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Joint marked moments and optimized zero-free half-planes for unitary Hecke L-functions

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Abstract

For every number field K and continuous unitary Hecke character ω we prove $L_K(s,\omega) \neq 0$ for $\operatorname{Re} s > B_* = 0.8733458561067185768\ldots$, with the possible norm-character pole permitted. The same bound holds for finite-dimensional continuous unitary Weil representations, with pole orders equal to the multiplicities of their norm-character constituents. We prove joint inverse and ordinary moments in the paired character family. The second-correlation terms of the ordinary moment are grouped by their Kummer parameter before applying the moment estimate. A divisor-weighted bound for the groups supplies the original scalar induction at all stages. We determine B_* as an algebraic number of degree nine and as the infimum of the specified positive comparison, allowing unequal ordinary moments and variable comparison lengths.

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1 Introduction

We improve the uniform zero-free bound for Hecke L-functions over number fields. The proof combines joint moments for inverse and ordinary character sums with the unitary theta and moment estimates of [2, 3]. At the second correlation, individual divided frequencies can have lower parameter exponents than the first side. In Subsection 8.9 we estimate their combined moment by grouping labels at the same Kummer parameter. Appendix A proves the reference, packing and prime-refinement estimates used in this step. The resulting moments give the limiting comparison at B_*.

1.1 The main theorem

In this subsection, we state the zero-free theorems proved in Sections 8 and 9.

Theorem 1.1 (One-stage bound)**.** *Let K be a number field, χ a ray class character of K and $t \in \mathbb{R}$. Then*

$$L_K(s, \chi(N_K\cdot)^{it}) \neq 0 \qquad (\operatorname{Re} s > B_1' = 0.87336475),$$

with the simple pole at $s = 1 + it$ permitted when χ is principal. The same holds for $L_K(s, \rho \otimes (N_K\cdot)^{it})$, ρ an Artin representation, with pole order $\dim \rho^{\mathrm{Gal}(\bar{K}/K)}$ at $1+it$; in particular $\zeta_K(s) \neq 0$ for $\operatorname{Re} s > B_1'$.

This improves $B_2 = 0.8733819666\ldots$ of [5, Theorem 1.1]: $B_2 - B_1' > 1.72 \cdot 10^{-5}$. It is Theorem 8.31 and Corollary 8.32. Its proof uses the joint moment induction of Theorem 8.21 for one stage $191/250 \to 37/50$ starting from the initial estimates, with the line of that stage in the comparison in place of G; the limit of this route, about 0.87336473, is discussed in Remark 8.33.

Let

$$B_* = 0.873345856106718576801184796065072356770 4\ldots. \tag{1}$$

The exact value is defined by (42) and (44). It is an algebraic number of degree nine; its minimal polynomial is given in Section 7.

Theorem 1.2 (Main theorem)**.** *Let K be a number field and let ω be a continuous unitary Hecke character of K. Then*

$$L_K(s, \omega) \neq 0 \qquad (\operatorname{Re} s > B_*),$$

with the simple pole at $s = 1 + it$ permitted when $\omega = (N_K\cdot)^{it}$. The boundary B_* is independent of K and ω.

In particular, $\zeta_K(s) \neq 0$ for $\operatorname{Re} s > B_*$, with its pole at $s = 1$ permitted. The finite-order and norm-twist case is Theorem 8.29. Theorem 9.4 treats general unitary characters by restricting the detector to Hu's oriented family. Both use Theorem 8.21. The zero-free half-plane is open; constants and thresholds in the estimates may depend on the fixed field and character.

Corollary 9.6 gives the corresponding conclusions for every finite-dimensional continuous unitary representation r of W_K:

$$L_K(s, r) \neq 0 \quad (\operatorname{Re} s > B_*), \qquad \operatorname{ord}_{s=1+it} L_K(s, r) = -\dim \operatorname{Hom}_{W_K}((N_K\cdot)^{it}, r),$$

where the possible poles are permitted in the nonvanishing statement.

1.2 Earlier bounds and the joint moments

In this subsection, we compare the earlier bounds with the joint moment estimates of Sections 2 and 8.

For finite-order Hecke characters over arbitrary number fields, [5] gives the boundary

$$B_2 = \frac{777 - \sqrt{1081}}{852} = 0.873381966613207519929\ldots.$$

Our method starts from the marked moment induction and Dirichlet comparison of [8], which gives the rational value

$$B_{137} = \frac{2183367}{2500000} = 0.8733468.$$

We adapt this induction to ideal-character sums over number fields and extend it to the paired unitary families. Sections 5–7 refine the finite construction and determine the infimum of the resulting positive comparison.

For the fixed construction in Section 5, Theorem 5.4 gives the comparison infimum θ_{240}, where

$$\theta_{240} = 0.8733466968679481026286\ldots, \qquad B_* < \theta_{240} < B_{137} < B_2.$$

The limiting comparison in Section 7 then gives (1).

The resulting joint estimates treat inverse and ordinary sums simultaneously. For the normalized sums and character families of Section 2, each constructed pair (R, z_R) gives

$$\sum_{Nu \ll Z} |M_P(u)Q(u)|^2 \ll Z^{1+\varepsilon},$$

$$\sum_{Nu \ll Z}^{*} |S_{n_1,1}(u)S_{n_2,2}(u)Q(u)|^2 \ll Z^{1+\varepsilon}, \qquad n_1 + n_2 = P,$$

$$P < R, \qquad S < (1-R)/2 + z_R.$$

Here S is the total exponent of the prime lengths in Q, and the star denotes the ordinary inducing-character exclusions. For each $\varepsilon > 0$, the original prime factors are sufficiently short; the mesh is fixed before the varying characters and actual primes. Test-function seminorms and polynomial height factors are suppressed here, with the conventions of Section 2. The inverse estimate is used in zero detection; the ordinary estimate improves both zero detection and the averaged Mellin numerator.

Theorem 8.21 proves the inductive enlargement of these ranges, retaining the original prime coefficients through selection, Poisson summation and Euler restoration. Proposition 9.2 verifies these inputs for the paired unitary family.

1.3 The method and the comparison infimum

In this subsection, we describe the induction and the comparison used in Sections 6–9.

The induction begins with the moment estimates of [6, 5]. Given earlier joint moment bounds and a supporting line, put

$$H = r/R, \qquad u = (1-H)/6, \qquad Y(r) = Hz_R + (1/k-2)u, \qquad 2/3 < r < R \le 7/9, \quad 0 < k < 1/2.$$

Under the hypotheses of Theorem 8.21, including the strict inequalities of Subsection 8.7, the new bounds hold on

$$P < r - \delta_0, \qquad S < (1-r)/2 + Y(r) - \Delta - \delta_0, \qquad \delta_0, \Delta > 0.$$

The positive decrements are retained in each finite construction. Section 8 proves this step with arbitrarily small fixed positive margins. Proposition 9.2 gives the paired unitary version. The number of steps and the prime mesh are fixed before the asymptotic scale tends to infinity.

Section 6 determines an algebraic upper bound G for the descending induction and approximates it by finite rational constructions. The tangent to G at the optimizing point r_* determines B_*. The exact comparison inequalities in Subsection 7.1 give every fixed boundary $B > B_*$ sufficiently close to B_*. The Mellin argument then proves Theorem 1.2 in the open half-plane stated above.

Theorem 7.2 identifies B_* as the infimum for the comparison class of Definition 7.1. This class allows unequal ordinary moments, variable comparison lengths and repeated use of earlier moment estimates. The lower bound concerns these specified positive inequalities; it does not exclude stronger arithmetic estimates using additional cancellation.

We use the earlier theta, moment and continuation theorems of [6, 5], together with the unitary results of [2, 3]. Their normalizations and the estimates needed below are recalled in Sections 2 and 9. The new joint moment estimates and their compatibility with the paired unitary families are proved in Sections 8 and 9.

Here is the structure of the paper. Section 2 fixes the arithmetic notation and joint moment ranges. Section 3 describes the main steps of the proof, and Section 4 optimizes the zero-detection estimate. Section 5 gives a finite rational construction. Sections 6 and 7 determine the limiting comparison and its infimum. Section 8 proves the joint moments and the finite-order zero-free theorem. Subsection 8.13 gives the separate calculation from one stage of the induction. Section 9 treats continuous unitary characters, abelian descent and unitary Weil representations.

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2 Arithmetic notation and joint moment estimates

In this section, we fix the arithmetic notation, recall the earlier estimates, define the joint moment ranges and collect the comparison and unitary parameters. Section 3 then outlines the argument. Section 4 uses these ranges in zero detection, and Section 8 proves their inductive enlargement.

2.1 Arithmetic notation and earlier estimates

In this subsection, we fix the notation and recall the estimates used in Sections 5 and 8.

The letter K denotes an arbitrary number field. For the moment and theta arguments we work over a fixed field $F \supset \mu_{12}$. Let r_F be its number of complex places. Thus

$$\mu_m = \{z \in \mathbb{C} : z^m = 1\}, \qquad \zeta_m = e^{2\pi i/m}, \qquad n_F = [F : \mathbb{Q}] = 2r_F, \qquad F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{C}^{r_F}.$$

Write $\mathcal{O}_F$ for its ring of integers. Unless stated otherwise, a, b, d, g, n, p denote nonzero integral ideals, with p prime; fraktur letters are used interchangeably. Put

$$N\mathfrak{a} = |\mathcal{O}_F/\mathfrak{a}|, \qquad \mathrm{rad}(\mathfrak{a}) = \prod_{p|\mathfrak{a}} p, \qquad \tau(\mathfrak{a}) = \sum_{d|\mathfrak{a}} 1,$$

$$\mu_F(\mathfrak{a}) = \begin{cases} (-1)^j, & \mathfrak{a} = p_1 \cdots p_j,\ p_i \text{ distinct}, \\ 0, & p^2 \mid \mathfrak{a} \text{ for some } p. \end{cases}$$

The notation (a, b) means the ideal greatest common divisor; $(a, b) = 1$ means coprimality. The symbol 1_E is the indicator of E, and $g \sqcup a$ means the product of coprime squarefree ideals when their prime factors are partitioned. Arithmetic indices in allowed-generator presentations are taken modulo the unit relations of [6]; their displayed norms are those of the corresponding ideals.

Let $\mathbb{A}_F$ be the adele ring, $C_F = F^\times \backslash \mathbb{A}_F^\times$ the idele class group, and W_F the global Weil group. Hecke characters are continuous characters of C_F; they are unitary when their values have absolute value one. For a character χ, $\mathfrak{f}_\chi$ denotes its finite conductor. Its ideal values are extended by zero:

$$(\mathfrak{a}, \mathfrak{f}_\chi) \neq 1 \quad \Longrightarrow \quad \chi(\mathfrak{a}) = 0.$$

The fixed finite set S of excluded places contains the archimedean places and the fixed primes required by the theta, ray-class and Poisson constructions of [6]. It may be enlarged for a fixed coefficient character. Ramification of a varying character is retained through its zero extension.

We write $L_F(s,\chi)$ for the full Hecke L-function and $L_F^S(s,\chi)$ for the function with Euler factors at the finite places in S omitted; ζ_F^S has the same convention. The superscript $(\mathfrak{r})$ omits the primes dividing an ideal $\mathfrak{r}$. Our norm-character convention is

$$L_F(s,\chi(N_F\cdot)^{it}) = L_F(s-it,\chi), \qquad t\in\mathbb{R}.$$

Thus the pole of $(N_F\cdot)^{it}$ is at $1+it$. The order $\mathrm{ord}_{s=s_0}$ is positive at a zero and negative at a pole. We use the same norm character on W_F through class field theory. The dual $\widehat{\mathrm{Gal}(F/K)}$ denotes the character group in abelian base change.

Fix a finite group Θ of ray-class characters: the subgroup generated by the coefficient ray-class characters of the earlier moment estimates, the fixed finite characters arising in the transformations, and the unit characters

$$\Xi = \{n\mapsto \chi_n(\varsigma)^{\pm1} : \varsigma\in\mathcal{O}_F^\times\}. \tag{2}$$

Ξ is finite and unramified outside 6∞ (Lemma 8.12), so Θ is a fixed finite group from the outset; all moment estimates below, including the earlier-stage ones and the exceptional inducing-character gap of Proposition 8.20, whose exponents do not depend on the finite group, are used with this Θ. Over $F\supset\mu_{12}$, $\chi_a(b)$ denotes the zero-extended sextic residue character in the ideal-class and ray presentations of [6]; χ_u is the corresponding varying character. In imprimitive residue presentations, zeros at primes of the original modulus are retained after primitive induction. A star on an ordinary moment sum retains the inducing-character exclusions of that estimate. In the paired unitary presentation $\lambda_u=\omega_0\xi_u$, this exclusion is imposed on the finite-order factor ξ_u as in (108).

For the finite-order comparison, write

$$\beta_*(F) = \sup(\{1/2\}\cup\{\mathrm{Re}\,\rho : L_F(\rho,\eta)=0,\ 0<\mathrm{Re}\,\rho<1,\ \eta \text{ finite order}\}).$$

The fixed exponent is permitted since

$$13/18<\kappa_1=374887/500000<3/4, \qquad \beta_*(F)\le B_2<\frac{1+\kappa_1}{2}=0.874887;$$

see [6, Lemma 12.7 and Theorems 10.7 and 11.2] for the corresponding finite-order prime and inverse estimates. We use reciprocity, the theta construction and radial reflection from [6, Sections 3–5], and Poisson summation and the transformed inverse families from [loc. cit., Sections 8 and 10–11]. The initial moments and common-divisor estimates are those of [loc. cit., Sections 10–12] and [5].

Zero detection, the Mellin comparison and continuation are treated in [6, Sections 13–14]. Their general-field formulations are [loc. cit., Sections 17–19]. The joint moment estimates are proved in Section 8; their paired unitary versions are proved in Section 9, using [2, 3].

2.2 Length exponents and test functions

In this subsection, we fix the scale and uniformity conventions used in Subsection 2.3 and in the induction of Section 8.

The asymptotic scale is $Z\ge2$. An exponent x denotes ideal norm length Z^x. For the original prime factors and their product, write

$$Np\asymp Z^{z_i}\quad(p\in\mathcal{P}_i), \qquad Q=\prod_i Q_i, \qquad S=\sum_i z_i, \qquad 0<z_i\le\eta.$$

Here $\eta > 0$ is the prime mesh. The sets $\mathcal{P}_i$ may overlap; the original coefficients $b_i(p)$ are fixed before amplitude selection. A complete-factor selection retains or removes an entire Q_i. For a finite set I of original positions, the injective mark is

$$M_I(n) = \sum_{\substack{(p_i)_{i\in I} \text{ distinct} \\ p_i \mid n \ (i\in I)}} \prod_{i\in I} b_i(p_i), \qquad M_\varnothing(n) = 1, \tag{3}$$

where b_i is extended by zero outside $\mathcal{P}_i$.

Definition 2.1 (The Θ-character class)**.** Let $j_{\max} \in \mathbb{N}$ and let Ψ be the completely multiplicative coefficient of Subsection 8.4. The original coefficients $(\mathcal{P}_i, b_i)_{i\in I_0}$, together with Ψ, lie in the Θ-*character class* (with exponent bound $j_{\max}$) if $\#I_0$, the widths z_i, the exponents j_i, the coefficients $c_{i,\theta}$ and Ψ are fixed before Z and are independent of the summation parameter, and

$$\begin{gathered} \mathcal{P}_i = \{p \notin S : \ C_0^{-1} Z^{z_i} \le Np \le C_0 Z^{z_i}\}, \qquad b_i(p) = \omega_0(p)^{-j_i} \sum_{\theta\in\Theta} c_{i,\theta}\, \theta(p) \quad (p \in \mathcal{P}_i), \\ j_i \in \mathbb{Z}, \quad |j_i| \le j_{\max}, \qquad \sum_{\theta\in\Theta} |c_{i,\theta}| \le 1, \end{gathered} \tag{4}$$

$$\Psi \in \mathcal{F}_\omega \quad (\text{eq. } (10)), \qquad \omega = 1 \text{ for finite-order presentations.} \tag{5}$$

The displayed sets $\mathcal{P}_i$ are the complete parent windows. We also use their fixed disjoint refinements from Lemmas A.5 and A.6, retaining the smooth parent profiles. For finite-order presentations $\omega_0 = 1$ and $j_i = 0$. In the canonical paired family the fixed factor is in $\Theta \cup \psi\Theta$, with $\psi \in \mathcal{F}_\omega$ as in (111). Each live mark has relative coefficient ω_0^{-1} times its finite coefficient, where $\omega_0 \in \{1, \psi\}$ is its block anchor. Conjugation changes both the relative coefficient and the block anchor together. The exponent bound $j_{\max}$ allows the formal coefficient powers in the decomposition. A repeated prime consumed by a collision contributes a scalar (Lemma 2.2(i)). Ray-class restrictions are expanded in Θ. Thus $|b_i(p)| \le 1$, and each component twists the summation character by $\omega_0^{-j_i}\theta$, $\theta \in \Theta$; this is the coefficient class of [6, Lemma 12.7 and Subsection 13.4].

For a dual parameter y and $\Psi' \in \Theta\Psi$ put $\psi_y(n) = \Psi'(n)\overline{\chi_n(y)}$. By Artin reciprocity for the Kummer extension $F(y^{1/6})/F$, $n \mapsto \chi_n(y)$ is a finite-order Hecke character of conductor norm $\ll_F Ny$. Hence, by (5),

$$\psi_y \in \mathcal{F}_\omega, \qquad \omega_0^j \theta \psi_y^{\pm 1} \text{ unitary Hecke}, \quad N\mathfrak{f}_{\omega_0^j\theta\psi_y^{\pm1}} \ll_{F,\Theta,\omega_0,\Psi} Ny \quad (\theta \in \Theta,\ |j| \le j_{\max}). \tag{6}$$

Let n_0 be the number of collision and conjugation operations performed along the finite induction; n_0 is fixed with the induction, before Z. Theorem 8.21 and Proposition 9.2 use the finite-order or synchronized paired live coefficients. Inside the induction the formal powers in Lemma 2.2 are covered by

$$j_{\max} = 2^{n_0}. \tag{7}$$

Lemma 2.2 (Operations preserving the class)**.** *Let $(\mathcal{P}_i, b_i)$ and Ψ satisfy (4) and (5) with exponent bound J. Then:*

(i) (collisions) for $i \ne j$ and $Z > C_0^{2/|z_i - z_j|}$, $\mathcal{P}_i \cap \mathcal{P}_j = \varnothing$ if $z_i \ne z_j$. If $z_i = z_j$, the complete parent windows coincide and we use the intersection of the actual lists.

$$b_i b_j = \omega_0^{-(j_i+j_j)} \sum_{\theta,\theta'} c_{i,\theta} c_{j,\theta'}\, \theta\theta', \qquad \theta\theta' \in \Theta, \quad |j_i + j_j| \le 2J, \quad \sum |c_{i,\theta} c_{j,\theta'}| \le 1,$$

so the coefficient lies in the class with bound $2J$. The collision prime enters as

$$Z^{-z_i} \sum_{p \in \mathcal{P}_i \cap \mathcal{P}_j} b_i(p) b_j(p) \lambda_u(p)^2 W_i W_j(Np/Z^{z_i}) \ll 1,$$

which is not a factor of the form Q_k; it is a collision component of the decomposition in Subsection 2.3;

(ii) (complete-factor selection) a selection retains or removes whole factors Q_i (Proposition 8.8) and keeps Ψ, so the retained family is a subfamily of $(\mathcal{P}_i, b_i)$ with the same Ψ;

(iii) (second Poisson summation) the retained marks in (53) carry the original b_i; the factor $\Psi'(p)\overline{\chi_p(y)}$ is part of the summation character ψ_y and does not change b_i;

(iv) (conjugation) $\overline{b_i} = \omega_0^{j_i} \sum_\theta \overline{c_{i,\theta}}\, \bar{\theta}$ with $\bar{\theta} \in \Theta$, that is $j_i \mapsto -j_i$, and $\bar{\Psi} \in \mathcal{F}_\omega$.

Excluded-ideal restoration (8), (56) changes only scalar coefficients. Consequently, a coefficient produced from top-level coefficients by $n \le n_0$ operations (i)–(iv) lies in the class with bound $2^n \le j_{\max}$.

