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A PROOF OF BRAUER'S $k(B)$-CONJECTURE AT THE PRIME TWO

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Abstract. We prove that every 2-block of a finite group has at most as many ordinary irreducible characters as the order of its defect groups.

1. Introduction

Let p be a prime, let B be a p-block of a finite group G, let D be a defect group of B, and write $k(B)$ for the number of ordinary irreducible characters belonging to B. Brauer's $k(B)$-conjecture asserts that

$$k(B) \leq |D|.$$

The conjecture relates the number of characters in a block to the order of its local defect group. In 1946 Brauer proved $k(B) \leq p^{d(d+1)/2}$, where $|D| = p^d$, and proposed the bound p^d in the paragraph following his Theorem 8 [7]. He also recorded the cases $d = 0, 1, 2$. The question later appeared as Problem 20 in his 1963 list [8]. We prove the conjecture at the prime two.

Theorem 1.1. *Let G be a finite group and let B be a 2-block of G with defect group D. Then $k(B) \leq |D|$.*

Brauer and Feit subsequently obtained the general estimate $k(B) \leq p^{2d}/4+1$ and the refinement $k(B) \leq p^{2d-2}$ for $d > 2$ [9, Theorems 1 and 1*]. Their Theorem 4 proves the conjectured bound when D is cyclic. For a nontrivial cyclic defect group, Dade's structure theorem [19, Theorem 1] gives the exact formula

$$k(B) = e + \frac{|D| - 1}{e}, \qquad l(B) = e \mid p - 1,$$

where e is the inertial index and $l(B)$ is the number of irreducible Brauer characters in B. Thus $e = 1$ and $k(B) = |D|$ for cyclic defect at two. The study of noncyclic 2-defect groups includes the dihedral, semidihedral and quaternion cases and their central extensions by cyclic 2-groups. Sambale's Cartan-matrix method proves the bound whenever the defect group is a central extension of a metacyclic 2-group by a cyclic 2-group [66, Theorem 2]. This form is used in the exceptional component calculations below.

The p-solvable case led to the coprime $k(GV)$ problem. Nagao [55] reduced the conjecture for p-solvable groups to the assertion

$$k(V \rtimes H) \leq |V|$$

for a finite p'-group H acting faithfully and irreducibly on an elementary abelian p-group V. Here $k(X)$ denotes the number of conjugacy classes of a finite group X. Robinson and Thompson [60] proved this assertion for $p > 5^{30}$. The remaining characteristics were resolved through character-theoretic and orbit estimates; the final characteristic-five case was settled by Gluck, Magaard, Riese and Schmid [30]. Their solution completed the p-solvable case for every prime. For arbitrary linear groups, the non-coprime problem was studied by Guralnick–Tiep [34] and

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Guralnick–Maróti [31]. Their results on quasisimple and symplectic-type linear groups provide representation-degree, support and class-number inputs for our affine argument.

Recent work on the non-coprime problem also treats wreath products. Hung, Maróti and Martínez Madrid [38, Theorem 1.1] prove $k(X \wr H) \leq k(X)^n$ for a nontrivial finite group X and a transitive semiprimitive group H of degree n, whenever n or $k(X)$ is sufficiently large. Here semiprimitive means that every normal subgroup is transitive or has trivial point stabilizers. The theorem concerns the full wreath product. Our imprimitive argument requires a bound for subgroups with relations between their coordinate projections; these relations enter the weighted permutation count in Section 5.

The classification of finite simple groups permits a separate analysis of quasisimple blocks. Malle [48] studied whether a block of a quasisimple group could be a minimal counterexample, comparing with blocks of all finite groups having the same defect-group isomorphism type and minimizing the index of the center. His Theorems 1 and 2 exclude primes at least five and abelian defect groups. His Theorem 3 confines a remaining quasisimple minimal counterexample to an isolated block of a group of Lie type in non-defining characteristic, at the prime two or three, with nonabelian defect. At three the remaining block is nonunipotent. Thus the exceptional isolated blocks at two are among the cases requiring a full ordinary-character count. For classical groups, the semisimple parametrization and generating functions give such counts; Brenner's work [10] supplies related estimates for 2-blocks of general linear and unitary groups.

Another line of progress concerns characters of height zero. Write $k_0(B)$ for their number. Kessar and Malle proved that abelian defect implies $k(B) = k_0(B)$ [41]. Ruhstorfer proved the Alperin–McKay conjecture for 2-blocks with almost abelian defect groups, in particular with abelian defect groups [63, Theorem F]. These results already give Brauer's bound for every abelian-defect 2-block. Indeed, if b is the Brauer correspondent of B in $N_G(D)$, they give

$$k(B) = k_0(B) = k_0(b) = k(b) \leq |D|.$$

The last inequality follows from the normal-defect description [44] and the coprime $k(GV)$ theorem. Ruhstorfer later proved the Alperin–McKay and height-zero conjectures at two in full [64]. For a block with nonabelian defect, Brauer's bound also requires control of the positive-height characters contributing to $k(B)$.

Jordan decomposition is central to the Lie-type analysis. Bonnafé–Rouquier [4] and Bonnafé–Dat–Rouquier [6] construct equivalences associated with semisimple centralizers and Levi subgroups. Ruhstorfer's equivariant form [65] provides comparison maps compatible with diagonal automorphisms. Kessar and Malle [43] describe the block distribution under Jordan correspondence, with the stated ambiguities for exceptional twin blocks. Their result on Robinson's conjecture concerns individual character defects. The block-distribution statements are the input used here to count all characters in the exceptional 2-series.

The proof of Theorem 1.1 combines an affine bound, a fixed-point comparison for covering blocks, and a proper-Levi descent. The affine bound is

$$k(V \rtimes H) \leq |V|\,|H|_2$$

for every faithful action of a finite group H on a finite $\mathbb{F}_2$-space V, where $|H|_2$ is the order of a Sylow 2-subgroup of H. Induction on the dimension handles arbitrary modules. The primitive case uses the generalized Fitting subgroup, support estimates and finite subgroup calculations. The imprimitive case uses a support-weighted inequality for permutation groups, with separate calculations over $\mathbb{F}_4$, $\mathbb{F}_8$ and $\mathbb{F}_{32}$.

For a G-stable 2-block b of $X \lhd G$, let $M \leq G$ consist of the elements acting innerly on the integral block algebra of b. When M/X is solvable, the character fibers in blocks of M covering b are bounded by relative defect orders. Suppose there is an $\mathbb{F}_2(G/M)$-module W of cardinality

equal to the defect order of b, whose action has a 2-group kernel and whose fixed-point numbers dominate those of $\mathrm{Irr}(b)$ for every subgroup. The affine inequality then bounds every covering block. In a minimum counterexample, the component estimates supply the required local modules. Their assembly proves the fixed-point comparisons, solvability and kernel conditions for the ambient group.

For a component orbit satisfying the hypotheses on lifted ordinary automorphisms in Theorem 10.1, with proper canonical minimal enclosing Levi subgroups, we construct an integral Morita equivalence with a block of an odd central extension of a proper ambient subgroup. The construction preserves the component-centralizer action, the character count and the defect order, while decreasing the index of the center. The remaining component families are counted explicitly. In particular, the nonprincipal unipotent E_8 calculation includes every semisimple layer contributing to the block.

Sections 2–4 establish the block-counting tools. The affine proof culminates in Theorem 9.1. Section 10 proves the proper-Levi descent. The classical counts and the component analysis follow, and Section 18 combines them by minimizing, in order, the defect order, the index of the center and the group order. The finite calculations are specified by their complete input ranges, algorithms and output tables.

2. Character counts and fixed points

All groups in this paper are finite unless an algebraic group is explicitly specified. We fix a splitting 2-modular system $(K, \mathcal{O}, k)$, where $\mathcal{O}$ is a complete discrete valuation ring, K has characteristic zero, and k is algebraically closed of characteristic two. It may be enlarged when a finite central extension is introduced. A block of G means a primitive central idempotent b of $\mathcal{O}G$, or the corresponding algebra $\mathcal{O}Gb$. We write $\mathrm{Irr}(b)$ and $\mathrm{IBr}(b)$ for its ordinary and irreducible Brauer characters, $k(b) = |\mathrm{Irr}(b)|$, $l(b) = |\mathrm{IBr}(b)|$, and $\delta(b) = |D_b|$ for the order of a defect group. For a group H, $k(H) = |\mathrm{Irr}(H)|$ is its number of conjugacy classes, and $B_0(H)$ denotes its principal 2-block. For a nonzero integer n and a prime p, let n_p be the largest power of p dividing $|n|$, put $n_{p'} = |n|/n_p$, and write $v_p(n) = \log_p(n_p)$. For $N \lhd G$ and $Y \subseteq \mathrm{Irr}(N)$, let $\mathrm{Irr}(G \mid Y)$ be the set of ordinary irreducible characters of G whose restrictions to N have a constituent in Y. For a block c of G, put $\mathrm{Irr}(c \mid Y) = \mathrm{Irr}(c) \cap \mathrm{Irr}(G \mid Y)$. A block b or a character θ in the right-hand argument means $Y = \mathrm{Irr}(b)$ or $Y = \{\theta\}$, respectively. We use the usual covering relation for blocks of normal subgroups. The Cartan matrix C_b is indexed by $\mathrm{IBr}(b)$; its largest elementary divisor is $\delta(b)$.

We write $O_p(H)$ and $O_{p'}(H)$ for the largest normal p-subgroup and normal p'-subgroup of H. The Fitting subgroup $F(H)$ is its largest nilpotent normal subgroup. A component is a subnormal quasisimple subgroup, where quasisimple means perfect with nonabelian simple central quotient. The layer $E(H)$ is the product of the components, and $F^*(H) = F(H)E(H)$ is the generalized Fitting subgroup. The solvable radical $\mathrm{Rad}(H)$ is the largest solvable normal subgroup, and $\mathrm{Soc}(H)$ is the product of the minimal normal subgroups. We write $H^{\mathrm{ab}} = H/[H, H]$.

A normalized scalar factor set on H is a map $\alpha : H \times H \to K^\times$ satisfying

$$\alpha(h, j)\alpha(hj, r) = \alpha(j, r)\alpha(h, jr), \qquad \alpha(1, h) = \alpha(h, 1) = 1 \quad (h, j, r \in H),$$

with trivial H-action on the scalars. For such a factor set, let $k_\alpha(H)$ be the number of irreducible α-projective characters. The center of the twisted group algebra has a basis indexed by the α-regular conjugacy classes, where x is α-regular if $\alpha(x, y) = \alpha(y, x)$ for every $y \in C_H(x)$. Therefore

$$k_\alpha(H) \le k(H). \tag{2.1}$$

In Clifford theory the factor set records the obstruction to extending the corresponding ordinary character.

For a finite H-set Y, write $H_y = \{h \in H : hy = y\}$, $Y^J = \{y \in Y : jy = y \text{ for all } j \in J\}$ for $J \le H$, and $Y^h = Y^{\langle h \rangle}$ for $h \in H$. Let $n(H, Y) = |Y/H|$ be the number of orbits.

Lemma 2.1. *If $H \le G$ and $N \lhd G$, then*

$$k(H) \le |G : H| k(G), \qquad k(G) \le k(N) k(G/N).$$

More generally, if a G-invariant block b of N is fixed, then

$$|\operatorname{Irr}(G \mid b)| \le k(b) k(G/N). \tag{2.2}$$

Proof. The intersection of a G-class with H is a union of at most $|G : H|$ H-classes: the classes are parametrized by suitable double cosets of a centralizer and H. This proves the first inequality. For each G-orbit representative θ in $\operatorname{Irr}(N)$, Clifford theory gives $k_{\alpha_\theta}(G_\theta/N)$ irreducible characters above its orbit. By (2.1) and the first inequality this is at most $|G : G_\theta| k(G/N)$. Summing the orbit sizes proves the second inequality. Restricting the same sum to $\operatorname{Irr}(b)$ proves (2.2). □

For a finite H-set Y, put

$$\kappa_H(Y) = \sum_{y \in Y/H} k(H_y), \tag{2.3}$$

where the sum is over orbit representatives. We call $|Y^J|$ the J-fixed-point number of Y.

Lemma 2.2. *For every finite H-set Y,*

$$\kappa_H(Y) = \frac{1}{|H|} \sum_{\substack{x,z \in H \\ xz = zx}} |Y^{\langle x,z \rangle}|. \tag{2.4}$$

Consequently, for a finite H-set Z and a real number $a \ge 0$, if $|Y^J| \le a|Z^J|$ for all two-generated abelian subgroups J of H, then $\kappa_H(Y) \le a\kappa_H(Z)$. For a finite-dimensional $\mathbb{F}_2 H$-module V, with dual module V^*,

$$\kappa_H(V) = k(V^* \rtimes H). \tag{2.5}$$

Proof. For an orbit represented by y, the number of commuting ordered pairs in H_y is $|H_y| k(H_y)$. Count triples (y, x, z) with $x, z \in H_y$ and $xz = zx$, first by orbits of y and then by (x, z), to obtain (2.4). The comparison follows term by term. The ordinary linear characters of the additive group V^* form the H-set V. Each such character extends to its inertia group in $V^* \rtimes H$ by making it trivial on the complement. Clifford theory therefore proves (2.5). □

We shall prove the following affine inequality in Section 9:

$$k(V \rtimes H) \le |V|\,|H|_2 \quad \text{if } H \text{ acts faithfully on the } \mathbb{F}_2\text{-space } V. \tag{2.6}$$

The following consequence permits a normal 2-kernel.

Lemma 2.3. *Assume (2.6). If the kernel R of the action of H on V is a 2-group, then*

$$\kappa_H(V) = k(V^* \rtimes H) \le |V|\,|H|_2.$$

Proof. The subgroup R is normal in $V^* \rtimes H$ and centralizes V^*. Apply Lemma 2.1, followed by (2.6) for H/R on V^*, to obtain

$$k(V^* \rtimes H) \le k(R) k(V^* \rtimes (H/R)) \le |R|\,|V|\,|H/R|_2 = |V|\,|H|_2.$$

□

Remark 2.4. The dual in (2.5) is necessary. In characteristic two the dimensions of common fixed spaces in V and V^* need not agree for arbitrary subgroups. The fixed-point comparison uses the natural identification $\operatorname{Irr}(V^*) \cong V$.

3. Linearization of normal 2-subgroups

This section retains individual blocks throughout the passage from a normal 2-subgroup to an elementary abelian group. Several of the arguments work at an arbitrary prime p; in those statements the coefficient system is understood to have residue characteristic p.

Lemma 3.1 (Central character fibers)**.** *Let $Z \le Z(X)$ be a p-subgroup, let c be a block of X, and let $\bar{c}$ be its dominated block in X/Z. For every $\lambda \in \mathrm{Irr}(Z)$,*

$$|\,\mathrm{Irr}(c \mid \lambda)| \le k(\bar{c}). \tag{3.1}$$

If D_λ consists of the decomposition rows belonging to $\mathrm{Irr}(c \mid \lambda)$, with all columns of $\mathrm{IBr}(c) = \mathrm{IBr}(\bar{c})$, then

$$D_\lambda^{\mathsf{t}} D_\lambda = C_{\bar{c}}. \tag{3.2}$$

In particular its row-column incidence graph is connected and every column occurs.

Proof. Multiplication by Z permutes the conjugacy-class-sum basis of $Z(\mathcal{O}X)$. The direct summand $cZ(\mathcal{O}X)$ is therefore an $\mathcal{O}Z$-permutation lattice. Each transitive permutation lattice $\mathcal{O}[Z/U]$ is indecomposable, since its endomorphism ring is $\mathcal{O}[Z/U]$, a local ring for a p-group Z/U. Krull–Schmidt over the complete order $\mathcal{O}Z$ implies that every direct summand of a permutation lattice is again a sum of such transitive lattices. Over K, the center of the block is the sum of the idempotent lines indexed by $\mathrm{Irr}(c)$, and the Z-character on the line of χ is the central character of χ. Every transitive permutation character of the abelian group Z contains the trivial character once and any other linear character at most once. Thus the multiplicity of λ is no larger than that of 1_Z, proving (3.1).

Column orthogonality proves (3.2) as follows; compare [59, Theorem 6.2]. Projection to X/Z bijects the p-regular conjugacy classes. For a p-regular x, one has $C_X(x)/Z = C_{X/Z}(xZ)$: an element centralizing xZ has a commutator with x in the central p-group Z, and the p'-order of x forces that commutator to be trivial. Central-character column orthogonality consequently gives the same column Gram matrix for every λ. The ordinary character table restricted to this central fiber is D_λ times the common Brauer character table. Cancel that invertible table and compare with the trivial fiber, which is the character table of X/Z. This gives the Gram identity, first for all blocks and then for the displayed block components. The Cartan matrix of a block is indecomposable and has positive diagonal entries. Its Gram realization by a nonnegative decomposition matrix proves the last assertion. □

Lemma 3.2 (Cartan matrices under linearization)**.** *Let $Q \lhd G$ be a p-group and choose a G-invariant central series $1 = Q_0 < Q_1 < \cdots < Q_t = Q$ with elementary abelian factors. Let $V = \bigoplus_{i=1}^{t} Q_i/Q_{i-1}$ with its induced $H = G/Q$-action, and put $L = V \rtimes H$. Under the common inflation labeling by $\mathrm{IBr}(H)$, the full Cartan matrices of G and L are equal. Their connected components give a bijection of blocks $B \leftrightarrow B^{\mathrm{lin}}$ with*

$$C_B = C_{B^{\mathrm{lin}}}, \qquad \delta(B) = \delta(B^{\mathrm{lin}}).$$

Proof. Every irreducible Brauer character is trivial on the normal p-subgroup. The p-regular classes of G and H correspond bijectively. If $h \in H$ is p-regular, choose a p-regular lift $x \in G$. Coprime conjugacy and fixed-point lifting give

$$|C_G(x)| = |C_Q(x)|\,|C_H(h)|.$$

Indeed, Schur–Zassenhaus identifies the p'-complements over $\langle h \rangle$ and gives both uniqueness of the lifted class and surjectivity of $C_G(x) \to C_H(h)$. Coprime fixed points are exact on the chosen central series, so

$$|C_Q(x)| = \prod_i |(Q_i/Q_{i-1})^h| = |V^h|.$$

The centralizer order at the corresponding p-regular element of $V \rtimes H$ is therefore identical. The Gram matrix of the irreducible Brauer characters for the scalar product on p-regular classes is the inverse Cartan matrix. Its entries, expressed as sums with denominators $|C_G(x)|$, are equal in the two groups. Thus their Cartan matrices and their indecomposable components agree. Equality of the largest elementary divisors gives equality of defect orders. □

Lemma 3.3 (A block assignment from modular induction)**.** *Let $A \lhd G$ be an abelian p-subgroup, let $\lambda \in \mathrm{Irr}(A)$, and put $T = G_\lambda$. For a block c of T/A, all characters in the λ-fiber of its corresponding block of $T/\ker\lambda$, inflated to T, induce to characters in one block of G. That block is determined solely by the modular induction matrix from T/A to G/A and by the block partition of $\mathrm{IBr}(G)$.*

Proof. The group $A/\ker\lambda$ is a central cyclic p-subgroup of $T/\ker\lambda$. Let $\hat{c}$ be the block corresponding to c. Let D_λ be its decomposition matrix in this central fiber, and let R be the nonnegative matrix of modular induction from T/A to G/A. Every row of R is nonzero. Clifford theory says that each character in $\mathrm{Irr}(\hat{c} \mid \lambda)$ induces irreducibly to G, and its decomposition row is the corresponding row of $D_\lambda R$. Such a row is supported on a single block. By (3.2), the columns of D_λ are connected by positive rows. Positivity now forces the rows of R indexed by $\mathrm{IBr}(c)$ all to be supported in the same block of G. This description uses only R and the indicated block partition. □

Proposition 3.4 (Blockwise linearization)**.** *In the notation of Lemma 3.2,*

$$k(B) \le k(B^{\mathrm{lin}}) \tag{3.3}$$

for every corresponding pair of blocks.

Proof. First let $A \lhd G$ be an abelian p-subgroup and compare G with $A \rtimes (G/A)$, writing $H = G/A$ for this comparison. The Cartan block partitions agree by Lemma 3.2. Choose orbit representatives $\lambda \in \mathrm{Irr}(A)/(G/A)$. The quotient inertia group is H_λ in both cases. For every block c of H_λ, Lemma 3.3 assigns the pair (λ, c) to the same block in both groups. In G its contribution is $|\mathrm{Irr}(\hat{c} \mid \lambda)| \le k(c)$ by Lemma 3.1. In the split group λ has the canonical extension, trivial on H_λ, and the contribution is exactly $k(c)$. The extension has p-power order and is one on p-regular elements, so its use changes neither the decomposition rows nor the block assignment. Sum over precisely the pairs assigned to B.

For general Q, put $A = Q_1 \le Z(Q)$. The abelian case replaces G by $A \rtimes (G/A)$, preserving the Cartan labels and increasing every block count. In this split group the complement-side subgroup Q/A is normal, since it acts trivially on A. Apply induction to Q/A. The quotient is $A \rtimes H$, and A acts trivially on all the remaining central factors of Q/A. Let V' be the direct sum of these remaining central factors. The resulting group is $V' \rtimes (A \rtimes H) \cong (A \oplus V') \rtimes H$. At every step the labels come from the same $\mathrm{IBr}(H)$, which proves the assertion. □

Proposition 3.5 (The kernel of an affine action)**.** *Let $L = V \rtimes H$, where V is an elementary abelian p-group, and put $K = C_H(V)$. For a block B of L, after passing to the inertia group of a covered block b of K by Fong–Reynolds, one has*

$$\delta(B) = |V|\delta(b)|H : K|_p, \tag{3.4}$$

$$k(B) \le k(b)k(V \rtimes (H/K)). \tag{3.5}$$

The quotient H/K in these formulas acts faithfully on V.

Proof. Fong–Reynolds preserves the count and the defect group, and the inertia group is again split over V. Thus assume b is H-invariant. Set $N = V \times K = C_L(V)$ and let $c = 1_V \otimes b$ be its block. It has defect order $|V|\delta(b)$. Write e_B and e_c for the corresponding block idempotents in the

residue-field group algebras. For a p-subgroup $P \le L$, the Brauer map $\mathrm{Br}_P : (kL)^P \to kC_L(P)$ retains exactly the coefficients supported on $C_L(P)$. There is a unique block above c. In fact V, being a normal p-subgroup, belongs to every block defect group of L. Hence $\mathrm{Br}_V(e_B) \neq 0$ for every covering block. The nonzero, orthogonal idempotents $\mathrm{Br}_V(e_B)$ sum to e_c, since e_c is supported in $N = C_L(V)$. They are central in kN, and the primitivity of e_c forces uniqueness.

Let P/N be a Sylow p-subgroup of L/N. The invariant block c has a unique covering block d in P, whose defect order is $\delta(c)|P : N|$. Its idempotent, viewed in kL, is $e_B = e_c$. The Brauer map at a defect group of d deletes the same basis coefficients in kP and in kL, so $\delta(B) \ge \delta(d)$. The normal-block intersection theorem gives the reverse inequality. This proves (3.4).

For $\lambda \in \mathrm{Irr}(V)/H$, ordinary Clifford theory identifies the characters above its orbit and b with the characters of H_λ above b. The canonical extension of λ is trivial on K. By (2.2),

$$k(B) = \sum_{\lambda \in \mathrm{Irr}(V)/H} |\mathrm{Irr}(H_\lambda \mid b)| \le k(b) \sum_{\lambda \in \mathrm{Irr}(V)/(H/K)} k((H/K)_\lambda).$$

The last sum is $k(V \rtimes (H/K))$ by Clifford theory. All projective factor sets in the first count have been bounded using (2.1). □

Corollary 3.6. *Assume (2.6). If B is a 2-block of a group G with $O_2(G) \neq 1$ and $k(B) > \delta(B)$, then a block of strictly smaller defect order also violates Brauer's inequality. In particular a counterexample of minimum defect order has $O_2(G) = 1$.*

Proof. Apply Proposition 3.4 to $O_2(G)$, then Proposition 3.5 to the resulting block. Equation (2.6) gives

$$\frac{k(B)}{\delta(B)} \le \frac{k(B^{\mathrm{lin}})}{\delta(B^{\mathrm{lin}})} \le \frac{k(b)}{\delta(b)}.$$

The exact defect formula gives $\delta(b) = \delta(B)/(|V||H : K|_2) < \delta(B)$. □

4. Integral inner action and relative character fibers

Let $X \lhd G$ and let b be a G-invariant block of X. Define

$$G[b] = \{g \in G : \text{conjugation by } g \text{ on } \mathcal{O}Xb \text{ is inner}\}. \tag{4.1}$$

This is a normal subgroup of G containing X. It is Dade's ramification subgroup in the integral formulation of [54, Lemma 2.1]. Here innerness is implemented by a unit of $\mathcal{O}Xb$; fixing ordinary characters concerns only the generic algebra.

Lemma 4.1 (Central specialization)**.** *Suppose $G[b] = G$, and let c be a block of G covering b. For every $\theta \in \mathrm{Irr}(b)$, the set $\mathrm{Irr}(c \mid \theta)$ is a single θ-block in the sense of [59], and is nonempty.*

Proof. Put $A = \mathcal{O}Xb$, $R = Z(A)$ and $H = G/X$. The commutative ring R is finite over $\mathcal{O}$, complete and local, with residue field k. Choose representatives g_h and units $u_h \in A^\times$ implementing their conjugation on A, with $g_1 = 1$ and $u_1 = b$. The homogeneous units $v_h = u_h^{-1} g_h b$ centralize A, and

$$v_h v_j = \gamma(h,j) v_{hj}, \qquad \gamma(h,j) \in R^\times.$$

Associativity makes γ a normalized 2-cocycle, with trivial H-action on R. Coefficientwise in the homogeneous decomposition,

$$\mathcal{O}Gb = \bigoplus_{h \in H} A v_h, \qquad C_{\mathcal{O}Gb}(A) = R_\gamma H, \qquad Z(\mathcal{O}Gb) = Z(R_\gamma H). \tag{4.2}$$

The last two identities follow by comparing coefficients in the homogeneous decomposition, since each v_h centralizes A.

Let $\mathfrak{m}$ be the maximal ideal of R and put $C = R_\gamma H$. Central primitive idempotents of C and of $C/\mathfrak{m}C$ correspond. By completeness, lift an idempotent modulo $\mathfrak{m}C$. If its image is central, then

$$eC(1-e) \subseteq \mathfrak{m}C, \qquad eC(1-e) = \mathfrak{m}\, eC(1-e).$$

Nakayama's lemma gives $eC(1-e) = 0$, and similarly $(1-e)Ce = 0$. Thus the lift is central. Central lifts are unique and preserve primitivity.

For $\theta \in \mathrm{Irr}(b)$, its central character $\omega_\theta : R \to \mathcal{O}$ reduces to the residue map of R. Specializing gives

$$C_\theta = \mathcal{O} \otimes_{R,\omega_\theta} C = \mathcal{O}_{\gamma_\theta} H, \qquad \gamma_\theta = \omega_\theta \gamma.$$

The image c_θ of c has the same nonzero central primitive reduction for every θ, and is consequently a primitive idempotent of this twisted order. Over K, the θ-part of A is a matrix algebra, so the θ-part of KGc is

$$\mathrm{Mat}_{\theta(1)}(K) \otimes_K K_{\gamma_\theta} H c_\theta.$$

Thus $\mathrm{Irr}(c \mid \theta)$ is the irreducible-character set of one twisted block.

We explain its agreement with the character-triple definition of a θ-block, including 2-primary factor sets. Replace γ_θ by a cohomologous cocycle with values in a finite cyclic group of roots of unity. The cochain can be taken unit-valued: the valuations of a cochain relating two unit-valued cocycles form a homomorphism from the finite group H to $\mathbb{Q}$, and hence vanish. The resulting central representation group $\widehat{H}$ realizes the twisted order as the central-character quotient of $\mathcal{O}\widehat{H}$. Its odd central part is selected by an integral central idempotent. For its central 2-part, reduction imposes $z = 1$ and the usual central-2 quotient preserves block idempotents. The single twisted block above is therefore precisely the fixed central-character fiber of one block of $\widehat{H}$. This is the definition of a θ-block; its independence of the representation group is [59, Theorem 4.3]. This also proves nonemptiness. The same compatibility is given by [49, Proposition 5.2] with the external coprime group trivial. □

Lemma 4.2 (The relative fiber bound)**.** *Under the hypotheses of Lemma 4.1, assume that G/X is 2-solvable. Then, for every $\theta \in \mathrm{Irr}(b)$,*

$$|\,\mathrm{Irr}(c \mid \theta)| \le \frac{\delta(c)}{\delta(b)}. \tag{4.3}$$

Proof. We recall the precise defect input from relative block theory. For a θ-block contained in a block c with defect group D, its θ-defect group D_θ/X, defined by the central representation group, is conjugate into DX/X [59, Theorem 5.1(a)]. Since $D \cap X$ is a defect group of b, this gives

$$|D_\theta/X| \le |DX/X| = \frac{\delta(c)}{\delta(b)}.$$

The representation group of G/X is 2-solvable. Remove its central 2-subgroup. Lemma 3.1 bounds the selected central-character fiber by the entire corresponding quotient block, and Brauer's inequality for 2-solvable groups applies to that block. This established case follows from the coprime $k(GV)$ theorem [30]. Selecting the prescribed character of the remaining odd central subgroup does not change the defect order. Equivalently, this is the proof of [59, Theorem 7.1] restricted to 2-solvable representation groups. It bounds the size of this single θ-block by $|D_\theta/X|$. Now apply Lemma 4.1. □

Lemma 4.3 (Full relative defect)**.** *Let $M = G[b]$, and let c be a G-invariant block of M above b. There is a unique block B of G covering c, and*

$$\delta(B) = \delta(c)|G : M|_2. \tag{4.4}$$

Proof. The uniqueness is [54, Theorem 3.5(i)]; its hypothesis is the whole subgroup $G[b]$. Hence $e_B = e_c$. Choose a Sylow 2-preimage P of G/M. The unique block of P above c has defect order $\delta(c)|P : M|$. Its idempotent is again e_c. The Brauer maps at its defect group delete the same coefficients in kP and in kG. Thus a defect group of B has at least that order. The normal-block intersection theorem bounds its order in the other direction by $\delta(c)|G : M|_2$, proving equality. □

Theorem 4.4 (A fixed-point criterion)**.** *Let $X \lhd G$, let b be a G-invariant 2-block of X, and put $M = G[b]$. Suppose M/X is 2-solvable. Assume that $Q = G/M$ has an $\mathbb{F}_2Q$-module W satisfying*

$$|W| \le \delta(b), \tag{4.5}$$

$$|\operatorname{Irr}(b)^J| \le |W^J| \quad \text{for every two-generated abelian } J \le Q, \tag{4.6}$$

$$\ker(Q \to \mathrm{GL}(W)) \text{ is a 2-group.} \tag{4.7}$$

If (2.6) holds, then every block B of G covering b satisfies $k(B) \le \delta(B)$.

Proof. Choose a block c of M below B and pass to its inertia group by Fong–Reynolds. This preserves $k(B)$ and its defect order. It also preserves the inner subgroup: if $I = G_c$, then $I[b] = I \cap G[b] = M$, since $M \le I$. Restrict W to I/M; all three hypotheses remain valid. Relabel I as G.

Now c is G-invariant. Since M acts innerly on $\mathcal{O}Xb$, each $\theta \in \operatorname{Irr}(b)$ is M-invariant. Restriction consequently gives a Q-equivariant map

$$r : \operatorname{Irr}(c) \longrightarrow \operatorname{Irr}(b),$$

sending a character to its unique irreducible constituent on X. By Lemma 4.2 every fiber has size at most $L = \delta(c)/\delta(b)$. Therefore

$$|\operatorname{Irr}(c)^J| \le L|\operatorname{Irr}(b)^J| \le L|W^J|$$

for the subgroups J occurring in (2.4). Clifford theory, with (2.1) for the inertia factor sets, gives

$$k(B) \le \kappa_Q(\operatorname{Irr}(c)) \le L\kappa_Q(W) \le L|W|\,|Q|_2.$$

The last inequality is Lemma 2.3. Equations (4.5) and (4.4) finish the calculation:

$$k(B) \le \delta(c)|Q|_2 = \delta(B).$$

□

4.1. **Tensor factors and component permutations.** We shall use the stronger form of (4.6) with all subgroups J. The following elementary algebra fact ensures that component modules descend through the group $G[b]$.

Lemma 4.5 (Innerness on a tensor factor)**.** *Let A and B be finite free $\mathcal{O}$-algebras with $Z(A)$ local. If a unit of $A \otimes_{\mathcal{O}} B$ induces an automorphism α on $A \otimes 1$, then α is inner on A.*

Proof. Write $u = \sum_i a_i \otimes b_i$ and $u^{-1} = \sum_j c_j \otimes d_j$ using bases in the second tensor factor. The identity $u(a \otimes 1) = (\alpha(a) \otimes 1)u$ gives $a_i a = \alpha(a)a_i$, and the inverse identity gives $c_j\alpha(a) = ac_j$. Hence $a_ic_j \in Z(A)$. Choose an $\mathcal{O}$-linear map $f : B \to \mathcal{O}$ with $f(1) = 1$; this is possible because 1 is primitive in the free lattice B. Apply $1 \otimes f$ to $uu^{-1} = 1$ to obtain

$$\sum_{i,j} f(b_id_j)a_ic_j = 1.$$

Since $Z(A)$ is local, some a_ic_j is a unit. Thus a_i has a right inverse in A, and is a unit: this follows after scalar extension to the finite-dimensional K-algebra, and its displayed right inverse is integral. The equation for a_i now shows $\alpha(a) = a_iaa_i^{-1}$. □

Proposition 4.6 (Assembly of component modules)**.** *Suppose $O_2(G) = 1$, $F(G) = Z(G)$, and $C_G(E(G)) = Z(G)$. Let $X = Z(G)E(G)$, and let b be a G-invariant block of X. Assume its component blocks b_i on the quasisimple components L_i have positive defect and at least two ordinary characters. Let A_i be the image in $\mathrm{Aut}(L_i)$ of the stabilizer of L_i in G, and let T_i be the subgroup acting innerly on $\mathcal{O}L_ib_i$. Suppose an $\mathbb{F}_2(A_i/T_i)$-module W_i is given with*

$$|W_i| = \delta(b_i), \quad |\operatorname{Irr}(b_i)^J| \le |W_i^J| \ (J \le A_i/T_i), \quad \ker(A_i/T_i \to \mathrm{GL}(W_i)) \text{ a 2-group.} \tag{4.8}$$

Then $M = G[b]$ has solvable quotient M/X, and $Q = G/M$ has a module W satisfying all three conditions of Theorem 4.4, with equality in (4.5) and with all subgroup fixed-point numbers.

Proof. The intersections between the components and with $Z(G)$ have odd order. In the central-character sector selected by b, the central product block is canonically the tensor product of the component block orders. Indeed the external tensor product descends through the central kernel, which acts trivially because its prescribed linear characters match. The additional central factor $Z(G)$ acts by its chosen scalar character. Consequently

$$\operatorname{Irr}(b) = \prod_i \operatorname{Irr}(b_i), \qquad \delta(b) = \prod_i \delta(b_i). \tag{4.9}$$

These identifications are in the central-character sector selected by b.

An element of M fixes every character in (4.9). It cannot permute component positions: each factor contains at least two choices, so moving one position moves a suitable tuple. Lemma 4.5 then shows that its automorphism on each component block is inner. The image of the stabilizer Q_i of a component in Q therefore acts on W_i through A_i/T_i. For one representative of each component orbit define

$$W = \bigoplus_{i \in \{\text{component orbits}\}} \operatorname{Ind}_{Q_i}^{Q} W_i. \tag{4.10}$$

Its cardinality is the product in (4.9).

Fix $J \le Q$. On each orbit of J on the component positions, a fixed character tuple is determined by one character fixed by the corresponding position stabilizer, including all return automorphisms. A fixed vector in (4.10) is determined in exactly the same way by a vector in the relevant W_i. Multiplying the local inequalities in (4.8) over the position orbits proves $|\operatorname{Irr}(b)^J| \le |W^J|$.

Let R be the kernel on W. If R had an element q of odd prime order ℓ, choose an ℓ-power-order lift $g \in G$ of q, using a Sylow subgroup of the preimage of $\langle q \rangle$. The induced module has nonzero summands at all component positions, so q fixes those positions. On every component, the image of g in A_i/T_i has ℓ-power order and lies in the local 2-kernel. It is trivial. Choose integral units implementing the component actions; their tensor product implements the action on $\mathcal{O}Xb$, including its scalar central factor. Thus $g \in M$, contrary to $q \ne 1$. Cauchy's theorem proves that R is a 2-group.

Finally, M fixes the component positions and $C_G(E(G)) = Z(G) \le X$. The quotient M/X therefore embeds in a product of the component outer automorphism groups. These are solvable by the Schreier theorem. This proves the remaining claim. □

5. A weighted permutation inequality

For a permutation g of a finite set Ω, write $s_\Omega(g) = |\Omega| - |\operatorname{Fix}_\Omega(g)|$, and write $g = g_2 g_o = g_o g_2$ for its commuting 2-part and odd part. Throughout this section $t = 2^{-1/2}$. If f is a nonnegative class function on the odd-order elements of a finite group G, set

$$\mathcal{P}(G; f) = \frac{1}{|G|} \sum_{\substack{a \in G \text{ odd} \\ b \in C_G(a)}} f(a) f(b_o), \qquad \mathcal{P}(G, \Omega) = \mathcal{P}(G; t^{s_\Omega}).$$

Here and below "odd" includes the identity. The assertion needed for the affine argument is the following.

Theorem 5.1. *For every faithful permutation action of a finite group G on a finite set Ω,*

$$\frac{1}{|G|}\sum_{\substack{a\in G\ \text{odd}\\ b\in C_G(a)}} 2^{-(s_\Omega(a)+s_\Omega(b_o))/2}\le 1. \tag{5.1}$$

The same assertion holds with $2^{-1/2}$ replaced by any real number in $(0,2^{-1/2}]$.

The almost simple input is stated in Proposition 5.7.

5.1. **Normal abelian subgroups.**

Lemma 5.2. *Let $L\trianglelefteq G$ be a 2-group. For an odd element $\bar a\in G/L$, let $\bar f(\bar a)=f(a)$, where a is any odd lift. This is well defined, and*

$$\mathcal{P}(G;f)=\mathcal{P}(G/L;\bar f). \tag{5.2}$$

Proof. Schur–Zassenhaus, applied to the inverse image of the cyclic group $\langle\bar a\rangle$, shows that all odd lifts of $\bar a$ are L-conjugate. Indeed their cyclic subgroups are complements, and a complement contains exactly one element over $\bar a$. Thus odd conjugacy classes correspond under the quotient map. Moreover $C_G(a)\longrightarrow C_{G/L}(\bar a)$ is onto: if a lift h centralizes $\bar a$, conjugate a^h back to a by an element of L. Every fibre has size $|C_L(a)|$. Quotients commute with taking odd parts. Consequently

$$\mathcal{P}(G;f)=\sum_{[a]\ \text{odd}}\frac{f(a)}{|C_G(a)|}\sum_{b\in C_G(a)}f(b_o)=\sum_{[\bar a]\ \text{odd}}\frac{\bar f(\bar a)}{|C_{G/L}(\bar a)|}\sum_{\bar b\in C_{G/L}(\bar a)}\bar f(\bar b_o).$$

□

Lemma 5.3. *If G has a nontrivial abelian normal subgroup, a faithful action (G,Ω) has a quotient E with a faithful action on a strictly smaller set Ω' such that $\mathcal{P}(G,\Omega)\le\mathcal{P}(E,\Omega')$.*

Proof. It suffices to treat an abelian normal r-subgroup for a prime r. First suppose $r=2$, and denote this subgroup by K. Its orbits inside each G-orbit are blocks of a common size b. On such a block the faithful image K_0 is regular abelian. The induced block stabilizer is $K_0\rtimes A_0$, where A_0 is the stabilizer of an origin and acts faithfully on K_0. Choose transports between the blocks; these embed the action into the corresponding imprimitive wreath product. For $b>1$, discard the translation base and replace each K_0-fibre by $K_0\setminus\{1\}$, with its A_0-action. For $b=1$, retain the original points and their action. The kernel L of the resulting homomorphism is a subgroup of the product of the translation bases. Thus L is abelian, normal and a 2-group, and $K\le L$. The action of G/L is faithful by construction and its degree decreases.

If an odd element moves a block, its old and new contributions to support are b and $b-1$. On a fixed block, its odd affine map is translation-conjugate to its linear part, again by Schur–Zassenhaus. Deleting the origin decreases the degree and the number of fixed points by the same amount. Its support is unchanged. The result now follows from Lemma 5.2 and positivity.

Suppose next that K has odd order. We supply the weighted estimate required for this case. For every permutation subgroup $D\le\mathrm{Sym}(\Omega)$ and every permutation $h\in\mathrm{Sym}(\Omega)$, where $|\Omega|=n$,

$$\sum_{x\in Dh}t^{s(x)}\le\sum_{x\in D}t^{s(x)}. \tag{5.3}$$

To see this, expand

$$t^{s(x)}=\sum_{U\subseteq\Omega}t^{n-|U|}(1-t)^{|U|}\,1_{\{x\ \text{fixes}\ U\ \text{pointwise}\}}.$$

The intersection of a coset with a pointwise stabilizer is empty or a coset of the corresponding subgroup intersection. All coefficients are nonnegative. If D is odd abelian and faithful, then

$$\sum_{x\in D} t^{s(x)} \le \sqrt{|D|}. \tag{5.4}$$

For a transitive action its faithful image is regular of odd order q, and $1+(q-1)t^q \le \sqrt{q}$. At $q=3$, the square of the left side is $3/2+\sqrt{2}<3$; for odd $q\ge 5$, $(q-1)2^{-q/2}$ decreases in steps of two and is at most $1/\sqrt{2}$. For several orbits, project to one orbit and apply (5.3) to each coset of the projection kernel on the remaining orbits. Induction gives (5.4), retaining all relations between orbits.

Write the K-orbits as B_i, identify B_i with its regular odd abelian image K_i, and put $q_i=|K_i|$. Let L consist of the elements of G which fix every fibre and act as translations on each. It is normal abelian of odd order and contains K. The quotient G/L acts by linear isomorphisms between the K_i. Replace a nontrivial fibre by

$$Y_i=(K_i\setminus\{0\})/(v\sim -v), \qquad |Y_i|=(q_i-1)/2,$$

and retain every singleton fibre. An automorphism of an odd abelian group fixing all pairs $\{v,-v\}$ is either 1 or -1: if it has opposite signs on nonzero v,w, its value on $v+w$ cannot have either allowed sign, since there is no 2-torsion. Thus the kernel R of this action of G/L is an elementary abelian 2-group. Set $E=(G/L)/R$. The new degree is strictly smaller, and its action is faithful. For odd $\bar g\in G/L$ write $w(\bar g)=t^{s_{\Omega'}(\bar g)}$.

We claim that for every subgroup $D\le L$ and every odd $g\in G$,

$$\sum_{x\in Dg} t^{s_\Omega(x)} \le \sqrt{|D|}\, w(gL). \tag{5.5}$$

The subgroup D need not be normalized by g. Let F,M be the fixed and moved fibres of g. On a fixed fibre let u_i be the linear part, $f_i=|C_{K_i}(u_i)|$, and $M_i=(1-u_i)K_i$, of order q_i/f_i. The fixed-point equation for a translation times g is solvable exactly when the translation lies in a specified coset c_i+M_i; then it has f_i solutions. Its weight on this fibre is therefore

$$t^{q_i}+(t^{q_i-f_i}-t^{q_i})1_{c_i+M_i}.$$

Expanding the product and using nonnegativity allows every c_i to be replaced by zero: each intersection with D is empty or a coset of a subgroup intersection. Let D_0 be the kernel of $D\to\prod_{i\in F}K_i/M_i$. After division by $t^{q_i-f_i}$, the fixed-fibre weight is 1 at the zero coset of K_i/M_i and t^{f_i} elsewhere. This is precisely the support weight for its regular action of degree f_i. Apply (5.4) to the image of D on the disjoint union of these quotient sets. Each image element has $|D_0|$ preimages, giving

$$\sum_{x\in Dg} t^{s(x)} \le \sqrt{|D|}\sqrt{|D_0|}t^{\sum_{i\in M}q_i+\sum_{i\in F}(q_i-f_i)}, \qquad |D_0|\le \prod_{i\in M} q_i \prod_{i\in F}(q_i/f_i).$$

For a moved nontrivial fibre, $\sqrt{q}\,t^q \le t^{(q-1)/2}$, equivalently $q\le 2^{(q+1)/2}$ for odd $q\ge 3$. For a fixed fibre,

$$\sqrt{q/f}\,t^{q-f}\le t^{(q-f)/2} \quad\Longleftrightarrow\quad q/f\le 2^{(q-f)/2}.$$

Writing $q=fr$, this is immediate if $r=1$ or $f\ge 3$. If $f=1$, it holds for odd $q\ge 7$; the apparent exceptions $q=3,5$ cannot occur, since an odd automorphism of their abelian groups is trivial. An odd automorphism fixes a pair $\{v,-v\}$ precisely when it fixes v. Hence the new fixed-fibre support is $(q-f)/2$. A moved singleton contributes one to each support. Multiplication proves (5.5).

For a finite group C and a class function f on its odd elements, put

$$Q(C;f)=\sum_{\substack{a,u\in C\text{ odd}\\ [a,u]=1}} f(a)f(u).$$

The unique decomposition $b=zu$, with z a 2-element, gives

$$\mathcal{P}(G;f)=\sum_{[z]\text{ a 2-element}}\frac{Q(C_G(z);f)}{|C_G(z)|}. \tag{5.6}$$

Because L is odd, Schur–Zassenhaus gives a bijection on 2-element classes under $G\to G/L$, and $C_G(z)\to C_{G/L}(zL)$ is onto, with kernel $D=C_L(z)$. Every odd coset in this centralizer consists of odd elements. For a commuting odd pair in the quotient, omit the additional commutation condition on its lifts and apply (5.5) twice. It follows that

$$Q(C_G(z);t^{s_\Omega})\le |D|Q(C_{G/L}(zL);w).$$

The factor $|D|$ cancels from the centralizer denominator in (5.6). Summing, and finally applying Lemma 5.2 to R, proves the claim. □

5.2. **Matrices for commuting pairs in prescribed cosets.** For $N\trianglelefteq H$ and commuting $\alpha,\beta\in H/N$, with α odd, define

$$U_N(\alpha,\beta;f)=\frac{1}{|N|}\sum_{\substack{aN=\alpha,\ a\text{ odd}\\ bN=\beta,\ [a,b]=1}} f(a)f(b_o). \tag{5.7}$$

Only N-conjugacy invariance of f is required.

