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GLOBAL TWISTOR SOURCES FOR ANALYTIC EINSTEIN SPACETIMES

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Abstract

We globalize the local analytic projective twistor correspondence for the gravitational googly problem, retaining its convergent nonlinear treatment of both gravitational chiralities. For each fixed Einstein coefficient, the correspondence identifies independently defined atlases of two analytic Čech classes of weights 2 and -6, structured operator sources on null monodromy presentations, and holomorphic oriented spin Einstein four-manifolds. Its inverse comparisons preserve the original source operations, Einstein scale, spin structure and complete point family, together with calibrated modules on parametrized incidence. For analytic Lorentz developments with a compact marked Cauchy surface, germs along the entire real development give the corresponding equivalence, and the maximal analytic Cauchy development determines a terminal source germ. A Nariai example admits no Hausdorff contact carrier containing the full complex skies over all points of the real development. In its product complexification and every allowed point restriction containing the entire real development, every real point sky has nonclosed saturation, so no ordinary locally finitely generated module has exactly that saturated support. These obstructions explain the use of null monodromy and parametrized incidence in the global correspondence, with compact point objects recovered by central specialization.

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1. Introduction and main results

Penrose's gravitational googly problem asks how to incorporate the interacting chirality absent from the nonlinear-graviton construction within the same twistor conventions (Penrose 1976; 2015, secs. 4–5). The local analytic correspondence in Isthmus (2026a), hereafter the *local paper*, treats the nonlinear coupling of both gravitational chiralities through two analytic Čech classes of weights 2 and -6 on a fixed flat twistor germ. It constructs every marked local analytic Einstein four-germ with a prescribed Einstein coefficient, proves convergence of its prescribed nonlinear construction, and establishes inverse correspondences on independently admitted geometric and operator domains. The present paper globalizes this local analytic solution over an entire point manifold and, with Lorentz reality, over an entire Cauchy development.

Several earlier constructions address the geometry beyond self-duality. LeBrun reconstructs complex conformal geometry locally from compact quadrics in a space of complex null geodesics and gives an ambitwistor recognition of conformally Einstein metrics (LeBrun 1983, 1985). The thickening approach of Baston and Mason has a fully curved extension due to LeBrun, with a conformal Einstein interpretation when the Weyl curvature is algebraically general (Baston and Mason 1987; LeBrun 1991). Sharma obtains a twistor action for Euclidean general relativity with zero cosmological constant, using partially integrable almost complex structures, and shows that this action reduces to the Plebański action (Sharma 2021). More recent approaches include Penrose's bi-twistors, which replace the role of α-planes by null geodesics, and the construction of a metric conformal to

Schwarzschild from holomorphic data in the twistor space of self-dual Taub–NUT (Penrose 2026; Adamo et al. 2026). Our starting point is the specific convergent two-class correspondence in Isthmus (2026a); the task here is to continue its source data, operations and inverse comparisons globally.

The operator formulation is also related to Penrose's palatial proposal, which discusses local twistor algebras, their patching and point recovery in connection with analytic Lorentz globally hyperbolic geometry (Penrose 2015, secs. 5–6). Marcolli and Penrose extend Donaldson–Friedman gluing in the Riemannian self-dual setting to several quantized twistor constructions by deformation theory (Marcolli and Penrose 2021). Here the global objects are independently admitted source atlases and their operator realizations on the null monodromy presentation of an entire point manifold. Their inverse correspondence preserves the original source classes and physical transition maps, as well as the supplied operator associators and framed point modules.

Global twistor geometry has established precedents in other signatures and settings. LeBrun and Mason construct a global correspondence for self-dual conformal structures of split signature near the standard structure on $S^2 \times S^2$, using holomorphic disks (LeBrun and Mason 2007). In Lorentz geometry, reconstruction from real light rays and skies has its own causal and sky-separation hypotheses (Bautista et al. 2015). Non-Hausdorff twistor spaces also occur in the Geroch-group construction of Woodhouse and Mason (Woodhouse and Mason 1988). The obstruction proved below concerns a specific stronger requirement: a Hausdorff complex carrier for the full complex null quadrics over every point of an entire real Einstein development.

A global source is an atlas of the original input classes with their nonlinear physical comparison law. Its point domains and comparison maps specify the topology, continuation domain and complete point family. They determine the complex null phase and its monodromy groupoid, which remembers null paths modulo endpoint-fixed homotopy. Structured operator algebroids descend on contact transversals of this groupoid. The point modules descend on Hausdorff parametrized incidence, retaining the ambient relative operator germs and their central point parameters; each point object is recovered by central specialization. This construction also accommodates the nonclosed point-sky saturations in the Nariai example.

The external local results are collected in §2, with precise references to the posted version v1 of the local paper. The new contributions are the global construction and independent inverse correspondence, their compatibility with spin structures and Lorentz reality, and the obstruction to stronger ordinary realizations. The null-geodesic incidence construction and structured local contact quantization are used with their established geometric and analytic conventions (Isthmus 2026a, Propositions 4.2–4.3; Kashiwara 1996, Theorem 2; Polesello 2005, Theorem 3.3).

Theorem 1.1 (Global source correspondence).
Fix $\Lambda \in \mathbb{C}$. Use the local correspondence and analytic conventions of §2. There are equivalences

$$\mathsf{SrcAtl}_{\Lambda}^{\mathrm{hol}} \simeq \mathsf{OpMon}_{\Lambda}^{\mathrm{hol}} \simeq \mathsf{MonRay}_{\Lambda}^{\mathrm{hol}} \simeq \mathsf{Ein}_{\Lambda}^{\mathrm{hol}}. \tag{1}$$

Here the source atlas consists of original two-class inputs with physical comparison germs as in Definition 5.1; the geometric and operator monodromy domains are independently admitted by their incidence, transport, optical and operator conditions. Metric objects are connected holomorphic oriented spin Einstein four-manifolds with coefficient Λ. Morphisms retain the specified spin lifts of point embeddings; operator morphisms are physical Hom classes modulo the stated normalized comparison isomorphisms.

Both inverse comparisons retain the entire chosen point manifold, its proper full quadric family, spin structure and Einstein scale. They preserve the original local source operations and prescribed analytic representatives, the projective spin, optical and infinity diagram, and the calibrated module family on parametrized incidence. The source unit recovers the original input classes and overlap instructions; the operator counit preserves the supplied associators and framed module comparisons. For the coherent holomorphic families specified in §5, these comparisons are holomorphic on conditioned local parents and preserve finite parameter jets on each retained finite diagram. The equivalence commutes with point restriction when monodromy is recomputed inside the restricted point domain.

Proof. Theorem 5.6 constructs the first equivalence and its inverse comparisons. Theorem 4.6 and Theorem 3.11 give the remaining equivalences. Proposition 5.7, Theorem 4.8 and §6 identify the preserved structures. The conditioned family assertions follow from Proposition 5.4 and the finite-jet argument of Proposition 5.7. Lemma 5.5 and the restriction comparisons of §§3–4, in particular Corollary 4.3, prove compatibility with holomorphic point restrictions. Proposition 7.1 gives its form for germs along the entire real development. □

Hausdorff contact carriers do not suffice in general for entire real Einstein developments. The following result concerns the complete real Nariai spacetime

$$M_a = \mathbb{R} \times S^1 \times S^2, \qquad g_a = \mathrm{d}t^2 - a^2 \cosh^2(t/a)\, \mathrm{d}\theta^2 - a^2 g_{S^2}, \qquad a > 0, \tag{2}$$

where g_{S^2} is the unit round metric and θ has period 2π. Its Einstein coefficient is $-a^{-2}$, and its Weyl tensor is nonzero.

Theorem 1.2 (Obstructions to ordinary realizations).
For the entire Nariai development (M_a, g_a), the following assertions hold.

(1) In any holomorphic Einstein complexification, no neighborhood of the full complex null quadrics over every real point admits a holomorphic map to a Hausdorff complex manifold whose differential has exactly

the null-geodesic line as kernel. In particular, a global Hausdorff contact carrier with those whole skies does not exist.

(2) In the explicit product complexification used in §8, every real point sky has nonclosed saturation after every allowed point restriction (defined in §7) retaining all of M_a. No ordinary locally finitely generated operator module on the resulting null monodromy presentation has exactly this saturated support. Every other compatible holomorphic Einstein complexification has an allowed point neighborhood on which this support obstruction holds for every real point and persists under every further allowed restriction.

The Hausdorff-carrier obstruction holds on every complexification. The support obstruction holds on every allowed product neighborhood and, for any other complexification, on a cofinal collection of representatives of its germ along the entire real development.

Proof. Theorem 8.2 proves the first assertion. Theorem 8.6, Lemma 8.7 and Corollary 8.8 prove the second. □

The module in Theorem 1.1 is a module over the inverse-image algebroid of ambient relative operator germs on incidence. Its support there is the full incidence manifold; its compact point fibers are central specializations. Thus it has the module action required by the local correspondence while respecting the support obstruction above. The theorem concerns the complete point families supplied in source admission, not a classification of all compact objects of a bare operator algebroid as spacetime points.

Corollary 1.3 (Maximal analytic Cauchy sources).
Fix real Λ and analytic Einstein constraint data (Σ, h, K, P_Σ) with compact connected Σ. Impose the compatible Lorentz reality, exact real point locus and initial markings defined in §7. After passing to germs on point neighborhoods of the entire real development, the correspondence gives

$$\mathsf{SrcAtl}_\Lambda^{\mathrm{germ}}(\Sigma, h, K) \simeq \mathsf{OpMon}_\Lambda^{\mathrm{germ}}(\Sigma, h, K) \simeq \mathsf{Ein}_\Lambda^{\omega}(\Sigma, h, K). \tag{3}$$

The maximal analytic spin Cauchy development determines a terminal source germ. This correspondence is natural under initial-data isomorphisms with their specified spin lifts and retains all the structures in Theorem 1.1.

Proof. Apply Theorem 7.4 and Corollary 7.5, using the analytic Cauchy construction of Appendix A. □

The real point manifold is the entire retained development; the neighborhoods defining its complex germ may have widths that vary from point to point. Section 7 defines point restriction by recomputing null monodromy. The maximal analytic development is constructed as an open subdevelopment of the smooth maximal globally hyperbolic development, with the scope stated in Remark A.9.

Sections 3 and 4 construct the geometric and operator global sources. Section 5 proves that they are obtained from, and recover, atlases of the original input classes. Section 6 treats the projective spin diagram and

Lorentz reality; §7 establishes the Cauchy correspondence. Section 8 proves the obstructions, and Appendix A supplies the analytic Cauchy facts needed for the maximality statement.

The computational supplement (Isthmus 2026b) provides universal metric-jet checks of the Gaussian differential identities, with explicit metric cross-checks from a separate implementation, as well as exact symbolic checks of selected Nariai formulas and the tractor and infinity-pair normalizations.

2. Local correspondence and conventions

We use the posted version v1 of the local paper, *A projective twistor correspondence for local analytic Einstein germs*, Hexagon 2611.00123v1, dated November 5, 2026 (Isthmus 2026a). Propositions 2.1–2.3 recall the forms of its original-source, holomorphic geometric and operator results needed below; their proofs belong to the local paper. Lemma 2.4 proves conjugation naturality for the independent geometric reconstruction. The common analytic convention for these local inputs is a bounded holomorphic family over a finite-dimensional complex manifold germ, with positive coordinate and conic buffers, bounded maps and inverses, and positive lower bounds for the finitely many coefficients inverted on a specified finite comparison diagram. Finite parameter jets are restrictions of such holomorphic families to actual finite Taylor neighborhoods. The global constructions use these bounds locally on compact subsystems.

Our curvature convention is $R(a,b) = [\nabla_a, \nabla_b] - \nabla_{[a,b]}$ and $\mathrm{Ric}(b,c) = \mathrm{tr}(a \mapsto R(a,b)c)$. The Einstein coefficient Λ is defined by $\mathrm{Ric}(g) = \Lambda g$. Holomorphic point manifolds have complex dimension four. Real spacetimes are oriented, time-oriented and four-dimensional, with signature $(+,-,-,-)$. Thus the de Sitter metric of radius a has $\Lambda = -3a^{-2}$ in these conventions. The Schouten tensor is $P = \frac{1}{2}(\mathrm{Ric} - \frac{1}{6}\,\mathrm{Scal}\; g)$.

All point manifolds and real spacetimes are Hausdorff and second countable. Groupoid arrow manifolds and explicitly identified contact leaf spaces may be non-Hausdorff. Point restrictions are taken componentwise when their domains are disconnected. Spin structures and specified full spin lifts are retained as data: the nontrivial central spin transformation remains distinct even when it acts trivially on tensor records or projective skies. An Einstein embedding is an open isometric embedding with its specified spin lift. The contact carrier has complex dimension five. A full complex sky is a compact quadric $\mathbb{P}^1 \times \mathbb{P}^1$ with ordered rulings. Completeness of the point family means the local four-dimensional completeness specified in Proposition 2.2, together with retention of the whole quadric over every point of the chosen domain.

Proposition 2.1 (Original source correspondence).
The following local statements are drawn from Definition 2.1, Theorems 3.1, 6.3, 8.3 and 9.10–9.12, and Proposition 8.4 of the local paper. Their proofs are in the local paper.

Let

$$T_0 = \mathrm{Tot}(\mathcal{O}(1)^{\oplus 2}) \longrightarrow \mathbb{P}(S')$$

be the marked flat seed as a germ along its entire zero line. For integer m, let $\mathscr{H}_m(T_0)$ be the two-chart analytic Čech group in the direct limit over neighborhoods of that line. With north coordinates (w, ζ) and south coordinates $(w/\zeta, 1/\zeta)$, the fixed differential is

$$(\delta a)_{NS}(w, \zeta) = \zeta^m a_S(w/\zeta, 1/\zeta) - a_N(w, \zeta). \tag{4}$$

At a fixed Einstein coefficient $\Lambda \in \mathbb{C}$, the original input domain is

$$\mathcal{A}_\Lambda = \mathscr{H}_2(T_0) \oplus \mathscr{H}_{-6}(T_0). \tag{5}$$

The fixed residue and middle-Laurent projections identify these classes with their unique canonical analytic representatives and with independent complete curvature records $U = (U^+, U^-)$. They give an analytic quotient correspondence, including analytic coboundaries and common smaller representatives for bounded holomorphic families. Adding a source coboundary leaves the canonical pair and its prescribed normalized output unchanged (Isthmus 2026a, Theorem 3.1 and §8.1).

Let $\mathfrak{O}_\Lambda$ be the independently admitted local operator domain recalled in Proposition 2.3. The original source constructor $\mathcal{S}_{\mathrm{op}}$ and its decoder $\mathcal{D}_{\mathrm{op}}$ satisfy

$$\mathcal{D}_{\mathrm{op}}\mathcal{S}_{\mathrm{op}} = 1_{\mathcal{A}_\Lambda}, \qquad \mathcal{S}_{\mathrm{op}}\mathcal{D}_{\mathrm{op}} \simeq 1_{\mathfrak{O}_\Lambda}. \tag{6}$$

The first equality is on the original marked input classes and their Einstein coefficient. The second is a comparison of operator presentations, with the supplied and reconstructed associators, framed module maps, torsor, optical root and point incidence retained. These comparisons are natural on physical point/full-spin Hom classes. Both domains contain arbitrary independent chiral records, every algebraic Weyl type and $\Lambda = 0$ (Isthmus 2026a, Theorems 9.10–9.11).

The physical source action is defined directly on the complete records by

$$R_{a,A}U = A^*\left(\mathrm{Fl}^1_{H_a}\, U\right), \tag{7}$$

where H_a is the original constant-direction Cartan record field, a is an admissible short marked displacement and A is an actual full spin element. Admissibility requires that the indicated flow and marked maps lie in their retained parents. If $V = R_{a,A}U$, the associated normal geometric map is

$$\Xi_{a,A} : g[V] \longrightarrow g[U], \qquad v \longmapsto \mathrm{Exp}^{g[U]}_{x_a}(\nu_a A v), \tag{8}$$

where $x_a = \mathrm{Exp}_o(a)$ and ν_a is initialized radial spin transport. The covariant source instruction $U \to V$ uses $\Xi^{-1}_{a,A}$. The geometry in this formula is derived from the source; it is not an additional admission datum.

For successive admissible instructions the law is

$$\begin{aligned} (b, B) \circ (a, A) &= (h_U(a, Ab), K_U(a, Ab)AB), \\ (a, A)^{-1} &= (-A^{-1}a, A^{-1}) \quad \text{at } V. \end{aligned} \tag{9}$$

Here h_U is the endpoint displacement read in the initial marked normal coordinates, and K_U is the initialized spin rotation comparing direct radial transport with the corresponding broken transport. These laws include identities and associativity wherever the finite diagram is admitted. Every physical local isometry with its specified full spin lift has a unique such short instruction in the marked representatives. The contact-affine comparisons obey the corresponding composition laws, and no independent vertical comparison remains after the physical map and full spin lift are specified (Isthmus 2026a, Proposition 8.1 and Theorem 8.3). Full spin labels remain distinct even when their action on tensor records is identical.

The original nonlinear constructor is the fixed finite source execution and prescribed normalizer specified in Proposition 5.9 and §6 of the local paper. Its outputs preserve the two original input classes, their nonlinear source operations and the six scale normalizations through the natural comparisons in Theorem 9.12 of the local paper. The negative class determines its dual lift; that lift is not an additional free input. The six scalar choices fix one value, four gradient components and the Einstein coefficient of the selected section. Their components are recomputed in each auxiliary scale and frame, with the Cauchy-state transformation; preserving them means preserving this selected-section comparison, rather than equality of raw chart components (Isthmus 2026a, Theorem 5.5 and Proposition 6.2).

For bounded holomorphic input families with finite-dimensional complex parameter bases and each fixed finite conditioned comparison diagram, the constructor, decoder, normalized representatives, their inverses and comparisons have common smaller analytic parents and the stated finite-jet dependence. The conditions retain positive coordinate and conic buffers, bounds for maps and inverses, positive unit margins, and parents containing every specified finite composite. No uniform estimate on all global charts or unbounded comparison histories is part of this input.

The Lorentz operation conjugates the complete records and exchanges their spin factors and chiralities, with Λ conjugated. The exact residue dictionaries transfer this operation to the original classes while retaining their weights 2 and -6. At real Λ, its fixed records and full real spin instructions give the Lorentz restriction. The operator comparison carries the specified weak square and module coherence of Proposition 6.5.

The source category retains original cohomology classes, their prescribed representatives and physical instructions. Its raw-representative and atlas conventions are specified in Definition 5.1; the Hamiltonian and parameter conventions are stated after Proposition 5.4. Local comparisons use common refinements near the whole retained compact sky.

Proposition 2.2 (Independent geometric recognition).

The independently admitted local objects used below are those of Definition 2.2 of the local paper. Such an object has a holomorphic contact fivefold germ with a complete regular four-dimensional family of embedded Legendrian

quadrics $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$, ordered rulings, and contact line of restriction $\mathcal{O}(1,1)$. Its incidence is $p : F \to U$, $q : F \to Z$, with one-dimensional q-fibers projecting nontrivially to U. It also has a quadratic rank-six holomorphic bundle $(\mathcal{V}, H)$, a section I with $H(\mathsf{I},\mathsf{I}) = -\Lambda/3$, and an affine line subbundle $\mathscr{A} \subset \mathcal{V}$. The bundle is trivial on the central sky, the translation line restricts to $\mathcal{O}(-1,-1)$, and

$$H(Y,Y) = 0, \qquad H(\mathsf{I},Y) = 1 \qquad (Y \in \mathscr{A}). \tag{10}$$

Complete-sky evaluation reconstructs a rank-six bundle $\mathcal{T}$ on U, a connection D, and a section X, with an evaluation isomorphism $p^*\mathcal{T} \simeq q^*\mathcal{V}$. Put $W = (\mathsf{I},\mathsf{X})^\perp$ and $\Theta(v) = D_v\mathsf{X}$. Admission requires $\Theta : TU \to W$ to be invertible, $F^D\mathsf{X} = 0$, and the Ricci contraction of

$$(v,w,u) \longmapsto \Theta^{-1}\,\mathrm{pr}_W\, F^D(v,w)\Theta(u) \tag{11}$$

to vanish. The connection in these conditions is reconstructed by inverse null transport; it is not freely supplied.

The local recognition theorem constructs $g(v,w) = H(\Theta(v),\Theta(w))$, proves $\mathrm{Ric}(g) = \Lambda g$, and identifies the supplied object with the canonical ray geometry of g. Evaluation intertwines D with the constant partial connection along q-fibers, takes X to the supplied affine position, and preserves H, I and the spin marking. Conversely, every holomorphic oriented spin Einstein four-germ has such a holomorphic whole-sky model, without restrictions on chirality or algebraic Weyl type. For the compatible Lorentz real form, Lemma 2.4 below proves reality of the reconstructed metric; its retained tangent calibration gives signature $(+,-,-,-)$ and real skies S^2. The converse retains this reality. Enriched comparison germs are in bijection with holomorphic local isometries carrying a specified spin lift, compatibly with restriction and composition; their compatible real restrictions give the corresponding analytic Lorentz comparisons.

The holomorphic assertions are the results of Theorem 2.4, Propositions 4.1–4.3 and Theorem 8.3 of the local paper, in the forms used here. The Lorentz converse and real-sky description use the constructions in §8.4 and Proposition I.3 of the local paper. In particular, Proposition 4.3 of the local paper proves recognition on independently supplied objects and both comparison directions. The hypotheses concern the supplied incidence, bundle, affine calibration and reconstructed normality conditions, rather than membership in a generated image.

Proposition 2.3 (Independent operator recognition).

The following local inputs are drawn from Definition H.1, Lemma H.2, Proposition H.3, Theorem H.4 and Theorem G.5 of the local paper.

An independently admitted operator object has analytic half-density algebras with their order filtrations, full graded-symbol identifications and complex-linear formal adjoints, including the specified square and degree-zero duality coherences. On presentation overlaps its algebra maps and

normalized units obey

$$\begin{aligned} \varphi_{ab}\varphi_{bc} &= \mathrm{Ad}(C_{abc})\varphi_{ac}, \\ C_{abc}C_{acd} &= \varphi_{ab}(C_{bcd})C_{abd}, \\ C^{*}_{abc}C_{abc} &= 1, \qquad \sigma_0(C_{abc}) = 1. \end{aligned} \tag{12}$$

Its symbols determine a separated contact germ Y along a compact characteristic quadric Q, with ordered rulings and contact line $\mathcal{L}|_Q = \mathcal{O}(1,1)$. The intrinsic first-unit sheaf is identified with $\mathcal{L}^{-1}$, and a torsor for this sheaf is retained as substantive affine data. The calibrated affine object satisfies the optical root, projective-extension and Einstein-scale recognition conditions in Definition 5.6 of the local paper. A compact locally standard order-zero simple module with reduced multiplicity-one characteristic Q determines a complete four-dimensional nearby point family. Its half-canonical root has its actual square identification and any retained base factor; its framing is a nonzero local-system vector along the calibrated cone for the whole parameter family. It is distinct from the optical density root.

Taking full symbols, the torsor and the characteristic gives an equivalence with the independently admitted calibrated affine whole-sky objects. Both inverse comparisons are comparisons of presentations. Physical Hom sets are taken modulo normalized comparison isomorphisms and retain the specified full spin lift. Actual structured comparisons retain both algebra associators and the semilinear framed-module maps; their defining equations are (35).

