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Hyperbolic Collision Invariants and Certified Odd–Even Tiers toward Markoff Unicity

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30 September 2026

Abstract

A collision would consist of two different ordered Markoff triples with the same largest entry M. We derive arithmetic conditions that every such collision must satisfy. A parametrization by hyperbolic functions factors a classical determinant identity and leads to two coupled quadratic equations. If M is odd, these equations require a factorization $M = FE$ with $F, E > 1$ and $\gcd(F, E) = 1$. Hand proofs and exact finite computations give increasingly strong bounds, ending with $F^4, E^4 \geq 300000M + 200000$. If M is even, division of the collision quantities by four gives the same inequalities for a coprime factorization of $M/2$. These bounds yield explicit sufficient conditions for a Markoff number to be unique. The finite computations cover a proved search domain and apply necessary congruence and continued-fraction conditions. Two different arguments reject the final odd candidate: one uses continued fractions, and the other bounds the integer solutions of two quadratic equations of Pell type. Separately written programs reproduce the computations. The constants measure the strength of collision inequalities; they are not ranges through which Markoff numbers were checked. We identify the classical ingredients and the quantitative additions. Factorizations with increasingly comparable factors remain unresolved, so the general unicity conjecture is not proved.

2020 Mathematics Subject Classification. Primary 11D25; Secondary 11D09, 11A55, 11Y50.

Keywords. Markoff numbers; Frobenius unicity conjecture; unitary divisors; continued fractions; exact finite certificate.

1 Introduction

A *Markoff triple* is a positive integral solution of

$$x^2 + y^2 + z^2 = 3xyz. \tag{1.1}$$

The equation was introduced by Markoff [12]. An integer occurring in such a triple is a Markoff number. The unicity conjecture posed by Frobenius [10] asserts that a Markoff triple is determined, up to permutation, by its largest entry. The conjecture remains open. It is known for prime powers, twice prime powers, and several other arithmetic families; see [2, 3, 4, 11, 16, 6, 14, 1].

Throughout, an *ordered occurrence with maximum* M means a triple (a, b, M) with $a \le b \le M$. An *ordered collision* is a pair of distinct such occurrences with the same maximum. We call M *unique* when it has only one ordered occurrence, equivalently when its Markoff triple is determined up to permutation.

Reading guide and notation. The argument uses algebra, calculus, prime factorization, and congruences. A congruence $u \equiv v \pmod{n}$ means that $n \mid u - v$. Integers are *coprime* when their greatest common divisor is one. A complete prime-power block of an integer is the factor p^e containing its entire power of a prime p. The notation $\mathrm{ord}_p(z)$ records the exponent of p in z; this is also called its p-adic valuation. A *quadratic residue* is a residue that is a square modulo the stated prime.

The words *packet*, *floor*, and *tier* will have specific meanings here. A packet is one of the displayed quadratic equations together with its integer variables and divisibility conditions. A floor is a proved lower bound for a product of its coordinates. A tier is the resulting level of the collision bound. A *relaxation* keeps only some necessary conditions, so it may include tuples that do not come from Markoff triples. Excluding every tuple in this larger set also excludes actual collisions. We identify explicitly which steps use a finite computer calculation.

The hyperbolic functions need no background beyond exponentials: $\cosh t = (e^t + e^{-t})/2$ and $\sinh t = (e^t - e^{-t})/2$. The notation log means the natural logarithm. Later, a continued fraction $[c_1; c_2, \dots, c_r]$ means $c_1 + 1/(c_2 + 1/(\cdots + 1/c_r))$. Its digits c_i are called partial quotients. Requiring the last digit to exceed one makes the finite expansion unique.

This paper studies the arithmetic forced by an ordered collision. Let (a, b, M) and (a', b', M) form an ordered collision. For odd M, the central chain is

$$\boxed{\text{hyperbolic four-point factorization}} \Longrightarrow \boxed{AB = M^2C} \Longrightarrow \boxed{M = FE \text{ unitary}}$$
$$\Longrightarrow \boxed{\text{two defect equations}} \Longrightarrow \boxed{\text{residual crossed-Pell coupling}}$$
$$\Longrightarrow \boxed{\text{four certified odd floors}}.$$

The first step supplies a geometric factorization and interpretation of Srinivasan's identity. The direct verification of that identity and all later algebraic reductions use integers.

The following table separates each product floor from the coefficient it gives in the collision inequality.

tier	defect floor	provenance	quartic consequence
99	33	hand proof	$F^4, E^4 \ge 99M + 66$
360	120	exact local enumeration	$F^4, E^4 \ge 360M + 240$
201411	67137	companion-prime certificate	$F^4, E^4 \ge 201411M + 134274$
300000	100000	crossed-Pell/continuant certificates	$F^4, E^4 \ge 300000M + 200000$

Table 1: The four certified odd tiers. The coefficient is three times the defect-product floor.

For even maxima, dividing the collision quantities by four gives the following parallel bounds.

In the odd table, 120 is sharp for the local relaxation and 67137 is sharp after the companion-prime condition. In the even table, 72 and 1014 are sharp for their corresponding relaxations. In both parities, 100000 is a certified cutoff, with no claim of sharpness. None of these numbers is a range through which Markoff unicity has merely been checked.

tier	defect floor	provenance	quartic consequence
216	72	hand proof	$F^4, E^4 \geq 216M + 144$
3042	1014	companion-prime certificate	$F^4, E^4 \geq 3042M + 2028$
300000	100000	continuant-root certificate	$F^4, E^4 \geq 300000M + 200000$

Table 2: The three certified tiers for even M, after normalization by $M_1 = M/2$.

The factorization needed for the argument keeps every complete prime-power block on one side. This is expressed by a unitary divisor. A divisor $d \mid N$ is *unitary*, written $d \parallel N$, when $\gcd(d, N/d) = 1$. Define

$$u_*(N) = \max\{d : d \parallel N,\ d \leq \sqrt{N}\}. \tag{1.2}$$

The set is nonempty because it contains 1.

Theorem 1.1 (Main collision and unicity theorem)**.** *If an odd Markoff number $M > 1$ admits an ordered collision, then there are coprime unitary factors $F, E > 1$, $M = FE$, for which*

$$F^4, E^4 \geq 300000M + 200000. \tag{1.3}$$

If an even Markoff number $M > 2$ admits an ordered collision, then $M_1 = M/2 \equiv 1 \pmod{16}$ has coprime proper unitary factors $F, E > 1$, $M_1 = FE$, satisfying the same bound (1.3)*. Consequently,*

$$u_*(M)^4 < 300000M + 200000 \qquad \Longrightarrow M \text{ is unique} \qquad (M \text{ odd}), \tag{1.4}$$

$$u_*(M/2)^4 < 300000M + 200000 \qquad \Longrightarrow M \text{ is unique} \qquad (M \text{ even}). \tag{1.5}$$

The corresponding weaker statements hold at every tier in Tables 1 and 2, with their stated proof provenance.

Corollary 1.2 (Prime-power tail)**.** *Let $M = kq^e$ be an odd Markoff number, where q is prime,* $\gcd(k, q) = 1$*, and $e \geq 1$. For any*

$$\mathcal{C} \in \{99, 360, 201411, 300000\},$$

the condition

$$k^3 \leq \mathcal{C} q^e \tag{1.6}$$

guarantees that M is unique, provided the tier with coefficient $\mathcal{C}$ is invoked with its stated hand or computer-assisted provenance.

The odd collision assertion of Theorem 1.1 is proved in Theorem 6.10, and its even counterpart in Theorem 7.4. Their unicity implications are Theorems 8.1 and 8.2. Corollary 1.2 follows from Theorem 8.3 by taking a single prime-power tail.

Historical position and quantitative comparison

Classical partial results are usually formulated as arithmetic-shape criteria. Baragar proved unicity when one of $M, 3M - 2, 3M + 2$ is a prime, twice a prime, or four times a prime [2]. Button proved the prime case and later developed a principal-ideal and continued-fraction method [3, 4]; Lang and Tan gave an independent proof for prime powers [11]. On the companion side, Zhang proved unicity

when one of $3M \pm 2$ is a prime power, four times a prime power, or eight times a prime power [16]. Chen and Chen subsequently treated $3M \pm 2 = kp^{\ell}$ for $1 \le k \le 10^{10}$ [6].

Srinivasan gave an elementary treatment of the prime-power and twice-prime-power cases and further criteria in terms of ambiguous classes and the prime divisors of $9M^2 - 4$ [14]. More specifically, in the present sign convention, Srinivasan's identity [14, Lemma 2.1] is exactly

$$(bb' - aa')(ba' - ab') = M^2(ab - a'b'),$$

that is, $AB = M^2C$. Her analysis also contains the underlying coprime allocation of complete prime-power blocks produced by a hypothetical nonunique maximum [14, Theorem 5.1]. Thus neither the determinant identity itself nor, in substance, the existence of a proper unitary split is asserted here as new.

The companion-discriminant allocation formulated below likewise has a classical genus-theoretic antecedent. Srinivasan's ambiguous-class analysis and Legendre-symbol criteria detect the same mutual quadratic-residue condition between complementary factors of $9M^2 - 4$ [14, Theorems 5.4 and 6.1]. The elementary residual equation below is used as a bridge from the present collision variables to that condition; priority is not claimed for the unitary or residue obstruction. The additional bounded crossed-Pell reconstruction couples this residual allocation back to the primitive collision slope and supplies a second exact rejection of the final odd certificate row.

For even Markoff numbers, Zhang proved the congruence $M \equiv 2 \pmod{32}$ and companion-number criteria [16, Theorems 1–2]. Srinivasan's treatment of twice prime powers already uses the determinant identity with its forced divisibility by 4, and her companion lemma states that every odd prime divisor of $9M^2 - 4$ is $1 \pmod 4$ [14, Theorem 2.2 and Lemma 4.2]. The even section below combines these antecedents with the complementary conics and an exact coefficient-6 enumeration; it does not claim priority for the congruence, the coprime splitting in substance, or the companion-prime condition.

Geometric reformulations also predate the present work. Tornero's fixed-M hyperbola

$$H_M : \quad x^2 - 3Mxy + y^2 + M^2 = 0$$

is exactly the conic used below [15]; a recent reformulation uses integral necklaces [9]. The narrower role of the hyperbolic viewpoint here is the closed-form Q^x parametrization and the explicit four-point factorization of the three terms in Srinivasan's identity.

Recent related work. O'Sullivan gives geometric growth and approximation estimates for real Markov trees [17, Section 3 and Proposition 8.1]. Yasutomi proves injectivity for substitution parameters $a \ge 2$, $b = a + 1$, and a partial ordering result in the classical case $a = 1$, $b = 2$ [18, Theorems 1.3 and 1.5(2)]. The full classical case is outside his injectivity theorem. Banaian extends Aigner's ordering results to generalized k-Markov numbers [19, Theorem 4]; Gyoda surveys the associated arithmetic, combinatorial, and geometric descriptions [20]. Our question is different: given a hypothetical classical collision, how large must each factor in its forced coprime split be? The quartic bounds in Theorem 1.1 address that question. They are not a replacement for these ordering or approximation results, and the proof does not depend on them.

Relation to the author's prior preprint

The author's preprint *A slope–scale factorization for prime-power and twice-prime-power Markoff unicity* (ResearchGate, 18 June 2026; https://doi.org/10.13140/RG.2.2.23525.56800) developed the

Q^x parametrization, the A, B, C, D identities, the collision line, the slope–scale formulas, the unitary allocation, and the coupled defect equations. The present article substantially extends and supersedes that preprint by isolating the complementary four-point conics and, principally, by establishing the four odd defect floors $33, 120, 67137, 100000$, the normalized even floors $72, 1014, 100000$, their unitary-gap consequences, the residual crossed-Pell reconstruction, and self-contained exact-arithmetic certificates. Claims of contribution below concern these quantitative extensions and their certificates, not the previously published or publicly circulated structural machinery.