Lemma 5.4. *If $\alpha=\beta_o=1$, let I be the β-stable odd N-classes, with identity class 0. There is a symmetric nonnegative matrix K, indexed by I, whose row sums are one and such that*

$$U_N(1,\beta;f)=f^{\mathsf{T}}Kf,\qquad K_{0i}\ge |C_N(x_i)|^{-1}.$$

Here x_i represents the class i. In general there is a rectangular nonnegative matrix on the odd N-classes in α and β_o, with every row and column sum in $\{0,1\}$, and with $U_N(\alpha,\beta;f)=f_\alpha^{\mathsf{T}}Kf_{\beta_o}$, where the two vectors restrict f to the row and column class sets. The number of nonzero rows equals the number of nonzero columns and is at most $k(N)$.

Proof. Write $\beta=\beta_2\beta_o$. In both cases set

$$K_{ij}=\frac{1}{|N|}\sum_{\substack{a\in i,\ u\in j\\ [a,u]=1}}|\{z\in H: zN=\beta_2, z\text{ a 2-element}, [z,a]=[z,u]=1\}|.$$

The substitution $b=zu$ is the unique 2/odd decomposition. For fixed a, the relevant b form either the empty set or a coset of $C_N(a)$, proving that its row sum is zero or one. For columns use $c=za$ instead. The image of c is $\beta_2\alpha$; these commuting factors have coprime orders, so the 2-part and odd part of c recover z and a in their prescribed cosets. The same centralizer argument proves the column assertion. Thus the two active counts agree.

For any x normalizing N, the number of N-classes in Nx equals the number of N-classes in N fixed by x. Indeed, Burnside's formula for conjugation on Nx sums $|C_{Nx}(n)|$ over $n\in N$; an x-stable N-class contributes exactly $|N|$ to that sum and an unstable class contributes zero. This proves the bound $k(N)$.

When both odd quotient labels are trivial, interchanging a,u makes K symmetric. The stable classes are exactly its nonzero rows. Moreover

$$K_{0i}=\frac{|\{z\in C_H(x_i): zN=\beta, z\text{ a 2-element}\}|}{|C_N(x_i)|}.$$

The numerator is positive: take the 2-part of any element of the nonempty centralizer coset. This proves all the assertions. □

Corollary 5.5. *Let $\beta \in H/N$ be a 2-element. Suppose $f(1) = 1$ and $0 \le f \le 1$. If*

$$(\sqrt{|C_N(x)|} + 1) f(x) \le 1 \tag{5.8}$$

for every nonidentity odd $x \in N$ whose N-class is β-stable, then $U_N(1, \beta; f) \le 1$. For general commuting labels $\alpha, \beta \in H/N$ with α odd, if all nonidentity odd weights are at most $\epsilon \ge 0$ and R is the active row count, then

$$U_N(\alpha, \beta; f) \le \begin{cases} R\epsilon^2, & \alpha \ne 1, \beta_o \ne 1, \\ \epsilon + (R-1)\epsilon^2, & \text{exactly one of } \alpha, \beta_o \ne 1. \end{cases} \tag{5.9}$$

In particular, for $q \ge k(N)$ and $d \ge 0$, the conditions

$$\epsilon \le (\sqrt{q} + 1)^{-1}, \qquad \sqrt{q}\,\epsilon \le t^d \tag{5.10}$$

make these two bounds at most t^{2d} and t^d, respectively.

Proof. The symmetric stochastic matrix satisfies the exact energy identity

$$f^{\mathsf{T}} K f = \sum_i f_i^2 - \frac{1}{2} \sum_{i,j} K_{ij} (f_i - f_j)^2.$$

Keeping the terms incident with the identity bounds this by $1 + \sum_{i \ne 0} (f_i^2 - K_{0i}(1 - f_i)^2)$. Each summand is nonpositive by (5.8). For the rectangular assertion use Lemma 5.4. If just one odd label is trivial, its identity row or column has mass exactly one; all the remaining mass is at most $R - 1$. Finally $1 + (q-1)\epsilon \le \sqrt{q}$ when $\epsilon \le (\sqrt{q} + 1)^{-1}$, which gives the last assertion. □

These estimates persist with t replaced by t^c, c any positive integer. In particular, if $\sqrt{q}\,\epsilon \le t^d$, then $\sqrt{q}\,\epsilon^c \le (\sqrt{q}\,\epsilon)^c \le t^{dc}$. The weights are functions on odd classes, including when an even return automorphism interchanges coordinate representations.

5.3. **Natural alternating groups.**

Lemma 5.6. *Let $A = A_m$, $m \ge 2$, in its natural action. For either sign $\varepsilon \in \{0, 1\}$ and $0 < u \le t$,*

$$\frac{1}{|A|} \sum_{\substack{x \in A \text{ odd}, \ y \in S_m \\ \operatorname{sgn}(y) = (-1)^\varepsilon, [x,y] = 1}} u^{s(x) + s(y_o)} \le 1. \tag{5.11}$$

For $m \ge 5$, the analogous bound $U_{A_m}(1, \beta; u^s) \le 1$ holds whenever $\beta \in \mathrm{Aut}(A_m)/A_m$ has 2-power order, including the identity. It holds for each natural representation, including either natural six-point representation of A_6.

Proof. We first take $u = t$. Put $P_m = \mathcal{P}(S_m, \{1, \dots, m\})$ and $P_0 = 1$. If the odd permutation x has r cycles of an odd length a, its centralizer on these cycles is $C_a \wr S_r$. A b-cycle of the permutation of these a-cycles has a total rotation $j \in \mathbb{Z}/a\mathbb{Z}$; there are a^{b-1} rotations with that total. On these ab points the second permutation has $\gcd(a, j)$ cycles of length $ab/\gcd(a, j)$. Its odd part fixes all of them exactly when b is a power of two and $j = 0$. Otherwise its odd part moves all ab points. The first permutation contributes weight 1 if $a = 1$, and t^{ab} if $a > 1$.

The cycle enumeration, including the averaging over rotations, therefore gives the following identity of formal power series:

$$F(z) := \sum_{m \ge 0} P_m z^m = \exp\left(\sum_{n \ge 1} \frac{h_n z^n}{n} \right), \quad h_n = \begin{cases} 1, & n \text{ a power of 2}, \\ 2t^n + (\sigma_1(n_{2'}) - 2)t^{2n}, & \text{otherwise}. \end{cases} \tag{5.12}$$

Here $\sigma_1(n) = \sum_{d|n} d$ is the divisor sum. To verify the coefficient explicitly before collection, put $u_1 = 1$ and $u_a = t^a$ for odd $a > 1$. The logarithm is

$$\sum_{a\geq 1 \text{ odd}} \sum_{b\geq 1} \frac{u_a^b z^{ab}}{ab} \sum_{j=0}^{a-1} v(a,b,j), \qquad v(a,b,j) = \begin{cases} 1, & b \text{ a 2-power}, j = 0, \\ t^{ab}, & \text{otherwise.} \end{cases}$$

For a non-2-power n, the divisor $a = 1$ contributes t^n, the divisor $a = n_{2'}$ contributes $t^n + (n_{2'} - 1)t^{2n}$, and each other odd divisor $a > 1$ contributes at^{2n}, proving (5.12).

We have $0 \leq h_n \leq 1$. The small non-2-powers give $h_3 = 1/\sqrt{2} + 1/4 < 1$ and $h_5 = 1/(2\sqrt{2}) + 1/8 < 1$. For $n \geq 6$, the bound $\sigma_1(n_{2'}) \leq n^2$ gives $h_n \leq 2^{1-n/2} + n^2 2^{-n} \leq 13/16$; both terms decrease thereafter. In particular $P_m \leq 1$ coefficientwise by comparison with $\exp(\sum z^n/n) = (1-z)^{-1}$.

To distinguish the two sign cosets, insert $\operatorname{sgn}(y)$ in the same enumeration. On the component above, this sign is $(-1)^{ab-\gcd(a,j)} = (-1)^{b-1} = (-1)^{ab-1}$. The signed series is therefore $R(z) = 1/F(-z)$. Let $J = F + R$ and $K = F - R$. Write $J(z) = \sum_{n\geq 0} J_n z^n$ and $K(z) = \sum_{n\geq 0} K_n z^n$. Their coefficients are nonnegative, and, for $m \geq 2$, are precisely the two expressions in (5.11). Write $H_{\text{odd}} = \sum_{n \text{ odd}} h_n z^{n-1}$ and $H_{\text{even}} = \sum_{n \text{ even}} h_n z^{n-1}$. Then

$$J' = H_{\text{odd}} J + H_{\text{even}} K, \qquad K' = H_{\text{odd}} K + H_{\text{even}} J.$$

The initial values are $J_0 = J_1 = 2$, $K_0 = K_1 = 0$, and $J_2 = K_2 = 1$. Inductively, in the coefficient recurrence for nJ_n or nK_n, the terms with $1 \leq j \leq n-2$ total at most $n-2$. Of the last two terms exactly one has a nonzero initial coefficient, equal to two. Thus $J_n, K_n \leq 1$ for every $n \geq 2$. Nonnegativity proves the result for every $u \leq t$.

In applications a return automorphism, or holonomy, is the composite of coordinate identifications around a closed coordinate path. For $m \neq 6$, $\operatorname{Aut}(A_m) = S_m$ for $m \geq 5$, so this proves the last assertion except at six. For A_6 the nonidentity odd classes have the following data in either chosen natural action:

class	$3A$	$3B$	$5A$	$5B$
support	3	6	5	5
$\lvert C_{A_6}(x)\rvert$	9	9	5	5.

The subgroup induced by S_6 is the stabilizer in $\operatorname{Aut}(A_6)$ of the support-three class and has index two. Every automorphism outside it interchanges $3A$ and $3B$. These finite facts can also be obtained by the explicit permutation calculation given below. If the even holonomy is induced by S_6, use the sign-coset result. Otherwise neither order-three class is an index of the symmetric matrix in Lemma 5.4. Every remaining nonidentity index has support five and centralizer order five, and

$$(\sqrt{5}+1)2^{-5/2} < 1.$$

Corollary 5.5 applies. This proof uses only the stable classes and therefore also applies when the even holonomy interchanges the two coordinate representation types. □

5.4. **The almost simple local estimates.** For each isomorphism type of nonabelian simple group S, fix a faithful nonempty $\operatorname{Out}(S)$-set Ψ_S. We use its regular set, with one point for the trivial outer group, except that for $S = \mathrm{PSL}_3(4)$ we use the faithful action of $\operatorname{Out}(S) \cong C_2 \times S_3 \cong D_{12}$, the dihedral group of order twelve, on the six vertices of a hexagon. Write $d_S = |\Psi_S|$. This choice is made once for each abstract S, including occurrences in different permutation degrees.

Proposition 5.7. *Let $S \leq A \leq S_m$ be primitive almost simple, with $A = N_{S_m}(S)$. Suppose the action is either natural alternating or is outside the alternating subset actions, allowing abstract twists. Identify A with its conjugation image in $\operatorname{Aut}(S)$. The following statements hold.*

(i) *$d_S < m$, and the support of every odd element of A is at least the support of its image on Ψ_S.*

(ii) *$k(S)t^{3(m-d_S)} \le 1$.*

(iii) *Let $S \trianglelefteq H \le \mathrm{Aut}(S)$, and suppose every odd-order element of H belongs to A. For commuting $\alpha, \beta \in H/S$ with α odd, and every integer $c \ge 1$, put $f(x) = t^{cs(x)}$ on the odd elements of H, using their coordinate action in A. Then*

$$U_S(\alpha, \beta; f) \le t^{c(s_{\Psi_S}(\alpha)+s_{\Psi_S}(\beta_o))}.$$

Here U_S is the prescribed-coset mass in (5.7), and H/S acts through $\mathrm{Out}(S)$ on Ψ_S. The hypothesis permits even return automorphisms that interchange coordinate representation types.

Proof. For a natural A_m, the outer group is a 2-group, so its odd supports are zero, and (iii) is Lemma 5.6. Its outer degree is two, except for A_6, where it is four. The general permutation class bound of Garonzi–Maróti [28, Theorem 1.1] is

$$k(U) \le 5^{(n-1)/3} < 2^{n-1} \qquad (U \le S_n,\ n \ge 4). \tag{5.13}$$

Since $3(m-2)/2 \ge m-1$ for $m \ge 5$, this proves (ii) except at six, where $k(A_6) = 7 \le 8 = 2^{3(6-4)/2}$.

Suppose $m \ge 128$, and put $x = \log_2 m$, $a = |\mathrm{Out}(S)|$ and $q = k(S)$. The following four published bounds are used with the indicated exceptions:

$$\begin{aligned} &s(u) \ge m/2 \qquad (1 \ne u \in A \text{ odd}), \\ &|S| < m^{1+\log_2 m}, \qquad a \le 3\log_2 m, \qquad q \le m^6. \end{aligned} \tag{5.14}$$

The first follows from Burness–Guralnick [14, Corollary 2]: outside the alternating subset/product family, an element of odd prime order r has fixed-point ratio at most $1/\sqrt{r+1} \le 1/2$. Take a nonidentity prime-order power of u. Maróti's order theorem [52, Theorem 1.1] gives the second bound. Its alternating product alternative reduces here to a subset action, and its Mathieu exceptions have degree at most 24. The bound on a is Guralnick–Maróti–Pyber [35, Lemma 7.7], for any nontrivial simple-group permutation representation. For the last bound, apply Garzoni–Gill [29, Theorem 1.2 and Table 1] to the primitive ambient A. Outside the subset family, their projective exception has $k(A) < 100m$, and every entry of their finite table satisfies this same bound; all other cases have $k(A) < m/2$. An A-class in S splits into at most $[A : S]$ S-classes. Hence

$$k(S) \le [A : S]k(A) < 100am \le 300m\log_2 m \le m^6.$$

These bounds imply the three local inequalities

$$(\sqrt{|S|}+1)t^{m/2} \le 1, \qquad t^{m/2} \le (\sqrt{q}+1)^{-1}, \qquad \sqrt{q}\,t^{m/2} \le t^a. \tag{5.15}$$

For completeness, at $x = 7$ the difference $2^x/4 - (x^2+x)/2$ is four and its derivative $(\log_e 2)2^x/4 - x - 1/2$ is positive thereafter. Thus $2^{m/4} \ge 16m^{(1+\log_2 m)/2}$, paying also the added one in the first inequality. The difference $m/4 - 3\log_2 m$ starts at eleven and increases, proving the second. Finally $m/2 \ge 9\log_2 m \ge a + \log_2 q$: the first comparison is $64 \ge 63$ at 128 and its difference increases. The inequalities remain true under $t \mapsto t^c$. Corollary 5.5 now proves (iii), and $d_S \le a < m$ proves (i). For (ii), one may use (5.13) and $a \le 3\log_2 m \le (m+2)/3$, valid for $m \ge 128$; this gives $3(m - d_S) \ge 2(m-1)$. The smaller outer action for $\mathrm{PSL}_3(4)$ only improves these estimates.

It remains to give the complete finite range $5 \le m < 128$. Its classification input is the primitive group classification [61, 18], in the permutation-isomorphism models of `PrimGrp` [37]. Select every primitive model in this range of almost simple type; omit only the natural A_m, S_m

models already proved. The selection has 292 models. The socle is computed inside the primitive ambient, so the selection includes socles that are imprimitive and the novelty maximal subgroups.

For every nonidentity odd socle class with centralizer order C and support s, the required internal test is the exact integer test

$$A_0 = 2^s - C - 1 \geq 0, \qquad A_0^2 \geq 4C. \tag{5.16}$$

This is equivalent to $(\sqrt{C}+1)2^{-s/2} \leq 1$. For 195 models, the test holds with $C = |S|$ and $s = \lceil m/2 \rceil$, using the first bound of (5.14). All alternating labels are excluded from this shortcut. The remaining 97 models give 887 odd-class records and 267 distinct pairs (C, s), all satisfying (5.16). The $A_5 \cong \mathrm{PSL}_2(5)$ label in degree six belongs to this exact branch. The exhaustive algorithm, complete model incidence, and integer records are given in the finite certificate accompanying this proof; the algorithm and its data format are also displayed in §5.8.

For odd outer labels the entire list in this range is Table 1. Except in its first row, the full primitive normalizer has minimum nonidentity odd-element support at least the displayed $\mu = \lceil m/2 \rceil$. The entries Q are upper bounds on $k(S)$; the first four use $9, 10, 16, 33$, respectively, and the other entries are simply $|S|$. Each row other than the degree-nine exception satisfies the integer tests

$$(2^\mu - Q - 1) \geq 0, \quad (2^\mu - Q - 1)^2 \geq 4Q, \quad 2^\mu \geq 2^{d_S} Q, \quad \mu \geq d_S. \tag{5.17}$$

They imply (5.10) and (i). At higher degrees of the same type the inequalities strengthen. If $\mathrm{Out}(S)$ is a 2-group there are no odd outer labels, and the internal test alone suffices. Every selected model also satisfies $3(m - d_S) \geq 2(m-1)$, so (5.13) proves (ii). The outer orders and four small class numbers can be recovered by the explicit procedures in §5.8.

TABLE 1. The finite almost simple types with odd outer automorphisms. The degree is the least selected degree.

S	m	d_S	μ	$Q \geq k(S)$
$\mathrm{PSL}_2(8)$	9	3	6	9
$\mathrm{PSL}_3(4)$	21	6	11	10
$\mathrm{PSL}_2(27)$	28	6	14	16
$\mathrm{PSL}_2(32)$	33	5	17	33
$\mathrm{PSU}_3(5)$	50	6	25	126000
$\mathrm{PSL}_3(7)$	57	6	29	1876896
$\mathrm{Sz}(8)$	65	3	33	29120
$\mathrm{PSL}_2(64)$	65	6	33	262080
$\mathrm{PSL}_3(8)$	73	6	37	16482816
$\mathrm{P}\,\Omega_8^+(2)$	120	6	60	174182400
$\mathrm{PSL}_2(125)$	126	6	63	976500

For $S = \mathrm{PSL}_2(8)$ in degree nine, the normalizer is $S : 3$, of order 1512. Each nontrivial outer coset has exactly two odd S-classes, with (order, support, class size) equal to $(3, 6, 84)$ and $(9, 9, 168)$. Every nonidentity odd element of the normalizer has support at least six. Thus the active row count in (5.9) is at most two whenever a nontrivial odd outer label is present. For the cth power weight put $e = 2^{-3c} \leq 1/8$. The nonidentity weight in the regular C_3 outer action is $\sqrt{e}$, and

$$e + e^2 \leq \sqrt{e}, \qquad 2e^2 \leq e.$$

These prove (iii) for one or two nontrivial labels. The internal test handles the remaining case. The class calculation is reproducible directly from the fractional linear transformations of $\mathbb{F}_8 \cup \{\infty\}$ and the Frobenius map, as specified below. This proves the proposition. □

5.5. **Smaller actions for the other primitive types.** If two faithful actions of the same group have $s_{\mathrm{new}}(g) \le s_{\mathrm{old}}(g)$ for all odd g, positivity gives $\mathcal{P}(G, \mathrm{old}) \le \mathcal{P}(G, \mathrm{new})$. We record replacements with this precise property.

Lemma 5.8. *Every primitive component of nonaffine, non-almost-simple type admits a faithful action of strictly smaller degree with no greater support on odd elements. The same is true of a non-natural alternating subset action. These replacements can be made inside an arbitrary block action, retaining all relations between its component kernels.*

Proof. First, $\mathrm{Aut}(S)$ has a faithful primitive action of degree at most $|S|/4$ for every nonabelian simple S. To construct one, let P be a Sylow 2-subgroup of S. The odd-order theorem [23] implies that $|S|$ is even. In fact $4 \mid |S|$: otherwise the regular permutation of an involution is odd, and its sign gives a nontrivial homomorphism from the perfect group S to C_2. The action of $\mathrm{Aut}(S)$ on the Sylow 2-subgroups is faithful. Its kernel intersects S trivially and hence centralizes S, whereas $C_{\mathrm{Aut}(S)}(S) = 1$. Let $J = N_{\mathrm{Aut}(S)}(P)$ and take a maximal proper subgroup M containing J. The Frattini argument gives $\mathrm{Aut}(S) = SJ$, so $S \not\le M$. The core of M again intersects S trivially and is trivial. The action on $\mathrm{Aut}(S)/M$ is therefore faithful primitive, of degree

$$d(S) \le [S : N_S(P)] \le |S|/|P| \le |S|/4. \tag{5.18}$$

Use one fixed such action for each abstract simple type.

Suppose a semisimple group $N = \prod_i S_i$ is regular and normal. Choosing an origin puts the full acting group in $\mathrm{Hol}(N) = N \rtimes \mathrm{Aut}(N)$. Here $\mathrm{inn}_b(z) = bzb^{-1}$. Since N is centerless,

$$(b, \varphi) \longmapsto (\mathrm{inn}_b \varphi, \varphi) \quad \text{in } \mathrm{Aut}(N) \times \mathrm{Aut}(N)$$

is injective. Each copy of $\mathrm{Aut}(N)$ acts faithfully on the disjoint union of the sets in (5.18), including its permutations of isomorphic factors. The two copies together have degree

$$2 \sum_i d(S_i) \le \frac{1}{2} \sum_i |S_i| \le \frac{1}{2} \prod_i |S_i| = |N|/2.$$

If an affine permutation $z \mapsto b\varphi(z)$ has a fixed point, its fixed set is a translate of $C_N(\varphi)$. For a nonidentity permutation this is a proper subgroup. A perfect group has no proper subgroup of index at most four, since its nontrivial coset image would be a perfect subgroup of the solvable group S_4. Thus a nonidentity old permutation moves at least $4|N|/5$ points. The new action has smaller degree and no greater support, even for elements of even order.

For a simple diagonal action put $N = S^r$, $r \ge 2$, $\Delta = \{(s, \dots, s) : s \in S\}$, and $\Omega = \Delta\backslash N$. Its degree is $n = |S|^{r-1}$. Centerlessness gives $N_N(\Delta) = \Delta$, so the elementary centralizer formula for a transitive coset action gives $C_{\mathrm{Sym}(\Omega)}(N) = 1$. The full normalizer W therefore embeds in $\mathrm{Aut}(S) \wr S_r$. Its stabilizer of Δ is $\mathrm{Aut}(S) \times S_r$: the coordinate automorphisms preserving Δ must all agree. In particular this normalizer is primitive. For a direct verification, a subgroup of S^r containing Δ is a product of aligned full diagonal subgroups, and so corresponds to a partition of the r factors. The full S_r preserves only the two trivial partitions. This proves maximality of the point stabilizer. The action is outside the alternating subset/product exceptions of [14, Corollary 2]: its stabilizer intersects every individual simple factor trivially, unlike a nontrivial subset stabilizer. Thus every nonidentity odd element of W moves at least $n/2$ points. On the other hand, the faithful imprimitive action induced by (5.18) has degree

$$rd(S) \le r|S|/4 \le |S|^{r-1}/2 = n/2.$$

The last inequality follows from $|S|^{r-2} \geq r/2$. Restriction to any subgroup of W supplies the replacement.

For a product action $B \leq L \wr S_r$ with $r \geq 2$, where L acts faithfully on $m \geq 2$ points, replace its m^r points by the mr points of the imprimitive coordinate action. For an odd element, each top cycle has odd length a. Its product-coordinate space has size m^a and at most m fixed points. For $a = 1$, its moved points are exactly the local imprimitive support. For $a \geq 3$, $m^a - m \geq ma$, which pays all its imprimitive support. For several cycles, the elementary inequality

$$\prod_i q_i - \prod_i f_i \geq \sum_i (q_i - f_i) \qquad (0 \leq f_i \leq q_i, q_i \geq 2)$$

combines the estimates. For two factors it is equivalent to $(q-1)(r-1) \geq (f-1)(g-1)$, including $f = 0$ or $g = 0$; induction proves the displayed version. The degree decreases except when $(m, r) = (2, 2)$. In nonabelian primitive product types the base degree is at least five. The compound diagonal types are product actions with a diagonal base and are included here.

For the action of S_m on j-subsets, $1 \leq j \leq m-1$, send the basis vector at a point to the sum of the basis vectors at the j-subsets containing it. This incidence map from the natural real permutation module has Gram matrix

$$\binom{m-2}{j-1} I + \binom{m-2}{j-2} J.$$

Here I, J are the $m \times m$ identity and all-ones matrices, and binomial coefficients with a negative lower entry are zero. The diagonal entries are $\binom{m-1}{j-1}$ and the off-diagonal entries are $\binom{m-2}{j-2}$; Pascal's identity gives the formula. It is positive definite, so the natural module embeds. In the orthogonal complement the trace of a permutation is at most its dimension; consequently subset support is at least natural support for every element. This applies to all subgroups and through any abstract identification of the component. A subset action inequivalent to the natural action has $2 \leq j \leq m-2$, so its degree decreases under replacement; this excludes complementary singleton actions too.

The O'Nan–Scott alternatives are affine, almost simple, regular nonabelian, simple diagonal, and product actions with almost simple or diagonal base; equivalently the eight usual types HA, AS, HS, HC, SD, CD, PA, TW, as in [18]. The constructions above cover every alternative claimed in the lemma. Finally, embed the block action into its imprimitive component wreath product and perform the replacement in that ambient action, then restrict to the original group. A moved block contributes its full old or new size; on a fixed block an odd element has an odd local action. Thus support domination and faithfulness persist. The weighted sum is then taken over the original subgroup. □

5.6. **Full diagonal subgroups with different coordinate actions.** A subdirect subgroup of a product of nonabelian simple groups is a product of full diagonal subgroups on disjoint sets of isomorphic factors. Here is the elementary induction behind this assertion. Project away the last factor. The kernel in that simple factor is either the whole factor, giving an independent factor, or trivial. In the latter case Goursat's lemma identifies the last factor with a simple quotient of the projected group. By induction that quotient is one of its diagonal factors, to which the last coordinate is attached by an isomorphism. We call the resulting full diagonal factors *strips*. Their coordinate representations need not be equivalent.

Lemma 5.9. *Let G act faithfully on Ω and preserve a partition of Ω. On a nonempty invariant set I of blocks, suppose its primitive components have nonabelian simple socles S_i and satisfy Proposition 5.7. Let $N \trianglelefteq G$ be subdirect in $\prod_{i \in I} S_i$, acting trivially on the other points. Replace*

each active block by Ψ_{S_i} and retain the inactive points. The quotient $E = G/N$ has a faithful action on this new set Ω', and

$$\mathcal{P}(G, \Omega) \leq \mathcal{P}(E, \Omega'). \tag{5.19}$$

In particular $|\Omega'| < |\Omega|$.

Proof. Choose an abstract parametrization $\tau_i : S \to S_i$ on each strip. An element normalizing N gives coordinate permutations and automorphisms between the parameters. Record their outer classes and use the fixed representation of $\mathrm{Out}(S)$ on Ψ_S. Changing a parameter conjugates the outer labels by an element of $\mathrm{Out}(S)$, implemented by a permutation of Ψ_S. Hence these labels define an action of G/N.

Each coordinate socle has trivial permutation centralizer. Indeed, that centralizer is normal in the full primitive normalizer. If nontrivial it is transitive, forcing the commuting transitive socle to be regular, contrary to the almost simple hypothesis.

A subdirect product of centerless simple groups is self-normalizing inside their direct product. On one strip an inner tuple normalizes it precisely when all induced automorphisms of its parameter agree; centerlessness then makes the tuple an element of that strip. Apply this separately to every strip. Thus an element in the kernel of the proposed action is in N: it fixes the active blocks, has inner coordinate labels there, and is trivial on the inactive points. The reverse inclusion is immediate. This proves faithfulness.

Fix a commuting pair (α, β) of E, with α odd. Decompose the strips into orbits of this pair. On one such orbit there are L strips of the same abstract type S. In either prescribed lift coset, variation by the kernel varies the inner transports freely through $S^L \cong \mathrm{Inn}(S)^L$. Regard the two permutations as the action of $\Gamma = \mathbb{Z}^2$ on the L strip positions. Choose a root and a rooted spanning tree of its Schreier graph. Conjugate by the subgroup of S^L whose root coordinate is one. Its action is free: the root value and successive tree edges determine every other conjugating coordinate. Each orbit therefore has $|S|^{L-1}$ elements and has a unique choice whose tree transports are the prescribed representatives. The transport of any remaining edge is then determined by the holonomy of the closed path obtained by adjoining the tree paths. The stabilizer $\Lambda \leq \Gamma$ of the root is a rank-two lattice. Choose a basis of Λ. Its holonomies lie in two prescribed S-cosets of $\mathrm{Aut}(S)$ and commute; these two values determine the holonomies of every closed path. Conversely these assignments reconstruct all edge transports satisfying the commuting-square relations. Specified coordinate permutations or representation identifications may impose further equations, which only reduce the possibilities. For the full strip system, a commutator of prescribed lifts lies in N. If its induced automorphism on every strip is trivial, it lies in $N \cap C_G(N) = Z(N) = 1$. Thus the commuting-square relations detect commutation of the lifts.

For two prescribed automorphism cosets, the number of commuting pairs divided by $|S|$ is at most $k(S)$. On any S-class in the first coset, the allowed second elements are empty or a centralizer coset, so that class contributes at most one. The number of classes in a coset is at most $k(S)$ by the Burnside calculation in Lemma 5.4. Therefore

$$\#\{\text{commuting lift pairs}\} \leq |S|^{L-1} \#\{\text{commuting holonomy pairs}\} \leq |S|^L k(S).$$

After division by $|S|^L$, the unweighted lift mass on the whole strip orbit is therefore at most $k(S)$. This bound applies to lattice-basis holonomies of every order.

Suppose first that α or β_o moves an original block in this orbit. A moved block of degree m_i gives excess $m_i - d_{S_i}$ between old and new support. On a fixed block, the old odd support dominates the new outer support by Proposition 5.7(i). An odd moved cycle has length at least three, and all its blocks have the same degree. Consequently the old pair weight is at most its quotient weight multiplied by $t^{3(m-d_S)}$, for one such cycle. The factor $k(S)$ just obtained is paid

by Proposition 5.7(ii). Thus this entire strip orbit has the required weighted bound, even if it also meets other degrees and representation types.

Now suppose that both odd quotient elements fix every original block on this orbit. Then α fixes every strip, and β permutes the L strips in one cycle of 2-power length. The preceding transport calculation leaves an odd holonomy x and a commuting holonomy y in two prescribed S-cosets. If a strip has h original coordinates, let s_j be odd support in its jth representation. The pair weight on these hL blocks is

$$\prod_{j=1}^{h} t^{L(s_j(x)+s_j(y_o))}. \tag{5.20}$$

These functions are defined even if the even holonomy permutes the coordinate representation types.

Use the strip normalizer generated by the coordinate-fixing subgroup and this cycle holonomy. Its permutation image on the coordinates is a 2-group. The conjugation map to $\mathrm{Aut}(S)$ has a 2-group kernel: a coordinate-fixing element centralizing S on every coordinate is trivial by the coordinate-centralizer argument above.

Every odd element of the automorphism image has an odd lift, obtained by taking the odd part of a preimage. Such a lift fixes all coordinates. It is unique, since two coordinate-fixing lifts with the same automorphism differ by the coordinatewise trivial centralizer. This defines all the support functions s_j, and they are S-class functions. In each coordinate the odd automorphism images belong to its coordinate normalizer, as required in Proposition 5.7(iii). The removed 2-kernel meets S trivially. Each prescribed S-coset therefore maps bijectively to its image; a commutator of two prescribed lifts lies in S and vanishes exactly when its image does. The first lift remains odd: its 2-part would lie in both S and the removed 2-kernel. Thus the prescribed holonomy count is preserved.

Also $(\beta^L)_o = \beta_o^L$. Since L is a power of two, taking this power does not change the support of the odd part, either in an original coordinate or in its outer action. This proves (5.20) with these support functions. Let μ be the common positive measure on the allowed holonomy pairs, of mass $1/|S|$ on each pair. If additional relations remain, enlarging to all commuting pairs in the prescribed cosets only increases the sums below. Put $F_j(x,y) = t^{L(s_j(x)+s_j(y_o))}$. Hölder's inequality gives

$$\int \prod_{j=1}^{h} F_j \, d\mu \le \prod_{j=1}^{h} \left(\int F_j^h \, d\mu \right)^{1/h}. \tag{5.21}$$

Apply Proposition 5.7(iii) to each coordinate with exponent Lh. Abstract changes of parameter conjugate outer labels, so preserve their supports on the chosen Ψ_S. Each factor on the right has the prescribed outer bound; their product is exactly the required quotient weight on these hL blocks. For alternating types both odd outer labels are trivial, and Lemma 5.6 includes every even A_6 holonomy. It can therefore be combined in the same Hölder inequality with non-natural representations of that simple group.

For the fixed quotient pair the equations on different strip orbits involve different direct factors of N. Their bounds multiply. On inactive points N is trivial, and the support is already the quotient support. Hence the normalized lift mass is at most $t^{s_{\Omega'}(\alpha)+s_{\Omega'}(\beta_o)}$. Averaging over the commuting quotient pairs proves (5.19). Every active degree decreases, proving the final assertion. □

5.7. **The degree induction.**

Proof of Theorem 5.1. Assume a counterexample exists and choose one of minimum degree. Remove global fixed points, which changes neither faithfulness nor the weighted sum. By

Lemma 5.3 its solvable radical is trivial: a nontrivial solvable radical contains an elementary abelian minimal normal subgroup of G.

On each transitive orbit choose a smallest block with more than one point and take all its translates; in a primitive orbit choose the whole orbit. Each resulting block component is primitive, since a smaller block in its stabilizer action would give a smaller original block. Lemma 5.8 and minimality imply that every component is affine or almost simple. They also remove every larger alternating subset representation, including any abstractly twisted instance.

Let K be the kernel on this entire block set. If $K = 1$, the block action is faithful and has degree at most half the original degree. Odd support can only decrease, contrary to minimality. Thus $K \neq 1$. Its solvable radical is characteristic in K and normal in G, so it is trivial. Consequently $N = \mathrm{Soc}(K)$ is a nontrivial direct product of nonabelian simple groups, characteristic in K and normal in G.

Its image N_i on any block is semisimple and normal in the primitive component. In an affine component it must be trivial: every nontrivial normal subgroup contains the unique elementary abelian minimal normal subgroup, impossible in a semisimple group. In an almost simple component with socle S_i it is either trivial or S_i. In fact a nontrivial N_i contains S_i; this normal simple subgroup is one direct factor of N_i, and all other factors centralize it, whereas its component centralizer is trivial.

Let I be the blocks on which $N_i = S_i$. The set I is nonempty and invariant. The faithful original action embeds N subdirectly in $\prod_{i\in I} S_i$, and N is trivial on all inactive points. The elementary subdirect description gives exactly the full strips of Lemma 5.9, possibly across different orbits, degrees and representations. Every active component satisfies Proposition 5.7. Lemma 5.9 produces a strictly smaller faithful quotient action with at least as large a weighted sum, the final contradiction.

All terms are nonnegative and nondecreasing in t, so the assertion for smaller parameters follows. For a 2-group every odd part is the identity and the weighted sum is one, showing sharpness of the constant. □

5.8. **The finite permutation certificate.** For reproducibility we specify the complete finite input to Proposition 5.7. The primitive database used here is GAP 4.14.0 with `PrimGrp` 3.4.4 and `CtblLib` [27, 11, 37]. Its completeness up to permutation isomorphism in the present range is the classification of [61], incorporated in the library described in [37]; [18] extends the range to 4095. We use only degrees 5 through 127. The official GAP type string `"2"` denotes almost simple, and the socle is obtained from the primitive model, including models with an imprimitive socle action.

The following executable test gives every internal, outer and moving inequality. For alternating groups, `OutOrder` uses $|\mathrm{Out}(A_n)| = 2$ for $n \neq 6$ and $|\mathrm{Out}(A_6)| = 4$. The linear and unitary formulas are given in [35, §7, proof of Lemma 7.7]; the symplectic and orthogonal orders occur in [50, proof of Theorem 2.9]. For $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_2, HS$ and $\mathrm{Sz}(8)$, the respective orders are $1, 2, 2, 1, 1, 2, 2, 3$, as recorded in their entries in [17]. In the library's notation `L` means $\mathrm{PSL}_n(q)$, `2A` with parameter $[n, q]$ means $\mathrm{PSU}_{n+1}(q)$, and `B` with parameter $[n, q]$ means $\mathrm{PSp}_{2n}(q)$ for the types that occur here. The only orthogonal labels present are $D_4(2)$ and ${}^2D_4(2)$. The formulas below are used only for the types in Table 2; unhandled series raise an error.

```
LoadPackage("primgrp");;
LoadPackage("ctbllib");;
ExactCriterion := function(c,s)
  local z;
  z := 2^s-c-1;
  return z >= 0 and z^2 >= 4*c;
```

```
end;;
OutOrder := function(st)
  local n,q,f,ser;
  ser := st.series;
  if ser="A" then
    if st.parameter=6 then return 4; else return 2; fi;
  elif ser="Spor" then
    if st.parameter in ["M(11)","M(23)","M(24)"] then
      return 1;
    else return 2; fi;
  elif ser="2B" then return LogInt(st.parameter,2);
  fi;
  n := st.parameter[1]; q := st.parameter[2];
  f := Length(FactorsInt(q));
  if ser="L" then
    if n=2 then return Gcd(n,q-1)*f;
    else return 2*Gcd(n,q-1)*f; fi;
  elif ser="2A" then return 2*Gcd(n+1,q+1)*f;
  elif ser="B" then
    if n=2 and q mod 2=0 then return 2*f; fi;
    return Gcd(2,q-1)*f;
  elif ser="D" and n=4 and q=2 then return 6;
  elif ser="2D" and n=4 and q=2 then return 2;
  fi;
  Error("Unrecognized simple type");
end;;
total := 0;; exact := 0;; triples := 0;; pairs := [];;
for m in [5..127] do
  for id in [1..NrPrimitiveGroups(m)] do
    g := PrimitiveGroup(m,id);
    if ONanScottType(g)="2" then
      st := SocleTypePrimitiveGroup(g);
      if not (st.series="A" and st.parameter=m) then
        total := total+1;
        s := Socle(g); sz := Size(s);
        key := [st.series,st.parameter];
        out := OutOrder(st); d := out;
        if key=["L",[3,4]] then d := 6; fi;
        mu := QuoInt(m+1,2);
        coarse := st.series<>"A" and ExactCriterion(sz,mu);
        if not coarse then
          exact := exact+1;
          for cl in ConjugacyClasses(s) do
            x := Representative(cl);
            if Order(x)>1 and IsOddInt(Order(x)) then
              c := sz/Size(cl); motion := NrMovedPoints(x);
              if not ExactCriterion(c,motion) then
                Error("Internal inequality failed");
              fi;
              triples := triples+1; AddSet(pairs,[c,motion]);
              Print("CLASS ",m," ",id," ",
                    [Order(x),motion,c],"\n");
```

```
            fi;
          od;
        fi;
        qbound := sz;
        if key=["L",[2,8]] then qbound := 9;
        elif key=["L",[2,27]] then qbound := 16;
        elif key=["L",[2,32]] then qbound := 33;
        elif key=["L",[3,4]] then qbound := 10; fi;
        if qbound<>sz and NrConjugacyClasses(s)<>qbound then
          Error("Small class number failed");
        fi;
        if out > 1 and ForAny(FactorsInt(out),r -> r<>2) then
          if not (key=["L",[2,8]] and m=9) then
            if not (ExactCriterion(qbound,mu) and
                    2^mu>=2^d*qbound and mu>=d) then
              Error("Outer inequality failed");
            fi;
          fi;
        fi;
        if not (d<m and 3*(m-d)>=2*(m-1)) then
          Error("Moving inequality failed");
        fi;
        Print("MODEL ",m," ",id," ",key," ",
              [sz,out,d,mu,qbound],"\n");
      fi;
    fi;
  od;
od;
Print([total,exact,triples,Length(pairs)],"\n");
# Output: [ 292, 97, 887, 267 ]
```

The supplementary files `verification/permutation/all_terminal_rows.json` and `all_socle_labels.json` retain every (m, id) and its simple type, order, outer order, chosen outer degree and method. The file `exact_class_certificate.json` contains, for each of the 97 exact models, every triple $[\mathrm{ord}(x), s(x), |C_S(x)|]$, with multiplicities. The file `integer_pairs.json` is its set of 267 distinct $(|C_S(x)|, s(x))$ pairs. The full generating script and output are included in the same directory. All comparisons use integers.

Table 2 records the complete type and degree incidence, with the number of models. The labels $L_2(5)$ and A_5 are kept separate to match the database; together they represent one abstract simple type. Thus the 80 labels give 79 simple types. Table 3 is a lossless reduction for checking the internal inequalities: for each centralizer order C occurring among the 267 pairs it gives the smallest support s_0. The test (5.16) strengthens as s increases, so these 109 boundary pairs suffice for its arithmetic. The full class records provide the incidence with the selected models.

Table 2: Complete finite type and degree incidence. Here $L_n(q) = \mathrm{PSL}_n(q)$ and $U_n(q) = \mathrm{PSU}_n(q)$.

S	$\lvert\mathrm{Out}(S)\rvert$	d_S	Models	Degrees
$L_2(5)$	2	2	2	6
$L_2(7)$	2	2	5	7, 8, 21, 28
$L_2(8)$	3	3	6	9, 28, 36
A_5	2	2	2	10

S	$\lvert\mathrm{Out}(S)\rvert$	d_S	Models	Degrees
A_6	4	4	13	10, 15, 36, 45
$L_2(11)$	2	2	7	11, 12, 55, 66
M_{11}	1	1	4	11, 12, 55, 66
M_{12}	2	2	2	12, 66
$L_3(3)$	2	2	3	13, 52, 117
$L_2(13)$	2	2	8	14, 78, 91
A_7	2	2	6	15, 21, 35, 120
A_8	2	2	9	15, 28, 35, 56, 105, 120
$L_2(16)$	4	4	9	17, 68, 120
$L_2(17)$	2	2	3	18, 102
$L_2(19)$	2	2	3	20, 57
$L_3(4)$	12	6	20	21, 56, 105, 120
M_{22}	2	2	4	22, 77
M_{23}	1	1	1	23
M_{24}	1	1	1	24
$L_2(23)$	2	2	2	24
$L_2(25)$	4	4	7	26, 65
$\mathrm{PSp}_4(3)$	2	2	10	27, 36, 40, 45
$U_3(3)$	2	2	8	28, 36, 63
$\mathrm{PSp}_6(2)$	1	1	4	28, 36, 63, 120
$L_2(27)$	6	6	4	28
$L_2(29)$	2	2	2	30
$L_3(5)$	2	2	1	31
$L_5(2)$	2	2	1	31
$L_2(31)$	2	2	2	32
$L_2(32)$	5	5	2	33
A_9	2	2	7	36, 84, 120, 126
$L_2(37)$	2	2	2	38
$L_4(3)$	4	4	4	40, 117
$L_2(41)$	2	2	2	42
$L_2(43)$	2	2	2	44
A_{10}	2	2	6	45, 120, 126
$L_2(47)$	2	2	2	48
$U_3(5)$	6	6	6	50, 126
$L_2(49)$	4	4	5	50
$L_2(53)$	2	2	2	54
A_{11}	2	2	2	55
$L_3(7)$	6	6	2	57
$L_2(59)$	2	2	2	60
$L_2(61)$	2	2	2	62
$L_6(2)$	2	2	1	63
$U_3(4)$	4	4	3	65
$\mathrm{Sz}(8)$	3	3	2	65
$L_2(64)$	6	6	4	65
A_{12}	2	2	2	66
$L_2(67)$	2	2	2	68
$L_2(71)$	2	2	2	72
$L_3(8)$	6	6	2	73
$L_2(73)$	2	2	2	74
A_{13}	2	2	2	78

S	$\lvert\mathrm{Out}(S)\rvert$	d_S	Models	Degrees
$L_2(79)$	2	2	2	80
$L_2(81)$	8	8	8	82
$L_2(83)$	2	2	2	84
$\mathrm{PSp}_4(4)$	4	4	4	85, 120
$L_4(4)$	4	4	2	85
$L_2(89)$	2	2	2	90
A_{14}	2	2	2	91
$L_3(9)$	4	4	2	91
$L_2(97)$	2	2	2	98
J_2	2	2	2	100
HS	2	2	2	100
$L_2(101)$	2	2	2	102
$L_2(103)$	2	2	2	104
A_{15}	2	2	2	105
$L_2(107)$	2	2	2	108
$L_2(109)$	2	2	2	110
$U_4(3)$	8	8	13	112, 126
$L_2(113)$	2	2	2	114
$\mathrm{P}\,\Omega_8^-(2)$	2	2	2	119
$\mathrm{PSp}_8(2)$	1	1	1	120
$\mathrm{P}\,\Omega_8^+(2)$	6	6	2	120
A_{16}	2	2	2	120
$L_5(3)$	2	2	1	121
$L_2(121)$	4	4	5	122
$L_2(125)$	6	6	4	126
$L_7(2)$	2	2	1	127

Table 3: The 109 exact internal boundary pairs.

C	s_0	C	s_0	C	s_0
3	6	90	60	3000	95
5	5	98	119	3240	90
6	9	105	102	3528	63
7	7	108	18	5400	76
9	9	126	104	5832	27
10	10	132	99	6480	51
11	11	150	75	7560	24
12	24	162	54	9720	81
13	13	180	15	12600	45
14	21	216	39	14580	105
15	15	252	85	17640	70
17	17	270	88	18000	105
18	9	300	35	22680	99
19	19	324	114	23328	114
20	40	375	105	37800	84
21	24	378	117	45360	57
23	23	405	91	58320	90
24	30	420	56	60480	27
25	30	450	117	100800	50

C	s_0	C	s_0	C	s_0
27	39	486	63	141120	77
30	25	500	25	181440	63
31	31	504	24	302400	92
33	88	540	81	362880	63
35	66	567	120	408240	99
36	12	600	85	544320	30
39	117	648	27	907200	55
42	21	660	110	1270080	84
45	52	810	119	3265920	69
49	63	900	68	5443200	33
50	65	972	66	9072000	60
54	9	1080	18	32659200	75
55	120	1176	56	59875200	36
60	20	1260	95	99792000	65
63	63	1800	40	718502400	39
75	78	1944	36	9340531200	42
81	36	2160	18		
84	49	2520	63		

There are two additional small automorphism computations. The data for A_6 in Lemma 5.6 are obtained by the following exact permutation procedure:

```
A := AlternatingGroup(6);;
cc := ConjugacyClasses(A);;
au := AutomorphismGroup(A);;
c3 := First(cc,c -> Order(Representative(c))=3 and
                        NrMovedPoints(Representative(c))=3);;
stab := Stabilizer(au,c3,function(c,a)
  return ConjugacyClass(A,Image(a,Representative(c)));
end);;
if Size(au)<>1440 or Size(stab)<>720 then
  Error("A6 automorphism assertion failed");
fi;
List(Filtered(cc,c -> IsOddInt(Order(Representative(c)))),
     c -> [Order(Representative(c)),
           NrMovedPoints(Representative(c)),Size(A)/Size(c)]);
# Up to order: [ [1,0,360], [3,3,9], [3,6,9], [5,5,5], [5,5,5] ]
```

The permutation-induced subgroup has order 720 and fixes the support-three class, so it is exactly this stabilizer. There are precisely two order-three classes, proving the asserted interchange. One may construct the exceptional automorphism independently through the six one-factorizations of the complete graph on six vertices.

