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A LAW OF LARGE NUMBERS FOR INFINITESIMAL CHARACTERS OF REAL REDUCTIVE LIE ALGEBRAS

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Abstract. We prove a law of large numbers for the infinitesimal characters of real reductive Lie algebras. Let π be a unitary representation of a connected Lie group with reductive Lie algebra. For product states on the tensor powers of π whose sites have uniformly bounded moments, for instance identical sites, the infinitesimal character divided by the number of sites concentrates at the spectrum of the limiting mean moment, with probabilities of deviation $O(n^{-m})$ for every m. The generators may be unbounded and the group non-compact. For compact groups this recovers the laws of large numbers of Keyl and Werner and of Botero, Christandl and Vrana, which enter the asymptotic analysis of linear programming bounds for binary codes; spherical designs require the non-compact case. We give concrete examples arising from Howe dual pairs on bosonic Fock space.

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1. Introduction

Suppose we want to estimate the eigenvalues of a density matrix ρ on $\mathbb{C}^m$, and that we can prepare the state $\rho^{\otimes n}$ on $(\mathbb{C}^m)^{\otimes n}$. The unitary group $U(m)$, acting on every factor at once, commutes with the symmetric group S_n, which permutes the factors, and by Schur–Weyl duality

$$(\mathbb{C}^m)^{\otimes n} = \bigoplus_{\lambda} V_\lambda \otimes S^\lambda,$$

over the partitions λ of n with at most m rows, where V_λ and S^λ are the irreducible representations of $U(m)$ and S_n with diagram λ. The projections P_λ onto the summands form a measurement, which returns λ with probability $\operatorname{tr}(\rho^{\otimes n} P_\lambda)$. A remarkable thing occurs: the outcomes concentrate, and λ/n converges to the eigenvalues of ρ in decreasing order. This is the theorem of Keyl and Werner [20], who also showed that deviations have exponentially small probability, with a rate given by a relative entropy. The measurement does not depend on ρ;

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it uses only the symmetry of the tensor power. Botero, Christandl and Vrana [1] extended the law of large numbers to every unitary representation of a compact connected Lie group on a finite-dimensional space, with the moment map of ρ in place of its eigenvalues, and the large deviation principle to faithful states.

As a special case of the theorem that we prove, we can answer the same question for orthogonal–symplectic Howe duality, where one member is not compact. Suppose we want to estimate the temperature of a heat bath. A harmonic oscillator of frequency ω in equilibrium with the bath is in the thermal state $\tau_b \propto \mathrm{e}^{-\beta\omega c^* c}$, with mean occupation $b = (\mathrm{e}^{\beta\omega} - 1)^{-1}$, from which the temperature can be read off. Suppose that we can prepare n such oscillators, but only through an unknown squeezing: we receive $V\tau_b^{\otimes n}V^*$, where $V = \pi(g)^{\otimes n}$ applies the same metaplectic operator $\pi(g)$, for an unknown $g \in Sp_2(\mathbb{R})$, to every oscillator. Counting quanta no longer works, since squeezing changes the mean occupation. What we can still do is use the POVM on Fock space $L^2(\mathbb{R})^{\otimes n} = L^2(\mathbb{R}^n)$ given by the isotypic decomposition under $O(n) \times Sp_2(\mathbb{R})$, where $O(n)$ rotates $\mathbb{R}^n$ and $Sp_2(\mathbb{R})$ acts by $g \mapsto \pi(g)^{\otimes n}$. Its components are the spaces of functions $h(x)f(|x|)$ with h a harmonic polynomial of degree k, the projections P_k onto them commute with V, and the outcome k has law $\mathrm{tr}(\tau_b^{\otimes n}P_k)$ whatever V is. Again a remarkable thing occurs: k/n converges to b. At zero temperature $\tau_b^{\otimes n}$ is the projection onto the radial Gaussian $\mathrm{e}^{-|x|^2/2}$, and $k = 0$.

With N modes per site, $L^2(\mathbb{R}^N)^{\otimes n} = L^2(\mathbb{R}^{n\times N})$, with $O(n)$ acting on the first index. Its isotypic components are labelled by partitions λ with at most N rows, and by Howe duality [16, 17] they correspond to irreducible highest weight modules of $\mathfrak{sp}_{2N}(\mathbb{R})$, which acts by quadratic operators on all sites at once. The one-site space is infinite-dimensional and the partner is not compact, so [1] does not apply. The role of the eigenvalues is played by symplectic eigenvalues. Let $\mathbf{M}_\rho$ be the $2N \times 2N$ matrix of second moments $\mathrm{tr}\big(\rho\, \frac{1}{2}(R_\alpha R_\beta + R_\beta R_\alpha)\big)$ of the positions and momenta $R = (x_1, \dots, x_N, -\mathrm{i}\partial_1, \dots, -\mathrm{i}\partial_N)$. By Williamson's theorem [47] there is a real symplectic T with $T\mathbf{M}_\rho T^{\mathsf{T}} = \mathrm{diag}(\nu, \nu)$, and the uncertainty principle gives $\nu_1 \geq \cdots \geq \nu_N \geq \frac{1}{2}$. For a Gaussian state centred at the origin, $\nu_r - \frac{1}{2}$ are the mean occupations of its normal modes, and the measurement below recovers them without knowing the symplectic frame.

Theorem A. *Let ρ be a density operator on $L^2(\mathbb{R}^N)$ with $\mathrm{tr}(\rho\, \mathsf{N}^t) < \infty$ for all t, where N is the number operator, and let ν be the symplectic eigenvalues of $\mathbf{M}_\rho$. Then the $O(n)$-type λ of $\rho^{\otimes n}$ satisfies $\lambda/n \to \nu - \frac{1}{2}$ in probability, and the probability that $\max_r |\lambda_r/n - (\nu_r - \frac{1}{2})| > \varepsilon$ is $O(n^{-s})$ for every $\varepsilon > 0$ and $s > 0$.*

For $N = 1$ the degree satisfies $k/n \to \sqrt{\det \mathbf{M}_\rho} - \frac{1}{2}$, which is b for the thermal state τ. Theorem A is the case of identical sites of Theorem 3.2 (identical sites satisfy its hypothesis by Remark 2.2), which allows the sites to differ.

The degree of spherical harmonics indexes the Gegenbauer polynomials in the linear programming bounds of Delsarte, Goethals and Seidel for spherical codes and designs [5], and asymptotic analyses of these bounds need to know in which degrees a function on $\mathbb{R}^n$ carries its mass. Corollary 3.3 computes the limit when finitely many sites carry coherent states and the others thermal states; the case of one coherent site gives the lower bound for spherical designs in [39]. For binary codes, the corresponding input is the theorem of Keyl and Werner for qubits [32, 38, 40].

The isotypic decomposition is rarely explicit, and the proof does not use it. Howe duality matches each $O(n)$-type with a highest weight of the partner, and the centre of the enveloping algebra of the partner acts on the corresponding component by the infinitesimal character of that weight. Central elements are polynomials in the collective observables, the sums over the sites of one-site observables, and in a product state such a polynomial of degree d, divided by n^d, approaches the same polynomial evaluated at the mean values of the observables. So

infinitesimal characters satisfy a law of large numbers for every reductive Lie algebra, and the type follows whenever the infinitesimal character determines it to first order.

To state this, let π be a unitary representation of a connected Lie group with real reductive Lie algebra $\mathfrak{g}$ on a Hilbert space $\mathcal{H}$, and let ρ be a density operator on $\mathcal{H}$ with finite moments (Definition 2.1). The centre of the enveloping algebra acts on $\mathcal{H}^{\otimes n}$ by commuting operators, and their joint spectral measure, evaluated in $\rho^{\otimes n}$, is the law of a random infinitesimal character χ_n, a point of $\mathfrak{h}^*/W$ for a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_{\mathbb{C}}$ with Weyl group W. The *moment map* sends ρ to the functional $\mu_\rho(X) = \operatorname{tr}(\rho\, \mathrm{d}\pi(X))$ on $\mathfrak{g}_{\mathbb{C}}$, the restriction to $\mathfrak{g}_{\mathbb{C}}$ of the state $u \mapsto \operatorname{tr}(\rho\, \mathrm{d}\pi(u))$ of ρ on the enveloping algebra. The *spectrum* $\operatorname{spec}(\mu)$ of a functional μ on $\mathfrak{g}_{\mathbb{C}}$ is its image under the quotient map $\mathfrak{g}_{\mathbb{C}}^* \to \mathfrak{g}_{\mathbb{C}}^*/\!/G_{\mathbb{C}} \cong \mathfrak{h}^*/W$ of invariant theory, where $G_{\mathbb{C}}$ is the adjoint group; for $\mathfrak{g} = \mathfrak{u}(m)$ it is the set of eigenvalues of the matrix representing μ.

Theorem B. *In this situation $\chi_n/n \to \operatorname{spec}(\mu_\rho)$ in probability, and the probability that χ_n/n lies at distance more than ε from $\operatorname{spec}(\mu_\rho)$ is $O(n^{-s})$ for every $\varepsilon > 0$ and $s > 0$.*

Theorem 2.8 proves this for product states whose sites may differ, under a bound on their fluctuations that holds for instance when the sites have uniformly bounded moments (Remark 2.2). When the tensor power decomposes discretely and the highest weights lie in a closed cone that meets the limiting orbit in one point, Corollary 2.10 turns Theorem B into a law of large numbers for the type. For compact groups it recovers the laws of large numbers of Keyl and Werner and of Botero, Christandl and Vrana (Example 2.11); their exponential bounds are stronger than our rates, but here the sites may differ. For non-compact groups the law of χ_n may be continuous, already for tensor products of principal series representations of $SL_2(\mathbb{R})$ [34, 36], and then it is not the image of a type. Besides the orthogonal pair, Section 3 treats the dual pairs $(U(n), \mathfrak{u}(p,q))$, whose case $q = 0$ with one particle per site is the theorem of Keyl and Werner, and $(Sp(n), \mathfrak{so}^*(2N))$.

