what the theorems establish and what Whitehead asked for.

Consider first what is constructed, and from what. In each stage the points and instants of the canonical model are not given; they are *defined*, as sets of regions, from the primitives of the abstract theory and nothing else. Space points are clans, or s-clans inside a time point. Time points are witness sets of clans, cliques of ultrafilters, or UTR-clusters. The time order and the snapshots are defined from the primitive relations. Vakarelov states this requirement explicitly. The canonical model must be extracted from the system itself, which means defining both space points and time points within it, and the construction may use only the axioms and standard set-theoretic constructions (Vakarelov, 2020, §1.4). The point-based model at the end of the construction is therefore an *output* of the point-free theory, not an input to it.

Consider next what the theorem adds. Without a representation theorem, the definitions of points would be mere proposals. The theorem shows that they are *adequate*: the defined points stand in exactly the relations to the regions that the points of the intended model stand in, so that everything true of the snapshot model, as far as the language can express it, is true of the reconstructed one. This is what Whitehead required. His claim was that "a point is merely the name for some peculiarity of the relations between matter" (Whitehead, 1917, ch. VIII, pp. 194–195). A representation theorem identifies the peculiarity and proves that the name fits.

The static contrast between ultrafilters and clans (Section 2.3) shows that this is not a formality. Both kinds of abstract point yield a representation of every contact algebra, but only clans yield contact as the sharing of a point. Representation theorems can succeed while delivering the wrong points. Their philosophical value lies in the *choice* of construction, which is justified by whether the defined points vindicate the intended reading of the primitives. The history of the dynamic program repeats the lesson on a larger scale. The witness sets of stage one and the small cliques of stage two both support representation theorems, but neither delivers time points with the features Whitehead's epochs should have: individuality in the first case, completeness in the second.

Three limits of the reconstruction should be acknowledged. First, the theorems are of *embedding* type: every algebra embeds into a standard model, but algebras and models are not in one-to-one correspondence. Vakarelov compares the situation with the representation of Boolean algebras as algebras of sets (Vakarelov, 2020, §4.4), and uses topology to obtain a sharper, Stone-style duality in later work. Second, the constructions are non-constructive. Ultrafilters, clans and clusters are obtained by appeal to the axiom of choice, or at least the Boolean prime ideal theorem. Whitehead's own abstractive classes were infinitary too, so this is not a departure from his program, but it means that the defined points are ideal objects of a fairly abstract kind. Third, the axioms themselves were chosen with the representation proof in view (Section 3.3). This is not a circularity: each axiom has an independent reading in the intended model, and the success of the construction is not guaranteed, as the shortcomings of

the second stage show. But it does mean that the abstract theory is designed, not discovered. It is the best available articulation of the intuitive model, not a neutral starting point from which that model was found.

7.2 The staged structure

We can now make good on the first half of our thesis. Vakarelov describes his strategy as starting from a simple system and refining it step by step, removing defects and extending its expressive power (Vakarelov, 2020, §1.4). Table 3 summarizes the result.

	Stage one (DM-I)	**Stage two** (DM-II)	**Stage three** (DM-III.1–c)
Primitives	$C^{\exists}$, $C^{\forall}$	C^s, C^t, $\mathcal{B}$	C^s, C^t, $\mathcal{B}$, TR, UTR, NOW
Time point	set of clans witnessing all stable contacts	small clique of contemporaneous ultrafilters	UTR-cluster: maximal class of contemporaneous ultrafilters
Time order	none	induced by $\mathcal{B}$	induced by $\mathcal{B}$; named by UTRs
Snapshots	non-trivial	trivial under some time axioms	maximal
Gains	moments; local and absolute space; decidable logic	order; correspondence theory; contemporaneity primitive	canonical time points; irreflexivity and trichotomy; tense; $C^{\forall}$ recovered
Left open	order; contemporaneity; individuated moments	non-trivial snapshots; (Irr), (Tri); reference to moments; $C^{\forall}$	TR-only theory; many time lines; nearness

Table 3: The three stages of the program.

