hexagon:2611.00424v1

AN ORDER-ELEVEN COUNTEREXAMPLE TO TRIANGULAR REALIZATION OF SIGN MATRIX PERMANENTS

OPENBMB

Abstract. We construct an 11×11 sign matrix with permanent $P = 2\,029\,568$. An exhaustive enumeration shows that $|\operatorname{per} B| \neq P$ for every sign matrix B of order eleven whose entries below the diagonal are all $+1$. This gives a negative answer to a question of Ingram and Razborov [1].

1. Introduction

The permanent of an $n \times n$ matrix $A = (a_{ij})$ is

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)},$$

where S_n is the symmetric group on n letters. With edge weights a_{ij}, each summand is the weight of a perfect matching between rows and columns [2, 3]. Let $\Omega_n = \{-1, +1\}^{n \times n}$ denote the set of all $n \times n$ sign matrices, that is, matrices whose entries are either -1 or $+1$. For each $A \in \Omega_n$, every perfect matching has weight ± 1. For any family $\mathcal{F} \subseteq \Omega_n$, write $\mathcal{P}(\mathcal{F}) = \{\operatorname{per} A : A \in \mathcal{F}\}$ for the set of permanent values attained on $\mathcal{F}$. Ingram and Razborov constructed a family of sign matrices for which the number of distinct permanents grows superpolynomially in n [1]. Hunter, Kwan and Sauermann subsequently proved that $|\mathcal{P}(\Omega_n)| \geqslant e^{cn}$ for some absolute constant $c > 0$ and all sufficiently large n [4, Corollary 1.3]. Ingram and Razborov [1, Problem 4, p. 283] also asked whether the same range can be attained within the family

$$\mathcal{U}_n = \{B \in \Omega_n : b_{ij} = +1 \text{ for } i > j\}.$$

Here the signs on and above the diagonal are unrestricted. The question is whether $\mathcal{P}(\mathcal{U}_n) = \mathcal{P}(\Omega_n)$ for every n (see also [5]). A matrix may share its permanent with a member of $\mathcal{U}_n$ even when no row and column permutations bring it into $\mathcal{U}_n$. A counterexample must therefore exclude every triangular representative of a given value. Negating the first row preserves each family and negates the permanent. Both ranges are therefore symmetric about zero, and it suffices to compare absolute values.

The triangular family contains representatives for several extremal values. For $n \geqslant 5$ and $1 \leqslant k \leqslant n$, Budrevich and Guterman proved that the largest absolute permanent among sign matrices of rank k is attained by changing $k - 1$ diagonal entries of the $n \times n$ all-ones matrix J_n to -1 [6]. Each of these matrices belongs to $\mathcal{U}_n$. At order eleven, an example of Wanless [7, §3] attains the smallest nonzero absolute permanent, 256. Its negative entries occupy precisely the positions with $j - i \in \{1, 2, 3, 8, 9\}$, so this Toeplitz matrix also belongs to $\mathcal{U}_{11}$.

Colbrook established by exact computation that the two ranges coincide for $1 \leqslant n \leqslant 10$ [8, Theorem 2]. At order ten the common range contains 3 865 distinct signed values. The following theorem establishes strict inclusion at order eleven.

Theorem 1.1. *There is a matrix $A \in \Omega_{11}$ with $\operatorname{per} A = 2\,029\,568$. Every $B \in \mathcal{U}_{11}$ satisfies $|\operatorname{per} B| \neq 2\,029\,568$.*

By Colbrook's result and Theorem 1.1, eleven is the least order at which the two ranges differ.

Set $P = 2\,029\,568$. Section 2 constructs the matrix $A = J_{11} - 2(C \oplus I_7)$, where J_{11} is the 11×11 all-ones matrix, I_7 is the 7×7 identity matrix, and C is the 4×4 biadjacency matrix of an eight-cycle. A direct calculation gives $\operatorname{per} A = P$.