Section 2 proves Theorem B and Section 3 applies it to the three dual pairs.

Related work. Several earlier results are special cases or relatives of Theorem B. For bounded observables, polynomials in averaged observables of a mean-field system converge to functions of the mean: Raggio and Werner [35] treat identical sites, and Duffield, Roos and Werner [7] inhomogeneous ones. Applied to the image of the enveloping algebra in a bounded site algebra, this gives Proposition 2.3 without the rate. For identical sites and unbounded generators, the moment bounds behind Proposition 2.3 follow from the algebraic central limit theorem of Giri and von Waldenfels [11]. For compact groups on finite-dimensional spaces and identical sites, Corollary 2.10 is contained in [20, 1], and for the multiplicities in tensor powers in [2, 8]. We are not aware of a law of large numbers for the infinitesimal character of a non-compact group, or for the types of the dual pairs of Section 3.

Dual pairs with a non-compact member appear in bosonic de Finetti theorems [24], in the tomography of Gaussian states [3, 37], and in tests of Gaussianity [10]; these works use the decomposition of a fixed number of copies. For identical gauge-invariant Gaussian sites and the unitary pair with $q = 0$, the type law is a Schur measure [31]. It is the law of the RSK shape of a $p \times n$ matrix of independent geometric variables, which for fixed p is a random walk conditioned to stay in the Weyl chamber [29, 28]; the law of large numbers is visible there too. The limit shapes of last-passage percolation [18] concern p proportional to n.

The dual pairs on an exterior algebra, such as $(O(n), \mathfrak{so}_{2N})$ on $\Lambda(\mathbb{C}^n \otimes \mathbb{C}^N) = (\Lambda\mathbb{C}^N)^{\otimes n}$, have two compact members. The partner acts through the compact group $Spin(2N)$ on the finite-dimensional space $\Lambda\mathbb{C}^N$, and the two types determine each other. So [1], applied to $Spin(2N)$, gives the law of large numbers for the $O(n)$-type of $\rho^{\otimes n}$ for every state ρ of N fermionic modes, and the large deviation principle for faithful ρ; Example 2.11 allows the sites to differ. Limit shapes for these pairs in other regimes are studied in [26].

Use of Generative AI

This article was written with Claude while exploring statements related to the concentration of types for the asymptotics of binary codes and spherical codes, as in [32, 38, 40]. The case of binary codes is an application of the Keyl–Werner theorem (a reference originally pointed out to the author by GPT-5.6 Sol), while the corresponding statements needed in the setting of spherical designs [39] turn out to concern an orthogonal–symplectic dual pair (as observed by Claude Opus 5.5). The common generalization needed for Delsarte asymptotics is the law of large numbers presented here. The main insights here were initiated by Claude Opus 5.5, with steering by the author.

2. A law of large numbers for infinitesimal characters

This section proves Theorem B. Section 2.1 shows that in a product state the collective observables, divided by n, become commuting variables with the values of the mean moment. Section 2.2 applies this to the centre of the enveloping algebra, whose joint spectrum is the infinitesimal character, and Section 2.3 passes from infinitesimal characters to types.

2.1. **Collective observables.** Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and complexification $\mathfrak{g}_{\mathbb{C}}$. Let π be a unitary representation of G on a Hilbert space $\mathcal{H}$, and let $\mathcal{H}^\infty$ be the space of smooth vectors. We write $\omega = \mathrm{d}\pi$ for the corresponding representation of the enveloping algebra $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ on $\mathcal{H}^\infty$.

Let $\dagger$ be the conjugate-linear anti-involution of $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ with $X^\dagger = -X$ for $X \in \mathfrak{g}$. Since π is unitary, $\omega(X)$ is skew-symmetric for $X \in \mathfrak{g}$, and $\omega(u)^* \supseteq \omega(u^\dagger)$ for every $u \in \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$.

On $\mathcal{H}_n = \mathcal{H}^{\otimes n}$ let π_n be the diagonal representation, $\mathcal{H}_n^\infty$ its space of smooth vectors and $\omega_n = \mathrm{d}\pi_n$. A tensor product of smooth vectors of π is smooth for π_n, and on such a product the *collective observable* $\omega_n(X)$, $X \in \mathfrak{g}_{\mathbb{C}}$, acts as

$$\omega_n(X) = \sum_{a=1}^{n} \omega(X)^{(a)},$$

where $A^{(a)}$ denotes A acting on the a-th factor. In this article we allow $\pi = \pi_1 \oplus \cdots \oplus \pi_k$ to decompose into representations that we view as attached to individual sites: the state of each site is then supported on one summand. In this way different sites may carry, for instance, a representation and its contragredient.

Definition 2.1. A *density operator* on $\mathcal{H}$ is a positive trace-class operator ρ of trace one. Write $\rho = \sum_j \mathsf{w}_j |\psi_j\rangle\langle\psi_j|$ for its spectral decomposition, with $\mathsf{w}_j > 0$ and (ψ_j) orthonormal. Then ρ has *finite moments* if every ψ_j lies in $\mathcal{H}^\infty$ and

$$\sum_j \mathsf{w}_j \|\omega(u)\psi_j\|^2 < \infty \qquad \text{for all } u \in \mathfrak{U}(\mathfrak{g}_{\mathbb{C}}).$$

If ρ has finite moments, then $\overline{\omega(u)}\rho^{1/2} = \sum_j \mathsf{w}_j^{1/2} |\omega(u)\psi_j\rangle\langle\psi_j|$ is a Hilbert–Schmidt operator, so $\rho^{1/2}\,\overline{\omega(u)}\rho^{1/2}$ is trace class. The *state* of ρ is the linear functional on $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$

$$\varphi_\rho(u) = \mathrm{tr}\big(\rho\,\omega(u)\big) = \sum_j \mathsf{w}_j \langle\psi_j, \omega(u)\psi_j\rangle,$$

where $\operatorname{tr}(\rho\,\omega(u))$ stands for the trace of $\rho^{1/2}\overline{\omega(u)}\rho^{1/2}$. It satisfies $\varphi_\rho(1) = 1$ and $\varphi_\rho(u^\dagger u) = \sum_j \mathsf{w}_j \|\omega(u)\psi_j\|^2 \geq 0$. Restricting a state to $\mathfrak{g}_\mathbb{C} \subset \mathfrak{U}(\mathfrak{g}_\mathbb{C})$ gives the *moment map*

$$\{\text{density operators with finite moments}\} \longrightarrow \{\text{states on } \mathfrak{U}(\mathfrak{g}_\mathbb{C})\} \longrightarrow \mathfrak{g}_\mathbb{C}^*,$$
$$\rho \longmapsto \varphi_\rho \longmapsto \mu_\rho = \varphi_\rho|_{\mathfrak{g}_\mathbb{C}},$$

so $\mu_\rho(X) = \operatorname{tr}(\rho\,\omega(X))$, which is imaginary on $\mathfrak{g}$. For compact G it is the map J of [1, §2.2], a normalization of the moment map. For a state φ we write $\mu_\varphi = \varphi|_{\mathfrak{g}_\mathbb{C}}$ and call it the *moment* of φ. Up to Proposition 2.3 the arguments use ρ only through φ_ρ; the spectral measures of Section 2.2 need ρ itself. We also write $\varphi(\omega(u)) = \varphi(u)$, and $\varphi(|A|^2) = \varphi(u^\dagger u)$ for $A = \omega(u)$, the squared Hilbert–Schmidt norm of $\overline{A}\rho^{1/2}$.

If $\rho_1, \dots, \rho_n$ have finite moments, then $\rho_1 \otimes \cdots \otimes \rho_n$ has an orthonormal eigenbasis of tensor products of eigenvectors of the factors, and it has finite moments for the representation $\omega^{\otimes n}$ of $\mathfrak{U}(\mathfrak{g}_\mathbb{C})^{\otimes n}$, with state $\varphi_{\rho_1} \otimes \cdots \otimes \varphi_{\rho_n}$ there. The collective observables are $\omega_n = \omega^{\otimes n} \circ \Delta^{(n)}$, where $\Delta^{(n)} \colon \mathfrak{U}(\mathfrak{g}_\mathbb{C}) \to \mathfrak{U}(\mathfrak{g}_\mathbb{C})^{\otimes n}$ is the iterated coproduct, $\Delta^{(n)}X = \sum_a X^{(a)}$ for $X \in \mathfrak{g}_\mathbb{C}$. So the state of $\rho_1 \otimes \cdots \otimes \rho_n$ for ω_n is $(\varphi_{\rho_1} \otimes \cdots \otimes \varphi_{\rho_n}) \circ \Delta^{(n)}$. We call it the *product state* $\varphi_n = \varphi_{n,1} \otimes \cdots \otimes \varphi_{n,n}$; on products of operators at different sites it is the product of the $\varphi_{n,a}$.

