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LANDAU'S PROBLEMS THROUGH RATIOS OF L-FUNCTIONS

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Abstract. Very recently, Lee announced a proof of the Twin Prime Conjecture assuming the Ratios Conjecture for the Riemann zeta function. We obtain conditional asymptotic formulas with power-saving errors for each of Landau's four prime problems. Similar results may be possible for other problems in analytic number theory.

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1. Introduction

In his invited address at the 1912 International Congress of Mathematicians in Cambridge, Edmund Landau surveyed the progress and outstanding challenges in the theory of prime numbers and the Riemann zeta function. He singled out four fundamental problems that he regarded as beyond the reach of the methods available at the time. These problems, now known as Landau's four problems, are the Goldbach Conjecture, the Twin Prime Conjecture, the existence of a prime between consecutive squares, and the infinitude of primes of the form n^2+1. For further historical background on these problems and an account of subsequent progress, see for instance [Pin09, Hel15, May19, TB26, GM25] and the references within.

It has long been known that Montgomery's Pair Correlation Conjecture can be derived from a uniform version of the Hardy–Littlewood Prime Pair Conjecture [Mon73, GM87, MS02]. Very recently, Lee [Lee26] announced a result in the other direction, showing that a non-uniform version of the Hardy–Littlewood Prime Pair Conjecture can be derived from the Ratios Conjecture of Conrey–Farmer–Zirnbauer [CFZ08]. Other recent consequences of the Ratios Conjecture can be found in [JT23, BGW26, WX25], with the work of Wang–Xu [WX25] being in retrospect closely related to Lee's work. In the present paper, we establish asymptotics for all four Landau problems under the Ratios Conjecture.[1]

Landau's four problems involve rather different arithmetic constraints. The purpose of this paper is to relate them through ratio statistics of three classical families of L-functions: Dirichlet L-functions, Riemann zeta shifts, and Hecke L-functions on $\mathbb{Q}(i)$. The ratios recipe of Conrey–Farmer–Zirnbauer [CFZ08], building on the moments recipe of [CFKRS05], predicts averages of quotients of products of L-functions.

We state the three ratios hypotheses and their uniformity ranges before giving the four theorems. Asymptotic formulas under weaker ratios hypotheses are given in Appendix A.

For $h \in \mathbb{Z}_{\geqslant 1}$, define

$$\mathfrak{S}(h)=\prod_p \sigma_p(h), \qquad \sigma_p(h)=\begin{cases} p/(p-1), & p \mid h, \\ p(p-2)/(p-1)^2, & p \nmid h. \end{cases} \tag{1.1}$$

[1] Strictly speaking we also assume GRH, but as in Lee's work, it is possible to formulate the Ratios Conjecture in a way that does not require GRH.

Thus $\mathfrak{S}(h)=0$ for odd h, and for even h it is the positive Hardy–Littlewood singular series. For the quadratic problem, write $\chi=\chi_{-4}$ and put

$$\mathfrak{S}_E=\prod_{p>2}\left(1-\frac{\chi(p)}{p-1}\right)=\frac{4}{\pi}\prod_{p\equiv 1\ (4)}\frac{p(p-2)}{(p-1)^2}\prod_{p\equiv 3\ (4)}\frac{p^2}{p^2-1}. \tag{1.2}$$

The first product is ordered by increasing primes; the second expression is absolutely convergent.

1.1. **Ratios conjectures.** We state three ratios conjectures below, for primitive Dirichlet L-functions, the Riemann zeta function, and angular Hecke L-functions over $\mathbb{Q}(i)$. The untwisted Dirichlet sum runs over all primitive characters modulo q. The Hecke average is unweighted. Section 2 gives the Euler products in the conjectured main terms. Section 3.6 defines $\mathcal{M}_q$, the sum of the two main terms for the Dirichlet conjecture. At a removable singularity, we evaluate the sum of the main terms by analytic continuation. On the indicated disks, the errors are holomorphic in the numerator shifts. Fix $\eta_*=1/16$ and $\rho=1/100$.

Hypothesis 1.1 (Ratios of primitive Dirichlet L-functions ($\mathrm{RC}_D(\omega_D)$)). Fix $0<\omega_D\leqslant 1/2$. There exist constants $\eta_0>0$ and $B\geqslant 0$ such that for every fixed $0<\eta<\eta_0$ and every $\varepsilon>0$,

$$\sideset{}{^*}\sum_{\chi \bmod q}\frac{L(a,\chi)L(b,\overline{\chi})}{L(s,\chi)L(t,\overline{\chi})}=\mathcal{M}_q(a,b;s,t)+O_{\eta,\varepsilon}\big(q^{1-\omega_D+\varepsilon}(1+T)^B\big),$$

uniformly for all conductors $q>1$, including composite conductors, and all shifts satisfying

$$\Re a=\Re b=\tfrac12,\qquad \Re s=\Re t=\tfrac12+\eta,\qquad T=\max\{|\Im a|,|\Im b|,|\Im s|,|\Im t|\}\leqslant 100\log(2q).$$

The starred sum is over primitive characters modulo q. The saving ω_D is independent of η.

Hypothesis 1.2 (Ratios of the Riemann zeta function ($\mathrm{RC}_Z(\omega_Z)$)). Let $0<\omega_Z\leqslant 1/2$. With $s=1/2+a$, $t_0=1/2+b$ and $\mathcal{M}_Z$ as in (2.14), assume

$$\int_0^Q\frac{\zeta(s+u+iy)\zeta(t_0+v-iy)}{\zeta(s+iy)\zeta(t_0-iy)}\,dy=\mathcal{M}_Z(Q;a,b;u,v)+O_\varepsilon(Q^{1-\omega_Z+\varepsilon}),$$

uniformly for

$$\frac{1}{\log Q}\leqslant \Re a,\Re b\leqslant \eta_*,\qquad |\Im a|,|\Im b|\leqslant Q^{3/4},\qquad |u|,|v|\leqslant\rho.$$

Hypothesis 1.3 (Ratios of angular Hecke L-functions ($\mathrm{RC}_H(\omega_H)$)). Let $0<\omega_H\leqslant 1/2$. For $L_k(s)=L(s,\xi_k)$, with $\xi_k((\alpha))=(\alpha/\overline{\alpha})^{2k}$ on $\mathbb{Z}[i]$, assume

$$\frac{1}{Q}\sum_{1\leqslant k\leqslant Q}\frac{L_k(1/2+a)}{L_k(1/2+c)}=\mathcal{M}_H(Q;a,c)+O_\varepsilon(Q^{-\omega_H+\varepsilon}),$$

uniformly for

$$\frac{2}{\log Q}\leqslant \Re c\leqslant\eta_*,\qquad |\Im c|\leqslant Q^{3/4},\qquad |a-c|\leqslant\rho,$$

where $\mathcal{M}_H$ is the sum of the main terms given in Section 2.5.

The last two error bounds are uniform in the imaginary parts of the shifts throughout the stated ranges. This uniformity is used to remove smoothing without a loss in the exponent. The upper bound $Q^{3/4}$ may be replaced by Q^θ for any fixed $1/2<\theta<1$. The uniform conjectures of Conrey–Snaith [CS07, Conjecture 2.1 and (2.11)] and Waxman [Wax21, Conjecture 3.3 and Section 3.6] predict square-root error bounds, corresponding to $\omega_Z=\omega_H=1/2$. Wang–Xu [WX25, Conjecture 2.1] conjecture a power saving for prime conductors. Hypothesis 1.1 assumes the formula for all primitive conductors and specifies the exponent ω_D. The case $\omega_D=1/2$ is the square-root error bound for this family.

1.2. **Four conditional results.** Each theorem below assumes only its own ratios input and the indicated Riemann hypothesis. The constants in the error terms may depend on the fixed ratios parameters.[2]

Theorem 1.1 (Goldbach)**.** *Assume GRH for Dirichlet L-functions, and* $\mathrm{RC}_D(\omega_D)$*. For every* $\varepsilon > 0$ *and every even* N*,*

$$\sum_{m+n=N} \Lambda(m)\Lambda(n) = N\mathfrak{S}(N) + O_\varepsilon(N^{1-\omega_D+\varepsilon}). \tag{1.3}$$

Theorem 1.2 (Prime pairs)**.** *Assume RH for* ζ *and* $\mathrm{RC}_Z(\omega_Z)$*. For every fixed integer* $h \geqslant 1$ *and every* $\varepsilon > 0$*,*

$$\sum_{n\leqslant X} \Lambda(n)\Lambda(n+h) = \mathfrak{S}(h)X + O_{h,\varepsilon}(X^{1-\omega_Z+\varepsilon}). \tag{1.4}$$

Theorem 1.3 (Legendre)**.** *Assume RH for* ζ *and* $\mathrm{RC}_Z(\omega_Z)$*. For every* $\varepsilon > 0$*,*

$$\psi((n+1)^2) - \psi(n^2) = 2n + 1 + O_\varepsilon(n^{1-\omega_Z/3+\varepsilon}). \tag{1.5}$$

Moreover $p_{j+1} - p_j \ll_\varepsilon p_j^{(1-\omega_Z)/(2-\omega_Z)+\varepsilon}$*. For* $\omega_Z = 1/2$*, the error is* $O_\varepsilon(n^{5/6+\varepsilon})$ *and the prime-gap exponent is* $1/3 + \varepsilon$*.*

Theorem 1.4 (Prime values of n^2+1)**.** *Assume GRH for the angular Hecke family and* $\mathrm{RC}_H(\omega_H)$*. For every* $\varepsilon > 0$*,*

$$\sum_{n\leqslant X} \Lambda(n^2+1) = \mathfrak{S}_E X + O_\varepsilon(X^{1-\omega_H+\varepsilon}). \tag{1.6}$$

The proofs are in Sections 3–6. Appendix A gives asymptotic formulas under Hypotheses 2.3 and 2.4 together with the relevant Riemann hypothesis. These ratios hypotheses allow errors of size $o(1)$ after normalization and shifts of logarithmic size.

1.3. **Overview of the paper.** The proof separates the arithmetic information in a ratios prediction from the operation used to recover a prime count. Differentiating the ratios with respect to the numerator shifts gives logarithmic derivatives, whose Dirichlet series have von Mangoldt coefficients. The resulting main terms contain explicit Euler products. Character orthogonality, Fourier inversion, and estimates for sums over zeros then relate these main terms to the prime counts.

Table 1 lists the ratios and the methods used in the proofs. Here 2/2 denotes two numerator and two denominator factors, and 1/1 denotes one of each.

Problem	Ratios	Method
Goldbach	Primitive Dirichlet L-functions, 2/2	The parity twist and summation over primitive conductors
Prime pairs	Riemann zeta function, 2/2	Fourier inversion for a fixed difference
Legendre	Riemann zeta function, 2/2	Pair correlation and second moments of primes in short intervals
Quadratic values	Angular Hecke L-functions, 1/1	Poisson summation over the Hecke characters

TABLE 1. Ratios of L-functions and the methods used to estimate the prime counts.

[2]For brevity, throughout the paper we refer to the asymptotic formulas in Theorems 1.1, 1.2, 1.3, and 1.4 as "Goldbach", "prime pairs", "Legendre", and "Prime values of n^2+1", respectively.

The arguments for Goldbach, fixed prime differences, and primes in short intervals use the functions D_1 and C_1 from Section 2. They satisfy

$$D_1(z) = \sum_{h\geqslant 1} \frac{\mathfrak{S}(h)}{h^z} \qquad (\Re z > 1), \tag{1.7}$$

$$C_1(z) = \Xi(z) D_1(z) = \zeta(1+z)\zeta(1-z) \prod_p \left(1 - \frac{(p^{-z}-1)^2}{(p-1)^2}\right),$$

where $\Xi(z) = 2(2\pi)^{-z}\Gamma(z)\cos(\pi z/2)$ is the archimedean factor from the zeta functional equation. A Fourier test selects the coefficient $\mathfrak{S}(h)$ for a fixed prime difference. The interval argument uses a positive test of the continued kernel C_1.

For Goldbach, summing over primitive conductors gives the exact projection

$$\frac{1}{\varphi(N)} \sum_{q|N} \varphi^*(q) q^{-z} D_q(z) = \sum_{N|h} \frac{\mathfrak{S}(h)}{h^z}, \qquad \Re z > 1. \tag{1.8}$$

Here $\varphi^*(q)$ counts primitive characters, with $\varphi^*(1) = 1$. Additive support then isolates $h = N$. Thus the prime-sum and prime-difference arguments recover coefficients of the same singular-series Dirichlet series by different tests.

The angular counterpart is $\mathcal{D}(z) = \sum_{b\geqslant 1} S_b b^{-z}$, where S_b is the Bateman–Horn singular series for $n^2 + b^2$; see (6.18). Its first coefficient is $S_1 = \mathfrak{S}_E$. The phases explain the parallel with fixed differences:

$$X \log(1 + h/n) = \frac{h}{n/X} + O_h(X^{-1}), \qquad X \arg(n + i) = \frac{1}{n/X} + O(X^{-2}) \qquad (n \asymp X).$$

Both tests distinguish an integer coefficient at reciprocal spatial scale. The interval argument instead applies a positive test to the continued zeta kernel; it does not isolate a single singular-series coefficient.

For Goldbach, character orthogonality introduces $\chi(-1)$, whereas the ordinary ratios average includes both parities. The numerator functional equations give a second ordinary average, from which we recover the parity difference. The inverse grows exponentially at opposite heights. Rational product weights control it and give distributional asymptotics. To retain the power saving in the sharp count, we instead use the fixed entire test

$$w(v) = e^{20(1-v)} \left(\frac{\sin \pi(v-1)}{\pi(v-1)}\right)^2, \qquad w(1) = 1, \quad w(k) = 0 \quad (k = 2, 3, \ldots).$$

The weight $w((m+n)/N)$ selects the target sum exactly after character orthogonality. Its double Mellin transform contains a beta factor which cancels the exponential loss of the parity inverse on the required logarithmic window. Outside that window the actual signed moment is used directly. Section 3.5 proves the resulting power-saving formula without approximating a sharp cutoff by rational product weights. Integrating the same argument against bounded real power weights removes the prime logarithms with the same saving.

Theorem 3.4 supplies the numerator disks from critical-line character data. Its interpolation loss tends to zero with the fixed denominator displacement, so the final error is $N^{1-\omega_D+\varepsilon}$. For prime differences and quadratic values, the uniform height-independent ratios error and bounded-variation cutoffs give Mellin L^1 norms of only logarithmic size. Proposition 4.3 therefore removes smoothing with no fixed loss in the exponent. For intervals, the localized second moment in Section 5 gives a quantitative monotonicity argument; optimizing the averaging width yields the saving $\omega_Z/3$ in the square parameter n. Appendix A retains the moving-line argument with $\eta_X = 4/\log X$ for the weaker qualitative variants.

Further remarks. The common kernels give a set of sufficient hypotheses for the four integer asymptotics. They do not assert reverse implications between the prime conjectures, nor do they replace the three family-dependent ratios inputs by a single zeta formula. Appendix B

gives a further application to two-point Möbius and Liouville correlations. That application uses a separate reciprocal-ratios hypothesis, with additional uniformity in small square twists.

Notation. The von Mangoldt function is $\Lambda(n)$ and the second Chebyshev function is $\psi(x) = \sum_{n\leqslant x}\Lambda(n)$. For $f \in C_c^\infty((0,\infty))$, the Mellin transform is

$$\widetilde{f}(s) = \int_0^\infty f(u)u^{s-1}\,du.$$

Contours (c) run upward on $\Re s = c$. Fourier conventions are stated with each detector. Implied constants may depend on fixed test functions and hypothesis parameters. Uniformity in conductors and summation lengths is stated explicitly.

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AI statement. After we saw Lee's preprint on the Twin Prime Conjecture [Lee26], which uses a case of the Ratios Conjecture analogous to that in the work of Harper and Wang–Xu on the Möbius Legendre Conjecture [Har23, WX25, Har26], we wondered whether there might exist a unified conditional treatment of Landau's problems, as well as the Power-saving Two-point Chowla Conjecture, via the Ratios Conjecture. The dynamical systems GPT-6 Pro and SAIR Open Math OS were used to convert our rough ideas into rigorous proofs and understanding, and to assist in the writing process, building on the existing literature on the Ratios Conjecture and Chowla's Conjecture.

2. Ratios models and arithmetic kernels

Fix $0 < \eta < 1/4$ and write $\sigma = 1/2 + \eta$. The fixed-line proofs use $0 < \omega \leqslant 1/2$ and $\omega > 2\eta$; the displacement may be chosen arbitrarily small under the introduction's hypotheses. We also record qualitative inputs on lines approaching the critical line for the variants in Appendix A.

2.1. The unitary ratios kernel.

For $q \geqslant 1$, let $\zeta_q(s) = \zeta(s)\prod_{p\mid q}(1-p^{-s})$, and define

$$Z_q(a,b;c,d) = \frac{\zeta_q(a+b)\zeta_q(c+d)}{\zeta_q(a+d)\zeta_q(b+c)}A_q(a,b;c,d), \tag{2.1}$$

$$A_q(a,b;c,d) = \prod_{p\nmid q}\frac{(1-p^{-c-d})(1-p^{-a-d}-p^{-b-c}+p^{-c-d})}{(1-p^{-a-d})(1-p^{-b-c})}. \tag{2.2}$$

Here the arguments are absolute complex arguments of L-functions, rather than shifts from $1/2$. The product is normally convergent in the neighborhoods used below. Its formal local interpretation is the constant Laurent coefficient in

$$\frac{(1-p^{-c}z)(1-p^{-d}z^{-1})}{(1-p^{-a}z)(1-p^{-b}z^{-1})}.$$

In an initial half-plane this gives

$$Z_q(a,b;c,d) = \prod_{p\nmid q}\frac{1-p^{-a-d}-p^{-b-c}+p^{-c-d}}{1-p^{-a-b}},$$

and (2.1) supplies the meromorphic continuation needed here.

For $\Re z > 0$, put

$$F_p(z) = \frac{1 - 2/p + p^{-1-z}}{(1-1/p)^2}, \tag{2.3}$$

$$D_q(z) = \left(\frac{q}{\varphi(q)}\right)^2 \zeta(z) \prod_{p\nmid q} F_p(z) \prod_{p\mid q}(1-p^{z-1}), \tag{2.4}$$

$$\Xi(z) = 2(2\pi)^{-z}\Gamma(z)\cos(\pi z/2), \tag{2.5}$$

$$C_q(z) = \Xi(z)D_q(z) = \left(\frac{q}{\varphi(q)}\right)^2 \zeta(1-z) \prod_{p\nmid q} F_p(z) \prod_{p\mid q}(1-p^{z-1}), \tag{2.6}$$

$$P_q(z) = \sum_{(n,q)=1} \frac{\Lambda(n)^2}{n^{1+z}}. \tag{2.7}$$

The identity in (2.6) is the zeta functional equation. Since $F_p(z) = 1 + O(p^{-1-\Re z} + p^{-2})$, the infinite product is absolutely convergent for $\Re z > 0$. For $q = 1$, D_1 has a simple pole of residue one at $z = 1$ and no other pole in this half-plane. If $q > 1$, the finite product cancels this pole.

