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A COUNTEREXAMPLE TO THE BARKER–LARMAN PROBLEM IN DIMENSION THREE

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Abstract. We construct an origin-symmetric convex body of revolution in $\mathbb{R}^3$, with real-analytic boundary and positive principal curvatures, whose planar sections tangent to a concentric inner ball all have the same area. The body is nonspherical and has the same section data as a centered ball. The construction bifurcates from the unit ball at height $1/\sqrt{3}$, where degree two is the only even Legendre resonance. We construct approximate solutions to arbitrary order and correct them by a Newton iteration on nested closed parameter sets. A finite-order reduction of the linearized section operator and measure estimates for its small divisors provide the inverse bounds needed for convergence.

1. Introduction and main results

Write B for the closed unit ball of $\mathbb{R}^3$ and set $H(u,h)=\{z\in\mathbb{R}^3 : z\cdot u=h\}$ for $u\in S^2$, $h>0$. The planes $H(u,h)$ are precisely the supporting planes of hB. Barker and Larman [1, Conjecture 2, p. 81] asked whether a convex body containing a prescribed inner body in its interior is determined by the section volumes on the supporting hyperplanes of the inner body. We consider the spherical-inner-body case.

Theorem 1.1. *For every $t>0$, there exist two distinct origin-symmetric convex bodies $K,L\subset\mathbb{R}^3$ with real-analytic boundaries and positive principal curvatures, both strictly containing tB, such that*

$$\operatorname{area}(K\cap H(u,t))=\operatorname{area}(L\cap H(u,t)) \qquad \text{for every } u\in S^2.$$

The body L is a centered ball and K is an origin-symmetric, nonspherical body of revolution. Such pairs can be chosen with radial functions arbitrarily close in any prescribed C^m norm. They satisfy $\operatorname{vol}(K)>\operatorname{vol}(L)$.

Yaskin proved uniqueness when the two outer bodies are polytopes [18, Theorem 2.1]. Matthews [14, Theorem 1.1] extended planar polygon reconstruction to sections supporting a piecewise-analytic hedgehog. In dimension three, Yaskin and Yaskina obtained uniqueness with a ball as the comparison body under an equal-volume assumption [19]. The volume ordering in

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Theorem 1.1 follows from a strict moment inequality. Kurusa and Ódor [12, Theorem 5.3] give rigidity when section data and cap volumes agree with those of a centered ball. Makai and Martini [13, Theorem 1] recover the first radial variation of a smooth outward deformation from the asymptotics of sections tangent to the body being deformed. Here the cutting sphere stays strictly inside both outer bodies, at a radius asymptotic to $1/\sqrt{3}$ times the comparison radius. Fish, Ryabogin, Sun and Yaskin [5, Theorem 6.2] prove uniqueness for fixed bodies when their section functions agree on infinitely many heights with an accumulation point inside the common radial interval. Theorem 1.1 concerns one prescribed height for each pair of bodies.

Morales-Amaya [16, Theorem 1] studies the shape of each section. The hypothesis there requires centrally symmetric sections tangent to a suitable inner ball which excludes the center of the outer body. Here the inner ball is concentric and the data are the areas of the sections.

Haddad and Ryabogin [7, Theorem 1] proved a rigidity result in dimension four for an arithmetically specified set of constant section volumes. Haddad [6, Theorem 1] constructed four-dimensional examples with section volumes constant to arbitrarily high finite order. Knowledge Sense subsequently gave exact constructions in all dimensions $d \geq 4$ [10, Theorems 1.1–1.3], and proved in a companion preprint [11, Theorem 1.1] that Haddad's specified four-dimensional series has zero radius of convergence. Part II of [10] treats the degree-two resonance for $d \geq 5$ using dimension-dependent normalizations and estimates for the effective diagonals. In dimension three the section derivative has Legendre multipliers at the ball. We prove isolation at $1/\sqrt{3}$ and use its exact weighted section operator to obtain an order-eight remainder after finite diagonal reduction.

The existence proof uses chord coordinates, finite-order diagonal reduction, and parameter exclusion. The strategy follows the approach of de Groot and Price [3, Sections 3–5] for the floating-body equation. Here we establish the estimates for the section-area operator and compute the critical equations for the degree-two resonance. The parameter a ranges over compact sets of asymptotically full relative measure. For each a in such a set the resulting boundary is real analytic in the spatial variables.

Theorem 1.2 (Volume-normalized solutions)**.** *Let $P_2(s) = (3s^2 - 1)/2$, let x be the meridian angle measured from the axis of revolution, and put*

$$h_* = \frac{1}{\sqrt{3}}, \quad a_* = \arccos h_*, \quad h_1 = \frac{1}{7\sqrt{3}}, \quad a_1 = \frac{3}{28\sqrt{2}}.$$

For every sufficiently small $\tau > 0$ there is a compact set $G_\tau \subset I_\tau = [a_* + \tau, a_* + 2\tau]$ with $|I_\tau \setminus G_\tau| = o(\tau)$ as $\tau \downarrow 0$. For each $a \in G_\tau$ there exist $h_a, \nu_a > 0$ and an origin-symmetric, nonspherical body of revolution K_a, with real-analytic boundary and positive principal curvatures, such that

$$\operatorname{vol}(K_a) = 4\pi/3, \qquad h_a B \subset \operatorname{int} K_a,$$
$$\operatorname{area}(K_a \cap H(u, h_a)) = \nu_a \qquad (u \in S^2).$$

The radial functions, expressed in the meridian angle, are analytic on a common positive strip. On a possibly smaller fixed strip, uniformly over $a \in G_\tau$, they satisfy

$$\begin{aligned} \rho_a(x) &= 1 + \varepsilon_a P_2(\cos x) + O(\tau^2), \\ h_a &= h_* + h_1 \varepsilon_a + O(\tau^2), \\ \varepsilon_a &= (a - a_*)/a_1 + O(\tau^2). \end{aligned} \tag{1.1}$$

Here ε_a is the degree-two Legendre coefficient of $\rho_a - 1$. The parameter a is half the increment between meridional endpoints of a section in the coordinates of Section 4.

The proof first constructs a polynomial seed and computes the drift of a from a_*. In chord coordinates the linearization splits into a normal operator of weighted spherical section type, a tangential difference operator, and two scalar equations. Two oscillatory factors of the normal operator are kept exact in the finite-order reduction. The resulting diagonal can be controlled on large closed parameter sets, and its inverse composed with the remainder is small. Section 7 gives the Newton iteration, including the strip allocation and the total measure discarded. The comparison with a ball is proved in Section 8.

2. The degree-two Legendre resonance

Write $P_l(1) = 1$ for the Legendre normalization, and put

$$h_* = 3^{-1/2}, \qquad a_* = \arccos h_*, \qquad \sigma_* = \sin a_* = \sqrt{2/3}.$$

We use Holt's change of variables [8] and the degree interpolation approach of Kadets and Mangoubi [9, Sections 2 and 4]. The calculation at the specific value h_* determines the zeros in each of four residue classes. Deng's preprint [4, Theorem 1.1] claims the irreducibility of all even Legendre polynomials over $\mathbb{Q}$, which would also give the isolation below. The special-height calculation supplies the arithmetic input directly.

Lemma 2.1. *For every integer $m \geq 0$,*

$$P_{2m}(1/\sqrt{3}) = 0 \quad \Longleftrightarrow \quad m = 1.$$

Moreover a_*/π is irrational, and there exist fixed constants $\gamma_* > 0$ and $\tau_* < \infty$ such that

$$|\sin(j a_*)| \geq \gamma_* (1 + |j|)^{-\tau_*} \qquad (0 \neq j \in \mathbb{Z}).$$

Proof. Define

$$H_m(z) = \sum_{k=0}^{m} \frac{\binom{m}{k}^2}{\binom{2k}{k}} z^k.$$

Holt's identity [8], in the normalization of [9, Section 2.2, (3)–(4)], gives

$$H_m(z) = \binom{2m}{m}^{-1} (z-4)^m P_{2m}\left(\sqrt{\frac{z}{z-4}}\right).$$

There is no choice of square root in this identity, since P_{2m} is even. In particular

$$H_m(-2) = \frac{(-6)^m}{\binom{2m}{m}} P_{2m}(1/\sqrt{3}). \tag{2.1}$$

Let v be the 2-adic Gauss valuation of a polynomial in u, normalized by $v(2) = 1$, and let $s_2(k)$ count the binary digits equal to one in k. For $r = 0, 1, 2, 3$ consider

$$F_r(u) = \sum_{k\geq 0} T_{r,k}(u), \qquad T_{r,k}(u) = \frac{(-2)^k}{\binom{2k}{k}} \binom{r+4u}{k}^2.$$

For $k \geq 1$, counting the factors divisible by 2 and by 4 in $\prod_{j=0}^{k-1}(r+4u-j)$ gives

$$v(T_{r,k}) = 2S_r(k) - k + s_2(k),$$

where

$$\begin{aligned} S_0(k) &= \lceil k/2 \rceil + \lceil k/4 \rceil, & S_1(k) &= \lfloor k/2 \rfloor + \lfloor (k+2)/4 \rfloor, \\ S_2(k) &= \lceil k/2 \rceil + \lfloor (k+1)/4 \rfloor, & S_3(k) &= \lfloor k/2 \rfloor + \lfloor k/4 \rfloor. \end{aligned}$$

Indeed a linear factor $4u + c$ has Gauss valuation $\min(2, v_2(c))$, also when $c = 0$, while $v_2(k!) = k - s_2(k)$ and $v_2\binom{2k}{k} = s_2(k)$. For $k = 4q + d$ with $0 \leq d \leq 3$, the valuation is $2q + s_2(k)$ plus the entry in the following table.

$r\backslash d$	0	1	2	3
0	0	3	2	3
1	0	−1	2	1
2	0	1	0	3
3	0	−1	0	−1

In particular, the valuations tend to $+\infty$ at least linearly in k. Thus each series converges in the Tate algebra $\mathbb{Q}_2\langle u \rangle$. For a nonnegative integer j, it specializes to $F_r(j) = H_{r+4j}(-2)$, since the binomial coefficients truncate.

We now determine the zeros on $\mathbb{Z}_2$ by explicit units. For $r = 0$, all terms with $k \geq 1$ have valuation at least three. For $r = 2$ they have valuation at least one. Consequently

$$F_0 \equiv 1 \pmod 8, \qquad F_2 \equiv 1 \pmod 2 \quad \text{in } \mathbb{Z}_2\langle u \rangle.$$

For $r = 3$, the terms with $k \geq 4$ have valuation at least three, and $1 + T_{3,1} = 1 - (3 + 4u)^2$ is divisible by eight. Directly,

$$T_{3,2} + T_{3,3} = \frac{4}{45}(3 + 4u)^2(1 + 2u)^2(7 - 4u - 8u^2).$$

It follows that $F_3/4 \equiv 1 \pmod 2$. These three functions therefore have no zeros in $\mathbb{Z}_2$.

