Growth criteria for reciprocal series with nonincreasing weights

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Abstract

Fix nonincreasing positive integer weights $w_0, \dots, w_{d-1}$. Define $c > 1$ by

$$(c-1)^2 \sum_{j=0}^{d-1} w_j c^j = w_0.$$

If the total weight is at least two, we prove irrationality of $\sum_{n\geq 1} 1/(a_n^{w_0} \cdots a_{n+d-1}^{w_{d-1}})$ whenever the positive integers a_n are strictly increasing and $\limsup_{n\to\infty} (\log a_n)/c^n = \infty$. For every nonconstant nonincreasing weight profile, this lowers the sufficient base in the general criterion of Barreto, Kang, Kim, Kovač and Zhang [1, Theorem 3], with the classical base-2 criterion included in the comparison. For weights $(3, 2, 1)$ the base becomes $\sqrt[3]{4}$; staircase weights $(d, d-1, \dots, 1)$ give $(d+1)^{1/d}$. Irregular growth is allowed. A more general form permits nondecreasing sequences and positive integer numerators under polynomial size bounds.

1 Introduction

Growth criteria for irrationality apply to families of reciprocal sums without requiring a closed form for the sum or information about the prime factors of the denominators. Their arithmetic starting point is simple: if the whole sum were rational, a positive tail could not become arbitrarily small after clearing the preceding partial sum. For products of overlapping sequence terms, the powers needed to clear that partial sum depend on the position of each factor. Keeping this dependence improves the growth condition.

Barreto, Kang, Kim, Kovač and Zhang [1] answered a question of Erdős and Graham [3, p. 64] about reciprocal products of consecutive terms and established criteria for general weight profiles. Their sufficient conditions and rational constructions leave a gap for many profiles with repeated factors [1, Section 5]. An earlier paper [4] improved their sufficient base from 2 to $\sqrt{3}$ for weights $(2, 1)$. Here we extend the endpoint argument to every nonincreasing positive weight profile. A positive combination of cumulative logarithms is what makes the extension work for irregular growth.

1.1 The result

Throughout the paper, $d \geq 1$ and

$$w_0 \geq w_1 \geq \cdots \geq w_{d-1} \geq 1 \tag{1.1}$$

are fixed integers. Write $s = \sum_{j=0}^{d-1} w_j$ and $f_{\mathbf{w}}(x) = \sum_{j=0}^{d-1} w_j x^j$. Define $c = c_{\mathbf{w}} > 1$ by

$$(c-1)^2 f_{\mathbf{w}}(c) = w_0. \tag{1.2}$$

The left-hand side is strictly increasing from zero to infinity for $c > 1$, so the definition is unambiguous. All logarithms are natural.

Theorem 1.1. *Suppose $s \geq 2$. If $(a_n)_{n\geq 1}$ is a strictly increasing sequence of positive integers and*

$$\limsup_{n\to\infty} \frac{\log a_n}{c^n} = \infty, \tag{1.3}$$

then

$$\sum_{n=1}^{\infty} \frac{1}{a_n^{w_0} a_{n+1}^{w_1} \cdots a_{n+d-1}^{w_{d-1}}} \tag{1.4}$$

is irrational.

Condition (1.3) is equivalent to $\limsup a_n^{1/c^n} = \infty$. It permits slow stretches and abrupt jumps: no ratio limit for successive logarithms is assumed. We prove the following form, which also permits repetitions and positive integer numerators.

Theorem 1.2. *Let (a_n) be nondecreasing positive integers and (b_n) positive integers. Suppose that, for some real numbers $0 \leq \eta < \tau$,*

$$b_n \leq n^{\eta}, \qquad x_n := \prod_{j=0}^{d-1} a_{n+j}^{w_j} \geq n^{1+\tau} \tag{1.5}$$

for all sufficiently large n. If (1.3) holds, then the convergent series $\sum_{n\geq 1} b_n/x_n$ is irrational.

For Theorem 1.1, take $b_n = 1$, $\eta = 0$ and $\tau = s - 1$, using $a_n \geq n$. The case $d = 1, w_0 = 1$ has $c = 2$ and retains the size hypothesis in Theorem 1.2; strict increase alone is not substituted for that hypothesis. The polynomial bounds in (1.5) control tails where the sequence grows slowly.

