A $\sqrt{3}$ growth criterion for weighted reciprocal series

twinheterodox
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Abstract

We prove that $\sum_{n\geq 1} 1/(a_n^2 a_{n+1})$ is irrational for every strictly increasing sequence of positive integers satisfying

$$\limsup_{n\to\infty} \frac{\log a_n}{(\sqrt{3})^n} = \infty.$$

This lowers the sufficient growth base from 2 to $\sqrt{3}$ in the criterion of Barreto, Kang, Kim, Kovač and Zhang [1, Theorem 3] for weights $(2,1)$. Irregular growth is allowed. The result also extends to nondecreasing sequences and positive integer numerators subject to polynomial size bounds.

1 Introduction

Growth criteria for irrationality ask how fast an integer sequence must grow before an associated reciprocal series is forced to have an irrational sum. Such a criterion applies to entire families of sums, without a closed form for the sum or information about the prime factors of the sequence. The underlying constraint is arithmetic: a rational partial sum cannot approach a different rational number more closely than their denominators permit.

Barreto, Kang, Kim, Kovač and Zhang [1] answered a question of Erdős and Graham [3, p. 64] about reciprocal products of successive sequence terms and developed criteria for more general products $a_n^{w_0} \cdots a_{n+d-1}^{w_{d-1}}$. When some weights exceed one, their sufficient growth condition and their rational examples can leave a gap [1, Section 5]. We improve the sufficient condition for the two adjacent factors with weights $(2,1)$.

Theorem 1.1. *Let $(a_n)_{n\geq 1}$ be a strictly increasing sequence of positive integers. If*

$$\limsup_{n\to\infty} \frac{\log a_n}{(\sqrt{3})^n} = \infty, \tag{1.1}$$

then

$$\sum_{n=1}^{\infty} \frac{1}{a_n^2 a_{n+1}}$$

is irrational.

All logarithms are natural. Condition (1.1) is equivalent to $\limsup_{n\to\infty} a_n^{1/(\sqrt{3})^n} = \infty$. It requires large values along a subsequence; it imposes no limit on the ratios of successive logarithms.

1.1 Comparison with the earlier bound

For weights $(2,1)$, the polynomial in [1, Theorem 3] is

$$(x-1)(2+x)-2x = x^2-x-2,$$

whose positive root is 2. That theorem therefore requires $\limsup a_n^{1/2^n} = \infty$. Theorem 1.1 replaces 2 by $\sqrt{3}$, and hence applies under a weaker growth condition. For unit numerators, the base-2 condition also follows from Erdős's criterion [2, Theorem 1], applied to $x_n = a_n^2 a_{n+1}$.

For a concrete comparison, suppose $\log a_n \sim \lambda^n$. The new criterion applies for every $\lambda > \sqrt{3}$, whereas the earlier criterion requires $\lambda > 2$ within this family. In particular,

$$a_n = \lceil \exp((9/5)^n) \rceil$$

is covered by Theorem 1.1, while $a_n^{1/2^n} \to 1$. The same theorem also covers sequences whose growth is irregular; the proof must allow long stretches of slower growth between the large values supplied by (1.1).

The rational examples in [1, Theorem 5] are governed here by

$$(x-1)(2+x)-x = x^2-2.$$

For every $C > 1$, that theorem gives a strictly increasing integer sequence for which $a_n^{1/(\sqrt{2})^n} \to C$ and the reciprocal series has a rational sum. Thus the sufficient base is now $\sqrt{3}$, while the rational constructions occur at $\sqrt{2}$. The optimal threshold remains open.

1.2 A general form

We prove a version that also permits repetitions in (a_n) and positive integer numerators.

Theorem 1.2. *Let $(a_n)_{n\ge1}$ be nondecreasing positive integers and let $(b_n)_{n\ge1}$ be positive integers. Suppose that, for some real $0 \le \eta < \tau$, one has*

$$b_n \le n^{\eta}, \qquad a_n^2 a_{n+1} \ge n^{1+\tau} \tag{1.2}$$

for all sufficiently large n. If (1.1) holds, then the convergent series

$$S = \sum_{n=1}^{\infty} \frac{b_n}{a_n^2 a_{n+1}} \tag{1.3}$$

is irrational.