Finally $1 + T_{1,1} = -8u - 16u^2$. Each $T_{1,k}$ for $k \geq 2$ is divisible by u^2. For $k = 2, 3, 4, 5$ the quotients $T_{1,k}/(8u^2)$ are respectively

$$\frac{(4u+1)^2}{3}, \qquad -\frac{(4u-1)^2(4u+1)^2}{45},$$

$$\frac{(2u-1)^2(4u-1)^2(4u+1)^2}{315}, \qquad -\frac{(2u-1)^2(4u-3)^2(4u-1)^2(4u+1)^2}{14175}.$$

All four are integral and congruent to one modulo two. For $k \geq 6$, $v(T_{1,k}) \geq 5$. Dividing by u^2 preserves the Gauss valuation. Hence

$$F_1(u) = -8u + 16u^2 G(u), \qquad G \in \mathbb{Z}_2\langle u \rangle.$$

The factor $1 - 2uG(u)$ is a unit at every $u \in \mathbb{Z}_2$, so F_1 has exactly the zero $u = 0$ there. Combining the four residue classes with (2.1) proves the isolation.

For the last assertion,

$$\zeta = e^{2ia_*} = \frac{-1 + 2i\sqrt{2}}{3}, \qquad \zeta + \zeta^{-1} = -\frac{2}{3}.$$

A root of unity and its inverse are algebraic integers, so their sum cannot be the nonintegral rational number $-2/3$. Hence a_*/π is irrational. Choose $q_j \in \mathbb{Z}$ so that $|ja_* - q_j\pi| \leq \pi/2$. With the fixed logarithms $\log \zeta = 2ia_*$ and $\log(-1) = i\pi$, the nonzero linear form $j \log \zeta - 2q_j \log(-1)$ has coefficients of size $O(1+|j|)$. Matveev's bound [15, Corollary 2.3] gives a polynomial lower bound for its absolute value. The inequality $|\sin y| \geq 2|y|/\pi$ on $[-\pi/2, \pi/2]$ transfers this bound to $|\sin(ja_*)|$. □

3. Formal solutions and the parameter drift

A zonal radial function is written as $\rho(\omega) = \rho(\omega_1)$ when its argument is a cosine. Write $A_\rho(u, h)$ for the section area of the radial body. We also write P_l for the zonal function $\omega \mapsto P_l(\omega_1)$ or its angular expression $P_l(\cos x)$. Denote by $[g]_l$ the coefficient of P_l in a zonal function g, and by $\operatorname{av} g$ the average of a periodic function in its angle variable. Set

$$f(x) = P_2(\cos x), \qquad \mu = \operatorname{av} f = \frac{1}{4}, \qquad f^2 = \frac{1}{5}P_0 + \frac{2}{7}P_2 + \frac{18}{35}P_4.$$

The last identity is in the Legendre basis. In particular, $[f]_0 = 0$ and $[f^2]_0 = 1/5$.

For h near h_* define

$$(S_h g)(u) = \int_{S^2} \delta(u\cdot\omega - h) g(\omega)\, \mathrm{d}\omega, \qquad S_h P_l = s_l(h) P_l, \qquad s_l(h) = 2\pi P_l(h).$$

Here δ is the Dirac distribution and $\mathrm{d}\omega$ is surface measure. The multiplier follows from the Funk–Hecke formula [2, Theorem 1.2.9]. In dimension three the coordinate $t = u\cdot\omega$ gives $\mathrm{d}\omega = \mathrm{d}t\, \mathrm{d}\varphi$, so the multiplier for a continuous zonal kernel $p(t)$ is $2\pi \int_{-1}^{1} p(t) P_l(t)\, \mathrm{d}t$. Approximating $\delta(t - h)$ gives the displayed multiplier. In particular $s_2'(h_*) = 2\pi\sqrt{3}$.

Lemma 3.1. *There is a unique formal, even zonal radial series with* $[\rho - 1]_2 = \varepsilon$, *beginning at* $\rho = 1 + \varepsilon f + O(\varepsilon^2)$, *such that* $\operatorname{vol}(K_\rho) = 4\pi/3$ *and*

$A_\rho(u,h)=\nu$ as formal series, with $h(0)=h_*$ and $\nu(0)=\pi(1-h_*^2)$. All radial coefficients are polynomials in the polar cosine. It satisfies

$$\begin{aligned} h &= h_* + h_1\varepsilon + O(\varepsilon^2),\\ [\rho-1]_0 &= -\tfrac{1}{5}\varepsilon^2 + O(\varepsilon^3),\\ [\rho-1]_4 &= \tfrac{27}{245}\varepsilon^2 + O(\varepsilon^3). \end{aligned} \tag{3.1}$$

It has a formal meridional reparametrization $x\mapsto x+\varepsilon w_1(x)+O(\varepsilon^2)$ by odd π-periodic trigonometric polynomials, and a formal parameter a, for which every chord with endpoints labeled $x\pm a$ has distance h from the origin. Their first terms are

$$a = a_* + a_1\varepsilon + O(\varepsilon^2), \qquad w_1(x) = -\frac{3}{16}\sin(2x). \tag{3.2}$$

For any positive integer N, truncation and formal inversion of (3.2) *give meridian, height and area seeds depending polynomially on $a-a_*$, with residual $O((a-a_*)^{N+1})$ in the chord equations of Section 4.*

Proof. The radial formula for the section area, interpreted by coarea, is

$$A_\rho(u,h) = \int_{S^2}\int_0^{\rho(\omega)} r^2\delta(r\,\omega\cdot u - h)\,\mathrm{d}r\,\mathrm{d}\omega. \tag{3.3}$$

The section of the reference sphere is transverse since $h_*<1$. The boundary of nearby sections can be solved for in polar coordinates centered at hu in the section plane. For a unit $v\perp u$, solve

$$\sqrt{h^2+r^2} - \rho\big((hu+rv)/\sqrt{h^2+r^2}\big) = 0.$$

At the ball the derivative with respect to the planar radius is $r>0$ at $h^2+r^2=1$. Thus derivatives of (3.3) at the ball may be computed by distributional differentiation of the moving radial upper limit. If $t=\omega\cdot u$, one differentiation gives S_h. For two directions g_1,g_2, the kernel at the second differentiation is $2\delta(t-h)+t\delta'(t-h)$. Using $(t-h)\delta'(t-h)=-\delta(t-h)$ gives

$$D^2_\rho A_1(u,h)[g_1,g_2] = (S_h - hS_h')(g_1g_2)(u). \tag{3.4}$$

Here and below a prime on S_h denotes differentiation in the physical height h. At any finite higher order the kernel in (3.3) is a finite linear combination of derivatives $\partial_h^j\delta(t-h)$ with scalar coefficients depending on h. This follows by repeatedly differentiating in r and using $(t-h)\delta^{(j)}(t-h)=-j\delta^{(j-1)}(t-h)$. The spherical multipliers therefore preserve polynomial inputs.

Subtract the spherical mean of the section equation and write $\rho=1+\varepsilon f+\varepsilon^2 g+\cdots$ and $h=h_*+h_1\varepsilon+\cdots$. The order-two equation is the nonconstant part of

$$S_{h_*}g + \frac{1}{2}(S_{h_*} - h_*S_{h_*}')(f^2) + h_1S_{h_*}'f = 0. \tag{3.5}$$

Its P_2 coefficient is $s_2'(h_*)(h_1 - h_*/7)$. The volume condition $\frac{1}{3}\int_{S^2}\rho^3\,\mathrm{d}\omega = 4\pi/3$ gives $[g]_0 = -[f^2]_0 = -1/5$. In degree four, inserting $P_4(h_*) = -7/18$ and $h_*P_4'(h_*) = -5/9$ in (3.5) gives $[g]_4 = 27/245$. For order $j \geq 2$, the new radial coefficient g_j enters the nonconstant equation by $S_{h_*}g_j$. The as yet undetermined height coefficient of order $j-1$ enters by a scalar multiple of $S_{h_*}'f$. The height coefficient of order j contributes only to the constant section coefficient. Lemma 2.1 solves each noncritical even Legendre equation. The critical equation determines the height coefficient of order $j-1$ since $s_2'(h_*) \neq 0$. Volume determines $[g_j]_0$, while $[g_j]_2 = 0$ fixes the chosen amplitude. Once the radial and height series are known, the spherical mean gives the area series ν. This proves the induction.

Put $e_r(x) = (\cos x, \sin x)$, $e_\theta(x) = (-\sin x, \cos x)$, and let $T_a g(x) = g(x+a)$. For the reference chord at x, the normal is $e_r(x)$ and the midpoint is $h_* e_r(x)$. Its first-order distance variation is the average of the normal components of the two endpoint variations. The radial variations give $\frac{h_*}{2}(T_{a_*} + T_{-a_*})f$. A tangent vector at $x + a_*$ has scalar component $-\sigma_*$ along $e_r(x)$, and the tangent vector at $x - a_*$ has component σ_*. The change in the two labels is $\pm a_1\varepsilon$. The distance equation at order one is consequently

$$\frac{h_*}{2}(T_{a_*} + T_{-a_*})f - \frac{\sigma_*}{2}(T_{a_*} - T_{-a_*})w_1 - \sigma_* a_1 = h_1.$$

Its mean gives $a_1 = (h_*\mu - h_1)/\sigma_* = 3/(28\sqrt{2})$. For the mode $\cos(2x)$, the equation uses $f = 1/4 + (3/4)\cos(2x)$, $\cos(2a_*) = -1/3$, and $(T_{a_*} - T_{-a_*})\sin(2x) = 2\sin(2a_*)\cos(2x)$. It gives the stated w_1. At each order the operator on the new reparametrization and on the new angle coefficient is the same linear difference operator and constant term above. The forcing is a finite even, π-periodic Fourier sum because the distance is obtained from the endpoints by dot products, a square root with nonzero reference value, and division by the chord length. The arithmetic lemma inverts the difference operator from odd functions onto the even functions of zero mean, frequency by frequency. The constant equation gives the angle coefficient. Since $a_1 > 0$, the formal inversion is possible. Taylor truncation gives the last assertion. At fixed order the analytic functions in the chord equations have a genuine Taylor expansion near the reference configuration, as also shown by the integral formulas in Section 4. □

4. Chord coordinates and the section derivative

4.1. **Coordinates and analytic norms.** We use a plane through the axis of revolution for meridians. Let $\gamma(x) = (X(x), Y(x))$ be close to $e_r(x)$, with X even, Y odd, and $\gamma(x+\pi) = -\gamma(x)$. On $(0,\pi)$ one has $Y > 0$ for meridians sufficiently close in C^1. Rotating (X, Y) about the first axis gives the radial body K_γ. Real meridians close enough in C^2 give convex bodies with positive principal curvatures, as verified at the end of Section 7. In a

fixed neighborhood of the reference meridian use the definitions

(4.1)

$$\gamma_\pm(x) = \gamma(x \pm a), \qquad L = |\gamma_+ - \gamma_-|, \quad n = \frac{((\gamma_+ - \gamma_-)_2, -(\gamma_+ - \gamma_-)_1)}{L},$$
$$q = n \cdot \gamma_+ = n \cdot \gamma_-, \quad S = |\gamma'|, \qquad N = (Y', -X')/S.$$

Square roots in the complex domain use the analytic branches at the reference values $L = 2\sin a$, $S = 1$. Write $n = (\cos\theta, \sin\theta)$ with $\theta - x$ small, and put $n^\perp = (-\sin\theta, \cos\theta)$. Thus $\gamma_+ - \gamma_- = Ln^\perp$. The section is taken with the three-dimensional normal $(n_1, n_2, 0)$. Let $A_\gamma(x)$ be its area. The equation for $z = (\gamma, h, \nu)$ is

$$F(z; a) = (F_q, F_A, F_W) = (q - h, A_\gamma - \nu, W(\gamma) - 4\pi/3),$$
$$W(\gamma) = -\pi \int_0^\pi Y^2 X' \,\mathrm{d}u. \tag{4.2}$$

We write displacements as $\delta\gamma = \psi N + w\gamma'$, with ψ even and w odd, both π-periodic. The volume derivative is

$$\delta W(\psi) = 2\pi \int_0^\pi \psi(u) S(u) Y(u) \,\mathrm{d}u. \tag{4.3}$$

Indeed a tangential displacement has zero normal speed, and the area element of the surface of revolution is $SY \,\mathrm{d}u \,\mathrm{d}\varphi$. Translations of the chord labels keep a fixed during differentiation in z. The equations of the preceding section use a as the external parameter.