The direct coefficient-1 antecedent of the present unitary-gap inequalities is Button's unitary-factor criterion [4, Corollary 6]: an odd Markoff number M is unique if every coprime factorization $M = rs$, $r < s$, satisfies $s > r^3$. In the notation of (1.2), this is equivalent to

$$u_*(M)^4 < M,$$

because the possible smaller factors r are precisely the unitary divisors below $\sqrt{M}$. Theorem 1.1 strengthens this coefficient-1 quartic gap to

$$u_*(M)^4 < \mathcal{C}M + \frac{2\mathcal{C}}{3}, \qquad \mathcal{C} \in \{99, 360, 201411, 300000\}.$$

Thus the strongest certificate raises the coefficient of M in this particular collision obstruction from 1 to 300000, while retaining an exact positive additive term. For $M = kq^e$, the corresponding cubic comparison changes from the coefficient-1 condition $k^3 < q^e$ to $k^3 \le \mathcal{C}q^e$. At fixed q^e, the final tier therefore expands the cofactor scale in this criterion by $300000^{1/3} = 66.943\ldots$.

This is not a blanket improvement on every earlier arithmetic family. Button also recorded, using the then-reported computation through 10^{140}, the separate conclusion for $M = kp^\ell$ with $1 \le k < 10^{35}$ [2, 4]; that result and the companion-number criteria above are nonnested with the present estimate. Within the present collision framework, however, the progression

$$99 \longrightarrow 360 \longrightarrow 201411 \longrightarrow 300000$$

is a strict sequence of certified lower-bound improvements. From the hand tier to the final tier the quartic coefficient increases by $300000/99 = 3030.30\ldots$, and the permitted k-scale in Corollary 1.2 increases by $(300000/99)^{1/3} = 14.470\ldots$ at fixed q^e. Against the antecedents above, the distinctive contribution of the present article is the quantitative strengthening: the explicit odd and normalized-even defect-product floors, their unitary-gap consequences, and transparent dual exact-arithmetic certificates, now including an independent residual crossed-Pell audit of the final odd row. The hyperbolic factorization is retained as a geometric interpretation and derivational bridge; it is not asserted as the first appearance of either the fixed conic or the identity $AB = M^2C$.

The comparison with Button is strict at the level of sufficient unitary-gap criteria. The Markoff number $51641 = 113 \cdot 457$ does not satisfy Button's condition $u_*(M)^4 < M$, whereas it satisfies the coefficient-201411 criterion proved here; see Example 8.5. This is a genuine strengthening of that collision obstruction, not a claim that the uniqueness of 51641 was unknown by all other methods.

The odd and even collision sectors are both treated, but their certified inequalities still leave increasingly balanced unitary allocations. No result below proves the full unicity conjecture.

2 The fixed-M conic and a hyperbolic factorization of Srinivasan's identity

Fix $M > 2$, and put

$$q = \frac{3M}{2}, \qquad R = \sqrt{q^2 - 1}, \qquad Q = q + R. \tag{2.1}$$

Thus $Q^{-1} = q - R$, and if $\theta = \log Q$, then $q = \cosh\theta$, $R = \sinh\theta$. Define

$$f(x) = \frac{M}{2R}(Q^x + Q^{-x}), \qquad g(x) = \frac{M}{2R}(Q^x - Q^{-x}). \tag{2.2}$$

Equivalently,

$$f(x) = \frac{M}{R}\cosh(x\theta), \qquad g(x) = \frac{M}{R}\sinh(x\theta).$$

The curve parametrized below is precisely Tornero's fixed-M hyperbola $H_M : X^2 - 3MXY + Y^2 + M^2 = 0$ [15, p. 596]. Tornero studies its integral points using the recurrence $B_{j+2} = 3MB_{j+1} - B_j$, whose characteristic roots are Q and Q^{-1}. Here we use the corresponding continuous Q^x parametrization to place the two occurrences of an ordered collision on one real component and factor their four-point determinants.

Lemma 2.1 (Fixed-M parametrization)**.** *For every real x,*

$$f(x)^2 + f(1-x)^2 - 3Mf(x)f(1-x) + M^2 = 0. \tag{2.3}$$

The map $x \in \mathbb{R} \mapsto (f(x), f(1-x))$ parametrizes the entire branch with both coordinates positive, once each. The ordering $f(x) \le f(1-x)$ holds precisely for $x \le 1/2$. If $(f(x), f(1-x), M)$ is an ordered Markoff triple with maximal entry $M > 2$, then $0 < x < 1/2$. On this interval the first coordinate increases and the second decreases.

Proof. Write $u = x\theta$. The addition formula gives

$$f(1-x) = \frac{M}{R}\cosh(\theta - u) = qf(x) - M\sinh u.$$

Consequently,

$$(f(1-x) - qf(x))^2 = R^2 f(x)^2 - M^2.$$

Expanding and using $q^2 - R^2 = 1$ proves (2.3).

To show that no positive point is missed, let (X, Y) satisfy the conic with $X, Y > 0$. Completing the square gives

$$(qX - Y)^2 = R^2X^2 - M^2.$$

Define $u = \operatorname{arsinh}((qX - Y)/M)$, where arsinh is the inverse of sinh. Then $\cosh u = RX/M$, since both sides are positive. Setting $x = u/\theta$ gives $X = f(x)$ and $Y = qX - M\sinh u = f(1-x)$. This also proves that the parameter is unique.

The function $\cosh t$ is even and increases with $|t|$. Thus $f(x) \le f(1-x)$ is equivalent to $|x| \le |1-x|$, or $x \le 1/2$. If $x \le 0$, then

$$f(1-x) \ge f(1) = \frac{3M^2}{2R} > M,$$

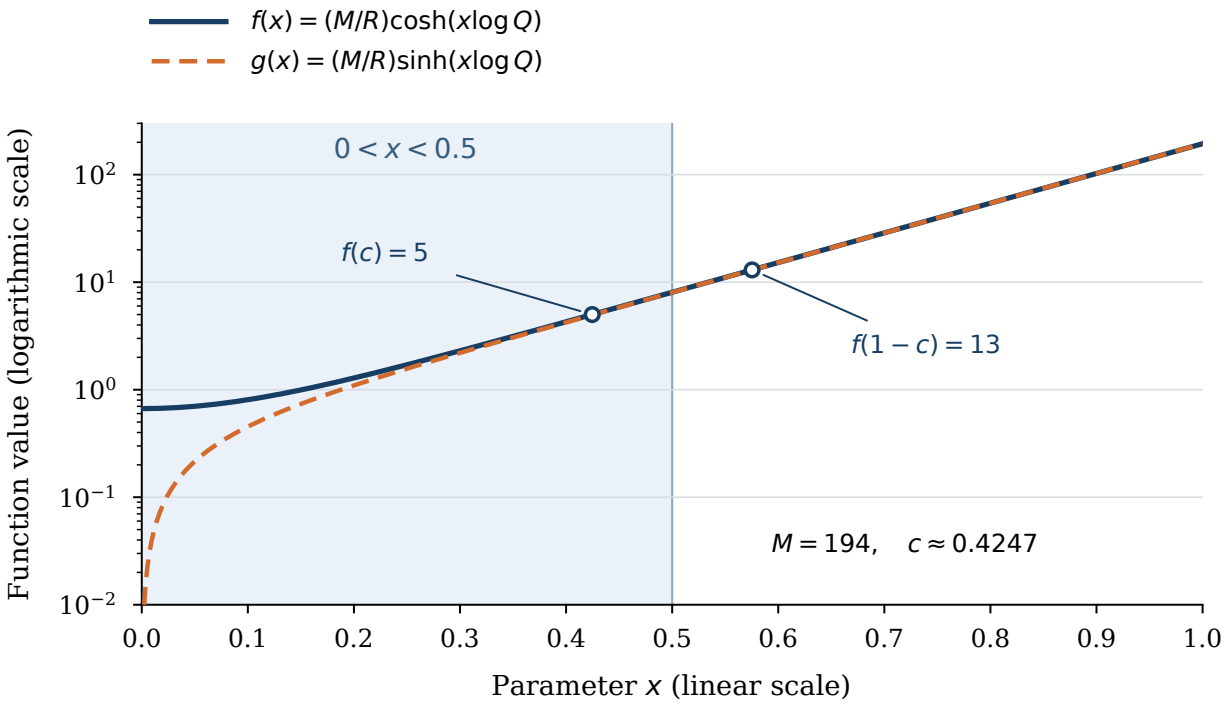


Fig. 1 The functions f and g for the Markoff maximum $M = 194$, which occurs in $(5, 13, 194)$. The horizontal axis is linear and the vertical axis is logarithmic. The shaded interval $0 < x < 1/2$ contains the parameter of the smaller entry of an ordered occurrence. The two marked points belong to this one known occurrence: $f(c) = 5$ and $f(1 - c) = 13$. They do not represent a collision. The solid and dashed curves become close because $f(x) - g(x) = (M/R)Q^{-x}$, while both approach $(M/(2R))Q^{x}$; hence their common large-x trend appears straight on these axes. A logarithmic axis displays only positive values. Thus $g(0) = 0$ and the negative values $g(x) < 0$ for $x < 0$ are omitted, and the dashed curve continues below the plotted range as $x \to 0^{+}$. The finite graph window stops at $x = 1$; the functions themselves are defined on all real x.

because $R < 3M/2$. Such a point cannot have maximum M. If $x = 1/2$, the two smaller integer entries agree. Pairwise coprimality, proved independently in Lemma 3.3, makes both equal to 1. The equation then gives $M = 1$ or $M = 2$. Hence $M > 2$ requires $0 < x < 1/2$.

Finally, $f'(x) = M\theta \sinh(x\theta)/R > 0$ on this interval, while $\frac{d}{dx} f(1-x) = -M\theta \sinh((1-x)\theta)/R < 0$. These signs prove the monotonicity assertions. □

Let (a, b, M) and (a', b', M) form an ordered collision. After interchanging the two ordered occurrences if necessary, there are

$$0 < c < d < \frac{1}{2} \tag{2.4}$$

such that

$$a = f(c), \quad a' = f(d), \quad b' = f(1-d), \quad b = f(1-c), \tag{2.5}$$

and hence

$$0 < a < a' < b' < b < M. \tag{2.6}$$

Define the three bilinear collision quantities

$$A = bb' - aa', \qquad B = ba' - ab', \qquad C = ab - a'b'. \tag{2.7}$$

The next formulas explain why these three quantities are useful. Put

$$\xi = d - c, \qquad \eta = 1 - c - d. \tag{2.8}$$

In the present sign convention, the identity (2.10) below is precisely Srinivasan's Lemma 2.1 [14]. The following proposition supplies the additional four-point geometric factorization of its three factors.

Proposition 2.2 (Hyperbolic four-point factorization)**.** *The four-point determinants factor as*

$$A = Mg(\eta), \qquad B = Mg(\xi), \qquad C = g(\eta)g(\xi). \tag{2.9}$$

In particular,

$$\boxed{AB = M^2 C}. \tag{2.10}$$

Proof. The identity $\cosh u \cosh v = (\cosh(u+v) + \cosh(u-v))/2$ lets us subtract the products in pairs. It gives

$$\cosh((1-c)\theta)\cosh((1-d)\theta) - \cosh(c\theta)\cosh(d\theta) = R\sinh(\eta\theta),$$

and

$$\cosh((1-c)\theta)\cosh(d\theta) - \cosh(c\theta)\cosh((1-d)\theta) = R\sinh(\xi\theta).$$

Each product of two f-values has the common factor M^2/R^2. Multiplying these identities by that factor gives $A = Mg(\eta)$ and $B = Mg(\xi)$.

For the third difference, the same product-to-sum identity gives

$$\cosh(c\theta)\cosh((1-c)\theta) - \cosh(d\theta)\cosh((1-d)\theta) = \sinh(\eta\theta)\sinh(\xi\theta).$$

After multiplication by M^2/R^2, this is $C = g(\eta)g(\xi)$. The first two formulas now multiply to $AB = M^2 C$. □

Thus Proposition 2.2 gives a hyperbolic factorization of a known arithmetic identity. This factorization was the author's original route to recognizing Srinivasan's identity in the present notation. For later arithmetic use, we also record a direct integral verification.