To pass from ratios to von Mangoldt coefficients, differentiate once in each numerator. For denominator arguments s, t with $\Re s, \Re t > 1/2$, the unswapped derivative is

$$\partial_u\partial_v Z_q(s+u, t+v; s, t)|_{u=v=0} = P_q(s+t-1).$$

The swapped kernel $Z_q(1-t-v, 1-s-u; s, t)$ contains

$$\zeta_q(1-u)^{-1}\zeta_q(1-v)^{-1} = \left(\frac{q}{\varphi(q)}\right)^2 uv + O(|uv|(|u|+|v|)).$$

Its mixed derivative is therefore $C_q(z)$, with $z = s+t-1$. Any derivative falling on an additional holomorphic factor leaves one of the two zeros uncanceled and contributes nothing. This is why the functional-equation factors remain undifferentiated in the character and continuous moment formulas.

2.2. **Singular series and conductor projection.** The kernel has two useful representations. In the coefficient half-plane,

$$D_1(z) = \sum_{h\geqslant 1} \frac{\mathfrak{S}(h)}{h^z} \qquad (\Re z > 1). \tag{2.8}$$

For its continuation toward zero, put $Z = \zeta'/\zeta$ and observe that

$$P_1(z) = Z'(1+z) - \sum_p \left(\frac{\log p}{p^{1+z}-1}\right)^2, \tag{2.9}$$

$$C_1(z) = \zeta(1+z)\zeta(1-z)\mathcal{A}(z), \qquad \mathcal{A}(z) = \prod_p \left(1 - \frac{(p^{-z}-1)^2}{(p-1)^2}\right). \tag{2.10}$$

The Euler products after removing the displayed zeta factors converge normally in $\Re z > -1/2$. The local identity

$$(1-p^{-1-z})F_p(z) = 1 - \frac{(p^{-z}-1)^2}{(p-1)^2}$$

proves (2.10). Expanding the two local geometric series proves (2.9); summing the local densities in (1.1) proves (2.8), including the factor at two. Thus the prime-difference and zero-correlation calculations use the same pair P_1, C_1.

The version with deleted primes records a conductor projection. For $q \geqslant 1$, write

$$\mathfrak{S}_q(h) = \prod_{p\nmid q} \sigma_p(h), \qquad c_q(h) = \sum_{d\mid(q,h)} d\mu(q/d), \qquad K_q(h) = \frac{q}{\varphi(q)^2} c_q(h)\mathfrak{S}_q(h).$$

We set $\varphi^*(1) = 1$. For $q > 1$, $\varphi^*(q)$ counts primitive characters modulo q.

Proposition 2.1 (Conductor projection)**.** *For $\Re z > 1$,*

$$D_q(z) = \sum_{h \geqslant 1} K_q(h)(h/q)^{-z}, \tag{2.11}$$

and for all positive integers N, h,

$$\sum_{q|N} \varphi^*(q) K_q(h) = \varphi(N)\mathfrak{S}(h)\mathbf{1}_{N|h}. \tag{2.12}$$

Consequently (1.8) holds.

Proof. The local identity

$$(1 - p^{-z}) \sum_{v \geqslant 0} \sigma_p(p^v) p^{-vz} = F_p(z)$$

also holds at $p = 2$, where $F_2(z) = 2^{1-z}$. Hence $\sum_{k \geqslant 1} \mathfrak{S}_q(k) k^{-z} = \zeta(z) \prod_{p \nmid q} F_p(z)$. Expand the finite product in (2.4), obtaining

$$D_q(z) = \left(\frac{q}{\varphi(q)}\right)^2 \sum_{d | \mathrm{rad}(q)} \frac{\mu(d)}{d} \sum_{k \geqslant 1} \mathfrak{S}_q(k)(k/d)^{-z}.$$

Set $h = qk/d$. The coefficient of $(h/q)^{-z}$ is

$$\left(\frac{q}{\varphi(q)}\right)^2 \mathfrak{S}_q(h) \sum_{\substack{d|q \\ q/d|h}} \frac{\mu(d)}{d} = K_q(h),$$

which proves (2.11).

For even h, divide (2.12) by $\mathfrak{S}(h)$ and use multiplicativity. For $p^a \| N$, with $v = v_p(h)$, the local sum is

$$1 + \sum_{j=1}^{a} \varphi^*(p^j) \frac{p^j c_{p^j}(h)}{\varphi(p^j)^2 \sigma_p(h)}.$$

Here $c_{p^j}(h)$ is $\varphi(p^j)$ for $j \leqslant v$, $-p^{j-1}$ for $j = v+1$, and zero thereafter; also $\varphi^*(p) = p - 2$ and $\varphi^*(p^j) = p^{j-2}(p-1)^2$ for $j \geqslant 2$. If $v = 0$, the first nonconstant summand is -1. If $v \geqslant 1$, the sum through $\min(v, a)$ is $\varphi(p^{\min(v,a)})$, and the next summand cancels it when $v < a$. This gives $\varphi(p^a)\mathbf{1}_{p^a|h}$, also at two since $\varphi^*(2) = 0$.

For odd h, terms with odd q vanish at the local factor two. If $2 \| q$, $\varphi^*(q) = 0$; if $4 \mid q$, then $c_q(h) = 0$. Thus both sides vanish. Finally insert (2.11) and use (2.12) to obtain (1.8). □

2.3. **Mellin coefficient extraction.** A ratios calculation first produces an integral of an arithmetic kernel. To recover a prime count, we move to the half-plane where its Dirichlet coefficients are visible. The following lemma isolates the analytic step; a zero moment of the test function removes the possible pole at one.

Lemma 2.2. *Let $0 < c < 1 < d$. Suppose D is meromorphic in $c \leqslant \Re z \leqslant d$, has polynomial vertical growth, and has no pole there except possibly a simple pole at one. Suppose $D(z) = \sum_{j \geqslant 1} a_j j^{-z}$ converges absolutely on (d). For $f, h \in C_c^\infty((0,\infty))$ with $\int h = 0$,*

$$\frac{1}{2\pi i} \int_{(c)} D(z) \widetilde{f}(z) \widetilde{h}(z)\, dz = \sum_{j \geqslant 1} a_j \int_0^\infty f(u) h(j/u) \frac{du}{u}. \tag{2.13}$$

If the product of the two supports contains just one positive integer j_0, only a_{j_0} contributes.

Proof. The equality $\widetilde{h}(1) = 0$ cancels the possible pole. Rapid Mellin decay permits a shift to (d), where the series may be inserted absolutely. Multiplicative Mellin convolution gives (2.13). The support assertion follows directly. □

For prime differences, $a_j = \mathfrak{S}(j)$. For angular ratios, the coefficients are the singular series S_j for $n^2 + j^2$, defined in (6.18). Goldbach uses the first sequence after the conductor projection (1.8).

2.4. **Continuous zeta ratios.** We specify qualitative variants of the continuous and angular ratios inputs; the quantitative main theorems use Hypotheses 1.2 and 1.3. Fix a sufficiently small absolute numerator radius $\rho > 0$, for example $\rho = 1/100$. The family parameter Q tends to infinity. Errors are uniform in the indicated ranges and holomorphic on the closed numerator disks. The real shifts lie in fixed intervals of multiples of $1/\log Q$.

For $s = 1/2 + a$, $t_0 = 1/2 + b$, define the continuous model

$$\mathcal{M}_Z(Q; a, b; u, v) = QZ_1(s + u, t_0 + v; s, t_0) + (2\pi)^{a+b+u+v} \frac{Q^{1-a-b-u-v}}{1 - a - b - u - v} Z_1(1 - t_0 - v, 1 - s - u; s, t_0). \tag{2.14}$$

Hypothesis 2.3 (Weak continuous ratios)**.** *For every fixed $A > 0$, uniformly for*

$$\frac{1}{\log Q} \leqslant \Re a, \Re b \leqslant \frac{8}{\log Q}, \qquad |\Im a|, |\Im b| \leqslant A \log Q, \qquad |u|, |v| \leqslant \rho,$$

one has

$$\int_0^Q \frac{\zeta(s + u + iy)\zeta(t_0 + v - iy)}{\zeta(s + iy)\zeta(t_0 - iy)}\, dy = \mathcal{M}_Z(Q; a, b; u, v) + o(Q). \tag{2.15}$$

The two main terms are interpreted together at removable singularities.

Conrey–Snaith [CS07, Conjecture 2.1 and (2.11)] predicts an error $O_\varepsilon(Q^{1/2+\varepsilon})$ uniformly on a domain containing these numerator disks and logarithmic height windows. Its restriction therefore implies Hypothesis 2.3. The weaker $o(Q)$ error is retained for the qualitative versions of the two zeta applications.

2.5. **Angular Hecke ratios.** The angular family records the argument of a Gaussian integer. This permits an average over characters to distinguish a fixed imaginary coordinate, just as a time average of zeta ratios distinguishes a fixed additive difference.

Let $F = \mathbb{Q}(i)$ and let $\chi = \chi_{-4}$. For a nonzero integral ideal $\mathfrak{a} = (\alpha)$ of $\mathbb{Z}[i]$, put

$$\xi_k(\mathfrak{a}) = \left(\frac{\alpha}{\overline{\alpha}}\right)^{2k}, \qquad L_k(s) = L(s, \xi_k), \qquad k \in \mathbb{Z}.$$

The character is independent of the generator. Conjugation gives $L_{-k} = L_k$, and L_k is entire for $k \neq 0$. These are the angular Hecke L-functions considered by Waxman [Wax21].

For complex a, c, define

$$A_3(a, c) = \prod_{p \equiv 3 \ (4)} \frac{1 - p^{-2-4c}}{1 - p^{-2-2a-2c}}, \tag{2.16}$$

$$A_1(a, c) = \prod_{p \equiv 1 \ (4)} \frac{(1 - 2p^{-1-a-c} + p^{-1-2c})(1 - p^{-1-2c})}{(1 - p^{-1-a-c})^2}, \tag{2.17}$$

$$G(a, c) = \frac{\zeta(1 + 2a)L(1 + 2c, \chi)}{\zeta(1 + a + c)L(1 + a + c, \chi)} A_1(a, c)A_3(a, c). \tag{2.18}$$

The products converge absolutely and locally uniformly near the diagonal and swapped diagonal used below. The factor at 2 is included in the zeta quotient.

The corresponding averaged ratios model is

$$\mathcal{M}_H(Q; a, c) = G(a, c) + \frac{1}{1 - 2a} \left(\frac{\pi}{2Q}\right)^{2a} G(-a, c).$$

Hypothesis 2.4 (Weak angular ratios)**.** *For some fixed $A > 0$, uniformly for*

$$\frac{2}{\log Q} \leqslant \Re c \leqslant \frac{8}{\log Q}, \qquad |\Im c| \leqslant A \log Q, \qquad |a - c| \leqslant \rho,$$

one has

$$\frac{1}{Q} \sum_{1 \leqslant k \leqslant Q} \frac{L_k(1/2 + a)}{L_k(1/2 + c)} = \mathcal{M}_H(Q; a, c) + o(1). \tag{2.19}$$

The combined model and error are holomorphic on the numerator disk.

Waxman [Wax21, Conjecture 3.3] predicts the displayed model with normalized error $O_\varepsilon(Q^{-1/2+\varepsilon})$. We use its uniform conjectural interpretation on the stated shift domain. The apparent pole at $a = 0$ cancels between $G(a, c)$ and the reflected term: their zeta residues are opposite, and the prefactor equals one at zero. Under the relevant Riemann hypotheses the remaining reciprocal factors are holomorphic on the numerator disks. Cauchy's formula thus differentiates the combined model and its error without a new conjectural input.

Corollary 2.5 (Square-root ratios implication)**.** *Under the relevant Riemann hypotheses and Hypotheses 1.1–1.3 with $\omega_D = \omega_Z = \omega_H = 1/2$, the Goldbach, fixed-difference, and quadratic counts, weighted or unweighted with the stated main terms, have errors $O_\varepsilon(N^{1/2+\varepsilon})$, $O_{h,\varepsilon}(X^{1/2+\varepsilon})$, and $O_\varepsilon(X^{1/2+\varepsilon})$, respectively. Moreover, (1.5) holds with error $O_\varepsilon(n^{5/6+\varepsilon})$.*

The continuous and angular hypotheses are restrictions of the cited uniform square-root predictions. For characters the all-conductor scope and the square-root saving are part of the assumption. If only a fixed positive character saving ω_D is available, Theorem 1.1 retains that saving up to N^ε.

2.6. **Zero-free margins.** For fixed-line arguments it suffices to assume a common $\beta < \sigma$ such that the relevant L-functions have no zeros in $\Re s > \beta$. RH or GRH gives $\beta = 1/2$. A standard logarithmic-derivative estimate in a closed zero-free half-plane gives, for primitive Dirichlet characters,

$$\frac{L'}{L}(\sigma + it, \chi) \ll_{\sigma - \beta} \log(2q(2 + |t|)). \tag{2.20}$$

The analogous estimates hold for ζ away from its pole and for the angular Hecke family with conductor parameter k. One obtains these bounds from the local zero expansion of L'/L: zeros at bounded distance contribute $O(\log(q(2 + |t|)))$, and their denominators are bounded below by the fixed margin; the remaining terms obey the same bound. We use also a fixed polynomial bound for $1/L$ on these lines, obtained by integrating the zero-free logarithm from a line of absolute convergence. These estimates justify the fixed-line Mellin shifts. The moving-line argument in Appendix A uses RH or GRH itself; see [IK04, Chapter 5].

Uniform holomorphic errors on numerator discs may be differentiated by Cauchy's formula. Discs of radius $r/\log Q$ cost a fixed power of $\log Q$; fixed discs cost a constant. Such logarithmic losses are absorbed in Q^ε.

2.7. **Smooth time and family weights.** The ratios predictions evaluate initial averages, while the arithmetic detectors use smooth time or family weights. Partial integration connects the two. A high-order zero of the weight at the origin makes the small-height contribution negligible, so no ratios formula at time zero is needed. Let $z = a + b$, $\Re a = \Re b = \eta$, and $L = 2 + |\Im a| + |\Im b|$. Suppose

$$\int_0^U V_{a,b}(t)\, dt = U A(z) + (2\pi)^z \frac{U^{1-z}}{1 - z} B(z) + O_\varepsilon(U^{1-\omega+\varepsilon} L^B) \tag{2.21}$$

whenever $L \leqslant cU^\theta$, for some fixed $\theta > 0$. Suppose the integrand and model have polynomial growth outside this range. If g is even and Schwartz, vanishes to sufficiently high order $2M$ at

zero, and $\widetilde{g}(w) = \int_0^\infty g(v)v^{w-1}\,dv$, then

$$\int_0^\infty g(t/T)V_{a,b}(t)\,dt = T\widetilde{g}(1)A(z) + T(2\pi/T)^z\widetilde{g}(1-z)B(z) + O_\varepsilon(T^{1-\omega+\varepsilon}L^{B'}) + O(T^{-2M}L^{C_M}). \tag{2.22}$$

Here B' is fixed and C_M depends on M. To prove this, subtract the displayed model primitive and integrate by parts against $T^{-1}g'(U/T)$. Above $U_0 = C\max(2, L^{1/\theta})$ use (2.21); below U_0 use polynomial growth and $g'(v) \ll v^{2M-1}$. The contribution below U_0 is bounded by T^{-2M} times a fixed power of L. Boundary terms vanish. On integrating against rapidly decreasing Mellin transforms, the last term in (2.22) is $O(T^{-2M})$ with finite seminorm dependence. Thus no uniform ratios formula at time zero is being assumed.

Lemma 2.6 (Removal of smoothing)**.** *Suppose one of the three coefficient problems has a smooth error $O(X^{1-\delta})$, and its error constant on upper and lower cutoffs of transition width v is $O(v^{-A})$, for fixed $\delta > 0$ and $A \geqslant 0$. Then its sharp weighted error is*

$$O_\varepsilon(X^{1-\delta/(A+1)+\varepsilon}).$$

This applies equally to $\Lambda(n)\Lambda(n+h)$, to Goldbach when such compact-test estimates are available, and to $\Lambda(n^2+1)$.

Proof. On a fixed dyadic interval, the transition contains $O(vX+1)$ integers. Each summand is $O(\log^2 X)$ in the first two cases and $O(\log X)$ in the quadratic case. Smooth majorants and minorants thus give errors $O(vX\log^2 X + X^{1-\delta}v^{-A} + \log^2 X)$; the Goldbach main-term transition also uses $\mathfrak{S}(N) \ll_\varepsilon N^\varepsilon$. Choose $v = X^{-\delta/(A+1)}$ and sum dyadic blocks when appropriate. In the applications we may decrease δ if necessary so that $\delta/(A+1) < 1$. □

The general lemma need not preserve the smooth saving. Proposition 4.3 does preserve it under the height-independent error in the introduction; the Goldbach proof uses a separate exact test.

3. Goldbach from character ratios

A character average naturally detects a multiplicative congruence, whereas Goldbach requires $m \equiv -n \pmod N$. For $(mn, N) = 1$, character orthogonality gives

$$\mathbf{1}_{N\mid m+n} = \frac{1}{\varphi(N)}\sum_{\chi \bmod N}\chi(-1)\chi(m)\overline{\chi(n)}. \tag{3.1}$$

The factor $\chi(-1)$ is therefore required by the arithmetic count. We recover its average from two ordinary ratios averages by applying the numerator functional equations. The resulting inverse can grow exponentially at opposite heights; rational test functions provide the Mellin decay needed to integrate it. Positivity gives compact-test asymptotics. An exact analytic test of the sum will also retain the power saving in sharp counting.

We first work with the numerator-disk input below. Section 3.5 gives the sharp power-saving estimate, and Section 3.6 derives the disk input from the critical-numerator hypothesis of Theorem 1.1.

Fix $0 < \eta < 1/4$, $\sigma = 1/2+\eta$, and $0 < r < \min(\eta/4, (1/4-\eta)/4)$. Write

$$R_q(a,b;c,d) = \sideset{}{^*}\sum_{\chi \bmod q}\frac{L(a,\chi)L(b,\overline{\chi})}{L(c,\chi)L(d,\overline{\chi})}.$$

All ratios formulas assumed in this section are untwisted and include both parities. Let $M_e(q)$ count primitive characters of parity $\chi(-1) = (-1)^e$, put $\varphi^*(q) = M_0(q) + M_1(q)$, and write

$$X_{q,e}(s) = \Big(\frac{q}{\pi}\Big)^{1/2-s}\frac{\Gamma((1-s+e)/2)}{\Gamma((s+e)/2)}.$$

Set

$$H_q(a,b)=\sum_{e=0}^{1}M_e(q)X_{q,e}(a)X_{q,e}(b).$$

This is an explicit average of functional-equation factors, not a parity-restricted ratios average.