For $\mathrm{PSL}_2(8)$, take $\mathbb{F}_8 = \mathbb{F}_2[z]/(z^3+z+1)$ and the nine-point projective line. The permutations

$$u \mapsto u+1, \qquad u \mapsto zu, \qquad u \mapsto u^{-1}$$

(with $0, \infty$ interchanged by inversion) generate its 504-element fractional linear group S. Adjoin $F : u \mapsto u^2$, fixing ∞, to obtain the 1512-element normalizer. Close the three generators under permutation multiplication; then on each of SF and SF^2 partition the odd elements into orbits

of conjugation by S. The output in each coset is

$\mathrm{ord}(u)$	$s(u)$	$\lvert u^S\rvert$
3	6	84
9	9	168.

This completely specified field and permutation construction is also provided in `verification/permutation/verify_permutation.py`, with output in `permutation_results.json`. It supplies the two-class refinement; the matrix row and column identities themselves were proved in Lemma 5.4.

Finally, the six-point outer action for $L_3(4)$ uses $\mathrm{Out}(L_3(4)) \cong D_{12}$; this standard automorphism structure [17] can be checked within the selected models by `PrimitiveGroup(105,6)`. Its order is 241920, its socle has order 20160, and the quotient has small-group identifier $[12, 4]$, the dihedral group of order twelve. Restrict this outer representation to the outer subgroup of the degree-21 component.

6. From permutation weights to affine groups

Write $\ell_2(H) = |\,\mathrm{IBr}(H)|$, equivalently the number of odd-order conjugacy classes of H. For a finite H-set Y, define

$$\Lambda_2(H, Y) = \sum_{y \in Y/H} \ell_2(H_y).$$

The analogue of (2.4) is

$$\Lambda_2(H, Y) = \frac{1}{|H|} \sum_{\substack{a \in H \text{ odd} \\ z \in C_H(a)}} |Y^{\langle a,z\rangle}|. \tag{6.1}$$

It follows by counting the same triples with the additional condition that a has odd order.

Lemma 6.1. *Let $A \lhd G$, put $T = G/A$ and $Y = \mathrm{Irr}(A)$. Then*

$$k(G) \le |T|_2 \Lambda_2(T, Y). \tag{6.2}$$

Suppose T acts faithfully on Ω, and $s(a)$ is the number of points moved by a. If there is a real number $M > 0$ such that

$$|Y^a| \le M 2^{-s(a)} \qquad (a \in T \text{ odd}), \tag{6.3}$$

then the permutation inequality of Theorem 5.1 gives $k(G) \le M|T|_2$.

Proof. Grouping the conjugacy classes of a group J by their odd parts gives $k(J) \le |J|_2 \ell_2(J)$. Over an odd class represented by a, the remaining choices are 2-element classes of $C_J(a)$; each meets a fixed Sylow subgroup and hence there are at most $|C_J(a)|_2$ of them. The number of irreducible characters above a T-orbit in Y is $k_\alpha(T_y) \le k(T_y) \le |T|_2 \ell_2(T_y)$. Summing proves (6.2).

Write z_o for the odd part of z. For every commuting pair occurring in (6.1),

$$|Y^{\langle a,z\rangle}| \le \sqrt{|Y^a|\,|Y^{z_o}|} \le M 2^{-(s(a)+s(z_o))/2}.$$

The normalized sum of these weights is at most one by Theorem 5.1. Now use (6.2). □

6.1. **Ordinary character counts on coordinate kernels.** Throughout this subsection, π_i denotes projection onto the ith coordinate. For an automorphism y of L, let $f_L(y) = |\operatorname{Irr}(L)^y|$. Equivalently this is the number of L-conjugacy classes in the coset Ly, and twisted Burnside gives

$$f_L(y) = \frac{1}{|L|} \sum_{x \in L} |C_L(xy)|. \tag{6.4}$$

If $K \lhd L$ is y-invariant, projecting the coset classes to L/K gives

$$f_L(y) \le f_{L/K}(y) \max_{x \in L} f_K(xy). \tag{6.5}$$

Indeed, over one quotient coset class every lifted class meets Kxy, whose K-classes give the asserted upper bound.

For a nonabelian simple group S, denote by $\widehat{S}$ its universal central cover with the central 2-part removed, and put

$$\kappa^+(S) = \max\left\{1, \frac{k(\widehat{S})}{|\widehat{S}|_2}\right\}. \tag{6.6}$$

Every perfect central extension of S with odd center is an equivariant quotient of $\widehat{S}$. Automorphisms lift by the universal property, and the central Sylow 2-subgroup removed from the universal cover is characteristic.

Lemma 6.2. *Let $M_i = L_i R_i$, where L_i is a central product of quasisimple groups with odd centers, $R_i = C_{M_i}(L_i)$ is solvable, and $[L_i, R_i] = 1$. Suppose B is a subdirect subgroup of $\prod_{i=1}^r M_i$ normalized by a coordinate automorphism y that permutes the coordinates in one cycle. Then, for every $i \in \{1, \dots, r\}$,*

$$f_B(y) \le |B|_2 \, |R_i|_{2'} \prod_{L \text{ a component of } L_i} \kappa^+(L/Z(L)). \tag{6.7}$$

The same bound holds when the coordinate images of B are normal subgroups of M_i.

Proof. Let $F = B^{(\infty)}$ be the last derived subgroup. Its projections are L_i. Modulo the odd centers, it is a subdirect product of nonabelian simple groups, and thus a product of full diagonal simple subgroups on disjoint sets of factor positions. This description follows directly by induction from Goursat's lemma: adjoining a simple coordinate either adds an independent factor or identifies it diagonally with one existing isomorphic factor. The components of F are therefore odd central quotients of the corresponding $\widehat{S}$.

Put $C = C_B(F)$. An element of B acts innerly on each coordinate L_i and preserves all factor positions. It consequently preserves each diagonal component and acts innerly on its simple quotient. An automorphism of a perfect central extension acting trivially on its central quotient is trivial: its deviation from the identity is a homomorphism to the center. Choose implementing elements in the components of F. Their product shows that

$$B = FC, \qquad C \le \prod_i R_i.$$

In particular C is solvable and y-invariant. There is a y-equivariant central cover

$$P = \left(\prod_j \widehat{S}_j\right) \times C \longrightarrow B$$

with odd kernel. Its kernel has odd order because the component cover kernels and $F \cap C = Z(F)$ have odd order. Inflation embeds $\operatorname{Irr}(B)$ equivariantly in $\operatorname{Irr}(P)$ and $|P|_2 = |B|_2$.

On an orbit of m component factors under y, a fixed character tuple is determined by one character, fixed by the composed return automorphism. Its count is at most $k(\widehat{S})$, whereas the

2-part of the order of all m factors is $|\widehat{S}|_2^m$. The normalized contribution is therefore at most $\kappa^+(S)$. Each orbit of diagonal components meets a factor position in any chosen coordinate; distinct orbits occupy disjoint positions. Assigning one such position to each orbit proves that their product is bounded by the product in (6.7). The maximum with one in (6.6) permits unused positions.

For C, each centralizer $C_C(cy)$ projects injectively into one coordinate: commutation determines all other coordinates successively around the cycle. Hence

$$f_C(y) \le |\pi_i(C)| \le |C|_2 |R_i|_{2'}$$

by (6.4). Multiplying proves the claimed bound. If a coordinate image is only normal in M_i, it is the product of those components that it contains and its intersection with R_i. Indeed a noncentral normal image on a simple factor contains the corresponding perfect component by taking commutators. Apply the proved case to these images; neither the odd order nor the product of the factors $\kappa^+(S) \ge 1$ increases. □

Suppose $O_2(E) = 1$. Its Fitting subgroup $F(E)$ has odd order and its components have odd centers. Define

$$c(E) = |E : F^*(E)|_{2'} \, |F(E)| \prod_{L \text{ a component of } E(E)} \kappa^+(L/Z(L)), \tag{6.8}$$

with $c(1) = 1$. The factors in $F^*(E) = F(E)E(E)$ commute, and $C_{F^*(E)}(E(E)) = F(E)$.

Lemma 6.3. *Let $K \le \prod_{i=1}^r E_i$ be subdirect, where $O_2(E_i) = 1$, and let y cyclically transport the coordinates and normalize K. Then $f_K(y) \le |K|_2 c(E_i)$ for every $i \in \{1, \dots, r\}$. The same conclusion holds for a subgroup K normal in a subdirect group on the E_i, provided y normalizes K.*

Proof. Put $M_i = F^*(E_i)$ and $A = K \cap \prod_i M_i$. The images of A are normal in M_i, so Lemma 6.2 applies to A and every twisted automorphism xy with $x \in K$. The quotient K/A embeds in $\prod_i E_i/M_i$. Again a centralizer of a cyclic coordinate automorphism is determined by one coordinate. Twisted Burnside gives

$$f_{K/A}(y) \le |\pi_i(K/A)| \le |K/A|_2 |E_i : M_i|_{2'}.$$

Combine the two bounds through (6.5); the quotient E_i/M_i enters only through its odd order part. For the last assertion, normality in the subdirect overgroup still makes every projection of A normal in E_i; all the same bounds therefore apply with the original M_i and $c(E_i)$. □

Proposition 6.4. *Let H act faithfully on $V = \bigoplus_{i \in \Omega} W_i$ over $\mathbb{F}_2$ and permute the summands. Let N be the kernel of that permutation action, and let E_i be its image on W_i. Suppose $O_2(E_i) = 1$ and*

$$c(E_i) \le |W_i|^2/8. \tag{6.9}$$

Assume also that $k(W_i \rtimes U) \le |W_i||U|_2$ for every subgroup U of the image in $\mathrm{GL}(W_i)$ of the stabilizer of W_i in H. Then $k(V \rtimes H) \le |V||H|_2$.

Proof. Put $q_i = |W_i|$ and $T = H/N$. For $g \in H$, filter $V \rtimes N$ by the kernels of its projections onto successive unions of coordinate cycles of g. Formula (6.5) applies at each step. On a moved cycle C, choose $i \in C$ and put $q_C = q_i$ and $E_C = E_i$, with $W_C = \bigoplus_{j \in C} W_j$. Its successive affine image has the form $W_C \rtimes K_C$ and contains the full additive group W_C. The linear image K_C is normal in the corresponding projection of N, which is subdirect on the original E_i in C. The last assertion of Lemma 6.3 therefore uses the original capacities. Any additive part of a coset twist is inner on $W_C \rtimes K_C$; removing it preserves the character trace. The remaining twist y normalizes K_C and has the same coordinate cycle. For $u \in W_C$ and $x \in K_C$, projection sends $C_{W_C \rtimes K_C}(uxy)$

into $C_{K_C}(xy)$. Its kernel has at most q_C elements, since a fixed vector is determined by one coordinate. Thus

$$|C_{W_C \rtimes K_C}(uxy)| \le q_C |C_{K_C}(xy)|.$$

Summing over u and x and dividing by $|W_C||K_C|$ in (6.4) gives

$$f_{W_C \rtimes K_C}(y) \le q_C f_{K_C}(y) \le q_C |K_C|_2 c(E_C).$$

For a fixed coordinate use the assumed lower-dimensional affine bound and $f_L(g) \le k(L)$. The orders of the successive linear factors telescope to $|N|$, so their 2-parts telescope to $|N|_2$. We obtain

$$|\operatorname{Irr}(V \rtimes N)^g| \le |N|_2 \prod_{i \text{ fixed}} q_i \prod_{C \text{ moved cycle}} q_C c(E_C). \tag{6.10}$$

The parameters on a cycle agree because g transports the coordinate actions. The projection kernels preserve the subdirect relations and the return automorphisms of the lift.

If $a \in T$ has odd order, all its moved cycles have odd length $r \ge 3$. By (6.9),

$$\frac{q_C c(E_C)}{q_C^r} \le \frac{q_C^{3-r}}{8} \le 2^{-r}.$$

At the identity (6.10) also gives $k(V \rtimes N) \le |V||N|_2$. Hence

$$|\operatorname{Irr}(V \rtimes N)^a| \le |V||N|_2 \, 2^{-s(a)}$$

for every odd $a \in T$. Apply Lemma 6.1 to the normal extension $V \rtimes N \lhd V \rtimes H$, with $M = |V||N|_2$. □

7. Primitive linear groups and local capacities

We prove two assertions needed in the affine reduction. Primitivity here means the absence of a nontrivial direct-sum decomposition permuted by the group. A module is quasiprimitive if its restriction to each normal subgroup is homogeneous. An irreducible primitive module is quasiprimitive: Clifford theory gives a permuted direct sum of isotypic components for each normal subgroup, and primitivity makes that sum have one term.

Proposition 7.1. *If E acts faithfully, irreducibly and primitively on a nonzero finite $\mathbb{F}_2$-space V, then*

$$k(V \rtimes E) \le |V| \, |E|_2.$$

The same conclusion holds for a faithful irreducible quasiprimitive action.

Proposition 7.2. *Let B act faithfully, irreducibly and primitively on a finite $\mathbb{F}_2$-space W, and let $E \lhd B$. Define $c(E)$ by (6.8). If $E(E) \ne 1$ or $F(E)$ is nonabelian, then*

$$c(E) \le |W|^2/8.$$

Here $O_2(E) = 1$, and W is homogeneous on every characteristic subgroup of E, by faithfulness and primitivity.

The proofs retain the scalar field, component degrees, central characters and homogeneous multiplicities. We first establish the structural and counting estimates, then give the classification inputs and the complete remaining numerical verification.

7.1. **The canonical field and the generalized Fitting subgroup.**

Lemma 7.3. *Let E be a finite group acting faithfully on a nonzero finite-dimensional $\mathbb{F}_2$-space V. Assume that V is homogeneous on every characteristic subgroup of E; in applications $E \lhd B$ and V is a faithful primitive irreducible B-module. Put $R = F(E)$, $L = E(E)$, and $F^* = RL$. Then $O_2(E) = 1$. The center K of $\mathrm{End}_{\mathbb{F}_2 F^*}(V)$ is an E-normalized finite field, say $K = \mathbb{F}_q$, $q = 2^f$, and V_{KF^*} is absolutely homogeneous. Moreover*

$$R = ZR_1 \cdots R_u, \qquad |R : Z| = n^2, \qquad n = \prod_{j=1}^{u} r_j^{a_j}, \tag{7.1}$$

where $Z = Z(R)$ is cyclic and acts by K-scalars, the r_j are distinct odd primes, and R_j is a characteristic extraspecial subgroup of the corresponding Sylow subgroup, of exponent r_j and order $r_j^{1+2a_j}$. The factors commute. If the components of L are $L_1, \dots, L_t$, there are absolute component degrees $m_i \geq 2$ and a positive integer b such that

$$\dim_K V = bn \prod_{i=1}^{t} m_i. \tag{7.2}$$

All the constituents have compatible central characters for the stated central product. Finally, with $N = C_E(L)$, one has

$$C_N(R) = Z, \qquad H = E/N \leq \mathrm{Aut}(S_1 \times \cdots \times S_t), \qquad F^*(H) = S_1 \times \cdots \times S_t, \tag{7.3}$$

where $S_i = L_i/Z(L_i)$. If $R = Z$, then $N \cap \mathrm{GL}_K(V) = Z$ and N/Z is cyclic of order dividing f, acting faithfully on Z.

Proof. Since $O_2(E)$ is characteristic in E, its restriction on V is homogeneous. Every irreducible module for a 2-group in characteristic two is trivial, so faithfulness gives $O_2(E) = 1$. Thus R has odd order. The central 2-parts of the components would generate a normal 2-subgroup of E, so their centers also have odd order.

Since F^* is characteristic in E, write $V_{F^*} = U^{\oplus b}$ with U irreducible. Then $\mathrm{End}_{\mathbb{F}_2 F^*}(V) = \mathrm{Mat}_b(K)$ by Schur's lemma and Wedderburn's theorem. Its center is intrinsic and is normalized by E. Centralizing this center leaves the same matrix algebra. Thus, when K is used as the scalar field, U is absolutely irreducible and the restriction remains homogeneous. The image of KF^* on U is $\mathrm{End}_K(U)$. The algebras generated by the commuting factors R, L_i are semisimple, by Clifford theory, and their centers commute with all of KF^*. Their centers are therefore K. Each is a central simple algebra over a finite field and hence a full matrix algebra. The usual commuting-algebras decomposition gives a tensor product over this *same* K. Its factors are faithful on the corresponding subgroups. This proves (7.2) and preserves the central identifications.

Every characteristic abelian subgroup of a Sylow subgroup of R is normal in E. Its faithful homogeneous representation over $\mathbb{F}_2$ has field image, so the subgroup is cyclic. Hall's theorem in the form [75, Theorem 1.1] says that an odd r-group with all characteristic abelian subgroups cyclic is a central product of a cyclic group and an extraspecial group of exponent r. The latter is characteristic in each noncyclic Sylow subgroup P_j. Indeed, write $P_j = Z(P_j)R_j$ with R_j extraspecial of exponent r_j. For $z \in Z(P_j)$ and $u \in R_j$ one has $(zu)^{r_j} = z^{r_j}$. The unique subgroup of order r_j in the cyclic center lies in R_j, so the elements of order dividing r_j are exactly R_j. Thus $R_j = \Omega_1(P_j) := \langle x \in P_j : x^{r_j} = 1 \rangle$ is characteristic. This proves (7.1). Its center belongs to the center of the algebra generated by F^*, and so is scalar over K. The faithful absolute degree of the extraspecial product is n. Also $Z(L) \leq R$, so $Z(RL) = Z(R) = Z$.

The generalized Fitting subgroup is self-centralizing. Therefore $C_N(R) = C_E(RL) = Z$. Conjugation on $L/Z(L)$ has kernel N: an automorphism of a perfect group acting trivially

modulo its center has central error a homomorphism to an abelian group and is the identity. The image of L is the full product of the S_i, with trivial centralizer in H. A nilpotent normal subgroup or an additional component of H would centralize that product. This proves (7.3). If $R = Z$, its centralizer in N is Z; in particular every K-linear element of N lies in Z. The remaining quotient embeds in $\mathrm{Gal}(K/\mathbb{F}_2)$, and its action on Z is faithful by the same centralizer equality. □

7.2. **Affine class counts, support and scalar centralizers.** For a linear group X on a space V, put $v = |V|$. The split-extension class formula and the ordinary little-groups formula are

$$k(V \rtimes X) = \sum_{x \in X/\mathrm{conj}} n(C_X(x), V/(x-1)V) = \sum_{\lambda \in V^*/X} k(X_\lambda). \tag{7.4}$$

The first formula uses a quotient space; it need not be equivariantly identified with $C_V(x)$. Their cardinalities are equal.

Lemma 7.4. *Let $r > 0$, and suppose $|C_V(x)| \le v^{1-1/r}$ for every $1 \ne x \in X$. Then*

$$k(V \rtimes X) \le \frac{v}{|X|} + \left(k(X) - \frac{1}{|X|}\right) v^{1-1/r}. \tag{7.5}$$

For $X \ne 1$, writing $P = |X|_2$, a sufficient threshold for $k(V \rtimes X) \le vP$ is

$$v^{1/r} \ge \frac{|X|k(X) - 1}{|X|P - 1}. \tag{7.6}$$

If $P \ge 2$, the right side is at most $k(X)/P + 1$. For $X = 1$ one has $k(V) = v$.

Proof. Burnside's lemma bounds the identity term of the first formula in (7.4) by $v/|X| + (1 - 1/|X|)v^{1-1/r}$. Each of the other $k(X) - 1$ terms is at most $v^{1-1/r}$. Rearrangement gives the assertions, including the additive correction in (7.6). □

Two elementary auxiliary consequences will be useful. If a cyclic odd central subgroup Z of X acts freely on $V \setminus \{0\}$, then

$$k(V \rtimes X) \le \left(z + \frac{v-1}{z}\right) k(X/Z), \qquad z = |Z|. \tag{7.7}$$

Indeed, $X_\lambda \cap Z = 1$ for $\lambda \ne 0$, and $k(X_\lambda) \le [X/Z : X_\lambda]k(X/Z)$; summing these contributions and using $k(X) \le zk(X/Z)$ proves the formula. If $v \ge (z+1)^2$, its coefficient is at most $2v/(z+1)$. In particular it is at most v.

Lemma 7.5. *If a solvable group X acts faithfully on an $\mathbb{F}_2$-space V, then $k(V \rtimes X) \le |V|\,|X|_2$, without a complete reducibility assumption.*

Proof. A Hall $2'$-subgroup J of a solvable X meets every odd-order conjugacy class. The X-orbits of pairs (x, w) with x odd and $xw = w$ are therefore images of the corresponding J-orbits. Consequently

$$\Lambda_2(X, V) \le k(V \rtimes J) \le v.$$

The last inequality is the coprime $k(GV)$ theorem [30]. Applying the same argument to V^* and using $k(X_\lambda) \le |X|_2 \ell_2(X_\lambda)$ in every dual-vector stabilizer gives

$$k(V \rtimes X) \le |X|_2 \Lambda_2(X, V^*) \le v|X|_2.$$

□

Lemma 7.6. *In Lemma 7.3, suppose $L \neq 1$ and $R = Z \neq 1$. Write $z = |Z|$, $N/Z = C_e$, and $P = e_2$. Then*

$$\frac{k(V \rtimes N)}{v|N|_2} \leq \frac{1}{2}, \qquad e > 1 \Longrightarrow \frac{k(V \rtimes N)}{v|N|_2} \leq \frac{5}{16} < \frac{1}{3}.$$

Every element of $N \setminus Z$ fixes at most $v^{1/2}$ vectors.

Proof. Here $q \geq 4$, $z \geq 3$, and $v = q^m$ with $m \geq 2$, since a nontrivial perfect layer cannot act on a one-dimensional multiplicity space. If $e = 1$, direct orbit counting gives

$$\frac{k(VZ)}{v} = \frac{1}{z} + \frac{z^2 - 1}{zv} \leq \frac{2}{z+1} \leq \frac{1}{2}.$$

We give the calculation for the possibly nonsplit extension when $e > 1$. For $d \mid e$, let z_d be the number of scalars fixed by the subgroup of order d in N/Z, and put $J_2(d) = d^2 \prod_{\ell \mid d,\ \ell \text{ prime}} (1 - \ell^{-2})$. Every character of Z extends to its inertia group, whose quotient over Z is cyclic: the value on a lift of a generator is obtained by taking a suitable complex root. In ordinary Clifford theory a stabilizer of order a in the cyclic quotient contributes $a^2 = \sum_{d \mid a} J_2(d)$. Averaging over $\mathrm{Irr}(Z)$ gives

$$k(N) = \frac{1}{e} \sum_{d \mid e} J_2(d) z_d \leq \frac{z}{e} + \left(e - \frac{1}{e}\right)(\sqrt{q} - 1). \tag{7.8}$$

The action on K has order d, so $z_d \leq q^{1/d} - 1$ for $d > 1$.

Every nonzero dual-vector stabilizer in N embeds in C_e and is cyclic. There are at most z subgroups of order d projecting isomorphically onto the subgroup of order d in C_e. If their generator x fixes a nonzero vector, then x^d is a scalar fixing that vector, hence is one. Galois descent gives $|C_V(x)| = v^{1/d}$. This descent is valid when d is even: if b_i, c_i are trace-dual bases of K/K^σ, the fixed vectors $\sum_{j=0}^{d-1} \sigma^j(c_i) x^j(w)$ span V over K, because $\sum_i b_i \sigma^j(c_i)$ is 1 for $j = 0$ and 0 otherwise. Fixed vectors independent over K^σ remain independent over K, by applying x to a minimal dependence. There is no division by d.

The little-groups formula, with squared stabilizer orders retained, now implies

$$k(VN) \leq k(N) + \frac{v - 1}{ze} + \frac{1}{e} \sum_{\substack{d \mid e \\ d > 1}} J_2(d)(v^{1/d} - 1). \tag{7.9}$$

Combining this with (7.8) gives

$$\frac{k(VN)}{vP} \leq \frac{1}{zeP} + \frac{z}{ePv} + e_{2'}(v^{-1/2} + \sqrt{q}/v). \tag{7.10}$$

For $q \geq 32$, use $eP \geq 3$, $z \geq 3$, $z < q$, $v \geq q^2$ and $e_{2'} \leq \log_2 q$. The right side is at most

$$\frac{1}{9} + \frac{1}{3q} + \frac{\log_2 q}{q} + \frac{\log_2 q}{q^{3/2}} < \frac{89}{288} < \frac{5}{16}.$$

Each nonconstant term decreases, and the displayed bound follows at $q = 32$ from $\sqrt{32} > 5$. At $q = 16$, $e = 2$ or 4, and (7.10) is at most $17/96$. At $q = 8$ the parameters are $(z, e) = (7, 3)$, and (7.9) gives $k(VN)/v \leq 1/21 + 16/(7v) + (8/3)v^{-2/3} \leq 1/4$. At $q = 4$ they are $(z, e) = (3, 2)$, and the bound is $1/12 + 2/(3v) + 3/(4\sqrt{v}) \leq 5/16$. Finally a semilinear transformation with nontrivial field action has fixed-space size at most $v^{1/2}$: commute it with a scalar of K not fixed by that field action and intersect the two conjugate fixed spaces. The scalar used for this argument need not belong to N. □

7.3. **An odd symplectic core.** Retain the notation of Lemma 7.3, and put $z = |Z|$. Write $Q_0 = \prod_j |\mathrm{Sp}_{2a_j}(r_j)|$ and $Q_o = (Q_0)_{2'}$. The restriction to R consists of m copies of its faithful absolute module of degree n, so $v = q^{nm}$. Let $R \le X \le N_{\Gamma L_K(V)}(R)$ and assume $C_{X\cap \mathrm{GL}_K(V)}(R) = Z$, where $\Gamma L_K(V)$ is the group of invertible maps semilinear over K. Then

$$|R| = zn^2, \quad (X \cap \mathrm{GL}_K(V))/R \le \prod_j \mathrm{Sp}_{2a_j}(r_j), \quad |X/R| \le fQ_0, \quad |X/R|_{2'} \le f_{2'}Q_o. \tag{7.11}$$

The kernel of the symplectic action consists of inner automorphisms of R: a center-fixing central error is identified with a linear functional on each symplectic quotient, and the commutator pairing is nondegenerate. The scalar-centralizer hypothesis then gives the asserted kernel.

For every nonzero $\lambda \in V^*$,

$$|R_\lambda| \le n. \tag{7.12}$$

The trace pairing identifies the additive dual with a K-dual. A stabilizer injects into R/Z, because no nonidentity scalar fixes a nonzero vector, and its image is isotropic for each extraspecial commutator form. Its order is therefore at most $\prod r_j^{a_j} = n$. This proof permits every homogeneous multiplicity.

Also

$$|C_V(x)| \le v^{2/3} \qquad (1 \ne x \in X). \tag{7.13}$$

If a K-linear x acts nontrivially on $R_j/Z(R_j)$, choose $g \in R_j$ with $[x, g]$ noncentral. Such an extraspecial element has all its r_j eigenvalues with equal multiplicity. Working over an algebraic closure $\overline{K}$ of K, let U be a λ-eigenspace of x on $\overline{K} \otimes_K V$. Its intersection with the corresponding eigenspace of x^g lies in the 1-eigenspace of $[x, g]$. Hence $2\dim_{\overline{K}} U - \dim_K V \le \dim_K V/r_j$, so every eigenspace of x has dimension at most $(r_j+1)\dim_K V/(2r_j) \le 2\dim_K V/3$. If all symplectic actions are trivial, remove an inner automorphism to put x in R; a noncentral element has a nonidentity scalar commutator, and the same argument gives the stronger bound $\dim_K V/2$. Nonidentity scalars are fixed-point free. For a nontrivial field action commute with a scalar of K; this also gives the bound $v^{1/2}$ over $\mathbb{F}_2$.

Burnside's lemma, (7.12), and the little-groups formula yield

$$k(VX) \le |X| + \frac{n}{|R|}v + n|X/R|v^{2/3}. \tag{7.14}$$

For example, the number of nonzero X-orbits is at most $v/|X| + v^{2/3}$, and each stabilizer has order at most $n|X/R|$.

Lemma 7.7. *Under the explicit core hypotheses above, $k(VX) \le v|X|_2$. This includes the scalar case $n = 1$.*

Proof. If $n = 1$, X is cyclic-by-cyclic and is solvable. Suppose $n > 1$. For $U = q^m$ put $T = fnQ_0/U^{n/3}$. Whenever $q^{2nm/3-1} \ge n^2$, (7.14) gives

$$\frac{k(VX)}{v} \le \frac{1}{n} + (1 + 1/n)T. \tag{7.15}$$

The absorption condition holds for $q \ge 16, n \ge 15, m \ge 1$, and for $q \ge 4, n \ge 5, m \ge 2$: check the boundary and differentiate $(2nm/3 - 1)\log q - 2\log n$ in n. The symplectic order formula gives

$$Q_0 < n^{1+2\log_3 n}. \tag{7.16}$$

Indeed, put $x_j = a_j \log r_j$; use $\log r_j \ge \log 3$ and $\sum x_j^2 \le (\sum x_j)^2$. For $U \ge 16, n \ge 35$,

$$T \le \frac{(\log_2 U)n^{2+2\log_3 n}}{U^{n/3}} \le \frac{4n^{2+2\log_3 n}}{16^{n/3}} < \frac{1}{2}.$$

Both monotonicities follow by differentiation. In n, the logarithmic derivative is $(2+4\log_3 n)/n-(\log 16)/3<0$ for $n\ge 35$. At 35 the exact comparisons $35^4<3^{13}$ and $35^{51}<2^{262}$ prove the bound. Equation (7.15) is then less than one.

If there are at least two distinct core primes, $q\ge 16$ and $n\ge 15$. Below 35 the only values are $15,21,33$, with $Q_0=2880,8064,31680$, respectively. Substitution in (7.15), using $T\le 4nQ_0/16^{n/3}$, gives bounds less than $243/1000, 51/1000, 31/1000$, respectively. Thus this entire case satisfies $k(VX)<v$.

For one prime and $m\ge 2$, $n=3$ gives a solvable group. For a prime $n=r\ge 11$, the condition $r\mid q-1$ gives

$$T<\frac{(\log_2(r+1))r^4}{(r+1)^{2r/3}}\le r^{5-2r/3}<\frac{1}{2}.$$

The remaining $n<35$ are covered by the following complete table. The last column bounds T; larger q or m improve it.

n	$q_{\min}$	Q_0	T bound
5	16	120	1/4
7	8	336	441/1024
9	16	51840	3645/32768
25	16	9360000	1/2
27	16	9170703360	1/2

The nonintegral exponents are checked by cubing; for example the first row is $9600^3<16^{10}$. This leaves $q=4, n=9$ or 27.

For multiplicity one and one prime, apply the following precise part of [31, Theorem 1.1]: if an odd group of symplectic type with quotient order r^{2a} has a faithful absolute module of degree r^a over a field K of characteristic two, every subgroup of its $\mathbb{F}_2$-linear normalizer containing it satisfies $k(VX)\le v$, except possibly $(r,a,|K|)=(3,2,4),(3,2,16),(3,3,4)$. When R has a larger cyclic scalar center, apply the theorem to its characteristic extraspecial factor R_0 from Lemma 7.3. The group X normalizes R_0 and contains it, so the theorem still applies to X itself. Thus only these three cases and their stated multiplicities remain.

Let R_a be extraspecial of exponent three and order 3^{2a+1}. On m copies of its degree 3^a module, put $U=q^m$ and

$$b_0(U)=(U+2)/3,\qquad b_a(U)=\frac{U^{3^a}+(3^{2a+1}-3)U^{3^{a-1}}+2}{3^{2a+1}}.$$

The exact formula is

$$k(VR_a)=b_a(U)+2+(3^{2a}-1)b_{a-1}(U). \tag{7.17}$$

Burnside proves the formula for b_a. There are two nonidentity central classes, each contributing one. For a noncentral class, $C_{R_a}(x)=\langle x\rangle\times R_{a-1}$, and its fixed eigenspace is the faithful R_{a-1}-module; this gives the last term. At $(q,a)=(4,2),(16,2),(4,3)$ the counts are 1624, 282813528, and 8237037448664.

The finite group data proved in Subsection 7.10 give, for nonsolvable J,

$$\frac{k(J)}{|J|_2}\le\begin{cases}9/8 & J\le \mathrm{Sp}_4(3),\\ 2187/4 & J\le \mathrm{Sp}_6(3).\end{cases}$$

The linear normalizer quotient has scalar kernel of order at most $(q-1)/3$, and the field quotient in these three cases is a 2-group. If its symplectic image is solvable, so is X. Otherwise Nagao's inequality and (7.17) give the normalized bounds

$$\frac{1827}{262144},\qquad \frac{1590826095}{68719476736},\qquad \frac{2251800112528521}{9007199254740992},$$

all less than one. The expression $k(VR_a)/U^{3^a}$ is a sum of positive multiples of nonpositive powers of U, and hence decreases. The same calculation therefore covers the multiplicities with $q = 4, n = 9, 27$ left above. This exhausts all cases. □

Lemma 7.8. *In Lemma 7.3, if $L \neq 1$ and R is nonabelian, then*

$$k(VN) < v|N|_2/6, \qquad N = C_E(L). \tag{7.18}$$

Proof. The core multiplicity is at least two: L centralizes R and acts nontrivially and perfectly on the multiplicity space. Thus $m \geq 2$, $q \geq 4$, and $z \geq 3$. Divide (7.14) by the $v|N|_2$, and put $T = f_{2'}nQ_o/q^{nm/3}$. Since $q^{2nm/3-1} \geq n^2$, one obtains

$$\frac{k(VN)}{v|N|_2} \leq \frac{1}{3n} + (1 + 1/n)T. \tag{7.19}$$

For $n \geq 37$, (7.16) and $m \geq 2$ give

$$T \leq \frac{2n^{2+2\log_3 n}}{16^{n/3}} < \frac{1}{8}.$$

The logarithmic derivative is negative on this range; at 37 use $37^3 < 3^{10}$ and $4096 \cdot 37^{26} < 16^{37}$. Hence the right side of (7.19) is less than $1/111 + 38/(8 \cdot 37) = 61/444 < 1/6$.

If $n = r$ is prime, write $R = ZR_0$ with $|R_0| = r^3$ and put $U = q^m$. Counting the scalar classes and the $r^2 - 1$ noncentral classes having eigenvalue one gives

$$k(VR) = \frac{z(r^2 + r - 1)}{r} + \frac{U^r + c_rU - c_r - 1}{zr^2}, \qquad c_r = r(r-1)(r+1)^2. \tag{7.20}$$

Indeed the identity orbit count is $[U^r + r(r^2-1)U + zr^2 - 1 - r(r^2-1)]/(zr^2)$; each of those noncentral classes contributes $(U + z - 1)/z$. This proves the displayed formula. For $N_1 = C_N(Z)$, the exact kernel calculation gives $N_1/R \leq \mathrm{SL}_2(r)$. Every subgroup $J \leq \mathrm{SL}_2(r)$ satisfies $k(J)/|J|_2 \leq r$. Let $\overline{J}$ be its image in $\mathrm{PSL}_2(r)$. If $-I \in J$, then $k(J) \leq 2k(\overline{J})$ and $|J|_2 = 2|\overline{J}|_2$. If $-I \notin J$, projection is an isomorphism; moreover J has odd order, since $-I$ is the only involution of $\mathrm{SL}_2(r)$. Dickson's classification [33, Appendix A] gives cyclic, dihedral, Borel, A_4, S_4, A_5, or the full group. The Borel ratio is bounded by $k(C_r : C_d) = d + (r-1)/d \leq r$; all the other ratios are at most r directly. For $r = 3$, intersect with the normal Q_8 in $\mathrm{SL}_2(3)$ to obtain the sharper bound three.

The quotient N/N_1 acts faithfully on Z, so its odd order factor is at most $\mathrm{ord}_z(2)_{2'}$. Here $\mathrm{ord}_z(2)$ is the multiplicative order of two modulo z, and ϕ is Euler's totient function. If $r = 3$, this is at most $\phi(z)/2 \leq z/3$, since $3 \mid z$ forces the order of two modulo z to be even. Since $U \geq (z+1)^2$, Nagao and (7.20) give

$$\frac{k(VN)}{v|N|_2} \leq \frac{1}{9} + \frac{11z^2}{3(z+1)^6} + \frac{32}{3(z+1)^4} \leq \frac{5929}{36864} < \frac{1}{6}.$$

For $r \geq 5$, use $\mathrm{ord}_z(2)_{2'} \leq \phi(z) \leq z(r-1)/r$ instead. The bound becomes

$$\frac{r-1}{r^2} + \frac{r(r-1)(r^2+r-1)}{(r+1)^{2r}} + \frac{(r-1)^2}{r(r+1)^{2r-4}} < \frac{4}{25} + \frac{1}{6^6} + \frac{1}{6^5} < \frac{1}{6}.$$

Here $z^2/(z+1)^{2r}$ decreases for $z \geq r$, and the two displayed numerators are bounded by $(r+1)^4$ and $r(r+1)$, respectively.

The composite odd $n < 37$ are exactly $9, 15, 21, 25, 27, 33, 35$. Apart from $(n, q) = (9, 4)$, the following table gives the endpoint for the decreasing bound $T \leq (\log_2 q)nQ_o/q^{2n/3}$:

n	9	15	21	25	27	33	35
$q_{\min}$	16	16	16	16	4	16	16
Q_o	405	45	63	73125	8955765	495	315

At every entry $(100fnQ_o)^3 < q^{2n}$, so $T < 1/100$ and (7.19) is less than $1/27 + 1/90 < 1/6$. For $(n, q) = (9, 4)$ the scalar order is three and the field quotient is a 2-group. The complete finite subgroup bound $k(J)/|J|_2 \le 27$ for every $J \le \mathrm{Sp}_4(3)$ is proved in Subsection 7.10. Formula (7.17) at $U \ge 16$ therefore gives

$$\frac{k(VN)}{v|N|_2} \le \frac{27 \cdot 282813528}{16^9} = \frac{954495657}{8589934592} < \frac{1}{6}.$$

All parameters have now been covered. □

7.4. **Component supports and the outer quotient.** For a nonabelian simple group S write

$$\beta(S) = \max_{S \le A \le \mathrm{Aut}(S)} \frac{k(A)}{|A|_2}.$$

If $F^*(H) = S_1 \times \cdots \times S_t$ and T is the permutation image on these factors, then

$$\frac{k(H)}{|H|_2} \le \frac{k(T)}{|T|_2} \prod_i \beta(S_i). \tag{7.21}$$

Indeed, the kernel J of that permutation action contains the full socle and embeds in $\prod_i \mathrm{Aut}(S_i)$. Let J_i be the subgroup of J acting trivially on the first i factors. The quotient J_{i-1}/J_i is an almost simple group with socle S_i. Repeated Nagao inequalities multiply their class numbers, while their two-parts multiply *exactly*. A final application to $J \lhd H$ proves the formula, including coupled outer actions.

By [28, Theorem 1.1], a permutation group of degree $t \ge 4$ has at most $5^{(t-1)/3}$ classes. Including the degrees $1, 2, 3$ gives $k(T) \le 2^{t-1}$. A sharper sufficient normalized bound for small orbits is

$$h(1) = h(2) = 1, \quad h(3) = h(4) = 3, \quad h(t) = 2^{t-1}\ (t \ge 5), \qquad \frac{k(T)}{|T|_2} \le h(t). \tag{7.22}$$

For degree four, an intransitive group either has a three-point orbit or is a 2-group. A transitive group has two-part at least four, so the class bound gives a ratio at most two. If components are partitioned into c nonempty unions of simple isomorphism types of sizes $t_1, \dots, t_c$, the same coordinate-kernel argument gives $k(T)/|T|_2 \le \prod_j h(t_j) \le 2^{t-c}$.

Lemma 7.9. *Let L_i be a component, with absolute factor degree m_i in (7.2), and let $d = \dim_{\mathbb{F}_2} V$. If x normalizes L_i but does not centralize it, then*

$$\mathrm{codim}_{\mathbb{F}_2} C_V(x) \ge d/(2m_i).$$

If x moves a component, the lower bound is $d/4$. Alternatively, suppose every prime-order automorphism of S_i has at most r_i conjugates by S_i generating a subgroup containing S_i. An element with nontrivial induced action on S_i has support at least d/r_i.

Proof. For the first assertion choose $y = [x, a] \ne 1$ in L_i. On every absolute tensor constituent, $y - 1$ has rank at least the total degree divided by m_i, because the component representation is faithful. Frobenius conjugate constituents have the same degree and kernel; summing over them and over the homogeneous multiplicity gives $\mathrm{rank}(y - 1) \ge d/m_i$. On the other hand $\mathrm{rank}(y - 1) \le 2\,\mathrm{rank}(x - 1)$, since y is a product of two conjugate matrices, one inverted. This argument is over $\mathbb{F}_2$ and also applies to transformations semilinear over K.

For a moved component choose $a, b \in L_i$ mapping to generators of S_i; they generate L_i, since it is perfect. With $v(u) = u^{-1}u^x$ the group generated by x, x^a, x^b, x^{ab} contains $v(a), v(b), v(ab)$ and

$$v(a)v(b)v(ab)^{-1} = a^{-1}b^{-1}ab.$$

The last element has nontrivial image in S_i. The intersection of $\langle v(a), v(b), v(ab)\rangle$ with L_i is normal under the inner actions of a, b, and therefore maps onto S_i. Perfectness puts L_i itself in the subgroup. The four conjugate fixed spaces have zero intersection, proving the asserted support.

For the last assertion, first pass to a power whose induced automorphism has prime order. Lift the conjugating elements to L_i. The resulting generators lie in one coset modulo L_i, so their generated group has cyclic quotient modulo its intersection with L_i. Its derived group maps onto S_i; hence that intersection contains L_i. Their common fixed space is zero. The intersection codimension inequality proves the bound, and taking the power can only increase fixed vectors. □

Lemma 7.10. *Suppose $L \neq 1$ in Lemma 7.3. Put $N = C_E(L)$ and $H = E/N$, and let $B \geq k(H)/|H|_2$. Let $0 < M \leq \prod_{i=1}^{t} m_i$, where the m_i are the absolute component degrees in (7.2); thus $\dim_K V \geq nM$. Suppose the support bounds just proved give $|C_V(x)| \leq v^{1-1/r}$ for $x \notin N$, with $r \geq 3$. Then the affine bound holds in either of the following situations:*

$$B \leq 1, \qquad \text{or} \qquad M/r \geq \log_2(2B).$$

For the second alternative with $B > 1$, it is enough to use the following overlapping ranges: $B \leq 6$ or $B \geq 4$ when R is nonabelian; $B \leq 2$ or $B \geq 2$ when $R \neq 1$ is abelian; and the ordinary support criterion when $N = 1$.

Proof. Lemma 7.7 applies to N: the core centralizer is Z, and the homogeneous multiplicity is unrestricted. Nagao gives $k(VE) \leq k(VN)k(H)$, proving the first alternative. Lemmas 7.6 and 7.8 prove the stated small-B ranges. All nonidentity elements of N satisfy the needed support bound, by (7.13) or Lemma 7.6.

For an abelian nontrivial core, $z \leq 2^f - 1$, $f \geq 2$, and

$$\frac{k(E)}{|E|_2} \leq z f_{2'} B \leq (2^f - 1) f B.$$

If $B \geq 2$, then $B^{f-1} \geq 2^{f-1} \geq f$, and the profile gives $v^{1/r} \geq (2B)^f \geq 2^f f B > (2^f - 1) f B + 1$. Use Lemma 7.4.

For a nonabelian core put $b = \log_2 B \geq 2$. Equations (7.11) and (7.16) give

$$\frac{k(E)}{|E|_2} < 2^f f n^2 Q_o B, \qquad \log_2(n^2 Q_o) < 4n \quad (n \geq 3).$$

For the latter inequality, use $\log n \leq (2/e)\sqrt{n} < (3/4)\sqrt{n}$, $\log 2 > 2/3$, $\log 3 > 1$, and $\sqrt{n} < (3/5)n$ for $n \geq 3$; the resulting coefficient is $297/80 < 4$. Therefore

$$f + \log_2(f n^2 Q_o) + b < 2f - 1 + 4n + b.$$

The profile gives $\log_2(v^{1/r}) \geq f n(1 + b)$, whose difference from the last upper bound is at least $2n(b - 1) - b - 3 \geq 1$. Thus $v^{1/r} > 2 \cdot 2^f f n^2 Q_o B$, which exceeds $k(E)/|E|_2 + 1$. Finally, when $N = 1$, the threshold follows from $v^{1/r} \geq 2B \geq B + 1$. This proves the lemma. □

7.5. **Complete simple-group numerical inputs.** We now record the specific published representation and class estimates used in the proof. They are stated for projective representations, so odd central covers are included. Write $L_n(q) = \mathrm{PSL}_n(q)$, $U_n(q) = \mathrm{PSU}_n(q)$, and use PSp and PΩ for the simple projective classical groups. Isomorphic small groups are assigned to a single row.

The half-class families. Except for

$$L_2(2^a), \quad L_3(2), \quad A_6, \tag{7.23}$$

every simple group of Lie type in characteristic two has $\beta(S) \le 1/2$. In the generic ranges, the estimates used from [50, Propositions 2.5–2.8 and proof of Theorem 2.9] give $k(A) \le k(U)$ for every almost simple A and a unipotent subgroup U of the following order, strictly below the Sylow order:

Type	$\lvert U\rvert$	$\lvert S\rvert_2$
A_h, $h \ge 3$	$q^{\lfloor (h+1)^2/4 \rfloor}$	$q^{h(h+1)/2}$
${}^2A_h$, $h \ge 3$	q^{2h-1}	$q^{h(h+1)/2}$
C_h, $h \ge 2$	$q^{h(h+1)/2}$	q^{h^2}
D_h, $h \ge 4$	$q^{h(h-1)/2}$	$q^{h(h-1)}$
${}^2D_h$, $h \ge 5$	$q^{2+(h-1)(h-2)/2}$	$q^{h(h-1)}$
${}^2D_4$	q^6	q^{12}
G_2, $q \ge 4$	q^3	q^6
${}^3D_4$	q^5	q^{12}
F_4	q^9	q^{24}
${}^2F_4$, $q \ge 8$	q^5	q^{12}
$E_6, {}^2E_6, E_7, E_8$	elementary abelian	nonabelian Sylow

In the last row the cited proof constructs an elementary abelian witness; it is proper because adjacent simple root groups have a nontrivial commutator. The small parameters in that comparison have the following upper bounds for every $k(A)$:

S	bound	$\lvert S\rvert_2/2$	S	bound	$\lvert S\rvert_2/2$
$L_4(2)$	22	32	$L_4(4)$	168	2048
$L_5(2)$	54	512	$U_4(2)$	25	32
$U_4(4)$	256	2048	$U_4(8)$	3612	131072
$U_5(2)$	47	512	$\mathrm{Sp}_6(2)$	30	256
$\mathrm{Sp}_4(4)$	64	128	$O_8^+(2)$	68	2048
$O_8^-(2)$	60	2048	$O_{10}^-(2)$	230	524288
${}^3D_4(2)$	49	2048			

For $O_8^+(4)$ the bound is $< 4^6 < 4^{12}/2$. For $\mathrm{Sp}_4(q)$, $q \ge 8$, the witness has order q^3. For $L_3(q), U_3(q)$, $q \ge 4$, the non-inner-diagonal bounds are q^2, apart from $U_3(4).2$ with 20 classes; the inner-diagonal formula is bounded by $q^2 + q + 4 \le q^3/2$. For $L_3(4)$ the all-extension bound is $28 < 32$. For the Suzuki groups, $\lvert U\rvert = 2q \le q^2/2$; the only extra small field extension is $\mathrm{Sz}(8).3$, with $17 < 32$ classes. The Tits group has $(k, \lvert S\rvert_2) = (22, 2048)$ and its extension has $(29, 4096)$. Under the convention $G_2(2)' = U_3(3)$, the two class counts are $14, 16$ and the two-parts $32, 64$. This proves the asserted half-class bound in every listed family.