Proposition 2.3 (Self-contained integral verification)**.** *If both* (a, b, M) *and* (a', b', M) *satisfy the Markoff equation, then*

$$M^2C = AB. \tag{2.11}$$

If

$$D = b^2 - b'^2 + a^2 - a'^2, \tag{2.12}$$

then also

$$D = 3MC, \qquad 9ABC = D^2. \tag{2.13}$$

Proof. Expand the product and collect the terms containing $a'b'$ and ab:

$$\begin{aligned} AB &= (bb' - aa')(ba' - ab') \\ &= a'b'(a^2 + b^2) - ab(a'^2 + b'^2). \end{aligned}$$

The two Markoff equations give

$$a^2 + b^2 = 3Mab - M^2, \qquad a'^2 + b'^2 = 3Ma'b' - M^2.$$

On substitution, the two terms containing $3Maba'b'$ cancel. What remains is $AB = M^2(ab - a'b') = M^2C$.

Next subtract the two Markoff equations. The terms M^2 cancel, leaving $D = 3MC$. Squaring gives $D^2 = 9M^2C^2$. Replace one factor M^2C by AB to obtain $D^2 = 9ABC$. □

The order (2.6) gives $A, B, C, D > 0$. In particular, $B = br + as > 0$ after the displacement notation introduced below, and the other signs follow from the identities. The strict inequality $b < M$ follows from pairwise coprimality: $b = M > 2$ would give a common divisor.

3 Collision line, slope-scale factorization, and unitary allocation

The slope–scale formulas and unitary allocation in this section were previously circulated in the author's earlier slope–scale preprint (https://doi.org/10.13140/RG.2.2.23525.56800); they are reproduced here to make the quantitative argument self-contained.

Put

$$r = a' - a > 0, \qquad s = b - b' > 0. \tag{3.1}$$

Then

$$C = as - br + rs, \qquad D = 2bs - s^2 - 2ar - r^2. \tag{3.2}$$

Combining these with $D = 3MC$ gives the collision line

$$(3Ms + 2r)a - (3Mr + 2s)b + (3Mrs + r^2 + s^2) = 0. \tag{3.3}$$

Set

$$P = 3Mr + 2s, \qquad S = 3Ms + 2r, \qquad d_0 = \gcd(P, S), \qquad K = r^2 + s^2 + 3Mrs. \tag{3.4}$$

Equation (3.3) is equivalent to

$$S(2a + r) = P(2b - s).$$

Because P/d_0 and S/d_0 are coprime, a positive integer j satisfies

$$2a + r = \frac{jP}{d_0}, \qquad 2b - s = \frac{jS}{d_0}. \tag{3.5}$$

Proposition 3.1 (Slope and scale identities)**.** *Every ordered collision satisfies*

$$C = \frac{j(s^2 - r^2)}{d_0}, \qquad B = \frac{jK}{d_0}, \tag{3.6}$$

$$A = \frac{M^2(s^2 - r^2)}{K}, \tag{3.7}$$

$$B^2 = \frac{K(K + 4M^2)}{9M^2 - 4}. \tag{3.8}$$

Proof. Solve (3.5) explicitly:

$$a = \frac{1}{2}\left(\frac{jP}{d_0} - r\right), \qquad b = \frac{1}{2}\left(\frac{jS}{d_0} + s\right).$$

Use $a' = a + r$, $b' = b - s$, and the definitions of B, C. Substitution gives

$$C = as - br + rs = \frac{j(sP - rS)}{2d_0} = \frac{j(s^2 - r^2)}{d_0},$$

$$B = br + as = \frac{j(rS + sP)}{2d_0} = \frac{jK}{d_0}.$$

Since $C, j, d_0 > 0$, we have $s^2 - r^2 > 0$. Dividing $M^2C = AB$ by the positive B proves (3.7).

The same substitution into $A = b(b - s) - a(a + r)$ yields

$$A = \frac{(s^2 - r^2)((9M^2 - 4)j^2 - d_0^2)}{4d_0^2}.$$

Equate this expression to (3.7) and cancel $s^2 - r^2 > 0$. The result is

$$((9M^2 - 4)j^2 - d_0^2)K = 4M^2 d_0^2.$$

Rearranging and multiplying by K/d_0^2, then using $B = jK/d_0$, gives (3.8). □

We now reduce the slope. Write

$$r = Gr_0, \qquad s = Gs_0, \qquad \gcd(r_0, s_0) = 1,$$

and define

$$T_0 = s_0^2 - r_0^2, \qquad K_0 = r_0^2 + s_0^2 + 3Mr_0s_0. \tag{3.9}$$

Equation (3.6) shows that $T_0 > 0$, and all quotients below are positive. Let

$$\Gamma = \gcd(K_0, T_0), \qquad K_0 = L\Gamma, \qquad T_0 = W\Gamma, \qquad \gcd(L, W) = 1. \tag{3.10}$$

Formula (3.7) becomes

$$A = \frac{M^2 W}{L}. \tag{3.11}$$

Thus $L \mid M^2$. Write

$$M^2 = LN, \qquad A = NW. \tag{3.12}$$

Lemma 3.2 (Scale divisor)**.** *If M is odd, then $L \mid B$.*

Proof. It is enough to compare the exponent of each prime in L and B. Let $p \mid L$. Because $L \mid M^2$ and M is odd, p is an odd divisor of M. Thus $9M^2 - 4 \equiv -4 \not\equiv 0 \pmod{p}$.

We have

$$K = G^2 L\Gamma, \qquad K + 4M^2 = L(G^2\Gamma + 4N).$$

Both factors on the right of $(9M^2 - 4)B^2 = K(K + 4M^2)$ are divisible by L. Write $\mathrm{ord}_p(z)$ for the exponent of p in a positive integer z. Comparing exponents gives

$$2\,\mathrm{ord}_p(B) = \mathrm{ord}_p(K) + \mathrm{ord}_p(K + 4M^2) \geq 2\,\mathrm{ord}_p(L).$$

Thus every prime occurs in B to at least its exponent in L, which proves $L \mid B$. □

We use two elementary facts about Markoff entries.

Lemma 3.3 (Coprimality and prime support)**.** *The entries of a Markoff triple are pairwise coprime. Every odd prime dividing a Markoff number is* $1 \pmod 4$*. In particular,* 3 *divides no Markoff number.*

Proof. We first prove coprimality by descent. If a prime divides two entries of a Markoff triple, the equation forces it to divide the third. Order the entries as $x \leq y \leq z$. Regarding the equation as a quadratic in its last entry gives the other integer root

$$z' = 3xy - z = \frac{x^2 + y^2}{z} > 0.$$

For $P(u) = u^2 - 3xyu + x^2 + y^2$,

$$P(y) = x^2 - y^2(3x - 2) < 0$$

unless $x = y = 1$. Since the leading coefficient is positive, $P(y) < 0$ places y strictly between its two roots. The root z is at least y, so the other root satisfies $0 < z' < y$.

Replacing z by z', and reordering, strictly decreases the maximum until $x = y = 1$. At that point the equation gives $z = 1$ or 2; the triple $(1, 1, 2)$ also moves to $(1, 1, 1)$. A prime dividing all three entries would remain a common divisor under every move, but it cannot divide the final triple. No such prime exists, proving pairwise coprimality.

Now let an odd prime p divide an entry M of (a, b, M). Coprimality allows division by a modulo p, giving

$$1 + (ba^{-1})^2 \equiv 0 \pmod{p}.$$

Thus -1 is a square modulo p. A square root of -1 has multiplicative order four, so four divides the order $p - 1$ of the nonzero residue group. Hence $p \equiv 1 \pmod 4$, which excludes $p = 3$. □

Lemma 3.4 (No split prime)**.** *If an odd prime $p \mid M$ divides both A and B, then the ordered collision is impossible.*

Proof. By pairwise coprimality, none of a, b, a', b' is divisible by p, so all four have inverses modulo p. Put $t = ba^{-1}$ and $t' = b'(a')^{-1}$. Dividing $B \equiv 0$ by aa' gives $t = t'$; dividing $A \equiv 0$ by aa' gives $tt' = 1$. Together these imply $t^2 = 1$.

The Markoff equation, divided by a^2 modulo p, instead gives $t^2 = -1$. This would imply $2 \equiv 0 \pmod{p}$, impossible for an odd prime. □

Theorem 3.5 (Odd unitary collision allocation)**.** *If an odd Markoff number M admits an ordered collision, then there are coprime integers $F, E > 1$ and positive integers H, W such that*

$$M = FE, \qquad A = F^2W, \qquad B = E^2H, \qquad C = HW. \tag{3.13}$$

Each of F, E is a unitary divisor of M.

Proof. Consider a complete prime-power factor p^e of M. Its square p^{2e} occurs in $M^2 = LN$. If a positive power of p occurred in both L and N, then $A = NW$ and $L \mid B$ would give $p \mid A, B$. Lemma 3.4 excludes this. Thus the entire power p^{2e} belongs to one of L, N.

Doing this for every prime gives

$$L = E^2, \qquad N = F^2, \qquad M = FE, \qquad \gcd(F, E) = 1.$$

Both factors exceed one. If $L = 1$, then $A = M^2W$, contradicting $0 < A < bb' < M^2$. If $N = 1$, then $M^2 = L \mid B$, contradicting $0 < B < ba' < M^2$.

Now $A = NW = F^2W$. Also $E^2 = L \mid B$, so $B = E^2H$ for an integer $H > 0$. Substitute these expressions in $M^2C = AB$ and cancel $M^2 = F^2E^2$ to obtain $C = HW$. The coprimality of F, E is exactly the assertion that they are complementary unitary divisors. □

Remark 3.6. Proper coprime factorizations of a hypothetical nonunique maximum underlie Srinivasan's Theorem 5.1 and Button's Corollary 6 [14, 4]. Theorem 3.5 records the exact allocation to A, B, C in the notation required for the defect packets.

4 Complementary hyperbolic factors and the defect packets

The coupled defect equations below were also circulated in the author's earlier slope–scale preprint (https://doi.org/10.13140/RG.2.2.23525.56800); they are reproved here from the complementary four-point conics in the form required by the certified tiers.

The same four hyperbolic points used to factor $AB = M^2C$ produce the two equations on which all four odd tiers rest. Define

$$\widehat{A} = 3Maa' - a'b - ab', \qquad \widehat{B} = 3Mab' - aa' - bb'. \tag{4.1}$$

Along with ξ, η from (2.8), put

$$\tau = c + d, \qquad \upsilon = 1 - d + c. \tag{4.2}$$

Then $0 < \tau, \upsilon < 1$, $\eta + \tau = 1$, and $\xi + \upsilon = 1$.

Lemma 4.1 (Complementary four-point conics)**.** *One has*

$$\widehat{A} = Mg(\tau), \qquad \widehat{B} = Mg(\upsilon), \tag{4.3}$$

and, for $0 < x < 1$,

$$g(x)^2 + g(1-x)^2 + 3Mg(x)g(1-x) = M^2. \tag{4.4}$$

Consequently,

$$A^2 + \widehat{A}^2 + 3MA\widehat{A} = M^4, \tag{4.5}$$

$$B^2 + \widehat{B}^2 + 3MB\widehat{B} = M^4. \tag{4.6}$$

Proof. We make the two product calculations explicit. Using $2q = 3M$ and $\cosh(\theta - u) = q\cosh u - R\sinh u$, we obtain

$$2q\cosh(c\theta)\cosh(d\theta) - \cosh(d\theta)\cosh((1-c)\theta) \\ - \cosh(c\theta)\cosh((1-d)\theta) = R\sinh((c+d)\theta).$$

The corresponding identity for the other factor is

$$2q\cosh(c\theta)\cosh((1-d)\theta) - \cosh(c\theta)\cosh(d\theta) \\ - \cosh((1-c)\theta)\cosh((1-d)\theta) = R\sinh((1-d+c)\theta).$$

Multiplication by M^2/R^2 proves $\widehat{A} = Mg(\tau)$ and $\widehat{B} = Mg(\upsilon)$. Both are positive because $\tau, \upsilon > 0$.

