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AN EDGE COMPARISON WITH CONSTANT 3/2 AND THE COEFFICIENT $6/\sqrt{2\pi}$ FOR THE AVERAGE SENSITIVITY OF POLYNOMIAL THRESHOLD FUNCTIONS

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Abstract. For every real polynomial threshold function f of degree at most d on the uniform cube $\{-1,1\}^n$ we prove

$$\mathrm{AS}(f) \le \min\Big\{n,\ 6\,d\,b_n\,\frac{n+d}{n+2d}\Big\} \le \frac{6}{\sqrt{2\pi}}\,d\sqrt{n} < 2.394\,d\sqrt{n},$$

where $b_n = \mathbb{E}|B_n - n/2|$ for $B_n \sim \mathrm{Bin}(n,1/2)$. The previous coefficient was $8/\sqrt{2\pi} \approx 3.1915$. The new ingredient is an edge comparison with constant $3/2$ in place of 2 in OpenAI's weighted-grading argument: for every positive weight and every sign function h, the number of ordered equal-sign edges of h is at most $\frac{3}{2}\|MH+HM\|_{\mathrm{HS}}^2$. The proof rests on the exact identity $(Z_iMZ_i-M)^2 = I-A_i^2$, where A_i is the grade-diagonal part of the multiplication by x_i. Combined with OpenAI's original coupling estimate alone, the lemma already gives $\mathrm{AS}(f) \le 6d\sqrt{n}$.

1. Result

For $f : \{-1,1\}^n \to \{-1,1\}$ let $\mathrm{AS}(f) = \sum_{i=1}^n \Pr\{f(X) \ne f(X^{\oplus i})\}$, with X uniform and $X^{\oplus i}$ obtained by reversing coordinate i. OpenAI [1] proved $\mathrm{AS}(\operatorname{sgn} p) \le 8d\sqrt{n}$ for every real polynomial p of degree at most d (with $\operatorname{sgn} 0 = 1$), uniformly in $1 \le d \le n$. The probability step of that argument was then sharpened by Lagdive, Aglave and Damre to 3.83 [2] and to $8/\sqrt{2\pi}$ [3]; Anchen gave a Lean proof of the coefficient 3.762 [4], and Q. Chen gave a self-contained reconstruction of the $8/\sqrt{2\pi}$ bound with the rational witness $399/125$ [5]. All these results keep OpenAI's edge comparison [1, §4], which carries a factor 2. We replace it by 3/2.

Theorem 1. *Let $n, d \ge 1$, let p be a real polynomial of degree at most d, and let $f = \operatorname{sgn} p$ on $\{-1,1\}^n$, with $\operatorname{sgn} 0 = 1$. Then*

$$\mathrm{AS}(f) \le \min\Big\{n,\ 6\,d\,b_n\,\frac{n+d}{n+2d}\Big\} \le \frac{6}{\sqrt{2\pi}}\,d\sqrt{n} < \frac{1197}{500}\,d\sqrt{n}.$$

Corollary 2. *With OpenAI's coupling estimate* [1, §2] *in place of the compensated one, Lemma 3 gives* $\mathrm{AS}(f) \le 6d\sqrt{n}$.

2. The weighted grading

We use the notation of [1, §3]. Put $\Omega = \{-1,1\}^n$, $N = 2^n$, and work in $\mathbb{R}^\Omega$ with the counting inner product, so that the point masses δ_x form an orthonormal basis; $\|B\|_{\mathrm{HS}}^2 = \sum_{x,y} B_{xy}^2$. Let V_k be the span of the characters χ_S with $|S| \le k$. Fix $w : \Omega \to (0,\infty)$, let D be the multiplication by $\sqrt{w}$, and let Π_k be the orthogonal projection onto $E_k = DV_k \cap (DV_{k-1})^\perp$. Then $\dim E_k = \binom{n}{k}$, the E_k are mutually orthogonal and span $\mathbb{R}^\Omega$, and

$$M = \sum_{k=0}^{n} \Big(k - \frac{n}{2}\Big)\Pi_k, \qquad \|M\|_{\mathrm{HS}}^2 = \frac{nN}{4}.$$

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Let Z_i be the multiplication by x_i. It is a self-adjoint unitary commuting with D, and $\Pi_s Z_i \Pi_r = 0$ for $|s-r| > 1$ [1, §3]. For an operator X and an integer t put $X^{(t)} = \sum_k \Pi_{k+t} X \Pi_k$; then $X = \sum_t X^{(t)}$, $(XY)^{(t)} = \sum_{r+s=t} X^{(s)} Y^{(r)}$, and $X = 0$ if and only if every $X^{(t)} = 0$. Write $\rho(x,y)$ for Hamming distance, x^i for x with coordinate i reversed, and

$$A_0 = \sum_x M_{xx}^2, \qquad F = \sum_{\rho(x,y)\geq 2} M_{xy}^2,$$

all sums being over ordered pairs.