Read down the columns, the table shows the problem–solution chain on which our reading depends. Stage one recovers moments, but cannot order them and probably cannot define contemporaneity; stage two takes contemporaneity and precedence as primitive. Stage two orders time, but its time points are minimal and its snapshots trivial, and it cannot name moments; stage three makes time points maximal and adds names for them. In each case the new primitives are introduced to repair a specific, identified gap, and Vakarelov's own retrospective, especially in DM-III.3 §4.2, makes the diagnosis explicit. Read along the rows, the table shows what the "layers of temporal structure" in our thesis are: first the existence of moments, then their order, then their identity and nameability. Only at the third layer do time points acquire the features that make them recognizable as Whitehead's epochs, which is why we regard the realization as staged rather than achieved at a stroke.

7.3 Which Whitehead? Events, processes and epochs

Whitehead wrote about time in two quite different idioms. In the books of 1919 and 1920 the basic entities are *events*: chunks of nature extended in both space and time, related by a primitive relation of extension (one event extends over another), and from which moments and points are obtained by the method of extensive abstraction (Whitehead, 1919, 1920). In the books of 1925 and 1929 the basic entities are *actual occasions*, the atomic units of becoming, and the temporal vocabulary shifts to durations, epochs, contemporaries and societies (Whitehead, 1925, 1929/1978). Vakarelov follows the later idiom, and does so deliberately. In a later survey he acknowledges that Whitehead's earliest sketch of a point-free theory took "event" as its basic notion and obtained points of space and time by a "principle of convergence," later renamed extensive abstraction. He records de Laguna's criticism of the 1919–1920

attempt and Whitehead's acceptance of it in *Process and Reality*, and he explicitly replaces extensive abstraction by the more tractable constructions of Stone duality and proximity theory: ultrafilters, clans and clusters (Vakarelov, 2020, §§1.3–1.4).

The choice is reasonable, since *Process and Reality* is where Whitehead's mature theory of extension lives. But the event theory remains the most developed account of time Whitehead gave, and setting the formalism beside it brings out four points that the dynamic papers do not make. They bear directly on what, exactly, has been realized.

Dynamic regions: changing objects or events? Vakarelov consistently describes the elements of a DCA as regions *changing* in time, and the snapshot picture encourages this: a dynamic region is a series of static regions, one per moment, like the frames of a film in which an object moves. That is object-talk. In *The Concept of Nature*, however, Whitehead separates sharply the events that pass from the objects that are recognized in them, and on his account change is not something an event undergoes, since each event simply is the stretch of nature it is (Whitehead, 1920). He held the same view of the atomic units of his later metaphysics: "actual entities perish, but do not change; they are what they are" (Whitehead, 1929/1978, p. 35). Formally, a dynamic region is a complete spatio-temporal history, a single element of a Boolean algebra whose identity is fixed by all its stages at once. That is much closer to a Whiteheadian event than to an enduring object.

We suggest that the ambiguity is not a defect of the formalism but a difference between its two layers. The *standard* models build time in from outside: the time structure $(T, \prec)$ is given first, and regions are functions on it, a "three-plus-one" picture in which space-at-a-time is primary. The *abstract* DCAs build nothing in: time points are extracted from the algebra, just as Whitehead extracted moments from events. Read at the abstract level, the elements of a DCA are best understood as events in Whitehead's early sense, and the language of "change" belongs to the snapshot model that the representation theorem reconstructs. This reading has a further advantage. It explains why Vakarelov can treat dynamic regions as formal analogs of Whitehead's *processes* (Vakarelov, 2016a, §1.2) without committing the formalism to any particular account of what persists through change.

Time points from events: an unacknowledged tradition. Constructing instants from temporally extended events has a history of its own that the dynamic papers do not mention. Following a suggestion of Whitehead's, Russell (1914) defined an instant as a maximal class of events all of which are simultaneous, in the sense of overlapping in time, with one another. The construction was refined by Wiener (1914) and Russell (1936), given an influential form by Walker (1947), and later systematized within the logic of time by Kamp (1979) and van Benthem (1983). Vakarelov's t-clusters belong recognizably to this family. A t-cluster is a maximal collection of regions any two of which are contemporaries, closed under the clan conditions, which play the role that the requirement of forming a coherent "slice" of nature plays in the Russell–Walker constructions. Placing dynamic mereotopology in this tradition is, we think, illuminating in both directions. It shows that the temporal half of the program rests on a well-tested idea, and it shows what is new: the instants are built not from events directly but from ultrafilters, the same atoms from which the spatial points are built. The "one stock of atoms, two modes of aggregation" picture of Figure 2 has no counterpart in the Russell–Walker tradition, where space plays no role.