For the conic, the addition formula for $\sinh(u+v)$ gives

$$\sinh^2 u + \sinh^2 v + 2\cosh(u+v)\sinh u \sinh v = \sinh^2(u+v).$$

Take $u = x\theta$, $v = (1-x)\theta$, so $u + v = \theta$. Multiply by M^2/R^2 and use $\sinh\theta = R$, $2\cosh\theta = 3M$. This proves (4.4). The pairs (η, τ) and (ξ, υ) each sum to one. Substituting them and multiplying by M^2 proves (4.5) and (4.6). □

Theorem 4.2 (Coupled defect packets)**.** *For every hypothetical odd collision there are positive integers n, m such that*

$$n^2 + 3MHn + H^2 = F^4, \tag{4.7}$$

$$m^2 + 3MWm + W^2 = E^4. \tag{4.8}$$

Moreover,

$$\gcd(F, H) = \gcd(H, n) = 1, \qquad \gcd(E, W) = \gcd(W, m) = 1. \tag{4.9}$$

Proof. First we show that the complementary factors have the required square divisors. Let $p^e \parallel F$, and suppose $v = \mathrm{ord}_p(\widehat{A}) < 2e$. In (4.5), the term $\widehat{A}^2$ has exponent $2v$. The exponents of A^2 and M^4 are at least $4e > 2v$, and that of $3MA\widehat{A}$ is at least $3e + v > 2v$.

This is impossible: dividing the equation by p^{2v} and reducing modulo p would leave exactly one nonzero term. Hence $\mathrm{ord}_p(\widehat{A}) \geq 2e$. For every prime of F, this proves $F^2 \mid \widehat{A}$. The identical argument with $E, B, \widehat{B}$ proves $E^2 \mid \widehat{B}$. Write

$$\widehat{A} = F^2 m, \qquad \widehat{B} = E^2 n.$$

Substitute these and (3.13) into the two conics. Dividing the first by F^4 gives (4.8); dividing the second by E^4 gives (4.7). The integers m, n are positive because the complementary factors are positive.

If a prime divided both F and H, it would divide both $A = F^2W$ and $B = E^2H$, contrary to Lemma 3.4. Thus $\gcd(F, H) = 1$; the same reasoning gives $\gcd(E, W) = 1$. A prime dividing H, n would, by (4.7), also divide F^4, hence F, which is impossible. This proves $\gcd(H, n) = 1$. Applying the same argument to the other equation proves $\gcd(W, m) = 1$. □

The quadratic equations also have a useful interpretation through conjugate numbers. In the ring of numbers of the form $u + vQ_M$, with $u, v \in \mathbb{Z}$, namely the real quadratic order

$$\mathcal{O}_M = \mathbb{Z}\left[\frac{3M + \sqrt{9M^2 - 4}}{2}\right],$$

one has

$$\mathrm{N}(x + yQ_M) = x^2 + 3Mxy + y^2, \qquad Q_M = \frac{3M + \sqrt{9M^2 - 4}}{2}.$$

Thus (4.7)–(4.8) are norm equations of fourth powers. Here the norm is the product obtained by replacing the square root by each of its two signs. This interpretation provides context; the bounds below use only the displayed integer equations.

5 The hand tier: 33 and 99

Abstract one defect equation as

$$x^2 + y^2 + 3TOxy = T^4. \tag{5.1}$$

For a collision packet, either

$$(T, O, x, y) = (F, E, H, n) \quad \text{or} \quad (T, O, x, y) = (E, F, W, m).$$

By Theorem 3.5, Lemma 3.3, and (4.9), the local block conditions are

$$T, O > 1, \quad \gcd(T, O) = 1, \quad 2 \nmid TO, \quad 3 \nmid TO, \quad p \mid TO \Longrightarrow p \equiv 1 \pmod 4, \quad \gcd(x, y) = 1. \tag{5.2}$$

Lemma 5.1 (The 33-lemma)**.** *Every solution of* (5.1) *satisfying* (5.2) *has*

$$xy \geq 33. \tag{5.3}$$

Proof. Reduce (5.1) modulo 3. Since $3 \nmid T$, it gives $x^2 + y^2 \equiv 1 \pmod 3$. The only square residues are zero and one, so exactly one of x, y is divisible by 3. In particular, $3 \mid xy$. Reduction modulo T also gives $T \mid x^2 + y^2$.

Suppose $xy < 33$, and use symmetry to take $x \leq y$. Listing the coprime factor pairs of the positive multiples of three below 33 gives

$$(1,3), (1,6), (1,9), (1,12), (1,15), (1,18), (1,21), (1,24), (1,27), (1,30),$$
$$(2,3), (2,9), (2,15), (3,4), (3,5), (3,7), (3,8), (3,10), (5,6).$$

Because T is odd, $T^4 \equiv 1 \pmod{16}$. Every integer whose prime divisors are $1 \pmod 4$ is itself $1 \pmod 4$, so TO must have residue $1, 5, 9$, or 13 modulo 16. For each listed pair, test these four residues in the equation, and then list the divisors $T > 1$ of $x^2 + y^2$ having the required prime divisors. The surviving six choices are

$\{x, y\}$	$x^2 + y^2$	T
$\{1, 9\}$	82	41
$\{1, 21\}$	442	13, 17, 221
$\{3, 7\}$	58	29
$\{3, 8\}$	73	73.

For each choice, the equation determines O:

$$O = \frac{T^3 - (x^2 + y^2)/T}{3xy}. \tag{5.4}$$

In the same order, these values are

$$\frac{68919}{27}, \quad \frac{2163}{63}, \quad \frac{4887}{63}, \quad \frac{10793859}{63}, \quad \frac{24387}{63}, \quad 5403.$$

The first five are not integers. The last is divisible by three, which the block conditions exclude. Thus every possible pair below 33 is rejected. □

Theorem 5.2 (Hand quartic tier)**.** *Every hypothetical odd collision satisfies*

$$Hn, Wm \geq 33, \qquad F^4, E^4 \geq 99M + 66. \tag{5.5}$$

It also satisfies the strict cubic imbalance

$$E < \frac{F^3}{99}, \qquad F < \frac{E^3}{99}. \tag{5.6}$$

Proof. Each of the two collision packets satisfies the hypotheses of Lemma 5.1. Therefore $Hn \geq 33$ and $Wm \geq 33$. The inequality $(x-y)^2 \geq 0$ gives $x^2 + y^2 \geq 2xy$, so either packet yields

$$T^4 = x^2 + y^2 + 3Mxy \geq (3M+2)xy \geq 33(3M+2).$$

This proves both quartic inequalities. For the first cubic inequality, use the strictly positive square terms to get

$$F^4 > 3MHn \geq 99M = 99FE.$$

Divide by $99F > 0$ to obtain $E < F^3/99$. Exchanging the two packets gives $F < E^3/99$. □

6 The exact finite certificate and the other three tiers

The finite reduction follows directly from (5.4). Fix a positive integer upper product bound B. By symmetry take $x \leq y$. There are finitely many pairs $xy < B$; for each pair, T must be a divisor of $x^2 + y^2$, and then O is forced by (5.4). More explicitly, $1 \leq x \leq \lfloor \sqrt{B-1} \rfloor$ and $x \leq y \leq \lfloor (B-1)/x \rfloor$. Listing those pairs, all divisors T, and each forced O covers the entire range. Thus no bound on T or O is guessed.

The primary certificate executes the following exact loop.

Listing 1: Core finite reduction (schematic).

```
for x <= y with lower <= x*y < upper:
    reject unless exactly one of x,y is divisible by 3
    for T in divisors(x*x + y*y):
        reject unless T is an admissible block
        O = (T**3 - (x*x+y*y)//T) / (3*x*y)
        retain only integral O>1 with gcd(T,O)=1
        retain only admissible O
```

The submitted source uses explicit failure checks rather than Python `assert`, so optimization flags cannot disable verification. It deliberately searches the larger domain in which $\gcd(x, y) = 1$ is not imposed; eliminating every point in this superset proves the required claim. A second program, written separately and runnable on its own, orders the search by the product $N = xy$, enumerates the factor pairs of N, and reimplements every arithmetic routine without importing the primary program. The primary program's `-mode both` option is an internal product-order cross-check that reuses its arithmetic helpers; it is not the standalone verifier.

6.1 The local floor 120 and coefficient 360

The exact search over $33 \le xy < 120$ leaves no admissible row. Before the final prime-support test on O, the sole row is

$$(x, y, T, O) = (1, 33, 109, 13081), \qquad 13081 = 103 \cdot 127,$$

and both prime factors are $3 \pmod 4$. At the boundary,

$$(x, y, T, O) = (5, 24, 601, 603005), \qquad xy = 120, \tag{6.1}$$

satisfies the equation and all local block conditions. Hence 120 is sharp for this relaxation.

Proposition 6.1 (Local exact tier)**.** *Every hypothetical odd collision satisfies*

$$Hn, Wm \ge 120, \qquad F^4, E^4 \ge 360M + 240. \tag{6.2}$$

Proof. Lemma 5.1 already excludes $xy < 33$. The finite search described above covers every pair with $33 \le xy < 120$, every possible divisor T, and the resulting forced value of O. It leaves no admissible row, even though its search domain is larger than the collision domain. Thus each collision packet must have $xy \ge 120$.

Apply this conclusion to $(x, y) = (H, n)$ and (W, m). In each equation, $x^2 + y^2 \ge 2xy$ then gives

$$T^4 \ge (3M + 2)xy \ge 120(3M + 2) = 360M + 240.$$

This is a computer-assisted step: the proved finite reduction and its exact exhaustive certificate together establish the product floor. □

6.2 The companion-prime filter and coefficient 201411

The next filter is an automatic property of a genuine Markoff entry.

Lemma 6.2 (Companion-prime necessity)**.** *If M is a Markoff number, every odd prime divisor of $(3M - 2)(3M + 2) = 9M^2 - 4$ is $1 \pmod 4$.*

Proof. Rearranging the Markoff equation gives two sum-of-squares identities:

$$(b - a)^2 + M^2 = ab(3M - 2), \qquad (a + b)^2 + M^2 = ab(3M + 2). \tag{6.3}$$

Let an odd prime p divide either companion $3M - 2$ or $3M + 2$. It cannot divide M, since that would also force $p \mid 2$. Reduce the corresponding identity modulo p and divide by M^2. It shows that -1 is a square modulo p. As in Lemma 3.3, this forces $p \equiv 1 \pmod 4$. □

The intentionally enlarged enumeration for $120 \le xy < 100000$ has nineteen rows, listed in Table 3. Eighteen satisfy $\gcd(x, y) = 1$; the row with $xy = 75000$ has $\gcd(x, y) = 25$ and is retained to make the superset search explicit. Of all nineteen rows, eighteen contain a displayed prime $\ell \equiv 3 \pmod 4$ dividing a companion $3TO - 2$ or $3TO + 2$. The sole companion-admissible row is

$$(x, y, T, O) = (23, 2919, 1145, 7453), \qquad xy = 67137. \tag{6.4}$$

It proves sharpness at 67137 for the local-plus-companion relaxation.

Proposition 6.3 (Companion exact tier)**.** *Every hypothetical odd collision satisfies*

$$Hn, Wm \geq 67137, \qquad F^4, E^4 \geq 201411M + 134274. \tag{6.5}$$

Proof. For each packet, the product TO is the actual common maximum M. Lemma 6.2 therefore supplies a necessary condition on both of its companions. The local tier already excludes products below 120. In the complete table for $120 \leq xy < 100000$, every row with $xy < 67137$ fails that necessary companion condition.

Hence both packets satisfy $xy \geq 67137$. Using $x^2 + y^2 \geq 2xy$ once more gives

$$T^4 \geq (3M + 2)xy \geq 67137(3M + 2) = 201411M + 134274.$$

The conclusion uses the exact finite enumeration; no assumption that an arbitrary surviving row is a Markoff collision is made. □

6.3 Residual companion allocation and a crossed Pell-type equation

The slope and scale formulas also divide the companion discriminant Δ_M into coprime factors. The word discriminant here refers to the square-root expression arising from the quadratic equation; we will use its explicit value. Put

$$\Delta_M = 9M^2 - 4, \qquad P_0 = 3Mr_0 + 2s_0, \qquad S_0 = 3Ms_0 + 2r_0.$$

Proposition 6.4 (Residual companion allocation)**.** *For every hypothetical odd collision,*

$$d_0 = G\Gamma, \qquad H = jG, \tag{6.6}$$

and Γ is a unitary divisor of Δ_M. More precisely, put

$$\Gamma^* = \frac{\Delta_M}{\Gamma}. \tag{6.7}$$

Then

$$\gcd(\Gamma, \Gamma^*) = 1, \qquad \Gamma^* j^2 - \Gamma G^2 = 4F^2. \tag{6.8}$$

Furthermore,

$$\gcd(\Gamma, M) = \gcd(\Gamma, j) = \gcd(\Gamma^*, G) = 1. \tag{6.9}$$

If

$$Z_F = 2n + 3MH - 2F^2, \tag{6.10}$$

then

$$Z_F = G^2\Gamma, \qquad G^2 = \gcd(Z_F, H^2). \tag{6.11}$$

Thus the residual data G, j, Γ, Γ^* are determined by the ordered packet (F, E, H, n), with $M = FE$.