For the three excluded types the exact sufficient bounds are

$$\beta(L_2(2^a)) \le 11/8, \qquad \beta(L_3(2)) = 3/4, \qquad \beta(A_6) = 7/8. \tag{7.24}$$

Here $k(L_2(q)) = q + 1$; all proper extensions have class number at most their two-part except $L_2(8).3$, whose ratio is $11/8$. The additional exception $L_2(16).4$ has 17 classes and two-part 64. For A_6, the $(k, 2\text{-part})$ pairs are $(7, 8)$, $(11, 16)$, $(11, 16)$, $(8, 16)$ and $(13, 32)$.

Let $p(n)$ be the number of partitions of n and $s_2(n)$ its binary digit sum. For alternating groups A_n, $n \ge 8$, the same half-class bound follows from $k(A_n) < k(S_n) = p(n)$ and

$$p(n) \le 2^{n - s_2(n) - 2} = \lvert A_n\rvert_2/2. \tag{7.25}$$

Indeed Euler's product gives $p(n) < \exp(\pi\sqrt{2n/3}) < 2^{4\sqrt{n}}$. For $n \geq 64$, $n - s_2(n) - 2 \geq n - \log_2(n+1) - 2 \geq n/2 \geq 4\sqrt{n}$. The complete range $8 \leq n \leq 63$ is covered by the integer partition recurrence displayed in Subsection 7.10; the minimum ratio of the right side of (7.25) to $p(n)$ is $32/30$ at $n = 9$.

For the sporadic groups the following are $k(S)/|S|_2$. Every nontrivial outer extension, whose index is two, has no larger ratio.

S	ratio	S	ratio
M_{11}	5/8	M_{12}	15/64
M_{22}	3/32	M_{23}	17/128
M_{24}	13/512	J_1	15/8
J_2	21/128	J_3	21/128
J_4	31/1048576	Co_1	101/2097152
Co_2	15/65536	Co_3	21/512
Fi_{22}	65/131072	Fi_{23}	49/131072
Fi'_{24}	27/524288	HS	3/64
McL	3/16	He	33/1024
Ru	9/4096	Suz	43/8192
$O'N$	15/256	HN	27/8192
Ly	53/256	Th	3/2048
B	23/274877906944	M	97/35184372088832

These are ordinary ATLAS character-table counts [17, 11]. Thus all sporadic groups except M_{11}, J_1 belong to the half-class families.

Cross-characteristic degree and generation bounds. For odd q, the following table gives a lower bound m for every nontrivial absolute projective characteristic-two degree, a bound r on the number of conjugates of a prime-order automorphism needed to generate a subgroup containing the socle, and an upper bound B on $\beta(S)$. The degree and generation assertions are [46, Proposition 2.1, Table 2.1, Propositions 2.3–2.6], which states the needed bounds including the small exceptions. The class estimates are those of [50, Propositions 2.5–2.8 and proof of Theorem 2.9]: in particular $k(A) \leq 100q^h$ for untwisted rank h, with the rank-one and rank-two refinements used below.

Socle and range	m	r	B
$L_3(q)$, $q \geq 3$	q^2+q-1	4	$q^3/16$
$U_3(q)$, $q \geq 5$	$q(q-1)$	3	$q^3/16$
$L_4(q), U_4(q)$, $q \geq 5$	$q^2(q-1)$	6	$100q^3/8$
$L_n(q), U_n(q)$, $n \geq 5$	$(q-1)q^{n-2}$	n	$100(q/2)^{n-1}$
$\mathrm{PSp}_4(q)$, $q \geq 9$	$(q^2-1)/2$	5	$100q^2/64$
$\mathrm{PSp}_6(q)$, $q \geq 5$	$(q^3-1)/2$	6	$100q^3/256$
$\mathrm{PSp}_{2h}(q)$, $h \geq 4$	$(q^h-1)/2$	$2h$	$100q^h/2^{3h-1}$
$\mathrm{P}\Omega_{2h+1}(q)$, $h \geq 3$	q^{2h-3}	$2h+1$	$100q^h/2^{3h}$
$\mathrm{P}\Omega^{\pm}_{2h}(q)$, $h \geq 4$	$q^{2h-3}-2$	$2h$	$25q^h$
$G_2(q)$, $q \geq 5$	q^3-1	5	$25q^2$
${}^3D_4(q)$, $q \geq 3$	$(q^5-1)/2$	7	$25q^4$
$F_4(q)$, $q \geq 3$	q^5	8	$25q^4$
$E_6(q), {}^2E_6(q)$, $q \geq 3$	q^5	9	$25q^6$
$E_7(q)$, $q \geq 3$	q^5	10	$25q^7$
$E_8(q)$, $q \geq 3$	q^5	11	$25q^8$
${}^2G_2(q)$, $q \geq 27$	$q(q-1)$	3	$25q^2$
$L_2(q)$, $q \geq 37$	$(q-1)/2$	4	$(q+2)/4$

All unqualified odd fields in the table start at three. The bounds for symplectic and orthogonal groups include the transvection and reflection exceptions to generic generation bounds. In particular the $q = 3$ odd-orthogonal degree is 27 at rank three. For the even-orthogonal minus type, the product bound in Lee's table implies the stated degree because

$$(q^h + 1)(q^{h-1} - q) - q^{2h-3}(q^2 - 1) = q^{h+1}(q^{h-4} - 1) + q^{h-1} - q \geq 0;$$

the plus-type formulas give at least the same lower bound.

The two-parts used in this table also follow directly from the order formulas. Use $\epsilon = 1$ for $L_n(q)$ and $\epsilon = -1$ for $U_n(q)$, and put $a = v_2(q - \epsilon)$, $b = v_2(q + \epsilon)$, $j = \lfloor n/2 \rfloor$. Then

$$v_2(|S|) = (n - 1)a + jb + v_2(j!) - \min(v_2(n), a) \geq n - 1.$$

At $n = 3$ one has $|S|_2 \geq 16$. For symplectic and odd-orthogonal rank h,

$$|S|_2 = 2^{hv_2(q^2-1)+v_2(h!)-1} \geq 2^{3h+v_2(h!)-1}.$$

For every other simple group $|S|_2 \geq 4$ suffices. The rank-two class bound q^3 includes the inner-diagonal formula $q^2 + q + 4 \leq q^3$ and the exception $k(L_3(3).2) = 17 < 27$. For $U_3(3)$ one instead uses its exact half-class bound.

The remaining odd rank-one profiles are

q	17	19	23	25	27	29	31
m	8	9	11	12	13	14	15
r	3	3	3	4	3	3	3
B	19/16	3	25/8	27/8	29/4	31/4	33/32

(7.26)

The refinement at 19 uses $k(L_2(19)) = 12$, $|L_2(19)|_2 = 4$ and $k(\mathrm{PGL}_2(19)) = 21$, with two-part eight. Odd field exponent at 27 permits $r = 3$. The small fields $5, 7, 9$ are $A_5, L_3(2), A_6$. For $q = 11, 13$ use the small table below.

Every row of the large tables satisfies

$$m \geq 8, \quad r \geq 3, \quad m \geq 2r, \quad m/r \geq \log_2(2B). \tag{7.27}$$

Here is a full numerical verification of the unbounded ranges. For fixed rank, write $F(q) = m(q)/r$ and $B = Cq^a$; then $qF'(q) > 3a/2$ at the first stated parameter and thereafter, proving $F(q) - \log_2(2B)$ increases. The boundary comparisons are $2^m \geq (2B)^r$ with the exact values in the tables. For the first four rows the pairs $(m/r, 2B)$ are $(11/4, 27/8)$, $(20/3, 125/8)$, $(100/6, 3125)$, $(54/5, 2025/2)$. The tight symplectic boundary is $2^8 > 2025/8$ at $\mathrm{PSp}_4(9)$. For variable rank it suffices to take $q = 3$. The second difference of $(q^j - c)/(aj + b)$, after multiplication by a positive denominator, has numerator

$$q^j\{u^2(q - 1)^2 + au(q - 1)(q - 3) + 2a^2\} - 2ca^2, \qquad u = aj + b,$$

which is positive in the ranges above. For orthogonal rows use the geometric ratio q^2. Thus the first differences increase. At the base rank the linear/unitary difference is $81/5 > \log_2(3/2)$; the symplectic and odd-orthogonal logarithmic slopes are negative; the even-orthogonal first difference exceeds $2 > \log_2 3$. For rank one, $(q - 1)/8 - \log_2((q + 2)/2)$ is positive at 37 and has positive derivative thereafter. The short exact-integer verification in Subsection 7.10 includes every boundary of these comparisons, including (7.26).

The small odd-field class data not assigned to these ranges are

S	$k(S)$	$\lvert S\rvert_2$	$k(S.2)$	$\lvert S.2\rvert_2$
$\mathrm{PSp}_4(3)$	20	64	25	128
$\mathrm{PSp}_4(5)$	34	64	47	128
$\mathrm{PSp}_4(7)$	52	256	77	512
$\mathrm{PSp}_6(3)$	74	512	76	1024
$G_2(3)$	23	64	28	128
$L_2(11)$	8	4	13	8
$L_2(13)$	9	4	15	8

The first, third, fourth and fifth rows have half-class bounds. The second does not: its ratio is $17/32$, and its profile $(m, r, B) = (12, 5, 17/32)$ satisfies (7.27). The group $L_4(3)$ has the all-extension bound $56 < 128/2$. For $U_4(3)$, $k(S) = 20$, $|S|_2 = 128$ and $\mathrm{Out}(S) = D_8$; Nagao's index factor cancels the two-part of every extension, so $\beta(S) \le 5/32$.

The remaining small profiles. After the half-class and large families, exactly the following seven numerical classes remain:

class	socle	m	r	B
R	$L_2(2^a)$ $(a \ge 2)$, $L_3(2)$	2	4	11/8
$A6$	A_6	3	5	7/8
$A7$	A_7	4	6	9/8
$J1$	J_1	10	8	15/8
$M11$	M_{11}	4	8	5/8
$L11$	$L_2(11)$	5	3	2
$L13$	$L_2(13)$	6	4	9/4

(7.28)

For the R and $A6$ generation bounds use [32, Lemma 3.1], equivalently [46, Proposition 2.3]; the value five is only needed for the involutory field automorphism at $q = 9$. The projective degree for $L_3(2)$ and A_6, including $3.A_6$, is at least three. For A_7 the characteristic-two degrees are $1, 4, 4, 6, 14, 20$; the additional faithful degrees of $3.A_7$ are $6, 6, 15, 15, 24, 24, 24, 24$. The central two-part acts trivially and cannot supply another faithful case. The generation bound is six by [32, Lemma 6.1]. Jansen's degree table [40] gives the stronger bounds $20, 10$ for J_1, M_{11}; their multipliers are trivial. Their prime-automorphism generation bounds are at most four by [58, Theorem 3], so the displayed weaker eight is safe. Dickson's classification implies that an absolute projective characteristic-two representation of degree two has simple image $L_2(2^a)$. Consequently every half-class component has degree at least three, and Lemma 7.9 gives it support denominator $2m$ without any generation classification.

7.6. **One component, including the small almost simple quotients.** We first give the small-group argument for an almost simple central quotient, over any field of definition of the module. For $S \lhd H$ with S simple put

$$\gamma_S(H) = \max_{J \le H,\ S \not\le J} \frac{|J| k(J)}{|H|}.$$

If a complete list of overgroups M covers the subgroups not containing S, then

$$\gamma_S(H) \le \max_M \frac{|M| k(M)}{|H|}, \tag{7.29}$$

since $k(J) \le [M : J] k(M)$. Let $E/Z(E) = H$, put $z = |Z(E)|$, and let V be faithful irreducible. For a nonzero dual vector, its stabilizer intersects the scalar center trivially and its image cannot contain S; otherwise perfectness would put the lifted component in the stabilizer. Thus

$$k(VE) \le z k(H) + \frac{v - 1}{z} \gamma_S(H). \tag{7.30}$$

Since $v \geq (z+1)^2$, the conditions $\gamma_S(H) \leq 1$ and $k(H) \leq 2|H|_2 + 1$ imply $k(VE) \leq v|E|_2$: at the endpoint the difference is $Pz^2 - 2z + P - 2 \geq 2P - 4 \geq 0$, with $P = |H|_2$.

The complete small capacities needed here are

H	A_7	J_1	$L_2(11)$	$\mathrm{PGL}_2(11)$	$L_2(13)$	$\mathrm{PGL}_2(13)$
$\gamma_S(H)$	1	4/133	7/12	11/12	4/7	13/14

For J_1 one may apply (7.30) directly when $z = 1$: $15 + (4/133)(v-1) \leq 8v$ for $v \geq 2$. When $z > 1$, (7.7) gives $30v/(z+1) \leq 15v/2 < 8v$. The analogous argument for A_7 gives $v + 8 \leq 8v$ when $z = 1$, and $18v/(z+1) < 8v$ otherwise. The overgroup lists establishing all these capacities are given in Subsection 7.10. The conjugacy classes of $\mathfrak{S}_7$ are parametrized by the partitions of seven, so $k(\mathfrak{S}_7) = 15$ and $|\mathfrak{S}_7|_2 = 16$. For $H = \mathfrak{S}_7$, (7.7) therefore gives

$$k(V \rtimes E) \leq 15\left(z + \frac{v-1}{z}\right) \leq 15v < 16v = v|E|_2.$$

For the rank-one defining-characteristic group $S = L_2(q)$, q even, its odd multiplier is trivial. A scalar central extension is therefore $Z \times S$. Here a direct torus calculation supplies the affine bound on every faithful module. If T_-, T_+ are the two torus types of orders $q-1, q+1$, every nonidentity odd element of S has torus centralizer and is fused only with its inverse. Hence

$$\Lambda_2(S,V) = n(S,V) + \frac{1}{2}\sum_{t \in T_- \setminus \{1\}} n(T_-, V^t) + \frac{1}{2}\sum_{t \in T_+ \setminus \{1\}} n(T_+, V^t) \leq \frac{k(VT_-) + k(VT_+)}{2} \leq v.$$

For $Z \times S$, sum the same identity over $z \in Z$ and replace $T_\pm$ by $Z \times T_\pm$. Both are faithful odd abelian acting groups, so the coprime bound applies. This proves the weak affine bound by applying $k(Y) \leq |Y|_2 \ell_2(Y)$ on the dual. All proper almost simple extensions except $L_2(8).3$ have $k(H) \leq |H|_2$, as recorded above, and (7.7) applies. For $H = L_2(8).3$, its maximal subgroups have $(|M|, k(M)) = (504, 9), (168, 8), (54, 10), (42, 7)$; the first is S. Since $[H : S] = 3$, S is the only proper overgroup containing the socle. Its maximal subgroups have pairs $(56, 8), (18, 6), (14, 5)$. Thus a stabilizer avoiding S has index at least nine and $k(J) \leq \sqrt{168 \cdot 8} < 37$, hence $k(J) \leq 36$. Each nonzero orbit has at least nine vectors, so the total nonzero-orbit contribution is at most $36(v-1)/9 = 4(v-1)$. It follows that $k(VH) \leq 11 + 4(v-1) = 4v + 7 \leq 8v$ for $v \geq 2$. For a nontrivial scalar center (7.7) instead gives $22v/(z+1) < 8v$.

These arguments, the half-class bound and the large support table prove the affine bound for every almost simple central quotient. For the support-table assertion with a central scalar extension, write $\mathrm{End}_E(V) = \mathbb{F}_{2^c}$. The scalar center satisfies $z \leq 2^c - 1$. The nontrivial component has degree at least two over this field, so $v \geq 2^{2c} \geq (z+1)^2$. If $B \leq 1$, then (7.7) gives

$$k(V \rtimes E) \leq v\, k(H) \leq v|H|_2 = v|E|_2.$$

For $B > 1$, the inequality $m/r \geq \log_2(2B)$ gives

$$v^{1/r} \geq (2B)^c \geq 2^c B \geq zB + 1 \geq k(E)/|E|_2 + 1.$$

Thus Lemma 7.4 applies over every finite field of definition.

Now suppose the layer consists of one component with an arbitrary Fitting subgroup. If the core is nonabelian, all non-large profiles have $B \leq 9/4 < 6$, and all large profiles are covered by Lemma 7.10. If $R = Z \neq 1$, every non-large profile except $L_2(13)$ has $B \leq 2$. For $L_2(13)$ use $B = 9/4$. A nontrivial field quotient is paid by the factor-three reserve. Otherwise $N = Z$, $m \geq 6$, and

$$v^{1/4} \geq q^{3/2} \geq \frac{9}{4}(q-1) + 1 \qquad (q \geq 4).$$

At four the values are 8 and 31/4, and the difference has positive derivative. The large profiles again use Lemma 7.10. Finally $R = 1$ implies $N = 1$, which is the already proved almost simple case. This completes every one-component action.

7.7. **Tensor profiles and their five residual cases.** For $t \geq 2$, let m_i be the component degrees and r_i their support denominators. For a half-class component use $r_i = 2m_i$. For the other components use the published generation bounds above. Put

$$M = \prod_i m_i, \quad R_0 = \max(4, r_1, \ldots, r_t), \quad B = \frac{k(T)}{|T|_2} \prod_i \beta_i,$$

where $\beta_i \geq 1/2$ are the sufficient bounds just proved; any smaller bound is replaced by $1/2$. The factor four covers moved components. We will prove either $B \leq 1$ or $M/R_0 \geq \log_2(2B)$, apart from explicitly resolved small profiles. Lemma 7.10 will then apply.

Set $a_i = \log_2(2\beta_i)$. Partition simple types into the following disjoint kinds: half-class factors G; large factors P satisfying (7.27); and the seven classes F in (7.28). The field ranges in a named simple type are kept together when applying the permutation estimate. Splitting into any of these disjoint unions of simple isomorphism types gives

$$\log_2(2B) \leq \sum_i a_i - (c-1), \tag{7.31}$$

where c is the number of nonempty type classes used. A G factor has $a = 0$, $m \geq 3$, $r = 2m$. A P factor has $m \geq 8$, $r \geq 3$, $m \geq 2r$ and $a \leq m/r$. Each F factor has

$$m \geq 2, \quad r \leq 2m, \quad a < 9/4, \quad a \leq m/2 \ (m \geq 3), \quad a < 3/2 \ (m = 2).$$

Every kind therefore satisfies $a \leq (3/2)(m-1)$. Once a partial product divided by the fixed R_0 is at least $3/2$, multiplying it by a further degree m increases it by at least the cost a. Choose first a component with maximal r_i; all later attachments leave the denominator R_0 unchanged.

If this component has kind P, put $u = m/R_0$. Then $u \geq 2$ and $a \leq 4u/3$; the slight weakening covers $r = 3 < R_0 = 4$. Pair it with any other factor. If the other degree $m' \geq 3$, the surplus is at least

$$um' - 4u/3 - (3/2)(m'-1) \geq m'/2 - 7/6 > 0.$$

If $m' = 2$, the factors have different types, so the one-unit saving in (7.31) gives surplus $(2/3)u - 1/2 > 0$. The pair has value at least four, and absorbs every remaining factor. If a P factor occurs but the maximizing component is of kind G or F, then $u \geq 1/2$, its cost is less than $9/4$, and pairing with the P factor, of degree $m' \geq 8$, gives surplus

$$m'/2 - m'/3 - 9/4 + 1 = m'/6 - 5/4 \geq 1/12 > 0.$$

Again the pair has value at least four. Thus every layer containing a large factor is covered.

If all factors have kind G, (7.21) gives $B \leq 1/2$. Otherwise, suppose a G factor and an F factor occur. If a maximizing factor is F, pair it with a G factor: the surplus is greater than $3/2 - 9/4 + 1 = 1/4$, and the pair has value at least $3/2$. If a maximizing factor is G, use

$$\frac{1}{2} \prod_{i \neq j} m_i \geq \sum_{i \neq j} a_i - 1. \tag{7.32}$$

For one factor this is immediate. For two degree-two factors it is $2 \geq 3 - 1$; for degrees $2, m \geq 3$ it is $m \geq m/2 + 1/2$; for two degrees at least three it follows from $m_1 m_2 \geq m_1 + m_2 - 2$. Thereafter the value is at least two, and the attachment inequality proves induction. Equation (7.31) proves the required profile inequality.

Only the seven finite classes remain. For $t \geq 7$, $R_0 \leq 8$, $M \geq 2^t$, and $\beta_i \leq 9/4$, so

$$M/R_0 \geq 2^{t-3} \geq (9/4)t > t \log_2(9/2) \geq \log_2(2B).$$

The middle inequality starts at $t = 7$ and is preserved by doubling; the strict logarithmic comparison is $(9/2)^4 < 2^9$. For $2 \leq t \leq 6$, the complete rational enumeration of the multisets in (7.28), using h separately on their seven multiplicities, gives

t	total	$B \le 1$	$2^M \ge (2B)^{R_0}$	residual
2	28	6	19	3
3	84	13	70	1
4	210	22	187	1
5	462	31	431	0
6	924	33	891	0
Total	1708	105	1598	5

The complete algorithm, with integer rather than logarithmic comparisons, is in Subsection 7.10. Its five residuals are

$$R^2, \quad RA6, \quad RA7, \quad R^3, \quad R^4. \tag{7.33}$$

We resolve these uniformly in every field, multiplicity and core.

Residual layers of types $L_2(2^a), L_3(2), A_6$. It is convenient to prove the assertion for every number of such components. For $t = 2, 3, 4$ use

$$B_2 = 121/64 < 2, \quad B_3 = 3993/512 < 8, \quad B_4 = 43923/4096 < 11;$$

for $t \ge 5$ use $B_t = 2^{t-1}(11/8)^t$. The support denominator is at most four in the absence of A_6 and at most five otherwise. Since an A_6 factor has degree at least three,

$$M/r \ge 2^{t-2}. \tag{7.34}$$

For $t \ge 5$, $2^{t-2} \ge t \log_2(11/4) = \log_2(2B_t)$; the base is $2^{18} > 11^5$, and doubling proves induction. Thus only $t = 2, 3, 4$ require more argument.

With a nonabelian core, $t = 2$ is within the factor-six reserve. For the other two values use the sharper elementary estimate

$$(3 + 2\log_3 n)\log_2 n < 3n \qquad (n \ge 3). \tag{7.35}$$

Here is a proof with explicit constants. Put $y = \log n \ge 1$. The inequalities $\log 2 > 9/13$, $\log 3 > 13/12$, obtained from the positive series for $2\,\mathrm{atanh}(1/3)$ and $2\,\mathrm{atanh}(1/2)$, bound the left side by $(13/3)y + (8/3)y^2$. Taylor's series for e^y bounds $9e^y - 13y - 8y^2$ below by $9 - 4y - (7/2)y^2 + (3/2)y^3 + (3/8)y^4$. Writing $y = 1 + s$ gives $27/8 - 5s + (13/4)s^2 + 3s^3 + (3/8)s^4 > 0$, since the quadratic has negative discriminant. This proves (7.35). It gives

$$\log_2(k(E)/|E|_2) < 2f - 1 + 3n + \log_2 B_t.$$

For $t = 4$, (7.34) gives $\log_2(v^{1/r}) \ge 4fn$; the sufficient comparison $4fn \ge 2f + 3n + 4$ holds already at $(f, n) = (2, 3)$ and increases. For $t = 3, n \ge 7$, use $2fn \ge 2f + 3n + 3$, starting at $(2, 7)$. The remaining $n = 3, 5$ have $Q_o = 3, 15$; directly,

$$k(E)/|E|_2 < 216f(q-1), \quad v^{1/r} \ge q^6; \qquad k(E)/|E|_2 < 3000f(q-1), \quad v^{1/r} \ge q^{10}.$$

The respective upper ratios $216\log_2 q/q^5 + q^{-6}$ and $3000\log_2 q/q^9 + q^{-10}$ decrease for $q \ge 4$ and are less than one there. The additive unit in the support criterion is thus retained in all cases.

If $N = 1$, the dimension over $\mathbb{F}_2$ is at least 2^{t+1} for $t \ge 2$. To justify the field factor, a nontrivial characteristic-two absolute module for $L_2(2^a)$ is, by Steinberg's tensor product theorem, a tensor over a nonempty set of binary positions. If the set has size b and Frobenius period e, then $b \ge a/e$ and $e2^b \ge e2^{a/e} \ge 2a$. The Frobenius period of a tensor constituent is a multiple of each labeled component period. Hence any such component supplies dimension at least $a2^t \ge 2^{t+1}$. If none occurs, all degrees are at least three, and $3^t \ge 2^{t+1}$. The exact comparisons

$$2^{8/5} > B_2 + 1, \qquad 2^{16/5} > B_3 + 1, \qquad 2^{32/5} > B_4 + 1$$

prove the support criterion. They follow by raising to the fifth power.

It remains to consider an abelian nontrivial core. For $t = 2$ use the factor-two reserve. For $t = 4$, $v^{1/r} \geq q^4 > 11(q-1)\log_2 q + 1$ for $q \geq 4$; dividing by q^4 proves monotonicity and reduces to four. For $t = 3$ one has $v^{1/r} \geq q^2$. If the field quotient $e > 1$, (7.8) gives $k(N)/|N|_2 \leq 3/2, 5, 6$ at $q = 4, 8, 16$, respectively; multiplying by $B_3 < 8$ and adding one is still below q^2. For $q \geq 32$ it gives at most $q/3 + f\sqrt{q}$, and $8(q/3 + f\sqrt{q}) + 1 < q^2$. After division by q^2 the terms decrease; at 32 use $\sqrt{32} > 5$. If $e = 1$ and $q \geq 8$, simply use $q^2 > 8(q-1) + 1$. Only $q = 4, z = 3, e = 1$ remains.

If one absolute component degree is at least three, then $M/r \geq 12/5$ and $v^{1/r} \geq 4^{12/5} > 25 > 3B_3 + 1$. If all three degrees are two, absolute homogeneity over the canonical $\mathbb{F}_4$ and Steinberg's field-period description force every socle to be A_5. We now handle all the outer groups of this cube.

If $M \lhd J$ has 2-group quotient A and an automorphism g of order three preserves M, then

$$|\operatorname{Irr}(J)^g| \leq |A|\,|\operatorname{Irr}(M)^g|. \tag{7.36}$$

Indeed each g-stable A-orbit in $\operatorname{Irr}(M)$ has size a power of two. Its stabilizer in $A : C_3$ contains a Sylow-three subgroup; Sylow conjugacy inside A supplies a label fixed by the specified g. Over that orbit there are at most $|A|$ projective irreducible characters of the inertia quotient. This proves the formula with the Clifford multiplier retained.

Now $A_5^3 \leq H \leq S_5 \wr S_3$. Let U be the image in $C_2^3 : S_3$, $A = U \cap C_2^3$, and J the preimage of A. If the permutation image is a 2-group, then $k(H)/|H|_2 \leq (5/4)^3 = 125/64 < 2$. Otherwise first take its subgroup above C_3 if needed. A Sylow-three subgroup of U projects isomorphically to the cycle group and is conjugate, by the base, to the pure coordinate cycle. The standard section $C_2 \leq S_5$ lifts this conjugation and this element. Since H contains the full A_5^3, it is the full preimage of U, so an order-three pure cycle g belongs to the resulting group. There are five g-fixed labels on $\operatorname{Irr}(A_5^3)$. Thus $k(J) \leq 125|A|$ and $|\operatorname{Irr}(J)^g| \leq 5|A|$. Cyclic Clifford theory gives

$$k(J : C_3) = \frac{k(J) + 8|\operatorname{Irr}(J)^g|}{3}, \qquad \frac{k(H)}{|H|_2} \leq \frac{125 + 40}{3 \cdot 64} = \frac{55}{64}.$$

Passing to an index-two extension preserves this normalized upper bound. The scalar reserve therefore closes every outer subgroup of A_5^3. This completes all residual layers of the three named types.

The remaining $RA7$ pair. The types are distinct, so $T = 1$ and $B \leq 99/64 < 2$. Both nontrivial core cases are covered by their reserves. If $N = 1$, the support denominator is six. The A_7 degree is at least four. If the R factor degree is at least three, $\dim_{\mathbb{F}_2} V \geq 12$. Otherwise it is a degree-two natural Frobenius twist for $L_2(2^a)$, $a \geq 2$, and its period forces $\dim_{\mathbb{F}_2} V \geq 2 \cdot 2 \cdot 4 = 16$. The group $L_3(2)$ has no degree-two projective constituent. In either case $v^{1/6} \geq 4 > 99/64 + 1$, proving the bound.

Proof of Proposition 7.1. Use Lemma 7.3; all its homogeneity hypotheses follow from quasiprimitivity. If the layer is trivial, Lemma 7.7 applies with $X = E$. One component was settled above. For at least two components the complete simple-group classification assigns every factor to G, P, or one of the seven classes in (7.28). The tensor argument handles every profile containing G or P, and every all-small profile of length at least seven. The complete finite enumeration leaves exactly (7.33), all of which have just been proved for every core and field. Lemma 7.10 completes the proof. □

7.8. **The capacity of one simple factor.** Recall the universal odd central cover $\widehat{S}$ and $\kappa^+(S)$ from (6.6). Put

$$\eta(S) = |\operatorname{Out}(S)|_{2'}\kappa^+(S).$$

The maximum with one in this definition will be kept throughout.

Lemma 7.11. *If a nonabelian simple group S has a nontrivial absolutely irreducible projective representation of degree a over $\mathbb{F}_Q$, where $Q = 2^f$, then*

$$\eta(S) \le Q^a/4. \tag{7.37}$$

The representation may be a nonfaithful representation of $\widehat{S}$; its projective image on S is faithful.

Proof. For every such image $Q^a \ge 8$. Indeed $a \ge 2$, and the only smaller possibility is $\mathrm{PGL}_2(2)$, a solvable group. Let μ be the order of the odd part of the Schur multiplier and $o = |\operatorname{Out}(S)|_{2'}$. For any upper bound $B \ge k(S)/|S|_2$, the central-kernel inequality gives

$$\eta(S) \le o \max(1, \mu B). \tag{7.38}$$

Alternating and sporadic groups. The odd outer part is one for every alternating group. Its odd multiplier is one except at A_6, A_7, where it is three. The class estimates above therefore give

$$\eta(A_n) \le 1 \ (n \ge 8), \qquad \eta(A_5) \le 5/4, \quad \eta(A_6) \le 21/8, \quad \eta(A_7) \le 27/8.$$

For A_6, A_7 one has $Q^a \ge 16$, since $|\mathrm{PGL}_3(2)| = 168$ is divisible by neither 360 nor 2520. This proves the required bounds. For all 26 sporadic groups the odd multiplier is at most three and the odd outer part is one; M_{11}, J_1 have trivial multiplier. These exact metadata are the ATLAS values [17]. The complete class-ratio table above then gives $\eta(S) \le 15/8 < 8/4$.

Groups in defining characteristic two. Let S be the simple quotient of a quasisimple simply connected rational-point group $\widetilde{S}$ in characteristic two, with its usual defining field of size $q_0 = 2^t$ and algebraic rank h. The small simple derived groups are treated under their isomorphism types below. For ordinary rational-point groups, the prime-to-two part of the multiplier of $\widetilde{S}$ is trivial [57, Theorem, p. 235]. For Suzuki and Ree groups in characteristic two the same conclusion follows from the projective lifting theorem [69, §12.4]: every class in $H^2(\widetilde{S}, \overline{\mathbb{F}}_2^{\times})$ has an irreducible projective realization, so lifting all such representations makes this group, which is dual to the odd multiplier, trivial. If $U \to S$ is the universal central cover, the kernel of $U \to \widetilde{S}$ is therefore a 2-group. Since $Z(\widetilde{S})$ has odd order, that kernel is $O_2(Z(U))$. Hence $\widetilde{S} \cong \widehat{S}$. Let D be the smallest absolute degree of a nontrivial irreducible defining-characteristic representation of $\widetilde{S}$.

The defining-characteristic restriction theorem [69, Theorems 7.4, 9.3 and 12.2] realizes each irreducible module as a restriction from the simple algebraic group. A nontrivial rational representation has finite kernel and image of rank h, contained in SL_D because a connected semisimple group has no nontrivial linear character. Thus $D \ge h + 1$. The precise field-sensitive input used here is the *first* assertion of [34, Lemma 4.2]: a nontrivial irreducible module U in defining characteristic for this finite group satisfies $|U| \ge q_0^D$, over any finite field of definition. The later $D \ge 6$ and Suzuki–Ree restrictions in that lemma concern its later alternatives, not this first inequality. Applied to the projective module by pullback, it gives, with $n = fa$,

$$tD \le n, \qquad D \le a \le n. \tag{7.39}$$

Write $z = |Z(\widetilde{S})|$. The diagonal, field and graph formulas give

$$z \le h + 1 \le D, \qquad |\operatorname{Out}(S)|_{2'} \le 3zt. \tag{7.40}$$

The factor three includes triality and the graph-field action in ${}^3D_4$.

For $S = L_2(q_0)$, $q_0 \ge 4$, one has $\kappa^+(S) = 1 + 1/q_0 \le 5/4$ and $o \le t$. Since $D = 2$, $n \ge 2t \ge 4$, so

$$\eta(S) \le 5n/8 \le 2^n/4.$$

The last inequality starts at $n = 4$ and is preserved by induction. For rank at least two, apart from $L_3(2)$, the half-class estimate gives $\kappa^+(S) \le \max(1, z/2) \le D/2$. Consequently, if $n \ge 9$,

$$\eta(S) \le (3/2)D^2 t \le (3/2)n^2 \le 2^n/4.$$

The last inequality is $6n^2 \le 2^n$, true at nine since $486 \le 512$, and the ratio increases thereafter. For $L_3(2)$, $\eta(S) = 1$ directly.

It remains to check the entire boundary $n \le 8$. Since $D \ge 3$, (7.39) forces $t \le 2$, so the odd field factor is one. The odd center has order at most three in this range. In linear and unitary type it divides q_0-1 or q_0+1; a center of order five at $q_0 = 4$ would force natural degree at least five, hence $tD \ge 10$. Other types have center at most three. Therefore $\eta(S) \le 3z \max(1, z/2) \le 27/2 \le 2^n/4$ for $6 \le n \le 8$. If $n \le 5$, then $t = 1$. A center of order three at $q_0 = 2$ could only arise in unitary natural degree divisible by three or in an E_6 type. The group $\mathrm{SU}_3(2)$ is not quasisimple; unitary degree at least six forces $D \ge 6$, and E_6 forces $D \ge 7$. Hence $z = 1$. At $n = 5$, $\eta(S) \le 3 \le 8$; at $n = 3, 4$, triality is impossible because D_4 has rank four and $D \ge 5$, so $\eta(S) = 1 \le 2^n/4$. This completes all admissible defining-two types. The Tits group is separate: its multiplier is trivial, its outer group has order two, and $k(S) = 22 < 2048 = |S|_2$, so $\eta(S) = 1$.

Odd-characteristic rank one. For $S = L_2(q)$, odd $q \ge 11$, the multiplier has no odd part and $\kappa^+(S) \le (q+5)/8$. The odd outer part is at most the defining field exponent, hence at most $\log_3 q \le q/3$. Every projective characteristic-two degree is at least $(q-1)/2$. Thus

$$\eta(S) \le \frac{q(q+5)}{24} \le \frac{2^{(q-1)/2}}{4} \le \frac{Q^a}{4}.$$

The middle inequality at eleven is $88/3 \le 32$; its logarithmic right-to-left derivative is $(\log 2)/2 - 1/q - 1/(q+5) > 0$. The smaller simple cases are $A_5, L_3(2), A_6$ and have already been treated.

The remaining odd-characteristic Lie groups. In this paragraph $q = r^t$ is the *defining* odd field, distinct from Q. The following exact sufficient multiplier and outer bounds follow from [71, 17] and the diagonal, graph and field formulas:

Family	μ	o	μo bound
$L_n(q), U_n(q)$, except $U_4(3)$	$\le n$	$\le nt$	n^2t
$\mathrm{PSp}_{2h}(q)$	1	$\le t$	t
$\mathrm{P\Omega}_{2h+1}(q)$	≤ 3	$\le t$	$3t$
$\mathrm{P\Omega}^{\pm}_{2h}(q)$	1	$\le 3t$	$3t$
Exceptional types	≤ 3	$\le 3t$	$9t$

The odd factor three for $O_7(3)$ is necessary: its multiplier is six. The exception $U_4(3)$ has odd multiplier nine and outer group D_8; $G_2(3)$ has odd multiplier three and outer group of order two. We retain these exceptions explicitly.

Use the projective degree bounds from the preceding table, with the weaker $m = q(q-1)$ for both $L_3(q), U_3(q)$, including $U_3(3)$. For $L_3(q)$ and $U_3(q)$, $B = q^3/16$ and $t \le q/3$ give $4\eta(S) \le (3/4)q^4$. At $q = 3$,

$$2^{q(q-1)} = 64 > 243/4,$$

and the logarithmic ratio has derivative $(2q-1)\log 2 - 4/q > 0$. For $n = 4, q \ge 5$, $4\eta(S) \le (800/3)q^4$; its base is $2^{100} > 500000/3$ and the derivative is $(3q^2 - 2q)\log 2 - 4/q > 0$. At $L_4(3)$, $\eta(S) = 1$ by $k(S) \le 56$, $|S|_2 = 128$ and $\mu = o = 1$. At $U_4(3)$, $\eta(S) \le \max(1, 9 \cdot 20/128) = 45/32 < 2 \le Q^a/4$. For $n \ge 5$,

$$4\eta(S) \le \frac{400n^2q^n}{3 \cdot 2^{n-1}}, \qquad m = (q-1)q^{n-2}. \tag{7.41}$$

At $(n, q) = (5, 3)$ this compares 2^{54} with 50625. The logarithm of the right-to-left ratio decreases on the full rectangle: for the logarithm of 2^m divided by the right side, the q-derivative multiplied by q is $\log 2\{(n-2)m + q^{n-1}\} - n > 0$, and the n-derivative is $\log 2\, m \log q - \log(q/2) - 2/n > 0$.

For the symplectic groups put $m = (q^h - 1)/2$ and $B = 100q^h/2^{3h-1}$. Retaining the maximum with one, the two required bounds are

$$4t \le 4q/3 \le 2^m, \qquad 4tB \le \frac{400q^{h+1}}{3 \cdot 2^{3h-1}} \le 2^m.$$

The group $\mathrm{PSp}_4(3) = U_4(2)$ is already covered. All the rest lie in the rectangles $h \ge 2, q \ge 5$ or $h \ge 3, q \ge 3$. At their bases the second right sides are $3125/6$ and $675/16$, and the exponents are $12, 13$. For the logarithmic ratio, $q\partial_q = \log 2\, hq^h/2 - (h+1) > 0$ and $\partial_h = (\log 2\, q^h/2 - 1)\log q + 3\log 2 > 0$.

For odd-orthogonal groups, $h \ge 3, q \ge 3$,

$$m = q^{2h-3}, \qquad 4\eta(S) \le \max\{4q, 400q^{h+1}/2^{3h}\}.$$

The first term is immediate from $m \ge q^3$. For the second the base is $2^{27} > 2025/32$; the two logarithmic derivatives are $\log 2(2h-3)m - (h+1) > 0$ after multiplying the q-derivative by q, and $(2\log 2\, m - 1)\log q + 3\log 2 > 0$ in h. For even-orthogonal groups use $m = q^{2h-3} - 2$ and $4\eta(S) \le 100q^{h+1}$. The base is $2^{241} > 24300$ at $(4, 3)$. The derivatives are $\log 2(2h-3)q^{2h-3} - (h+1) > 0$ and $(2\log 2\, q^{2h-3} - 1)\log q > 0$.

For exceptional groups, the class bound $B = 25q^h$ and $\mu o \le 9t \le 3q$ give $4\eta(S) \le 300q^{h+1}$. The following three rows exhaust the generic parameters:

Family	range	m	rank used	base right side
$G_2(q)$	$q \ge 5$	$q^3 - 1$	2	37500
${}^3D_4, F_4, E_6, {}^2E_6, E_7, E_8$	$q \ge 3$	$(q^5 - 1)/2$	8	5904900
${}^2G_2(q)$	$q \ge 27$	$q(q-1)$	2	5904900

Their base exponents are $124, 121, 702$. The logarithmic derivative, multiplied by q, is $\log 2\, qm'(q) - (h+1) > 0$ at the base and increases thereafter. The common degree in the middle row is weaker than each degree in Lee's table; for ${}^3D_4$ use $q^5 - q^3 + q - 1 \ge (q^5 - 1)/2$. Finally at $G_2(3)$, $\eta(S) \le 69/64 < 2$, from $(\mu, o, k(S), |S|_2) = (3, 1, 23, 64)$. The small Ree derived group is $L_2(8)$, already covered. All nonabelian finite simple groups have now been included. □

7.9. **Assembly of the local capacity.**

Lemma 7.12. *In Lemma 7.3, retain $Q_o = \prod_j |\mathrm{Sp}_{2a_j}(r_j)|_{2'}$ and n as the extraspecial degree. If $n \ge 3$, then*

$$f(Q-1)n^2 Q_o \le Q^{2n}/8, \qquad Q = 2^f. \tag{7.42}$$

For every $n \ge 1$ the same assertion with denominator two holds, where $Q_o = 1$ if $n = 1$.

Proof. For $n = 3$, $Q_o = 3$, and it is enough that $8fn^2Q_o \le Q^{2n-1}$. At $f = 2$ this is $432 \le 1024$; the ratio $2^{5f}/f$ increases thereafter. For $n \ge 5$, (7.16) gives $Q_o \le n^{2\log_3 n + 1}$, so it suffices to prove

$$8fn^{2\log_3 n + 3} \le 2^{f(2n-1)}.$$

At $(n, f) = (5, 2)$, $\log_3 5 < 3/2$ gives left side $< 16 \cdot 5^6 = 250000 < 4^9 = 262144$. The logarithm of the right-to-left ratio has derivatives

$$(2n-1)\log 2 - 1/f > 0, \qquad 2f\log 2 - \frac{4\log_3 n + 3}{n} > 0$$

on $n \ge 5, f \ge 2$. The subtracted function in the second derivative decreases; at five it is below $9/5$, whereas $4\log 2 > 2$. This proves the whole rectangle. If $n = 1$, use $2f \le 2^f = Q$ to obtain $f(Q-1) < Q^2/2$. □

Proof of Proposition 7.2. Let $F^*(E) = RL$, with $R = F(E)$ and components $L_1, \dots, L_t$. Restrictions to characteristic subgroups of E are homogeneous because those subgroups are normal in B; also $O_2(E) = 1$, as in Lemma 7.3. That lemma supplies the normalized field $K = \mathbb{F}_Q$, an extraspecial degree n, and absolute component degrees $a_1, \dots, a_t \geq 2$. Put $A = \prod_i a_i$ and $D = \sum_i a_i$. The module has

$$|W| \geq Q^{nA}, \qquad |R| \leq (Q-1)n^2. \tag{7.43}$$

Its homogeneous multiplicity can only increase the first lower bound.

Remove first the field action on K, then the component permutation action. The remaining image acts on each extraspecial symplectic quotient and each simple component. An element trivial on all of these acts by inner automorphisms on R and on the components. Correct by those inner automorphisms; the remaining element centralizes $F^*(E)$ and hence lies in it. This proves the kernel assertion in

$$|E : F^*(E)|_{2'} \leq f_{2'} Q_o (t!)_{2'} \prod_{i=1}^{t} |\operatorname{Out}(S_i)|_{2'}. \tag{7.44}$$

The component-permutation image is a subgroup of $\mathrm{Sym}(t)$, so its order divides $t!$; the bound uses the successive kernels of these actions.

If $t = 0$ and R is nonabelian, then $n \geq 3$ and (7.44), (7.43), and Lemma 7.12 give $c(E) \leq |W|^2/8$. Suppose $t \geq 1$. By Lemma 7.11,

$$c(E) \leq f(Q-1)n^2 Q_o\, t! \prod_i \eta(S_i) \leq \frac{Q^{2n}}{2}\, t! \frac{Q^D}{4^t}.$$

It remains to prove

$$4t! \leq 4^t Q^{2n(A-1)-D}. \tag{7.45}$$

The exponent is minimized by $n = 1$ and every $a_i = 2$: increasing a_i increases $2A - D$, and increasing n increases it. Its minimum is $2^{t+1} - 2 - 2t \geq 0$. Since $Q \geq 2$, the desired inequality follows from $t! \leq 2^{2^{t+1}-4}$. Equality holds at $t = 1$; the ratio on the right from t to $t+1$ is $2^{2^{t+1}} \geq t+1$, proving induction. Thus $c(E) \leq Q^{2nA}/8 \leq |W|^2/8$. All central characters and multiplicities are those of the stated tensor factors. □

7.10. **Finite data and complete verification procedures.** The finite checks concern the subgroup lists and bounded numerical ranges specified below. The preceding monotonicity and tensor inequalities cover the unbounded ranges. Character-table names in the code are the identifiers of the GAP Character Table Library [11]; subgroup enumeration uses GAP [27]. The group orders and the numbers of subgroup classes are included as checks on the groups being enumerated.