Instants or epochs? The two Whiteheads also disagree about what a time point is. In the event theory a moment is an ideal limit: an abstractive class of durations, nested and converging, which picks out "all nature at an instant" without itself having any temporal thickness (Whitehead, 1920). In the later theory the temporal units are epochs, atomic but not instantaneous; in a phrase Vakarelov quotes, "time is atomic (i.e. epochal)" (Whitehead, 1925). The formalism takes a side here, and it is the later one. A t-cluster is obtained not by convergence but by *maximality*: it is the largest coherent collection of contemporaneous material, not the limit of a shrinking sequence. Nothing in the construction makes a t-cluster durationless, and Vakarelov himself warns that his time points should not be read as durationless instants (Vakarelov, 2016a, Remark 1.1). We take this to be a substantive vindication of his choice of texts. Had the program followed the early Whitehead, it would have had to reproduce the method of extensive abstraction; by following the later Whitehead, it obtains, from the standard algebraic notion of a cluster, something that behaves exactly like an epoch.

Actual occasions, societies and atoms. DM-III.3 pairs Whitehead's vocabulary with the formalism as follows: actual occasions correspond to dynamic regions, contemporaneity to time contact, and durations to t-clusters (Vakarelov, 2016b, §4.2). The last two pairings are apt; the first, we suggest, is not. Actual occasions are atomic, and each belongs to its epoch. Dynamic regions are neither atomic nor confined to a single epoch, since a region in general exists at many time points. The formal objects that do have both features are *ultrafilters*. They are the atoms of the construction, and, because the Efremovich axiom makes time contact an equivalence relation on them, each ultrafilter belongs to exactly one t-cluster.[1] Whitehead himself supplies the notion needed for the other side of the mapping. In *Process and Reality* he uses "event" in "the more general sense of a nexus of actual occasions, interrelated in some determinate fashion in one extensive quantum," and adds that "an actual occasion is the limiting type of an event with only one member" (Whitehead, 1929/1978, p. 73). That is almost exactly the relation between a dynamic region and its atoms. On this re-mapping, dynamic regions correspond not to actual occasions but to aggregates of them, and the natural Whiteheadian counterpart is a *society*: a nexus of occasions with a common character, the paradigm being an enduring object whose occasions are serially ordered (Whitehead, 1929/1978, Part II, ch. III). This in turn shows that Vakarelov's "society of contemporaries" (Vakarelov, 2016a, §3.1.1) borrows Whitehead's word for a rather different notion. A Whiteheadian enduring object is a society whose members are, for the most part, *not* contemporaries of one another, since they succeed one another in time.

The re-mapping also sharpens the reading of durations given in DM-III.3. Whitehead defines a duration as a complete set of actual occasions that are mutually contemporary, a locus of occasions in "unison of becoming" (Whitehead, 1929/1978, Part IV, ch. IV, p. 320; 1929 ed., p. 491). If occasions are ultrafilters, a duration is precisely an equivalence class of ultrafilters under the canonical time-contact relation, and the union of such a class is a t-cluster. Vakarelov's identification of durations with t-clusters therefore holds more exactly at the level of atoms than at the level of regions.

[1] Only t-clusters containing a universal time representative count as time points in DM-III.3, so some ultrafilters may lie in t-clusters that are not time points. In a full standard model with infinitely many moments, for instance, there are ultrafilters not concentrated on any single moment. Such atoms behave rather like points at infinity: ideal elements that the construction must allow but that are not located at any epoch. The status of these atoms deserves further study.