Fix a small positive upper strip width r_0, and an integer $s_0 \geq 2$. For a complex, 2π-periodic function $g(x) = \sum_m g_m e^{imx}$ put

$$\|g\|_{r,s} = \sum_m e^{r|m|}(1 + |m|)^s |g_m|, \qquad 0 < r \leq r_0.$$

We use the product norm for vectors, scalars and several angle variables, with the double norm having weights $e^{r(|m|+|n|)}(1+|m|+|n|)^s$. Analogously, the zonal coefficient norm is

$$\|g\|^L_{r,s} = \sum_{l \geq 0} e^{rl}(1 + l)^s |[g]_l|.$$

The family norms on a closed set $G \subset I_\tau$ replace each coefficient modulus by $|c|^\#_G = \sup_G |c| + \tau \operatorname{Lip}_G(c)$, where $\operatorname{Lip}_G(c) = \sup_{a \neq b \in G} |c(a) - c(b)|/|a - b|$. The analogous norm of a scalar is denoted by the same symbol. The superscript # on a norm denotes this convention, including for double series and matrix coefficients. Coefficient multiplication satisfies $|cd|^\#_G \leq |c|^\#_G |d|^\#_G$. All widths below are positive and bounded by r_0.

Here are the estimates used in changing coordinates. Fourier multiplication is bounded at width r by submultiplicativity of the weights, and d derivatives cost $C_d\delta^{-d}$ across a width gap $\delta \leq 1$. A holomorphic function bounded on a strip of width r has Fourier coefficients bounded by its supremum times $e^{-r|m|}$. Thus for $r' < r$, passage between supremum and

Wiener norms costs at most a fixed power of $r-r'$, also in two variables and for coefficientwise parameter differences. Even series are Chebyshev series through $T_p(\cos x) = \cos(px)$. Each $P_l(\cos x)$ has Fourier bandwidth l and real supremum at most one. Conversely T_p has Legendre bandwidth p and

$$|[T_p]_l| \le \frac{2l+1}{2} \|T_p\|_{L^2[-1,1]} \|P_l\|_{L^2[-1,1]} \le (2l+1)^{1/2}.$$

Both changes of coefficients are therefore bounded across a strip gap with a fixed polynomial loss. A translated family $g(x+a)$ incurs at most one extra derivative for the Lipschitz part, since $|e^{ima} - e^{imb}| \le |m||a-b|$. Below we write $\|\cdot\|_r$ for the angular norm with the fixed index s_0, and we indicate the Legendre version explicitly when it is needed.

4.2. **An exact change of the azimuthal coordinate.**

Lemma 4.1. *Put $\xi = \cos x$, $y = \cos u$, $H = \sin x \sin u$, and $h_0 = \cos a$. There are functions $e(\xi, y)$ and $o(\xi, y)$, analytic in the two cosine coordinates and close to one and zero respectively, such that*

$$\frac{n_1 X(u) + n_2 Y(u) - q}{\cos(u-x) - h_0} = e + Ho. \tag{4.4}$$

With $\chi = e^2 - H^2 o^2$, the shape derivative at the fixed physical plane is

$$(\mathcal{T}_\gamma \psi)(x) = \int_{S^2} b_\gamma(\xi, y) \psi(u) \delta(\omega \cdot (e_r(x), 0) - h_0) \, \mathrm{d}\omega, \qquad b_\gamma = \frac{S(u) Y(u)}{\sin u \sqrt{\chi}}. \tag{4.5}$$

The amplitude is positive on the real square and close to one. It satisfies $b_\gamma(-\xi, -y) = b_\gamma(\xi, y)$.

Proof. At $u = x \pm a$ the numerator in (4.4) vanishes by the definition of q. The two roots of the denominator are simple and separated uniformly as a ranges in a compact subinterval of $(0, \pi)$ about a_*. The zeros hold for complex x, so the quotient extends analytically in x, u. Under the simultaneous reflection $(x, u) \mapsto (-x, -u)$ it is unchanged. Taking its part even in each variable yields e, and dividing its part odd in each variable by $\sin x \sin u$ yields o. An odd periodic function vanishes at 0 and π, so these divisions are removable. The two functions are even in each angle separately and consequently have double Chebyshev expansions. Reflecting x in (4.4) and comparing the even and odd parts gives

$$n_1 X - q = e(\xi y - h_0) + H^2 o, \qquad n_2 Y = H\{e + o(\xi y - h_0)\}. \tag{4.6}$$

Write a reference unit sphere point as $\omega = (\cos u, \sin u \, \eta, \sin u \, \eta_\perp)$ with $\eta^2 + \eta_\perp^2 = 1$ and $0 \le u \le \pi$. Let $s = \xi y + H\eta$ and $J = e - Ho\eta$. At fixed x, u use the map on the azimuthal circle given by

$$\eta_{\mathrm{old}} = \frac{e\eta - Ho}{J}, \qquad \eta_{\perp,\mathrm{old}} = \frac{\sqrt{\chi}}{J} \eta_\perp. \tag{4.7}$$

The denominator is positive. The sum of the squares of these coordinates is one, and the map is a diffeomorphism of the circle with length Jacobian

$\sqrt{\chi}/J$. For example, differentiating the first coordinate in the azimuthal angle and using the second gives this factor away from its two zeros. Continuity gives it there as well. Substitution into (4.6) yields

$$n \cdot (X, Y\eta_{\mathrm{old}}) - q = \frac{\chi}{J}(s - h_0). \tag{4.8}$$

The physical surface element before changing azimuth is $SY\,\mathrm{d}u\,\mathrm{d}\varphi_{\mathrm{old}}$. At a fixed plane the first variation is the integral of normal speed ψ times the Dirac distribution of the plane excess against this element. Equation (4.8) contributes J/χ and (4.7) contributes $\sqrt{\chi}/J$. The reference surface element is $\sin u\,\mathrm{d}u\,\mathrm{d}\varphi$. This proves (4.5). Parity makes the division by $\sin u$ in $Y/\sin u$ removable. Positivity follows from closeness to the circle. Shifting x, u by π preserves the quotient in (4.4) and H, as well as S and $Y/\sin u$. It proves the joint sign symmetry. □

4.3. **Estimates for the nonlinear equation.** For clarity, division in (4.4) and the analytic estimates on the section area can be obtained without composition of functions at complex, moving endpoints. In a fixed small neighborhood of a zero $u = x \pm a$ use $d = u - x \mp a$. The quotient by d is $\int_0^1 \partial_u G(x, x \pm a + td)\,\mathrm{d}t$, where G is the numerator. For real a the segment in its second argument remains in the same strip in u as u and x. The other factors of the denominator have a uniform lower bound on this neighborhood. On the complement the original denominator is bounded away from zero. This gives a bound in supremum norm across a strip gap, then in Wiener norm by the Fourier estimate above. To control parameter differences, write $c(a) = \pm a$ and for two parameters compare their quotients on a common smaller neighborhood of $u = x + c(a)$ whenever $|a - b|$ is smaller than the fixed neighborhood radius. The integral expression uses the corresponding G_a, G_b and $c(a), c(b)$. A single extra u-derivative bounds their difference divided by $|a - b|$. Outside the root neighborhoods use the quotient rule for differences. Pairs of parameters separated by at least the neighborhood radius are bounded by the two supremum norms. Division by $\sin x$ or $\sin u$ uses the same argument at fixed zeros. The estimates apply to the shape derivatives, since the identities $G_a(x, x \pm a) = 0$ hold identically for nearby meridians, and hence for each such derivative. Products, square roots and inverses on a fixed nonvanishing neighborhood have the same estimates by their convergent analytic series.

We give an explicit integral formula to control F_A itself. The physical cap volume V_γ and the boundary cap integral $C_\gamma(g)$ are

$$\begin{aligned} V_\gamma &= \int_{\partial K_\gamma \cap \{z \cdot n > q\}} (z \cdot n - q)(n \cdot N)\,\mathrm{d}S, \\ C_\gamma(g) &= \int_{\partial K_\gamma \cap \{z \cdot n > q\}} g\,\mathrm{d}S, \\ qA_\gamma &= C_\gamma(z \cdot N) - 3V_\gamma. \end{aligned}$$

In the first identity apply the divergence theorem to $(z\cdot n-q)n$. The planar face gives zero flux. For the last identity apply it to the field z, whose flux on the planar face of the cap is $-qA_\gamma$. In these formulas the revolved unit normal at a boundary point is $(N_1, N_2\eta_{\mathrm{old}}, N_2\eta_{\perp,\mathrm{old}})$. In particular $z\cdot N = XN_1 + YN_2$. Set $t = s - \xi y$. The transformed expressions use

$$J = e - ot, \qquad n\cdot N = n_1N_1 + \frac{n_2N_2}{\sin x\sin u}\,\frac{et - H^2o}{J}.$$

The quotient $n_2N_2/(\sin x\sin u)$ is analytic by parity. The cap restriction is the fixed spherical cap $s > h_0$.

Here is a direct bound for the cap integrals on angular strips. The reference density after the substitution is $b_0\sqrt{\chi}/J$, where $b_0 = SY/\sin u$. For each real $s \in [h_0, 1]$ the quantities $t = s - \xi y$, $H^2 = (1-\xi^2)(1-y^2)$ are bounded on a fixed pair of complex strips. The denominator J is uniformly close to one there. Hence the transformed amplitudes, including the factor $\chi(s-h_0)/J$ in V_γ, have double Chebyshev series with coefficient estimates uniform in s. Their shape derivatives are obtained by the product and quotient rules. For a term $g_{pq}(s)T_p(\xi)T_q(y)$ in such a series, the cap integral is $T_p(\xi)$ times the zonal transform of $T_q(y)$ with kernel $1_{[h_0,1]}(s)g_{pq}(s)$. On degree $l \le q$ its multiplier is $2\pi\int_{h_0}^1 g_{pq}(s)P_l(s)\,\mathrm{d}s$. Its size is at most $C\|g_{pq}\|_\infty$. A parameter difference of the multiplier is bounded by the difference of the amplitudes and the length of the interval between the two lower limits times their supremum. The same reasoning for a fixed spherical section uses the value at h_0, multiplied by $2\pi P_l(h_0)$, and a fixed number of derivatives in s or h_0 costs a fixed polynomial in $1+l$. Converting between Chebyshev and Legendre coefficients costs only powers of the degrees as established above. Thus the sum of the transformed integrals on the angular strip has bounds with only polynomial losses in the strip gap.