Proof. For $z_i > z_j$, $C_0 Z^{z_j} < C_0^{-1} Z^{z_i}$ once $Z^{z_i - z_j} > C_0^2$. Θ is a group, ω_0 is unitary and $\mathcal{F}_\omega$ is closed under conjugation and under multiplication by Θ; this gives (i) and (iv). The collision sum is bounded by $Z^{-z_i} \#\{p : Np \le C_0 Z^{z_i}\} \|W_i W_j\|_\infty \ll 1$. (ii) is the definition of complete-factor selection; (iii) is the form of the earlier bound in Subsection 8.4, where $M_{I \setminus J}(n)$ is built from the original b_i by (3). Each operation (i) at most doubles the exponent bound and (ii)–(iv) do not increase it; induction on n gives the last assertion. □

Norm test functions W, W_i lie in fixed smooth families, usually $C_c^\infty((0,\infty))$; Schwartz functions occur in Poisson summation. We use

$$\widehat{W}(s) = \int_0^\infty W(x) x^s \frac{dx}{x}, \qquad e(t) = e^{2\pi i t}, \qquad \widehat{f}(y) = \int_{F_\infty} f(x) e(-\operatorname{Tr}(xy))\, dx,$$

with self-dual additive measure. At a complex place $|z|_v = |z|^2$. A fractional-ideal lattice $L \subset F_\infty$ has trace-dual L^* and additive covolume $\operatorname{covol}(L)$. The notation $\mathcal{S}(F_\infty)$ denotes the Schwartz space.

The symbols $p_J(W)$ and $\mathfrak{S}_J$ denote finite homogeneous seminorms of the specified test functions; H_{in} denotes their incoming norm-twist heights. For these prescribed families, an abbreviated moment estimate suppresses factors of the form

$$\mathfrak{S}_J(\mathcal{W}, \mathbf{W})(1 + H_{\text{in}})^{J'}, \qquad J, J' < \infty,$$

where $\mathcal{W} \ge 0$ is a common parameter weight. Implied constants may depend on F, the fixed excluded places and coefficient characters, the test supports, and the requested $\varepsilon > 0$.

Fix the field, the numerical length margins and the finite induction before choosing the prime mesh. The mesh is then uniform in all later data: for every $\varepsilon > 0$ there is $\eta_0 = \eta_0(F, \varepsilon, \text{margins}) > 0$ such that for all $0 < \eta \le \eta_0$, all supports of the test functions W, W_j, W_i of Subsection 2.3 in a compact subset of $(0,\infty)$, all character presentations λ_u, all constants C_0 in

$$C_0^{-1} Z^{z_i} \le Np \le C_0 Z^{z_i} \qquad (p \in \mathcal{P}_i),$$

and all prime sets and coefficients $(\mathcal{P}_i, b_i)$ in the Θ-character class (4), there are J, J', C, Z_0 for which the moment bounds hold for $Z \ge Z_0$. For each fixed character presentation, common Fourier and Mellin measures are chosen before the summation and divisor indices; seminorm orders, polynomial height degrees and thresholds may depend on that presentation.

2.3 The joint moment ranges

In this subsection, we define the two moment bounds enlarged in Theorem 8.21 and used in the zero-detection estimate of Section 4.

With $Nu \ll Z$, the joint moment bounds at (R, z_R) have polynomial length exponent P and total prime length exponent S in

$$P < R, \qquad S < \frac{1-R}{2} + z_R.$$

For the inverse sum, P is its length exponent; for the ordinary product, P is the sum of the two length exponents. For one common character presentation λ_u, extended by zero at ramified ideals, set

$$M_r(u) = Z^{-r/2} \sum_n \mu_F(n)\lambda_u(n) W(Nn/Z^r), \qquad S_{n,j}(u) = Z^{-n/2} \sum_a \lambda_u(a) W_j(Na/Z^n),$$

$$Q_i(u) = Z^{-z_i/2} \sum_{p \in \mathcal{P}_i} b_i(p)\lambda_u(p) W_i(Np/Z^{z_i}), \qquad Q = \prod_i Q_i, \qquad S = \sum_i z_i.$$

The joint moment bounds at (R, z_R) are

$$\sum_{Nu \ll Z} |M_P(u)Q(u)|^2 \ll Z^{1+\varepsilon}, \qquad \sum_{Nu \ll Z}^{*} |S_{n_1,1}(u) S_{n_2,2}(u) Q(u)|^2 \ll Z^{1+\varepsilon}, \quad n_1 + n_2 = P,$$

with the ordinary inducing-character exclusions of the earlier theorem, finitely many test-function seminorms and polynomial height losses understood. Inverse positions have the original coefficients in the Θ-character class (Definition 2.1), independent of the summation parameter, and the original sets of primes. Ordinary positions are the original complete smooth prime ranges with fixed finite ray-character coefficients; in the unitary case they have the paired relative coefficient (107). Ray-class restrictions are expanded as finite linear combinations of characters. The original prime sets may overlap; their collision decomposition retains whole original factors and natural zeros before creating injective marked components.

These bounds hold uniformly with an excluded ideal. For an integral ideal $\mathfrak{r}$ let $M_P^{(\mathfrak{r})}$ and $S_{n,j}^{(\mathfrak{r})}$ denote the sums above restricted to $(n, \mathfrak{r}) = 1$, respectively $(a, \mathfrak{r}) = 1$. Since λ_u is completely multiplicative, Lemma 8.7 and the Möbius expansion of $1_{(a,\mathfrak{r})=1}$ give

$$M_P^{(\mathfrak{r})}(u) = \sum_{\mathrm{rad}(d) \mid \mathfrak{r}} \frac{\lambda_u(d)}{\sqrt{Nd}} M_{P - \log_Z Nd}(u), \qquad S_{n,j}^{(\mathfrak{r})}(u) = \sum_{d \mid \mathfrak{r}} \frac{\mu_F(d)\lambda_u(d)}{\sqrt{Nd}} S_{n - \log_Z Nd, j}(u),$$
$$\sum_{\mathrm{rad}(d) \mid \mathfrak{r}} \frac{1}{\sqrt{Nd}} + \sum_{d \mid \mathfrak{r}} \frac{1}{\sqrt{Nd}} \ll_{F,\varepsilon} (N\mathfrak{r})^{\varepsilon}. \tag{8}$$

Every length on the right is at most the original one, and terms with Nd beyond the support of W vanish. Cauchy's inequality with the weights $(Nd)^{-1/2}$ gives

$$\sum_{Nu \ll Z} |M_P^{(\mathfrak{r})}(u)Q(u)|^2 \ll (N\mathfrak{r})^{2\varepsilon} \max_{P' \le P} \sum_{Nu \ll Z} |M_{P'}(u)Q(u)|^2 \ll (N\mathfrak{r})^{2\varepsilon} Z^{1+\varepsilon},$$

and the same bound for the ordinary product, with $(N\mathfrak{r})^{4\varepsilon}$ and $n_1' + n_2' \le P$. Hence the joint bounds at (R, z_R) hold for every $\mathfrak{r}$ with

$$N\mathfrak{r} \le Z^{\varrho_0}, \qquad P < R, \qquad S < \frac{1-R}{2} + z_R,$$

for any fixed ϱ_0; the loss $Z^{4\varrho_0\varepsilon}$ is absorbed by decreasing ε. No condition coupling $\log_Z N\mathfrak{r}$ with P or S is needed. Here Q is unrestricted; when marked prime factors lie inside the restricted variable, the coprimality of the marked primes to $\mathfrak{r}$ is removed by (56), with the same loss $(N\mathfrak{r})^{\varepsilon}$. The excluded ideals below are those of the first canonical reduction (Subsection 8.5, [6, proof of Theorem 11.2]), of the coprimality conditions in Proposition 8.6 and of the small cores in Subsection 8.6. Each divides a product of a fixed ideal and of the variables of one block on the bounded exponent box $N' + V \leq 2$, $K' \leq 4$ of Subsection 8.4; hence $N\mathfrak{r} \leq Z^{\varrho_0}$ with ϱ_0 depending only on that box.

No arbitrary new bounded ordinary prime coefficient is introduced. The Fourier and Mellin measures are fixed before the arithmetic indices.

The induction gives the recursion

$$H = \frac{r}{R}, \qquad u = \frac{1-H}{6}, \qquad Y(r) = Hz_R + \left(\frac{1}{k} - 2\right) u. \tag{9}$$

For $2/3 < r < R$ and $0 < k < 1/2$, we have $Y(r) > Hz_R$. Theorem 8.21 proves the two new moment estimates under the strict induction inequalities, using selection, Poisson summation and restoration of Euler factors.

2.4 Comparison and transformation parameters

In this subsection, we collect the parameters used in Sections 4–9. Their precise ranges and hypotheses are stated with the corresponding results.

Notation	Meaning
P, S	Polynomial length exponent and total prime length exponent. For an ordinary product, $P = n_1 + n_2$.
(R, z_R)	Joint moment parameters, giving $P < R$ and $S < (1-R)/2 + z_R$.
$H, u, Y(r)$	Relative parameter scale $H = r/R$, selected length $u = (1-H)/6$, and improvement in (9).
k, ν, e	Prime-amplitude cutoff, zero slope and prime-product slope in Subsection 8.6; there $k - 16e_b \leq \nu \leq 2B_2 - 1$ and $e \leq \nu + 16e_b$ (65).
$g(r), \lambda(w)$	Affine bounds for earlier moment improvements, written $g_0 - g_1 r$ and $a_0 - a_1 w$.
R_o, R_n, R_g	Count exponents, with positive denominators J_o, J_n, J_g, in Subsection 5.2. The notation $R(\delta, x)$ denotes their permitted minimum.
B_C, B_P, L_C, L_P	Moment parameters for the count and numerator comparisons, specified in Section 5.
δ, x	Amplitude parameters of the comparison in Section 5; $V = 5/6 - \delta$ in zero detection.
θ, B, h	Proposed zero-free boundary and outer parameter length; $Nu \ll Z^h$. The letters θ and B are used in the fixed and limiting comparisons, respectively.
l_x, l_y, ℓ	Comparison length exponents and total exponent of the original prime factors in the Mellin comparison; see (14).
B_2, B_{137}	Earlier comparison values recalled in the introduction.

Notation	Meaning
θ_{240}, B_*	Fixed-comparison and full positive-comparison infima, defined by (35) and (44).
$G, \widehat{S}, \mathcal{G}$	Limiting improvement, upper bound for all earlier prime lengths, and convex improvement majorant; see (36), (37) and (41).
$\mathcal{C}(P), C(P)$	The initial prime-length bound $(1-P)/2$. These functions are distinct from the Gauss coefficient $C(n)$.
$M_{\mathrm{ref}}, \nu_{\mathrm{ref}}$	Reference total-length bound and its cutoff in Proposition 6.3. The radii R_c, R_+ delimit the cases of $\widehat{S}$.
L_0, L_K	Direct comparison bounds in Subsection 6.2; the additional numerator bounds are defined in Subsection 7.3.
r_*, δ_*, h_*	Optimizing point and limiting parameters in (42) and (45).
m_D, c_*, R_*	Lower-bound parameters $m_D = r_*/2$, $c_* = m_D + G(r_*)$ and $R_* = 1 - \delta_*(1/2 + c_*)$.
N, V, M, z	In a transformed inverse sum: total polynomial, auxiliary ideal, parameter and remaining prime length exponents. The free exponent is $N - z$ and $F_0 = N + V$.
N', K', H_c	Exponents after selection and the second Poisson summation: $N' = N - u'$, $K' = M + 4u'$ and $H_c = 2(N' + V) - K' + \kappa_{\mathrm{P}}$.
$C(n), \Lambda, \gamma_2$	Normalized cubic Gauss coefficient and its presentation $C = \bar{\Lambda}\gamma_2$, used in Sections 8 and 9.
$p(P, s, M)$	Excess exponent defined in Subsection 8.7 for the sixth-power summation and ordinary induction.
a	Lower integration limit in Subsection 8.10, where $1/2 < a \leq 51/100$.
β, Δ, γ	Locally specified positive endpoint decrements and exponent reductions. The zero supremum β in Section 9 is separately defined below.

The symbol S denotes a set when it specifies excluded places and a real exponent in a length inequality. Likewise, u and k denote ideal parameters, in their fixed presentations, in Nu, Nk and real parameters in the induction formulas. Letters such as M, R, V, C are reused locally with the displayed definitions above; all auxiliary coefficients and margins are defined where they enter the argument. In the continuation argument, η denotes an exponent reduction rather than the prime mesh, and H denotes the external height cutoff rather than r/R. The polynomial $Q(\theta, \delta, x)$ in (34) is distinct from the prime product Q.

2.5 Unitary families and angular parameters

In this subsection, we fix the unitary-family notation used in Section 3 and in the continuation argument of Section 9.

For a fixed continuous unitary Hecke character ω of F, put

$$\mathcal{F}_\omega = \{\theta\omega^j(N_F\cdot)^{it} : j \in \{-1, 0, 1\},\ \theta \text{ finite order},\ t \in \mathbb{R}\}, \tag{10}$$

$$\beta_\omega = \sup\big(\{1/2\} \cup \{\operatorname{Re}\rho : L_F(\rho, \vartheta) = 0,\ \vartheta \in \mathcal{F}_\omega,\ 0 < \operatorname{Re}\rho < 1\}\big). \tag{11}$$

In the unitary contradiction argument, $\beta = \beta_\omega$. The finite-order supremum $\beta_*(F)$ above is distinct. The paired inverse class $\mathcal{X}_u$ is defined in (110); its fixed factors are $\xi \in \Theta \cup \psi\Theta$. For ordinary factors, $\lambda_u = \omega_0 \xi_u$ has ξ_u finite order and paired relative coefficients given by (107).

At a complex place, write

$$\psi_v(z) = (z/|z|)^{k_v} |z|_v^{i\tau_v}, \qquad k_v \in \mathbb{Z}, \quad \tau_v \in \mathbb{R}.$$

The fixed theta character has angular vector $\boldsymbol{\lambda} \in \{1, -1\}^{r_F}$; $\boldsymbol{k} = (k_v)_v$ is the angular vector of the target. The notation CT denotes a theta constant term and w in wv the Weyl action on a test vector. The multiplier $\mathcal{K}_{m,\tau}$ of (109) uses angular mode m and radial unitary parameter τ.

3 Outline of the proof

In this section, we describe the moment induction, the limiting comparison and the passage to unitary characters used to prove Theorem 1.2. The analytic estimates are proved in Sections 8 and 9. We first work over a fixed number field $F \supset \mu_{12}$.

Joint inverse and ordinary moments. The two estimates of Section 2 are proved together. Given their validity at (R, z_R), an earlier supporting line and the strict inequalities of Subsection 8.7, put

$$H = r/R, \qquad u = (1-H)/6, \qquad Y(r) = Hz_R + (1/k-2)u, \qquad 2/3 < r < R \le 7/9, \quad 0 < k < 1/2.$$

For these parameters, we have $Y(r) > Hz_R$. Theorem 8.21 gives both moments for

$$P < r - \delta_0, \qquad S < (1-r)/2 + Y(r) - \Delta - \delta_0, \qquad \delta_0, \Delta > 0.$$

The positive decrements are fixed at each step. Proposition 8.20 supplies the ordinary terminal at the packed parameter radius. Proposition 9.2 verifies closure of the paired unitary family.

Selection and Poisson summation. The first inverse transformation gives $M \le 2(N+V) - 1$. Enlarging the positive parameter sum allows $M = 2(N+V) - 1 + O(\eta)$. Here N, V, M are the exponents of the polynomial length, auxiliary ideal length and parameter range. Selecting complete prime factors of total length $u' = u + O(\eta)$ gives

$$\begin{aligned} N' &= N - u', & K' &= M + 4u', \\ H_0 &= 2(N'+V) - K' = 1 - 6u' + O(\eta), & H_c &= H_0 + \kappa_{\mathrm{P}}, \quad \kappa_{\mathrm{P}} > 0. \end{aligned}$$

The fourth power comes from sextic reciprocity on coprime squarefree ideals:

$$C(dn) = C(d)C(n)\chi_n(d)^4 \omega(d,n), \qquad (d,n) = 1,$$

where C is the normalized cubic Gauss coefficient and ω is a finite linear combination of ray-class characters.

The second Poisson summation converts the cubic Gauss coefficients to Möbius coefficients. Coprimality expansion and (56) retain the remaining prime coefficients. The normalization satisfies

$$(K' - N' - V) + (H_c + u') = N + V + \kappa_{\mathrm{P}}.$$

For a fixed reduction $\gamma > 0$ and requested loss $\varepsilon > 0$, choose the auxiliary parameters so that

$$\kappa_{\mathrm{P}} + C\eta + 2\varepsilon_1 < \min\{\gamma/4, \varepsilon\}, \qquad \kappa_{\mathrm{P}}, \eta, \varepsilon_1 > 0,$$

as in (58). If the prime length is insufficient for the selection, the initial inverse estimate applies.

For the ordinary moment we group the second-correlation frequencies at $y = V_C j$. The label multiplicities at each y have subpower total weight. This gives a single bound at the first-side cap $M_C = 2a - k_C + q_C \geq H - O(\eta)$. The two sides can then be estimated with the previous-stage moments and the detector at their own respective packed caps (Proposition 8.20).

The limiting comparison. For descending sequences beginning with $z_0 = 0$ and $2/3 < R_0 < 7/9$, Proposition 6.2 gives the upper bound

$$G(R) = \frac{11R}{16} - \frac{7}{24} - \frac{5}{16}\{7R^4(R - 2/3)\}^{1/5}, \qquad 2/3 < R \leq 7/9.$$

It satisfies

$$G(7/9) = 0, \qquad G'(R) = \frac{(45R - 24)G(R) + 9R - 7}{R(45R - 30)}, \qquad G''(R) > 0.$$

Proposition 6.3 allows earlier estimates to be used in any order. Its bound $\widehat{S}$ agrees with $(1 - R)/2 + G(R)$ for $R_+ \leq R \leq 7/9$, and

$$R_+ < 367/500 < r_* < 7/9.$$

Proposition 6.4 approximates G by finite rational sequences on every compact subinterval of $(367/500, 7/9)$. Subsection 7.1 supplies the short extension below $367/500$ by truncating the last moment range.

For an affine bound $g(R) = g_0 - g_1 R$ valid for both moment classes, the reductions in the zero-detection exponents, on their respective moment ranges, are

$$\begin{aligned} &\delta(A + Kr), \qquad \delta(A + 2Km), \\ A = x(1 + 2g_0), \qquad &K = 1 - x(1 + 2g_1). \end{aligned}$$

They agree at $r = 2m = 2t/3$, where $r + m = t$, provided both moment ranges permit this split. For $\delta > 0$, $K > 0$ and $V = 5/6 - \delta > 0$, Proposition 4.1 minimizes

$$\max\{1 - \delta(A + 2Kt/3),\ 1/6 + Vt\}, \qquad a \leq t \leq b,$$

at

$$t = \min\left\{b, \max\left\{a, \frac{5/6 - \delta A}{V + 2\delta K/3}\right\}\right\}.$$

The tangent at r_* gives

$$B_* = \frac{99r_* - 14}{12(9r_* - 1)}.$$

Subsection 7.1 proves the full comparison for each fixed nearby $B > B_*$. Subsections 7.2 and 7.3 prove the matching lower bound within Definition 7.1.

The unconditional stage. One stage from $(R_0, z_0) = (191/250, 0)$ to $R_1 = 37/50$, with $k = 460859/10^6$, gives

$$Y(r) = c\Big(1 - \frac{r}{R_0}\Big), \qquad c = \frac{1/k - 2}{6} = \frac{13047}{460859},$$

from (I1)–(I4). The comparison with $g = Y - 10^{-9}$ on $[37/50, 7/9]$, using the initial estimates beyond R_0, holds at $B_1' = 0.87336475$ with strict margins (Theorem 8.31).