1.2 What improves

Let $\rho_{\mathbf{w}}$ be the positive root used in [1, Theorem 3]. For the profiles (1.1), it solves

$$(x-1)f_{\mathbf{w}}(x) - w_0 x^{d-1} = 0. \tag{1.6}$$

The classical base-2 criterion of Erdős [2, Theorem 1] also applies to the integer product denominators with unit numerators; the same fallback for the general setting follows from [1, Theorem 3, $d = 1$]. Thus the relevant comparison is with $\min(\rho_{\mathbf{w}}, 2)$, as emphasized in [1, Remark 4(4)]. We prove in Proposition 4.1 that

$$c_{\mathbf{w}} < \min(\rho_{\mathbf{w}}, 2) \tag{1.7}$$

for every nonconstant profile satisfying (1.1). Constant profiles recover the earlier base.

Weights	Previous base $\min(\rho_{\mathbf{w}}, 2)$	Base in this paper
$(2, 1)$	2	$\sqrt{3}$
$(3, 2)$	$(1+\sqrt{7})/2$	$(1+\sqrt{33})/4$
$(3, 2, 1)$	2	$\sqrt[3]{4}$

The first row recovers [4], as do the profiles $(2k, k)$ after replacing a_n by a_n^k. For staircase weights $(d, d-1, \ldots, 1)$, equation (1.2) reduces to

$$c = (d+1)^{1/d}. \tag{1.8}$$

For example, $a_n = \lceil \exp((8/5)^n) \rceil$ is covered for weights $(3, 2, 1)$ because $8/5 > \sqrt[3]{4}$, while $(\log a_n)/2^n \to 0$. More generally, any regular growth rate $\log a_n \sim \theta^n$ with $c_{\mathbf{w}} < \theta < \min(\rho_{\mathbf{w}}, 2)$ distinguishes the two criteria. These examples illustrate the improvement; the theorem does not require regular growth. The optimal thresholds remain open in general.

1.3 Why the endpoint matters

For weights $(3, 2, 1)$, the first N terms are cleared by

$$D_N = \left(\prod_{i=1}^{N} a_i^3 \right) a_{N+1}^2 a_{N+2}.$$

The last two factors need powers two and one, respectively, rather than three. With $u_n = \log a_n$ and $V_N = \sum_{i=1}^{N} u_i$, this gives

$$\log \frac{D_N}{a_{N+1}^3 a_{N+2}^2 a_{N+3}} = 4V_N - V_{N+3}.$$

The recurrence $V_{N+3} = 4V_N$ explains the cube-root base. The harder question is whether the saving persists for the entire tail when growth is irregular. For general nonincreasing weights, we construct a positive combination W_N of $V_N, \dots, V_{N+d-1}$ for which the same comparison becomes $cW_N - W_{N+1}$. A carefully selected first peak then controls the near tail, and a later large term controls the far tail using the same denominator.

We adapt the positive-integer tail argument, uniform tail estimates, peak selection and fast/slow splitting in [1, Lemmas 8–10 and Proposition 12], whose antecedents include Erdős's arguments [2, pp. 3–6]. The endpoint refinement and cumulative-log approach for $(2, 1)$ come from [4]. The new ingredient is the positive factorization for every profile (1.1), together with control of its forward shifts at the last slow index. The proof below supplies all required estimates.

2 Uniform tail estimates

We need one estimate valid under polynomial growth and a sharper one on intervals of exponential growth. Both intervals will move during the proof, so their constants must be independent of the endpoints. These are the relevant forms of [1, Lemma 10].

Lemma 2.1. *Let (x_n) be nondecreasing positive real numbers, and fix $0 \le \eta < \tau$. Let A be a positive integer and suppose $0 \le y_n \le n^\eta$ on the indices being summed.*

1. *If $x_n \ge n^{1+\tau}$ for all $n \ge A$, then*

$$\sum_{n \ge A} \frac{y_n}{x_n} \le C_{\mathrm{pol}} x_A^{-\alpha}, \qquad \alpha = \frac{\tau - \eta}{1 + \tau} > 0. \tag{2.1}$$

   Here C_{pol} depends only on η, τ.

2. *If $x_n \ge e^n$ for $A \le n \le B$, where $B \ge A$ is an integer or $B = \infty$, then*

$$\sum_{n=A}^{B} \frac{y_n}{x_n} \le C_{\exp} \frac{(\log x_A)^{1+\eta}}{x_A}. \tag{2.2}$$

   Here $C_{\exp}$ depends only on η.

