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IRRATIONALITY OF $\chi_2(1/q)$, $\mathrm{Ti}_2(1/q)$ AND $\chi_2(2/q)$

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ABSTRACT. Let $\chi_2(z) = \sum_{n\geq 0} z^{2n+1}/(2n+1)^2$ be Legendre's chi function and $\mathrm{Ti}_2(z) = \sum_{n\geq 0}(-1)^n z^{2n+1}/(2n+1)^2$ the inverse tangent integral. We prove three theorems: $\chi_2(1/q)$ is irrational for every integer $q \geq 3$ with $q \neq 4$; $\mathrm{Ti}_2(1/q)$ is irrational for every integer $q \geq 5$; and $\chi_2(2/q)$ is irrational for every odd integer $q \geq 175$. All three come from one construction: the Hankel determinants of a moment functional with half-integer poles and geometric weight w, read at $w = q^{-2}$, $w = -q^{-2}$ and $w = 4q^{-2}$. Their denominators are bounded prime by prime, their size by a discrete log-gas argument against explicit dual certificates, and their nonvanishing by a reduction modulo an auxiliary prime. The three cases differ at the prime 2, where each needs its own lemma, in the certificates, and in how the nonvanishing lemma is transferred. For large q the three statements follow from the linear independence criteria of David, Hirata-Kohno and Kawashima; for the first two, a 1990 criterion of Hata, which we could not consult, also applies from a threshold unknown to us. Below these ranges the results are new as far as we know. All three theorems are formalised in Lean 4 with Mathlib and checked by the Lean kernel, using only Lean's three standard axioms.

CONTENTS

1. Introduction 3
1.1. Results 3
1.2. Context 3
1.3. The method 5
1.4. The formalisation 6
1.5. Organisation and notation 7
2. The machine 7
2.1. Data 7
2.2. The polynomial part 8
2.3. The pole expansion 8
2.4. Change of basis 9
3. The criterion 9

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4. Denominators I: the symbolic machine and the generic primes 10
4.1. The free ring 10
4.2. All odd primes: the Lagrange basis 11
4.3. One-level primes: a basis adapted to residue classes 12
4.4. The class count 13
4.5. The generic primes 14
4.6. The twist 14
5. Denominators II: the odd primes dividing q and the pole factor 15
5.1. The pole order 15
5.2. The q-adic gain 15
5.3. The odd part of the denominator 17
6. The size of $\Delta_h(\theta)$ 18
6.1. Dual certificates 18
6.2. Node expansion 18
6.3. A lattice energy inequality 19
6.4. Discretization 20
6.5. Proof of Theorem 6.2 21
7. Certificates and the shift lemma 22
7.1. The shift lemma 22
7.2. The certificates used 22
8. Nonvanishing: Lemma P 24
8.1. Statement 24
8.2. Steps 1–2: the Lagrange Gram matrix 25
8.3. Step 3: splitting by progressions 25
8.4. Step 4: integrality and block triangularity 25
8.5. Step 5: the diagonal blocks 26
8.6. Step 6: conclusion 27
9. Theorem A: $\chi_2(1/q)$ 27
9.1. The prime 2 at odd q 27
9.2. The prime 2 at even q: a Gauss bound 29
9.3. The prime 2 at even q: the weighted Gauss valuation 29
9.4. Proof of Theorem A 31
10. Theorem B: $\mathrm{Ti}_2(1/q)$ 32
10.1. The prime 2 at odd q 32
10.2. The prime 2 at odd q: the binomial basis 33
10.3. The prime 2 at even q 35
10.4. Nonvanishing 35
10.5. Proof of Theorem B 36
11. Theorem C: $\chi_2(2/q)$ 36
11.1. The prime 2 36
11.2. Nonvanishing 37
11.3. Proof of Theorem C 38
12. Limits of the method 38
13. The formalisation 39
13.1. What is formalised 39

13.2. The statements 39
13.3. Axioms 40
13.4. Correspondence with the paper 40
13.5. Differences from the written proof 40
13.6. How the certificates are checked 42
13.7. Building and replaying 43
13.8. What has to be trusted 44
Acknowledgements and disclosure 44
References 45

1. Introduction

1.1. **Results.** For $|z| \le 1$ let

$$\chi_2(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)^2} = \tfrac{1}{2}\big(\mathrm{Li}_2(z) - \mathrm{Li}_2(-z)\big),$$
$$\mathrm{Ti}_2(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)^2} = \mathrm{Im}\,\mathrm{Li}_2(iz)$$

be Legendre's chi function and the inverse tangent integral [14, §1.8, Ch. 2].

Theorem A. *For every integer $q \ge 3$ with $q \ne 4$, the number $\chi_2(1/q)$ is irrational.*

Theorem B. *For every integer $q \ge 5$, the number $\mathrm{Ti}_2(1/q)$ is irrational.*

Theorem C. *For every odd integer $q \ge 175$, the number $\chi_2(2/q)$ is irrational.*

The cases $q = 3$ and $q = 5$ of Theorem A were proved in [12], whose construction we extend here. For even q, $\chi_2(2/q) = \chi_2(1/(q/2))$ is covered by Theorem A unless $q/2 \in \{1, 2, 4\}$, so Theorem C is stated for odd q. With finer ball-arithmetic certificates, which are not part of this paper's proof, the argument would reach $q = 173$ (Remark 11.4).

1.2. **Context.** In Hata's notation put $L(\alpha, k; x) = \sum_{n\ge1} x^n/(n-\alpha)^k$. Then (1.1)

$$\chi_2(\tfrac{1}{q}) = \tfrac{q}{4} L(\tfrac{1}{2}, 2; \tfrac{1}{q^2}), \quad \mathrm{Ti}_2(\tfrac{1}{q}) = -\tfrac{q}{4} L(\tfrac{1}{2}, 2; -\tfrac{1}{q^2}), \quad \chi_2(\tfrac{2}{q}) = \tfrac{q}{8} L(\tfrac{1}{2}, 2; \tfrac{4}{q^2}),$$

so each of our numbers is a single shifted dilogarithm (a value of Lerch's transcendent with shift $\frac{1}{2}$) at a rational point of size about q^{-2}.

- *The criterion of David, Hirata-Kohno and Kawashima.* Their Theorem 2.1 in [5] (shifts $\frac{1}{2}$, depths ≤ 2, one rational point α/β) gives the linear independence of 1, $L(\frac{1}{2}, 1; \alpha/\beta)$, $L(\frac{1}{2}, 2; \alpha/\beta)$ over $\mathbb{Q}$ when an explicit quantity V is positive. By our evaluation, at $\alpha/\beta = \pm q^{-2}$ the condition is $\log q^2 > 2\log 2 + 6\log 3 + 8$, i.e. $q \ge 2949$; this covers

$\chi_2(1/q)$ and $\mathrm{Ti}_2(1/q)$ for $q \geq 2949$. At $\alpha/\beta = 4/q^2$ the criterion as printed gives $q \geq 11{,}795$ for $\chi_2(2/q)$. As far as we can see, the last step of their proof gives only the weaker form of the criterion, with $(\rho m + 1)\log\|\alpha\|$ in place of $\rho m \log\|\alpha\|$, which gives $q \geq 23{,}587$; we compare with the printed threshold 11,795. (For odd q both thresholds are the first odd integers above 11,793.2 and 23,586.4.) These specialisations are ours; [5] does not mention χ_2 or Ti_2. The other criteria of these authors [3, 4] give larger thresholds here.

- *Hata's criterion.* Hata [9] proved linear independence measures for 1 and values $L(\alpha_j, k_j; 1/Q)$ with rational shifts α_j, for integers $|Q| > Q_0$ with an explicit Q_0. We have not been able to consult this paper; what follows is secondhand, from its zbMATH review and from later citations [3, 5]. Negative Q are allowed, and the stated application is $\mathrm{Li}_2(1/Q) \notin \mathbb{Q}$ for $Q \geq 12$ or $Q \leq -8$; Hata himself attributes this application to [9] in [10, §1]. Applied with $Q = q^2$ and $Q = -q^2$, the theorem therefore gives $\chi_2(1/q) \notin \mathbb{Q}$ and $\mathrm{Ti}_2(1/q) \notin \mathbb{Q}$ for all sufficiently large q, by (1.1). We do not know its threshold for the shift $\frac{1}{2}$, at either sign of Q. A heuristic reconstruction of what we take to be Hata's construction at shift $\frac{1}{2}$ and $Q = q^2$ (calibrated so that it reproduces his unshifted constant $11.837\ldots$) suggests a threshold near $q \approx 55$–85 for $\chi_2(1/q)$; this is an estimate, not a statement about his paper, and it says nothing about $Q = -q^2$, where his unshifted thresholds are better. The point $4/q^2$ of $\chi_2(2/q)$ is not of the form $1/Q$, so Theorem C lies outside the theorem as it is reviewed.
- *Single dilogarithm values do not suffice.* Hata [10, Thm. 1.1] proved $\mathrm{Li}_2(1/k) \notin \mathbb{Q}$, and the linear independence of 1, $\log(1 - 1/k)$, $\mathrm{Li}_2(1/k)$, for every integer $k \geq 7$ or $k \leq -5$, with Padé-type approximations to the unshifted dilogarithm built from Legendre-type polynomials; it does not treat shifted values. Recent work proves $\mathrm{Li}_2(1/n) \notin \mathbb{Q}$ for every integer $n \neq 0$ [25], after special cases [21, 17, 15], and $\mathrm{Li}_2(p/q) \notin \mathbb{Q}$ for $q > (4e^4/27)p^3$ [20] (all recent, unrefereed). Two irrational numbers can have a rational difference, so none of these says anything about $\chi_2(z) = \frac{1}{2}(\mathrm{Li}_2(z) - \mathrm{Li}_2(-z))$ or $\mathrm{Ti}_2(z) = \mathrm{Im}\,\mathrm{Li}_2(iz)$. The independence results of Rhin–Viola [18] and Viola–Zudilin [24] concern other pairs of points.
- *Integer arguments.* Bahçıvan [2] (unrefereed) proves, from the criterion of [5] and the reciprocity formulas, that $\mathrm{Ti}_s(n)$, $\mathrm{Re}\,\chi_2(n)$ and related values are irrational for a set of positive integers n of natural density one. These are different numbers: $\mathrm{Ti}_2(n) = \mathrm{Ti}_2(1/n) + \frac{\pi}{2}\log n$ and $\mathrm{Re}\,\chi_2(n) = \frac{\pi^2}{4} - \chi_2(1/n)$ for $n > 1$. His Corollary 6.2 (1, $\mathrm{Ti}_2(n)$, $\pi \log n$ linearly independent for a density-one set of n) does imply $\mathrm{Ti}_2(1/n) \notin \mathbb{Q}$ on a set of density one. That

set is not explicit: independence is obtained only through the criterion of [5] at points $1/(qs)$ with q sufficiently large, and no individual n is decided.

- *Identities do not suffice.* Landen's identity [14, App. A.2.1, (24)] relates $\chi_2(x)$ to $\chi_2\big(\frac{1-x}{1+x}\big)$ up to a product of logarithms, and $\mathrm{Ti}_2(x) - \mathrm{Ti}_2(1/x) = \frac{\pi}{2}\log x$. Neither moves our numbers to values known to be irrational.

Novelty. To our knowledge, and relative to the criteria we have read, the following ranges are new: Theorem A for $6 \le q \le 2948$ (the cases $q = 3, 5$ are in [12]); Theorem B for $5 \le q \le 2948$; and Theorem C for $175 \le q < 11{,}795$. For Theorems A and B this is subject to the threshold of [9] for the shift $\frac{1}{2}$, which we could not check: if that threshold is below q, the corresponding values are covered by Hata's theorem. By the heuristic estimate above, the range that is genuinely new for Theorem A may be much shorter than $6 \le q \le 2948$, perhaps $q \lesssim 85$ or less; for Theorem B we have no estimate. Theorem C does not depend on [9] in this way. Our search for prior work covered the web, arXiv, Zenodo and GitHub (repository and code search), but not MathSciNet or zbMATH beyond individual reviews.

1.3. **The method.** Fix a real number w with $0 < |w| < 1$. For $h \ge 1$ let $x_k := -(k + \frac{1}{2})$ for $0 \le k < h$, $D(u) := \prod_{k<h}(u - x_k)$ and

$$\Lambda(g) := \sum_{t \ge 1} w^t \Big(\frac{g(u)\,u}{D(u)}\Big)'\Big|_{u=t} \qquad (g \in \mathbb{Q}[u]). \tag{1.2}$$

For $w = q^{-2}$, $w = -q^{-2}$ and $w = 4q^{-2}$, partial fractions give $\Lambda(g) = A(g) + \theta B(g)$ with rational functionals A, B and $\theta = \chi_2(1/q)$, $\mathrm{Ti}_2(1/q)$ and $\chi_2(2/q)$ respectively (Lemma 2.2). So the Hankel determinant $\det(\Lambda(u^{i+j}))_{i,j<h}$ is the value at $X = \theta$ of a polynomial $\Delta_h(X) \in \mathbb{Q}[X]$ of degree at most h. This is the construction of [12], a Padé-type construction of the kind used by Q. Tang for $\mathrm{Li}_2(1/2)$ [21]. The proof has three inputs.

(1) *Denominators* (Sections 4–5 and the prime 2 in Sections 9–11). There are rationals $d_h > 0$ with $d_h\Delta_h \in \mathbb{Z}[X]$ and $\log d_h \le (\alpha + o(1))h^2$, where $\alpha = \frac{133}{60} + c_{\mathrm{odd}} + c_2$. The term $\frac{133}{60}$ comes from the odd primes not dividing $q(q^2-1)$, $q(q^2+1)$ or $q(q^2-4)$ respectively; c_{odd} collects the odd primes dividing q (a gain) and those dividing q^2-1, q^2+1 or q^2-4 (a cost); c_2 is the prime 2.

(2) *Size* (Sections 6–7). For every dual certificate, $\limsup h^{-2}\log|\Delta_h(\theta)| \le \Phi$, an explicit exponent that depends on w only through $|w|$. A shift lemma moves a certificate from $|w|$ to any smaller $|w'|$ and gains $\frac{1}{4}\log(|w|/|w'|)$.

(3) *Nonvanishing* (Section 8). For every rational a/b, $\Delta_h(a/b) \ne 0$ for infinitely many h, by a reduction modulo a prime $\ell = h/3$.

If $\alpha + \Phi < 0$, the standard argument (Proposition 3.1) concludes. Table 1 lists the arithmetic constants.

	w	odd primes c_{odd}	prime 2: c_2	lemma
A, odd q	q^{-2}	$\frac{1}{4}\log \mathrm{odd}(q^2-1) - \frac{1}{3}\log q$	$\frac{\nu-2}{4}\log 2$	9.1
A, even q	q^{-2}	$\frac{1}{4}\log(q^2-1) - \frac{1}{3}\log \mathrm{odd}(q)$	$(\frac{1}{2} - \frac{e}{3})\log 2$	9.3
			or $(c(e) - \frac{e}{3})\log 2$	9.4
B, odd q	$-q^{-2}$	$\frac{1}{4}\log \frac{q^2+1}{2} - \frac{1}{3}\log q$	$\frac{1}{4}\log 2$	10.1
			or $\frac{1}{8}\log 2$	10.2
B, even q	$-q^{-2}$	$\frac{1}{4}\log(q^2+1) - \frac{1}{3}\log \mathrm{odd}(q)$	$(c(e) - \frac{e}{3})\log 2$	10.4
C, odd q	$4q^{-2}$	$\frac{1}{4}\log(q^2-4) - \frac{1}{3}\log q$	$\frac{4}{3}\log 2$	11.1

TABLE 1. The arithmetic constants per h^2 for Theorems A–C, in addition to $\frac{133}{60}$. Here $\mathrm{odd}(n)$ is the odd part of n, $\nu = v_2(q^2-1)$, $e = v_2(q)$ and $c(e) = (6-4e)/(3(3e+8))$.

The odd primes behave in the same way in all three cases, because the three determinants are specialisations of one symbolic determinant over $\mathbb{Q}[q, E]$, where E stands for $1/(q^2-1)$: at $(q, E) = (q, 1/(q^2-1))$ for Theorem A, at $(q/2, 4/(q^2-4))$ for Theorem C, and, through a coefficient twist that realises $q \mapsto iq$, $X \mapsto -iX$, for Theorem B (Proposition 4.6). The prime 2 is different in each case, and needs a separate lemma in each:

- for odd q in Theorem A, the bound of [12] at general $\nu = v_2(q^2-1) \geq 3$;
- for even q, a weighted Gauss valuation on the reflected expansion of the functional, which gives the constant $c(e)$; it is what brings $q = 6, 8, 10, 12$ within reach;
- for Ti_2 at odd q, $\nu = v_2(q^2+1) = 1$, so the 2-part of q^2+1 saves only $\frac{1}{4}\log 2$ and $\mathrm{Ti}_2(1/q)$ pays $\frac{1}{2}\log 2$ more in total than $\chi_2(1/q)$ (entry bounds in the anchored basis cost $\frac{1}{4}\log 2$); a change to the binomial basis halves the cost of the prime 2 to $\frac{1}{8}\log 2$ (Lemma 10.2), which brings $q = 5$ within reach ($q = 3$ stays out);
- for $\chi_2(2/q)$ the numerator costs at least $\log 2$ per h^2 (we prove $n_2 \geq h^2 - h$) and at most $\frac{4}{3}\log 2$, which the analytic side pays only from $q \approx 175$ on.

1.4. **The formalisation.** All three theorems and their complete proofs are formalised in Lean 4 [6] with Mathlib [22] and checked by the Lean kernel, in the repository `https://github.com/telaviv/chi2-irrationality` [13], which also contains the formalisation of [12]. The proofs depend only on Lean's three standard axioms (`propext`, `Classical.choice`, `Quot.sound`); they contain no `sorry` and no `native_decide`, and every numerical certificate is evaluated by the kernel. Section 13 maps each step to its Lean declaration. In a few places the formal proof differs from the written one (cruder lower-order terms, and the organisation into separate files); these are listed in Section 13.5.

The written proof below is self-contained, with two exceptions that are only described: the dual certificates (Section 7), which are decidable statements about rational data proved by evaluation in the Lean kernel, and the closed forms of the block determinants in Section 8, which are identities between explicit rational functions.

1.5. **Organisation and notation.** Section 2 sets up the machine at a general weight and proves the pole expansion. Section 3 states the criterion. Sections 4 and 5 bound the odd part of the denominators for all three cases at once. Sections 6 and 7 bound the size and prove the shift lemma, and Section 8 proves the nonvanishing lemma. Sections 9, 10 and 11 specialise to the three theorems: the prime 2, the certificates, the nonvanishing data and the margins. Section 12 discusses the limits of the method, and Section 13 the formalisation.

Throughout, p and ℓ denote primes; v_p is the p-adic valuation on $\mathbb{Q}$ and $|x|_p = p^{-v_p(x)}$. For an integer $n \neq 0$, $\mathrm{odd}(n) = n/2^{v_2(n)}$. $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ are the floor and ceiling, $(x)_+ = \max(x, 0)$, and $[\,P\,]$ is 1 if P holds and 0 otherwise. For a sequence f on $\mathbb{Z}_{\geq 0}$, $\Delta f(t) = f(t+1) - f(t)$. $s_2(n)$ is the binary digit sum, so $v_2(n!) = n - s_2(n)$. We write $\theta(x) = \sum_{p \leq x} \log p$ only in Section 4; elsewhere θ is the target number.

2. The machine

2.1. **Data.** We work with three kinds of *point*. Each is given by a rational weight w with $0 < |w| < 1$, a real target θ, and rationals c_k, s_k $(k \geq 0)$:

case	w	θ	c_k	s_k
(A) $q \geq 2$	q^{-2}	$\chi_2(1/q)$	q^{2k+1}	$S_k(q)$
(B) $q \geq 2$	$-q^{-2}$	$\mathrm{Ti}_2(1/q)$	$(-1)^k q^{2k+1}$	$S_k^{\mathrm{Ti}}(q)$
(C) $q \geq 3$ odd	$4q^{-2}$	$\chi_2(2/q)$	ζ^{2k+1}, $\zeta = q/2$	$S_k(\zeta)$

Here, for a rational y,

$$S_k(y) := \sum_{j=0}^{k} \frac{y^{2(k-j)}}{(2j+1)^2}, \qquad S_k^{\mathrm{Ti}}(y) := \sum_{j=0}^{k} \frac{(-1)^{k-j} y^{2(k-j)}}{(2j+1)^2} = S_k(iy).$$

For $h \geq 1$ we use the h poles $x_k = -(k + \frac{1}{2})$, $0 \leq k < h$, and put

$$D(u) := \prod_{k<h} (u - x_k), \qquad \omega_k := \frac{x_k}{D'(x_k)}, \qquad \varphi(u) := \frac{u}{D(u)},$$

where $D'(x_k) = \prod_{m \neq k} (m-k) = (-1)^k k!\,(h-1-k)!$. For a rational function F regular at the positive integers and of polynomial growth put

$$\mu(F) := \sum_{t \geq 1} w^t F'(t),$$

an absolutely convergent series. The functional (1.2) is $\Lambda(g) = \mu(g\varphi)$.