The *mean moment* of a product state is $\mu_n = \frac{1}{n}\sum_a \mu_{\varphi_{n,a}} = \frac{1}{n}\mu_{\varphi_n}$. Fix a basis $X_1, \dots, X_{\dim \mathfrak{g}}$ of $\mathfrak{g}$. The normalized fluctuation of the i-th collective observable is

$$F_i = n^{-1/2}\big(\omega_n(X_i) - n\mu_n(X_i)\big) = n^{-1/2}\sum_{a=1}^{n}\big(\omega(X_i) - \mu_{\varphi_{n,a}}(X_i)\big)^{(a)},$$

which depends on n. It is $\omega_n(g_i)$ for $g_i = n^{-1/2}(X_i - n\mu_n(X_i)) \in \mathfrak{U}(\mathfrak{g}_\mathbb{C})$, and $g_i^\dagger = -g_i$ since $\mu_n(X_i)$ is imaginary. Our hypothesis is that the fluctuations have bounded mixed moments:

(F) the mean moments μ_n are bounded, and $\sup_n |\varphi_n(F_{i_1} \cdots F_{i_s})| < \infty$ for every s and all $i_1, \dots, i_s$.

Hypothesis (F) does not depend on the basis. It holds whenever the sites have uniformly bounded moments, and also when a bounded number of sites have moments that grow with n, such as the coherent sites of Corollary 3.3.

Remark 2.2 (a sufficient condition for (F)). For a state φ let $Y_i = \omega(X_i) - \mu_\varphi(X_i)$ be its centred generators, and put $\beta_s(\varphi) = \max_{i_1,\dots,i_s} |\varphi(Y_{i_1} \cdots Y_{i_s})|$, so that $\beta_1(\varphi) = 0$. Product states satisfy (F) if

(M) the mean moments μ_n are bounded, and $\sum_{a=1}^n \beta_s(\varphi_{n,a}) = O(n^{s/2})$ for every $s \geq 2$.

This covers identical sites with finite moments, since $n \leq n^{s/2}$, sites with uniformly bounded β_s and bounded mean moments, and in addition a bounded number of sites with $\beta_s(\varphi_{n,a}) = O(n^{s/2})$ and $\mu_{\varphi_{n,a}} = O(n)$. For identical sites the argument is the moment count behind the algebraic central limit theorem of Giri and von Waldenfels [11, Theorem 1]; see also [46]. Let $Y_{a,i}$ be the centred generators of $\varphi_{n,a}$, so that $F_i = n^{-1/2}\sum_a Y_{a,i}^{(a)}$. Expanding, $\varphi_n(F_{i_1} \cdots F_{i_s})$ is $n^{-s/2}$ times a sum over sequences of sites $a_1, \dots, a_s$. As φ_n is a product state, each term factorizes over the sites that occur, and it vanishes if some site occurs only once, because $\beta_1 = 0$. Grouping the other terms by the partition $\mathcal{P}$ of $\{1, \dots, s\}$ into the sets on which a_j is constant, and dropping the condition that different blocks sit at different sites, the terms of $\mathcal{P}$ contribute at most $n^{-s/2}\prod_{B\in\mathcal{P}}\sum_a \beta_{|B|}(\varphi_{n,a}) = O(1)$, since $\mathcal{P}$ has no singletons.

Let $\mathfrak{U}_d \subset \mathfrak{U}(\mathfrak{g}_\mathbb{C})$ be the subspace of elements of degree at most d. We identify the symmetric algebra $S(\mathfrak{g}_\mathbb{C})$ with the polynomial functions on $\mathfrak{g}_\mathbb{C}^*$. By the Poincaré–Birkhoff–Witt theorem, $\mathfrak{U}_d/\mathfrak{U}_{d-1} \cong S^d(\mathfrak{g}_\mathbb{C})$, and we write $\sigma_d \colon \mathfrak{U}_d \to S^d(\mathfrak{g}_\mathbb{C})$ for the resulting *symbol map*: $\sigma_d(X_1 \cdots X_d)$ is

the commutative product $X_1 \cdots X_d$ for $X_1, \dots, X_d \in \mathfrak{g}_{\mathbb{C}}$, and σ_d vanishes on $\mathfrak{U}_{d-1}$. So $\sigma_d(u)(\mu)$ is the top-degree part of u, with every X replaced by $\mu(X)$.

A sequence of operators $R_n \in \omega_n(\mathfrak{U}(\mathfrak{g}_{\mathbb{C}}))$ has *bounded moments* if $\sup_n \varphi_n(|R_n^m|^2) < \infty$ for every $m \geq 1$.

Proposition 2.3 (collective observables become classical)**.** *Assume* (F)*, and let $u \in \mathfrak{U}_d$. Then*

$$n^{-d}\omega_n(u) = \sigma_d(u)(\mu_n) + n^{-1/2}R_n,$$

where the operators R_n have bounded moments.

Proof. Write u as a linear combination of words $X_{i_1} \cdots X_{i_k}$ of length $k \leq d$. In each word substitute $n^{-1}\omega_n(X_i) = \mu_n(X_i) + n^{-1/2}F_i$ and expand. For a word of length d, the term without any F_i is $\prod_j \mu_n(X_{i_j})$, and these terms add up to $\sigma_d(u)(\mu_n)$; every other term has a factor $n^{-1/2}$. A word of length $k < d$ carries the further factor $n^{k-d} \leq n^{-1}$. Hence R_n is a polynomial of degree at most d in the F_i whose coefficients are bounded independently of n, since the μ_n are bounded. Since $g_i^\dagger = -g_i$, the quantity $\varphi_n(|R_n^m|^2)$ is a combination of mixed moments of length at most $2md$ of the F_i, with coefficients bounded independently of n, and these moments are bounded by (F). □

In terms of states on $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$, Proposition 2.3 and the Cauchy–Schwarz inequality give $n^{-d}(\varphi_{n,1} \otimes \cdots \otimes \varphi_{n,n})(\Delta^{(n)}u) = \sigma_d(u)(\mu_n) + O(n^{-1/2})$: the iterated coproduct, read through a product state, becomes the symbol evaluated at the mean moment.

For bounded single-site algebras, Proposition 2.3 without the rate is the mean-field limit of [35, Proposition II.2] and [7, §3]; that commutators of averaged observables cost $1/n$ is also the mechanism of [13] and, at the scale of the central limit theorem, of [23]. The proposition holds for every $u \in \mathfrak{U}_d$. We now apply it to central elements, whose values are polynomials in the infinitesimal character.

2.2. **The infinitesimal character.** From now on $\mathfrak{g}$ is reductive. Let $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$ be a Cartan subalgebra, with Weyl group W, and fix a system of positive roots with half-sum δ. Let $\mathfrak{Z}$ be the centre of $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$. We regard $\mathfrak{h}^*$ as the subspace of $\mathfrak{g}_{\mathbb{C}}^*$ of functionals that vanish on all root spaces, so that restricting a function from $\mathfrak{g}_{\mathbb{C}}^*$ to $\mathfrak{h}^*$ corresponds to the projection $S(\mathfrak{g}_{\mathbb{C}}) \to S(\mathfrak{h})$ that kills the root vectors. Two classical theorems relate central elements to invariant polynomials.

(i) *Chevalley* [6, Theorem 7.3.7]. Restriction to $\mathfrak{h}^*$ is an isomorphism from the invariants $S(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{g}}$ of the adjoint action onto $S(\mathfrak{h})^W$.

(ii) *Harish-Chandra* [6, Theorem 7.4.5]. There is an isomorphism $\gamma\colon \mathfrak{Z} \to S(\mathfrak{h})^W$ such that $z \in \mathfrak{Z}$ acts on every highest weight module of highest weight $\Lambda \in \mathfrak{h}^*$ by the scalar $\chi_\Lambda(z) = \gamma(z)(\Lambda + \delta)$.

The two are compatible in top degree: for $z \in \mathfrak{Z} \cap \mathfrak{U}_d$,

$$\gamma(z) = \sigma_d(z)\big|_{\mathfrak{h}^*} + (\text{terms of degree} < d). \tag{1}$$

Thus γ is a filtered map whose associated graded map is the Chevalley restriction (i). This is the standard step in the proof of Harish-Chandra's theorem [6, proof of Theorem 7.4.5]; for reductive $\mathfrak{g}$ it follows from the semisimple case, since $\mathfrak{Z}$ is generated by the centre of $\mathfrak{g}_{\mathbb{C}}$ and the centre of $\mathfrak{U}([\mathfrak{g},\mathfrak{g}]_{\mathbb{C}})$.

Let $G_{\mathbb{C}}$ be the adjoint group of $\mathfrak{g}_{\mathbb{C}}$, acting on $\mathfrak{g}_{\mathbb{C}}^*$ by the coadjoint action. Its invariants $S(\mathfrak{g}_{\mathbb{C}})^{G_{\mathbb{C}}} = S(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{g}}$ are the coordinate ring of the categorical quotient $\mathfrak{g}_{\mathbb{C}}^* /\!\!/ G_{\mathbb{C}}$ of geometric invariant theory, and the quotient map $\mathfrak{g}_{\mathbb{C}}^* \to \mathfrak{g}_{\mathbb{C}}^* /\!\!/ G_{\mathbb{C}}$ sends μ to the point at which every invariant polynomial takes its value at μ. Two functionals have the same image if and only if the closures of their orbits meet, and each fibre contains exactly one closed orbit [27, Corollary 1.2].

Similarly $\mathfrak{h}^*/W$, with coordinate ring $S(\mathfrak{h})^W$, is the quotient of $\mathfrak{h}^*$ by the finite group W, and its points are the W-orbits. By (i), the inclusion $\mathfrak{h}^* \subset \mathfrak{g}_{\mathbb{C}}^*$ induces an isomorphism $\mathfrak{h}^*/W \cong \mathfrak{g}_{\mathbb{C}}^*/\!\!/G_{\mathbb{C}}$.