Neither constant depends on the endpoints or on the sequences. No numerator bound before A is required.

Proof. Put $X = x_A$ and group indices by $2^j X \le x_n < 2^{j+1} X$, $j \ge 0$. In the first case every index in the jth group is at most $(2^{j+1}X)^{1/(1+\tau)}$. Summing n^η up to that bound shows that the group's contribution is at most

$$C \frac{(2^{j+1}X)^{(1+\eta)/(1+\tau)}}{2^j X} \le C' X^{-\alpha} 2^{-\alpha j}.$$

The geometric series converges, proving (2.1). In the second case, every index in the jth group is at most $\log X + (j+1)\log 2$. Its contribution is at most

$$C\,\frac{(\log X + j + 1)^{1+\eta}}{2^j X}.$$

Since $\log X \geq A \geq 1$, summing over j proves (2.2). The same estimate applies if a finite interval ends partway through a group. □

3 Proof of the irrationality criterion

3.1 Endpoint clearing and a positive combination

Retain x_n from (1.5), and put $u_n = \log a_n$, $V_0 = 0$, $V_N = \sum_{i=1}^{N} u_i$. For $N \geq 1$, define

$$D_N = \left(\prod_{i=1}^{N} a_i^{w_0}\right) \prod_{r=1}^{d-1} a_{N+r}^{w_r}, \qquad E_N = \log D_N, \qquad F_N = \log x_{N+1}. \tag{3.1}$$

The integer D_N clears every term through N. Factors of index at most N are allotted the largest weight w_0. At index $N + r > N$, an earlier term can use only a weight w_j with $j \geq r$, hence $w_j \leq w_r$. Thus divisibility holds even when some a_i coincide or share prime factors. We do not claim that D_N is reduced.

Set $w_d = 0$ and

$$\lambda_r = w_{r-1} - w_r \quad (1 \leq r \leq d).$$

Then $\lambda_r \geq 0$, $\lambda_d = w_{d-1} > 0$, and direct subtraction gives

$$E_N - F_N = w_0 V_N - \sum_{r=1}^{d} \lambda_r u_{N+r}, \tag{3.2}$$

$$E_{N+1} - E_N = \sum_{r=1}^{d} \lambda_r u_{N+r} \leq F_N. \tag{3.3}$$

To use (3.2) at peaks, we need a positive combination of cumulative logarithms. Introduce

$$q_{\mathbf{w}}(x) = (x-1)\sum_{r=1}^{d} \lambda_r x^{r-1} - w_0.$$

The identity

$$x q_{\mathbf{w}}(x) = (x-1)^2 f_{\mathbf{w}}(x) - w_0 \tag{3.4}$$

shows that $q_{\mathbf{w}}(c) = 0$ for the base in (1.2). Division by this root has a useful sign property:

$$\begin{aligned} q_{\mathbf{w}}(x) &= (x-c)H(x), \\ H(x) &= \sum_{r=1}^{d} \lambda_r \left[x^{r-1} + (c-1)\sum_{k=0}^{r-2} c^{r-2-k} x^k\right] = \sum_{j=0}^{d-1} h_j x^j. \end{aligned} \tag{3.5}$$

The inner sum is empty for $r = 1$. To verify the formula, subtract the value at c from each $(x-1)x^{r-1}$, divide by $x - c$, and use $q_{\mathbf{w}}(c) = 0$. Every h_j is positive: the term with $r = d$ supplies all lower degrees, and $h_{d-1} = \lambda_d = w_{d-1} > 0$. For $d = 1$ the formula simply gives $H = w_0$.

Define

$$W_N = \sum_{j=0}^{d-1} h_j V_{N+j}, \qquad Z_N = \frac{W_N}{c^N}. \tag{3.6}$$

Substituting $u_{N+r} = V_{N+r} - V_{N+r-1}$ in (3.2) and using (3.5) gives the exact identity

$$E_N - F_N = cW_N - W_{N+1}. \tag{3.7}$$

This identity preserves the endpoint saving and allows us to compare positive quantities at successive peaks.