Where the single time line enters. The last point concerns a departure from Whitehead that Vakarelov himself acknowledges. Whitehead, influenced by relativity, assumed many time-systems, and in *Process and Reality* contemporaneity is a relation of causal independence that is not, in general, transitive (Whitehead, 1929/1978, Part II, ch. IV, pp. 123–126). Vakarelov's snapshot construction yields a single time line, the clock of a single observer, and he leaves the incorporation of many time lines as a serious open problem (Vakarelov, 2016b, §4.2). The remapping above suggests where, formally, the single time line comes from. It is the Efremovich axiom for time contact. That axiom is what makes the canonical time-contact relation on ultrafilters transitive (Vakarelov, 2014, Lemma 2.8), and hence what makes every atom belong to exactly one t-cluster. Without it, time contact would still be a contact relation and the canonical relation on ultrafilters would still be reflexive and symmetric, but its maximal cliques, and hence the maximal t-clans, could overlap. A single atom could then lie in several maximal collections of mutually contemporaneous material: in Whitehead's terms, a single occasion could lie in several durations, one for each time-system. This is no more than a suggestion. Without the Efremovich axiom much of the representation theory of DM-III.3 would have to be rebuilt, starting with the definition of the time order between overlapping maximal t-clans. But it locates Vakarelov's open problem precisely, at a single axiom, and it indicates that the right model for a relativistic dynamic mereotopology may be the early Whitehead's many time-systems as much as the late Whitehead's epochs.

Summary. The comparison with the event theory refines rather than undermines our thesis. The program realizes the *late* Whitehead's demand: its time points are epochs, not instants, and its central temporal notion, the t-cluster, corresponds closely to Whitehead's duration. But the elements of an abstract DCA are best read as events in the early sense, the construction of time points belongs to a tradition of building instants from events that the early Whitehead inspired, and the most natural candidates for actual occasions are the atoms of the construction rather than its regions. On this reading, the deepest structural result of the program, that spatial and temporal points are different aggregations of one stock of atoms, becomes a formal counterpart of Whitehead's later view that both space and time are derivative from the relations among atomic occasions.

7.4 Departures and open worries

The comparison with Whitehead in Section 7.3 already identified the main structural departure, the single time line. Four further points deserve mention. Vakarelov acknowledges most of them, and none seems to us to threaten the thesis, but together they mark the limits of the claim that the program formalizes *Whitehead's* theory rather than a theory in his spirit.

Contact is weaker than connection. As noted in Section 2.2, Whitehead's connection is extensional, so that part-of is definable from it, and his theory of extension has neither a null region nor Boolean complements. Contact algebras take the Boolean structure as primitive and assume no extensionality (Vakarelov, 2014, §2.1). The choice follows Tarski rather than Whitehead, and it pays for itself many times over in the representation theory. But it means that the zero region, on which the predicate of existence depends, and the complement of a dynamic region, "everything else, at every moment," are features of the formalism with no direct counterpart in *Process and Reality*. Since the existence predicate underlies every temporal relation in the theory, this is not a minor technicality.

Contemporaneity as simultaneity. Time contact formalizes contemporaneity in its ordinary sense: existing at the same time. Whitehead's contemporaneity, in *Process and Reality*, is causal independence, introduced informally and tied to his relativistic notions of causal past and causal future (Whitehead, 1929/1978, Part II, ch. IV, pp. 123–126). Vakarelov points out the difference himself and lists formal analogs of the causal relations as desirable (Vakarelov, 2016b, §4.2). The two notions coincide only under the single-time-line assumption discussed in Section 7.3.

Universal time representatives. The representation theory of the third stage depends on regions that, in the intended model, fill the whole of space during exactly one epoch. Vakarelov concedes that such regions are "rather abstract" (Vakarelov, 2016b, §4.2). The worry can be sharpened. A universal time representative is a cross-section of the whole world at a single time. That is precisely the kind of global simultaneity slice that relativistic considerations, which motivated Whitehead's integration of space and time in the first place, call into question. Ordinary time representatives, such as "the epoch of Leonardo," are local, and a theory that used them alone would be both closer to linguistic practice and more hospitable to relativistic extensions. Vakarelov poses exactly this as an open problem; our point is that its solution matters philosophically as well as technically.

Ontology bracketed. Finally, Vakarelov declines to discuss the ontological questions raised by his models, about what it is for a dynamic region to exist, or for two regions to be identical (Vakarelov, 2020, §3.2). The formalism nonetheless takes a stand on them. Existence is non-emptiness at a time, identity is identity of whole histories, and regions may exist intermittently and coincide temporarily. These are substantive choices, broadly in line with a four-dimensionalist ontology of events rather than an endurantist ontology of things. They are compatible with our reading of DCAs as algebras of events (Section 7.3), but they are choices, and a full philosophical assessment would need to defend them.