Hypothesis 3.1 (Untwisted Dirichlet ratios)**.** *For some fixed $0<\omega\leqslant 1/2$ and $B\geqslant 0$, uniformly for all $q>1$, $\Re s=\Re t=\sigma$, $|\Im s|,|\Im t|\leqslant 10\log(2q)$, and $|u|,|v|\leqslant r$, one has*

$$\begin{aligned}R_q(a,b;s,t)=\varphi^*(q)Z_q(a,b;s,t)+H_q(a,b)Z_q(1-b,1-a;s,t)\\+O_\varepsilon\big(q^{1-\omega+\varepsilon}(1+|\Im s|+|\Im t|)^B\big),\end{aligned}\tag{3.2}$$

for both $(a,b)=(s+u,t+v)$ and $(a,b)=(1-t-v,1-s-u)$. The error is holomorphic in u,v and uniform on the indicated closed discs.

These are two numerator specializations of the ordinary unitary two-over-two ratios prediction. In particular there is no factor $\chi(-1)$ and no average restricted to one parity. The numerator real parts lie in the usual interval $(1/4,3/4)$; the denominator real parts are to the right of $1/2$. The input still concerns every primitive conductor, including composite conductors. We do not deduce that conductor uniformity from the prime-modulus formulation in [WX25]. The two-term model follows from the general ratios recipe [CFZ08]: the balanced functional-equation choices give the diagonal and double swap, and the exact finite average of the archimedean factors is H_q.

Theorem 3.2. *Assume GRH for Dirichlet L-functions and Hypothesis 3.1, with $\omega>2\eta$. Uniformly for even N,*

$$\sum_{m+n=N}\Lambda(m)\Lambda(n)=N\mathfrak{S}(N)+O_\varepsilon(N^{1-\omega+2\eta+\varepsilon}).$$

The ordered prime representation count is

$$\mathfrak{S}(N)\int_2^{N-2}\frac{dt}{\log t\log(N-t)}+O_\varepsilon(N^{1-\omega+2\eta+\varepsilon}).$$

The denominator displacement η is a proof parameter. A saving $\omega>0$ that is uniform for sufficiently small fixed displacements allows the choice $2\eta<\omega$. Theorem 3.4 makes this reduction from critical numerator shifts precise.

3.1. **Recovering parity.** The two numerator functional equations give a second linear combination of the even and odd contributions. Their root numbers cancel because the characters are conjugate. The only issue is then the size of the inverse linear system.

For $a=s+u$, $b=t+v$, put

$$R=R_q(a,b;s,t),\qquad R^\vee=R_q(1-b,1-a;s,t),\qquad h_e=X_{q,e}(1-b)X_{q,e}(1-a).$$

If R_e denotes the contribution of parity e to R, the two numerator functional equations give the exact identities

$$R=R_0+R_1,\qquad R^\vee=h_0R_0+h_1R_1.$$

The root numbers cancel in their product. Hence

$$\sideset{}{^*}\sum_{\chi\bmod q}\chi(-1)\frac{L(a,\chi)L(b,\overline{\chi})}{L(s,\chi)L(t,\overline{\chi})}=\frac{2R^\vee-(h_0+h_1)R}{h_0-h_1}.\tag{3.3}$$

Writing $z=a+b-1$, duplication and reflection give

$$h_0-h_1=-4q^z(2\pi)^{-z-1}\Gamma(a)\Gamma(b)\sin(\pi z/2),\tag{3.4}$$

which is nonzero on the indicated discs. Substitution of the two models in (3.2) into (3.3) gives exactly

$$\begin{aligned}&\Delta(q)Z_q(a,b;s,t)\\&\quad+\{M_0(q)X_{q,0}(a)X_{q,0}(b)-M_1(q)X_{q,1}(a)X_{q,1}(b)\}Z_q(1-b,1-a;s,t),\end{aligned}\tag{3.5}$$

where $\Delta(q) = M_0(q) - M_1(q)$. For example, the coefficient of $Z_q(a,b;s,t)$ is $\{2(M_0h_0 + M_1h_1) - (h_0+h_1)\varphi^*\}/(h_0-h_1) = \Delta$. For the swapped coefficient, the identity $X_{q,e}(a)X_{q,e}(b) = h_e^{-1}$ reduces the coefficient of M_0 to h_0^{-1} and that of M_1 to $-h_1^{-1}$, giving the displayed expression.

Formula (3.3) does not give a polynomially uniform pointwise bound for the signed error at arbitrary heights. We use it only after Mellin testing, where its conditioning can be controlled.

3.2. **Rational weights.** Compact smooth tests have rapid polynomial Mellin decay, which does not control the exponential growth of the inverse in (3.3). We first work with rational tests whose Mellin transforms contain two Gamma factors. One compensates for the inverse; the other leaves exponential decay for integration.

Fix an integer $A > 3$ and consider finite linear combinations of

$$f_j(u) = \frac{u^j}{(1+u)^{A+j}}, \qquad j = 0, 1, 2, \dots. \tag{3.6}$$

Their Mellin transforms are

$$\widetilde{f}_j(s) = \frac{\Gamma(s+j)\Gamma(A-s)}{\Gamma(A+j)} = \frac{\Gamma(s)(s)_j\Gamma(A-s)}{\Gamma(A+j)}, \qquad -j < \Re s < A. \tag{3.7}$$

Here $(s)_j$ is the rising factorial. The factor $\Gamma(A-s)$ supplies the additional decay.

Let $E, E^\vee$ be the two errors in (3.2). The error in (3.3) is

$$\frac{2E^\vee - (h_0+h_1)E}{h_0-h_1}.$$

For rational weights f, g from (3.6), Stirling's formula and (3.4) give, on the two numerator discs,

$$\left|\frac{\widetilde{f}(s)\widetilde{g}(t)}{h_0-h_1}\right| + \left|\frac{(h_0+h_1)\widetilde{f}(s)\widetilde{g}(t)}{h_0-h_1}\right| \ll_{f,g,\eta,r} (1+|\Im s|+|\Im t|)^C e^{-c(|\Im s|+|\Im t|)} \tag{3.8}$$

for fixed $c > 0$, uniformly in q. To verify the estimate, first use (3.4) to cancel $\Gamma(a)\Gamma(b)$ against $\Gamma(s)\Gamma(t)$ in (3.7); the resulting Gamma ratios have polynomial growth because $a-s, b-t$ are bounded. The first quotient has a factor $q^{-\Re z} \leqslant q^{-2\eta+2r} \leqslant 1$. For the second quotient use also

$$h_0 + h_1 = 4q^z(2\pi)^{-z-1}\Gamma(a)\Gamma(b)\cos(\pi(a-b)/2).$$

Its possible growth is $e^{\pi\min(|\Im s|,|\Im t|)}$ up to fixed polynomial factors. The two rational Mellin transforms decay as $e^{-\pi(|\Im s|+|\Im t|)}$. This proves (3.8), including bounded perturbations by u, v.

Apply Cauchy's formula in u, v before the two Mellin integrations. Equation (3.8) shows that the differentiated, Mellin-tested error at conductor q is

$$O_{f,g,\varepsilon}(q^{1-\omega+\varepsilon}). \tag{3.9}$$

Initially restrict $|\Im s|, |\Im t| \leqslant 5\log(2q)$ to apply the ratios formula. Outside this window use the actual signed logarithmic-derivative moment and its explicit signed model directly. By GRH they are $O_\eta(q^{1+\varepsilon}(1+|\Im s|+|\Im t|)^C)$. The rational Mellin factors give a tail $O_{f,g,\varepsilon}(q^{1-5\pi+\varepsilon})$. Its normalized conductor sum costs $O_{f,g,\varepsilon}(N^{2\eta+\varepsilon})$.

3.3. **Counting all multiples of the target.** Rational tests are not compactly supported, so orthogonality initially counts every positive multiple of N. We retain those multiples in the model. This gives a positive reference estimate that will later justify compact approximation.

For these weights, define

$$\mathcal{B}_N(f,g) = \sum_{\substack{m,n\geqslant 1\\ N\mid m+n}} \Lambda(m)\Lambda(n)f(m/N)g(n/N), \qquad F = f * g,$$

where the convolution is additive:

$$(f*g)(x) = \int_0^x f(u)g(x-u)\,du.$$

All sums converge absolutely. We claim that, for even N,

$$\mathcal{B}_N(f,g) = N\sum_{k\geqslant 1}\mathfrak{S}(kN)F(k) + O_{f,g,\varepsilon}(N^{1-\omega+2\eta+\varepsilon}). \tag{3.10}$$

There are three parts to the calculation: differentiation, the character projection, and extraction of the multiples of N.

Step 1: The differentiated model. Mixed numerator differentiation of (3.5) gives the model

$$\Delta(q)P_q(z) + \{M_0(q)X_{q,0}(s)X_{q,0}(t) - M_1(q)X_{q,1}(s)X_{q,1}(t)\}C_q(z), \qquad z = s+t-1. \tag{3.11}$$

The two reciprocal zeta zeros in the swapped kernel explain why derivatives of the archimedean factors do not enter. The parity counts satisfy

$$\Delta(q) = \sum_{d\mid(q,2)}\mu(q/d)\varphi(d), \qquad |\Delta(q)| \leqslant 2,$$

by Möbius inversion of character orthogonality. Split the coefficient of C_q into $\varphi^*(q)/2$ times the difference of the two Gamma products and $\Delta(q)/2$ times their sum. The latter and the diagonal term contribute $O_{f,g,\varepsilon}(N^{2\eta+\varepsilon})$ after the normalized conductor sum.

Duplication and reflection give the identity

$$\frac{X_{q,0}(s)X_{q,0}(t) - X_{q,1}(s)X_{q,1}(t)}{2}\,\Xi(z) = q^{-z}\frac{\Gamma(1-s)\Gamma(1-t)}{\Gamma(1-z)}. \tag{3.12}$$

Indeed $X_{q,e}(s) = 2q^{1/2-s}(2\pi)^{s-1}\Gamma(1-s)\sin(\pi(s+e)/2)$; the sine-product difference is $\sin(\pi z/2)$, and Euler reflection completes the calculation. For $F = f*g$, Mellin–Barnes inversion gives

$$\frac{1}{2\pi i}\int_{(\sigma)}\widetilde{f}(s)\widetilde{g}(1+z-s)\frac{\Gamma(1-s)\Gamma(s-z)}{\Gamma(1-z)}\,ds = \widetilde{F}(z), \qquad \Re z = 2\eta. \tag{3.13}$$

This follows by integrating the Barnes representation of $(u+v)^{z-1}$ against $f(u)g(v)$; its conditions are $\Re z < \sigma < 1$. The rational Gamma factors justify absolute convergence. Thus the parity difference converts the swapped multiplicative kernel into the additive convolution kernel needed for Goldbach.

Step 2: Character orthogonality. Use (3.1) and group the characters by primitive conductor. The congruence detects $N \mid m+n$ without restricting the multiple. The excluded noncoprime terms have $m = p^a, n = p^b$ with $p \mid N$. Since $|f(u)|, |g(u)| \ll (1+u)^{-A}$ and

$$\sum_{a\geqslant 1}(1+p^a/N)^{-A} \ll_A 1 + \frac{\log N}{\log p},$$

their total weight is $O_{f,g}(\log^3 N)$. Replacing an induced character by its primitive source still changes its smoothed prime sum by $O_f(\log^2 N)$. The principal character contributes $N^2\widetilde{f}(1)\widetilde{g}(1)/\varphi(N)$, with error $O_{f,g,\varepsilon}(N^{\sigma+\varepsilon})$, by GRH and Mellin inversion.

Using (3.11)–(3.13) and (3.9) therefore gives

$$\mathcal{B}_N(f,g) = \frac{N^2}{\varphi(N)}\widetilde{f}(1)\widetilde{g}(1) + \frac{1}{\varphi(N)}\sum_{\substack{q\mid N\\ q>1}}\varphi^*(q)\frac{N}{2\pi i}\int_{(2\eta)}(N/q)^z D_q(z)\widetilde{F}(z)\,dz$$
$$+ O_{f,g,\varepsilon}(N^{1-\omega+2\eta+\varepsilon}).$$

The Mellin–Barnes integrals converge absolutely by the rational Gamma factors. The additive convolution F is smooth on $(0,\infty)$, is $O(x)$ at zero and $O((1+x)^{-A})$ at infinity. Repeated integration by parts gives rapid vertical decay of $\widetilde{F}$ on each closed substrip of $-1 < \Re z < A$.

Step 3: The conductor projection. For $q > 1$, move the D_q contour to $\Re z = 3/2$ and use its Ramanujan coefficient expansion in Proposition 2.1. The exact conductor selection identity (2.12)

then gives

$$\mathcal{B}_N(f,g) = \frac{N^2}{\varphi(N)}\widetilde{f}(1)\widetilde{g}(1) + N\sum_{k\geqslant 1}\mathfrak{S}(kN)F(k) \\ - \frac{N}{\varphi(N)}\sum_{h\geqslant 1}\mathfrak{S}(h)F(h/N) + O_{f,g,\varepsilon}(N^{1-\omega+2\eta+\varepsilon}).$$

Moving the background D_1 contour to $1/4$ shows that the first and third terms cancel within $O_{f,g,\varepsilon}(N^{1/4+\varepsilon})$. This proves (3.10).

3.4. **From rational weights to sharp counting.** The approximation must control the entire positive sum, including its tails. The rational test $w_A(u) = (1+u)^{-A}$ bounds that mass on the arithmetic side, and an elementary comparison of singular series gives the same control on the model side. For even N,

$$\mathfrak{S}(kN) \leqslant \mathfrak{S}(N)\tau(k), \qquad k \geqslant 1. \tag{3.14}$$

Indeed each new odd prime divisor of k contributes at most two. Splitting the convolution integral at $k/2$ gives

$$(w_A * w_A)(k) \ll_A (1+k)^{-A}.$$

Since $A > 3$, the series in (3.10) for w_A, w_A is $O_A(N\mathfrak{S}(N))$. Positivity, the lower bound $\mathfrak{S}(N) \gg 1$ for even N, and $\omega > 2\eta$ consequently give the uniform dominating estimate

$$\mathcal{B}_N(w_A, w_A) \ll_A N\mathfrak{S}(N). \tag{3.15}$$

Thus no additional sieve hypothesis is needed for approximation.

For every continuous compactly supported function f_0 on $[0,\infty)$ and every $\epsilon > 0$, there is a rational test f of the above form such that

$$|f(u) - f_0(u)| \leqslant \epsilon w_A(u), \qquad u \geqslant 0. \tag{3.16}$$

To see this, set $x = u/(1+u)$. The function $(1-x)^{-A}f_0(x/(1-x))$, extended by zero near $x = 1$, is continuous on $[0,1]$. Approximate it uniformly by a polynomial $P(x)$ and take $f(u) = (1+u)^{-A}P(u/(1+u))$.

Approximate f_0, g_0 in this way. Their rational approximants are bounded by fixed multiples of w_A. Equation (3.15) bounds the error on the arithmetic side by $O_{f_0,g_0,A}(\epsilon N\mathfrak{S}(N))$. Equations (3.14) and the convolution bound give the same bound for the model side. Applying (3.10) to the fixed rational approximants, letting $N \to \infty$ first and then $\epsilon \to 0$, proves

$$\mathcal{B}_N(f_0, g_0) = N\sum_{k\geqslant 1}\mathfrak{S}(kN)(f_0 * g_0)(k) + o_{f_0,g_0}(N\mathfrak{S}(N)). \tag{3.17}$$

If f_0, g_0 are supported in $(0,1)$, only $k = 1$ survives; this is the smooth Goldbach asymptotic under the untwisted hypothesis.

Finally fix $0 < \delta < 1/4$ and choose nonnegative continuous cutoffs f_- and f_+. The first is supported in $[\delta, 1-\delta]$, is at most one and equals one on $[2\delta, 1-2\delta]$. The second is one on $[0,1]$, at most one everywhere and supported in $[0, 1+\delta]$. The lower count $\mathcal{B}_N(f_-, f_-)$ contains only the multiple $k = 1$ and is bounded above by the sharp Goldbach count. The upper count $\mathcal{B}_N(f_+, f_+)$ dominates it. In the upper model only $k = 1, 2$ occur; their convolution values are respectively $1 + O(\delta)$ and $O(\delta)$, and $\mathfrak{S}(2N) = \mathfrak{S}(N)$. Equation (3.17) therefore bounds the normalized sharp count between $1 - O(\delta) + o_\delta(1)$ and $1 + O(\delta) + o_\delta(1)$. Letting $\delta \to 0$ proves the qualitative weighted conclusion; the power-saving error in Theorem 3.2 is proved next. Proper prime powers contribute $O(N^{1/2}\log^2 N)$. The compact-test result controls the distribution of prime representations in the relative position m/N. On $\delta N \leqslant m \leqslant (1-\delta)N$ both prime logarithms are $(1+o_\delta(1))\log N$. Representations with a prime below $N^{1/2}$ contribute only $O(N^{1/2})$ to the unweighted count. In the remaining endpoint ranges both logarithms are at least $(1/2+o(1))\log N$, and the compact-test upper bound gives a contribution $O(\delta N\mathfrak{S}(N)/(\log N)^2)$. Let $N \to \infty$ and then $\delta \to 0$ to obtain the asserted unweighted asymptotic.

Remark 3.3. The argument uses only the logarithmic height range stated in Hypothesis 3.1. A uniform ordinary ratios formula with a polynomial height window therefore supplies more than is needed. The functional equation is used to recover the Mellin-tested parity information required in this proof; no polynomially uniform pointwise parity-refined formula is asserted.

3.5. **An exact test of the target sum.** The qualitative approximation above does not by itself give a power-saving sharp error. We avoid this loss by using a nonseparable weight. Put

$$w(v) = e^{20(1-v)} \left(\frac{\sin \pi(v-1)}{\pi(v-1)} \right)^2, \qquad W(u,v) = w(u+v), \tag{3.18}$$

with the removable value $w(1) = 1$. Since $w(k) = 0$ at all other positive integers,

$$\sum_{\substack{m,n \geqslant 1 \\ N \mid m+n}} \Lambda(m)\Lambda(n) W(m/N, n/N) = \sum_{m+n=N} \Lambda(m)\Lambda(n). \tag{3.19}$$

The weight is fixed, independently of N.

Step 1: Mellin decay and the parity inverse. Write $\widetilde{w}(z) = \int_0^\infty w(v) v^{z-1}\, dv$. The function w is entire, is $O(v^2)$ at zero, and decays exponentially in the closed sector $|\arg v| \leqslant 1$. Indeed $20 \cos 1 > 2\pi \sin 1$. Rotation of the Mellin ray therefore gives, uniformly for b in a fixed compact subset of $(-2, \infty)$,

$$|\widetilde{w}(b+iy)| \ll_b e^{-|y|}. \tag{3.20}$$

For $\Re s, \Re t > 0$, the double Mellin transform of W is

$$\mathcal{W}(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \widetilde{w}(s+t). \tag{3.21}$$

This follows by setting $u = xv$, $u' = x(1-v)$ in the defining double integral. On $\Re s = \Re t = \sigma$, Stirling's formula and (3.20) imply

$$|\mathcal{W}(s,t)| \ll (1 + |\Im s| + |\Im t|)^C e^{-(|\Im s| + |\Im t|)}.$$

Here and below fixed polynomial factors may change.

