AIGNER'S THREE MONOTONICITIES FROM FIBONACCI MATRICES AND SIX SNAKE PIECES

ELLEARD FELIX WEBSTER HEFFERN

Abstract. Starting from the established fact that Markov numbers count perfect matchings of their Markov snake graphs, we give a self-contained proof of the fixed-sum, fixed-numerator, and fixed-denominator orderings proposed by Aigner. The author's Fibonacci-matrix normalization begins with $(AC)_{11} - (B^2)_{11} = 2$ for consecutive odd-indexed Fibonacci blocks. Fibonacci matrices prove normalization and fixed-numerator monotonicity. The author's six-piece decomposition proves the remaining fixed-sum comparison and fixed-denominator monotonicity. We credit the matching correspondence to its original 2002–2003 REACH development and distinguish that history from the later ordering theorems.

hexagon:2611.00434v1

Version note. This edition is synchronized with the Aigner arguments in Section 5 of *Meridians on the Markov Surface*, version 21 (9 November 2026). Figures 1 and 2 use the identical figure files.

1. The starting correspondence and the three orderings

Write $m_{p/q}$ for the Markov number with the Farey label used in Aigner's conjectures. A snake graph is a chain of square tiles, each attached to the right of or above the preceding tile. Shared edges are counted once. A perfect matching covers every vertex exactly once. The Markov snake associated with a reduced $0 < p/q < 1$ is built as follows. Take the lower staircase from $(0, 0)$ to (q, p): at each horizontal coordinate j, step right and then up to height $\lfloor pj/q \rfloor$. Write right steps as a and up steps as b. Write this step sequence as $w_{p/q} = azb$. Remove its first and last letters; start with one square and replace each a of z by two added squares to the right and each b by two upward. The resulting snake has $2(p + q) - 3$ tiles.

Write $\mu(w)$ for its number of perfect matchings. The established Markov matching correspondence gives

$$\mu(w_{p/q}) = m_{p/q}. \tag{1}$$

For example, $w_{2/5} = a^3ba^2b$, its trimmed word is $aabaa$, and the eleven-tile snake has 194 matchings. Auxiliary rearranged words also specify snakes, counted by μ; they need not be canonical Markov snakes.

The matching interpretation originated in the REACH work of 2002–2003. Musiker's memorandum, dated 27 June 2002, proposed the recursive graph model [1]; Itsara, Musiker, Propp, and Viana's manuscript of 10 January 2003 stated and proved the snake-matching correspondence [2, Theorem 3.1]. Setting all edge weights to one gives the ordinary perfect-matching counts. Propp's later exposition, first posted in 2005, records the broader collaborative development [3]. These original contributions precede the later proofs of Aigner's orderings.

Aigner formulated these orderings in [4, Chapter 10]. Fixed-numerator monotonicity was proved by Rabideau and Schiffler [5]; all three orderings were proved by LPTV, Lee–Li–Rabideau–Schiffler, and McShane [6, 7, 8]. The author's Fibonacci matrices prove normalization and the fixed-numerator ordering. His six pieces prove the remaining fixed-sum comparison and the fixed-denominator ordering, using normalization to return to a canonical Markov snake. Every argument after the matching correspondence is proved below.

Theorem 1 (Aigner's three monotonicities)**.** *For $i > 0$, whenever both fractions are reduced and lie in $(0, 1)$,*

$$m_{p/q} > m_{(p+i)/(q-i)} \qquad \textit{(fixed sum)},$$
$$m_{p/q} < m_{p/(q+i)} \qquad \textit{(fixed numerator)},$$
$$m_{p/q} < m_{(p+i)/q} \qquad \textit{(fixed denominator)}.$$

Date: Version 4 — 11 November 2026.
2020 *Mathematics Subject Classification.* Primary 11D25; Secondary 05C70, 11B39.
Key words and phrases. Markov numbers, Aigner conjectures, snake graphs, perfect matchings, Fibonacci matrices.