We also need to control the forward shifts in W_N. Since all u_i are nonnegative and V_i is nondecreasing, with $H_* := \sum_j h_j$,

$$\begin{aligned} w_{d-1}V_{N+d-1} \le E_N \le w_0 V_{N+d-1}, \\ w_{d-1}V_{N+d-1} \le W_N \le H_* V_{N+d-1}. \end{aligned} \tag{3.8}$$

For the first line one may also use $E_N = \sum_r \lambda_r V_{N+r-1}$. Writing $C_E = w_0/w_{d-1}$ and $C_W = H_*/w_{d-1}$, we obtain

$$E_N \le C_E W_N, \qquad W_N \le C_W E_N, \qquad \log x_N \le w_0 V_{N+d-1} \le C_E W_N. \tag{3.9}$$

These constants depend only on the fixed weights.

3.2 The jump bound forced by rationality

Suppose for a contradiction that $S = \sum b_n/x_n = p/q$, where q is a positive integer. The series converges by (1.5). Put $T_N = \sum_{n>N} b_n/x_n$. Since D_N clears the first N terms and the tail is positive, $qD_N T_N$ is a positive integer. Thus

$$D_N T_N \ge 1/q \qquad (N \ge 1). \tag{3.10}$$

This is the elementary criterion in [1, Lemma 8].

The denominators x_n are nondecreasing. Applying (2.1) to T_N and combining it with (3.10) yields, for all sufficiently large N,

$$\alpha F_N \le E_N + \log(C_{\mathrm{pol}} q).$$

Monotonicity and (1.5) imply $a_n \to \infty$, so $E_N \to \infty$. Consequently,

$$F_N \le \frac{2}{\alpha} E_N, \qquad E_{N+1} \le A E_N, \qquad A = 1 + \frac{2}{\alpha} \tag{3.11}$$

for all large N, the second inequality following from (3.3). This coarse bound prevents an uncontrolled jump immediately after a slow index under the rationality assumption. All constants below may depend on the fixed weights, η, τ, the sequence and q, but not on the indices chosen later.

3.3 Peaks and eventual exponential growth

By (1.3) and $h_0 > 0$, the sequence Z_N is unbounded. There are therefore infinitely many indices Q such that

$$Z_Q > (1 + Q^{-2}) \max(1, Z_1, \dots, Z_{Q-1}). \tag{3.12}$$

Otherwise the successive running maxima would eventually be bounded by a fixed constant times

$$\prod_{j \ge 1} (1 + j^{-2}) \le \exp\left(\sum_{j \ge 1} j^{-2} \right) \le e^2. \tag{3.13}$$

This is the summable-increment peak argument used in [1, Lemma 9]. At any such peak, (3.7) gives

$$E_{Q-1} - F_{Q-1} = cW_{Q-1} - W_Q \le -\frac{W_Q}{Q^2+1}, \qquad W_Q > c^Q. \tag{3.14}$$

Suppose first that $a_n \ge e^n$ for every sufficiently large n. Then $x_n \ge a_n^{w_0} \ge e^n$ there. At large peaks Q, (2.2), (3.14) and (3.9) give

$$\log(D_{Q-1}T_{Q-1}) \le -\frac{W_Q}{Q^2+1} + (1+\eta)\log(C_E W_Q) + O(1).$$

For large Q, the right-hand expression is a decreasing function of W_Q throughout $W_Q \ge c^Q$. It is therefore at most

$$-\frac{c^Q}{Q^2+1} + (1+\eta)(Q\log c + \log C_E) + O(1),$$

which tends to minus infinity. This contradicts (3.10). Notice that no upper bound on the size of a peak was needed in this case.

3.4 A first peak after a slow index

It remains to allow $a_n < e^n$ at arbitrarily large indices. Choose R tending to infinity with

$$\mu_R = \frac{u_R}{c^R} \longrightarrow \infty. \tag{3.15}$$

For large R one has $a_R \ge e^R$. Let $P = P(R) < R$ be the last index before R with $a_P < e^P$. For every fixed slow index t, the selected R eventually exceed t, and then $P(R) \ge t$. Thus $P(R) \to \infty$, and

$$a_n \ge e^n \qquad (P < n \le R). \tag{3.16}$$

The quantity W_P involves cumulative sums as far as $P+d-1$, whose size is not supplied by $a_P < e^P$. This is where the jump estimate (3.11) is needed. All factors of D_{P-d+1} have index at most P, so, once P is large enough,

$$E_{P-d+1} \le w_0 V_P \le w_0 P u_P < w_0 P^2.$$

Applying (3.11) exactly $d-1$ times, and then (3.9), gives

$$E_P \le w_0 A^{d-1} P^2, \qquad W_P \le C_W w_0 A^{d-1} P^2. \tag{3.17}$$

For $d = 1$ there are no jump steps. In every case $Z_P < 1$ for large P.