As a consequence there exist fixed C, P such that

$$\|F(z+v) - F(z) - D_zF(z)v\|^{\#}_{r-\delta} \le C\delta^{-P}(\|v\|^{\#}_r)^2. \tag{4.9}$$

This holds with the displayed geometric denominators bounded away from their zeros on the intermediate strips. One can obtain the estimate by taking the second derivative along $z+tv$ and integrating with weight $1-t$. For the parameter difference, take the difference of these second derivatives at the two parameters. The estimate is bounded by the two terms in which one v is differenced and by the third shape derivative times the difference of the base meridians, together with the translated-function and cap-limit differences above. The base need only have bounded coefficientwise family norm. A finite division of the gap δ among the estimates gives a fixed exponent P. The same proof gives a local increment estimate for b_γ in the double Chebyshev norm. When estimates are centered at an already controlled tuple of quotients and other geometric factors, their increments are bounded by $C\delta^{-P}\|\Delta\gamma\|^{\#}_r$ on the smaller strip. For an inverse or square root this

follows by expanding about the old value under the condition $\delta^{-P}\|\Delta\gamma\|_r^{\#}$ sufficiently small. This local form will be used for shrinking strips.

The physical height derivative A_h, at fixed normal, has the same bounds. Indeed translating the entire body in the direction n changes the effective height from q to $q-t$. Its normal speed is $n\cdot N$, giving

$$A_h(x) = -\int_{\partial K_\gamma} (n\cdot N)\delta(z\cdot n - q)\,\mathrm{d}S. \tag{4.10}$$

The transformed amplitude contains the analytic expression for $n\cdot N$ above, evaluated at $s=h_0$. In particular A_h can be computed without taking a derivative in a.

4.4. **Splitting the linearization.**

Lemma 4.2. *Set $q_\theta = q'/\theta'$, $t_\pm = n^\perp\cdot\gamma_\pm - q_\theta$, and $c_q = \theta' t_+ t_-/L$. For a displacement $\delta\gamma = \psi N + w\gamma'$ put*

$$\zeta_\psi = -\frac{t_-}{L} n\cdot N_+\psi_+ + \frac{t_+}{L} n\cdot N_-\psi_-.$$

After subtracting $A_h\, D_z F_q$ from the section row, the derivative is the sum of

$$\mathcal{L}_z(\psi,w,\delta h,\delta\nu) = \big(\zeta_\psi + c_q(T_a - T_{-a})w - \delta h,\ \mathcal{T}_\gamma\psi + A_h\delta h - \delta\nu,\ \delta W(\psi)\big) \tag{4.11}$$

and the error

$$\left(\frac{F_q'}{\theta'}\delta\theta,\ \frac{F_A' - A_h F_q'}{\theta'}\delta\theta,\ 0\right). \tag{4.12}$$

At the circle $c_q = -\sin a/2$ and $\zeta_\psi = \frac{\cos a}{2}(T_a + T_{-a})\psi$. The angle variation satisfies $\delta\theta = n\cdot(\delta\gamma_- - \delta\gamma_+)/L$.

Proof. The equations $n\cdot\gamma_\pm = q$ give

$$\delta q - q_\theta\delta\theta = n\cdot\delta\gamma_\pm + t_\pm\delta\theta, \qquad n\cdot\gamma_\pm' = -\theta' t_\pm.$$

Taking the difference of the two equations and using $t_+ - t_- = L$ gives the formula for $\delta\theta$. Eliminating $\delta\theta$ from their common value gives $\delta q - q_\theta\delta\theta = \zeta_\psi + c_q(T_a - T_{-a})w$. With height and normal varied independently the section derivative is $\mathcal{T}_\gamma\psi + A_h\delta q + A_\theta\delta\theta$. Differentiating the section area with respect to x gives $A_\theta = (A_\gamma' - A_h q')/\theta'$. Subtracting the row $A_h D_z F_q$ gives the formulas. The circle has $\theta'=1$, $t_\pm = \pm\sin a$, and $n\cdot N_\pm = \cos a$, proving the reference values. □

5. Finite-order diagonal reduction

5.1. **Matrices on the Legendre coefficients.**

We use the Legendre coefficient model

$$\mathcal{H}P_l = e^{ik_l x}, \qquad k_l = l + \tfrac{1}{2}, \qquad De^{ik_l x} = k_l e^{ik_l x}.$$

The index set is initially $l\ge 0$. A periodic multiplier $b(x) = \sum_m b_m e^{imx}$ gives the Toeplitz matrix $(T_b)_{jl} = b_{j-l}$ on this half-lattice. The notation

T_a with scalar angle a still denotes translation on periodic functions. For a matrix R, define

$$|R|_{r,s;N} = \sup_{l\ge 0}(1+l)^N \sum_{j\ge 0} e^{r|j-l|}(1+|j-l|)^s |R_{jl}|, \qquad N \ge 0. \tag{5.1}$$

Finite norm for $N = 0$ implies boundedness on the Legendre coefficient norm $\|\cdot\|^L_{r,s}$. For every $\delta > 0$ with $r - \delta > 0$,

$$|AB|_{r,s;N} \le |A|_{r,s;0}|B|_{r,s;N}, \tag{5.2}$$

$$|BA|_{r-\delta,s;N} \le C_N \delta^{-N} |B|_{r,s;N}|A|_{r,s;0} \qquad (0 < \delta \le 1). \tag{5.3}$$

Indeed, in the second product the intermediate column index i of B can be replaced by the input index l using $(1+l)/(1+i) \le 1 + |l-i|$. Absorb the resulting factor $(1+|i-l|)^N$ using $e^{-\delta|i-l|}$. The first product leaves the input index unchanged. In particular, $|\cdot|_{r,s;0}$ is submultiplicative. Each assertion also holds with the coefficientwise family norm. Throughout this section the exponents in losses can depend on the chosen finite orders and on s_0.

Lemma 5.1 (Polynomial multiplication and section asymptotics)**.** *For a polynomial B_0 of degree p, on input indices $l \ge p$ the exact multiplication formula is*

$$\mathcal{H} M_{B_0(\cos x)} \mathcal{H}^{-1} = \sum_{j=0}^{p} \binom{-1/2}{j} (-iM_{\sin x} D^{-1})^j M_{B_0^{(j)}(\cos x)}. \tag{5.4}$$

Here M_b means multiplication in the displayed Fourier representation. Lower columns are handled by the actual polynomial multiplication matrix on the half-lattice.

For an analytic amplitude $b(\xi, y)$ with a double Chebyshev expansion, let $\mathcal{T}_b$ be the operator in (4.5) *with amplitude b, and put*

$$V_\pm e^{ik_l x} = e^{\pm ik_l a} e^{ik_l x}, \qquad A(a) = \sqrt{2\pi/\sin a},$$
$$B_b = A(a)^{-1} \mathcal{H} \mathcal{T}_b \mathcal{H}^{-1} D^{1/2}.$$

For any fixed $J \ge 2$, this matrix has an expansion

$$B_b = \sum_{\sigma=\pm 1} \sum_{j=0}^{J-1} T_{b_{\sigma,j}} D^{-j} V_\sigma + R_J. \tag{5.5}$$

The leading profiles are

$$b_{\sigma,0}(x) = e^{-\sigma i\pi/4} b(\cos x, \cos(x + \sigma a)). \tag{5.6}$$

Every profile and the remainder, the latter measured with $N = J-1$ in (5.1)*, satisfy linear bounds in the double analytic norm of b with fixed polynomial strip losses, including in the coefficientwise family norms. The same expansion and remainder estimates hold for finite products involving Toeplitz matrices and fixed nonpositive powers of D, with the phase in a product kept exact.*

Proof. For the algebraic identity, use the usual Gegenbauer polynomials C_l^λ and write $\mathcal{H}_\lambda C_l^\lambda = e^{i(l+\lambda)x}$. Off the lower boundary the recurrence in [17, Section 18.9] gives

$$\mathcal{H}_\lambda M_{\cos x} \mathcal{H}_\lambda^{-1} = M_{\cos x} - i(\lambda - 1) M_{\sin x} D^{-1}.$$

In fact the coefficients of the upward and downward transfers are $(k - \lambda + 1)/(2k)$ and $(k + \lambda - 1)/(2k)$ with $k = l + \lambda$. For positive integers m, use

$$(C_l^m)' = 2m C_{l-1}^{m+1}, \qquad C_l^m = \frac{m}{l+m}(C_l^{m+1} - C_{l-2}^{m+1}).$$

These are the derivative and parameter change identities in [17, Section 18.9]. For $l \geq 1$, converting differentiation in the polynomial variable by $\mathcal{H}_{m+1}$ gives $2m\mathcal{H}_m$. The second identity gives $\mathcal{H}_{m+1}\mathcal{H}_m^{-1} = 2imM_{\sin x}D^{-1}$. Write $A_B^m = \mathcal{H}_m M_B \mathcal{H}_m^{-1}$. Apply the product rule to the derivative of Bf and these two conversion formulas to get, for sufficiently high inputs,

$$A_B^{m+1} = A_B^m - iM_{\sin x} D^{-1} A_{B'}^m.$$

At $m = 1$ the matrix for multiplication by the coordinate is ordinary Toeplitz multiplication away from the lower boundary, so the same holds for a polynomial on sufficiently high inputs. Induction with Pascal's identity gives the right side of (5.4) with $\binom{m-1}{j}$. For each transfer, both sides are polynomials in $\lambda - 1$ with coefficients rational in k, by the displayed three-term matrix and the finite formula on the right. At any fixed positive integer λ, equality holds for infinitely many k, so the rational functions agree. Equality at all positive integers λ identifies their polynomial coefficients. At $\lambda = 1/2$ the three-term matrix applies on every path from $l \geq p$, since the path has length at most p. This gives the exact identity there and $C_l^{1/2} = P_l$.

We detail the estimates needed to use this identity for analytic amplitudes. For the Chebyshev polynomial T_p, the Markov inequality on $[-1, 1]$, applied repeatedly, gives $\|T_p^{(j)}\|_\infty \leq p^{2j}$ for $j \geq 1$. The Fourier sum of $T_p^{(j)}(\cos x)$ is bounded by $(2p + 1)p^{2j}$ and has bandwidth at most $p - j$. For $p \leq c\sqrt{k_l}$ with a sufficiently small constant $c > 0$, every frequency in (5.4) is at least $k_l/2$. In the column sum with the weights of (5.1) at $N = 0$, the terms with $j \geq J$ are bounded by

$$C_J e^{rp}(1 + p)^{s+1} k_l^{-J}(1 + p)^{2J}. \tag{5.7}$$

Indeed the absolute value of $\binom{-1/2}{j}$ is at most one, and the remaining geometric series has ratio at most $2p^2/k_l \leq 1/2$. For the terms $j < J$, expand every denominator $(k_l + \nu)^{-1}$, $|\nu| \leq p$, in inverse powers of k_l. Taylor's formula on $|\nu/k_l| \leq 1/2$ gives a remainder with the same type of bound as (5.7), with a possibly larger fixed power of $1 + p$. Every new coefficient is a finite sum of weighted Fourier convolutions, where powers of ν cost finitely many derivatives.