2. Six pieces and two connector states

Figure 1 fixes the five three-tile pieces and the three one-square variants of type 6. In a cut between pieces, a matching uses both yellow connector edges or neither: the cut leaves an even number of vertices on either side. Denote these states by 1 and 0. At a selected port remove the two vertices already covered by its connector pair. Direct counting within the pieces gives

(2)

piece	00	01	10	11
1	5	2	2	1
2	5	3	3	2
3	5	2	3	1
4	5	3	2	1
5	5	2	2	1
6i	2	1	1	1

The state order is incoming, outgoing. The one-port end squares 6ii and 6iii have counts $(2, 1)$.

For the ordering surgeries, use a fixed one-ended chop: retain tile triples $(1, 2, 3), (5, 6, 7), \ldots$, with tiles $4, 8, \ldots$ providing paired connectors. A one-square remainder occurs exactly when $p + q$ is even and is counted with the closed end piece. For every connector sequence σ, multiply the applicable row entries in (2); then sum over σ, with outer ports closed. Each matching determines one sequence and one internal completion per piece, and gluing reverses the construction. This proves the counting rule, without an independence assumption. Equivalently it is a sum of terms $5^{k(\sigma)}3^{j(\sigma)}2^{\ell(\sigma)}$, where the exponents count factors of the indicated size. For the $2/5$ snake the piece list $2, 1, 2$ gives

$$5^3 + 5 \cdot 2 \cdot 3 + 3 \cdot 2 \cdot 5 + 3^2 = 194.$$

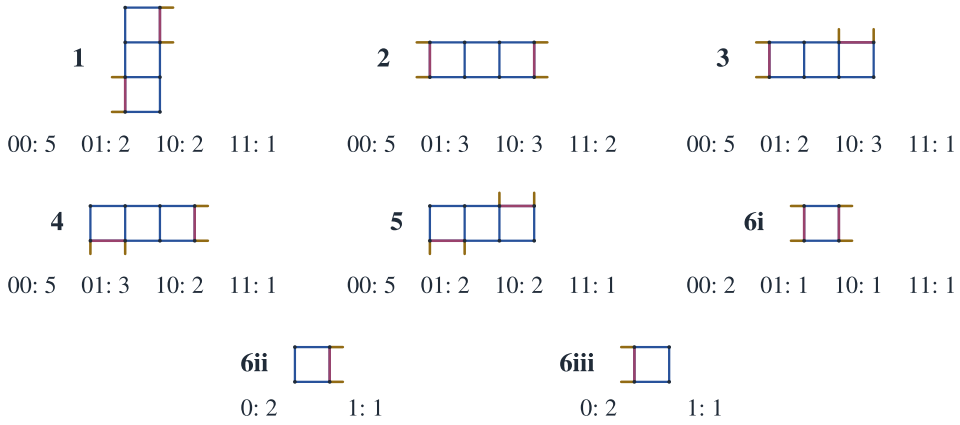


Figure 1. The six physical pieces. Gold edges are paired connectors, magenta marks ports, and dots mark tile vertices. Two-port counts use the order $00, 01, 10, 11$. The central square 6i has counts $(2, 1, 1, 1)$; the one-port end squares have $(2, 1)$. Types 1 and 5 have equal count rows but different physical orientations.

3. Normalization with Fibonacci matrices

The author proposed the Fibonacci-matrix formulation and normalization approach developed here. We begin with the comparison of three consecutive blocks and then apply it to complete products of run blocks.

With $F_0 = 0, F_1 = 1$ and $F_{-j} = (-1)^{j+1}F_j$, the basic odd-index identity is

(3)
$$F_{2k-1}F_{2k+3} - F_{2k+1}^2 = 1.$$

For example, $5 \cdot 34 - 13^2 = 1$. Consider the author's three consecutive Fibonacci blocks:

$$A = \begin{pmatrix} F_{2k+3} & F_{2k+1} \\ F_{2k+1} & F_{2k-1} \end{pmatrix},$$
$$B = \begin{pmatrix} F_{2k+1} & F_{2k-1} \\ F_{2k-1} & F_{2k-3} \end{pmatrix}, \tag{4}$$
$$C = \begin{pmatrix} F_{2k-1} & F_{2k-3} \\ F_{2k-3} & F_{2k-5} \end{pmatrix}, \qquad k \geq 2.$$