Unitary characters and Mellin continuation. For a fixed unitary twist ω_0, write $\lambda_u = \omega_0 \xi_u$, where ξ_u is finite order. Include the ramified places of ω_0 in the fixed set S. For $p \notin S$, the paired coefficients satisfy

$$b_i(p) = \omega_0(p)^{-1} \sum_j c_{ij} \vartheta_{ij}(p), \qquad b_i(p)\lambda_u(p) = \xi_u(p) \sum_j c_{ij} \vartheta_{ij}(p).$$

The characters ϑ_{ij} belong to the fixed finite ray-class family. Thus the prime estimates still concern finite-order characters. Up to coefficient two the mixed ordinary moment (108) of [2] uses $(\omega_1, \omega_2) = (\omega_0, \omega_0)$ for detection and $(1, 1)$ for the finite-order numerator. Larger marked widths use the preceding joint rectangles. Subsection 9.2 checks the character orientations, vanishing theta constant terms and transformations needed for the new moments.

Fix ω and let

$$\mathcal{F}_\omega = \{\theta \omega^j (N\cdot)^{it} : j \in \{-1, 0, 1\},\ \theta \text{ finite order},\ t \in \mathbb{R}\}.$$

Let β be the supremum of its nontrivial zero real parts, with $1/2$ adjoined. The theorem of [2] gives $\beta \le B_2$. If $\beta > B_*$, choose $B_* < B' < B < \beta$, with B' sufficiently close to B_*. The finite construction at B' and the Mellin comparison give an $\eta > 0$, independent of $\psi \in \mathcal{F}_\omega$, such that

$$e^{(s-5/6)^2} \frac{H_\psi^S(s)}{L_F^S(s, \psi)} \quad \text{is holomorphic for } \operatorname{Re} s > \sigma_0, \qquad \sigma_0 = \max\{B, \beta - \eta\} < \beta.$$

The numerator is nonzero there. The omitted finite Euler factors have no zero or pole in $\operatorname{Re} s > 0$. Consequently

$$L_F(s, \psi) \ne 0 \ (\operatorname{Re} s > \sigma_0) \quad \text{for every } \psi \in \mathcal{F}_\omega \quad \Longrightarrow \quad \beta \le \sigma_0 < \beta.$$

This contradiction proves the unitary bound over F with the possible norm-character poles permitted. Constants and thresholds may depend on ψ; the exponent reduction η is common to the family.

Descent and Weil representations. For an arbitrary number field K, take $F = K(\zeta_{12})$. Theorem 9.5 applies abelian base change with all ramified local factors retained. For a finite-dimensional continuous unitary representation r of W_K, unitary induction gives finite extensions E_j/K and unitary Hecke characters ω_j such that

$$[r] = \sum_j n_j [\operatorname{Ind}_{W_{E_j}}^{W_K} \omega_j], \qquad L_K(s, r) = \prod_j L_{E_j}(s, \omega_j)^{n_j}, \qquad n_j \in \mathbb{Z}.$$

Each Hecke factor is zero-free away from its norm-character pole in $\operatorname{Re} s > B_*$. Frobenius reciprocity gives

$$\operatorname{ord}_{s=1+it} L_K(s, r) = -\dim \operatorname{Hom}_{W_K}((N_K\cdot)^{it}, r) \le 0.$$

This also excludes zeros at the possible poles and proves Corollary 9.6. For an Artin representation, Brauer induction gives ω_j of finite order.

4 Zero detection from an affine moment bound

In this section, we optimize the zero-detection estimate supplied by the joint moments of Section 2. The resulting count is used in the comparisons of Sections 5 and 7.

Let $g(r) = g_0 - g_1 r$ and set

$$A = x(1 + 2g_0), \qquad K = 1 - x(1 + 2g_1), \qquad V = \frac{5}{6} - \delta, \qquad D = V + \frac{2\delta K}{3}.$$

The inverse and ordinary detector savings at widths r and m are respectively

$$\delta(A + Kr), \qquad \delta(A + 2Km).$$

At the common split $r + m = t$ they agree at $r = 2m = 2t/3$.

Proposition 4.1 (Exact two-branch minimax)**.** *Suppose $\delta > 0$, $K > 0$, $V > 0$, and $a \le b$. Put*

$$f(t) = 1 - \delta\left(A + \frac{2K}{3}t\right), \qquad q(t) = \frac{1}{6} + Vt, \qquad t_* = \frac{5/6 - \delta A}{D}.$$

Then

$$\operatorname*{argmin}_{a \le t \le b} \max\{f(t), q(t)\} = \{\min(b, \max(a, t_*))\}.$$

If $a \le t_* \le b$, the minimum equals

$$C_* = \frac{1}{6} + V\frac{5/6 - \delta A}{D}. \tag{12}$$

Proof. $f' = -2\delta K/3 < 0$ and $q' = V > 0$. Their unique crossing is t_*. To its left the maximum is f; to its right it is q. □

Corollary 4.2 (Sensitivity at an interior crossing)**.** *Under the hypotheses of Proposition 4.1, with δ, x fixed and $a < t_* < b$,*

$$\frac{\partial C_*}{\partial g_0} = -\frac{2\delta x V}{D}, \qquad \frac{\partial C_*}{\partial g_1} = \frac{4\delta x V t_*}{3D}.$$

Thus $\partial C_*/\partial g_0 < 0$ when $x > 0$. These derivatives hold on the open region where the moment bounds and the zero-detection inequalities used in Proposition 4.1 hold.

Proof. Differentiate (12), using $\partial_{g_0} A = 2x$ and $\partial_{g_1} K = -2x$. □

Lemma 4.3 (Detector savings)**.** *Let $g(R) = g_0 - g_1 R$ with $g_1 > -1/2$, and suppose that joint moments hold for every $R \in [a_-, b]$ with prime width $z(R) = (1 - R)/2 + g(R)$, decreased by an arbitrarily small amount. Let $U = Z^d$, let ℓ be the comparison length of (14), and fix a buffered class $\delta = 2a - 1$ with prime-amplitude ratio x. Assume*

$$\text{(S1) } 1 \le t \le \tfrac{3}{2}, \qquad \text{(S2) } a_- \le \tfrac{2t}{3}, \qquad \text{(S3) } \tfrac{1 - a_-}{2} + g(a_-) < \frac{\ell}{d}. \tag{13}$$

Then, for $a_- \le r \le b$ and $a_-/2 \le m \le b/2$, the inverse and ordinary detectors save $\delta(A + Kr)$ and $\delta(A + 2Km)$, and the class has count $\ll U^{\max\{f(t), q(t)\} + \varepsilon}$ in the notation of Proposition 4.1.

Proof. By [6, Lemma 13.7, (13.14)] every nonfloor class has a pair with $r \le t$, $r+m \ge t$, $r \ge t-1/2$, $m \le 1/2$ and $|M_r|^2 \ge U^{\delta r-\varepsilon}$, $|S_m|^2 \ge U^{\delta m-\varepsilon}$. Selecting original primes in decreasing amplitude order gives $|Q|^2 \ge U^{2\delta xz-o(1)}$ for every $z \le \ell/d$ [5, after (6.6)]. Since $z'(R) = -1/2 - g_1 < 0$, (S3) gives $z(R) < \ell/d$ for $R \ge a_-$. Chebyshev's inequality with $\sum |M_rQ|^2 \ll U^{1+\varepsilon}$ and $\sum |S_m^2Q|^2 \ll U^{1+\varepsilon}$ (joint moment at $R = 2m$; [5, (6.17)]) gives the savings $\delta(r + 2xz(r))$ and $\delta(2m + 2xz(2m))$, where

$$r + 2xz(r) = r + x(1 - r) + 2x(g_0 - g_1 r) = A + Kr,$$
$$2m + 2xz(2m) = 2m + x(1 - 2m) + 2x(g_0 - 2g_1 m) = A + 2Km.$$

If $2t/3 \le r \le b$, the inverse saving is at least $\delta(A+2Kt/3)$, by $K > 0$. If $r < 2t/3$ and $2m \le b$, then $m \ge t - r > t/3$ and $2m > 2t/3 \ge a_-$ by (S2), so the ordinary saving is at least the same. Pairs with $r > b$ or $2m > b$ are covered by the range conditions (I6)–(I9) of Theorem 5.2. For $r \ge 1$, [6, Lemma 10.10] gives $e(r) - \delta r = 1/6 + Vr \le q(t)$ [5, proof of Lemma 6.2]. The split $r = 2m = 2t/3$ satisfies $r \ge t - 1/2$ and $m \le 1/2$ precisely when $t \le 3/2$; this is (S1). □

5 A finite rational construction and its comparison

In this section, we give a finite rational moment construction and determine the infimum of its comparison inequalities. The construction uses Section 4; Sections 6 and 7 then allow the moment ranges and the comparison lengths to vary.[1]

Write θ for a proposed boundary and use x for the amplitude ratio. The comparison lengths are denoted by l_x, l_y:

$$\ell = \frac{11}{3} - 4\theta, \quad l_x = 1 + \ell - h, \quad l_y = h - 2\ell, \quad \kappa = 2\theta - 1. \tag{14}$$

Subsection 7.1 uses the independently known parameter κ_1 in the analytic application. The comparison parameters and fixed losses are

$$B_C = B_P = 1 - 10^{-12}, \quad L_C = L_P = 2 + 10^{-11}, \quad c_C = \frac{3}{20}, \quad c_P = \frac{3}{25}, \quad c_N = \frac{3}{40}, \quad \rho = 10^{-5}.$$

These parameters agree with the finite comparison in [8]. The construction uses the coefficient families of Section 2.

5.1 The rational moment induction

In this subsection, we specify the finite rational induction used in Subsection 5.2. Its comparison infimum is computed in Subsection 5.4.

Set

$$R_j = \frac{311}{400} - j\Delta R \quad (0 \le j \le 240), \qquad R_{240} = \frac{1467}{2000}, \qquad \Delta R = \frac{11}{60000},$$
$$H_j(r) = r/R_j, \qquad u_j(r) = (1 - H_j(r))/6, \qquad \Delta_z = \frac{1}{25000000}, \quad p = 10^{-7}.$$

[1] All numerical assertions in Sections 5–7 are finite comparisons of rational numbers, or sign conditions for polynomials with rational coefficients on closed intervals with rational endpoints. Each sign condition is proved by subdividing the interval finitely many times and checking that all coefficients in the Bernstein basis of every subinterval have the required sign; every quantity is computed exactly in $\mathbb{Q}$.

Start with $z_0 = a_{0,0} = a_{1,0} = 0$. At stage j put

$$w_j = \frac{R_j/2 + z_j - a_{0,j}}{1/2 - a_{1,j}}, \tag{15}$$

$$k_j = \max\left\{\frac{45457}{125000},\ 10^{-6}\left\lceil 10^6\left(\frac{5w_j/4 - 5/6}{3w_j/2 - (1+R_j)/2 - z_j} + \frac{18}{10^5}\right)\right\rceil\right\}, \tag{16}$$

$$Y_j(r) = \frac{r}{R_j}z_j + \frac{1/k_j - 2}{6}\left(1 - \frac{r}{R_j}\right), \tag{17}$$

$$z_{j+1} = 10^{-10}\left\lfloor 10^{10}(Y_j(R_{j+1}) - \Delta_z)\right\rfloor, \tag{18}$$

$$a_{1,j+1} = -\frac{Y_j(R_j) - Y_j(R_{j+1})}{\Delta R}, \tag{19}$$

$$a_{0,j+1} = Y_j(R_{j+1}) - \Delta_z - 2p + a_{1,j+1}R_{j+1}. \tag{20}$$

All constants are rational. The explicit lower cutoff in (16) is essential: without it the final stage violates the lower cutoff in the induction inequalities.

Proposition 5.1 (The finite moment sequence)**.** *The 240-stage chain satisfies all fourteen inequalities of Subsection 8.7, including both branches of its thirteenth condition, at both endpoints of every stage. It also satisfies*

$$u_j(R_{j+1}) \le 10^{-4}, \quad R_j \le w_j, \quad \frac{45457}{125000} \le k_j < \frac{1}{2}, \quad z_{j+1} \ge 0.$$

Each incoming line is strictly below every preceding stage improvement with the required margin. The lowered line of every stage is a global minorant of all 240 stage improvements. Precisely, with $\lambda_j(w) = a_{0,j} - a_{1,j}w$ and $c_\lambda = \Delta_z + p = 1.4 \cdot 10^{-7}$,

$$\begin{gathered}\min_{1\le j\le 239}\ \min_{0\le i<j}\ \min_{r\in\{R_{i+1},R_i\}}\big(Y_i(r) - c_\lambda - \lambda_j(r)\big) = p = 10^{-7},\\ 0 \le a_{1,j} \le 0.166, \qquad \lambda_j(R_0) \le -2.4\cdot 10^{-7} \le -c_\lambda \qquad (1 \le j \le 239),\end{gathered} \tag{21}$$

the minimum being attained at $j = 1$, $i = 0$, where $\lambda_{j+1} = Y_j - \Delta_z - 2p$ identically. Thus (82) holds with $c_\lambda = \Delta_z + p$ and $0 \le a_1 \le 2/5$.

Proof. Apply the rational recursion to the inequalities of Subsection 8.7. Each condition is evaluated at both endpoints. For each incoming or supporting line λ_j, evaluate its difference from $Y_i - \Delta_z - p$, $0 \le i < j$, at both endpoints of $[R_{i+1}, R_i]$, and evaluate $a_{1,j}$ and $\lambda_j(R_0)$; exact rational arithmetic gives (21). These differences are affine in r, so endpoint positivity suffices. The chain enters the scalar Theorem 5.4. □

5.2 The complete comparison inequalities

In this subsection, we combine the moment sequence of Subsection 5.1 with the count of Section 4. We verify the comparison at all four conductor boundaries; Subsection 5.3 gives the corresponding lower bound.

For a count exponent $R(\delta,x)$ and conductor exponent d, set

$$E(d,C;\delta,x)=\frac{1}{2}+\frac{13h}{75}-\frac{l_y}{2}-\frac{\ell}{2}-\frac{9d}{100}+\frac{\delta}{2}+\delta x\left(\ell-\frac{C}{2}\right), \tag{22}$$

$$C_{\mathrm{old}}(d)=\frac{\min(d,2l_y-d)}{6\kappa}, \tag{23}$$

$$C_{\mathrm{new}}(d)=\min\left(\frac{3d}{40},\frac{2l_y-(2-B_P)d}{L_P}\right). \tag{24}$$

The four conductor edges are

$$(d,C)=\left(\frac{1}{200},C_{\mathrm{old}}(\frac{1}{200})\right),\quad (\frac{19}{25},C_{\mathrm{old}}(\frac{19}{25})),\quad (\frac{19}{25},C_{\mathrm{new}}(\frac{19}{25})),\quad (h+10^{-10},C_{\mathrm{new}}(h+10^{-10})).$$

The marked-length bound C_{old} is that of [6, Proposition 14.3]: for $d>l_y$ it is case (2) there, with numerator shift $\delta x/(6\kappa)$, which requires the gate [6, (14.15)]

$$R(\delta,x)\le(6\kappa-2\delta x)\frac{\ell}{d}. \tag{25}$$

For $d\le 6\kappa\ell$ case (2) coincides with case (1) there, and (25) is not needed; this includes $d\le 19/25$. Lemma 8.27 verifies (25) for $\delta\ge 1/8$ up to the top edge; for $\delta<1/8$ and $d>19/25$ it replaces C_{old} by the bound C_{aff} of case (1). The bound C_{new} is justified by Proposition 8.24; its additional inequalities are verified in Proposition 8.26. Put $V=5/6-\delta$. The old and initial count functions are

$$\begin{aligned}
D_o&=3-x-2x/(3\kappa), & C_o&=2(1-x)(1-x/(3\kappa)),\\
J_o&=VD_o+\delta C_o, & R_o&=1-\delta+\frac{\delta VC_o}{2J_o},\\
D_n&=3-x-4x/L_C, & A_n&=2-4x/L_C,\\
C_n&=1-2xB_C/L_C, & J_n&=VD_n+\delta(1-x)A_n,\\
& & R_n&=1-\delta+\frac{\delta V(1-x)C_n}{J_n}.
\end{aligned}$$

For an affine improvement $g(r)=g_0-g_1r$, Section 4 gives

$$\begin{aligned}
J_g&=3V+2\delta\{1-x(1+2g_1)\},\\
R_g&=1-\delta+\frac{\delta V\{1-x(1+6g_0-4g_1)\}}{J_g}.
\end{aligned}$$

The symbols J_{old} and R_{old} below mean J_o and R_o. Each branch is used only where its range conditions hold. With $c_0=27/1000$, $\nu_0=(1-\frac{9}{2}c_0)/2$, $\eta_0=10^{-7}$, $U_n=(1-x)C_n$, $U_g=1-x(1+6g_0-4g_1)$, $K=1-x(1+2g_1)$,

$$t_n=1+\frac{\delta U_n}{J_n},\quad r_n=\frac{A_nt_n+x(2B_C/L_C-1)}{D_n},\quad m_n=t_n-r_n,\qquad t_g=1+\frac{\delta U_g}{J_g},$$

$$m_-=\max\{\tfrac{3}{10},\tfrac{B_C-L_C(c_C-\eta_0)}{2}\}+\eta_0,\quad m_+=\min\{\tfrac{11}{25},\nu_0,\tfrac{B_C-L_C(c_0+\eta_0)}{2}\}-\eta_0,$$

these conditions are as follows:

(O) $J_o>0$;

(N) on the initial range: $D_n, A_n, C_n, J_n > 0$; $1 < t_n < 1.2499$; $23/37 < r_n < 1$; $m_- < m_n < m_+$; $R_n \geq 1 - 2\delta\nu_0 - 2\delta x c_0$; $\partial_x(U_n/J_n) < 0$; $11/25 - \nu_0 > \eta_0$ and $B_C - 2\nu_0 - L_C(c_0 + \eta_0) > 2\eta_0$;

(I) on the improved range, for a line with domain $[a_-, b]$: (I1) $K > 0$; (I2) $J_g > 0$; (I3) $t_g > 1$; (I4) $t_g < 1.2499$; (I5) $2t_g/3 > a_- + 10^{-6}$; (I6) $R_g \geq 1 - \delta(x + (1-x)b)$; (I7) $R_g \geq 1 - \delta(b + 2(B_C - b)x/L_C)$; (I8) $R_g \geq 1 - \delta(b + 2(c_C - \eta_0)x)$; (I9) $R_g \geq 1 - 2\delta\nu_0 - 2\delta x c_0$; (I10) $(1 + 6g_0 - 4g_1)J_g - 2\delta(1 + 2g_1)U_g > 0$.

Here (I3)–(I5), and t_n, r_n, m_n in (N), are (13) at the interior minimax point (Lemma 4.3). (N), with $m_\pm$, keeps the ordinary prime width $(B_C - 2m)/L_C$ in $[c_0, c_C]$. (I6)–(I9) and the R_n-condition say that detected lengths outside the line's domain, namely $r > b$, $2m > b$, or an ordinary length at the cap ν_0, give a count at most the branch value. (I10) and $\partial_x(U_n/J_n) < 0$ fix the sign of the x-derivative. The constants c_0, ν_0, η_0, $23/37$ and 1.2499 are fixed choices of the present scalar model; they are not taken from [6] or [5]. In (A), $b = 311/400$, $a_- = 1467/2000$, $B_C = 1 - 10^{-12}$, $L_C = 2 + 10^{-11}$; in (B), $b \uparrow 7/9$, $B_C = 1$, $L_C = 2$.

The count envelope is the minimum of the old count, the initial count range, and the improved ranges supplied by the lowered own lines of stages 132, 133 and 134. Each branch is used only inside its range of validity. The improved range is

$$[189/512, 15/32] \times [15/32, 1/2],$$

and the initial range is $[15/64, 9/16] \times [9/32, 1/2]$.

Theorem 5.2 (A rational comparison bound)**.** *With*

$$\theta_+ = \frac{873346697}{10^9}, \qquad h_+ = \frac{81449508391}{10^{11}},$$

all four edge inequalities

$$E(d, C; \delta, x) + \frac{3d}{4} R(\delta, x) < \theta_+ - 10^{-12} \tag{26}$$

hold throughout $[0, 3/4] \times [0, 1/2]$. *All* 45 *conditions (O), (N), (I) hold strictly: one, fourteen, and ten for each of the three lines. The comparison geometry and its floor margin also hold.*

Proof. For a branch $R = N/J$, multiply the margin in (26) by its positive denominator J. This gives a bivariate polynomial over $\mathbb{Q}$. On a box, positivity of every tensor Bernstein coefficient implies positivity of the polynomial. Successive dyadic subdivisions use respectively $1, 2, 2, 57$ subrectangles for the four edges, with maximal depth 25. The conditions on the parameter ranges are strict without subdivision on their respective ranges.