Both associators occur in these equations. Normalization of algebra comparison units is separate from module framing: the module maps are semilinear isomorphisms carrying the whole local-system vector, and are not additionally restricted to principal scalar one. Over a fixed admitted physical map and actual full spin label, the structured comparison groupoid is contractible. Its uniqueness includes the framed module comparison (Isthmus 2026a, Lemma H.2, Theorem H.4, Proposition G.4 and Theorem G.5). These statements commute with restriction and common refinement and hold on the conditioned finite diagrams and bounded local holomorphic families of the cited domain.

These assertions are external inputs, not new quantization or module existence results. The admissible point modules are the locally standard framed modules specified above; arbitrary coherent or flat relative modules are not included. In particular, Theorem H.4 of the local paper does not turn an arbitrary contact germ into a physical map of complete point objects.

Lemma 2.4 (Conjugation and Lorentz recognition).
Independent whole-sky recognition commutes with compatible antiholomorphic comparison germs of the incidence and quadratic-affine data, including their actual spin identifications. In particular, if such a comparison is a reality involution over $\sigma : U \to U$ at real Λ, the reconstructed metric satisfies

$$\sigma^{*}\overline{g} = g. \tag{13}$$

If the reality has the prescribed Lorentz spin/tangent calibration at a fixed mark, its real restriction has signature $(+,-,-,-)$ near that mark and its fixed sky is S^2. Every marked real analytic Lorentz Einstein germ has a compatible whole-sky model with this calibration.

Proof. Apply the reconstruction in Isthmus (2026a, Proposition 4.3) to a model and its conjugate. On each complete compact sky, conjugation carries holomorphic sections of $q^*\mathcal{V}$ to holomorphic sections on the conjugate sky. Hence it induces the comparison of $\mathcal{T} = p_* q^*\mathcal{V}$, and evaluation intertwines this comparison with the supplied one on $q^*\mathcal{V}$. The descended H and I are therefore preserved. The affine torsor on a sky has translation line $\mathcal{O}(-1,-1)$, whose H^0 and H^1 vanish. Its unique section is taken to the unique section on the conjugate sky, so the reconstructed X is preserved as well.

The constant partial connection along the q-fibers commutes with the comparison, since it is pulled back from the ray bundle. In an evaluation frame, inverse null transport reconstructs its coefficients as the unique sections of $\mathrm{End}(\mathcal{T}) \otimes \mathcal{O}(1,1)$ representing linear functions of the tangent direction. Conjugating the reconstructed connection gives a connection with these same prescribed restrictions on every null direction. Uniqueness of that reconstruction therefore intertwines D with the conjugate connection. This also shows naturality for antiholomorphic comparisons between distinct models and for their composites.

For an involution, use these induced bundle identifications to write

$$\sigma^*\overline{H} = H, \qquad \sigma^*\overline{\mathsf{I}} = \mathsf{I}, \qquad \sigma^*\overline{\mathsf{X}} = \mathsf{X}, \qquad \sigma^*\overline{D} = D.$$

Consequently $\Theta = D\mathsf{X}$ is equivariant and $g = H(\Theta,\Theta)$ gives the asserted identity. Thus g restricts to a real nondegenerate metric on the fixed point locus. At a fixed mark, the prescribed Lorentz spin/tangent calibration identifies its real tangent form with $\eta = \mathrm{diag}(1,-1,-1,-1)$. Nondegeneracy makes this signature constant after restriction. Recognition identifies the complete sky with the projective null quadric; its fixed locus is the projectivized real Lorentz null cone, namely S^2. The spin identifications, including their central signs, are retained throughout.

Conversely, the construction in §8.4 of the local paper realizes every marked real analytic Lorentz Einstein germ with this tangent calibration. Its canonical whole-sky model in Proposition 4.2 of the local paper is constructed by null geodesic flow and quadratic-affine transport. The initialized holomorphic ODEs commute with conjugation, so their uniqueness gives the required reality on that model. The fixed-sky description agrees with the Lorentz spinor calculation in Proposition I.3 of the local paper. □

For the Lorentz Cauchy application, fix a connected oriented compact analytic three-manifold Σ without boundary, an analytic positive-definite metric h, a symmetric analytic tensor K, and a specified spatial spin structure P_Σ. With $K(X,Y) = -g(\nabla_X n, Y)$ for the future unit normal, the constraints are

$$\mathrm{Scal}(h) + (\mathrm{tr}_h K)^2 - |K|_h^2 = -2\Lambda, \qquad \mathrm{div}_h K - \mathrm{d}\,\mathrm{tr}_h K = 0. \tag{14}$$

A Cauchy hypersurface meets every inextendible nonconstant causal curve exactly once. The category $\mathsf{Ein}^{\omega}_{\Lambda}(\Sigma, h, K)$ consists of analytic spin globally hyperbolic developments with the prescribed initial identification; its morphisms preserve that identification. The spin structure P_{Σ} is suppressed only in the category notation. Appendix A defines this category precisely, constructs an analytic collar, and proves the maximal analytic development statement used in §7. The local correspondence above is applied to the analytic Einstein germs of these developments.

The principal notation is as follows.

Symbol	Meaning
T_0, $\mathcal{A}_{\Lambda}$	The fixed local flat germ and its two original input classes
$R_{a,A}$	The nonlinear physical change of marked point and full spin frame
B, M	A holomorphic point manifold and its designated Lorentz real locus
$p : F \to B$, K, E	The full compact sky family, its null characteristic line and its even-contact hyperplane
$\mathrm{Mon}(K)$	Null paths modulo homotopy with endpoints fixed
$\mathcal{T}$, H, I, X	The rank-six quadratic bundle, its form, Einstein scale and affine position
$\mathcal{T}_{\mathrm{LT}}$	The rank-four normal local-twistor bundle with spin lift
$\mathscr{A}$	The calibrated affine line of positions along a ray
$\mathscr{E}_{\mathfrak{X}}$, $\mathscr{E}^{\mathrm{inc}}_{F}$, $\mathscr{M}_{F}$	The contact operator algebroid, its relative ambient germs on incidence, and their point-module family
$\mathsf{SrcAtl}^{\mathrm{hol}}_{\Lambda}$	Original-source atlases with their physical comparison germs
$\mathsf{MonRay}^{\mathrm{hol}}_{\Lambda}$, $\mathsf{OpMon}^{\mathrm{hol}}_{\Lambda}$	Independently admitted geometric and operator monodromy sources
M_{ω}	The maximal analytic Cauchy subdevelopment of the smooth maximal development

Local inputs to the global results. The table records the main dependencies on the local paper; the cited propositions above and the proofs below specify their hypotheses and uses. Entries for the global theorems include the local inputs used by their intermediate lemmas.

Global result	Results in the local correspondence	Role
Theorem 3.11	Theorem 2.4, Propositions 4.1–4.3, Theorem 8.3	Whole-sky recognition and the local spin-isometric Hom correspondence
Theorems 4.2 and 4.6	Lemmas F.8 and H.2, Proposition H.3, Theorem H.4	Analytic structured quantization comparisons and independent operator recognition
Theorem 4.8	Lemmas G.1–G.2, Proposition G.4, Theorem G.5, Proposition H.3	Relative simple modules, framed comparisons and central specialization
Lemma 5.3 and Proposition 5.4	Theorems 3.1 and 3.3; Theorem 5.5 and Proposition 5.9; Propositions 6.1–6.2 and Theorem 6.3; Proposition 8.1 and Theorem 8.3; Theorem G.5, Proposition H.5 and Lemma K.16	Source dictionaries, nonlinear execution, finite-diagram convergence and rebasing; Lemma K.16 supplies the zero-scalar amplitude model
Theorem 5.6 and Proposition 5.7	Theorems 2.4, 6.3 and 8.3; Proposition 9.9 and Theorems 9.10–9.12	Original-source construction, decoding and the two presentation-level comparisons, using the preceding analytic lemmas
Theorem 6.3 and Propositions 6.4–6.5	Proposition I.3, Theorem J.2, Propositions J.3–J.4 and J.7, Theorem J.6, Theorem G.5	Spin, optical, infinity and Lorentz comparisons, including the module square
Theorem 7.4 and Corollary 7.5	The preceding local inputs, including §8.4 and Proposition 8.4	Lorentz admission and the real restrictions of the source and operator comparisons

The global integration and descent arguments, including the correspondence of germs along the entire real development, are carried out in Sections 3–7. Section 8 proves the Nariai obstructions directly; Appendix A constructs

the analytic Cauchy category and its maximal development, using the smooth Cauchy theorem stated there.

3. Global null incidence and monodromy

We construct the geometric object on which the operator and original-input atlases will be defined. The point manifold and its full quadric incidence remain Hausdorff and second countable. The arrow space of a Lie groupoid may be non-Hausdorff; quotient notation denotes that presentation with its equivariant data. The local recognition input is Proposition 2.2, in the form proved in Propositions 4.2–4.3 of the local paper.

Proposition 3.1.
Let (M, g) be a Hausdorff second-countable real analytic oriented spin Lorentz four-manifold with $\mathrm{Ric}(g) = \Lambda g$. It has a Hausdorff second-countable complexification B with an antiholomorphic involution whose fixed locus is exactly M, and a nondegenerate holomorphic Einstein metric g_B extending g. The orientation, specified spin structure and Einstein scale extend. These data are unique as germs along all of M.

The full projective null bundle

$$F = \{(x, [k]) \in \mathbb{P}(TB) : g_B(k, k) = 0\}, \qquad p : F \longrightarrow B, \tag{15}$$

is a proper holomorphic submersion with fiber $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Its nonsingular null-geodesic line field $K \subset TF$ projects to the tautological null direction. Its real fixed locus $F_{\mathbb{R}}$ is the real projective null phase of M.

Proof. The complexification theorem of Whitney and Bruhat (1959, sec. 1, Proposition 1 and the following remark) supplies a complexification, unique as a germ, with the stated real involution. Choose a countable representative and shrink around the entire closed real locus. Analytic tensor coefficients extend in complexified coordinate charts, and their transition identities extend by the identity theorem on the maximal totally real slice. A common shrinking gives a holomorphic metric. Its determinant is nonzero on M, so it remains nonzero on a smaller neighborhood of all of M. The holomorphic Einstein tensor vanishes there by the same identity theorem.

For the spin extension, choose local analytic lifts of the oriented orthonormal frame transitions. Their identities extend holomorphically, including the prescribed discrete central signs. They glue a spin double covering of the extended frame bundle. Conjugation extends with its Lorentz action on the two chiral factors. Local uniqueness followed by a common shrinking proves uniqueness of the tensor, spin and reality data as germs along M.

Nondegeneracy of the metric gives the locally trivial smooth quadric bundle in the statement; compactness of its fiber makes p proper. The Levi-Civita connection supplies the projective geodesic spray. Its projection is a nonzero representative of the null direction, so K is nonsingular. Conjugation preserves it, and a fixed projective direction has a real representative. All

shrinkings concern neighborhoods of the entire real manifold; no uniform complex radius or geodesic completeness is required. □

For an object of the analytic Cauchy category of Definition A.1, with its specified compact Cauchy hypersurface Σ, let n be the future unit normal and $h = -g|_{T\Sigma}$. A real ray is a maximal future-oriented null geodesic modulo orientation-preserving affine reparametrization. Write $\mathcal{N}$ for their set. The smooth contact identification with a Cauchy cosphere bundle is classical (Chernov 2018, sec. 1); the affine domain will also be needed.

Proposition 3.2.

The ray space, with its quotient topology from $F_{\mathbb{R}}$, is a compact analytic contact five-manifold canonically identified with $S^*_h\Sigma$. Normalize each ray by

$$\gamma(0) \in \Sigma, \qquad g(\dot\gamma(0), n) = 1. \tag{16}$$

If I_γ is its maximal affine interval, then

$$D_M = \{(\gamma, s) \in \mathcal{N} \times \mathbb{R} : s \in I_\gamma\} \tag{17}$$

is open and the geodesic flow is an analytic diffeomorphism

$$\Phi : D_M \longrightarrow F_{\mathbb{R}}, \qquad (\gamma, s) \longmapsto (\gamma(s), [\dot\gamma(s)]). \tag{18}$$

Each sky $Q_x = \{\gamma : x \in \gamma\}$ is an embedded analytic Legendrian two-sphere.

Proof. A maximal null geodesic is inextendible as a causal curve. Indeed, near a putative causal endpoint choose a coordinate with timelike time differential. Parametrized by this time, its directions satisfy a regular projective geodesic equation on a compact direction sphere. Bounded derivatives give a limiting direction, and the logarithm of its affine speed has bounded time derivative. Its affine tangent thus has a finite nonzero limit, extending the geodesic. The Cauchy property now gives one intersection with Σ.

The normalized initial tangent is $n + v$ with $h(v, v) = 1$; associate the covector $h(v, \cdot)$. Analytic dependence of the geodesic equation gives the open maximal domain D_M. Unique Cauchy intersection and geodesic uniqueness make Φ bijective; a return to the same phase point would give a closed causal curve. Its inverse is analytic locally: the finite geodesic segment to Σ is covered by analytic flow maps, and its hit on Σ is transverse because $g(k, n) > 0$ for future null k. The normalization and affine travel time are analytic. These local inverses agree. Projection $D_M \to S^*_h\Sigma$ is an open submersion with interval fibers, proving the quotient-topology assertion and compactness.

The normalized tangent defines $\alpha = g(k, \mathrm{d}p\,\cdot)$ on the real phase. For a variation J through normalized affine null geodesics,

$$\frac{\mathrm{d}}{\mathrm{d}s} g(\dot\gamma, J) = g(\dot\gamma, \nabla_J \dot\gamma) = \tfrac{1}{2} J g(\dot\gamma, \dot\gamma) = 0.$$

Consequently $\Phi^*\alpha = -\operatorname{pr}^*_{\mathcal{N}} \lambda$, with λ the tautological contact form on $S^*_h\Sigma$. The sign follows from the Lorentz signature and the chosen covector. The ray projection on $F_{\mathbb{R},x} \simeq S^2$ is injective, since a ray cannot return to x. Its differential is injective because its kernel projects nontrivially to TM,

whereas $TF_{\mathbb{R},x}$ is vertical. Compactness gives an embedding, and α vanishes on this vertical tangent space, proving the Legendrian assertion. □

In the specified Einstein scale, the canonical rank-six bundle is $\mathcal{T} = \mathbb{R} \oplus TM \oplus \mathbb{R}$, with

$$\begin{aligned} H((\sigma,\mu,\rho),(\sigma',\mu',\rho')) &= \sigma\rho' + \rho\sigma' + g(\mu,\mu'), \\ D_v(\sigma,\mu,\rho) &= \big(v\sigma - g(v,\mu), \\ &\qquad \nabla_v\mu + ((\Lambda/6)\sigma + \rho)v, \\ &\qquad v\rho - (\Lambda/6)g(v,\mu)\big), \\ \mathsf{X} = (0,0,1), &\qquad \mathsf{I} = (1,0,-\Lambda/6). \end{aligned} \tag{19}$$

These are the normal standard-tractor formulas of the local correspondence. They also define the holomorphic data after complexification.

Lemma 3.3.
The connection D preserves H and I. With $\Theta(v) = D_v\mathsf{X} = (0,v,0)$,

$$\begin{aligned} H(\mathsf{I},\mathsf{I}) &= -\Lambda/3, \\ H(\mathsf{I},\mathsf{X}) &= 1, \\ H(\mathsf{X},\mathsf{X}) &= 0, \\ D_v\Theta(w) &= \Theta(\nabla_v w) - g(v,w)\mathsf{I} - (\Lambda/3)g(v,w)\mathsf{X}. \end{aligned} \tag{20}$$

Along every affine null geodesic, $D_k^2\mathsf{X} = 0$ and $D_k\mathsf{X} \neq 0$.

Proof. Substitution in the derivative of H cancels the scalar and middle-component terms, proving metric compatibility. Substitution of I gives $D\mathsf{I} = 0$. The pairings follow directly. Differentiating $(0,w,0)$ gives $(-g(v,w), \nabla_v w, -\Lambda g(v,w)/6)$, which is the stated identity. For an affine null tangent, both $\nabla_k k$ and $g(k,k)$ vanish, while Θ is injective. □

Theorem 3.4.
There exist an analytic quadratic rank-six bundle $(\mathcal{V}, H)$ on $\mathcal{N}$, a scale section I, and an affine line subbundle $\mathscr{A} \subset \mathcal{V}$ satisfying $H(Y,Y) = 0$ and $H(\mathsf{I},Y) = 1$. Affine position gives an open embedding

$$j : F_{\mathbb{R}} \longrightarrow \mathscr{A} \tag{21}$$

whose image $\mathscr{D}_M$ retains exactly the intervals I_γ. The lifted skies $j(F_{\mathbb{R},x})$ are disjoint embedded two-spheres with normal bundle $T_xM \otimes \mathbb{R}_{S^2}$.

Proof. Pull $\mathcal{T}|_\Sigma$ back by the initial-intersection map $e_\Sigma : \mathcal{N} \to \Sigma$. Let $P_\gamma(s)$ be tractor transport from zero to s. Finite analytic transport equations along a compact parameter segment show that it is analytic jointly on D_M. By Lemma 3.3,

$$P_\gamma(s)^{-1}\mathsf{X}(\gamma(s)) = \mathsf{X}_0(\gamma) + sV_\gamma, \qquad V_\gamma = \Theta(\dot\gamma(0)) \neq 0. \tag{22}$$

Define $\mathscr{A}_\gamma = \mathsf{X}_0(\gamma) + \mathbb{R}V_\gamma$. This gives an analytic trivialization $a : \mathcal{N} \times \mathbb{R} \to \mathscr{A}$. The tractor pairings imply its calibration, since V_γ is null and orthogonal to X_0 and I. Set $j = a\Phi^{-1}$. Its image is $a(D_M)$, so j is an open embedding. The lifted skies are the images of the distinct fibers of $F_{\mathbb{R}} \to M$. Their normal

bundles follow from the differential of that submersion. The complete affine fiber is larger than its realized part whenever $I_\gamma \neq \mathbb{R}$. □

Proposition 3.5.
A spin isometric Cauchy embedding $f : M \to M'$ preserving orientation and time orientation and identifying the specified initial hypersurfaces induces a contact diffeomorphism of their real ray spaces and a quadratic affine-bundle isomorphism preserving I. Under the Cauchy normalization,

$$(f_{\mathcal{N}} \times \mathrm{id})(D_M) \subset D_{M'}, \qquad f_{\mathscr{A}} \circ j = j' \circ F(f). \tag{23}$$

These maps preserve lifted skies and composition, with the specified spin lift retained.

Proof. The Cauchy isometry identifies cosphere bundles, future normals and normalized initial tangents. Geodesic uniqueness gives the interval inclusion. The map $(\sigma, \mu, \rho) \mapsto (\sigma, \mathrm{d}f(\mu), \rho)$ intertwines the canonical tractor connections and the initial affine generators. Transport uniqueness gives the second identity and composition. The full spin lift remains separate morphism data, including its central sign. □

For a holomorphic Einstein fourfold B, retain the entire phase F, not only its real locus. The tautological even-contact hyperplane distribution $E \subset TF$ has characteristic line K, and its Levi form is nondegenerate on E/K. This follows from the characteristic construction on the null hypersurface in T^*B.

Definition 3.6.
The null monodromy groupoid $\mathcal{G} = \mathrm{Mon}(K) \rightrightarrows F$ has as arrows the endpoint-preserving homotopy classes of piecewise smooth paths in a single complex null leaf. Composition concatenates paths and inversion reverses them. Its analytic charts are the path-continuation charts of the foliation groupoid. Distinct path classes remain distinct even if their transverse holonomy germs coincide.

Proposition 3.7.
The groupoid $\mathcal{G}$ is holomorphic of complex dimension seven, with source-connected, source-simply-connected fibers. A countable disjoint union T of local holomorphic transversals meeting every leaf gives an étale groupoid

$$\mathcal{G}_T = T \times_{F,t} \mathcal{G} \times_{F,s} T \rightrightarrows T$$

with an invariant contact structure on its five-dimensional object space. Complete transversal systems give equivalent presentations through their common union.

Proof. The source fiber at u is the universal cover of its leaf. This is the foliation case of source-simply-connected integration (Crainic and Fernandes 2003, sec. 0, pp. 577–579). The line algebroid K has injective anchor $K \hookrightarrow TF$. Its integration is holomorphic by the integration theorem (Laurent-Gengoux et al. 2009, Theorem 3.17), and has dimension $\dim F + \operatorname{rank} K = 7$. Explicitly, a finite chain of holomorphic flow boxes along a path

gives holomorphic continuation charts, multiplication and inversion, without requiring the arrow space to be Hausdorff.

Choose a countable cover by flow boxes and one transversal in each. Transversality makes the source and target maps of $\mathcal{G}_T$ local biholomorphisms. The distribution induced by E on each transversal is contact because its Levi form is the nondegenerate form on E/K; leafwise continuation preserves it. In a common union of two systems, each inclusion is fully faithful and meets every orbit. Flow boxes supply the local submersions required for the equivalence of presentations. □

Theorem 3.8.

Every holomorphic oriented spin Einstein fourfold has the contact monodromy presentation above, an equivariant quadratic rank-six bundle with scale section, and an invariant calibrated affine line whose translation line is the inverse contact line. Its proper full-quadric incidence has an affine-position morphism

$$j : F \longrightarrow [A_F/\mathcal{G}] \simeq [A_T/\mathcal{G}_T] \tag{24}$$

that is étale in local groupoid charts. The prescribed point spin structure and local whole-sky models, with full-spin comparisons, are retained. The construction is natural for holomorphic spin isometries. No separated leaf quotient, trivial leafwise holonomy or injective affine position is assumed.

Proof. Put $\mathcal{E} = p^*\mathcal{T}$ with its partial connection along K. Its curvature vanishes because K has complex rank one. Parallel transport therefore depends only on the endpoint-preserving leafwise homotopy class (Crainic and Fernandes 2003, sec. 1.4, Proposition 1.6), and gives a representation

$$P_\gamma : \mathcal{E}_{s(\gamma)} \longrightarrow \mathcal{E}_{t(\gamma)}.$$

It is holomorphic by solving the linear transport equation in a finite chain of holomorphic flow boxes, and preserves H and I.

For $u = (x, [k])$, set

$$(A_F)_u = \mathsf{X}_x + \mathbb{C}D_k\mathsf{X}_x. \tag{25}$$

The line is independent of the representative k, and its nonzero translation vector and calibration follow from Lemma 3.3. Since $D_k^2\mathsf{X} = 0$ along affine null geodesics, transport preserves the entire affine line. Thus it is $\mathcal{G}$-invariant.

The pairing $H(D_k\mathsf{X}, D_v\mathsf{X}) = g(k, v)$ identifies the translation line with the inverse contact line. It is invariant under null transport: for a commuting variation v through affine null geodesics, its derivative is $g(k, \nabla_v k) = \frac{1}{2}vg(k, k) = 0$. This is the tautological contact pairing, so it supplies a nonzero duality on each transverse chart. Restriction to T gives the asserted equivariant bundles.