Lemma 2.1 (pole series)**.** *In each case, for every $k \geq 0$,*

$$\mu\Big(\frac{1}{u-x_k}\Big) = -4\big(c_k\,\theta - s_k\big).$$

Proof. We have

$$\mu\Big(\frac{1}{u-x_k}\Big) = -\sum_{t\geq 1}\frac{w^t}{(t+k+\frac{1}{2})^2} = -4\sum_{t\geq 1}\frac{w^t}{(2t+2k+1)^2};$$

put $s = 2t+2k+1$, an odd number $\geq 2k+3$. In case (A), $w^t = q^{-2t} = q^{2k+1}q^{-s}$, so the last sum is

$$q^{2k+1}\Big(\chi_2(1/q) - \sum_{s\leq 2k+1,\ s\text{ odd}}\frac{q^{-s}}{s^2}\Big) = q^{2k+1}\chi_2(1/q) - S_k(q).$$

Case (C) is the same computation with q replaced by $\zeta = q/2$, since $4q^{-2} = \zeta^{-2}$ and $\chi_2(2/q) = \chi_2(1/\zeta)$. In case (B), $(-1)^t = (-1)^k\chi_4(s)$ with $\chi_4(s) = (-1)^{(s-1)/2}$, so the sum is

$$(-1)^k q^{2k+1}\Big(\mathrm{Ti}_2(1/q) - \sum_{s\leq 2k+1,\ s\text{ odd}}\frac{\chi_4(s)q^{-s}}{s^2}\Big) = (-1)^k q^{2k+1}\,\mathrm{Ti}_2(1/q) - S_k^{\mathrm{Ti}}(q).$$

All series converge absolutely. □

2.2. **The polynomial part.** For a polynomial R, $\mu(R)$ is rational. Indeed $R'(t) = \sum_b \Delta^b[R'](0)\binom{t}{b}$ gives the Newton form

(2.1)

$$\mu(R) = \sum_{b\geq 0}\Delta^b[R'](0)\,\Omega_b, \qquad \Omega_b := \sum_{t\geq 1}\binom{t}{b}w^t = \begin{cases}\dfrac{w}{1-w}, & b=0,\\[2mm] \dfrac{w^b}{(1-w)^{b+1}}, & b\geq 1,\end{cases}$$

where $\Delta^b[R'](0) = \sum_{k\leq b}(-1)^{b-k}\binom{b}{k}R'(k)$ is an integer combination of $R'(0),\dots,R'(b)$, and the sum is finite ($b < \deg R$). With $y = q$, iq or $q/2$ in cases (A), (B), (C), so that $w = y^{-2}$,

$$\Omega_0 = \frac{1}{y^2-1}, \qquad \Omega_b = \frac{y^2}{(y^2-1)^{b+1}} \quad (b\geq 1). \tag{2.2}$$

2.3. **The pole expansion.**

Lemma 2.2 (pole expansion)**.** *For $g \in \mathbb{Q}[u]$ write $g(u)u = D(u)R_g(u) + r_g(u)$ with $\deg r_g < h$. Then $\Lambda(g) = A(g) + \theta\,B(g)$, where*

$$A(g) := \mu(R_g) + 4\sum_{k<h} g(x_k)\,\omega_k\,s_k, \qquad B(g) := -4\sum_{k<h} g(x_k)\,\omega_k\,c_k.$$

Both are rational and $\mathbb{Q}$-linear in g.

Proof. By Lagrange interpolation at the nodes x_k,

$$r_g(u) = \sum_k \frac{r_g(x_k)\,D(u)}{D'(x_k)\,(u-x_k)}, \qquad r_g(x_k) = g(x_k)\,x_k.$$

Hence $g\varphi = R_g + \sum_k g(x_k)\omega_k/(u-x_k)$, and Lemma 2.1 gives the pole terms. □

For an indeterminate X put $\Lambda_X := A + X\,B$ and

$$H(X) := \big(\Lambda_X(u^{i+j})\big)_{0\le i,j<h}, \qquad \Delta_h(X) := \det H(X) \in \mathbb{Q}[X].$$

The entries are linear in X, so $\deg \Delta_h \le h$, and by Lemma 2.2

$$\Delta_h(\theta) = \det\big(\Lambda(u^{i+j})\big)_{0\le i,j<h}. \tag{2.3}$$

When the case matters we write $\Delta_h^{\chi}(X;q)$, $\Delta_h^{\mathrm{Ti}}(X;q)$ and $\Delta_h^{\chi}(X;q/2)$.

2.4. **Change of basis.** If $a_0,\dots,a_{h-1}$ and $b_0,\dots,b_{h-1}$ are polynomials of degree $<h$ with coefficient matrices C_a and C_b, then by linearity of Λ_X

$$\big(\Lambda_X(a_ib_j)\big)_{i,j} = C_a\,H(X)\,C_b^{\mathsf T}, \qquad \det\big(\Lambda_X(a_ib_j)\big) = \det C_a \det C_b\,\Delta_h(X). \tag{2.4}$$

We use three bases. The *Lagrange basis* $f_i := D/(u-x_i)$ has $f_i(x_k) = [\,i=k\,]D'(x_i)$; with the Vandermonde matrix $V = (x_k^d)_{d,k<h}$ we have $C_fV = \mathrm{diag}(D'(x_i))$ and $\prod_i D'(x_i) = \pm(\det V)^2$, where $\det V = \pm\prod_{k<k'}(k'-k)$. The *anchored basis*

$$a_i := \prod_{m<i}(u-x_m), \qquad b_j := \prod_{h-j\le m<h}(u-x_m)$$

consists of monic polynomials of degrees i and j, so $\det(\Lambda_X(a_ib_j)) = \Delta_h(X)$. The third basis, adapted to residue classes, is introduced in Section 4.3.

3. The criterion

Proposition 3.1 (Lemma E)**.** *Let $\xi\in\mathbb{R}$, and for $h\ge1$ let $P_h\in\mathbb{Q}[X]$ with $\deg P_h\le h$ and let $d_h>0$ be rationals with $d_hP_h\in\mathbb{Z}[X]$. Suppose that there is $c>0$ with $d_h|P_h(\xi)|\le e^{-ch^2}$ for all large h, and that for every $a\in\mathbb{Z}$ and $b\ge1$ we have $P_h(a/b)\ne0$ for infinitely many h. Then ξ is irrational.*

Proof. Suppose $\xi=a/b$. For every h, $N_h := b^hd_hP_h(a/b)$ is an integer. Choose h with $P_h(a/b)\ne0$, $d_h|P_h(\xi)|\le e^{-ch^2}$ and $b^he^{-ch^2}<1$. Then $1\le|N_h|<1$, a contradiction. □

Theorem 3.2. *Fix a point as in Section 2.1. Suppose there are rationals $d_h>0$ and real numbers α,β with $\alpha+\beta<0$ such that*

1. *$d_h\Delta_h\in\mathbb{Z}[X]$ for every h, and for every $\varepsilon>0$, $\log d_h\le(\alpha+\varepsilon)h^2$ for all large h;*
2. *for every $\varepsilon>0$, $|\Delta_h(\theta)|\le e^{(\beta+\varepsilon)h^2}$ for all large h;*
3. *for every $a\in\mathbb{Z}$ and $b\ge1$, $\Delta_h(a/b)\ne0$ for infinitely many h.*

Then θ is irrational.

Proof. Apply Proposition 3.1 with $P_h = \Delta_h$ and $c = -(\alpha+\beta)/2$, using (i) and (ii) with $\varepsilon = c/2$. □

4. Denominators I: the symbolic machine and the generic primes

4.1. **The free ring.** We treat q and $1/(q^2-1)$ as independent indeterminates. Let $\mathcal{R} := \mathbb{Q}[q, E]$. In $\mathcal{R}$ put $\Omega_0 := E$ and $\Omega_b := q^2 E^{b+1}$ for $b \geq 1$, and for $R \in \mathbb{Q}[u]$ define $\mu^{\mathcal{R}}(R) := \sum_b \Delta^b[R'](0)\,\Omega_b \in \mathcal{R}$ as in (2.1). Replacing μ by $\mu^{\mathcal{R}}$, s_k by $S_k(q) \in \mathbb{Q}[q]$ and c_k by q^{2k+1}, Lemma 2.2 defines functionals $A^{\mathcal{R}}, B^{\mathcal{R}}$ and

$$\Delta_h^{\mathcal{R}}(X) := \det\big(A^{\mathcal{R}}(u^{i+j}) + X\,B^{\mathcal{R}}(u^{i+j})\big)_{i,j<h} \in \mathcal{R}[X].$$

For a rational y with $y^2 \neq 1$ let $\mathrm{ev}_y\colon \mathcal{R} \to \mathbb{Q}$ be the ring homomorphism with $q \mapsto y$ and $E \mapsto 1/(y^2-1)$. By (2.2) it maps $\mu^{\mathcal{R}}$ to μ and $\Delta_h^{\mathcal{R}}$ to Δ_h in case (A) with $y = q$ and in case (C) with $y = q/2$:

$$\mathrm{ev}_q(\Delta_h^{\mathcal{R}}) = \Delta_h^{\chi}(X; q), \qquad \mathrm{ev}_{q/2}(\Delta_h^{\mathcal{R}}) = \Delta_h^{\chi}(X; q/2). \tag{4.1}$$

Case (B) is the specialisation at $y = iq$ followed by $X \mapsto -iX$; Section 4.6 makes this precise over $\mathbb{Q}$.

Gauss norms. For a prime p and $f \in \mathcal{R}$ let $\|f\|_p$ be the maximum of $|c|_p$ over the rational coefficients c of f, and extend this coefficientwise to $\mathcal{R}[X]$. Then $\|f+g\|_p \leq \max(\|f\|_p, \|g\|_p)$ and $\|fg\|_p \leq \|f\|_p \|g\|_p$. If $|v_0|_p \leq 1$ and $|v_1|_p \leq 1$, the evaluation $q \mapsto v_0$, $E \mapsto v_1$ does not increase the norm. By the Leibniz expansion:

Lemma 4.1. *Let M be an $h \times h$ matrix over $\mathcal{R}[X]$ (or over $\mathbb{Q}[X]$) with $\|M_{ij}\|_p \leq \rho_i \kappa_j$ for some $\rho_i, \kappa_j \geq 0$. Then $\|\det M\|_p \leq \prod_i \rho_i \prod_j \kappa_j$.*

Two facts carry all the dependence on q:

- (F1) $\|\Omega_b\|_p = 1$ for all b and p, so for $R \in \mathbb{Q}[u]$ we have $\|\mu^{\mathcal{R}}(R)\|_p \leq \sup_{t \in \mathbb{Z}_{\geq 0}} |R'(t)|_p$;
- (F2) $\|q^{2k+1}X - S_k(q)\|_p \leq p^{2\#\{e \geq 1 :\, p^e \leq 2k+1\}}$, since the coefficients of S_k are $1/s^2$ with s odd, $s \leq 2k+1$, and $v_p(s) \leq \#\{e \geq 1 : p^e \leq s\}$.

The Gram identity. Write $\Lambda_X^{\mathcal{R}} := A^{\mathcal{R}} + XB^{\mathcal{R}}$. By the argument of (2.4), for any family $(e_m)_{m<h}$ of polynomials of degree $< h$ with coefficient matrix C_e,

$$\det\big(\Lambda_X^{\mathcal{R}}(e_m e_{m'})\big)_{m,m'<h} = (\det C_e)^2\,\Delta_h^{\mathcal{R}}(X). \tag{4.2}$$

For a pole x_l we have $\Lambda_X^{\mathcal{R}}(g) = \mu^{\mathcal{R}}(R) - 4c\,(q^{2l+1}X - S_l(q))$ whenever $g\varphi = R + c/(u - x_l)$ with R a polynomial; more generally the pole parts add up.

Below, $\mathrm{lev}(p) := \#\{e \geq 1 : p^e \leq 2h-1\}$. For $t \in \mathbb{Z}_{\geq 0}$ and $m < h$ we have $t - x_m = (2t+2m+1)/2$, and for p odd $v_p(t - x_m) = v_p(2t+2m+1)$. The numbers $2t+2m+1$, $m < h$, are h consecutive odd integers, so for each level $d = p^e$ those divisible by d are the m in one residue class modulo d.

4.2. **All odd primes: the Lagrange basis.**

Lemma 4.2 (Lemma 9)**.** *For every odd prime p and every $h \geq 1$, $\|\Delta_h^{\mathcal{R}}\|_p \leq p^{2h\,\mathrm{lev}(p)}$.*

Proof. Use the Lagrange basis $f_i = D/(u - x_i)$ and put $G_{ij} := \Lambda_X^{\mathcal{R}}(f_i f_j)$. Since $C_f V = \mathrm{diag}(D'(x_i))$ and $\prod_i D'(x_i) = \pm(\det V)^2$, we get $(\det C_f)^2 = \pm\prod_i D'(x_i)$, so by (4.2)

$$\|\Delta_h^{\mathcal{R}}\|_p = \|\det G\|_p \Big/ \prod_i |D'(x_i)|_p.$$

By Lemma 4.1 it suffices to prove the *row claim* $\|G_{ij}\|_p \leq |D'(x_i)|_p\, p^{2\,\mathrm{lev}(p)}$. For a level $e \geq 1$ put $N_e(i) := \#\{m < h : m \neq i,\ p^e \mid m - i\}$; then $v_p(D'(x_i)) = \sum_e N_e(i)$, and $N_e(i) = 0$ unless $p^e \leq h - 1$. So it suffices to show that every value $R'(t)$ entering (F1) has

$$v_p(R'(t)) \geq \sum_e \big(N_e(i) - 2[\,N_e(i) \geq 1\,]\big), \tag{4.3}$$

and that the pole part obeys the same bound.

Fix $t \in \mathbb{Z}_{\geq 0}$ and a level $d = p^e$, and let m_0 be the residue class mod d of the indices m with $d \mid 2t + 2m + 1$. The class of i has $N_e(i) + 1$ elements in $[0, h)$, and residue classes mod d in $[0, h)$ differ in size by at most one, so the class m_0 has at least $N_e(i)$ elements.

Off-diagonal entries ($i \neq j$). Here $f_i f_j \varphi = F_{ij} := u\prod_{m \neq i,j}(u - x_m)$ is a polynomial, so $G_{ij} = \mu^{\mathcal{R}}(F_{ij})$. Each term of $F'_{ij}(t)$ is a product of all factors but one. At level d the factors $t - x_m$ divisible by d are those with $m \equiv m_0$, $m \neq i, j$; omitting one factor leaves at least $N_e(i) - 2$ of them in both cases $m_0 \equiv i$ and $m_0 \not\equiv i$. Summing over levels (v_p of a product is $\sum_e$ of the number of factors divisible by p^e) gives (4.3).

Diagonal entries. Put $g_i := u\prod_{m \neq i}(u - x_m)$. Then $f_i^2\varphi = g_i/(u - x_i) = Q_i + g_i(x_i)/(u - x_i)$ with $Q_i := (g_i(u) - g_i(x_i))/(u - x_i)$ and $g_i(x_i) = x_i D'(x_i)$. The pole part is $-4x_i D'(x_i)(q^{2i+1}X - S_i)$; by (F2), $|x_i|_p \leq 1$ and $2i + 1 \leq 2h - 1$, its norm is at most $|D'(x_i)|_p\, p^{2\,\mathrm{lev}(p)}$. For the polynomial part, list the h roots $\{x_m : m \neq i\} \cup \{0\}$ of g_i as $\alpha_1, \ldots, \alpha_h$; telescoping $(u - \alpha_r) - (x_i - \alpha_r) = u - x_i$ gives

$$Q_i(u) = \sum_{r=1}^{h} \prod_{s<r}(x_i - \alpha_s) \prod_{s>r}(u - \alpha_s).$$

Order the roots by decreasing $v_p(m - i)$, with 0 last. Then for every level e simultaneously, the $\alpha_s = x_m$ with $p^e \mid m - i$ form an initial segment of length $N_e(i)$ (note $x_i - x_m = m - i$). Fix t, a level $d = p^e$ and a term r of $Q'_i(t)$ (a product with one more factor omitted). If $r > N_e(i)$, the first product already has $N_e(i)$ factors divisible by d. If $r \leq N_e(i)$, it has $r - 1$; if moreover $m_0 \equiv i$, the remaining $N_e(i) - r$ class members sit at positions $s > r$ and at most one of them is omitted, giving at least $N_e(i) - 2$ in total; if

$m_0 \not\equiv i$, the whole class m_0 (at least $N_e(i)$ members, all with $v_p(m-i) < e$) sits at positions $s > N_e(i) \geq r$, giving at least $N_e(i) - 1$. So (4.3) holds for $Q_i'(t)$ as well. Finally $\#\{e : N_e(i) \geq 1\} \leq \mathrm{lev}(p)$. □

4.3. **One-level primes: a basis adapted to residue classes.** For a modulus d and $m < h$ write $r_m := \lfloor m/d \rfloor$ (the rank of m in its class), n_m for the size of the class of m modulo d in $[0, h)$, and put

$$w_d(l) := 2[\, d \leq 2l+1 \,] - [\, d \mid 2l+1 \,],$$
$$M_m := \max\{w_d(l) : l \leq m,\ l \equiv m\ (d)\},$$
$$\mathrm{cb}(h,d) := \sum_{m<h} \big(2r_m + 1 + M_m - n_m\big)_+.$$

Proposition 4.3 (class bound)**.** *Let p be an odd prime with $p^2 > 2h$. Then $\|\Delta_h^{\mathcal{R}}\|_p \leq p^{\mathrm{cb}(h,p)}$.*

Proof. Since $p^2 > 2h$, every difference $m-l$ of distinct indices and every odd $s \leq 2h-1$ has $v_p \leq 1$; so $v_p(m-l) = [\, p \mid m-l \,]$ and $v_p(2l+1) = [\, p \mid 2l+1 \,]$. For $m < h$ let $\mathrm{seg}(m) := \{l \leq m : l \equiv m\ (p)\}$ (it has $r_m + 1$ elements), and put

$$e_m := s_m \frac{D(u)}{\prod_{l \in \mathrm{seg}(m)} (u - x_l)}, \qquad s_m := \prod_{l \in \mathrm{seg}(m),\, l<m} (x_m - x_l).$$

Then $\deg e_m = h - r_m - 1 < h$ and $v_p(s_m) = r_m$. The polynomial e_m vanishes at x_k unless $k \in \mathrm{seg}(m)$, and $e_m(x_m) = D'(x_m)$. So the evaluation matrix $(e_m(x_k)) = C_e V$ is lower triangular with diagonal $D'(x_m)$, hence $\det C_e \det V = \prod_m D'(x_m) = \pm(\det V)^2$ and $\det C_e = \pm \det V$. By (4.2), $\|\Delta_h^{\mathcal{R}}\|_p = \|\det G^e\|_p / |\det V|_p^2$ with $G^e_{mm'} := \Lambda_X^{\mathcal{R}}(e_m e_{m'})$, and $2v_p(\det V) = 2\#\{k < k' : p \mid k' - k\} = 2\sum_m r_m$.

Let $\mathcal{A} := \mathrm{seg}(m)$, $\mathcal{B} := \mathrm{seg}(m')$, $r = r_m$, $r' = r_{m'}$. Then

$$e_m e_{m'} \varphi = s_m s_{m'}\, u\, \frac{\prod_{l \notin \mathcal{A} \cup \mathcal{B}} (u - x_l)}{\prod_{l \in \mathcal{A} \cap \mathcal{B}} (u - x_l)}.$$

If m and m' lie in different classes, $\mathcal{A} \cap \mathcal{B} = \emptyset$, the right side is $s_m s_{m'}$ times a polynomial in $\mathbb{Z}[\frac{1}{2}][u]$, and (F1) gives $v_p(G^e_{mm'}) \geq r + r'$. If they lie in the same class, of size n, say with $r \leq r'$, then $\mathcal{A} \subseteq \mathcal{B}$. Division by the monic polynomial $\prod_{l \in \mathcal{A}} (u - x_l) \in \mathbb{Z}[\frac{1}{2}][u]$ leaves a p-integral polynomial part, again of valuation $\geq r + r'$. The residue at x_l, $l \in \mathcal{A}$, is

$$c_l = s_m s_{m'}\, x_l\, \frac{\prod_{k \notin \mathcal{B}} (x_l - x_k)}{\prod_{k \in \mathcal{A},\, k \neq l} (x_l - x_k)},$$
$$v_p(c_l) = r + r' + [\, p \mid 2l+1 \,] + (n - 1 - r') - r,$$

because exactly the $n - 1 - r'$ class members after m' lie outside $\mathcal{B}$, and the r other elements of $\mathcal{A}$ lie in the class. By (F2) and the single level, the pole

factor $q^{2l+1}X - S_l$ has norm at most $p^{2[p\le 2l+1]}$. Hence

$$v_p(G^e_{mm'}) \ge r + r' + \beta(r, r'), \qquad \beta(a,b) := \min\bigl(0,\ n-1-a-b-\widehat{M}_{\min(a,b)}\bigr),$$

where $\widehat{M}_a$ is the largest weight $w_p(l)$ among the class members of rank $\le a$. Since $\widehat{M}$ is nondecreasing and $x \mapsto \min(0,x)$ is concave, $\beta(a,b) \ge \frac{1}{2}(\beta(a,a)+\beta(b,b))$. So every entry, in the same class or not, satisfies $v_p(G^e_{mm'}) \ge \rho_m + \rho_{m'}$ with $\rho_m := r_m + \frac{1}{2}\beta(r_m, r_m) \le r_m$. Lemma 4.1 gives $v_p(\det G^e) \ge 2\sum_m \rho_m$, and therefore

$$v_p(\Delta^{\mathcal{R}}_h) \ge \sum_m \beta(r_m, r_m) = -\sum_m (2r_m + 1 + M_m - n_m)_+ = -\operatorname{cb}(h,p). \quad \square$$

4.4. **The class count.** Let φ_1 be the continuous, piecewise linear, nonincreasing function on $[2/5, \infty)$ with

$$\varphi_1(x) = \begin{cases} 3 - \frac{5}{2}x, & \frac{2}{5} \le x \le \frac{1}{2}, \\ 2 - \frac{1}{2}x, & \frac{1}{2} \le x \le \frac{2}{3}, \\ 3 - 2x, & \frac{2}{3} \le x \le 1, \\ 2 - x, & 1 \le x \le 2, \end{cases}$$

and $\varphi_1(x) = 0$ for $x \ge 2$. Then $\int_{2/5}^2 \varphi_1 = \frac{3}{16} + \frac{41}{144} + \frac{4}{9} + \frac{1}{2} = \frac{17}{12}$.

Lemma 4.4 (Lemma 8A, class count)**.** *Let $h \ge 2$ and let d be odd. If $2h \le 5d$ and $d < 2h$ then $\operatorname{cb}(h,d) \le h\,\varphi_1(d/h)$; more precisely $\operatorname{cb}(h,d) = h\varphi_1(d/h) - \frac{3}{2}$ if $d \le (2h-1)/3$ and $\operatorname{cb}(h,d) = h\varphi_1(d/h)$ otherwise. If $d \ge 2h$ then $\operatorname{cb}(h,d) = 0$. (The formal version assumes $h \ge 13$, which is needed only in Theorem 4.5.)*

Proof. Write $K = h$ and $\kappa = (d-1)/2$. If $d \ge 2K$, every atom $l < K$ has $2l+1 < d$, so all weights vanish, every class is a singleton and every term is $(1+0-1)_+ = 0$. Now let $2K \le 5d$ and $d < 2K$; as d is odd, $5d \ge 2K+1$ and $3d \ne 2K$. The odd multiples of d below $2K$ are d and possibly $3d$. Hence an atom l has weight 0 if $l < \kappa$ (we call it *small*), weight 1 if $l = \kappa$ or $l = \kappa + d$, and weight 2 otherwise. Every atom after the first in its class is $\ge d > \kappa$, so only the first atom c of a class can be small. We evaluate cb class by class; a class $\{c, c+d, \dots\}$ of size n contributes $\sum_r (2r+1+M-n)_+$ over its atoms.