Theorem 1.1 follows by taking $b_n = 1$, $\eta = 0$ and $\tau = 2$, since strict increase gives $a_n \ge n$. The size assumptions in Theorem 1.2 are the same type as those in [1, Theorem 3]: they give a uniform bound for tails even where growth is slow.

1.3 The source of $\sqrt{3}$

The improvement begins with the denominator of a partial sum. The first three terms of (1.3) have denominators

$$a_1^2 a_2, \qquad a_2^2 a_3, \qquad a_3^2 a_4.$$

Their sum is cleared by $a_1^2 a_2^2 a_3^2 a_4$: only the first power of the last factor is needed. For the first N terms, the corresponding common denominator is

$$D_N = \left(\prod_{i=1}^{N} a_i^2 \right) a_{N+1}. \tag{1.4}$$

Writing $u_n = \log a_n$ and $V_n = \sum_{i=1}^{n} u_i$, we obtain

$$\log \frac{D_N}{a_{N+1}^2 a_{N+2}} = 3V_N - V_{N+2}. \tag{1.5}$$

The relation $V_{N+2} = 3V_N$ has growth scale $(\sqrt{3})^N$. This explains the base in the theorem.

Identity (1.5) only compares the common denominator with the first omitted term. For irregular growth, the rest of the tail need not be negligible. We adapt the tail estimates and the argument for selecting a peak after a slow index in [1, Lemmas 9–11 and Proposition 12]. These build on the tail and peak arguments of Erdős [2, pp. 3–6]. The two changes are the endpoint denominator (1.4) and the use of peaks of the cumulative logarithms V_n. These changes preserve the saving in (1.5) while controlling the entire tail. We give all the required estimates below.

2 Two uniform tail estimates

The proof uses a coarse estimate valid everywhere and a sharper estimate on intervals of fast growth. The endpoints of those intervals will be chosen later, so the constants must be independent of them. The following estimates adapt [1, Lemma 10]; the corresponding bounds with unit numerators appear in [2, pp. 3–4].

Lemma 2.1. *Let (x_n) be a nondecreasing sequence of positive real numbers and let $0 \le y_n \le n^\eta$. Fix $0 \le \eta < \tau$, $c_0 > 0$ and a positive integer A.*

1. *If $x_n \ge c_0 n^{1+\tau}$ for every $n \ge A$ and $X = x_A \ge 1$, then*

$$\sum_{n \ge A} \frac{y_n}{x_n} \le C_{\mathrm{pol}} X^{-\alpha}, \qquad \alpha = \frac{\tau - \eta}{1+\tau} > 0. \tag{2.1}$$

   The constant C_{pol} depends only on η, τ, c_0.

2. *If $x_n \ge e^n$ for $A \le n \le B$, where $B \ge A$ is an integer or $B = \infty$, then*

$$\sum_{n=A}^{B} \frac{y_n}{x_n} \le C_{\exp} \frac{(\log x_A)^{1+\eta}}{x_A}. \tag{2.2}$$

   The constant $C_{\exp}$ depends only on η.

Neither constant depends on the endpoints or on the particular sequences.

Proof. Put $X = x_A$ and group the relevant indices according to $2^j X \le x_n < 2^{j+1} X$, for $j \ge 0$. In the first case, every index in the jth group is at most $(2^{j+1}X/c_0)^{1/(1+\tau)}$. Since $\eta \ge 0$, summing n^η up to this bound shows that the group's contribution is at most

$$C \frac{(2^{j+1}X)^{(1+\eta)/(1+\tau)}}{2^j X}.$$

The sum over j is geometric with ratio $2^{-\alpha} < 1$, proving (2.1).

In the second case, every index in the jth group is at most $\log X + (j+1)\log 2$. Its contribution is therefore at most

$$C \frac{(\log X + j + 1)^{1+\eta}}{2^j X}.$$

Here $\log X \ge A \ge 1$. Summing the convergent series $\sum_{j \ge 0} (j+2)^{1+\eta}/2^j$ proves (2.2). The same bounds apply when an interval ends partway through a group. □

3 Proof of Theorem 1.2

3.1 What rationality would force

Write

$$x_n = a_n^2 a_{n+1}, \qquad T_N = \sum_{n>N} \frac{b_n}{x_n}.$$

The series converges because (1.2) bounds its terms by $n^{-1-(\tau-\eta)}$ for all large n. We use the elementary criterion in [1, Lemma 8]. Suppose, for a contradiction, that $S = p/q$ with q a positive integer. The integer D_N in (1.4) clears each term of the partial sum through N. Thus qD_NT_N is a positive integer, and

$$D_N T_N \geq \frac{1}{q} \qquad (N \geq 1). \tag{3.1}$$

We will find indices at which this same cleared tail tends to zero.