3. The edge comparison

Lemma 3. *Let $w > 0$ and $h : \Omega \to \{-1,1\}$, let H be the multiplication by h, and let $\mathcal{E}_h$ be the set of ordered pairs (x,y) with $\rho(x,y) = 1$ and $h(x) = h(y)$. Then*

$$|\mathcal{E}_h| \leq \frac{3}{2} \|MH + HM\|_{\mathrm{HS}}^2.$$

More precisely, $|\mathcal{E}_h| \leq \|MH + HM\|_{\mathrm{HS}}^2 + 4\sum M_{xy}^2$, the sum running over the ordered pairs with $\rho(x,y) \geq 3$ and $h(x) \neq h(y)$.

Proof. Step 1: an exact identity. Put $A_i = Z_i^{(0)}$ and $B_i = Z_i^{(1)}$, so that $Z_i = A_i + B_i + B_i^*$ and A_i is self-adjoint. The parts of $Z_i^2 = I$ of shift 2 and 0 give $B_i^2 = 0$ and $A_i^2 + B_i B_i^* + B_i^* B_i = I$. Since $[M, \Pi_{k+1} Z_i \Pi_k] = \Pi_{k+1} Z_i \Pi_k$, we have $C_i := [M, Z_i] = B_i - B_i^*$, hence

$$C_i^* C_i = B_i B_i^* + B_i^* B_i - B_i^2 - (B_i^*)^2 = I - A_i^2.$$

Put $G_i = Z_i C_i = Z_i M Z_i - M$. It is self-adjoint, and $G_i^2 = G_i^* G_i = C_i^* C_i$. Therefore

$$G_i^2 = I - A_i^2, \qquad (G_i)_{xy} = (x_i y_i - 1) M_{xy} = -2\,\mathbf{1}_{\{x_i \neq y_i\}} M_{xy}. \tag{1}$$

Step 2: edge deficits. The diagonal entry of (1) at x reads $4\sum_{y:\, x_i \neq y_i} M_{xy}^2 = 1 - (A_i^2)_{xx}$. Separating the neighbour $y = x^i$,

$$\frac{1}{4} - M_{x\,x^i}^2 = \frac{1}{4}\, a_i(x) + \sum_{\substack{y:\ x_i \neq y_i \\ \rho(x,y)\geq 2}} M_{xy}^2, \qquad a_i(x) := (A_i^2)_{xx} = \|A_i \delta_x\|^2 \geq 0. \tag{2}$$

Summing (2) over x and i, and using $\sum_{x,i}(\frac{1}{4} - M_{x\,x^i}^2) = \frac{nN}{4} - \sum_{\rho=1} M_{xy}^2 = A_0 + F$,

$$A_0 = \frac{1}{4}\sum_i \|A_i\|_{\mathrm{HS}}^2 + \sum_{\rho(x,y)\geq 2} (\rho(x,y) - 1) M_{xy}^2. \tag{3}$$

Step 3: counting. Call (x,i) an equal slot if $h(x) = h(x^i)$; there are $|\mathcal{E}_h|$ of them. Put $A_h = \sum_{(x,y)\in\mathcal{E}_h} M_{xy}^2$ and, for $\rho(x,y) \geq 2$, $\nu(x,y) = \#\{i : x_i \neq y_i,\ h(x^i) \neq h(x)\}$. Writing $\frac{1}{4} = M_{x\,x^i}^2 + (\frac{1}{4} - M_{x\,x^i}^2)$ on each equal slot and using (2), then $a_i \geq 0$ and (3),

$$\begin{aligned} \frac{|\mathcal{E}_h|}{4} &= A_h + \frac{1}{4}\sum_{\text{equal slots}} a_i(x) + \sum_{\rho(x,y)\geq 2} M_{xy}^2 \big(\rho(x,y) - \nu(x,y)\big) \\ &\leq A_h + A_0 + \sum_{\rho(x,y)\geq 2} M_{xy}^2 \big(1 - \nu(x,y)\big). \end{aligned}$$