To exclude P from the permanents of the 2^{66} matrices in $\mathcal{U}_{11}$, we enumerate suffixes consisting of the last r rows. In such a suffix the first $11 - r$ columns are identical. The permanent of an $r \times r$ submatrix is therefore determined by the subset of the last r columns that it contains. Section 3 derives a recurrence for these submatrix permanents and uses them to evaluate Colbrook's upper bound on $|\operatorname{per} B|$ for every completion B [8, Lemma 6]. The enumeration identifies states related by a permutation of the last r columns and discards any state whose upper bound is less than P. At $r = 10$ this leaves 34 099 distinct sorted vectors of absolute first-row cofactors. For each retained suffix, the possible first rows yield precisely the signed sums of the entries of its cofactor vector. Section 4 verifies that no such sum equals P. Appendix A gives the C++ implementation of this enumeration.

2. Construction of the order-eleven witness

View an $n \times n$ zero-one matrix $N = (n_{ij})$ as a board on which a square (i, j) is marked if $n_{ij} = 1$. The marked squares are precisely the positions of the negative entries of $J_n - 2N$. A placement of k nonattacking rooks is a set of k marked squares, no two in the same row or column. The entries in the selected squares therefore occur together in at least one term of the permanent expansion.

The number of such placements is the kth *rook number* $r_k(N)$ [9]. There is one empty placement, so $r_0(N) = 1$, and $r_1(N)$ is the number of marked squares. Collecting the counts by placement size gives the *rook polynomial*

$$R_N(x) = \sum_{k=0}^{n} r_k(N) x^k.$$

Lemma 2.1. *For every zero-one board N of order n,*

$$\operatorname{per}(J_n - 2N) = \sum_{k=0}^{n} (-2)^k (n-k)!\, r_k(N).$$

Proof. Expand each permutation product in $\operatorname{per}(J_n - 2N)$ by selecting either 1 or the corresponding entry of $-2N$ from each factor. In a nonzero term, the k positions at which the second summand is selected form a placement of nonattacking rooks on N. Each such placement extends to $(n-k)!$ permutations and contributes $(-2)^k$ in each extension. Summation over the placements proves the formula. □

Proposition 2.2. *Let C be the 4×4 zero-one matrix with ones on the diagonal and on the cyclic superdiagonal. Then $A = J_{11} - 2(C \oplus I_7)$ has permanent 2 029 568.*

Proof. By Lemma 2.1 it suffices to compute the rook polynomial of $N = C \oplus I_7$, where

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

The matrix $A = J_{11} - 2N$ has entries in $\{-1, +1\}$.

Associate to each zero-one board a bipartite graph whose vertex classes are the rows and columns, with an edge for each entry equal to 1. Placements of nonattacking rooks correspond bijectively to matchings. The graph associated to C is an eight-cycle. On a path of m edges, a matching of size k is specified by indices $i_1 < \cdots < i_k$ of edges with successive gaps at least two. The map

$i_j \mapsto i_j - (j-1)$ gives a bijection with the k-element subsets of $\{1, \dots, m-k+1\}$. Hence the path has $\binom{m-k+1}{k}$ matchings of size k.

Fix an edge of the eight-cycle. Deleting this edge leaves a seven-edge path, whose matchings account for those that omit the edge. For a matching containing the edge, removal of both endpoints leaves a matching of size $k-1$ in a five-edge path. Consequently,

$$r_k(C) = \binom{8-k}{k} + \binom{7-k}{k-1} \qquad (0 \leqslant k \leqslant 4),$$

with the convention that a binomial coefficient vanishes if its lower index is negative or exceeds its upper index. Thus

$$R_C(x) = 1 + 8x + 20x^2 + 16x^3 + 2x^4.$$

The graph of N is the disjoint union of this cycle and the seven isolated edges arising from I_7. The matching generating polynomial of a disjoint union is the product of those of its components. It follows that

$$R_N(x) = (1 + 8x + 20x^2 + 16x^3 + 2x^4)(1+x)^7,$$

with coefficient vector

$$\big(r_0(N), \dots, r_{11}(N)\big) = (1, 15, 97, 359, 849, 1351, 1477, 1107, 554, 174, 30, 2).$$

Substitution in Lemma 2.1 yields $\operatorname{per} A = \sum_{k=0}^{11} (-2)^k (11-k)!\, r_k(N) = 2\,029\,568$. □

3. Permanental minors of triangular suffixes

For $n \geqslant 2$, consider a matrix in $\mathcal{U}_n$ whose last r rows have been specified. The permanents of the square submatrices on these rows determine the permanent values attainable by completion of the suffix.