Applying (3) twice gives precisely the upper-left comparison used in normalization:

$$\begin{aligned}(AC)_{11} - (B^2)_{11} = &\left(F_{2k+3}F_{2k-1} - F_{2k+1}^2\right) \\ &+ \left(F_{2k+1}F_{2k-3} - F_{2k-1}^2\right) = 2.\end{aligned} \tag{5}$$

The same identities in the other entries give

$$AC - B^2 = \begin{pmatrix} 2 & 6 \\ 0 & 2 \end{pmatrix}, \qquad CA - B^2 = \begin{pmatrix} 2 & 0 \\ 6 & 2 \end{pmatrix}.$$

Thus replacing AC or CA by B^2 decreases the upper-left entry even inside a longer product: the surrounding Fibonacci blocks have nonnegative entries and positive upper-left entries. For $k = 3$ this reads $(AC)_{11} = 34 \cdot 5 + 13 \cdot 2 = 196 > 194 = 13^2 + 5^2 = (B^2)_{11}$.

3.1. **Run counts and general index shifts.** To apply the same comparison at arbitrary run lengths, index the blocks by

$$\mathcal{A}_t = \begin{pmatrix} F_{2t+3} & F_{2t+1} \\ F_{2t+1} & F_{2t-1} \end{pmatrix}, \qquad t \geq 0. \tag{6}$$

Thus $A = \mathcal{A}_k$, $B = \mathcal{A}_{k-1}$ and $C = \mathcal{A}_{k-2}$ in (4). The four boundary counts of a straight block of $2t+1$ tiles are $F_{2t+3}, F_{2t+1}, F_{2t+1}, F_{2t-1}$: these follow by conditioning on the first edge of a ladder, giving the Fibonacci recurrence; at $t = 0$ they are $(2, 1, 1, 1)$. Gluing successive runs with their paired connector states therefore gives, for $w = a^{h_1}b \cdots a^{h_p}b$ with $h_1 \geq 1$ and $h_j \geq 0$,

$$\mu(w) = [\mathcal{A}_{h_1-1}\mathcal{A}_{h_2} \cdots \mathcal{A}_{h_p}]_{11}. \tag{7}$$

The first run is shortened by the removed initial a; the final b closes the other end. Equivalently, a block-index list $(t_1, \ldots, t_p)$ represents $w = a^{t_1+1}ba^{t_2}b \cdots a^{t_p}b$, with $t_1 + \cdots + t_p = q - 1$.

Fibonacci addition gives the scalar index-shifting identity

$$F_{2r+1}F_{2s+1} - F_{2r-1}F_{2s+3} = F_{2(r-s)-2}.$$

Apply it to the two summands of each product entry to obtain

$$\mathcal{A}_r\mathcal{A}_s - \mathcal{A}_{r-1}\mathcal{A}_{s+1} = 2\begin{pmatrix} F_{2(r-s)-2} & F_{2(r-s)} \\ F_{2(r-s)-4} & F_{2(r-s)-2} \end{pmatrix}, \quad r \geq 1,\ s \geq 0. \tag{8}$$

Consequently, whenever two adjacent indices differ by at least two, moving one unit from the larger to the smaller strictly decreases the whole product's upper-left entry. The case $r = s + 2$ is exactly (5). The only remaining local comparison has adjacent indices differing by one; it is an interchange of the two blocks.

3.2. **Comparing the surrounding Fibonacci blocks.** For a finite list of block indices, define the ratio of its two open counts by

$$\rho(u_1, \ldots, u_m) = \frac{[\mathcal{A}_{u_1} \cdots \mathcal{A}_{u_m}]_{21}}{[\mathcal{A}_{u_1} \cdots \mathcal{A}_{u_m}]_{11}}, \qquad \rho(\varnothing) = 0.$$

All these ratios lie in $[0, 1)$. Direct multiplication gives

$$\rho(t, U) = \frac{F_{2t+1} + F_{2t-1}\rho(U)}{F_{2t+3} + F_{2t+1}\rho(U)}. \tag{9}$$

Its right-hand side increases strictly with $\rho(U)$, since (3) gives $\det \mathcal{A}_t = 1$. Moreover, another application of that identity gives

$$\frac{F_{2t+1}}{F_{2t+3}} - \frac{F_{2t+3} + F_{2t+1}}{F_{2t+5} + F_{2t+3}} = \frac{1}{F_{2t+3}(F_{2t+5} + F_{2t+3})} > 0.$$

Hence $\rho(t+1, U) < \rho(t, V)$ for any two finite tails U, V. Two lists can therefore be compared by cancelling equal initial indices: at the first unequal pair, the larger index gives the smaller ratio; an exhausted list has the smaller ratio than any nonempty remainder.