The floor margin is

$$\theta - \frac{51}{100} - \frac{h}{3} - \frac{53\ell}{100} - \frac{67}{100} 10^{-256} \geq \rho. \tag{27}$$

Indeed $h_+ = (234\theta_+ - 184)/25 - 3\rho - 10^{-11}$, giving margin $\rho + 10^{-11}/3 - (67/100)10^{-256} > \rho$. The geometry in (14) also gives

$$1/6 < \ell < 1/5, \qquad \ell < l_x < l_y < 1/2,$$
$$2l_x - l_y - \ell > 0, \qquad 2(l_x + l_y) - 1 - 3\ell > 0,$$
$$l_y/2 - 13h_+/75 - 1/50 > 7/100.$$

The additional numerator inequalities of Proposition 8.26 (A) hold at the same parameters. At the edges $(1/200, C_{\mathrm{old}}(1/200))$ and $(19/25, C_{\mathrm{old}}(19/25))$, $d < 6\kappa\ell$ by Lemma 8.27 (i), so case (2) of [6, Proposition 14.3] is case (1) and (25) is not needed; for $19/25 < d \leq d_{\mathrm{top}}$ and $\delta < 1/8$ the comparison with C_{aff} holds by Lemma 8.27 (iii). □

5.3 A lower bound for the fixed comparison

In this subsection, we bound the infimum of the fixed comparison of Subsection 5.2 from below. The exact infimum is determined in Subsection 5.4.

Let $\mathcal{S}_{240}$ be precisely the preceding scalar system, allowing all 240 lowered supporting lines in its detector envelope and allowing h to vary, but retaining the displayed parameter ranges, numerator estimate, four-edge requirements, floor margin, and $0.873 \le \theta \le 0.873347$. No other moment or signed comparison is included. Let $B(\mathcal{S}_{240})$ be its infimum.

Theorem 5.3 (Bounds for the fixed comparison)**.**

$$0.87334669685 < B(\mathcal{S}_{240}) \le 0.873346697. \tag{28}$$

This interval has length $1.5 \cdot 10^{-10}$.

Proof. The upper bound is Theorem 5.2. For the lower bound, fix $\delta_0 = 42767/10^5$, $x = 1/2$. Allow every supporting line without its restrictions on the parameters. Exact rational comparison gives the smallest supplied count q_0 at stage 133. Both the initial count and the old count at $\kappa = 0.746$ exceed q_0; the old count increases with κ on the permitted range. Hence $R \ge q_0$ even in this optimistic relaxation. Put

$$s_0 = \frac{5}{12} + \frac{\delta_0}{8} - \frac{3q_0}{4} > 0.$$

Remove the positive additional error exponent and extension of the parameter range, and enlarge the numerator marked-length bound to $(2l_y - h)/2$. The resulting necessary endpoint inequality is

$$(3 + 4\delta_0)\theta - \frac{7}{3} - \frac{25\delta_0}{6} + hs_0 \ge 0.$$

The floor condition implies $h \le (234\theta - 184)/25 - 3\rho$. Therefore

$$\theta \ge \frac{7/3 + 25\delta_0/6 + (184/25 + 3\rho)s_0}{3 + 4\delta_0 + (234/25)s_0} = 0.8733466968582541\ldots > 0.87334669685. \tag{29}$$

□

5.4 The exact algebraic infimum

In this subsection, we determine the algebraic infimum of the fixed comparison in Subsection 5.2. Section 7 treats the larger class with variable moment ranges and comparison lengths.

Let (g_0, g_1) be the lowered supporting line of stage 133 in the 240-stage recursion. Define

$$H(\theta) = \frac{234\theta - 184}{25} - 3\rho - \frac{201}{100}10^{-256}, \tag{30}$$

$$J(\delta, x) = 3(5/6 - \delta) + 2\delta\{1 - x(1 + 2g_1)\}, \tag{31}$$

$$U(x) = 1 - x(1 + 6g_0 - 4g_1), \tag{32}$$

$$N(\delta, x) = (1 - \delta)J(\delta, x) + (5/6 - \delta)\delta U(x). \tag{33}$$

In the following definitions use $h = H(\theta)$ and the geometry (14). Put

$$d_\theta = H(\theta) + 10^{-10}, \qquad C_\theta = \frac{2l_y - (2 - B_P)d_\theta}{L_P},$$

and

$$Q(\theta,\delta,x)=J(\delta,x)\{\theta-10^{-12}-E(d_\theta,C_\theta;\delta,x)\}-\frac{3d_\theta}{4}N(\delta,x). \tag{34}$$

At $x=1/2$ this is a quadratic in δ:

$$Q(\theta,\delta,1/2)=A(\theta)\delta^2+B(\theta)\delta+C(\theta),$$

whose coefficients are affine rational functions of θ. The polynomial $\mathfrak{D}(\theta)=B(\theta)^2-4A(\theta)C(\theta)$ has degree two. Let θ_{240} be its unique root in

$$0.8733466968<\theta_{240}<0.8733466970. \tag{35}$$

The rational recursion and (34) define this algebraic number without a rounded input. Numerically,

$$\boxed{\theta_{240}=0.873346696867948102628637768733828847 2\ldots.}$$

Theorem 5.4 (The infimum of the fixed comparison)**.** *For the system $\mathcal{S}_{240}$ defined above,*

$$B(\mathcal{S}_{240})=\theta_{240}.$$

Every fixed $\theta>\theta_{240}$ sufficiently close to it supplies strict comparison inequalities, with $h=H(\theta)$. At the infimum, the only zero margin among the four comparison inequalities is a double contact at

$$x=1/2,\qquad \delta=\delta_{240}:=-\frac{B(\theta_{240})}{2A(\theta_{240})}=0.4276734\ldots.$$

At $\theta=\theta_{240}$, the strict inequality (26) fails at this contact; the floor inequality (27) holds with equality.

Proof. On the interval (35), $A(\theta)>0$, the discriminant has one root by Sturm's theorem, and

$$27/64<\delta_{240}<57/128,\qquad (\delta_{240},1/2)\in\mathcal{C}:=[27/64,57/128]\times[31/64,1/2].$$

Thus

$$Q(\theta_{240},\delta,1/2)=A(\theta_{240})(\delta-\delta_{240})^2.$$

Exact Bernstein bounds give, throughout $\mathcal{C}$ and the entire θ interval in (35),

$$\partial_x Q<0,\qquad \partial_\theta Q>0.$$

Hence the stage-133 branch is nonnegative on $\mathcal{C}$ at the endpoint, vanishing only at the displayed contact, and is strictly positive there whenever $\theta>\theta_{240}$.

Outside $\mathcal{C}$, a uniform three-variable Bernstein cover in (θ,δ,x) proves strict positivity. It has respectively $1,2,2,41$ ordinary subrectangles on the four edges, and one further subrectangle on the fourth edge treated by the contact argument. The old branch's denominators are cleared by the positive factor $(2\theta-1)^2$. All range conditions (O), (N), (I) remain the same as before. The branches of the marked-length bound and parameter geometry are valid on the whole θ interval. This proves attainability arbitrarily close above θ_{240}.

For necessity, evaluate at $(\delta_{240},1/2)$. Exact interval polynomial comparisons show that all other 239 counts from the supporting lines, even with their restrictions on the parameters removed, exceed the stage-133 count R_{133}. The initial count does also. The old count at $\kappa=0.746$ exceeds it, and its derivative for $x=1/2$ is

$$\partial_\kappa R_{\mathrm{old}}=\frac{\delta(5/6-\delta)^2}{24\kappa^2J_{\mathrm{old}}^2}>0.$$

Thus the full supplied count is at least R_{133} for every $\theta \geq 0.873$. The affine upper bound for the numerator marked length grants at least as much as the actual minimum defining that bound. The corresponding margin has positive coefficient of h:

$$\frac{5}{12} + \frac{\delta_{240} B_P}{4 L_P} - \frac{3}{4} R_{133} > 0.$$

The floor inequality gives $h \leq H(\theta)$. Therefore the actual top-comparison margin is no larger than $Q(\theta, \delta_{240}, 1/2)/J(\delta_{240}, 1/2)$. The latter is affine and strictly increasing in θ, and is zero at θ_{240}. No $\theta \leq \theta_{240}$ supplies the required strict comparison. □

6 The limiting moment range

In this section, we determine an algebraic upper bound for the moment recursion of Section 2. We also treat repeated use of earlier estimates and prove finite approximation on the interval needed in Section 7.

Define, for $2/3 < R \leq 7/9$,

$$G(R) = \frac{11R}{16} - \frac{7}{24} - \frac{5}{16} \left(7R^4(R - 2/3)\right)^{1/5}. \tag{36}$$

The fifth root is positive. We first bound the recursion from above. Subsection 6.1 constructs finite moment ranges approaching this bound on the required interval.

Lemma 6.1 (Differential identity and convexity)**.** *We have* $G(7/9) = 0$ *and*

$$G'(R) = \frac{(45R - 24)G(R) + 9R - 7}{R(45R - 30)}, \qquad G''(R) > 0.$$

Proof. Put $H(R) = R^{4/5}(R - 2/3)^{1/5}$. Then

$$\frac{H'}{H} = \frac{4}{5R} + \frac{1}{5(R - 2/3)}, \qquad \frac{H''}{H} = -\frac{16}{25R^2(3R - 2)^2}, \qquad G''(R) = \frac{7^{1/5}}{45R^{6/5}(R - 2/3)^{9/5}} > 0.$$

Substitution gives the differential equation and $G(7/9) = 0$. □

Proposition 6.2 (An upper bound for the moment recursion)**.** *Suppose a moment sequence starts at* $2/3 < R_0 \leq 7/9$ *with* $z_0 \leq G(R_0)$*. For a step* $R \mapsto r < R$*, assume*

$$w \geq R, \quad 0 < k < 1/2, \quad \frac{5}{6} - \left(\frac{5}{4} - \frac{3k}{2}\right) w - k\frac{1 + R}{2} - kz \geq 0,$$

and $2/3 < r < R \leq R_0 \leq 7/9$*. If* $z \leq G(R)$*, every resulting improvement no larger than* (9) *satisfies* $z_{\text{new}} \leq G(r)$*.*

Proof. Since $5/4 - 3k/2 > 0$, replace w by R to obtain

$$k(R - 1/2 - z) \geq 5R/4 - 5/6 > 0.$$

Consequently $k \geq k_*(R, z) := (5R/4 - 5/6)/(R - 1/2 - z)$. The propagated improvement decreases with k. At $k = k_*$ it is

$$Y_*(r; R, z) = z + (r - R)\{a(R)z + b(R)\}, \quad a(R) = \frac{45R - 24}{R(45R - 30)}, \quad b(R) = \frac{9R - 7}{R(45R - 30)}.$$

The expression is increasing in z, since

$$(R-r)a(R) < (R-2/3)a(R) = 1 - \frac{8}{15R} < 1.$$

Lemma 6.1 and convexity therefore give

$$Y(r) \le Y_*(r;R,z) \le G(R) + (r-R)G'(R) \le G(r).$$

Positive decrements and downward rounding can only decrease the resulting improvement. □

Proposition 6.3 (An upper bound for all earlier moment ranges)**.** *Allow arbitrary reuse of proved moment bounds, nonlinear reference boundaries, and every cutoff admitted by the short- and long-sum exponents for large values. Allow every positive finite step $0 < u < 1/6$, remove the cutoff floor, and enlarge the prime-length bound for the new estimate to $c = 1/6$. Start with all initial joint rectangles $0 < R \le 1$, $S \le C(R) := (1-R)/2$. Put*

$$M_{\mathrm{ref}} = \frac{48+\sqrt{909}}{90} = 0.86832918737069634234\ldots, \qquad \nu_{\mathrm{ref}} = \frac{1}{6M_{\mathrm{ref}}-1}, \qquad R_c = M_{\mathrm{ref}} - 1/6.$$

There is a unique $R_+ \in (0.7286, 0.7287)$ on the increasing branch of $F_G(R) = (1+R)/2 + G(R)$ with $F_G(R_+) = M_{\mathrm{ref}}$. Every available joint marked length is at most

$$\widehat{S}(R) = \begin{cases} C(R), & 0 \le R \le 2/3, \\ 1/6, & 2/3 \le R \le R_c, \\ M_{\mathrm{ref}} - R, & R_c \le R \le R_+, \\ C(R) + G(R), & R_+ \le R \le 7/9, \\ C(R), & 7/9 \le R \le 1. \end{cases} \tag{37}$$

In particular the limiting moment bound is unchanged throughout the critical interval for zero detection $R \ge 0.734$. The conclusion is preserved by free-width truncations of any proved rectangle.

Proof. The defining equations and rational interval arithmetic give

$$0.7016 < R_c < 0.7017, \quad 0.7286 < R_+ < 0.7287, \quad G'(0.728) > -1/4, \quad F_G(R_c) < M_{\mathrm{ref}}.$$

Thus $\widehat{S}$ is continuous and nonincreasing. Its total-length bound $F(R) = R + \widehat{S}(R)$ is nondecreasing, is constant M_{ref} on $[R_c, R_+]$, and has derivative at least $1/4$ above R_+. Convexity of F_G gives $F_G(R) \le M_{\mathrm{ref}}$ on $[R_c, R_+]$.

Put $A_\nu = 5/4 - 3\nu/2$. The quadratic identity $180M_{\mathrm{ref}}^2 - 192M_{\mathrm{ref}} + 31 = 0$ gives

$$A_{\nu_{\mathrm{ref}}} R_c + \nu_{\mathrm{ref}} M_{\mathrm{ref}} = 5/6, \qquad 1 + 1/\nu_{\mathrm{ref}} = 6M_{\mathrm{ref}}. \tag{38}$$

For a stage write $r = HR$, $H = 1-6u$, let S' be its proposed marked length, and put $s = S' - u$. If $S' < u$, then $S' < C(r)$ since $C(r) \ge 3u$, so the initial moment domain suffices. We may therefore assume $s \ge 0$.

Here is a general envelope argument which removes every small-step restriction. Suppose a previous total-length bound is $F(R)$ and $\nu_0 \in (0, 3/4]$ satisfies

$$A_{\nu_0} R + \nu_0 F(R) \ge 5/6. \tag{39}$$

Then every admitted stage obeys

$$r + S' \le HF(R) + u(1 + 1/\nu_0). \tag{40}$$

If $k > \nu_0$, this is immediate from propagation and the previous bound. If $k \le \nu_0$, use its class of large amplitudes and write $F' = r + S'$. Its short and long inequalities imply, since $\min\{s/H, \sigma\} \le \sigma$,

$$\nu_0 H(w + \sigma) \ge \nu_0 F' - u(1 + \nu_0), \qquad A_{\nu_0} Hw + \nu_0 F' \le 5H/6 + u(1 + \nu_0).$$

For $w \le R$, monotonicity of the total-length bound gives $w + \sigma \le F(R)$, proving (40) by the first inequality. For $w > R$, use the second inequality and (39) to obtain

$$\nu_0 F' \le H\{5/6 - A_{\nu_0} w\} + u(1 + \nu_0) \le H\nu_0 F(R) + u(1 + \nu_0).$$

Thus the same bound holds for all cutoffs. This argument also includes endpoint decrements, since they only decrease the advertised S'.

Induct on the number of completed stages. Initial rectangles satisfy (37). If $r \le R_c$, the enlarged mark cap proves the ceiling directly. For a previous $R \in [R_c, R_+]$, use $\nu_0 = \nu_{\mathrm{ref}}$. The identity $A_{\nu_{\mathrm{ref}}} R_c + \nu_{\mathrm{ref}} M_{\mathrm{ref}} = 5/6$ and $R \ge R_c$ prove (39), while (40) gives $r + S' \le M_{\mathrm{ref}}$. If $R \ge 7/9$, use $\nu_0 = 1/2$. Indeed

$$A_{1/2} R + F(R)/2 = 3R/4 + 1/4 \ge 5/6,$$

and the resulting bound is $H(1 + R)/2 + 3u = (1 + r)/2 \le r + \widehat{S}(r)$. For $R \in [R_+, 7/9]$, put

$$\nu_0 = k_*(R, G(R)) = \frac{5R/4 - 5/6}{R - 1/2 - G(R)} \in (0, 1/2].$$

The denominator is positive because $R - 1/2 - G(R) = R/2 - \widehat{S}(R) > 1/5$. The upper bound follows from $9R + 6G(R) \le 7$, since its derivative is positive on this interval and its value at $7/9$ is 7. Here (39) is an equality and the ODE gives

$$1 + 1/\nu_0 = 6\{F_G(R) - RF_G'(R)\}.$$

If $r < R_c$, the marked-length bound gives the conclusion directly. For $r \ge R_c > 2/3$, the finite-step inequality is

$$r + S' \le F_G(R) + (r - R)F_G'(R) \le F_G(r).$$

For $r \ge R_+$ this is the bound for the limiting moment range. For $R_c \le r \le R_+$, use $F_G(r) \le M_{\mathrm{ref}}$. This closes the induction for arbitrary finite step sizes. Finally $\widehat{S}$ is nonincreasing, which proves preservation under free-width truncation. □

For the simultaneous ordinary obstruction we need a convex improvement majorant rather than the exact ceiling. Define

$$\mathcal{G}(P) = \begin{cases} M_{\mathrm{ref}} - (1 + P)/2, & 2/3 \le P \le R_+, \\ G(P), & R_+ \le P \le 7/9, \\ 0, & 7/9 \le P \le 1. \end{cases} \tag{41}$$

It is continuous and convex: its successive derivatives are $-1/2$, $G'(P) \ge G'(R_+) > -1/4$, and zero. Moreover $\widehat{S}(P) \le C(P) + \mathcal{G}(P)$ and $\widehat{S}(P) \le \max\{C(P), 1/6\}$. Thus the repeated use of the shorter moment ranges is included without changing the critical improvement at r_*.

For the descending chains used below, Proposition 6.2 applies at every finite step with $r > 2/3$. These chains start with $z_0 = 0$, while $G(311/400) > 0$. Thus G is an upper bound for those chains. The upper bound for all earlier estimates above also covers arbitrary reuse and large steps, with the same G on the critical interval for zero detection. The infimum of the resulting comparison is determined in Section 7, with a variable lower integration limit in place of the fixed choice of Section 5.

6.1 Finite approximation of the limiting moment range

In this subsection, we approximate the algebraic bound of Section 6 by finite rational moment sequences. Subsection 7.1 uses these sequences at each fixed boundary above the limiting infimum.

Here the margins in the scalar inequalities are parameters: the large-value exponent margin, supporting-line padding, endpoint decrement, and rounding mesh may be arbitrarily small positive numbers. This is a larger scalar system than $\mathcal{S}_{240}$.

Proposition 6.4 (Finite approximation)**.** *Put $a = 367/500$. For every compact interval $I \subset (a, 7/9)$ and every $\varepsilon > 0$, there is a finite rational chain with $R_0 = b < 7/9$, $z_0 = 0$, $R_n = a$, and $I \subset [a, b]$, such that*

$$|G(R) - z^{(n)}(R)| < \varepsilon \qquad (R \in I),$$

where $z^{(n)}$ is its polygonal interpolation. The first condition is $HF_{\mathrm{high}} > 0$; conditions 2–14 retain their displayed strict margins. The cutoff $k \geq 45457/125000$ is retained. The supporting lines λ_j satisfy (82) with $c_\lambda = 2q^3$, and $0 \leq a_{1,j} \leq 2/5$.