(IV) $K < d < 2K$. All classes are singletons, contributing their weights: $2(K-\kappa) - 1 = 2K - d$.

(III) $(2K+1)/3 \le d \le K$. Classes $c < K - d$ are pairs, with $c < \kappa$ small; they contribute $0 + (1+2) = 3$. The other classes $c \in [K-d, d-1]$ are singletons; those with $c \ge \kappa$ ($(d+1)/2$ of them, one of weight 1) contribute d in total. The atom $\kappa + d$ is $\ge K$. Total $3(K-d) + d = 3K - 2d$.

(II) $K/2 \le d \le (2K-1)/3$. Classes $c < K - d$ are pairs and $c \in [K - d, d-1]$ singletons, all of weight 2 (as $\kappa < K - d$). A pair contributes $(w(c)-1)_+ + 1 + M$: this is 3 for $c < \kappa$, 2 for $c = \kappa$ (weights $1, 1$),

and 4 for $\kappa < c < K-d$. Total $3\kappa+2+4(K-d-\kappa-1)+2(2d-K) = 2K-\frac{d}{2}-\frac{3}{2}$.

(I) $(2K+1)/5 \le d \le K/2$. Classes $c < K-2d$ are triples, with $c < \kappa$ (as $K-2d \le \kappa$), contributing $0+2+4=6$. Classes $c \in [K-2d, d-1]$ are pairs, contributing 3 for $c < \kappa$ ($\frac{5d-1}{2} - K$ of them), 2 for $c = \kappa$, and 4 for $c > \kappa$ ($\frac{d-1}{2}$ of them). Total $3K - \frac{5}{2}d - \frac{3}{2}$.

In the four regions $K\varphi_1(d/K)$ equals $2K-d$, $3K-2d$, $2K-\frac{d}{2}$ and $3K-\frac{5}{2}d$. □

4.5. **The generic primes.** For odd p put

$$e(h,p) := \begin{cases} \mathrm{cb}(h,p), & p^2 > 2h \text{ and } 5p \ge 2h, \\ 2h\,\mathrm{lev}(p), & \text{otherwise.} \end{cases}$$

Theorem 4.5 (generic primes)**.** *For every odd prime p and every h, $\|\Delta_h^{\mathcal{R}}\|_p \le p^{e(h,p)}$. For every $\varepsilon > 0$ and all large h,*

$$\sum_{p \text{ odd}} e(h,p)\log p \le \Big(\frac{133}{60} + \varepsilon\Big)h^2.$$

Proof. The first statement is Proposition 4.3 or Lemma 4.2. For the sum let $h \ge 13$. If $5p \ge 2h$ then $p^2 \ge 4h^2/25 > 2h$, so $e(h,p) = \mathrm{cb}(h,p)$, which vanishes for $p \ge 2h$ and is at most $h\varphi_1(p/h)$ for $2h/5 \le p < 2h$ (Lemma 4.4). So the sum is at most

$$\sum_{p<2h/5} 2h\,\mathrm{lev}(p)\log p + \sum_{2h/5\le p<2h} h\,\varphi_1(p/h)\log p.$$

For the first sum, $\mathrm{lev}(p) \le 1$ when $p^2 > 2h-1$ and $\mathrm{lev}(p)\log p \le \log(2h)$ always, so it is at most $2h\big(\theta(2h/5) + (\sqrt{2h}+1)\log(2h)\big) = (\frac{4}{5} + o(1))h^2$ by the prime number theorem $\theta(x) \sim x$. For the second, cut $[2/5, 2]$ into M equal cells. On each cell φ_1 is at most its value at the left end, and $\sum_{p\in hI}\log p = h|I| + o(h)$ for each fixed cell I. Hence the second sum is at most $h^2\big(\int_{2/5}^2 \varphi_1 + \frac{16}{5M}\big) + o(h^2)$, and M is arbitrary. □

4.6. **The twist.** For case (B) let $\mathrm{ev}_q^{\mathrm{Ti}}\colon \mathcal{R} \to \mathbb{Q}$ be the homomorphism with $q \mapsto q$ and $E \mapsto 1/(q^2+1)$, for an integer $q \ge 2$.

Proposition 4.6 (the twist)**.** (1) *Every monomial $q^{m_0}E^{m_1}X^j$ occurring in $\Delta_h^{\mathcal{R}}$ has $m_0 + j$ even.*

(2) *Let $\Delta_h^{\mathcal{R},\mathrm{Ti}} \in \mathcal{R}[X]$ be obtained from $\Delta_h^{\mathcal{R}}$ by multiplying the coefficient of each monomial $q^{m_0}E^{m_1}X^j$ by $\tau(m,j) := (-1)^{m_1+j+(m_0+j)/2}$. Then $\mathrm{ev}_q^{\mathrm{Ti}}(\Delta_h^{\mathcal{R},\mathrm{Ti}}) = \Delta_h^{\mathrm{Ti}}(X;q)$.*

Proof. (i) The automorphism $q \mapsto -q$, $X \mapsto -X$ of $\mathcal{R}[X]$ fixes every entry $A^{\mathcal{R}}(u^n) + XB^{\mathcal{R}}(u^n)$: Ω_b and $S_k(q)$ are even in q, and q^{2k+1} is odd. So it fixes $\Delta_h^{\mathcal{R}}$.

(ii) Over $\mathbb{C}$, the homomorphism $q \mapsto iq$, $E \mapsto -1/(q^2+1) = 1/((iq)^2 - 1)$ maps Ω_b to the values (2.2) with $y = iq$, i.e. to those of case (B); it maps $S_k(q)$ to $S_k(iq) = S_k^{\mathrm{Ti}}(q)$, and q^{2k+1} to $i(-1)^k q^{2k+1}$. So it maps $A^{\mathcal{R}}$ to the functional A of case (B) and $B^{\mathcal{R}}$ to i times its B, hence $\Delta_h^{\mathcal{R}}(X)$ to $\Delta_h^{\mathrm{Ti}}(iX;q)$. Replacing X by $-iX$, the monomial $c\,q^{m_0}E^{m_1}X^j$ goes to $c\,i^{m_0}(-1)^{m_1}(-i)^j q^{m_0}(q^2+1)^{-m_1}X^j$, and by (i), $i^{m_0}(-i)^j(-1)^{m_1} = \tau(m,j)$. □

So the coefficients of $\Delta_h^{\mathcal{R},\mathrm{Ti}}$ are, up to sign, those of $\Delta_h^{\mathcal{R}}$: the two have the same Gauss norms, the same E-degrees and the same q-degrees, coefficient by coefficient.

5. Denominators II: the odd primes dividing q and the pole factor

5.1. **The pole order.** For $f \in \mathcal{R}$ let $\deg_E f$ be its degree in E. Put

$$\rho_i := (i + 1 - \lceil h/2 \rceil)_+, \qquad \kappa_j := (j - \lfloor h/2 \rfloor)_+,$$
$$\beta_h := \sum_{i<h} \rho_i + \sum_{j<h} \kappa_j = \lfloor h^2/4 \rfloor \le \frac{h^2+2h}{4}.$$

Lemma 5.1 (Lemma 5)**.** *Every coefficient of $\Delta_h^{\mathcal{R}} \in \mathcal{R}[X]$ has E-degree at most β_h.*

Proof. In the monomial basis, the entry (i,j) is $\mu^{\mathcal{R}}(R)$ plus pole terms that do not involve E, where R is the quotient of u^{i+j+1} by D, of degree $i+j+1-h$ (and $R=0$ if $i+j+1<h$). In $\mu^{\mathcal{R}}(R) = \sum_{b<\deg R} \Delta^b[R'](0)\Omega_b$ each Ω_b has E-degree $b+1 \le \deg R$. So the entry has E-degree at most $(i+j+1-h)_+ \le \rho_i + \kappa_j$, and the Leibniz expansion gives the claim. □

Clearing E. For $f = \sum c_{a,b} q^a E^b \in \mathcal{R}$ with $\deg_E f \le \beta$ and $\sigma \in \{\pm 1\}$ put $\mathrm{cl}_\beta^\sigma(f) := \sum c_{a,b}\, q^a (q^2-\sigma)^{\beta-b} \in \mathbb{Q}[q]$, and $\mathrm{cl}_\beta := \mathrm{cl}_\beta^+$. Since $(q^2 \mp 1)^k$ has integer coefficients, $\|\mathrm{cl}_\beta^\sigma f\|_p \le \|f\|_p$ for every p. If $v_1(v_0^2 - \sigma) = 1$, then evaluating at $q = v_0$, $E = v_1$ gives

$$\mathrm{ev}\big(\mathrm{cl}_\beta^\sigma f\big) = (v_0^2 - \sigma)^\beta \,\mathrm{ev}(f). \tag{5.1}$$

By the computation in the proof of Proposition 4.6(ii), for $f = [X^j]\Delta_h^{\mathcal{R}}$ and $f^{\mathrm{Ti}} = [X^j]\Delta_h^{\mathcal{R},\mathrm{Ti}}$, the coefficient of q^n in $\mathrm{cl}_\beta^-(f^{\mathrm{Ti}})$ is $\pm$ the coefficient of q^n in $\mathrm{cl}_\beta(f)$: indeed $\mathrm{cl}_\beta^-(f^{\mathrm{Ti}})(q)$ is a unit of $\{\pm 1, \pm i\}$ times $\mathrm{cl}_\beta(f)(iq)$.

5.2. **The q-adic gain.** Let $\psi_q \colon \mathcal{R} \to \mathbb{Q}[[q]]$ be the ring homomorphism with $q \mapsto q$ and $E \mapsto 1/(q^2-1) = -\sum_{n\ge 0} q^{2n}$. Put

$$a_h := \max\Big(0, \Big\lfloor \frac{h(h-1)-6h}{3} \Big\rfloor\Big) \ge \frac{h^2-7h-2}{3}.$$

Lemma 5.2 (Lemma 4′)**.** *Every coefficient of $\psi_q(\Delta_h^{\mathcal{R}}) \in \mathbb{Q}[[q]][X]$ is divisible by q^{a_h}. Consequently, for every k, all monomials of $\mathrm{cl}_{\beta_h}\big([X^k]\Delta_h^{\mathcal{R}}\big) \in \mathbb{Q}[q]$ have degree at least a_h.*

Proof. Reflection. From $\psi_q(\Omega_b) = (-1)^{b+1}q^2(1-q^2)^{-b-1}$ for $b \geq 1$ and $\psi_q(\Omega_0) = -(1-q^2)^{-1}$, the coefficient of q^{2m} in $\psi_q(\Omega_b)$ is $-(-1)^b\binom{m+b-1}{b}$, and odd powers do not occur. Newton's backward formula $f(-m) = \sum_b (-1)^b\binom{m+b-1}{b}\Delta^b f(0)$ for polynomials f then gives

$$\psi_q\big(\mu^{\mathcal{R}}(R)\big) = -\sum_{m\geq 0} q^{2m} R'(-m) \qquad (R \in \mathbb{Q}[u]). \tag{5.2}$$

Recombination. For a polynomial g write $g\varphi = R + \sum_k c'_k/(u-x_k)$, $c'_k = g(x_k)\omega_k$. Since $\big((u-x_k)^{-1}\big)'(-m) = -4/(2k+1-2m)^2$ and

$$\sum_{m\geq 0} \frac{q^{2m}}{(2k+1-2m)^2} = S_k(q) + q^{2k+1}\chi_2(q),$$

where $\chi_2(q) = \sum_{j\geq 0} q^{2j+1}/(2j+1)^2 \in \mathbb{Q}[[q]]$, (5.2) and Lemma 2.2 combine to

$$\psi_q\big(\Lambda_X^{\mathcal{R}}(g)\big) = -\sum_{m\geq 0} q^{2m}(g\varphi)'(-m) - 4\,(X+\chi_2(q))\sum_{k<h} g(x_k)\,\omega_k\,q^{2k+1}. \tag{5.3}$$

Slots. For $g = u^{i+j}$ we have $(g\varphi)'(-m) = \varphi'(-m)\,t^it^j + \varphi(-m)\big(it^{i-1}\big)t^j + \varphi(-m)\,t^i\big(jt^{j-1}\big)$ at $t=-m$, and $g(x_k) = x_k^i x_k^j$. So the matrix $\psi_q(H)$ is a sum of rank-one matrices $c\,xy^{\mathsf{T}}$, where the row vector x is either a *value row* $(t^i)_i$ or a *derivative row* $(it^{i-1})_i$ at a node $t=-m$, or a pole row $(x_k^i)_i$; the same holds for y. A node $-m$ carries the factor q^{2m} and a pole k carries q^{2k+1}. Give these row types the *keys* (m,val), (m,der) and k, with *prices* $2m$, $2m$ and $2k+1$.

Truncation and Andreief. Let $M := a_h + 1$ and let H_M be the same sum restricted to the nodes $m < M$. Then $\psi_q(H) \equiv H_M$ entrywise modulo q^{2M}, so $\psi_q(\Delta_h^{\mathcal{R}}) = \det\psi_q(H) \equiv \det H_M \pmod{q^{2M}}$. For a finite sum of rank-one matrices, the Andreief (Cauchy–Binet) identity [1, 8] reads

$$h!\,\det\Big(\sum_\iota c_\iota x_\iota y_\iota^{\mathsf{T}}\Big) = \sum_{f\colon [h]\to I}\Big(\prod_i c_{f(i)}\Big)\det\big(x_{f(i)}(j)\big)_{i,j}\det\big(y_{f(i)}(j)\big)_{i,j}. \tag{5.4}$$

Here the rank-one terms are indexed by $\iota \in I$; the row vector x_ι and the power of q in c_ι depend only on the key of ι (the two terms $\varphi'(-m)\,t^it^j$ and $\varphi(-m)\,t^i(jt^{j-1})$ of a node share the key (m,val)). A term of (5.4) vanishes unless the rows $x_{f(i)}$ are pairwise distinct, so unless the h keys of $f(0),\dots,f(h-1)$ are pairwise distinct. The term is then divisible by q to the sum of their prices. The map $(m,\mathrm{val}) \mapsto 3m$, $(m,\mathrm{der}) \mapsto 3m+2$, $k \mapsto 3k+1$ is injective and satisfies $2\cdot(\text{image}) \leq 3\cdot\text{price} + 6$. The images of h distinct keys are h distinct naturals, with sum $\geq h(h-1)/2$; hence the prices sum

to at least $(h(h-1)-6h)/3 \geq a_h$. Since $h!$ is invertible and $2M > a_h$, q^{a_h} divides every coefficient of $\psi_q(\Delta_h^{\mathcal{R}})$.

Clearing. Let $f = [X^k]\Delta_h^{\mathcal{R}}$. By Lemma 5.1 and (5.1) (with $v = (q, 1/(q^2-1))$ in $\mathbb{Q}[[q]]$), $\mathrm{cl}_{\beta_h} f = (q^2-1)^{\beta_h}\psi_q(f)$ in $\mathbb{Q}[[q]]$, which is divisible by q^{a_h}. As $\mathrm{cl}_{\beta_h} f$ is a polynomial in q, its monomials of degree $< a_h$ vanish. □

5.3. **The odd part of the denominator.** For the three cases put

$$(y, Y) = (q,\, q^2-1)\ \text{(A)}, \qquad (q,\, q^2+1)\ \text{(B)}, \qquad (q/2,\, q^2-4)\ \text{(C)},$$

so that Y is y^2-1, $-(y^2-1)$ with $y = iq$, or $4(y^2-1)$, and put

$$\alpha_{\mathrm{odd}} := \frac{133}{60} + \frac{1}{4}\log \mathrm{odd}(Y) - \frac{1}{3}\log \mathrm{odd}(q).$$

Theorem 5.3 (the odd primes)**.** *Fix a point of case (A) or (B) with $q \geq 3$, or of case (C) with $q \geq 5$ odd. There are positive rationals d_h^{odd}, with $v_2(d_h^{\mathrm{odd}}) = 0$, such that $|d_h^{\mathrm{odd}} c|_p \leq 1$ for every coefficient c of Δ_h and every odd prime p, and for every $\varepsilon > 0$ and all large h,*

$$\log d_h^{\mathrm{odd}} \leq (\alpha_{\mathrm{odd}} + \varepsilon)h^2.$$

Consequently, if $n_2(h) \in \mathbb{Z}$ satisfy $|c|_2 \leq 2^{n_2(h)}$ for every coefficient c of Δ_h, then $d_h := 2^{n_2(h)} d_h^{\mathrm{odd}}$ satisfies $d_h\Delta_h \in \mathbb{Z}[X]$ and $\log d_h = n_2(h)\log 2 + \log d_h^{\mathrm{odd}}$.

Proof. Let $c = [X^k]\Delta_h$. In cases (A) and (C), $c = \mathrm{ev}_y(f)$ with $f = [X^k]\Delta_h^{\mathcal{R}}$ by (4.1); in case (B), $c = \mathrm{ev}_q^{\mathrm{Ti}}(f^{\mathrm{Ti}})$ with $f^{\mathrm{Ti}} = [X^k]\Delta_h^{\mathcal{R},\mathrm{Ti}}$ by Proposition 4.6. Let p be odd. Then y is p-integral, with $v_p(y) = v_p(q)$, and v_p of the value of E is $-v_p(Y)$. We use the clearing cl^σ with $\sigma = +1$ in cases (A), (C) and $\sigma = -1$ in case (B), so that $\mathrm{ev}(\mathrm{cl}^\sigma f) = (y^2-\sigma)^{\beta_h} c$ and $v_p(y^2-\sigma) = v_p(Y)$. By Proposition 4.6 and the remark after (5.1), the Gauss norms and the q-degrees of the monomials of the cleared coefficient are the same as for $\mathrm{cl}_{\beta_h}(f)$ in case (A).

- $p \mid q$: then $p \nmid Y$, and by Lemma 5.2 every monomial of the cleared coefficient has degree $\geq a_h$, so $|c|_p \leq \|f\|_p\, |q|_p^{a_h} \leq p^{2h\,\mathrm{lev}(p)}|q|_p^{a_h}$ by Lemma 4.2.
- $p \mid Y$: by Lemma 5.1 and (5.1), $|c|_p \leq \|f\|_p\, |Y|_p^{-\beta_h} \leq p^{2h\,\mathrm{lev}(p)+v_p(Y)\beta_h}$, by Lemma 4.2.
- $p \nmid qY$: y and the value of E are p-integral, so $|c|_p \leq \|f\|_p \leq p^{e(h,p)}$ by Theorem 4.5.

Let $N_h := \prod_p p^{e_p}$ over the odd primes, where $e_p = 2h\,\mathrm{lev}(p) + e(h,p) + v_p(Y)\beta_h$ for $p \mid qY$ and $e_p = e(h,p)$ otherwise. Only primes $p < 2h+q^2$ occur. Put $d_h^{\mathrm{odd}} := N_h/\,\mathrm{odd}(q)^{a_h}$; since $|\,\mathrm{odd}(q)|_p = |q|_p$ for odd p, $|d_h^{\mathrm{odd}} c|_p \leq 1$. Finally the terms $2h\,\mathrm{lev}(p)\log p \leq 2h\log(2h)$ of the finitely many $p \mid qY$ are $O(h\log h)$, $\beta_h \log \mathrm{odd}(Y) \leq \frac{h^2+2h}{4}\log \mathrm{odd}(Y)$, $a_h \geq (h^2-7h-2)/3$, and Theorem 4.5 bounds $\sum_p e(h,p)\log p$. □

Remark 5.4. In [12] the gain at $p = q$ used that q is prime. Here it is applied at each odd prime $p \mid q$ separately, with multiplicity $v_p(q)$; the 2-adic share of the gain, for even q, is counted inside the 2-adic lemmas of Section 9, and only there.

6. The size of $\Delta_h(\theta)$

6.1. **Dual certificates.** Let $\psi(u) := u \log|u| - u$ (with $\psi(0) = 0$), an antiderivative of $\log|u|$, so that $\int_\alpha^\beta \log|x-y|\,dy = \psi(x-\alpha) - \psi(x-\beta)$. Put for $0 < W < 1$ and $x \geq 0$

$$V_W(x) := x\log W - \psi(x+1) + \psi(x) = x\log W - (x+1)\log(x+1) + x\log x + 1.$$

We write $V = V_W$ when W is fixed. A *step density* is $\mu = \sum_{j<J} c_j \mathbf{1}_{[a_j, a_{j+1})}$ with $0 = a_0 \leq a_1 \leq \cdots \leq a_J$, $0 \leq c_j \leq 2$ and $\int \mu = 1$. Its potential and energy are

$$U^\mu(x) := \int \log|x-y|\,\mu(y)\,dy, \qquad E(\mu) := \iint \log|x-y|\,\mu(x)\mu(y)\,dx\,dy.$$

Definition 6.1. A *dual certificate* for a weight $W \in (0,1)$ consists of a step density μ, a real number λ, integers $L, X, N \geq 1$ with $a_J \leq N \leq X$, and real numbers S_k $(0 \leq k < XL)$ such that

(1) $F(x) := 2U^\mu(x) + V(x) \leq S_k$ for $x \in [k/L, (k+1)/L]$, for every $k < XL$;
(2) $1/X + \log W < 0$ and $\log X + X\log W < \lambda$.

Its *exponent* is $\Phi := -E(\mu) + \lambda + \frac{2}{L}\sum_{k<XL}(S_k - \lambda)_+$.

Theorem 6.2 (the size bound)**.** *Fix a point of case (A), (B) or (C) and put $W := |w|$. For every dual certificate for W, as $h \to \infty$,*

$$\log|\Delta_h(\theta)| \leq \Phi h^2 + O(h\log h).$$

The constant in $O(h\log h)$ depends on the certificate. The heuristic behind the theorem: by (2.3), $\Delta_h(\theta)$ is a Hankel determinant of a functional on the nodes $t = 1, 2, \ldots$ with the derivative structure of $(g\varphi)'$ and weights of size W^t, i.e. a discrete log-gas in the external field V with at most two particles per node. The dual certificate is a feasible point of the dual of the corresponding constrained equilibrium problem [19, Ch. I], and the proof below is a discrete version of weak duality.