Proof. We first identify the common divisor in the slope formulas. Since $P = GP_0$ and $S = GS_0$, write $d_0 = G\delta$, where $\delta = \gcd(P_0, S_0)$. At least one of P_0, S_0 is odd, because M is odd and the coprime integers r_0, s_0 cannot both be even. Thus δ is odd. The identities

$$s_0P_0 - r_0S_0 = 2T_0, \qquad s_0P_0 = K_0 + T_0, \qquad r_0S_0 = K_0 - T_0$$

show that $\delta \mid T_0$, and then $\delta \mid K_0$. Hence $\delta \mid \Gamma$.

For the reverse divisibility, K_0 is odd for either possible parity pattern of r_0, s_0, so Γ is odd. Also $\gcd(\Gamma, r_0 s_0) = 1$: a prime dividing Γ and one slope coordinate would divide the other by $T_0 = s_0^2 - r_0^2$. The last two displayed identities therefore allow cancellation of r_0, s_0 modulo Γ, giving $\Gamma \mid P_0, S_0$. Thus $\delta = \Gamma$ and $d_0 = G\Gamma$.

Next, direct linear combinations give

$$3MP_0 - 2S_0 = \Delta_M r_0, \qquad 3MS_0 - 2P_0 = \Delta_M s_0. \tag{6.12}$$

Both left sides are divisible by Γ. Since r_0, s_0 are coprime, an integer combination of them is one, so $\Gamma \mid \Delta_M$. As M is odd, $\gcd(M, \Delta_M) = \gcd(M, 4) = 1$. In particular, $\gcd(\Gamma, M) = 1$.

Use $K = G^2E^2\Gamma$ and $d_0 = G\Gamma$ in the formula for B:

$$B = \frac{jK}{d_0} = jGE^2 = HE^2.$$

Cancel E^2 to obtain $H = jG$. Substitute these same quantities into (3.8). Cancellation gives

$$\Delta_M j^2 = \Gamma(G^2\Gamma + 4F^2).$$

Writing $\Delta_M = \Gamma\Gamma^*$ and dividing by Γ proves (6.8).

If a prime divided both Γ, Γ^*, that equation would force it to divide $4F^2$. It is odd and coprime to $F \mid M$, a contradiction. Hence the allocation is unitary. The same reduction modulo a common prime of Γ, j, or of Γ^*, G, proves the remaining coprimalities.

Finally, complete the square in the first defect equation:

$$(2n + 3MH)^2 = 4F^4 + H^2\Delta_M.$$

The right side exceeds $4F^4$, so $Z_F = 2n + 3MH - 2F^2 > 0$, and

$$H^2\Delta_M = Z_F(Z_F + 4F^2).$$

The residual equation and $H = jG$ give the same product as

$$(G^2\Gamma)(G^2\Gamma + 4F^2).$$

The function $t(t + 4F^2)$ increases strictly for $t > 0$. Its two positive inputs must therefore agree: $Z_F = G^2\Gamma$. Since $H^2 = G^2j^2$ and $\gcd(\Gamma, j) = 1$, taking a greatest common divisor gives $\gcd(Z_F, H^2) = G^2$. Thus the packet determines G, then $j = H/G$, then $\Gamma = Z_F/G^2$, and finally $\Gamma^* = \Delta_M/\Gamma$. □

Remark 6.5 (Relation with the principal-genus obstruction). For every prime $p \mid \Gamma$, equation (6.8) shows that Γ^* is a quadratic residue modulo p; for every $p \mid \Gamma^*$, it shows that $-\Gamma$ is a quadratic residue modulo p. By Lemma 6.2, all these primes are $1 \pmod 4$, so the second condition may equivalently be stated with Γ in place of $-\Gamma$. These are the quadratic-residue conditions, written using Legendre symbols, occurring in Srinivasan's principal-genus method [14, Theorem 5.1, Definition 3.1, and Theorems 5.4 and 6.1]; no priority is claimed for this residue obstruction. The additional ingredient below is its simultaneous coupling to the primitive collision slope.

Factor the two coprime companions according to the residual allocation:

$$3M-2=\alpha\bar{\alpha},\qquad 3M+2=\beta\bar{\beta},\qquad \Gamma=\alpha\beta,\qquad \Gamma^*=\bar{\alpha}\bar{\beta},\tag{6.13}$$

The bars in this notation denote complementary integer factors, not complex conjugates. Here $\alpha=\gcd(\Gamma,3M-2)$ and $\beta=\gcd(\Gamma,3M+2)$. The allocation is unitary within each companion.

Theorem 6.6 (Crossed slope equation)**.** *For every hypothetical odd collision there are positive integers λ,μ such that*

$$r_0+s_0=\alpha\lambda,\qquad s_0-r_0=\beta\mu,\tag{6.14}$$

$$W=\lambda\mu<E^2,\qquad \gcd(\lambda,\mu)\in\{1,2\},\tag{6.15}$$

and

$$\boxed{\alpha\bar{\beta}\lambda^2-\beta\bar{\alpha}\mu^2=4E^2.}\tag{6.16}$$

The positive solution satisfies the following collision inequalities

$$\alpha\lambda>\beta\mu,\qquad \bar{\beta}\lambda>\bar{\alpha}\mu.\tag{6.17}$$

Moreover,

$$1<\Gamma<\Delta_M,\qquad \alpha<3M-2,\qquad \beta<3M+2.\tag{6.18}$$

Proof. Put $u=r_0+s_0$ and $v=s_0-r_0$; both are positive. From $\Gamma\mid P_0,S_0$, we get

$$\Gamma\mid S_0+P_0=(3M+2)u,\qquad \Gamma\mid S_0-P_0=(3M-2)v.$$

The odd numbers $3M-2$ and $3M+2$ are coprime: a common divisor would divide four. Thus $\alpha\mid 3M-2$ is coprime to $3M+2$, and $\beta\mid 3M+2$ is coprime to $3M-2$. Cancellation gives $\alpha\mid u$ and $\beta\mid v$. Write $u=\alpha\lambda$, $v=\beta\mu$, with $\lambda,\mu>0$.

Since $T_0=uv$ and $\Gamma=\alpha\beta$,

$$W=T_0/\Gamma=\lambda\mu.$$

Also $A=F^2W<M^2=F^2E^2$, so $W<E^2$. Coprimality of r_0,s_0 implies $\gcd(u,v)\in\{1,2\}$. Dividing u,v by the odd numbers α,β cannot introduce a common prime and cannot remove their common factor two. Thus $\gcd(\lambda,\mu)=\gcd(u,v)\in\{1,2\}$.

Now expand K_0 in terms of u,v:

$$4K_0=(3M+2)u^2-(3M-2)v^2,\qquad K_0=E^2\Gamma.$$

Substitute the four companion factors and $u=\alpha\lambda$, $v=\beta\mu$, then divide by $\Gamma=\alpha\beta$. This proves (6.16). The first inequality in (6.17) is $u>v$. The second follows from

$$2P_0=(3M+2)u-(3M-2)v=\Gamma(\bar{\beta}\lambda-\bar{\alpha}\mu)>0.$$

To prove that the companion split is proper, use the original ordering:

$$G(r_0+s_0)=r+s=(b-a)-(b'-a')<M,\qquad G(s_0-r_0)=s-r<M.$$

In particular, $G<M$. If $\Gamma=1$, then $\Gamma^*=\Delta_M$, and (6.8), with $j\geq 1$, gives

$$\Delta_M\leq G^2+4F^2<5M^2.$$

But $\Delta_M = 9M^2 - 4 > 5M^2$ for $M > 1$, a contradiction. Thus $\Gamma > 1$. Also

$$\alpha \le r_0 + s_0 < M/G \le M < 3M - 2,$$

and $\beta \le s_0 - r_0 < M/G \le M < 3M + 2$. These prove the two proper-factor bounds. They exclude $\Gamma = \Delta_M$ as well. □

The second packet also recovers the primitive slope. This removes an otherwise possible ambiguity about which packet realizes a row of the finite certificate.

Lemma 6.7 (Slope reconstruction from the second packet)**.** *For every hypothetical odd collision,*

$$m = \frac{r_0 P_0}{\Gamma}, \qquad E^2 + W = \frac{s_0 P_0}{\Gamma}. \tag{6.19}$$

Consequently, if

$$k = \gcd(m, E^2 + W), \tag{6.20}$$

then

$$k = \frac{P_0}{\Gamma}, \qquad r_0 = \frac{m}{k}, \qquad s_0 = \frac{E^2 + W}{k}. \tag{6.21}$$

Thus (E, W, m) determines r_0, s_0, and hence $\Gamma = \gcd(K_0, T_0)$ and $\Gamma^* = \Delta_M/\Gamma$.

Proof. The identity $s_0 P_0 = K_0 + T_0$, divided by Γ, gives

$$\frac{s_0 P_0}{\Gamma} = E^2 + W.$$

For the other formula, substitute (3.5) into the definition of $\widehat{A}$. Recall that $d_0 = G\Gamma$, $P = GP_0$, and $S = GS_0$. Expansion yields

$$\begin{aligned} 4\widehat{A} &= \frac{j^2 P_0(3MP_0 - 2S_0)}{\Gamma^2} - G^2 r_0 P_0 \\ &= r_0 P_0 \left(\frac{j^2 \Delta_M}{\Gamma^2} - G^2 \right) \\ &= \frac{4F^2 r_0 P_0}{\Gamma}. \end{aligned}$$

The second equality uses (6.12); the third uses (6.8). Since $\widehat{A} = F^2 m$, cancellation gives $m = r_0 P_0/\Gamma$.

The two quantities m and $E^2 + W$ are therefore the coprime integers r_0, s_0 multiplied by the same integer P_0/Γ. Their greatest common divisor is exactly this multiplier, giving all formulas in (6.21). □

Proposition 6.8 (Residual rejection of the exceptional odd row)**.** *The certificate row*

$$(x, y, T, O) = (23, 2919, 1145, 7453) \tag{6.22}$$

cannot be either packet of an odd ordered collision, in either ordering of its two defect coordinates.

Proof. The row forces $M_0 = 1145 \cdot 7453 = 8533685$. Its two companions factor as

$$3M_0 - 2 = 137 \cdot 186869, \qquad 3M_0 + 2 = 3137 \cdot 8161, \tag{6.23}$$

with four distinct prime factors. There are four cases: either packet can give the row, and its two positive coordinates can occur in either order. We reduce all four cases to the same two equations.

If the row is the first packet, then $(F, E) = (1145, 7453)$ and $\{H, n\} = \{23, 2919\}$. Apply the greatest-common-divisor extraction in Proposition 6.4:

(H, n)	G	j	Γ
$(23, 2919)$	1	23	$586208053 = 186869 \cdot 3137$
$(2919, 23)$	7	417	$1525037909 = 186869 \cdot 8161$.

For these companion allocations, Theorem 6.6 gives, respectively,

$$1525037909X^2 - 429769Y^2 = 4 \cdot 7453^2, \tag{6.24}$$

$$586208053X^2 - 1118057Y^2 = 4 \cdot 7453^2, \tag{6.25}$$

where $(X, Y) = (\lambda, \mu)$, $X, Y > 0$, and $XY < 7453^2$.