For $a = s + u$, $b = t + v$ on the numerator disks, use (3.4) and the displayed formula for $h_0 + h_1$ to obtain

$$\left| \frac{\mathcal{W}(s,t)}{h_0 - h_1} \right| + \left| \frac{(h_0 + h_1)\mathcal{W}(s,t)}{h_0 - h_1} \right| \ll (1 + |\Im s| + |\Im t|)^C e^{-|\Im s + \Im t|}. \tag{3.22}$$

For the first quotient the two numerator Gamma factors cancel, while the exponential factors of $\Gamma(s+t)$ and $\sin(\pi(a+b-1)/2)$ cancel each other. For the second quotient the remaining exponential factor is bounded by

$$\exp\Big\{ \frac{\pi}{2} \big(|\Im s - \Im t| - |\Im s| - |\Im t| \big) \Big\} \leqslant 1.$$

The bounded perturbations u, v do not change these estimates, and the sine is bounded away from zero at bounded height because $r < \eta/4$.

Unlike (3.8), estimate (3.22) need not decay in the difference of the two heights. We only integrate the recovered ratios error on $|\Im s|, |\Im t| \leqslant 5 \log(2q)$. Its integral there has a fixed polynomial logarithmic cost. Cauchy differentiation on the numerator disks consequently gives total conductor error $O_\varepsilon(q^{1-\omega+\varepsilon})$. Outside that box we use the actual signed logarithmic-derivative moment and its explicit model, whose growth is polynomial by GRH, without using the parity inverse. The exponential decay of $\mathcal{W}$ bounds this tail by $O_\varepsilon(q^{-4+\varepsilon})$. After the normalized conductor sum it contributes $O_\varepsilon(N^{2\eta+\varepsilon})$.

Step 2: Orthogonality and the main term. Double Mellin inversion and character orthogonality now apply to W, just as they applied to product weights. The principal double residue is $N^2 \widetilde{w}(2)/\varphi(N)$. The noncoprime terms contribute $O(\log^3 N)$, since a surviving noncoprime pair is $m = p^a, n = p^b$ with $p \mid N$ and $m + n = N$. Replacing induced characters by their primitive sources costs $O_\varepsilon(N^{\sigma+\varepsilon})$. To justify this bound for the nonseparable weight, fix an omitted prime

power $d = p^a$, $p \mid N$, and apply the GRH smoothed character estimate to $v \mapsto w(d/N + v)$. Its smooth seminorms are $O_A((1 + d/N)^{-A})$, uniformly in d. Thus each nonprincipal sum is $O_{A,\varepsilon}(N^{\sigma+\varepsilon}(1 + d/N)^{-A})$; summing the omitted powers costs only logarithms. For the principal character, the cross term has size $O(N \log^2 N/\varphi(N))$, and the single-residue errors have size $O_\varepsilon(N^{1+\sigma+\varepsilon}/\varphi(N))$. These errors are absorbed below.

Set $z = s + t - 1$. The beta factor in (3.12) and Barnes's first lemma give

$$\frac{1}{2\pi i}\int_{(\sigma)} \mathcal{W}(s, 1 + z - s)\frac{\Gamma(1 - s)\Gamma(s - z)}{\Gamma(1 - z)}\,ds = \widetilde{w}(1 + z), \qquad \Re z = 2\eta. \tag{3.23}$$

Indeed the integral of $\Gamma(s)\Gamma(s-z)\Gamma(1-s)\Gamma(1+z-s)$ is $\Gamma(1-z)\Gamma(1+z)$, and these two factors cancel the denominators. All four Gamma arguments have positive real part on the chosen line. The terms involving $\Delta(q)$ again contribute $O_\varepsilon(N^{2\eta+\varepsilon})$. Consequently the left side of (3.19) equals

$$\frac{N^2}{\varphi(N)}\widetilde{w}(2) + \frac{N}{\varphi(N)}\sum_{\substack{q\mid N\\ q>1}} \varphi^*(q)\frac{1}{2\pi i}\int_{(2\eta)} (N/q)^z D_q(z)\widetilde{w}(1 + z)\,dz \\ + O_\varepsilon(N^{1-\omega+2\eta+\varepsilon}).$$

The conductor sum of errors is bounded using $\sum_{q\mid N} q^{1-\omega+\varepsilon} \ll_\varepsilon N^{1-\omega+2\varepsilon}$ and $N/\varphi(N) \ll_\varepsilon N^\varepsilon$.

Step 3: Exact coefficient selection. Move the $q > 1$ contours to $\Re z = 3/2$ and apply Proposition 2.1. Mellin inversion gives

$$\frac{N^2}{\varphi(N)}\widetilde{w}(2) + N\sum_{k\geqslant 1} \mathfrak{S}(kN)kw(k) \\ - \frac{N}{\varphi(N)}\sum_{h\geqslant 1} \mathfrak{S}(h)(h/N)w(h/N) + O_\varepsilon(N^{1-\omega+2\eta+\varepsilon}).$$

Moving the background D_1 contour to $1/4$ cancels the first and third terms with error $O_\varepsilon(N^{1/4+\varepsilon})$. The middle sum is exactly $N\mathfrak{S}(N)$. This proves the weighted assertion of Theorem 3.2. Proper prime powers cost $O(N^{1/2}\log^2 N)$, which is absorbed because $\omega \leqslant 1/2$.

Step 4: Removing the logarithmic weights. For $0 \leqslant \lambda, \mu \leqslant \omega \leqslant 1/2$, repeat the argument with

$$W_{\lambda,\mu}(u, v) = u^{-\lambda}v^{-\mu}w(u + v), \qquad \mathcal{W}_{\lambda,\mu}(s, t) = \frac{\Gamma(s - \lambda)\Gamma(t - \mu)}{\Gamma(s + t - \lambda - \mu)}\widetilde{w}(s + t - \lambda - \mu).$$

The Mellin and parity-inverse bounds remain uniform on this compact parameter range, since $\sigma - \lambda, \sigma - \mu \geqslant \eta$. The weighted version of (3.23), again by Barnes's first lemma, is

$$\frac{1}{2\pi i}\int_{(\sigma)} \mathcal{W}_{\lambda,\mu}(s, 1 + z - s)\frac{\Gamma(1 - s)\Gamma(s - z)}{\Gamma(1 - z)}\,ds \\ = \mathrm{B}(1 - \lambda, 1 - \mu)\widetilde{w}(1 + z - \lambda - \mu),$$

where $\mathrm{B}(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$. The principal residue and the background term still cancel, while the selected coefficient is multiplied by $\mathrm{B}(1 - \lambda, 1 - \mu)$. After multiplying by $N^{-\lambda-\mu}$, the induced-character error is still $O_\varepsilon(N^{\sigma+\varepsilon})$: an omitted power d contributes $d^{-\lambda}N^{\sigma-\mu}$ times a function decreasing faster than any power of $1 + d/N$, and the symmetric term is identical. All other errors are bounded as before. Thus, uniformly in λ, μ,

$$\sum_{m+n=N} \frac{\Lambda(m)\Lambda(n)}{m^\lambda n^\mu} = \mathfrak{S}(N)N^{1-\lambda-\mu}\mathrm{B}(1 - \lambda, 1 - \mu) + O_\varepsilon(N^{1-\omega+2\eta+\varepsilon}). \tag{3.24}$$

Integrate over $[0, \omega]^2$, writing

$$K_\omega(t) = \int_0^\omega t^{-\lambda}\,d\lambda = \frac{1 - t^{-\omega}}{\log t}, \qquad K_\omega(1) = \omega.$$

The prime contribution becomes $\sum_{p_1+p_2=N}(1-p_1^{-\omega})(1-p_2^{-\omega})$. It differs from the unweighted count by $O(N^{1-\omega})$, since $\sum_{m\leqslant N} m^{-\omega} \ll N^{1-\omega}$; proper prime powers are absorbed as above. The integrated main term is $\mathfrak{S}(N)\int_0^N K_\omega(t)K_\omega(N-t)\,dt$. On $[2,N-2]$ its difference from the stated integral is $O(\mathfrak{S}(N)N^{1-\omega})$. The two endpoint integrals are $O(\mathfrak{S}(N)/\log N)$, since K_ω is integrable at zero. As $\mathfrak{S}(N)\ll_\varepsilon N^\varepsilon$, these errors prove the unweighted assertion of Theorem 3.2.

Finally, in (3.27), take $r=\eta/8$ and let the fixed η be sufficiently small in terms of a prescribed $\varepsilon>0$. Then $\kappa\to\omega_D$ and $2\eta\to 0$, so the exponent $1-\omega+2\eta$ can be made smaller than $1-\omega_D+\varepsilon/2$. The remaining logarithmic losses are absorbed into $N^{\varepsilon/2}$. This proves Theorem 1.1.

3.6. **Interpolation from critical numerators.** We now derive both numerator disks in Hypothesis 3.1 from critical-line data. The error is small on the critical numerator lines and has a polynomial bound on a wider strip. Interpolating in the two numerator variables preserves a positive saving when the denominator displacement is sufficiently small. Write

$$\mathcal{M}_q(a,b;s,t)=\varphi^*(q)Z_q(a,b;s,t)+H_q(a,b)Z_q(1-b,1-a;s,t)$$

for the ordinary two-term model in (3.2), and put $\ell_q=\log(2q)$.

The hypothesis concerns ordinary averages, with both parities included. Its restriction to prime conductors has the critical-numerator form of [WX25, Conjecture 2.1]. That conjecture uses an equal-weight parity average, whereas H_q uses the exact counts. Their archimedean coefficients differ by $O(1)$ on critical numerator lines. The difference vanishes on $a+b=1$, canceling the pole of the swapped kernel. The resulting model difference is absorbed by the error on the stated logarithmic window, after decreasing ω_D to at most $1/2$ if necessary. Uniformity over all primitive conductors is assumed explicitly here. Only a logarithmic height range is used.

Theorem 3.4. *Assume GRH and Hypothesis 1.1. For sufficiently small fixed $\eta>0$, choose $0<r<\eta/4$. Then, for some $\omega>2\eta$, (3.2) holds uniformly on both numerator discs in Hypothesis 3.1, with $|\Im s|,|\Im t|\leqslant 10\ell_q$. In particular, Theorem 3.2 applies. Letting the fixed displacement tend to zero in terms of the desired error exponent gives Theorem 1.1.*

Proof. Step 1: A holomorphic error on a fixed strip. Fix $\delta=1/8$ and decrease η_0 so that $\eta_0<1/64$. Keep the denominator arguments s,t fixed, with their heights at most $10\ell_q$, and set

$$F_q(a,b)=q^{-1}\{R_q(a,b;s,t)-\mathcal{M}_q(a,b;s,t)\}.$$

This function is holomorphic on $|\Re a-1/2|,|\Re b-1/2|\leqslant\delta$, and on a neighborhood of that closed region. Indeed, GRH excludes zeros of the denominator zeta factors in the model, whose real parts are at least $1-\delta+\eta>1/2$. The Euler products converge normally. The only apparent numerator pole is on $a+b=1$. On this hyperplane the two arithmetic kernels have opposite residues and $X_{q,e}(a)X_{q,e}(1-a)=1$, so that $H_q(a,1-a)=\varphi^*(q)$; the residues therefore cancel.

There are absolute constants K,D such that, throughout this region,

$$|F_q(a,b)|\ll_\eta q^K(1+|\Im a|+|\Im b|+|\Im s|+|\Im t|)^D. \tag{3.25}$$

Here $K=2$ is more than sufficient and is independent of η. To check this, convexity bounds the two numerator L-functions by a fixed power of their conductors. Under GRH, for each fixed $\eta>0$, $L(1/2+\eta+iv,\chi)^{-1}\ll_{\eta,\epsilon}(q(2+|v|))^\epsilon$. The same bounds for the model follow from Stirling's formula, normal convergence of the arithmetic products, and the finite Euler factors, which cost q^ϵ. Cauchy's estimate in a slightly larger strip handles the canceled pole. These are the usual conditional bounds for L-functions in fixed strips; see [IK04, Chapter 5].

Step 2: Interpolation in the first numerator. Put $\tau=\eta+r$ and $\lambda=\tau/\delta$. First let $\Re b=1/2$, $|\Im b|\leqslant 40\ell_q$, and fix a target height $|v|\leqslant 20\ell_q$. For either sign consider

$$G(w)=F_q(\tfrac12\pm w+iv,b)e^{w^2},\qquad 0\leqslant\Re w\leqslant\delta.$$

This function is bounded on the strip. On $\Re w=0$, the critical-line input applies while $|\Im(1/2\pm w+iv)|\leqslant 100\ell_q$. Outside this range $|\Im w|\geqslant 80\ell_q$, and (3.25) multiplied by $e^{-(\Im w)^2}$ is smaller

than every fixed negative power of q. Consequently the two boundary bounds, absorbing powers of ℓ_q into q^{ε}, are

$$\sup_{\Re w=0} |G(w)| \ll_{\eta,\varepsilon} q^{-\omega_D+\varepsilon}, \qquad \sup_{\Re w=\delta} |G(w)| \ll_{\eta,\varepsilon} q^{K+\varepsilon}.$$

The three-lines theorem gives, uniformly for $|\Re a - 1/2| \leqslant \tau$, $|\Im a| \leqslant 20\ell_q$,

$$|F_q(a,b)| \ll_{\eta,\varepsilon} q^{-\omega_1+\varepsilon}, \qquad \omega_1 = \omega_D(1-\lambda) - K\lambda. \tag{3.26}$$

Step 3: Interpolation in the second numerator. Now fix such an a and interpolate in b, using $F_q(a, 1/2 \pm w + iv)e^{w^2}$ with $|v| \leqslant 12\ell_q$. On the central boundary, (3.26) applies up to $|\Im b| = 40\ell_q$. Beyond that height $|\Im w| \geqslant 28\ell_q$, so the same Gaussian argument supplies the boundary bound. The other boundary is again controlled by (3.25). A second application of the three-lines theorem yields

$$|F_q(a,b)| \ll_{\eta,\varepsilon} q^{-\kappa+\varepsilon}, \qquad \kappa = \omega_D(1-\lambda)^2 - K(2\lambda - \lambda^2), \tag{3.27}$$

for both numerator real parts within τ of $1/2$ and both heights at most $12\ell_q$. Since $r < \eta/4$, choosing η sufficiently small gives $\kappa > \omega_D/2 > 2\eta$.

The original and reflected numerator discs lie in this final region when the denominator heights are at most $10\ell_q$. Multiplying (3.27) by q proves the asserted ratios estimate with any $2\eta < \omega < \min(\kappa, 1/2)$. The error is holomorphic and the bound holds on the closed discs, as required for Cauchy differentiation. The logarithmic-height Goldbach argument now applies. □

4. Fixed prime differences

A time average of a short Dirichlet polynomial can distinguish a fixed additive difference. We choose a signed Fourier filter that retains n and $n+h$ while removing the diagonal. After the ratios prediction evaluates the average, the same filter extracts the coefficient $\mathfrak{S}(h)$ from D_1.

We first keep the Mellin line fixed to track the saving. Proposition 4.3 below removes smoothing at the same exponent under $\mathrm{RC}_Z(\omega_Z)$. Proposition A.2, in Appendix A, gives the qualitative variant under Hypothesis 2.3.

Hypothesis 4.1 (Quantitative fixed-line ratios)**.** *For fixed $0 < \eta < 1/4$, $0 < \omega \leqslant 1/2$, and some $\theta > 0, B \geqslant 0$, the formula (2.15) holds with its $o(Q)$ replaced by $O_\varepsilon(Q^{1-\omega+\varepsilon}(1+|\Im a|+|\Im b|)^B)$, uniformly on $\Re a = \Re b = \eta$, $|\Im a|, |\Im b| \leqslant Q^\theta$, and $|u|, |v| \leqslant r/\log(2Q)$ for a sufficiently small fixed $r > 0$.*

For any fixed $0 < \theta < 1$, this is a restriction of the uniform continuous $2/2$ prediction in [CS07, Conjecture 2.1 and (2.11)]; its square-root error gives $\omega = 1/2$. The application uses no parity or conductor refinement.

Theorem 4.2. *Assume the zeta zero-free margin and Hypothesis 4.1. If $\omega > 2\eta$, then for each fixed $h \geqslant 1$ and $\Phi \in C_c^\infty((0,\infty))$,*

$$\sum_n \Lambda(n)\Lambda(n+h)\Phi(n/X) = \mathfrak{S}(h)X\int_0^\infty \Phi(u)\,du + O_{\Phi,h,\varepsilon}(X^{1-\omega+2\eta+\varepsilon}). \tag{4.1}$$

The implied constant has polynomial dependence on finitely many seminorms of Φ on each fixed compact support.

4.1. **A signed Fourier filter.** The filter has Fourier support near h and $-h$. A narrow spatial support then excludes all other integral differences. Its two zero moments have separate purposes: one removes the Dirichlet-polynomial diagonal, and the other removes the pole of the singular-series Dirichlet series.

Use the Fourier convention $\widehat{g}(v) = \int_{\mathbb{R}} g(t)e^{-2\pi ivt}\,dt$. Choose a nonzero real even Schwartz function k with smooth Fourier transform supported in $(-\epsilon_0, \epsilon_0)$, where $2\epsilon_0 < 1/8$. Let M be a

sufficiently large fixed integer and set

$$r_0(t) = t^M k(t), \qquad H = \widehat{r_0^2}, \qquad g_h(t) = r_0(t)^2 \cos(2\pi h t),$$
$$\phi_h(v) = \widehat{g}_h(v) = \frac{H(v-h) + H(v+h)}{2}.$$

Here g_h is even and Schwartz, vanishes to order $2M$ at zero, and

$$\begin{aligned} \phi_h(0) &= 0, \qquad & \int_0^\infty g_h(t)\,dt &= 0, \\ g_h(0) &= 0, \qquad & \int_0^\infty \phi_h(v)\,dv &= 0. \end{aligned} \tag{4.2}$$

Since $H(0) = \int_{\mathbb{R}} r_0(t)^2\,dt > 0$, there is a sufficiently small interval $J_h = (1, 1+\epsilon_h)$ on which

$$a_h(u) = \tfrac{1}{2} H(h/u - h) > 0.$$

Shrink it so that $\phi_h(j/u) = 0$ for every integer $j \geqslant 1$, $j \neq h$, and every $u \in J_h$.

For real coefficients $c(n)$ and $W \in C_c^\infty(J_h)$ real, put

$$P_{c,W,X}(t) = \sum_n \frac{c(n)}{n^{1/2+it}} W(n/X), \qquad T = 2\pi X, \qquad I_c(X) = \int_0^\infty g_h(t/T) |P_{c,W,X}(t)|^2\,dt.$$

Fourier inversion, using evenness of $|P|^2$, gives an exact sum over differences. The diagonal vanishes by $\phi_h(0) = 0$. For all sufficiently large X, only the difference h remains, and

$$I_c(X) = \frac{T}{2} \sum_n \frac{c(n)c(n+h)}{\sqrt{n(n+h)}} W(n/X) W((n+h)/X) \times H\big(X \log(1 + h/n) - h\big). \tag{4.3}$$

Indeed the positive Fourier lobe is near h, and $X\log(1+j/n)$ is below this lobe for $j < h$ and above it for $j > h$, by monotonicity and the choice of J_h. If $c(n) \ll_\varepsilon n^\varepsilon$, Taylor expansion in (4.3) yields

$$\frac{I_c(X)}{T} = \sum_n \frac{c(n)c(n+h)}{n} W(n/X)^2 a_h(n/X) + O_{W,h,\varepsilon}(X^{-1+\varepsilon}). \tag{4.4}$$

For $c = \Lambda$ the error is $O(X^{-1}\log^2 X)$.