6.2. **Node expansion.** We have $\Lambda(u^n) = \sum_{s\geq 1} w^s\big(ns^{n-1}\varphi(s) + s^n\varphi'(s)\big)$. Let $H^{(T)}$ be the matrix of truncated sums over $1 \leq s \leq T$. Then $H^{(T)} \to (\Lambda(u^{i+j}))_{i,j}$ entrywise, so by continuity of the determinant $|\Delta_h(\theta)| \leq \sup_T |\det H^{(T)}|$. Expanding $(i+j)s^{i+j-1} = (is^{i-1})s^j + s^i(js^{j-1})$,

$$H^{(T)} = \sum_{s\leq T}\Big(w^s\varphi'(s)\,v_s v_s^{\mathsf T} + w^s\varphi(s)\,v_s' v_s^{\mathsf T} + w^s\varphi(s)\,v_s v_s'^{\mathsf T}\Big),$$

with the value row $v_s = (s^j)_{j<h}$ and the derivative row $v_s' = (js^{j-1})_{j<h}$. We call (s, val) and (s, der) the two *slots* of the node s. Since $\varphi'/\varphi =$

$1/s - \sum_k 1/(s-x_k)$ is a difference of two numbers in $[0,h]$ for $s \geq 1$, all three coefficients are bounded by

$$b(s) := W^s\, h\, \varphi(s),$$

since $|w^s| = W^s$.

Lemma 6.3 (minors with derivative rows)**.** *Let $h \geq 2$ and let $(s_i,\tau_i)_{i<h}$ be slots at integer nodes $s_i \geq 0$. Then*

$$\bigl|\det(\textit{row of slot } (s_i,\tau_i))_{i<h}\bigr| \leq h^h\Bigl(1+\frac{2}{h}\Bigr)^{h^2}\prod_{i<j}\mathrm{pf}(s_i,s_j),$$

where $\mathrm{pf}(a,b) = |a-b|$ for $a \neq b$ and $\mathrm{pf}(a,a) = 1$.

Proof. Let ζ be a primitive h-th root of unity and $r = 1/h$. For $j < h$, $\sum_{l<h}\zeta^{l(h-1)}(t+r\zeta^l)^j = h\,r\,jt^{j-1}$ (only the binomial term with $h \mid h-1+m$, i.e. $m=1$, survives; this needs $h \geq 2$). Since $hr = 1$, the derivative row at t is the sum $\sum_l \zeta^{l(h-1)}$(value row at $t+r\zeta^l$). By multilinearity the determinant is a sum of at most $h^{\#\mathrm{der}}$ Vandermonde determinants at points z_i with $|z_i - s_i| \leq r$, each with a coefficient of modulus 1. Since the nodes are integers, each Vandermonde factor satisfies $|z_j - z_i| \leq |s_j - s_i| + 2r \leq (1+2r)\,\mathrm{pf}(s_i,s_j)$. So the determinant is at most $h^h(1+2/h)^{h(h-1)/2}\prod_{i<j}\mathrm{pf}(s_i,s_j)$. □

Apply (5.4) to $H^{(T)}$, indexing the rank-one terms by triples (node, type). A term with two equal row vectors vanishes, so only assignments f whose *x-slots* are distinct contribute; in particular each node carries at most two of the h indices. With Lemma 6.3 for both determinants,

(6.1)

$$|\det H^{(T)}| \leq \frac{K_h^2}{h!}\sum_f \prod_{i<h} b(s_i)\prod_{i<j}\mathrm{pf}(s_i,s_j)^2, \qquad K_h := h^h(1+2/h)^{h^2},$$

where f runs over the assignments $i \mapsto (s_i,\tau_i)$ with distinct x-slots and s_i is the node of $f(i)$.

6.3. **A lattice energy inequality.**

Lemma 6.4 (Pólya-type inequality)**.** *For $M \geq 1$ and real $\nu_0,\dots,\nu_{M-1}$,*

$$\sum_{i\neq j}\nu_i\nu_j\log|i-j| \leq \log 2\sum_i \nu_i^2 + \log M\Bigl(\sum_i \nu_i\Bigr)^2.$$

Proof. For $M = 1$ the claim reads $0 \leq \log 2\,\nu_0^2$. Let $M \geq 2$. We follow Pólya's criterion for positive definite functions [16] in a finite form. Let $f(0) = \log M + \log 2$, $f(d) = \log(M/d)$ for $1 \leq d \leq M$, and $f(d) = 0$ for $d > M$. Then f is convex on $\mathbb{Z}_{\geq 0}$ (this uses $\log(k-1)+\log(k+1) \leq 2\log k$ for $2 \leq k < M$, $f(M-1) - 2f(M) + f(M+1) = \log\frac{M}{M-1} \geq 0$, and $f(0) - 2f(1) + f(2) = 0$ when $M \geq 2$). Hence $f(d) = \sum_{k=1}^{M}\Delta^2 f(k-1)\,(k-d)_+$ with $\Delta^2 f \geq 0$. Each triangle kernel $(k-|i-j|)_+$ is the number of

windows of length k containing both i and j, hence positive semidefinite. So $\sum_{i,j}\nu_i\nu_j f(|i-j|)\ge 0$, which rearranges to the claim. □

For a reference sequence $m=(m_s)_{s\ge 0}$ with finite support, $\sum_s m_s = h$, put

$$P_m(s):=\sum_{s'\neq s} m_{s'}\log|s-s'|,\qquad E_m:=\sum_{s\neq s'} m_s m_{s'}\log|s-s'|.$$

Lemma 6.5 (discrete weak duality)**.** *Let m be as above with $\sum_s m_s^2\le 2h$, and let $\lambda'\in\mathbb{R}$. Put $g(s):=\log b(s)+2P_m(s)-\lambda'$. For every assignment f in* (6.1)*,*

$$\prod_i b(s_i)\prod_{i<j}\mathrm{pf}(s_i,s_j)^2\le\exp\Big(-E_m+\lambda' h+8h\log 2+\sum_i g(s_i)\Big).$$

Proof. Let $n_s:=\#\{i:s_i=s\}\le 2$, so $\sum_s n_s=h$ and $\sum_s n_s^2\le 2h$. Pairs at the same node contribute pf $=1$, so $2\sum_{i<j}\log\mathrm{pf}(s_i,s_j)=\sum_{s\neq s'}n_sn_{s'}\log|s-s'|$. With $\nu:=n-m$ (total mass 0),

$$\sum_{s\neq s'}n_sn_{s'}\log|s-s'|=-E_m+2\sum_s n_sP_m(s)+\sum_{s\neq s'}\nu_s\nu_{s'}\log|s-s'|,$$

and by Lemma 6.4 the last sum is at most $\log 2\sum\nu_s^2\le 2\log 2(\sum n_s^2+\sum m_s^2)\le 8h\log 2$. Finally $\sum_s n_sP_m(s)=\sum_i P_m(s_i)$. □

6.4. **Discretization.** Given a step density μ supported in $[0,N]$, let $m_s:=h\int_{(s-1)/h}^{s/h}\mu$ for $s\ge 1$ and $m_0:=0$ (cell averages). Then $\sum m_s=h$, $0\le m_s\le 2$ (so $\sum m_s^2\le 2h$), and $m_s=0$ for $s>Nh$.

Lemma 6.6. *With these m_s, for all $h\ge 2$ and $s\ge 1$:*

(1) $\log b(s)\le\log h+\log s-h\log h+h\,V(s/h)$*;*
(2) $P_m(s)\le h\,U^\mu(s/h)+h\log h+4+2\log(Nh+1)$*;*
(3) $E_m\ge h^2E(\mu)+h^2\log h-4h\big(1+\log(Nh+1)\big)$*.*

Proof. (i) By concavity of log, $\log(c+\frac12)\ge\int_c^{c+1}\log y\,dy=\psi(c+1)-\psi(c)$ for $c\ge 0$, so $\log D(s)=\sum_{k<h}\log(s+k+\frac12)\ge\psi(s+h)-\psi(s)=h\big(\psi(x+1)-\psi(x)\big)+h\log h$ with $x=s/h$. Insert this into $\log b(s)=s\log W+\log h+\log s-\log D(s)$.

(ii) We have $h\,U^\mu(s/h)=\sum_{s'}\int_{I_{s'}}\log|s-u|\,\rho(u)\,du-h\log h$, where $I_{s'}=[s'-1,s']$ and $\rho(u):=\mu(u/h)$; this density has mass $m_{s'}$ on $I_{s'}$, total mass h, and is bounded by 2. For $s'<s$ and $u\in I_{s'}$, $\log|s-u|\ge\log(s-s')$, so these cells contribute with the right sign. For $s'=s$ the discrete term is 0 and $\int_{I_s}(-\log|s-u|)\,\rho\le 2$. For $s'=s+d$, $d\ge 1$, the deficit is at most $2\int_{d-1}^{d}(\log d-\log u)\,du=2\big(1-(d-1)\log\frac{d}{d-1}\big)\le 2/d$. Summing over $d\le Nh$ gives (ii).

(iii) Similarly $h^2E(\mu)+h^2\log h=\sum_{s,s'}\iint_{I_s\times I_{s'}}\log|u-u'|\,\rho(u)\rho(u')$. The diagonal cells contribute ≤ 0 because $|u-u'|\le 1$ there. For $|s-s'|=d\ge 1$

we have $\log|u-u'| \le \log(d+1) \le \log d + 1/d$. Since $m_{s'} \le 2$, summing $m_s m_{s'}/d$ over the pairs gives at most $\sum_s m_s \cdot 2 \cdot 2\sum_{d\le Nh} 1/d \le 4h(1+\log(Nh+1))$. □

6.5. **Proof of Theorem 6.2.** Fix a certificate and $h \ge 2$. Take m as above and $\lambda' := h\lambda + h\log h$. By Lemma 6.6, with $\epsilon_h := 4 + 2\log(Nh+1)$ and $x = s/h$,

$$g(s) \le h\big(F(x)-\lambda\big) + \log h + \log s + 2\epsilon_h, \tag{6.2}$$
$$-E_m + \lambda' h \le h^2\big(\lambda - E(\mu)\big) + O(h\log h).$$

Near and far nodes. Let $S_0 := Xh$. Since each node carries at most two indices, $\sum_i g(s_i) \le 2\sum_{1\le s\le S_0} g(s)_+ + \sum_{i:\, s_i > S_0} g(s_i)$. Summing over f in (6.1) with Lemma 6.5 therefore gives

$$|\det H^{(T)}| \le \frac{K_h^2}{h!}\, e^{-E_m+\lambda' h + 8h\log 2 + 2\sum_{s\le S_0} g(s)_+} \Big(3S_0 + 3\sum_{s>S_0} e^{g(s)}\Big)^h.$$

Near nodes. Each $x = s/h \le X$ lies in a cell $[k/L, (k+1)/L]$ with $k < XL$, where $F \le S_k$; each cell contains at most $h/L+1$ points s/h. By (6.2), $\sum_{s\le S_0} g(s)_+ \le h(h/L+1)\sum_k (S_k-\lambda)_+ + S_0(\log h + \log S_0 + 2\epsilon_h)$.

Far nodes. Put $\delta := -\max\big(1/X + \log W,\ (\log X + X\log W - \lambda)/X\big) > 0$. For $x \ge N$, $U^\mu(x) \le \log x$, and $V(x) \le x\log W - \log x$ (because $1 - x\log(1+1/x) \le 1/(x+1) \le \log(1+1/x)$). Hence, using $\log x \le \log X + (x-X)/X$, for $x \ge X$

$$\begin{aligned} F(x) - \lambda &\le \log x + x\log W - \lambda \\ &\le (\log X + X\log W - \lambda) + (x-X)(1/X + \log W) \le -\delta x. \end{aligned}$$

So $e^{g(s)} \le h\, s\, e^{2\epsilon_h} e^{-\delta s} \le \frac{2}{\delta} h\, e^{2\epsilon_h} e^{-\delta s/2}$ for $s > S_0$, and the far sum is $e^{O(\log h)}$.

Assembly. $\log(K_h^2/h!) = O(h\log h)$, and $\log\big(3S_0 + 3\sum_{s>S_0} e^{g(s)}\big) = O(\log h)$. Collecting,

$$\log|\det H^{(T)}| \le h^2\Big(-E(\mu) + \lambda + \frac{2}{L}\sum_k (S_k-\lambda)_+\Big) + O(h\log h)$$

uniformly in T, and the theorem follows from $|\Delta_h(\theta)| \le \sup_T |\det H^{(T)}|$. □

Remark 6.7. Let $C(W) := \sup\{E(\rho) + \int V\rho\}$ over densities ρ on $[0,\infty)$ with compact support, $0 \le \rho \le 2$ and $\int \rho = 1$ (the constrained equilibrium constant). For such ρ, $E(\rho-\mu) \le 0$ [19, Lemma I.1.8], so $E(\rho) + \int V\rho \le -E(\mu) + \int F\rho \le -E(\mu) + \lambda + 2\int_0^X (F-\lambda)_+ \le \Phi$, using $\rho \le 2$, $F < \lambda$ beyond X and $F \le S_k$ on the cells. Hence $\Phi \ge C(W)$. In [12] the finite-h values $h^{-2}\log|\Delta_h(\chi_2(1/q))|$ for $q = 3, 5$, computed in ball arithmetic up to $h = 320$, were observed to approach $C(1/9) \approx -2.1003$ and $C(1/25) \approx -2.4565$, so for $w > 0$ Theorem 6.2 seems essentially sharp for this construction. For $w < 0$ it is not: it sees only $|w|$, and numerically $|\Delta_h^{\mathrm{Ti}}(\mathrm{Ti}_2(1/q))|$ is smaller than $|\Delta_h^{\chi}(\chi_2(1/q))|$ by a factor of roughly e^{-h^2/q^2} for small q. Only the form with an explicit certificate is used.

7. Certificates and the shift lemma

7.1. **The shift lemma.** A dual certificate depends on the point only through W, and only through the term $x \log W$ of V_W. Lowering W therefore tilts the external field, and the cap $\mu \le 2$ turns the tilt into a gain.

Lemma 7.1 (shift lemma)**.** *Let $0 < W' \le W < 1$ and $\tau := \log(W/W')$. For every dual certificate for W with exponent Φ and every $\delta > 0$ there is a dual certificate for W' with exponent at most $\Phi - \tau/4 + \delta$.*

Proof. Keep μ, X, N, and put $\lambda' := \lambda - \tau/2$. Since $V_{W'}(x) = V_W(x) - \tau x$, the new F' is $F - \tau x$. Refine the grid by a factor $2r$: $L' = 2rL = 2m$ with $m = rL$, and for $k' < XL'$ put $S'_{k'} := S_k - \tau k'/L'$ with $k = \lfloor k'/(2r) \rfloor$; on $[k'/L', (k'+1)/L'] \subseteq [k/L, (k+1)/L]$ the term $-\tau x$ is at most its value at the left end, so condition (i) holds. For (ii), $1/X + \log W' \le 1/X + \log W < 0$, and $\log X + X \log W' - \lambda' = (\log X + X \log W - \lambda) - (X - \frac{1}{2})\tau < 0$. Finally $(S_k - \lambda + \tau(\frac{1}{2} - k'/L'))_+ \le (S_k - \lambda)_+ + \frac{\tau}{2}(1 - k'/m)_+$, each k is repeated $2r$ times, and $\sum_{k' \ge 0}(1 - k'/m)_+ = (m+1)/2$, so

$$\Phi' \le -E(\mu) + \lambda - \frac{\tau}{2} + \frac{2}{L}\sum_k (S_k - \lambda)_+ + \frac{2}{L'} \cdot \frac{\tau}{2} \cdot \frac{m+1}{2} = \Phi - \frac{\tau}{4} + \frac{\tau}{4m}.$$

Choose r with $\tau/(4rL) \le \delta$. □

For $W = q_0^{-2}$ and $W' = q^{-2}$ the gain is $\tau/4 = \frac{1}{2}\log(q/q_0)$; the same holds for $W = 4q_0^{-2}$, $W' = 4q^{-2}$. Without the cap (μ of arbitrary height) there would be no gain: the cap forces every admissible density to have mean at least $\frac{1}{4}$.

7.2. **The certificates used.**

Theorem 7.2. *There are dual certificates with the following exponents:*

weight W	$\Phi \le$	*used for*
$1/3^2$	-2.078	*Thm. A, odd q*
$1/14^2$	-3.03	*Thm. A, even $q \ge 14$*
$1/6^2$, $1/8^2$	-2.56, -2.72	*Thm. A, $q = 6, 8$*
$1/10^2$, $1/12^2$	-2.85, -2.95	*Thm. A, $q = 10, 12$*
$1/6^2$ *(finer)*	-2.5671	*Thm. B, all $q \ge 6$*
$1/5^2$	-2.44	*Thm. B, $q = 5$*
$4/175^2$	-4.003	*Thm. C, odd $q \ge 175$*

The certificates are explicit rational data (Table 2). The density μ is constant on the J cells $[js/L, (j+1)s/L)$ with dyadic heights $c_j \in [0, 2]$ (denominator 2^{16}) and exact mass $\sum_j c_j s/L = 1$. The bounds S_k are rationals. All checks are inequalities between explicit rational numbers and finitely many values of log at rational points, and are verified as follows.

- *Logarithms.* A table of dyadic intervals of width $2\cdot 2^{-40}$ for $\log n$, $n \le n_{\max}$, each checked against the series of $\log(1-x)$ with an explicit tail bound, after reducing the argument by powers of 2; $\log 2$ is enclosed to width 2^{-100}. The weight enters through $\log W = -2\log q$, or $2\log 2 - 2\log 175$.
- *Potential.* By summation by parts, $2U^\mu(x) + V(x) = \sum_j D_j\,\psi(x - \alpha_j/L) + x\log W$, with $D_j = 2(c_j - c_{j-1})$ at $\alpha_j = js$, plus the terms $\psi(x)$ and $-\psi(x+1)$ of V. On each grid cell $[k/L, (k+1)/L]$ every term is bounded above by an affine function: by the chord where the term is convex, and by a tangent (at the midpoint, or at the endpoint away from α_j/L) where it is concave. Summing the affine bounds and taking the larger endpoint value bounds F on the cell. All values are rational enclosures.
- *Energy.* $E(\mu) = \sum_{j,j'} c_j c_{j'} \iint \log|x-y|$ over pairs of cells, and the double integral over $[a_1,b_1]\times[a_2,b_2]$ equals $-\Psi(b_1-b_2)+\Psi(b_1-a_2)+\Psi(a_1-b_2)-\Psi(a_1-a_2)$ with $\Psi(u) = \frac{1}{2}u^2\log|u| - \frac{3}{4}u^2$. A rational lower bound $E_{\mathrm{lo}} \le E(\mu)$ is computed in interval arithmetic.
- *Tail.* $1/X + \log W < 0$ and $\log X + X\log W < \lambda$, from the enclosures of the logarithms.

The certified exponent is the rational number $\Phi_{\mathbb{Q}} := -E_{\mathrm{lo}} + \lambda + \frac{2}{L}\sum_k (S_k - \lambda)_+ \ge \Phi$.

W	$1/L$	$[0,X]$ (XL cells)	$J\times$ width	N	log table	$\Phi_{\mathbb{Q}}$
$1/3^2$	$1/100$	$[0,3]$ (300)	$30 \times \frac{1}{20}$	2	804	$-2.0783260\ldots$
$1/14^2$	$1/200$	$[0,2]$ (400)	$40 \times \frac{1}{50}$	1	1204	$-3.0372394\ldots$
$1/6^2$	$1/400$	$[0,2]$ (800)	$54 \times \frac{1}{50}$	2	2404	$-2.5635463\ldots$
$1/8^2$	$1/200$	$[0,2]$ (400)	$50 \times \frac{1}{50}$	1	1204	$-2.7264892\ldots$
$1/10^2$	$1/200$	$[0,2]$ (400)	$46 \times \frac{1}{50}$	1	1204	$-2.8528812\ldots$
$1/12^2$	$1/200$	$[0,2]$ (400)	$46 \times \frac{1}{50}$	1	1204	$-2.9535971\ldots$
$1/6^2$ (finer)	$1/800$	$[0,2]$ (1600)	$108 \times \frac{1}{100}$	2	4804	$-2.5671799\ldots$
$1/5^2$	$1/200$	$[0,2]$ (400)	$115 \times \frac{1}{100}$	2	1204	$-2.4435517\ldots$
$4/175^2$	$1/1200$	$[0,1]$ (1200)	$70 \times \frac{1}{100}$	1	4804	$-4.0033242\ldots$

TABLE 2. The dual certificates (decimal values truncated). The $q = 3$ certificate is that of [12].

The data are generated by a script that optimises the density in floating point, rounds it to dyadic rationals, and then replicates the interval arithmetic exactly in rational arithmetic. The script is not trusted: in the formalisation every check above is a decidable statement about the data, proved by evaluation in the Lean kernel (Section 13). The certificates are not optimal. Finer independent certificates in ball arithmetic give, for example, $\Phi \le -2.5704$ at $W = 1/6^2$ and $\Phi \le -4.0005$ at $W = 4/173^2$; they are used only in the remarks.

8. Nonvanishing: Lemma P

8.1. **Statement.** In this section we work in case (A) with an integer $q \geq 2$, so $z = 1/q$, $w = q^{-2}$ and $\Delta_h = \Delta_h^{\chi}(X; q)$; Sections 10 and 11 transfer the result to cases (B) and (C). For $r \in \{0,1,2\}$ put $c_r := 1/\prod_{a\in\{0,1,2\},a\neq r}(a-r)$, so $(c_0, c_1, c_2) = (\frac{1}{2}, -1, \frac{1}{2})$. Let $T(M) = \sum_{s\leq M,\, s \text{ odd}} z^s/s^2$, with $T(M) = 0$ for $M < 1$, so that $S_k(q) = q^{2k+1}T(2k+1)$. Define three 3×3 rational matrices, indexed by $r, s \in \{0,1,2\}$:

$$M_S(q)_{rs} = \frac{c_r}{1-w} + [\, r = s \,]\, 4q^{2r+1}T(2r-1),$$
$$M_L(q)_{rs} = \frac{w\, c_r}{1-w} + [\, r = s \,]\, 4q^{2r+1}T(2r+1),$$
$$M_D(q)_{rs} = w\, c_r\Big(\frac{e_{rs}}{1-w} + \frac{2w}{(1-w)^2}\Big) - [\, r = s \,]\, 4\big(r+\tfrac{1}{2}\big)q^{2r+1}T(2r+1),$$

where $e_{rs} = \frac{11}{2} - 2r$ if $r = s$ and $e_{rs} = (3 - r - s) + \frac{5}{2}$ if $r \neq s$.