7.5 Comparisons

Dynamic mereotopology occupies a distinctive position among formal theories of space and time, and a brief comparison with its neighbors helps to locate what is new in it.

Timed contact structures. The closest relatives are the "moving spaces" and timed contact algebras of Düntsch and Winter (2008, 2009). These are Boolean algebras equipped with a family of contact relations indexed by an explicitly given set of moments, with a representation theory in models similar to Vakarelov's coherent standard algebras. They are point-free with respect to space but not with respect to time, since the moments are primitives. Vakarelov's stable and unstable contact are definable in them, but, curiously, some of the axioms of DM-I fail in timed contact structures even though they hold in the models presented for them, a mismatch that he flags for further study (Vakarelov, 2010, §5).

Modal approaches. Vakarelov's own modal contact logic reads stable and unstable contact as necessitated and possible contact over an S5 modality (Vakarelov, 2008). It contains DCL, but its possible worlds play the role of moments and are presupposed rather than constructed. The large literature combining spatial and temporal logics, surveyed by Kontchakov et al. (2007), is similar in this respect. It combines a spatial formalism, such as RCC-8 or topological

modal logic, with a temporal one, such as linear temporal logic, and studies expressivity and complexity, but with time points given. Vakarelov credits this literature with the idea of the snapshot model (Vakarelov, 2016b, §4.1).

Point-free time from intervals. On the temporal side, the closest relatives are theories of time based on intervals or events rather than instants: Allen's interval relations (Allen, 1983) and the logical theories of periods and events of Kamp (1979) and van Benthem (1983), in the tradition of Russell, Wiener and Walker discussed in Section 7.3. These derive instants from extended entities, as Vakarelov does, but they are purely temporal. Vakarelov's own remark is that interval structures are usually understood as pairs of time points, and so do not escape the criticism that motivates the point-free approach (Vakarelov, 2020, §1.1). That is fair of interval-based temporal logic as usually presented, though less so of the Russell–Walker constructions, whose point is precisely to obtain instants from events. What dynamic mereotopology adds is the integration: temporal and spatial points from the same primitives.

Causal theories of space-time. Robb's axiomatization of Minkowski space-time takes as primitive a single "after" relation between space-time points, interpreted causally, and defines all spatial notions from it (Robb, 1914). Goldblatt, Shehtman, and Shapirovsky and Shehtman have studied modal logics of the Minkowski causal and chronological orders (Goldblatt, 1980; Shehtman, 1983; Shapirovsky and Shehtman, 2003, 2005). These theories take relativity seriously in a way the snapshot construction does not, but they are point-based: their primitives relate points, not regions. Vakarelov observes that neither classical nor relativistic space-time theories have yet received point-free reformulations (Vakarelov, 2020, §1.1). A dynamic mereotopology with many time lines, of the kind Section 7.3 suggests, would be a step toward a region-based counterpart of these causal theories.

Operator algebras. The duality between algebras and spaces on which the program rests has an analytic counterpart. By the Gelfand–Naimark theorem, commutative unital C*-algebras correspond exactly to compact Hausdorff spaces, and the points of the space are recovered from the algebra as its maximal ideals (Gelfand and Naimark, 1943). Contact has a direct expression in these terms. Represent a closed region F of a compact Hausdorff space X by the ideal of continuous functions on X that vanish on F. Then two closed regions meet iff the sum of their ideals is a proper ideal: iff the two ideals do not, together, generate the whole algebra. A region exists iff its ideal is proper. The bridge can also be made at the level of contact algebras. De Vries algebras, complete Boolean algebras with a proximity relation satisfying axioms that include an Efremovich-type condition, are dual to the same compact Hausdorff spaces (de Vries, 1962), and the relation between de Vries duality and Gelfand duality has been worked out explicitly (Bezhanishvili et al., 2013). Contact algebras whose canonical spaces are Hausdorff are thus, in effect, commutative C*-algebras in another guise. This gives Vakarelov's question of when the coordinate spaces of a dynamic contact algebra are Hausdorff (Vakarelov, 2012, §6) a functional-analytic meaning: it asks when each snapshot is, up to duality, a commutative C*-algebra.