4.2. **The swapped moment.** We now evaluate the filtered average for $c = \Lambda$. Mellin inversion expresses its Dirichlet polynomial in terms of $-\zeta'/\zeta$. The unswapped moment vanishes against the time weight; the swapped moment becomes the coefficient integral of Lemma 2.2.

Cauchy differentiation of (2.15) in the quantitative form of Hypothesis 4.1, using the kernels in Section 2, gives

$$\int_0^U \frac{\zeta'}{\zeta}(1/2 + a + it) \frac{\zeta'}{\zeta}(1/2 + b - it)\,dt = U P_1(z) + (2\pi)^z \frac{U^{1-z}}{1-z} C_1(z) + O_\varepsilon(U^{1-\omega+\varepsilon} L^{B'}), \tag{4.5}$$

where $z = a + b$, $L = 2 + |\Im a| + |\Im b|$. The error satisfies the required holomorphic uniformity because the differentiated discs are contained in the assumed discs after decreasing r.

For $c = \Lambda$, shifting a Mellin contour gives

$$\begin{aligned} P_{\Lambda,W,X}(t) &= Q_X(t) + R_X(t), \\ Q_X(t) &= -\frac{1}{2\pi i} \int_{(\eta)} \frac{\zeta'}{\zeta}(1/2 + a + it) X^a \widetilde{W}(a)\,da, \\ R_X(t) &= X^{1/2-it} \widetilde{W}(1/2 - it). \end{aligned} \tag{4.6}$$

The residue R_X comes from the pole of ζ at one. No zero is crossed under the zero-free margin. The bounds of Section 2 give

$$Q_X(t) \ll_W X^\eta \log(2 + |t|), \qquad R_X(t) \ll_{W,A} X^{1/2}(1 + |t|)^{-A}.$$

Since $g_h(t/T) \ll (t/T)^{2M}$, the terms containing R_X contribute $O_W(X^{-1})$ after taking M and A large enough.

Write $\widetilde{g}_h$ for the Mellin transform of the time weight. Apply (2.22) to (4.5) inside the two Mellin integrals for $|Q_X|^2$. The first cancellation in (4.2) gives $\widetilde{g}_h(1) = 0$. Combining the Mellin variables, and using $T = 2\pi X$, leaves

$$I_\Lambda(X) = \frac{T}{2\pi i} \int_{(2\eta)} \widetilde{W^2}(z) C_1(z) \widetilde{g}_h(1-z)\, dz + O_{W,h,\varepsilon}(X^{1-\omega+2\eta+\varepsilon}). \tag{4.7}$$

Here the Mellin convolution of $\widetilde{W}(a)$ and $\widetilde{W}(z-a)$ is $\widetilde{W^2}(z)$. The factor $X^{2\eta}$ in the error is the cost of the two Mellin lines.

Define $\Phi_h(z) = \int_0^\infty \phi_h(v) v^{z-1}\, dv$. The cosine Mellin transform gives

$$\Phi_h(z) = \Xi(z)\widetilde{g}_h(1-z), \qquad 0 < \Re z < 1, \tag{4.8}$$

initially with an exponential convergence factor. Continuation extends the identity to $0 < \Re z < 2M+1$. In addition, $\Phi_h(1) = 0$ by (4.2). Thus $C_1(z)\widetilde{g}_h(1-z) = D_1(z)\Phi_h(z)$ and the pole of D_1 at one is canceled.

The Dirichlet-series expansion (2.8) identifies the coefficients of D_1 with the Hardy–Littlewood singular series, including the local factor at two. Lemma 2.2, with $D = D_1$, now gives the exact model contribution

$$T \sum_{j \geqslant 1} \mathfrak{S}(j) \int_0^\infty W(u)^2 \phi_h(j/u)\, \frac{du}{u} = T\mathfrak{S}(h) \int_0^\infty W(u)^2 a_h(u)\, \frac{du}{u}.$$

The support construction selects $j = h$. Equations (4.4) and (4.7) now imply

$$\sum_n \frac{\Lambda(n)\Lambda(n+h)}{n} W(n/X)^2 a_h(n/X) = \mathfrak{S}(h) \int_0^\infty W(u)^2 a_h(u)\, \frac{du}{u} + O_{W,h,\varepsilon}(X^{-\omega+2\eta+\varepsilon}). \tag{4.9}$$

To obtain an arbitrary real test supported in J_h, take $W(u) = f(u)\sqrt{u/a_h(u)}$. This gives (4.1) for tests f^2. Every smooth compactly supported real test is a difference of two such squares: if $\chi = 1$ on its support, use $((\chi+\Phi)^2 - (\chi-\Phi)^2)/4$. A finite partition of unity and rescaling of X treat any fixed compact support in $(0,\infty)$. Complex tests follow by linearity. These operations use only finitely many derivatives, proving the seminorm assertion and Theorem 4.2.

Under Hypothesis 4.1 with a general polynomial height factor, Lemma 2.6 gives a sharp error with an explicitly positive saving once the cutoff seminorm cost is specified. The following refinement exploits the height-independent error of the standard uniform prediction and also applies to Section 6.

Proposition 4.3 (Sharp cutoffs without loss of the ratios saving)**.** *Assume the relevant Riemann hypothesis and either* $\mathrm{RC}_Z(\omega)$ *or* $\mathrm{RC}_H(\omega)$*. For sufficiently small fixed* $\eta > 0$*, set* $d = \omega - 2\eta > 0$*. The corresponding sharp weighted error is*

$$O_\varepsilon(X^{1-d+\varepsilon}).$$

Consequently it is $O_\varepsilon(X^{1-\omega+\varepsilon})$ *after choosing* η *in terms of* ε*. In the continuous case the implied constants may also depend on the fixed difference* h*.*

Proof. We give the dependence on the cutoff, rather than using an unspecified smooth seminorm. On a fixed compact interval in $(0,\infty)$, choose upper and lower smooth cutoffs with transition width $v = X^{-d}$. The radial weights W_v in either detector, after multiplication by fixed smooth factors, may be chosen to satisfy

$$\|W_v\|_\infty + \|W_v'\|_1 \ll 1, \qquad \|W_v^{(j)}\|_1 \ll_j v^{1-j} \quad (j \geqslant 1).$$

For prime pairs one can use squares of smooth cutoffs and a fixed partition into squares, so that taking a square root causes no loss in these bounds. Integration by parts in logarithmic coordinates gives, on each of the fixed Mellin lines used in the detectors,

$$|\widetilde{W}_v(b+it)| \ll_A \frac{1}{1+|t|} \min\{1, (v|t|)^{-A}\}, \qquad \int_{\mathbb{R}} |\widetilde{W}_v(b+it)|\, dt \ll \log(2/v). \tag{4.10}$$

The minimum is interpreted as one at $t = 0$.

Fix $d < b_0 < 3/4$ and then $b_0/(3/4) < a_0 < 1$. Truncate the Mellin heights at X^{b_0}. Above family height $Q = X^{a_0}$ the introduction's uniform ratios estimate applies to all the retained shifts. Cauchy differentiation has a fixed or logarithmic cost, and integration by parts against the fixed time or family weight gives a weighted error $O_\varepsilon(X^{1-\omega+\varepsilon})$, uniformly in those shifts. Below X^{a_0} use the direct logarithmic-derivative and model bounds. Choosing the fixed order of vanishing of the time or family weight sufficiently large makes this omitted range $O(X^{-A})$ for any prescribed A. The same choice suppresses the pole residue of ζ in (4.6).

The Mellin tails are also negligible: by (4.10) each additional integration by parts supplies a factor $X^{-(b_0-d)}$, which beats the fixed polynomial growth of the actual moments and their models. The two Mellin integrals in the continuous case cost $O(\log^2 X)$, and the one integral in the angular case costs $O(\log X)$. Thus (4.7) and (6.24) retain the error $O_\varepsilon(X^{1-\omega+2\eta+\varepsilon})$ for these varying cutoffs.

The elementary Taylor errors in the detectors are at most $O(v^{-1}\log^2 X)$ after conversion to the coefficient sums. This is absorbed, since $d \leqslant 1/2$ and $v^{-1} = X^d \leqslant X^{1-d}$. The endpoint intervals contain $O(vX+1)$ integers, each with weight $O(\log^2 X)$ in the prime-pair problem and $O(\log X)$ in the quadratic problem. Their contribution and the change in the model integral are $O_\varepsilon(X^{1-d+\varepsilon})$. A fixed partition of unity and a dyadic sum finish the argument. Choose $2\eta < \varepsilon/2$ and use the remaining $\varepsilon/2$ for logarithmic factors. □

For prime pairs, proper prime powers contribute $O_h(X^{1/2}\log^3 X)$, which is absorbed. Partial summation with the exact weight $1/(\log t \log(t+h))$ therefore gives

$$\#\{p \leqslant X : p+h \text{ prime}\} = \mathfrak{S}(h)\int_2^X \frac{dt}{\log t \log(t+h)} + O_{h,\varepsilon}(X^{1-\omega_Z+\varepsilon}) \tag{4.11}$$

for fixed even h, proving Theorem 1.2. The two cancellations in the detector retain their separate roles: $\phi_h(0) = 0$ removes the diagonal, whereas $\Phi_h(1) = 0$ removes the singular-series pole.

5. Primes between consecutive squares

For prime differences, a Fourier test selects one coefficient of D_1. For intervals, we use the same differentiated ratios model with a positive Fourier transform. Positivity localizes a zero sum in a short spatial interval, and the explicit formula converts that estimate into a second moment of $\psi(x+h) - \psi(x) - h$. Monotonicity then gives a pointwise prime number theorem when $h \asymp \sqrt{x}$.

The relation between zero correlations and primes in short intervals follows Goldston–Montgomery and Heath-Brown [GM87, HB82]; see also [Baz00, LPZ16]. Here the test varies with the height, so its uniformity must be tracked. We first obtain a quantitative prime-spacing bound from the published continuous ratios prediction. Proposition 5.4 then shows that the logarithmic-height, qualitative input is sufficient for Legendre's asymptotic.

Write $Z = \zeta'/\zeta$ and retain P_1, C_1 from (2.9)–(2.10).

We specify the restricted ratios input used in this section. For $s = 1/2 + it$, write

$$\mathcal{R}_H(\alpha,\beta;\gamma,\delta) = \int_H^{2H} \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)}\,dt.$$

Let

$$Q(\alpha,\beta;\gamma,\delta) = Z_1(1/2+\alpha, 1/2+\beta; 1/2+\gamma, 1/2+\delta).$$

Thus the standard formula is

$$\mathcal{R}_H(\alpha,\beta;\gamma,\delta) = \int_H^{2H} \Big\{Q(\alpha,\beta;\gamma,\delta) + (t/2\pi)^{-\alpha-\beta}Q(-\beta,-\alpha;\gamma,\delta)\Big\}\,dt + \Delta_H(\alpha,\beta;\gamma,\delta). \tag{5.1}$$

This is the dyadic difference of [CS07, Conjecture 2.1]. Its published uniform domain contains

$$\gamma = \sigma_H, \quad \delta = \sigma_H + ir, \quad \alpha = \sigma_H + u, \quad \beta = \sigma_H + ir + v, \qquad \sigma_H = (\log H)^{-1}, \quad |u|, |v| \leqslant \rho, \tag{5.2}$$

where $\rho = 1/100$ and $|r| \leqslant H^{1/2}$. Indeed, the numerator real parts lie in $(-1/4, 1/4)$, the denominator real parts are $\gg 1/\log H$ and less than $1/4$, and the imaginary shifts are $\ll H^{1-\varepsilon}$.

Theorem 5.1. *Assume RH and the standard uniform continuous 2/2 ratios conjecture* [CS07, Conjecture 2.1 and (2.11)]. *Then*

$$p_{j+1} - p_j \ll_\varepsilon p_j^{1/3+\varepsilon},$$

and the error in (1.5) is $O_\varepsilon(n^{5/6+\varepsilon})$. More generally, suppose that, for some $0 < q \leqslant 1/2$ and every $\varepsilon > 0$,

$$\sup_{\substack{|r|\leqslant H^q \\ |u|,|v|\leqslant\rho}} |\Delta_H(\sigma_H + u, \sigma_H + ir + v; \sigma_H, \sigma_H + ir)| \ll_{q,\varepsilon} H^{1-q+\varepsilon}. \tag{5.3}$$

Then

$$\psi((n+1)^2) - \psi(n^2) = 2n + 1 + O_{q,\varepsilon}(n^{1-q/3+\varepsilon}), \tag{5.4}$$

$$\pi((n+1)^2) - \pi(n^2) = \int_{n^2}^{(n+1)^2} \frac{dt}{\log t} + O_{q,\varepsilon}(n^{1-q/3+\varepsilon}),$$

and $p_{j+1} - p_j \ll_{q,\varepsilon} p_j^{(1-q)/(2-q)+\varepsilon}$. In particular,

$$\psi((n+1)^2) - \psi(n^2) \sim 2n, \qquad \pi((n+1)^2) - \pi(n^2) \sim \frac{n}{\log n}. \tag{5.5}$$

Lemma 5.2 (Differentiated ratios)**.** *Under RH, the input (5.1) on (5.2) gives*

$$\begin{aligned} &\int_H^{2H} Z(\tfrac{1}{2} + \sigma_H + it) Z(\tfrac{1}{2} + \sigma_H - it + ir)\, dt \\ &\quad = \int_H^{2H} \{P_1(a_H + ir) + (t/2\pi)^{-a_H - ir} C_1(a_H + ir)\}\, dt + E_H(r), \end{aligned} \tag{5.6}$$

where $a_H = 2/\log H$, and

$$|E_H(r)| \leqslant \rho^{-2} \sup_{|u|=|v|=\rho} |\Delta_H(\sigma_H + u, \sigma_H + ir + v; \sigma_H, \sigma_H + ir)|. \tag{5.7}$$

Proof. The integrand and the combined main term are holomorphic in the two closed numerator disks. The original denominator arguments have real part $1/2 + \sigma_H > 1/2$. The reciprocal zeta factors in the main term have arguments of real part at least $1 + 2\sigma_H - 2\rho > 1/2$. Thus RH excludes their zeros. The possible poles on $\alpha + \beta = 0$ cancel between the two main terms: their analytic coefficients agree there and their residues have opposite signs. The Euler products converge locally uniformly. Differentiate in u and v and set $u = v = 0$. The local calculation is precisely [CS07, Theorem 2.5], and gives (5.6). Cauchy's formula on the fixed disks gives (5.7). □

The zero sum will be tested against an entire function with nonnegative Fourier transform. The triangular factor localizes its Fourier mass, while Gaussian convolution permits the contour shifts without losing positivity. For $M \geqslant 2$, put

$$T_M(r) = (1 - |r|/M)_+, \quad g(r) = \pi^{-1/2} e^{-r^2}, \quad \varphi_M(r) = \frac{(T_M * g)(r)}{(T_M * g)(0)}.$$

With $\widehat{f}(v) = \int_{\mathbb{R}} f(r) e^{-irv}\, dr$, one has

$$\widehat{\varphi}_M(v) = c_M M \operatorname{sinc}^2(Mv/2) e^{-v^2/4} \geqslant 0, \qquad c_M = (T_M * g)(0)^{-1} \asymp 1. \tag{5.8}$$

In particular, $\widehat{\varphi}_M(v) \gg M$ for $|v| \leqslant 1/M$. The convolution is entire, and

$$|\varphi_M(r + ib)| \leqslant e^{b^2} \varphi_M(r), \qquad \int_{\mathbb{R}} (1 + |r|)^j \varphi_M(r)\, dr \ll_j M^{j+1}. \tag{5.9}$$

Outside $[-2M, 2M]$ these functions, with bounded imaginary translates, have Gaussian decay. All these assertions follow directly from the Gaussian convolution.

Lemma 5.3 (A uniform zero-correlation estimate)**.** *Assume RH. Fix $K_0 > 1$, $0 < c < 1/2$, and $\varepsilon > 0$. Uniformly for $H/4 \leqslant V \leqslant H^{K_0}$ and $\log H \leqslant M \leqslant H^{1/2}/4$, one has*

$$\begin{aligned} P_H(V,M) &:= \sum_{H<\gamma,\gamma'\leqslant 2H} V^{i(\gamma-\gamma')}\varphi_M(\gamma-\gamma') \\ &= \frac{1}{2\pi}\int_H^{2H} \log(t/2\pi)\,dt \\ &\quad + O_{K_0,c,\varepsilon}\left(M \sup_{|r|\leqslant 2M}|E_H(r)| + HV^{-c}M^{1+\varepsilon} + H(H/V)^c M^{\varepsilon} + M^2(\log H)^B\right), \end{aligned} \tag{5.10}$$

where B is an absolute constant and zeros are counted with multiplicity. In particular, the coefficient of the ratios error is independent of M. The gain over a pointwise estimate of the swapped arithmetic term comes from integrating in the common height before taking absolute values.

Proof. Step 1: The double-contour identity and its endpoint errors. Put $a = 2/\log H$, $f(r) = \varphi_M(r)\cos(r\log V)$, and $\mathcal{K}(z,t) = P_1(z) + (t/2\pi)^{-z}C_1(z)$. The double argument principle gives

$$\begin{aligned} P_H(V,M) &= \frac{1}{4\pi^2}\int_H^{2H}\log^2(t/2\pi)\,dt\int_{\mathbb{R}} f(r)\,dr \\ &\quad + \frac{1}{2\pi^2}\int_{\mathbb{R}} f(r-ia)\int_H^{2H} Z(\tfrac12+\sigma_H+it)Z(\tfrac12+\sigma_H-it+ir)\,dt\,dr \\ &\quad + O_{K_0}\big(M^2(\log H)^B\big). \end{aligned} \tag{5.11}$$

We give the uniformity check, since an $O(M^3)$ remainder would not suffice for the endpoint $M \asymp H^{1/2}$.

Choose horizontal sides at good heights within $O(1)$ of H and $2H$. The logarithmic derivatives on these sides and on the vertical lines a fixed distance from the critical line are $O((\log H)^B)$. The replacement of the good heights by the original endpoints costs $O(M(\log H)^B)$: each short endpoint interval contains $O(\log H)$ zeros and the remaining ordinate is effectively confined to a window of length $O(M)$.