Retain $u_n = \log a_n$ and $V_n = \sum_{i=1}^{n} u_i$, with $V_0 = 0$, and put

$$E_N = \log D_N = V_N + V_{N+1}, \qquad F_N = \log x_{N+1} = 2u_{N+1} + u_{N+2}. \tag{3.2}$$

The endpoint identity becomes

$$E_N - F_N = 3V_N - V_{N+2}. \tag{3.3}$$

By (2.1), with $c_0 = 1$ and $\alpha = (\tau - \eta)/(1 + \tau)$, the full tail satisfies $T_N \leq C_{\mathrm{pol}} e^{-\alpha F_N}$ for all large N. Combining this with (3.1) gives

$$\alpha F_N \leq E_N + \log(C_{\mathrm{pol}} q).$$

Monotonicity and (1.2) imply $a_n \to \infty$, so $E_N \to \infty$. Consequently, for all sufficiently large N,

$$F_N \leq \frac{2}{\alpha} E_N, \qquad E_{N+1} \leq H E_N, \qquad H = 1 + \frac{2}{\alpha}. \tag{3.4}$$

Indeed, $E_{N+1} - E_N = u_{N+1} + u_{N+2} \leq F_N$.

The bound (3.4) is deliberately coarse. Under the assumption of rationality, it prevents a slow index from being followed immediately by an uncontrolled jump. It also gives $E_N = O(H^N)$ and hence

$$\log(F_N + 1) = O(N + 1). \tag{3.5}$$

All constants from this point may depend on the fixed sequences, η, τ and the hypothetical denominator q, but not on the indices that will be selected.

3.2 Eventual exponential growth

Set $c = \sqrt{3}$ and $W_n = V_n/c^n$. Suppose first that $a_n \geq e^n$ for every sufficiently large n. Then $x_n \geq e^n$ there, and (2.2) applied to the whole tail gives

$$T_N \leq C_{\exp} F_N^{1+\eta} e^{-F_N}.$$

Using (3.1), (3.3) and (3.5), we obtain

$$V_{N+2} \leq 3V_N + (1 + \eta) \log F_N + O(1) \leq 3V_N + C(N + 1)$$

for a fixed $C > 0$ and all large N. Since $c^2 = 3$,

$$W_{N+2} \leq W_N + \frac{C(N + 1)}{c^{N+2}}.$$

The errors have a finite sum on each parity class. Iterating separately on the even and odd indices therefore bounds W_n. But $V_n \geq u_n$, so this contradicts (1.1).

3.3 A first peak after a slow index

It remains to treat sequences with $a_n < e^n$ at infinitely many indices. We will split a tail inside a fast stretch. A first peak will control the near part, and a later large value will control the part beyond the stretch.

Choose increasing indices R such that

$$\mu_R = \frac{u_R}{c^R} \longrightarrow \infty. \tag{3.6}$$

For large R, these indices are fast: $a_R \geq e^R$. Let $P = P(R) < R$ be the last index before R with $a_P < e^P$. It exists for all large R. For every fixed slow index t, the selected R eventually exceed t, and then $P(R) \geq t$. Since the slow indices are unbounded, $P(R) \to \infty$. By definition,

$$a_n \geq e^n \qquad (P < n \leq R). \tag{3.7}$$

Monotonicity gives $V_P \leq Pu_P < P^2$ and $E_{P-1} \leq 2P^2$. For all large P, (3.4) therefore yields

$$V_{P+1} \leq E_P \leq 2HP^2.$$

It follows that $W_P < 1$ and $W_{P+1} < 1$. On the other hand, $W_R \geq \mu_R \to \infty$, so $R \geq P + 2$ eventually.