The section $\mathsf{X} : F \to A_F$ defines j. Its base change against the atlas A_F is represented by

$$\Psi : \mathcal{G} \longrightarrow A_F, \qquad \Psi(\gamma) = P_\gamma \mathsf{X}_{p(s(\gamma))} \in (A_F)_{t(\gamma)}. \tag{26}$$

This map covers t. The one-dimensional kernel of $\mathrm{d}t$ is identified by $\mathrm{d}s$ with K at the source, since the anchor is injective. Varying the source while fixing the target differentiates Ψ to the transport of $D_k\mathsf{X}$, which is nonzero. Together with the identity on the six-dimensional base this makes $\mathrm{d}\Psi$ invertible. Hence Ψ is locally biholomorphic and j is étale. This argument gives neither injectivity nor the monomorphism property.

Near a point of B, the original local correspondence supplies a whole-sky short-ray model. Inclusion of its paths gives a functor from its short-leaf monodromy groupoid to $\mathcal{G}$. Its affine comparison is the same parallel transport, so its recognition data agree with the retained global data. Extra arrows can come from paths leaving the chart. The point spin bundle and its comparison cocycle are retained on B. A spin isometry sends null paths and homotopies to their images and intertwines tractor transport; therefore all these constructions, including the full spin lift, are natural. □

Proposition 3.9.

For a holomorphic extension of an analytic globally hyperbolic development, the designated real path groupoid is

$$\mathcal{G}_{\mathbb{R}} = \mathrm{Mon}(K_{\mathbb{R}}) \simeq F_{\mathbb{R}} \times_{\mathcal{N}} F_{\mathbb{R}}.$$

Inclusion of real paths defines a conjugation-compatible functor $\mathcal{G}_{\mathbb{R}} \to \mathcal{G}$. Its quadratic and affine descent is exactly Theorem 3.4.

Proof. The real leaves are intervals by Proposition 3.2, so there is one real path class between any two points on a leaf. The real ray submersion gives the displayed analytic groupoid. Restricting the canonical transport to these paths is precisely the Cauchy-normalized transport used in Theorem 3.4. □

The designated groupoid uses paths in the real spacetime. The full restriction of $\mathcal{G}$ to real objects can also contain nonreal paths and additional isotropy, and need not be this real path groupoid.

We now define the independent geometric source. Let $\mathsf{Ein}^{\mathrm{hol}}_{\Lambda}$ denote holomorphic oriented spin Einstein fourfolds with connected Hausdorff second-countable point manifolds and holomorphic spin isometric embeddings as morphisms.

Definition 3.10.

An independent monodromy source consists of the following data.

(1) A connected Hausdorff second-countable complex fourfold B, a proper holomorphic submersion $p : F \to B$ with quadric fibers Q and ordered rulings, a line subbundle $K \subset TF$, and a hyperplane distribution $E \subset TF$. The map $\mathrm{d}p|_K$ is nonzero; E contains K and $\ker \mathrm{d}p$, has characteristic line K, and has nondegenerate Levi form on E/K.

(2) A holomorphic Lie groupoid $\mathcal{G} \rightrightarrows F$ with Hausdorff connected simply connected source fibers and Lie algebroid identified by its anchor with K. It has a rank-six representation $\mathcal{V}_F$, an invariant nondegenerate quadratic form H, an invariant section I of norm $-\Lambda/3$, and an invariant affine line subbundle A_F. Its calibration is $H(Y, Y) = 0$

and $H(\mathsf{I}, Y) = 1$. A holomorphic position section $\mathsf{X} : F \to A_F$ is supplied; it need not be invariant.

(3) An atlas of independently admitted whole-sky models from Proposition 2.2. Over $U_i \subset B$, each model identifies its full incidence with $F_i = p^{-1}U_i$, its distributions with K, E, and its pulled-back quadratic, affine and position data with the supplied data on F_i. Writing $q_i : F_i \to Z_i$ for its ray projection and C_i for its contact distribution, these identifications give $K|_{F_i} = \ker \mathrm{d}q_i$ and $E|_{F_i} = (\mathrm{d}q_i)^{-1}C_i$. Its point family is complete and four-dimensional near each retained quadric. Actual enriched whole-sky comparison germs on $U_i \cap U_j$ cover its identity and are compatible with the supplied identifications of incidence, distributions, quadratic-affine data and position. They obey identity, inverse and triple laws, including their actual full spin lifts. Restriction and common refinement identify presentations.

(4) The representation agrees with each local short-path action. Precisely, the canonical integrated functor

$$\eta_i : \mathrm{Mon}(K|_{F_i}) \longrightarrow \mathcal{G}$$

takes a path in F_i to an action which, under $\mathcal{V}_F|_{F_i} \simeq q_i^*\mathcal{V}_i$, is the identity on the common fiber over its local ray. On common subdomains these comparisons agree under the retained whole-sky transition. Equivalently, the representation's infinitesimal partial connection is the constant relative connection of $q_i^*\mathcal{V}_i$ along $\ker \mathrm{d}q_i$.

The functor η_i exists by integration of the line-algebroid inclusion; the equivalence in clause 4 follows from source connectedness. The clause does not identify the local groupoid with the full restriction of $\mathcal{G}$: arrows whose paths leave F_i are retained.

A morphism consists of a point embedding $f : B \to B'$, a holomorphic incidence map $\widehat{f} : F \to F'$ over f that is an isomorphism on each quadric and preserves its rulings and K, E, and a holomorphic groupoid functor $\Phi : \mathcal{G} \to \mathcal{G}'$ over $\widehat{f}$. A Φ-equivariant quadratic isomorphism $b : \mathcal{V}_F \to \widehat{f}^*\mathcal{V}'_{F'}$ preserves I, affine lines and position. These maps have admitted local whole-sky representatives with specified full spin lifts, compatible with the source and target comparison cocycles; the groupoid functor commutes with the local integrated functors. Equality includes the full spin lifts. Write $\mathsf{MonRay}^{\mathrm{hol}}_{\Lambda}$ for this category.

No metric is supplied. The optical and affine local recognition conditions are tested on the given models, and source connectedness and source simple connectedness impose global completeness of the arrows.

Theorem 3.11.
The local independent correspondence induces an equivalence

$$\mathsf{MonRay}^{\mathrm{hol}}_{\Lambda} \simeq \mathsf{Ein}^{\mathrm{hol}}_{\Lambda}. \tag{27}$$

It retains the chosen complex point manifold, complete quadric family, quadratic and affine data, and specified spin structure. It agrees locally with

the original correspondence and is fully faithful for the stated embeddings with their full spin lifts.

Proof. Recognition reconstructs a holomorphic Einstein metric on each U_i. The admitted comparisons cover the identity on each point overlap and respect the supplied global identifications. They therefore identify these metrics on the fixed overlap, so the metrics glue on B; nondegeneracy and the Einstein equation are local properties. The spin comparison cocycle glues the full spin frame bundles and orientation.

Recognition identifies every supplied quadric with the whole projective null quadric of the resulting metric g. Thus

$$\varphi : F \longrightarrow F_g, \qquad u \longmapsto \big(p(u), [\mathrm{d}p_u(K_u)]\big) \tag{28}$$

is a global holomorphic isomorphism, with inverse given on each whole-sky chart. It identifies K, E with their canonical counterparts. An injective anchor forces the algebroid bracket to be the vector-field bracket on K. The uniqueness of source-connected, source-simply-connected integration (Crainic and Fernandes 2003, sec. 0, pp. 578–579) therefore supplies the unique isomorphism

$$\Xi : \mathcal{G} \xrightarrow{\sim} \mathrm{Mon}(K_g) \tag{29}$$

inducing φ and its algebroid map. It is holomorphic by the flow-box construction, or by the holomorphic integration result (Laurent-Gengoux et al. 2009, Proposition 3.20). This identifies all arrows, including those leaving any individual point chart.

Complete-sky evaluation gives local isomorphisms from the canonical $p^*\mathcal{T}_g$ to $\mathcal{V}_F$. Their naturality for the comparisons glues them to

$$\beta : p^*\mathcal{T}_g \xrightarrow{\sim} \mathcal{V}_F, \tag{30}$$

preserving the quadratic form, scale and position, and taking $\mathsf{X}_g + \mathbb{C}D_k\mathsf{X}_g$ to A_F. Local recognition and clause 4 intertwine the infinitesimal partial connections. This also identifies the full groupoid actions: join a unit to any arrow by a path in its connected source fiber. Under its target map, both actions solve the same leafwise transport equation with the same initial value. Uniqueness gives equality at that arrow. Hence β is Ξ-equivariant on all of $\mathcal{G}$, and together with φ and the local full-spin comparisons identifies the entire supplied source with the canonical one.

Conversely, Theorem 3.8 gives the canonical data for any object of $\mathsf{Ein}^{\mathrm{hol}}_{\Lambda}$. Proposition 3.7 gives the source-fiber and anchor conditions. Original whole-sky models provide the independent local admission, and identity metric comparisons with the specified spin structure give their cocycles by local full faithfulness. Evaluation identifies canonical short-path transport with the constant relative connection, proving clause 4. Recognition recovers the given metric.

For morphisms, a spin isometric embedding f gives its differential map on phase and the functor $[\gamma] \mapsto [\widehat{f} \circ \gamma]$. Image homotopies prove well-definedness. This functor need not be faithful: a path can become null-homotopic in

the larger point domain. Its tractor map intertwines transport and affine position. Local full faithfulness gives the unique admitted comparisons with the specified spin lift, so these maps define a source morphism and respect composition.

An admitted source morphism decodes locally to a spin isometry; its point and spin maps glue to its supplied point embedding and full spin lift. To prove uniqueness, fix this decoded morphism. The phase identification forces $\widehat{f}$ to be its differential on null directions. Evaluation and local full faithfulness force its quadratic map and comparison germs. Its differentiated groupoid functor is then fixed. Lie II gives a unique integration because the source groupoid is source connected and source simply connected. Thus decoding is faithful and the preceding construction makes it full. Uniqueness of evaluation and integration also makes both inverse comparisons natural.□

Changing complete transversals changes only a presentation. Shrinking the point manifold to $B_0 \subset B$ instead requires recomputing $\mathrm{Mon}(K|_{p^{-1}B_0})$; it need not be the restriction of the old arrows to those endpoints. This second operation is used for germs along the entire real development in §7.

For the obstruction argument we record when an ordinary quotient can be used. Here a possibly non-Hausdorff complex manifold means a second-countable space with the usual local complex manifold charts, without the global Hausdorff axiom. A leaf relation $R \subset F \times F$ is *embedded* when it is a complex submanifold with the subspace topology, so its inclusion is an injective holomorphic immersion and a homeomorphism onto its image.

Lemma 3.12.

Let $R \subset F \times F$ be the maximal complex null-leaf relation. If R is embedded, $Z = F/R$ is a possibly non-Hausdorff complex contact fivefold and $q : F \to Z$ is an open holomorphic submersion with connected fibers. If R is also closed, Z is Hausdorff. The canonical quadratic bundle descends with its prescribed null transport exactly when transport around every leafwise loop is the identity. In that case the calibrated affine lines descend, their translation line is the inverse contact line, and affine position $j : F \to A$ is locally biholomorphic. It is an open embedding exactly when it is injective on every leaf.

Proof. At a diagonal pair the projections of R are submersions, since the diagonal is a section of each. Its fiber is one leaf and its tangent relation is $\{(v, w) : w - v \in K\}$. Finite compositions of local leafwise flows in one factor carry this conclusion to every related pair. A small transversal T has $R \cap (T \times T)$ equal to its diagonal after shrinking, because R is embedded. Thus it meets each global leaf at most once. Saturations of open sets are open, being unions of their images under local flows. The transversals consequently give quotient charts with holomorphic holonomy transitions and a countable base. The distribution E/K supplies their contact structure. If R is closed, unrelated points have product neighborhoods disjoint from R; their saturations are disjoint, proving Hausdorffness.

Necessity of trivial loop transport follows for a pulled-back bundle with constant relative connection. For sufficiency choose local holomorphic sections of q. Transport between any two chosen representatives of one fiber is unique, because the fiber is connected and loop transport is trivial. A finite chain of flow boxes along a connecting path proves holomorphic dependence. Concatenation proves the cocycle identities, yielding the bundle and its unique transport-compatible comparison. The preserved form and scale descend. Lemma 3.3 makes the affine lines invariant, and the contact pairing used in Theorem 3.8 identifies their translation line. The derivative of affine position is nonzero on K, and it covers q, so it is locally biholomorphic in all six dimensions. Equal affine images lie over the same ray; hence global injectivity is equivalent to leafwise injectivity. An injective local biholomorphism is an open embedding. □

4. Operator algebroids and parametrized incidence modules

We now retain the operator objects of the local correspondence on the global monodromy presentation. There are two descent problems. First, contact transformations along arbitrary null paths act on the structured microdifferential algebroid. Second, the calibrated simple modules near complete compact skies are compared along point overlaps and then descend on the parametrized incidence. Keeping these constructions separate allows the point family to remain complete even when its projected skies fail to be closed in a ray quotient.

4.1. Quantization of the contact groupoid. Let $\mathcal{H} \rightrightarrows T$ be an étale holomorphic Lie groupoid. Its object space is a Hausdorff second-countable complex contact fivefold, with contact distribution C and contact line $\mathcal{L} = TT/C$. Its arrow manifold may be non-Hausdorff, and every local bisection induces a contact local isomorphism. Write $\mathcal{H}_n$ for its manifold of composable n-tuples. Sheaves and stacks on the arrow spaces are understood on their ordinary open-set sites; their local analytic models are defined on Hausdorff coordinate neighborhoods.

Definition 4.1.
A structured quantization of $\mathcal{H}$ is a microdifferential algebroid $\mathscr{E}_T$ with a weak $\mathcal{H}$-action. It retains its order filtration, the full graded-symbol identification

$$\operatorname{gr} \mathscr{E}_T \simeq \left(\bigoplus_{m \in \mathbb{Z}} \mathcal{L}^m \right)^+ ,$$

and the complex-linear formal-adjoint anti-involution, including its square and degree-zero duality coherences. The superscript + denotes the algebroid associated with a sheaf of algebras. Locally this is the structured algebroid of analytic half-density microdifferential operators on a projective cotangent chart.

The weak action consists of a structured equivalence

$$\Phi : s^{-1}\mathscr{E}_T \longrightarrow t^{-1}\mathscr{E}_T$$

on $\mathcal{H}_1$, multiplication comparisons $\mu_{g,h} : \Phi_g\Phi_h \Longrightarrow \Phi_{gh}$ on $\mathcal{H}_2$, and unit comparisons. They satisfy the associativity and unit equations on the groupoid nerve. Comparisons preserve all the displayed structures and coherences. Local algebra representatives and their comparisons may use common refinements of object and arrow charts.

The quantization inputs are Kashiwara's existence theorem (Kashiwara 1996, Theorem 2) and the structured uniqueness theorem (Polesello 2005, Theorem 3.3). With the filtration, all graded symbols and the compatible adjoint coherences retained, a structured equivalence is unique up to a unique structured isomorphism. The algebra-level normalization is

$$\{P : \operatorname{ord} P = 0,\ \sigma_0(P) = 1,\ P^*P = 1\} \xrightarrow{\mathrm{Ad}} \mathrm{Aut}_{\mathrm{gr},*}(\mathcal{E}), \tag{31}$$

an isomorphism of sheaves, where the automorphisms induce the identity on all graded symbols (Polesello 2005, Lemma 1.5). Lemmas F.8 and H.2 of the local paper supply the analytic relative representatives and their conditioned parameter estimates. These are local statements; refinements permit their use without requiring one quantizer on an entire previously chosen overlap.

We use effective 2-descent for algebroid stacks, their equivalences and modules on the open-set sites of these spaces. Local algebra maps and their inner associators give the algebroid gluing data; their compatible semilinear module maps give its modules. The bimodule formulation of this construction applies without a paracompactness hypothesis (Kashiwara and Schapira 2012, sec. 2.1, Proposition 2.1.13). For structured quantization comparisons, fix the underlying contact data; for physical source comparisons, fix the physical map and spin lift, as in Proposition 2.3. In each application the relevant comparison groupoid is contractible: it is nonempty and any two comparisons admit a unique structured isomorphism. Hence any selected overlap comparisons have unique coherence isomorphisms on triples, and the two coherence composites on a quadruple overlap agree. Effective 2-descent then produces the global object and comparison, uniquely up to the prescribed structured isomorphisms. The same argument applies to equivalences of two fixed stacks, whose functors and natural transformations satisfy descent.

Theorem 4.2.

Every étale holomorphic contact groupoid as above has a structured quantization. It is unique up to a structured equivariant equivalence, and that equivalence is unique up to a unique structured equivariant isomorphism. The construction is natural for étale contact functors and invariant under common complete-transversal refinement of a holomorphic contact foliation. Neither properness of the groupoid nor a Hausdorff coarse quotient is required.

Proof. Choose the canonical structured algebroid on each component of T. On every Hausdorff arrow chart, the contact action identifies the

contact structures and contact lines pulled back by s and t. The two pulled-back algebroids therefore quantize the same contact data. Local quantized contact transformations give structured equivalences on sufficiently small arrow charts. On intersections, structured uniqueness supplies their unique comparison. The two comparison composites on a triple intersection agree by the same uniqueness. The effective descent convention above consequently glues these local equivalences to Φ on $\mathcal{H}_1$.

On $\mathcal{H}_2$, the equivalences $\Phi_g\Phi_h$ and Φ_{gh} have the same prescribed contact symbols and adjoint structure. Their unique structured comparison defines $\mu_{g,h}$ and agrees across arrow-chart refinements. On $\mathcal{H}_3$, both composites in

$$\mu_{gh,k} \circ (\mu_{g,h} * 1_{\Phi_k}) = \mu_{g,hk} \circ (1_{\Phi_g} * \mu_{h,k}) \tag{32}$$

are structured comparisons of the same equivalences, so they are equal. Here $*$ denotes horizontal composition; the formal adjoint is a separate operation. Unit and inverse comparisons follow in the same way. The argument uses ordinary sheaf and stack descent and does not require a locally finite cover of a non-Hausdorff arrow space.

For a second quantization, structured uniqueness first gives an equivalence on T. Its two composites with the actions have the same contact symbols, and hence a unique comparison on $\mathcal{H}_1$. Uniqueness on $\mathcal{H}_2$ gives its equivariance, and uniqueness again compares any two such equivariant equivalences. This proves the assertion for weak actions without replacing them by strict actions on chosen algebra sheaves.

An étale contact functor pulls back all the structures. Uniqueness gives the comparison with the quantization constructed on its source, and also its identity and composition laws. For two complete transversal systems, take their common union as in Proposition 3.7. Restriction from that groupoid gives both quantizations. The complete-transversal inclusions are fully faithful and essentially surjective; their local arrows transport objects and morphisms between the presentations, and the action coherence makes this transport independent of its local choices. Descent therefore identifies the represented algebroids and module stacks.

Analytic estimates are needed only on the chosen local representatives. Each specified compact finite diagram admits the common conditioned refinements of Lemmas F.8 and H.2 of the local paper. The construction imposes no uniform positive radius or uniform symbol bound on all arrows of a noncompact development. □

On local algebra representatives, the action and its associator have the form

$$\begin{aligned} \varphi_g\varphi_h &= \mathrm{Ad}(C(g,h))\varphi_{gh}, \\ C(g,h)C(gh,k) &= \varphi_g(C(h,k))C(g,hk). \end{aligned} \tag{33}$$

The units $C(g,h)$ have order zero, symbol one and unitary formal adjoint. These are analytic equations on common refinements of the nerve. They remain part of the object.

Corollary 4.3.
The contact transversal groupoid $\mathcal{G}_T$ of an independent monodromy source has a canonical structured operator algebroid. Its intrinsic first-unit sheaf is $\mathcal{L}^{-1}$, with a strict contact action. The source's affine line bundle on T is an equivariant torsor for this sheaf. These constructions are natural for source embeddings, complete-transversal changes and the functors induced by restricting the point domain and recomputing its monodromy.

Proof. Apply Theorem 4.2. Locally the first-unit group is

$$\mathscr{U}_1 = \{1 + \bar{A} : \bar{A} \in F_{-1}\mathcal{E}/F_{-2}\mathcal{E}\} \simeq \mathcal{L}^{-1}. \tag{34}$$

Products of negative-order classes vanish in this quotient. Normalized inner changes act trivially, since the commutator of an order-zero operator with an order-minus-one operator has order at most minus two. The weak action thus induces a strict action on $\mathscr{U}_1$, and its graded identification is precisely the inverse-contact-line action. The affine line of Theorem 3.8 has this translation representation, so its supplied affine transport makes it the required torsor.

A point embedding, or the inclusion of a restricted point domain, induces the functor taking each null path to the same path in the target. Locally its phase map identifies open phase domains. Images of source transversals can therefore be adjoined to the target transversal system, reducing the comparison to the étale contact functors of Theorem 4.2. Its uniqueness makes the resulting comparisons compatible with composition. □

The algebroid action can identify the effects of distinct null-path classes having the same transverse contact germ. The monodromy groupoid and its quadratic-affine representation remain supplied data and retain the corresponding geometric transport. Likewise the affine torsor is additional structure, as in the independent local operator domain. Its existence does not produce a neutralization of the algebroid. Such a neutralization would require a global invertible object with the full groupoid descent; a simple module supported on a Legendrian quadric is a different kind of object (Kashiwara 1996, Theorem 2 and the following paragraph; Polesello 2005, Corollary 3.4).

4.2. Independent operator sources. Use the independent local operator domain and its inverse comparisons from Proposition 2.3. In particular, the optical density root and the module half-canonical root remain distinct. Whole-family module framing fixes a vector in the rank-one local-system factor on the calibrated cone; it does not require a nowhere-zero section of the negative root line on a compact quadric. Algebra comparison units are normalized separately from these framed semilinear module maps.

Definition 4.4.
An independent operator monodromy source consists of the following data.

(1) The point and phase data B, $p : F \to B$, K, E and the source-connected, source-simply-connected groupoid $\mathcal{G} \rightrightarrows F$ of clauses 1–2 of Definition 3.10. Its source fibers are Hausdorff and its anchor

identifies its Lie algebroid with K. The quadratic-affine representation $(\mathcal{V}_F, H, \mathsf{I}, A_F)$ and position section X are retained. No metric is supplied.

(2) A structured analytic operator algebroid on a complete contact transversal presentation of $\mathcal{G}$, with the first-unit identification of Corollary 4.3. The restriction $A_F|_T$ is retained as its equivariant torsor, including its supplied affine translations. Changes of transversal are compared by Theorem 4.2.

(3) A point atlas $\{U_i\}$ of independently admitted local operator objects $\mathcal{O}_i$. Their derived complete point families and full quadric incidences are identified with U_i and $F_i = p^{-1}(U_i)$, preserving the ordered rulings, K and E. Their calibrated affine and complete-sky evaluation data identify with $(\mathcal{V}_F, H, \mathsf{I}, A_F, \mathsf{X})|_{F_i}$. The infinitesimal groupoid action becomes the constant relative connection of their short-ray projections $q_i : F_i \to Y_i$. After shrinking the local point chart, these projections have the contractible short fibers of the local whole-sky construction.