Proof. Write a_1, b_1 for the coefficients in the differential equation of Lemma 6.1. On $I_0 = [a, 7/9]$, set

$$C_0 = 1 + \max_{I_0}(|a_1| + |a_1'| + |b_1'| + |G''|), \qquad c_0 = \min_{I_0} G'' > 0.$$

Choose rational $b \in (\max I, 7/9)$ and $q = (b-a)/n$. For $R_j = b - jq$, choose positive supporting-line separations, rounding errors and total endpoint decrements $d_j \in [q^3, 2q^3]$, and a positive large-value margin comparable to q^2. Choose the cutoff above its exact threshold with rounding $O(q^2)$. The incoming line then gives

$$w_j - R_j = O_{I_0}(q^3), \qquad k_j - k_*(R_j, z_j) = O_{I_0}(q^2).$$

The propagation formula therefore becomes

$$z_{j+1} = z_j - q\{a_1(R_j)z_j + b_1(R_j)\} + O_{I_0}(q^3).$$

Write d_j for the total downward error, so that $z_{j+1} = Y_j(R_{j+1}) - d_j$. If $e_j = G(R_j) - z_j$, Taylor's formula gives

$$0 \leq e_{j+1} \leq (1 - qa_1(R_j))e_j + C_1 q^2, \qquad \max_j e_j \leq G(b) + C_1(b - 367/500)q.$$

Here C_1 is independent of b and n. Choose

$$G(b) < \varepsilon/3, \qquad C_1(b-a)q + C_0 q^2 < 2\varepsilon/3.$$

Linear interpolation on each interval gives

$$\sup_{[a,b]} |G - z^{(n)}| \leq \max_j e_j + (q^2/8) \max_{I_0} |G''| < \varepsilon.$$

For completeness the supporting-line condition is stable under this construction. Put $v(R,z) = a_1(R)z + b_1(R)$. Along the graph of G,

$$v_R + v_z v = G'' \geq c_0.$$

In a sufficiently small neighbourhood it is at least $c_0/2$. Consequently, for sufficiently small q,

$$v(R_{j+1}, z_{j+1}) \leq v(R_j, z_j) - c_0 q/4.$$

Choose the rational supporting lines

$$\lambda_j(R) = Y_j(R) - d_j - 4q^3.$$

For small enough q, consecutive slopes differ by at least $c_0 q/8$, whereas $d_j \leq 2q^3$. Hence

$$Y_\ell(R) - d_\ell - \lambda_j(R) \geq 2q^3 \quad (R \in [R_{\ell+1}, R_\ell],\ 0 \leq j, \ell < n).$$

Indeed, at the next junction the difference is $4q^3 - d_{j+1}$; across each further interval it increases by at least $c_0 q^2/8 - 2q^3 > 0$. In the opposite direction both the slope difference and the junction increment are positive. Affineness reduces each interval to its endpoints. At the next entry,

$$z_{j+1} - \lambda_j(R_{j+1}) = 4q^3, \qquad w_{j+1} - R_{j+1} = O_{I_0}(q^3).$$

Moreover,

$$\sup_{R \in [R_{j+1}, R_j]} |G(R) - \lambda_j(R)| \leq \max_\ell e_\ell + C_0 q^2 + 6q^3.$$

This tends to zero after choosing $b \uparrow 7/9$ and then $q \downarrow 0$. The first incoming line is zero.

The slopes satisfy $0 \leq a_{1,j} \leq 2/5$. Indeed $Y_j(R_j) = z_j$ and $Y_j(R_{j+1}) = z_j - q v(R_j, z_j) + O_{I_0}(q^3)$, so $a_{1,j} = -v(R_j, z_j) + O_{I_0}(q^2)$, and $|v(R_j, z_j) - G'(R_j)| \leq C_0 e_j \leq C_0(G(b) + C_1 q)$. Since $G'' \geq c_0$, $|G''| \leq C_0$ and $G(7/9) = G'(7/9) = 0$,

$$-G'(R_j) \geq -G'(b) \geq c_0\Big(\frac{7}{9} - b\Big), \qquad G(b) \leq C_0\Big(\frac{7}{9} - b\Big)^2, \qquad -G'(R_j) < \frac{1}{6}.$$

Hence, for $7/9 - b < c_0/(2C_0^2)$ and then q small,

$$0 < \frac{c_0}{2}\Big(\frac{7}{9} - b\Big) - O(q) \leq a_{1,j} \leq \frac{1}{6} + C_0^2\Big(\frac{7}{9} - b\Big)^2 + O(q) < \frac{2}{5}.$$

Since $b < 7/9$, its cutoff is strictly below $1/2$ once q is small enough.

The other conditions have uniform strict limiting margins. On $367/500 \leq R \leq 7/9$,

$$0 \leq G(R) < 31/10^4, \quad -1/6 < G'(R) \leq 0, \quad k_*(R, G(R)) > 0.3644 > 45457/125000,$$

Indeed, $G'' > 0$, $G'(7/9) = 0$, and exact endpoint bounds give $G(a) < 31/10^4$ and $G'(a) > -1/6$. Furthermore,

$$\frac{d}{dR} k_*(R, G(R)) = \frac{33R - 48G(R) - 14}{72R\{R - 1/2 - G(R)\}^2} > 0,$$

so the cutoff lower bound follows from its value at a. Also

$$S = (1-R)/2 + G(R) < 0.1361 < 0.137, \quad R + S < 0.892, \quad R - 0.7 - G(R) > 0.0309.$$

The limiting expressions in conditions 2–14 have the following strict rational lower bounds:

j	2	3	4	5	6	7	8
L_j	0.0009	0.0039	0.008	0.018	0.015	0.243	0.0309
j	9	10	11	12	13	14	
L_j	0.130	0.039	0.067	0.012	0.147	0.012	

These follow from the preceding bounds with $H = 1$, $u = 0$, $w(k) = w_+ = R$, and $s = S$. Continuity supplies all strict inequalities for sufficiently small q. Rational rounding preserves them. The analytic estimate permitting these variable margins is Proposition 8.18. □

6.2 An algebraic lower bound for the comparison

In this subsection, we derive a lower bound for the comparison from the algebraic function in (36). Section 7 extends the argument to unequal ordinary moments and variable comparison lengths.

Let r_* be the unique root in

$$\frac{7523220538}{10^{10}} < r_* < \frac{7523220539}{10^{10}}$$

of

$$\left\{\frac{19(3r-2)(9r+2)}{3(45r-14)}\right\}^5 = 7r^4(r-2/3). \tag{42}$$

Define

$$\delta_* = \frac{45r_*-14}{6+54r_*}, \qquad B_{\mathrm{rel}} = \frac{14+25\delta_*}{18+24\delta_*} = 0.8733458561067\ldots. \tag{43}$$

Clearing denominators in (42) and removing the factor $3r-2$ gives a degree-nine polynomial. Sturm's theorem isolates r_* in the displayed interval. The expression inside the fifth power is positive there.

Proposition 6.5 (A lower-bound identity)**.** *Let $M, \ell \in \mathbb{R}$, set $\delta = \delta_*$, and put*

$$L_0 = \frac{5}{6} + \frac{M}{12} - \frac{\ell}{6}, \quad L_K = \frac{1}{3} + \frac{7M}{12} + \frac{\ell}{3}, \quad H = 1 + \frac{3\delta}{4} - \left(\frac{1}{2} + \frac{\delta}{4}\right) M + \frac{3\delta\ell}{4}.$$

If a comparison requires $B \geq L_0, L_K, H$, then $B \geq B_{\mathrm{rel}}$.

Proof. The positive weights

$$u = \frac{25\delta+8}{6(4\delta+3)}, \quad v = \frac{4-\delta}{6(4\delta+3)}, \quad w = \frac{1}{4\delta+3}$$

satisfy $u+v+w = 1$ and the exact identity

$$uL_0 + vL_K + wH = \frac{14+25\delta}{18+24\delta}.$$

□

The connection with the relaxed detector is explicit. Put $F(r) = (1+r)/2 + G(r)$ and $t_* = 3r_*/2$. Equations (36)–(42) give

$$1 - \delta_* F(r_*) = \frac{1}{6} + \left(\frac{5}{6} - \delta_*\right) t_* = \frac{10+3\delta_*}{18} =: R_*.$$

Thus

$$\frac{5}{12}+\frac{\delta_*}{8}-\frac{3R_*}{4}=0.$$

This is the amplitude at which the optimistic endpoint comparison loses its dependence on h. It produces the plane H of Proposition 6.5 when the numerator retains its coefficient-two fourth-moment/Hölder estimate.

This obstruction has a specified scope: the limiting moment range of Proposition 6.2, the supplied short/long detector alternatives, and the unchanged positive numerator estimate. It does not exclude an improved long inverse estimate, a new mixed moment, a signed comparison, or a numerator estimate beyond those inequalities. Section 7 proves finite attainability and the lower bound allowing unequal ordinary moments.

7 The infimum of the positive comparison

In this section, we determine the infimum of the positive comparison using the moment ranges of Section 6. We prove the upper bound by finite constructions in Subsection 7.1 and the lower bound for unequal ordinary moments and variable lengths in Subsections 7.2 and 7.3.

We now allow the whole limiting moment range and all unequal two-ordinary moments, rather than retaining only the numerator estimate used in Section 6.

Definition 7.1 (The positive comparison class)**.** Let $\mathcal{S}_{\mathrm{pos}}$ consist of the positive comparisons using the earlier direct majorant, pointwise amplitude bounds, ordinary reflection, Hölder's inequality and the long inverse exponent

$$e(r)=\max\{1,(1+5r)/6\}.$$

The comparison parameters satisfy

$$l_x,l_y>0,\qquad \ell\geq 0,\qquad b=l_y-l_x\geq 0,\qquad 1/2<a\leq 51/100.$$

Allow every finite joint moment construction satisfying the fourteen strict inequalities of Subsection 8.7, every permitted zero-detection cutoff and all unequal two-ordinary moments supplied by these constructions. Earlier moment estimates may be used in any order, with nonlinear or piecewise reference boundaries. The short- and long-sum estimates for large values and the bounds for small values are retained. Only the auxiliary positive margins, supporting-line and cutoff decrements, endpoint decrements and rounding errors may tend to zero, after fixing a strict target.

For the lower bound, enlarge the ordinary prime-length bound to $S<1/6$ and the new reference prime-length bound to $1/6$. Also allow every finite step $0<u<1/6$, remove the reference cutoff floor, and omit the restrictions on the number of available prime factors and on reflection. Proposition 6.3 bounds these enlarged ranges by $\widehat{S}$ and the corresponding convex majorant $\mathcal{G}$. Allow the initial range

$$S\leq \mathcal{C}(P):=(1-P)/2,\qquad 0\leq P\leq 1,$$

and every improvement permitted by $\widehat{S}$ and $\mathcal{G}$, with the same bounds for independent fixed unitary twists.

Theorem 7.2 (The comparison infimum)**.** *With r_* defined by (42), the infimum for Definition 7.1 is*

$$\boxed{B_*=\frac{99r_*-14}{12(9r_*-1)}=0.873345856106718576801184796065072356 77\ldots.} \tag{44}$$

The lower bound allows arbitrary finite step sizes and repeated use of earlier estimates, reference mark cap $1/6$*, removal of the cutoff floor, and all granted unequal ordinary moments. In particular* θ_{240} *is not the infimum of this larger comparison class. The obstruction includes positive interpolation of all unequal two-ordinary moments and arbitrary* $b \geq 0$*; it does not apply to a new mixed estimate using cancellation between inverse and ordinary sums.*

The number B_* has algebraic degree nine. Its primitive minimal polynomial is

$$\begin{aligned}\mathcal{P}(B) = {} & 487371101862528B^9 - 4126408710053184B^8 \\ & + 15517896164111592B^7 - 34020380857037616B^6 \\ & + 47916748658170215B^5 - 44964545308854309B^4 \\ & + 28111661257098222B^3 - 11291220876239226B^2 \\ & + 2643837682856883B - 274959111914881.\end{aligned}$$

It is obtained from the degree-nine polynomial for r_* by $r = (12B - 14)/(108B - 99)$. Exact bisection isolates the indicated root to more than 70 decimal places. Irreducibility follows modulo 11: if $f \in \mathbb{F}_{11}[X]$ is the monic reduction, then

$$X^{11^9} \equiv X \pmod{f}, \qquad \gcd(f, X^{11^3} - X) = 1.$$

These are the Rabin conditions for degree nine.

7.1 The limiting inequalities and finite constructions

In this subsection, we combine the finite approximation of Subsection 6.1 with the exact comparison inequalities. This proves the upper bound for the infimum in Theorem 7.2.

Set

$$\begin{aligned}\delta_* &= \frac{45r_* - 14}{6(9r_* + 1)}, \qquad h_* = \frac{2(9r_* + 2)(45r_* - 14)}{(9r_* - 1)(117r_* - 14)}, \\ g_0 &= \frac{17 - 18r_*}{3(45r_* - 14)}, \quad g_1 = \frac{135r_*^2 - 42r_* - 38}{6r_*(45r_* - 14)}.\end{aligned} \tag{45}$$

Then $g_0 - g_1 r$ is the tangent to G at r_*, and

$$h_* = 0.815324503858647590779765597 5\ldots, \qquad \delta_* = 0.425830047444893341432112395 1\ldots.$$

Use the expressions of Theorem 5.2, with

$$\kappa = \kappa_1 = 374887/500000, \qquad B_C = B_P = 1, \qquad L_C = L_P = 2.$$

Here κ is fixed independently of B in the old count and old numerator estimate; all auxiliary losses tend to zero. Use the original initial range and also

$$[15/64, 17/32] \times [1/4, 1/2]. \tag{46}$$

Conditions (N) are strictly positive on this larger range. The ideal tangent satisfies conditions (I1)–(I10) with $b = 7/9$; since (I6)–(I8) are strict at $b = 7/9$, they persist for $b < 7/9$ close to $7/9$.

We verify the limiting comparison as follows. Clear positive denominators and write Q_j for its numerator. At the top parameter and $x = 1/2$,

$$Q_3(\delta, 1/2) = A_*(\delta - \delta_*)^2, \qquad A_* > 0. \tag{47}$$

On the contact rectangle $[27/64, 57/128] \times [31/64, 1/2]$, $-\partial_x Q_3 > 0$. Away from this rectangle an exact rational Bernstein cover gives strict positivity on all four boundary segments:

edge	0	1	2	3
ordinary subrectangles	1	2	2	37
contact subrectangles	0	0	0	1

The maximum subdivision depth is 13. The coefficients are uniform on a rational interval of width less than 10^{-35} isolating r_*. Thus the limiting comparison margin is nonnegative, with its only possible zero at $(\delta_*, 1/2)$ on the top edge. The inequalities of Proposition 8.26 (B) other than $\Sigma_N - \mathcal{E}_0 \geq 0$ have strictly positive margins on compact subrectangles, and $\Sigma_N - \mathcal{E}_0$ is bounded below by the third and fourth edge margins. They therefore persist for the perturbed lines and for $B > B_*$ close to B_* (Corollary 8.28).

For $B > B_*$ close to B_*, fix $h = h_*$ and the tangent line. At the top parameter,

$$\partial_B\{J(B - E) - 3hN/4\} = J(3 + 8\delta x) > 0.$$

The denominator and range inequalities have strict margins. Compactness, the finite cover and Proposition 6.4 therefore give a finite rational chain with positive edge margins for every such fixed target. The direct inequalities are made strict by using an intermediate boundary $B' \in (B_*, B)$, as in Theorem 8.29. Put $a_- = 1467/2000$, $a = 367/500$, and $\lambda_*(R) = g_0 - g_1 R$. Exact endpoint bounds give

$$\lambda_*(a_-) < G(a) - (a - a_-)/2.$$

Since $R \mapsto (1 - R)/2 + \lambda_*(R)$ is decreasing,

$$\frac{1 - R}{2} + \lambda_*(R) < \frac{1 - a}{2} + G(a) \qquad (a_- \leq R \leq a).$$

Choose the finite approximation error smaller than this strict gap. Truncating the free length of its last moment range supplies this extension. On $[a, b]$, choose the supporting-line approximation with $|\lambda_j - G| < \varepsilon$. Then

$$\lambda_* - 2\varepsilon \leq \lambda_j - \varepsilon \qquad (R \in [R_{j+1}, R_j]).$$

Choose rational coefficients for a line $\lambda_{\mathbb{Q}}$ with

$$\lambda_* - 3\varepsilon \leq \lambda_{\mathbb{Q}} \leq \lambda_* - 2\varepsilon \qquad (R \in [a_-, b]).$$

The denominators in the earlier estimates have a positive minimum on their compact ranges. All count functions are therefore uniformly continuous in the line coefficients. Choosing ε after the strict comparison margin proves the finite rational assertion.

The floor imposes no limiting restriction after retuning. Its general exponent at scale $Nu \asymp Z^v$ is

$$E_a(v) = \frac{4}{25} l_x + a - \frac{33}{50} - a l_y + (a + 4/25)v + (a - 33/50)\ell. \tag{48}$$

Let $C(B) = B - (4 + l_y - l_x)/6$ be the principal Mellin exponent. Using (14),

$$\begin{aligned} C(B) &= B - 1/2 - h/3 + \ell/2, \\ E_{51/100}(h) &= 1/100 + (103/100)\ell, \\ E_a(h) - E_{51/100}(h) &= (a - 51/100)(1 + 3\ell), \\ C(B) - E_a(h) &= B - 51/100 - h/3 - (53/100)\ell \\ &\quad + (51/100 - a)(1 + 3\ell). \end{aligned}$$

At the limiting parameters, $a = 509/1000$ gives a floor margin greater than 10^{-3} by rational interval bounds. Proposition 8.22 gives the corresponding integration path and the estimate for the compensated cubic product.

7.2 Amplitudes satisfying all unequal ordinary moments

In this subsection, we give simultaneous amplitudes satisfying the permitted unequal ordinary moments. Subsection 7.3 uses these amplitudes to prove the lower bound in Theorem 7.2.

For the necessary direction put

$$m_D = r_*/2, \quad z_* = G(r_*), \quad c_* = m_D + z_*, \quad R_* = 1 - \delta_*(1/2 + c_*).$$

Use the convex majorant for all earlier estimates $\mathcal{G}$ of (41), which agrees with G throughout $[R_+, 7/9]$. For the necessary direction, allow every marked length up to $Z(P) := \widehat{S}(P)$, including the initial rectangles. Then

$$Z(P) \le \mathcal{C}(P) + \mathcal{G}(P) \quad (2/3 < P \le 1), \qquad Z(P) = \mathcal{C}(P) \quad (0 \le P \le 2/3). \tag{49}$$

For the enlarged class used in the obstruction, the granted marked cap gives the global bound

$$Z(P) \le \max\{\mathcal{C}(P), 1/6\} \le \mathcal{C}(P) + 1/6 \qquad (0 \le P \le 1). \tag{50}$$

This necessary direction grants mark cap $1/6$, removes the cutoff floor, and allows every finite step $0 < u < 1/6$. Proposition 6.3 includes arbitrary repeated use of earlier estimates and free-width truncation. It therefore grants more reference marked length than the original fourteen-condition construction.

On a set of $\lfloor U^{R_*} \rfloor \asymp U^{R_*}$ parameters, give every prime subproduct squared amplitude $U^{\delta_* z}$ and put

$$s(y) = \begin{cases} 0, & 0 \le y < 1/3, \\ \min\left\{ \dfrac{\delta_* y}{2}, \dfrac{\delta_*}{4}(y + c_* - \mathcal{G}(2y)) \right\}, & 1/3 \le y \le 1/2. \end{cases} \qquad |S_y| = U^{s(y)}. \tag{51}$$

On $[1/3, R_+/2]$, $y + \mathcal{G}(2y) = M_{\mathrm{ref}} - 1/2 < c_*$. On $[R_+/2, m_D]$,

$$\frac{d}{dy}\{y + G(2y)\} = 1 + 2G'(2y) > 0, \qquad m_D + G(2m_D) = c_*.$$

Therefore $s(y) = \delta_* y/2$ for $1/3 \le y \le m_D$. If both factors are at least $1/3$, convexity gives

$$\mathcal{G}(y_1 + y_2) \le \frac{\mathcal{G}(2y_1) + \mathcal{G}(2y_2)}{2},$$

and hence

$$R_* + 2s(y_1) + 2s(y_2) + \delta_* Z(y_1 + y_2) \le 1. \tag{52}$$

If one factor is below $1/3$, the excess is at most

$$-\delta_*\{c_*/2 + \min(y_1, y_2)/2 - 1/6\} < 0.$$

If both are below $1/3$, it is at most $-\delta_* c_* < 0$. Thus every unequal ordinary moment is respected simultaneously. Assigning the same length-dependent amplitude to every fixed unitary twist also respects all ordinary moments with two fixed unitary twists at these bounds on marked length. Pointwise ordinary bounds hold by construction. For $1/2 < y \le 1$ use the reflected amplitude at

$1-y$; for $y>1$ assign zero amplitude to the empty-dual limiting model. For $0\le y_i\le 1$, put $\widetilde{y}_i=\min(y_i,1-y_i)$. Then $\widetilde{y}_1+\widetilde{y}_2\le y_1+y_2$, and

$$Z(y_1+y_2)\le Z(\widetilde{y}_1+\widetilde{y}_2).$$

Thus the inequalities already proved apply to every reflected pair. Set $Z(P)=0$ for $P>1$. The pointwise and unmarked estimates are also preserved. Apply the same extension to the small-numerator amplitude below.