We need a peak that produces a definite saving in (3.3), while keeping the earlier denominator under control. Let $L = e^2$ and choose the first $Q \in [P + 2, R]$ such that

$$W_Q > (1 + Q^{-2}) \max(1, W_P, \ldots, W_{Q-1}). \tag{3.8}$$

Such a Q exists for all large selected R. Otherwise the successive running maxima, starting from 1, could increase only by the factors $1 + j^{-2}$, and would remain at most

$$\prod_{j \geq 1} (1 + j^{-2}) \leq \exp\left(\sum_{j \geq 1} j^{-2} \right) \leq L.$$

This contradicts $W_R \geq \mu_R \to \infty$. The summable-increment argument appears in [2, p. 4]; we follow its use for weighted series in [1, Lemma 9 and Proposition 12].

Because Q is the first such peak,

$$W_j \leq L \quad (P \leq j < Q), \qquad E_{Q-2} = V_{Q-2} + V_{Q-1} \leq 2Lc^Q. \tag{3.9}$$

Also $V_Q > c^Q$, and comparison with $Q - 2$ in (3.8) gives

$$E_{Q-2} - F_{Q-2} = 3V_{Q-2} - V_Q \leq -\frac{V_Q}{Q^2 + 1} \leq -\frac{c^Q}{Q^2 + 1}. \tag{3.10}$$

Here $Q - 2 \geq P$, so the earlier index is included in the maximum. Moreover, $Q \to \infty$ because $Q \geq P + 2$.

3.4 Both parts of the cleared tail tend to zero

Split the tail after $Q - 2$ as

$$D_{Q-2} T_{Q-2} = D_{Q-2} \sum_{n=Q-1}^{R} \frac{b_n}{x_n} + D_{Q-2} T_R. \tag{3.11}$$

The bound on E_{Q-2} in (3.9) has two uses: it controls the logarithmic factor in the near-tail estimate, and it limits the clearing cost for the far tail.

For the near part, (3.7) gives $x_n \geq e^n$ throughout $Q-1 \leq n \leq R$. By (3.4) and (3.9),

$$F_{Q-2} \leq \frac{2}{\alpha} E_{Q-2} \leq \frac{4L}{\alpha} c^Q.$$

The finite-interval bound (2.2) and (3.10) now give

$$\begin{aligned} \log\left(D_{Q-2} \sum_{n=Q-1}^{R} \frac{b_n}{x_n} \right) &\leq E_{Q-2} - F_{Q-2} + (1+\eta) \log F_{Q-2} + O(1) \\ &\leq -\frac{c^Q}{Q^2+1} + (1+\eta) Q \log c + O(1) \longrightarrow -\infty. \end{aligned} \tag{3.12}$$

The constants are independent of the moving endpoints P, Q, R.

For the far part, use (2.1) and $x_{R+1} = a_{R+1}^2 a_{R+2} \geq a_R^3$. Together with $Q \leq R$, (3.9) and (3.6), this gives

$$\begin{aligned} D_{Q-2} T_R &\leq C_{\mathrm{pol}} \exp(E_{Q-2} - \alpha \log x_{R+1}) \\ &\leq C_{\mathrm{pol}} \exp(2Lc^Q - 3\alpha u_R) \\ &\leq C_{\mathrm{pol}} \exp\big((2L - 3\alpha\mu_R) c^R\big) \longrightarrow 0. \end{aligned} \tag{3.13}$$

Thus both terms in (3.11) tend to zero along the same selected indices. This contradicts (3.1) and proves Theorem 1.2, hence also Theorem 1.1. □

Verification

An independent mathematical audit (13 November 2026) found no error or missing hypothesis in the underlying proof. This was not formal verification.

References

[1] K. Barreto, J. Kang, S.-h. Kim, V. Kovač and S. Zhang, *Irrationality of rapidly converging series: A problem of Erdős and Graham*, Bull. Lond. Math. Soc. **58** (2026), no. 8, e70457. doi:10.1112/blms.70457. arXiv:2601.21442v3.

[2] P. Erdős, *Some problems and results on the irrationality of the sum of infinite series*, J. Math. Sci. **10** (1975), 1–7. Full text.

[3] P. Erdős and R. L. Graham, *Old and new problems and results in combinatorial number theory*, Monographies de L'Enseignement Mathématique, vol. 28, Université de Genève, L'Enseignement Mathématique, Geneva, 1980. Full text.