(4) Actual physical operator comparisons on $U_i \cap U_j$ near each whole retained sky, covering its identity and compatible with the supplied identifications of incidence, distributions, quadratic-affine data and position. They retain the actual full spin lifts, root square identifications, torsor maps and whole-family module frames. Their classical incidence and full spin maps satisfy strict identity, inverse and triple laws. Operator representatives have normalized comparison isomorphisms between a composite and the representative of the composite, satisfying the quadruple coherence law.

(5) Structured comparisons between each local chart algebroid and the pullback of the contact-groupoid algebroid along its short-ray presentation. They preserve full symbols and the first-unit torsor, and their overlap diagrams commute through normalized comparison isomorphisms.

In clause 5, the short-path functor $\mathrm{Mon}(K|_{F_i}) \to \mathcal{G}$ integrates the inclusion of the line algebroid. Its short-ray quotient presents Y_i. Thus the compatibility is a comparison of these groupoid descent data; it does not apply the physical whole-sky correspondence to an arbitrary long null path. All operator maps and their inverse comparisons may use common local refinements.

For a comparison between source and target operator presentations, write k_a for its algebra maps, T_{ab} for its normalized units and F_a for its framed semilinear module maps. The descent equations are

$$\begin{aligned} k_a \varphi^s_{ab} &= \mathrm{Ad}(T_{ab}) \varphi^t_{ab} k_b, \\ T_{ab} \varphi^t_{ab}(T_{bc}) C^t_{abc} &= k_a(C^s_{abc}) T_{ac}, \\ F_a \tau^s_{ab} &= T_{ab} \tau^t_{ab} F_b. \end{aligned} \tag{35}$$

Both source and target associators occur. A morphism contains the point embedding, phase map, groupoid functor and equivariant quadratic-affine comparison of an independent monodromy-source morphism, together with compatible structured algebroid comparisons and the above physical comparisons on whole-sky charts. Its full spin lift is retained. Two representatives of a morphism are identified only by normalized comparison isomorphisms over the same underlying maps and spin label. Composition uses the retained coherences. Write $\mathsf{OpMon}_{\Lambda}^{\mathrm{hol}}$ for the resulting category of physical Hom classes.

These admission conditions concern operator symbols, calibrated torsors, intrinsic optical recognition, complete point families and specified groupoid actions. They can be checked on the supplied source. Atlases related by restriction and common refinement represent the same source presentation, with all structures and comparison maps restricted together. A globally supported module on a ray quotient is not an admission condition.

Lemma 4.5.

The quadratic-affine representation and position section in Definition 4.4 can equivalently be recovered from the local operator charts, their symbolic comparison cocycle and the supplied source-complete groupoid. The recovery uses whole-sky evaluation and inverse null transport, without supplying a global Einstein metric.

Proof. Apply the full-symbol and calibrated-affine functor of Proposition 2.3. The independently admitted local ray presentation and complete-sky evaluation give a rank-six bundle $\mathcal{T}_i$, connection D_i, form H_i and sections $\mathsf{I}_i, \mathsf{X}_i$ on U_i. Their connection is the one reconstructed by inverse null transport in Proposition 2.2.

Actual symbolic comparisons cover the identity of each point overlap, respect the supplied incidence identifications and preserve all these objects. Normalized operator comparison isomorphisms leave the symbols and first-unit torsor unchanged. Consequently the weak operator cocycle induces a strict cocycle on the evaluation bundles, their connections and the point spin bundles. These data glue on B.

Put $\mathcal{V}_F = p^*\mathcal{T}$ and restrict its pulled-back connection to K. This partial connection is flat because K has complex rank one. On every short-ray chart, complete-sky evaluation identifies it with the constant relative connection on the local ray bundle. The affine lines and position sections agree on point overlaps and therefore give A_F and X globally. Transport preserves H, I and these affine lines.

The anchor and source-completeness hypotheses identify the supplied groupoid with $\mathrm{Mon}(K)$ by Theorem 3.11. Flat partial transport hence integrates to a holomorphic representation of that groupoid. Any representation with this infinitesimal action agrees with it on each source fiber by initialized transport uniqueness. The same argument proves uniqueness of the

completion and naturality for physical embeddings, spin comparisons and restrictions. □

Theorem 4.6.
Full symbols and calibrated affine data give an equivalence

$$\mathsf{OpMon}_{\Lambda}^{\mathrm{hol}} \simeq \mathsf{MonRay}_{\Lambda}^{\mathrm{hol}}. \tag{36}$$

It preserves the point manifold, complete quadric incidence, affine position, selected scale and spin structure. On each whole-sky chart it is the local operator correspondence. It is fully faithful for the embedding morphisms of Definition 3.10. Both inverse comparisons retain the operator associators and framed point-module comparisons.

Proof. Taking full symbols and the calibrated torsor of each $\mathcal{O}_i$ gives an independently admitted local ray model. Clause 3 identifies its whole-sky incidence with the supplied phase data, and clause 4 gives a symbolic and full-spin comparison cocycle over the identity on point overlaps, compatible with those global identifications. The infinitesimal-action condition is precisely the short-path condition of the independent monodromy-source definition. The global phase, groupoid and quadratic-affine data are already retained, so the resulting object is an admitted ray source.

For morphisms, full symbols and first-unit torsor maps are the admitted physical ray comparisons. Normalized comparison isomorphisms preserve their symbolic data and spin label. The operation therefore descends to physical Hom classes. The two-associator equations above show that it preserves composition, giving a functor $\mathcal{F}$.

Conversely, start with an independent ray source. Theorem 4.2 quantizes its contact transversal groupoid. Retain the supplied affine representation under the first-unit identification of Corollary 4.3. On each whole-sky chart, apply the inverse local operator correspondence to its independently admitted affine model. This gives an operator object with its compact module family, root and frame data.

The short-ray presentation maps to the full monodromy presentation by including paths. Its local chart algebroid and the pullback of the groupoid algebroid quantize the same contact data. Structured uniqueness compares them on refined contact charts, with unique normalized comparisons on intersections. Their first-unit torsor is the same supplied affine torsor, giving clause 5 of Definition 4.4.

Lift each physical ray comparison over the identity on a point overlap by the local operator theorem, with its specified spin lift and the same supplied global identifications. A composite of lifts and the lift of the composite cover the same physical map and spin label. The local structured comparison groupoid over those fixed data is contractible, so there is a unique normalized comparison between them. On a quadruple overlap the two comparison composites agree by uniqueness. The framed module comparisons satisfy the same coherence through their semilinear descent equations and whole-family framing. Uniqueness also makes these comparisons compatible with the

contact-groupoid algebroid. This constructs the inverse functor $\mathcal{Q}$, up to its specified normalized comparisons.

For a noncompact point manifold, use relatively compact point charts contained in larger analytic parents. Properness of p retains every full compact sky. Each finite comparison diagram can then be placed in finitely many buffered parents after refinement, where the conditioned local estimates apply. This gives analytic representatives and their inverses for every required local diagram. The unique comparisons glue them over the entire cover and the local charts of the possibly non-Hausdorff arrow space. No estimate uniform over all global charts is needed.

The local inverse comparisons identify $\mathcal{FQ}$ with the supplied ray source, preserving its global phase, groupoid and transport. They also preserve its spin cocycle, so they glue globally.

For $\mathcal{QF}$, apply the local counit to each supplied operator chart on a common refinement of the supplied and reconstructed presentations. Its algebra and module components satisfy the displayed comparison equations with both associators. On a point overlap, the two counit composites cover the same physical comparison and spin label; unique normalized comparison identifies them. The same uniqueness gives coherence on triples. Theorem 4.2 compares the global algebroids and makes that comparison compatible with the chart counits. Thus the counit is a comparison of the full operator presentations, rather than only of their decoded geometric objects.

To prove fullness, fix a ray-source embedding. Its local whole-sky comparisons lift by the local operator correspondence, while its contact-groupoid map has the comparison constructed in Corollary 4.3. Structured uniqueness identifies the two constructions on short-ray charts, and the comparison equations glue the lifts on point overlaps. This gives an operator-source morphism. The induced functor between monodromy groupoids need not be faithful; its supplied phase and contact action are sufficient.

For faithfulness, two lifts of the same ray-source morphism have the same physical local maps and spin lifts. The unique normalized operator and framed module comparisons identify them on every chart. These comparisons agree on overlaps by uniqueness, and the global algebroid comparisons agree as well. Hence the lifts have the same physical Hom class. Distinct spin labels are never identified. The unit, counit and composition diagrams commute through these same comparisons and therefore as equalities of physical Hom classes. □

4.3. The module on parametrized incidence. Let $\mathfrak{X} = [T/\mathcal{G}_T]$ denote the contact quotient presentation, let $\mathscr{E}_{\mathfrak{X}}$ be its structured algebroid, and let $q : F \to \mathfrak{X}$ be the canonical phase morphism. The proper point map $p : F \to B$ remains a map of Hausdorff manifolds. Here B is the physical point parameter, denoted by S in the relative-module construction in Appendix G of the local paper; it is not that appendix's additional external parameter base.

The operators below retain ambient contact germs along incidence, including operators transverse to a point sky. They are not intrinsic differential operators on F.

Proposition 4.7.
There is an analytic relative structured algebroid $\mathscr{E}_{\mathfrak{X}\times B/B}$. Its local symbols are jointly holomorphic in contact coordinates and the passive parameter $b \in B$, with the local factorial bounds of the analytic operator domain, and its operators differentiate only in the contact model. The inverse-image algebroid

$$\mathscr{E}_F^{\mathrm{inc}} = (q,p)^{-1}\mathscr{E}_{\mathfrak{X}\times B/B} \tag{37}$$

is well defined on F and contains $p^{-1}\mathcal{O}_B$ centrally. It is independent, up to structured comparison, of transversals and local representatives, and natural for physical embeddings and recomputed point restrictions.

Proof. On a projective cotangent chart, take analytic relative half-density operators with passive parameter in a point chart. Their composition, filtration and adjoint are the corresponding relative operator formulas; the adjoint fixes the passive parameter. Jointly holomorphic symbols and the factorial estimates on smaller product parents give the analytic relative algebra constructed in §§F.1, G.1 and H.1 of the local paper.

Every contact-groupoid comparison and associator extends with the point parameter passive. The exact operator equations remain valid coefficientwise. Changes of point coordinates act by substitution. These data give the relative algebroid and its weak action on $\mathcal{G}_T \times B$. Theorem 4.2 and the relative comparisons of Proposition 2.3 establish independence of the presentations.

For an explicit construction on F, choose phase neighborhoods V_i with holomorphic short-path choices into contact transversals. They give maps $\chi_i : V_i \to T$. On an overlap, two choices are related by the groupoid arrow obtained by composing one chosen path with the inverse of the other. These arrows satisfy the groupoid cocycle. Pull back the relative operator germs along (χ_i, p) and glue their algebroids using the weak operator comparisons and their associators. The groupoid-nerve coherence is exactly the required descent coherence. A change of short-path choices gives the corresponding comparison.

This construction takes inverse images of ambient ring germs. It takes neither a quotient by functions vanishing on incidence nor a tensor product with $\mathcal{O}_F$ over ambient functions. Thus transverse operator germs remain available. The central passive functions give $p^{-1}\mathcal{O}_B$. Physical embeddings and recomputed restrictions carry the local paths to their images; the comparisons of Corollary 4.3 and the same descent equations give naturality. □

Theorem 4.8.
The calibrated whole-sky module families of an independent operator monodromy source descend effectively to a locally finitely generated algebroid module

$$\mathscr{M}_F \in \mathrm{Mod}(\mathscr{E}_F^{\mathrm{inc}}). \tag{38}$$

Its support on F is all of F. Its local realizations retain the root $R^2 = K_{F/B}$, optical calibration, whole-family frames and operator associators. It is unique up to the unique compatible calibrated comparison and natural for physical source embeddings and recomputed point restrictions. Its Lorentz comparison is the weak comparison established in §6.

For $x \in B$, let $i_x : F_x \hookrightarrow F$ and let $\mathfrak{m}_x \subset \mathcal{O}_{B,x}$ be the maximal ideal. The point fiber is the central specialization

$$\begin{aligned} \mathscr{E}_x^{\mathrm{inc}} &= i_x^{-1}\mathscr{E}_F^{\mathrm{inc}}/\mathfrak{m}_x i_x^{-1}\mathscr{E}_F^{\mathrm{inc}}, \\ \mathscr{M}_x &= i_x^{-1}\mathscr{M}_F/\mathfrak{m}_x i_x^{-1}\mathscr{M}_F. \end{aligned} \tag{39}$$

The compact quadric F_x with these germs is locally the calibrated simple sky object of the original correspondence at x.

Proof. On a whole-sky point chart U_i, the relative construction of Proposition H.3 of the local paper, recalled in Proposition 2.3, gives a module family $\mathcal{N}_i$ on $Y_i \times U_i/U_i$. Its support is the incidence embedded by

$$\jmath_i = (q_i, p) : F_i \longrightarrow Y_i \times U_i.$$

The differential is injective because $\ker \mathrm{d}q_i = K$ and $\mathrm{d}p$ is nonzero on K. The map is injective since each sky is embedded and its point parameter is retained. It is proper as a map into the retained product neighborhood: the inverse image of a compact set is closed in the compact inverse image under p of that set's point projection. Hence $\jmath_i$ is a proper holomorphic embedding. This argument concerns the local whole-sky representative, independently of the global ray quotient.

Clause 5 of Definition 4.4 identifies the inverse-image relative chart algebroid with $\mathscr{E}_F^{\mathrm{inc}}|_{F_i}$. Thus $\jmath_i^{-1}\mathcal{N}_i$ is a module over that restriction. Its finite local presentations remain finite presentations after inverse image, which is exact on sheaves and can be checked stalkwise. Each stalk is nonzero by the locally standard simple incidence model.

On a point overlap, the physical operator comparison intertwines the two relative families. After transport to the fixed incidence algebroid, its algebra and module components satisfy precisely the comparison equations (35), including the supplied and reconstructed associators. On triple overlaps these give the module descent equation through the specified algebroid associator; the unique normalized algebra and framed module comparisons give quadruple coherence.

More explicitly, refine by algebra and module charts. The local module sheaves have semilinear transition maps τ_{ab} satisfying $\tau_{ab}\tau_{bc} = C_{abc}\tau_{ac}$. This is the descent datum defining a module for the algebroid represented by φ_{ab} and C_{abc}. Sheaf descent glues the local objects with these equations. Finite generation and nonzero stalks are preserved by the transition equivalences, giving the asserted support. Root square identifications, any retained parameter-base factors and their frames descend through their comparison maps. The framing remains on the calibrated cone, as specified in Proposition 2.3.

Any two choices have unique local calibrated comparisons. Their restrictions agree on overlaps by uniqueness and hence glue. The same argument proves naturality for embeddings and recomputed restrictions. The square equations in Proposition 6.5 descend through these same comparisons, giving the asserted Lorentz compatibility.

It remains to identify the fiber. Inverse image to F_x retains passive parameter germs, so the quotient by $\mathfrak{m}_x$ is necessary. In a product chart centered at x, evaluation of relative operators has kernel generated by the point coordinates. Indeed, for each holomorphic coefficient symbol,

$$a(y,b) - a(y,x) = \sum_{\ell} (b^{\ell} - x^{\ell}) \int_0^1 \partial_{b^{\ell}} a(y, x + t(b-x))\, \mathrm{d}t.$$

On a smaller convex parameter parent, one Cauchy gap bounds the integral coefficients with the same factorial symbol order. Applying the formula coefficientwise gives division inside the analytic relative algebra, and therefore identifies its central specialization with the ordinary contact operator germ. The calculation commutes with the algebra comparisons, so it also applies to the algebroid.

The relative flattenings and finite defining-ideal presentations in Lemmas G.1–G.2 and Proposition H.3 of the local paper specialize to the fixed-parameter simple-module presentation. Hence ordinary central specialization gives precisely the local sky module. This uses the admitted locally standard presentation, not a flatness claim about arbitrary relative modules. Roots, frames and comparison equations specialize simultaneously, completing the identification of the compact point object. □

Corollary 4.9.

Theorem 4.6 retains the global incidence module $\mathscr{M}_F$ and all specialized compact point objects $(F_x, \mathscr{O}_x^{\mathrm{inc}}, \mathscr{M}_x)$ naturally. Equivalently, the local point-module component of Definition 4.4 can be presented by this incidence module together with its locally standard realizations, root identifications and frame comparisons. This change of presentation preserves physical Hom classes.

Proof. Theorem 4.8 constructs the incidence module from the admitted local families and their comparisons. Conversely, the specified locally standard realizations recover those local families and their comparisons. On a closed embedded local incidence, a supported sheaf is determined as a germ by its inverse image: direct image by that embedding recovers its local supported sheaf. This stalkwise identification respects the ambient operator action. The specified local realizations retain the required finite operator presentations, so the statement does not enlarge the admissible module class.

The two constructions are inverse through the unique calibrated comparisons of Theorem 4.8 and commute with physical maps. Their compatibility with the functors and both counits of Theorem 4.6 proves the assertion, including its naturality under recomputed point restriction. □

The module constructed here lives on the Hausdorff parametrized incidence, with its ambient relative operator germs. Its support there is all of F, and its compact point labels are the fibers of p. The construction supplies no descent of that module along every null ray and no globally supported coherent module on a projected sky in the coarse quotient. This distinction will be essential in the Nariai example.

5. Atlases of the original source classes

We now globalize the original input itself. The chart data are the two cohomology classes of Proposition 2.1, and the transition data are their nonlinear physical comparisons. The definition records the topology and comparison law; the analytic and descent arguments then construct their global geometric and operator realizations. We use the descent-data and common-refinement conventions of the Stacks Project (The Stacks project authors 2026, Tag 02ZC).

Use the original source domain and analytic constructor recalled in §2. An original marked input is a pair

$$\mathbf{a}_i = (a_i^+, a_i^-) \in \mathscr{H}_2(T_{0,i}) \oplus \mathscr{H}_{-6}(T_{0,i}) \tag{40}$$

at the common Einstein coefficient Λ, with its full spin mark. Each $T_{0,i}$ is a marked copy of the germ of $\mathrm{Tot}(\mathcal{O}(1)^{\oplus 2})$ along the entire zero line. The fixed residue and middle-Laurent projection selects its canonical representatives. Raw representatives differing by a Čech coboundary represent the same input; an arbitrary choice of a coboundary primitive is not an additional physical arrow.

The convergent original constructor supplies a complete local point domain P_i and its full compact sky incidence. A *retained point domain* means a representative inside the constructor's analytic parents, with the whole compact sky over every retained point. Further restriction and recentering are allowed. The six normalization requirements belong to that prescribed construction; they are not six additional freely chosen source classes.

On a common point domain, a physical source comparison is specified locally after recentering at a point of that domain. It is the original short instruction (a, A) and its source comparison, with the full spin element A retained. Its point map uses the covariant convention in Theorem 8.3 of the local paper: the instruction $\mathbf{a} \to R_{a,A}\mathbf{a}$ is represented by $\Xi_{a,A}^{-1}$. Comparisons between other retained representatives are those of Isthmus (2026a, Proposition 8.1). Their composition uses the source-dependent physical law, not addition of displacement vectors or multiplication of tensor labels alone.

The action on the two source classes is computed by the exact residue dictionaries, the original Cartan flow on whole records, and the inverse dictionaries. It need not be the pullback of both classes by a map between the old flat seeds. In particular, the global input below is descent data for this nonlinear physical comparison law.

Definition 5.1.
An original-source atlas consists of the following data.

(1) A connected Hausdorff second-countable complex four-manifold B.
(2) A covering $\{U_i\}$ of B, original marked inputs $\mathbf{a}_i$ as above, retained derived point domains P_i, and biholomorphic identifications $e_i : P_i \to U_i$.
(3) On every point overlap, physical source comparison germs c_{ji} from the retained i-presentation to the retained j-presentation, covering $e_j^{-1}e_i$. They are germs near every whole compact sky in the overlap, with holomorphic representatives after the conditioned local restrictions of Proposition 5.4. The central points of the old charts need not belong to the overlap: recenter both charts near a common point before specifying a short comparison.
(4) These comparisons satisfy identity, inverse and triple laws as original physical arrows, including their full spin lifts:

$$c_{ii} = 1, \qquad c_{ij} = c_{ji}^{-1}, \qquad c_{kj} \circ c_{ji} = c_{ki}. \tag{41}$$

Equality here is equality of the original physical class and spin label on the common germ domain. The contact-affine representatives obey the corresponding strict comparison laws in Proposition 8.1 of the local paper. Operator lifts, when constructed, retain their normalized inner comparison isomorphisms.

Restriction of charts, recentering and common refinement give equivalent presentations through specified physical comparison maps. Every refinement still covers the same retained point manifold B. Its comparison maps retain the original classes and their full spin lifts. These maps are changes of presentation; distinct physical morphisms are not identified merely because their underlying point maps agree. In particular, a refinement does not add arbitrary raw Čech zero-cochains as new arrows.

Restricting the retained point manifold itself to an open subset $U \subset B$ is a separate operation producing a source object over U. It is not a change of presentation of the original object over B.

A morphism consists of a holomorphic point embedding $f : B \to B'$ and physical source comparison germs on common source and target point charts covering f. They have full spin lifts and commute with the source and target overlap comparisons. Morphisms are compared on a common refinement using the same specified source and target refinement identifications, and are identified only when their germs and spin lifts agree there. Independent changes of spin identification cannot turn the negative central spin morphism into the identity. Composition is the original physical composition on representatives, followed by common refinement. Denote the resulting category by $\mathsf{SrcAtl}_{\Lambda}^{\mathrm{hol}}$.

Admission in this definition is tested on the two original analytic cohomology classes, their record-flow action and physical composition law, and the identifications of their derived complete point domains. These data are

defined before the geometric decoder; the global metric is constructed in Theorem 5.6.

Equivalently, the point topology may be specified by the generated domains P_i and their physical transition germs: require the usual full atlas identity, inverse and cocycle laws, including the spin comparisons, and require the quotient of the disjoint union by these identifications to be connected, Hausdorff and second countable, with each P_i mapping by an open embedding. The quotient then has a unique complex structure for which these maps are holomorphic charts, since the supplied transition maps are biholomorphic and agree as germs on overlaps. This recovers the manifold B and the identifications in the definition. The topology and the full spin transition law remain source-atlas data.

Remark 5.2.

The point topology and continuation data in this definition cannot be recovered from one original germ pair. For example, the flat developments $\mathbb{R}\times(\mathbb{R}^3/L\mathbb{Z}^3)$ with metric $\mathrm{d}t^2-\sum_j(\mathrm{d}x^j)^2$ have zero scalar and zero curvature records, hence the same original pair $(0,0)$ at a chosen full spin mark. Their Cauchy volumes are L^3, so different L give distinct marked global data. For fixed L, a proper time collar and the full flat development have the same local pair but different realized point domains. An atlas retains precisely the additional identifications and domains involved in these examples.

Lemma 5.3.