If the row is the second packet, then $(E, F) = (1145, 7453)$ and $\{W, m\} = \{23, 2919\}$. The slope reconstruction of Lemma 6.7 gives

(W, m)	k	(r_0, s_0)	Γ
$(23, 2919)$	417	$(7, 3144)$	429769
$(2919, 23)$	23	$(1, 57128)$	1118057.

The corresponding complementary divisors Γ^* are 1525037909 and 586208053. Therefore the residual equation (6.8) is again (6.24) or (6.25), now with $(X, Y) = (j, G)$. Since $B = E^2H < M_0^2 = E^2F^2$, we have $XY = jG = H < F^2 = 7453^2$. All four cases have the same stated bounds.

It remains to exclude positive solutions of these two equations. Put $N = 7453$, and write either equation as $RX^2 - SY^2 = 4N^2$. In both cases $R > 8N^2$. Since $X \geq 1$,

$$SY^2 = RX^2 - 4N^2 > \frac{R}{2}X^2.$$

Multiply by X^2 and use $XY < N^2$. This gives the finite bound

$$RX^4 < 2SN^4. \tag{6.26}$$

For the first equation,

$$1525037909 \cdot 1149^4 > 2 \cdot 429769 \cdot 7453^4,$$

so $X \leq 1148$. Reducing that equation modulo $S = 429769 = 137 \cdot 3137$, and inverting R modulo S, gives

$$X^2 \equiv 275216 \pmod{429769}.$$

Its four roots between zero and 429768 are

$$85941, \quad 104763, \quad 325006, \quad 343828.$$

None lies in $1 \leq X \leq 1148$, so this case is impossible.

For the second equation,

$$586208053 \cdot 1853^4 > 2 \cdot 1118057 \cdot 7453^4,$$

so $X \le 1852$. Reduction modulo $1118057 = 137 \cdot 8161$ gives

$$X^2 \equiv 391856 \pmod{1118057}.$$

Its four roots in the standard residue interval are

$$129197, \quad 471959, \quad 646098, \quad 988860.$$

Again none is small enough. Each root list is complete because each of its two distinct odd prime moduli has exactly two roots, and the Chinese remainder theorem combines their four choices into exactly four residues. The supplementary programs verify the prime factors, all four residues, and the finite bounds independently. □

6.4 An independent canonical-continuant filter and coefficient 300000

Proposition 6.8 already eliminates the sole odd companion-admissible row below 100000. We retain the following different necessary test as an independent audit of that rejection and because the even 100000 tier uses it. We use only a necessary part of the classical Cohn–Christoffel description of a Markoff occurrence [7, 8, 13]; the precise finite continued-fraction form needed here is supplied by [5, Theorem 4.1].

Lemma 6.9 (Canonical continuant necessity)**.** *For every occurrence of a Markoff number $M > 2$, there is an integer ρ satisfying*

$$\rho^2 \equiv -1 \pmod{M}, \qquad \frac{M}{3} < \rho < \frac{M}{2}, \tag{6.27}$$

such that every partial quotient of the canonical finite regular continued fraction of M/ρ (whose last partial quotient is greater than 1) belongs to $\{1, 2\}$.

Proof. The Markoff-tree construction assigns a reduced slope to each ordered occurrence with $M > 2$. Theorem 4.1 of Çanakçı and Schiffler [5] gives that occurrence an even palindrome of continued-fraction digits

$$[a_r, \dots, a_1, a_1, \dots, a_r], \qquad a_i \in \{1, 2\}, \quad a_r = 2.$$

Call the digits $c_1, \dots, c_{2r}$. Each matrix associated with a digit is symmetric. Transposing their product reverses the digits, and the palindrome leaves the product unchanged. The continued-fraction matrix is therefore

$$\prod_{i=1}^{2r} \begin{pmatrix} c_i & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} M & \rho \\ \rho & \sigma \end{pmatrix}. \tag{6.28}$$

Each factor has determinant -1. The number of factors is even, so $M\sigma - \rho^2 = 1$, proving $\rho^2 \equiv -1 \pmod{M}$.

The usual matrix rule for a continued fraction gives

$$M/\rho = [c_1; c_2, \dots, c_{2r}].$$

Its last digit is $2 > 1$, so it is the canonical finite expansion. Its first digit is also two, and the tail $[c_2; \dots, c_{2r}]$ is strictly greater than one. Hence $2 < M/\rho < 3$, or $M/3 < \rho < M/2$. Every digit is one or two, as required. □

For the exceptional row (6.4),

$$M_0 = TO = 8533685 = 5 \cdot 29 \cdot 229 \cdot 257,$$

and

$$3M_0 - 2 = 137 \cdot 186869, \qquad 3M_0 + 2 = 3137 \cdot 8161.$$

All four displayed factors are primes congruent to 1 (mod 4). The four prime factors of M_0 give sixteen CRT roots of -1. Exactly three lie in $(M_0/3, M_0/2)$:

$$2863767, \qquad 2981473, \qquad 3368483.$$

Their continued fractions are

$$[2, 1, 48, 1, 2, 2, 1, 1, 1, 1, 2, 2, 1, 48, 1, 2],$$
$$[2, 1, 6, 3, 1, 6, 3, 1, 1, 3, 6, 1, 3, 6, 1, 2],$$
$$[2, 1, 1, 6, 1, 76, 76, 1, 6, 1, 1, 2].$$

Each contains a digit larger than 2, so Lemma 6.9 rejects the last row. The exact enumeration has no other companion-admissible row below 100000. This continuant rejection is independent of the residual crossed-Pell rejection above.

Theorem 6.10 (Final odd exact tier)**.** *Every hypothetical odd collision satisfies*

$$Hn, Wm \geq 100000, \qquad F^4, E^4 \geq 300000M + 200000. \tag{6.29}$$

Proof. Each collision packet satisfies the local block conditions (5.2) and the companion condition of Lemma 6.2. The hand and local arguments exclude products below 120. In the complete enlarged-domain enumeration for $120 \leq xy < 100000$, only the row (6.4) passes the companion test.

Proposition 6.8 excludes that row for either packet and either coordinate order. There is also an independent rejection: Lemma 6.9 requires a root with only one and two as continued-fraction digits, and the three displayed possible roots all fail this test. Thus no collision packet can have product below 100000.

Apply the resulting floor to each defect equation:

$$T^4 = x^2 + y^2 + 3Mxy \geq (3M + 2)xy \geq 100000(3M + 2) = 300000M + 200000.$$

These are the claimed inequalities for F and E. □

Remark 6.11 (Nested status). The 99 tier is fully hand proved. The 360, 201411, and 300000 tiers are computer-assisted theorems based on an exact finite reduction. The 201411 tier is not conditional for actual odd Markoff collisions: Lemma 6.2 makes its added hypothesis automatic. The final row has two independent exact rejections: the residual crossed-Pell test and the canonical-continuant test. Each is a necessary condition for an actual collision or occurrence in the role where it is used. The tiers are retained separately to expose exactly where each gain enters.

Table 3: All survivors in the intentionally enlarged domain with $120 \le xy < 100000$. Eighteen have $\gcd(x, y) = 1$. The last column records a domain or companion obstruction; RP/CF denotes the sole row passed to, and then rejected independently by, the residual crossed-Pell and continued-fraction filters.

xy	x	y	T	O	obstruction
120	5	24	601	603005	$\ell = 7$
624	16	39	1777	2997481	$\ell = 7$
7161	31	231	173	241	$\ell = 19$
8601	1	8601	377	2069	$\ell = 7$
9189	9	1021	101	37	$\ell = 11$
10080	1	10080	269513	647377152493	$\ell = 6359$
10512	9	1168	9409	26413369	$\ell = 7$
26145	9	2905	19093	88738609	$\ell = 191$
41496	56	741	552217	1352701049849	$\ell = 23$
41565	15	2771	197	61	$\ell = 47$
42837	1	42837	141481	22037080561	$\ell = 7$
43557	3	14519	6305	1918121	$\ell = 7$
48960	9	5440	163501	29757646189	$\ell = 7$
49749	23	2163	137	17	$\ell = 11$
67137	23	2919	1145	7453	RP/CF
68880	41	1680	2824081	108997629688121	$\ell = 11$
74520	5	14904	1073	5525	$\ell = 7$
75000	125	600	3005	120601	$\gcd(x, y) = 25$; also $\ell = 7$
93093	3	31031	2305	43849	$\ell = 43$

7 Even normalization and three certified tiers

The preceding unitary allocation was stated for odd maxima because the prime 2 cannot be inserted into Lemma 3.4. The even case is nevertheless accessible after a forced normalization by 4. The congruence used below is classical [16, Theorem 1]; we include the short derivation needed for the packet construction.

Lemma 7.1 (Parity of an even maximum)**.** *Let $M > 2$ be an even Markoff number and write $M = 2M_1$. Then*

$$M \equiv 2 \pmod{32}, \qquad M_1 \equiv 1 \pmod{16}. \tag{7.1}$$

In every occurrence (a, b, M), the entries a, b are $1 \pmod 4$.

Proof. Pairwise coprimality makes a, b odd. If $4 \mid M$, reducing the Markoff equation modulo four would give $1 + 1 \equiv 0$, which is false. Thus $M = 2M_1$ with M_1 odd.

Every odd prime dividing an entry is $1 \pmod 4$, by Lemma 3.3. It follows that $a, b, M_1 \equiv 1 \pmod 4$,

including when one of them is one. Write

$$a = 1 + 4u, \qquad b = 1 + 4v, \qquad M_1 = 1 + 4w.$$

Substitute these expressions in $a^2 + b^2 + 4M_1^2 = 6M_1ab$ and reduce modulo 32. The result is

$$16((u^2 - u) + (v^2 - v)) - 24w \equiv 0 \pmod{32}.$$

Both $u(u-1)$ and $v(v-1)$ are even, so the first term vanishes modulo 32. Hence $32 \mid 24w$, or $4 \mid w$. Therefore $M_1 = 1 + 4w \equiv 1 \pmod{16}$, and doubling gives $M \equiv 2 \pmod{32}$. □

For an even ordered collision, put

$$A_0 = \frac{A}{4}, \quad B_0 = \frac{B}{4}, \quad C_0 = \frac{C}{4}, \quad \widehat{A}_0 = \frac{\widehat{A}}{4}, \quad \widehat{B}_0 = \frac{\widehat{B}}{4}. \tag{7.2}$$

These are integers. Indeed, all four smaller entries are $1 \pmod 4$, so the definitions of A, B, C give $4 \mid A, B, C$; and, since $M \equiv 2 \pmod 4$,

$$\widehat{A} = 3Maa' - a'b - ab' \equiv 2 - 1 - 1 \equiv 0 \pmod{4},$$

with the same calculation for $\widehat{B}$. Dividing the determinant identity and the complementary conics by 16 gives

$$A_0B_0 = M_1^2C_0, \tag{7.3}$$

$$A_0^2 + \widehat{A}_0^2 + 3MA_0\widehat{A}_0 = M_1^4, \tag{7.4}$$

$$B_0^2 + \widehat{B}_0^2 + 3MB_0\widehat{B}_0 = M_1^4. \tag{7.5}$$

Theorem 7.2 (Normalized even collision allocation)**.** *If an even Markoff number $M > 2$ admits an ordered collision, then there are coprime proper unitary factors $F, E > 1$ of $M_1 = M/2$ and positive integers H, W, n, m such that*

$$M_1 = FE, \qquad A = 4F^2W, \quad B = 4E^2H, \quad C = 4HW, \tag{7.6}$$

and

$$\widehat{A} = 4F^2m, \qquad \widehat{B} = 4E^2n. \tag{7.7}$$

Moreover,

$$n^2 + 3MHn + H^2 = F^4, \tag{7.8}$$

$$m^2 + 3MWm + W^2 = E^4, \tag{7.9}$$

and

$$\gcd(F, H) = \gcd(H, n) = 1, \qquad \gcd(E, W) = \gcd(W, m) = 1. \tag{7.10}$$

Proof. We allocate each complete prime-power factor of M_1 to one side of the normalized identity. Let $p^e \parallel M_1$. The prime p is odd. Lemma 3.4 says it cannot divide both A_0 and B_0. Comparing prime exponents in (7.3) gives

$$\operatorname{ord}_p(A_0) + \operatorname{ord}_p(B_0) = 2e + \operatorname{ord}_p(C_0).$$

One exponent on the left is zero, so the other is at least $2e$. Allocate p^e to F if its square divides A_0, and to E if its square divides B_0. This gives

$$M_1 = FE, \quad \gcd(F, E) = 1, \quad A_0 = F^2 W, \quad B_0 = E^2 H,$$

and the normalized identity then gives $C_0 = HW$.