Algebraic quantum field theory. The closest physical relative of dynamic mereotopology is algebraic quantum field theory (Haag and Kastler, 1964; Haag, 1996). There the basic objects are not points of space-time but C*-algebras of observables assigned to space-time *regions*. A

larger region receives a larger algebra (isotony). The algebras of spacelike separated regions commute (Einstein causality). And the algebra of an arbitrarily thin neighborhood of a time-slice determines the algebra of the whole (the time-slice axiom). Two features bear directly on our discussion. First, Einstein causality identifies causal independence with commutativity. Causal independence is precisely what Whitehead meant by contemporaneity, the notion that time contact does not formalize (Section 7.4), so the theory offers a worked-out mathematical model of Whitehead's relation, though a quantum one. Second, the time-slice axiom has a recognizable affinity with a Whiteheadian duration: a temporally thin but spatially complete slab of nature that carries everything there is to know about the world. There is also a reconstruction program in the spirit of Vakarelov's. Buchholz, Summers and collaborators recover the symmetries of space-time from the algebras of observables and a state, by a condition of "geometric modular action" (Buchholz et al., 2000). The limitation, from the point of view of this paper, is that algebraic quantum field theory still indexes its algebras by the regions of a given Minkowski space-time. It is point-free in its observables but not in its geometry. A version in which the index set is itself an abstract contact or causal algebra of regions, in Vakarelov's style, would be a natural meeting point of the two programs.

Noncommutative geometry pushes the point-free idea further than either. It treats noncommutative C*-algebras as "spaces" with too few points to be described by any point set (Connes, 1994). In region-based terms, noncommutativity would mean that the algebra of regions ceases to be Boolean, becoming something closer to the orthomodular lattice of projections of an operator algebra: regions that cannot all be sharply delimited together. Whether a dynamic mereotopology over such structures makes sense is an open question. If it did, point-freeness would mean not that points are derivable, as for Whitehead and Vakarelov, but that, in general, there are not enough of them to derive.

Relational dynamic mereology. Finally, Nenchev (2011, 2013) develops dynamic mereology and mereotopology on a purely relational basis, with stable and unstable relations as primitives and no Boolean operations on regions. This suggests that the ideas of the first stage do not depend essentially on the Boolean structure, although the later stages, which rely heavily on ultrafilters and clans, plainly do.

8 Conclusion and open problems

Whitehead called the relational definition of points of space and instants of time "an unwritten chapter of mathematics," and predicted that "many ways of effecting these definitions will have to be tried and compared" (Whitehead, 1917, ch. VIII, pp. 194–195). Vakarelov closes the five papers by quoting the same sentence (Vakarelov, 2016b, §4.2), and it describes his program well. We have argued that the program is best read as a staged realization of Whitehead's demand. Each stage takes the snapshot model as its intended interpretation, selects spatio-temporal relations between dynamic regions, axiomatizes them, and proves that the abstract theory reconstructs the model, extracting both space points and time points from the regions alone. Each stage recovers a further layer of temporal structure, and each is motivated by a specific gap in its predecessor. The third stage obtains time points that are canonical, maximal and nameable, and that behave like Whitehead's epochs. In doing so it reveals the structural result we have emphasized throughout: spatial and temporal points are different aggregations of one stock of atoms.

We have also argued that the representation theorems are best understood as delivering reconstruction rather than reduction, and that reconstruction is what Whitehead asked for. And we have suggested that the comparison with Whitehead's earlier, event-based theory sharpens the picture. The elements of an abstract dynamic contact algebra behave like events rather than changing objects, the construction of time points belongs to a tradition of building instants from events that Whitehead inspired, and the atoms of the construction, rather than its regions, are the natural counterparts of actual occasions.