Theorem 8.1 (Lemma P)**.** *Let $q \geq 2$ and let $\ell \geq 7$ be a prime with $\ell \nmid q(q^2-1)$ such that $\det M_S(q)$, $\det M_L(q)$ and $\det M_D(q)$ are ℓ-adic units. Then $\Delta_{3\ell}(a/b) \neq 0$ for all $a \in \mathbb{Z}$ and $b \geq 1$ with $\ell \nmid b$. More precisely, $|K_\ell\, \Delta_{3\ell}(a/b)|_\ell = 1$, where $K_\ell \in \mathbb{Q}^\times$ depends only on ℓ (not on q, a, b).*

Proposition 8.2 (the block determinants)**.** *With $Q = q^2$, $P_L(Q) = 16200Q^3 - 10575Q^2 - 6352Q + 905$ and $P_D(Q) = 16200Q^3 - 51075Q^2 + 28523Q + 420$,*

$$\det M_S(q) = \frac{8Q^3(9Q+1)}{9(Q-1)}, \qquad \det M_L(q) = \frac{8Q\, P_L(Q)}{2025(Q-1)},$$
$$\det M_D(q) = -\frac{Q^2 P_D(Q)}{135(Q-1)^2}$$

as rational functions of q. The numerators $9Q+1$, $P_L(Q)$ and $P_D(Q)$ do not vanish for real $Q \geq 4$, nor for real $Q \leq -1$.

Proof. The identities follow by expanding the three 3×3 determinants, whose entries are explicit rational functions of q ($w = 1/Q$, $T(1) = z$, $T(3) = z + z^3/9$, $T(5) = T(3) + z^5/25$, $T(-1) = 0$); the formalisation carries out this expansion exactly. For $Q \geq 4$: $9Q + 1 > 0$; $P_L(Q) \geq Q(16200Q^2 - 10575Q - 6352) + 905 > 0$ since $16200Q \geq 10575 + 6352$; and $P_D(Q) \geq Q^2(16200Q - 51075) + 28523Q + 420 > 0$. For $Q = -x$ with $x \geq 1$: $9Q + 1 < 0$; $P_L = -16200x^3 - 10575x^2 + 6352x + 905 < 0$ because $10575x^2 \geq 10575x > 6352x + 905$; and $P_D = -16200x^3 - 51075x^2 - 28523x + 420 < 0$. □

Corollary 8.3 (nonvanishing, case (A))**.** *For every integer $q \geq 2$ and all $a \in \mathbb{Z}$, $b \geq 1$, $\Delta_h^{\chi}(a/b; q) \neq 0$ for infinitely many h.*

Proof. By Proposition 8.2 the three determinants are nonzero rationals. Every prime $\ell \geq 7$ that divides none of their numerators and denominators,

nor $q(q^2-1)b$, satisfies the hypotheses of Theorem 8.1. There are infinitely many such primes, and each gives $h = 3\ell$. □

The rest of this section proves Theorem 8.1. Fix q, ℓ as there and $h = 3\ell$; then $\ell^2 \geq 7\ell > 2h$. Let $\mathbb{Z}_{(\ell)}$ be the ring of ℓ-integral rationals and $\bar{\cdot}\colon \mathbb{Z}_{(\ell)} \to \mathbb{F}_\ell$ the reduction. Put $\kappa := (\ell-1)/2$. The indices $i < h$ fall into the classes $C_\gamma = \{\gamma, \gamma+\ell, \gamma+2\ell\}$, $\gamma < \ell$; we write $i = \gamma + r\ell$ with $r \in \{0,1,2\}$. The class γ has *type* S if $\gamma < \kappa$, D if $\gamma = \kappa$, and L if $\gamma > \kappa$. Note that $\ell \mid 2i+1$ iff $i \in C_\kappa$; put $\delta_i := [\, i \in C_\kappa \,]$.

8.2. **Steps 1–2: the Lagrange Gram matrix.** For $r_0 \in \mathbb{Q}$ let $\Lambda_{r_0} := A + r_0 B$ and $G(r_0)_{ij} := \Lambda_{r_0}(f_i f_j)$, with the Lagrange basis $f_i = D/(u-x_i)$. By (2.4), $\det G(r_0) = (\det C_f)^2 \Delta_h(r_0)$. Put

$$\mathcal{E}(r_0)_{ij} := \nu_i\, G(r_0)_{ij}\, \ell^{-\delta_j}, \qquad \nu_i := \ell^2/D'(x_i).$$

Then $\det \mathcal{E}(r_0) \neq 0$ implies $\Delta_h(r_0) \neq 0$; no value of $\det C_f$ is needed. As in the proof of Lemma 4.2, for $i \neq j$ we have $G(r_0)_{ij} = \mu(F_{ij})$ with $F_{ij} = u\prod_{m\neq i,j}(u-x_m)$, and $G(r_0)_{ii} = \mu(Q_i) + 4x_i D'(x_i)\big(S_i(q) - r_0 q^{2i+1}\big)$. Hence

$$\mathcal{E}(r_0) = \mathcal{E}(0) + r_0\, \mathrm{diag}(d_i), \qquad d_i = -4\ell^{2-\delta_i} x_i q^{2i+1}, \qquad v_\ell(d_i) = 2.$$

If $\mathcal{E}(0)$ is ℓ-integral and $\ell \nmid b$, then $\mathcal{E}(a/b)$ is ℓ-integral with the same reduction as $\mathcal{E}(0)$. So it suffices to show that $\mathcal{E}(0)$ is ℓ-integral and $\det \overline{\mathcal{E}(0)} \neq 0$.

8.3. **Step 3: splitting by progressions.** For $|v| < 1$ and a polynomial g, $\sigma(v,g) := \sum_{j\geq 0} v^j g(j) = \sum_b \Delta^b g(0)\, v^b/(1-v)^{b+1}$. Since w, w^ℓ and $1-w^\ell \equiv 1-w = (q^2-1)/q^2$ are ℓ-units, we get the following. If g has ℓ-integral values on $\mathbb{Z}_{\geq 0}$, then $\sigma(w^\ell, g)$ is ℓ-integral. If moreover $\bar{g}(j) = g_0(j)$ for all j, for a polynomial g_0 over $\mathbb{F}_\ell$, then $\overline{\sigma(w^\ell, g)} = \sum_b \Delta^b g_0(0)\, \bar{w}^b/(1-\bar{w})^{b+1}$ (use $\bar{w}^\ell = \bar{w}$). Splitting $t \geq 1$ into the residue classes modulo ℓ,

$$\mu(F) = \sum_{\gamma<\ell} w^{t_\gamma}\, \sigma\big(w^\ell,\ j \mapsto F'(t_\gamma + j\ell)\big),$$

where the *progression of* γ is $\{t \geq 1 : \ell \mid 2t + 2\gamma + 1\}$ and t_γ is its least element: $t_\gamma = \kappa - \gamma$ (type S), $\kappa - \gamma + \ell$ (type L), and ℓ (type D). On the progression of γ, $\ell \mid t - x_m$ iff $m \in C_\gamma$, and $\ell \mid t$ iff $\gamma = \kappa$. Since $\ell^2 > 2h$, all these divisibilities are simple.

8.4. **Step 4: integrality and block triangularity.** We have $D'(x_i) = \prod_{m\neq i}(m-i) = \ell^2\, c_r^{-1}\, U_i$ with $U_i := \prod_{m\notin C_\gamma}(x_i - x_m)$ an ℓ-unit, so $\nu_i = c_r U_i^{-1}$ is a unit. Let γ_i, γ_j be the classes of i, j and consider the contribution of the progression of γ.

Off-diagonal entries. At t on the progression of γ, the factors of F_{ij} divisible by ℓ number $N_\gamma = 3 - [\,\gamma = \gamma_i\,] - [\,\gamma = \gamma_j\,] + [\,\gamma = \kappa\,]$, and every term of $F'_{ij}(t)$ keeps at least $N_\gamma - 1$ of them. So the contribution has valuation $\geq N_\gamma - 1 - \delta_j \geq 0$, with equality only if $\gamma = \gamma_i = \gamma_j$ (inside a diagonal block) or $\gamma = \gamma_i \neq \gamma_j = \kappa$ (a non-D row against a D column).

Diagonal entries. Off its own progression, $|t-x_i|_\ell = 1$ and $Q_i'(t) = \big(g_i'(t)(t-x_i) - g_i(t) + g_i(x_i)\big)/(t-x_i)^2$ with g_i having at least three roots ℓ-adically close to t, so $v_\ell(Q_i'(t)) \geq 2$ and the contribution has valuation $\geq 2-\delta_i \geq 1$. The pole term $4\ell^{2-\delta_i}x_iS_i(q)$ is integral, since $v_\ell(x_i)=\delta_i$ and the denominators s^2 of S_i have $v_\ell(s)\leq 1$. The own-progression contributions are integral by Step 5.

Hence $\mathcal{E}(0)$ is ℓ-integral. Ordering the classes with κ last, $\overline{\mathcal{E}(0)}$ is block upper triangular, with 3×3 diagonal blocks $\overline{\mathcal{E}(0)}_{C_\gamma}$, and each diagonal block receives only its own progression and the pole terms.

8.5. **Step 5: the diagonal blocks.** Fix a class γ and $i=\gamma+r\ell$, $j=\gamma+s\ell$. On the progression of γ we have $t\equiv x_i\equiv x_\gamma \pmod\ell$.

For $r\neq s$ let $c:=3-r-s$ be the rank of the third class member, and for $r=s$ let c_1,c_2 be the ranks of the other two members; we write $x_{(c)}:=x_{\gamma+c\ell}$. For a polynomial P let $Q[P]:=(P(u)-P(x_i))/(u-x_i)$, so that $Q[PR]=Q[P]\,R+P(x_i)\,Q[R]$ and $Q_i=Q[g_i]$.

Types S and L. If $r\neq s$, then $F_{ij}=(u-x_{(c)})\cdot u\prod_{m\notin C_\gamma}(u-x_m)$, and modulo ℓ, $F_{ij}'(t)\equiv t\prod_{m\notin C_\gamma}(t-x_m)\equiv x_iU_i$. If $r=s$, write $g_i=P\cdot R$ with $P=(u-x_{(c_1)})(u-x_{(c_2)})$ and $R=u\prod_{m\notin C_\gamma}(u-x_m)$. Then $Q_i'=Q[P]'R+Q[P]R'+P(x_i)Q[R]'$ with $Q[P]=u+x_i-x_{(c_1)}-x_{(c_2)}$. At t we have $Q[P](t)\equiv 0 \pmod\ell$, and $\ell^2\mid P(x_i)$, so $Q_i'(t)\equiv R(t)\equiv x_iU_i \pmod\ell$. In both cases $\nu_i(\cdot)\equiv c_r\bar{x}_\gamma$, a constant along the progression. Summing with Step 3, and using $\bar{w}^\kappa=\bar{q}^{-(\ell-1)}=1$ and $\bar{w}^\ell=\bar{w}$, the progression contributes $\bar{w}^{t_\gamma}c_r\bar{x}_\gamma/(1-\bar{w})$, where $\bar{w}^{t_\gamma}=\bar{w}^{-\gamma}$ (S) or $\bar{w}^{1-\gamma}$ (L). For the pole term only the odd $s=\ell s'$ survive in ℓ^2S_i, and $q^\ell\equiv q$, so $\overline{\ell^2S_i}=\bar{q}^{2i+1}\,T(M)$ with M the largest odd number $\leq(2i+1)/\ell$: $M=2r-1$ (S) or $2r+1$ (L). Since $\bar{q}^{2i+1}=\bar{q}\,\bar{w}^{-\gamma}\bar{w}^{-r}$, the diagonal block is

$$\overline{\mathcal{E}(0)}_{C_\gamma}=\bar{x}_\gamma\,\bar{w}^{-\gamma}\,\overline{M_{\mathrm{type}(\gamma)}(q)}.$$

Type D ($\gamma=\kappa$). Here $x_i=-\ell(r+\frac12)$, the progression is $t=\ell(1+j')$, $j'\geq 0$, and we put $y=t/\ell+\frac12=j'+\frac32$; then $t=\ell(y-\frac12)$ and $t-x_{(c)}=\ell(y+c)$. Let $R:=\prod_{m\notin C_\kappa}(u-x_m)$, so $R(t)\equiv R(x_i)=U_i \pmod\ell$. If $r\neq s$, two roots of $F_{ij}=u(u-x_{(c)})R$ are close to t, and $F_{ij}'(t)=(2t-x_{(c)})R(t)+t(t-x_{(c)})R'(t)\equiv \ell(2y+c-\frac12)\,U_i \pmod{\ell^2}$. If $r=s$, write $g_i=PR$ with $P=u(u-x_{(c_1)})(u-x_{(c_2)})$. Then $\ell^3\mid P(x_i)$; $Q[P](t)$ is a quadratic form in $t,x_i,x_{(c_1)},x_{(c_2)}$, all divisible by ℓ, so $Q[P](t)\equiv 0 \pmod{\ell^2}$; and $Q[P]'(t)=2t+x_i-x_{(c_1)}-x_{(c_2)}=\ell(2y+\frac52-2r)$ (using $c_1+c_2=3-r$). Hence $Q_i'(t)\equiv\ell(2y+\frac52-2r)\,U_i \pmod{\ell^2}$. With $\nu_i=c_rU_i^{-1}$ and the column factor ℓ^{-1}, the residues are $c_r(2y+c-\frac12)=c_r(2j'+e_{rs})$ for $r\neq s$ and $c_r(2y+\frac52-2r)=c_r(2j'+e_{rr})$ for $r=s$. Summing $c_r(2j'+e_{rs})$ against $\bar{w}^{1+j'}$ over $j'\geq 0$ gives $\bar{w}\,c_r\big(e_{rs}/(1-\bar{w})+2\bar{w}/(1-\bar{w})^2\big)$. The pole term $4\ell x_iS_i=-4(r+\frac12)\ell^2S_i$ reduces to $-4(r+\frac12)\bar{q}^{2r+1}T(2r+1)$. So $\overline{\mathcal{E}(0)}_{C_\kappa}=\overline{M_D(q)}$.

8.6. **Step 6: conclusion.** By Steps 4 and 5,

$$\det \overline{\mathcal{E}(0)} = \prod_{\gamma \neq \kappa} \left(\bar{x}_\gamma \bar{w}^{-\gamma}\right)^3 \overline{\det M_{\mathrm{type}(\gamma)}(q)} \cdot \overline{\det M_D(q)}.$$

For $\gamma \neq \kappa$, $\bar{x}_\gamma = -(2\gamma+1)/2 \neq 0$, and the three determinants are units by hypothesis. So $\det \overline{\mathcal{E}(a/b)} = \det \overline{\mathcal{E}(0)} \neq 0$ in $\mathbb{F}_\ell$, hence $\det \mathcal{E}(a/b) \neq 0$ and $\Delta_{3\ell}(a/b) \neq 0$. Moreover $\mathcal{E}(a/b)$ is ℓ-integral with a unit determinant, and $\det \mathcal{E}(a/b) = \prod_i \nu_i \cdot \ell^{-3} \cdot (\det C_f)^2 \, \Delta_{3\ell}(a/b)$, since exactly three columns ($j \in C_\kappa$) carry the factor ℓ^{-1}. The factor $K_\ell := \ell^{-3} (\det C_f)^2 \prod_i \nu_i$ depends only on ℓ. This proves Theorem 8.1. □

Remark 8.4. The same analysis works for every h with $h \bmod \ell \leq \kappa$ and three or more atoms per class. One then needs a fourth block type with four atoms, and an explicit prime-gap theorem to find suitable ℓ near $h/3$ for all large h (we have not written this out). For irrationality, infinitely many h suffice, and $h = 3\ell$ avoids both complications. With at most two atoms per class the analogous block constants of type S vanish identically in our computations, so primes $\ell > h/2$ did not give a usable nonvanishing statement.

9. Theorem A: $\chi_2(1/q)$

In this section we are in case (A): $w = q^{-2}$, $\theta = \chi_2(1/q)$, $\Delta_h = \Delta_h^\chi(X; q)$ with an integer $q \geq 3$. It remains to bound the prime 2. We write $L := \lfloor \log_2 h \rfloor$ and use $|\cdot|_2$ coefficientwise on $\mathbb{Q}[X]$.

9.1. **The prime 2 at odd q.**

Lemma 9.1 (Lemma 10)**.** *Let $q \geq 3$ be odd and $\nu := v_2(q^2-1) \geq 3$. For every $h \geq 1$ and $k \geq 0$,*

$$\left|[X^k]\Delta_h\right|_2 \leq 2^{(\nu-2)\beta_h + h(L+6)}.$$

For $q = 3, 5$ ($\nu = 3$) this is [12, Lemma 5.3]; the proof there uses $\nu = 3$ only in the bound for the entries of type (L) below. We work with the anchored basis, for which $\det(\Lambda_X(a_i b_j)) = \Delta_h(X)$.

Lemma 9.2. *Let $T\colon \mathbb{Z}_{\geq 0} \to \mathbb{Q}$ satisfy $T(l+1) = Q\,T(l) + e\,(s+2l)^{-2}$ with $v_2(e) \geq 0$, $v_2(T(0)) \geq 0$ and s odd, and put $\eta := Q - 1$, $g(l) := (s+2l)^{-2}$. Then $\Delta^m T(0) = \eta^m T(0) + e \sum_{k<m} \eta^{m-1-k} \Delta^k g(0)$ and $v_2(\Delta^k g(0)) \geq k + v_2(k!)$. Consequently:*

1. *if $v_2(\eta) \geq 2$, then $v_2(\Delta^m T(0)) \geq v_2(m!) + m - 2 - \lfloor \log_2 m \rfloor$ for $m \geq 1$;*
2. *if $v_2(\eta) \geq 1$, then $v_2(\Delta^m T(0)) \geq m - 1$ for $m \geq 1$.*

Proof. $\Delta T = \eta T + eg$, and the formula follows by iterating. By induction on k, $\Delta^k g(0) = (-2)^k k!\, H_k(s)/P_k(s)$ with $P_k(s) = \prod_{a \leq k}(s+2a)$ odd and $H_k(s) = \sum_{a \leq k}(s+2a)^{-1}$ 2-integral, so $v_2(\Delta^k g(0)) \geq k + v_2(k!)$. (i) The term $k = 0$ has valuation $\geq 2(m-1)$, which is enough. For $k \geq 1$, using

$k-1-\lfloor\log_2 k\rfloor \le v_2(k!)$ and $v_2(m!)\le m-1$, each term has valuation at least $2(m-1-k)+2k-1-\lfloor\log_2 k\rfloor \ge v_2(m!)+m-2-\lfloor\log_2 m\rfloor$, and $v_2(\eta^m T(0))\ge 2m$ is larger. (ii) Each term has valuation at least $(m-1-k)+k$, and $v_2(\eta^m T(0))\ge m$. □

Proof of Lemma 9.1. We show that each entry satisfies

$$v_2\big(\Lambda_X(a_ib_j)\big) \ge h-(i+j+1)-(\nu-2)(i+j+1-h)_+ - L - 6. \tag{9.1}$$

Entries of type (L), $\pi := i+j+1-h \ge 1$. Then $a_ib_j = D\cdot P$ with $P=\prod_{h-j\le m<i}(u-x_m)$, so $a_ib_j\varphi = uP$ is a polynomial: $B(a_ib_j)=0$ and $A(a_ib_j)=\mu(uP)$. We have $uP = 2^{-\pi}H(2u)$ with $H\in\mathbb{Z}[T]$ of degree π. Since $\Delta^b[(2t)^n](0) = 2^n b!\,S(n,b)$ (Stirling numbers of the second kind) vanishes for $n<b$,

$$v_2\big(\Delta^b[(uP)'](0)\big) \ge 1-\pi+b+v_2(b!). \tag{9.2}$$

By (2.2), $v_2(\Omega_b) = -\nu(b+1)$. By (2.1) and $v_2(b!)\ge b-1-\lfloor\log_2 b\rfloor$,

$$v_2(A(a_ib_j)) \ge \min_{b<\pi}\big(1-\pi-\nu-(\nu-1)b+v_2(b!)\big) \ge -(\nu-1)\pi-2-\lfloor\log_2\pi\rfloor,$$

which implies (9.1).

Entries of type (U), $N := h-1-i-j\ge 0$. Then $a_ib_j\varphi = u/\prod_{l=0}^N(u-\xi_l)$ with $\xi_l := x_{i+l} = -c/2-l$, $c:=2i+1$. Since $\xi_l-\xi_m = m-l$, its partial fractions are a constant (which μ kills) plus $\sum_l (-1)^l\binom{N}{l}\xi_l/(N!\,(u-\xi_l))$. By Lemma 2.2, both $A(a_ib_j)$ and $B(a_ib_j)$ have the form

$$\pm\frac{4}{N!}\sum_l(-1)^l\binom{N}{l}\xi_l T(l) = \pm\frac{4}{N!}\Delta^N[\xi_l T(l)](0), \qquad T(l)=s_{i+l}\text{ or }c_{i+l}. \tag{9.3}$$

Both satisfy the recursion of Lemma 9.2 with $Q=q^2$: $S_{i+l+1} = q^2S_{i+l} + (2i+3+2l)^{-2}$ ($e=1$, $s=2i+3$) and $q^{2(i+l+1)+1} = q^2\cdot q^{2(i+l)+1}$ ($e=0$); here $v_2(Q-1)=\nu\ge 2$. From $\Delta^N[l\,T(l)](0) = N\,\Delta^{N-1}[T(\cdot+1)](0)$, and since $T(\cdot+1)$ satisfies a recursion of the same kind, Lemma 9.2(i) gives

$$\begin{aligned} &v_2\Big(\frac{4}{N!}\Delta^N[\xi_l T(l)](0)\Big) \ge 2-v_2(N!)\\ &+\min\Big(-1+v_2(N!)+N-2-\lfloor\log_2 N\rfloor,\ v_2(N!)+N-3-\lfloor\log_2(N-1)\rfloor\Big)\end{aligned}$$

for $N\ge 2$, which is at least $N-1-\lfloor\log_2 N\rfloor$. For $N\le 1$ the entry is $\pm4\,\xi_0T(0)$ or $\pm4\big(\xi_0\Delta T(0)-T(1)\big)$, of valuation ≥ 1, since $v_2(\xi_0)=-1$ and $T(0),T(1),\Delta T(0)$ are 2-integral. This implies (9.1).