For adjacent blocks in positions $j, j+1$, the outward lists are $(t_{j-1}, \dots, t_1)$ on the left and $(t_{j+2}, \dots, t_p)$ on the right. At $r = s+1$, (8) becomes

$$\mathcal{A}_{s+1}\mathcal{A}_s - \mathcal{A}_s\mathcal{A}_{s+1} = 2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

After multiplication by the surrounding blocks, its upper-left entry has the sign of

$$\rho(t_{j+2}, \dots, t_p) - \rho(t_{j-1}, \dots, t_1). \tag{10}$$

Indeed, it equals this difference times twice the positive upper-left entries of the two surrounding products. This proves the needed surrounding-block test entirely in the same Fibonacci matrices.

Lemma 1 (Fibonacci-matrix normalization)**.** *For coprime positive p, q, every word $w = azb$ with q letters a and p letters b satisfies $\mu(w_{p/q}) \le \mu(w)$.*

Proof. For $p = 1$ there is only the word $a^q b$. Assume $p > 1$ and use the block-index list in (7). Its prefix discrepancies are

$$\delta_j = qj - p(1 + t_1 + \cdots + t_j) \quad (1 \le j \le p), \qquad \delta_0 = \delta_p = 0.$$

Changing (t_j, t_{j+1}) to $(t_j + 1, t_{j+1} - 1)$ decreases only δ_j, by p; the reverse change increases only δ_j, by p. We show that an appropriate change always decreases the matching count until $-p \le \delta_j \le 0$ for every j.

The sign checks use just a prefix-sum identity. Put $h_1 = t_1 + 1$ and $h_i = t_i$ for $i > 1$, the actual run lengths. Whenever $j - m \ge 0$ and $j + m \le p$,

$$\delta_{j-m} + \delta_{j+m} = 2\delta_j + p\left(\sum_{i=j-m+1}^{j} h_i - \sum_{i=j+1}^{j+m} h_i\right). \tag{11}$$

In the block indices, the left sum has the extra 1 precisely when it includes t_1.

First remove positive discrepancies. Choose the rightmost largest $\delta_j > 0$ and transfer one unit from t_{j+1} to t_j. Comparison with its two neighboring discrepancies gives $h_j < q/p < h_{j+1}$; coprimality makes these inequalities strict. Thus the transfer is allowed and $t_{j+1} \ge t_j + 1$; for $j = 1$ the gap is at least two. Gaps of at least two are handled by (8).

If $t_{j+1} = t_j + 1$, compare the outward lists. A first unequal pair with the left index larger would make the parenthesis in (11) nonnegative: the central difference is -1, the equal outward indices cancel, and the first unequal difference is at least 1. This contradicts maximality and the choice of the rightmost maximum. If the left list were exhausted first, its extra initial 1 would instead give $\delta_{2j} = 2\delta_j > \delta_j$, again impossible (including simultaneous exhaustion, when $2j = p$). The left outward ratio is therefore strictly larger than the right one. Reversing the interchange in (10) shows that the selected transfer strictly decreases the count. It also decreases $\sum_{j=1}^{p-1} \delta_j$. There are only finitely many nonnegative block-index lists of sum $q - 1$, so this phase ends with all $\delta_j \le 0$.

Then remove discrepancies below $-p$. Choose the leftmost smallest $\delta_j < -p$ and transfer one unit from t_j to t_{j+1}. For $j > 1$, the neighboring comparisons give $h_j > q/p > h_{j+1}$; for $j = 1$, the inequality $\delta_1 < -p$ gives $t_1 > q/p$, while $t_2 < q/p$. Thus $t_j \ge t_{j+1} + 1$ and the transfer is allowed. Again a gap of at least two is settled by (8).