After using the functional equation on the left sides, the two mixed terms are written in common-height and difference variables $r = t - u$. Extending their common-height intervals to $[H, 2H]$ costs, by (5.9),

$$(\log H)^B\int_{\mathbb{R}}(1+|r|)\varphi_M(r)\,dr \ll M^2(\log H)^B.$$

For the two same-side terms, move the two real coordinates together. If $F(z,w) = Z(z)Z(w)f(-i(z-w))$ and $\mathcal{I}(x) = \int_H^{2H}\int_H^{2H} F(x+it, x+iu)\,dt\,du$, then holomorphy gives

$$\begin{aligned} \mathcal{I}'(x) &= \frac{1}{i}\int_H^{2H}\{F(x+2iH, x+iu) - F(x+iH, x+iu)\}\,du \\ &\quad + \frac{1}{i}\int_H^{2H}\{F(x+it, x+2iH) - F(x+it, x+iH)\}\,dt. \end{aligned}$$

The test does not change under a common real translation. Thus only height-endpoint integrals occur, and they are $O(M(\log H)^B)$. On a translated line to the right of 1, both logarithmic derivatives have absolutely convergent Dirichlet series; integration in the common height gives $O(1/\log(mn))$ for their product. The resulting sums cost only powers of $\log H$, times $O(M)$ for the difference variable. The terms with one gamma logarithmic derivative obey the same endpoint estimate after integration by parts. For the gamma–gamma term, Stirling's formula and

$$\log((t+r)/2\pi) - \log(t/2\pi) \ll |r|/H \qquad (|r| \leqslant 2M)$$

show that replacing the two heights by the common height costs $O(M^2(\log H)^B)$; the higher Taylor remainder is $O(M^3H^{-1}(\log H)^B)$ and is absorbed since $M \leqslant H^{1/2}/4$. This proves (5.11). No factor $V^{1/2}$ appears, because the two real coordinates are translated together.

Step 2: Inserting differentiated ratios. The first term of (5.11) is negligible, because $\int f = \widehat{\varphi}_M(\log V) \ll Me^{-(\log V)^2/4}$. Use (5.6) on $|r| \leqslant 2M$; the omitted tails are negligible by Gaussian decay. Moreover, $|\cos((r-ia)\log V)| \ll_{K_0} 1$ and $\int \varphi_M(r)\,dr \ll M$. Consequently the ratios error contributes at most $O_{K_0}(M \sup_{|r|\leqslant 2M}|E_H(r)|)$. The arithmetic expression is

$$\frac{1}{2\pi}\int_H^{2H}\{J_+(t)+J_-(t)\}\,dt, \qquad J_\pm(t)=\frac{1}{2\pi i}\int_{(a)}\varphi_M(iz)V^{\pm z}\mathcal{K}(z,t)\,dz. \tag{5.12}$$

This is the right-boundary contour identity; the zero diagonal is already included in it.

Step 3: Integrating the height before estimating the swapped term. At zero,

$$P_1(z)=z^{-2}+O(1), \qquad C_1(z)=-z^{-2}+O(1), \qquad \mathcal{K}(z,t)=\frac{\log(t/2\pi)}{z}+O_t(1).$$

Under RH, zero is the only pole of $\mathcal{K}$ in $|\Re z| \leqslant c$. On the boundary lines, the identities in (2.9)–(2.10), absolute convergence of the remaining Euler products, and RH give

$$P_1(\pm c+ir),\ C_1(\pm c+ir) \ll_{c,\varepsilon} (1+|r|)^{\varepsilon}.$$

Introduce the exact common-height integral

$$W_H(z) := \int_H^{2H}(t/2\pi)^{-z}\,dt = (2\pi)^z\frac{(2H)^{1-z}-H^{1-z}}{1-z}. \tag{5.13}$$

For $\Re z = \pm c$, this satisfies

$$|W_H(\pm c+ir)| \ll_c \frac{H^{1\mp c}}{1+|r|}. \tag{5.14}$$

We also have, uniformly for bounded $|b|$,

$$\int_{\mathbb{R}}\varphi_M(r+ib)(1+|r|)^{\varepsilon}\,dr \ll_\varepsilon M^{1+\varepsilon}, \qquad \int_{\mathbb{R}}\varphi_M(r)(1+|r|)^{-1+\varepsilon}\,dr \ll_\varepsilon M^{\varepsilon}, \tag{5.15}$$

where the first integral may be replaced by its absolute value. The second estimate is the relevant improvement over the unweighted $O(M^{1+\varepsilon})$ bound.

Integrate $J_\pm(t)$ over $H \leqslant t \leqslant 2H$ first. Moving the J_+ line to $\Re z = -c$ crosses the simple pole of the combined kernel, whose residue is $\log(t/2\pi)$. The unswapped term is bounded using the first estimate in (5.15); the swapped term uses (5.14) and the second. Hence

$$\int_H^{2H}J_+(t)\,dt = \int_H^{2H}\log(t/2\pi)\,dt + O_{c,\varepsilon}\left(V^{-c}HM^{1+\varepsilon}+V^{-c}H^{1+c}M^{\varepsilon}\right).$$

Moving the J_- line to $\Re z = c$ crosses no pole and gives

$$\int_H^{2H}J_-(t)\,dt \ll_{c,\varepsilon} V^{-c}HM^{1+\varepsilon}+V^{-c}H^{1-c}M^{\varepsilon}.$$

Insert these two estimates in (5.12) and then (5.11). Since $H \geqslant 2$, the H^{1-c} contribution is absorbed by the H^{1+c} contribution, proving (5.10). □

Write

$$B_H(u)=\sum_{H<\gamma\leqslant 2H}u^{i\gamma}, \qquad S_t(u)=\sum_{0<\gamma\leqslant t}u^{i\gamma}.$$

The positive Fourier transform in (5.8) gives both $P_H(V,M) \geqslant 0$ and the useful localization

$$\int_V^{V+Y}|B_H(u)|^2\,du \ll \frac{V}{M}P_H(V,M) \qquad (0<Y\leqslant V/(2M)). \tag{5.16}$$

Indeed, $P_H = (2\pi)^{-1}\int \widehat{\varphi}_M(v)|B_H(Ve^v)|^2\,dv$, and $0 \leqslant \log(u/V) \leqslant 1/M$ on the required interval.

For $1 \ll h \leqslant Y \leqslant X/10$, put

$$J(X,Y,h) = \int_X^{X+Y} |\psi(x+h) - \psi(x) - h|^2 \, dx, \qquad U = X/h.$$

We use the following elementary transfer from zero norms. If, on every interval $I = [V, V+Y]$ with $X \leqslant V \leqslant X+h$, and uniformly for $U \leqslant t \leqslant Z \asymp X$,

$$\int_I |S_t(u)|^2 \, du \leqslant \sum_{j=1}^{k} A_j t^{b_j} + D_0, \qquad A_j, D_0 \geqslant 0, \quad 0 < b_j < 2, \tag{5.17}$$

then

$$J(X,Y,h) \ll_{\{b_j\},k} \frac{h^2}{X} \left(\sum_j A_j U^{b_j} + D_0 \right) + Y \log^4 X. \tag{5.18}$$

To prove this, truncate the explicit formula at a good height $Z \in [X, 2X]$:

$$\psi(x) = x - \sum_{|\gamma| \leqslant Z} \frac{x^{1/2+i\gamma}}{1/2+i\gamma} + O(\log^2 X) \qquad (x \asymp X).$$

The usual endpoint convention changes the last error by at most $O(\log X)$; see [IK04]. For $0 < \gamma \leqslant U$, express each increment as $\int_x^{x+h} u^{-1/2+i\gamma} \, du$ and use Minkowski's inequality. For the remaining zeros use Abel summation:

$$\left\| \sum_{U < \gamma \leqslant Z} \frac{u^{i\gamma}}{1/2+i\gamma} \right\|_2 \ll \frac{\|S_Z\|_2}{Z} + \frac{\|S_U\|_2}{U} + \int_U^Z \frac{\|S_t\|_2}{t^2} \, dt.$$

Each summand in (5.17) contributes $O_{b_j}(A_j^{1/2} U^{b_j/2-1})$, because $b_j < 2$. Multiplying by $u^{1/2}$ and taking the two endpoints gives (5.18), since $X/U^2 = h^2/X$. Negative ordinates are handled by complex conjugation.

Proof of Theorem 5.1. Step 1: A localized second moment. Assume (5.3), now with $0 < q \leqslant 1/2$. For each dyadic block $(H, 2H]$ put

$$M_H = H^q/100, \qquad H_0 = X^{1/100}, \qquad Y = 10X^{1-q}.$$

Only blocks with $H \geqslant H_0$ require ratios. For such blocks, $M_H \geqslant \log H$ and $M_H \leqslant H^{1/2}/4$ once X is sufficiently large. Also $Y \leqslant V/(2M_H)$ for $V \asymp X$ in the intervals used below, provided $H \leqslant 2X$; the fixed constant 10 ensures that the square interval $h \asymp 2\sqrt{X}$ is covered when $q = 1/2$. The condition $V \leqslant H^{K_0}$ holds with $K_0 = 200$. The zeros below H_0 contribute at most $O(H_0 \log X)$ pointwise.

Apply Lemmas 5.2 and 5.3, followed by (5.16). The resulting estimate on any $I = [V, V+Y]$, with $V \asymp X$, is

$$\int_I |B_H(u)|^2 \, du \ll_{q,c,\varepsilon} X^{\varepsilon} \left(XH^{1-q} + X^{1-c}H + X^{1-c}H^{1+c-q} + XH^q \right).$$

Here the term XH^{1-q} absorbs both the zero-density term and the tested ratios error; the final term comes from the M_H^2 endpoint remainder. Choose $c < 1/2$ with $c > q - \varepsilon/10$. Such a choice is available also at $q = 1/2$, by taking c sufficiently close to $1/2$. Since $H \leqslant 2X$ and $q \leqslant 1/2$, all four terms are $O_{q,\varepsilon}(X^{1+\varepsilon}H^{1-q})$, after adjusting ε.

Decompose $(0, t]$ into complete dyadic blocks and apply Minkowski's inequality. The square roots of the positive powers of H sum geometrically. Uniformly for $X/h \leqslant t \leqslant Z \asymp X$, we obtain

$$\int_I |S_t(u)|^2 \, du \ll_{q,\varepsilon} X^{1+\varepsilon} t^{1-q} + Y H_0^2 (\log X)^2.$$

Equation (5.18) therefore yields

$$J(X,Y,h) \ll_{q,\varepsilon} X^{1-q+\varepsilon}h^{1+q} + \frac{h^2Y}{X}H_0^2(\log X)^2 + Y\log^4 X, \tag{5.19}$$

for $1 \ll h \leqslant Y$ in the range needed here.

Step 2: The bound on consecutive primes. Let $h = X^\theta$ with

$$\frac{1-q}{2-q} < \theta < \frac{1}{2}.$$

The first term in (5.19) is $o(h^3)$ because $1-q+\theta(1+q) < 3\theta$; the two remaining terms are also $o(h^3)$. Thus $J(X,Y,h) = o(h^3)$. If $(X, X+2h]$ contained no primes, then for every $x \in [X, X+h]$ the increment $\psi(x+h)-\psi(x)$ would contain only proper prime powers, of total weight $O(\log^2 X) = o(h)$. Consequently $J(X,Y,h) \geqslant (1-o(1))h^3$, a contradiction. This proves the exponent $(1-q)/(2-q)+\varepsilon$. For the standard square-root error in the continuous ratios conjecture, take $q = 1/2$ to obtain $1/3+\varepsilon$. *Step 3: A quantitative pointwise estimate between squares.* For $h \asymp \sqrt{X}$ and $H \asymp h$, (5.19) gives, uniformly in nearby starting points,

$$J(X,Y,H) \ll_{q,\varepsilon} X^{3/2-q/2+\varepsilon}.$$

The terms involving $H_0 = X^{1/100}$ and $Y\log^4 X$ are smaller. Let $0 < \delta < 1/2$, now allowed to depend on X. For $y \in [X, X+\delta h]$ one has $[y, y+(1-\delta)h] \subseteq [X, X+h]$, and for $y \in [X-\delta h, X]$ one has $[X, X+h] \subseteq [y, y+(1+\delta)h]$. Average these inclusions for the increasing function ψ. Cauchy–Schwarz on each averaging interval, and the uniform second-moment bound for $H = (1\pm\delta)h$, imply

$$|\psi(X+h)-\psi(X)-h| \ll \delta h + \left(\frac{X^{3/2-q/2+\varepsilon}}{\delta h}\right)^{1/2}.$$

Choose $\delta = X^{-q/6}$ and decrease the ε used in the second moment. It follows that

$$\psi(X+h)-\psi(X) = h + O_{q,\varepsilon}(X^{1/2-q/6+\varepsilon}) \qquad (h \asymp \sqrt{X}). \tag{5.20}$$

Taking $X = n^2$ and $h = 2n+1$ gives the first formula in (5.4). Proper prime powers in this interval have total weight $O(\log^2 X)$: each fixed exponent contributes $O(1)$ terms. Replacing the prime logarithms by $\log X$ costs $O(1)$ in the count, and

$$\int_X^{X+h} \frac{dt}{\log t} = \frac{h}{\log X} + O\left(\frac{h^2}{X\log^2 X}\right).$$

This proves the second formula. The standard square-root error gives $q = 1/2$, while $\mathrm{RC}_Z(\omega_Z)$ gives (5.3) with $q = \omega_Z$ by taking the dyadic difference of the two initial averages. Thus Theorem 1.3 follows as well.

□

Proposition 5.4 (A weaker input for the square intervals)**.** *Assume RH. Formula (5.5) already follows if, for every fixed $A > 0$, the error in (5.1) is $o_A(H)$ uniformly on (5.2) with $|r| \leqslant A\log H$. Under this weaker assumption, $p_{n+1} - p_n = o(\sqrt{p_n})$.*

Proof. Fix $L \geqslant 1$ and use $M_H = L\log H$ on blocks with $H \geqslant H_0$; put $Y = X/(100L\log X)$. If $|E_H(r)| \leqslant \eta H$ for $|r| \leqslant 2L\log H$, the preceding localization and dyadic summation give, for $h \asymp \sqrt{X}$,

$$J(X,Y,h) \leqslant CXh(L^{-1}+\eta) + o_L(Xh),$$

with an absolute C on each fixed compact range of $h/\sqrt{X}$. Indeed, the zero-density term $H\log H$ is divided by M_H, the tested error contributes $O(X\eta H)$ to a block norm, and all arithmetic and endpoint terms in Lemma 5.3 give $o(Xh)$ after (5.18). The geometric block sums preserve the independence of C from L. Equation (5.7) allows any fixed $\eta > 0$ for all sufficiently large H. First let $X \to \infty$, then $\eta \to 0$ and $L \to \infty$. Thus $J = o(h^3)$ on this scale, and the same monotonicity

sandwich proves (5.5). If a fixed positive multiple of $\sqrt{X}$ were prime-free, its second moment would instead be $\gg h^3$; hence the stated little-o bound on consecutive primes. □

6. Quadratic prime values

The angular characters replace the logarithmic phase used for prime differences by the argument of a Gaussian integer. A Fourier window at angular distance $1/X$ from an axis selects Gaussian integers $n \pm i$ with $n \asymp X$. The resulting ideal von Mangoldt sum is exactly a weighted sum of $\Lambda(n^2+1)$. The swapped angular ratios term then extracts the first coefficient of the Dirichlet series for the quadratic singular series.

As in the prime-difference argument, we first prove a quantitative fixed-line version. Proposition 4.3 then retains its saving in the sharp count under $\mathrm{RC}_H(\omega_H)$. Proposition A.3 records the weaker qualitative variant.

Recall the angular characters ξ_k, the functions L_k, and the arithmetic model G from Section 2.5. We first prove a quantitative version on a fixed Mellin line.

Fix parameters

$$0 < \eta < \tfrac{1}{4}, \qquad 2\eta < \omega \leqslant \tfrac{1}{2}, \qquad \theta > 0, \qquad \sigma = \tfrac{1}{2} + \eta. \tag{6.1}$$

Hypothesis 6.1 (Uniform angular Hecke ratios)**.** *There are fixed $B \geqslant 0$ and $0 < \rho < \min(\eta/2, 1/4 - \eta)$ such that, for all real $Y \geqslant 2$, all $c = \eta + it$ with $|t| \leqslant Y^\theta$, and all $|a - c| \leqslant \rho$,*

$$\frac{1}{Y} \sum_{1 \leqslant k \leqslant Y} \frac{L_k(1/2+a)}{L_k(1/2+c)} = G(a,c) + \frac{1}{1-2a}\left(\frac{\pi}{2Y}\right)^{2a} G(-a,c) + O_\varepsilon\big(Y^{-\omega+\varepsilon}(1+|t|)^B\big). \tag{6.2}$$

The error is holomorphic in a and uniform on the indicated disks. All constants are independent of Y, a, c.

The two main terms in (6.2) are those of [Wax21, Conjecture 3.3]. When $0 < \theta < 1$, its shift domain (3.3) contains the present disks and height window. We use the uniformity discussed in its Section 3.6. No angularly twisted ratio is assumed. We also suppose that there is a fixed $\beta < \sigma$ for which

$$L_k(s) \neq 0 \quad (\Re s > \beta,\ k \geqslant 1). \tag{6.3}$$

GRH for this family suffices. The fixed margin in (6.3), together with the standard zero-counting estimate for L_k, gives

$$\frac{L_k'}{L_k}(u+iv) \ll_{\sigma,\beta} \log^2(3+k+|v|), \qquad u \geqslant \sigma, \tag{6.4}$$

with the usual harmless modification for u in a fixed bounded strip. This follows, for example, by writing the logarithmic derivative as a sum over zeros with $|\Im\rho - v| \leqslant 1$ and a logarithmic conductor error. Each denominator has modulus at least $\sigma - \beta$, and the number of such zeros is $O(\log(3+k+|v|))$.

Theorem 6.2. *Assume Hypothesis 6.1 and (6.3). For each fixed $\Phi \in C_c^\infty((0,\infty))$,*

$$\sum_{n \geqslant 1} \Lambda(n^2+1)\Phi(n/X) = \mathfrak{S}_E X \int_0^\infty \Phi(u)\,du + O_{\Phi,\varepsilon}(X^{1-\omega+2\eta+\varepsilon}), \tag{6.5}$$

where $\mathfrak{S}_E$ is defined in (1.2). For weights supported in a fixed compact subinterval of $(0,\infty)$, the error constant is bounded by a fixed finite collection of smooth seminorms of Φ.

6.1. **An angular detector.** Poisson summation in the character index converts a Fourier window into a restriction on the angle of an ideal. A narrow radial support makes the imaginary coordinate an integer close to one, and hence exactly one. We also require zero moments to remove the unswapped average and the pole at one.

We use the Fourier transform $\widehat{g}(v) = \int_{\mathbb{R}} g(t) e^{-2\pi i t v}\,dt$.