The factors are proper. If $F = 1$, then $M_1^2 = E^2 \mid B_0$, whereas $0 < B_0 < M^2/4 = M_1^2$. This is impossible. If $E = 1$, the same argument applies to A_0. Multiplying the normalized quantities by four proves (7.6).

For a prime block $p^e \parallel F$, suppose $v = \mathrm{ord}_p(\widehat{A}_0) < 2e$. In (7.4), the term $\widehat{A}_0^2$ has exponent $2v$. The exponents of A_0^2 and M_1^4 are at least $4e$, and that of $3MA_0\widehat{A}_0$ is at least $3e + v$. All exceed $2v$. Division by p^{2v} would leave one nonzero term modulo p, a contradiction. Hence $F^2 \mid \widehat{A}_0$. The other conic gives $E^2 \mid \widehat{B}_0$. Write $\widehat{A}_0 = F^2 m$, $\widehat{B}_0 = E^2 n$.

Substitution in the normalized conics, followed by division by F^4 and E^4, gives the two defect equations. If a prime divided F, H, or E, W, it would divide both A, B, which is excluded. Finally, a prime dividing both coordinates H, n of the first defect equation would divide F, contradicting $\gcd(F, H) = 1$. The other equation proves $\gcd(W, m) = 1$ in the same way. □

Combining Zhang's 2-adic congruence with Srinivasan's determinant splitting thus gives the normalized packet above. The underlying coprime factorization and the twice-prime-power consequence are not asserted as new [14]. To the best of our knowledge, the coefficient- $6TO$ packet and the explicit floors below have not previously been recorded.

Abstract either defect equation as

$$x^2 + y^2 + 6TOxy = T^4, \qquad TO = M_1 \equiv 1 \pmod{16}, \tag{7.11}$$

where $T, O > 1$ are coprime, neither is divisible by 3, every prime dividing TO is $1 \pmod 4$, and $\gcd(x, y) = 1$.

Lemma 7.3 (The sharp local floor 72)**.** *Every solution of* (7.11) *satisfying these block conditions has* $xy \geq 72$*. The value* 72 *is sharp for this local relaxation.*

Proof. Reduction modulo three shows that exactly one of x, y is divisible by three. Since $T^4 \equiv 1 \pmod{16}$ and $TO \equiv 1 \pmod{16}$, reduction modulo 16 gives

$$x^2 + 6xy + y^2 \equiv 1 \pmod{16}. \tag{7.12}$$

Take $x \leq y$. Under $xy < 72$, the coprime pairs satisfying these two congruence conditions are exactly

$$(1, 18), (1, 24), (1, 42), (1, 48), (1, 66),$$
$$(2, 3), (2, 9), (2, 27), (2, 33),$$
$$(3, 4), (3, 10), (3, 20), (5, 6), (5, 12), (6, 7).$$

For each pair, the equation forces $T \mid x^2 + y^2$ and

$$O = \frac{T^3 - (x^2 + y^2)/T}{6xy}. \tag{7.13}$$

List the divisors $T > 1$ having only prime divisors $1 \pmod 4$, and substitute each in this formula. The only integer values $O > 1$ are the following; every one fails a block condition.

x	y	T	O	obstruction
1	18	325	317853	$3 \mid O$
1	24	577	1334028	$2 \mid O$
1	42	1765	21818937	$3 \mid O$
1	48	2305	42522648	$2 \mid O$
1	66	4357	208865877	$3 \mid O$
2	3	13	61	$TO \equiv 9 \pmod{16}$
3	4	25	217	$7 \mid O$
5	6	61	1261	$TO \equiv 9 \pmod{16}$
6	7	85	2437	$TO \equiv 9 \pmod{16}$

Thus $xy < 72$ is impossible. To check sharpness for the stated local conditions, take

$$(x, y, T, O) = (8, 9, 145, 7057), \qquad 145 = 5 \cdot 29.$$

The number 7057 is prime and $1 \pmod 4$; the two blocks are coprime and have product $1 \pmod{16}$. Moreover,

$$8^2 + 9^2 = 145, \qquad 1 + 432 \cdot 7057 = 145^3.$$

Multiplying the second identity by 145 verifies the defect equation. This admissible local row has product 72, proving sharpness for the relaxation. □

The separate even certificate enumerates the intentionally enlarged domain $72 \le xy < 100000$ twice, first by pairs and then by products. It retains 65 local rows, of which 63 satisfy $\gcd(x, y) = 1$. Applying Lemma 6.2 to the actual candidate maximum $M = 2TO$ leaves exactly four primitive rows:

xy	x	y	T	O	$M = 2TO$
1014	3	338	61	37	4514
7140	84	85	14281	67987081	1941847007522
74256	272	273	148513	7352086561	2183760862867586
96624	9	10736	613	397	486722

(7.14)

Since $M = 2M_1$, one has $3M \pm 2 = 2(3M_1 \pm 1)$; hence the odd-prime companion test is equivalently applied to $3M_1 \pm 1$. The first row proves sharpness for that relaxation: with $M_1 = 2257$,

$$3M_1 - 1 = 2 \cdot 5 \cdot 677, \qquad 3M_1 + 1 = 2^2 \cdot 1693,$$

so all odd companion primes are $1 \pmod 4$.

The canonical test of Lemma 6.9 uses the even maximum $M = 2TO$, not the half-maximum. For $M = 4514$, the interval roots are $1597, 1819$, with continued fractions

$$[2, 1, 4, 1, 3, 3, 1, 4, 1, 2], \qquad [2, 2, 13, 13, 2, 2].$$

The middle two rows in (7.14) have no root of -1 in $(M/3, M/2)$. For $M = 486722$, the interval roots are $191291, 193673$, with continued fractions

$$[2,1,1,5,7,1,2,2,1,7,5,1,1,2], \quad [2,1,1,18,1,1,3,3,1,1,18,1,1,2].$$

Every displayed expansion has a digit greater than 2. Thus all four rows fail the canonical-continuant necessity.

Theorem 7.4 (Certified even tiers)**.** *Every hypothetical even collision satisfies the hand tier*

$$Hn, Wm \geq 72, \qquad F^4, E^4 \geq 216M + 144. \tag{7.15}$$

The exact companion-prime certificate strengthens this to

$$Hn, Wm \geq 1014, \qquad F^4, E^4 \geq 3042M + 2028, \tag{7.16}$$

and the canonical-continuant certificate gives

$$Hn, Wm \geq 100000, \qquad F^4, E^4 \geq 300000M + 200000. \tag{7.17}$$

The floors 72 *and* 1014 *are sharp for their stated relaxations;* 100000 *is a certified cutoff, not a sharpness claim.*

Proof. Each packet in Theorem 7.2 has $(T, O) = (F, E)$ or (E, F), with actual maximum $M = 2TO$. Lemma 7.3 gives the hand floor $xy \geq 72$. In the complete search above this floor and below 100000, the companion filter leaves exactly the four displayed rows, whose smallest product is 1014. Thus the necessary companion condition improves the floor to 1014.

The continued-fraction condition must be applied modulo the full maximum M. Two of those four rows have no root in its required interval, and the other two have only the displayed continued fractions, each containing a digit above two. Thus none is a Markoff occurrence, and the final floor is 100000.

For each established floor B_0, either packet satisfies

$$T^4 = x^2 + y^2 + 3Mxy \geq (3M + 2)xy \geq B_0(3M + 2).$$

Substituting $B_0 = 72, 1014, 100000$ gives the three claimed quartic bounds. The rows at products 72 and 1014 prove sharpness only for their specified relaxations; no row establishes sharpness at 100000. □

Remark 7.5 (Provenance of the even tiers). The 72 tier is proved entirely in the article. The 1014 and 100000 tiers are computer-assisted theorems certified by two separately implemented exact-integer traversals supplied in the supplementary archive. The companion and canonical filters are necessary for every actual Markoff occurrence by Lemmas 6.2 and 6.9.

8 Intrinsic unitary-gap and tail consequences

For the odd case, it is useful to state all four tiers at once. Let B_0 be one of

$$33, \quad 120, \quad 67137, \quad 100000,$$

and put $\mathcal{C} = 3B_0$. The corresponding tier says

$$F^4, E^4 \geq \mathcal{C}M + \frac{2\mathcal{C}}{3}. \tag{8.1}$$

Theorem 8.1 (Intrinsic unitary-gap criterion)**.** *Let M be an odd Markoff number. At any certified coefficient $\mathcal{C} \in \{99, 360, 201411, 300000\}$,*

$$u_*(M)^4 < \mathcal{C}M + \frac{2\mathcal{C}}{3} \quad \Longrightarrow \quad M \text{ is unique.} \tag{8.2}$$

Proof. The maximum $M = 1$ has the single ordered occurrence $(1, 1, 1)$. For $M > 1$, suppose a collision existed and set $d = \min(F, E)$ in Theorem 3.5. This is a unitary divisor of M and satisfies $d \le \sqrt{M}$, so it belongs to the set defining $u_*(M)$. Hence $d \le u_*(M)$.

The chosen collision tier gives $d^4 \ge \mathcal{C}M + 2\mathcal{C}/3$. Together with the assumed inequality, this would give

$$\mathcal{C}M + 2\mathcal{C}/3 \le d^4 \le u_*(M)^4 < \mathcal{C}M + 2\mathcal{C}/3,$$

a contradiction. Thus no collision exists. □

Theorem 8.2 (Even intrinsic unitary-gap criterion)**.** *Let $M > 2$ be an even Markoff number and put $M_1 = M/2$. At any certified even coefficient*

$$\mathcal{C} \in \{216, 3042, 300000\},$$

with respectively hand, companion-certificate, and continuant-certificate provenance,

$$u_*(M_1)^4 < \mathcal{C}M + \frac{2\mathcal{C}}{3} \quad \Longrightarrow \quad M \text{ is unique.} \tag{8.3}$$

Proof. Suppose a collision existed. Theorem 7.2 gives a unitary split $M_1 = FE$. Its smaller factor $d = \min(F, E)$ is a unitary divisor at most $\sqrt{M_1}$, so $d \le u_*(M_1)$.

The chosen tier in Theorem 7.4 bounds each factor's fourth power below by $\mathcal{C}M + 2\mathcal{C}/3$. Therefore

$$\mathcal{C}M + 2\mathcal{C}/3 \le d^4 \le u_*(M_1)^4 < \mathcal{C}M + 2\mathcal{C}/3,$$

which is impossible. Notice that the divisor function uses M_1, while the quartic bound uses the full maximum M. □

For a composite tail, define

$$\mathcal{G}(N) = \frac{N}{u_*(N)^4} = \min_{\substack{UV=N \\ \gcd(U,V)=1}} \frac{\max(U, V)}{\min(U, V)^3}. \tag{8.4}$$

The equality follows by putting $d = \min(U, V)$: the displayed ratio is N/d^4, minimized by the largest unitary $d \le \sqrt{N}$.

Theorem 8.3 (Composite unitary tail)**.** *Let $M = kN$ be an odd Markoff number with* $\gcd(k, N) = 1$. *At any certified coefficient $\mathcal{C}$, the condition*

$$k^3 u_*(N)^4 \le \mathcal{C}N, \quad \textit{equivalently} \quad k^3 \le \mathcal{C}\mathcal{G}(N), \tag{8.5}$$

guarantees that M is unique.

Proof. Assume there is a collision, with unitary split $M = FE$. Because $\gcd(k, N) = 1$, every prime-power block of N lies wholly in one of these two factors. Write its resulting split as $N = XY$, with $\gcd(X, Y) = 1$, and choose the names so that $X \le Y$. Then $X \le u_*(N)$.