On the other hand $T = MH + HM$ has entries $(h(x) + h(y)) M_{xy}$, so $\frac{1}{4}\|T\|_{\mathrm{HS}}^2 = A_0 + A_h + F_h$ with $F_h = \sum_{\rho\geq 2,\, h(x)=h(y)} M_{xy}^2$. Hence

$$\frac{|\mathcal{E}_h| - \|T\|_{\mathrm{HS}}^2}{4} \leq \sum_{\{x,y\}:\, \rho\geq 2} M_{xy}^2 \big(2 - c(x,y) - 2\cdot \mathbf{1}_{\{h(x)=h(y)\}}\big), \qquad c(x,y) = \nu(x,y) + \nu(y,x),$$

the sum running over unordered pairs. A pair with $h(x) = h(y)$ contributes at most 0. Let $h(x) \neq h(y)$ and $\rho(x,y) = 2$, with x, y differing in i, j. The edges counted by $c(x,y)$ are $\{x, x^i\}$, $\{x, x^j\}$, $\{y, y^i\} = \{y, x^j\}$ and $\{y, y^j\} = \{y, x^i\}$, the four edges of the square spanned by x and y. Each of the edge-disjoint paths x, x^i, y and x, x^j, y contains an edge on which h changes, so $c(x,y) \geq 2$ and the contribution is at most 0. A pair with $h(x) \neq h(y)$ and $\rho \geq 3$ contributes at most $2M_{xy}^2$. This proves the second assertion.

Step 4. By (3), $2\sum_{\rho\geq 3} M_{xy}^2 \leq \sum_{\rho\geq 2}(\rho-1)M_{xy}^2 \leq A_0 \leq \frac{1}{4}\|T\|_{\mathrm{HS}}^2$. Therefore $|\mathcal{E}_h| \leq \|T\|_{\mathrm{HS}}^2 + 2A_0 \leq \frac{3}{2}\|T\|_{\mathrm{HS}}^2$. □

Remark 4. The edge comparison of [1, §4] states $|\mathcal{E}_h| \leq 2\|MH + HM\|_{\mathrm{HS}}^2$. Identity (3) refines its inequality $F \leq A_0$, and (2) identifies each edge deficit exactly.

4. Proof of Theorem 1

The reduction is OpenAI's [1, §5]. After a harmless perturbation p has no zero on Ω; put $w = |p|$, $h = \chi_{[n]} f$. Then $\mathrm{AS}(f) = |\mathcal{E}_h|/N$, and the random grades (R, S) with $\Pr\{R = r, S = s\} = N^{-1}\|\Pi_s H \Pi_r\|_{\mathrm{HS}}^2$ have both marginals $\mathrm{Bin}(n, 1/2)$, satisfy $R + S \geq n - d$, and $N^{-1}\|T\|_{\mathrm{HS}}^2 = \mathbb{E}(R + S - n)^2$. With $U = S$, $V = n - R$ (same law, values in $[0, n]$, $V - U \leq d$), Lemma 3 gives

$$\mathrm{AS}(f) \leq \frac{3}{2}\,\mathbb{E}(U - V)^2.$$

The compensated coupling estimate of Lagdive, Aglave and Damre [3] (see also [5, Lemma 2]) states that equal-law variables in an interval of length L, with $V - U \leq a$ and $b = \mathbb{E}|U - \mathbb{E}U|$, satisfy $\mathbb{E}(U-V)^2 \leq 4ab\,(L+a)/(L+2a)$. With $L = n$, $a = d$, $b = b_n$ this gives $\mathrm{AS}(f) \leq 6db_n(n+d)/(n+2d)$; also $\mathrm{AS}(f) \leq n$ trivially. The estimate $b_n(n+d)/(n+2d) \leq \sqrt{n/(2\pi)}$ for all $n, d \geq 1$ is proved in [5, §6] from the Wallis inequalities; it gives the bound $\frac{6}{\sqrt{2\pi}} d\sqrt{n}$.

For the rational witness: $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{\pi}{4}$ and $\arctan x > x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$ for $0 < x \leq 1$ give $\pi > \frac{1538665}{489888}$ [5, (20)]. Hence

$$\frac{36}{2\pi} = \frac{18}{\pi} < \frac{18 \cdot 489888}{1538665} = \frac{8817984}{1538665} < \frac{1432809}{250000} = \Big(\frac{1197}{500}\Big)^2,$$

the last inequality being $8817984 \cdot 250000 = 2204496000000 < 2204613059985 = 1432809 \cdot 1538665$. □

5. Verification

The ancillary script `VERIF_AS_EDGE.py` (Python and numpy only) checks, for random weights and random sign functions with $3 \leq n \leq 7$, the identities (1), (2) and (3), the exact form of Step 3 and the inequality of Lemma 3; it checks Theorem 1 on random polynomial threshold functions, the binomial estimate in exact arithmetic for $1 \leq n \leq 2000$, and the rational witness in integer arithmetic. These checks are verifications of the statements above, not part of their proofs.

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References

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