The symplectic finite inputs. The complete enumeration of subgroup conjugacy classes gives

G	subgroup classes	nonsolvable classes	$\max_{H \text{ nonsolvable}} k(H)/\lvert H\rvert_2$	$\max_{H \leq G} k(H)/\lvert H\rvert_2$
$\mathrm{Sp}_4(3)$	162	8	9/8	27
$\mathrm{PSp}_4(3)$	116	8	5/4	not needed
$\mathrm{GL}_3(3)$	154	2	3/4	not needed

(7.46)

The maxima in the nonsolvable column occur at groups isomorphic to $\mathrm{SL}_2(5), A_5, L_3(3)$, respectively. The maximum 27 is attained by an abelian subgroup of order 27. The following exact-arithmetic algorithm enumerates every subgroup conjugacy class:

```
Part2 := n -> 2^Length(Filtered(FactorsInt(n),x->x=2));;
VecPerm := function(G,d)
  local vs,phi;
  vs:=Filtered(Elements(GF(3)^d),v->not IsZero(v));
```

```
  phi:=ActionHomomorphism(G,vs,OnRight);
  return Image(phi,G);
end;;
TestSubgroups := function(G,num,nonnum,bnd,allbnd)
  local cs,c,H,rat,best,allbest,non;
  cs:=ConjugacyClassesSubgroups(G);
  Assert(0,Length(cs)=num);
  best:=0; allbest:=0; non:=0;
  for c in cs do
    H:=Representative(c);
    rat:=NrConjugacyClasses(H)/Part2(Size(H));
    allbest:=Maximum(allbest,rat);
    if not IsSolvableGroup(H) then
      non:=non+1; best:=Maximum(best,rat);
    fi;
  od;
  Assert(0,non=nonnum and best=bnd);
  if allbnd<>fail then Assert(0,allbest=allbnd); fi;
end;;
G4:=VecPerm(Sp(4,3),4);;
Assert(0,Size(G4)=51840);;
TestSubgroups(G4,162,8,9/8,27);
P4:=FactorGroup(G4,Centre(G4));;
Assert(0,Size(P4)=25920);;
TestSubgroups(P4,116,8,5/4,fail);
G3:=VecPerm(GL(3,3),3);;
Assert(0,Size(G3)=11232);;
TestSubgroups(G3,154,2,3/4,fail);
```

The finite ambient group $\mathrm{Sp}_6(3)$ has eleven conjugacy classes of maximal subgroups. The complete data needed are

$\lvert M\rvert$	$k(M)$	$\lvert\mathrm{Rad}(M)\rvert$	$\lvert M/\mathrm{Rad}(M)\rvert$	quotient or status
25194240	252	972	25920	$\mathrm{PSp}_4(3)$
2519424	264	2519424	1	solvable
8188128	124	1458	5616	$L_3(3)$
1244160	238	48	25920	$\mathrm{PSp}_4(3)$
82944	140	82944	1	solvable
22464	30	2	11232	
58968	29	2	29484	
48384	46	4	12096	
120	9	2	60	
2184	17	2	1092	
2184	17	2	1092	

For reproducibility, obtain the list by `MaximalSubgroupClassReps(Sp(6,3))`; for each entry compute `Size`, `NrConjugacyClasses`, and `SolvableRadical`. Assert that the sorted triples are exactly those in this table. For each of the three named nonsolvable radical quotients, form `FactorGroup(M,SolvableRadical(M))` and test `IsomorphismGroups` against `P4` above or `VecPerm(SL(3,3),3)`, respectively. The latter group has order 5616, which is asserted before testing. The quotient identifications are checked by explicit isomorphisms.

Here is why these finite data imply the all-subgroup bound needed in Lemma 7.7:

$$H \le \mathrm{Sp}_6(3),\ H \text{ nonsolvable} \quad\Longrightarrow\quad k(H)/|H|_2 \le 2187/4. \tag{7.47}$$

For the full group use $k = 141$ and two-part 1024. Otherwise choose a maximal overgroup M. In the first, third and fourth rows put $K = H \cap \mathrm{Rad}(M)$. The quotient H/K is nonsolvable. Since $k(K)/|K|_2 \le |\mathrm{Rad}(M)|_{2'}$, Nagao and (7.46) give the respective bounds 243(5/4), 729(3/4), and 3(5/4) for $k(H)/|H|_2$. For the remaining nonsolvable rows,

$$k(H) \le \min\{|H|, [M : H]k(M)\} \le \sqrt{|M|k(M)}.$$

A nonsolvable group has two-part at least four: groups of odd order are solvable, and a group with Sylow two-subgroup of order two has a normal odd complement. Each remaining product $|M|k(M)$ is less than 2187^2. Together these maximal-overgroup estimates prove (7.47) for every subgroup.

Maximal overgroups for the small capacities. The complete maximal-subgroup pairs $(|M|, k(M))$ for A_7 are

$$(72, 9),\ (120, 7),\ (360, 7),\ (168, 6),\ (168, 6).$$

For J_1 they are

$$(660, 8),\ (168, 8),\ (120, 10),\ (114, 9),\ (110, 11),\ (60, 12),\ (42, 7).$$

Their respective isomorphism types are $L_2(11)$, $2^3 : 7 : 3$, $2 \times A_5$, $19 : 6$, $11 : 10$, $D_6 \times D_{10}$ and $7 : 6$. Formula (7.29) gives 1 and 4/133. For the two odd rank-one pairs the complete overgroup data are as follows. Repeated conjugacy classes with the same pair are immaterial to the maximum, but all subgroup types are covered.

H	allowed maximal-overgroup pairs $(\|M\|, k(M))$
$L_2(11)$	$(60, 5)$ twice, $(55, 7)$, $(12, 6)$
$\mathrm{PGL}_2(11)$	$(110, 11)$, $(24, 5)$, $(24, 9)$, $(20, 8)$, and the preceding proper socle overgroups
$L_2(13)$	$(78, 8)$, $(14, 5)$, $(12, 6)$, $(12, 4)$
$\mathrm{PGL}_2(13)$	$(156, 13)$, $(28, 10)$, $(24, 9)$, $(24, 5)$, and the preceding proper socle overgroups

Taking $|M|k(M)/|H|$ gives $7/12, 11/12, 4/7, 13/14$ in order. Here $[H : S] = 2$ for each projective general linear group, so a proper maximal overgroup containing S is S itself. Subgroups inside it and avoiding S are covered by the listed maximal subgroups of S.

These lists can be reproduced using `MaximalSubgroupClassReps(AlternatingGroup(7))` and the complete `Maxes(CharacterTable("J1"))` attribute. The latter attribute means the complete list of maximal-subgroup conjugacy classes in the Character Table Library. Missing data must cause an error, not an empty list. For $q = 11, 13$, construct the projective action on the $q + 1$ lines directly. For example:

```
ProjectivePair:=function(q)
  local F,pts,G,phi,S,H;
  F:=GF(q);
  pts:=Concatenation(List(Elements(F),x->[One(F),x]),
                     [[Zero(F),One(F)]]);
  G:=GL(2,q); phi:=ActionHomomorphism(G,pts,OnLines);
  H:=Image(phi,G); S:=Image(phi,SL(2,q));
  Assert(0,Size(H)=q*(q^2-1));
  Assert(0,Size(S)=q*(q^2-1)/2);
  return [H,S];
end;;
for q in [11,13] do
  pair:=ProjectivePair(q); H:=pair[1]; S:=pair[2];
  over:=Filtered(MaximalSubgroupClassReps(H),
```

```
                    M->not IsSubgroup(M,S));
  Append(over,MaximalSubgroupClassReps(S));
  cap:=Maximum(List(over,M->
       Size(M)*NrConjugacyClasses(M)/Size(H)));
  Assert(0,cap=([11/12,13/14][Position([11,13],q)]));
od;
```

For $L_2(8).3$, use the lines over $\mathbb{F}_8$, the projective image of $\mathrm{SL}_2(8)$ of order 504, and adjoin the permutation squaring all line coordinates. Assert the resulting order 1512 and the normality of the socle. Its four maximal pairs and the three proper socle maximal pairs are exactly those displayed in the preceding subsection. This construction checks both the group and the required class-covering list, including the subgroups lying inside the socle.

Ordinary tables and small projective degrees. All ordinary class counts in the sporadic and small Lie tables are obtained by applying `NrConjugacyClasses` to `CharacterTable(name)`, with order obtained by `Size`. The sporadic names are

```
["M11","M12","M22","M23","M24","J1","J2","J3","J4",
 "Co1","Co2","Co3","Fi22","Fi23","Fi24'","HS","McL",
 "He","Ru","Suz","ON","HN","Ly","Th","B","M"]
```

The groups with outer order two are exactly

$$M_{12},\ M_{22},\ J_2,\ J_3,\ Fi_{22},\ Fi'_{24},\ HS,\ McL,\ He,\ Suz,\ O'N,\ HN.$$

Append `".2"` to each of these identifiers and require the table to exist. For every one, its normalized ratio is at most that of its simple socle. The small odd Lie table uses identifiers `"U4(2)"`, `"S4(5)"`, `"S4(7)"`, `"S6(3)"`, `"G2(3)"`, `"L2(11)"`, and `"L2(13)"`, together with their `".2"` extensions. The characteristic-two A_7 degrees used above are reproduced by the following commands, with `name` equal to `"A7"` or `"3.A7"`:

```
ct := BrauerTable(CharacterTable(name),2);;
List(Irr(ct),x->x[1]);
```

These are complete Brauer character tables, including the nontrivial odd central characters.

Exact profile and partition verification. The following Python program contains the entire finite seven-profile calculation, the 24 support-table endpoints, and the partition boundary. It uses only integers and rational numbers. Together with the displayed monotonicity proofs, these assertions verify the remaining finite inequalities in the tensor argument.

```
from fractions import Fraction as F
from itertools import combinations_with_replacement
from collections import Counter
from math import prod
P = [("R",2,4,F(11,8)), ("A6",3,5,F(7,8)),
     ("A7",4,6,F(9,8)), ("J1",10,8,F(15,8)),
     ("M11",4,8,F(5,8)), ("L11",5,3,F(2)),
     ("L13",6,4,F(9,4))]
def h(n):
    return 1 if n<=2 else 3 if n<=4 else 2**(n-1)
counts=Counter(); residual=[]; bylength={}
for t in range(2,7):
    bylength[t]=Counter()
    for ids in combinations_with_replacement(range(7),t):
        A=[P[i] for i in ids]
        B=prod(p[3] for p in A)*prod(h(v)
                                     for v in Counter(ids).values())
        M=prod(p[1] for p in A); R=max(4,*(p[2] for p in A))
        s="budget" if B<=1 else "support" if F(2**M)>=(2*B)**R \
```

```
          else "residual"
        counts[s]+=1; bylength[t][s]+=1
        if s=="residual": residual.append([p[0] for p in A])
assert counts=={"budget":105,"support":1598,"residual":5}
assert residual==[["R","R"],["R","A6"],["R","A7"],
                  ["R","R","R"],["R","R","R","R"]]
rows=[(18,4,F(39,4)),(8,3,F(19,16)),(9,3,F(3)),
 (11,3,F(25,8)),(12,4,F(27,8)),(13,3,F(29,4)),
 (14,3,F(31,4)),(15,3,F(33,32)),(11,4,F(27,16)),
 (20,3,F(125,16)),(100,6,F(3125,2)),(54,5,F(2025,4)),
 (40,5,F(2025,16)),(62,6,F(3125,64)),(40,8,F(2025,512)),
 (27,7,F(675,128)),(241,8,F(2025)),(124,5,F(625)),
 (121,7,F(2025)),(243,8,F(2025)),(243,9,F(18225)),
 (243,10,F(54675)),(243,11,F(164025)),(702,3,F(18225))]
for m,r,B in rows:
    assert F(2**m)>=(2*B)**r
assert F(2**12)>=(2*F(17,32))**5
parts=[0]*64; parts[0]=1
for j in range(1,64):
    for n in range(j,64): parts[n]+=parts[n-j]
for n in range(8,64):
    assert parts[n]<=2**(n-n.bit_count()-2)
assert min(F(2**(n-n.bit_count()-2),parts[n])
           for n in range(8,64))==F(32,30)
assert 37**3<3**10 and 4096*37**26<16**37
for n,q,Qo in [(9,16,405),(15,16,45),(21,16,63),
               (25,16,73125),(27,4,8955765),
               (33,16,495),(35,16,315)]:
    f=q.bit_length()-1
    assert (100*f*n*Qo)**3<q**(2*n)
for d,B in [(8,F(121,64)),(16,F(3993,512)),
            (32,F(43923,4096))]:
    assert F(2**d)>(B+1)**5
print(dict(counts),dict(bylength),residual)
```

The comparisons use exact integer and rational arithmetic. The support tests retain their denominators and the additive unit in (7.6).

8. The small semilinear fields

The three scalar fields left by the capacity estimate are treated by ordinary character counts. We prove a slightly more general statement for odd prime field degree. The prime denoted by r in this section is a field-automorphism prime; the vector-space characteristic remains two. We use $\Gamma L_1(q) = \mathbb{F}_q^\times \rtimes \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_2)$ in its natural action on the additive group of $\mathbb{F}_q$.

Proposition 8.1. *Let $q = 4$ or $q = 2^r$ for an odd prime r, let $V = \mathbb{F}_q^n$, and let $H \le \Gamma L_1(q) \wr S_n$ act naturally on V. Then*

$$k(V \rtimes H) \le q^n |H|_2. \tag{8.1}$$

In particular this holds for $q = 4, 8, 32$, for every scalar code, field code and permutation image, including nonsplit lifts of the permutation image.

8.1. **Scalar codes, norms, and shortening.** Put $m = q - 1$ and identify the scalar torus with $(C_m)^n$. For $A \le (C_m)^n$, write $M_A = \mathbb{F}_q^n \rtimes A$. A coordinate permutation π normalizing A has cycles of lengths $\ell_1, \dots, \ell_c$. A cycle is called good when $\gcd(\ell_i, m) = 1$, and bad otherwise.

Lemma 8.2. *Let $z \in (C_m)^n$ and let $z\pi$ normalize A. If g, b are the numbers of good and bad cycles of π, then*

$$|\operatorname{Irr}(M_A)^{z\pi}| \le q^g[2(q-1)]^b \le q^{n-s(\pi)/2}. \tag{8.2}$$

The first inequality holds for every subgroup $A \le (C_m)^n$. For one cycle of length ℓ its bound is q if $\gcd(\ell, m) = 1$, and $2m$ otherwise.

Proof. Since every scalar has odd order, $V = C_V(s) \oplus [V, s]$ for $s \in A$. Conjugation by V removes the second summand. The ordinary classes of M_A are therefore the A-orbits of

$$Y_A = \{(v, s) : s \in A, v \in V, sv = v\}.$$

This parametrization is equivariant under every linear normalizer of the displayed scalar group. The number of fixed ordinary characters equals the number of fixed ordinary classes: the character table is an invertible intertwining matrix for their two permutation actions. We use this equality only for the trace of one automorphism. The fixed-orbit form of Burnside's lemma consequently gives

$$|\operatorname{Irr}(M_A)^{z\pi}| = \frac{1}{|A|} \sum_{a \in A} |Y_A^{az\pi}|. \tag{8.3}$$

The fixed scalar labels form $S = C_A(\pi)$, identified with a subgroup of $(C_m)^c$ by their constant values on cycles. Let $R = N_\pi A \le (C_m)^c$ be the image of the cycle-product map, and let u be the vector of cycle products of z. All norm fibres have the same size. On cycle i a simultaneous fixed vector has q possibilities if both its scalar label s_i and cycle product w_i are one, and exactly one possibility otherwise. Thus

$$|\operatorname{Irr}(M_A)^{z\pi}| = \frac{1}{|R|} \sum_{w \in uR} \sum_{s \in S} \prod_{i=1}^{c} (1 + m1_{\{w_i = s_i = 1\}}). \tag{8.4}$$

The fixed and norm codes satisfy the essential relation

$$(s_i^{\ell_i})_{i=1}^c \in R \qquad (s \in S). \tag{8.5}$$

For subgroups $P, Q \le (C_m)^j$ and cosets xP, yQ, define

$$Z(xP, yQ) = \sum_{w \in xP,\ s \in yQ} \prod_i (1 + m1_{\{w_i = s_i = 1\}}).$$

Expanding over subsets J of coordinates gives positive terms $m^{|J|}|\{w \in xP : w_J = 1\}|\,|\{s \in yQ : s_J = 1\}|$. Each cardinality is zero or the corresponding kernel size. Therefore

$$Z(xP, yQ) \le Z(P, Q). \tag{8.6}$$

If $Q \le P$, then

$$Z(P, Q) \le Z(P, P) = |P| k(\mathbb{F}_q^j \rtimes P) \le |P| q^j. \tag{8.7}$$

Here the last inequality is elementary. Project P to one coordinate, with cyclic image of order $d \mid m$, and use the actual kernel on the other coordinates. The corresponding affine quotient has $d + m/d \le m + 1 = q$ classes. Induction and Lemma 2.1 give $k(\mathbb{F}_q^j \rtimes P) \le q^j$. This proof also covers a trivial projection or a proper subdirect P.

By (8.6) the norm coset in (8.4) may first be replaced by R. On the good coordinates apply the automorphism $s_i \mapsto s_i^{\ell_i}$ to S, leaving the bad coordinates unchanged; call the resulting code S'. The weights are unchanged. Let R_0, S_0' be their shortened codes on the good coordinates, that is, kernels of projection onto the bad coordinates. Relation (8.5) gives $S_0' \le R_0$. Fix any projected bad-coordinate values. The remaining good-coordinate fibres are cosets of R_0, S_0', so (8.6)–(8.7) bound their weighted sum by $|R_0| q^g$. Division by $|R| = |R_0||\pi_{\mathrm{bad}}(R)|$ leaves

$$\frac{q^g}{|\pi_{\mathrm{bad}}(R)|} \sum_{w \in \pi_{\mathrm{bad}}(R),\ s \in \pi_{\mathrm{bad}}(S')} \prod_{i\ \mathrm{bad}} (1 + m1_{\{w_i = s_i = 1\}}).$$

For each w, enlarge the s-domain to the whole bad-coordinate torus and replace w by one. Each such coordinate costs $q + (m-1) = 2m$. This proves the first inequality of (8.2).

A bad cycle has length at least three because m is odd, and $2(q-1) \le q^{3/2}$ for $q \ge 4$. A nontrivial good cycle of length ℓ costs $q \le q^{\ell/2}$; a fixed coordinate costs q. Multiplying gives the second bound. For $q = 2$ the scalar group is trivial and the same assertion follows directly. □

The coset in (8.4) accounts for lifts with nontrivial scalar parts. Bad cycles require the larger bound: at $q = 4$, the diagonal $C_3 \le C_3^3$ on a three-cycle has $4 + 1 + 1 = 6$ fixed classes. The shortening argument includes both features.

8.2. **The field of four elements.**

Proof of Proposition 8.1 for $q = 4$. Let $A = H \cap (C_3)^n$, $Q = H/A$, and $X = \mathrm{Irr}(V \rtimes A)$. Put $\Omega = \{1, \dots, n\}$ and recall $t = 2^{-1/2}$. The quotient Q embeds in $C_2 \wr S_n$. Let T be its image on Ω and R its kernel; then R is a normal 2-group. For every odd $a \in Q$ we claim

$$|X^a| \le 4^n 2^{-s_\Omega(a)}. \tag{8.8}$$

Choose a lift $g \in H$. The group $\langle A, g\rangle$ has order $|A|\,|\langle a\rangle|$, which is odd, so g is odd. Every coordinate cycle of g has odd length, and the product of its field labels in C_2 is trivial, including on a fixed coordinate. Successive diagonal field gauges around the cycle therefore conjugate g to a scalar times a pure permutation. The consistency condition for the gauges is precisely that trivial product. This conjugation transports A to another scalar code. Applying Lemma 8.2 to that code proves (8.8); it also gives $|X| \le 4^n$ at the identity.

In the commuting formula (6.1), a common fixed set is contained in X^a and in X^{b_o}. Its size is at most their geometric mean. Hence

$$\Lambda_2(Q, X) \le 4^n\, \mathcal{P}(Q; t^{s_\Omega}).$$

The weight factors through $T = Q/R$, so Lemma 5.2 and Theorem 5.1 give $\mathcal{P}(Q; t^{s_\Omega}) = \mathcal{P}(T, \Omega) \le 1$. Lemma 6.1, applied to the extension of $V \rtimes A$ by Q, yields

$$k(V \rtimes H) \le |Q|_2 \Lambda_2(Q, X) \le 4^n |H|_2,$$

since A is odd. Thus the possible field 2-kernel is retained through the precise quotient identity. □

8.3. **Odd prime field degree and equivariant extensions.** Now let $q = 2^r$ with r an odd prime and $m = q - 1$. Fermat's theorem gives $m \equiv 1 \pmod r$, so $(r, 2m) = 1$.

Lemma 8.3. *Let $W = \mathbb{F}_q^\ell$, let $K \le (C_m : C_r)^\ell$, and let g be a semilinear monomial map normalizing K whose coordinate permutation is one ℓ-cycle. Put*

$$c_\ell = \begin{cases} q, & \gcd(\ell, m) = 1, \\ 2m, & \gcd(\ell, m) > 1. \end{cases}$$

Then

$$|\,\mathrm{Irr}(W \rtimes K)^g| \le c_\ell + 2(r-1) \le 2(q + r - 2). \tag{8.9}$$

Proof. Put $A = K \cap (C_m)^\ell$. Its field image is a subgroup $B \le (C_r)^\ell$. Schur–Zassenhaus provides a complement to A in K. In the ambient inverse image $(C_m)^\ell : B$, all complements are conjugate by the scalar torus. Thus, after a scalar conjugation, we may write $K = A \rtimes B$ with B a group of pure field maps. Scalar conjugation does not alter A. Since g normalizes K, conjugacy of complements within K allows multiplication of g by an element of A so that it normalizes B. This inner change does not change its action on $\mathrm{Irr}(W \rtimes K)$. It still has the same coordinate cycle. Conjugation by g on B permutes its field exponents by this cycle. Consequently

$$D = C_B(g) \le \text{the diagonal } C_r, \qquad |D| \in \{1, r\}. \tag{8.10}$$

Let $M = W \rtimes A$ and $X = \mathrm{Irr}(M)$. The little-group parametrization over the abelian normal group W shows that every $\theta \in X$ has degree $|A : A_\lambda|$ for some $\lambda \in \mathrm{Irr}(W)$: the scalar stabilizer is abelian. Thus $r \nmid \theta(1)$. We now justify the required extensions and their equivariance.

The obstruction to extending θ to $M \rtimes B_\theta$ is a class $[\alpha] \in H^2(B_\theta, \mathbb{C}^*)$. It satisfies $|B_\theta|[\alpha] = 0$. If T_b are intertwining matrices, taking determinants in $T_b T_c = \alpha(b,c) T_{bc}$ shows that the obstruction is killed by $\theta(1)$, since

$$\alpha(b,c)^{\theta(1)} = \frac{\det T_b \det T_c}{\det T_{bc}}.$$

These two coprime annihilators force it to vanish. Thus θ extends. Its extensions differ by the linear characters of the abelian r-group B_θ. There is a unique extension whose determinant restricted to this specified complement is trivial: twisting changes that determinant by the $\theta(1)$st power, an automorphism of its character group. This choice is preserved by every automorphism normalizing B.

It follows that $\mathrm{Irr}(M \rtimes B)$ is parametrized equivariantly by B-orbits of pairs (θ, η) with $\theta \in X$, $\eta \in \mathrm{Irr}(B_\theta)$. A g-stable orbit of θ has $|B : B_\theta|$ members, and for each member exactly $|B_\theta|$ choices of $b \in B$ make bg fix it. Since B is abelian, bg and g induce the same automorphism on B_θ. Its fixed linear-character count is $|C_{B_\theta}(g)|$, by duality for finite abelian groups. Expanding this number over $d \in D$ gives the exact identity

$$|\mathrm{Irr}(M \rtimes B)^g| = \frac{1}{|B|} \sum_{b \in B} \sum_{d \in D} |X^{bg,d}|. \tag{8.11}$$

The equivariant extension parametrization therefore gives the common fixed-character count in (8.11).

We compute first the terms with $d \neq 1$. Such a d is the same nontrivial Frobenius power on every coordinate. Its fixed field is $\mathbb{F}_2$, so $C_A(d) = 1$ and $d - 1$ is an automorphism of A and of every d-stable subgroup and quotient of A. Use the explicit little-group labels for X: they are the A-orbits of pairs (λ, ψ) with $\lambda \in \mathrm{Irr}(W)$ and $\psi \in \mathrm{Irr}(A_\lambda)$. The extension of λ is the canonical character $(v, a) \mapsto \lambda(v)$ on $W \rtimes A_\lambda$, so this parametrization is equivariant under the displayed semilinear maps. Each d-stable orbit on $\mathrm{Irr}(W)$ has a unique d-fixed member. Indeed A is abelian, so every point of the orbit has the same stabilizer. Its d-stability therefore makes A_λ d-stable. The fixed-point equation on the torsor A/A_λ has a unique solution because $d - 1$ is invertible. For that member, the only d-fixed character of A_λ is the trivial one, since $C_{A_\lambda}(d) = 1$. We obtain a bijection

$$X^d \longleftrightarrow \mathrm{Irr}(W)^d. \tag{8.12}$$

The uniqueness just proved makes this bijection equivariant for every semilinear map normalizing A and commuting with d, including every bg in (8.11).

The set $\mathrm{Irr}(W)^d$ is the product of ℓ binary fixed spaces, one in each dual coordinate. Each has one nonzero vector. An invertible coordinate map commuting with d sends this vector to the unique nonzero fixed vector of the next coordinate. Thus bg acts on these binary letters as its coordinate ℓ-cycle, and

$$|X^{bg,d}| = 2 \qquad (d \neq 1). \tag{8.13}$$

It remains to bound $|X^{bg}|$ for $d = 1$. Put $y = bg$ and let σ be the total field map around its coordinate cycle. If $\sigma \neq 1$, a fixed scalar label is determined by an element of $\mathbb{F}_q^*$ fixed by σ, so $C_A(y) = 1$. Apply the ordinary class parametrization Y_A from the proof of Lemma 8.2. In the fixed-orbit average over ay, the scalar label must be one. A scalar times a nontrivial field map $u \mapsto u^{2^j}$ has exactly two fixed vectors, because

$$\gcd(2^j - 1, 2^r - 1) = 1 \qquad (0 < j < r).$$

The cycle fixed vector is determined by one such vector, so every summand in the average is two and $|X^y| = 2$. If $\sigma = 1$, successive field gauges conjugate y to a scalar-permutation map and transport A to another scalar code. Lemma 8.2 gives $|X^y| \le c_\ell$. Combining these bounds with (8.10), (8.11) and (8.13) proves (8.9). □

8.4. **Fixed coordinates and arbitrary permutation tops.**

Lemma 8.4. *For every $E \le \Gamma L_1(2^r)$, with r an odd prime, $k(\mathbb{F}_{2^r} \rtimes E) \le 2^r$.*

Proof. Keep $q = 2^r$ and $m = q-1$. If E is cyclic scalar of order d, then $k(\mathbb{F}_q \rtimes E) = d + m/d \le q$. Otherwise its field image is C_r, and a scalar conjugation puts $E = C_d : C_r$, with $d \mid m$. Each nonidentity field element acts fixed-point-freely on $C_d \setminus \{1\}$, so ordinary Clifford theory gives $k(E) = r + (d-1)/r$. On the nonzero additive dual, every complement C_r has exactly one fixed vector, and distinct complements have distinct fixed vectors because a nonidentity scalar fixes none. The resulting d vectors form one E-orbit with stabilizer C_r; all other nonzero orbits have trivial stabilizer. This also holds when $d = 1$. The affine little-group formula therefore gives

$$\begin{aligned} k(\mathbb{F}_q \rtimes E) &= 2r + \frac{d-2}{r} + \frac{m}{rd} \\ &\le 2r + \frac{q-2}{r} \le q. \end{aligned}$$

The first comparison uses $d + m/d \le m + 1$; the last is equivalent to $2(r+1) \le 2^r$, valid for $r \ge 3$. □

Proof of Proposition 8.1 for odd prime field degree. Let $\Omega = \{1, \dots, n\}$, let N be the kernel of the action of H on Ω, and put $T = H/N$. Since $\Gamma L_1(q)$ has odd order mr, N is odd. Put $Z = \mathrm{Irr}(V \rtimes N)$. Let $C_1, \dots, C_s$ be the coordinate cycles of a lift g of an odd $a \in T$. Put $L_0 = V \rtimes N$, and let L_j be the kernel of projection onto $C_1 \cup \cdots \cup C_j$. These kernels are g-invariant, $L_s = 1$, and

$$L_{j-1}/L_j = W_j \rtimes K_j, \qquad W_j = \mathbb{F}_q^{C_j}, \qquad K_j \le (C_m : C_r)^{|C_j|}.$$

Every such image contains its full additive group W_j. The twisted normal-subgroup inequality (6.5) takes the maximum over the prescribed coset twists at each step. The additive part of a twist acts by inner conjugation because the image contains the full W_j, so removing it preserves the character trace. The remaining twist is semilinear with the same coordinate cycle. Lemma 8.3 therefore applies to every moved-cycle image. For a fixed coordinate, its trace is at most its class number, bounded by Lemma 8.4. Consequently

$$|Z^a| \le q^{\#\text{fixed coordinates}} \prod_{\text{moved cycles}} 2(q + r - 2). \tag{8.14}$$

Every moved cycle has odd length $\ell \ge 3$. Since $q \ge 8$ and $r \le q/2$,

$$2(q + r - 2) \le 3q - 4 \le q^3/8 \le (q/2)^\ell.$$

For the middle inequality, $q^3 - 24q + 32$ is positive at eight and increasing thereafter. Thus

$$|Z^a| \le q^n 2^{-s_\Omega(a)}. \tag{8.15}$$

At the identity, the same coordinate filtration and Lemma 8.4 give $|Z| \le q^n$. Now T acts faithfully on the coordinate set. Lemma 6.1 and Theorem 5.1 yield

$$k(V \rtimes H) \le |T|_2 \Lambda_2(T, Z) \le q^n |T|_2 = q^n |H|_2,$$

because N is odd. This proves the proposition for all subgroups stated, and in particular for the two remaining fields of orders eight and thirty-two. □

9. The affine inequality in characteristic two

Theorem 9.1. *Let V be a finite-dimensional vector space over $\mathbb{F}_2$ and let $H \le \mathrm{GL}(V)$. Then*

$$k(V \rtimes H) \le |V|\,|H|_2. \tag{9.1}$$

Proof. We induct on $\dim_{\mathbb{F}_2} V$. The zero-dimensional case has $H = 1$. Suppose first that $0 < W < V$ is H-invariant, put $U = V/W$, and let $C = C_H(U)$. The subgroup $W \rtimes C$ is normal in $V \rtimes H$ because $[C, V] \le W$, and its quotient is $U \rtimes (H/C)$ with faithful action on U.

Let $J = C \cap C_H(W)$. Relative to W and V/W, each element of J is the identity plus a map $U \to W$. Thus J is a 2-group. It centralizes W, is normal in $W \rtimes C$, and $(W \rtimes C)/J \cong W \rtimes (C/J)$ acts faithfully on W. Applying Lemma 2.1 and the induction hypothesis gives

$$k(W \rtimes C) \le k(J)k(W \rtimes (C/J)) \le |J|\,|W|\,|C/J|_2 = |W|\,|C|_2.$$

Apply the induction hypothesis also to $U \rtimes (H/C)$ and then Lemma 2.1 to the normal subgroup $W \rtimes C$. This proves (9.1) in the reducible case.

We may assume V irreducible. The primitive case is Proposition 7.1. Otherwise choose an H-invariant direct-sum decomposition

$$V = W_1 \oplus \cdots \oplus W_t, \qquad t > 1,$$

with the smallest possible positive summand dimension. The action on the summands is transitive. The full stabilizer image $B_i \le \mathrm{GL}(W_i)$ is irreducible: an invariant proper subspace would induce over a block transversal to an invariant proper subspace of V. It is primitive, since any finer decomposition would contradict the choice of the W_i.

Let N be the kernel of the permutation action and E_i its image on W_i. Then $E_i \lhd B_i$. Every characteristic subgroup of E_i is normal in B_i. By Clifford theory its restriction on W_i is completely reducible, and its isotypic summands are permuted by B_i. Primitivity forces a single isotypic summand; the restriction is therefore homogeneous. Also $O_2(E_i) = 1$: this subgroup is normal in B_i, and a normal 2-subgroup acts trivially on an irreducible module in characteristic two. Faithfulness of B_i gives the assertion.

For every subgroup of B_i, the induction hypothesis applies on W_i, whose dimension is smaller than that of V. Thus the fixed-coordinate hypothesis of Proposition 6.4 holds without requiring that a particular subgroup act irreducibly. By Proposition 7.2, if E_i has a nontrivial layer or a nonabelian Fitting subgroup, then $c(E_i) \le |W_i|^2/8$. The property is constant on the coordinate orbit, so this case follows from Proposition 6.4.

The remaining abelian Fitting subgroup is cyclic: its faithful homogeneous representation has field image. Thus it remains to treat $F^*(E_i) = F(E_i)$ cyclic. Write $E = E_i$ and $W = W_i$. If $E \ne 1$, put $R = F(E)$; the Fitting subgroup is nontrivial because it is self-centralizing. The homogeneous scalar algebra $Q = \mathbb{F}_2[R]$ is a finite field, say of order $q = 2^f$. The group R embeds in $Q^\times$, and $C_E(R) = R$. Consequently

$$E/R \hookrightarrow \mathrm{Gal}(Q/\mathbb{F}_2), \qquad c(E) = |E : R|_{2'}|R| \le f_{2'}(q-1).$$

If $m = \dim_Q W \ge 2$, use $2f \le 2^f = q$ to obtain

$$c(E) \le f(q-1) \le q^2/2 \le q^4/8 \le |W|^2/8.$$

If $m = 1$, the entire B_i normalizes R and its scalar algebra Q, so $B_i \le \Gamma L_1(q)$. For $f = 4$ the displayed capacity is at most $15 \le q^2/8$. For $f \ge 6$, the elementary inequality $f \le 2^{f-3}$ gives the same conclusion. Since R is nontrivial, $q \ne 2$. Only $q = 4, 8, 32$ remain. Transporting the scalar fields along a block transversal embeds the original H in $\Gamma L_1(q) \wr \mathrm{Sym}(t)$ on its natural direct sum. Proposition 8.1 is a theorem for every subgroup of that wreath product and supplies (9.1) for these three field sizes.

Finally, if $E = 1$, then $c(E) = 1$ and the capacity inequality holds whenever $|W| \geq 4$. The only remaining possibility is $\dim_{\mathbb{F}_2} W = 1$. Now $B_i \leq \mathrm{GL}_1(2) = 1$, so H acts by permutation matrices on these one-dimensional summands. It fixes their nonzero all-ones vector. Since $t > 1$, this contradicts irreducibility. All cases have been covered, completing the dimension induction. □

10. Descent through a proper minimal Levi subgroup

Throughout this section the coefficient system is a sufficiently large splitting 2-modular system $(K, \mathcal{O}, k)$, with $\mathcal{O}$ complete and k algebraically closed. Algebraic groups are defined over an algebraic closure of a finite field of odd characteristic. For a finite group T, a block is identified with its central idempotent in $\mathcal{O}T$. For a connected reductive group $\mathbf{H}$ with Frobenius F, write $H = \mathbf{H}^F$ and let H^* be its finite dual group. For an odd-order semisimple element $s \in H^*$, put

$$\mathcal{E}_2(H, s) = \bigcup_{\substack{t \in C_{H^*}(s) \\ t \text{ a 2-element}}} \mathcal{E}(H, st),$$

where $\mathcal{E}(H, st)$ is the rational Lusztig series. The idempotent e_s^H is the sum of the block idempotents in this 2-series. For an algebraic group $\mathbf{H}$, the superscript $\circ$ denotes its identity component. For a torus $\mathbf{T}$, write $X^*(\mathbf{T}) = \mathrm{Hom}(\mathbf{T}, \mathbb{G}_m)$ and $Y(\mathbf{T}) = \mathrm{Hom}(\mathbb{G}_m, \mathbf{T})$ for its character and cocharacter lattices.

An *ordinary* finite group of Lie type in this section is a central quotient of the fixed points of a simply connected algebraic group with endomorphism $F = \gamma F_r^f$, where γ is a pinned diagram automorphism and F_r is the standard field Frobenius. Its ordinary automorphisms are the inner, diagonal, field and pinned diagram automorphisms, with their relations on the finite group. This terminology excludes the exceptional inseparable graph automorphism of $G_2(3^f)$ and exceptional finite Schur covers. Those cases are treated by the component estimates.

Theorem 10.1 (Proper minimal-Levi descent)**.** *Let $D \lhd G$ be a central product of one G-orbit of ordinary quasisimple finite groups of Lie type in odd characteristic, and let b be a G-stable block of D. Suppose the action induced by G lifts to the product of the ordinary simply connected covering groups, preserves its original central kernel, and consists of ordinary automorphisms and component permutations. For each component, let $\mathbf{S}$ be its simply connected algebraic covering group, let $\mathbf{S}^*$ be the dual, and let s be the odd semisimple label of its covered block. Set*

$$\mathbf{L}^* = C_{\mathbf{S}^*}\big(Z(C_{\mathbf{S}^*}(s)^\circ)^\circ\big). \tag{10.1}$$

Assume that each of these canonical minimal Levi subgroups is proper.

Then there exist a proper subgroup $H < G$ containing $C_G(D)$, a finite central extension $\pi : \widehat{H} \longrightarrow H$ with odd kernel, and an $\widehat{H}$-stable block c of $\widehat{J} = \pi^{-1}(D \cap H)$ such that

$$\mathcal{O}Gb \ \simeq_{\mathrm{Mor}} \ \mathcal{O}\widehat{H}c. \tag{10.2}$$

The extension has a canonical copy of $C_G(D)$, and the equivalence intertwines the action of every element of this copy with the action of the same element of $C_G(D)$. Consequently, every block B of G covering b corresponds to a block $\widehat{B}$ of $\widehat{H}$ satisfying

$$k(\widehat{B}) = k(B), \qquad |D_{\widehat{B}}| = |D_B|, \qquad |\widehat{H} : Z(\widehat{H})| < |G : Z(G)|. \tag{10.3}$$

Any specified block of $C_G(D)$ covered by B is retained.

The proof constructs the extension and its block after selecting a primitive summand of a cohomology lattice. The geometric input is the connected-centralizer Bonnafé–Rouquier theorem and the full diagonal comparison for different parabolics. The latter is the comparison of [6, Theorem 7.2], in the diagonal form of [65, Lemma 3.1]. Numbering for the latter reference is that of arXiv:2010.04468v1. We give the remaining order and equivariance arguments explicitly.

10.1. **The relative normalizer and the action.** We first work on an ordinary simply connected cover $S = \mathbf{S}^F$ and write $L = \mathbf{L}^F$, where $\mathbf{L}$ is dual to (10.1). Put

$$N = N_S(\mathbf{L}, e_s^L), \qquad A = N/L.$$

The normalizer here preserves both the algebraic Levi and its rational series idempotent.

Lemma 10.2. *The group $C_{\mathbf{S}^*}(s)/C_{\mathbf{S}^*}(s)^\circ$ is abelian of odd order. The group A is abelian of odd order. A finite 2-group of inner-diagonal algebraic automorphisms which preserves $(\mathbf{L}, e_s^L)$ acts trivially on $Y(Z(\mathbf{L})^\circ) \otimes_{\mathbb{Z}} \mathbb{R}$.*

Proof. Choose a finite central isogeny $\nu : \mathbf{S}^\sharp \longrightarrow \mathbf{S}^*$ whose source has simply connected derived subgroup. A point of an algebraic group over the algebraic closure of a finite field has finite order. We may therefore lift s to an odd-order semisimple element $s^\sharp$: replace a finite-order lift by its odd part. For $x \in C_{\mathbf{S}^*}(s)$, choose a lift $x^\sharp$ and set

$$d(x) = x^\sharp s^\sharp (x^\sharp)^{-1} (s^\sharp)^{-1}.$$

This lies in the central kernel of ν, is independent of the lift, and is multiplicative in x. Its order divides $|s^\sharp|$, since $d(x)$ commutes with $s^\sharp$. Its kernel is the image of $C_{\mathbf{S}^\sharp}(s^\sharp)$, which is connected by the connected-centralizer theorem for groups with simply connected derived subgroup [70, §§8–9]. This kernel is exactly $C_{\mathbf{S}^*}(s)^\circ$. The first assertion follows.

An element of a relative normalizer preserving the $\mathbf{L}^*$-class of s has a representative in $C_{\mathbf{S}^*}(s)$: multiply its dual representative by a suitable element of $\mathbf{L}^*$. Thus its relative Weyl image belongs to

$$\big(C_{\mathbf{S}^*}(s) \cap N_{\mathbf{S}^*}(\mathbf{L}^*)\big)\mathbf{L}^*/\mathbf{L}^*. \tag{10.4}$$

This is a quotient of a subgroup of the odd component group, because $C_{\mathbf{S}^*}(s)^\circ \leq \mathbf{L}^*$. Rational Levi duality identifies A with the corresponding rational series stabilizer in this relative normalizer; see [6, §7.A]. Hence A is abelian of odd order. In particular, the relative image of a 2-group of inner automorphisms is trivial. Elements of $\mathbf{L}$ act trivially on its center, giving the asserted cocharacter action.

The inner-diagonal assertion can be checked on finite groups as well. Take a compatible regular embedding $\mathbf{S} \hookrightarrow \widetilde{\mathbf{S}}$ and put $\widetilde{\mathbf{L}} = \mathbf{L}Z(\widetilde{\mathbf{S}})$. The quotient tori $\widetilde{\mathbf{S}}/\mathbf{S}$ and $\widetilde{\mathbf{L}}/\mathbf{L}$ are naturally isomorphic. The Lang–Steinberg theorem [70, Theorem 10.1] for the connected kernel $\mathbf{L}$ gives

$$\widetilde{S} = S\widetilde{L}, \qquad \widetilde{S} = \widetilde{\mathbf{S}}^F, \quad \widetilde{L} = \widetilde{\mathbf{L}}^F. \tag{10.5}$$

An inner-diagonal representative normalizing the pair can be written $g\widetilde{l}$, with $g \in S$ and $\widetilde{l} \in \widetilde{L}$. The diagonal Levi operator preserves e_s^L and acts trivially on the central cocharacter space. Thus $g \in N$, and the relative action is that of its odd-order image in N/L. This proves the assertion. □

Automorphisms pass between central forms as follows. If $T \twoheadrightarrow T/Z$ is a central quotient of a perfect group, two lifts of the same automorphism differ by a homomorphism $T \to Z$. Such a homomorphism is trivial. Consequently the existence of the ordinary lifts implies their uniqueness and their compatibility with multiplication. This observation applies to the whole product of simply connected covering groups. In particular, an element of $C_G(D)$ lifts to the identity automorphism of that product.

The field automorphisms admit the following algebraic realization. Write $F = \gamma F_r^f$ and replace $\mathbf{S}$ by f copies. Set

$$\tau(x_0, \dots, x_{f-1}) = (\gamma(x_{f-1}), x_0, \dots, x_{f-2}).$$

Then

$$\tau^f = \gamma_{\mathrm{diag}}, \qquad \overline{F} = \tau \ \mathrm{diag}(F_r). \tag{10.6}$$

The inverse to projection onto the first coordinate is

$$\iota(x) = (x, F_r(x), \dots, F_r^{f-1}(x)) \qquad (x \in S).$$

Use the same formula on the adjoint group. An inner-diagonal automorphism represented by $x \in \mathbf{S}_{\mathrm{ad}}^F$ is realized by conjugation with $\iota(x)$. On the fixed-point group, $F_r^f = \gamma^{-1}$. A compatible pinned diagram automorphism δ satisfies $\delta\gamma = \gamma\delta$, whence $\delta_{\mathrm{diag}}\tau = \tau\delta_{\mathrm{diag}}$ and

$$\tau^{-1}\iota(x) = \iota(F_r(x)), \qquad \delta_{\mathrm{diag}}\iota(x) = \iota(\delta(x)).$$

Consequently these algebraic operators satisfy

$$\tau^{-1}\,\mathrm{Ad}(\iota(x))\tau = \mathrm{Ad}(\iota(F_r(x))),$$
$$\delta_{\mathrm{diag}}\,\mathrm{Ad}(\iota(x))\delta_{\mathrm{diag}}^{-1} = \mathrm{Ad}(\iota(\delta(x))), \qquad (\tau^{-1})^f = \gamma_{\mathrm{diag}}^{-1}.$$

Together with the inner-diagonal and diagram multiplication laws, these are the ordinary finite automorphism relations [36, §2.1]. Thus the actual ordinary automorphism group acts algebraically on the product, commuting with $\overline{F}$; its field generator is τ^{-1}. For a component orbit, use these copies simultaneously and retain the literal permutations of component factors.

The product Levi is

$$\mathbf{L} \times F_r(\mathbf{L}) \times \cdots \times F_r^{f-1}(\mathbf{L}).$$

It is $\overline{F}$-stable because $F(\mathbf{L}) = \mathbf{L}$, and the first-coordinate projection identifies its fixed points with L. Its native dual label represents the original rational series. This compatibility follows from the fixed-point projection and series comparison of [73, §§8.2–8.8]; the same comparison applies to the Levi itself. We apply cohomology directly to this product datum. Component permutations act on this product variety. Under Künneth they are the graded tensor permutations, which satisfy the symmetric-group relations.

Lemma 10.3. *In the product realization* (10.6)*, an actual Sylow* 2*-subgroup of the stabilizer of the Levi, its rational series, and the selected block fixes a parabolic whose Levi is the selected product Levi.*

Proof. Let P be that Sylow subgroup and let P_j be the stabilizer of an algebraic factor. A field power which returns to that factor is already a diagram holonomy by $\tau^f = \gamma_{\mathrm{diag}}$. Thus P_j acts on the factor by inner-diagonal and pinned diagram automorphisms; there is no remaining inseparable field map in this factor action. It preserves the geometric class of the label inside the selected dual Levi, since it preserves the product Levi series.

Write $Y_j = Y(Z(\mathbf{L}_j)^\circ) \otimes \mathbb{R}$. The kernel of the diagram projection of P_j acts trivially on Y_j by Lemma 10.2. If the diagram group has odd order, the whole P_j acts trivially on Y_j, and an integral point outside the finitely many relative-root hyperplanes gives the desired parabolic on this factor.

The ordinary diagram groups with even order occur in types A, D and E_6 (including the full diagram group $\mathfrak{S}_3$ of D_4). In these cases properness and minimality imply

$$C_{\mathbf{S}_j^*}(s_j)^\circ = \mathbf{L}_j^*. \tag{10.7}$$

Indeed, in type A every semisimple connected centralizer is a Levi. For an odd-order semisimple element in an orthogonal or symplectic realization, there is no -1-eigenspace: paired nontrivial eigenvalues give general linear factors, and the 1-eigenspace gives the one residual classical factor. This is the standard Levi form. The assertion passes through central isogenies by taking an odd-order lift. Every proper Levi in E_6 has only A- and D-type derived components. Therefore $C_{\mathbf{L}_j^*}(s_j)^\circ = C_{\mathbf{S}_j^*}(s_j)^\circ$ is a Levi of $\mathbf{L}_j^*$, hence of $\mathbf{S}_j^*$; the canonical minimal choice forces (10.7).