Let $u=(U^+,U^-;\Lambda)$ be an original analytic input with its marked full spin frame. Fix a compact amplitude disk $|\varepsilon|\le T_b$, where $T_b>1$ lies strictly inside a retained larger disk, and a finite local star of point and full-spin instructions, together with its specified inverses and composites. After choosing sufficiently small passive displacement neighborhoods and smaller point parents, these instructions have compatible bounded holomorphic representatives over the entire compact amplitude disk. Every vertex is obtained by rebasing the same central physical family

$$u_\varepsilon=(\varepsilon U^+,\varepsilon U^-;\varepsilon\Lambda). \tag{42}$$

For passive instructions (a_i,A_i), the vertex input is

$$v_i(\varepsilon)=R_{a_i,A_i}u_\varepsilon=\big(V_i^+(\varepsilon),V_i^-(\varepsilon);\varepsilon\Lambda\big). \tag{43}$$

Its prescribed source constructor uses a separate internal amplitude s on

$$\big(sV_i^+(\varepsilon),sV_i^-(\varepsilon);s\varepsilon\Lambda\big), \tag{44}$$

and is evaluated at $s=1$ before the physical vertex comparisons are made. The resulting analytic contact, affine, operator and point-module comparisons retain their full spin labels, both calibrated roots, associators and whole-family frames. The assertion includes $\Lambda=0$.

Proof. The local analytic inputs are the results of Theorem 3.3, Propositions 6.1–6.2, Theorem 6.3, Theorem 8.3 and Theorem G.5 of the local paper. The compatible-amplitude construction is also displayed for zero scalar in

Lemma K.16 of the local paper. We give the finite-star argument with the Einstein coefficient retained.

Propositions 6.1–6.2 of the local paper construct the central family g_ε and its marked normal coordinates, initialized spin transports, prescribed ray charts and augmented stored tuples on common smaller parents over the compact amplitude disk. The metric and inverse, the finitely many required derivatives, and the relevant inverse maps have common bounds there.

Let $\nu_i(\varepsilon)$ be the initialized full-spin transport to $x_i(\varepsilon) = \operatorname{Exp}_o^{g_\varepsilon}(a_i)$. The exact record action makes $v_i(\varepsilon)$ the whole curvature record of the same g_ε at that point in the frame $\nu_i(\varepsilon)A_i$. Theorem 3.3 of the local paper gives a common factorial bound on a smaller record radius for sufficiently small displacements. The fixed finite spin parameters and their inverses are bounded after restriction to compact parameter subcores. The two residue encoders therefore give bounded holomorphic representatives of these vertex inputs on common smaller annular neighborhoods of the whole reference line.

The initialized metric comparisons are

$$\Xi_i^\varepsilon : g[v_i(\varepsilon)] \longrightarrow g_\varepsilon, \qquad y \longmapsto \operatorname{Exp}_{x_i(\varepsilon)}^{g_\varepsilon}\big(\nu_i(\varepsilon)A_i y\big). \tag{45}$$

At zero displacement their point differentials are the invertible vector representations of the retained spin maps. Finitely many fixed spin frames can be used as reference values; their central signs remain distinct spin labels. The analytic inverse theorem, compactness of the amplitude disk and the finite number of specified maps give smaller common domains for Ξ_i^ε, their inverses and the requested composites. The passive neighborhoods can be chosen so that their images have positive margins inside the assigned parents. The required determinants and scalar coefficients used as units then have positive modulus minima. This condition concerns the inverted coefficients and determinants, not every individual matrix entry.

On these domains put

$$f_{ij}^\varepsilon = (\Xi_i^\varepsilon)^{-1}\Xi_j^\varepsilon. \tag{46}$$

Their inverse and triple laws are identities of maps, with the same initialized spin comparisons. Expressing them as original source instructions gives the composition law in Isthmus (2026a, Theorem 8.3), including its radial-versus-broken-path spin factor. This is a coordinate comparison law; it does not declare spin parallel transport path independent.

The rebased inputs are bounded holomorphic families in ε and the passive parameters. Apply the prescribed source construction to each such family using the independent variable s in the statement. On a compact internal disk of radius greater than one, Propositions 6.1–6.2 and Theorem 6.3 of the local paper supply its analytic representative and inverse components. At $s = 1$ it is the source associated with $v_i(\varepsilon)$. The physical comparison f_{ij}^ε is then exactly a comparison to which Theorem G.5 of the local paper applies. Its proof already uses the nonlinear outer family and evaluates each vertex's independent internal constructor at one. It gives the analytic structured algebra maps, normalized comparison units and framed module

maps, with both associators retained. Normalized-inner uniqueness and framed-module uniqueness give all the stipulated finite comparison equations. The Einstein coefficient is $\varepsilon\Lambda$ throughout the outer family and $s\varepsilon\Lambda$ in the internal constructor. The cited formulas have no division by this scalar.

Finally, the full sky remains present under these restrictions. The central sky over the compact retained parameter set is compact. A finite cover of it by buffered direction charts and short hit-map domains gives a common smaller point neighborhood on which these charts still cover the full proper quadric family. All the finite comparison data, root maps and module charts are restricted together. Thus the finite minima above are positive without intersecting an uncontrolled infinite collection of neighborhoods. □

The two amplitudes have different roles. In general,

$$R_{a,A}(\varepsilon U^{+}, \varepsilon U^{-}; \varepsilon\Lambda) \neq \big(\varepsilon(R_{a,A}u)^{+}, \varepsilon(R_{a,A}u)^{-}; \varepsilon\Lambda\big). \tag{47}$$

Consequently the lemma supplies physical comparisons after each internal constructor is evaluated at one. It does not compare separately linearly scaled endpoint inputs at every internal amplitude.

The central physical family in this lemma belongs to one local comparison star. Different stars need not share an outer amplitude family. In particular, the lemma does not deform a whole global source atlas, or an arbitrary global loop of point charts, to one flat source.

Proposition 5.4.

Consider a collection of original local source presentations with physical comparison germs, on a Hausdorff second-countable point manifold. Assume the supplied holomorphic families and specified finite local comparison diagrams have the conditioned parent, inverse and unit margins of Definition 2.1 of the local paper. On every retained compact point and passive-parameter subsystem, the collection may be refined to finitely many representatives on which the original prescribed nonlinear construction and its physical comparisons are analytic, including their whole compact skies. Their Einstein coefficient and six scale normalizations are preserved through the calibrated comparisons.

The construction may therefore be used on a locally finite source atlas over a noncompact point manifold. Its estimates are local on compact subsystems and finite comparison diagrams. It requires no common positive radius or common norm bound over the whole point manifold.

Proof. The two source representatives determine their complete factorially bounded records by the exact residue dictionaries. Conversely, the same dictionaries recover unique original cohomology classes from those records, with representative and coboundary comparisons on smaller buffered domains (Isthmus 2026a, Theorem 3.1). To rebase at a point of an overlap, join the old mark to that point by a piecewise analytic path inside the existing constructor's analytic representative. Use the metric already derived by that local constructor. A compact path is covered by finitely many relatively compact point and spin charts inside larger analytic parents. The chiral curvature,

transported in parallel spin frames along sufficiently short complex null geodesics, is holomorphic in the geodesic parameter on a common positive disk for normalized spinor directions. Its rth derivatives give diagonal evaluations of the free records, so Cauchy estimates give $Mr!R^{-r}$. Polarization and the fixed spinor norms cost only an exponential factor in r, absorbed by one radius decrease. Thus the complete records at the intermediate marks have refreshed factorial bounds, uniformly on each retained compact passive-parameter subsystem.

Subdivide the path so that successive marks lie in common short normal parents. Each displacement and specified spin-frame change is then an admitted R_{a_j,A_j} as in Theorems 3.3 and 8.3 of the local paper. Their finitely many local source constructions and comparisons have common smaller parents by the finite-word assertion. The exact dictionaries return original source classes at every step, and local inverse uniqueness identifies the resulting recentered germ with the given analytic representative. This gives rebasing by a finite chain of short instructions. The full spin labels are retained even when their action on tensor records has a kernel.

For the coherent point-overlap diagrams used in atlas descent, work near a common overlap point and refine to a local star with one reference mark. Express its finitely many retained marks by admissible short instructions, with its comparisons induced by the same initialized identifications. Lemma 5.3 gives coherent physical families and analytic representatives on smaller parents. It is not used to interpolate arbitrary independently specified automorphism or loop relations to a flat family. Other admitted finite diagrams of completed analytic objects use the conditioned finite-diagram assertion of Theorem 9.10 of the local paper directly. When the starting data already include a bounded coherent passive family, use that family's comparison identities and apply the analytic construction locally with those parameters retained. In each case it is only necessary to reserve the buffers for the finitely many specified maps, inverse maps and composite images.

The nonlinear execution in each point chart is the one prescribed in §§5–6 of the local paper. Its correction at a weight $m \geq 2$ is

$$\Delta c_{m+1} = -L_m a_m + \iota_{m+1} Z_{m-2}. \tag{48}$$

Here Π_{m+1} is the free-record map from the closed contact correction classes at that weight to $W^+_{m-2} \oplus W^-_{m-2}$, ι_{m+1} is its fixed right inverse, and Z_{m-2} is the prescribed minus current free-record component. The map o_m takes a closed contact correction class to its optical obstruction class, and $L_m = (o_m|_{\ker \Pi_{m+1}})^{-1}$ on the compatible optical defects. Thus the first term removes the defect without changing the free record, and the second supplies its prescribed record correction, with $\Pi_{m+1}\iota_{m+1} = 1$.

The actual optical defect a_m and current record are read from that chart's freshly recomputed incidence. The remaining flat scalar freedom is $a + b \cdot x + \gamma\eta(x, x)$. In the lifted lower Einstein scale the already selected section is represented by 1; a, b, γ describe the highest-stage correction (Isthmus 2026a,

Theorem 5.5). The value and four first derivatives fix its first five coefficients, and the numerical coefficient fixes

$$\gamma = (\lambda_{\mathrm{target}} - \lambda_*)/12. \tag{49}$$

In this formula $\lambda_{\mathrm{target}}$ and λ_* are the prescribed and currently computed numerical-scalar coefficients in the highest stage being corrected; the original Einstein coefficient remains Λ. These are the original six normalizations, including their transverse scalar condition, rather than only central sky data. The deterministic stored normalizer uses the selected geometry and section, initialized normal and spin development, its prescribed contact Moser maps, the same graph evaluations and fresh optical recomputation.

Theorem 6.3 of the local paper identifies every finite prescribed stored tuple with the corresponding internal-amplitude Taylor polynomial of one analytic tuple. For each of finitely many selected vertices, write this tuple as $Y_i(s)$ and its degree-N polynomial as $Y_{i,N}(s)$. After the finite restrictions, choose $T_* > 1$ smaller than the internal Cauchy radii and a common finite bound M for these finitely many classical augmented tuples. Their analytic sums obey

$$\max_i \|Y_i(1) - Y_{i,N}(1)\| \le \frac{M T_*^{-N-1}}{1 - T_*^{-1}}. \tag{50}$$

All required inverse components are included in these tuples. The analytic operator and supported-module realizations use the relative constructions of Appendices F–H of the local paper. In those constructions the finite active operator presentations are compared through normalized inner equivalences; the displayed classical tuple estimate does not assert a literal equality of arbitrary raw quantum representatives.

Physical comparisons preserve the same selected section and its calibration by Proposition 8.1 and Theorem 9.12 of the local paper. Its value and gradient are transported and recomputed in the chosen auxiliary density frame. In particular, after an auxiliary scale change they transform with the Cauchy-state matrix; their old coordinate entries are not held fixed. The Einstein coefficient is preserved. Hence the six normalizations agree geometrically under the comparison, even when the stored coordinate tuples differ because their presentation choices differ. The correction law and its obstruction are the same local source law, including the preservation correction identified in Proposition H.5 of the local paper.

A compact point subsystem lies in finitely many smaller chart cores with larger parents. At each point of a retained overlap, the finite diagram under consideration has representatives on a smaller neighborhood by the preceding argument. Compactness yields a finite refinement covering the required overlap subcores. Properness of the sky family makes its inverse image over each compact core compact, so finitely many direction charts and common finite buffer minima retain every whole sky. Repeating this on a locally finite refinement gives source presentations on the entire point manifold. Local

identities are tested after common finite refinement and agree as germs; no global minimum of all radii is taken.

Every constructed map is jointly holomorphic in the retained passive parameters. Restriction to an actual finite Taylor neighborhood of a coherent passive family preserves its comparison equations. This proves the stated local analytic control and the compatibility used for source-atlas descent.□

The proposition preserves the two original source operations and their nonlinear correction as local executions joined by physical comparisons. It does not identify the Hamiltonian representatives in distinct reference charts as one global Hamiltonian. Nor does it give an exact atlas cocycle for the polynomials obtained by truncating each vertex's independent internal amplitude separately. Exact descent concerns the analytic evaluations at internal amplitude one. Finite-jet descent concerns a coherent passive or outer physical family, such as the common family of Lemma 5.3, whose comparison identities already hold before the jet restriction.

Lemma 5.5.

An original-source atlas determines a proper holomorphic quadric bundle $p : F \to B$, its ordered rulings, a characteristic line K, a hyperplane distribution E, and the point spin structure. These are the point and phase data of Definition 3.10. The source-complete null-path groupoid is canonically

$$\mathcal{G} = \mathrm{Mon}(K) \rightrightarrows F. \tag{51}$$

Complete-sky evaluation and inverse null transport determine its quadratic-affine representation and position. These constructions use the source comparisons, are natural for source-atlas morphisms, and commute with point restriction when the path groupoid is recomputed.

Proof. Apply the original analytic source constructor on each retained chart. Its output has the full proper quadric incidence $p_i : F_i \to P_i$, line K_i and hyperplane E_i. The comparisons in Propositions 8.1 and 9.9 and Theorem 9.12 of the local paper induce holomorphic incidence identifications over the specified point maps, preserving the ordered rulings and these distributions. The source cocycle (41) makes their classical incidence and full spin cocycles strict. Thus they glue the F_i over B to a holomorphic quadric bundle, and glue K_i, E_i to K, E. Each local bundle is a bundle of the entire compact quadric, so p is proper. The total space is Hausdorff and second countable by its local bundle charts over B.

The conditions $\mathrm{d}p|_K \neq 0$, $K + \ker \mathrm{d}p \subset E$, characteristic line K and nondegenerate transverse Levi form are local conditions and hold in every constructed chart. The full spin cocycle also glues the local spin bundles and their frame maps. No spin sign is inferred from its action on a projective fiber.

Apply Proposition 3.7 to the resulting holomorphic characteristic line. Its monodromy integration has the required source-fiber and anchor properties. It retains paths through arbitrarily many point charts; the construction has not identified those paths with a single short-ray overlap.

On each whole-sky chart, evaluation and inverse null transport reconstruct $(\mathcal{T}_i, D_i, H_i, \mathsf{I}_i, \mathsf{X}_i)$. The source comparisons preserve these objects. Their strict classical cocycle glues them on B. The proof of Lemma 4.5 now applies: pull $\mathcal{T}$ to F, restrict its connection to K, and integrate this flat partial connection on $\mathrm{Mon}(K)$. It preserves the quadratic form, scale and affine lines, with position retained as an incidence section. This determines the entire representation from the source atlas.

A source-atlas morphism preserves the local incidence, evaluation and spin maps. They glue to its phase and quadratic-affine maps, and mapping paths by its phase map gives the groupoid functor. The local transport equations prove equivariance and composition. For $U \subset B$, the restricted atlas glues the restricted phase $F|_U$ and line $K|_{F|_U}$. Its completion is consequently $\mathrm{Mon}(K|_{F|_U})$, with the integrated inclusion into the old groupoid. This is precisely the recomputed restriction of Proposition 7.1. □

Theorem 5.6.
Use the original local source results (Isthmus 2026a, Theorems 2.4, 6.3, 8.3 and 9.10–9.12) with their stated analytic-domain and physical-arrow conventions, together with the contact-groupoid quantization proved in Theorem 4.2. The original source constructor and decoder extend to an equivalence

$$\mathsf{SrcAtl}_\Lambda^{\mathrm{hol}} \simeq \mathsf{OpMon}_\Lambda^{\mathrm{hol}}. \tag{52}$$

It retains the chosen complex point manifold, full compact point family, spin structure, selected Einstein scale and the completed null-path representation. On each chart it is the original two-input source correspondence. Both inverse comparisons preserve the source presentations through their initialized physical comparisons, and preserve the operator associators and framed local module comparisons. The equivalence is fully faithful for the stated embedding morphisms.

Proof. Start with a source atlas. Lemma 5.5 constructs its phase, groupoid and quadratic-affine data. On the i-chart apply the source operator constructor $\mathcal{S}_{\mathrm{op}}$ of Theorem 9.10 of the local paper to $\mathbf{a}_i$. Its full symbols and first-unit torsor are the original constructed contact-affine tuple, and its locally standard compact modules are the original complete sky family. Thus it supplies an admitted local operator chart on the retained point domain.

The physical source comparisons c_{ji} cover $e_j^{-1}e_i$ and hence, under the retained identifications e_i, the identity of $U_i \cap U_j$. Their operator and framed module lifts preserve the completed incidence and quadratic-affine identifications by Proposition 9.9 and Theorem 9.10 of the local paper. On a triple overlap, the lift of $c_{kj}c_{ji}$ and the chosen lift of c_{ki} cover the same physical map and spin label. The unique normalized comparison isomorphism identifies these two operator representatives. Its module component obeys the semilinear comparison equations (35), with both associators. On a quadruple overlap, the two composites of these comparison isomorphisms have the same

source, target and normalization, so they agree. This is the weak operator cocycle required by Definition 4.4.

Choose complete contact transversals of the global characteristic foliation. Theorem 4.2 supplies their structured groupoid algebroid. On each short-ray chart, its pullback and $\mathcal{S}_{\mathrm{op}}(\mathbf{a}_i)$ are structured quantizations of the same contact symbols. Their structured uniqueness supplies the comparison in clause 5 of Definition 4.4, and the source affine torsor is retained under their first-unit identification. Unique normalized comparisons make these identifications compatible on intersections and with the physical operator lifts. All clauses of the independent operator source are therefore satisfied. A source-atlas morphism yields the corresponding phase and groupoid maps by Lemma 5.5, and its local physical operator lifts glue by these same equations. This constructs the functor $\mathcal{Q}$ on physical Hom classes.

Conversely, let $\mathcal{O}$ be an independent operator monodromy source over B. Near every point choose one of its admitted whole-sky operator models, recentered at that point when necessary. Apply the independent decoder $\mathcal{D}_{\mathrm{op}}$ from Theorem 9.10 of the local paper. It gives an original pair $\mathbf{a}_i$ at the same Einstein coefficient and with the same mark. The counit

$$\epsilon_i : \mathcal{S}_{\mathrm{op}}\mathcal{D}_{\mathrm{op}}(\mathcal{O}_i) \longrightarrow \mathcal{O}_i \tag{53}$$

identifies complete point families, so it gives an identification e_i of a retained derived source point domain with a neighborhood $U_i \subset B$.

These neighborhoods can be chosen to cover all of B. If a local counit is initially represented only near its central sky, recenter the supplied operator chart at each other point and use its local theorem there. Then choose a point-cover refinement by such neighborhoods. This refines the atlas and does not shrink the retained global point manifold. Relatively compact subcharts inside larger parents supply the conditioned local representatives when needed.

On a common point neighborhood, compose the supplied operator comparison with the counits and their inverses. Since the supplied comparison covers the identity of $U_i \cap U_j$, this composite covers $e_j^{-1}e_i$ and respects the supplied global identifications. Local full faithfulness in Isthmus (2026a, Theorem 9.10) gives a unique original physical source comparison, with its specified full spin lift, between the two decoded pairs after the necessary recentering. These local comparisons agree as germs where two choices overlap. Indeed, their operator images are the same physical Hom class, and the local source-to-operator map is faithful on that class with its full spin label.

The supplied operator cocycle commutes through its normalized comparison isomorphisms. In physical Hom classes it is an equality. Faithfulness of the original source correspondence therefore gives the strict identity, inverse and triple laws of the source atlas in (41). The source comparisons are holomorphic on their common parents by the analytic family assertion of Theorem 9.10 of the local paper. Their contact-affine representatives are

those of Isthmus (2026a, Proposition 8.1), hence obey the retained comparison equations. We obtain an independently admitted original-source atlas without using a metric-image criterion. The same local decoding applied to operator source morphisms defines a functor $\mathcal{D}$.

We verify the two inverse comparisons at the stated presentation level. For a source atlas, the local equality

$$\mathcal{D}_{\mathrm{op}}\mathcal{S}_{\mathrm{op}}(\mathbf{a}_i) = \mathbf{a}_i \tag{54}$$

is the equality of the two original input classes, Einstein coefficient and mark in Theorem 9.11 of the local paper. The fixed canonical projection gives the same canonical representatives. If the comparison uses recentered charts or different retained parents, Isthmus (2026a, Proposition 8.1) gives the initialized contact-affine comparison on a common refinement. The full-spin physical-arrow bijection identifies the decoded overlap instruction with the original c_{ji}. Consequently these local comparisons glue to the source-atlas unit. This recovers the original source data and comparisons.

For the operator composite, use the counits ϵ_i from the counit formula (53). On a point overlap their two composites have the same physical comparison and full spin lift. The unique normalized comparison identifies them with their module components and both operator associators. On triples those identifications are coherent for the same reason. Thus the local counits define a comparison of the complete operator-chart descent data.

The phase identification supplied by these counits glues globally. The reconstructed groupoid is the monodromy integration of that phase line, while the supplied groupoid has the same anchor and source-completeness conditions. The integration uniqueness of Theorem 3.11 identifies the two groupoids. The whole-sky evaluation comparisons identify their quadratic-affine bundles and partial connections; initialized transport then identifies their full actions. Finally, structured uniqueness in §4.1 identifies the global contact-groupoid algebroids and makes that identification compatible with the local counits. This is a counit in $\mathsf{OpMon}_{\Lambda}^{\mathrm{hol}}$, retaining its module and associator data.

For fullness, take an operator source embedding between two objects constructed from source atlases. On every common source and target point chart the original local correspondence decodes its physical Hom class to a source comparison with the given point map and spin lift. On intersections these local decoded comparisons agree by local faithfulness. They therefore define a source-atlas morphism. Its image has exactly the original local operator Hom classes. The global phase map is forced by the full incidence, the groupoid functor by its differentiated map and source simple connectedness, and the quadratic-affine map by evaluation and transport. The global algebroid comparison is fixed up to its unique structured isomorphism. Hence this image is the given operator source morphism.

For faithfulness, two source-atlas morphisms with the same operator image have equal local physical arrows, including their full spin labels, by

the physical Hom bijection of Theorem 9.10 of the local paper. They agree after common refinement and hence are the same source-atlas morphism. Identity and composition are preserved by their original physical laws. The local unit and counit naturality diagrams commute by Theorem 9.10 of the local paper; their normalized operator comparisons glue by uniqueness. This proves the equivalence and its stated presentation-level naturality. □

A *coherent holomorphic family of global sources* is specified over a finite-dimensional complex manifold P by a holomorphic submersion $\pi : B_P \to P$ with four-dimensional point fibers and local presentations of the relevant source type. Each point fiber carries an admitted object of that source category. In an original-source presentation, the chart inputs $\mathbf{a}_i$, derived point-domain identifications e_i and physical overlap comparisons c_{ji} depend holomorphically on P in the conditioned sense of §2, and their identity, inverse and cocycle equations hold before any parameter truncation. Operator presentations retain holomorphic associators, normalized comparison maps, roots and framed module maps satisfying their stated equations. The geometric presentations retain the corresponding holomorphic incidence and transport data. On each retained compact subsystem and specified finite comparison diagram, the common parents, inverse bounds and unit margins are those of Proposition 5.4. The null monodromy groupoid is formed fiberwise; the parameter assertions concern these local presentations and finite comparison diagrams.