Assembly. Since $(i+j+1-h)_+\le \rho_i+\kappa_j$, (9.1) gives $|\Lambda_X(a_ib_j)|_2 \le \big(2^{i+1+(\nu-2)\rho_i+L+6-h}\big)\big(2^{j+(\nu-2)\kappa_j}\big)$, and Lemma 4.1 over $\mathbb{Q}[X]$ gives $|[X^k]\Delta_h|_2 \le 2^{\sum_i(i+1)+\sum_j j+(\nu-2)\beta_h+h(L+6)-h^2} = 2^{(\nu-2)\beta_h+h(L+6)}$. □

So for odd q the prime 2 costs $\frac{\nu-2}{4}\log 2$ per h^2, and with Theorem 5.3 the arithmetic constant is

$$\begin{aligned}\alpha(q) &= \frac{133}{60} + \frac{1}{4}\log \operatorname{odd}(q^2-1) - \frac{1}{3}\log q + \frac{\nu-2}{4}\log 2 \\ &= \frac{133}{60} - \frac{1}{2}\log 2 + \frac{1}{4}\log(q^2-1) - \frac{1}{3}\log q.\end{aligned} \tag{9.4}$$

9.2. **The prime 2 at even q: a Gauss bound.** For even q the lemma above has nothing to work with: $1-w$ is a 2-adic unit. The 2-adic size of q itself enters instead, through the q-adic gain of Lemma 5.2.

Lemma 9.3 (Lemma 10_0)**.** *Let $q \geq 4$ be even and $e := v_2(q)$. For every $h \geq 1$ and $k \geq 0$, $\big|[X^k]\Delta_h\big|_2 \leq 2^{\lfloor h^2/2\rfloor - e\,a_h}$.*

Proof. First, $\|\Delta_h^{\mathcal{R}}\|_2 \leq 2^{\lfloor h^2/2\rfloor}$. Indeed, in the anchored basis over $\mathcal{R}[X]$ (the Gram identity (4.2) with $\det C = 1$) every entry has $\|\cdot\|_2 \leq 2^{|i+j+1-h|}$. For type (L), $B^{\mathcal{R}} = 0$ and $A^{\mathcal{R}} = \mu^{\mathcal{R}}(uP)$; the coefficient of u^d in $uP = 2^{-\pi}H(2u)$ has $v_2 \geq d - \pi$, and $\mu^{\mathcal{R}}(u^d)$ is an integer combination of the Ω_b, which have Gauss norm 1. For type (U), the residue weights $\xi_l(-1)^l/(l!\,(N-l)!)$ have $v_2 \geq -1 - v_2(N!) \geq -N$, against the factor 4 and the pole factors $q^{2k+1}X - S_k(q)$ of Gauss norm ≤ 1 (S_k has odd denominators). Since $|i+j+1-h| \leq |i-\frac{h-1}{2}| + |j-\frac{h-1}{2}|$, Lemma 4.1 gives $\|\Delta_h^{\mathcal{R}}\|_2 \leq 2^{2\sum_i |i-(h-1)/2|}$, and $2\sum_{i<h}|i-\frac{h-1}{2}| = \lfloor h^2/2\rfloor$.

Let $c = [X^k]\Delta_h = \mathrm{ev}_q(f)$. Since q^2-1 is odd, (5.1) gives $|c|_2 = |\operatorname{ev}(\mathrm{cl}_{\beta_h} f)|_2$. Every monomial of $\mathrm{cl}_{\beta_h} f$ has degree $\geq a_h$ (Lemma 5.2), and $|q|_2 = 2^{-e}$, so $|c|_2 \leq \|\mathrm{cl}_{\beta_h} f\|_2\, 2^{-e a_h} \leq \|f\|_2\, 2^{-e a_h}$. □

So Lemma 9.3 costs $(\frac{1}{2} - \frac{e}{3})\log 2$ per h^2; it includes the 2-adic share of the q-adic gain.

9.3. **The prime 2 at even q: the weighted Gauss valuation.** Lemma 9.3 loses twice: the Leibniz expansion in the anchored basis does not see the cancellation between its terms, and the Gauss norm forgets that q is 2-adically small. Both losses disappear if we bound the 2-adic size of the *reflected* slot expansion of Lemma 5.2 directly, giving q the weight e.

For $e \geq 1$ define v_e on $\mathbb{Q}[[q]][X]$ by $v_e\big(\sum c_{n,j} q^n X^j\big) := \inf_{n,j}\big(v_2(c_{n,j}) + en\big)$. It is ultrametric and satisfies $v_e(FG) \geq v_e(F) + v_e(G)$.

Lemma 9.4 (Lemma 10_e)**.** *Let $q \geq 4$ be even and $e := v_2(q)$. Put $m_e := e - \frac{2(e+1)^2}{3e+8}$. Then for every $h \geq 2$ and $k \geq 0$,*

$$-\log_2\big|[X^k]\Delta_h\big|_2 \geq v_e\big(\psi_q(\Delta_h^{\mathcal{R}})\big) \geq m_e h^2 - O(h\log h),$$

with an implied constant depending only on e. So the prime 2 costs $-m_e\log 2 = (c(e) - \frac{e}{3})\log 2$ per h^2, where $c(e) = \frac{6-4e}{3(3e+8)}$: $c(1) = \frac{2}{33}$, $c(2) = -\frac{1}{21}$, $c(3) = -\frac{2}{17}$.

Proof. Step 1: reduction. Let $f = [X^k]\Delta_h^{\mathcal{R}}$ and $c = \mathrm{ev}_q(f)$. As q^2-1 is odd, $v_2(c) = v_2(\mathrm{ev}(\mathrm{cl}_{\beta_h} f))$, and for a polynomial $N = \sum_n c_n q^n$ we have $v_2(N(q)) \geq \min_n(v_2(c_n) + en) = v_e(N)$. In $\mathbb{Q}[[q]]$, $\mathrm{cl}_{\beta_h} f = (q^2-1)^{\beta_h}\psi_q(f)$ and $v_e(q^2-1) = 0$, so $v_2(c) \geq v_e(\psi_q(f)) \geq v_e(\psi_q(\Delta_h^{\mathcal{R}}))$.

Step 2: the slot expansion. By the proof of Lemma 5.2, for every M, the coefficients of $q^n X^j$ with $n < 2M$ in $\psi_q(\Delta_h^{\mathcal{R}})$ and in $\det H_M$ agree, where H_M is the sum of the rank-one matrices

$$-q^{2m}\varphi'(-m)\, v_m v_m^{\mathsf{T}}, \quad -q^{2m}\varphi(-m)\, v'_m v_m^{\mathsf{T}}, \quad -q^{2m}\varphi(-m)\, v_m v_m'^{\mathsf{T}} \qquad (m < M),$$
$$-4q^{2k+1}\omega_k(X + \chi_2(q))\, p_k p_k^{\mathsf{T}} \qquad (k < h),$$

with value rows $v_m = ((-m)^i)_i$, derivative rows $v'_m = (i(-m)^{i-1})_i$ and pole rows $p_k = (x_k^i)_i$. So it suffices to bound $v_e(\det H_M)$ from below uniformly in M, and by (5.4) and $v_2(h!) \leq h$ it suffices to bound each Andreief term, $\prod_i c_{f(i)} \det X_f \det Y_f$, with pairwise distinct row keys in X_f and in Y_f.

Step 3: the factors. $D(-m) = \prod_k (k + \frac{1}{2} - m) = 2^{-h} \cdot (\text{odd})$, and $D'(-m)$ is a sum of products of $h-1$ such factors. Hence $v_2(\varphi(-m)) \geq h$ and $v_2(\varphi'(-m)) \geq h$. Next $v_2(4\omega_k) = 1 - v_2(k!) - v_2((h-1-k)!) \geq -h$, and $v_e(X + \chi_2(q)) \geq 0$ because $\chi_2(q) = \sum q^{2j+1}/(2j+1)^2$ has 2-integral coefficients. So a node slot at m contributes $v_e \geq h + 2em$ and a pole slot at k contributes $v_e \geq -h + e(2k+1)$.

Step 4: the minors. Each Andreief term attaches to every $i < h$ one point z_i, a node $-m$ or a pole x_k, and X_f, Y_f are both determinants with rows $(z_i^j)_j$ (value rows) or $(jz_i^{j-1})_j$ (derivative rows) at these same points. Distinct keys allow at most two rows at a node (one value, one derivative) and one at a pole, and each rank-one term has at most one derivative row among its two, so X_f and Y_f together have at most I derivative rows, where I is the number of node slots. We use the following bound. *Let R have value rows at the points z_i, $i \notin U$, and derivative rows at z_i, $i \in U$, and suppose $|z_i - z_{i'}|_2 \geq 2^{-\Lambda}$ whenever $i \in U$ and $z_{i'} \neq z_i$. Then $|\det R|_2 \leq \prod_{i<i',\, z_i \neq z_{i'}} |z_i - z_{i'}|_2 \cdot 2^{\Lambda|U|}$.* Indeed $\det R = (\partial_U V)(z)$ with $V = \prod_{i<i'}(X_{i'} - X_i) \in \mathbb{Z}[X_0, \ldots, X_{h-1}]$ and $\partial_U = \prod_{i \in U} \partial/\partial X_i$, since only row i involves X_i. By the product rule $\partial_U V$ is an integer combination of the products of all linear factors $X_{i'} - X_i$ except those in $c(U)$, for injective maps c from U to the pairs with $i \in c(i)$. At z, such a product vanishes unless every pair with equal points lies in $c(U)$, and then it is at most the product over pairs with distinct points times $2^{\Lambda|U|}$ (one removed factor per $i \in U$, each of size $\geq 2^{-\Lambda}$). The ultrametric inequality gives the claim. We apply it with $\Lambda := \lfloor \log_2 M_f \rfloor$, where M_f is the largest node of the term (if there is no node slot, or only node 0, there is no derivative row with another node to compare with, and we take $\Lambda = 0$, $M_f = 1$): node differences are nonzero integers of absolute value $\leq M_f$, and node–pole differences $k + \frac{1}{2} - m$ have $|\cdot|_2 = 2$.

Step 5: counting. Let the term have I node slots and $P = h - I$ pole slots. Then:

- prices: each node carries at most two slots, so $\sum 2m \ge 2\sum_{j<I}\lfloor j/2\rfloor + (2M_f-h)_+ \ge \frac{I(I-1)}{2} - I + (2M_f-h)_+$; the poles are distinct, so $\sum(2k+1) \ge P^2$;
- differences: for $L' \ge 1$ let $\mathrm{cnt}(a) = \#\{l < L' : 2^{l+1} \mid a\} \le v_2(a)$. Over ordered pairs of node slots at distinct nodes, $\sum \mathrm{cnt}(m-m') \ge \sum_{l<L'}\big(I^2/2^{l+1} - 2I\big) = I^2(1-2^{-L'}) - 2L'I$, by Cauchy–Schwarz over the residues modulo 2^{l+1} (at most $2I$ ordered pairs share a node). Likewise the pole pairs give at least $P^2(1-2^{-L'}) - L'P$, and the $2IP$ ordered node–pole pairs give exactly $-2IP$;
- derivatives: at most $I \le h$ rows lose Λ each, and $h\lfloor \log_2 M\rfloor \le e(2M-h)_+ + h(2\lfloor \log_2 h\rfloor + 1)$ for all $M \ge 1$ (if $M \le h^2$ use $\lfloor \log_2 M\rfloor \le 2\lfloor \log_2 h\rfloor + 1$; if $M > h^2$ then $h\log_2 M \le 2M - h$).

Adding Steps 3–5 with $L' = \lceil \log_2 h\rceil$, the term has

$$v_e \ge hI - hP + e\Big(\frac{I^2}{2} + P^2\Big) + I^2 + P^2 - 2IP - O(h\log h) = h^2 f_e(I/h) - O(h\log h),$$

where $f_e(x) = e(\frac{x^2}{2} + (1-x)^2) + (2x-1) + (2x-1)^2 = (\frac{3e}{2}+4)x^2 - (2e+2)x + e$. Its minimum on $[0,1]$ is at $x^* = \frac{2e+2}{3e+8}$ and equals m_e. This bound is uniform in M, which proves the lemma; and $-m_e = \frac{2(e+1)^2}{3e+8} - e = c(e) - \frac{e}{3}$. □

Remark 9.5. The minimising slot sets are intervals: nodes $0, 1, \dots$ and poles $0, \dots, P-1$, with about a third of the slots at nodes when $e = 1$. A similar count with $e = 0$ (the plain Gauss valuation, applied to the polynomial $\mathrm{cl}_{\beta_h} f$, whose q-degree is $O(h^2)$, so that only nodes $M = O(h^2)$ and losses $\log M = O(\log h)$ occur) gives the cost $\frac{1}{4}\log 2$ in place of the $\frac{1}{2}\log 2$ of Lemma 9.3; we do not use this. Exact computations up to $h = 256$ suggest that $c(e)$ is the true asymptotic constant, approached slowly from above; we do not need this.

9.4. **Proof of Theorem A.** Let $q \ge 3$, $q \ne 4$. Hypothesis (iii) of Theorem 3.2 is Corollary 8.3. For (i) we use Theorem 5.3 with the 2-adic bound of Lemma 9.1, 9.3 or 9.4; for (ii), Theorem 6.2 with a certificate of Theorem 7.2, shifted by Lemma 7.1 where needed, with $\delta = 10^{-3}$. Write $\log(q^2-1) < 2\log q$ throughout.

- *Odd* $q \ge 3$, base $W = 1/9$: by (9.4),

$$\begin{aligned}\alpha + \beta &< \frac{133}{60} - \frac{1}{2}\log 2 + \frac{1}{2}\log q - \frac{1}{3}\log q - 2.078 - \frac{1}{2}\log\frac{q}{3} + \delta \\ &\le \frac{133}{60} - 2.078 - \frac{1}{2}\log 2 + \frac{1}{6}\log 3 + \delta < -0.023.\end{aligned}$$

- *Even* $q \ge 14$, Lemma 9.3, base $W = 1/196$: here $\alpha = \frac{133}{60} + \frac{1}{2}\log 2 + \frac{1}{4}\log(q^2-1) - \frac{1}{3}\log q$, so

$$\alpha+\beta < \frac{133}{60} + \frac{1}{2}\log 2 - 3.03 + \frac{1}{6}\log 14 + \delta < \frac{133}{60} - 3.03 + \frac{7}{6}\log 2 + \delta < -0.003.$$

- $q = 6, 8, 10, 12$, Lemma 9.4, direct certificates: $\alpha_e(q) + \Phi$ with $\alpha_e(q) = \frac{133}{60} + \frac{1}{4}\log(q^2-1) - \frac{1}{3}\log q + c(e)\log 2$ is at most -0.0097, -0.2422, -0.2101, -0.3539 (with $e = 1, 3, 1, 2$ and $\Phi \le -2.56, -2.72, -2.85, -2.95$).

The numerical values follow from $0.6931471803 < \log 2 < 0.6931471808$ and elementary bounds such as $\log 3 < \frac{27}{17}\log 2$ ($3^{17} < 2^{27}$); the formalisation proves each inequality this way. Theorem 3.2 gives Theorem A. □

Remark 9.6. For even $q \ge 14$ Lemma 9.4 would give larger margins; we use Lemma 9.3 there because that is how the formal proof is organised. The case $q = 6$ is the thinnest: with $c(1) = \frac{2}{33}$ it needs $\Phi < -2.5502$, and the plain bound $c = \frac{1}{4}$ would not suffice. For $q = 4$ see Section 12.

10. Theorem B: $\mathrm{Ti}_2(1/q)$

In this section we are in case (B): $w = -q^{-2}$, $\theta = \mathrm{Ti}_2(1/q)$, $\Delta_h = \Delta_h^{\mathrm{Ti}}(X; q)$ with an integer $q \ge 3$. By Proposition 4.6, Δ_h^{Ti} is the image of the twisted symbolic determinant, so Theorem 5.3 already bounds the odd primes, with $Y = q^2 + 1$. In terms of the formal identity $\Delta_h^{\mathrm{Ti}}(X; q) = \Delta_h^{\chi}(-iX; iq)$, $\mathrm{Ti}_2(1/q)$ is χ_2 at the non-real point i/q; Theorem A says nothing about it.

10.1. The prime 2 at odd q.

Lemma 10.1 (Lemma 10_1)**.** *Let $q \ge 3$ be odd. For every $h \ge 1$ and $k \ge 0$, $\left|[X^k]\Delta_h^{\mathrm{Ti}}\right|_2 \le 2^{\beta_h}$, $\beta_h = \lfloor h^2/4 \rfloor$.*

Proof. We follow the proof of Lemma 9.1. Here $q^2 + 1 \equiv 2 \pmod 4$, so by (2.2) with $y = iq$, $v_2(\Omega_b) = -(b+1)$: the analogue of ν is 1. For entries of type (L), (9.2) gives $v_2(A(a_i b_j)) \ge \min_{b<\pi}(1 - \pi + b + v_2(b!) - b - 1) \ge -\pi$, and $B(a_i b_j) = 0$. For entries of type (U), (9.3) holds with $T(l) = S_{i+l}^{\mathrm{Ti}}(q)$ or $(-1)^{i+l} q^{2(i+l)+1}$, which satisfy the recursion of Lemma 9.2 with $Q = -q^2$, $\eta = -(q^2+1)$, $v_2(\eta) = 1$, and 2-integral initial values. By Lemma 9.2(ii), for $N \ge 2$, $v_2(\Delta^N[\xi_l T(l)](0)) \ge \min(-1 + N - 1,\ N - 2) = N - 2$, so the entry has $v_2 \ge 2 - v_2(N!) + N - 2 = s_2(N) \ge 1$; for $N \le 1$ the entry is $\pm 4\xi_0 T(0)$ or $\pm 4(\xi_0 \Delta T(0) - T(1))$, of valuation ≥ 1. So $|\Lambda_X(a_i b_j)|_2 \le 2^{(i+j+1-h)_+} \le 2^{\rho_i} 2^{\kappa_j}$, and Lemma 4.1 gives 2^{β_h}. □

So for odd q the prime 2 costs $\frac{1}{4}\log 2$, and the arithmetic constant is

(10.1)
$$\alpha^{\mathrm{Ti}}(q) = \frac{133}{60} + \frac{1}{4}\log\frac{q^2+1}{2} - \frac{1}{3}\log q + \frac{1}{4}\log 2 = \frac{133}{60} + \frac{1}{4}\log(q^2+1) - \frac{1}{3}\log q.$$

Compared with (9.4), $\mathrm{Ti}_2(1/q)$ pays $\frac{1}{2}\log 2 + \frac{1}{4}\log\frac{q^2+1}{q^2-1}$ more than $\chi_2(1/q)$ at odd q.

10.2. **The prime 2 at odd q: the binomial basis.** With Lemma 10.1 the margin at $q = 5$ is positive ($+0.055$ with the certificate $\Phi \le -2.44$ of Theorem 7.2, $+0.043$ with the ball-arithmetic certificate of the remark after the proof of Theorem B), and no bound on the entries of the anchored basis can do better: their tropical maximum is $\approx h^2/4$, and the true 2-adic size of Δ_h^{Ti} comes from cancellation. Exact computations give $-\min_k v_2([X^k]\Delta_h^{\mathrm{Ti}}) \approx h^2/8$. The next lemma proves this leading constant by changing to the binomial basis, where the cancellation becomes a factorisation.

Lemma 10.2 (Lemma M)**.** *Let $q \ge 3$ be odd. Then $n_2(h) := -\min_k v_2\big([X^k]\Delta_h^{\mathrm{Ti}}\big)$ satisfies $n_2(h) \le \frac{h^2}{8} + O(h\log h)$. So the prime 2 costs at most $\frac{1}{8}\log 2$ per h^2.*

Notation. $C_s(u) = u(u-1)\cdots(u-s+1)/s!$ is the binomial polynomial: it has degree s and leading coefficient $1/s!$, it is integer-valued on $\mathbb{Z}_{\ge 0}$, and $C_{s+1}(u+1) = C_{s+1}(u) + C_s(u)$ (Pascal). For $F\colon \mathbb{Z}_{\ge 0} \to \mathbb{Q}$, $\Delta^b F(0)$ is the iterated forward difference. Put $B_0 := 16h + 16$,

$$\mathbb{M}(F) := \sum_{b<B_0} \Omega_b\, \Delta^b F(0), \qquad f_k(t) := -\frac{1}{(t-x_k)^2},$$

$$\tilde\theta := \frac{4 - \mathbb{M}(f_0)}{4q} \in \mathbb{Q}.$$

Here $\tilde\theta$ is an auxiliary rational number, not the target $\mathrm{Ti}_2(1/q)$. By (2.1), $\mu(R) = \mathbb{M}(R')$ for every polynomial R of degree $\le B_0$. (The written proof in our working notes uses the 2-adic limit θ_2 of $\tilde\theta$ and the untruncated sum; the truncation keeps everything rational.)

Proof. Step 1: the binomial Gram matrix. By (2.4) with $a_i = b_i = C_i$, $\Delta_h^{\mathrm{Ti}} = (\prod_{i<h} i!)^2 \det(\Lambda_X(C_iC_j))_{i,j<h}$. For $k < h$, Newton's formula gives $C_i(x_k) = \sum_{n<h} \binom{k}{n}\mathcal{A}_{ni}$ with $\mathcal{A}_{ni} := \Delta_k^n[C_i(x_k)](0)$. Since $B(g) = \sum_k \beta_k g(x_k)$ with $\beta_k = -4\omega_k(-1)^k q^{2k+1}$, we get $B(C_iC_j) = (\mathcal{A}^{\mathsf T}\mathcal{O}\mathcal{A})_{ij}$ with $\mathcal{O}_{nn'} := \sum_{k<h} \beta_k \binom{k}{n}\binom{k}{n'}$. As $\Lambda_X = (A + \tilde\theta B) + (X - \tilde\theta)B$,

$$\Lambda_X(C_iC_j) = E_{ij} + (X-\tilde\theta)(\mathcal{A}^{\mathsf T}\mathcal{O}\mathcal{A})_{ij}, \qquad E_{ij} := A(C_iC_j) + \tilde\theta B(C_iC_j).$$

By the Schur complement, $\det(E + (X-\tilde\theta)\mathcal{A}^{\mathsf T}\mathcal{O}\mathcal{A}) = \det\mathcal{N}$ for the $3h \times 3h$ matrix $\mathcal{N} = \begin{pmatrix} E & \mathcal{A}^{\mathsf T} & 0 \\ 0 & 1 & -(X-\tilde\theta)\mathcal{O} \\ -\mathcal{A} & 0 & 1 \end{pmatrix}$ over $\mathbb{Q}[X]$.