Definition 2.4. The *spectrum* $\operatorname{spec}(\mu) \in \mathfrak{h}^*/W$ of $\mu \in \mathfrak{g}_{\mathbb{C}}^*$ is its image under

$$\mathfrak{g}_{\mathbb{C}}^* \longrightarrow \mathfrak{g}_{\mathbb{C}}^*/\!\!/G_{\mathbb{C}} \cong \mathfrak{h}^*/W.$$

Thus every $q \in S(\mathfrak{h})^W$ is a function on $\mathfrak{h}^*/W$, and $q(\operatorname{spec}(\mu))$ is the value at μ of the invariant polynomial on $\mathfrak{g}_{\mathbb{C}}^*$ that restricts to q.

Identify $\mathfrak{g}_{\mathbb{C}}^*$ with $\mathfrak{g}_{\mathbb{C}}$ by an invariant nondegenerate form. The closed orbits are then those of the semisimple elements [22], and $\operatorname{spec}(\mu)$ is the W-orbit of the points of $\mathfrak{h}^*$ conjugate to the semisimple part of μ; a nilpotent part does not change the spectrum. For $\mathfrak{g} = \mathfrak{u}(p)$, where μ is represented by a matrix, $\operatorname{spec}(\mu)$ is the set of orderings of its eigenvalues.

We fix a W-invariant norm on $\mathfrak{h}^*$, and measure distances on $\mathfrak{h}^*/W$ by $\operatorname{dist}(Wx, Wy) = \min_{w\in W} |x - wy|$; for $t > 0$ we write $t \cdot Wx = W(tx)$, and for $x \in \mathfrak{h}^*$ we write $\operatorname{dist}(x, \cdot) = \operatorname{dist}(Wx, \cdot)$. This metric gives the quotient topology. Since W is finite, this topology is also the one induced by the invariant polynomials: the sets $\{x : |q(x) - q(p)| < \varepsilon \text{ for } q \in F\}$, with F a finite set of homogeneous elements of $S(\mathfrak{h})^W$ and $\varepsilon > 0$, form a base of neighbourhoods of $p \in \mathfrak{h}^*/W$.

For a non-compact group the tensor power need not decompose into irreducible pieces: the tensor product of two principal series representations of $SL_2(\mathbb{R})$ is a direct integral with a continuous part [34, 36]. The infinitesimal character is nevertheless a random variable, distributed according to the joint spectral measure of the commutative algebra $\omega_n(\mathfrak{Z})$. Its existence rests on a theorem of Segal.

Lemma 2.5 (Segal)**.** *For every $z \in \mathfrak{Z}$ with $z^\dagger = z$ the operator $\omega_n(z)$ on $\mathcal{H}_n^\infty$ is essentially self-adjoint, and the spectral projections of its closure commute with $\pi_n(G)$. The closures of these operators commute strongly.*

Proof. Essential self-adjointness is a theorem of Segal [42] (see [44, p. 160]), valid for every unitary representation of a connected Lie group. Segal works on a dense invariant domain of smooth vectors; since $\omega_n(z)$ is symmetric on $\mathcal{H}_n^\infty$, which contains that domain, the closure is the same. Each $\pi_n(g)$ preserves $\mathcal{H}_n^\infty$, and there $\pi_n(g)\omega_n(z)\pi_n(g)^{-1} = \omega_n(\operatorname{Ad}(g)z) = \omega_n(z)$, since G is connected and z is central. So $\pi_n(g)$ commutes with the closure and with its spectral projections. A bounded operator commuting with $\pi_n(G)$ preserves $\mathcal{H}_n^\infty$ and commutes there with every $\omega_n(z')$, $z' \in \mathfrak{Z}$, hence with the closure of $\omega_n(z')$, because $\mathcal{H}_n^\infty$ is a core. Applied to the spectral projections of $\overline{\omega_n(z)}$, this gives the strong commutation. □

Let $\widehat{\mathfrak{Z}}$ be the set of characters of $\mathfrak{Z}$, that is, of algebra homomorphisms $\zeta\colon \mathfrak{Z} \to \mathbb{C}$, with the topology of pointwise convergence. A character is *hermitian* if $\zeta(z^\dagger) = \overline{\zeta(z)}$ for all z. By Harish-Chandra's theorem (ii), $x \mapsto \big(z \mapsto \gamma(z)(x)\big)$ is a homeomorphism from $\mathfrak{h}^*/W$ onto $\widehat{\mathfrak{Z}}$, and we use it to identify the two.

Proposition 2.6. *The representation ω_n of the commutative algebra $\mathfrak{Z}$ has a unique spectral measure: there is a unique projection-valued measure $\mathcal{E}_n$ on $\widehat{\mathfrak{Z}} = \mathfrak{h}^*/W$ such that*

$$\omega_n(z)\psi = \int_{\mathfrak{h}^*/W} \gamma(z)\, \mathrm{d}\mathcal{E}_n\, \psi \qquad (z \in \mathfrak{Z},\ \psi \in \mathcal{H}_n^\infty). \tag{2}$$

It is supported on the hermitian characters.

Proof. Existence. The anti-involution † preserves $\mathfrak{Z}$, so $\mathfrak{Z}$ is generated as an algebra by finitely many elements $h_1, \dots, h_r$ with $h_j^\dagger = h_j$, for instance the real and imaginary parts $\frac{1}{2}(w + w^\dagger)$

and $\frac{1}{2\mathrm{i}}(w - w^\dagger)$ of finitely many generators w, which exist since $\mathfrak{Z} \cong S(\mathfrak{h})^W$. By Lemma 2.5 the closures T_j of the $\omega_n(h_j)$ are self-adjoint and commute strongly; let F be their joint spectral measure on $\mathbb{R}^r$ [41, Theorem 5.23]. Since $\mathcal{H}_n^\infty$ is invariant under $\omega_n(\mathfrak{Z})$ and the joint functional calculus satisfies $f(T)g(T) \subset (fg)(T)$, induction on monomials gives $\omega_n(P(h))\psi = P(T)\psi$ for every polynomial P and $\psi \in \mathcal{H}_n^\infty$. If $P(h) = 0$ in $\mathfrak{Z}$, then $\int |P|^2 \,\mathrm{d}\langle \mathsf{F}\psi, \psi\rangle = \|P(T)\psi\|^2 = 0$ on the dense subspace $\mathcal{H}_n^\infty$, so F is supported on the common real zeros of these P. A real common zero t is the same as a hermitian character ζ, with $\zeta(h_j) = t_j$, and $\zeta \mapsto (\zeta(h_j))_j$ is a homeomorphism of the hermitian characters onto this closed set. So F is a projection-valued measure $\mathcal{E}_n$ on the hermitian characters, and $\omega_n(z)\psi = \int \zeta(z)\,\mathrm{d}\mathcal{E}_n(\zeta)\psi$, which is (2).

Uniqueness. Let $\mathcal{E}'$ also satisfy (2). The operator $\int \zeta(h_j)\,\mathrm{d}\mathcal{E}'(\zeta)$ is normal and extends $\omega_n(h_j)$, hence its closure T_j; a normal extension of a self-adjoint operator equals it, since $\operatorname{dom} T' = \operatorname{dom} T'^* \subset \operatorname{dom} T_j^* = \operatorname{dom} T_j$. So $\zeta(h_j)$ is real $\mathcal{E}'$-almost everywhere, $\mathcal{E}'$ is supported on the hermitian characters, and its image under $\zeta \mapsto (\zeta(h_j))_j$ is F. As this map is injective, $\mathcal{E}' = \mathcal{E}_n$. □

Definition 2.7. Let φ_n be the state of a density operator ρ_n on $\mathcal{H}_n$ with finite moments. The *law of the infinitesimal character* is the probability measure

$$\mathsf{P}_n(B) = \operatorname{tr}\big(\rho_n\, \mathcal{E}_n(B)\big) \qquad (B \subset \mathfrak{h}^*/W \text{ Borel}),$$

and we write χ_n for a random variable with this law.

By (2), the law of χ_n determines the second moments of the central elements: for $z, z' \in \mathfrak{Z}$,

$$\varphi_n\big(\omega_n(z^\dagger)\omega_n(z')\big) = \int_{\mathfrak{h}^*/W} \overline{\gamma(z)}\,\gamma(z')\,\mathrm{d}\mathsf{P}_n. \tag{3}$$

Indeed, writing $\rho_n = \sum_j \mathsf{w}_j |\psi_j\rangle\langle\psi_j|$, the left side is $\sum_j \mathsf{w}_j \langle \omega_n(z)\psi_j, \omega_n(z')\psi_j\rangle$, because $\omega_n(z)^* \supseteq \omega_n(z^\dagger)$.