Following Colbrook [8, §5.4], we normalize the last row of $B \in \mathcal{U}_n$, which has the form $(1, \dots, 1, b_{nn})$. Multiplication of the last column by b_{nn} makes this an all-ones row, preserves membership in $\mathcal{U}_n$, and multiplies $\operatorname{per} B$ by b_{nn}. It therefore preserves $|\operatorname{per} B|$. For fixed rows $2, \dots, n$, let p_j denote the permanent of the submatrix obtained by deleting column j from these rows. Expansion along the first row shows that the set of attainable permanents is

$$\left\{ \sum_{j=1}^{n} \varepsilon_j p_j : (\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n \right\}.$$

The sign choices are independent, so this set is invariant under permutations and sign changes of the cofactors. In particular, it is determined by the multiset of their absolute values. Negating the first row negates the permanent and preserves the normalized last row because $n \geqslant 2$.

Definition 3.1. Let $1 \leqslant r < n$. The last r rows of a matrix in $\mathcal{U}_n$ form an r-row *suffix*. Its first $n-r$ columns consist entirely of ones and are called the *common columns*. Its last r columns, called the *active columns*, are labelled $0, \dots, r-1$. A subset S of the active labels is *valid* if $|S| \geqslant \max(0, 2r-n)$, or equivalently $r - |S| \leqslant n - r$. For each valid S, let q_S denote the permanent of the submatrix formed by the suffix rows, the active columns in S, and any $r - |S|$ common columns. Since the common columns are identical, this permanent is independent of their choice. The vector $q = (q_S)$ is the *state* of the suffix. A state is *feasible* if it arises from a suffix. Its *completion values* are the permanents obtained by adjoining admissible rows above the suffix.

For the normalized last row the initial state is $q_\varnothing = q_{\{0\}} = 1$. For $1 \leqslant r < n-1$, let q' denote the state after the addition of one row above the suffix. Let b_* be the entry of the new row in the newly active column and b_j its entry in the active column previously labelled j. These entries are

arbitrary signs. All entries in the remaining common columns are $+1$. In the following lemma T ranges over the valid active subsets of the extended suffix.

Lemma 3.2. *Let $R = r + 1$, denote the newly active column by $*$, and set $T_0 = T \setminus \{*\}$. Then*

$$q'_T = (R - |T| + \mathbf{1}_{*\in T} b_*) q_{T_0} + \sum_{j \in T_0} b_j q_{T_0 \setminus \{j\}}. \tag{3.1}$$

Two feasible states that differ only by a permutation of their active coordinates have the same set of completion values.

Proof. Expand the permanent indexed by T along the new row. Each of the $R - |T|$ common columns contributes q_{T_0}. If $* \in T$, the newly active column contributes $b_* q_{T_0}$, since it was a common column in the original suffix. Each previously active column $j \in T_0$ contributes $b_j q_{T_0 \setminus \{j\}}$. The resulting minors require either $r - |T_0|$ or $r - |T_0| + 1$ original common columns. The validity condition $R - |T| \leqslant n - R$ ensures that neither number exceeds $n - r$. Summing the contributions gives (3.1).

Let π be a permutation of the original active labels, extended to fix $*$. Under the substitutions $q_S \mapsto q_{\pi^{-1}(S)}$ and $b_j \mapsto b_{\pi^{-1}(j)}$, the recurrence (3.1) sends q'_T to $q'_{\pi^{-1}(T)}$. In each subsequent row the entries in these columns are arbitrary signs, so the same permutation induces a bijection on all choices of the remaining rows. In the expansion along the first row, it permutes the cofactors together with the corresponding signs and thus preserves the permanent. The bijection identifies the completion values of the two states. Moreover, even if a relabelled state is expressed in a column order that does not display the triangular restriction, the inverse permutation of the free signs carries each of its continuations back to a triangular representative. It therefore suffices to retain a single representative from each orbit. □

For $n = 11$, consider the first extension of the normalized last row. Indexed by $\varnothing, \{*\}, \{0\}, \{*, 0\}$, the entries of the new state are

$$2, \quad 1 + b_*, \quad 1 + b_0, \quad b_* + b_0.$$

The two assignments with $b_* = -b_0$ are related by an exchange of active labels. The four assignments thus determine three distinct states up to relabelling.