For large transfers we use the exact multiplication matrix. Orthogonality and $\|T_p\|_\infty \le 1$ give

$$|[T_pP_l]_j| \le \left(\frac{2j+1}{2l+1}\right)^{1/2}, \qquad [T_pP_l]_j = 0 \text{ if } |j-l| > p.$$

The lower-bandwidth bound follows by moving T_p to the other side of the inner product. Thus the column sum has a polynomial bound in $1+p$, times e^{rp}, uniformly in l. A truncated multiplication matrix has the same type of bound, using $k_l \ge 1/2$. If $p > c\sqrt{k_l}$, then $k_l^N \le C_N(1+p)^{2N}$ for any fixed N. After a strip gap δ, all the additional powers are absorbed using $\sup_{p\ge 0} e^{-\delta p}(1+p)^d \le C_d\delta^{-d}$. In particular the columns where a projected product meets the lower boundary are controlled by this estimate.

For the middle factor of the section matrix, the Legendre asymptotics in [17, Section 18.15(iii), (18.15.12)–(18.15.13)] give, for $k = l + 1/2$,

$$d_l^*(a) := A(a)^{-1}2\pi k^{1/2}P_l(\cos a) = \sum_{\sigma=\pm1} e^{\sigma ika}\sum_{j=0}^{J-1} p_{\sigma,j}(a)k^{-j} + r_J(k,a). \tag{5.8}$$

Here $p_{\sigma,0} = e^{-\sigma i\pi/4}$ and $p_{-,j} = \overline{p_{+,j}}$. The remainder satisfies $|r_J| \le C_Jk^{-J}$ and $|\partial_a r_J| \le C_Jk^{1-J}$ uniformly on a fixed compact interval about a_*. For the expansion, in the m-th term of the cited formula write

$$(-1)^l\binom{m-1/2}{l} = \frac{\Gamma(l+1/2-m)}{\Gamma(1/2-m)\Gamma(l+1)}.$$

The fixed-shift Stirling expansion [17, Section 5.11(iii)] of the ratio has leading power $l^{-m-1/2}$. The factor $(-1)^l$ cancels that of the cosine in (18.15.13), leaving phases $\pm k_l a$ and smooth coefficients. Expanding the powers of $l = k_l - 1/2$ gives (5.8). To obtain the derivative statement from the undifferentiated expansion, one can first use one additional order at l and $l-1$ in

$$\frac{\mathrm{d}}{\mathrm{d}a}P_l(\cos a) = \frac{l}{\sin a}\big(\cos a, P_l(\cos a) - P_{l-1}(\cos a)\big).$$

There is also a direct remainder argument. Denote the finite expansion in (5.8) by F_J. The recurrence for the derivative gives an expansion for $(d_l^*)' - F_J'$ with error $O(k^{1-J})$, whose possible terms of larger size are finite sums of smooth coefficients times $k^p e^{\sigma ika}$. Integrate this difference on $[a, a+b/k]$ for a fixed $0 < b < \pi$. The integral is $O(k^{-J})$ because $d_l^* - F_J = O(k^{-J})$ at both ends. For the highest possible power $p > 1-J$, the leading terms in the integral are $k^{p-1}e^{\sigma ika}$ times their coefficients and $(e^{\sigma ib}-1)/(\sigma i)$. Use two different choices of b in $(0,\pi)$ to obtain an invertible two-by-two matrix. Letting $k\to\infty$ forces both coefficients to vanish. Continuing in decreasing powers proves the derivative estimate on an interior compact interval. The leading factor $A(a)$ follows from the cosine term of the standard expansion.

For an amplitude term $b_{pq}T_p(\xi)T_q(y)$ the exact operator is

$$b_{pq}M_{T_p}S_{\cos a}M_{T_q}.$$

With $p+q \le c\sqrt{k_l}$, insert the multiplication expansions on both sides of (5.8). If the right multiplication transfers by ν, the phase is exactly $e^{\sigma i(k_l+\nu)a}$, and the power normalization includes $(k_l/(k_l+\nu))^{1/2}$. Expand this ratio and all the inverse powers of $k_l+\nu$ in powers of k_l^{-1}. The resulting leading right multiplier is $T_q(\cos(x+\sigma a))$. The nonleading terms and their remainders are bounded by the appropriate power of k_l^{-1} times the weight $e^{r(p+q)}(1+p+q)^d$. Differencing an external phase costs at most one power of $k_l+|\nu|$. All profile differences have polynomial transfer bounds. If $p+q > c\sqrt{k_l}$, use the exact multiplication bounds, the bounded normalized diagonal d_l^*, and $(k_l/(k_l+\nu))^{1/2} \le C(1+p+q)^{1/2}$. Here one may trade k_l^N for $C_N(1+p+q)^{2N}$. The same estimate applies to each truncated term. Summing against the double Chebyshev coefficients with one strip gap gives (5.5) and (5.6), with $N = J-1$ covering the loss of one input power in the family norms.

For use in the finite recursion, the composition estimate can also be seen directly from one matrix entry. For a product $(T_bD^{-j}V_\sigma)(T_cD^{-m}V_\kappa)$, allow $V_0 = I$. The contribution of transfers ν on the right and μ on the left to column l has coefficient

$$b_\mu c_\nu (k_l+\nu)^{-j} k_l^{-m} e^{i\sigma\nu a} e^{i(\sigma+\kappa)k_l a}. \tag{5.9}$$

It is present when the intermediate and final indices belong to the half-lattice. For $|\nu| < k_l/2$, expand only $(1+\nu/k_l)^{-j}$. This gives the convolution profiles, including differentiations from powers of ν. The remainder is bounded by a fixed transfer moment times the first omitted inverse power of k_l. When $|\nu| \ge k_l/2$, analyticity absorbs the input powers in the same way as above. Any omitted intermediate index has $|\nu| \ge k_l/2$ except for bounded l. The corresponding bound covers both cases. For terms with an existing matrix remainder, use (5.3). The proofs for family differences use $|e^{i\sigma\nu a} - e^{i\sigma\nu b}| \le |\nu||a-b|$ and the corresponding inequality with k_l, using one order in the remainder. This proves the composition assertion and gives polynomial loss estimates for differences in each input. □

5.2. **Balancing the phases and removing the off-diagonal terms.** Let $\mathcal{G}_h \subset I_\tau$ be the closed set where

$$|2\sin(ma)| \ge \gamma_0(1+|m|)^{-\tau_h} \qquad (0 \ne m \in 2\mathbb{Z}) \tag{5.10}$$

with $\tau_h > \tau_* + 1$ and sufficiently small fixed $\gamma_0 > 0$. The operator $T_a - T_{-a}$ on mean-zero, π-periodic functions has multiplier $2i\sin(ma)$. Its inverse costs a fixed number of powers of $1+|m|$. Its parameter differences cost at most $C(1+|m|)^{2\tau_h+1}$, using the product of the two lower bounds for the denominators. We use the inverse between the opposite reflection parities whenever the data have a given parity.

Proposition 5.2 (Finite reduction)**.** *For $a \in \mathcal{G}_h$ and $b = b_\gamma$ near the reference amplitude of Lemma 4.1, there are invertible order-zero matrices $T_w, T_{1/e}, Q, E$, each close to I, a real diagonal $\mathcal{D}$ and a matrix R such*

that

$$T_{1/e}\, B_b\, T_w Q = E\mathcal{D} + R. \tag{5.11}$$

They preserve the real, even Legendre coefficient sequences. At $\gamma = e_r$*, these four matrices are* I*,* $R = 0$*, and the diagonal entries are* $d_l^*(a)$ *from* (5.8)*. On a smaller strip* R *has finite norm in* (5.1) *with* $N = 8$*.*

The construction uses a fixed number of operations on the coefficients. Starting with bounded old reduction data on a strip of width r*, a meridian increment of norm* η *there changes the data by at most* $C\delta^{-P}\eta$ *on width* $r - \delta$*, provided* $\delta^{-P}\eta$ *is small and the old data satisfy the nonvanishing margins. The data include all finite profiles, the matrices in* (5.11)*, and the order-eight remainder. These bounds hold for* # *norms. For all sufficiently large* l *the entries have the exact form*

$$d_l = e^{ik_l a} M_+(k_l, a) + e^{-ik_l a} M_-(k_l, a), \qquad M_- = \overline{M_+}. \tag{5.12}$$

The same increment bound holds for $\sup_k k|\Delta M_+(k,a)|_G^{\#}$*.*

Proof. Put $\beta_\sigma(x) = b(\cos x, \cos(x + \sigma a))$. By Lemma 4.1 these functions are positive on the real line, π-periodic, and $\beta_-(x) = \beta_+(-x)$. Solve

$$(T_a - T_{-a})v = \log \beta_- - \log \beta_+, \qquad \operatorname{av} v = 0,$$
$$w = \exp(v), \qquad e = \beta_+ T_a w = \beta_- T_{-a} w.$$

The right side is odd, so v is even. Uniqueness, reality and the reflection relation show that w, e are real and even. They are close to one on a smaller strip, where their Toeplitz matrices and $T_{1/e}$ are invertible by a Neumann series. The leading profiles of $C_b = T_{1/e} B_b T_w$ are therefore the constants $c_\pm = e^{\mp i\pi/4}$. This follows from the order-zero case of (5.9). At the reference amplitude $b = 1$, C_b is the exact diagonal d_l^*.

We perform the recursion through degree $J - 1$ in D^{-1}, where $J = 16$ suffices. At step $j \geq 1$ write $Q_{<j} = I + \sum_{1 \leq m < j} T_{q_m} D^{-m}$ and $E_{<j} = I + \sum_{1 \leq m < j} T_{e_m} D^{-m}$. Let $D_{<j}$ have branches $c_\sigma(1 + \sum_{1 \leq m < j} m_{\sigma,m} k^{-m})$ with the phases V_σ. The residual at the first untreated order of $C_b Q_{<j} - E_{<j} D_{<j}$ has profiles $c_\sigma F_{\sigma,j}(x)$. Adding $T_{q_j} D^{-j}$ and $T_{e_j} D^{-j}$ in the two factors and $c_\sigma m_{\sigma,j} k^{-j}$ in the diagonal branch changes those profiles to

$$c_\sigma \big(F_{\sigma,j} + T_{\sigma a} q_j - e_j - m_{\sigma,j} \big).$$

This follows from (5.9) at leading order of the new terms. Products of new terms with already nonconstant inverse powers enter at orders $> j$. Set

$$m_{\sigma,j} = \operatorname{av} F_{\sigma,j}, \qquad \operatorname{av} q_j = 0,$$
$$(T_a - T_{-a}) q_j = -F_{+,j} + F_{-,j} + \operatorname{av}(F_{+,j} - F_{-,j}),$$
$$e_j = F_{+,j} + T_a q_j - m_{+,j}.$$

These equations cancel the profiles at order j in both branches. At the circle $q_j = e_j = 0$ for all j, and $c_\sigma m_{\sigma,j}^* = p_{\sigma,j}$.