Theorem 2.8 (law of large numbers)**.** *Let $\mathfrak{g}$ be reductive, and let φ_n be product states with finite moments satisfying* (F) *whose mean moments converge, $\mu_n \to \mu$. Then $\chi_n/n \to \operatorname{spec}(\mu)$ in probability: for every neighbourhood U of $\operatorname{spec}(\mu)$ in $\mathfrak{h}^*/W$ and every $m \geq 1$, $\operatorname{Prob}(\chi_n/n \notin U) = O(n^{-m})$. In particular, for every $\eta > 0$,*

$$\operatorname{Prob}\Big(\operatorname{dist}\big(\tfrac{1}{n}\chi_n,\ \operatorname{spec}(\mu)\big) > \eta\Big) = O\big(n^{-m}\big).$$

Proof. Fix $m \geq 1$. Let $q \in S(\mathfrak{h})^W$ be homogeneous of degree d, and let $z = \gamma^{-1}(q)$. By (1), $z \in \mathfrak{U}_d$, and $\sigma_d(z)$ is the invariant polynomial on $\mathfrak{g}_{\mathbb{C}}^*$ that restricts to q, so $\sigma_d(z)(\mu) = q(\operatorname{spec}(\mu))$. Put $y = n^{-d}z - q(\operatorname{spec}\mu_n) \in \mathfrak{Z}$. By Proposition 2.3, $\omega_n(y) = n^{-1/2}R_n$ with R_n of bounded moments, so $\varphi_n(|\omega_n(y)^m|^2) \leq Cn^{-m}$. By homogeneity $\gamma(y)(\chi) = q(\chi/n) - q(\operatorname{spec}\mu_n)$, and (3) with $z = z' = y^m$ gives

$$\int \big|q(\chi/n) - q(\operatorname{spec}\mu_n)\big|^{2m}\,\mathrm{d}\mathsf{P}_n(\chi) = \varphi_n\big(|\omega_n(y)^m|^2\big) \leq \frac{C}{n^m}.$$

As $\mu \mapsto q(\operatorname{spec}(\mu))$ is a polynomial function on $\mathfrak{g}_{\mathbb{C}}^*$, $q(\operatorname{spec}\mu_n) \to q(\operatorname{spec}(\mu))$, and Markov's inequality gives $\operatorname{Prob}\big(|q(\chi_n/n) - q(\operatorname{spec}(\mu))| \geq \varepsilon\big) = O(n^{-m})$ for every $\varepsilon > 0$. As the invariant polynomials induce the topology of $\mathfrak{h}^*/W$, every neighbourhood U of $\operatorname{spec}(\mu)$ contains a set $\{x : |q(x) - q(\operatorname{spec}(\mu))| < \varepsilon \text{ for } q \in F\}$ with F finite, and summing over $q \in F$ gives $\operatorname{Prob}(\chi_n/n \notin U) = O(n^{-m})$. The ball of radius η about $\operatorname{spec}(\mu)$ is such a neighbourhood. □

2.3. **Types.** When the tensor power decomposes discretely and $\mathfrak{Z}$ acts on each piece by a scalar, the law of χ_n is the image of a law on the pieces.

Definition 2.9. Suppose that $\mathcal{H}_n$ is the orthogonal sum of subspaces H_λ, indexed by a countable set, whose projections P_λ commute with $\pi_n(G)$. Suppose that there are a system of positive roots, with half-sum δ', and points $\Lambda(\lambda) \in \mathfrak{h}^*$ such that every $z \in \mathfrak{Z}$ acts on $P_\lambda \mathcal{H}_n^\infty$ by the scalar $\chi_\lambda(z) = \gamma(z)(\Lambda(\lambda) + \delta')$. The *type law* of φ_n is $\mathsf{T}_n(\lambda) = \operatorname{tr}(\rho_n P_\lambda)$.

The subspaces, the index set and the points $\Lambda(\lambda)$ depend on n. By (ii), applied to the given positive system, χ_λ is the scalar by which $\mathfrak{Z}$ acts on a highest weight module of highest weight $\Lambda(\lambda)$ for that system; the map γ does not depend on the positive system, since its values are W-invariant.

Corollary 2.10 (concentration of types)**.** *In the situation of Definition 2.9:*

(a) *The P_λ preserve $\mathcal{H}_n^\infty$, and $\omega_n(z)\psi = \sum_\lambda \chi_\lambda(z) P_\lambda \psi$ for $z \in \mathfrak{Z}$ and $\psi \in \mathcal{H}_n^\infty$. The law of χ_n is the image of T_n under $\lambda \mapsto W(\Lambda(\lambda) + \delta')$, and*

$$\varphi_n\big(\omega_n(z^\dagger)\omega_n(z')\big) = \sum_\lambda \mathsf{T}_n(\lambda)\, \overline{\chi_\lambda(z)}\, \chi_\lambda(z') \qquad (z, z' \in \mathfrak{Z}).$$

(b) *Under the hypotheses of Theorem 2.8, $\mathsf{T}_n\{\lambda : \operatorname{dist}(\frac{1}{n}\Lambda(\lambda), \operatorname{spec}(\mu)) > \eta\} = O(n^{-m})$ for every $\eta > 0$ and $m \geq 1$.*

(c) *If moreover the $\Lambda(\lambda)$ lie in a closed cone $\mathcal{C} \subset \mathfrak{h}^*$ that meets the orbit $\operatorname{spec}(\mu)$ in a single point μ^+, then $\Lambda(\lambda)/n \to \mu^+$ in probability: $\mathsf{T}_n\{\lambda : |\frac{1}{n}\Lambda(\lambda) - \mu^+| > \eta\} = O(n^{-m})$ for every $\eta > 0$ and $m \geq 1$.*

Proof. (a) Since the P_λ commute with $\pi_n(G)$, they preserve $\mathcal{H}_n^\infty$ and commute there with every $\omega_n(u)$. As $\sum_\lambda P_\lambda = 1$, this gives the formula for $\omega_n(z)\psi$. Let h_j be the hermitian generators of $\mathfrak{Z}$ from the proof of Proposition 2.6. The scalar $\chi_\lambda(h_j)$ is real, because the symmetric operator $\omega_n(h_j)$ acts by it on $P_\lambda \mathcal{H}_n^\infty$, which is dense in H_λ. Hence $\sum_\lambda \chi_\lambda(h_j) P_\lambda$ is a self-adjoint extension of $\omega_n(h_j)$, hence its closure T_j. The projection-valued measure $\sum_\lambda P_\lambda\, \delta_{W(\Lambda(\lambda)+\delta')}$ therefore satisfies (2) on the generators, hence on $\mathfrak{Z}$, and by uniqueness it is $\mathcal{E}_n$. This gives the law of χ_n, and with (3) the displayed identity.

(b) This is Theorem 2.8, since $\delta'/n \to 0$.

(c) The points $\Lambda(\lambda)/n$ lie in $\mathcal{C}$. Since $\mathcal{C}$ is closed and $\operatorname{spec}(\mu)$ is finite, the number $r = \min\{\operatorname{dist}(s, \mathcal{C}) : s \in \operatorname{spec}(\mu),\ s \neq \mu^+\}$ is positive (put $r = \infty$ if there is no such s). For $\eta' < \min(\eta, r)/2$, a point of $\mathcal{C}$ within η' of the orbit $\operatorname{spec}(\mu)$ is within η' of μ^+. Apply (b) with η'. □

Example 2.11 (compact groups). Let $G = K$ be a compact connected Lie group, and let π be a unitary representation on a finite-dimensional Hilbert space $\mathcal{H}$. Then $\mathcal{H}^\infty = \mathcal{H}$, every density matrix has finite moments, and (M) holds for all product states, since β_s is bounded by an s-fold product of operator norms; so (F) holds by Remark 2.2. Let $\mathfrak{h}$ be the complexification of the Lie algebra of a maximal torus, and decompose $\mathcal{H}^{\otimes n} = \bigoplus_\lambda H_\lambda$ into isotypic components, labelled by dominant weights λ. Then $\mathfrak{Z}$ acts on H_λ by χ_λ, so Definition 2.9 applies with $\Lambda(\lambda) = \lambda$ and $\delta' = \delta$, and $\mathsf{T}_n(\lambda) = \operatorname{tr}(\rho_1 \otimes \cdots \otimes \rho_n\, P_\lambda)$. The moment $X \mapsto \operatorname{tr}(\rho\, \mathrm{d}\pi(X))$ is imaginary on $\mathfrak{g}$, so it is conjugate under K to a unique point of the closed dominant chamber. If the mean moments converge to μ, Corollary 2.10(c) says that λ/n converges in probability to the dominant point μ^+ of $\operatorname{spec}(\mu)$. For identical sites this is the law of large numbers for the isotypic type that Botero, Christandl and Vrana derive from [1, Theorem 1.2], a statement about a finer measure on $\mathfrak{ig}^*$, by passing to the dominant chamber.

For $K = U(p)$ acting on $\mathcal{H} = \mathbb{C}^p$, Schur–Weyl duality labels the isotypic components of $(\mathbb{C}^p)^{\otimes n}$ by the Young diagrams λ with n boxes and at most p rows. In the coordinates E_{ij} of

$\mathfrak{gl}_p$ the moment of ρ is the matrix ρ itself. So if $\frac{1}{n}\sum_a \rho_{n,a} \to \bar{\rho}$, then λ/n converges to the decreasingly ordered spectrum of $\bar{\rho}$. For identical sites this is the law of large numbers in the theorem of Keyl and Werner [20, §II, eq. (10)], here for every ρ; it also follows from the bounds of [12, Lemma 5] and [4, Theorem 1]. In this case exact formulas are known: the expectations of the normalized central characters $p^\sharp$ of λ in the sense of [30] are falling-factorial multiples of the power sums of the spectrum of ρ [30, Proposition 2.34]. The proof of Theorem 2.8 uses only the leading order of such identities, for any reductive $\mathfrak{g}$.

Remark 2.12 (what the infinitesimal character does not see). Theorem 2.8 locates the orbit χ_n/n, not a point of it. In Corollary 2.10 a point is selected by a cone containing the types; in Section 3 the cone comes from the dominant weights of the compact member of a dual pair. Nor does χ determine the representation. For a linear connected reductive group only finitely many irreducible unitary representations share an infinitesimal character [21, Corollary 10.37], but they may belong to different series. For instance, normalize the Cartan subalgebra of $\mathfrak{sl}_2(\mathbb{R})$ so that the roots are ± 2; then the trivial representation of $SL_2(\mathbb{R})$ and the two discrete series representations with lowest K-types ± 2 all have $\chi = \{\pm 1\}$ [21, Chapter II]. The limit orbit does see the series at scale n: for $\mu = \mathrm{i}\,\mathrm{tr}(Y\,\cdot\,)$ with $Y \in \mathfrak{sl}_2(\mathbb{R})$, the orbit $\mathrm{spec}(\mu)$ is real and nonzero if Y is elliptic, as for the discrete series, and imaginary and nonzero if Y is hyperbolic, as for the unitary principal series.