First-row cofactors. When $r = n - 1$, the suffix has precisely one common column. Let $E = \{0, \dots, n-2\}$ be the set of active labels. Deleting the common column or an active column j gives the first-row cofactors

$$q_E, \qquad q_{E \setminus \{j\}} \quad (j \in E).$$

The set of completion values consists of all signed sums of these cofactors. Hence the sorted vector of absolute cofactors determines the completion values at this stage. The recurrence for intermediate states uses the signed minors.

Colbrook's complementary-minor bound [8, Lemma 6] can be expressed in the suffix coordinates by grouping terms according to their active columns.

Lemma 3.3. *Every completion B of a feasible state q satisfies*

$$|\operatorname{per} B| \leqslant L_r(q) := (n - r)! \sum_S \binom{n - r}{r - |S|} |q_S|. \tag{3.2}$$

Here S ranges over all valid active subsets.

Proof. Fix a completion B. For a set D of r columns, let H_D denote the submatrix on the last r rows and the columns in D, and let K_{D^c} be its complementary submatrix on the first $n - r$ rows.

Grouping the permutation terms by the columns assigned to the suffix gives

$$\operatorname{per} B = \sum_{\substack{D \subseteq \{1,\dots,n\} \\ |D| = r}} \operatorname{per} H_D \operatorname{per} K_{D^c}.$$

The permanent of K_{D^c} is a sum of $(n-r)!$ signs and thus has absolute value at most $(n-r)!$. If S is the set of active labels in D, then $\operatorname{per} H_D = q_S$. There are $\binom{n-r}{r-|S|}$ choices of D for a given S. Applying the triangle inequality and collecting the terms with the same S proves (3.2). Each weight depends only on $|S|$, so a permutation of the active coordinates preserves the bound. □

In particular, a state satisfying $L_r(q) < P$ admits no completion of permanent P or $-P$.

4. Exclusion from the triangular family at order eleven

Fix $n = 11$ and $P = 2\,029\,568$.

Proposition 4.1. *No matrix $B \in \mathcal{U}_{11}$ has $|\operatorname{per} B| = 2\,029\,568$.*

Proof. Enumeration and pruning. By the normalization of Section 3, Algorithm 1 need only enumerate suffixes with an all-ones last row. The initial state has $q_\varnothing = q_{\{0\}} = 1$. For each retained state of size r, where $1 \leqslant r \leqslant 9$, the recurrence (3.1) is evaluated for all 2^{r+1} assignments of signs to the active entries of the next row. A successor q' is retained precisely when $L_{r+1}(q') \geqslant P$. Lemma 3.3 ensures that every suffix admitting a completion of absolute permanent P satisfies this condition.

Table 1. Enumeration of normalized suffixes at order eleven.

Suffix size r	Transitions before merging		After merging
	Examined	Pruned ($L_r < P$)	
2	4	0	3
3	24	0	12
4	192	0	71
5	2 272	0	697
6	44 608	972	12 335
7	1 578 880	1 156 707	73 940
8	18 928 640	17 835 206	134 568
9	68 898 816	66 475 145	254 960
10	261 079 040	254 731 228	34 099

The last column counts distinct canonical states for $2 \leqslant r \leqslant 9$ and distinct sorted absolute cofactor profiles for $r = 10$.

Canonical representatives. For suffixes of size $r < 10$, the enumeration retains one representative from each orbit under permutations of the active coordinates $0, \dots, r-1$. Define the signature of coordinate j by concatenating, in increasing order of k, the sorted lists

$$\big(q_S : |S| = k,\ j \in S\big), \qquad \max(0, 2r-11) \leqslant k \leqslant r.$$

The lists consist of signed minor values. Partition the coordinates by signature and arrange the classes in lexicographic order of their signatures. Among the full state vectors obtained by ordering the coordinates within the classes, denote the lexicographically least by $\mathrm{CAN}_r(q)$. The entries of this vector are indexed by subsets S in increasing order of $\sum_{j \in S} 2^j$, with a fixed zero at each invalid index.