At $p_0 \in P$, write $P_{p_0,N}$ for the finite Taylor neighborhood with local algebra $\mathcal{O}_{P,p_0}/\mathfrak{m}_{p_0}^{N+1}$. Preservation of finite parameter jets means that the completed local constructor and its comparisons commute with restriction to $P_{p_0,N}$ on every retained finite diagram. Thus for the source-to-operator constructor $\mathcal{Q}$,

$$\mathcal{Q}(\mathfrak{A})|_{P_{p_0,N}} \simeq \mathcal{Q}\Big(\mathfrak{A}|_{P_{p_0,N}}\Big), \tag{55}$$

where the right side denotes the Taylor restriction of the prescribed analytic local construction and its descent equations.

Proposition 5.7.

The equivalence of Theorem 5.6 retains the full local source-operation diagram in Theorem 9.12 of the local paper on every atlas chart. In particular, it uses the same two original classes, corrected nonlinear source constructor, six normalization requirements and selected analytic representatives. These local constructions and their physical comparisons determine one global operator-incidence source. Its rank-four spin, optical and infinity data of §6, and its calibrated incidence module of §4.3, are natural images of the same source atlas.

For a supplied coherent holomorphic family of source atlases on conditioned common domains, these comparisons are holomorphic and preserve its finite parameter jets on every retained finite diagram. The assertion concerns the completed analytic local constructors and coherent passive families. It does

not identify independently truncated internal construction-amplitude stages on different point charts.

Proof. On each chart, use the source constructor of Theorem 5.6. The comparison of Theorem 9.12 of the local paper identifies its operations, correction cochains, fixed inverses, normalizations and analytic representatives with those of the independent operator chart. On an overlap, the comparison is the physical source comparison required in Definition 5.1. Theorem 9.12 of the local paper identifies its contact, affine, optical, operator and framed module components. The comparison and associator equations used in Theorem 5.6 therefore glue these completed local outputs to one global source.

The selected scale and roots are transported by the physical comparisons. Recomputing the six normalization components in a different auxiliary chart represents this same selected calibration; it is not an equality of six chart-independent raw numbers. The analytic normalization and local finite-diagram comparison of Proposition 5.4 give their common representatives.

The normal spin diagram is natural for these comparisons by Theorem 6.3. The same is true of the global incidence module by Theorem 4.8 and Corollary 4.9. Hence both are carried through the source-atlas unit and operator counit, with their spin and root data.

For an admitted coherent parameter family, fix a supplied finite conditioned comparison diagram. Proposition 5.4 provides its local analytic representatives. A cocycle equation is checked near a common overlap point, using the finite local star there and the compatible construction of Lemma 5.3. All composition and inverse equations are equalities of these holomorphic maps or of their specified normalized comparison maps. Passing to finite Taylor neighborhoods preserves these equations. A compact set can be covered by finitely many such local parents, retaining their separate bounds. Gluing uses these local representatives and their germs, with the finite-star and local-bound conventions of Lemma 5.3 and Proposition 5.4.

Physical comparisons use each original constructor at its completed analytic value, or an already supplied coherent passive family. The distinction from independent internal-amplitude scaling is expressed by (47) in the discussion following Lemma 5.3. □

6. The projective spin diagram and Lorentz reality

Let an independent monodromy source be given, and let (B, g) be its recognized holomorphic spin Einstein manifold. Write $p : F \to B$ for its full null phase, K for the characteristic line and $\mathcal{G} = \mathrm{Mon}(K)$ for its recognized groupoid. The complete-sky evaluation data are $(\mathcal{T}, D, H, \mathsf{I}, \mathsf{X})$, and $\Theta(v) = D_v\mathsf{X}$. The Einstein coefficient is fixed by $\mathrm{Ric}(g) = \Lambda g$.

The local exterior-square, flag, optical and Lorentz comparisons are those of Proposition I.3, Theorem J.2 and Propositions J.4 and J.7 of the local paper, assembled in Theorem 9.12 of that paper. Their tensor formulas

are equivariant for the specified full spin maps. We use them to construct the bundles on the point manifold and prove their continuation along all null-path classes. The point spin structure is retained before any quotient by rays.

Proposition 6.1.

The point spin structure determines a rank-four holomorphic bundle $\mathcal{T}_{\mathrm{LT}}$, its volume-preserving normal spin connection and a parallel identification

$$\mathcal{F}_{\mathrm{spin}} : \mathcal{T} \xrightarrow{\sim} \Lambda^2 \mathcal{T}_{\mathrm{LT}}^*. \tag{56}$$

The pullback $\mathbb{L} = p^*\mathcal{T}_{\mathrm{LT}}$ is a holomorphic representation of $\mathcal{G}$ preserving its volume and a flag $\ell \subset h \subset \mathbb{L}$ of ranks one and three. For the contact-line representation $\mathcal{L}_F = TF/E$,

$$\ell \otimes \mathrm{Ann}(h) \simeq \mathcal{L}_F^{-1}. \tag{57}$$

On each point quadric Q_x, $\mathbb{L}$ is constant with fiber $(\mathcal{T}_{\mathrm{LT}})_x$, while $\ell|_{Q_x} = \mathcal{O}(0,-1)$ and $\mathrm{Ann}(h)|_{Q_x} = \mathcal{O}(-1,0)$ in the specified order of rulings.

The alternating form $I_{\mathrm{cov}} = \mathcal{F}_{\mathrm{spin}}(\mathsf{I})$ and its volume dual I_{contra} are invariant. With $c = \Lambda/6$, their normalization is

$$I_{\alpha\beta} I^{\beta\gamma} = c\, \delta_\alpha{}^\gamma, \qquad 2\,\mathrm{Pf}(I_{\mathrm{cov}}) = -\Lambda/3. \tag{58}$$

Both forms have rank four for $\Lambda \neq 0$ and rank two for $\Lambda = 0$.

Proof. In the selected Einstein scale, the spin bundle supplies the two determinant-one spinor bundles and their rank-four sum. The local normal spin connection is the Levi-Civita spin connection plus the algebraic soldering term with coefficient c. The exterior-square comparison of Proposition I.3 of the local paper is spin-equivariant and identifies its induced connection with D. These formulas therefore glue through the spin cocycle. Their trace is zero and their transition determinants are one, giving the parallel volume and the displayed global identification.

Pull the connection back to F and restrict it to K. Its curvature on this complex line distribution vanishes. Its transport along a leafwise path consequently depends only on its endpoint-fixed homotopy class. Denote it by $\Pi_\gamma : \mathbb{L}_u \to \mathbb{L}_v$ for $\gamma : u \to v$. A representative path is covered by finitely many holomorphic flow boxes. In each box, transport solves a holomorphic linear matrix equation; finite composition gives holomorphic dependence in a path-continuation chart. Homotopy invariance and ODE uniqueness identify these expressions on chart intersections and give the identity, inverse and composition laws. This constructs a representation on the full, possibly non-Hausdorff, arrow manifold.

The parallel exterior-square comparison gives

$$\mathcal{F}_{\mathrm{spin},v}(P_\gamma Y) = (\Pi_\gamma^{-1})^* \mathcal{F}_{\mathrm{spin},u}(Y), \tag{59}$$

where P_γ is the rank-six transport of the geometric source. Transport around a nontrivial loop is retained in this equation.

For a null tangent $k^{AA'} = \kappa^A \pi^{A'}$, the local flag is $\ell = \mathbb{C}(0, \pi)$ and $h = \ker(\kappa_A, 0)$ in the spin conventions of the local comparison. Rescaling either factor leaves these subspaces unchanged. Along an affine null geodesic choose parallel spin factors. The skew-spinor identities make both $(0, \pi)$ and $(\kappa_A, 0)$ parallel for the normal spin connection and its dual. Thus every Π_γ preserves the flag. Their tensor product is the tautological null-covector line, identified with $\mathcal{L}_F^{-1}$ by the characteristic pairing $g(k, \mathrm{d}p\,\cdot)$. Parallelism preserves this identification. Restriction to Q_x gives the two stated tautological ruling lines.

Finally I is parallel. In a compatible spin splitting, the fixed volume and the selected section give

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad I_{\mathrm{cov}} = \mathrm{diag}(-cJ_2, J_2), \qquad I_{\mathrm{contra}} = \mathrm{diag}(J_2, -cJ_2).$$

The local exterior-square normalization is $2\,\mathrm{Pf}\,\mathcal{F}_{\mathrm{spin}}(Y) = H(Y, Y)$. The displayed matrices therefore give both stated identities and ranks. They glue by the same volume-preserving spin comparisons. This construction keeps the selected Einstein section and the normalization $H(\mathsf{I}, \mathsf{X}) = 1$; replacing its infinity form by an arbitrary projective multiple would discard that scale information. No inverse infinity form is used, so $\Lambda = 0$ is included. □

Proposition 6.2.
There is a holomorphic totally null rank-two subrepresentation

$$\mathbb{V}_u = \mathrm{span}\{\mathsf{X}_x, \Theta_x(k)\} \subset (p^*\mathcal{T})_u, \qquad u = (x, [k]). \tag{60}$$

Its dual is the monodromy continuation of the intrinsic optical solution bundle, with calibrated Wronskian $\Lambda^2\mathbb{V}^* \simeq \mathcal{L}_F$. The affine line is exactly

$$(A_F)_u = \{v \in \mathbb{V}_u : H(\mathsf{I}, v) = 1\}. \tag{61}$$

Taking the kernel of the exterior-square form gives an equivariant isomorphism from $\mathbb{P}(\mathbb{V}_u)$ to the pencil of two-planes L with $\ell_u \subset L \subset h_u$. The position selects

$$L_u^{\mathrm{point}} = \ker \mathcal{F}_{\mathrm{spin}}(\mathsf{X}_x). \tag{62}$$

This selected incidence section records the point's position in the invariant pencil and need not itself be groupoid-invariant.

Proof. Rescaling k leaves the plane unchanged. The identities $H(\mathsf{I}, \mathsf{X}) = 1$, $H(\mathsf{I}, \Theta(k)) = 0$ and $\Theta(k) \neq 0$ give independence of its generators. Their three mutual H-pairings vanish. Along an affinely parametrized null geodesic the canonical Einstein-scale connection satisfies

$$D_k\mathsf{X} = \Theta(k), \qquad D_k\Theta(k) = 0.$$

Thus $\mathbb{V}$ is parallel along K, and its affine slice is $\mathsf{X} + \mathbb{C}\Theta(k)$.

On a simply connected short leaf segment, a parallel covector $\lambda \in \mathbb{V}^*$ determines $y(t) = \lambda(\mathsf{X}(t))$. It satisfies $y'' = 0$, and its initial value and derivative determine λ. This is the local Cauchy identification with the intrinsic optical equation. In an auxiliary conformal parameter it has the

projective optical equation and density transformation proved in Propositions 5.3 and J.4 of the local paper. Since the identification uses evaluation and parallel transport, it agrees on overlaps and extends through every finite chain of leafwise continuation. It therefore identifies representations, including their possible loop actions.

The functional $H(\mathsf{I},\cdot)$ has value one on the affine slice and kernel the translation line. Replacing X by another point in that slice leaves $\mathsf{X}\wedge\Theta(k)$ unchanged. Hence $\det\mathbb{V}$ is canonically the translation line $\mathcal{L}_F^{-1}$, and its dual gives the calibrated value-slope determinant on the optical solutions.

The kernel-pencil identity is the algebraic identity in Proposition J.4 of the local paper. For completeness, every nonzero form in the image of the totally null plane $\mathbb{V}_u$ has zero Pfaffian and rank two. Its kernel contains ℓ_u and lies in h_u. A rank-two alternating form on a four-dimensional space is determined up to scalar by its kernel. Hence the kernel map is injective on $\mathbb{P}(\mathbb{V}_u)$; as a nonconstant holomorphic map between compact projective lines it is surjective and therefore an isomorphism. This calculation is intrinsic under the exterior-square comparison. The transport identity (59) makes it equivariant on every arrow. Evaluation at X_x gives the selected point plane. Since $P_\gamma\mathsf{X}_x$ generally differs from the position at the endpoint, invariance is asserted for the pencil and not for that selected member. □

Theorem 6.3.

The rank-four, flag, infinity, optical and point-incidence constructions form a natural diagram on independent monodromy sources. On each whole-sky chart it is the diagram in Theorem J.6 of the local paper. It is natural for physical spin embedding morphisms, complete-transversal changes and recomputed point-neighborhood restrictions. Consequently it gives a diagram on the source germs along the entire real development defined in §7, independent of the representative complex extension under the specified real comparison.

Normalized operator comparison isomorphisms over the same physical map and spin label induce identity comparison on this finite-rank diagram. The central negative spin element acts by -1 on $\mathbb{L}$.

Proof. Complete-sky evaluation identifies the source data with the canonical rank-six data on B, while the spin comparisons identify their normal spin lifts. The local comparisons of Theorem J.6 of the local paper therefore glue on the point manifold.

A holomorphic spin isometric embedding $f : B \to B'$ induces a map f_{LT} intertwining the normal spin connections: it preserves the Levi-Civita connection, soldering and numerical coefficient c. Initialized transport uniqueness on a finite flow-box chain gives

$$f_{\mathrm{LT},v}\Pi_\gamma = \Pi'_{F(f)\gamma} f_{\mathrm{LT},u}. \tag{63}$$

This proves naturality on every path class even if the induced groupoid functor identifies some classes. It preserves volume, flag, infinity forms and optical planes. Evaluation and the kernel construction also preserve the

selected incidence maps. The comparisons respect composition because the point-bundle maps and initialized transport do.

Changing complete transversals restricts these same equivariant objects to their common presentation. On restricting the point domain, the groupoid is recomputed and maps to the original one by inclusion of paths. Pullback of Π is exactly the transport of the restricted connection. All the bundles and maps therefore restrict to the newly constructed diagram, without an assertion that the restriction functor is faithful or a Morita equivalence. The comparison of complexifications along the entire real development and the germ correspondence proved in §7 then give the stated independence.

A normalized operator comparison isomorphism preserves the same physical symbols, torsor, root and spin label. Its finite-rank comparison is identity by Proposition J.3 of the local paper. Transport uniqueness propagates that identity along every arrow. This does not remove the operator associators or their module actions. Finally the central negative spin element acts by -1 on both spinor summands. It is invisible on their exterior square and projective flags but remains -1 on $\mathbb{L}$. □

Assume now that Λ is real and that the source has compatible Lorentz reality, with its real point spin lift. Write $\bar{u}$ and $\bar{\gamma}$ for the conjugate phase point and path. When the designated real locus is a globally hyperbolic development M, its real-path groupoid is $\mathrm{Mon}(K_{\mathbb{R}})$. It consists of paths in the real null phase, rather than all complex arrows between real objects.

Proposition 6.4.

The rank-four representation has an antilinear duality

$$J_u : \mathbb{L}_u \longrightarrow \mathbb{L}_{\bar{u}}^*, \qquad J_v \Pi_\gamma = (\Pi_{\bar{\gamma}}^{-1})^* J_u. \tag{64}$$

Its dual companion has the retained normalized square. It exchanges ℓ with $\mathrm{Ann}(h)$, pairs a point plane with the annihilator of its conjugate point plane, and exchanges the covariant and contravariant infinity forms. These comparisons commute with Theorem 6.3. On the designated real-path groupoid they descend to a rank-four bundle with this duality over the real null-ray space.

Proof. The compatible point spin reality gives the local duality in Proposition J.7 of the local paper. In normalized Lorentz spin frames it is $J(Z) = Z^\dagger H_{\mathrm{LT}}$, with

$$H_{\mathrm{LT}} = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}.$$

This is a covector-valued antilinear map. The local normal spin connection preserves it, and its dual companion has square one in these frames, with the retained root and spin square globally. These tensor identities glue through the spin cocycle and extend to paired complex points. Integrating parallelism along a finite flow-box chain gives the displayed transport equation. Consequently the local flag, point-plane and infinity-form comparisons extend over every path class. The latter use the volume dual of the infinity form and remain valid when $\Lambda = 0$.

All the local identities are natural for the specified spin maps, and initialized transport proves their compatibility with embedding and restriction. On a globally hyperbolic real development, maximal real null leaves are intervals. Real-path transport consequently has one class between two points of the same real ray and descends through its real ray submersion. This gives the final assertion for the real-path groupoid of Proposition 3.9. □

Proposition 6.5.
The operator equivalence in Theorem 4.6 retains compatible Lorentz reality and the designated real-path groupoid. On operator representatives, reality is a structured antilinear operation with its specified square coherence. For a local object $\mathcal{O}$ and its conjugate $c\mathcal{O}$,

$$\begin{aligned} J_{c\mathcal{O}}\overline{J_{\mathcal{O}}} &= \mathrm{Ad}(Z_{\mathcal{O}}), \\ J_{\mathcal{O}}(\overline{Z_{\mathcal{O}}}) &= Z_{c\mathcal{O}}, \\ j_{c\mathcal{O}}\overline{j_{\mathcal{O}}} &= Z_{\mathcal{O}} \cdot (-). \end{aligned} \tag{65}$$

Here $Z_{\mathcal{O}}$ is the normalized square unit and $j_{\mathcal{O}}$ is the compatible framed module map. This antilinear operation is distinct from the complex-linear formal adjoint.

Proof. Apply the coupled Lorentz construction of Theorem G.5 of the local paper on each operator chart, retaining the root square and whole-family framing. It gives exactly the displayed equations. On a point overlap, the conjugate comparison and the comparison of conjugates cover the same classical Lorentz map and spin lift. The unique normalized comparison identifies them.

Transporting a square relation across such a comparison gives two normalized units implementing the same inner automorphism. Normalized-inner injectivity makes them equal. Framed semilinear Hom uniqueness gives the module equation and its coherence. The same uniqueness on triple and quadruple overlaps makes these comparisons compatible with both operator associators.

Choose contact transversals together with their conjugates. The conjugate algebroid, pulled back by the antiholomorphic contact involution, is another structured quantization of the same groupoid. Theorem 4.2 gives its comparison with the original algebroid and the unique comparison of its square with the identity. Its restrictions to whole-sky charts agree with the preceding local construction through the unique structured comparisons. Thus the antilinear action and its square glue globally. The finite-rank square in Proposition 6.4 does not strictify the chosen operator representatives.

Finally the original, conjugate and reconstructed whole-sky objects are related by the local counits, which preserve these same reality data. Their uniqueness makes both global inverse comparisons of Theorem 4.6 compatible with the antilinear action. All constructions retain the designated real-path groupoid and use only its real paths for real descent. □

The same equations give a weak Lorentz comparison on the incidence module $\mathscr{M}_F$: conjugation preserves the central point parameters and sends each local incidence presentation to its conjugate one; the framed maps $j_{\mathcal{O}}$ glue by the module descent equations of Theorem 4.8. Their square is the action of the retained units $Z_{\mathcal{O}}$. Central specialization sends the point object at x to that at $\bar{x}$, with its root and frame comparisons. Thus the full parametrized point family, its finite-rank spin diagram and its operator coherence have the same Lorentz continuation.

7. Germ correspondence and maximal Cauchy development

Fix the analytic Cauchy data of §2, including the orientations and prescribed initial spin structure. Their development category is Definition A.1. We now remove the choice of complex point extension from the independent source-atlas, operator and monodromy correspondences.

An *extension object* has a connected point domain (B, σ) whose fixed locus is exactly a connected real four-manifold M. Require compatible antiholomorphic reality on its incidence, groupoid and quadratic-affine data, and actual spin reality on its whole-sky charts. The chart reality must have the Lorentz real form described in §2 and in §8.4 of the local paper: its retained spin/tangent calibration at each real mark has tangent form $\eta = \mathrm{diag}(1, -1, -1, -1)$ in the selected Einstein scale. By Lemma 2.4, the locally recognized metrics are real with this signature, the fixed incidence $F_{\mathbb{R}} \to M$ has real two-sphere fibers, and the lines $\mathrm{d}p(K_{\mathbb{R}})$ form their null cone. Supply a time orientation of that cone, an orientation of M, and an analytic embedding $\iota : \Sigma \to M$ that is spacelike and Cauchy for its causal structure. In particular every inextendible real causal curve must meet $\iota(\Sigma)$ exactly once. Require that local recognition along this embedding gives the prescribed forms (h, K) and orientation convention of Definition A.1, together with an actual identification of P_Σ with the initial adapted-frame spin reduction. These are conditions on the supplied source and its local models; no global Einstein metric is supplied. Morphisms preserve all these markings. The designated real-path groupoid is $\mathrm{Mon}(K_{\mathbb{R}})$, with its intrinsic functor into the complex monodromy groupoid, as in Proposition 3.9.

For an original-source atlas, an extension structure means the preceding reality and initial markings on the phase constructed in Lemma 5.5, with the local whole-sky models and their physical overlap comparisons compatible with that reality. The Lorentz operation on the original pair is the residue-transferred operation in §8.4 of the local paper; its two classes retain their weights 2 and -6. For an operator source, use its supplied phase and the same markings, requiring the structured local comparisons and their descent maps to preserve them. On operator representatives the induced reality has the normalized inner square, framed module square and overlap coherence of Proposition 6.5. These are equations on structured representatives, considered through their normalized comparisons. Morphisms in both extension

categories are their physical morphisms preserving this data, including the specified real spin and initial identifications. Thus the extension structures are defined on the independently constructed source phases before global metric recognition.

An *allowed point restriction* is a connected conjugation-invariant open neighborhood of all of M in B. Every such restriction retains the entire real development, its initial hypersurface, and the full complex quadric over every retained point.

Proposition 7.1.

Let $\mathcal{S}$ be an extension monodromy source on B, and let $B_0 \subset B$ be an allowed point neighborhood. Its canonical recomputed restriction $\mathcal{S}^{\mathrm{mon}}_{B_0}$ has point space B_0, incidence $F_0 = p^{-1}(B_0)$ and groupoid

$$\mathcal{G}_0 = \mathrm{Mon}(K|_{F_0}). \tag{66}$$

The remaining data restrict to F_0, with action pulled back along the integrated inclusion $\mathcal{G}_0 \to \mathcal{G}$. The result is an admitted extension source with the same real development and markings, and inclusion gives a source morphism $r_{B_0} : \mathcal{S}^{\mathrm{mon}}_{B_0} \to \mathcal{S}$. This operation is canonically compatible with further restrictions and with source morphisms whose point maps take the smaller source domain into the smaller target domain.

Proof. The supplied characteristic line restricts without using a metric. Its monodromy integration is source connected and source simply connected. The Lie algebroid inclusion integrates uniquely into the supplied source groupoid, by the uniqueness of integration used in Proposition 3.7. Pullback along this functor gives the vector and affine actions. Whole-sky conditions persist, since point restriction retains every full quadric; properness of p is stable under this base restriction. The short-path compatibility equations restrict and are preserved by composition of the integrated inclusions. Reality and all initial markings remain unchanged because every real point is retained.