For the exact diagonal retain the spherical eigenvalue and add only the changes computed in the recursion:

$$d_l = d_l^* + \sum_{\sigma=\pm1} c_\sigma e^{\sigma i k_l a} \sum_{j=1}^{J-1} (m_{\sigma,j} - m_{\sigma,j}^*) k_l^{-j}. \tag{5.13}$$

This definition applies to all l, including the low modes. For large k_l, split the real reference remainder $r_J(k_l, a)$ of (5.8) by assigning $r_J(k_l, a)e^{-ik_l a}/2$ and its conjugate to the two branches. This gives (5.12), with reference positive branch $M_+^* = e^{-i\pi/4} + O(k^{-1})$ and $\partial_a M_+^* = O(k^{-1})$. The new branch terms begin at k^{-1}. Reality here is conjugation of the coefficients at the same transfer: $F_{-,j,\nu} = \overline{F_{+,j,\nu}}$. This relation is preserved by (5.9), with the real initial matrix and the two opposite phases. The difference in the equation for q_j is purely imaginary at each nonzero transfer and division uses $2i\sin(\nu a)$. Thus the coefficients of q_j are real and the two formulas for e_j give conjugate coefficients, so that its coefficients are real too. The joint sign symmetry of b implies that only even transfers occur. In particular the inversion at every step uses exactly (5.10).

Define R by (5.11). The canceled coefficients and Lemma 5.1 give the order-eight bound, including the low columns. To see the local difference estimate, expand the logs and exponentials about the old bounded profiles using the algebra inequality. Apply the homogeneous inverse once to their increment. For each following recursion the coefficient operations are the finite convolutions of (5.9), derivatives, translations and the homogeneous inverse. A difference in a product is a sum of products with one differenced factor. Apply the strip estimates to these factors, using the old norms and the smallness condition for the increments. For the matrix remainders, use the actual remainders in the high- and low-transfer estimates of Lemma 5.1. Each is multilinear in its profiles before taking the norms. This also controls the change in R after cancellation: one subtracts the retained coefficients before estimating the difference so that the first omitted input power is retained. The change introduced by replacing the truncated reference diagonal with the exact one is E times a diagonal remainder of order at least $J-1$ in the family norm. The first inequality in (5.3) preserves that order. Finite initial columns are estimated by the same polynomial bounds on matrix entries. Taking the same fraction of the gap for each of the finitely many operations gives one fixed exponent P. All profiles, the exact diagonal and the remainders are defined by coefficient formulas independent of the strip used for estimation. This is why the increment estimate can be centered at old data when the width is decreased. □

If the perturbative data of Proposition 5.2 are $O^{\#}(\tau)$, then for all sufficiently large l,

$$\begin{aligned} &d_l = 2A_l \cos(k_l a + \phi_l), \qquad 0 < c \le A_l \le C, \\ &\operatorname{Lip}_G \phi_l \le C/k_l, \qquad \sup_G |d_l - d_l^*| \le C\tau/k_l. \end{aligned} \tag{5.14}$$

Indeed take $A_l = |M_+|$ and the argument near $-\pi/4$, using the bounds for M_+^* and $k_l|M_+ - M_+^*|_G^{\#}$. The phase estimate does not require differentiating the family away from G. It follows that $\mathrm{Lip}_G\, d_l \le C(1+l)$. The finitely many lower modes satisfy this last bound by (5.13).

6. The inverse of the linearized equations

6.1. **Eliminating the two low modes.** Let $S_\gamma g$ denote the normal component ψ of the solution of the augmented problem

$$\mathcal{T}_\gamma \psi - c - \eta P_2 = g, \qquad \delta W(\psi) = 0, \qquad [\psi]_2 = 0. \tag{6.1}$$

The constants c, η are auxiliary unknowns. We first reduce this equation algebraically, then give conditions for inversion.

Lemma 6.1. *The restriction of* (6.1) *to the even Legendre coefficients reduces, for indices $l \neq 0, 2$, to*

$$(\mathcal{D}_H + \widehat{R}) u_H = \widetilde{g}.$$

On a smaller strip $\widehat{R}$ has order -8 in (5.1)*, vanishes at the circle, and obeys the local increment estimate of Proposition 5.2. The source and recovery maps have fixed polynomial strip losses.*

Proof. Set

$$\psi = \mathcal{H}^{-1} D^{1/2} T_w Q u. \tag{6.2}$$

Apply $E^{-1} T_{1/e} \mathcal{H}/A(a)$ to the equation in (6.1) to obtain

$$(\mathcal{D} + R_1) u - \mathcal{B}\lambda = g_1, \qquad \mathcal{L} u = 0, \qquad R_1 = E^{-1} R, \quad \lambda = (c, \eta).$$

Here g_1 is the transformed source, $\mathcal{B}$ has the transformed columns $1, P_2$, and $\mathcal{L}$ has the two constraints after (6.2). Let P be the projection onto $l = 0, 2$ and $H = I - P$. At the circle the constraint rows are $4\pi k_0^{1/2} u_0$ and $k_2^{1/2} u_2$, and the target columns are nonzero multiples of the corresponding coordinate vectors. Hence $\mathcal{L}_P$ and $\mathcal{B}_P$ stay invertible. In this block notation a subscript P or H means restriction of the source for $\mathcal{L}$, projection of the target for $\mathcal{B}$, and the indicated restriction to a diagonal block for $\mathcal{D}$. Solve the constraints for $u_P = T u_H$ and the low output for $\lambda = U u_H - \mathcal{B}_P^{-1} g_{1,P}$, where

$$T = -\mathcal{L}_P^{-1} \mathcal{L}_H, \qquad U = \mathcal{B}_P^{-1} \big((\mathcal{D}_P + P R_1 P) T + P R_1 H \big).$$

The high equation becomes the asserted equation with

$$\widehat{R} = H R_1 H + H R_1 P T - \mathcal{B}_H U, \qquad \widetilde{g} = g_{1,H} - \mathcal{B}_H \mathcal{B}_P^{-1} g_{1,P}. \tag{6.3}$$

These identities use only inversions of two-by-two matrices.

We verify the decay needed for the finite-rank products. For the volume row, the analytic test function in the spherical inner product is $SY/\sin u$. For the degree-two constraint it is a constant multiple of P_2. If an analytic test has Legendre coefficients b_l, its pairings against P_l are proportional to

$b_l/(2l+1)$. Thus the rows before (6.2) have exponential decay. The factor $D^{1/2}$ costs just a half power of the index. Multiplying a row by an order-zero matrix with transfer weight $e^{r|i-l|}$ preserves its exponential decay on a smaller strip by $l \le i + |i-l|$. Indeed a row with $\sup_i e^{ri}|b_i| < \infty$ composed with such a matrix has l-th coefficient bounded by $\sup_i e^{ri}|b_i|\, e^{-rl}|A|_{r,0;0}$. The row of R_1 at any fixed output index satisfies the same estimate by (5.1). The columns of $\mathcal{B}$ have exponential decay by boundedness of the order-zero maps on the coefficient space. Products of these rows and columns have arbitrary fixed negative order after a strip gap: use $e^{(r-\delta)|j-l|} \le e^{(r-\delta)(j+l)}$ and absorb each needed polynomial in l or j in the remaining factors. These estimates also hold in family norms and for differences. At the circle $R_1 = 0$, $T = 0$ and $\mathcal{B}_H = 0$. Taking differences in the displayed formulas proves the claimed increment bound. Source and recovery have only the basis-change and fixed derivative losses already described. □

Proposition 6.2. *Suppose the perturbative data of Proposition 5.2 and the norm of $\widehat{R}$ of order -8, on a working strip, are bounded by $C_0\tau$ in the family norms. There is a constant M, depending on C_0, such that on closed parameter sets satisfying*

$$|d_l(a)| \ge M\tau(1+l)^{-2} \qquad (l \ge 4,\ l \text{ even}) \tag{6.4}$$

the solution of (6.1) *is unique on smaller strips. Its inverse has norm at most $C\tau^{-P}\delta^{-P}$ across a strip gap δ. Across any fixed positive gap, uniformly for the indicated families,*

$$\tau\|S_\gamma\|^{\#} \longrightarrow 0 \qquad (\tau \downarrow 0). \tag{6.5}$$

The two equalities $S_\gamma 1 = S_\gamma P_2 = 0$ hold exactly.

Proof. The inverse diagonal satisfies the coefficient estimates

$$\sup_G |d_l^{-1}| \le \frac{(1+l)^2}{M\tau}, \qquad \tau\,\mathrm{Lip}_G(d_l^{-1}) \le \frac{C(1+l)^5}{M^2\tau}.$$

The second bound uses the quotient identity for the difference of two inverses and $\mathrm{Lip}_G d_l \le C(1+l)$. In the product $\widehat{R}\mathcal{D}_H^{-1}$ each inverse diagonal entry multiplies its own input column of $\widehat{R}$. The latter contributes $C_0\tau(1+l)^{-8}$, also for the family norm. Hence

$$|\widehat{R}\mathcal{D}_H^{-1}|^{\#}_{r,s_0;0} \le C_1/M + C_2/M^2 < \tfrac{1}{2}$$

with M large. The inverse is the exact product

$$\mathcal{D}_H^{-1}(I + \widehat{R}\mathcal{D}_H^{-1})^{-1}.$$

The Neumann series for the second factor is bounded on the working strip. The left factor uses a strip gap to absorb the input powers. This gives existence and the claimed bounds after the source and recovery maps of Lemma 6.1. For uniqueness, an analytic solution of the homogeneous high equation can be restricted to a positive working strip. Applying $\mathcal{D}_H$ to it

gives a vector in the kernel of the invertible Neumann factor. This vector vanishes, and the nonzero diagonal entries force the solution to vanish.

For fixed $l \geq 4$ even, Lemma 2.1 gives $d_l^*(a_*) \neq 0$. The exact formula (5.13) and the perturbative bounds give $d_l \to d_l^*(a_*)$ uniformly over the families under consideration. The Lipschitz constant is bounded for each fixed l. Therefore $\tau |d_l^{-1}|^{\#} \to 0$ for each such index. After inserting the exponential of a fixed strip gap, the polynomial bounds above give a uniform tail tending to zero. This proves (6.5), since the Neumann factor and the source and recovery maps have uniform bounds across fixed inner gaps. Finally, for $g = 1$ or $g = P_2$, taking $\psi = 0$ and choosing the auxiliary constants solves (6.1). Uniqueness gives the claimed annihilation. □

6.2. **The two scalar equations.** We consider meridians in the first-order regime

(6.6)
$$\gamma = e_r + \varepsilon(f e_r + w_1 e_\theta) + O^{\#}(\tau^2), \qquad a - a_* = a_1 \varepsilon + O^{\#}(\tau^2), \qquad c\tau \leq \varepsilon \leq C\tau,$$

where $|\varepsilon|^{\#} \leq C\tau$. We require the bounds on a fixed positive inner strip. With the first-order calculation for the chord distance, (6.6) implies

$$q(x) = h_* + h_1 \varepsilon + O^{\#}(\tau^2), \qquad \theta(x) = x + O^{\#}(\tau) \tag{6.7}$$

on a smaller fixed strip.