The inverse amplitudes may be assigned as in the known obstruction: $|M_{r_*}|^2=U^{\delta_* r_*}$, $|M_r|^2=U^{\delta_* r}$ for $3r_*/2\le r\le 3/2$, and 1 elsewhere. At the two critical splits,

$$R_*+\delta_* r_*+\delta_* Z(r_*)=1,\qquad R_*+\delta_*\frac{3r_*}{2}=\frac{1+5(3r_*/2)}{6}.$$

The other short inverse inequalities are strict; the long inverse margin is $(5/6-\delta_*)(r-3r_*/2)\ge 0$. Put $t_0=3r_*/2$. For any split $r+m=t$,

$$t\le t_0\implies r\le r_*\text{ or }m\le m_D,\qquad t\ge t_0\implies \frac{1}{6}+(5/6-\delta_*)t\ge R_*.$$

The short-moment endpoint equalities and the long inverse inequality therefore permit count exponent R_* for every supplied cutoff. The selected operators inherited from the earlier paper have prime-amplitude thresholds above $1/2$; all positions in this model have squared amplitude slope $\delta_*<1/2$, so those operators impose no additional saving. The same observation applies to the $\delta_0=37/100$ model below. These are models of inequalities, not assertions that arithmetic parameters realize the assigned amplitudes.

7.3 Variable comparison lengths and the numerator estimate

In this subsection, we allow the comparison lengths to vary and complete the lower bound in Theorem 7.2. The proof uses the amplitudes of Subsection 7.2 and separates the possible numerator lengths.

Since $8741/10000>B_*$, it suffices to consider $B\le 8741/10000$. Let $l_x,l_y>0$ and $\ell\ge 0$, and put

$$M=l_x+l_y,\quad b=l_y-l_x\ge 0,\quad h=1-(M-b)/2+\ell,\quad n=1-l_y/h.$$

The direct majorant [5, Proposition 6.1(4), (6.13)] gives the three planes [5, (6.51)] (its $h=1-l_x+\ell$ agrees with h here, since $M-b=2l_x$):

$$B\ge L_0,\quad B\ge L_K,\quad B\ge L_0+G_0,\qquad G_0=2b/3-M/4.$$

For $\mu=37/100$, the identity [5, (6.52)] is

$$h-l_y-\mu h=\frac{11(4-3\mu)}{6}-\left(\frac{22}{3}-\frac{25\mu}{4}\right)L_0-\left(\frac{2}{3}+\frac{\mu}{4}\right)L_K-\frac{3\mu}{4}(L_0+G_0).$$

All three subtracted coefficients are positive. If $B\le 0.8741$, the right side is at least $41671/12000000>0$. Also $M\le(12/7)(B-1/3)<1$ and $h\ge 1-M+\ell>0$. Hence $n>0.37$. If $7/18\le n<1/2$, then $\mathcal{G}(2n)=0$ and $c_*<19/50<7/18\le n$. Consequently (51) gives

$$R_*+s(n)=\frac{3R_*+1-\delta_*\mathcal{C}(2n)}{4}.$$

Thus all the additional unequal moments permit the same adverse numerator value as the earlier fourth-moment estimate. The plane H in Proposition 6.5 is necessary, proving $B \ge B_*$. For $n \ge 1/2$ the estimate from the approximate functional equation uses its half-length endpoint. The same model has $R_* + 4s(1/2) = 1$ and therefore permits exactly the unmarked endpoint value $(3R_* + 1)/4$ of the no-prime fourth moment [5, proof of Lemma 6.3]. The unmarked endpoint obstruction gives the stronger bound. Explicitly, put $\delta = \delta_*$,

$$H_{1/2} = 1 + 5\delta/8 - (1/2 + \delta/16)M + \delta b/16 + 5\delta\ell/8, \qquad D_e = h - 2l_y \ge 0,$$

$$q_e = 19 - 21\delta/2, \quad v_e = 13/q_e, \quad u_e = (6 - 21\delta/2)/q_e, \quad w_e = u_e/3 + 5\delta v_e/8, \quad z_e = \delta v_e/16 + w_e/2.$$

The exact dual identity is

$$B - v_e = u_e(B - L_K) + v_e(B - H_{1/2}) + w_e D_e + z_e b.$$

All weights are positive since $21/50 < \delta_* < 1/2$. Therefore

$$B \ge \frac{13}{19 - (21/2)\delta_*} > 0.89.$$

It remains to exclude $0.37 < n < 7/18$, where the new numerator moments may genuinely be stronger. Set $c = 1/250$. At $\delta_0 = 37/100$ its balanced contact is rational:

$$r_0 = \frac{24257}{33000}, \quad m_0 = r_0/2 < 0.37, \quad R_0 = \frac{4471723}{6600000}.$$

Use $\lfloor U^{R_0} \rfloor$ parameters, prime squared amplitudes $U^{\delta_0 z}$, and the inverse assignment above with (r_*, δ_*, R_*) replaced by (r_0, δ_0, R_0). The endpoint identities are

$$R_0 + \delta_0\{r_0 + \mathcal{C}(r_0) + c\} = 1, \qquad R_0 + \delta_0(3r_0/2) = \{1 + 5(3r_0/2)\}/6.$$

To respect even the stronger limiting moment estimates at shorter lengths, use the convex majorant

$$\widetilde{G}_0(P) = \begin{cases} \mathcal{G}(P) + c - \mathcal{G}(r_0), & 2/3 \le P \le r_0, \\ c, & r_0 \le P \le 1. \end{cases}$$

Here $0 < \mathcal{G}(r_0) = G(r_0) < c$, and $\mathcal{G}'(r_0) < 0$, so the function is convex and majorizes the whole available improvement. Use the amplitude function

$$s_0(y) = \min\{\delta_0 y/2, \delta_0[y + m_0 + c - \widetilde{G}_0(2y)]/4\}$$

for $1/3 \le y \le 1/2$, and zero below $1/3$; extend by reflection for $1/2 < y \le 1$ as in Subsection 7.2. The same convexity calculation as in (52) proves all unequal moment bounds. For a single shorter factor, the inequality is strict because $(m_0 + c)/2 > 1/6$. The endpoint inequality $m_0 + G(r_0) > M_{\mathrm{ref}} - 1/2$ ensures $s_0(y) = \delta_0 y/2$ for $1/3 \le y \le m_0$. At the detector length $s_0(m_0) = \delta_0 m_0/2$; at every numerator length $n > 0.37 > m_0$, $s_0(n) = \delta_0(n + m_0)/4$. Thus its adverse numerator value is precisely that obtained from the marked-length bound $\mathcal{C}(2n) + c$. Its necessary numerator plane is

$$H_c = \frac{1 + \delta_0}{2} - \frac{l_y}{2} + h\left(\frac{3R_0 + 1}{4} - \frac{1}{6}\right) + \frac{\delta_0 - 1}{2}\ell - \frac{\delta_0(2l_y - h)}{8} - \frac{h\delta_0 c}{4}.$$

Set $q = 7/18$ and define the nonnegative differences

$$A = B - L_0, \quad B' = B - L_K, \quad C = B - H_c, \quad D = qh - (h - l_y).$$

Exact coefficient comparison gives

$$B - \frac{72703411}{83204812} = \frac{57303411}{83204812}A + \frac{10501401}{83204812}B' + \frac{3850000}{20801203}C + \frac{3553203}{166409624}D.$$

All coefficients are positive. Hence this entire range forces

$$B \geq \frac{72703411}{83204812} = 0.873788537614867755 48\ldots > B_*.$$

Together with the preceding cases, this proves the lower bound in Theorem 7.2. The limiting comparison inequalities and finite approximation prove the upper bound.

Theorem 7.2 concerns the stated positive inequalities. Section 8 proves the analytic induction and the choice of the lower integration limit, yielding Theorem 8.29.

8 The joint moment induction over number fields

In this section, we prove the joint moment induction underlying the constructions of Sections 5–7. The argument combines Gauss-sum identities, Poisson summation, complete-factor selection and the reference ordinary moment estimate. Subsection 8.12 applies these estimates to the finite-order zero-free theorem.

Let $F \supset \mu_{12}$ be fixed and totally complex. All implied constants below may depend on F, the excluded places S, and fixed coefficient characters. Polynomial height exponents must remain finite, and the prime mesh depends only on F, ε and the numerical margins, uniformly in the test supports, characters and the constant C_0 of Subsection 2.2.

8.1 Gauss sums and ideal sums

In this subsection, we record the Gauss-sum and ideal-sum identities used in Subsections 8.2–8.5. The identities hold over the fixed number field without a restriction on its residue degrees.

Lemma 8.1 (Fourth-power Gauss identity)**.** *Let k be a finite field of order q, ψ a nontrivial additive character, and χ a character of exact order three. Extend χ by zero and put*

$$\tau = \sum_{a \in k} \chi(a)\psi(a), \qquad J = \sum_{a \in k} \chi(a)\chi(1-a), \qquad g = \tau/\sqrt{q}.$$

Then

$$\tau^3 = qJ, \quad |J| = \sqrt{q}, \quad g^4 = \frac{J}{\sqrt{q}}\, g.$$

Proof. The change of variables $a + b = t$ and Gauss conjugation give

$$\tau(\chi)^2 = J\tau(\chi^2), \qquad \tau(\chi)\tau(\chi^2) = \chi(-1)q = q, \qquad |\tau(\chi)| = |\tau(\chi^2)| = \sqrt{q}.$$

Consequently

$$\tau^3 = qJ, \qquad |J| = \sqrt{q}, \qquad g^4 = \tau^4/q^2 = (J/\sqrt{q})g.$$

□

This is the local power identity used to select remaining prime positions. For a number field, its phase $J/\sqrt{q}$ must be retained in the actual coefficient character; it cannot be replaced by its absolute value before the transformed polynomial is identified.

Lemma 8.2 (Ideal sums and injective marking)**.** *Let $\{\mathfrak{p}_i\}$ be distinct nonzero prime ideals outside S. The map*

$$U \longmapsto \prod_{i\in U} \mathfrak{p}_i$$

is injective on finite subsets. For $X \geq 1$, its image of norm at most X has $O_F(X)$ elements, and

$$\sum_{N\mathfrak{a}\leq X} \frac{1}{N\mathfrak{a}} \ll_F 1 + \log X.$$

Fixed-degree divisor multiplicities are $O_{F,\varepsilon}(X^{\varepsilon})$.

Proof. Unique factorization of nonzero ideals proves injectivity. The ideal counting bound and partial summation prove the second and third assertions. The last is the usual fixed-degree ideal-divisor bound. □

Lemma 8.3 (Norm-scaled archimedean Poisson homogeneity)**.** *Let $n = [F : \mathbb{Q}]$, $Y > 0$, and let f be a Schwartz function on $F \otimes \mathbb{R}$. For $f_Y(v) = f(Y^{-1/n}v)$, with self-dual additive measure,*

$$\widehat{f_Y}(\xi) = Y\widehat{f}(Y^{1/n}\xi).$$

Proof. Substitute $v = Y^{1/n}w$ in the n-dimensional Fourier integral. The real Jacobian is $(Y^{1/n})^n = Y$. □

The lemma explains why the leading norm-volume exponent need not grow with the degree. It does not by itself prove the required dual-parameter estimate. In higher degree, unit orbits and the full archimedean shape must be included. Replacing $N\alpha$ by one complex embedding would be invalid when the unit rank is positive.

8.2 Conversion of cubic Gauss coefficients

In this subsection, we convert cubic Gauss coefficients to Möbius coefficients in every residue degree. Subsection 8.4 uses the identity in the transformed inverse second moment.

Let $\mathbb{F}_q$ have odd characteristic, $q \equiv 1 \pmod 6$, and let χ have order six. Fix a nontrivial additive character and put $g_j = q^{-1/2}\sum_x \chi(x)^j\psi(x)$ for $1 \leq j \leq 5$, with every nontrivial multiplicative character zero at zero. Define

$$A = -g_2^3, \qquad C = \overline{A}\, g_2, \qquad G = \overline{\chi(4)}g_3.$$

Lemma 8.4 (Cubic-to-Möbius conversion)**.** *For every such finite field,*

$$|A| = |C| = |G| = 1, \quad g_1g_2 = Gg_2^3, \quad -g_1\overline{G} = \overline{C}, \quad -g_5G = \chi(-1)C.$$

Proof. The Gauss norm and conjugation identities give $|g_j| = 1$ and $g_4 = \overline{g_2}$. The duplication identity is

$$g_1g_4 = \overline{\chi(4)}g_2g_3.$$

For clarity it follows directly from

$$J(\chi,\chi) = \sum_y \chi(y)\{1 + \chi^3(1-4y)\} = \overline{\chi(4)}J(\chi,\chi^3)$$

and $J(\alpha,\beta)=\sqrt{q}\,g(\alpha)g(\beta)/g(\alpha\beta)$. Thus $g_1=Gg_2^2$. Since $A=-g_2^3$, $C=-\overline{g_2}^{\,2}$, which proves the first conversion. Finally $g_5=\chi(-1)\overline{g_1}$ proves the other orientation. No degree-one residue-field assumption occurs. □

For squarefree coprime ideal supports, CRT gives the cross factor $\chi_a(b)\chi_b(a)$, whose sixth power is one. Consequently the multiplicative extension of $A=-g_2^3$ is compatible with the Jacobi character in the normalization of [6, Lemma 3.4]. After the conversion of Lemma 8.4, the only cross factor is

$$\Omega(a,b)=\chi_a(b)^3\chi_b(a)^3.$$

By [6, Lemma 3.4], allowed generators satisfy $\chi_a(b)=\chi_b(a)$ and $\chi_a(\epsilon b)=\chi_a(\epsilon)\chi_b(a)$ for $\epsilon\in U_S$. Hence, for allowed a,b, $\Omega(a,b)=\chi_b(a)^6=1$. A general generator differs from an allowed one by a unit class in U_S/U_S^6, and this contributes $\chi_b(\epsilon)$, a character of b of conductor supported on S. Locally $G=\overline{\chi(4)}g_3$ with $g_3^2=\chi(-1)^3=\pm1$ (quadratic Gauss identity), and the global $G(a)$ is a finite-order ray-class character by [6, Lemma 4.2 and its proof]. Explicitly, on a fixed finite ray group $\mathcal{R}$ one has finite expansions

$$G(a)=\sum_{\xi\in\widehat{\mathcal{R}}}c_\xi\xi(a),\qquad \Omega(a,b)=\sum_{\xi,\eta\in\widehat{\mathcal{R}}}c_{\xi,\eta}\xi(a)\eta(b).$$

The sums of absolute coefficients are constants depending on F,S. The cubic Gauss factor itself is never replaced by an arbitrary bounded coefficient or by a finite ray phase.

8.3 Poisson summation with coprimality restrictions

In this subsection, we impose the coprimality conditions before applying lattice Poisson summation. The resulting identity is used in the second-moment estimate of Subsection 8.4.

Let L be a fixed fractional-ideal lattice in F_∞, and let $\mathfrak{q},\mathfrak{e}$ be integral ideals prime to its fixed presentation denominators. Write L^* for the trace-dual lattice, and use the Fourier convention $\widehat{f}(y)=\int f(x)e(-\operatorname{Tr}(xy))\,dx$. For $x\in L$, put $\mathfrak{a}_x=(x)L^{-1}\subseteq\mathcal{O}_F$. Thus $\mathfrak{e}\mid\mathfrak{a}_x$ if and only if $x\in\mathfrak{e}L$. On the fixed ideal-class presentation, let ϑ be a character modulo $\mathfrak{q}$, and define

$$\mathcal{G}_{\mathfrak{e},L}(y)=\sum_{a\in\mathfrak{e}L/\mathfrak{q}\mathfrak{e}L}\vartheta(a)e(\operatorname{Tr}(ay)).$$

All sums below use the fixed ideal-class and ray charts of the unit-orbit Poisson formula [6, Proposition 8.1].

Lemma 8.5 (Coprimality-restricted pair Poisson). *For $f\in\mathcal{S}(F_\infty)$ and squarefree $\mathfrak{g}$ prime to $\mathfrak{q}$,*

$$\sum_{x\in L}f(x)\vartheta(x)1_{(\mathfrak{a}_x,\mathfrak{g})=1}$$
$$=\sum_{\mathfrak{e}\mid\mathfrak{g}}\frac{\mu_F(\mathfrak{e})}{\operatorname{covol}(\mathfrak{q}\mathfrak{e}L)}\sum_{y\in(\mathfrak{q}\mathfrak{e}L)^*}\widehat{f}(y)\mathcal{G}_{\mathfrak{e},L}(y).$$

If ϑ is primitive, every nonzero Gauss coefficient has absolute value $\sqrt{N\mathfrak{q}}$. For two squarefree sextic ideal variables $U_1=ga$, $U_2=gb$, with $g=(U_1,U_2)$, so that $(a,b)=(g,ab)=1$, this identity applies with $\vartheta=\overline{\chi_a}\chi_b$ and $\mathfrak{q}=ab$.

Proof. Expand the coprimality indicator as $\sum_{\mathfrak{e}\mid\mathfrak{g}}\mu_F(\mathfrak{e})1_{x\in\mathfrak{e}L}$, split each lattice into its $\mathfrak{q}\mathfrak{e}L$ cosets, and apply the lattice Poisson formula on each coset. All series converge absolutely. The primitive Gauss norm follows from finite additive orthogonality. The shared-prime factor in $\overline{\chi_{U_1}(x)}\chi_{U_2}(x)$ is precisely the coprimality indicator at g; the difference supports are disjoint. Fixed charts and the finite family of S-unit characters affect only the constant. □

8.4 The second Poisson estimate after selection

In this subsection, we estimate the inverse second moment after complete-factor selection. We combine the conversion identity of Subsection 8.2 with the Poisson formula of Subsection 8.3; Subsection 8.5 then gives the required length exponents.

We apply the pair identity to the second moment. Let $X = Z^{N'}$, $T = Z^{K'}$ and $F' = N' + V$. For a squarefree ideal n let $C(n)$ be the canonical normalized cubic Gauss coefficient, and let $M_I(n)$ be an injective mark on the original, possibly overlapping, prime lists, with the convention (3) and $|b_i(p)| \le 1$. The fixed completely multiplicative coefficient Ψ satisfies $|\Psi(n)| \le 1$. Proposition 8.6 uses only these two properties; in the induction, and in Proposition 8.9, Ψ is in addition a Hecke character in $\mathcal{F}_\omega$ fixed before Z, (5), so that the summation characters $\psi_y = \Psi'\overline{\chi_\cdot(y)}$ satisfy (6). Put

$$c_{f,m}(n) = C(n)\Psi(n)1_{(n,m)=1}\chi_n(f)^4,$$

with the same natural zero extensions as in the known canonical inverse family. For one compactly supported norm test function W, define

$$\mathcal{E}_{\mathrm{sel}} = Z^{-F'}\sum_f D_f \sum_{0<Nk\le T}\Bigg|\sum_{n\ \text{squarefree}} c_{f,m}(n)\chi_n(k)M_I(n)W(Nn/X)\Bigg|^2,$$
$$\mathcal{D} = X^{-1}\sum_n |M_I(n)W(Nn/X)|^2.$$

The k-sum is on allowed unit orbits. Assume $Nf \le Z^V$, $0 \le D_f \le D_{\max}$, $N', V, K' \ge 0$, $N'+V \le 2$ and $K' \le 4$.