Put $J = R - d + 1$. The last term in W_J gives

$$Z_J \ge \frac{w_{d-1}u_R}{c^{R-d+1}} = w_{d-1}c^{d-1}\mu_R \longrightarrow \infty. \tag{3.18}$$

We have $J > P$ eventually. Indeed, otherwise $P < R \le P + d - 1$, so $W_P \ge w_{d-1}u_R$. Equation (3.17) would imply $\mu_R \le CP^2/c^R \to 0$, contradicting (3.15).

Choose the first $Q \in [P+1, J]$ for which

$$Z_Q > (1+Q^{-2})\max(1, Z_P, \dots, Z_{Q-1}). \tag{3.19}$$

It exists for all large R: if no such peak occurred, the product bound (3.13) would give $Z_J \le e^2$, contrary to (3.18). Because this is the first peak, all preceding running maxima are bounded by $L = e^2$. In particular,

$$E_{Q-1} \le C_E L c^Q, \qquad W_Q > c^Q, \qquad Q \le R - d + 1 \le R. \tag{3.20}$$

Also $Q \to \infty$, since $Q \ge P + 1$ and $P \to \infty$. The saving (3.14) remains valid because $Q - 1 \ge P$.

3.5 Controlling the same cleared tail on both sides

Split the tail at R, retaining one multiplier:

$$D_{Q-1}T_{Q-1} = D_{Q-1}\sum_{n=Q}^{R}\frac{b_n}{x_n} + D_{Q-1}T_R. \tag{3.21}$$

For the near part, $x_n \geq a_n^{w_0} \geq e^n$ on $Q \leq n \leq R$. Only this first factor is needed; other factors of x_n may lie beyond the fast stretch. By (3.11) and (3.20),

$$F_{Q-1} \leq \frac{2C_E L}{\alpha}c^Q.$$

The finite-interval estimate (2.2) and the saving (3.14) now imply

$$\begin{aligned}\log\left(D_{Q-1}\sum_{n=Q}^{R}\frac{b_n}{x_n}\right) &\leq E_{Q-1} - F_{Q-1} + (1+\eta)\log F_{Q-1} + O(1)\\ &\leq -\frac{c^Q}{Q^2+1} + (1+\eta)Q\log c + O(1) \longrightarrow -\infty.\end{aligned} \tag{3.22}$$

The constants are independent of P, Q, R.

For the far part, monotonicity gives $x_{R+1} \geq a_R^s$. Using (2.1), (3.20) and (3.15),

$$\begin{aligned}D_{Q-1}T_R &\leq C_{\mathrm{pol}}\exp(E_{Q-1} - \alpha\log x_{R+1})\\ &\leq C_{\mathrm{pol}}\exp(C_E L c^Q - \alpha s u_R)\\ &\leq C_{\mathrm{pol}}\exp\big((C_E L - \alpha s\mu_R)c^R\big) \longrightarrow 0.\end{aligned} \tag{3.23}$$

Both terms in (3.21) tend to zero along the same selected indices, contradicting (3.10). This proves Theorem 1.2 and hence Theorem 1.1. □

4 Comparison of the growth bases

The comparisons below separate the improvement in the sufficient base from the question of sharpness. They also show that a natural class of product substitutions into the earlier theorem does not recover the new base.

Proposition 4.1. *Let $\psi_d > 1$ solve $\psi_d^d - \psi_d^{d-1} = 1$, and let $\rho_{\mathbf{w}}$ be the root in (1.6). For constant profiles, $c_{\mathbf{w}} = \rho_{\mathbf{w}} = \psi_d$. For nonconstant profiles satisfying (1.1),*

$$\psi_d < c_{\mathbf{w}} < \min(\rho_{\mathbf{w}}, 2).$$

Proof. Write $T_d(x) = 1 + x + \cdots + x^{d-1}$. For $x > 1$, $f_{\mathbf{w}}(x)/w_0 \leq T_d(x)$, strictly unless the profile is constant. At $x = \psi_d$, one has $(x-1)^2 T_d(x) = 1$. The monotonicity in (1.2) therefore gives $\psi_d \leq c$, with the asserted equality case.