The collision factor containing X has the form dX, where d is a unitary divisor of k. It is therefore at most $ku_*(N)$. Apply the collision tier to this factor and the assumed bound to its upper estimate:

$$\mathcal{C}kN + \frac{2\mathcal{C}}{3} \le (dX)^4 \le k^4 u_*(N)^4 \le \mathcal{C}kN.$$

The positive term $2\mathcal{C}/3$ makes this chain impossible. It also explains why a non-strict inequality is allowed in the hypothesis. □

Taking $N = q^e$ makes $u_*(N) = 1$, proving Corollary 1.2. More generally, if $N = XY$ has exactly two coprime primary blocks with $X < Y$, then (8.5) is $(kX)^3 \le \mathcal{C}Y$.

The even analogue applies to the odd core of the maximum.

Theorem 8.4 (Even composite unitary tail)**.** *Let $M = 2kN$ be an even Markoff number with $\gcd(k, N) = 1$. At any certified even coefficient $\mathcal{C} \in \{216, 3042, 300000\}$, the condition*

$$k^3 u_*(N)^4 \le 2\mathcal{C}N \tag{8.6}$$

guarantees that M is unique.

Proof. Suppose there is a collision. Its unitary factorization is now of $M/2 = kN$. Split the complete prime-power blocks of N according to the two collision factors, writing $N = XY$ with $X \le Y$. As before, $X \le u_*(N)$, and the collision factor containing X is dX for a unitary divisor d of k. Thus $dX \le ku_*(N)$.

The even tier uses $M = 2kN$. It and the assumed bound give

$$2\mathcal{C}kN + \frac{2\mathcal{C}}{3} \le (dX)^4 \le k^4 u_*(N)^4 \le 2\mathcal{C}kN,$$

a contradiction. Hence the maximum is unique. □

Example 8.5 (The tier transition at 51641)**.** The Markoff triple $(13, 1325, 51641)$ shows directly that

$$51641 = 113 \cdot 457$$

is a Markoff number. It has $u_*(51641) = 113$ and $113^4 = 163047361$. The right sides of (8.2) are

$$5112525, \quad 18591000, \quad 10401199725, \quad 15492500000$$

for the four coefficients in order. Thus this intrinsic test does not close the number at tiers 99 or 360, but it does at tier 201411, before either final-row rejection is used. The example illustrates the logical role of the intermediate tier and proves that the present sufficient unitary-gap criterion is strictly stronger than Button's coefficient-1 criterion. It is not a claim that the uniqueness of 51641 was previously unknown by all other methods.

9 Certificate protocol and reproducibility

Use of generative AI in the research methodology. OpenAI ChatGPT and Codex were used for mathematical exploration, provisional proof drafting and review, exact-arithmetic code development and testing, reference checking, and manuscript preparation. This assistance included substantive research and writing, beyond copy editing. AI outputs were treated as proposals requiring mathematical justification; the proofs and the supplied executable checks provide the stated evidence. The author directed the research and is responsible for the final text, arguments, references, and code. The supplied checks are exact computational certificates, not a formal proof-assistant verification or an external referee report.

The supplementary archive contains six current proof programs. Four certify the odd and normalized-even defect-product enumerations: the two primary programs enumerate pairs first, emit deterministic JSON/CSV/text objects, and include internal product-order cross-checks; the two standalone programs enumerate products first, import no primary code, and independently implement factorization, divisors, admissibility, companion filtering, CRT, and continued fractions. Two further, independently structured programs certify the odd residual crossed-Pell rejection. Specifically, the files are

1. `verify_four_tiers.py` and `verify_four_tiers_product_first.py` for the four odd tiers;
2. `verify_odd_crossed_pell.py` and `verify_odd_crossed_pell_divisor_first.py` for the residual allocation, the four packet/coordinate orientations, the complete modular root lists, and the product-chamber bounds used in the final odd rejection;
3. `verify_even_tiers.py` and `verify_even_tiers_product_first.py` for the three normalized even tiers;
4. deterministic machine-readable results, exact tables, and reference transcripts in the following four directories.

   `generated/` `generated_even/`
   `generated_crossed_pell/` `generated_crossed_pell_divisor_first/`

5. `MANIFEST.sha256`, hashing every archived source and result; and
6. `legacy/verify_defect_120.py` and `legacy/verify_defect_global_100000.py`, retained as provenance for the original odd-tier programs from which the unified odd certificate was reconstructed.

The principal commands are

```
python3 verify_four_tiers.py --mode primary --output-dir odd_replay
python3 verify_four_tiers_product_first.py
python3 verify_four_tiers.py --mode both
python3 verify_odd_crossed_pell.py --output-dir crossed_pell_replay
python3 verify_odd_crossed_pell_divisor_first.py \
  --output-dir crossed_pell_divisor_replay
python3 verify_even_tiers.py --mode primary --output-dir even_replay
python3 verify_even_tiers_product_first.py
python3 verify_even_tiers.py --mode both
```

All six current verifiers are also replayed under `python3 -O`, so no mathematical check depends on optimization-sensitive assertions. Python 3.10 or later is required; the six proof programs use only the standard library. The separate plotting program `draw_fg.py` uses NumPy and Matplotlib. No floating-point value enters a certificate test. The new figure is a numerical illustration of the stated functions and is not used in the proof. Environment and timing metadata appear only in the run logs and are not part of the deterministic certificate files.

The odd primary program records the exhaustive counts in Table 4. The modulo-3 gate is the proved condition that exactly one of x, y is divisible by 3.

count	$33 \le xy < 120$	$120 \le xy < 100000$
pairs before modulo-3 gate	236	583217
pairs after modulo-3 gate	91	246960
divisors of $x^2 + y^2$ examined	530	2942721
admissible target divisors T	209	1307435
survivors in enlarged domain	0	19
of which $\gcd(x, y) = 1$	0	18

Table 4: Exact enumeration counts emitted by the primary verifier.

Table 5 records the corresponding pair-first counts for the even coefficient-$6TO$ traversal over $72 \le xy < 100000$.

count	value
pairs before modulo-3 gate	583354
pairs after modulo-3 gate	247013
divisors of $x^2 + y^2$ examined	2943053
admissible target divisors T	1307570
survivors in enlarged domain	65
of which $\gcd(x, y) = 1$	63
companion-prime survivors	4
canonical-continuant survivors	0

Table 5: Exact counts emitted by the primary even verifier.

The source hashes at the time of submission preparation are

file	SHA-256
verify_four_tiers.py	46b0d81367e8a94684a32f6dc08fce484ccd8ab005d4c0946d61b3b1b9a79cc5
verify_four_tiers_product_first.py	b908dfd28df6c540088f0caf5c83f5ea85d23752f638f95d938c9bb2f3cc8231
verify_odd_crossed_pell.py	6388d0e8a50c65a3f5e741ede96e2314931c991324185fd3ca3d5f6178ce18aa
verify_odd_crossed_pell_divisor_first.py	d7143b2be8471c77a1721775b41743fcd4e0be3b7303a96c584e9b7a3519ab72
verify_even_tiers.py	01f8bb050d3e16bcd9b424ec1d9fb6123c4f60f8feb446ff521df9e130df140c
verify_even_tiers_product_first.py	10c111181f945c49c0a4d3e72766f88a712c467ea7476d43a260c7cfef279ed9
legacy/verify_defect_120.py	8cd0ab00f3546e2340a2380b780fbb0415331fadc93264d99ed7078863a93a27
legacy/verify_defect_global_100000.py	805e94175bf906f6ba70b0e0da0ae6486d72476c6fc24a8dee42f825baff3874

The archive manifest is authoritative if any filename is moved or any source is revised.

To reproduce the mathematical certificate, a reader can use the following checks.

(i) verify the manifest before executing code;

(ii) run all six current verifiers under a standard interpreter and under `python3 -O`;

(iii) compare the nineteen-row enlarged-domain CSV against Table 3 and verify that eighteen rows have $\gcd(x, y) = 1$;

(iv) compare the sixty-five-row even tables, verify that sixty-three rows are primitive, and verify the standalone program's canonical four-column row-serialization digest against the reference transcript;

(v) independently multiply each row into its odd or even defect equation;

(vi) verify every displayed companion witness and factorization in both parity sectors;

(vii) recompute the odd and even CRT roots, canonical-interval roots, and continued fractions;

(viii) compare the two residual-orientation tables, verify all four packet/coordinate cases, and recompute the two crossed equations, their product-chamber caps, and their complete modular square-root lists.

10 Scope and remaining problem

The argument gives a rigorous partial theorem, not a proof of full unicity. It establishes that a hypothetical odd collision must admit a proper unitary factorization of M whose two factors satisfy strong quartic lower bounds. For an even collision the corresponding proper unitary factorization is of the odd core $M/2$, with the same final quartic bound. In either parity the prime-power blocks therefore cannot be allocated too unevenly in this quantified sense. No fixed coefficient alone can exclude every increasingly balanced unitary allocation.

The determinant argument does give a precise blockwise switch for elementary occurrence roots. For a collision, set

$$t \equiv ba^{-1}, \qquad t' \equiv b'(a')^{-1} \pmod{M}.$$

Both satisfy $u^2 \equiv -1 \pmod{M}$. If $p^e \parallel F$, then $F^2 \mid A$ gives $tt' \equiv 1 \pmod{p^e}$, and hence $t' \equiv -t \pmod{p^e}$. If $p^e \parallel E$, then $E^2 \mid B$ gives $t' \equiv t \pmod{p^e}$. Thus a collision switches exactly the F-blocks for these ratio roots. The present paper does not identify this ratio root with the canonical interval root of Lemma 6.9 in a form uniform enough to finish the conjecture.

The crossed-Pell reconstruction in Section 6.3 recovers additional primitive-slope information and couples the two defect packets. For the final-row certificate, this coupling is more restrictive than the companion unitary and quadratic-residue conditions alone. It does not, however, uniformly identify an elementary occurrence root with the canonical interval root, nor does it preclude every balanced unitary allocation.

The determinant and continuant viewpoints therefore suggest a narrow next problem.

Problem 10.1 (Unitary switch separation)**.** For an odd ordered collision, let $K = M$; for an even ordered collision, let $K = M/2$. In either case write the collision split as $K = FE$. Relate the two ratio roots above to the canonical roots ρ, ρ' supplied by Lemma 6.9. In the even case, K is only

the split modulus: the canonical roots remain roots of -1 modulo the full maximum M. Prove that no two canonical roots attached to distinct occurrences can satisfy

$$\rho' \equiv -\rho \pmod{F}, \qquad \rho' \equiv \rho \pmod{E}$$

while both lie in the canonical interval and have only $\{1, 2\}$-digits.

The certified tiers treat both parity sectors and eliminate every collision whose relevant unitary allocation is sufficiently unbalanced. Problem 10.1 asks for both the missing canonical-root bridge and the universal separation needed in the balanced range.

Statements and Declarations

Data and code availability. Online Resource 1 (`ESM_1.zip`) supplies the six current proof programs, machine-readable certificate results, enlarged-domain row tables, reference transcripts, and SHA-256 manifest supporting this article. The programs require Python 3.10 or later and use only its standard library. No external empirical dataset is used.

Supplementary information. Online Resource 1: Exact finite certificates for the odd and normalized-even collision tiers, including two independent implementations of the residual crossed-Pell rejection, the program and parameters for Figure 1, and instructions for reproduction.

Generative AI assistance. The substantive uses of OpenAI ChatGPT and Codex in mathematical exploration, proof drafting, code development, testing, and writing are described in Section 9. The author is responsible for the article and its accompanying code and results.

Author contribution. Elleard Felix Webster Heffern formulated the research questions and framework, directed the investigation, and is responsible for the article and its accompanying materials.

Funding. This research received no funding.

Competing interests. The author declares no competing interests.

Related work by the author. The separate manuscript *Chebyshev Reference Bounds and Certified Spectral Refinement for Markov Numbers* develops bounds for matrix products and localization of possible collisions among Farey labels. The present article proves arithmetic constraints on a shared maximum and uses separate finite certificates. Neither manuscript requires the other's results. Earlier versions of the present arithmetic work circulated under the titles *Quantitative Unitary-Factor Bounds for Collisions of Markov Triples* and *Unitary-Factor Obstructions to Collisions of Markov Triples*; these are related versions of this work, not independent companion contributions.

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