Proposition 6.3. *Under the conditions of Proposition 6.2 and* (6.6)*, the operator $\mathcal{L}_z$ in* (4.11) *admits a right inverse with norm at most $C\tau^{-P}\delta^{-P}$ across a strip gap δ. The constants are uniform when the working widths decrease to a fixed positive limit.*

Proof. Write $\mathsf{V}\psi = \delta W(\psi)$. Let

$$v_0 = (\mathsf{V}1)^{-1}, \qquad v_2 = f - (\mathsf{V}f)v_0,$$
$$p_j = v_j - S_\gamma \mathcal{T}_\gamma v_j \quad (j = 0, 2), \qquad p_h = -S_\gamma A_h.$$

The augmented conditions imply

$$\mathsf{V}p_0 = 1, \quad [p_0]_2 = 0, \qquad \mathsf{V}p_2 = 0, \quad [p_2]_2 = 1, \qquad \mathsf{V}p_h = 0, \quad [p_h]_2 = 0.$$

For a source (g_q, g_A, g_W), set $\psi_b = g_W p_0 + S_\gamma g_A$ and seek $\psi = \psi_b + \alpha p_2 + \delta h\, p_h$. This solves the volume equation, and $\mathcal{T}_\gamma \psi + A_h \delta h - g_A$ belongs to $\operatorname{span}(1, f)$ by (6.1). Its degree-two coefficient must be zero, and its constant coefficient will be $\delta\nu$.

For the first row of $\mathcal{L}_z$, divide by the nonvanishing even coefficient c_q. The right side of the resulting difference equation for w must have angular mean zero. Define

$$\Lambda_\gamma \psi = \frac{\operatorname{av}(\zeta_\psi / c_q)}{\operatorname{av}(1/c_q)}, \qquad g_q^0 = \frac{\operatorname{av}(g_q/c_q)}{\operatorname{av}(1/c_q)}.$$

The denominator stays away from zero by the circle value $c_q = -\sin a/2$. Thus the two equations in α and δh have coefficient matrix

$$\begin{pmatrix} [\mathcal{T}_\gamma p_2]_2 & [A_h + \mathcal{T}_\gamma p_h]_2 \\ \Lambda_\gamma p_2 & \Lambda_\gamma p_h - 1 \end{pmatrix} \tag{6.8}$$

and right side $\big([g_A - \mathcal{T}_\gamma \psi_b]_2, g_q^0 - \Lambda_\gamma \psi_b\big)$. The circle limit of Λ_γ is $\Lambda_* \psi = h_* \operatorname{av} \psi$.

We now compute the leading terms. The bounds for the section operator give $\mathcal{T}_\gamma = S_{h_*} + O^\#(\tau)$ across fixed gaps, by (4.5) and the change of height from h_* to $\cos a$ in the reference transform. Since $v_2 = f + O^\#(\tau)$ and $S_{h_*} f = 0$, $\mathcal{T}_\gamma v_2 = O^\#(\tau)$. Equation (6.5) gives $p_2 = f + o^\#(1)$. With (6.7), the radial first variation of the physical height derivative gives

$$A_h = \text{constant in } x + \varepsilon s_2'(h_*) f + O^\#(\tau^2). \tag{6.9}$$

For example, at a reference sphere the height derivative is $-2\pi q$. The radial variation is $S'_{h_*} f$ and the first change in the chord distance is constant in x. Moving the normal by $O^\#(\tau)$ changes the first-order radial contribution only at order two. To justify the expansions in the family norm, compare the actual meridian and angle with their first-order polynomial expressions from (6.6) on fixed inner strips. The shape and parameter estimates of Section 4 bound the differences by $O^\#(\tau^2)$. On the polynomial expressions ordinary Taylor expansion computes the displayed first derivative. The two exact annihilation identities and (6.5) now give $p_h = o^\#(\tau)$.

For the remaining coefficient, a normal displacement f in meridian label coordinates gives, at fixed physical polar angle, radial displacement

$$f - \varepsilon w_1 f' + O^\#(\tau^2).$$

Indeed the old label corresponding to a fixed polar angle x is $x - \varepsilon w_1(x) + O^\#(\tau^2)$, and the radial direction has dot product $1 + O^\#(\tau^2)$ with the physical exterior normal. The radial displacement is the normal displacement divided by this dot product. Use (3.4) and (6.7) in the fixed-plane shape derivative. To first order it is

$$\mathcal{T}_\gamma f = \varepsilon\big((S_{h_*} - h_* S'_{h_*})(f^2) + h_1 S'_{h_*} f - S_{h_*}(w_1 f')\big) + O^\#(\tau^2).$$

The output-coordinate variation at order zero differentiates $S_{h_*} f = 0$ and hence contributes zero. The projection $[S_{h_*} g]_2$ is zero for every g. Using $[f^2]_2 = 2/7$ and $h_1 = h_*/7$ gives

$$[\mathcal{T}_\gamma f]_2 = -\varepsilon h_1 s_2'(h_*) + O^\#(\tau^2).$$

In the first column of (6.8), replacing f by p_2 changes this coefficient by $o^\#(\tau)$ because its reference projection vanishes and $\mathcal{T}_\gamma - S_{h_*} = O^\#(\tau)$. The term $[\mathcal{T}_\gamma p_h]_2$ is $o^\#(\tau)$ as well. This proves

$$\begin{pmatrix} -\varepsilon h_1 s_2'(h_*) + o^\#(\tau) & \varepsilon s_2'(h_*) + o^\#(\tau) \\ h_* \mu + o^\#(1) & -1 + o^\#(1) \end{pmatrix}. \tag{6.10}$$

Its determinant is $-\varepsilon s_2'(h_*)(h_*\mu - h_1) + o^{\#}(\tau) = -3\pi\varepsilon/14 + o^{\#}(\tau)$. Its absolute value is bounded below by $c\tau$. The inverse matrix thus has a polynomial bound in τ^{-1}, also for the family norm by the quotient formula for its determinant.

Solve this matrix equation, choose the constant section output as $\delta\nu$, and solve

$$(T_a - T_{-a})w = (g_q - \zeta_\psi + \delta h)/c_q.$$

The right side is even with zero mean. Indeed reflection in x interchanges the two endpoints, reverses the signs of t_+ and t_-, and leaves c_q and ζ_ψ even. All these functions are π-periodic by their central parity. The homogeneous divisor condition gives an odd solution with fixed strip losses. Reconstruction by $\delta\gamma = \psi N + w\gamma'$ uses the same analytic bounds.

For the expansions in (6.10), take a finite sequence of fixed strips strictly below the limiting working width. They are used only to estimate the scalar entries and their inverses. To construct the functions ψ, w from the sources at a given Newton step, the same formulas use the variable strip gap of that step. This gives uniform constants without a cumulative fixed loss. □

7. Parameter exclusion and Newton convergence

7.1. **Measure of the excluded parameters.**

Lemma 7.1. *The homogeneous conditions* (5.10) *exclude a subset of I_τ of measure $o(\tau)$. For a diagonal satisfying* (5.14) *on a closed subset $G \subset I_\tau$ and a sufficiently large index l, one has*

$$\left|\{a \in G : |d_l(a)| < \Gamma(1+l)^{-2}\}\right| \le C\Gamma(\tau + k_l^{-1})(1+l)^{-2} \tag{7.1}$$

whenever $\Gamma(1+l)^{-2}$ is sufficiently small.

Proof. By Lemma 2.1, $|2\sin(ma_*)| \ge 2\gamma_*(1+|m|)^{-\tau_*}$. For $a \in I_\tau$ the left side changes by at most $4|m|\tau$. Taking a sufficiently small constant in $J_\tau \asymp \tau^{-1/(\tau_*+1)}$ and $\gamma_0 \le \gamma_*$ makes every $m \le J_\tau$ safe. For a single higher positive m, the interval I_τ meets at most $C(m\tau+1)$ zero neighborhoods. Each has length at most $C\gamma_0 m^{-1}(1+m)^{-\tau_h}$. Summing gives

$$|I_\tau \setminus \mathcal{G}_h| \le C\gamma_0(\tau J_\tau^{1-\tau_h} + J_\tau^{-\tau_h}) = o(\tau),$$

since $\tau_h > \tau_* + 1$.

For the second assertion extend the real ϕ_l linearly across complementary intervals of G, and constantly to the endpoints of I_τ if needed. The Lipschitz constant is still at most C/k_l. Thus $k_l a + \phi_l(a)$ is increasing, with difference quotients in $[k_l/2, 2k_l]$ for large l. The phase range meets at most $C(k_l\tau+1)$ neighborhoods of zeros of cosine. By the lower bound on A_l, each has phase length at most $C\Gamma(1+l)^{-2}$, hence parameter length at most $C\Gamma k_l^{-1}(1+l)^{-2}$. This proves (7.1). Only the real phase is extended for this estimate. □

7.2. **The iteration.**

Proof of Theorem 1.2. We first fix the arithmetic constants in (5.10), the finite reduction order, the index s_0, and a small positive width r_0' with a larger initial strip available. The first formal perturbation in (3.2) has size $O^{\#}(\tau)$. The fixed-gap difference estimates in Sections 4 and 5, and Lemma 6.1, give a bound $C_0\tau$ for the perturbed data on width r_0'. The data kept at a step consist of the geometric denominators and factors in Section 4, including A_h, the double amplitude, the coefficients and matrices of the finite reduction, and the Schur rows, columns and remainder. In particular we include B_b in its order-zero matrix norm. Its increment bound is the order-zero case of the exact multiplier estimates in Lemma 5.1. It therefore gives an estimate at the working width, while $D^{-1/2}$ and the changes of coefficient basis give the stated strip estimates for $\mathcal{T}_\gamma$ in angular coordinates. The reference values are taken at the circle with the same a. The bound $C_0\tau$ also holds for higher-order formal seeds for sufficiently small τ, since their remaining terms have size $O^{\#}(\tau^2)$ on the larger initial strip, for any fixed seed order. Increase C_0 once to include the leading meridian perturbation. Fix M as in Proposition 6.2 to handle the bound $2C_0\tau$. Fix the strip losses and the large exponent used below, then choose the order N of the formal seed. Finally shrink τ.