Algorithm 1 Target exclusion at order eleven

Require: $n = 11$, $P = 2\,029\,568$
Ensure: Retained profiles Γ and target profiles H
Let $q^{(1)}_{\varnothing} = q^{(1)}_{\{0\}} = 1$; set $Q \leftarrow \{q^{(1)}\}$
for $r = 1, \dots, 9$ **do**
  $R \leftarrow r + 1$; $N \leftarrow \varnothing$
  for all $q \in Q$ **do**
    for all $b = (b_0, \dots, b_r) \in \{-1, +1\}^R$ **do**
      Compute only valid entries v_T by (3.1), with $* = r$
      if $L_R(v) < P$ **then**
        Discard v and continue with the next sign vector b
      end if
      if $R < 10$ **then**
        $N \leftarrow N \cup \{\mathrm{CAN}_R(v)\}$
      else
        $F \leftarrow \{0, \dots, 9\}$
        $a \leftarrow \mathrm{sort}(|v_F|, |v_{F\setminus\{0\}}|, \dots, |v_{F\setminus\{9\}}|)$
        $N \leftarrow N \cup \{a\}$
      end if
    end for
  end for
  $Q \leftarrow N$
end for
$\Gamma \leftarrow Q$; $H \leftarrow \{a \in \Gamma : \mathrm{HIT}_P(a) = \mathrm{true}\}$
return (Γ, H)

Relabelling the coordinates preserves the signature of each corresponding coordinate and hence the set of candidate vectors. Thus states in the same orbit have the same representative. Conversely, two states with a common representative differ by a coordinate permutation. By Lemma 3.2, they have the same completion values. In particular, a transition from a canonical representative corresponds to a transition from a triangular suffix under the same relabelling of the free signs, even if the representative is expressed in a different column order. In the implementation, hash values identify candidates for comparison. Two states are merged only if their full integer vectors are equal.

Within each signature class, fix its first coordinate i and test, for every other coordinate j in the class, whether

$$q_S = q_{S\triangle\{i,j\}} \quad \text{for every valid } S \text{ containing exactly one of } i, j.$$

If these equalities hold for all j, each transposition $(i\ j)$ fixes the state. These transpositions generate the symmetric group on the class, so a single ordering suffices. Otherwise the algorithm examines every ordering of the class.

Terminal cofactor test. At $r = 10$, let $F = \{0, \dots, 9\}$. The eleven first-row cofactors are q_F and $q_{F\setminus\{j\}}$ for $j \in F$, by Section 3. Let $a = (a_1, \dots, a_{11})$ be the nondecreasing sequence of their absolute values. Expansion along the unrestricted first row gives the set of completion values

$$\left\{ \sum_{i=1}^{11} \varepsilon_i a_i : (\varepsilon_1, \dots, \varepsilon_{11}) \in \{-1, +1\}^{11} \right\}.$$

It therefore suffices to retain one copy of each such profile. Let $\mathrm{HIT}_P(a)$ denote the predicate that one of these sums equals P. The search starts with the partial sum 0. At step i, a partial sum s has two successors, $s + a_i$ and $s - a_i$. At depth eleven the search compares each sum with P and

terminates if equality holds. It examines at most 2^{11} leaves, with a separate sign choice for each occurrence of a repeated cofactor magnitude. Since the set of sums is symmetric about zero, the predicate also detects every profile that admits a completion with permanent $-P$.

In Algorithm 1, the newly active coordinate $*$ is assigned label r, so $b_* = b_r$. At size $R = r + 1$, a subset $T \subseteq \{0, \dots, R-1\}$ is valid precisely when $|T| \geqslant \max(0, 2R - 11)$. The set Γ consists of the distinct retained terminal profiles, and H is the subset satisfying HIT_P. The operation sort arranges its arguments in nondecreasing order. Set membership uses equality of the entire vector. The transition recurrence retains the signs of all intermediate minors. Absolute cofactor profiles are formed at $R = 10$.