At c, the polynomial in (1.6) has value

$$(c-1)f_{\mathbf{w}}(c) - w_0 c^{d-1} = w_0\left(\frac{1}{c-1} - c^{d-1}\right).$$

It vanishes for constant profiles and is negative otherwise. The uniqueness of the source's positive root [1, Remark 4(1)] gives the comparison with $\rho_{\mathbf{w}}$. Finally, for $d \geq 2$ one has $f_{\mathbf{w}}(2) > w_0$, hence $c < 2$. For $d = 1$, equation (1.2) gives $c = 2$. □

The rational examples in [1, Theorem 5] are governed by the largest positive root $r_{\mathbf{w}}$ of

$$(x-1)f_{\mathbf{w}}(x) - x^{d-1}.$$

Because every weight is at least one, this polynomial is at least $x^d - x^{d-1} - 1$ for $x > 1$. It is strictly larger there if some weight exceeds one. Thus $r_{\mathbf{w}} \le \psi_d \le c$, with $r_{\mathbf{w}} < \psi_d$ if a weight exceeds one. When all weights are one, the examples have finite $\limsup a_n^{1/\psi_d^n}$ and do not meet the infinite-limsup hypothesis here. For $(2,1)$ the familiar gap between $\sqrt{2}$ and $\sqrt{3}$ remains. For $(3,2,1)$ the rational-example polynomial is $x^3 + x - 3$, whereas the sufficient base here is $\sqrt[3]{4}$. The present theorem does not close these gaps.

One possible way to apply an earlier criterion is to replace a_n by a finite product of its shifts. The following comparison treats those substitutions explicitly.

Proposition 4.2. *Suppose*

$$f_{\mathbf{w}}(x) = V(x)G(x), \qquad V(x) = \sum_{i=0}^{k-1} v_i x^i, \qquad G(x) = \sum_{j=0}^{g} q_j x^j,$$

where the coefficients are nonnegative integers, $v_0 q_0 = w_0 > 0$, $v_{k-1} > 0$ and $q_g > 0$. Let $v_{\max} = \max_i v_i$, and let ρ_v be the sufficient base in [1, Theorem 3] for the profile $(v_0, \dots, v_{k-1})$. Then $c_{\mathbf{w}} \le \rho_v$, with equality only if both G and the coefficient profile of V are constant. In particular, the inequality is strict for every nonconstant profile $\mathbf{w}$ satisfying (1.1).

To interpret the factorization, set $y_n = \prod_{j=0}^{g} a_{n+j}^{q_j}$. Then $f_{\mathbf{w}} = VG$ says exactly that $\prod_i y_{n+i}^{v_i} = x_n$: the profile v applied to the derived sequence y_n reproduces the original denominators.

Proof. At $x = \rho_v$ one has $(x-1)V(x) = v_{\max} x^{k-1}$. Since $V(x) \le v_{\max} T_k(x)$ for $x > 1$, it follows that $x^{k-1} \le x^k - 1$, or $(x-1)x^{k-1} \ge 1$. Consequently,

$$\frac{(\rho_v - 1)^2 f_{\mathbf{w}}(\rho_v)}{w_0} = \big((\rho_v - 1)\rho_v^{k-1}\big)\frac{v_{\max}}{v_0}\frac{G(\rho_v)}{q_0} \ge 1. \tag{4.1}$$

Monotonicity in (1.2) proves $c_{\mathbf{w}} \le \rho_v$. Equality forces $\rho_v = \psi_k$, equality in $V \le v_{\max} T_k$, and $G(\rho_v) = q_0$. Hence all v_i are equal and G is constant. Conversely those conditions give equality and a constant profile $\mathbf{w}$. □

For every fixed $t > 1$, the bounds

$$q_0 \log a_n \le \log y_n \le G(1) \log a_{n+g}$$

show that $\limsup (\log y_n)/t^n = \infty$ is equivalent to the same condition for a_n; the fixed shift changes the normalization by only the factor t^g. Thus these direct applications of the earlier theorem do not recover the smaller sufficient base. This conclusion concerns the stated finite product substitutions, not every possible method of deciding irrationality for an individual series.

Verification

An independent audit found no gap in the stated theorems and propositions. This was not formal verification.

References

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