Let $z_0 = (\gamma_0, h_0^{\mathrm{seed}}, \nu_0)$ be this seed, expressed as a polynomial in $a - a_*$. By Lemma 3.1, its residual on width r_0' is $O^{\#}(\tau^{N+1})$. Indeed a Taylor remainder of order $N+1$ on a fixed neighborhood has a first parameter derivative of order N, which is multiplied by τ in the family norm. The seed is in the regime (6.6) with $\varepsilon = (a - a_*)/a_1$. For $j \geq 0$ set

$$\Gamma_j = M\tau(1 + 2^{-j}). \tag{7.2}$$

On $\mathcal{G}_h$ form the seed diagonal $d_{0,l}$ and define the compact set

$$G_0 = \{a \in \mathcal{G}_h : |d_{0,l}(a)| \geq \Gamma_0(1+l)^{-2} \text{ for all even } l \geq 4\}.$$

Every fixed even $l \geq 4$ is safe for all sufficiently small τ, by Lemma 2.1 and (5.13). For the larger indices, the right side of (7.1), divided by τ, is bounded by $CM((1+l)^{-2} + (1+l)^{-3})$. This sequence is summable. Combining the fixed-index observation, this tail estimate, and Lemma 7.1 gives

$$|I_\tau \setminus G_0| = o(\tau). \tag{7.3}$$

Take $\delta_j = d/(j+1)^2$, with $d > 0$ so small that $4\sum_j \delta_j < r_0'/2$. Let r_j be the working widths, starting at r_0' and with $r_{j+1} = r_j - 4\delta_j$. Their limit r_∞ is positive. Choose the fixed inner strips for the scalar calculations of Proposition 6.3 below $r_\infty/4$. At step j the meridian, residual and data are measured on r_j and are defined for $a \in G_j$. The perturbative bounds are at most $2C_0\tau$, the first-order regime (6.6) holds, and $|d_{j,l}| \geq \Gamma_j(1+l)^{-2}$ for even $l \geq 4$. Write $e_j = \|F(z_j)\|^{\#}_{r_j, G_j}$.

Use the right inverse of Proposition 6.3, with one gap δ_j, on

$$(-F_q, -F_A + A_h F_q, -F_W).$$

Let $v_j = (\delta\gamma, \delta h, \delta\nu)$ be the resulting correction, and put $z_{j+1} = z_j + v_j$ on G_j. In the update the row operation is frozen at z_j. By Lemma 4.2, the error in the row-transformed linear cancellation is a sum of terms containing one residual derivative and $\delta\theta$. The latter is a linear combination of $\delta\gamma(x \pm a)$ with uniformly bounded geometric coefficients. Differentiation of the residual and the family loss in translating $\delta\gamma$ fit in a further gap δ_j. Undoing the frozen row operation multiplies by the bounded A_h. Across an additional gap, the Taylor remainder is bounded by (4.9). Thus before any restriction of parameters the new residual at width $r_j - 3\delta_j$ is bounded by

$$C\delta_j^{-P}\big(e_j\|v_j\|^{\#}_{r_j-\delta_j,G_j} + (\|v_j\|^{\#}_{r_j-\delta_j,G_j})^2\big).$$

The old tuple and v_j are both available on $r_j - \delta_j$. The local increment estimates, applied with a gap at most $3\delta_j$, control all the new tuple on r_{j+1}. This reuses the old bounds for the geometric quotients, the diagonal and the remainder. Applied to $\Delta\gamma$, the estimates also control $\sup_k k|\Delta M_+(k,a)|^{\#}_{G_j}$. The exact definition (5.13) gives the corresponding low-mode change.

Enlarging one fixed exponent P and constant C, put $B_j = C\tau^{-P}\delta_j^{-P}$. Let Δ_j be the maximum of the function, matrix and branch norms of the data increments on r_{j+1} and G_j. The estimates above give

$$\|v_j\|^{\#}_{r_j-\delta_j,G_j} + \Delta_j \le B_j e_j, \qquad e_{j+1} \le B_j e_j^2, \tag{7.4}$$

provided the correction is small enough for the local estimates. We increase B_j so that $B_j e_j$ also bounds the quantities required to be small for these estimates. The new residual estimate holds on G_j before it is used on G_{j+1}.

To see the bounds close with the prescribed widths, let $\kappa > 4$ be fixed and large. Since $\log B_j \le C' + P\log(1/\tau) + 2P\log(j+1)$,

$$\sum_{j=0}^{\infty} 2^{-j-1}\log B_j \le C'' + P\log(1/\tau), \qquad \sup_j 2^{-j}\log B_j \le C'' + P\log(1/\tau).$$

Taking logarithms in (7.4), or replacing e_j by its positive recursive upper bound when it vanishes, gives

$$\log e_j \le 2^j\left(\log e_0 + \sum_{m<j} 2^{-m-1}\log B_m\right).$$

Choose N so large that the right side, even after adding $\log B_j$, is at most $-\kappa 2^j\log(1/\tau)$ for all j and small τ. Consequently the residual, correction and each data increment in (7.4) are bounded by

$$\tau^{\kappa 2^j}. \tag{7.5}$$

The estimates apply inductively because this bound makes the local smallness hypotheses hold at every step. Their sum is $O(\tau^\kappa)$. Thus the original

tuple of size $C_0\tau$ remains bounded by $2C_0\tau$ as the width decreases. The non-vanishing margins persist, the regime (6.6) holds on the fixed inner strips, and the determinant estimate in (6.10) is uniform. This also proves uniform phase bounds (5.14) on G_j for the new diagonal.

Define $G_{j+1} \subset G_j$ by $|d_{j+1,l}| \geq \Gamma_{j+1}(1+l)^{-2}$ for every even $l \geq 4$. These sets are closed by the parameter estimates. To bound the new loss, (7.5) gives

$$\sup_{G_j} |d_{j+1,l} - d_{j,l}| \leq C\tau^{\kappa 2^j}(1+l)^{-1}. \tag{7.6}$$

The bound uses (5.12) at high indices and (5.13) on the remaining finite set. Put $K_j = \lfloor \tau^{-\kappa 2^j/4} \rfloor$. For $l \leq K_j$, multiplying (7.6) by $(1+l)^2$ gives at most $C'\tau^{3\kappa 2^j/4}$. This is less than $\Gamma_j - \Gamma_{j+1} = M\tau 2^{-j-1}$ for all j when τ is small. No new parameter can therefore be lost to a mode $l \leq K_j$. For the other modes apply (7.1) to $d_{j+1,l}$ on G_j and sum the tail:

$$|G_j \setminus G_{j+1}| \leq CM\big(\tau^2/K_j + \tau/K_j^2\big).$$

The sum over j is $o(\tau)$. Together with (7.3) this proves that $G_\tau = \bigcap_{j\geq 0} G_j$ is compact and that its complement in I_τ has measure $o(\tau)$.

For $a \in G_\tau$ the iterates converge on a common positive strip since the correction norms are summable and the widths are bounded below by r_∞. The formulas of Section 4 give continuity of F to a smaller strip. There $F(z) = 0$ since the residuals tend to zero. In particular the limiting volume is $4\pi/3$. We verify explicitly that the limiting meridian gives the claimed body. Its polar angle equals $x + \omega(x)$ where

$$\omega(x) = \frac{1}{2i}\log\left(e^{-2ix}\frac{X(x)+iY(x)}{X(x)-iY(x)}\right)$$

is small and analytic on a fixed inner strip. The logarithm uses the branch at one. The inverse angle is obtained by solving $x = y - \omega(x)$. On a smaller fixed strip, uniform smallness of ω and ω' gives a contraction in a disk of fixed radius about each y. The iterates converge holomorphically and periodically, and $\rho(y) = (X^2+Y^2)^{1/2}(x(y))$. This also gives uniform control of differences of radial functions across fixed strips, by subtracting the two fixed point equations.

On the real upper meridian, $Y/\sin x > 0$ and $X'/\sin x < 0$ by parity, division at the simple zeros and closeness to the circle. The signed meridian curvature $(X'Y'' - Y'X'')/S^3$ is positive by C^2-closeness. Thus the radius Y as a function of the axial coordinate X is concave: its second derivative is $(X'Y'' - Y'X'')/(X')^3 < 0$. The corresponding family of disks of concave radius defines a convex body by the triangle inequality. At each pole Y has a simple zero and X is even about the pole, so locally X is analytic in Y^2. Rotation yields a real-analytic surface there. The rotational principal curvature is $-X'/(SY) > 0$, with a positive limit at each pole by the removable

ratios just described. The principal curvatures remain positive on the entire surface. The parity of γ gives origin symmetry.

The normal angle θ has derivative close to one and covers all meridional directions. Rotation about the axis therefore gives every unit normal in S^2. Since $q = h_a$ and $A_\gamma = \nu_a$, all sections in the theorem have the asserted area. The reference value of the area is $\pi(1 - h_*^2) > 0$. The total correction is $O(\tau^\kappa)$, so the first-order expansions of the seed give (1.1), including the indicated Legendre normalization of ε_a. The nonzero degree-two coefficient and origin symmetry imply the body is nonspherical. Finally $h_a \to h_* < 1$ and $\rho_a \to 1$ uniformly, giving strict containment of $h_a B$. □

8. Comparison with a ball

Use $\langle g \rangle = (4\pi)^{-1} \int_{S^2} g \, \mathrm{d}\omega$ for the spherical mean. For a body with radial function $\rho > h > 0$, averaging the parallel section areas gives

$$\frac{1}{4\pi} \int_{S^2} A_\rho(u, h) \, \mathrm{d}u = \pi\big(\langle \rho^2 \rangle - h^2\big). \tag{8.1}$$

This is the three-dimensional case of [12, Lemma 4.2, (4.4)]. To see the normalization directly, integrate the kernel in (3.3) over the normals u. For $r > h$, this integral is $2\pi/r$ by the coordinate $t = u \cdot \omega$, and for $r < h$ it is zero. The formula follows by integrating $2\pi r$ over $[h, \rho(\omega)]$ and averaging in ω. The same calculation follows by differentiating the directional average of the cap volume, whose integrand for fixed r is $2\pi(1 - h/r)_+$. This also justifies the exchange of integrals in the distributional formula.

Proof of Theorem 1.1. For a body from Theorem 1.2, set

$$R_a = (h_a^2 + \nu_a/\pi)^{1/2} = \langle \rho_a^2 \rangle^{1/2}.$$

The comparison ball $R_a B$ has section area ν_a in each plane $H(u, h_a)$, and $R_a > h_a$. Since ρ_a is nonconstant, the strict convexity of $s \mapsto s^{3/2}$ gives

$$R_a^3 = \langle \rho_a^2 \rangle^{3/2} < \langle \rho_a^3 \rangle = 1.$$

Thus $\mathrm{vol}(R_a B) < 4\pi/3 = \mathrm{vol}(K_a)$. Dilate both bodies by t/h_a. Their areas in corresponding sections scale by $(t/h_a)^2$, and the cutting sphere becomes tB. The dilation preserves the strict containment, regularity and volume ordering. Because $\rho_a \to 1$ on a common positive strip, their radial functions approach each other in every fixed C^m norm after dilation. □

The leading size of the volume discrepancy follows from the same identity. Write $\rho_a = 1 + \varepsilon_a f + g$ with $g = O(\varepsilon_a^2)$. Expanding $\langle \rho_a^3 \rangle = 1$ gives $\langle g \rangle = -\varepsilon_a^2/5 + O(\varepsilon_a^3)$. It follows from (8.1) that

$$\begin{aligned} \nu_a &= \pi(1 - h_a^2) - \frac{\pi}{5}\varepsilon_a^2 + O(\varepsilon_a^3), \\ R_a &= 1 - \frac{1}{10}\varepsilon_a^2 + O(\varepsilon_a^3). \end{aligned} \tag{8.2}$$

AI use declaration

OpenAI's ChatGPT and Codex were used to develop and draft the argument and to revise the manuscript. Their mathematical contributions include the specialization of the Legendre root calculation in Section 2, the section-area construction and its linearization in Sections 3–4, and the finite-order reduction, inverse estimates and Newton argument in Sections 5–7. They also assisted in checking the calculations and references in the present revision.

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