In the gap-one case, a first unequal outward pair with the left index smaller would make the parenthesis in (11) nonpositive, contradicting minimality and the leftmost choice. If that pair reaches t_1, its extra 1 instead gives $\delta_{2j} \le 2\delta_j + p < \delta_j$, the same contradiction. If the right list were exhausted first, the identity would give $\delta_{2j-p} = 2\delta_j + p < \delta_j$. Simultaneous exhaustion would give $\delta_j = -p$. Thus the left outward ratio is strictly smaller than the right one, and (10) proves a strict decrease in count. The changed discrepancy is $\delta_j + p < 0$, so positivity is not reintroduced. This phase increases the bounded sum of discrepancies by p each time and terminates with $-p \le \delta_j \le 0$.

Finally, $\delta_j \equiv qj \pmod p$. For $1 \le j < p$, coprimality leaves exactly one possible value in $[-p, 0]$, forcing

$$1 + t_1 + \cdots + t_j = \left\lceil \frac{jq}{p} \right\rceil .$$

These are precisely the canonical run lengths. Every transfer has decreased the upper-left product entry, proving the assertion. □

For the canonical word, $h_j = \lceil jq/p \rceil - \lceil (j-1)q/p \rceil$. With $k = 1 + \lfloor (q-1)/p \rfloor$, the factors are the author's A, B from (4), and the end factors are B. For example, $m_{3/8} = [BAB]_{11}$ with $k = 3$. A similar normalization result appears in [6, Proposition 21]; the proof above is self-contained in the author's Fibonacci matrices.

4. The three ordering proofs

4.1. **Fixed numerator: Fibonacci matrices.**

Proof. Fibonacci recurrence gives, for $t \ge 1$,

$$\mathcal{A}_t - \mathcal{A}_{t-1} = \begin{pmatrix} F_{2t+2} & F_{2t} \\ F_{2t} & F_{2t-2} \end{pmatrix}.$$

All entries are nonnegative and the upper-left is positive. Start with $w_{p/(q+i)}$ and delete i internal letters a, retaining the first. Each deletion changes one factor $\mathcal{A}_t$ to $\mathcal{A}_{t-1}$ with $t \ge 1$. Surrounding factors are positive, so the matching count strictly falls. The resulting word ends at (q, p); normalization gives $m_{p/q} \le \mu(w_*) < m_{p/(q+i)}$. □

4.2. **Fixed sum: six pieces.**

Proof. At the same physical cuts, the changes $2 \to 3$ or 4, then 3 or $4 \to 5$, increase no entry of (2); the changed factors are $3 \to 2$ or $2 \to 1$. Rotation $5 \leftrightarrow 1$ preserves the row. Selecting the changed connector and no others exhibits a strictly smaller positive summand whenever an equal-direction join becomes unequal.

The trimmed canonical word begins and ends with a, contains no bb, and has $q - p - 1$ equal joins, all aa. At its rightmost equal join, interchange a, b strictly to the right. That suffix is alternating, so only the chosen join changes. Exactly one prescribed piece changes by the stated rules, while any rotated pieces keep their rows. Each surgery strictly decreases the count. Odd-numbered operations replace one horizontal step by a vertical step in net; even-numbered operations keep the totals. After $2i$ operations the endpoint is $(q - i, p + i)$. There are enough joins because $p + i < q - i$. For the resulting auxiliary word w_*, normalization gives $m_{(p+i)/(q-i)} \le \mu(w_*) < m_{p/q}$. A similar tail comparison appears in [6, Proposition 22]; here the comparison is proved using the author's six physical pieces. □