The program leaves a rich set of open problems. Some are Vakarelov's own:

- a representation theory using only ordinary time representatives, or a better intuitive motivation for universal ones (Vakarelov, 2016b, §4.2);
- new spatio-temporal relations sensitive to the order of time: *stabilized contact*; principles of *continuous change* (Vakarelov, 2010, §5); and a relation of temporal *nearness*, perhaps with an Archimedean property, supporting notions such as the near future and the remote past (Vakarelov, 2016b, §4.2);
- conditions under which the coordinate spaces are connected, Hausdorff or Euclidean (Vakarelov, 2012, §6);
- a dynamic mereotopology with many time lines, corresponding to many observers (Vakarelov, 2016b, §4.2);
- the decidability and complexity of the logics with the rule for transitivity (Vakarelov, 2012, §5).

Others arise from our discussion:

- whether dropping the Efremovich axiom for time contact, so that maximal sets of contemporaneous atoms may overlap, yields a workable theory of many time-systems, and whether time contact then behaves like Whitehead's contemporaneity as causal independence (Section 7.3);
- the status of atoms that lie in no time point, such as ultrafilters not concentrated at any moment of an infinite time structure, which behave like ideal points at infinity in the temporal dimension;
- whether universal time representatives survive the requirement that change be continuous. In a full product of snapshots, a region that is the whole world at one moment and nothing at any other is always available. In the C*-algebraic analog of a dynamic model of space, an algebra fibred continuously over a space of times, the time points are recovered from the *center* of the algebra (Dixmier, 1977), and the corresponding objects are minimal central projections. When time is a continuum and sections are required to vary continuously, there are no such projections. This suggests that universal time representatives are an artifact of admitting discontinuous trajectories, and that Vakarelov's problem of a theory with ordinary time representatives alone and his problem of formulating principles of continuous change may turn out to be the same problem;
- the relation between t-clusters and the Russell–Walker constructions of instants from events, and whether results from that tradition, for instance on when the resulting time is dense or continuous, transfer to the dynamic setting;

- whether an algebra of regions with Whitehead's extensional connection alone, without Boolean complements or a null region, can support a comparable theory of time, and what then replaces the predicate of existence;
- a dynamic counterpart of Whitehead's method of extensive abstraction, which would yield durationless instants as limits within t-clusters, and so reconcile the early and late theories within one framework.

The last of these seems to us the most interesting. If it could be done, the formalism would contain both of Whitehead's conceptions of time point, the instant and the epoch, and the relation between them could be studied exactly. That would be a fitting addition to the chapter of mathematics Whitehead left unwritten, and which Vakarelov has done more than anyone to write.

Appendix A: The axioms at a glance

This appendix collects the axioms of the systems discussed in the paper, in a uniform notation. Throughout, $(B, 0, 1, \leq, +, \cdot, ^*)$ is a non-degenerate Boolean algebra, and a bar over a relation symbol denotes its negation.

A.1 Contact, precontact and the Efremovich axiom

(C1) if $a \; C \; b$ then $a \neq 0$ and $b \neq 0$
(C2) if $a \; C \; b$, $a \leq a'$ and $b \leq b'$ then $a' \; C \; b'$
(C3′) if $a \; C \; (b + c)$ then $a \; C \; b$ or $a \; C \; c$
(C3″) if $(a + b) \; C \; c$ then $a \; C \; c$ or $b \; C \; c$
(C4) if $a \; C \; b$ then $b \; C \; a$
(C5) if $a \cdot b \neq 0$ then $a \; C \; b$
(CE) if $a \; \overline{C} \; b$ then for some c: $a \; \overline{C} \; c$ and $c^* \; \overline{C} \; b$

A *precontact* relation satisfies (C1)–(C3″); a *contact* relation satisfies (C1)–(C5). Source: Vakarelov (2014, Defs. 2.1, 2.4).

A.2 Stage one: dynamic contact algebras of DM-I

A DCA is $(B, C^{\exists}, C^{\forall})$ where $C^{\exists}$ is a contact relation and

($C^{\forall}1$) $1 \; C^{\forall} \; 1$
($C^{\forall}2$) not $0 \; C^{\forall} \; 0$
($C^{\forall}3$) if $p \; C^{\forall} \; q$ and not $a \; C^{\forall} \; b$, then $(p \cdot a^*) \; C^{\exists} \; (q \cdot a^*)$ or $(p \cdot b^*) \; C^{\exists} \; (q \cdot b^*)$

Source: Vakarelov (2010, Def. 2.1).