Work at this point with the simply connected primal group and its adjoint dual. The center of a dual Levi is connected: its character lattice is the quotient of the adjoint root lattice

by the sublattice generated by a subset of the simple roots, and this quotient is torsion-free. Equation (10.7) places s_j in this connected center. Its Levi-conjugacy class is a singleton, so P_j fixes it as a point.

Choose a Weyl-and-diagram-invariant positive definite form $(-,-)_j$ on the real cocharacter space of a maximal torus $\mathbf{T}_j \leq \mathbf{L}_j$. Under $X^*(\mathbf{T}_j^*) = Y(\mathbf{T}_j)$, the real character space of $Z(\mathbf{L}_j^*)$ is the quotient by the span of the Levi coroots. The space Y_j is the orthogonal complement of that span. Thus

$$\langle y, \overline{u}\rangle_j = (y,u)_j \quad (y \in Y_j,\ \overline{u} \in X^*(Z(\mathbf{L}_j^*)) \otimes \mathbb{R})$$

is a nondegenerate invariant pairing. Suppose an outside root α vanished on $Y_j^{P_j}$. Averaging its restriction over P_j gives zero on Y_j. The root-coroot proportionality and this pairing then give

$$\sum_{g \in P_j} {}^g\alpha^\vee \,|_{Z(\mathbf{L}_j^*)} = 0 \quad \text{in } X^*(Z(\mathbf{L}_j^*)) \otimes \mathbb{R}. \tag{10.8}$$

The proportionality factor between a root and its coroot is constant on its orbit; in the diagram-primary cases here the root system is simply laced. Different torus-normalizer representatives differ by Levi Weyl elements, which act trivially on the central quotient and on Y_j. Thus these are genuine actions on the two central spaces. The sum in (10.8) is integral, and the character lattice of the connected center is torsion-free, so it is the zero integral character. Evaluating at P_j-fixed s_j yields

$$\alpha^\vee(s_j)^{|P_j|} = 1.$$

The left-hand base has odd order and $|P_j|$ is a power of two. Hence $\alpha^\vee(s_j) = 1$, which would put the outside dual root in $C_{\mathbf{S}_j^*}(s_j)^\circ = \mathbf{L}_j^*$. This contradiction proves that no outside-root hyperplane contains $Y_j^{P_j}$.

The fixed space is rational. Choose an integral fixed cocharacter y_j outside all these hyperplanes. More explicitly, for a basis $v_1, \dots, v_t$ of its fixed lattice, each outside-root value on $v_1 + uv_2 + \cdots + u^{t-1}v_t$ is a nonzero polynomial. Some integer u therefore avoids their finite sets of zeros. For each P-orbit of algebraic factors choose such a representative and put $y_{gj} = gy_j$. The stabilizer property makes this well defined. The product cocharacter y is P-fixed and regular relative to the selected product Levi. Its parabolic $\mathbf{P}(y)$ has precisely that Levi and is P-stable. Frobenius stability of this parabolic is not required. □

10.2. **Cohomology and the full prescribed diagonal subgroup.** Suppress the product notation and write $\mathbf{S}, F, \mathbf{L}, S, L$ for the data just obtained. For a parabolic $\mathbf{P} = \mathbf{LU}$, put

$$Y_{\mathbf{U}} = \{g\mathbf{U} \in \mathbf{S}/\mathbf{U} : g^{-1}F(g) \in \mathbf{U}F(\mathbf{U})\}, \qquad d = \dim Y_{\mathbf{U}}, \qquad M = H_c^d(Y_{\mathbf{U}}, \mathcal{O})e_s^L.$$

Here H_c^d denotes compactly supported étale cohomology in the middle degree. It has commuting left S and right L actions and the prescribed full diagonal Levi action. For a regular embedding as in (10.5), write

$$X = (S \times L^{\mathrm{op}})\Delta\widetilde{L}. \tag{10.9}$$

Here $\Delta\widetilde{L} = \{(t, t^{-1}) : t \in \widetilde{L}\}$ inside $\widetilde{S} \times \widetilde{L}^{\mathrm{op}}$. The regular-embedding center acts trivially diagonally. After selecting a left block b_S, let X_{b_S} denote its diagonal stabilizer.

The comparison theorem [6, Theorem 7.2], together with [65, Lemma 3.1 and its proof], supplies an isomorphism of the full prescribed diagonal modules for any two parabolics with this Levi. Its hypothesis is $C_{\mathbf{S}^*}(s)^\circ \leq \mathbf{L}^*$. Applying a pair-preserving automorphism first sends $Y_{\mathbf{U}}$ to $Y_{a(\mathbf{U})}$, after which this comparison gives an intertwiner for M. We establish the multiplication law for these intertwiners below.

For the parabolic of Lemma 10.3, P and the prescribed diagonal Levi operators act on the same variety. Let $H_0 \leq G$ be the stabilizer of the product datum $(\mathbf{L}, e_s^L, b_S)$, allowing its component

permutations, and let $\rho_0 : H_0 \to \mathrm{Aut}(S)$ be its uniquely lifted action on the covering product. In this automorphism group put

$$\mathcal{A} = \rho_0(H_0), \qquad D_0 = \{\mathrm{Ad}(t) : t \in \widetilde{L}, {}^t b_S = b_S\}, \qquad \Gamma = D_0\mathcal{A}.$$

Intrinsically, D_0 consists of the rational adjoint-Levi automorphisms preserving b_S. Every element of $\mathcal{A}$ preserves the Levi and the block, so it normalizes D_0. Thus D_0 is normal in Γ. To record their intersection relations, form the finite groups

$$\mathcal{U} = (S \times L^{\mathrm{op}}) \rtimes \Gamma, \quad K_0 = \{(t^{-1}, t, \mathrm{Ad}(t)) : t \in L\}, \quad \mathcal{Z} = \mathcal{U}/K_0. \tag{10.10}$$

The group K_0 is normal: it centralizes $S \times L^{\mathrm{op}}$, and an actor sends its t-element to its $a(t)$-element. Its geometric operator is the identity. Let $\overline{X}_{b_S}$ be the subgroup of $\mathcal{Z}$ above D_0. The homomorphism

$$X_{b_S} \longrightarrow \overline{X}_{b_S}, \qquad (s, \ell)\Delta t \longmapsto [(s, \ell, \mathrm{Ad}(t))]$$

is onto and has kernel $\Delta Z(\widetilde{S})$, the trivially acting diagonal regular-embedding center. The relations in K_0 make this map independent of the chosen expression $(s, \ell)\Delta t$. The action on $M_b = b_S M$ factors through $\overline{X}_{b_S}$, and the two groups give the same endomorphism order. Thus

$$R = \mathcal{Z}/\overline{X}_{b_S} = \Gamma/D_0, \tag{10.11}$$

and the actor maps onto R. The internal subgroup $A = N/L$ is a normal subgroup of R.

Let $\overline{P}$ be the image in R of a Sylow 2-subgroup P of $\mathcal{A}$. It is a Sylow 2-subgroup of R. The groups $S \times L^{\mathrm{op}}$, D_0 and P act on the same variety, and K_0 acts trivially. Hence the prescribed $\overline{X}_{b_S}$-module extends to *the whole inverse image* of $\overline{P}$ in $\mathcal{Z}$. Indeed, the formula

$$\rho(xp) = \rho_X(x)\rho_P(p)$$

is well defined because the two actions coincide on their intersection and satisfy the geometric conjugation relation. The extension obstruction therefore restricts to zero on the Sylow subgroup $\overline{P}$ of R.

10.3. **Endomorphisms before a normalizer extension.**

Proposition 10.4. *The lattice M is projective on both the S and L sides and is a left $\mathcal{O}Se_s^S$-progenerator. For each $a \in A$, choose $n_a \in N$ representing a and a full-X-compatible diagonal intertwiner U_a for $\mathrm{Ad}(n_a)$. Write $L_x(m) = xm$ and $R_y(m) = my$ for the left and right operators, and put $T_a = L_{n_a^{-1}} U_a$. Then T_a is S-linear, $T_a R_\ell = R_{n_a \ell} T_a$ for $\ell \in L$, and*

$$\mathrm{End}_{KS}(KM) = \bigoplus_{a \in A} (KLe_s^L) T_a. \tag{10.12}$$

The integral order $E = \mathrm{End}_{\mathcal{O}X}(M)$ is commutative. For a selected left block b_S, put $M_b = b_S M$ and $E_b = \mathrm{End}_{\mathcal{O}X_{b_S}}(M_b)$. This is commutative, and E_b^A is local. Consequently A acts transitively on the primitive idempotents of E_b, whose number is odd.

Proof. We recall the regular-embedding calculation, including the parts needed over $\mathcal{O}$. Put $\widetilde{N} = N\widetilde{L}$ and $\widetilde{X} = \widetilde{S} \times \widetilde{L}^{\mathrm{op}}$. The series idempotents above e_s^L have free A-orbits [6, Lemma 7.4]. Let e_0 be the sum of one representative from each such orbit. The lifted dual centralizers are connected, so the connected-centralizer Bonnafé–Rouquier theorem makes $\widetilde{M}e_0$ an integral Morita lattice [4, Theorem B′]. Here

$$\widetilde{M} = \mathrm{Ind}_X^{\widetilde{X}} M = H_c^d(Y_{\mathbf{U}}^{\widetilde{\mathbf{S}}}, \mathcal{O})e_s^L.$$

Induction along the free orbit extends this upstairs module to $\widetilde{N}$ and gives an upstairs Morita equivalence with $\mathcal{O}\widetilde{N}e_s^L$. The idempotents are orthogonal elements of $\mathcal{O}\widetilde{L}$; the connected-centralizer Morita theorem holds over $\mathcal{O}$; and the induction along the orbit is an identity of integral modules. In particular the calculation in [6, §7.C] gives

$$\mathrm{End}_{\mathcal{O}\widetilde{X}}(\widetilde{M}) \cong Z(\mathcal{O}\widetilde{L}e_0)^{[N:L]}. \tag{10.13}$$

It is obtained by separating the free right idempotents and using the Morita equivalence on each representative summand.

Write $d_{\mathrm{reg}} = [\widetilde{S} : S] = [\widetilde{L} : L]$. On the left,

$$\widetilde{M} \cong \mathrm{Ind}_S^{\widetilde{S}} M, \qquad \mathrm{Res}_S^{\widetilde{S}} \widetilde{M} \cong M^{\oplus d_{\mathrm{reg}}}. \tag{10.14}$$

The second identity uses the prescribed diagonal $\widetilde{L}$ action and (10.5). Restriction of the upstairs projective module makes $M^{\oplus d_{\mathrm{reg}}}$ projective, hence so is M. The upstairs regular module $\mathcal{O}\widetilde{S}e_s^S$ is a direct summand of finitely many copies of $\widetilde{M}$. Its restriction is $(\mathcal{O}Se_s^S)^{\oplus d_{\mathrm{reg}}}$, so $\mathcal{O}Se_s^S$ is a direct summand of finitely many copies of M. This proves the left generator property. On the right, restriction of the upstairs projective module to L, followed by the analogous direct-sum decomposition, proves right L-projectivity. These statements also imply that M is an $\mathcal{O}$-lattice.

Frobenius reciprocity and the upstairs Morita equivalence give the following dimension calculation:

$$\begin{aligned} d_{\mathrm{reg}} \dim_K \mathrm{End}_{KS}(KM) &= \dim_K \mathrm{End}_{K\widetilde{S}}(K\widetilde{M}) \\ &= \dim_K(K\widetilde{N}e_s^L) = d_{\mathrm{reg}} \dim_K(KNe_s^L). \end{aligned} \tag{10.15}$$

Use the common left-induced model

$$K\widetilde{M} = K\widetilde{S} \otimes_{KS} KM, \qquad \widetilde{T}_a(g \otimes m) = g \otimes T_a m.$$

Each operator is well defined because T_a is S-linear. Let $\delta(t)$ be the prescribed diagonal operator for $t \in \widetilde{L}$. In this model the right action is

$$(g \otimes m)t = gt \otimes \delta(t)^{-1} m.$$

For $n = n_a$, put $u = ntn^{-1}$ and $h = t^{-1}u \in L$. Full-X compatibility gives $U_a\delta(t)^{-1} = \delta(u)^{-1}U_a$. Since $hu^{-1}n^{-1}u = n^{-1}$, it follows that

$$\begin{aligned} (g \otimes T_a m)u &= gt \otimes L_h \delta(u)^{-1} L_{n^{-1}} U_a m \\ &= gt \otimes L_{n^{-1}} \delta(u)^{-1} U_a m = \widetilde{T}_a((g \otimes m)t). \end{aligned}$$

Thus $\widetilde{T}_a$ permutes the free right lifted idempotents by a. If $\sum_a R_{u_a} T_a = 0$ with $u_a \in KLe_s^L$, induction of these S-linear maps gives $\sum_a \widetilde{R}_{u_a}\widetilde{T}_a = 0$; here $\widetilde{R}_{u_a}$ is the upstairs right action of u_a. For each lifted source idempotent separately, the terms have mutually orthogonal target idempotents. Projection to those targets shows that every induced u_aT_a vanishes on that source summand. Doing this for all source summands makes each induced u_aT_a zero. Faithfulness of the upstairs right Levi action, and invertibility of T_a, give $u_a = 0$. The resulting independent summands have total dimension $|A| \dim_K(KLe_s^L) = \dim_K(KNe_s^L)$; (10.15) proves (10.12).

Induction of endomorphisms is faithful, so it embeds E into (10.13). Hence E is commutative. If b_S is not fixed by the whole diagonal group, decompose M by the orbit of b_S under that group. Induction from its diagonal stabilizer identifies the orbit endomorphism order with $\mathrm{End}_{\mathcal{O}X_{b_S}}(b_S M)$; different left blocks are orthogonal. Thus E_b is the corresponding direct factor and is commutative. Two full-X_{b_S} intertwiners for the same automorphism differ by a unit of E_b. Their conjugation actions on E_b therefore agree. The prescribed subgroup acts trivially on its endomorphism order, so these actions define an action of $R = \Gamma/D_0$ on E_b. The projection of (10.12) onto b_S still spans the full generic left endomorphism algebra.

An A-fixed element of E_b commutes with right L and with all internal intertwiners. It therefore centralizes the full algebra in (10.12), after projection onto b_S. Intersecting integrally, and using that M_b is a progenerator, gives

$$E_b^A \cong Z(\mathcal{O}Sb_S)^{\text{diagonal stabilizer}}. \tag{10.16}$$

Conversely every fixed central element has the stated commutation properties, proving equality. The center of a block order over $\mathcal{O}$ is local; an invariant unit has invariant inverse, so the invariant subring in (10.16) is local as well. The sum of an A-orbit of primitive idempotents is an idempotent of this local ring. There can therefore be only one such orbit. Its size divides the odd order $|A|$. □

10.4. **One scalar extension of the full stabilizer.** We isolate the obstruction argument in a form which keeps the prescribed subgroup action.

Lemma 10.5. *Let $X_0 \lhd Z_0$ be finite groups, let $R_0 = Z_0/X_0$, and let U_0 be a Z_0-stable $\mathcal{O}X_0$-lattice. Suppose $E_0 = \mathrm{End}_{\mathcal{O}X_0}(U_0)$ is a commutative complete local order. If the prescribed X_0-action extends to the full preimage of a Sylow 2-subgroup of R_0, then U_0 extends to a finite central odd cover of Z_0 pulled back from R_0. Its central kernel acts by a single scalar character.*

Proof. Choose intertwiners for the elements of R_0. Because E_0 is commutative, their conjugation action on E_0 is independent of the choices and is a genuine group action. Their multiplication obstruction is

$$\omega \in H^2(R_0, E_0^\times).$$

The residue field of E_0 is k, and R_0 acts trivially on it because all operators are $\mathcal{O}$-linear. Put $V_0 = 1 + \mathfrak{m}_0$, where $\mathfrak{m}_0$ is the maximal ideal of E_0, and put $m = |R_0|_{2'}$. The image of ω in $H^2(R_0, k^\times)$ is killed by m: positive-degree cohomology is killed by $|R_0|$, while squaring is an automorphism of $k^\times$ and hence of its cohomology. Let μ_m be the group of mth roots of unity in $k^\times$. The exact sequence

$$1 \longrightarrow \mu_m \longrightarrow k^\times \xrightarrow{x \mapsto x^m} k^\times \longrightarrow 1$$

therefore supplies a μ_m-valued cocycle representing that image. Lift the roots to $\mathcal{O}^\times$ and choose this one normalized scalar cocycle on the whole R_0. Its pullback defines the required finite central odd cover.

Subtract its class from ω. The remainder comes from a class $v \in H^2(R_0, V_0)$. Let $P_0 \in \mathrm{Syl}_2(R_0)$. The original obstruction restricts to zero on P_0 by the stipulated extension on its whole preimage. The scalar cocycle also has zero restriction, since its coefficient group has odd order. Moreover $H^1(P_0, k^\times) = 0$, so the exact coefficient sequence makes

$$H^2(P_0, V_0) \longrightarrow H^2(P_0, E_0^\times)$$

injective. Thus v restricts to zero in $H^2(P_0, V_0)$. Restriction followed by corestriction multiplies v by $[R_0 : P_0]$. Hensel's lemma makes the power map for this odd integer an automorphism of V_0, and hence of its cohomology. It follows that $v = 0$. An integral cochain adjustment now gives a coherent action on the cover, extending the originally prescribed X_0-action. □

Apply Proposition 10.4 to the whole component product, with its block and diagonal stabilizer. The strict Sylow action constructed above permutes the odd number of primitive idempotents of E_b. Hence some primitive idempotent e is fixed. Put

$$M_e = eM_b, \qquad E_e = eE_be, \qquad R_e = \mathrm{Stab}_R(e), \qquad A_e = A \cap R_e. \tag{10.17}$$

The ring E_e is commutative and local. The chosen Sylow image $\overline{P}$ lies in R_e and is still Sylow there. The module M_e retains the strict action on its whole preimage. Thus Lemma 10.5 constructs one scalar odd cover of its full stabilizer, including all component permutations.

All primitive summands of M_b are isomorphic as left block modules, since the internal intertwiners giving A-transitivity commute with the left group. Therefore M_e is still a left block progenerator. Cornering the generic span gives

$$\mathrm{End}_{KSb_S}(KM_e) = \sum_{a\in A_e} (KLe_s^L)eT_a. \tag{10.18}$$

Indeed, $eT_ae = 0$ if $a(e) \neq e$, and e commutes with right L. This remains true after the cochain adjustments. A unit of eE_be extends by the identity to E_b and then over its diagonal block orbit to a unit of the original full-X endomorphism order. Consequently the adjusted operators are restrictions of a new arbitrary full-X intertwiner family, to which (10.12) again applies.

10.5. **An integral support criterion.** The following symmetric-order argument constructs the target block. Its conclusion includes the integrality of the generic support. The proof uses the adjunction argument of [12, Theorems 1.1 and 2.4].

Lemma 10.6. *Let A_0 and B_0 be symmetric $\mathcal{O}$-orders with split semisimple generic fibers. Let ${}_{A_0}M_{0B_0}$ be projective of finite type on both sides. Suppose KM_0 is faithful as a KA_0-module and the right-action homomorphism*

$$(KB_0)^{\mathrm{op}} \longrightarrow \mathrm{End}_{KA_0}(KM_0)$$

is surjective. There is a central idempotent $c \in B_0$ with $M_0 = M_0c$, and M_0 induces a Morita equivalence between A_0 and B_0c. If A_0 is indecomposable, c is primitive.

Proof. Put $N_0 = \mathrm{Hom}_{\mathcal{O}}(M_0, \mathcal{O})$, $P_0 = M_0 \otimes_{B_0} N_0$ and $Q_0 = N_0 \otimes_{A_0} M_0$. Symmetry identifies N_0 both with $\mathrm{Hom}_{A_0}(M_0, A_0)$ and with $\mathrm{Hom}_{B_0}(M_0, B_0)$. The functors $\mathcal{F} = M_0 \otimes_{B_0} -$ and $\mathcal{G} = N_0 \otimes_{A_0} -$ are therefore adjoint in both orders. Denote the unit and counit for $\mathcal{F} \dashv \mathcal{G}$ by

$$\eta_B : B_0 \longrightarrow Q_0, \qquad \epsilon_A : P_0 \longrightarrow A_0,$$

and those for $\mathcal{G} \dashv \mathcal{F}$ by

$$\eta_A : A_0 \longrightarrow P_0, \qquad \epsilon_B : Q_0 \longrightarrow B_0.$$

All four are bimodule maps and satisfy their triangle identities. The bimodules P_0, Q_0 are $\mathcal{O}$-free: projectivity expresses each as a direct summand of finitely many copies of M_0 or N_0.

The generic hypotheses give a semisimple Morita equivalence between KA_0 and a central summand KB_0c_K. Thus $K\eta_A$ and $K\epsilon_A$ are isomorphisms. Tensor η_A on the left by N_0 and on the right by M_0. The resulting map

$$N_0\eta_AM_0 : Q_0 \longrightarrow Q_0 \otimes_{B_0} Q_0$$

has left inverse $\epsilon_B \otimes 1_{Q_0}$ by the triangle identity for $\mathcal{G} \dashv \mathcal{F}$. Its generic fiber is an isomorphism, so its complementary integral summand is free of rank zero and vanishes. It is therefore an integral isomorphism. Similarly

$$N_0\epsilon_AM_0 : Q_0 \otimes_{B_0} Q_0 \longrightarrow Q_0$$

has right inverse $\eta_B \otimes 1_{Q_0}$ by the triangle identity for $\mathcal{F} \dashv \mathcal{G}$, and is an integral isomorphism.

The composite $\epsilon_A\eta_A$ is multiplication by an element $z \in Z(A_0)$. Its action on Q_0 is the composite of the preceding two isomorphisms. Consider

$$Z(A_0) \longrightarrow T_0 = \mathrm{End}_{B_0\text{-}B_0}(Q_0), \qquad z \longmapsto [n \otimes m \mapsto nz \otimes m].$$

This map becomes an isomorphism over K, by the generic Morita equivalence, and is injective. The order T_0 is commutative since its generic fiber is commutative; it is finite over $\mathcal{O}$, hence integral over $Z(A_0)$. The image of z is a unit, so z is a unit already in $Z(A_0)$. Indeed, a monic relation

$$z^{-r} + a_{r-1}z^{-(r-1)} + \cdots + a_0 = 0, \qquad a_i \in Z(A_0),$$

gives $z^{-1} = -(a_{r-1} + a_{r-2}z + \cdots + a_0 z^{r-1}) \in Z(A_0)$. It follows that $z^{-1}\epsilon_A$ is a left inverse of η_A. The same free-rank-zero argument proves that η_A, and then ϵ_A, is an integral isomorphism.

Set $T = \eta_B\epsilon_B \in \mathrm{End}_{B_0\text{-}B_0}(Q_0)$. The triangle identities and the isomorphisms just proved show that $M_0\eta_B$ and $M_0\epsilon_B$ are isomorphisms. Furthermore $Q_0 = \mathcal{G}(M_0)$ belongs to the essential image of the fully faithful functor $\mathcal{G}$. On this image $\mathcal{F}$ detects isomorphisms. Hence T is an automorphism; its inverse is also a bimodule map. The maps

$$p = \eta_B : B_0 \longrightarrow Q_0, \qquad j = \epsilon_B T^{-1} : Q_0 \longrightarrow B_0$$

satisfy $pj = 1_{Q_0}$. The projector jp on the regular B_0-bimodule is multiplication by an integral central idempotent c. Thus $Q_0 \cong B_0 c$.

Under the canonical identification $Q_0 \cong \mathrm{End}_{A_0}(M_0)^{\mathrm{op}}$, p is the right-action map; hence $B_0 c \cong \mathrm{End}_{A_0}(M_0)^{\mathrm{op}}$ as algebras. The isomorphism $M_0\eta_B$ also gives $M_0 c = M_0$. Finally $P_0 \cong A_0$ and left B_0-projectivity of N_0 show that A_0 is a direct summand of finitely many copies of M_0. Thus M_0 is an A_0-progenerator, proving Morita equivalence. The induced center isomorphism makes c primitive when A_0 is indecomposable. □

Let J be the inverse image of A_e in the internal normalizer N, and restrict the same cover to $\widehat{J}$. Write $\pi : \widehat{J} \to J$ for the projection and ρ for its coherent action on M_e. For $\widehat{n} \in \widehat{J}$, put $n = \pi(\widehat{n})$ and $T_{\widehat{n}} = L_{n^{-1}}\rho(\widehat{n})$. Since $\rho(\widehat{n})$ commutes with L_n, the right operator is

$$R_{\widehat{n}} = L_n \rho(\widehat{n})^{-1} = T_{\widehat{n}}^{-1}.$$

For $\widehat{u} \in \widehat{J}$, put $u = \pi(\widehat{u})$. Semilinearity gives

$$T_{\widehat{n}} T_{\widehat{u}} = L_{n^{-1}} L_{nu^{-1}n^{-1}} \rho(\widehat{n}\widehat{u}) = T_{\widehat{nu}}.$$

Hence $R_{\widehat{n}} R_{\widehat{u}} = R_{\widehat{un}}$, as required for a right action. The right-action image contains these operators and their inverses; thus (10.18) gives its generic surjectivity. The cover is canonically trivial on L, and $[\widehat{J} : L]$ is odd. Since M_e is right L-projective, averaging over this odd index shows that it is right $\widehat{J}$-projective. It is left block projective as a summand of M_b. Let λ be the scalar character of the covering kernel on the right. In the notation of Lemma 10.6, take

$$A_0 = \mathcal{O}Sb_S, \qquad B_0 = \mathcal{O}\widehat{J}e_\lambda e_s^L.$$

These are symmetric orders. Generic left faithfulness follows from the progenerator property. The lemma therefore supplies one block c of $\widehat{J}$ and an equivalence

$$\mathcal{O}Sb_S \ \simeq_{\mathrm{Mor}} \ \mathcal{O}\widehat{J}c. \tag{10.19}$$

The block c is stable under the whole covering actor. Indeed, a semilinear operator sends $M_e = M_e c$ to $M_e = M_e{}^{\widehat{h}} c$. If the two primitive central idempotents differed, their orthogonality would force $M_e = 0$. Thus the target block in (10.19) is the same block throughout the ambient construction.

10.6. **The subgroup, the central quotient, and ambient transport.** The construction on the simply connected component product carries the unique lifts of the G-action. The subgroup $H_0 \leq G$ defined above stabilizes the tuple of Levi and Levi-series data. Then

$$G = DH_0. \tag{10.20}$$

Here is the rational conjugacy justification. An ambient automorphism preserves the global series, so its dual sends s to a rationally conjugate element and sends the canonical minimal Levi to a rationally conjugate Levi. Rational Levi duality allows an inner adjustment by an element of the original finite group which normalizes the primal Levi. After this adjustment, a rational conjugator of the two labels normalizes the dual Levi, because both canonical minimal Levis are now the same. The relative normalizer correspondence supplies the matching primal normalizer

element. A second inner adjustment therefore preserves the Levi-series idempotent itself. This is the rational conjugacy argument of [65, Lemma 2.10, Corollary 2.13, and the initial argument of Proposition 4.8]. Apply it to every component, with the same component permutation, to obtain (10.20).

Let N now denote the image in D of the product normalizer; then $H_0 \cap D = N$. The homomorphism

$$\eta : H_0 \xrightarrow{\rho_0} \mathcal{A} \hookrightarrow \Gamma \longrightarrow R = \Gamma/D_0$$

satisfies $\eta(N) = A$ and $C_G(D) \leq \ker \eta$. Put $H = \eta^{-1}(R_e)$ and $J = D \cap H$. Thus $\eta(J) = A_e$, and H is the stabilizer of the chosen primitive idempotent e. The conjugation action on E_b of an internal right normalizer intertwiner and of the corresponding actual diagonal normalizer element is the same: their ratio is a prescribed left group operator, which commutes with E_b. Transitivity of A on the primitive idempotents therefore gives

$$H_0 = NH, \qquad G = DH, \qquad J = D \cap H. \tag{10.21}$$

The local groups used above are their full inverse images in the simply connected product. The identity $G = DH$ retains every ambient coset. The centralizer $C = C_G(D)$ acts trivially on the data and lies in H.

The group H is proper. For one factor, N/L is abelian. If $N = S$, perfectness would give $S = [S, S] \leq L$. But if r is the defining characteristic and $q = r^f$, let Φ^+ be a positive root system of $\mathbf{S}$ and Φ_L^+ its Levi subset. The group-order formulas give

$$|L|_r = q^{|\Phi_L^+|} < q^{|\Phi^+|} = |S|_r$$

for a proper Levi, a contradiction. Hence $N < S$; the same argument on a proper factor applies to the product. Thus $J < D$, and (10.21) implies $H < G$.

We next impose the original central kernel Z_0 of the simply connected product mapping onto D. Write $(Z_0)_2$ and $(Z_0)_{2'}$ for its Sylow 2-subgroup and odd Hall subgroup. If ζ_i is the prescribed central character on the ith component, select their tensor-product sector with

$$\prod_i \zeta_i(z_i) = 1 \qquad \big((z_i)_i \in (Z_0)_{2'}\big).$$

The geometric lattice has identical left and right action of each $z \in Z_0$:

$$zm = mz. \tag{10.22}$$

The primitive cut and the scalar extension preserve this identity. The cocycle is trivial on the prescribed Levi and hence on Z_0. The local Morita equivalence consequently matches the two central ideals generated by $z - 1$, $z \in Z_0$. Quotienting by those corresponding ideals yields a Morita equivalence of the actual quotient group-block orders. To see this directly, restrict the Morita functors to the full subcategories annihilated by these ideals. They are exactly the module categories of the quotient orders. The quotient equivalence makes the quotient bimodule projective and a generator on both sides; it is therefore $\mathcal{O}$-free.

For the odd part of Z_0, the selected trivial sector is already the quotient group algebra. For its central 2-part, the usual central-2 block bijection gives one quotient block on each side. Their defect orders both lose the same factor $|(Z_0)_2|$. An actor need only normalize Z_0: it sends $(z - 1)m$ to $({}^h z - 1)\rho(h)m$, and so preserves the quotient module. The covers descend as well, with odd scalar kernel. Properness is retained because $Z_0 \leq L \leq J$. From this point on, M denotes this quotient Morita lattice, $J = D \cap H$, and $\widehat{J}$ is the corresponding covered subgroup of the group. The lattice is a Morita bimodule for $\mathcal{O}Db$ and one block $\mathcal{O}\widehat{J}c$.

With $n_{\mathrm{odd}} = |R_e|_{2'}$, let $\omega : R_e \times R_e \to \mu_{n_{\mathrm{odd}}}$ be the normalized scalar cocycle defining the cover above. Its pullback to H is

$$(h, h') \longmapsto \omega(\eta(h), \eta(h')).$$

Since $\eta(C) = 1$, this cocycle is 1 on every pair with one entry in C. Consequently the resulting central odd cover $\widehat{H} \to H$ has a canonical embedded copy of C, and its coherent operator satisfies $\rho(c_0) = 1$ for every $c_0 \in C$. Its restriction to $\widehat{J}$ is exactly the descended cover used in the local Morita equivalence.

The resulting local Morita lattice M carries semilinear operators. For $\widehat{h} \in \widehat{H}$, $a \in \mathcal{O}Db$, $m \in M$ and $j \in \widehat{J}$, they satisfy

$$\begin{aligned} \rho(\widehat{h})(amj) &= {}^{\pi(\widehat{h})}a\,\rho(\widehat{h})m\,{}^{\widehat{h}}j, \\ \rho(\widehat{j})m &= \pi(\widehat{j})m\widehat{j}^{-1} \qquad (\widehat{j} \in \widehat{J}). \end{aligned} \tag{10.23}$$

The second identity is the original intersection relation in (10.10). Equivalently, the right operator is $R_{\widehat{j}} = L_{\pi(\widehat{j})}\rho(\widehat{j})^{-1}$. Thus the scalar kernel acts by λ on the right and by λ^{-1} under ρ.

For a left $\mathcal{O}Gb$-module V, define

$$\Phi(V) = \mathrm{Hom}_{\mathcal{O}Db}(M, V), \qquad (\widehat{h}f)(m) = \pi(\widehat{h})f(\rho(\widehat{h})^{-1}m). \tag{10.24}$$

For $\widehat{j} \in \widehat{J}$, relation (10.23) reduces this to $f(m\widehat{j})$, the ordinary local Morita action. Thus $\Phi(V)$ is an actual $\mathcal{O}\widehat{H}c$-module in the prescribed scalar sector. Conversely, for such a module W, put

$$\Psi(W) = M \otimes_{\mathcal{O}\widehat{J}c} W, \qquad h(m \otimes w) = \rho(\widehat{h})m \otimes \widehat{h}w, \quad \pi(\widehat{h}) = h. \tag{10.25}$$

The two scalar kernel actions cancel, so the formula is independent of the lift. On $J = D \cap H$ it agrees with the left D-action by (10.23). It also respects conjugation by H. The resulting representation of $D \rtimes H$ kills the exact kernel $\{(j^{-1}, j) : j \in J\}$ of multiplication onto DH. It is therefore a representation of the group $G = DH$.

Local Morita evaluation is equivariant. Explicitly, $m \otimes f \mapsto f(m)$ sends $\rho(\widehat{h})m \otimes \widehat{h}f$ to $\pi(\widehat{h})f(m)$. The inverse coevaluation is equivariant by the same formulas. Thus (10.24) and (10.25) are inverse integral equivalences. For $c_0 \in C$, the operator $\rho(c_0)$ is 1; both functors therefore retain its original operator on the module factor. In particular, every C-block idempotent retains its action.

Finally, integral Morita equivalence of the sums in (10.2) bijects their primitive central idempotents. It sends the specified B to one $\widehat{B}$. Extension of scalars preserves the number of generic simple modules, giving $k(\widehat{B}) = k(B)$. The Cartan elementary divisors are preserved, and their largest member is the defect order of a group block; hence $|D_{\widehat{B}}| = |D_B|$. Since $Z(G) \le C$ and the cocycle has value 1 on every mixed pair involving C, the full preimage of $Z(G)$ is central in $\widehat{H}$. Therefore

$$|\widehat{H} : Z(\widehat{H})| \le |H : Z(G)| < |G : Z(G)|.$$

This proves Theorem 10.1.

Remark 10.7. The canonical minimal Levi in (10.1) makes the odd label central in types A, D and E_6 in Lemma 10.3. For types with odd-order diagram group, the proof uses the trivial relative action of the factor stabilizer.

11. Principal blocks in type A

We count all constituents on restriction through the determinant quotient. A character with scalar stabilizer of order s contributes s^2 to the orbit sum. Let q be an odd prime power, let

$\epsilon \in \{1, -1\}$, and set

$$T_n = \begin{cases} \mathrm{GL}_n(q) & \epsilon = 1, \\ \mathrm{GU}_n(q) & \epsilon = -1, \end{cases} \qquad N_n = \begin{cases} \mathrm{SL}_n(q) & \epsilon = 1, \\ \mathrm{SU}_n(q) & \epsilon = -1. \end{cases}$$

We also write $\mathrm{GL}_n(-q) = \mathrm{GU}_n(q)$ and $\mathrm{SL}_n(-q) = \mathrm{SU}_n(q)$. Write $a = v_2(q - \epsilon)$, $b = v_2(q + \epsilon)$, and $A = 2^a$. Exactly one of a, b is one and the other is at least two. Let $K(n) = k(B_0(T_n))$, with $K(0) = 1$, and let $P(t) = \prod_{i \geq 1}(1 - t^i)^{-1}$.

Lemma 11.1. *One has*

$$F_{a,b}(t) := \sum_{n \geq 0} K(n)t^n = P(t)^A P(t^2)^{2^{a-1}(2^b-1)} \prod_{r \geq 2} P(t^{2^r})^{2^{a+b-2}}. \tag{11.1}$$

For $n \geq 1$, put $m = \min\{a, v_2(n)\}$. Then

$$k(B_0(N_n)) = \frac{S(n)}{A}, \qquad S(n) = K(n) + \sum_{j=1}^{m} 3 \cdot 4^{j-1} K(n/2^j). \tag{11.2}$$

Proof. Ordinary Jordan labels in $B_0(T_n)$ are partitions attached to the orbits of ϵq on the roots of unity of 2-power order. A root orbit of length d carries one partition and gives the factor $P(t^d)$. The lifting-the-exponent formula gives A orbits of length one, $2^{a-1}(2^b - 1)$ of length two, and 2^{a+b-2} of each length 2^r for $r \geq 2$. This proves (11.1); see also [10, Lemma 1.2].

Here are the block facts needed to pass to N_n. For an odd semisimple label s, the set $\mathcal{E}_2(T_n, s)$ is a single block. The block with $s = 1$ covers just the principal block of N_n, and the other blocks covering that block are its odd scalar twists. These are the ordinary covering statements in [24, Sections 4 and 6, Theorem 6.8 and Remark 6.9]. Consequently, if $X = \mathrm{Irr}(B_0(T_n))$ and C is the cyclic group of determinant characters of 2-power order, then $|C| = A$, each full determinant-twist orbit meeting X meets it in one C-orbit, and every stabilizer is contained in C.

Restriction through a cyclic quotient is multiplicity free. An orbit with stabilizer order s has A/s members and supplies s distinct constituents on N_n. Hence

$$k(B_0(N_n)) = \frac{1}{A} \sum_{\chi \in X} |C_\chi|^2.$$

For the subgroup $C_j \leq C$ of order 2^j, write $X_j = X^{C_j}$. Counting ordered pairs in a cyclic stabilizer by the subgroup they generate gives

$$\sum_{\chi \in X} |C_\chi|^2 = |X| + \sum_{j=1}^{a} 3 \cdot 4^{j-1} |X_j|. \tag{11.3}$$

We prove that the generating function for $|X_j|$ is $F_{a,b}(t^{2^j})$.

Suppose first $a \geq 2$, so $b = 1$. Scalar multiplication of order 2^j has orbits of length 2^j on the A root labels of degree one. For each $r \geq 1$, the $A/2$ root orbits of degree 2^r arise from primitive roots of order 2^{a+r}. The subgroup generated by ϵq on these roots is the group of units congruent to one modulo 2^a. Thus scalar multiplication has orbit length $2^{\max\{j-r,0\}}$ on their Frobenius orbits. Equal partitions are required along a scalar orbit. The resulting exponent of $P(t^{2^j})$ is

$$\frac{A}{2^j} + \sum_{r=1}^{j-1} 2^{a+r-j-1} + \frac{A}{2} = A,$$

and that at every subsequent weight 2^r, $r > j$, is $A/2$. This is exactly $F_{a,1}(t^{2^j})$.

If $a = 1$, only $j = 1$ occurs. Put $B = 2^b$. The labels $1, -1$ are interchanged, giving $P(t^2)$. Among the $B - 1$ degree-two root orbits, precisely the orbit of primitive fourth roots is fixed by multiplication by -1; the others occur in pairs. Their contribution is $P(t^2)P(t^4)^{B/2-1}$. Every

root orbit of degree 2^r, $r \geq 2$, is fixed by -1: on roots of order 2^{b+r} the even powers of ϵq generate the units congruent to one modulo 2^{b+1}, including the exponent change for multiplication by -1. They contribute $\prod_{r\geq 2} P(t^{2^r})^{B/2}$. Combining the factors again gives $F_{1,b}(t^2)$. Thus

$$|X_j| = \begin{cases} K(n/2^j) & 2^j \mid n, \\ 0 & 2^j \nmid n. \end{cases}$$

Substitution in (11.3) proves the lemma. □

11.1. **A uniform coefficient estimate.** We give the elementary partition estimates used below. If $p_2(n)$ counts partitions into powers of two, then

$$p_2(n) \leq 2^{\lfloor n/2 \rfloor}. \tag{11.4}$$

Indeed $p_2(2u+1) = p_2(2u)$ and $p_2(2u) = p_2(2u-2) + p_2(u) \leq 2p_2(2u-2)$ for $u \geq 2$. The assertion follows from $p_2(0) = p_2(1) = 1$, $p_2(2) = 2$. If $k(c,n) = [t^n]P(t)^c$, then

$$k(c,n) \leq c^n \quad (c \geq 3), \qquad k(2,n) \leq 3^n. \tag{11.5}$$

The quantity $k(c,n)$ is a polynomial in c with nonnegative coefficients and degree at most n. Therefore $k(c,n)/c^n$ is nonincreasing for positive c, and it suffices to take $c = 3$. With $\sigma_1(j) = \sum_{d|j} d$, logarithmic differentiation gives

$$nk(3,n) = 3\sum_{j=1}^{n} \sigma_1(j)k(3,n-j).$$

The cases $n = 1, 2$ are $3, 9$. For the induction at $n \geq 3$, use

$$\sum_{j\geq 1} \frac{\sigma_1(j)}{3^j} = \sum_{d\geq 1} \frac{d3^{-d}}{1-3^{-d}} \leq \frac{1}{2} + \frac{9}{8}\left(\frac{3}{4} - \frac{1}{3}\right) = \frac{31}{32} < 1.$$

This proves (11.5).

The order formula and the lifting-the-exponent formula give

$$\Delta_n := |T_n|_2 = 2^{an+(b-1)\lfloor n/2 \rfloor + n - s_2(n)}, \tag{11.6}$$

where $s_2(n)$ is the binary digit sum. We claim

$$2^m S(n) < \Delta_n \qquad (n \geq 3). \tag{11.7}$$

If $a \geq 2$, expand (11.1) by binary decompositions of n and use (11.5). Every term is at most A^n, so $K(n) \leq p_2(n)A^n$. For $n \geq 4$ this yields

$$S(n) \leq p_2(n)A^n(1 + (n^2-1)A^{-n/2}) \leq 2p_2(n)A^n,$$

because $n^2 \leq 2^n$. Since $2^m \leq n$ and $2^{s_2(n)} \leq n+1$,

$$\frac{2^m S(n)}{\Delta_n} \leq 2n(n+1)2^{-n/2} < 1 \quad (n \geq 20). \tag{11.8}$$

The last expression is $840/1024$ at 20 and decreases thereafter. For $3 \leq n \leq 19$, enlarge m to $v_2(n)$. Every coefficient $K(w)$ is a polynomial in A with nonnegative coefficients and degree at most w. Thus the resulting ratio is largest at $A = 4$.

If $a = 1$, put $B = 2^b \geq 4$, $M = \lfloor n/2 \rfloor$, and enlarge the exponent $B-1$ of $P(t^2)$ to B. Denote the enlarged coefficients by $\widetilde{K}(n)$. Their dependence on B is polynomial with nonnegative coefficients and degree at most M. For a binary decomposition with w_0 parts of size one, the other multiplicities w_r satisfy $\sum_{r\geq 1} w_r \leq M - \lfloor w_0/2 \rfloor$. Equations (11.4)–(11.5) give

$$\frac{K(n)}{\Delta_n} \leq \frac{p_2(n)3^n}{2^{2n+M-s_2(n)}} \leq (n+1)(3/4)^n.$$

Here we used $3^{w_0}/4^{\lfloor w_0/2\rfloor} \le 3^n/4^M$, since w_0 and n have the same parity. The sequence $K(n)$ is nondecreasing, so $S(n) \le 4K(n)$ and $2^m \le 2$. In particular

$$\frac{2^m S(n)}{\Delta_n} \le 8(n+1)(3/4)^n < 12(n+1)(3/4)^n < 1 \quad (n \ge 20). \tag{11.9}$$

The last inequality at 20 is $252\cdot 3^{20} < 4^{20}$, and the expression decreases thereafter. For $3 \le n \le 19$, replace K by $\widetilde{K}$ and set $B = 4$: after division by B^M each polynomial term is nonincreasing in B.

The remaining finite calculation is displayed in Table 4. The plus columns mean $K_+(n) = [t^n]F_{2,1}(t)$ and $S_+(n) = K_+(n) + \sum_{j=1}^{v_2(n)} 3\cdot 4^{j-1}K_+(n/2^j)$. The minus columns use $\widetilde{K}_-(n)$ at $a = 1, B = 4$ and $\widetilde{S}_-(n) = \widetilde{K}_-(n) + 3\widetilde{K}_-(n/2)$ for even n, with no correction for odd n. The inequalities to check in each row are precisely

$$\frac{2^{v_2(n)}S_+(n)}{2^{3n-s_2(n)}} < 1, \qquad \frac{2^{\min\{1,v_2(n)\}}\widetilde{S}_-(n)}{2^{2n+\lfloor n/2\rfloor - s_2(n)}} < 1.$$

Their largest value is $9/16$. The entries can be reproduced using integer arithmetic alone: if $F(t) = \prod_{d\ge1}(1-t^d)^{-e_d} = \sum K(n)t^n$, put $h_j = \sum_{d|j} de_d$ and use $nK(n) = \sum_{j=1}^n h_j K(n-j)$, starting from $K(0) = 1$.

TABLE 4. The complete finite boundary for (11.7).

n	$K_+(n)$	$S_+(n)$	$\widetilde{K}_-(n)$	$\widetilde{S}_-(n)$
3	48	48	18	18
4	140	236	56	83
5	360	360	108	108
6	896	1040	273	327
7	2080	2080	510	510
8	4696	5500	1145	1313
9	10132	10132	2076	2076
10	21344	22424	4281	4605
11	43536	43536	7560	7560
12	87024	90288	14665	15484
13	169736	169736	25290	25290
14	325376	331616	46761	48291
15	611680	611680	78948	78948
16	1132852	1150156	140526	143961
17	2064664	2064664	232758	232758
18	3714192	3744588	401470	407698
19	6591664	6591664	653570	653570

Proposition 11.2. *Every principal* 2*-block of* N_n/Z*, where* $Z \le Z(N_n)$*, satisfies* $k(B_0(N_n/Z)) \le |N_n/Z|_2$*.*

Proof. An odd central quotient retains exactly the principal central sector and does not change the defect order. We may therefore first consider a central 2-subgroup Z. For $n \ge 3$ its order is at most 2^m, while $|N_n|_2 = \Delta_n/A$. Inflation and (11.7) give

$$k(B_0(N_n/Z)) \le k(B_0(N_n)) = \frac{S(n)}{A} < \frac{\Delta_n}{A2^m} \le |N_n/Z|_2.$$

The case $n = 1$ is trivial. We include $n = 2$ explicitly, since discarding the central-character condition is too costly there.

For $a \geq 2$, $K(2) = A(A+4)/2$ and $S(2) = A(A+10)/2 \leq 2A^2 = \Delta_2$. For $a = 1$, $K(2) = B+4$ and $S(2) = B+10 \leq 4B = \Delta_2$. This proves the unquotiented assertion. The unique nontrivial central 2-subgroup has order two. Let x_0 be the number of principal T_2 labels trivial on this central involution and let x_1 be the number among them fixed by the scalar involution. The same restriction calculation gives $(x_0 + 3x_1)/A$ characters on the central quotient. The ordinary torus pairing identifies the central character of a Jordan label with its determinant label; this follows already from the common central action on the two sides of Deligne–Lusztig induction. Thus the following count is on the required central sector.