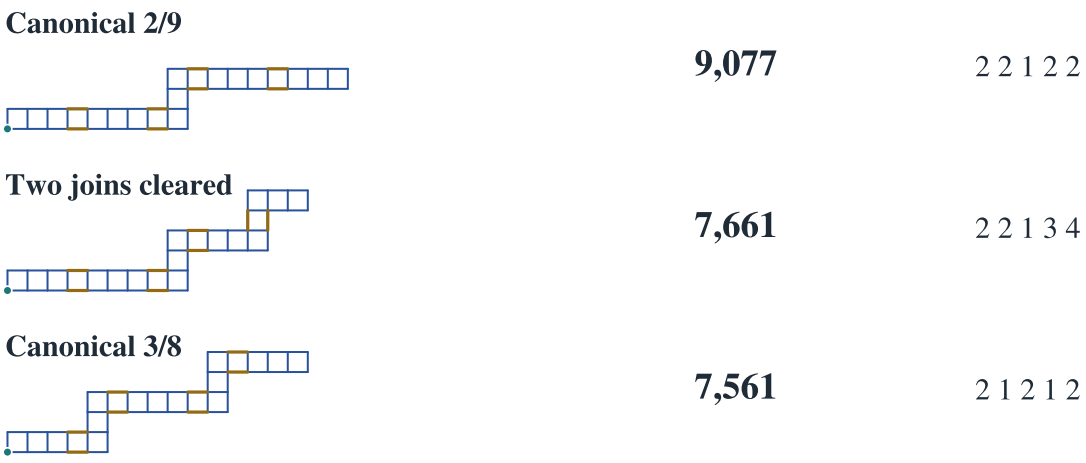


Figure 2. Fixed-sum surgery on the same five pieces. The canonical 2/9 snake has 9077 matchings; after two changes the auxiliary snake has 7661; normalization gives the 3/8 snake with 7561. The intermediate first change, not shown, has 8263. Gold pairs mark cuts and teal dots the starting corners.

4.3. **Fixed denominator: six pieces.**

Proof. For the fixed-denominator comparison, refine the same six-piece chop into single-square pieces of type 6, retaining odd-numbered tiles and using the even tiles as paired connectors. A retained square has counts $(2,1,1,1)$ for opposite ports and $(2,1,1,0)$ for adjacent ports. The zero forbids covering their common corner twice. Summing an intervening state regroups these squares into the original three-tile pieces.

Insert one internal b, adding two vertical squares. The affected local counts are the following complete table, with outside ports fixed:

	before				after			
directions	00	01	10	11	00	01	10	11
a,a	2	1	1	1	5	2	2	1
b,b	2	1	1	1	5	3	3	2
a,b	2	1	1	0	5	3	2	1
b,a	2	1	1	0	5	2	3	1

.

The endpoint caps have the corresponding rows with an outer port closed; the empty trimmed word has the increase $2 \to 5$. No outside count falls, and the all-zero choice witnesses a strict increase. Thus the two added squares compensate for every changed direction.

Starting from $w_{(p+i)/q}$, delete i internal b's and reverse this comparison. At least i are available. Normalizing the resulting word to (q,p) proves $m_{p/q} \le \mu(w_*) < m_{(p+i)/q}$. □

Author contributions and assistance. The author devised the Fibonacci-matrix formulation and six-piece approach. Generative AI assisted with proof development, exact checking, reference retrieval, and typesetting. The author is responsible for the final claims. No funding was received; the author declares no competing interests.

References

[1] G. Musiker, *A conjectured combinatorial interpretation for Markov numbers*, REACH memorandum, 27 June 2002. Original memorandum; REACH archive.

[2] A. Itsara, G. Musiker, J. Propp, and R. Viana, *Combinatorial interpretations for the Markov numbers*, REACH manuscript, 10 January 2003, Theorem 3.1. Original manuscript.

[3] J. Propp, *The combinatorics of frieze patterns and Markoff numbers*, arXiv:math/0511633, first posted 25 November 2005; published in *Integers* **20** (2020), A12.

[4] M. Aigner, *Markov's Theorem and 100 Years of the Uniqueness Conjecture*, Springer, 2013. doi:10.1007/978-3-319-00888-2.

[5] M. Rabideau and R. Schiffler, *Continued fractions and orderings on the Markov numbers*, Adv. Math. **370** (2020), 107231. doi:10.1016/j.aim.2020.107231.

[6] C. Lagisquet, E. Pelantová, S. Tavenas, and L. Vuillon, *On the Markov numbers: fixed numerator, denominator, and sum conjectures*, Adv. Appl. Math. **130** (2021), 102227. arXiv:2010.10335.

[7] K. Lee, L. Li, M. Rabideau, and R. Schiffler, *On the ordering of the Markov numbers*, Adv. Appl. Math. **143** (2023), 102453. arXiv:2010.13010.

[8] G. McShane, *Convexity and Aigner's conjectures*, preprint (2021), arXiv:2101.03316.

Unaffiliated, St. Louis, Missouri, USA

Email address: soldier@alum.mit.edu