3. Dual pairs on Fock space

We apply Corollary 2.10 to three Howe dual pairs on bosonic Fock space [16]. A site carries finitely many modes, a compact group K_n' acts linearly on the n sites, and its commutant is generated by the quadratic operators of a real reductive Lie algebra $\mathfrak{g}$, applied to all sites at once. The moment of a state of one site is then a matrix of second moments, Howe duality matches the K_n'-types with highest weights of $\mathfrak{g}$, and the limit of the normalized type is read off from the eigenvalues of the limiting moment matrix. The number of modes per site stays fixed as $n \to \infty$, and partitions are padded with zeros to the number of rows needed.

3.1. **Fock space and type laws.** Let $\mathcal{F}$ be the bosonic Fock space of finitely many modes, with number operator N, and let $\mathcal{S} \subset \mathcal{F}$ be the Schwartz space of vectors ψ with $\|(\mathsf{N}+1)^t\psi\| < \infty$ for all t. A polynomial A of degree at most $2t$ in the creation and annihilation operators maps $\mathcal{S}$ into itself and satisfies $|\langle\psi, A\psi\rangle| \le C_A\|(\mathsf{N}+1)^{t/2}\psi\|^2$. Applied to A^*A, this gives

$$\|A\psi\| \le C_A'\|(\mathsf{N}+1)^t\psi\| \qquad (\psi \in \mathcal{S}).$$

The Fock space of n sites, each carrying the modes of $\mathcal{F}$, is $\mathcal{F}_n = \mathcal{F}^{\otimes n}$, with total number operator N_n, and the Schwartz space $\mathcal{S}_n$. For a density operator ρ with finite moments and an operator A on $\mathcal{S}$ we write $\rho(A) = \mathrm{tr}(\rho A)$, as in Section 2.1.

In each of the three cases below, ω maps $\mathfrak{g}_{\mathbb{C}}$ to quadratic polynomials in the creation and annihilation operators of one site, and it integrates to a unitary representation π of a connected Lie group G with Lie algebra $\mathfrak{g}$ [19, 16]. The total number operator is a collective observable up to an additive constant: $\mathsf{N}_n = \mathrm{i}\,\omega_n(X_0) + \mathrm{const}$, where X_0 is a multiple of the sum of the Cartan elements below and lies in the centre of the Lie algebra of a maximal compact subgroup K of G. As N_n has finite-dimensional eigenspaces, the space $\mathcal{D}_n \subset \mathcal{F}_n$ of K-finite vectors of π_n consists of the vectors with finitely many particles. Its elements are analytic vectors, and for $X \in \mathfrak{g}$ the generator of $t \mapsto \pi_n(\exp tX)$ is the closure of $\omega_n(X)$ on $\mathcal{D}_n$.

Lemma 3.1. *In this situation:*

(a) $\mathcal{H}_n^\infty = \mathcal{S}_n$. *So a density operator ρ on $\mathcal{F}_n$ has finite moments in the sense of Definition 2.1 if and only if $\mathrm{tr}(\rho\,\mathsf{N}_n^t) < \infty$ for all t. Gaussian states, density operators*

supported on finitely many eigenspaces of N_n*, and products of states with finite moments have finite moments.*

(b) *A unitary operator on* $\mathcal{F}_n$ *that preserves* $\mathcal{S}_n$ *and commutes there with* $\omega_n(\mathfrak{g})$ *commutes with* $\pi_n(G)$*.*

(c) *Let* $H \subset \mathcal{F}_n$ *be a closed subspace whose projection* P *commutes with* N_n*. If* $z \in \mathfrak{Z}$ *acts on the* K*-finite vectors in* H *by a scalar, it acts by the same scalar on* $P\mathcal{S}_n$*.*

(d) *Let a compact group* K' *act on* $\mathcal{F}_n$ *by unitary operators that commute with* N_n*, preserve* $\mathcal{S}_n$ *and commute there with* $\omega_n(\mathfrak{g})$*, and let* P_λ *be its isotypic projections. If every* $z \in \mathfrak{Z}$ *acts on the* K*-finite vectors of* $P_\lambda\mathcal{F}_n$ *by* $\gamma(z)(\Lambda(\lambda)+\delta')$*, for a system of positive roots with half-sum* δ' *and points* $\Lambda(\lambda) \in \mathfrak{h}^*$*, then the law* $\mathsf{T}_n(\lambda) = \operatorname{tr}(\rho_n P_\lambda)$ *of the* K'*-type is a type law in the sense of Definition 2.9.*

Proof. (a) The space $\mathcal{S}_n$ consists of the smooth vectors of the oscillator representation of the metaplectic group, through which π_n factors [9, Chapter 4], so $\mathcal{S}_n \subset \mathcal{H}_n^\infty$. Conversely, smooth vectors lie in the domain of the generator $\overline{\omega_n(X_0)} = -\mathrm{i}(\overline{\mathsf{N}_n} - \text{const})$ and are mapped to smooth vectors by it, so $\mathcal{H}_n^\infty$ lies in the domain of every power of $\overline{\mathsf{N}_n}$, that is, in $\mathcal{S}_n$.

The remaining statements follow from the bound on $|\langle\psi, A\psi\rangle|$.

(b) For $X \in \mathfrak{g}$, the space $\mathcal{H}_n^\infty = \mathcal{S}_n$ contains the core $\mathcal{D}_n$ of the generator $\overline{\omega_n(X)}$ of $t \mapsto \pi_n(\exp tX)$ and is contained in its domain, so it is a core. A unitary commuting with $\omega_n(X)$ on a core commutes with its closure and with the one-parameter group it generates, and these groups generate $\pi_n(G)$.

(c) The projection P preserves $\mathcal{S}_n$. For $\psi \in \mathcal{S}_n$ the truncations of $P\psi$ to finitely many eigenspaces of N_n converge to $P\psi$ in each of the norms $\|(\mathsf{N}_n+1)^t \cdot\|$, and by the bound $\|A\psi\| \le C'_A\|(\mathsf{N}_n+1)^t\psi\|$ the operator $\omega_n(z)$ is continuous from $\mathcal{S}_n$, with one of these norms, to $\mathcal{F}_n$.

(d) By (b) the elements of K' commute with $\pi_n(G)$, and so do the P_λ, which are averages of them. The P_λ also commute with N_n, so by (c) z acts on $P_\lambda\mathcal{S}_n = P_\lambda\mathcal{H}_n^\infty$ by $\gamma(z)(\Lambda(\lambda)+\delta')$. □

3.2. **Three dual pairs.** In each case a state ρ of one site has a *moment matrix*, a matrix of second moments on which its moment depends linearly, and there is a fixed matrix J such that the coadjoint action of G conjugates the product of the moment matrix with J. The Cartan subalgebra $\mathfrak{h}$ is spanned by elements H_r whose collective images are the total occupations of the modes, shifted by the vacuum, and a point of $\mathfrak{h}^*$ is given by its coordinates $x_r = x(H_r)$.

(O) *The orthogonal pair* $(O(n), \mathfrak{sp}_{2N}(\mathbb{R}))$. A site carries N modes c_r. The quadratures $R_r = (c_r + c_r^*)/\sqrt{2}$ and $R_{N+r} = (c_r - c_r^*)/(\mathrm{i}\sqrt{2})$ satisfy $[R_\alpha, R_\beta] = \mathrm{i}\Omega_{\alpha\beta}$ with $\Omega = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. For a complex symmetric matrix A of size $2N$ put $\omega(A) = -\frac{\mathrm{i}}{2}\sum_{\alpha,\beta} A_{\alpha\beta}S_{\alpha\beta}$, with $S_{\alpha\beta} = \frac{1}{2}(R_\alpha R_\beta + R_\beta R_\alpha)$. Then

$$[\omega(A), R_\gamma] = \textstyle\sum_\alpha (A\Omega)_{\alpha\gamma} R_\alpha, \qquad [\omega(A), \omega(A')] = \omega(A\Omega A' - A'\Omega A), \tag{4}$$

so the real symmetric matrices, with this bracket, form $\mathfrak{g} \cong \mathfrak{sp}_{2N}(\mathbb{R})$, by $A \mapsto A\Omega$, and ω is the oscillator representation of the metaplectic group. The moment matrix is $\mathbf{M}_\rho = (\rho(S_{\alpha\beta}))$, with $\mu_\rho(A) = -\frac{\mathrm{i}}{2}\operatorname{tr}(A\mathbf{M}_\rho)$, and $J = \Omega$. The Cartan elements have $\omega(H_r) = c_r^* c_r + \frac{1}{2}$; the point x is the moment with $\mathbf{M} = \operatorname{diag}(x, x)$, where $\mathbf{M}\Omega$ has the eigenvalues $\pm\mathrm{i}x_r$. The group $O(n)$ rotates the sites, $c_{ar} \mapsto \sum_{a'} g_{a'a} c_{a'r}$. Its types are the partitions λ with at most N rows whose first two columns have total length at most n; the highest weights are $\Lambda_n(\lambda) = \lambda + \frac{n}{2}$, in the cone $x_1 \ge \cdots \ge x_N \ge 0$, for the positive system whose root vectors have the images $\sum_a c_{ar}c_{as}$, for all r, s, and $\sum_a c_{ar}^* c_{as}$, for $r < s$ [19, 16, 17].