Exhaustive calculation. The implementation enumerates the sign vectors in binary reflected Gray-code order [10]. Changing one sign from ε to $-\varepsilon$ changes each affected entry of (3.1) by -2ε times the corresponding minor of the preceding state. Updating these entries and their contributions to the bound in integer arithmetic yields precisely the transitions in Algorithm 1. Table 1 gives the counts at each stage. At termination, $|\Gamma| = 34\,099$ and $H = \varnothing$.

Suppose that a matrix in $\mathcal{U}_{11}$ had absolute permanent P. Normalizing its last row would preserve this absolute value. Each successive suffix would satisfy the pruning bound, and its state would be retained up to a permutation preserving all completion values. The resulting terminal profile would therefore belong to H, contradicting $H = \varnothing$. □

Proof of Theorem 1.1. Proposition 2.2 gives a sign matrix of order eleven with permanent $2\,029\,568$, and Proposition 4.1 shows that $|\operatorname{per} B|$ differs from this value for every $B \in \mathcal{U}_{11}$. □

Appendix A. Reproducing the enumeration

The C++17 program below implements the exhaustive search in Section 4. Its arguments are the matrix order and the target absolute permanent. Save the listing as tensor_target.cpp in the working directory. On Linux or macOS, use a C++17 compiler supporting the GCC or Clang bit-counting intrinsics. The following commands create the required output directory, compile the program, and execute the search for $n = 11$ and $P = 2\,029\,568$.

```
mkdir -p results/pm1_permanent_range
c++ -O3 -std=c++17 tensor_target.cpp -o tensor_target
./tensor_target 11 2029568
```

The program writes the terminal profiles in lexicographic order to results/pm1_permanent_range/target_11_2029568_profiles.txt. This ordering makes the file independent of the order in which the hash tables are traversed. The first and last lines of the standard output are

```
n=11 target=2029568
final_profiles=34099 target_hits=0 verdict=EXCLUDED
```

The intervening lines report the counts in Table 1 for each suffix size, together with the machine-dependent elapsed times.

The permanent of any $r \times r$ submatrix of a sign matrix has absolute value at most $r!$. Hence the weighted sum in the completion bound (3.2), including its factorial factor, is bounded by

$$(11 - r)! \binom{11}{r} r! = 11! = 39\,916\,800.$$

At order eleven, all state entries, signed sums of first-row cofactors, and completion bounds are therefore representable as signed 32-bit integers. The program uses unsigned 64-bit counters for transitions and tests equality of states by comparing their full vectors.