Lemma 6.3. *For every integer $J \geqslant 1$ and every sufficiently small fixed $\delta > 0$, there is a real even Schwartz function g such that $H = \widehat{g}$ is smooth,*

$$\operatorname{supp} H \subset (1-\delta, 1+\delta) \cup (-1-\delta, -1+\delta), \qquad H(1) \neq 0, \tag{6.6}$$

and $g^{(j)}(0) = 0$ for $0 \leqslant j < J$. Moreover,

$$\int_0^\infty g(u)\,du = 0, \qquad \int_0^\infty H(u)\,du = 0. \tag{6.7}$$

Proof. Choose $H_+ \in C_c^\infty((1-\delta, 1+\delta))$ with $H_+(1) \neq 0$. By adding finitely many bumps supported away from 1, impose $\int_0^\infty u^{2j} H_+(u)\,du = 0$ for $0 \leqslant j \leqslant J$. The required moment matrix is invertible if the added bumps are sufficiently concentrated at distinct positive points: its limiting matrix is Vandermonde in their squares. Extend H_+ evenly and take the inverse Fourier transform. The moments give the asserted zero at the origin. Also $\widehat{g}(0) = 0$ and $g(0) = 0$, which imply (6.7) by evenness. □

Choose $\delta' > 0$ sufficiently small in terms of δ, and let $W \in C_c^\infty((1-\delta', 1+\delta'))$. Write Λ_F for the von Mangoldt function on ideals, and put

$$K = \frac{\pi X}{2}, \qquad P_X(k) = \sum_{\mathfrak{a} \neq 0} \frac{\Lambda_F(\mathfrak{a})\xi_k(\mathfrak{a})}{\sqrt{N\mathfrak{a}}} W\left(\frac{\sqrt{N\mathfrak{a}}}{X}\right), \qquad I_X = \sum_{k \geqslant 1} g(k/K) P_X(k). \tag{6.8}$$

Lemma 6.4. *For all sufficiently large X,*

$$I_X = K \sum_{n \geqslant 1} \frac{\Lambda(n^2+1)}{\sqrt{n^2+1}} W\left(\frac{\sqrt{n^2+1}}{X}\right) H\left(X \arctan\frac{1}{n}\right). \tag{6.9}$$

Proof. Conjugation gives $P_X(-k) = P_X(k)$. Since g is even and $g(0) = 0$, the sum defining I_X is half the corresponding sum over $k \in \mathbb{Z}$. If ϑ is a generator angle of $\mathfrak{a}$, Poisson summation gives

$$\sum_{k \in \mathbb{Z}} g(k/K) e^{4ik\vartheta} = K \sum_{j \in \mathbb{Z}} H\left(K\left(j - \frac{2\vartheta}{\pi}\right)\right). \tag{6.10}$$

A surviving ideal therefore has angular distance $(1+O(\delta))/X$ from an axis. Choose its generator $a+bi$ with angle in $[-\pi/4, \pi/4)$. The radial support gives $\sqrt{a^2+b^2}/X = 1 + O(\delta')$, whence $|b| = 1 + O(\delta + \delta' + X^{-1})$. Small fixed supports force $|b| = 1$ and $a = n > 0$. The surviving ideals are $(n+i)$ and $(n-i)$. They are distinct on the support for large X and have equal weights; their multiplicity two cancels the preceding factor $1/2$.

It remains to identify their weights. For even n, the ideals $(n+i)$ and $(n-i)$ are coprime. Since their norm is odd, $(n+i)$ is a prime ideal power if and only if n^2+1 is a rational prime power; the prime ideal has norm equal to that rational prime. Hence its ideal von Mangoldt weight is $\Lambda(n^2+1)$. For odd $n > 1$, the norm is divisible by 2 exactly once and is larger than 2, so both weights vanish. The exceptional value $n = 1$ lies outside the support. This proves (6.9). □

Because $H(1) \neq 0$, there is a fixed neighborhood of 1 on which $H(1/u)$ does not vanish. For Φ supported in this neighborhood, set

$$W(u) = \frac{u\Phi(u)}{H(1/u)}. \tag{6.11}$$

For $n \asymp X$,

$$\frac{\sqrt{n^2+1}}{X} = \frac{n}{X} + O(X^{-2}), \qquad X \arctan(1/n) = \frac{X}{n} + O(X^{-2}).$$

Taylor expansion in (6.9), using $\Lambda(n^2+1) \ll \log X$, consequently gives

$$I_X = \frac{K}{X} \sum_{n \geqslant 1} \Lambda(n^2+1)\Phi(n/X) + O_\Phi(\log X). \tag{6.12}$$

This identity uses no assertion about the distribution of quadratic prime values.

6.2. **The differentiated angular average.** Differentiating a numerator ratio supplies the ideal von Mangoldt coefficients. As in the continuous family, the swapped term has a reciprocal-zeta zero on the diagonal. Its derivative retains the arithmetic coefficient but removes derivatives of the functional-equation prefactor.

Define

$$\mathcal{B}(r) = -\,\partial_a G(a,c)|_{a=c=r}\,, \qquad \mathcal{C}(r) = \zeta(1-2r)\frac{L(1+2r,\chi)}{L(1,\chi)}A_1(-r,r)A_3(-r,r). \tag{6.13}$$

Cauchy's formula applied to (6.2) gives, uniformly for $r=\eta+it$, $|t|\leqslant Y^\theta$,

$$\sum_{1\leqslant k\leqslant Y} -\frac{L_k'}{L_k}(1/2+r) = Y\mathcal{B}(r) + \Big(\frac{\pi}{2}\Big)^{2r}\frac{Y^{1-2r}}{1-2r}\mathcal{C}(r) + O_\varepsilon\big(Y^{1-\omega+\varepsilon}(1+|t|)^B\big). \tag{6.14}$$

Indeed $G(r,r)=1$, while $G(-r,r)=0$. In the latter expression the zero comes from $1/\zeta(1-a+c)$, whose derivative with respect to a at $a=c$ is -1. Differentiating the prefactor of the swapped term contributes nothing because it multiplies this zero. This proves the sign and the coefficient in (6.14).

Lemma 6.5. *Fix $0<\tau<\theta$. If the order J in Lemma 6.3 is sufficiently large, then, uniformly for $r=\eta+it$ with $|t|\leqslant K^\tau$,*

$$\sum_{k\geqslant 1} g(k/K)\left(-\frac{L_k'}{L_k}(1/2+r)\right) = K\left(\frac{\pi}{2K}\right)^{2r}\mathcal{C}(r)\widetilde{g}(1-2r) + O_{g,\varepsilon}\big(K^{1-\omega+\varepsilon}(1+|t|)^{B'}\big), \tag{6.15}$$

where B' is fixed.

Proof. Let $A(y,r)$ denote the partial sum on the left of (6.14). Abel summation gives

$$\sum_{k\geqslant 1} g(k/K)\left(-\frac{L_k'}{L_k}(1/2+r)\right) = -\frac{1}{K}\int_0^\infty A(y,r)g'(y/K)\,dy.$$

The contribution from $y\mathcal{B}(r)$ is $K\mathcal{B}(r)\int_0^\infty g(u)\,du = 0$. Integration by parts transforms the swapped term into

$$\Big(\frac{\pi}{2}\Big)^{2r}\mathcal{C}(r)\int_0^\infty g(y/K)y^{-2r}\,dy = K\left(\frac{\pi}{2K}\right)^{2r}\mathcal{C}(r)\widetilde{g}(1-2r).$$

The ratios estimate is available for $y\geqslant y_0 = \max(2,(1+|t|)^{1/\theta})$. On this range its error, integrated against $K^{-1}|g'(y/K)|$, is bounded by the error in (6.15).

For $y<y_0$, (6.4) bounds $A(y,r)$ by y times a fixed power of $\log(3+y+|t|)$. The two proposed main terms have polynomial growth in t and at most linear growth in $y\geqslant 1$. Since $g'(u)\ll_g u^{J-1}$ near zero, their combined contribution over this omitted range is

$$\ll K^{-J}y_0^{J+1}(1+|t|)^C.$$

Here $y_0\ll K^{\tau/\theta}$, so this is smaller than any prescribed negative power of K after increasing J. The interval $0<y<1$ is handled in the same way; $y^{1-2\eta}$ is integrable against the displayed weight. Schwartz decay handles the upper endpoint. This proves the lemma. □

6.3. **The quadratic singular series.** The arithmetic factor in the swapped average has a Dirichlet-series expansion whose coefficients are the Bateman–Horn singular series for n^2+b^2. The detector will select $b=1$, so we need the first coefficient rather than the residue at one. Put

$$\mathcal{D}(z) = \zeta(z)\frac{L(1+z,\chi)}{L(1,\chi)}A_1(-z/2,z/2)A_3(-z/2,z/2). \tag{6.16}$$

For $\Re z>0$, local simplification yields

$$\mathcal{D}(z) = \mathfrak{S}_E\zeta(z)\prod_{p\equiv 1\ (4)}\left(1+\frac{p^{-z}}{p-2}\right)\prod_{p\equiv 3\ (4)}(1-p^{-1-z}). \tag{6.17}$$

Indeed the respective local factors outside $\zeta(z)$ are

$$\frac{1-2/p+p^{-1-z}}{1-1/p}=\frac{p-2}{p-1}\left(1+\frac{p^{-z}}{p-2}\right),\qquad \frac{1-p^{-1-z}}{1-1/p}=\frac{p}{p-1}(1-p^{-1-z}).$$

The constant factors, ordered by increasing primes, give (1.2); equivalently one may factor out $L(1,\chi)^{-1}=4/\pi$ to obtain the absolutely convergent expression there. There is no extra local factor at 2.

Both products in (6.17) converge absolutely for $\Re z>0$. Thus $\mathcal{D}$ is meromorphic there with its only pole at $z=1$. For $\Re z>1$ it has the absolutely convergent expansion

$$\mathcal{D}(z)=\sum_{b\geqslant 1}\frac{S_b}{b^z},\qquad S_b=\mathfrak{S}_E\prod_{\substack{p\mid b\\ p\equiv 1\ (4)}}\frac{p-1}{p-2}\prod_{\substack{p\mid b\\ p\equiv 3\ (4)}}\frac{p-1}{p}. \tag{6.18}$$

In particular $S_1=\mathfrak{S}_E$. These coefficients are exactly the Bateman–Horn singular series for n^2+b^2. At an odd prime not dividing b, the root count is $1+\chi_{-4}(p)$; at an odd prime dividing b it is one. At two the root count is always one. Comparing the corresponding local densities with those for $b=1$ gives the two products in (6.18). Retaining the factor $\zeta(z)$ in (6.16) is essential: the first Dirichlet coefficient, rather than a residue, is what the angular detector will extract.

6.4. **Extracting the first coefficient.** The Fourier–Mellin identity converts the differentiated angular model into the coefficient integral of Lemma 2.2. Its pole at one is canceled by the zero moment of H, leaving the coefficient selected by the angular support.

Evenness and the Fourier convention give the Fourier–Mellin identity

$$\widetilde{H}(z)=2(2\pi)^{-z}\Gamma(z)\cos(\pi z/2)\widetilde{g}(1-z),\qquad 0<\Re z<1. \tag{6.19}$$

One obtains it by inserting $H(u)=2\int_0^\infty g(t)\cos(2\pi tu)\,dt$ into the Mellin integral, initially with a damping factor, and applying the Mellin transform of the cosine kernel. Analytic continuation extends the identity to the strip needed below because g has a sufficiently high-order zero at zero. Moreover,

$$\widetilde{H}(1)=\int_0^\infty H(u)\,du=0. \tag{6.20}$$

First suppose that Φ is supported sufficiently close to 1, and choose W by (6.11). Mellin inversion on an absolute-convergence line and then a shift to $\Re z=2\eta$ give

$$P_X(k)=\frac{1}{2\pi i}\int_{(2\eta)}-\frac{L_k'}{L_k}\left(\frac{1+z}{2}\right)X^z\widetilde{W}(z)\,dz. \tag{6.21}$$

The zero-free margin and (6.4) justify the shift. They also justify summation against the Schwartz weight $g(k/K)$ and truncation at $|\Im z|\leqslant K^\tau$, for any fixed $0<\tau<\theta$: the tail is $O_{W,g,A}(X^{-A})$ for each fixed A, after using sufficiently many integrations by parts in $\widetilde{W}$.

Apply Lemma 6.5 with $r=z/2$. The integrated error is

$$\ll_{W,g,\varepsilon}X^{2\eta}K^{1-\omega+\varepsilon}\int_{\mathbb{R}}|\widetilde{W}(2\eta+it)|(1+|t|)^{B'}\,dt\ll_{W,g,\varepsilon}X^{1-\omega+2\eta+\varepsilon}. \tag{6.22}$$

The main term is

$$\frac{K}{2\pi i}\int_{(2\eta)}\widetilde{W}(z)\mathcal{C}(z/2)\widetilde{g}(1-z)\left(\frac{\pi X}{2K}\right)^z\,dz.$$

Our choice $K=\pi X/2$ removes the last factor. The zeta functional equation and (6.19) give

$$\mathcal{C}(z/2)\widetilde{g}(1-z)=\mathcal{D}(z)\widetilde{H}(z). \tag{6.23}$$

Consequently,

$$I_X=\frac{K}{2\pi i}\int_{(2\eta)}\widetilde{W}(z)\widetilde{H}(z)\mathcal{D}(z)\,dz+O_{\Phi,\varepsilon}(X^{1-\omega+2\eta+\varepsilon}). \tag{6.24}$$

Move the contour to $\Re z=2$. The only possible pole crossed is at $z=1$, and (6.20) cancels it, as in Lemma 2.2. The two compactly supported Mellin transforms decrease rapidly on vertical

lines, while (6.17) gives polynomial growth for $\mathcal{D}$ in the intervening strip. Expanding (6.18) now gives

$$I_X = K \sum_{b \geqslant 1} S_b \int_0^\infty W(u) H(b/u) \, \frac{du}{u} + O_{\Phi,\varepsilon}(X^{1-\omega+2\eta+\varepsilon}). \tag{6.25}$$

The supports of W and the positive part of H are sufficiently close to 1 that the integral vanishes for every integer $b \neq 1$. Using $S_1 = \mathfrak{S}_E$ and (6.11), we obtain

$$I_X = K\mathfrak{S}_E \int_0^\infty \Phi(u) \, du + O_{\Phi,\varepsilon}(X^{1-\omega+2\eta+\varepsilon}).$$

Comparison with (6.12), and $K \asymp X$, proves (6.5) for such Φ.

For general compactly supported Φ, use a fixed finite smooth partition of unity into intervals of sufficiently small relative width. On a piece centered at $u_0 > 0$, replace X by $u_0 X$ and the weight by $v \mapsto \Phi(u_0 v)$ times its cutoff. The preceding proof applies to each piece. Summing proves the theorem. All estimates use only finitely many derivatives of these weights; on a fixed compact support this also proves the seminorm assertion.

Under $\mathrm{RC}_H(\omega_H)$, Proposition 4.3 applied to (6.5) gives

$$\sum_{n \leqslant X} \Lambda(n^2+1) = \mathfrak{S}_E X + O_\varepsilon(X^{1-\omega_H+\varepsilon}).$$

There is no loss from proper prime powers here. Lebesgue's theorem [Leb50] states that $x^2+1 = y^a$ has no solution in positive integers x, y with $a \geqslant 2$. Thus every nonzero $\Lambda(n^2+1)$ comes from a prime value. Partial summation with $1/\log(t^2+1)$ gives

$$\#\{n \leqslant X : n^2+1 \text{ prime}\} = \mathfrak{S}_E \int_1^X \frac{dt}{\log(t^2+1)} + O_\varepsilon(X^{1-\omega_H+\varepsilon}),$$

proving Theorem 1.4. Under the general fixed-line hypothesis with a polynomial height loss, Lemma 2.6 still gives a positive sharp saving determined by the cutoff cost. Under Hypothesis 2.4 alone, only the qualitative asymptotic is asserted.

Appendix A. Ratios with qualitative errors

For prime differences and quadratic values, the power-saving ratios errors in the main theorems can be weakened if only qualitative asymptotics are required. The condition $\omega > 2\eta$ compensates for the fixed Mellin-line factor $X^{2\eta}$. Taking η of order $1/\log X$ keeps that factor bounded. Uniform $o(Q)$ primitive errors then give $o(X)$ after smoothing and Mellin inversion. The character argument continues to use a positive power saving.

Lemma A.1 (Moving Mellin lines)**.** *Assume RH for the relevant L-functions and the weak logarithmic ratios Hypothesis 2.3 or 2.4. Put $c = 4$ and*

$$\eta_X = \frac{c}{\log X}. \tag{A.1}$$

Let $W \in C_c^\infty((0,\infty))$ be a fixed radial weight allowed in the corresponding detector. In the continuous case use the time weight g_h of Section 4 and $T = 2\pi X$; in the angular case use the weight g of Lemma 6.3 and $K = \pi X/2$. Choose their fixed vanishing orders at zero sufficiently large. Then (4.7) and (6.24), respectively, hold on $\Re z = 2\eta_X$ with error $o(X)$ in place of their displayed error terms.

Proof. Step 1: Bounds near the critical line. The local zero expansion, together with RH, gives the bounds

$$\left| \frac{L'}{L}(\tfrac{1}{2} + \eta_X + it) \right| \ll_c (\log X) \log^2(3 + Q_L + |t|), \tag{A.2}$$

where $Q_L = 1$ for ζ and $Q_L = k$ for the angular character of index k. The pole of ζ contributes a bounded term on this line. Indeed, the zeros within unit distance contribute at most their number, $O(\log(3+Q_L+|t|))$, times η_X^{-1}; the remaining local expansion is bounded by a logarithmic conductor term. We have allowed an extra logarithm in (A.2). Thus logarithmic derivatives

have only a fixed power of $\log X$ as growth when the other parameters are bounded by fixed powers of X.

The differentiated arithmetic models enjoy the same type of bound, with polynomial growth in the Mellin heights. For the continuous kernel this follows by factoring

$$\prod_p F_p(z) = \zeta(1+z)\mathcal{E}(z), \qquad \mathcal{E}(z) = \prod_p (1-p^{-1-z})F_p(z).$$

The latter product is uniformly absolutely convergent on $0 \leqslant \Re z \leqslant 1/4$, since its local factors are $1+O(p^{-2})$ there. Consequently $P_1(z)$ and $C_1(z)$ on $\Re z = 2\eta_X$ are bounded by a fixed power of $\log X$ times a fixed polynomial in $|\Im z|$. For the angular model, the products $A_1(-r,r)$ and $A_3(-r,r)$ are uniformly absolutely convergent down to $\Re r = 0$; the zeta and Dirichlet factors in (6.13) give the same bound for $\mathcal{B}(r)$ and $\mathcal{C}(r)$ on $\Re r = \eta_X$.

Step 2: Truncation of the Mellin variables. For any fixed $W \in C_c^\infty((0,\infty))$ and every integer $A \geqslant 0$, integration by parts gives

$$|\widetilde{W}(u+iv)| \ll_{W,A} (1+|v|)^{-A}, \qquad 0 \leqslant u \leqslant 1/4. \tag{A.3}$$

The constants are uniform in u. We can therefore truncate every Mellin integral at $|v| \leqslant D\log X$, with any fixed $D > 0$. Equations (A.2) and (A.3) show that the omitted parts, after integration against the fixed time or family weight, are $o(X)$: choosing A sufficiently large beats every fixed logarithmic loss. The same argument permits the eventual extension of the explicit model integrals back to complete vertical lines.