(U) *The unitary pair* $(U(n), \mathfrak{u}(p,q))$. A site carries $p+q$ modes $\mathsf{a}_1, \dots, \mathsf{a}_p, \mathsf{b}_1, \dots, \mathsf{b}_q$. Put $\Phi = (\mathsf{a}_1, \dots, \mathsf{a}_p, \mathsf{b}_1^*, \dots, \mathsf{b}_q^*)$ and $I_{p,q} = \operatorname{diag}(1_p, -1_q)$, so that $[\Phi_\alpha, \Phi_\beta^*] = (I_{p,q})_{\alpha\beta}$, and for a complex

matrix A of size $p+q$ put $\omega(A) = \sum_{\alpha,\beta} A_{\alpha\beta} \cdot \frac{1}{2}(\Phi_\alpha^* \Phi_\beta + \Phi_\beta \Phi_\alpha^*)$. Then $[\omega(A), \omega(A')] = \omega(AI_{p,q}A' - A'I_{p,q}A)$, and the skew-Hermitian matrices form $\mathfrak{g} \cong \mathfrak{u}(p,q)$, by $A \mapsto AI_{p,q}$. The moment matrix is $\Gamma_\rho = \big(\rho(\frac{1}{2}(\Phi_\alpha \Phi_\beta^* + \Phi_\beta^* \Phi_\alpha))\big)$, with $\mu_\rho(A) = \operatorname{tr}(A\Gamma_\rho)$, and $J = I_{p,q}$. The Cartan elements are the diagonal matrix units, $\omega(H_i) = \mathsf{a}_i^* \mathsf{a}_i + \frac{1}{2}$ and $\omega(H_{p+i'}) = \mathsf{b}_{i'}^* \mathsf{b}_{i'} + \frac{1}{2}$; the point x is the moment with $\Gamma = \operatorname{diag}(x)$, where $\Gamma I_{p,q}$ has the eigenvalues $\theta = (x_1, \dots, x_p, -x_{p+1}, \dots, -x_{p+q})$. The group $U(n)$ acts by g on the a_{ai}^* and by $\bar{g}$ on the $\mathsf{b}_{ai'}^*$. Its types are the pairs of partitions $[\lambda^+, \lambda^-]$ with $\ell(\lambda^+) \le p$, $\ell(\lambda^-) \le q$ and $\ell(\lambda^+) + \ell(\lambda^-) \le n$, labelling the highest weight $(\lambda_1^+, \dots, 0, \dots, -\lambda_1^-)$ of $U(n)$; the highest weights have $\theta(\Lambda_n(\lambda)) = (\lambda^+ + \frac{n}{2}, -\lambda^- - \frac{n}{2})$, in the cone $\theta_1 \ge \dots \ge \theta_p \ge 0 \ge \theta_{p+q} \ge \dots \ge \theta_{p+1}$, for the positive system whose root vectors have the images $\mathsf{a}_i^* \mathsf{a}_j$ $(i<j)$, $\mathsf{b}_{i'}^* \mathsf{b}_{j'}$ $(i'<j')$ and $\mathsf{a}_i \mathsf{b}_{j'}$ [19, 16]. For $p=q=1$ this is the natural action of $U(n)$ on $L^2(\mathbb{C}^n)$.

(Sp) *The symplectic pair* $(Sp(n), \mathfrak{so}^*(2N))$. A site carries $2N$ modes $\mathsf{a}_r, \mathsf{b}_r$, and $Sp(n) \subset U(2n)$ acts on the $2n$ creation operators $(\mathsf{a}_{ar}^*, \mathsf{b}_{ar}^*)_{a \le n}$, for each r. Under $Sp(1) = SU(2)$ the pairs $D_r = (\mathsf{a}_r^*, \mathsf{b}_r^*)$ and $D_{N+r} = (-\mathsf{b}_r, \mathsf{a}_r)$ transform alike, and the antisymmetrized invariants $M_{AB} = \frac{1}{2}(\langle D_A, D_B\rangle - \langle D_B, D_A\rangle)$, with $\langle D, D'\rangle = D^1 D'^2 - D^2 D'^1$, span the image of $\mathfrak{g}_\mathbb{C} \cong \mathfrak{so}_{2N}(\mathbb{C})$, the Lie algebra of the form $\mathbf{K} = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$; we write $\omega(m_{AB}) = M_{AB}$. The real form $\mathfrak{g} \cong \mathfrak{so}^*(2N)$ consists of the X with $\omega(X)$ skew-symmetric. The moment matrix is the antisymmetric matrix $\mathbf{X}_\rho = (\rho(M_{AB}))$, and $J = \mathbf{K}$. The Cartan elements are $H_r = m_{r,N+r}$, with $\omega(H_r) = \mathsf{a}_r^* \mathsf{a}_r + \mathsf{b}_r^* \mathsf{b}_r + 1$; the point x is the moment with $\mathbf{X} = \left(\begin{smallmatrix} 0 & \operatorname{diag}(x) \\ -\operatorname{diag}(x) & 0 \end{smallmatrix}\right)$, where $\mathbf{XK} = \operatorname{diag}(x, -x)$. The types of $Sp(n)$ are the partitions λ with at most $\min(n, N)$ rows; the highest weights are $\Lambda_n(\lambda) = \lambda + n$, in the cone $x_1 \ge \dots \ge x_N \ge 0$, for the positive system with roots $e_r - e_s$ and $-e_r - e_s$, $r < s$ [16, 15]. For $N = 2$ this is the natural action of $Sp(n)$ on $L^2(\mathbb{H}^n)$.

(H) *Howe duality.* In each case the K-finite vectors of $\mathcal{F}_n$ decompose under $K_n' \times \mathfrak{g}_\mathbb{C}$ as $\bigoplus_\lambda V_\lambda \otimes \mathcal{M}_\lambda$, where V_λ is the irreducible representation of K_n' of type λ and $\mathcal{M}_\lambda$ is an irreducible highest weight module with highest weight $\Lambda_n(\lambda)$.

By (H), $z \in \mathfrak{Z}$ acts on $\mathcal{M}_\lambda$ by $\gamma(z)(\Lambda_n(\lambda) + \delta')$ [33], where δ' is the half-sum of the chosen positive roots. By Lemma 3.1(d), the compact type distribution is a type law.

Theorem 3.2. *Let $\rho_n = \rho_{n,1} \otimes \dots \otimes \rho_{n,n}$ be product states on $\mathcal{F}_n$ with finite moments satisfying* (F)*, for instance* (M) *of Remark 2.2, whose mean moment matrices converge to* $\mathbf{M}$*,* Γ *or* $\mathbf{X}$*. Let $P_{n,\lambda}$ be the K_n'-isotypic projections. Give the type λ the law $\mathsf{T}_n(\lambda) = \operatorname{tr}(\rho_n P_{n,\lambda})$. Then, in probability with deviation probabilities $O(n^{-m})$ for every $m \ge 1$:*

(a) (O) *the eigenvalues of* $\mathbf{M}\Omega$ *are* $\pm \mathrm{i}\nu_r$ *with* $\nu_1 \ge \dots \ge \nu_N \ge \frac{1}{2}$*, and the $O(n)$-type satisfies* $\lambda/n \to \nu - \frac{1}{2}$*;*

(b) (U) $\Gamma I_{p,q}$ *has p eigenvalues* $\nu_1^+ \ge \dots \ge \nu_p^+ \ge \frac{1}{2}$ *and q eigenvalues* $-\nu_1^- \le \dots \le -\nu_q^- \le -\frac{1}{2}$*, and the $U(n)$-type satisfies* $\lambda^\pm/n \to \nu^\pm - \frac{1}{2}$*;*

(c) (Sp) *the eigenvalues of* $\mathbf{XK}$ *are* $\pm\vartheta_r$ *with* $\vartheta_1 \ge \dots \ge \vartheta_N \ge 1$*, and the $Sp(n)$-type satisfies* $\lambda/n \to \vartheta - 1$.

In particular, by Remark 2.2, this holds for $\rho^{\otimes n}$ for every state ρ of one site with finite moments.

Proof. Apply Theorem 2.8 to the diagonal tensor powers of the fixed representation π of G, with Lie algebra $\mathfrak{g}$. By Corollary 2.10(a), the law of χ_n is the image of the type law under $\lambda \mapsto W(\Lambda_n(\lambda) + \delta')$. Let $\Sigma = \operatorname{spec}(\mu) \subset \mathfrak{h}^*$, where μ is the limiting mean moment. Since $\delta'/n \to 0$, Corollary 2.10(b) gives, for every $\eta > 0$ and $m \ge 1$,

$$\mathsf{T}_n\{\lambda : \operatorname{dist}(\Lambda_n(\lambda)/n, \Sigma) > \eta\} = O(n^{-m}).$$

Put

$$B = \begin{cases} \mathbf{M}\Omega & \text{in case (O)},\\ \Gamma I_{p,q} & \text{in case (U)},\\ \mathbf{X}\mathbf{K} & \text{in case (Sp)}. \end{cases}$$

The coadjoint action conjugates B, so its characteristic coefficients are invariant polynomials. They agree at μ and every $x \in \Sigma$. The Cartan moment matrices above therefore give the eigenvalues of B as

$$\begin{cases} \{\pm \mathrm{i} x_r : 1 \le r \le N\} & \text{in case (O)},\\ \{\theta_\alpha(x) : 1 \le \alpha \le p+q\} & \text{in case (U)}, \qquad x \in \Sigma,\\ \{\pm x_r : 1 \le r \le N\} & \text{in case (Sp)}, \end{cases}$$

with eigenvalues counted with multiplicity.