For nested restrictions, the integrated functors send a path to the same path in the larger domain. They agree by uniqueness of integration, as do their representation comparisons. A source morphism sends paths and their homotopies in a smaller source domain to paths and homotopies in the smaller target domain whenever the stated image condition holds. This proves functoriality. □

The groupoid $\mathcal{G}_0$ is generally not the full subgroupoid of $\mathcal{G}$ on F_0. Such a full subgroupoid includes paths leaving F_0, while homotopies in the larger leaf may identify classes that remain distinct in the smaller leaf. The integrated inclusion need not be faithful or a Morita equivalence.

Operator restriction simultaneously restricts symbols, affine torsors, whole-family modules, roots, frames, associators and their comparison maps, and pulls the groupoid action back along the same integrated inclusion. All their defining equations persist. The structured comparison of Corollary 4.3 identifies this with the quantization of the new contact monodromy presentation. Original-source restriction restricts every point chart and

physical comparison germ. Lemma 5.5 identifies the resulting completed groupoid with $\mathrm{Mon}(K|_{F_0})$. The fixed-domain source-atlas and operator equivalences therefore commute with these restrictions through their inverse comparisons.

Definition 7.2.

The category $\mathsf{MonRay}^{\mathrm{germ}}_{\Lambda}(\Sigma, h, K)$ has the extension monodromy sources as objects. A morphism from $\mathcal{S}$ on (B, M) to $\mathcal{S}'$ on (B', M') is represented by an allowed neighborhood $B_0 \subset B$ and an enriched source morphism $f : \mathcal{S}^{\mathrm{mon}}_{B_0} \to \mathcal{S}'$. Its real point map must preserve orientation, time orientation, the initial embedding and the prescribed initial spin identification. Two representatives define the same morphism when they agree after recomputed restriction to a common allowed neighborhood of all of M. Equality includes the spin lift. Equivalently, a morphism is a roof

$$\mathcal{S} \xleftarrow{r_{B_0}} \mathcal{S}^{\mathrm{mon}}_{B_0} \xrightarrow{f} \mathcal{S}', \tag{67}$$

modulo common point restriction. The source-atlas and operator categories $\mathsf{SrcAtl}^{\mathrm{germ}}_{\Lambda}(\Sigma, h, K)$ and $\mathsf{OpMon}^{\mathrm{germ}}_{\Lambda}(\Sigma, h, K)$ are defined by the same construction with their respective physical morphisms and restrictions. Atlas refinement and normalized operator comparison have the conventions of Definition 5.1 and Definition 4.4. In particular operator equality is equality of the physical Hom classes with their specified normalized comparisons, while the full spin labels remain distinct.

For composition, if the second map is represented on $B'_0 \subset B'$, restrict the first domain to the component containing M of

$$B_0 \cap f^{-1}(B'_0). \tag{68}$$

This component contains all of connected M and is invariant under conjugation. Recompute the source groupoid there, use the induced functor into the recomputed groupoid over B'_0, and compose. Common restriction proves independence of representatives and associativity; identity source maps give identities. The restriction arrow r_{B_0} is invertible in this category, with inverse represented by the identity on the restricted object viewed as a germ from the original object.

These categories are precisely the localizations by the specified restriction arrows. Indeed, any functor L on extension objects that inverts those arrows must send a roof to $L(f)L(r_{B_0})^{-1}$. Common restriction makes this expression well defined, and the composition formula makes it functorial. Every germ arrow has a roof, so this factorization is unique. This is a neighborhood localization of enriched sources, defined before global metric recognition; it does not declare the integrated restriction functors to be groupoid equivalences.

Lemma 7.3.

Let (B, g, σ) and (B', g', σ') be compatible holomorphic oriented spin extensions with fixed loci exactly M and M'. Every real analytic spin

isometric embedding $f : M \to M'$ has a unique germ along all of M of a compatible holomorphic spin isometric embedding into B'. This allows noncompact M and nonproper f, and preserves composition and full spin lifts.

Proof. Equal dimensions make $f(M)$ open in M', with f an analytic diffeomorphism onto its image. The fixed locus M' is closed in the Hausdorff manifold B', and $M' \setminus f(M)$ is therefore closed in B'. The invariant open complex manifold

$$\Omega = B' \setminus (M' \setminus f(M))$$

has fixed locus exactly $f(M)$. Via f, the manifolds B and Ω are complexifications of the same real analytic manifold. Germ uniqueness of complexification (Whitney and Bruhat 1959, sec. 1, Proposition 1) gives a biholomorphism between neighborhoods of all of M and all of $f(M)$ extending f. This gives injectivity on a suitable source neighborhood, including when the real embedding is nonproper.

The metric tensors agree on the maximal totally real slice and hence, by holomorphic uniqueness, on a neighborhood of that entire slice. Comparing the extension with its conjugate makes it equivariant after an invariant restriction. The spin lift extends by the local holomorphic double-cover construction of Proposition 3.1. Its sheet choices agree on real overlaps and therefore on smaller complex overlaps, retaining the prescribed sign. Connected neighborhoods containing all of M can be retained throughout.

Two extensions have equal holomorphic germs by uniqueness on the totally real slice. Their spin lifts have the same sheet choices there, so their germs also agree. Applying this uniqueness to composites proves the composition assertion. The argument requires no uniform complex radius. □

Theorem 7.4.
Under the local inputs of §2, real recognition gives equivalences

$$\begin{aligned} \mathsf{SrcAtl}_{\Lambda}^{\mathrm{germ}}(\Sigma, h, K) &\simeq \mathsf{OpMon}_{\Lambda}^{\mathrm{germ}}(\Sigma, h, K) \\ &\simeq \mathsf{MonRay}_{\Lambda}^{\mathrm{germ}}(\Sigma, h, K) \\ &\simeq \mathsf{Ein}_{\Lambda}^{\omega}(\Sigma, h, K). \end{aligned} \tag{69}$$

They retain the proper complete four-dimensional point family, each full complex sky, the quadratic-affine representation, the Einstein scale, Lorentz reality, the designated real-path groupoid and the spin and initial identifications. On each whole-sky chart they agree with the original local correspondence, including its two input classes, source-operation comparisons and normalized operator and module data. The projective spin diagram and parametrized incidence-module family are carried by the same natural comparisons. Two compatible complex extensions of one real development give canonically isomorphic source germs under the specified real spin identification.

Proof. First consider monodromy sources. Theorem 3.11 recognizes an independent source as a holomorphic Einstein manifold and identifies its source morphisms with holomorphic spin isometric embeddings. By Lemma

2.4, local recognition intertwines the supplied conjugation and spin reality, and $\sigma^*\overline{g} = g$. Its fixed metric on M is real, and the retained Lorentz tangent calibration gives signature $(+,-,-,-)$. Its null directions are precisely $\mathrm{d}p(K_{\mathbb{R}})$. The admission conditions therefore make $\iota(\Sigma)$ Cauchy for this metric and give the prescribed (h, K), orientations and initial spin identification. Thus its real restriction is an analytic spin development. Recomputed restriction changes none of its real data. Real recognition defines the required functor to $\mathsf{Ein}^{\omega}_{\Lambda}(\Sigma, h, K)$.

Every analytic spin development has a compatible holomorphic extension with precisely that real fixed locus, by Proposition 3.1. The canonical monodromy source on this extension is independently admitted by Theorem 3.11, and its recognition returns the original development. This proves essential surjectivity.

For fullness, take a development morphism $f : M \to M'$, with its spin lift. Lemma 7.3 extends it to a holomorphic spin isometric embedding from a neighborhood of all of M into the target extension. Fixed-domain full faithfulness then supplies the corresponding enriched source morphism on the recomputed source domain. Its germ recognizes to f. The induced groupoid functor sends each path class to the class of its image; no injectivity on arrows is required.

For faithfulness, suppose two source-morphism germs recognize to the same real spin embedding. The lemma makes their holomorphic point maps and spin lifts agree on a common neighborhood of all of M. Fixed-domain full faithfulness identifies their enriched morphisms on the recomputed source there. This comparison controls every path in that smaller domain: each path is a finite composite of short paths, and the source connected and source simply connected integration has a unique functor inducing the fixed infinitesimal comparison. Thus the source germs agree. The monodromy recognition functor is fully faithful and essentially surjective.

Now use Theorem 4.6 and Theorem 5.6 on the fixed point domains. Their local reality comparisons are those of Proposition 8.4 and Theorems 9.10 and G.5 of the local paper. The source cocycles and full spin lifts make these comparisons coherent on overlaps. On operator representatives they preserve the normalized inner square and the module square, with the supplied associators present; the original class-level involution does not change these into strict equations on arbitrary representatives. The designated real-path groupoid is the integration of the real characteristic line throughout.

Both fixed-domain equivalences commute with recomputed restriction, as established after Proposition 7.1. They consequently induce functors on the roof categories. To verify fullness, represent a target germ arrow after one allowed point restriction and apply the fixed-domain full Hom correspondence on that domain. For faithfulness, equality of two target germs is witnessed after a further common point restriction; fixed-domain faithfulness then identifies their lifted physical Hom classes there. For essential surjectivity,

construct or decode a representing object on its point domain by the fixed-domain inverse. Transport the supplied phase reality and initial markings through these inverse comparisons; their local reality compatibility and the normalized operator square of Proposition 6.5 give exactly the extension structures specified above. This proves both remaining equivalences without any separate assumption about groupoid equivalence of restriction arrows.

The inverse comparisons used here are the local evaluation, source-operation and operator-module comparisons, extended over paths by unique integration and glued with their specified associators. Their naturality follows from the established Hom bijections. Proposition 5.7, Theorem 6.3, and Theorem 4.8 show that the indicated projective and parametrized-module data are carried by these same comparisons. They commute with restriction and therefore pass to the germ categories.

Finally, apply fullness to the identity real spin isometry between two complex extensions, and faithfulness to the composites of the resulting arrows. This gives their unique source-germ isomorphism under the specified real identification. All restrictions retain every real point and its whole complex sky, so the equivalence retains the entire development. □

Corollary 7.5.

Each source germ category in Theorem 7.4 has a terminal object whose recognized real point development is M_ω. It is unique up to the unique initial-data-preserving physical spin isomorphism. An analytic isomorphism of the initial data, with its specified spatial spin lift, induces the unique corresponding isomorphism of maximal source germs, compatibly with identities and composition.

Proof. By Corollary A.8, the prescribed analytic constraint data have the terminal analytic spin development M_ω. Choose a compatible holomorphic extension and a corresponding object in each source category. Any other source germ recognizes to an analytic development N. Its unique development morphism $N \to M_\omega$ lifts to a unique physical source-germ arrow by the full faithfulness of Theorem 7.4. This is the terminal property, and it also proves independence of the extension and source representatives.

For an initial-data isomorphism, transport the initial embedding and spin identification as in Corollary A.8. Its unique maximal real spin isometry lifts uniquely through the equivalences. Uniqueness proves the identity and composition laws. These are the same comparisons that preserve the original source operations and global point-object diagram. In particular they preserve curvature tensors and all their derivatives; no independently specified abstract Cauchy record is required. □

The terminal object belongs to the analytic Cauchy germ category of Remark A.9. Its source presentation retains the full point family, spin lifts and parametrized module under the restriction conventions of Proposition 7.1 and Definition 7.2.

8. Nariai obstructions

The equal-radius Nariai development shows why the preceding correspondence retains a monodromy presentation and parametrized incidence. First, no Hausdorff holomorphic ray carrier can contain all complex null directions over its entire real development. Second, in its product complexification the exact ray image of a point is not the support of an ordinary locally finitely generated global module, even after restriction around all real points. The first assertion applies to every complexification. The second holds for every real point on every allowed product neighborhood and hence on a cofinal collection of representatives of the germ of any compatible complexification.

Let $a > 0$ and set

$$\begin{aligned} &H_1 = \mathrm{d}x_0^2 - \mathrm{d}x_1^2 - \mathrm{d}x_2^2, \qquad H_2 = -\mathrm{d}y_1^2 - \mathrm{d}y_2^2 - \mathrm{d}y_3^2, \\ &Q_i = \{v : H_i(v,v) = -a^2\}, \quad B_0 = Q_1 \times Q_2, \qquad g_0 = H_1|_{TQ_1} + H_2|_{TQ_2}. \end{aligned} \tag{70}$$

Write M for the conjugation-fixed locus. The product geometry is standard (Beyer 2009, sec. 3.2.1); the obstructions are proved below.

Proposition 8.1.
The real metric is

$$M = \mathbb{R} \times S^1 \times S^2, \qquad g = \mathrm{d}t^2 - a^2 \cosh^2(t/a)\, \mathrm{d}\theta^2 - a^2 g_{S^2}. \tag{71}$$

It is analytic, geodesically complete and globally hyperbolic, with $\mathrm{Ric}(g) = \Lambda g$ for $\Lambda = -a^{-2}$ in our curvature convention, and with nonzero Weyl tensor. Its compact Cauchy data at $t = 0$ are $h = a^2(\mathrm{d}\theta^2 + g_{S^2})$ and $K = 0$. With either prescribed spin structure on $S^1 \times S^2$, it is the maximal analytic spin Cauchy development of these data.

Proof. Parametrize the real factors by

$$x_1 = (a \sinh(t/a), a \cosh(t/a) \cos\theta, a \cosh(t/a) \sin\theta),$$
$$x_2 = a\omega, \quad \omega \in S^2.$$

The induced metric is the displayed one. On each factor,

$$R_i(v,w)u = -a^{-2}\big(g_i(w,u)v - g_i(v,u)w\big).$$

Since both factors have dimension two, their Ricci tensors are $-a^{-2}g_i$, proving the Einstein equation. Mixed product curvature vanishes, whereas for nonzero $v \in TQ_1$ and non-null $w \in TQ_2$,

$$W(v,w)w = -\frac{\Lambda}{3} g(w,w)v \neq 0. \tag{72}$$

The function t is temporal. On every bounded time interval the spatial metric is bounded below by a fixed positive metric on the compact spatial manifold. A causal curve parametrized by t thus has bounded spatial speed. A finite endpoint of its time range would give a spacetime endpoint and an extension. Every inextendible causal curve therefore meets every time slice

exactly once. At zero the derivative of the spatial metric vanishes, giving $K = 0$.

Each affine factor geodesic satisfies

$$x_i'' = (\varepsilon_i/a^2)x_i, \qquad \varepsilon_i = g_i(x_i', x_i') \text{ constant}. \tag{73}$$

The linear equation has solutions for all real parameters and preserves the quadric and tangency constraints. All product geodesics are complete. Such a manifold has no proper connected isometric open extension: choose a convex normal neighborhood of a boundary point of a proposed image and a point of the image in that neighborhood. The ambient geodesic joining them must be the image of the complete source geodesic with the same initial data, placing the boundary point in the image, a contradiction. Its embedding supplied by the maximal analytic development theorem (Theorem A.7) is consequently onto. Finally, $S^1 \times S^2$ has two spin structures, parametrized by $H^1(S^1 \times S^2; \mathbb{Z}_2) = \mathbb{Z}_2$, and each extends over the product with $\mathbb{R}$. The spin comparison in that theorem gives the stated identification. □

Theorem 8.2.

Let (B, g_B) be any holomorphic extension of this real Nariai metric, and let $F \to B$ be its full projective null bundle with line field K. There is no triple consisting of an open subset $U \subset F$ containing the entire quadric F_x for every $x \in M$, a Hausdorff complex manifold Z, and a holomorphic map

$$q : U \longrightarrow Z, \qquad \ker \mathrm{d}q = K|_U. \tag{74}$$

The obstruction is independent of projected-sky embeddedness, affine enrichment and spin structure.

Proof. At a real point, the tangent space of every complexification identifies with the complexified real tangent space. We can therefore use the following directions over real points without choosing any nonreal extension. Fix

$$x_1 = (0, a, 0), \qquad x_2 = ae_3, \qquad x = (x_1, x_2).$$

For a real parameter η near zero put

$$\begin{aligned} k_\eta &= (a\cosh\eta, 0, a\sinh\eta), \\ y_{1,\eta} &= x_1\cosh\pi + k_\eta\sinh\pi, \\ k_\eta' &= x_1\sinh\pi + k_\eta\cosh\pi, \qquad y_\eta = (y_{1,\eta}, -x_2). \end{aligned} \tag{75}$$

The initial and final first-factor velocities have squared length a^2. For $z \in \mathbb{C}^*$ let

$$u(z) = \frac{a}{2}((z + z^{-1})e_1 - i(z - z^{-1})e_2).$$

This is tangent at x_2, has squared length $-a^2$, and is real when $|z| = 1$. For these real directions,

$$\gamma_{\eta,z}(s) = \big(x_1\cosh s + k_\eta\sinh s, x_2\cos s + u(z)\sin s\big), \qquad 0 \le s \le \pi, \tag{76}$$

is an affine null geodesic from x to y_η, with initial and final velocities $(k_\eta, u(z))$ and $(k_\eta', -u(z))$. Its phase lift lies in U, so its endpoints have the same image under q.

These endpoint directions extend over the whole parameter curve $\mathbb{P}^1$. Put

$$v(S,T) = a(S^2+T^2)e_1 - ia(S^2-T^2)e_2$$

and define

$$\begin{aligned} \alpha_\eta([S:T]) &= (x, [2STk_\eta, v(S,T)]), \\ \beta_\eta([S:T]) &= (y_\eta, [2STk'_\eta, -v(S,T)]). \end{aligned} \tag{77}$$

The vectors never vanish and their two squared lengths are $4a^2S^2T^2$ and its negative. Thus these are holomorphic maps into the full fibers retained in U. For $z = S/T$ on the unit circle, the real geodesics give $q\alpha_\eta = q\beta_\eta$. The diagonal of the Hausdorff complex manifold Z is a closed analytic subset. Hence the equalizer is a closed analytic subset of $\mathbb{P}^1$ containing the unit circle, and is all of $\mathbb{P}^1$.

At $[S:T] = [1:0]$, let $v_+ = a(e_1 - ie_2)$. Projective sign cancellation gives

$$q(x, [0, v_+]) = q(y_\eta, [0, v_+]) \qquad (\eta \text{ real near } 0). \tag{78}$$

Differentiate the right-hand side at zero, obtaining $w \in TF$ with $\mathrm{d}q(w) = 0$ and

$$\mathrm{d}p(w) = (a \sinh \pi\, e_2, 0) \neq 0,$$

where e_2 in the first factor is its third ambient coordinate vector. At $(y_0, [0, v_+])$, however, $\mathrm{d}p(K)$ is the pure second-factor line $\mathbb{C}(0, v_+)$. The condition $\ker \mathrm{d}q = K$ is incompatible with this nonzero first-factor derivative.□

In particular, no such extension has a closed embedded null-leaf relation, by Lemma 3.12. The obstruction already precedes the descent of quadratic or operator data. The real affine incidence and the monodromy construction of §3 remain defined.

To analyze supports, now use the particular product extension B_0, its phase F, and its maximal leaf relation R. The following explicit geometry controls every restriction of this extension around the entire real development.

Proposition 8.3.

Every complex null leaf of B_0 is a properly embedded copy of $\mathbb{C}$. Its partial tractor connection has trivial holonomy and its affine position is injective. Nevertheless, R is not closed in $F \times F$.

Proof. For initial null tangent (k_1, k_2) at (x_1, x_2) write $\varepsilon_i = H_i(k_i, k_i)$, so $\varepsilon_2 = -\varepsilon_1$. If $\varepsilon_1 = 0$, the affine solution is the nonconstant proper embedded line $(x_1 + sk_1, x_2 + sk_2)$. Otherwise choose r with $r^2 = \varepsilon_1/a^2$. The entire solution is

$$\gamma(s) = (x_1 \cosh(rs) + r^{-1}k_1 \sinh(rs), x_2 \cos(rs) + r^{-1}k_2 \sin(rs)). \tag{79}$$

In each factor x_i, k_i are independent: they are orthogonal and x_i has nonzero squared length. A return to the starting point forces both $rs \in 2\pi i\mathbb{Z}$ and $rs \in 2\pi\mathbb{Z}$, hence $s = 0$. Shifting the affine origin proves injectivity. A bounded set of image points bounds $\cosh(rs), \sinh(rs), \cos(rs), \sin(rs)$, which bounds both real and imaginary parts of rs. This proves properness. The phase lift is also proper and injective. Its tangent spans K, and its image is both

open and closed in its connected leaf; it is therefore the entire leaf. Simple connectivity gives trivial partial holonomy. In a parallel tractor frame its affine position is $\mathsf{X}_0 + sV$ with $V \neq 0$, proving injectivity.

For nonclosedness fix $x_1 = (0, a, 0)$, $x_2 = ae_3$, $u_2 = ae_1$, and for positive $r \to 0$ put

$$k_{1,r} = \frac{a}{2}(1 + r^2, 0, 1 - r^2), \qquad k_{2,r} = ru_2, \qquad s_r = 2\pi i/r. \tag{80}$$

The tangent is null with $\varepsilon_1 = a^2 r^2$. At time s_r the first factor and its tangent return to $x_1, k_{1,r}$, while the second factor and tangent are

$$y_2 = x_2 \cosh(2\pi) + iu_2 \sinh(2\pi), \qquad r\big(u_2 \cosh(2\pi) - ix_2 \sinh(2\pi)\big).$$

The related initial and final phase points converge to

$$\big((x_1, x_2), [k_{1,0}, 0]\big), \qquad \big((x_1, y_2), [k_{1,0}, 0]\big), \qquad k_{1,0} = \tfrac{a}{2}(1, 0, 1).$$

Their pure first-factor null leaves keep the second point constant. Since $y_2 \neq x_2$, the limits are unrelated. □

Proposition 8.4.

Let $B' \subset B_0$ be any connected conjugation-invariant open neighborhood of all of M, and put $F' = F|_{B'}$. Let R' relate points in the same connected component of an original leaf's intersection with F'. Then R and R' are embedded, and

$$q' : F' \longrightarrow Z' = F'/R' \tag{81}$$

is an open holomorphic submersion onto a non-Hausdorff contact fivefold. The quadratic and affine data descend to Z', and affine position embeds F' as a Hausdorff open incidence domain in the affine bundle. Its lifted full-quadric point family is proper and locally complete in this Hausdorff incidence domain. For either prescribed real spin structure, B' can be chosen so that the prescribed spin structure extends.

Proof. We first prove the following separation property, for every $x \in B_0$,

$$\overline{R} \cap (F_x \times F_x) = \Delta_{F_x}. \tag{82}$$

Let $(u_n, v_n) \in R$ converge to $(u, v) \in F_x \times F_x$. Normalize representatives k_n of the initial directions by a fixed linear functional nonzero at the limiting direction. Thus $k_n \to k \neq 0$, and the initial and final base points x_n, y_n tend to x. Proposition 8.3 gives a unique affine parameter s_n from u_n to v_n.

For nonzero first-factor energy, set $r_n^2 = H_1(k_{1,n}, k_{1,n})/a^2$ and $z_n = r_n s_n$. Taking the ambient inner products of the endpoint equations with the initial points gives

$$\cosh z_n = -a^{-2} H_1(x_{1,n}, y_{1,n}) \longrightarrow 1, \qquad \cos z_n = -a^{-2} H_2(x_{2,n}, y_{2,n}) \longrightarrow 1.$$

These bounds bound $\operatorname{Re} z_n$ and $\operatorname{Im} z_n$, respectively. Every limit is in $2\pi i\mathbb{Z} \cap 2\pi\mathbb{Z} = \{0\}$, so $z_n \to 0$. Choose a factor with $k_i \neq 0$. Its endpoint equation is

$$y_{i,n} - x_{i,n} f_i(z_n) = k_{i,n} s_n h_i(z_n),$$
$$(f_1, h_1) = (\cosh, \sinh(z)/z), \quad (f_2, h_2) = (\cos, \sin(z)/z).$$

Both removable values $h_i(0)$ equal one. Apply a fixed linear functional nonzero on k_i to see $s_n \to 0$. For zero energy the same follows directly from $y_n - x_n = s_n k_n$. Short-flow continuity now gives $v = u$, proving the separation property.