Complete C++17 program.

```
#include <algorithm>
#include <chrono>
#include <cstdint>
#include <cstdlib>
#include <fstream>
#include <iostream>
#include <string>
#include <unordered_set>
#include <utility>
#include <vector>

using namespace std;
using V = vector<int>;

// Hashes locate candidates; unordered_set compares full vectors.
struct Hash {
  size_t operator()(V const &v) const {
    uint64_t h = 0x98251203;
    for (int x : v) {
      h ^= uint64_t(x) + 0x9e3779b97f4a7c15ULL + (h << 6) + (h >> 2);
    }
    return h;
  }
};

int n;
// In an unpacked state, q[s] is the minor for active-subset mask s.
// valid[r] contains precisely the admissible masks at suffix size r.
vector<int> valid[14];
int choose_[13][13], factorial_[13];

// Pack valid coordinates; unpacking fills other entries with zero.
V pack(const V &q, int r) {
  V p;
  p.reserve(valid[r].size());
  for (int s : valid[r])
    p.push_back(q[s]);
  return p;
}

V unpack(const V &p, int r) {
  V q(1 << r);
  for (int j = 0; j < (int)p.size(); ++j)
    q[valid[r][j]] = p[j];
  return q;
}

// Test whether interchanging coordinates i and j fixes the state.
bool twins(const V &q, int i, int j, int r) {
  for (int s : valid[r])
    if (((s >> i) ^ (s >> j)) & 1)
      if (q[s] != q[s ^ (1 << i) ^ (1 << j)])
        return false;
  return true;
}
```

```
V permuteq(const V &q, const V &order, int r) {
  V v(1 << r), map(1 << r);
  for (int s = 1; s < (1 << r); ++s) {
    int b = __builtin_ctz((unsigned)s);
    map[s] = map[s ^ (1 << b)] | (1 << order[b]);
  }
  for (int s : valid[r])
    v[s] = q[map[s]];
  return v;
}

// Enumerate orderings within each tied signature class.
// If the class consists of twins, every such ordering is equivalent.
void canonrec(const V &q, const vector<V> &groups, int gi,
              V &order, V &best, int r) {
  if (gi == (int)groups.size()) {
    V v = permuteq(q, order, r);
    if (best.empty() || v < best)
      best = move(v);
    return;
  }
  V g = groups[gi];
  bool alltwins = true;
  for (int j = 1; j < (int)g.size(); ++j)
    if (!twins(q, g[0], g[j], r)) {
      alltwins = false;
      break;
    }
  do {
    int old = order.size();
    order.insert(order.end(), g.begin(), g.end());
    canonrec(q, groups, gi + 1, order, best, r);
    order.resize(old);
    if (alltwins)
      break;
  } while (next_permutation(g.begin(), g.end()));
}

// Invariant signatures fix class order before exact canonicalization.
V canon(const V &q, int r) {
  vector<pair<V, int>> signatures;
  for (int j = 0; j < r; ++j) {
    V sig;
    for (int k = max(0, 2 * r - n); k <= r; ++k) {
      V block;
      for (int s : valid[r])
        if (__builtin_popcount((unsigned)s) == k && (s & (1 << j)))
          block.push_back(q[s]);
      sort(block.begin(), block.end());
      sig.insert(sig.end(), block.begin(), block.end());
    }
    signatures.emplace_back(move(sig), j);
  }
  sort(signatures.begin(), signatures.end());
```

```
  vector<V> groups;
  for (int j = 0; j < r; ++j) {
    if (!j || signatures[j].first != signatures[j - 1].first)
      groups.emplace_back();
    groups.back().push_back(signatures[j].second);
  }
  V order, best;
  canonrec(q, groups, 0, order, best, r);
  return best;
}

// Choose each sign recursively; accept when the final sum is target.
bool hits(const V &q, int target, size_t i = 0, int sum = 0) {
  if (i == q.size())
    return sum == target;
  return hits(q, target, i + 1, sum + q[i]) ||
         hits(q, target, i + 1, sum - q[i]);
}

int main(int argc, char **argv) {
  n = stoi(argv[1]);
  int target = stoi(argv[2]);
  if (n < 3 || n > 12)
    return 2;
  factorial_[0] = 1;
  for (int i = 1; i <= n; ++i)
    factorial_[i] = i * factorial_[i - 1];
  for (int i = 0; i <= n; ++i) {
    choose_[i][0] = choose_[i][i] = 1;
    for (int j = 1; j < i; ++j)
      choose_[i][j] = choose_[i - 1][j - 1] + choose_[i - 1][j];
  }
  for (int r = 0; r <= n; ++r)
    for (int s = 0; s < (1 << r); ++s)
      if (__builtin_popcount((unsigned)s) >= max(0, 2 * r - n))
        valid[r].push_back(s);
  unordered_set<V, Hash> states;
  // Normalize the bottom row to all +1: q_empty = q_{0} = 1.
  states.insert(V{1, 1});
  auto start = chrono::steady_clock::now();
  cout << "n=" << n << " target=" << target << "\n";
  for (int r = 1; r < n - 1; ++r) {
    int R = r + 1;
    unordered_set<V, Hash> next;
    uint64_t pruned = 0, generated = 0;
    vector<V> affected(R);
    V weights(1 << R);
    for (int s : valid[R]) {
      weights[s] = choose_[n - R][R - __builtin_popcount((unsigned)s)];
      for (int j = 0; j < R; ++j)
        if (s & (1 << j))
          affected[j].push_back(s);
    }
    for (auto const &packed : states) {
      V q = unpack(packed, r), v(1 << R);
```

```
    int norm = 0;
    // Start with all new row signs positive. The weighted norm
    // is L_R divided by factorial_[n-R].
    for (int s : valid[R]) {
      int t = s & ((1 << r) - 1), k = __builtin_popcount((unsigned)s);
      int p = (R - k + ((s & (1 << r)) ? 1 : 0)) * q[t];
      for (int z = t; z; z &= z - 1) {
        int bit = z & -z;
        p += q[t ^ bit];
      }
      v[s] = p;
      norm += weights[s] * abs(p);
    }
    for (int b = 0; b < (1 << R); ++b) {
      ++generated;
      // Consecutive Gray words flip exactly one new row sign.
      // Update only minors containing that coordinate.
      if (b) {
        int j = __builtin_ctz((unsigned)b), bit = 1 << j,
            gray = b ^ (b >> 1), delta = (gray & bit) ? -2 : 2;
        for (int s : affected[j]) {
          int t = s & ((1 << r) - 1),
              coefficient = q[j == r ? t : (t ^ bit)], old = v[s];
          v[s] += delta * coefficient;
          norm += weights[s] * (abs(v[s]) - abs(old));
        }
      }
      // Discard bounds strictly below the target; retain equality.
      if (factorial_[n - R] * norm < target) {
        ++pruned;
        continue;
      }
      // Retain sorted magnitudes of all n first-row cofactors.
      if (R == n - 1) {
        V profile;
        profile.push_back(abs(v[(1 << R) - 1]));
        for (int j = 0; j < R; ++j)
          profile.push_back(abs(v[((1 << R) - 1) ^ (1 << j)]));
        sort(profile.begin(), profile.end());
        next.insert(move(profile));
      }
      else
        next.insert(pack(canon(v, R), R));
    }
  }
  states = move(next);
  cout << "r=" << R << " generated=" << generated
       << " pruned=" << pruned << " states=" << states.size()
       << " seconds="
       << chrono::duration<double>(
            chrono::steady_clock::now() - start).count() << endl;
}
vector<V> profiles(states.begin(), states.end());
// Sort profiles so output is independent of hash-table order.
// Each row has trailing spaces and a final newline (LF on POSIX).
```

```
  sort(profiles.begin(), profiles.end());
  ofstream out("results/pm1_permanent_range/target_" + to_string(n)
               + "_" + to_string(target) + "_profiles.txt");
  int found = 0;
  for (const V &q : profiles) {
    for (int x : q)
      out << x << " ";
    out << "\n";
    if (hits(q, target)) {
      ++found;
      cout << "HIT_PROFILE ";
      for (int x : q)
        cout << x << " ";
      cout << endl;
    }
  }
  cout << "final_profiles=" << profiles.size()
       << " target_hits=" << found << " verdict="
       << (found ? "ATTAINED" : "EXCLUDED") << endl;
}
```