Let $\mathcal{C}$ be the cone of highest weights specified for the pair. The formulas for $\Lambda_n(\lambda)$ show that $\Lambda_n(\lambda)/n$ lies in a closed subset $D \subset \mathcal{C}$: in case (O) every coordinate is at least $\frac{1}{2}$; in case (U), $\theta_i \ge \frac{1}{2}$ for $i \le p$ and $\theta_{p+i'} \le -\frac{1}{2}$ for $i' \le q$; in case (Sp) every coordinate is at least 1. The concentration bound forces $\Sigma \cap D \neq \varnothing$. Indeed, if it were empty, the finite set Σ would have positive distance from the closed set D. For η smaller than that distance the probability in the display would be 1 for every n, contradicting the bound. Choose $y \in \Sigma \cap D$.

The eigenvalue lists determine at most one point of $\mathcal{C}$. In (O) and (Sp) its coordinates are ordered and nonnegative. In (U) the positive eigenvalues and magnitudes of negative eigenvalues are ordered separately. Thus $\Sigma \cap \mathcal{C} = \{y\}$. Writing

$$y = \nu \quad \text{in (O)}, \qquad \theta(y) = (\nu^+, -\nu^-) \quad \text{in (U)}, \qquad y = \vartheta \quad \text{in (Sp)},$$

the spectral identities give the asserted eigenvalues. Membership in D gives the lower bounds.

Corollary 2.10(c) now gives $\Lambda_n(\lambda)/n \to y$ with the same $O(n^{-m})$ bounds for fixed deviations. In cases (O) and (Sp), subtracting the vacuum shifts $\frac{1}{2}$ and 1 gives (a) and (c). In case (U), the identity

$$\theta(\Lambda_n(\lambda)/n) = \left(\lambda^+/n + \tfrac{1}{2}, -\lambda^-/n - \tfrac{1}{2}\right)$$

gives (b). □

In case (O) the ν_r are the symplectic eigenvalues of $\mathbf{M}$: by Williamson's theorem [47] there is a real symplectic T with $T\mathbf{M}T^{\mathsf{T}} = \operatorname{diag}(\nu, \nu)$, and $\nu_r \ge \frac{1}{2}$ is the uncertainty principle in the form of [45]. For a centred Gaussian state, $\nu_r - \frac{1}{2}$ are the occupations of its normal modes; displacing a state by a mean quadrature vector $\mathbf{m}$ replaces $\mathbf{M}$ by $\mathbf{M} + \mathbf{m}\mathbf{m}^{\mathsf{T}}$. The shift $\frac{1}{2}$ is the weight $\frac{n}{2}$ of the vacuum, the zero-point energy; it is the reason why the group dual to $O(n)$ is the metaplectic cover of $Sp_{2N}(\mathbb{R})$ when n is odd. In case (U) with $q = 0$, the matrix $\Gamma - \frac{1}{2}$ is the transpose of the one-particle density matrix $(\rho(\mathsf{a}_i^* \mathsf{a}_j))$. If each site carries one particle, then ρ_n lives on $(\mathbb{C}^p)^{\otimes n}$, the weight space of weight $(1, \dots, 1)$ of the diagonal torus of $U(n)$, which meets the $U(n)$-isotypic component of type λ in the Schur–Weyl component of type λ; so (b) contains Example 2.11, for any number of particles per site. For $q > 0$ the entries $\rho(\mathsf{a}_i \mathsf{b}_{j'})$ of Γ are pairing amplitudes; if they vanish in the limit, $\nu^{\pm} - \frac{1}{2}$ are the spectra of the two one-particle density matrices. In case (Sp), thermal sites with $\rho(\mathsf{a}_r^* \mathsf{a}_s) = \rho(\mathsf{b}_r^* \mathsf{b}_s) = b_r \delta_{rs}$ and no pairing give $\vartheta_r = 2b_r + 1$.

We next consider, in case (O), finitely many coherent sites among thermal ones. In the Schrödinger model $\mathcal{F} = L^2(\mathbb{R}^N)$, the vacuum $\psi_0(x) = \pi^{-N/4} \mathrm{e}^{-|x|^2/2}$ spans the lowest K-type, and the *coherent states* are its images under the Heisenberg group; we use the position translates $\psi_v(x) = \psi_0(x - v)$, $v \in \mathbb{R}^N$. The *thermal states* are the Gibbs states of the maximal torus of K: for $s_1, \dots, s_N > 0$, the state $\mathrm{e}^{-H} / \operatorname{tr} \mathrm{e}^{-H}$, where $H = \sum_r s_r (c_r^* c_r + \frac{1}{2}) = \mathrm{i}\,\omega(T)$ for an element

T of the Lie algebra of the maximal torus of K, has occupations $b_r = (\mathrm{e}^{s_r} - 1)^{-1}$. Conjugating by the unitary operators that rotate the modes by elements of $O(N)$, and allowing $s_r = \infty$, we obtain for every real symmetric positive semidefinite $N \times N$ matrix $\mathbf{B}$ with eigenvalues $b_1 \geq \cdots \geq b_N \geq 0$ a state $\tau_{\mathbf{B}}$ with $\tau_{\mathbf{B}}(c_r^* c_s) = \mathbf{B}_{rs}$ and moment matrix $\mathrm{diag}(\mathbf{Q}, \mathbf{Q})$, where $\mathbf{Q} = \mathbf{B} + \frac{1}{2} I_N$; note $b_r + \frac{1}{2} = \frac{1}{2} \coth \frac{s_r}{2}$.

Corollary 3.3. *Fix $k \geq 1$ and $\xi^{(1)}, \ldots, \xi^{(k)} \in \mathbb{R}^N$, and let Ξ be the $N \times k$ matrix with these columns. Let ρ_n be the product state with the coherent states $\psi_{\sqrt{2n}\,\xi^{(l)}}$, $l = 1, \ldots, k$, at k sites and the thermal state $\tau_{\mathbf{B}}$ at the other $n - k$ sites. Let $\nu_1^2 \geq \cdots \geq \nu_N^2$, $\nu_r > 0$, be the eigenvalues of $\mathbf{Q}^2 + 2\,\mathbf{Q}^{1/2} \Xi \Xi^{\mathsf{T}} \mathbf{Q}^{1/2}$. Then:*

(a) *the $O(n)$-type λ of ρ_n satisfies $\lambda / n \to \nu - \frac{1}{2}$;*

(b) *the type λ' of the subgroup $O(n-k)$ acting on the thermal sites satisfies $\lambda'/n \to b$;*

both in probability, with probabilities of deviation $O(n^{-m})$ for every m. Moreover $\nu_r \geq b_r + \frac{1}{2}$ for $1 \leq r \leq N$, and $b_r + \frac{1}{2} \geq \nu_{r+k}$ for $1 \leq r \leq N - k$. For $k = 1$ this is the interlacing $\nu_1 \geq b_1 + \frac{1}{2} \geq \nu_2 \geq b_2 + \frac{1}{2} \geq \cdots \geq \nu_N \geq b_N + \frac{1}{2}$.

Proof. At the l-th coherent site, write $R = \sqrt{n}\,\mathbf{v}_l + \tilde{R}$, where $\mathbf{v}_l = (\sqrt{2}\,\xi^{(l)}, 0)$ and $\tilde{R}$ has the law of ψ_0. Its moment matrix is $n\,\mathbf{v}_l \mathbf{v}_l^{\mathsf{T}} + \frac{1}{2}$. Its centred generators are $\sqrt{n}$ times linear forms in $\tilde{R}$ plus centred quadratic forms in $\tilde{R}$, so $\beta_s = O(n^{s/2})$. The thermal sites have $\mathbf{M} = \mathrm{diag}(\mathbf{Q}, \mathbf{Q})$, since $\mathbf{B}$ is real, and bounded β_s. So (M) holds, hence (F) by Remark 2.2, and the mean moment matrix tends to $\mathbf{M} = \mathrm{diag}(\mathbf{Q} + 2\,\Xi\Xi^{\mathsf{T}}, \mathbf{Q})$.

For $\mathbf{M} = \mathrm{diag}(\mathbf{A}_1, \mathbf{A}_2)$ one has $(\mathbf{M}\Omega)^2 = -\,\mathrm{diag}(\mathbf{A}_1\mathbf{A}_2, \mathbf{A}_2\mathbf{A}_1)$. So the squared symplectic eigenvalues of $\mathbf{M}$ are the eigenvalues of $(\mathbf{Q} + 2\,\Xi\Xi^{\mathsf{T}})\mathbf{Q}$, which is similar to $\mathbf{Q}^2 + 2\,\mathbf{Q}^{1/2}\Xi\Xi^{\mathsf{T}}\mathbf{Q}^{1/2}$, and Theorem 3.2(a) gives (a).

The group $O(n-k)$ acts only on the thermal sites, so the law of λ' in ρ_n is its law in $\tau_{\mathbf{B}}^{\otimes(n-k)}$. Theorem 3.2(a) on these $n - k$ sites, where $\mathbf{M} = \mathrm{diag}(\mathbf{Q}, \mathbf{Q})$, gives $\lambda'/(n-k) \to b$, hence (b).

The matrix $\mathbf{Q}^2 + 2\,\mathbf{Q}^{1/2}\Xi\Xi^{\mathsf{T}}\mathbf{Q}^{1/2}$ is a positive semidefinite perturbation of rank at most k of $\mathbf{Q}^2$, whose eigenvalues are the $(b_r + \frac{1}{2})^2$. The inequalities are Weyl's [14, Theorem 4.3.1]; for $k = 1$ this is the interlacing theorem [14, Corollary 4.3.9]. □

If the modes split into blocks, $\mathbf{B}$ is block diagonal for this splitting, and each $\xi^{(l)}$ is supported in its own block, then the matrix of Corollary 3.3 is block diagonal. In that case ν is the decreasing rearrangement of the values given by the case $k = 1$ on the blocks with a coherent site, together with the $b_r + \frac{1}{2}$ of the other blocks.

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