A.3 Stage two: dynamic contact algebras of DM-II

A DCA is $(B, C^s, C^t, \mathcal{B})$ where C^s and C^t are contact relations, $\mathcal{B}$ is a precontact relation, and

$$(C^s \to C^t) \quad \text{if } a \; C^s \; b \text{ then } a \; C^t \; b$$

Optional time axioms are the right-hand conditions of Table 2, except (irr) and (tri), which DM-II does not include. In the logics, (tr) is replaced by the rule: from $\alpha \Rightarrow (a \; \mathcal{B} \; p \vee p^* \; \mathcal{B} \; b)$, with p not occurring in α, a, b, infer $\alpha \Rightarrow a \; \mathcal{B} \; b$. Source: Vakarelov (2012, Def. 3.1, §5).

A.4 Stage three: dynamic contact algebras of DM-III.3

A DCA is $(B, C^s, C^t, \mathcal{B}, TR, UTR, \mathrm{NOW})$ satisfying the axioms of A.3 together with the following. In (UTR$\mathcal{B}$), $c(i)$ and $c(j)$ range over UTR, and $c(k)$ is existentially quantified over UTR.

(C^tE)	if $a\ \overline{C^t}\ b$ then for some c: $a\ \overline{C^t}\ c$ and $c^*\ \overline{C^t}\ b$		
(TR1)	$c \in TR$ iff $c \neq 0$ and, for all a, b: if $a\ C^t\ c$ and $b\ C^t\ c$ then $a\ C^t\ b$		
(TR2)	$c \in UTR$ iff $c \in TR$ and $c\ \overline{C^t}\ c^*$		
(TRC^t)	if $a\ C^t\ b$ then for some $c \in UTR$: $a\ C^t\ c$ and $b\ C^t\ c$		
(TRC^s)	if $a\ C^s\ b$ then for some $c \in UTR$: $(a \cdot c)\ C^s\ b$		
(TR$\mathcal{B}$1)	if $c \in TR$, $c\ \mathcal{B}\ b$ and $a\ C^t\ c$, then $a\ \mathcal{B}\ b$		
(TR$\mathcal{B}$2)	if $d \in TR$, $a\ \mathcal{B}\ d$ and $b\ C^t\ d$, then $a\ \mathcal{B}\ b$		
(TR$\mathcal{B}$3)	if $a\ \mathcal{B}\ b$ then for some $c \in UTR$: $c\ \mathcal{B}\ b$ and $a\ C^t\ c$		
(TR$\mathcal{B}$4)	if $a\ \mathcal{B}\ b$ then for some $d \in UTR$: $a\ \mathcal{B}\ d$ and $b\ C^t\ d$		
(UTR$\mathcal{B}$11)	for all p: $p\ \mathcal{B}\ c(i)$ or $p^*\ \mathcal{B}\ c(j)$	iff	$c(k)\ \mathcal{B}\ c(i)$ and $c(k)\ \mathcal{B}\ c(j)$
(UTR$\mathcal{B}$12)	for all p: $p\ \mathcal{B}\ c(i)$ or $c(j)\ \mathcal{B}\ p^*$	iff	$c(k)\ \mathcal{B}\ c(i)$ and $c(j)\ \mathcal{B}\ c(k)$
(UTR$\mathcal{B}$21)	for all p: $c(i)\ \mathcal{B}\ p$ or $p^*\ \mathcal{B}\ c(j)$	iff	$c(i)\ \mathcal{B}\ c(k)$ and $c(k)\ \mathcal{B}\ c(j)$
(UTR$\mathcal{B}$22)	for all p: $c(i)\ \mathcal{B}\ p$ or $c(j)\ \mathcal{B}\ p^*$	iff	$c(i)\ \mathcal{B}\ c(k)$ and $c(j)\ \mathcal{B}\ c(k)$
(UTRNOW)	$\mathrm{NOW} \in UTR$		

Optional time axioms are the right-hand conditions of Table 2, including (irr) and (tri). Sources: Vakarelov (2016b, §1.1); Vakarelov (2016a, Lemmas 1.4, 3.3, 3.7).

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