Step 2: the entries of $\mathcal{A}$: $|\mathcal{A}_{ni}|_2 \le 2^{2(i-n)}$, and $\mathcal{A}_{ni} = 0$ for $n > i$. Since $x_{k+1} = x_k - 1$, Pascal gives $C_i(x_{k+1}) - C_i(x_k) = -C_{i-1}(x_{k+1})$, so $\mathcal{A}_{ni} = (-1)^n C_{i-n}(x_n)$ for $n \le i$ and $\mathcal{A}_{ni} = 0$ for $n > i$. And $C_s(x_k) = \prod_{l<s}\big(-\frac{2k+2l+1}{2}\big)/s!$ has $|\cdot|_2 \le 2^{2s}$.

Step 3: the entries of $\mathcal{O}$: $|\mathcal{O}_{nn'}|_2 \le 2^{n+n'-1}$. With $K = h-1$ and $Q = q^2$, $D'(x_k) = (-1)^k k!(K-k)!$ gives $\beta_k = \frac{2q}{K!}\binom{K}{k}(2k+1)Q^k$. From

$\sum_k \binom{K}{k}\binom{k}{s}Q^k = \binom{K}{s}Q^s(1+Q)^{K-s}$ and $k\binom{k}{s} = s\binom{k}{s}+(s+1)\binom{k}{s+1}$, the number $T_s := \sum_k \beta_k \binom{k}{s}$ is $\frac{2q}{K!}$ times an integer combination of $\binom{K}{s}Q^s(1+Q)^{K-s}$ and $2\binom{K}{s+1}Q^{s+1}(1+Q)^{K-s-1}$ (with coefficients $2s+1$ and $s+1$); as $v_2(1+Q)=1$ and $v_2(K!) \le K$, $v_2(T_s) \ge 1-s$. Finally $\binom{k}{n}\binom{k}{n'} = \sum_{s\le n+n'} c_s \binom{k}{s}$ with $c_s \in \mathbb{Z}$ (the forward differences of an integer-valued polynomial of degree $n+n'$), so $v_2(\mathcal{O}_{nn'}) \ge 1-n-n'$.

Step 4: the rational $\tilde\theta$. As $f_{k+1}(t) = f_k(t+1)$, $\Delta^b f_{k+1}(0) = \Delta^b f_k(0) + \Delta^{b+1} f_k(0)$, and $\Omega_{b-1} = (w^{-1}-1)\Omega_b$ $(b \ge 2)$, $\Omega_0 = (w^{-1}-1)\Omega_1 - 1$ give

$$\mathbb{M}(f_{k+1}) = -q^2\,\mathbb{M}(f_k) + \frac{4}{(2k+3)^2} + (-q^2-1)\,\Omega_{B_0}\,\Delta^{B_0} f_k(0).$$

The pole values $D_k := 4S_k^{\mathrm{Ti}}(q) - 4\tilde\theta(-1)^k q^{2k+1}$ satisfy $D_{k+1} = -q^2 D_k + 4/(2k+3)^2$, and $D_0 = \mathbb{M}(f_0)$ by the choice of $\tilde\theta$. Here $v_2(-q^2-1)=1$, $v_2(\Omega_{B_0}) = -(B_0+1)$, and $v_2(\Delta^{B_0} f_k(0)) \ge B_0 + 2 + v_2(B_0!)$ (the proof of Lemma 9.2, as $f_k(t) = -4(2k+1+2t)^{-2}$), with $v_2(B_0!) \ge B_0/2$. By induction on k, $|D_k - \mathbb{M}(f_k)|_2 \le 2^{-(8h+8)}$ for all k. Also $v_2(\mathbb{M}(f_0)) \ge 1$, so $|\tilde\theta|_2 \le 2$.

Step 5: the entries of E: $|E_{ij}|_2 \le 2^{i+j+2-h}$. Let $g = C_i C_j$ and $R_g = (gu)\operatorname{div} D$. By definition $E_{ij} = \mu(R_g) + \sum_k g(x_k)\omega_k D_k$. Split $D_k = \mathbb{M}(f_k) + e_k$. The error terms have $|g(x_k)\omega_k e_k|_2 \le 2^{2i+2j}\cdot 2^h \cdot 2^{-8h-8} \le 2^{i+j+2-h}$ (Step 2, $v_2(\omega_k) \ge -h$, Step 4). The rest is $\mu(R_g) + \sum_k g(x_k)\omega_k \mathbb{M}(f_k) = \mathbb{M}(F)$, where $F(t) = (gu/D)'(t)$ by partial fractions; here $(F_1/F_2)'(t)$ means $(F_1'(t)F_2(t) - F_1(t)F_2'(t))/F_2(t)^2$, so no analysis is needed. Since $|\Omega_b|_2 = 2^{b+1}$, it suffices to show $|\Delta^b F(0)|_2 \le 2^{i+j+1-h-b}$ for $b < B_0$. Write $D(u) = 2^{-h}P(2u)$ with $P(T) = \prod_{k<h}(T+2k+1)$, $P(0)$ odd. With $U := 1 - P/P(0) = TV$ and $Y := P(0)^{-1}\sum_{m<M} U^m$ one has $PY = 1 - T^M V^M$, with 2-integral coefficients, so

$$\frac{g(u)\,u}{D(u)} = 2^h\, g(u)\,u\,Y(2u) + 2^{h+M}\,\frac{g(u)\,u^{M+1}V(2u)^M}{P(2u)}.$$

The polynomial part is $\sum_m 2^{h+m}\sigma_m u^{m+1} g$ with σ_m 2-integral. For an integer-valued polynomial G of degree $\le d$, $|\Delta^b G'(0)|_2 \le 2^{d-b}$: write $G = \sum_s c_s C_s$ with $c_s \in \mathbb{Z}$, and use $\Delta^b C_s'(0) = C_{s-b}'(0) = (-1)^{s-b-1}/(s-b)$ for $b < s$ (Pascal, differentiated), and 0 for $b \ge s$. With $d = m+1+i+j$ this gives $2^{i+j+1-h-b}$. For the tail take $M = B_0 + 2h + 1$: at a node t, $P(2t)$ is odd; the numerator $g(u)\,u^{M+1}V(2u)^M$ is an integer there, and its derivative has $|\cdot|_2 \le 2^{2h}$: the factor $u^{M+1}V(2u)^M$ has integer coefficients, $g(t) \in \mathbb{Z}$, and $|g'(t)|_2 \le 2^{i+j} \le 2^{2h}$ by the derivative bound just proved (with $b=0$, applied to $G(x) = g(t+x)$). So the tail contributes at most $2^{2h-h-M} \le 2^{i+j+1-h-b}$.

Step 6: one Leibniz bound with potentials. Put $\alpha = -\lfloor h/2\rfloor$, $\gamma = 2 - h + \lfloor h/2\rfloor$ and $\beta = \alpha - \lfloor h/4\rfloor$, and give the rows of $\mathcal{N}$ the exponents $\max(i+\alpha, 2i+\beta)$ (E-rows, $i<h$), $\max(2n+\beta, n)$ (middle rows) and $-2n-\beta$ (bottom rows), and the columns the exponents $\max(j+\gamma, 2j+\beta)$ (E-columns, $j<h$),

$-2n-\beta$ (middle columns) and $\max(2n+\beta, n)$ (right columns). By Steps 2–5, and since the coefficients of $-(X-\tilde{\theta})\mathcal{O}_{nn'}$ are $-\mathcal{O}_{nn'}$ and $\tilde{\theta}\mathcal{O}_{nn'}$, every entry of $\mathcal{N}$ has coefficients of 2-adic size at most $2^{\text{row+column}}$. Lemma 4.1 bounds the coefficients of $\det\mathcal{N}$ by 2^{S_h}, where S_h is the sum of all the exponents. Writing $\max(x,y) = y + (x-y)_+$ and summing, $2S_h \le 4h(h-1) - 4hr + p(p+1) + p'(p'+1) + 2r(r+1)$ with $r = -\beta$, $p = \lfloor h/4 \rfloor$, $p' = \gamma - \beta$, and this is at most $\frac{9}{4}h^2 + 13$. With $v_2(i!) \ge i - 1 - \log_2 i$, $n_2(h) \le S_h - 2\sum_{i<h} v_2(i!) \le \frac{h^2}{8} + O(h\log h)$. □

Remark 10.3. The crossover of the potentials at $h/4$ is the extremal configuration of the proof in our working notes: a quarter of the slots at node levels, three quarters at pole levels. With the sharper bound $|E_{ij}|_2 \le 2^{i+j+1-h}$ of those notes, the tropical value of the Leibniz bound for $\mathcal{N}$ equals their subset bound exactly (we checked this for $h \le 48$); the version above is weaker by $O(h)$. Exact computations at $q = 3, 5$ give $n_2(h) = h^2/8 + O(h)$ for $h \le 80$, so the constant $\frac{1}{8}$ appears to be sharp.

10.3. **The prime 2 at even q.**

Corollary 10.4 (Lemma 10_e for Ti_2)**.** *Let $q \ge 4$ be even and $e = v_2(q)$. Then $-\log_2 |[X^k]\Delta_h^{\mathrm{Ti}}|_2 \ge m_e h^2 - O(h\log h)$, with m_e as in Lemma 9.4.*

Proof. Let $c = [X^k]\Delta_h^{\mathrm{Ti}} = \mathrm{ev}_q^{\mathrm{Ti}}(f^{\mathrm{Ti}})$. Since q^2+1 is odd, $v_2(c) = v_2(\mathrm{ev}(\mathrm{cl}^-_{\beta_h} f^{\mathrm{Ti}})) \ge v_e(\mathrm{cl}^-_{\beta_h} f^{\mathrm{Ti}})$. By the remark after (5.1), $\mathrm{cl}^-_{\beta_h} f^{\mathrm{Ti}}$ and $\mathrm{cl}_{\beta_h} f$ have the same coefficients up to sign, so $v_e(\mathrm{cl}^-_{\beta_h} f^{\mathrm{Ti}}) = v_e(\mathrm{cl}_{\beta_h} f) \ge v_e(\psi_q(\Delta_h^{\mathcal{R}}))$, which Lemma 9.4 bounds. □

10.4. **Nonvanishing.** Let $M_S^{\mathrm{Ti}}(q)$, $M_L^{\mathrm{Ti}}(q)$, $M_D^{\mathrm{Ti}}(q)$ be the matrices of Section 8 with $w = -q^{-2}$ and with $q^{2r+1}T(M)$ replaced by $(-1)^r q^{2r+1} T^\natural(M)$, where $T^\natural(M) = \sum_{s \le M,\, s \text{ odd}} \chi_4(s) q^{-s}/s^2$, so that $(-1)^k q^{2k+1} T^\natural(2k+1) = S_k^{\mathrm{Ti}}(q)$. Their determinants are the right-hand sides of Proposition 8.2 at $Q = -q^2$; for instance $\det M_S^{\mathrm{Ti}}(q) = -8q^6(9q^2-1)/(9(q^2+1))$. By Proposition 8.2 (the case $Q \le -1$) they are nonzero rationals.

Proposition 10.5 (nonvanishing, case (B))**.** *Theorem 8.1 holds for $\Delta_{3\ell}^{\mathrm{Ti}}$ with $\ell \nmid q(q^2+1)$ and M_S^{Ti}, M_L^{Ti}, M_D^{Ti} in place of M_S, M_L, M_D. Hence for every integer $q \ge 3$ and all $a \in \mathbb{Z}$, $b \ge 1$, $\Delta_h^{\mathrm{Ti}}(a/b) \ne 0$ for infinitely many h.*

Proof. The proof of Theorem 8.1 uses w only through the facts that w, w^ℓ and $1-w$ are ℓ-adic units and $\bar{w}^\ell = \bar{w}$, which hold for $w = -q^{-2}$ when $\ell \nmid q(q^2+1)$, and through two congruences in Step 5, which change by signs. First, $\bar{w}^\kappa = (-1)^\kappa \bar{q}^{-(\ell-1)} = \chi_4(\ell)$ instead of 1. Second, in $\ell^2 S_i^{\mathrm{Ti}}$ only the terms with $\ell \mid s$ survive, and $\chi_4(\ell s') = \chi_4(\ell)\chi_4(s')$. In the blocks of types S and L both changes occur together, and the block becomes the unit $(-1)^{\kappa+\gamma}$ times $\bar{x}_\gamma \bar{q}^{2\gamma}\, \overline{M_{\mathrm{type}}^{\mathrm{Ti}}(q)}$; in the block of type D the two signs cancel (in the pole term $(-1)^i = (-1)^\kappa(-1)^r$), and the block is $\overline{M_D^{\mathrm{Ti}}(q)}$. So $\det\overline{\mathcal{E}(0)}$ is a unit

times the product of the three determinants, and Step 6 concludes as before. The last statement follows as in Corollary 8.3. □

10.5. **Proof of Theorem B.** Let $q \geq 5$. Theorem 6.2 applies with $W = |w| = q^{-2}$: the certificates see only $|w|$. For $q = 5$ we use Lemma 10.2 and the certificate at $W = 1/25$ of Theorem 7.2 ($\Phi \leq -2.44$):

$$\frac{133}{60} + \frac{1}{4}\log 13 - \frac{1}{3}\log 5 + \frac{1}{8}\log 2 - 2.44 = -0.0319\ldots.$$

For $q \geq 6$ we use the finer certificate at $W = 1/36$ of Theorem 7.2 ($\Phi \leq -2.5671$) directly at $q = 6$ and shifted by Lemma 7.1 for $q \geq 7$, $\beta = -2.5671 - \frac{1}{2}\log\frac{q}{6} + \delta$. With $\log(q^2+1) \leq 2\log q + q^{-2}$:

- $q = 6$ ($e = 1$, Corollary 10.4): $\frac{133}{60} + \frac{1}{4}\log 37 - \frac{1}{3}\log 6 + \frac{2}{33}\log 2 - 2.5671 = -0.00294\ldots$;
- odd $q \geq 7$ (Lemma 10.1): by (10.1), $\alpha+\beta \leq \frac{133}{60} - 2.5671 + \frac{1}{2}\log 6 + \frac{1}{4q^2} - \frac{1}{3}\log q + \delta$, which decreases in q and is < -0.097 at $q = 7$;
- even $q \geq 8$ (Corollary 10.4, $c(e) \leq c(1)$): the same bound plus $\frac{2}{33}\log 2$, < -0.10 at $q = 8$.

With $\delta = 10^{-4}$ all margins are negative, and Theorem 3.2 gives Theorem B. □

Remark 10.6. For odd $q \geq 7$ Lemma 10.1 suffices, and the formal proof uses it there; Lemma 10.2 would give larger margins. At $q = 5$ a finer ball-arithmetic certificate ($\Phi \leq -2.4516816$) gives -0.0436. The margin at $q = 6$ is about -0.003 per h^2 (only Theorem C at $q = 175$, about -0.0014, is thinner). It rests on the exact leading constant $c(1) = \frac{2}{33}$ of Lemma 9.4; with $c = \frac{1}{4}$ the margin would be $+0.13$. A finer certificate in ball arithmetic ($\Phi \leq -2.5704$) gives -0.0063.

11. Theorem C: $\chi_2(2/q)$

In this section we are in case (C): $q \geq 5$ odd, $\zeta = q/2$, $w = \zeta^{-2} = 4q^{-2}$, $\theta = \chi_2(2/q)$ and $\Delta_h = \Delta_h^{\chi}(X;\zeta) = \mathrm{ev}_\zeta(\Delta_h^{\mathcal{R}})$. This is the machine of Theorem A read at the rational point ζ, and Theorem 5.3 bounds the odd primes with $Y = q^2 - 4$. At $p = 2$, however, ζ is not 2-integral: $v_2(\zeta) = -1$ and $v_2(\zeta^2 - 1) = -2$.

11.1. The prime 2.

Lemma 11.1 (Lemma 10″)**.** *For odd $q \geq 3$, every $h \geq 1$ and $k \geq 0$, $\big|[X^k]\Delta_h\big|_2 \leq 2^{n_2(h)}$ with*

$$n_2(h) = 4hm - 3m^2 + 2m, \qquad m = \Big\lfloor \frac{2h+2}{3} \Big\rfloor, \qquad 3n_2(h) \leq 4h^2 + 4h + 4.$$

So the prime 2 costs at most $\frac{4}{3}\log 2$ per h^2.

Proof. Use the anchored basis. *Type (L)*, $\pi = i+j+1-h \geq 1$: $B = 0$, and by (2.2) with $y = \zeta$, $\Omega_0 = 4/(q^2-4)$ and $\Omega_b = \zeta^2\Omega_0^{b+1}$ have $v_2(\Omega_b) = 2b + [b = 0]\cdot 2 \geq 0$, so by (9.2) $v_2(A) \geq 1-\pi \geq -\pi$. *Type (U)*, $N = h-1-i-j \geq 0$: by (9.3) and the triangle inequality, with $v_2(\xi_l) = -1$, $v_2(\zeta^{2k+1}) = -(2k+1)$ and $v_2(S_k(\zeta)) \geq -2k$, each term $\xi_l T(l)$ has $v_2 \geq -2(i+l)-2 \geq -2i-2N-2$, and with $v_2(N!) \leq N$ the entry has $v_2 \geq -2i - 3N$. So $v_2(\Lambda_X(a_i b_j)) \geq -c(i,j)$ with $c(i,j) = i+j+1-h$ on (L) and $c(i,j) = 2i + 3(h-1-i-j)$ on (U).

Put $a = \lfloor (4h+2)/3 \rfloor$ and

$$U_i = \begin{cases} -i, & 3i < 2h, \\ i-a, & 3i \geq 2h, \end{cases} \qquad V_j = \begin{cases} 3h-3j, & 3j < 2h, \\ j+1-h+a, & 3j \geq 2h, \end{cases}$$

integer versions of the potentials $u(x) = \max(-x, x - \frac{4}{3})$, $v(y) = \max(3 - 3y, y + \frac{1}{3})$. A check of the four cases $3i \gtrless 2h$, $3j \gtrless 2h$ shows $c(i,j) \leq U_i + V_j$ for all $i, j < h$. Lemma 4.1 with $\rho_i = 2^{U_i}$, $\kappa_j = 2^{V_j}$ gives $|[X^k]\Delta_h|_2 \leq 2^{\sum_i (U_i+V_i)}$, and the sum equals $4hm - 3m^2 + 2m$. Its continuous version is $h^2 \int_0^1 (u+v) = \frac{4}{3}h^2$. □

The constant $\frac{4}{3}$ cannot be lowered much by any 2-adic argument for this machine:

Proposition 11.2 (the price of the numerator)**.** *For odd q and every $h \geq 1$, $[X^h]\Delta_h = \pm 4^h \zeta^{h^2} \prod_{k<h} x_k$, which has $v_2 = h - h^2$. Hence every d_h with $d_h\Delta_h \in \mathbb{Z}[X]$ has $v_2(d_h) \geq h^2 - h$.*

Proof. $[X^h]\Delta_h = \det(B(u^{i+j}))$, and $B(u^{i+j}) = \sum_k b_k x_k^{i+j}$ with $b_k = -4\omega_k\zeta^{2k+1}$, so $\det(B(u^{i+j})) = \prod_k b_k \cdot (\det V)^2$. Since $(\det V)^2 = \prod_{k<l}(l-k)^2 = \prod_k |D'(x_k)|$, this is $\pm 4^h \prod_k x_k \cdot \zeta^{\sum_k (2k+1)} = \pm 4^h \zeta^{h^2} \prod_k x_k$, of valuation $2h - h^2 - h$. □

For the numerator 3 we observed numerically that the 3-adic loss is exactly that of the leading coefficient, $\log 3$ per h^2; at $p = 2$ the measured loss (about $1.18 \log 2$ at $h = 64$) is larger than $\log 2$, presumably because of the half-integer poles.

11.2. **Nonvanishing.**

Proposition 11.3 (nonvanishing, case (C))**.** *For every odd $q \geq 5$ and all $a \in \mathbb{Z}$, $b \geq 1$, $\Delta_h^{\chi}(a/b; q/2) \neq 0$ for infinitely many h.*

Proof. We transfer Theorem 8.1 from an integer point by ℓ-adic continuity. Let $\ell \geq 7$ be a prime that divides neither $30\, b\, q(q^2-4)$ nor the integers $9q^2+4$, $64P_L(q^2/4)$, $64P_D(q^2/4)$, which are nonzero by Proposition 8.2 ($q^2/4 \geq 4$). Fix $M \geq 1$ and put $q' := q(\ell^M + 1)/2$, an integer with $2q' \equiv q \pmod{\ell^M}$. Then $4q'^2 \equiv q^2$, so $\ell \nmid q'(q'^2 - 1)$, and by Proposition 8.2 the numerators and denominators of $\det M_\bullet(q')$ are congruent modulo ℓ to units times the integers above; so q' and ℓ satisfy the hypotheses of Theorem 8.1, and $|K_\ell\, \Delta_{3\ell}^{\chi}(r; q')|_\ell = 1$ for $r = a/b$.

Now let $h = 3\ell$ and let $f \in \mathcal{R}$ be a coefficient of $\Delta_h^{\mathcal{R}}$; by Lemma 4.2, $\|f\|_\ell \le \ell^{2h\,\mathrm{lev}(\ell)}$. The points $y_1 = q/2$ and $y_2 = q'$ are ℓ-adic units with $y_1^2 - 1$, $y_2^2 - 1$ units and $y_1 \equiv y_2 \pmod{\ell^M}$, so $q^{m_0}E^{m_1}$ takes values at y_1 and y_2 that are congruent modulo ℓ^M, and $|\operatorname{ev}_{y_1}(f) - \operatorname{ev}_{y_2}(f)|_\ell \le \ell^{2h\,\mathrm{lev}(\ell)-M}$. As r is ℓ-integral, $|\Delta_h(r; q/2) - \Delta_h(r; q')|_\ell \le \ell^{2h\,\mathrm{lev}(\ell)-M}$. Taking M so large that $|K_\ell|_\ell\, \ell^{2h\,\mathrm{lev}(\ell)-M} < 1$, we get $|K_\ell\, \Delta_h(r; q/2)|_\ell = 1$, so $\Delta_h(r; q/2) \neq 0$. Infinitely many primes ℓ qualify. □

11.3. **Proof of Theorem C.** Let $q \ge 175$ be odd. By Theorem 5.3 and Lemma 11.1, the arithmetic constant is

$$\alpha(q) = \frac{133}{60} + \frac{1}{4}\log(q^2 - 4) - \frac{1}{3}\log q + \frac{4}{3}\log 2.$$

Theorem 6.2 with the certificate at $W = 4/175^2$ ($\Phi \le -4.003$), shifted by Lemma 7.1 to $W = 4/q^2$, gives $\beta = -4.003 - \frac{1}{2}\log\frac{q}{175} + \delta$. With $\log(q^2-4) < 2\log q$,

$$\alpha + \beta < \frac{133}{60} + \frac{4}{3}\log 2 - 4.003 + \frac{1}{6}\log 175 + \delta < -0.0012 + \delta,$$

using $175^{104} < 2^{775}$ and $\log 2 < 0.6931471808$. With $\delta = 10^{-4}$ and Proposition 11.3, Theorem 3.2 gives Theorem C. □

Remark 11.4 ($q = 173$). Two independent ball-arithmetic computations (Arb), which are not included in this paper and not formalised, give $\Phi \le -4.0005336$ and $\Phi \le -4.0004556$ at $W = 4/173^2$. With either of them the margin at $q = 173$ would be at most -0.00082 (respectively -0.00074), and the argument would reach every odd $q \ge 173$. We do not claim this: in the certificate format of Section 7, a grid of about $L = 3200$ cells would be needed at $W = 4/173^2$, which we have not run. The value $q = 171$ is out of reach for Lemma 11.1 even with the exact equilibrium constant (numerically the margin is $+0.0016$).