In particular, related pairs approaching a diagonal pair have joining parameter tending to zero. In a flow box their relation is therefore exactly the embedded local leaf relation, with no additional branches. Finite compositions of local leafwise flows in one factor transfer this to every related pair, so R is embedded.

The relation R' is open in $R \cap (F' \times F')$. A path connecting a related pair inside F' is compact and has a finite flow-box cover there; its endpoint comparison persists for nearby related pairs. Hence R' is embedded too. Lemma 3.12 gives the quotient charts, open submersion and contact structure. This quotient cannot be Hausdorff by Theorem 8.2.

Every restricted leaf is an open subset of an original affine-parameter line. Its partial connection extends over that entire line, so all restricted loop transports are trivial, and its affine position stays injective. Lemma 3.12 therefore descends the bundles and embeds F' into the affine bundle. Properness of the lifted point family is that of $F' \to B'$. In this incidence domain its fiber has normal bundle $T_x B' \otimes \mathcal{O}_Q$. To see local completeness there directly, a small compact deformation stays over a coordinate neighborhood of x; each coordinate of its point projection is holomorphic on a compact connected manifold and is constant. Equal dimensions force the deformation to be the entire fiber over that constant point. Evaluation on a local section shows that this point depends holomorphically in a holomorphic family.

Finally, Proposition 3.1 extends either specified real spin structure to a holomorphic metric extension. Germ uniqueness identifies a smaller conjugation-invariant neighborhood of all of M with one in B_0, preserving metric and reality. The selected spin structure thus extends to a suitable B'. □

For $x \in B'$ let $S_x = q'(F_x)$. An embedding into a possibly non-Hausdorff manifold is an injective holomorphic immersion that is a homeomorphism onto its image; the image need not be closed.

Lemma 8.5.

The map $q'|_{F_x}$ is a holomorphic embedding and S_x is Hausdorff in its subspace topology.

Proof. By the separation property (82), distinct $u, v \in F_x$ have neighborhoods $U, V \subset F'$ with $(U \times V) \cap R' = \varnothing$. The open sets $q'(U), q'(V)$ are disjoint, making S_x Hausdorff. Proposition 8.3 shows that no original null geodesic returns to its base point, so $q'|_{F_x}$ is injective. It is a continuous bijection from a compact quadric to its Hausdorff image, hence a homeomorphism. Its differential is injective because $K \cap \ker \mathrm{d}p = 0$. □

Theorem 8.6.

For every neighborhood B' in Proposition 8.4 and every real point $x \in M$, the compact embedded sky S_x is not closed in Z'. Equivalently,

$$\mathrm{Sat}_{R'}(F_x) = \{u \in F' : \text{the restricted leaf through } u \text{ meets } F_x\} = q'^{-1}(S_x) \tag{83}$$

is not closed in the Hausdorff phase manifold F'.

Proof. First take $x = ((0, a, 0), ae_3)$. Set $k_0 = (a, 0, 0)$ and use the real endpoint and velocity

$$y_1 = x_1 \cosh \pi + k_0 \sinh \pi, \qquad k' = x_1 \sinh \pi + k_0 \cosh \pi, \qquad y = (y_1, -x_2).$$

The direction conics from Theorem 8.2 at $\eta = 0$ are

$$\alpha([S : T]) = (x, [2STk_0, v(S, T)]), \qquad \beta([S : T]) = (y, [2STk', -v(S, T)]). \tag{84}$$

They lie in F' because their entire fibers are over real points. For $|S/T| = 1$ the real geodesic between their endpoints stays in M, hence in B', so $q'\beta = q'\alpha \in S_x$ there.

If S_x were closed, Lemma 8.5 would make it a closed analytic subset of Z': use submanifold coordinates at its image points and its open complement elsewhere. The inverse image $(q'\beta)^{-1}(S_x)$ would be a closed analytic subset of $\mathbb{P}^1$ containing the unit circle, and therefore all of $\mathbb{P}^1$.

But at $[1 : 0]$ the final phase is $(y, [0, v_+])$, with $v_+ = a(e_1 - ie_2)$. Its original geodesic is pure second-factor null and keeps its first base point equal to $y_1 \neq x_1$. It cannot meet F_x even in the full product extension, and neither can its restricted leaf. Thus $q'\beta([1 : 0]) \notin S_x$, a contradiction.

The group $SO_0(1, 2) \times SO(3)$ acts transitively on M. Each of its elements extends by the same real matrices to a holomorphic isometry of B_0 commuting with conjugation. For an arbitrary real point x, choose such an isometry A carrying the distinguished point $x_* = ((0, a, 0), ae_3)$ to x. The neighborhood $A^{-1}(B')$ is again allowed. The induced phase isomorphism preserves the restricted null-leaf relation and identifies its contact quotient with Z', carrying the sky at x_* to S_x. The distinguished-point argument applied to $A^{-1}(B')$ therefore proves nonclosedness for S_x. Finally, q' is a quotient map, so closedness of a subset of Z' is equivalent to closedness of its inverse image.□

Lemma 8.7.

The nonzero-stalk support of a locally finitely generated module over a sheaf of rings is closed. The same holds chartwise for an algebroid module and after inverse image to a presentation atlas.

Proof. At a point where the stalk vanishes, choose finitely many local generators. Each vanishes on a smaller neighborhood, and on their finite intersection the module is zero. Its zero-stalk locus is open. Algebroid comparisons are equivalences and preserve zero modules, so supports agree on overlaps. Inverse image preserves finite local generation and has the original stalk at each point above a base point. □

Corollary 8.8.
For every neighborhood B' in Proposition 8.4 and every real point $x \in M$, no ordinary locally finitely generated module over an operator algebroid on Z' has support exactly S_x. On the corresponding null monodromy contact-groupoid presentation, no such module has phase-atlas support exactly $\mathrm{Sat}_{R'}(F_x)$. In particular, there is no ordinary coherent simple point module with that exact incidence support.

Every compatible holomorphic Einstein extension of the entire real Nariai development has an allowed point neighborhood U such that these nonexistence assertions hold on U and on every further allowed point restriction, for every real point. These representatives are cofinal among its neighborhoods of the entire real development. The conclusion holds for either prescribed real spin structure.

Proof. The first assertion contradicts Lemma 8.7 and Theorem 8.6. For the groupoid version, pull the module and algebroid to the Hausdorff phase atlas F'. This inverse image of operator germs preserves local finite generation. Its support must therefore be closed, whereas the required saturation is not.

Let $(\widetilde{B}, \widetilde{g}, \widetilde{\sigma})$ be any compatible extension of the entire real Nariai development. By Lemma 7.3, applied to the identity real isometry, there is a biholomorphism from an allowed neighborhood $U \subset \widetilde{B}$ onto an allowed neighborhood in B_0, preserving the metric, conjugation and the specified real identification. If a real spin structure is prescribed, Proposition 8.4 first supplies a product neighborhood carrying its extension, and the lemma retains the specified spin identification after a further restriction. Every allowed neighborhood inside U is identified in the same way with a product neighborhood. The induced phase maps identify the recomputed monodromy presentations and their point-sky saturations, so the first part applies to every real point on every such neighborhood. Given any other allowed neighborhood $W \subset \widetilde{B}$, the component containing all of connected M of $U \cap W$ is an allowed neighborhood contained in both. This proves cofinality. □

For nested allowed product neighborhoods $B'' \subset B'$, the recomputed saturations satisfy

$$\mathrm{Sat}_{R''}(F_x) \subseteq \mathrm{Sat}_{R'}(F_x) \cap F'', \qquad F'' = F|_{B''}, \tag{85}$$

and the inclusion need not be equality: a joining null path in B' may leave B''. Ordinary restriction of a module on a larger extension therefore need not have the recomputed exact support. The corollary concerns the stated cofinal representatives; it neither excludes isolated modules on arbitrary larger complexifications nor asserts a neutralization obstruction for the operator algebroid. Local whole-sky simple modules remain available, and their parametrized realization on the Hausdorff incidence is the construction of §4.3.

Appendix A. The analytic Cauchy category

Fix a connected compact oriented real analytic three-manifold Σ without boundary, an analytic positive definite metric h, an analytic symmetric tensor K, and a specified spatial spin structure P_Σ. Let $\Lambda \in \mathbb{R}$ be fixed. The data satisfy the Einstein constraints (14), namely $\mathrm{Scal}(h)+(\mathrm{tr}_h K)^2-|K|_h^2=-2\Lambda$ and $\mathrm{div}_h K-\mathrm{d}\,\mathrm{tr}_h K=0$. We use signature $(+---)$, $\mathrm{Ric}(g)=\Lambda g$, and $K(v,w)=-g(\nabla_v\nu,w)$ for the future unit normal ν.

The required Cauchy input concerns these analytic constraint data. We give the analytic existence and maximality argument explicitly, including the relative spin identification used by the source correspondence.

Definition A.1.

A smooth globally hyperbolic development of (Σ,h,K) is a connected oriented and time-oriented smooth Lorentzian four-manifold (M,g) satisfying $\mathrm{Ric}(g)=\Lambda g$, with a spacelike embedding $\iota:\Sigma\to M$ inducing (h,K) and having Cauchy image. Here every inextendible causal curve must meet the image exactly once. The spacetime orientation is the one for which $(\nu_\iota,\mathrm{d}\iota(v_1),\mathrm{d}\iota(v_2),\mathrm{d}\iota(v_3))$ is positive whenever (v_1,v_2,v_3) is positive on Σ. An analytic development has an analytic manifold structure, metric and initial embedding. A development morphism is an orientation- and time-orientation-preserving isometric embedding f satisfying $f\circ\iota=\iota'$; it is analytic in the analytic category.

A spin development also has a Lorentz spin structure with a specified identification of P_Σ with its reduction along the initial hypersurface to frames adapted to ν_ι. This identification covers

$$(e_1,e_2,e_3)\longmapsto(\nu_\iota,\mathrm{d}\iota(e_1),\mathrm{d}\iota(e_2),\mathrm{d}\iota(e_3)). \tag{86}$$

Spin morphisms include a spin lift preserving that identification. The resulting analytic spin category is denoted by $\mathsf{Ein}^{\omega}_{\Lambda}(\Sigma,h,K)$, with the orientation and P_Σ understood. A maximal development means a terminal object: it receives a unique morphism from every development in the category.

Proposition A.2.

The specified data admit an analytic spin globally hyperbolic development on $(-\varepsilon,\varepsilon)\times\Sigma$, with initial embedding $x\mapsto(0,x)$ and metric $g=\mathrm{d}t^2-h_t$.

Proof. Put $\lambda=-\Lambda$ and solve the Gaussian evolution equations

$$\begin{aligned}\partial_t h_t &= 2K_t,\\ \partial_t K_t &= -\mathrm{Ric}(h_t)-(\mathrm{tr}_{h_t}K_t)K_t+2K_t h_t^{-1}K_t+\lambda h_t,\\ (h_0,K_0)&=(h,K).\end{aligned} \tag{87}$$

Eliminating K_t gives a system solved for $\partial_t^2 h_t$. Its right-hand side is analytic in the invertible metric, its first time derivative, and tangential derivatives of order at most two; the coefficient of the highest normal derivative is the identity. The analytic Cauchy theorem in normal form therefore gives a unique local analytic solution in every spatial chart. This is the same normal

form and constraint-propagation calculation used in Appendix A.1 of the local paper, now applied directly to the prescribed constraint data.

For completeness, set $\widehat{g} = -\mathrm{d}t^2 + h_t$ and $\tau = \mathrm{tr}_{h_t} K_t$. Its time-containing Christoffel symbols are $\widehat{\Gamma}^0_{ij} = (K_t)_{ij}$ and $\widehat{\Gamma}^i_{0j} = (K_t)^i{}_j$. Consequently

$$\mathrm{Ric}(\widehat{g})_{ij} = \mathrm{Ric}(h_t)_{ij} + \partial_t(K_t)_{ij} + \tau(K_t)_{ij} - 2(K_t)_i{}^q(K_t)_{qj},$$
$$\mathrm{Ric}(\widehat{g})_{00} = -\partial_t\tau - |K_t|^2_{h_t}, \qquad \mathrm{Ric}(\widehat{g})_{0i} = \nabla^j(K_t)_{ij} - \partial_i\tau.$$

Thus the spatial components of $E = \mathrm{Ric}(\widehat{g}) - \lambda\widehat{g}$ vanish. Contracted Bianchi, with $H = E_{00}$ and $M_i = E_{0i}$, gives

$$\partial_t H = 2\nabla^i M_i - 2\tau H, \qquad \partial_t M_i = \tfrac{1}{2}\partial_i H - \tau M_i. \tag{88}$$

Tracing the evolution equations shows that initially $H = \mathrm{Scal}(h) + (\mathrm{tr}_h K)^2 - |K|^2_h - 2\lambda = 0$; the momentum constraint gives $M = 0$. Uniqueness for this homogeneous analytic Cauchy system gives $E = 0$. Constant reversal of metric sign leaves the Levi-Civita connection and covariant Ricci tensor unchanged, so $g = -\widehat{g}$ satisfies $\mathrm{Ric}(g) = \Lambda g$.

Time-independent analytic spatial coordinate changes preserve the evolution equations and initial data. Analytic uniqueness therefore identifies the local solutions on overlaps near the initial hypersurface. Choose a finite cover by smaller spatial charts with closures in their construction charts. Compactness gives a common positive time interval on which all solutions and overlap identifications exist. They glue on the fixed product with Σ; uniqueness under complex conjugation gives real coefficients. After decreasing the interval, h_t is positive definite and extends to its closed time interval with uniform upper and lower bounds relative to h.

The function t is temporal. A future-directed causal curve, parametrized by t, satisfies $h_t(\mathrm{d}x/\mathrm{d}t, \mathrm{d}x/\mathrm{d}t) \le 1$ almost everywhere. Uniform comparison with h bounds its spatial speed. If its time range had a finite endpoint inside the interval, its spatial component would be Cauchy as that endpoint is approached, and completeness of the compact Riemannian manifold (Σ, h) would give a limit. The curve could then be continued through that spacetime point by a vertical timelike segment. Therefore an inextendible causal curve has the entire time interval as its time range and meets every slice once. In particular the initial slice is Cauchy.

The prescribed orientations extend over the product. Lemma A.6 extends P_Σ to this development with its required initial identification and analytic covering structure. □

Theorem A.3 (Smooth Cauchy theorem).
Smooth Einstein constraint data for fixed Λ admit a smooth globally hyperbolic development $(M_{\mathrm{sm}}, g_{\mathrm{sm}}, \iota_{\mathrm{sm}})$ receiving a unique initial-data- and time-orientation-preserving isometric embedding from every smooth globally hyperbolic development of those data.

For $\Lambda = 0$ this is the maximal-development result of Sbierski (2016, Theorems 2.6 and 2.8, Corollary 3.2); the embedding and common-development

arguments are his Lemma 3.3 and Theorem 3.4. The hypotheses are connected smooth developments and smooth constraint data. Before Definition 2.1 he notes that the same proof applies with a cosmological constant once local existence and uniqueness hold. In the harmonic reduction, Λg is a smooth zeroth-order term, so the reduced equations have the same quasilinear wave principal part and the same local existence and uniqueness argument. The contracted Bianchi identity gives homogeneous gauge and constraint propagation because $\nabla g = 0$ and $\mathrm{d}\Lambda = 0$; the constraint calculation in Proposition A.2 is its Gaussian form. These observations give the local input needed for the same maximal-development construction. Signature reversal changes the Einstein coefficient and preserves the argument. The required orientation extends from the initial hypersurface by the product decomposition proved below. An initial-data- and time-orientation-preserving embedding preserves that orientation on the initial hypersurface, hence everywhere on its connected domain.

Lemma A.4.

A smooth local isometry between real analytic pseudo-Riemannian manifolds of the same dimension is analytic. A fixed smooth metric consequently has at most one compatible maximal analytic atlas in which it is analytic.

Proof. Analytic dependence for the geodesic equation makes the exponential maps analytic near their zero vectors, with analytic local inverses. Preservation of the Levi-Civita connection gives

$$f \circ \exp_p^{g_N} = \exp_{f(p)}^{g_M} \circ \mathrm{d}f_p. \tag{89}$$

This formula proves analyticity near each p. Apply it to the identity between two proposed analytic structures, in both directions, to obtain compatibility of their maximal atlases. □

Lemma A.5.

Let (M, g) be globally hyperbolic with connected Cauchy hypersurface S. If a nonempty set of open subsets $U_\alpha \subset M$ all contain S and have S as a Cauchy hypersurface, then S is Cauchy in $U = \bigcup_\alpha U_\alpha$. The union is a connected globally hyperbolic open subspacetime.

Proof. For an inextendible causal curve $\gamma : I \to U$, choose a parameter value whose image belongs to some U_α, and take the component J of $\gamma^{-1}(U_\alpha)$ containing it. The restriction $\gamma|_J$ is inextendible in U_α: an endpoint of J inside I has image outside U_α, whereas an extension at an endpoint of I would extend γ in U. Thus $\gamma|_J$ meets S. Two intersections of γ with S would give an ambient causal segment that extends to an ambient inextendible causal curve meeting S twice, contrary to the ambient Cauchy property. Each component of each U_α meets S, by applying the same argument to an inextendible timelike curve. Connectedness of S implies connectedness of the union. The existence of its Cauchy hypersurface gives global hyperbolicity.□

This is a union inside a fixed ambient development, as in the causal argument in Sbierski (2016, Theorem 3.4). It requires no uniform analytic radius.

Lemma A.6.
The prescribed P_Σ extends to every smooth development, uniquely up to a unique isomorphism with prescribed initial identification. On an analytic development the extension has a unique analytic covering structure. Every analytic development morphism has a unique analytic spin lift extending a prescribed compatible initial lift.

Proof. Choose a future-directed timelike vector field and rescale it to have bounded norm in a complete auxiliary Riemannian metric. Its flow is complete, and each of its inextendible timelike orbits meets the initial Cauchy hypersurface exactly once. The resulting map $\mathbb{R} \times \Sigma \to M$ is bijective and a local diffeomorphism, since the field is transverse to the initial hypersurface. It is therefore a diffeomorphism and exhibits that hypersurface as a deformation retract.

Choose a smooth future unit timelike field T agreeing with the initial normal. On $E = T^\perp$, endowed with $-g|_E$, parallel transport for a metric connection along the product lines identifies E with the pullback of $(T\Sigma, h)$, relative to the identity on the initial slice. Pull back P_Σ and extend its structure group to $\mathrm{Spin}^+(1,3)$. This constructs an extension without requiring an analytic product decomposition.

The two spin structures in a comparison differ by a principal $\mathbb{Z}_2$-bundle on M. The prescribed initial identification is a section of that difference bundle on the deformation retract. Covering-space lifting extends the section over M; two extensions differ by a locally constant sign fixed on the initial hypersurface, so they coincide on connected M. This proves relative uniqueness.

For an analytic metric the orthonormal frame bundle is analytic. Its spin cover has the unique analytic atlas making the covering projection a local analytic diffeomorphism. The spin action and initial identification are analytic by local lifting. For a morphism, pull back the target spin structure and apply relative uniqueness on the source. The frame map is analytic, hence so is its covering lift. Uniqueness proves the composition law. Initial lifts differing by the central sign have distinct extensions. □

Theorem A.7 (Maximal analytic development).
Assume that the analytic initial data admit an analytic globally hyperbolic development. There is an analytic spin development $(M_\omega, g_\omega, \iota_\omega, P_\omega)$ receiving a unique analytic spin morphism from every analytic spin development of the same oriented data and specified initial spin structure. It is unique up to the unique analytic spin isometry preserving the initial identification. Its underlying smooth spacetime is an open subspacetime of the smooth maximal development.

Proof. Fix the smooth maximal development and put $S = \iota_{\mathrm{sm}}(\Sigma)$. Let $\mathscr{U}_\omega$ consist of the open subsets $U \subset M_{\mathrm{sm}}$ containing S, with S Cauchy in U, that admit a compatible analytic structure making the restricted metric and initial embedding analytic. This is a set of subsets of a fixed manifold; Lemma A.4 gives at most one compatible maximal analytic atlas on each member. It is nonempty: the assumed analytic development embeds smoothly in M_{sm}, and its analytic atlas pushes forward to its open image.

Set

$$M_\omega = \bigcup_{U \in \mathscr{U}_\omega} U, \qquad g_\omega = g_{\mathrm{sm}}|_{M_\omega}, \qquad \iota_\omega = \iota_{\mathrm{sm}} : \Sigma \longrightarrow M_\omega. \tag{90}$$

This open subspace is Hausdorff and second countable. On the intersection of two members the identity is a smooth local isometry, so Lemma A.4 makes their analytic atlases compatible. The union atlas, with a countable subatlas if desired, makes the metric and initial embedding analytic. Lemma A.5 makes S Cauchy in M_ω.

Every analytic development N has a unique smooth initial-data-preserving embedding into M_{sm}. Its open image, with its pushed-forward analytic atlas, belongs to $\mathscr{U}_\omega$; the embedding therefore factors analytically through M_ω. Any second factor would give a second such embedding into M_{sm}, contrary to the smooth theorem. Lemma A.6 extends the prescribed spin structure to M_ω and uniquely lifts each factor with its initial identification. Two objects with this universal property admit unique morphisms in both directions, whose composites are identities. This also removes dependence on the chosen smooth maximal development. □

Corollary A.8.

The analytic constraint data fixed above have a maximal analytic spin development, and their analytic Cauchy collar has a unique spin embedding into it. Every analytic isomorphism of such initial data with a specified spatial spin lift extends uniquely to an analytic spin isometry between their maximal analytic developments. These extensions preserve identities and composition.

Proof. Proposition A.2 supplies the existence hypothesis of Theorem A.7. For an initial-data isomorphism, reparametrize the target initial embedding and transport its initial spin identification by the specified lift. The universal property gives a morphism for these identified data. Apply it also to the inverse data isomorphism. The two composites are identities by uniqueness; the same argument proves the identity and composition laws. □

Remark A.9.

Maximality here is within the analytic globally hyperbolic category of the fixed Cauchy data. The construction proves $M_\omega \subset M_{\mathrm{sm}}$ as an open subspacetime. Equality would require a separate analytic-regularity propagation argument. Neither geodesic completeness nor maximality among extensions in which the initial hypersurface ceases to be Cauchy is asserted.

AI-use disclosure. AI systems were used extensively throughout this research and the preparation of this manuscript. The models used were GPT-6 Pro, gpt-6.1-sol (xhigh), and Claude Opus 5.5 (xhigh). Selected symbolic identities and explicit metric examples were checked using SageMath and SymPy; the code, inputs, computational environment and recorded results are provided in the computational supplement (Isthmus 2026b).

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