References

[1] D. Ingram and A. Razborov, On the range of the permanent of (±1)-matrices. *Linear Algebra and its Applications* **743** (2026), 271–285. doi:10.1016/j.laa.2026.04.027.

[2] H. Minc, *Permanents*. Encyclopedia of Mathematics and its Applications, vol. 6, Cambridge University Press, 1984. doi:10.1017/CBO9781107340688.

[3] R. A. Brualdi and H. J. Ryser, *Combinatorial Matrix Theory*. Cambridge University Press, 1991, Ch. 7, pp. 198–249. doi:10.1017/CBO9781107325708.008.

[4] Z. Hunter, M. Kwan and L. Sauermann, *Exponential anticoncentration of the permanent*. Preprint, 2025. arXiv:2509.22577.

[5] *Realizing every sign-matrix permanent with negative entries confined on or above the diagonal*. Available online.

[6] M. V. Budrevich and A. E. Guterman, Kräuter conjecture on permanents is true. *Journal of Combinatorial Theory, Series A* **162** (2019), 306–343. doi:10.1016/j.jcta.2018.11.009.

[7] I. M. Wanless, Permanents of matrices of signed ones. *Linear and Multilinear Algebra* **53** (2005), 427–433. doi:10.1080/03081080500093990.

[8] M. J. Colbrook, *Exact finite cases of two sign-matrix permanent problems*, 11 September 2026. Available online.

[9] I. Kaplansky and J. Riordan, The problem of the rooks and its applications. *Duke Mathematical Journal* **13** (1946), 259–268. doi:10.1215/S0012-7094-46-01324-5.

[10] C. Savage, A survey of combinatorial Gray codes. *SIAM Review* **39** (1997), 605–629. doi:10.1137/S0036144595295272.