12. Limits of the method

The values below are numerical unless stated otherwise. "Margin" means $\alpha + \Phi$ per h^2, with the proved arithmetic constants and the best certificate (or the equilibrium constant $C(W)$, which no certificate can beat).

- $\chi_2(1/4)$. With Lemma 9.4 ($c(2) = -\frac{1}{21}$) and the certificate $\Phi \le -2.3088$ the margin is $+0.090$, and the equilibrium constant does not change this. Closing it would need $c(2) < -0.177$, while exact computations give about -0.025 at $h = 160$. So $q = 4$ needs a different construction.
- $\chi_2(1/2)$. The margin is about $+0.52$ [12, §11.4]; the analytic constant $C(\frac{1}{4}) \approx -1.744$ is the obstruction.
- $\mathrm{Ti}_2(1/q)$ *for* $q \le 4$. With Lemma 10.2 the margin at $q = 3$ is $+0.241$, and at $q = 4$ (even, where Lemma 9.4 already applies) it is $+0.121$. Improving the 2-adic bound cannot close either gap:

at $q = 3$ removing the prime 2 altogether ($\frac{1}{8}\log 2 \approx 0.087$) leaves $+0.154$, and at $q = 4$ the even constant $c(2)$ would have to drop far below what exact computations show, as for $\chi_2(1/4)$ above. The size bound sees only $|w|$, and the true determinants at $w < 0$ are smaller (Section 6), so a signed version of Theorem 6.2 is the remaining lever; $q = 2$ fails for χ_2 already.

- $\chi_2(2/q)$ *below* 173. Proposition 11.2 is proved: no 2-adic lemma can give less than $\log 2$ per h^2 for this machine. With $c_2 = \log 2$ and the equilibrium constant, the margin is positive for odd $q \lesssim 87$ (it changes sign between 81 and 101); with the measured 2-adic constant (about $1.18 \log 2$ at $h = 64$) the sign change is between 121 and 141. So, numerically and assuming that $\frac{133}{60}$ is the true generic constant, this machine does not reach odd q below about 87, and the gap between 87 and 175 is the gap between the 2-adic floor and Lemma 11.1 (plus the coarseness of our certificates).
- *Effectivity.* Every bound is effective except the prime number theorem in Theorem 4.5, which is used without an error term. The theorems are not affected.

13. The formalisation

13.1. **What is formalised.** The repository `https://github.com/telaviv/chi2-irrationality` [13], at commit `cef469e` of branch `main` (full hash `cef469ef846a2a3967d2d441e331254bd82898e2`), contains proofs of Theorems A, B and C in Lean 4 [6] (version v4.35.0) with Mathlib [22] (version v4.35.0 [23]). The same repository contains the formalisation of [12] ($q = 3, 5$), on which the present one builds, and an intermediate theorem ($\chi_2(1/q) \notin \mathbb{Q}$ for $q \geq 3$, $q \notin \{4, 6, 8, 10, 12\}$), which the proof of Theorem A reuses. It has 148 Lean files under `Chi2Irrational/`, including the certificate data. Paths below are relative to `Chi2Irrational/`, and all declarations are in the namespace `Chi2Irr`. Every step of the proofs of Theorems A–C is formalised, including the prime number theorem, the Andreief identity, the lattice energy inequality, all dual certificates and the block determinants.

13.2. **The statements.** The definitions that the reader has to check are:

```
-- Defs.lean
noncomputable def chi2Term (z : ℝ) (n : ℕ) : ℝ :=
    z ^ (2 * n + 1) / (2 * (n : ℝ) + 1) ^ 2
noncomputable def chi2 (z : ℝ) : ℝ := ∑' n : ℕ, chi2Term
z n
-- TiQ/Defs.lean (namespace TiQ)
noncomputable def ti2Term (z : ℝ) (n : ℕ) : ℝ :=
    (-1) ^ n * z ^ (2 * n + 1) / (2 * (n : ℝ) + 1) ^ 2
noncomputable def ti2 (z : ℝ) : ℝ := ∑' n : ℕ, ti2Term
z n
-- Q/Even/Defs.lean, Q/Even/Interface.lean (namespace Q)
```

```
def TheoremQ4 :  Prop := ∀ q :  ℕ, 3 ≤ q → q ≠ 4 →
    Irrational (chi2 (1 / (q :  ℝ)))
theorem theoremQ4 :  TheoremQ4
-- TiQ/Interface5.lean (namespace TiQ)
def TheoremT5 :  Prop := ∀ q :  ℕ, 5 ≤ q → Irrational (ti2
(1 / (q :  ℝ)))
theorem theoremT5 :  TheoremT5
-- QZ/Interface.lean (namespace QZ)
theorem theoremQ4num :  ∀ q :  ℕ, Odd q → 175 ≤ q →
    Irrational (chi2 (2 / (q :  ℝ)))
```

So Theorem A is `Q.theoremQ4`, Theorem B is `TiQ.theoremT5` (which uses `TiQ.theoremT`, the case $q \geq 6$, from `TiQ/Interface.lean`) and Theorem C is `QZ.theoremQ4num`. Here $\sum'$ is Mathlib's unconditional sum (`tsum`); `summable_chi2Term` and `TiQ.summable_ti2Term` prove that the series converge absolutely for $|z| \leq 1$, so `chi2 z` and `ti2 z` are the genuine values. `Irrational x` is Mathlib's predicate $x \notin \mathbb{Q}$, and $1/(q : \mathbb{R})$, $2/(q : \mathbb{R})$ are real divisions after the cast $\mathbb{N} \to \mathbb{R}$. The statements do not involve the machine, so a wrong internal definition could make a proof fail but could not change what is proved.

13.3. **Axioms.** The command `lake env lean Axioms.lean` prints, for 72 declarations (the three theorems above, the theorems of [12], and the main ingredients of each),

```
'Chi2Irr.Q.theoremQ4' depends on axioms:  [propext,
Classical.choice, Quot.sound]
'Chi2Irr.TiQ.theoremT5' depends on axioms:  [propext,
Classical.choice, Quot.sound]
'Chi2Irr.QZ.theoremQ4num' depends on axioms:  [propext,
Classical.choice, Quot.sound]
```

and the same list for each of the other 69 (run on 2026-11-13 on the tree of the commit above; the output is `scripts/axioms_t5.txt`). These are Lean's three standard axioms, which Mathlib also uses. The development contains no `sorry`, no added axiom, no `native_decide` and no `implemented_by`.

13.4. **Correspondence with the paper.** Tables 3–6 list the formal counterparts of the main steps.

13.5. **Differences from the written proof.** The formal proofs follow the written ones. The differences are these; none weakens a stated theorem.

- *Organisation.* For odd q and even $q \geq 14$, `Q.theoremQ4` calls the intermediate theorem `Q.theoremQ`; for $q = 6, 8, 10, 12$ it uses Lemma 9.4. The machines of cases (B) and (C) are separate copies of the machine of case (A) (`TiQ/Machine`, `QZ/Machine`), tied to the symbolic determinant by `TiQ.map_DeltaFT` and `QZ.map_DeltaF_Z`; no file of the earlier formalisation was edited. Lemma 2.2, the size bound and Theorem 8.1 are copied with the needed sign or weight changes.

paper	Lean declaration (file)
Lemma 2.2, case (A)	`pole_series` (`PoleSeries`); `Lam_eq`, `aeval_chi2_Delta` (`PoleExpansion`)
Prop. 3.1, Thm. 3.2	`irrational_of_poly_approx` (`Criterion`); `irrational_of_bounds` (`MainCond`)
Section 4.1	`map_DeltaF` (`Arith/Free`); `det_GramF` (`Arith/Gauss`)
Lemma 4.2	`lemma9_DeltaF` (`Arith/Lemma9`)
Prop. 4.3, Lemma 4.4	`classBound_DeltaF` (`Arith/ClassBound`); `classBd_le_phi1` (`Arith/ClassCount`)
Thm. 4.5	`eventually_primeSum_phi1_le`, `eventually_primeSum_lev_le` (`Arith/PrimeSums`); `PNT.theta_div_tendsto` (`PNT/Theta`)
Prop. 4.6	`TiQ.parity_DeltaF`, `TiQ.map_DeltaFT`, `TiQ.coeff_clearET_twF` (`TiQ/Free`)
Lemma 5.1, (5.1)	`EDeg_DeltaF`, `aeval_clearE` (`Arith/PoleOrder`)
Lemma 5.2	`ord_DeltaP`, `qdeg_clearE_DeltaF` (`Arith/QAdic`); `factorial_smul_det_sum_rankOne` (`Analytic/Andreief`)
Thm. 5.3	`Q.arith_q` (`Q/Arith`); `TiQ.arith_t` (`TiQ/Arith`); `QZ.arith_z` (`QZ/Arith`)
Def. 6.1, Thm. 6.2	`DualCert`, `DualCert.analytic` (`Analytic/TheoremB`); `TiQ.analytic_t_proof` (`TiQ/Analytic`); `QZ.DualCertW`, `QZ.DualCertW.analyticZ` (`QZ/TheoremB`)
Lemma 7.1	`Q.DualCert.shift` (`Q/Shift`); `QZ.DualCertW.shift` (`QZ/Shift`)
Thm. 7.2	`Cert.CData.Valid`, `Cert.CData.Phi_le` (`Cert/Instance`); `QZ.CDataZ.Valid` (`QZ/CertInstance`)
Thm. 8.1	`Delta_eval_ne_zero_of_det` (`NonVan/Gram`); `lemmaP` (`NonVan/LemmaP`); `nonvan_of` (`NonVan/NonVanQ`)

TABLE 3. Formal counterparts of the shared steps.

- *Lower-order terms.* Lemma 9.3 is formalised with $h^2/2 + h$ in place of $\lfloor h^2/2 \rfloor$; Lemma 10.1 with weaker entry bounds ($v_2 \geq -\pi - 2 - \lfloor \log_2 \pi \rfloor$ on (L), $v_2 \geq 0$ on (U)), giving $n_2 \leq \beta_h + h(\lfloor \log_2 h \rfloor + 2)$; Lemma 9.4 with the explicit bound $n_2 \leq h - \lceil m_e h^2 - h(\frac{3e}{2} + 4\lfloor \log_2 h \rfloor + 4) \rceil$. The leading constants are those of the paper. The value $\beta_h = \lfloor h^2/4 \rfloor$ is not formalised; the proofs use $4\beta_h \leq h^2 + 2h$.
- *Shift and margins.* The shift lemma is formalised in the discretised form of Lemma 7.1. The margins are proved with elementary bounds for logarithms, so the formal margins are slightly weaker: for instance -0.00904 at $q = 6$ for Theorem A, about -0.0025 at $q = 6$ and -0.019 at $q = 5$ for Theorem B, and -0.00126 for Theorem C.

paper	Lean declaration (file)
Lemma 9.1	`Q.twoAdic_Delta`, `Q.twoAdic_odd_proof` (`Q/TwoAdic`); `Q.V2_entry` (`Q/TwoAdicEntries`)
Lemma 9.3	`Q.GLeX_DeltaF_two`, `Q.twoAdic_even_proof` (`Q/TwoAdicEven`)
Lemma 9.4	`Q.CoreDet.padicNorm_detR_le` (`Q/Even/CoreDet`); `Q.sum_cntL_ge`, `Q.large_node` (`Q/Even/Count`); `Q.twoAdic_even_sharp_proof` (`Q/Even/TwoAdicSharp`)
Prop. 8.2, Cor. 8.3	`Q.nonvan_q_proof` (`Q/NonVan`)
certificates	`Cert.Q3.analytic` (`Cert/Q3`); `Cert.Q14.phi_le` (`Cert/Q14`); `Q.cert_newQ_proof` (`Q/Even/Certs`; data `Cert/Q6`, …, `Cert/Q12`)
margins	`Q.margin_odd`, `Q.margin_even` (`Q/Interface`, proofs in `Q/Margin`); `Q.margin_newQ` (`Q/Even/Interface`, `Q/Even/Margin`)
Theorem A	`Q.theoremQ` (`Q/Interface`; odd q, even $q \geq 14$); `Q.theoremQ4` (`Q/Even/Interface`)

TABLE 4. Formal counterparts for Theorem A.

- *Lemma 10.2.* The formal proof is Steps 1–6 above (rational $\tilde{\theta}$, binomial basis, the block matrix $\mathcal{N}$ and one Leibniz bound with potentials). These differ from our working notes, not from the Lean: the notes use the 2-adic θ_2 and a Cauchy–Binet expansion over $2h$ rows, with the sharper bound $|E_{ij}|_2 \leq 2^{i+j+1-h}$; with that bound the tropical value of the Leibniz bound for $\mathcal{N}$ equals the notes' bound. Step 5 proves $2^{i+j+2-h}$ (one weaker), so the proved exponent exceeds the notes' by $O(h)$; the limit constant $\frac{1}{8}\log 2$ is the same. Lemma 10.2 is used formally only at $q = 5$.
- *Not formalised.* Proposition 11.2, the remarks (including Remark 11.4 on $q = 173$) and Section 12; none of them is used in the proofs of the theorems.

13.6. **How the certificates are checked.** A dual certificate is a record of rational data (for example `Cert/DataQ14.lean`, `TiQ/Cert/DataQ6L800.lean`, `Cert/DataQ5L200.lean`, `QZ/CertDataQ175.lean`): the log table as a list of integers in units of 2^{-40}, the heights c_j, the bounds S_k, λ and E_{lo}. A Boolean predicate (`Cert.CData.Valid`, `QZ.CDataZ.Valid`) collects every check of Section 7, and the certificate files prove it by `decide +kernel`, i.e. by evaluation in the Lean kernel; soundness lemmas turn it into a `DualCert`. The data were produced by the scripts `scripts/gen_cert.py` and `scripts/q2num/gen_cert_z.py`, which are not trusted: a wrong value makes the build fail but cannot make a false statement provable. The block determinants of Proposition 8.2 are evaluated as rational functions of q (`Q/NonVan`, `TiQ/NonVan/NonVanT`) and at $q/2$ (`QZ/NonVan`) by exact computation.

paper	Lean declaration (file)
case (B) of Lemma 2.2	`TiQ.pole_seriesT` (`TiQ/PoleSeries`); `TiQ.Lam_eqT`, `TiQ.aeval_ti2_DeltaT` (`TiQ/PoleExpansion`)
Lemma 10.1	`TiQ.V2_fd_seq1`, `TiQ.V2_entryT` (`TiQ/TwoAdicEntries`); `TiQ.twoAdic_DeltaT`, `TiQ.twoAdic_odd_t_proof` (`TiQ/TwoAdicNu1`)
Cor. 10.4	`TiQ.twoAdic_even_t_proof` (`TiQ/TwoAdicEven`)
Prop. 10.5	`TiQ.block_SL`, `TiQ.block_D` (`TiQ/NonVan/BlockMat`); `TiQ.lemmaP` (`TiQ/NonVan/LemmaP`); `TiQ.nonvan_t_proof` (`TiQ/NonVan/NonVanT`)
Thm. 6.2 at $w < 0$	`TiQ.abs_WT`, `TiQ.analytic_t_proof` (`TiQ/Analytic`)
certificate $W = 1/36$	`TiQ.cert_t6_proof` (`TiQ/Certs`; data and checks `TiQ/Cert/`)
Lemma 10.2, Steps 1–2	`TiQ.M.DeltaT_eq_Cb`, `TiQ.M.BfunT_Cb`, `TiQ.M.HM_eq`, `TiQ.M.det_block` (`TiQ/M/Gram`); `TiQ.M.Am_zero`, `TiQ.M.Am_bound` (`TiQ/M/Binom`)
Lemma 10.2, Steps 3–5	`TiQ.M.Om_bound` (`TiQ/M/BoundB`); `TiQ.M.fd_deriv_intval` (`TiQ/M/Binom`); `TiQ.M.MB_poly`, `TiQ.M.err_bound`, `TiQ.M.theta_bound` (`TiQ/M/Trunc`); `TiQ.M.boundA` (`TiQ/M/BoundA`)
Lemma 10.2, Step 6	`TiQ.M.det_PNz`, `TiQ.M.Nblk_entry`, `TiQ.M.twoAdic_DeltaT_M`, `TiQ.M.eventually_n2M_le` (`TiQ/M/Final`); `TiQ.twoAdic_odd_M` (`TiQ/Interface5`)
certificate $W = 1/25$	`TiQ.cert_t5` (`TiQ/Interface5`; data and checks `Cert/DataQ5L200`, `Cert/Q5L200`)
margins	`TiQ.margin_t5` (`TiQ/Interface5`); `TiQ.margin_t6_proof`, `TiQ.margin_t_odd_proof`, `TiQ.margin_t_even_proof` (`TiQ/Margin`)
Theorem B	`TiQ.irrational_of_boundsT` (`TiQ/MainCond`); `TiQ.theoremT` (`TiQ/Interface`, $q \geq 6$); `TiQ.theoremT5` (`TiQ/Interface5`)

TABLE 5. Formal counterparts for Theorem B.

13.7. **Building and replaying.** With elan installed:

```
git clone https://github.com/telaviv/chi2-irrationality
cd chi2-irrationality && git checkout cef469e
lake exe cache get
LEAN_NUM_THREADS=4 lake build
lake env lean Axioms.lean
```

The build has 9196 jobs. Most of its time is spent in the kernel checks of the certificates: a few minutes each for $W = 1/14^2$, $W = 1/6^2, \ldots, 1/12^2$ and $W = 1/5^2$ (461 s), about three hours for $W = 4/175^2$ (10,697 s wall time on a shared machine), and several hours for the finer certificate at $W = 1/36$, whose cells are split over four files. To avoid trusting the elaborator's

paper	Lean declaration (file)
case (C), (4.1)	`QZ.DeltaZ`, `QZ.map_DeltaF_Z` (`QZ/Machine`); `QZ.pole_seriesZ`, `QZ.aeval_chi2_DeltaZ` (`QZ/PoleExpansion`); `QZ.chi2_two_div` (`QZ/Interface`)
Lemma 11.1	`QZ.V2_entryZ` (`QZ/TwoAdicEntries`); `QZ.twoAdic_DeltaZ`, `QZ.twoAdic_num_proof` (`QZ/TwoAdic`)
Prop. 11.3	`QZ.lemmaP_norm`, `QZ.nonvan_z_proof` (`QZ/NonVan`)
certificate	`QZ.Cert.Q175.valid` (`QZ/CertQ175`; data
$W = 4/175^2$	`QZ/CertDataQ175`)
margin	`QZ.margin_z_proof` (`QZ/Margin`)
Theorem C	`QZ.irrational_of_boundsZ` (`QZ/MainCond`); `QZ.theoremQ4num` (`QZ/Interface`)

TABLE 6. Formal counterparts for Theorem C.

handling of the stored environment, each module can be replayed through the kernel with `leanchecker`:

```
for m in $(find Chi2Irrational -name "*.lean" | sed
's/\.lean$//; s#/#.#g'); do
    lake env leanchecker "$m"; done
```

Each module is replayed against the compiled environment of its imports, so Mathlib itself is not re-checked. For the formalisation of [12] and the intermediate theorem (91 modules), all modules passed on 2026-11-09. For the modules of Theorems B and C (`TiQ/`, `QZ/`), 40 modules passed on 2026-11-12 (3189 s, log `scripts/leanchecker_qz_tiq.log`); the seven modules that contain the `decide +kernel` proofs of the two certificates of Theorems B ($q \geq 6$) and C (`QZ/CertQ175.lean`, `TiQ/Certs.lean` and the five files of `TiQ/Cert/` other than the data) were not replayed, because each would repeat hours of kernel evaluation; they are checked by the kernel during `lake build`. The modules added for Lemma 10.2 and $q = 5$ (`TiQ/M/`, `TiQ/Interface5.lean`, `Cert/DataQ5L200.lean`, `Cert/Q5L200.lean`) have not been replayed.

13.8. **What has to be trusted.** Beyond the Lean kernel and the three axioms, the reader has to trust that the definitions displayed above express Theorems A–C, and the Mathlib definitions they use (real numbers, `tsum`, `Irrational`, `Odd`). Nothing else, in particular no numerical data and no generating script, needs to be trusted. Apart from the check by the Lean kernel, all verification of the formalisation so far was carried out by AI systems; no human expert has yet reviewed the Lean code.

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produced largely by AI models (Claude, Anthropic), working under the direction of the author. Apart from the check by the Lean kernel, all verification so far, including the internal reviews of the argument by AI referees, was carried out by AI systems. The written proofs of the new ingredients (the extension of the construction to general q, Lemma 9.4, the case of Ti_2 including Lemma 10.2, and the case of $\chi_2(2/q)$) were each checked by one AI referee, at a single gate, with independent implementations of the machine. Several written proofs here follow the formal proofs rather than those working notes (Lemma 9.4, Lemma 11.1, Proposition 11.3, and Theorem 6.2 at $w < 0$); as written proofs these were checked by the Lean kernel and by the whole-paper referee only. Lemma 10.2 (added after the whole-paper review) also follows the formal proof; as written it was checked by the Lean kernel and by one AI referee of the changed sections, and its source proof by one AI gate referee. The fidelity of each Lean statement was checked by an independent AI referee, and the text of this paper was reviewed as a whole by one further AI referee. No MathSciNet search or expert query has been made. No human expert has yet reviewed the mathematics or the Lean code; we would be grateful for corrections. The prime number theorem in the formalisation is ported from the PrimeNumberTheoremAnd project of A. Kontorovich, T. Tao and collaborators [11], through the `zeta5` repository of César Del Solar [7].

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