If $a \geq 2$, the degree-one labels are the A roots with exponents modulo A. The unordered pairs of distinct exponents with even sum number $A(A-2)/4$; the repeated labels each carry two partitions, adding $2A$. Degree-two root orbits have odd determinant charge and do not occur. Hence

$$x_0 = A(A+6)/4, \qquad x_1 = A/2.$$

The second equality counts pairs $\{\alpha, -\alpha\}$; their determinant charge is even because $4 \mid A$. Therefore $2(x_0 + 3x_1) = A(A+12)/2 \leq 2A^2$. If $a = 1$, the two repeated degree-one labels contribute four characters, and the allowed degree-two root orbits contribute $B/2 - 1$. Exactly the primitive-fourth-root orbit is scalar fixed. Thus

$$x_0 = B/2 + 3, \qquad x_1 = 1, \qquad 2(x_0 + 3x_1) = B + 12 \leq 4B.$$

Division by A proves the bound against the quotient defect $\Delta_2/(2A)$ in both cases. Finally take any odd central quotient, which cannot increase the count or change its 2-part. □

Lemma 11.3. *For either sign ϵ and every odd q,*

$$\frac{k(B_0(N_3))}{|N_3|_2} \leq \frac{1}{2}, \qquad \frac{k(B_0(N_6))}{|N_6|_2} \leq \frac{1}{16}.$$

The same estimates hold after an odd central quotient.

Proof. For $n = 3$, the quotient is $K(3)/\Delta_3$. If $a \geq 2$, then $K(3) = A(A^2 + 12A + 8)/6$ and $\Delta_3 = 2A^3$; the ratio is largest at $A = 4$ and is at most $1/2$. If $a = 1$, the two quantities are $2B + 8$ and $8B$, again giving at most $1/2$ for $B \geq 4$.

For $n = 6$ the numerator is $K(6) + 3K(3)$, including the scalar stabilizer term. For $a \geq 2$, polynomial positivity shows that the ratio is largest at $A = 4$. There $K(6) = 896$, $K(3) = 48$, and $\Delta_6 = 16 \cdot 4^6$, giving $65/4096 < 1/16$. For $a = 1$, use the enlarged series above. Its corresponding coefficients at $B = 4$ are 273 and 18, and $\Delta_6 = 128 \cdot 4^3$. The ratio is at most $327/8192 < 1/16$. An odd central quotient only removes characters. □

12. Principal blocks of classical groups in odd characteristic

We prove the principal-block bounds for classical components, including the stronger estimate for $P\Omega_8^+(q)$ needed for triality.

Proposition 12.1. *The principal 2-blocks of the simple symplectic and orthogonal groups in odd characteristic satisfy $k(b) \leq \delta(b)$. For $S = P\Omega_8^+(q)$, q an odd prime power, the stronger estimate*

$$k(B_0(S)) \leq \frac{149}{1024}|S|_2 < \frac{1}{4}|S|_2 \tag{12.1}$$

holds. The same Brauer inequality holds for every quasisimple central quotient of the corresponding simply connected finite groups.

12.1. **The series to be counted.** Put $A = \max\{(q-1)_2, (q+1)_2\} \geq 4$, and define the following formal power series:

$$\begin{aligned} P(t) &= \prod_{m\geq 1}(1-t^m)^{-1}, & H_A(t) &= P(t)^{A/2-1}\prod_{r\geq 1}P(t^{2^r})^{A/4}, \\ \Psi(t) &= \sum_{j\geq 0} t^{j(j+1)}, & \Theta(t) &= \sum_{j\in\mathbb{Z}} t^{j^2}, \\ B(t) &= P(t)^2\Psi(t), & O(t) &= P(t)^2\Theta(t), \\ F(t) &= P(t^2). \end{aligned} \tag{12.2}$$

The factor H_A records all primary factors other than the eigenvalues 1 and -1 in a semisimple 2-element. A reciprocal pair of polynomials gives a general linear centralizer factor; a self-reciprocal polynomial gives a unitary factor. In both cases the unipotent characters are indexed by partitions of the multiplicity. There are $A/2-1$ inverse–Frobenius orbits of rank weight one, and $A/4$ of rank weight 2^r for each $r \geq 1$. Indeed, if $A = 2^c$, the subgroup generated by q and -1 on the primitive 2^j-roots, $j > c$, has order 2^{j-c+1}. This proves (12.2). At weight one the factor is linear for $q \equiv 1 \pmod 4$ and unitary for $q \equiv -1 \pmod 4$. The latter changes the orthogonal sign by the parity of the multiplicity. Thus the signed primary series is $H_A(t)$ or $H_A(-t)$, respectively.

The standard symbol parametrization of unipotent characters gives $B(t)$ for either symplectic or odd special orthogonal factors: a symbol of odd defect $2j+1$ has a bipartition after a staircase of rank $j(j+1)$. Ordered even-defect symbols similarly give $O(t)$ for the sum of the two full orthogonal types. These are the parametrizations in [16, Section 13.8] and [74, Sections 4–6]. Writing O_+ and O_- for the two types, Jacobi's product identity gives

$$O_+ + O_- = O, \qquad O_+ - O_- = P(t)^2\sum_{j\in\mathbb{Z}}(-1)^j t^{j^2} = F. \tag{12.3}$$

The determinant-twist-fixed full orthogonal characters arise from the pairs of degenerate special orthogonal symbols. Their series is F, entirely of plus type. At rank zero its formal value is one.

For two full orthogonal factors with unipotent counts x_1, x_2 and determinant-twist-fixed counts f_1, f_2, restriction to the determinant-one subgroup gives

$$\frac{x_1x_2 + 3f_1f_2}{2}. \tag{12.4}$$

In fact a nonfixed pair of characters has one irreducible restriction, whereas a determinant-fixed character splits into two. When one rank is zero this is the familiar special orthogonal formula $(x+3f)/2$. When both ranks vanish the formal value two counts the two special orthogonal semisimple classes into which the full orthogonal class splits. Thus the same expression counts the entire Jordan series, including that boundary case. By (12.3), its sum over the two total signs is $(O^2+3F^2)/2$, and its signed difference is $2F^2$.

Jordan decomposition, including the disconnected centralizers in (12.4), now gives the following exact counts of the full 2-series at the identity:

$$\begin{aligned} K_C(n,A) &= |\mathcal{E}_2(\mathrm{Sp}_{2n}(q),1)| = [t^n]BOH_A, \\ K_B(n,A) &= |\mathcal{E}_2(\mathrm{SO}_{2n+1}(q),1)| = [t^n]B^2H_A. \end{aligned} \tag{12.5}$$

For the even special orthogonal group of sign ε, call the sign good when $q^n \equiv \varepsilon \pmod 4$ and bad otherwise. For $n \geq 1$ the two series counts are

$$K_g(n,A) = [t^n]\frac{H_A}{4}(O^2 + 7F^2),$$
$$K_b(n,A) = [t^n]\frac{H_A}{4}(O^2 - F^2). \tag{12.6}$$

For $q \equiv -1 \pmod 4$, the sign at degree n is multiplied by $(-1)^n$, because F^2 contains only even powers. This gives the stated convention for the good sign. The constant terms are formal boundary terms; the formulas are used for $n \geq 1$.

12.2. **Uniform coefficient bounds.** Let $s_2(n)$ be the number of nonzero binary digits of n, and $n_2 = 2^{v_2(n)}$. The Sylow order for the symplectic and odd special orthogonal groups is

$$\Delta_n = \prod_{i=1}^{n}(q^{2i}-1)_2 = \frac{(4A)^n}{2^{s_2(n)}}. \tag{12.7}$$

We prove

$$2K_C(n,A) \leq \Delta_n, \qquad 4K_B(n,A) \leq \Delta_n \qquad (n \geq 2), \tag{12.8}$$
$$16K_g(n,A) \leq \Delta_n, \qquad 4An_2K_b(n,A) \leq \Delta_n \qquad (n \geq 4). \tag{12.9}$$

For B and C, the coefficients in (12.5) are polynomials in A with nonnegative coefficients and degree at most n. To see the positivity, combine the four factors P from B, O with $P^{A/2-1}$, obtaining $P^{A/2+3}$, and use the nonnegative logarithmic coefficients of the other Euler factors. Therefore $K_C(n,A)/A^n$ and $K_B(n,A)/A^n$ decrease with A.

For D, replace H_A coefficientwise by

$$\widehat{H}_A = PH_A = P^{A/2}\prod_{r\geq 1} P(t^{2^r})^{A/4},$$

and denote the resulting upper counts by $\widehat{K_g}$ and $\widehat{K_b}$. They also have nonnegative polynomial coefficients in A. Moreover $(O^2 - F^2)/4 = O_+O_-$ has zero constant term, so the degrees of these two polynomials are at most n and $n-1$, respectively. Hence both $\widehat{K_g}(n,A)/A^n$ and $A\widehat{K_b}(n,A)/A^n$ decrease with A. It suffices to establish all four inequalities at $A = 4$.

Set $t = 1/8$ and $J(t) = \prod_{r\geq 1} P(t^{2^r})$. The elementary bounds

$$\log P(x) \leq \frac{x}{(1-x)^2}, \qquad P(1/8) < \frac{6}{5}, \qquad J(1/8) < \frac{48}{47}$$

follow by expanding the logarithm. For the last bound, use $\log J(t) \leq t^2/(1-t^2)^3 < 1/48$ and $e^x \leq (1-x)^{-1}$. Also

$$\Psi(1/8) \leq \frac{64}{63}, \qquad \Theta(1/8) \leq \frac{9}{7}, \qquad F(1/8) < \frac{6}{5}.$$

At $A = 4$ the four series in question are

$$P^5\Psi\Theta J, \quad P^5\Psi^2 J, \quad \frac{J}{4}(P^6\Theta^2 + 7P^2F^2), \quad \frac{J}{4}(P^6\Theta^2 - P^2F^2).$$

The displayed rational bounds show that all four have value less than eight at $t = 1/8$. Their coefficients are nonnegative, so each degree-n coefficient is less than $8 \cdot 8^n$. Since $2^{s_2(n)} \leq n+1$ and $n_2 \leq n$, every normalized left side of (12.8)–(12.9) is at most

$$\frac{128n(n+1)}{2^n}.$$

At $n = 15$ this is $15/16$, and the ratio of consecutive values is $(n+2)/(2n) < 1$. This proves the inequalities for $n \geq 15$.

The remaining coefficients are obtained by multiplying the displayed Euler products modulo t^{15}:

n	$K_C(n,4)$	$K_B(n,4)$	$\widehat{K}_g(n,4)$	$\widehat{K}_b(n,4)$
2	32	23	28	12
3	119	80	88	56
4	386	256	298	208
5	1134	731	866	694
6	3089	1968	2492	2092
7	7918	4963	6632	5896
8	19302	11985	17168	15632
9	45098	27701	42274	39542
10	101587	61927	101144	95864
11	221642	134048	233464	224344
12	470124	282575	525162	508480
13	972372	580875	1148968	1120888
14	1966088	1168480	2458196	2408996

For example, the coefficients of an Euler product $U(t) = \prod_{m\ge1}(1-t^m)^{-a_m}$ may be generated from $u_0 = 1$ by

$$nu_n = \sum_{j=1}^{n} \left(\sum_{d\mid j} d a_d \right) u_{n-j}.$$

Together with the sparse series Ψ, Θ, this recurrence independently specifies every entry of the table. Substitution in (12.7) proves the remaining 48 inequalities (the D columns are only used for $n \ge 4$). The largest ratios are $1/2$, $23/32$, $149/1024$ and $13/32$, respectively.

Proof of Proposition 12.1. The principal block is contained in the counted 2-series. For $P\,\mathrm{Sp}_{2n}(q)$ the Sylow order is $\Delta_n/2$, and quotient characters form a subset of the symplectic characters. For $\Omega_{2n+1}(q)$ it is again $\Delta_n/2$. The index-two inclusion $\Omega \lhd \mathrm{SO}$ gives a unique block of SO covering $B_0(\Omega)$; it contains the trivial character, so it is $B_0(\mathrm{SO})$. Every character of $B_0(\Omega)$ occurs in a restriction from this block, and each such restriction has at most two distinct constituents. Thus restriction costs at most a factor two, giving (12.8).

For even orthogonal groups,

$$|P\Omega^{\varepsilon}_{2n}(q)|_2 = \begin{cases} \Delta_n/8, & q^n \equiv \varepsilon \pmod 4, \\ \Delta_n/(2An_2), & q^n \equiv -\varepsilon \pmod 4. \end{cases}$$

The same unique-covering-block argument makes restriction to Ω cost at most two, and the subsequent central quotient only removes characters. Thus (12.9) proves the assertion. For $n = 4$, plus type is good and the same comparison, using the table and the monotonicity in A, gives the ratio $149/1024$ in (12.1). Lower ranks are covered by the standard type-A isomorphisms. Finally, a central 2-extension multiplies the defect order by its kernel size, and Lemma 3.1, summed over its central characters, increases the block count by at most that same factor. Odd central quotients in the principal sector preserve the count. This proves the assertion for these central quotients. □

The principal block is the only unipotent 2-block for a classical derived component in odd characteristic; the precise statement used here is recorded in [25, Section 5] from the Cabanes–Enguehard classification. Noncentral odd semisimple labels are handled by the proper-Levi descent, not by the principal-series estimate alone.

13. Local modules and integral ramification

Throughout this section L is quasisimple with center of odd order, b is a 2-block of L, and P is a nonabelian defect group of b. An algebraic central cover is a central quotient of the fixed points of a simply connected algebraic group; for Suzuki and Ree groups we use the defining Steinberg endomorphism. Exceptional odd covers are the additional odd parts of the finite Schur cover. Write $\delta = |P| = 2^d$ and $\mathrm{Aut}(L,b) = \{a \in \mathrm{Aut}(L) : {}^a b = b\}$. If

$$\mathrm{Inn}(L) \le A \le \mathrm{Aut}(L,b), \qquad T = \{a \in A : a \text{ is inner on } \mathcal{O}Lb\}, \qquad R = A/T,$$

then R acts on $\mathrm{Irr}(b)$. Innerness is implemented by a unit of the integral block algebra $\mathcal{O}Lb$.

We shall construct $\mathbb{F}_2R$-modules W with

$$|W| = \delta, \qquad |\mathrm{Irr}(b)^J| \le |W^J| \quad (J \le R), \qquad \ker(R \longrightarrow \mathrm{GL}(W)) \text{ a 2-group.} \tag{13.1}$$

The use of every subgroup in this condition includes subgroups of even order and mixed field–graph subgroups. By Schreier's theorem, R is solvable, since T contains $\mathrm{Inn}(L)$.

Lemma 13.1 (The intrinsic ramification bound)**.** *With the notation above, let E_b be the inertial quotient of a maximal b-Brauer pair. Then*

$$|R|_{2'} \le \frac{|\mathrm{Aut}(P)|_{2'}}{|E_b|}\,|E_b^{\mathrm{ab}}| \le |\mathrm{Aut}(P)|_{2'}. \tag{13.2}$$

Proof. We give the argument for an invariant block b of an arbitrary normal subgroup $N \lhd H$. Fix a maximal pair (P,e) and put

$$J = N_H(P,e), \quad J_0 = N_N(P,e), \quad C = C_H(P)_e, \quad C_0 = C_N(P), \quad H_1 = NC.$$

Conjugacy of maximal pairs gives $H = NJ$. If I, I_0 denote the images of J, J_0 in $\mathrm{Aut}(P)$, respectively, then

$$H/H_1 \cong I/I_0, \qquad |I_0|_{2'} = |E_b|, \qquad E_b = J_0/(PC_0).$$

In particular,

$$|H : H_1|_{2'} \le |\mathrm{Aut}(P)|_{2'}/|E_b|.$$

The canonical character of e is the inflation of the unique character of the defect-zero block of $C_0/Z(P)$ dominated by e. It determines e, so an element centralizing P stabilizes this character exactly when it stabilizes e. Thus C retains the canonical-character stabilizer required in Murai's pairing. Murai's pairing theorem, in the form of [54, Lemma 3.12 and Theorem 3.13], gives a homomorphism

$$C \longrightarrow \mathrm{Hom}(E_b, K^\times)$$

with kernel $C_!$ such that $C_0 \le C_!$ and

$$H[b] = NC_! \le H_1.$$

Here $H[b]$ is the integral ramification group. Since $N \cap C = C_0 \le C_!$, we have $H_1/H[b] \cong C/C_!$, the image of this homomorphism. Its order is odd and at most $|E_b^{\mathrm{ab}}|$. Multiplication of the two indices proves the first inequality; the second follows from $|E_b^{\mathrm{ab}}| \le |E_b|$.

Apply this argument to the semidirect product $H = L \rtimes A$ and $N = L$. An element (x,a) acts on $\mathcal{O}Lb$ as the composition of conjugation by x and a. Consequently $H[b] = L \rtimes T$ and $H/H[b] \cong A/T$. This proves the assertion for every central character of b, as well as for centreless L. □

Lemma 13.2 (Two elementary module constructions)**.** *The following conditions each imply* (13.1).

(i) *R is a 2-group and $k(b) \le \delta$.*

(ii) $|R|_{2'} \leq 3$ *and* $k(b) \leq \delta/4$.

Proof. In the first case take the trivial module $\mathbb{F}_2^d$. For the second case put $U = R/O_2(R)$. If $U \neq 1$, solubility implies $F(U) \neq 1$. Since $O_2(U) = 1$ and $|U|_{2'} \leq 3$, its Fitting subgroup is C_3. The self-centralizing property of the Fitting subgroup yields $U/C_3 \leq \mathrm{Aut}(C_3) = C_2$. Hence

$$U \in \{1, C_3, S_3\}.$$

Use the natural two-dimensional module $E = \mathbb{F}_2^2$ for $S_3 = \mathrm{GL}_2(2)$, restricted to C_3 when appropriate, and inflate it to R. If $U \neq 1$, take $W = E \oplus \mathbb{F}_2^{d-2}$. Its kernel is $O_2(R)$, and every fixed space has order at least $2^{d-2} = \delta/4$. This dominates $|\mathrm{Irr}(b)^J| \leq k(b)$. For $U = 1$ use the first construction. □

Lemma 13.3 (Defect groups with small odd automorphism part)**.** *The following assertions hold.*

(i) *If $Z(P)$ is cyclic and $P/Z(P)$ is nonabelian dihedral, then* $\mathrm{Aut}(P)$ *is a 2-group. In particular this applies to $C_{2^a} \wr C_2$ and $\mathrm{SD}_{2^{a+2}}$, $a \geq 2$.*

(ii) *If P is a product of $r \leq 3$ wreathed or semidihedral groups of these orders, then* $|\mathrm{Aut}(P)|_{2'} \leq (r!)_{2'}$.

(iii) *Suppose P contains a normal subgroup $D \cong C_{2^a} \times C_{2^a}$, $a \geq 2$, and $P/D \cong V_4 = C_2 \times C_2$ acts faithfully on D by the matrices $-I$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then* $\mathrm{Aut}(P)$ *is a 2-group.*

Proof. For (i), the map

$$\mathrm{Aut}(P) \longrightarrow \mathrm{Aut}(Z(P)) \times \mathrm{Aut}(P/Z(P))$$

has 2-group image. Indeed, an automorphism of the dihedral group of order 2^n, $n \geq 3$, maps its characteristic rotation generator to an odd power and a reflection to a rotation times a reflection. Its automorphism group has order 2^{2n-3}. An element α of the displayed kernel has the form $\alpha(x) = xf_\alpha(xZ(P))$, where $f_\alpha \in \mathrm{Hom}(P/Z(P), Z(P))$. This embeds the kernel in a 2-group. For the wreathed group $\langle x, y, t \mid x^{2^a} = y^{2^a} = t^2 = 1, [x, y] = 1, x^t = y\rangle$, the center is $\langle xy\rangle$ and the quotient is dihedral of order 2^{a+1}. The semidihedral presentation

$$\langle x, t \mid x^{2^{a+1}} = t^2 = 1, x^t = x^{-1+2^a}\rangle$$

has center $\langle x^{2^a}\rangle$ and the same dihedral quotient.

Write $\Phi(P)$ for the Frattini subgroup, the intersection of the maximal subgroups of P. For (ii), each factor is directly indecomposable because its center is cyclic, and its center lies in its Frattini subgroup. For a wreathed factor this follows from $xy \in \langle x^2, y^2, xy^{-1}\rangle = \Phi(P)$; the semidihedral assertion follows immediately from its presentation. The central-automorphism form of the Remak theorem [51, Theorem 12.1 and the proof of Theorem 12.8] says that automorphisms of the product are generated by central automorphisms, factor automorphisms and permutations of isomorphic direct factors. Central automorphisms form a normal subgroup and act trivially on $P/\Phi(P)$, so they belong to the 2-group kernel in Burnside's basis theorem. Modulo this normal 2-subgroup, the automorphism group is an image of the product of the factor automorphism groups extended by permutations of isomorphic factors. The factor automorphism groups are 2-groups by (i). Thus only factor permutations contribute to the odd part, giving the claimed bound.

For (iii), let B be an abelian subgroup not contained in D and write $V = BD/D \neq 1$. Then

$$B \cap D \leq C_D(V), \qquad |B| = |B \cap D|\,|V|.$$

For $V = \langle -I\rangle$ this gives $|B| \leq 8$; for either reflection subgroup it gives $|B| \leq 2^{a+1}$; for $V = V_4$ it again gives $|B| \leq 8$. Each bound is strictly smaller than $|D| = 2^{2a}$. Thus D is the unique

abelian subgroup of largest order and is characteristic. The restriction image in $\mathrm{GL}_2(\mathbb{Z}/2^a\mathbb{Z})$ normalizes the given V_4. Modulo 2 it normalizes a nontrivial transvection subgroup of $\mathrm{GL}_2(2)$, whose normalizer has order 2. The congruence kernel is a 2-group, so the restriction image is a 2-group. An automorphism fixing D pointwise acts trivially on P/D, because its action on D is faithful. Its errors on a fixed transversal form a 1-cocycle in $Z^1(V_4, D)$. The restriction kernel therefore embeds in a 2-group. This proves (iii). □

We shall also use the following published defect-group statements with their exact scope. A block with defect a central extension of a metacyclic 2-group by a cyclic 2-group satisfies $k(b) \le |P|$ [66, Theorem 2]. A metacyclic 2-group has 2-group automorphism group unless it is homocyclic abelian or Q_8 [67, Lemma 1]. Thus dihedral, wreathed and semidihedral defects in the preceding lemma give the first construction of Lemma 13.2 for every block stabilizer.

14. Components in defining characteristic two

Let S be a simple group of Lie type in characteristic 2, written over $\mathbb{F}_q$ with $q = 2^f$, and let L be a perfect algebraic central cover with odd center. The three simple derived groups arising from the nonsimple fixed-point groups $\mathrm{Sp}_4(2)$, $G_2(2)$ and ${}^2F_4(2)$ are treated separately in the last table below; until that point they are excluded. The defining-characteristic block theorem of Humphreys, recalled in [47, Proposition 8.1], has the following precise form. The Steinberg character gives a defect-zero block; every other block has full Sylow defect, and there is one such block for each character of $Z(L)$. For $L = S$ this says

$$\mathrm{Irr}(B_0(S)) = \mathrm{Irr}(S) \setminus \{\mathrm{St}\}. \tag{14.1}$$

We first check the integral quotient on which our modules must act.

Lemma 14.1. *For every positive-defect block b of L under the preceding hypotheses, the subgroup of* $\mathrm{Aut}(L, b)$ *acting innerly on $\mathcal{O}Lb$ is* $\mathrm{Inn}(L)$.

Proof. If $L = S$ is simple, an integral-inner automorphism fixes every character of b. The Steinberg character is fixed by every automorphism, so (14.1) shows that it fixes all ordinary irreducible characters. The character table then shows that it fixes every conjugacy class. The theorem that every class-preserving automorphism of a finite simple group is inner [22, Theorem C] proves the assertion.

A nontrivial algebraic odd center occurs only in types $A_r, {}^2A_r, E_6, {}^2E_6$. The following argument applies to every central-character sector. Apply Murai's theorem to $L \rtimes A$, where $A = \mathrm{Aut}(L, b)$. Since a defect group P is Sylow, that theorem gives

$$(L \rtimes A)[b] \le LC_{L\rtimes A}(P).$$

Thus every element of $T/\mathrm{Inn}(L)$ has a representative which centralizes P pointwise. We prove that such a representative is inner.

Work first with a simply connected algebraic group $\mathbf{S}$ and Frobenius F, with a pinned Borel subgroup $\mathbf{B} = \mathbf{TU}$. Let $\mathbf{U}_2$ be the connected subgroup generated by positive root groups of height at least two. The commutator relations imply $[\mathbf{U}, \mathbf{U}] \le \mathbf{U}_2$, and

$$\mathbf{U}/\mathbf{U}_2 = \prod_{\alpha \text{ simple}} \mathbf{U}_\alpha.$$

Lang's theorem for the connected group $\mathbf{U}_2$ gives a surjection $\mathbf{U}^F \to (\mathbf{U}/\mathbf{U}_2)^F$. We use this height quotient.

An automorphism normalizing $\mathbf{U}^F$ has the standard pinned form $\mathrm{Int}(ut)\sigma$, with $u \in \mathbf{U}^F$, t in the rational adjoint torus, and σ an ordinary field–diagram operator. This follows from the standard generators [36, §2.1] and $N_{\mathbf{S}^F_{\mathrm{ad}}}(\mathbf{U}^F) = \mathbf{B}^F_{\mathrm{ad}}$. In the split case the simple-root coordinates

of the height quotient are copies of $\mathbb{F}_q$. Pointwise triviality first forces the diagram permutation to be trivial and then gives

$$a_\alpha x^{2^e} = x \qquad (x \in \mathbb{F}_q)$$

in each coordinate. Taking $x = 1$ yields $a_\alpha = 1$; the remaining equality yields $e \equiv 0 \pmod f$. The simple roots form a basis of the character lattice of the adjoint torus, so $t = 1$. The factor u is inner.

In a twisted group write $F = \gamma F_2^f$. On the finite group the basic field map $v = F_2$ satisfies

$$v^f = \gamma^{-1}, \qquad v^{2f} = 1.$$

The height quotient contains a coordinate $\mathbb{F}_{q^2}$ from a pair of diagram-interchanged roots. Surjectivity from Lang's theorem includes the adjacent-root pair in type A_2. Trivial action on this coordinate gives $ax^{2^e} = x$ for all $x \in \mathbb{F}_{q^2}$, hence $a = 1$ and $e \equiv 0 \pmod{2f}$. Triviality on all the remaining coordinates again forces the adjoint torus factor to be one.

Finally, let $\pi : \widetilde{L} \to L$ be the simply connected presentation, with odd central kernel, and let $a \in \mathrm{Aut}(L)$. The pinned inner-diagonal, field and diagram generators lift its induced automorphism on S to an automorphism $\widetilde{a}$ of $\widetilde{L}$. The homomorphisms $\pi\widetilde{a}$ and $a\pi$ agree modulo $Z(L)$, so their pointwise quotient is a homomorphism from the perfect group $\widetilde{L}$ to $Z(L)$. It is trivial. Hence $\pi\widetilde{a} = a\pi$, which proves both kernel preservation and existence of the lift; the same argument proves uniqueness. If a centralizes P pointwise and $\widetilde{P}$ lies above P, then $\pi^{-1}(P) = \widetilde{P} \times \ker\pi$. Its unique Sylow 2-subgroup $\widetilde{P}$ is $\widetilde{a}$-stable, and $\widetilde{P} \to P$ is an isomorphism. Thus $\widetilde{a}$ centralizes $\widetilde{P}$ pointwise. The preceding root-coordinate argument proves the assertion for every algebraic odd central quotient. □

For a block in the central character ζ, the complex ζ-sector of L is a twisted group algebra of $S = L/Z(L)$. Only cocycle-regular conjugacy classes contribute to its center. Consequently

$$k(b) \le k_\alpha(S) \le k(S). \tag{14.2}$$

The bound is independent of $|Z(L)|$. We use the simple-group class estimate

$$k(S) \le \frac{76}{5} q^r, \tag{14.3}$$

where r is the absolute Lie rank [26, Corollary 1.2(1)]. The small ranks where this estimate is too coarse will be calculated separately.

14.1. **Outer modules.** Write $f = 2^a m$, with m odd. For odd h, put

$$U_h = \{(x_0, \dots, x_{h-1}) \in \mathbb{F}_2^h : \textstyle\sum_i x_i = 0\}.$$

Cyclic rotation gives a faithful C_h-module of dimension $h - 1$: the permutation module is the direct sum of U_h and its trivial line, so an element trivial on U_h is trivial on the whole faithful permutation module. We allow $U_1 = 0$. Let E be the natural module of $S_3 = \mathrm{GL}_2(2)$, with generators

$$c = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \qquad t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad c^t = c^{-1}.$$

Here are two modules that retain diagonal automorphisms. For an odd divisor d of $q - 1$, the split group $C_d \rtimes (C_f \times C_2)$ has a faithful representation on $\mathbb{F}_q \oplus \mathbb{F}_q$ given by

$$z(u, v) = (\zeta u, \zeta^{-1} v), \quad \phi(u, v) = (u^2, v^2), \quad \tau(u, v) = (v, u), \tag{14.4}$$

where ζ has order d. The field map raises z to its square and τ inverts it. A transformation acting trivially cannot interchange the two summands; evaluation at 1 and then at all elements

of $\mathbb{F}_q$ forces both its scalar and field parts to be trivial. For $d \mid q+1$ the twisted group $C_d \rtimes C_{2f}$ acts faithfully on $\mathbb{F}_{q^2}$ by

$$z(u) = \zeta u, \qquad \phi(u) = u^2. \tag{14.5}$$

The relation $\phi^f z \phi^{-f} = z^{-1}$ is the actual field–graph relation. These representations both have $\mathbb{F}_2$-dimension $2f$.

If an active module has dimension $e \le D$ and kernel a 2-group, adjoining $D-e$ trivial dimensions gives a module W of order 2^D for which

$$|W^J| \ge 2^{D-e} \qquad \text{for every subgroup } J. \tag{14.6}$$

By Lemma 14.1, these are modules on the integral quotient; restricting from the full outer group to any block stabilizer preserves the kernel condition. The standard outer-group descriptions used below are realized by pinned diagonal, field and diagram operators [36, Section 2.1 and Lemma 2.1].

14.2. **Ranks one, two and three.** For $S = \mathrm{PSL}_2(q)$, $q \ge 4$, take $W = \mathbb{F}_q$ with its Frobenius action of C_f. For $J \le C_f$, write $\mathbb{F}_{q_0} = \mathbb{F}_q^J$. Every nonzero trace labels one conjugacy class in $\mathrm{SL}_2(q)$; trace zero gives the identity and the single nonidentity unipotent class. Thus there are exactly q_0+1 classes fixed by a generator of J. The permutation representations on conjugacy classes and on irreducible characters have equal traces, by the invertibility of the character table. Since J is cyclic, exactly q_0+1 irreducible characters are J-fixed. Removing the Steinberg character gives

$$|\operatorname{Irr}(B_0(S))^J| = q_0 = |W^J|.$$

The action on W is faithful, and $|W| = |S|_2 = q$.

For rank two of type A, write $d = (3, q-\varepsilon)$, with $\varepsilon = 1$ in the linear case and $\varepsilon = -1$ in the unitary case. The exact class numbers are

	$d=1$	$d=3$
$\mathrm{PSL}_3(q)$	q^2+q	$(q^2+q+10)/3$
$\mathrm{PSU}_3(q)$	q^2+q+2	$(q^2+q+12)/3.$

(14.7)

These formulas are given in [50, Proposition 2.6] away from its excluded linear cases $q = 2, 4$. In those cases the ordinary character tables give $k(\mathrm{PSL}_3(2)) = 6$ and $k(\mathrm{PSL}_3(4)) = 10$ [11], agreeing with (14.7). They also follow from the degree-three Macdonald formula: the general linear and unitary coefficients are respectively $(q-1)(q^2+q)$ and $(q+1)(q^2+q+2)$, and the two divisor terms for 3 add $2+8=10$ before division by the central factor. In particular the unitary numerator is q^2+q+12.

If $d = 1$, project the field part to C_m and use U_m, ignoring the graph involution. The kernel is a 2-group, whether the outer group is $C_f \times C_2$ or C_{2f}. Here $D = 3f$, and

$$2^{D-(m-1)} \ge 2q^2 \ge k(S).$$

For the linear $d = 3$ case, f is even. Map the diagonal generator to c, and both the field generator and graph involution to t. Also map the field generator to a generator of C_m. The resulting module $E \oplus U_m$ has dimension $m+1$. Its kernel is generated by $\phi^m\tau$ and has order 2^a. Since $m \le f-1$,

$$2^{3f-m-1} \ge q^2 \ge (q^2+q+10)/3 \qquad (q \ge 4).$$

For the unitary $d = 3$ case, f is odd. The outer group is $C_3 \rtimes C_{2f}$ with the field generator inverting C_3. Send this generator to t and to a generator of C_f, and the diagonal generator to c. On $E \oplus U_f$ this action is faithful. Since $q = 2$ is nonsimple, $q \ge 8$, and

$$2^{3f-(f+1)} = q^2/2 \ge (q^2+q+12)/3,$$

because $q^2 - 2q - 24 \geq 0$ for $q \geq 8$. Equation (14.2) shows that these same estimates include every odd central sector of the algebraic central covers.

For A_3 and ${}^2A_3$ the diagonal center has no odd part when q is even. Use U_m and pad to $D = 6f$. For $q \geq 4$ its fixed-space lower bound is at least $2q^5$, exceeding $(76/5)q^3$. At $q = 2$ the exact class numbers are 14 and 20, respectively, both less than 2^6. More generally the class formulas here are $q^3 + q^2 + q$ and $q^3 + q^2 + 3q + 2$; only their values at two are needed.

14.3. **The remaining families.** For $A_r, {}^2A_r$ with $r \geq 4$, use (14.4) or (14.5), with the diagonal order $d = (r + 1, q - \varepsilon)$, and pad to $D = fN$, $N = r(r + 1)/2$. Since $N - r - 2 \geq 4$,

$$2^{D-2f} = q^{N-2} \geq 16q^r > (76/5)q^r.$$

The same two constructions work for $E_6, {}^2E_6$, where $r = 6$, $N = 36$ and $d = (3, q - \varepsilon)$; the lower bound is q^{34}.

In characteristic two, $B_r(q) \cong C_r(q)$. In the following table the outer group has a cyclic odd quotient C_m and a 2-group kernel. Use U_m and pad to fN dimensions.

Type	Range	N	Field–diagram group
C_r	$r \geq 3$	r^2	C_f
D_r	$r \geq 5$	$r(r-1)$	$C_f \times C_2$
${}^2D_r$	$r \geq 4$	$r(r-1)$	C_{2f}
G_2	$q \geq 4$	6	C_f
F_4		24	C_{2f}
E_7		63	C_f
E_8		120	C_f

For every row, $N - r - 1 \geq 3$. Its fixed-space floor therefore satisfies

$$2^{fN-m+1} \geq 2q^{N-1} \geq 16q^r > (76/5)q^r.$$

The cyclic C_{2f} entry records the exceptional graph–field relation in F_4.

For C_2 with $q \geq 4$, the outer group is likewise the actual cyclic group C_{2f}. For $q \geq 8$ the construction with U_m has floor $2q^3 \geq (76/5)q^2$. At $q = 4$, $m = 1$, the outer group is a 2-group and the trivial module of order $q^4 = 256$ suffices, since $(76/5)q^2 < 256$.

For D_4, the outer group in even characteristic is $S_3 \times C_f$. Use $E \oplus U_m$, of dimension $m + 1$, and pad to $12f$. Its kernel is the 2-part of the field group, and its fixed-space floor is at least $q^{11}/2 > (76/5)q^4$. For ${}^3D_4$, the outer group is cyclic of order $3f$. Put $h = (3f)_{2'}$ and use U_h. Here $h \leq 3f$, so the floor in $12f$ dimensions is at least $2q^9 > (76/5)q^4$. This keeps the relation between triality and field automorphisms.

For $S = {}^2B_2(q)$, $q = 2^f \geq 8$ with f odd, take U_f and pad to $2f$ dimensions. The fixed-space floor is $2q$, whereas $k(B_0(S)) = q + 2 \leq 2q$. For $S = {}^2F_4(q)$, $q = 2^f \geq 8$ with f odd, use U_f and pad to $12f$ dimensions. The full class formulas give

$$k(B_0(S)) \leq q^2 + 4q + 16 \leq 2q^{11}.$$

These class counts are included in the small-rank exceptional class data used in [48, Section 3] and [26]. The remaining small derived groups have the following principal-block counts:

S	$\lvert S\rvert_2$	$k(B_0(S))$	$\mathrm{Out}(S)$
$\mathrm{Sp}_4(2)' \cong A_6$	8	5	V_4
$G_2(2)' \cong \mathrm{PSU}_3(3)$	32	12	C_2
${}^2F_4(2)'$	2048	20	C_2

In each group the principal block is the only positive-defect 2-block; the other two blocks have defect zero. The complete lists of defect exponents and ordinary character counts are

$$(3,0,0),\ (5,1,1); \qquad (5,0,0),\ (12,1,1); \qquad (11,0,0),\ (20,1,1),$$

respectively. They are obtained from the ordinary character tables [11] by the following commands; the last output identifies the principal block by its trivial character.

```
LoadPackage("ctbllib");;
for nam in ["A6", "U3(3)", "2F4(2)'"] do
  t := CharacterTable(nam);;
  r := PrimeBlocks(t,2);;
  counts := List([1..Length(r.defect)],
                 i -> Number(r.block,j -> j=i));;
  principal := r.block[Position(Irr(t),TrivialCharacter(t))];;
  Print(nam," ",r.defect," ",counts," ",principal,"\n");
od;
```

Thus every positive-defect block has full defect and satisfies $k(b) \le \delta(b)$. Since each outer group is a 2-group, the trivial modules of dimensions $3, 5, 11$ satisfy (13.1). Equations (14.2) and (14.6), together with all the preceding cases, prove the following statement.

Proposition 14.2. *Every positive-defect block of a quasisimple algebraic central cover in characteristic two with odd center satisfies* (13.1) *for its integral quotient. The assertion includes every nontrivial central character sector and every block stabilizer. It also holds for the three small simple derived groups in the preceding table.*

15. Principal blocks in odd characteristic

We first establish integral innerness for the odd diagonal and field automorphisms of a principal block. Let $N \lhd H$, assume H/N has odd order, and let $P \in \mathrm{Syl}_2(N)$. If

$$H = NC_H(P), \tag{15.1}$$

the Alperin–Dade theorem gives a restriction bijection

$$\mathrm{Irr}(B_0(H)) \longrightarrow \mathrm{Irr}(B_0(N))$$

[1, 20]. The implication (iii)⇒(i) of [54, Theorem 4.1] then gives

$$H = H[B_0(N)]. \tag{15.2}$$

We use the integral conclusion (15.2).

Here is the diagonal verification of (15.1). Suppose $H = NT$, where T is abelian, P normalizes T and H/N has odd order. The odd Hall subgroup $T_{2'}$ maps onto H/N. Its quotient action by P is trivial, because $P \le N$. Coprime averaging on the abelian group $T_{2'}$, or equivalently $H^1(P, T_{2'} \cap N) = 1$, shows that

$$C_{T_{2'}}(P) \longrightarrow H/N$$

is surjective. For a finite reductive group, a rational maximal torus in the adjoint group maps onto the diagonal quotient: this follows from the central-isogeny cohomology sequence and Lang's theorem for its covering torus. Choose a Sylow torus normalized by P. The preceding argument supplies (15.1) for the odd diagonal extension.

For an odd field extension $q = q_0^h$, the identities

$$(q_0^{hi} - 1)_2 = (q_0^i - 1)_2, \qquad (q_0^{hi} + 1)_2 = (q_0^i + 1)_2 \qquad (h \text{ odd}) \tag{15.3}$$

show that the corresponding same-type smaller-field group contains a full Sylow 2-subgroup. The central divisors in the order formula have unchanged 2-parts as well. Thus an odd field

subgroup centralizes such a P. For an order-two twist, the same argument uses $F = F_0^h$, with the twist unchanged because h is odd. The torus used for the diagonal step can be chosen over this smaller field. Consequently the whole odd diagonal–field subgroup satisfies (15.1), including when it is nonabelian.

The cyclic field–triality group of ${}^3D_4(q)$ requires a separate order check. Write $q = r^f$, $f = 2^a u$ with u odd. Its full odd outer subgroup has order $3u$ and a fixed subgroup containing $G_2(r^{2^a})$. Since

$$|{}^3D_4(q)|_2 = (q^2-1)_2^2 = |G_2(r^{2^a})|_2,$$

it also centralizes a full Sylow subgroup. Here the full field–triality group is cyclic of order $3f$. For small Ree groups the analogous fixed subgroup ${}^2G_2(3)$ has the full Sylow order 8; all their defects are abelian in any case.

For $G_2(3^f)$ the exceptional graph–field generator τ satisfies $\tau^2 = F_3$, and the full outer group is C_{2f}. If $f = 2^a u$ with u odd, its odd subgroup is generated by $\tau^{2^{a+1}} = F_3^{2^a}$. It is therefore an ordinary odd field subgroup, already contained in the integral-inner kernel.

For every ordinary simple group in odd characteristic other than split D_4, the outer group has a normal odd diagonal–field subgroup with 2-group quotient. The preceding argument puts that subgroup in the integral-inner kernel. The principal block bounds of Propositions 11.2 and 12.1, and the exceptional principal block theorem [48, Section 6, introduction and Proposition 6.1], therefore give (13.1) by the trivial-module construction. The exceptional theorem cited here is specifically the principal-block theorem; the nonprincipal E_8 calculation will be given separately below.

15.1. **The split triality module.** Let $S = P\Omega_8^+(q)$ with q odd. The diagonal–graph part of its outer group is $V_4 \rtimes S_3 \cong S_4$, and the field part commutes with it. Let $H \le S_3$ be the graph image of the block stabilizer modulo inner automorphisms. Odd field automorphisms are integral inner by the preceding argument. All unipotent characters belong to the principal 2-block; diagonal and field automorphisms fix them individually, whereas triality has two orbits of length three [39, p. 1138, before Theorem 5.4].

If H has no element of order three, then A/T is a 2-group. Otherwise $H = C_3$ or S_3. The graph image of T is normal in H and contains no element of order three, because such an element would move a unipotent character. Neither C_3 nor S_3 has a nontrivial normal subgroup of 2-power order, so this image is trivial. The natural $E = \mathbb{F}_2^2$ representation of $H \le \mathrm{GL}_2(2)$ therefore descends to $R = A/T$. Its kernel is a 2-group: the remaining diagonal group is V_4 and the remaining field group has 2-power order. Pad E to dimension $d = \log_2 |S|_2$. The fixed-space floor is $\delta/4$, while (12.1) gives

$$k(B_0(S)) \le (149/1024)\delta < \delta/4.$$

This proves every subgroup inequality. For an algebraic central cover with odd center, the principal sector is canonically the principal block algebra of the simple quotient; this identification is equivariant for the stabilizer and preserves the defect order.

Proposition 15.1. *Every principal 2-block of a quasisimple algebraic central cover of a simple group of Lie type in odd characteristic, with odd center and nonabelian defect, satisfies* (13.1).

16. Isolated odd labels

Let $\mathbf{G}$ be a simple simply connected algebraic group with Frobenius F in odd characteristic, let $(\mathbf{G}^*, F^*)$ be dual to $(\mathbf{G}, F)$, and let $s \in (\mathbf{G}^*)^{F^*}$ be semisimple of odd order. The element s is isolated if $C_{\mathbf{G}^*}(s)^\circ$ is not contained in any proper Levi subgroup of $\mathbf{G}^*$. Put $A(s) = C_{\mathbf{G}^*}(s)/C_{\mathbf{G}^*}(s)^\circ$. The isolated-element classification [5, Proposition 4.9 and Tables II–III],

together with the 2-block tables in [41] and their correction, gives the following noncentral odd isolated types:

$$\begin{array}{c|l} G_2 & A_2 \\ F_4 & A_2A_2 \\ E_6, {}^2E_6 & A_2^3 \quad (A(s) \cong C_3) \\ E_7 & A_5A_2 \\ E_8 & A_8, A_4A_4, E_6A_2. \end{array} \tag{16.1}$$

The rational forms, including factor permutations, are retained. The E_6 rational three-cycle entries consist of three blocks, as recorded in the correction [42, Supplement, Section 2.1], also recorded in [43, Remark 6.8]. In the other rows the centralizer is connected. For E_7 this follows also because its component group has exponent dividing both the odd label order and the 2-power algebraic fundamental group. For G_2, F_4, E_8 the dual group is simply connected. The odd quasi-isolated D_4 centralizer in E_6 is not isolated and belongs to the proper-Levi alternative.

16.1. **A bound for the full two-series of type E_6.** Let $(\mathbf{H}, F)$ and $(\mathbf{H}^*, F^*)$ be dual connected reductive groups over $\mathbb{F}_q$, where q is odd, and put $H = \mathbf{H}^F$ and $H^* = (\mathbf{H}^*)^{F^*}$. Assume that the centralizer in $\mathbf{H}^*$ of every semisimple 2-element is connected. Let $n_2(H^*)$ be the number of semisimple 2-classes in H^*. Choose an F^*-stable maximal torus $\mathbf{T}^*$ of $\mathbf{H}^*$, let $W = N_{\mathbf{H}^*}(\mathbf{T}^*)/\mathbf{T}^*$, and write $F^* = q\varphi$ on its character lattice, with φ of finite order. Then

$$n_2(H^*) = \frac{1}{|W|} \sum_{w \in W} |\det(q\varphi w - I)|_2. \tag{16.2}$$

Indeed, geometric semisimple 2-classes are W-orbits on the 2-power torsion of $\mathbf{T}^*$. An F^*-stable orbit contributes exactly $|W|$ pairs (w, x) with $wF^*(x) = x$, and a nonstable orbit contributes none. The fixed sets are finite; Smith normal form gives their orders as the determinant 2-parts in (16.2). Connectedness of the centralizers and Lang's theorem give one rational class in each stable geometric class. Suppose that $C_{\mathbf{H}^*}(t)^{F^*}$ has at most u unipotent characters for every semisimple 2-element $t \in H^*$. Jordan decomposition then gives

$$|\mathcal{E}_2(H, 1)| \le u\, n_2(H^*) \le u \max_{w \in W} |\det(q\varphi w - I)|_2. \tag{16.3}$$

For type E_6 one can take $u = 30$. Here are details of the finite root-system calculation. Label the affine diagram by the edges

$$(0,2), (2,4), (1,3), (3,4), (4,5), (5,6),$$

with highest-root marks $(1, 1, 2, 2, 3, 2, 1)$ at nodes $0, 1, \ldots, 6$. The alcove description of a semisimple centralizer [5, Proposition 3.13(a)] says that its roots are generated by the zero-coordinate nodes. For a 2-element, the sole omitted node cannot be node 4, since its mark is three. Thus one examines the 126 subsets other than the full affine diagram and the subset obtained by omitting only node 4. Deleting vertices in the displayed tree gives the following maxima:

Semisimple rank	Maximum unipotent count	Types attaining it
0	1	1
1	2	A_1
2	4	A_1^2
3	8	A_1^3
4	16	A_1^4
5	20	$D_5, A_3A_1^2$
6	30	E_6

Let $p(n)$ be the number of partitions of n. To reproduce this table, multiply the factor counts $u(A_i) = p(i+1)$, $u(D_4) = 14$, $u(D_5) = 20$ and $u(E_6) = 30$ for each connected component of an induced subgraph. The only allowed rank-six types are E_6 and A_5A_1, the latter having $11 \cdot 2 = 22$ characters. At rank five D_4A_1 does not occur in this tree; the other possible products are bounded by 20. All lower ranks are bounded by 2^r. These statements can also be checked by choosing the omitted vertices successively along each of the three arms of the displayed tree. The values are the usual partition and symbol counts; in particular D_4 triality has eight, the nonsplit D_i counts do not exceed the split counts, and both E_6 and ${}^2E_6$ have thirty unipotent characters [16, 21]. Frobenius permutation of equal factors replaces a product by one factor count and cannot increase it. This proves $u = 30$ for every rational form.