A bound from an earlier moment estimate $\mathcal{B}$ means the following uniform bound for the actual Möbius polynomial and its whole conjugate:

$$\sum_{0<Ny\le Z^{H_c}}\Bigg|\sum_n \frac{\mu_F(n)\Psi'(n)\overline{\chi_n(y)}}{\sqrt{Nn}}1_{(n,m')=1}M_{I\setminus J}(n)\Phi(Nn/X_c)\Bigg|^2 \le \mathcal{B}(1+\|\mathbf{t}\|)^{2d}.$$

Here Ψ' ranges over the finite family of ray-class characters times Ψ; the whole conjugate conjugates this coefficient and its character orientation together. The bound is required uniformly in m' with $Nm' \le Z^{\varrho_0}$, on the range of Subsection 2.3 normalized by the parameter exponent H_c:

$$\frac{\log_Z X_c - \sum_{i\in I\setminus J} z_i}{H_c} < R, \qquad \frac{\sum_{i\in I\setminus J} z_i}{H_c} < \frac{1-R}{2} + z_R. \tag{53}$$

The marks $M_{I\setminus J}(n)$ lie inside n, so $\log_Z X_c$ already contains $\sum_{i\in I\setminus J} z_i$; the free length is the first numerator. Since $n \mapsto \Psi'(n)\overline{\chi_n(y)}$ is completely multiplicative, (56) with $g = \mathrm{rad}(m')$ writes the restricted polynomial as a combination, of absolute mass $\ll (Nm')^{\varepsilon} \le Z^{\varrho_0\varepsilon}$, of unrestricted polynomials of scales X_c/Nd with marks $I \setminus (J \cup J')$. Their free lengths and marked lengths do not exceed those in (53), so uniformity in m' follows from the case $m' = 1$ by Cauchy's inequality. Here $J \subseteq I$ consists of removed original positions. Their extracted prime product d_J satisfies $X_c N d_J \ll X$. The remaining mark $M_{I\setminus J}$ is injective; repeated primes in removed positions occur only in the scalar restoration coefficients. Common independent norm Fourier modes are included in $\mathbf{t}$. The test functions belong to a finite family chosen from W before the arithmetic indices.

Proposition 8.6 (Selected second Poisson)**.** *For every $\varepsilon, \kappa_{\mathrm{P}} > 0$, there are finite derivative and height orders and a constant C such that the bound from an earlier moment estimate at*

$$H_c = 2(N' + V) - K' + \kappa_{\mathrm{P}}$$

implies

$$\mathcal{E}_{\mathrm{sel}} \le CZ^{K'-F'+\varepsilon} D_{\max}(1+\|\mathfrak{t}_0\|)^A \mathcal{B} + CZ^{K'+\varepsilon} D_{\max}\left(Z^{-V}\#\{f\}\right)\mathcal{D}. \tag{54}$$

This implication is valid over every fixed $F \supset \mu_{12}$. It does not assume finite order for Ψ; the earlier moment bounds must still be supplied for its character family.

Proof. First majorize the whole positive second moment by the nonnegative average over fibers of the norm map described below, and only then expand its square. Separate its diagonal pairs; they are estimated by the diagonal below. For every other pair put $U_1 = ga$, $U_2 = gb$; the primitive parameter modulus is $ab \neq 1$. Use the unit-orbit Poisson formula [6, Proposition 8.1] on norm fibers [6, Proposition 8.1], equivalently the orbit average of Lemma 8.5. A fixed nonnegative Gaussian test function has integral over norm fibers $\Psi_\Phi(x) \gg 1$ for $0 < x \le 1$: keep all but one archimedean radius in fixed positive intervals and let the last radius supply the norm x. Hence it majorizes the whole set of ideal parameters, after fixed normalization. The logarithmic small-norm bound, rapid large-norm decay and ideal counting also give

$$\sum_{[k]\neq 0} \Psi_\Phi(Nk/T) \ll_{F,\Phi} T.$$

Thus the contribution from identical ideal variables is bounded by the same diagonal estimate. The primitive character is already zero at the additive origin, so the convention $\Phi(0) = 0$ is unnecessary on these nontrivial-character terms. Both origin terms vanish before absolute values. The resulting nonzero norm kernel is

$$K_\Phi\left(\frac{TNk}{NeNaNb}\right), \qquad K_\Phi = \Psi_{\widehat{\Phi}},$$

with the known logarithmic small-norm bounds and arbitrary large-norm decay for every fixed Euler derivative. The Poisson factor is

$$\frac{T\mu_F(e)}{Ne\sqrt{NaNb}}$$

and the dual kernel, with $e \mid g$, $g = eg'$. The canonical Gauss coefficients are converted before absolute values, using Lemma 8.4 on each exterior prime and CRT. Apart from the fixed finite family of quadratic characters, the two surviving factors are exactly

$$\mu_F(a)\Psi'(a)1_{(a,m')=1}\overline{\chi_a(y)}, \qquad \mu_F(b)\Psi''(b)1_{(b,m')=1}\overline{\chi_b(y)}, \qquad y = f^2eg'^2k.$$

The cubic Gauss coefficients have canceled; the Möbius signs have not been discarded. The factors $1/\sqrt{NaNb}$ are exactly the paired normalizations at the earlier moment estimate.

The radius computation is especially useful. On the original ideal sum annulus $N(ga), N(gb) \asymp X$,

$$\frac{TNk}{NeNaNb} \asymp \frac{T\,Ny}{(Nf)^2X^2}, \qquad y = f^2eg'^2k. \tag{55}$$

Thus outside $Ny \le Z^{2(N'+V)-K'+\kappa_{\mathrm{P}}}$ the Fourier kernel has argument at least a fixed multiple of Z_{P}^{κ}.

For the marks use the literal finite identity

$$M_I(g \sqcup a) = \sum_{J\subseteq I} M_J(g)M_{I\setminus J}(a).$$

It is valid on overlapping lists because every tuple is injective. A fixed tuple divides g, so its contribution is included through the smaller summation scale $X_c \asymp X/Ng$. The scalar factors

depending only on g are bounded by fixed divisor powers. They introduce no factor coefficient depending on the summation parameter.

The exterior coprimality is retained until it is separated exactly. After evaluating the primitive Gauss identities on the original coprime pairs, extend their finite reciprocity character expansion by its ray-character formula to all pairs and insert

$$1_{(a,b)=1} = \sum_{d|a,\ d|b} \mu_F(d), \qquad a = da', \quad b = db'.$$

No primitive Gauss identity is applied on the extended noncoprime locus. The paired central normalization is

$$\frac{1}{\sqrt{NaNb}} = \frac{1}{Nd}\frac{1}{\sqrt{Na'Nb'}}.$$

The resulting sums have scale $X_c \asymp X/(NgNd)$ and retain $1_{(a',gd)=1}$ and $1_{(b',gd)=1}$. Split the original marks between gd and the exterior ideal sum before freezing a tuple. The outer character factors have modulus at most one. Their common factor in the ideal parameter $|\chi_d(y)|^2$ is its natural coprimality indicator; it may be bounded by one after forming the positive resulting second moment. The ideal parameter remains $y = f^2eg'^2k$, and the new divisor sum is harmonic:

$$\sum_{Nd \ll X/Ng} \frac{1}{Nd} \ll_F 1 + \log(2X).$$

Thus d introduces no additional power from the multiplicity of the ideal parameter. Restore coprimality conditions in the resulting sum by (56). The remaining marks have their original coefficients; all factors at the extracted ideals are in the scalar coefficients $B_J(d)$. No new restriction is imposed on the quotient.

The unit-orbit average over norm fibers has already integrated the archimedean shape coordinates. Separate only the actual independent ideal-norm ratios and K_Φ, with the common Sobolev–Mellin measure of [6, Section 8], before fixing g, e, f, k, d or a fixed tuple. Its density is integrable after every required fixed height weight and contributes only a fixed power of $\log Z$. No extra angular or relative-radial character is introduced. Group terms by the subset of remaining marks, the finite ray characters, and dyadic summation scales before applying the moment bound.

The grouping is exact. Let $\operatorname{supp}\Phi \subseteq [\alpha_0, \beta_0]$, write $X_c = 2^\ell\theta_c$ with $\ell \in \mathbb{Z}$, $1 \le \theta_c < 2$, and fix $\Phi_1 \in C_c^\infty((\alpha_0/2, 4\beta_0))$ with $\Phi_1 = 1$ on $[\alpha_0, 2\beta_0]$. Then

$$\Phi\Big(\frac{Nn}{X_c}\Big) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\Phi}(i\tau)\theta_c^{i\tau}\,\Phi_1\Big(\frac{Nn}{2^\ell}\Big)\Big(\frac{Nn}{2^\ell}\Big)^{-i\tau} d\tau, \qquad \int_{\mathbb{R}} |\widehat{\Phi}(i\tau)|(1+|\tau|)^{J_0} d\tau < \infty.$$

The factor $\theta_c^{i\tau}$ is unimodular and joins the divisor scalars. The test functions $\Phi_1(x)x^{-i\tau}$ depend only on Φ and τ, are fixed before the arithmetic indices, and have seminorms $\ll (1+|\tau|)^{J_0}$. If $2^\ell < (4\beta_0)^{-1}$ the term vanishes, since $Nn \ge 1$. For the finitely many $\ell < 0$ that remain, the polynomial has length exponent 0: it has $O_{\beta_0}(1)$ terms of modulus $O(1)$, so

$$\sum_{0<Ny\le Z^{H_c}} \Big| \sum_{Nn<4\beta_0} \frac{\mu_F(n)\Psi'(n)\overline{\chi_n(y)}}{\sqrt{Nn}} 1_{(n,m')=1} M_{I\setminus J}(n)\Phi_1\Big(\frac{Nn}{2^\ell}\Big)\Big(\frac{Nn}{2^\ell}\Big)^{-i\tau}\Big|^2 \ll_{\beta_0} Z^{H_c} \le Z^{H_c+u'},$$

which is within the bound $\mathcal{B} \ll Z^{H_c+u'+\varepsilon_1}$ used after (58).

Within each such group, separation of the bounded norm ratios leaves polynomials $\mathcal{M}_{\alpha,\mathbf{t}}(y)$ independent of the divisor indices. Use

$$2|\mathcal{M}_{\alpha,\mathbf{t}}(y)\overline{\mathcal{M}_{\alpha',\mathbf{t}'}(y)}| \le |\mathcal{M}_{\alpha,\mathbf{t}}(y)|^2 + |\mathcal{M}_{\alpha',\mathbf{t}'}(y)|^2$$

within the original earlier character families. For fixed nonzero ideal parameter y, the ideals f, g', e divide y in the displayed powers; there are $O_{\varepsilon,I}((Ny)^{\varepsilon})$ presentations. The finite unit classes and ray-class character expansions have fixed absolute mass. The d-sum has the harmonic bound above. Since $1/Ne \le 1$ and the restoration coefficients have the preceding small-power bound, the off-diagonal contribution is at most

$$Z^{K'-F'+\varepsilon} D_{\max} \sum_{\alpha} \int w_{\alpha}(\mathbf{t}) \sum_{0<Ny\le Z^{H_c}} |\mathcal{M}_{\alpha,\mathbf{t}}(y)|^2 \, d\mathbf{t},$$

where $w_{\alpha} \ge 0$ and

$$\sum_{\alpha} \int w_{\alpha}(\mathbf{t})(1+\|\mathbf{t}\|)^{2d} \, d\mathbf{t} \ll Z^{\varepsilon}(1+\|\mathbf{t}_0\|)^A.$$

Apply the stated earlier moment bound and decrease the subsidiary ε to obtain

$$\mathcal{E}_{\text{off}} \ll Z^{K'-F'+\varepsilon} D_{\max}(1+\|\mathbf{t}_0\|)^A \mathcal{B}.$$

This also explains why the number of auxiliary ideals is not counted again on this term.

The diagonal term separated at the start has total at most

$$CTZ^{-F'} \sum_{f} D_f \sum_{n} |M_I(n) W(Nn/X)|^2 \le CZ^{K'} D_{\max}(Z^{-V} \#\{f\}) \mathcal{D}.$$

For the off-diagonal pairs the primitive Gauss sum at dual frequency zero is zero, so there is no further principal contribution. For the tail in (55), use the bound $(1+Z_{\mathrm{P}}^{\kappa})^{-J}$ after common separation of test functions. All untransformed polynomial, auxiliary-ideal and divisor counts are polynomial in Z on the stated bounded exponent box. Choose $J > (D_0+1)/\kappa_{\mathrm{P}}$, where D_0 is their fixed total degree, and use Cauchy's inequality on the two original ideal sums. The tail is bounded by the same diagonal estimate, with an arbitrarily small Z^{ε} loss.

Every scaling above uses the field norm: the real Jacobian of multiplication by $x \in F^{\times}$ is $|N_{F/\mathbb{Q}} x|$. Ideal counting on a fixed compact unit-orbit shape is $O_F(H)$ up to powers of $\log H$. Hence increasing $[F:\mathbb{Q}]$ changes only finite derivative orders and constants, not any displayed power of Z. This proves (54) with the required order of quantifiers. □

Lemma 8.7 (Reciprocal Euler restoration)**.** *Let $a(n)$ be a completely multiplicative coefficient with $|a(p)| \le 1$, including all natural zero extensions. Let $W \in C_c^{\infty}((0,\infty))$. For a squarefree ideal g, put*

$$M_y(X) = X^{-1/2} \sum_{n} \mu_F(n) a(n) W(Nn/X).$$

Put

$$M_y^{(g)}(X) = X^{-1/2} \sum_{(n,g)=1} \mu_F(n) a(n) W(Nn/X).$$

Then

$$M_y^{(g)}(X) = \sum_{\mathrm{rad}(d)|g} \frac{a(d)}{\sqrt{Nd}} M_y(X/Nd).$$

Its absolute coefficient mass is

$$\prod_{p|g} (1-(Np)^{-1/2})^{-1} \ll_{F,\varepsilon} (Ng)^{\varepsilon}.$$

For the injective marks $M_I(n)$, put

$$\mathcal{M}_I(X) = X^{-1/2} \sum_n \mu_F(n) a(n) M_I(n) W(Nn/X),$$

and let $\mathcal{M}_I^{(g)}$ impose $(n, g) = 1$. Write $\mathbf{T}^J = \prod_{i \in J} T_i$. In the ring $\mathbb{C}[\mathbf{T}]/(T_i^2 : i \in I)$, define

$$B_J(d) = [\mathbf{T}^J] \prod_{p \mid d} \left(1 + \sum_{i \in I} b_i(p) T_i\right)^{v_p(d)}.$$

Then

$$\mathcal{M}_I^{(g)}(X) = \sum_{J \subseteq I} \sum_{\mathrm{rad}(d) \mid g} \frac{a(d) B_J(d)}{\sqrt{Nd}} \mathcal{M}_{I \setminus J}(X/Nd), \tag{56}$$

and, for $|b_i(p)| \le 1$,

$$\sum_{J \subseteq I} \sum_{\mathrm{rad}(d) \mid g} \frac{|a(d) B_J(d)|}{\sqrt{Nd}} \ll_{F, \varepsilon, I} (Ng)^{\varepsilon}.$$

The coefficients on the remaining prime factors are unchanged.

Proof. At each $p \mid g$ multiply the full inverse Euler factor $1 - a(p)(Np)^{-s}$ by its geometric reciprocal; elsewhere leave it unchanged. This is a coefficient identity, including $a(p) = 0$. Compact support makes the smoothed sum locally finite. For the mass, exclude the finitely many primes for which $-\log(1 - (Np)^{-1/2}) > \varepsilon \log Np$ and absorb them into the constant. The remaining product is at most $(Ng)^{\varepsilon}$.

For the marked identity, first work on $\mathrm{Re}\, s > 1$. The generating Dirichlet series is

$$\mathscr{D}(\mathbf{T}, s) = \prod_p \left\{1 - a(p) \left(1 + \sum_{i \in I} b_i(p) T_i\right) (Np)^{-s}\right\}.$$

Its coefficient of $\mathbf{T}^I$ is $\sum_n \mu_F(n) a(n) M_I(n) (Nn)^{-s}$. Removing the factors at $p \mid g$ gives

$$\mathscr{D}^{(g)}(\mathbf{T}, s) = \mathscr{D}(\mathbf{T}, s) \sum_{\mathrm{rad}(d) \mid g} \frac{a(d)}{(Nd)^s} \prod_{p \mid d} \left(1 + \sum_i b_i(p) T_i\right)^{v_p(d)}.$$

Coefficient extraction and smoothing prove (56). For a fixed integer A_I,

$$\sum_{J \subseteq I} |B_J(d)| \le \tau(d)^{A_I}, \qquad \prod_{p \mid g} \sum_{j \ge 0} (j+1)^{A_I} (Np)^{-j/2} \ll_{F, \varepsilon, I} (Ng)^{\varepsilon}.$$

Here $\tau(d) = \#\{e : e \mid d\}$. The last estimate follows by the same large-prime argument as above. Every individual nonzero contribution to $B_J(d)$ extracts a product $d_J \mid d$, with multiplicities included. Hence, if z_J is its total marked length,

$$\log_Z(X/Nd) - (z - z_J) \le \log_Z X - z + O(\eta).$$

Thus restoration cannot increase the remaining free length. Repeated primes occur only in the scalar coefficient; the remaining mark is still injective.

The quotient in each restored Möbius polynomial is unrestricted; adding another coprimality condition would invalidate the identity. □

8.5 Fourth-power selection and the exact exponent identity

In this subsection, we select complete prime factors and retain their original character coefficients. The resulting exponent identity is used in the estimates of Subsections 8.6 and 8.9.

Proposition 8.8 (Complete-factor selection)**.** *Consider one canonical inverse block over F, with polynomial length exponent N, auxiliary ideal exponent V, parameter exponent M and remaining marked length z. Its free length exponent is $N-z$. Write $F_0 = N+V$. Suppose $z \ge u \ge 0$. Whole original factors can be selected with total width $u' = u + O(\eta)$ so that their removal gives*

$$N' = N - u', \qquad K' = M + 4u', \qquad z' = z - u'.$$

Up to $Z^{O(\eta)+\varepsilon}$ and one common test function measure, the original normalized second moment is bounded by the selected second moment at these new scales. Its residual free width is unchanged:

$$N' - z' = N - z.$$

Proof. Select whole factors greedily, stopping just below u; each factor width is at most η. Fix an injective selected tuple with product d. CRT and allowed-generator sextic reciprocity give, on the natural coprime support,

$$C(dn) = C(d)C(n)\chi_n(d)^4\omega(d,n),$$

where ω belongs to a fixed finite family of ray-class characters. Thus the remaining Gauss parameter is evaluated at kd^4, not at kd or kd^2. All terms where $(d,n) \ne 1$ already vanish. Restoring the old prime lists after removing the selected tuple is exact, since any restored term reusing a prime of d is killed by $\chi_n(kd^4) = 0$.