Step 3: Transfer of the qualitative error. We transfer the ratios error to the smooth time or family weight. Take $\kappa = 1/2$ and $R = 2$. On

$$X^\kappa \leqslant Q \leqslant X^R, \qquad |v_j| \leqslant D\log X, \tag{A.4}$$

the denominator shift η_X belongs to $[2/\log Q, 8/\log Q]$, and $|v_j| \leqslant 2D\log Q$. Choose the fixed $D > 0$ small enough for the height window in Hypothesis 2.4. Hypothesis 2.3 permits every fixed logarithmic window. Their uniformity, followed by Cauchy's formula on the fixed numerator disks, therefore gives a differentiated error

$$E_X(Q;\boldsymbol{v}) = o(Q) \tag{A.5}$$

uniformly on (A.4). No logarithmic differentiation loss occurs. The two main terms are interpreted together at removable singularities before differentiation.

Let $T \asymp X$ and let g be a fixed Schwartz weight with a zero of order J at the origin. In the continuous case, subtract the explicit primitive in (4.5) from the actual primitive. In the angular case use the corresponding difference in (6.14). Integration by parts, or Abel summation, expresses the weighted error as

$$-\frac{1}{T}\int_0^\infty g'(Q/T)E_X(Q;\boldsymbol{v})\,dQ. \tag{A.6}$$

On the middle range (A.4), its absolute value is at most

$$o(1)\,T\int_0^\infty v|g'(v)|\,dv = o(X),$$

uniformly in the retained Mellin variables. Their Mellin transforms have uniformly bounded L^1 norms by (A.3), so this estimate is preserved by the remaining integrations.

Step 4: The remaining family ranges. Outside the middle range, the ratios hypothesis is replaced by the direct bounds from Step 1. There the actual primitive and its explicit model are bounded by Q times fixed logarithmic factors for $Q \geqslant 1$, by (A.2) and the model bounds above. For $0 < Q < 1$ the model can instead have the harmless power $Q^{1-2\eta_X}$. Since $g'(v) \ll v^{J-1}$ at zero, the contribution below X^κ to (A.6) is

$$\ll X^{-J+\kappa(J+1)}(\log X)^B + X^{-J}(\log X)^B.$$

Increasing the fixed vanishing order J makes it negligible. Above X^R, Schwartz decay of g' gives a negligible contribution by the same argument. Boundary terms vanish. These estimates also justify the integration by parts when the angular primitive is a step function.

Finally, the radial Mellin factors cost at most $X^{2\eta_X} = e^{2c}$. Thus the weighted error $o(X)$ obtained above remains $o(X)$ after Mellin inversion. This proves the lemma. □

Proposition A.2 (The two zeta applications)**.** *Assume RH and Hypothesis 2.3. For every fixed $h \geqslant 1$ and $\Phi \in C_c^\infty((0,\infty))$,*

$$\sum_n \Lambda(n)\Lambda(n+h)\Phi(n/X) = \mathfrak{S}(h)X\int_0^\infty \Phi(u)\,du + o_{\Phi,h}(X). \tag{A.7}$$

Moreover,

$$\pi((n+1)^2) - \pi(n^2) \sim \frac{n}{\log n}, \qquad p_{n+1} - p_n = o(\sqrt{p_n}). \tag{A.8}$$

Proof. For prime pairs, use the exact Fourier detector (4.3) and move the two Mellin lines to $\Re a = \Re b = \eta_X$. RH justifies these shifts. The pole residue in (4.6) is negligible against the high-order zero of the time weight, exactly as in the fixed-line argument. Lemma A.1 now evaluates the moment with error $o(X)$. The diagonal term vanishes because $\widetilde{g}_h(1) = 0$. The swapped term is still the same integral of $D_1(z)\Phi_h(z)$; its only pole in $\Re z > 0$ is canceled by $\Phi_h(1) = 0$. Moving the z-line to the right and inserting (2.8) select the coefficient $\mathfrak{S}(h)$, independently of the location of the original line. The polarization and partition of unity following (4.9) give (A.7) for every fixed Φ.

For the interval assertion, take denominator shifts $\gamma = 1/\log H$ and $\delta = 1/\log H + ir$. Hypothesis 2.3 gives the $o_A(H)$ error in Proposition 5.4 for $|r| \leqslant A\log H$ and fixed numerator disks. Passing from the initial integral to $[H, 2H]$ uses its two endpoint formulas; the same denominator shifts lie in the permitted fixed ranges at both endpoints. Proposition 5.4 therefore proves (A.8). □

Proposition A.3 (The angular application)**.** *Assume GRH for the angular Hecke family and Hypothesis 2.4. For every fixed $\Phi \in C_c^\infty((0,\infty))$,*

$$\sum_{n\geqslant 1} \Lambda(n^2+1)\Phi(n/X) = \mathfrak{S}_E X\int_0^\infty \Phi(u)\,du + o_\Phi(X). \tag{A.9}$$

Proof. In the angular detector set $K = \pi X/2$ and move the Mellin line to $\Re z = 2\eta_X$. Lemma A.1 replaces the weighted logarithmic-derivative sum by its explicit swapped term, with total error $o(X)$. The unswapped term vanishes since $\int_0^\infty g = 0$. The identity (6.23), the canceled pole at $z = 1$, and the coefficient extraction in (6.25) are exact arithmetic calculations and are unchanged by the moving line. They select $S_1 = \mathfrak{S}_E$. The exact detector and a finite partition of unity prove (A.9). □

Both (A.7) and (A.9) yield their sharp weighted asymptotics without a rate of convergence in the ratios hypotheses. A fixed smooth majorant of $[1/2, 1]$ first bounds each dyadic block by $O(X)$. Summing blocks bounds the part below ϵX by $O(\epsilon X) + o(X)$. Positivity and fixed smooth upper and lower approximations on $[\epsilon, 1]$ then allow $X \to \infty$ first and $\epsilon \to 0$ afterward. Removing proper prime powers and partial summation give the unweighted prime counts. For Goldbach, the positive rational reference weight in Section 3 supplies the analogous endpoint control directly from the ordinary character ratios formula.

The preceding propositions prove the qualitative counterparts of Theorems 1.2–1.4. The quantitative assertions require the stronger hypotheses stated in the introduction and are proved in Sections 4–6; the Goldbach saving is obtained separately in Section 3.5.

Appendix B. Reciprocal ratios and Chowla

The signed filter also applies to Möbius and Liouville coefficients, provided a reciprocal-ratios prediction is available. This is an additional hypothesis, separate from the three inputs

to Theorems 1.1–1.4. Its model has only a diagonal, which the filter removes. For Liouville, truncating the square-divisor expansion reduces the required uniformity to small square twists.

Put

$$D_\mu(z) = \frac{\zeta(1+z)}{\zeta(2+2z)} = \sum_{n\geqslant 1} \frac{\mu(n)^2}{n^{1+z}}, \qquad \Re z > 0.$$

Retain $\sigma = 1/2 + \eta$ and the zeta zero-free margin from Section 2.

Hypothesis B.1 (Square-twisted reciprocal ratios)**.** *There are fixed $0 < \omega_C \leqslant 1/2$, $\theta_C, \nu > 0$, and $A, B \geqslant 0$ such that, for every $\varepsilon > 0$, uniformly for $U \geqslant 2$, $\Re a = \Re b = \eta$, $|\Im a|, |\Im b| \leqslant U^{\theta_C}$, and integers $1 \leqslant k, l \leqslant U^\nu$,*

$$\int_0^U \frac{(l/k)^{2it}}{\zeta(1/2+a+it)\zeta(1/2+b-it)}\,dt = \mathbf{1}_{k=l}\, U D_\mu(a+b) + O_\varepsilon\big(U^{1-\omega_C+\varepsilon}(kl)^A(1+|\Im a|+|\Im b|)^B\big). \tag{B.1}$$

The diagonal is the denominator-only prediction of the ratios recipe [CFZ08, CS07]. Indeed its formal coefficient constraint is $k^2 m = l^2 n$. If $k = gk_0$, $l = gl_0$, and $(k_0, l_0) = 1$, then $m = l_0^2 j$ and $n = k_0^2 j$. Unless $k = l$, one Möbius coefficient vanishes; when $k = l$ the diagonal is $D_\mu(a+b)$. The stated square-twist range and error cost are additional uniformity assumptions, not consequences here of a published untwisted formula.

Theorem B.2. *Assume the zeta zero-free margin and Hypothesis B.1, with*

$$\Delta_C = \omega_C - 2\eta > 0, \qquad \rho_C = \min\left\{\frac{\nu}{4}, \frac{\Delta_C}{2(2A+1)}\right\}.$$

For each fixed $h \geqslant 1$ and $\Phi \in C_c^\infty((0,\infty))$,

$$\sum_n \mu(n)\mu(n+h)\Phi(n/X) \ll_{\Phi,h,\varepsilon} X^{1-\Delta_C+\varepsilon}, \tag{B.2}$$
$$\sum_n \lambda(n)\lambda(n+h)\Phi(n/X) \ll_{\Phi,h,\varepsilon} X^{1-\rho_C+\varepsilon}.$$

On each fixed compact support the constants depend polynomially on finitely many seminorms of Φ. For each $c \in \{\mu, \lambda\}$ there is $\kappa > 0$ such that

$$\sum_{n\leqslant X} c(n)c(n+h) \ll_h X^{1-\kappa}. \tag{B.3}$$

For the Möbius conclusions only the untwisted case $k = l = 1$ of Hypothesis B.1 is needed.

Proof. Use g_h, W, $P_{c,W,X}$, $T = 2\pi X$, and $I_c(X)$ as in Section 4; choose the vanishing order $2M$ of g_h sufficiently large in terms of the fixed hypothesis parameters. Mellin inversion and the zero-free margin give

$$P_{\mu,W,X}(t) = \frac{1}{2\pi i}\int_{(\eta)} \frac{X^a \widetilde{W}(a)}{\zeta(1/2+a+it)}\,da. \tag{B.4}$$

There is no pole residue, since $1/\zeta$ vanishes at one. The polynomial bound for $1/\zeta$ justifies the contour shift. Insert (B.1) with $k = l = 1$ into the two Mellin integrals and apply the weighted transfer (2.22). Its model vanishes because $\widetilde{g}_h(1) = \int_0^\infty g_h(v)\,dv = 0$, by (4.2). Therefore

$$I_\mu(X) \ll_{W,h,\varepsilon} X^{1-\Delta_C+\varepsilon}.$$

Equation (4.4), followed by the same polarization and finite partition of unity used for prime pairs, proves the first line of (B.2).

For Liouville use

$$\lambda(n) = \sum_{d^2\mid n} \mu(n/d^2), \qquad \lambda_D(n) = \sum_{\substack{d^2\mid n\\ d\leqslant D}} \mu(n/d^2).$$

There is exactly one d for which n/d^2 is squarefree. Hence $\lambda_D(n)$ is either $\lambda(n)$ or zero; in particular $|\lambda_D(n)| \leqslant 1$. Moreover

$$\sum_{n\leqslant Y} |\lambda(n) - \lambda_D(n)| \leqslant \sum_{d>D} \left\lfloor \frac{Y}{d^2} \right\rfloor \ll \frac{Y}{D}.$$

For each fixed smooth Φ this implies

$$\sum_n \big(\lambda(n)\lambda(n+h) - \lambda_D(n)\lambda_D(n+h)\big)\Phi(n/X) \ll_{\Phi,h} X/D. \tag{B.5}$$

Writing $Z_D(w) = \sum_{d\leqslant D} d^{-w}$, the Mellin formula is

$$P_{\lambda_D,W,X}(t) = \frac{1}{2\pi i}\int_{(\eta)} \frac{Z_D(1+2a+2it)}{\zeta(1/2+a+it)} X^a \widetilde{W}(a)\, da.$$

On this line $|Z_D(1+2a+2it)| \leqslant \zeta(1+2\eta)$ uniformly in D. Expanding the two finite numerators produces (B.1), with weights $k^{-1-2a}l^{-1-2b}$. Every diagonal model again vanishes against g_h, while the ratios error contributes at most

$$X^{1-\Delta_C+\varepsilon}\left(\sum_{k\leqslant D} k^{-1-2\eta+A}\right)^2 \ll_\eta X^{1-\Delta_C+\varepsilon} D^{2A}. \tag{B.6}$$

For completeness, the twist restriction does not require a uniform formula down to time zero. Put $L = 2 + |\Im a| + |\Im b|$ and $U_0 = C\max(2, L^{1/\theta_C}, D^{1/\nu})$. Above U_0 the hypothesis applies. Below U_0, the integrand and the model have polynomial growth, uniformly in the real twists. The integration-by-parts argument of Section 2.7 bounds this part, after the two Mellin integrals, by

$$O_W\left(X^{2\eta-2M}\{1 + D^{(2M+C_0+1)/\nu}\}\right)$$

for a fixed C_0; powers of L are absorbed by rapid Mellin decay. Take $D = \lfloor X^{\rho_C} \rfloor$. Since $\rho_C \leqslant \nu/4$, choosing M large makes the last display $O_W(X^{-2})$. Thus (B.6) bounds I_{λ_D} up to this negligible error. The Taylor error in (4.4) is uniform in D, because $|\lambda_D| \leqslant 1$. Polarization, partition of unity, and (B.5) now give the Liouville estimate, since

$$\Delta_C - 2A\rho_C \geqslant \rho_C + \Delta_C/2.$$

Only finitely many integrations by parts and a fixed vanishing order were used, proving the seminorm assertion.

Finally approximate the indicator of a fixed dyadic interval by a smooth cutoff with transition width ϵ. Since $|c| \leqslant 1$, the endpoint error is $O(\epsilon X + 1)$. If the smooth seminorm cost is $O(\epsilon^{-K})$, choose $\epsilon = X^{-a}$ with $a > 0$ sufficiently small, and choose the ε in (B.2) smaller than the saving. This yields a power saving on each dyadic interval. Summing these intervals proves (B.3). □

Partial summation of (B.3) shows that, for $c = \mu$ or $c = \lambda$, the series $\sum_{n\geqslant 1} c(n)c(n+h)/n$ converges, with a power-saving tail. No mean value for $\zeta(2s)/\zeta(s)$ at mixed heights is assumed: the numerator needed for Liouville was truncated before the spectral average was taken.

References

[Baz00] D. Bazzanella, *Primes between consecutive squares*, Arch. Math. (Basel) **75** (2000), no. 1, 29–34. doi:10.1007/s000130050469. 21

[BGW26] T. D. Browning, J. Glas and V. Y. Wang, *Sums of three cubes over a function field*, Forum Math. Sigma **14** (2026), Paper No. e112, 58 pp. doi:10.1017/fms.2026.10259; arXiv:2402.07146. 1

[CFKRS05] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein and N. C. Snaith, *Integral moments of L-functions*, Proc. London Math. Soc. (3) **91** (2005), 33–104. doi:10.1112/S0024611504015175; arXiv:math/0206018. 1

[CFZ08] J. B. Conrey, D. W. Farmer and M. R. Zirnbauer, *Autocorrelation of ratios of L-functions*, Commun. Number Theory Phys. **2** (2008), 593–636. arXiv:0711.0718. 1, 11, 34

[CS07] J. B. Conrey and N. C. Snaith, *Applications of the L-functions ratios conjectures*, Proc. Lond. Math. Soc. (3) **94** (2007), 594–646. doi:10.1112/plms/pdl021. 2, 8, 18, 21, 22, 34

[GM87] D. A. Goldston and H. L. Montgomery, *Pair correlation of zeros and primes in short intervals*, in *Analytic Number Theory and Diophantine Problems*, Progr. Math. **70**, Birkhäuser, Boston, 1987, 183–203. 1, 21

[GM25] L. Grimmelt and J. Merikoski, *On the greatest prime factor and uniform equidistribution of quadratic polynomials*, arXiv:2505.00493. 1

[Har23] A. J. Harper, *The typical size of character and zeta sums is* $o(\sqrt{x})$, arXiv:2301.04390. 5

[Har26] A. J. Harper, *Better than squareroot cancellation in number theory*, arXiv:2512.23681; to appear in the *Proceedings of the ICM 2026*. 5

[HB82] D. R. Heath-Brown, *Gaps between primes, and the pair correlation of zeros of the zeta-function*, Acta Arith. **41** (1982), 85–99. 21

[Hel15] H. A. Helfgott, *The ternary Goldbach problem*, arXiv:1501.05438, 2015, 317 pp. 1

[IK04] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, Amer. Math. Soc. Colloq. Publ. **53**, American Mathematical Society, Providence, RI, 2004. 9, 17, 25

[JT23] O. Järviniemi and J. Teräväinen, *Composite values of shifted exponentials*, Adv. Math. **429** (2023), Paper No. 109187. doi:10.1016/j.aim.2023.109187; arXiv:2010.01789. 1

[Lee26] S. Lee, *The Hardy–Littlewood asymptotic for prime pairs of fixed even difference from a uniform zeta-ratios conjecture*, preprint, revised 3 November 2026. arXiv:2610.32672v2. 1, 5

[LPZ16] A. Languasco, A. Perelli and A. Zaccagnini, *An extension of the pair-correlation conjecture and applications*, Math. Res. Lett. **23** (2016), 201–220. arXiv:1308.3934. 21

[May19] J. Maynard, *The twin prime conjecture*, Jpn. J. Math. **14** (2019), 175–206. doi:10.1007/s11537-019-1837-z. 1

[Mon73] H. L. Montgomery, *The pair correlation of zeros of the zeta function*, in *Analytic Number Theory* (Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, Mo., 1972), Amer. Math. Soc., Providence, RI, 1973, 181–193. 1

[Leb50] V.-A. Lebesgue, *Sur l'impossibilité, en nombres entiers, de l'équation* $x^m = y^2 + 1$, Nouv. Ann. Math. (1) **9** (1850), 178–181. https://www.numdam.org/item/NAM_1850_1_9__178_0/. 31

[MS02] H. L. Montgomery and K. Soundararajan, *Beyond pair correlation*, in *Paul Erdős and his mathematics, I* (Budapest, 1999), Bolyai Soc. Math. Stud. **11**, János Bolyai Math. Soc., Budapest, 2002, 507–514. 1

[Pin09] J. Pintz, *Landau's problems on primes*, J. Théor. Nombres Bordeaux **21** (2009), 357–404. doi:10.5802/jtnb.676. 1

[TB26] C. L. Turnage-Butterbaugh, *A decades-long breakthrough in zero-density estimates and primes in short intervals*, arXiv:2607.04632; to appear in Bull. Amer. Math. Soc. 1

[WX25] V. Y. Wang and M. W. Xu, *Harper's beyond square-root conjecture*, Int. Math. Res. Not. IMRN **2025** (2025), no. 18, rnaf279. arXiv:2405.04094. 1, 2, 5, 11, 17

[Wax21] E. Waxman, *Lower order terms for the one-level density of a symplectic family of Hecke L-functions*, J. Number Theory **221** (2021), 447–483. arXiv:1905.10362. 2, 8, 9, 27

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