We also need a bound on the multiplicity of -1 in the reflection representation of $W(E_6)$. It is at most four. For if that multiplicity were at least five, orthogonality would make the remaining subspace, of dimension at most one, have eigenvalue 1 or -1. The element would be an involution. Carter's involution lemma [15, Lemmas 2 and 5] expresses it as the product of as many mutually orthogonal root reflections as the dimension of its -1-space. The roots perpendicular to any root of E_6 form A_5: for the highest root this is seen by deleting its adjacent node 2 in the displayed affine diagram, and all roots are Weyl conjugate. In the realization $A_5 = \{e_i - e_j : 1 \le i \ne j \le 6\}$, orthogonal roots have disjoint pairs of indices, so there are at most three. Thus E_6 has at most four mutually orthogonal roots, a contradiction.

Put $Q = \varepsilon q$, where q is an odd prime power and $\varepsilon = 1$ in the split case and $\varepsilon = -1$ in the twisted case. Write $E_6(Q)_{\mathrm{sc}}$ for $E_6(q)_{\mathrm{sc}}$ or ${}^2E_6(q)_{\mathrm{sc}}$, respectively, and let $A = \max\{(Q-1)_2, (Q+1)_2\} \ge 4$. The diagram action in the twisted case is $-w_0$, where w_0 is the longest Weyl-group element. Let Φ_j denote the jth cyclotomic polynomial. Replacing q by Q gives the same torus determinants. For odd Q, only $\Phi_1(Q), \Phi_2(Q)$ and $\Phi_{2^j}(Q)$ contribute to the 2-part; the latter have 2-part two for $j \ge 2$. If $Q \equiv 1 \pmod 4$, every torus has 2-part at most A^6. If $Q \equiv -1 \pmod 4$, the multiplicity bound just proved gives at most $A^4 \cdot 2^2 = 4A^4$. The group orders are

$$|E_6(Q)_{\mathrm{sc}}|_2 = \begin{cases} 128A^6, & Q \equiv 1 \pmod 4, \\ 512A^4, & Q \equiv -1 \pmod 4. \end{cases} \tag{16.4}$$

For example these follow at once from degrees $2, 5, 6, 8, 9, 12$ in the order formula and the lifting-the-exponent identity. The central divisors have odd order. Equations (16.3) and (16.4) prove

Lemma 16.1. *Let* $\mathbf{H}$ *be a connected semisimple algebraic group of type* E_6*, and let* F *define a split or twisted* $\mathbb{F}_q$*-structure, where* q *is an odd prime power. Then*

$$|\mathcal{E}_2(\mathbf{H}^F, 1)| \le \frac{15}{64}|\mathbf{H}^F|_2.$$

The central isogenies between these forms have odd degree. Their dual semisimple 2-centralizers are connected, since a component group would have exponent dividing both a power of two and the odd fundamental group. The rational 2-classes and the unipotent-character counts are therefore unchanged.

16.2. **The three rational configurations in type** E_6**.** Let $G = E_6(\varepsilon q)_{\mathrm{sc}}$, q odd, and let b have a noncentral isolated odd label. Its connected dual centralizer has geometric type A_2^3 and component group C_3. The rational action on the three factors is the identity, a three-cycle, or a transposition. Each rational A_2 factor has a wreathed or semidihedral Sylow subgroup of order at least sixteen.

We spell out how restriction through a regular embedding is counted. Choose a one-dimensional regular embedding $G \le X$ with connected center. The group X/G is cyclic of order $q - \varepsilon$; the map

$Z(X) \to X/G$ has odd kernel and odd cokernel, each of order dividing three. Let $Z_2 = O_2(Z(X))$ and let Y be the preimage of the odd Hall subgroup of X/G. Then

$$X = Z_2 \times Y, \quad |Y : G| \text{ odd}, \quad |Y : Z(Y)G| \leq 3. \tag{16.5}$$

Indeed Z_2 maps isomorphically onto the Sylow 2-subgroup of X/G. If $\widetilde{b}$ is the connected-center block over b, then

$$\widetilde{b} = \mathcal{O}Z_2 \otimes c, \qquad k(\widetilde{b}) = |Z_2|k(c), \qquad \delta(\widetilde{b}) = |Z_2|\delta(b).$$

Connected-center Jordan decomposition identifies its full ordinary count with the principal 2-series count of the connected centralizer. Multiplication of its central torus and its simply connected A_2 factors has odd degree, so the same connected-centralizer argument as above identifies the 2-labels and their counts. Removing Z_2 leaves precisely the product of the rational A_2 principal counts.

By Lemma 11.3, each rational factor has ratio at most $1/2$. For a factor over $\mathbb{F}_{q_0}$, put $Q = q_0$ in the linear case and $Q = -q_0$ in the unitary case. Writing $a = v_2(Q-1)$ and $b_1 = v_2(Q+1)$, its general linear or unitary count is

$$K_3(Q) = \begin{cases} A(A^2 + 12A + 8)/6, & a \geq 2, A = 2^a, \\ 2B + 8, & a = 1, B = 2^{b_1}, \end{cases}$$

and its Sylow orders are $2A^3$ and $8B$, respectively. Since no nontrivial scalar 2-element fixes a degree-three label, the special linear ratio is the same. The respective ratios are at most $3/8$ and $1/4 + 1/B \leq 1/2$.

In the three-cycle case, the corrected block table gives three G-blocks, permuted transitively by diagonal automorphisms. Fix one, and fix the central character μ of c on $Z(Y)$. A character of b extends uniquely to $Z(Y)G$ in the sector μ; its inertia group in Y is exactly $Z(Y)G$. Induction gives a bijection with $\mathrm{Irr}(c)$. Hence $k(b)/\delta(b) \leq 1/2$. In the transposition case the diagonal quotient in (16.5) is trivial; the same central-product argument gives $k(b) = k(c)$ and ratio at most $1/4$.

In the separate-factor case there is one invariant block and $Y/Z(Y)G = C_3$. Write

$$K = k(B_0(\mathrm{SL}_3(\varepsilon q))), \qquad D = |\mathrm{SL}_3(\varepsilon q)|_2.$$

After fixing μ and removing Z_2 we have $k(c) = K^3$ and $\delta(b) = D^3$. The quotient linear characters of $Y/Z(Y)G$ fix c: under central duality their order-three labels preserve the lifted odd class and cyclically permute the three A_2 factors. The cube of a rational conjugating element is in the connected centralizer, so the returning action on a factor's ordinary Jordan labels is inner. More explicitly, identify the three label sets with a set Ω of size K. The action has the form

$$(x_1, x_2, x_3) \longmapsto (f_3(x_3), f_1(x_1), f_2(x_2)), \qquad f_3 f_2 f_1 = 1.$$

Its fixed set is $\{(x, f_1(x), f_2 f_1(x)) : x \in \Omega\}$. The one-dimensional central torus is fixed: the induced automorphism has order dividing three, whereas the automorphism group of a one-dimensional torus has order two. Thus there are exactly K fixed characters of c. Before removing Z_2 the number is $|Z_2|K$, which explains why that removal was necessary.

If a block c in a cyclic extension $Y/N = C_3$ is fixed by quotient linear characters, it is the unique block over its invariant block b of N. Indeed the sum of the idempotents in any linear-twist orbit belongs to $\mathcal{O}N$; primitivity of b makes that orbit the whole covering set. If a is the number of invariant characters of b and t the number of its orbits of length three, cyclic Clifford theory gives

$$k(c) = 3a + t, \qquad k(b) = a + 3t.$$

The t induced characters are exactly those fixed by a nontrivial quotient linear character. Here $t = K$, so

$$k(b) = \frac{K^3 + 8K}{3}, \qquad \frac{k(b)}{D^3} \le \frac{1}{24} + \frac{4}{3D^2} \le \frac{3}{64}, \qquad D \ge 16. \tag{16.6}$$

The count uses the central-twist and conjugation compatibility of connected-center Jordan decomposition [62, Theorem 1.1(3),(7) and Theorem 1.2].

The defects in the three configurations are products of three, one and two wreathed or semidihedral factors, respectively. Lemmas 13.3 and 13.1 show that their integral quotients have odd parts at most $3, 1, 1$. In the first case, (16.6) gives $k(b) < \delta/4$, so Lemma 13.2(ii) applies; in the other cases use its trivial-module construction. Every odd central quotient of an algebraic central cover is included: the corresponding central sector has unchanged defect and its ordinary characters are those of the covering block with the quotient kernel in their kernel.

16.3. **The connected isolated cases.** Consider the connected centralizers in (16.1), excluding the E_6 row. For $E_8 : E_6A_2$, this subsection concerns the maximal-defect semisimple block. Its other blocks will be treated below. The block distribution and defect statements from [41, 43] have the following consequences. The $G_2 : A_2$, $F_4 : A_2^2$, $E_7 : A_5A_2$, $E_8 : A_8$ and $E_8 : A_4^2$ series each consist of one block; the maximal $E_8 : E_6A_2$ block has full Sylow defect in its associated connected centralizer. Ordinary Jordan decomposition counts the indicated full series, so its count also bounds any selected block in it.

For factors of type A_2, A_4, A_8, E_6, the fundamental groups have odd order. Thus all semisimple 2-centralizers are connected, and odd central isogenies identify their rational 2-classes and unipotent-character counts. This is the odd-isogeny case of [48, Proposition 4.6]; it also follows directly by lifting the unique 2-part through the odd kernel and applying Lang's theorem to the connected centralizer. It follows that the relevant series-to-defect ratios are products of the type-A principal ratios and, when needed, Lemma 16.1. Factor permutations are treated as single rational factors over an extension field. Thus

type	$k(b)/\delta(b)$ at most
$G_2 : A_2$	$1/2$
$F_4 : A_2^2$	$1/2$
$E_8 : A_8$	1
$E_8 : A_4^2$	1
$E_8 : E_6A_2$ (maximal)	$15/128$.

(16.7)

For F_4 the two factors give $1/4$, which implies the displayed $1/2$ bound. The last E_8 row uses the complete E_6 two-series, so it bounds every allocation among cuspidal or twin blocks.

For $E_7 : A_5A_2$ the even center of A_5 must be retained. Work first with the simply connected group $G = E_7(q)_{\mathrm{sc}}$, and let $\mathbf{C}$ be the connected reductive group *dual to* $C_{\mathbf{G}^*}(s)$ in the adjoint dual group. Put $C = \mathbf{C}^F$. Jordan decomposition on G and on $\mathbf{C}$ gives $k(b) = k(B_0(C))$ and $\delta(b) = |C|_2$; this is [48, Lemma 4.4], which allows disconnected centralizers of subsequent 2-elements. The character lattice of $\mathbf{C}$ is the E_7 weight lattice. Write $Q(\Phi)$ and $P(\Phi)$ for the root and weight lattices of a root system Φ, and C_Φ for its Cartan matrix. With the common root-length normalization,

$$[Q(E_7) : Q(A_5A_2)]^2 = \frac{\det C_{A_5} \det C_{A_2}}{\det C_{E_7}} = \frac{6 \cdot 3}{2} = 9.$$

Consequently

$$[P(E_7) : Q(A_5A_2)] = 2 \cdot 3 = 6, \qquad [P(A_5A_2) : P(E_7)] = \frac{18}{6} = 3.$$

Hence the simply connected covering of $\mathbf{C}$ has kernel of order three, and its rational points are $\mathrm{SL}_6(\varepsilon q) \times \mathrm{SL}_3(\varepsilon q)$. Its image H has index at most three in C. Both this kernel and index are odd. This identifies the group being counted; the adjoint-dual centralizer itself can have an additional even covering kernel. The principal block of H has the product character count. For a cyclic quotient of order at most three, Clifford theory gives

$$k(B_0(C)) \le 3\, k(B_0(H)), \qquad |C|_2 = |H|_2.$$

Lemma 11.3 gives

$$\frac{k(B_0(\mathrm{SL}_6(\varepsilon q)))}{|\,\mathrm{SL}_6(\varepsilon q)|_2} \le \frac{1}{16}, \qquad \frac{k(B_0(\mathrm{SL}_3(\varepsilon q)))}{|\,\mathrm{SL}_3(\varepsilon q)|_2} \le \frac{1}{2}.$$

The first estimate includes the scalar-stabilizer correction in (11.2). The Jordan count therefore gives ratio at most $3/32$ for the simply connected E_7 block. Its quotient by the central involution has no more characters and half the defect order, so

$$k(b)/\delta(b) \le 3/16 \tag{16.8}$$

for the ordinary simple group relevant to odd-centred components.

Lemma 16.2 (Integral innerness for connected isolated labels)**.** *Let L be an algebraic central cover with odd center in one of the connected rows of (16.1), and let b be a block of maximal defect in that series. For every $\mathrm{Inn}(L) \le A \le \mathrm{Aut}(L, b)$, put $T = \{a \in A : a \text{ acts innerly on } \mathcal{O}Lb\}$. Then A/T is a 2-group.*

Proof. Work first with $G = \mathbf{G}^F$, where $\mathbf{G}$ is simply connected, and let $s \in (\mathbf{G}^*)^{F^*}$ be the odd label of the covering block. As in [62, Section 2.2], choose a compatible identification of $\mathbf{G}_{\mathrm{ad}}$ with $\mathbf{G}^*$, let $\pi : \mathbf{G} \to \mathbf{G}_{\mathrm{ad}}$ be the adjoint quotient, and put

$$G(s) = \left(\pi^{-1}(C_{\mathbf{G}^*}(s))\right)^F.$$

The local structure theorem [62, Proposition 6.1] supplies a maximal pair (P, e) for which $P \in \mathrm{Syl}_2(G(s))$ and

$$N_G(P, e) = N_{G(s)}(P). \tag{16.9}$$

The construction uses the odd linear torus character $\widehat{s}$ and the uniquely corresponding block at $C_G(P)$. In particular it is compatible with automorphisms fixing that torus and s. The two exceptions to the cited proposition, the rational $A_2(\varepsilon q^3).3$ centralizer and the triality case, are absent from the present list.

The simply connected rational factors of $G(s)$ have type A of rank at least two, or type E_6. For these factors [13, Corollary 2.4 and Remark 2.2] gives

$$N(P) = PC(P),$$

and the property descends through a central quotient. Any further rational diagonal index is odd. A maximal torus normalized by P maps onto it, and the averaging argument preceding (15.1) shows that its P-fixed part still maps onto that quotient. To see the normalizer equality explicitly, let H_0 be the image of the product of the simply connected rational factors and put $H = G(s)$. The averaging argument gives $H = H_0 C_H(P)$, whence

$$N_H(P) = N_{H_0}(P) C_H(P) = P C_{H_0}(P) C_H(P) = P C_H(P).$$

Combining this with (16.9) gives

$$E_b = 1. \tag{16.10}$$

Let an odd field subgroup of order h stabilize the block. Write $F = F_0^h$ and $F^* = (F_0^*)^h$ for the corresponding primal and dual Frobenius maps. We may assume that s is F_0^*-fixed. Indeed, an F_0^*-stable geometric semisimple class has an F_0^*-fixed representative by Lang's theorem in

the connected ambient dual group. Since $C_{\mathbf{G}^*}(s)$ is connected, this representative is conjugate to the original one by an element of $(\mathbf{G}^*)^{F^*}$. In dual adjoint type E_7 its rational conjugator can be chosen in the image of the simply connected group: an F-stable maximal torus of $C(s)$ maps onto the diagonal quotient, by the torus cohomology sequence; multiply the conjugator by an appropriate element of this centralizing torus. Thus the change is implemented by an inner automorphism on the original ordinary presentation.

All diagram twists and factor permutations in these centralizers have 2-power order; none has triality or an odd factor cycle. Their central covering divisors are odd. Equation (15.3) therefore shows that $C_{G(s)}(F_0)$ and $G(s)$ have equal 2-parts. Choose P in the smaller-field Sylow torus normalizer. It is a full Sylow subgroup of the larger group and is centralized pointwise by F_0.

We next determine the second component of the Brauer pair. In the torus construction of (16.9), the torus is defined over the smaller field and its 2-torsion is unchanged by the odd extension. Its canonical odd character $\widehat{s}$ is F_0-fixed because s is. The uniquely corresponding block e of $C_G(P)$ is therefore F_0-fixed. For the field extension H this proves

$$H = GC_H(P)_e.$$

By (16.10), Murai's pairing is trivial, so $H = H[b]$ integrally. The odd part of the ordinary outer group in each of these types is precisely its normal odd field subgroup; the remaining quotient has 2-power order. Hence A/T is a 2-group.

For the E_7 simple quotient the conclusion descends through the central involution. A unit implementing an automorphism of the covering integral block maps to a unit implementing it in the dominated quotient block. Thus the argument gives integral innerness on the quotient, in addition to the count in (16.8). □

Equations (16.7) and (16.8), followed by Lemma 16.2, give trivial modules for all these maximal blocks.

16.4. **Nonmaximal blocks in the isolated E_8 series.** The remaining nonabelian-defect rows in the isolated E_6A_2 series are rows 4 and 6 of the 2-block portion of [41, Table 5], with their Ennola forms. Row 4 has a wreathed defect group. Lemma 13.3(i), the intrinsic bound and the cited central-extension theorem of Sambale give its trivial module.

Row 6 has a metacyclic defect group [43, Proposition 7.5 and Table 11]. We check that it is neither abelian nor Q_8, the two exceptions to the automorphism lemma. Write $\mathbf{L}$ for its cuspidal Levi in the ambient algebraic group $\mathbf{G}$ of type E_8. Its derived subgroup has type E_7, its center is connected of dimension one, and its relative Weyl group has type A_1. Connectedness of its center follows because the seven Levi simple roots span a primitive summand of the E_8 character lattice. Its intersection with the simply connected derived E_7 is that group's central subgroup of order two. For $q \equiv \varepsilon \pmod 4$, put $A = (q-\varepsilon)_2 \geq 4$. Let λ be the cuspidal character in this row, and put $W_{\mathbf{G}^F}(\mathbf{L},\lambda) = N_{\mathbf{G}^F}(\mathbf{L},\lambda)/\mathbf{L}^F$. The defect formula [41, Theorem 1.2(b) and the proof of Proposition 6.4, case 6] gives

$$|P| = |Z(\mathbf{L})^F|_2 \left|\mathbf{L}^F/(O_2(Z(\mathbf{L})^F)[\mathbf{L},\mathbf{L}]^F)\right|_2 |W_{\mathbf{G}^F}(\mathbf{L},\lambda)|_2. \tag{16.11}$$

The first factor is A. Lang's theorem for the derived subgroup makes the map to the rational one-dimensional quotient torus surjective. The image of $Z(\mathbf{L})^F$ has index two there, as its kernel is the central involution. Thus the middle factor is two, in both the split and nonsplit rank-one torus cases. The last factor is two, so

$$|P| = 4A \geq 16.$$

The nonabelianness assertion in [41, Theorem 1.2(c)] applies since the relative Weyl group has even order. Hence P is neither homocyclic abelian nor Q_8. By [67, Lemma 1], $\mathrm{Aut}(P)$ is a

2-group. The metacyclic block bound and Lemma 13.1 once again give the trivial module for the quotient, including every field stabilizer.

17. The remaining component blocks

17.1. **Nonprincipal unipotent blocks.** The complete list of nonprincipal unipotent 2-blocks of exceptional groups with nonabelian defect is [43, Section 7, immediately before Table 10]. Only E_7 and E_8 occur, for both congruences modulo four. The E_7 defects are nonabelian dihedral. Their automorphism groups are 2-groups by [67, Lemma 1]. The intrinsic bound and the dihedral block theorem therefore give the required trivial module. The E_8 case requires a full ordinary-character estimate, which we now prove.

Proposition 17.1. *Let b be a nonprincipal unipotent 2-block of $E_8(q)$ with nonabelian defect, where q is an odd prime power. Put $A = \max\{(q-1)_2, (q+1)_2\} \geq 4$. Then*

$$k(b) \leq \frac{2}{3}A^2 + 8A + \frac{10}{3} < 4A^2 = \delta(b). \tag{17.1}$$

Its integral quotient is a 2-group, so the trivial module of order $\delta(b)$ satisfies (13.1).

Proof. Let $\mathbf{G}$ be the algebraic group of type E_8 with Frobenius F defining $E_8(q)$, and identify it with its dual. The quasi-central cuspidal source has derived type E_6 (the twisted rational form for $q \equiv -1 \pmod 4$), a connected central torus of rank two, and one of the two cuspidal characters labelled θ, θ^2, where θ is a primitive cube root of unity. Its geometric relative Weyl group, and its rational relative Weyl group, is $S_3 \times C_2$, acting faithfully on that central torus. The defect order is $4A^2$. These assertions and the following single-source unipotent counts are the local data of [62, Lemma 7.1(ii), the proof of Lemma 7.2, and the first paragraph of the proof of Lemma 7.4]. For a semisimple 2-element $t \in \mathbf{G}^F$, the contributions are

Rational derived type of $C_{\mathbf{G}}(t)$	Contribution from one source
E_6 or ${}^2E_6$	1
E_6A_1 or ${}^2E_6A_1$	2
E_7	2
E_7A_1	4
E_8 $(t = 1)$	6

The E_7A_1 contribution is four per source. The cited calculation counts all ordinary characters in these layers before restricting to height zero. Moreover, [43, Section 4.2 and Table 1] retains a twin-block ambiguity when the order of q modulo four is two. We therefore allow both sources for every nonidentity t: their union contributes at most eight. At $t = 1$ the unipotent distribution gives six in the selected individual block.

We next justify the source assertion needed to count the possible t. If $\chi \in \mathrm{Irr}(b) \cap \mathcal{E}(\mathbf{G}^F, t)$, let c_t be the unipotent block containing its Jordan correspondent in $C_{\mathbf{G}}(t)^F$. The block/source comparison [43, Theorem 1, Lemma 3.4, Definition 3.5, Propositions 3.6 and 4.1] gives a quasi-central cuspidal pair $(\mathbf{L}_t, \lambda_t)$ for c_t whose Levi is also an ambient Levi of $\mathbf{G}$. Uniqueness of the quasi-central cuspidal source of a unipotent block, as used in the proof of Proposition 4.1(b) of that paper, identifies its derived source with that of b or its indicated twin. Definition 3.5 and Proposition 3.6(a) preserve this derived subgroup up to rational conjugacy; the exception in Proposition 3.6(b) is only at the block prime three. Both present twins have geometric derived source E_6. Thus $\mathbf{L}_t$ is an F-stable ambient E_6 Levi. Since it is a Levi of $C_{\mathbf{G}}(t)$, it contains the center of that connected group. In particular $t \in \mathbf{L}_t$, and because t centralizes it,

$$t \in Z(\mathbf{L}_t). \tag{17.2}$$

This membership uses the Levi property, in addition to the fact that $\mathbf{L}_t \le C_{\mathbf{G}}(t)$.

Fix a split E_6 Levi $\mathbf{L}$ and put $\mathbf{S} = Z(\mathbf{L})$. Every ambient Levi with derived type E_6 is geometrically conjugate to $\mathbf{L}$: the E_8 Dynkin diagram has a unique subdiagram of type E_6. The six Levi simple roots form a primitive summand of the E_8 character lattice, whose simple roots are a basis. Thus $\mathbf{S}$ is a connected torus of rank two. For a torus $\mathbf{T}$, write $\mathbf{T}[2^\infty]$ for its subgroup of elements of 2-power order. Consider geometric conjugacy classes of marked pairs

$$(t, \mathbf{L}'), \qquad t \in Z(\mathbf{L}')[2^\infty], \quad \mathbf{L}' \text{ conjugate to } \mathbf{L}.$$

After conjugating $\mathbf{L}'$ to $\mathbf{L}$, the residual equivalence relation is the action of

$$W = N_{\mathbf{G}}(\mathbf{L})/\mathbf{L} = S_3 \times C_2$$

on $\mathbf{S}[2^\infty]$. This action is faithful, because $C_{\mathbf{G}}(\mathbf{S}) = \mathbf{L}$. It commutes with Frobenius. By (17.2), each rational semisimple class contributing to b is the image of an F-stable geometric marked-pair class. All semisimple centralizers in simply connected E_8 are connected; consequently Lang's theorem gives exactly one rational class in each F-stable geometric semisimple class. Hence the number n_b of contributing rational 2-classes is at most the number N of F-stable W-orbits on $\mathbf{S}[2^\infty]$. Thus N counts stable geometric orbits with all rational torus twists.

The twisted Burnside argument used in (16.2) gives

$$N = \frac{1}{12} \sum_{w \in W} |\det(qw - I)|_2.$$

It remains valid on the infinite torsion set, since every fixed-point set is finite and each stable orbit contributes twelve pairs. The faithful rank-two representation of W is the dihedral reflection representation of order twelve. Let ω have order three. The spectra and multiplicities are

	number	eigenvalues
1	1	$(1, 1)$
$-I$	1	$(-1, -1)$
reflections	6	$(1, -1)$
order-three rotations	2	(ω, ω^{-1})
order-six rotations	2	$(-\omega, -\omega^{-1})$.

Their determinant 2-parts are $A^2, 4, 2A, 1, 1$, respectively, with the first two possibly interchanged. Replacing q by $-q$ for the twisted source does not change the sum, because $-I \in W$. Thus

$$N = \frac{A^2 + 4 + 12A + 4}{12} = \frac{A^2}{12} + A + \frac{2}{3}. \tag{17.3}$$

There is one marked orbit above $t = 1$. The single-block identity contribution and the two-source bound at nonidentity classes give

$$k(b) \le 6 + 8(n_b - 1) \le 6 + 8(N - 1) = \frac{2}{3}A^2 + 8A + \frac{10}{3}.$$

Writing $A = 4 + x$, $x \ge 0$, the difference from $4A^2$ is

$$18 + \frac{56}{3}x + \frac{10}{3}x^2 > 0.$$

This proves the count for every valuation, and includes positive-height characters.

Return to the F-stable cuspidal source Levi $\mathbf{L}_b$ of b. It has rational type $E_6\Phi_1^2$ when $q \equiv 1 \pmod 4$ and ${}^2E_6\Phi_2^2$ when $q \equiv -1 \pmod 4$. Put $D = O_2(Z(\mathbf{L}_b)^F) \cong C_A^2$. Choose a defect group P of b containing D. The proof of [62, Lemma 7.2] gives $D \lhd P$, $P/D \cong V_4$, and generators of P/D acting on D by inversion and interchange of its two cyclic factors. The odd central intersection with the derived E_6 subgroup does not change this 2-torsion. Lemma 13.3(iii) applies

without assuming that the extension splits. Thus $\mathrm{Aut}(P)$, and by Lemma 13.1 also R, are 2-groups. The trivial module completes the proof. □

17.2. **The exceptional graph in G_2.** We prove (13.1) directly for every nonprincipal 2-block of $G_2(q)$ with nonabelian defect, where q is an odd prime power. This includes the exceptional graph action when q is a power of three.

The block list in [68, Appendix B.3.1] is

$$B_0; \quad B_3\ (3 \nmid q); \quad B_{1a}, B_{1b}, B_{2a}, B_{2b}; \quad BX_1, BX_2, BX_a, BX_b; \quad \text{defect-zero blocks.}$$

Section 12.2 of that source expressly includes $q = 3$. The B_1, B_2 blocks have two Brauer characters, B_3 has three, and the BX blocks have one. The BX_a, BX_b defects are cyclic. Write $q \equiv \varepsilon \pmod 4$ and $(q-\varepsilon)_2 = 2^a$, $a \geq 2$. The defect groups in [68, Remark 12.22] are

$$P \cong \begin{cases} C_{2^a} \wr C_2, & B_{1a}, B_{1b},\ \varepsilon = 1,\ \text{or } B_{2a}, B_{2b},\ \varepsilon = -1, \\ \mathrm{SD}_{2^{a+2}}, & \text{the other congruence combinations.} \end{cases} \tag{17.4}$$

We show that BX_1 and BX_2 also have abelian defect. First observe the following rank-one fact. Let $\mathbf{M}$ be connected reductive with connected center and derived type A_1, and let F define an $\mathbb{F}_q$-structure with q odd. Then

$$l(B_0(\mathbf{M}^F)) \geq 2. \tag{17.5}$$

Indeed Lang's theorem for the center gives a surjection onto $\mathrm{PGL}_2(q)$. Inflate its permutation action on $\mathbb{P}^1(q)$. The ordinary augmentation character is the irreducible Steinberg character of degree q. Modulo two its augmentation lattice contains the constant vector, since $q+1$ is even, so its reduction has the trivial composition factor and belongs to the principal block. It also has a nontrivial composition factor: an odd-order nonidentity root element fixes exactly one point of $\mathbb{P}^1(q)$ and has Steinberg value zero. If all composition factors were trivial, the Brauer value there would be q. This proves (17.5).

By the unipotent list stated above, a nonprincipal unipotent block of $G_2(q)$ has abelian defect. For a non-unipotent block let $s \neq 1$ be its odd dual label. The dual centralizer is connected. Its only noncentral isolated possibility for an odd label is A_2, with an order-three label; the A_1A_1 centralizer requires an even label. The isolated series is a single block [41, Lemma 6.13], and it is exactly B_3 in the above notation. One can check this identification from the character degrees: if $q \equiv \eta \pmod 3$ with $\eta \in \{1, -1\}$, the Jordan semisimple character has degree

$$(q+\eta)\Phi_6(\eta q),$$

which for $\eta = -1$ is $(q-1)\Phi_3(q)$. This is the degree of the character denoted X_{32} in [68, Table B.2]; Appendix B.3.1 places it in B_3. For $q \geq 5$ this degree is unique in that table. In fact every nontrivial unipotent degree is at least $q(q-1)^2\Phi_6(q)/6$, which is larger than $(q+1)\Phi_3(q)$ because

$$\Phi_6(q) > \tfrac{2}{3}\Phi_3(q), \qquad q(q-1)^2 > 9(q+1).$$

The other non-unipotent families, apart from the two strict multiples X_{31}, X_{33}, have degree at least $\Phi_3(q)\Phi_6(q)$, also larger. In characteristic three there is no such semisimple order-three label, so B_3 is absent.

The remaining odd centralizers are proper Levis of semisimple rank one or zero. A rank-one Levi in G_2, and its dual, have connected center: the Cartan determinant is one, every root and coroot is Weyl-conjugate to a simple one, and hence is primitive in its lattice. Ordinary Bonnafé–Rouquier equivalence sends the block to a block of that dual Levi. Its central odd label makes it a linear twist of a unipotent block; the principal 2-block is the only unipotent block for a classical derived component, as recalled in [43, the proof of Lemma 2.6]. Morita equivalence and linear twisting preserve $l(b)$, so (17.5) excludes this case for BX_1, BX_2. In rank zero the

correspondent is a torus block. The abelian-defect preservation theorem [41, Theorem 1.3] for this correspondence gives abelian defect for the original block. Thus BX_1 and BX_2 have abelian defect.

It follows that every nonprincipal nonabelian-defect block is either the isolated B_3 case already treated or one of (17.4). For the latter, the intrinsic bound and Lemma 13.3(i) make R a 2-group, and the wreathed or semidihedral block bound gives its trivial module. This argument includes the full exceptional graph–field stabilizer of $G_2(3^f)$.

17.3. **Nonprincipal triality blocks.** For $S = {}^3D_4(q)$ with q odd, put $(q^2-1)_2 = 2^{a+1}$, $a \geq 2$. Every nonabelian defect group is isomorphic to one of the following:

$$S_2, \qquad C_{2^a} \wr C_2, \qquad \mathrm{SD}_{2^{a+2}}, \qquad S_2 \in \mathrm{Syl}_2(S), \tag{17.6}$$

by [3, p. 274]. The same source uses $C_S(S_2) = Z(S_2)$ on p. 287. This centralizer is a 2-group and has just one block. Thus every block with defect S_2 has the same maximal Brauer pair as the principal block and is principal. For a nonprincipal block only the last two groups in (17.6) remain. Their automorphism groups are 2-groups by Lemma 13.3(i), and the central-extension block theorem gives $k(b) \leq |P|$. Lemmas 13.1 and 13.2 provide the trivial module for every stabilizer. The construction applies to the full stabilizer with its cyclic field–triality action.

17.4. **Alternating, sporadic and exceptional covering groups.** Let S be nonabelian simple and let L be quasisimple with $L/Z(L) \cong S$. The map $\mathrm{Aut}(L) \to \mathrm{Aut}(S)$ is injective. An automorphism trivial on S has a central error homomorphism from L to its center and is therefore the identity. An automorphism inducing an inner automorphism of S is inner on L after lifting the conjugating element. Hence $\mathrm{Out}(L)$ embeds in $\mathrm{Out}(S)$. If this latter group is a 2-group, every quotient A/T is a 2-group.

The scalar block inequalities for alternating groups, sporadic covers, the Tits group and exceptional covers are [48, Proposition 2.3 and Theorem 2.4]. The outer groups of alternating and sporadic simple groups are 2-groups. After removing exceptional multiplier 2-parts, the exceptional odd-cover list is

Simple quotient	Exceptional odd multiplier	Outer group
A_6	C_3	V_4
A_7	C_3	C_2
$\mathrm{PSU}_4(3)$	$C_3 \times C_3$	D_8
$\Omega_7(3)$	C_3	C_2
$G_2(3)$	C_3	C_2

The multiplier list is [72, Introduction]; the outer orders are the corresponding entries of [17]. Every quotient of the displayed multiplier is allowed, including both triple covers of $\mathrm{PSU}_4(3)$. The exceptional multiplier parts of $\mathrm{PSL}_3(4)$, $\mathrm{PSU}_6(2)$ and ${}^2E_6(2)$ are 2-primary. Their remaining odd central covers arise from the simply connected algebraic presentations and were included in Proposition 14.2. In every displayed exceptional case the scalar bound and the outer 2-group give the trivial module.

17.5. **The local alternative.**

Theorem 17.2 (Component alternative)**.** *Let L be a quasisimple group with odd center and let b be a 2-block of L with nonabelian defect. Let $\mathrm{Inn}(L) \leq A \leq \mathrm{Aut}(L, b)$, and put*

$$T = \{a \in A : a \text{ acts innerly on } \mathcal{O}Lb\}, \qquad R = A/T.$$

At least one of the following holds.

(i) *There is an ordinary presentation* $L = \mathbf{S}^F/Z_0$ *in odd characteristic, with* $\mathbf{S}$ *simply connected and* $Z_0 \leq Z(\mathbf{S}^F)$*, such that the* A*-action lifts to* $\mathbf{S}^F$ *by ordinary automorphisms preserving* Z_0*. Let* b_S *be the block of* $\mathbf{S}^F$ *dominating* b*, let* $\mathbf{S}^*$ *be dual to* $\mathbf{S}$*, and let* s *be the odd semisimple label of* b_S*. The canonical minimal Levi*

$$\mathbf{M}^* = C_{\mathbf{S}^*}\big(Z(C_{\mathbf{S}^*}(s)^\circ)^\circ\big)$$

is proper.

(ii) *The group* R *has an* $\mathbb{F}_2R$*-module* W *satisfying* (13.1)*.*

Proof. The preceding propositions cover algebraic central covers in defining characteristic two, their principal blocks in odd characteristic, all noncentral isolated odd labels, nonprincipal unipotent blocks with nonabelian defect, all G_2 and triality endpoints, and all alternating, sporadic and exceptional odd covers. The small Ree groups have elementary abelian Sylow 2-subgroups of order eight and have no nonabelian-defect case. The nonsimple small presentations are treated through their simple derived groups, already listed above.

For completeness we check the remaining ordinary-action boundary. Every block of a finite group of Lie type in nondefining characteristic belongs to an odd semisimple 2-series. At a central label the resulting unipotent cases are precisely those already listed, allowing their central linear twists in a regular presentation. At a noncentral non-isolated label the canonical minimal Levi in the statement is proper by definition. In classical types the connected odd centralizer is a proper Levi. In E_6 its possible component group of order three is retained in the actual normalizer of that Levi. The classification in [5, 41, 43] gives exactly (16.1) when the canonical Levi is the whole group. Thus these alternatives exhaust all remaining labels.

The standard automorphism theorem realizes the remaining finite automorphisms by adjoint conjugations, field maps and pinned diagram maps [36, Section 2.1]. Adjoint conjugation acts canonically on the simply connected cover, independently of a central choice of lift. Field and pinned diagram maps act on its Chevalley generators with their relations. The original quotient kernel is preserved: in type A and E_6 it is a subgroup of a cyclic center; for the odd-centred orthogonal and E_7 forms the entire 2-primary center has already been removed. In twisted E_6, if $F = \gamma F_r^f$ and $u = F_r$ on fixed points, retain $u^f = \gamma^{-1}$ and $u^{2f} = 1$. Perfectness makes the lift unique, so the ambient relations are preserved. On a component orbit fixed identifications give the literal permutation action, and the original central gluing is preserved by the actual conjugation action.

The exceptional inseparable graph families are B_2 and F_4 in characteristic two and G_2 in characteristic three [36, Lemma 2.1]. The first two have direct defining-two modules; the last has the direct G_2 endpoint proved above. The triality nonprincipal cases also have their direct endpoint. Thus none of these operators is inserted into an ordinary Levi descent. For every other remaining proper Levi, the hypotheses of Theorem 10.1 are exactly the properties just verified. This proves the first alternative, and completes the exhaustion. □

18. The minimum counterexample

We now combine the affine inequality, the component estimates and the block descent. The order of the minimization is relevant: the normal-2 argument decreases the defect order even when an intermediate Morita equivalence changes the size of the ambient group.

Lemma 18.1 (Reduction of normal subblocks)**.** *Assume* (2.6)*, and choose a counterexample* (G, B) *for which*

$$(\delta(B),\ |G : Z(G)|,\ |G|) \tag{18.1}$$

is lexicographically minimal. Then B is quasiprimitive, meaning that every block of a normal subgroup covered by B is G-invariant. Moreover

$$O_2(G) = 1, \qquad O_{2'}(G) = F(G) = Z(G), \qquad C_G(E(G)) = Z(G). \tag{18.2}$$

The center is cyclic of odd order. If B covers a nilpotent block of $N \lhd G$, then $N \le Z(G)$.

Proof. Corollary 3.6 gives $O_2(G) = 1$. If a covered normal block has proper inertia group I, the Fong–Reynolds correspondent has the same character count and defect group. Since $Z(G) \le I$,

$$|I : Z(I)| \le |I : Z(G)| < |G : Z(G)|,$$

contrary to minimality. Thus B is quasiprimitive.

We spell out the use of the nilpotent-block reduction [45]. Suppose B covers a nilpotent block of $N \lhd G$. The Külshammer–Puig construction, applied also to the central factor, gives a Morita equivalent block $\widehat{B}$ with the same defect group in a finite group $\widehat{L}$. There is a central odd subgroup $\widehat{Z}$ and a normal subgroup $\widehat{R} \cong D_B \cap Z(G)N$ such that

$$\widehat{L}/(\widehat{Z}\widehat{R}) \cong G/Z(G)N. \tag{18.3}$$

This is the usual reduced-pair construction; its precise form for arbitrary defect groups is also recorded in [2, Proposition 6.1 and its proof]. If $\widehat{R} \ne 1$, apply Corollary 3.6 to $\widehat{B}$. It produces a counterexample of smaller defect order, which is impossible. Thus $\widehat{R} = 1$. If $N \not\le Z(G)$, (18.3) gives

$$|\widehat{L} : Z(\widehat{L})| \le |G : Z(G)N| < |G : Z(G)|,$$

again contradicting (18.1). This proves the assertion about nilpotent covered blocks.

Every block of the odd normal subgroup $O_{2'}(G)$ has defect zero and is nilpotent. It follows that $O_{2'}(G) \le Z(G)$. Conversely $Z(G)$ is odd because $O_2(G) = 1$, so equality holds. The nilpotent Fitting subgroup is therefore $F(G) = Z(G)$. The generalized Fitting subgroup is $F^*(G) = Z(G)E(G)$ and is self-centralizing. Moreover $Z(E(G))$ is an abelian normal subgroup, so $Z(E(G)) \le F(G) = Z(G)$. Hence

$$C_G(E(G)) \le Z(G)E(G) \cap C_G(E(G)) = Z(G)Z(E(G)) = Z(G),$$

and equality follows.

The block selects one linear character of the odd center. If its kernel Z_0 is nontrivial, quotienting by Z_0 preserves the block's character count and defect group, does not increase the second entry of (18.1), and decreases the third. Its central character is thus faithful, so $Z(G)$ is cyclic. □

Lemma 18.2 (Abelian normal subblocks)**.** *In the minimum counterexample of Lemma 18.1, a covered block of a noncentral normal subgroup cannot have abelian defect.*

Proof. The defect-zero case is covered by the last assertion of Lemma 18.1. Let b be a covered block of $N \lhd G$ with nontrivial abelian defect group P. Write $\mathrm{Irr}_0(b) = \{\chi \in \mathrm{Irr}(b) : \chi(1)_2 = |N|_2/\delta(b)\}$ for the height-zero characters. Brauer's height-zero theorem [41] gives $\mathrm{Irr}(b) = \mathrm{Irr}_0(b)$. The inductive Alperin–McKay condition at the prime two holds for all nonabelian simple groups [64]. We may therefore apply the normal-subgroup theorem [56, Theorem B].

Let b^* be the Brauer correspondent of b in $N_N(P)$. It has the same abelian defect group, so $\mathrm{Irr}(b^*) = \mathrm{Irr}_0(b^*)$. Conjugacy of the defect groups of the G-invariant block b gives $G = NN_G(P)$. The normal-subgroup theorem gives the equivariant bijection on the lower height-zero characters and, for each corresponding lower character, a bijection of the *full* ordinary upper block fibers. The identity $G = NN_G(P)$ identifies the lower orbit sets. Let Θ be a set of G-orbit representatives

in $\mathrm{Irr}(b)$, and let θ^* correspond to $\theta \in \Theta$. Since all lower characters have height zero, the full upper fibers give

$$\mathrm{Irr}(B) = \coprod_{\theta\in\Theta} \mathrm{Irr}(B \mid \theta) \;\longleftrightarrow\; \coprod_{\theta\in\Theta} \mathrm{Irr}(B^* \mid \theta^*) = \mathrm{Irr}(B^*),$$

where B^* is the corresponding block of $N_G(P)$. The two upper blocks have a common defect group: this is the Harris–Knörr correspondence used in [56, Corollary 3.10]. Hence

$$k(B^*) = k(B), \qquad \delta(B^*) = \delta(B).$$

But $P \lhd N_G(P)$ is a nontrivial 2-subgroup. Corollary 3.6 again produces a counterexample of smaller defect order, a contradiction. □

Proof of Theorem 1.1. The affine inequality (2.6) has been proved in Theorem 9.1. Suppose the conclusion is false and choose (G, B) as in (18.1). Lemma 18.1 applies. Put $X = Z(G)E(G)$, and let b be the unique block of X covered by B. Write $L_1, \dots, L_t$ for the components of G, and b_i for the corresponding component blocks. There is at least one component: otherwise the self-centralizing property of $F^*(G)$ would give $G = Z(G)$, whose 2-blocks have defect zero.

For any G-orbit of components, their central product is a normal subgroup. If a component block in that orbit had defect zero, the product block would be nilpotent, contrary to Lemma 18.1. If its defect were nontrivial abelian, the product defect would be abelian as well, contrary to Lemma 18.2. Thus every b_i has nonabelian defect. All central overlaps have odd order.

Since b is G-stable, $N_G(L_i)$ stabilizes b_i. For each i, let A_i be its image in $\mathrm{Aut}(L_i, b_i)$, and apply Theorem 17.2 to (L_i, b_i, A_i). If its first alternative holds for one component, it holds throughout its G-orbit. Fix a representative L_0 of this orbit and its presentation $\pi_0 : S_0 \twoheadrightarrow L_0$ from that alternative. For each component L_i in the orbit choose $g_i \in G$ with ${}^{g_i}L_0 = L_i$, take a copy S_i of S_0, and set $\pi_i = c_{g_i}\pi_0$, where c_g denotes conjugation by g. If ${}^gL_i = L_j$, define the map $S_i \to S_j$ by the unique ordinary lift of the automorphism of L_0 induced by $g_j^{-1}gg_i \in N_G(L_0)$. Two lifts differ by a homomorphism from the perfect group S_0 to $\ker\pi_0$, so the lift is unique. For ${}^hL_j = L_k$, the identity

$$(g_k^{-1}hg_j)(g_j^{-1}gg_i) = g_k^{-1}hgg_i$$

therefore makes these maps a G-action on $\prod_i S_i$. By the defining relation for a lift, the product map onto the original central product is G-equivariant, so its central kernel is G-stable. The lifted action satisfies the hypotheses of Theorem 10.1. Applied to the central product of that orbit, the theorem supplies a block $\widehat{B}$ of a finite central odd cover $\widehat{H}$ with

$$k(\widehat{B}) = k(B), \qquad \delta(\widehat{B}) = \delta(B), \qquad |\widehat{H} : Z(\widehat{H})| < |G : Z(G)|.$$

This contradicts the second entry of (18.1). Thus every component satisfies the second alternative for its induced group A_i. Let T_i be the subgroup of A_i acting innerly on $\mathcal{O}L_ib_i$, and choose the resulting $\mathbb{F}_2(A_i/T_i)$-module W_i. These modules satisfy (4.8). Each b_i has at least two ordinary characters. Indeed, Brauer's subsection formula expresses $k(b_i)$ as the sum of $l(c)$ over conjugacy classes of b_i-subsections (u, c), where c is a block of $C_{L_i}(u)$ belonging to b_i. The identity subsection and a subsection at a nonidentity element of the center of a defect group each contribute at least one. Proposition 4.6 constructs a module W for $Q = G/G[b]$ with

$$|W| = \delta(b), \qquad |\mathrm{Irr}(b)^J| \le |W^J| \quad (J \le Q), \qquad \ker(Q \to \mathrm{GL}(W)) \text{ a 2-group},$$

and shows that $G[b]/X$ is solvable. Theorem 4.4 therefore gives $k(B) \le \delta(B)$, the desired contradiction. □

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