Separate the smooth ratios $Nd/Z^{u'}$ and their compact shapes before tuple selection, with a common measure. The central Cauchy inequality is

$$\sum_k \left| \sum_d b_d X(kd^4) \right|^2 \le \left(\sum_d |b_d|^2 \right) \max_y \#\{(d,k) : kd^4 = y\} \sum_y |X(y)|^2.$$

Here $\sum_d |b_d|^2 \ll Z^{u'+\varepsilon}$, and the fibre is bounded by a fixed divisor power, hence by Z^{ε}. The norm of the new parameter is at most $Z^{M+4u'+O(\eta)}$. The normalization changes by

$$Z^{-F_0} Z^{u'} = Z^{-(N'+V)}.$$

The remaining coefficients are still the original factor coefficients, the same Ψ, and the canonical Gauss coefficient. No arbitrary coefficient moment has been used. □

We fix the notation of the first canonical reduction, which is the initialization in the proof of [6, Theorem 10.7] applied after the tuple assignment in the proof of [6, Theorem 11.2]. Let I_0 be the set of original marks of the target block, with total width $S = \sum_{i \in I_0} z_i$. Its summation variable is $m\, p_{I_0}$, where $Nm \le Z^r$ and $p_{I_0} = \prod_{i \in I_0} p_i$ with distinct $p_i \in \mathcal{P}_i$; thus $\log_Z N(m\, p_{I_0}) \le r + S + O(\eta)$. Normalize the parameter exponent to 1. The assigned part $\mathfrak{c}' = (m, p_{I_0})$, of exponent G, leaves a squarefree residual of exponent $D' = r + S - 2G$ [6, proof of Theorem 11.2]. Coprimality-restricted Poisson summation with divisor $d' \mid C$ [6, proof of Theorem 10.7] writes the residual as $C\, s\, n$, with new label $f = d's$ and cofactor $\mathfrak{c} = C/d'$ of exponent $P_1 = B - \theta \ge -\eta$; $n, \mathfrak{c}, \mathfrak{c}'$ are squarefree and pairwise coprime, and $Nn \asymp Z^N$. With m replaced by 1 and r by D' in *loc. cit.*, the formal exponents are

$$N = D' - B - v, \qquad V = \theta + v, \qquad F_0 = N + V = D' - P_1, \qquad M = 2D' - 1 - 2P_1 = 2F_0 - 1,$$

up to $O(\eta)$ for the actual scales ($|F' - F_c| \le 5\eta$ there). The excluded ideal is

$$\mathfrak{r} = \mathfrak{r}_0 \mathfrak{c}_E, \qquad \mathfrak{c}_E = \mathfrak{c}\,\mathfrak{c}',$$

where $\mathfrak{r}_0 = R_\rho$ is the fixed coprimality modulus of the family $\mathcal{E}(N, V, M; \rho)$ [6, Subsection 10.2]; it divides a fixed ideal determined by S and the fixed ray-class data. The modulus $\mathrm{rad}(\mathfrak{c}\mathfrak{c}')$ is the fixed coprimality modulus $\mathrm{rad}((C/d')j)$ of [6, proof of Theorem 11.2]. Since $\theta, v \ge 0$,

$$N + \log_Z N\mathfrak{c} + \log_Z N\mathfrak{c}' = D' - \theta - v + G \le r + S - G \le r + \sum_{i \in I_0} z_i + O(\eta). \tag{57}$$

The exponents N, V, M do not depend on how the marks are distributed. Since $P_1 \ge -\eta$ and $G \ge 0$, $F_0 \le r + S - 2G + \eta$, and therefore

$$F_0 \le r + S, \qquad M \le 2F_0 - 1.$$

The marks are distributed by the injective identity for coprime squarefree factors,

$$M_{I_0}(n\mathfrak{c}\mathfrak{c}') = \sum_{I_1 \sqcup E_1 \sqcup E_2 = I_0} M_{I_1}(n)\, M_{E_1}(\mathfrak{c})\, M_{E_2}(\mathfrak{c}').$$

The term (I_1, E_1, E_2) is a block with remaining marks I_1 and scalars $M_{E_1}(\mathfrak{c})M_{E_2}(\mathfrak{c}')$. We keep the indicators $1[M_{E_1}(\mathfrak{c}) \ne 0]\, 1[M_{E_2}(\mathfrak{c}') \ne 0]$ and do not bound them by divisor powers. On their support there are distinct $p_i \in \mathcal{P}_i$ with $p_i \mid \mathfrak{c}$ ($i \in E_1$) and $p_i \mid \mathfrak{c}'$ ($i \in E_2$), so that, by $Np \ge C_0^{-1} Z^{z_i}$,

$$\log_Z N\mathfrak{c} \ge \sum_{i \in E_1} z_i - O(\eta), \qquad \log_Z N\mathfrak{c}' \ge \sum_{i \in E_2} z_i - O(\eta).$$

By (57), with $z = \sum_{i \in I_1} z_i$,

$$N \le r + \sum_{i \in I_0} z_i - \sum_{i \in E_1 \sqcup E_2} z_i + O(\eta) = r + z + O(\eta), \qquad \text{i.e.} \qquad N - z \le r + O(\eta).$$

Hence every term satisfies, before small endpoint errors,

$$N - z \le r, \qquad F_0 \le r + S, \qquad M \le 2F_0 - 1.$$

If $\mathfrak{c}' = 1$, only $E_2 = \varnothing$ occurs, since $M_E(1) = 0$ for $E \ne \varnothing$; if moreover $\mathfrak{c} = 1$, then $I_1 = I_0$. The overlap of original prime sets decreases both N and the remaining width by the same amount; it cancels from the first inequality. Enlarging a positive sum over ideal parameters, when needed, lets us use the upper parameter cap $M = 2F_0 - 1 + O(\eta)$. The selected sum then satisfies

$$\begin{aligned} H_0 &= 2(N' + V) - K' = 1 - 6u' + O(\eta), \\ H_c &= H_0 + \kappa_{\mathrm{P}}, \\ (K' - N' - V) + (H_c + u') &= F_0 + \kappa_{\mathrm{P}}, \\ K' &\le F_0 \quad \text{if } F_0 + 4u' + O(\eta) < 1. \end{aligned} \tag{58}$$

Thus $\mathcal{B} \ll Z^{H_c + u' + \varepsilon_1}$ gives

$$Z^{K' - N' - V + \varepsilon_1} \mathcal{B} \ll Z^{F_0 + \kappa_{\mathrm{P}} + 2\varepsilon_1}.$$

For a fixed strict margin $\gamma > 0$ and a prescribed loss $\varepsilon > 0$, choose

$$0 < \kappa_{\mathrm{P}}, \qquad \kappa_{\mathrm{P}} + C\eta + 2\varepsilon_1 < \min\{\gamma/4, \varepsilon\}.$$

The diagonal is $\ll Z^{F_0 + \varepsilon}$ under the last condition in (58). Apply Lemma 8.7 with the same subsidiary losses.

8.6 The estimate for large values

In this subsection, we prove the estimate for large values with arbitrarily small fixed positive margins. Subsection 8.7 verifies the remaining induction inequalities and sums the sixth-power factors.

Let $\lambda(w) = a_0 - a_1 w$ be a supporting line from earlier moment estimates, $D = 1/2 - a_1 > 0$, and put

$$0 < H < 1, \quad r = HR, \quad u = (1-H)/6, \quad s = H\{(1-R)/2 + z_R\} + u/k.$$

The remaining marked length is $s = S - u$ before the endpoint decrement. For $k \le \nu \le 3/4$, define

$$g(\nu) = \frac{r + s - H/2 - Ha_0 - u/\nu}{D}, \quad w(\nu) = g(\nu)/H + \upsilon,$$

where $\upsilon > 0$ is a padding. Let

$$\sigma(w) = \max\{0, (1-w)/2 + \lambda(w) - \upsilon'\}, \quad t(w) = 3w/2 + \tau,$$

with $0 < \upsilon' < \upsilon/20$ and $\tau > 0$ chosen sufficiently small. Let $\mathfrak{r}$ be the excluded ideal of the block, $N\mathfrak{r} \le Z^{\varrho_0}$. The original coefficients and Ψ lie in the Θ-character class (4), (5) with $j_{\max}$ as in (7). *Zero slopes.* Fix the target anchor ψ_0, chosen in the orientation of Subsection 9.2; use $\psi_0 = 1$ in the finite-order case. The zero maximum is taken over the paired set

$$\mathcal{X}_y = \{\theta\overline{\chi_\bullet(y)}, \psi_0\theta\overline{\chi_\bullet(y)} : \theta \in \Theta\} \ \cup \ \text{their whole conjugates.} \tag{59}$$

The live prime coefficients in a block with fixed factor $\omega_0 \in \{1, \psi_0\}$ are ω_0^{-1} times finite coefficients. The actual prime characters are finite twists of the moving sextic character. If an attaining zero belongs to the other anchor, rebasing the coefficients to that anchor keeps the actual prime characters unchanged. Whole conjugation transports the factor, all its tests and the theta coefficient together. A slot consumed in a collision contributes a bounded scalar. The detector uses the actual prime amplitudes. On a disjoint refinement of a complete window the pointwise upper bound follows from the complete window and its subpower discrepancy, by Lemmas A.5 and A.6. The height variable v denotes a shift $s \mapsto s + iv$ in the L-function, so the tested character is $\psi(N\cdot)^{-iv}$. Let $\mathcal{V}_y \ni 0$ contain both signs of the finitely many shift centers of the blocks, prime twists and detector frequencies. Then $|v| \le V_0 + \|\mathbf{t}\|$ for $v \in \mathcal{V}_y$. Fix $a_f = 51/100$, a buffer e_b, a height exponent ε_h, and $T_1 = Z^{\tau_1}$, where

$$\begin{gathered} 0 < e_b \le \min\Big\{\frac{k}{200}, \ \frac{\varepsilon}{10^3(H+1)}\Big\}, \qquad 0 < \varepsilon_h \le \frac{\varepsilon}{8n_F}, \qquad J' \ge \max\Big\{J'_{<j}, \ \frac{8n_F C_{\mathrm{tr}}}{\varepsilon}\Big\}, \\ 0 < \tau_1 \le \min\Big\{\frac{\varepsilon}{4J'}, \ \frac{\varepsilon_h}{2}\Big\}, \qquad J\tau_1 > 100H + 2. \end{gathered} \tag{60}$$

Here J' is the polynomial height order in (D3) below, $J'_{<j}$ is the largest height order of the earlier-stage estimates used in (D3) at the requested loss ε, C_{tr} is an exponent of the trivial bounds for the fixed bounded lengths (every sum estimated below is $\ll Z^{C_{\mathrm{tr}}}$), and J is the Mellin order of Lemma 8.11. The order of choices is

$$\tau \ \to \ \varepsilon \ \to \ e_b \ \to \ \varepsilon_h \ \to \ J' \ \to \ \tau_1 \ \to \ J,$$

where τ is the padding in $t(w)$ and

$$0 < \varepsilon < \min\{H\tau, \ 2Hc_\lambda, \ H\upsilon', \ H(1-R_0)\}, \tag{61}$$

with c_λ, R_0 as in Lemma 8.14; omit $2Hc_\lambda$ at $j=0$. The constants $\upsilon, \upsilon', \tau$ are fixed before ε, and τ_1 depends on τ through ε.

Height reduction. For Z large, $V_0 \le Z^{\varepsilon_h}/2$. If $V_0 + \|\mathbf{t}\| > Z^{\varepsilon_h}$, then $1 + H_{\mathrm{in}} \ge Z^{\varepsilon_h}/2$, and every estimate of this subsection follows from its trivial bound:

$$Z^{C_{\mathrm{tr}}} \le 2^{C_{\mathrm{tr}}/\varepsilon_h}(1+H_{\mathrm{in}})^{C_{\mathrm{tr}}/\varepsilon_h} \ll_\varepsilon (1+H_{\mathrm{in}})^{J'}, \qquad \frac{C_{\mathrm{tr}}}{\varepsilon_h} \le J' \quad \text{by (60) with } \varepsilon_h = \frac{\varepsilon}{8n_F}. \tag{62}$$

We take $\varepsilon_h = \varepsilon/(8n_F)$. Hence it suffices to treat

$$V_0 + \|\mathbf{t}\| \le Z^{\varepsilon_h}, \tag{63}$$

and then all heights below satisfy $|v| \le Z^{\varepsilon_h} + (3I+2)T_1 \le 2Z^{\varepsilon_h}$, since $\tau_1 \le \varepsilon_h/2$ and I is fixed before Z tends to infinity. The earlier-stage estimates are used at all heights, with their own factor $(3+|v|)^{J'_{<j}}$; no cutoff is nested along the chain. For $1 \le j \le I := \lceil 1/e_b \rceil + 2$ put

$$B_j(y) = \max\big(\{a_f\} \cup \{\operatorname{Re}\rho : \ L_F(\rho,\psi)=0,\ \psi \in \mathcal{X}_y,\ \operatorname{Re}\rho \ge a_f,\ \min_{v\in\mathcal{V}_y} |\operatorname{Im}\rho - v| \le 3jT_1\}\big).$$

Then $a_f \le B_1 \le \cdots \le B_I \le 1$ and $(I-1)e_b > 1 - a_f$, so there is a least $i = i(y) \le I-1$ with $B_{i+1} - B_i \le e_b$. Put

$$a(y) = e_b\lfloor B_i(y)/e_b \rfloor, \qquad \nu(y) = 2a(y) - 1, \qquad a \le B_i < a + e_b, \quad B_{i+1} < a + 2e_b. \tag{64}$$

This is the buffered zero bin of [6, Lemma 13.7]. For a parameter y outside the set $\mathcal{Y}_0$ of small sixth-power cores below, write $y \in \mathcal{C}(\nu, e)$ when

$$\nu(y) = \nu, \qquad |Q(y)|^2 \asymp Z^{es} \quad \text{(dyadic precision)}.$$

Thus ν is the zero slope and e the slope of the prime product. By Lemma 8.11 below, ν is an upper bound for the slope of the Möbius amplitude, $|M_r^{(\mathfrak{r})}(y)|^2 \ll Z^{(\nu+14e_b)r+\varepsilon}$; this upper bound is all that the pointwise exponent uses. Parameters with $|Q(y)|^2 \le Z^{ks}$ are the small values treated at the end of this subsection; the others form the large-value case. In the large-value case Lemma 8.11 gives

$$k - 16e_b \le \nu \le 2B_2 - 1 < \tfrac{3}{4}, \qquad e \le \nu + 16e_b, \qquad a > a_f. \tag{65}$$

For $k - 16e_b \le \nu < k$ put $g(\nu) = g(k)$ and $w(\nu) = w(k)$. Put

$$\sigma^*(w') = \max\{0, (1-w')/2 + \lambda(w')\}, \qquad \sigma^*_{\upsilon'}(w') = \max\{0, \sigma^*(w') - \upsilon'/2\} \ge \sigma(w'),$$

and, for a set K of complete factors, $Q_K = \prod_{i\in K} Q_i$ and $s_K = \sum_{i\in K} z_i$. Put $\delta_1 = \varepsilon/H$.

Orientation. Write each presentation in (59) as

$$\psi = \nu_\psi \chi_\bullet(y)^{-\epsilon_\psi}, \qquad \nu_\psi = \theta\psi_0^{b\epsilon_\psi}, \qquad \omega' = \psi_0^{b\epsilon_\psi}, \qquad \omega^\natural = (\omega')^{\epsilon_\psi}, \qquad b \in \{0,1\}, \quad \epsilon_\psi \in \{1,-1\}.$$

Here $\theta \in \Theta$. When $\epsilon_\psi = -1$, whole conjugation puts the moving character in the $\overline{\chi_\bullet(y)}$ orientation and changes the fixed factor to $\overline{\nu_\psi}$. Thus $\omega^\natural = \psi_0^b$ in canonical orientation. With $\boldsymbol{\lambda} \in \{1,-1\}^{r_F}$ the angular vector of the cubic theta coefficient (Subsection 2.5), the presentation is *oriented* if

$$\operatorname{ang}(\omega^\natural) \ne \boldsymbol{\lambda}, \tag{66}$$

that is, $\operatorname{ang}_v(\omega^\natural) \ne \lambda_v$ at some complex place v; this is the orientation of [3, Proposition 3.2]. There the orientation is $m_v = k_v - b_v \ne 0$, with k_v the angular exponent of the character and b_v the

weight of the theta coefficient Λ; here $\lambda_v = b_v$, so (66) is exactly $m_v \neq 0$ at some v. Finite-order characters have trivial angular vector, so (66) holds whenever ω' has finite order.

All infinite-order members of (59) have canonical factor ψ_0 up to finite twists. They satisfy (66). For this set the unmarked estimate needed below is

$$\sum_{0<Nk\le C_0H'} \left| D^{-1/2} \sum_n \mu_F(n)\nu_\psi(n) N n^{-iv} \chi_n(k)^{-\epsilon_\psi} W(Nn/D) \right|^2 \ll_{c,\varepsilon} H'(H'D)^{\varepsilon}(3+|v|)^{J'}, \qquad H' \ge \max\{2, D^{1+c}\}. \tag{67}$$

It is [6, Theorem 10.7] for finite-order fixed factors and Lemma 8.13 for oriented infinite-order ones.

The large-value estimate uses the following detection properties.

Proposition 8.9 (Detection at a variable length)**.** *For the blocks of this subsection, with original coefficients $(\mathcal{P}_i, b_i)$ and Ψ in the Θ-character class (4), (5), (7), uniformly in excluded ideals $\mathfrak{r}$ with $N\mathfrak{r} \le Z^{\varrho_0}$, assume:*

(H1) $\varepsilon_h = \varepsilon/(8n_F)$ and (63); by the height reduction (62) this is no restriction;

(H2) e_b and τ_1 satisfy (60), in the order of choices stated there;

(H3) λ is the supporting line of stage j of a descending chain as in Lemma 8.14, the joint moment estimates of the stages $i < j$ hold, and $\delta_1 = \varepsilon/H < \min\{2c_\lambda, \upsilon', 1-R_0\}$, with $2c_\lambda$ omitted at $j = 0$, as in (61);

(H4) The live block and prime characters are in (59), with the two anchors and prime coefficients synchronized as there.

Then the following hold for every class $\mathcal{C}(\nu, e)$ of the large-value case, restricted to $Z^{H/5} \le Ny \le Z^H$, with $w = w(\nu)$ and $t = Ht(w)$.

(D2) There is a set $\mathcal{T}(\nu, e)$ of data $\mathfrak{d} = (\psi, \iota, \mathcal{V}, r', m')$, consisting of a presentation $\psi \in \mathcal{X}_y$ with $(b, \theta, \epsilon_\psi)$ from a fixed finite set, indices ι of test functions in the prescribed families, a height box $\mathcal{V}$ and a dyadic pair (r', m'), with

$$\#\mathcal{T}(\nu, e) \ll_\varepsilon Z^\varepsilon, \qquad \mathcal{C}(\nu, e) = \bigcup_{\mathfrak{d}\in\mathcal{T}(\nu,e)} \mathcal{C}_\mathfrak{d}(\nu, e),$$

such that for $y \in \mathcal{C}_\mathfrak{d}(\nu, e)$, at a height $v \in \mathcal{V}$ with

$$|v| \le V_0 + \|\mathbf{t}\| + (3I+1)T_1, \tag{68}$$

$$\begin{gathered} r' \le t + \varepsilon, \qquad r' + m' \ge t - \varepsilon, \qquad 0 \le m' \le H/2 + \varepsilon, \\ |M_{r'}(\psi(N\cdot)^{-iv})|^2 \ge Z^{\nu r' - \varepsilon}, \qquad |S_{m'}(\psi(N\cdot)^{-iv})|^2 \ge Z^{\nu m' - \varepsilon} \quad (S_0 = 1). \end{gathered} \tag{69}$$

Here $M_{r'}, S_{m'}$ are the Möbius and ordinary polynomials of Subsection 2.3 at the character $\psi(N\cdot)^{-iv}$, excluded ideal 1 and test functions of index ι; the index ι contains a real part $\sigma \in [a_f, 1]$, and $\mathfrak{B}_\mathfrak{d} = [a_f, 1] \times \mathcal{V}$ is the parameter box of [6, Lemma 13.9]. V_0 is fixed and $\|\mathbf{t}\|$ is the sum of the absolute heights carried by the block, on an ordinary side those of both factors. On an ordinary side the parameter is twisted by a unit ς; then the bins B_j, the slope ν and the class $\mathcal{C}(\nu, e)$ are defined at ςy, through $\mathcal{X}_{\varsigma y}$, which lies in the family $\mathcal{X}_y$ formed with $\Theta \supset \Xi$ (2).

(D3) *For $\mathfrak{d}\in\mathcal{T}(\nu,e)$, with the height factor $(3+|v|)^{J'}$ of the abbreviated convention, and every set K of complete factors,*

$$\sum_{Ny\le Z^H}\sup_{\mathfrak{B}_\mathfrak{d}}|M_{r'}(\psi(N\cdot)^{-iv})Q_K(y)|^2\ll Z^{H+\varepsilon}\quad \text{if } Hw\le r'\le H(1-\delta_1),\ s_K\le H\sigma^*_{v'}(r'/H),$$

$$\sum_{Ny\le Z^H}^{*}\sup_{\mathfrak{B}_\mathfrak{d}}|S_{m'}(\psi(N\cdot)^{-iv})^2Q_K(y)|^2\ll Z^{H+\varepsilon}\quad \text{if } r'<Hw,\ s_K\le H\sigma^*_{v'}(\min\{2m'/H,1\}),$$

the supremum being over $(\sigma,v)\in\mathfrak{B}_\mathfrak{d}$, with the conventions of Subsection 2.2 and the ordinary inducing-character exclusions. Moreover (67) holds for ν_ψ and every v in (68).

Proposition 8.9 is proved after Lemma 8.14: (D2) is Lemma 8.12, a recentred form of [6, Lemma 13.7]; the two bounds in (D3) are the inverse and ordinary joint moments of the earlier stages at the dual parameters; and (67) is [6, Theorem 10.7], or Lemma 8.13, according to the fixed factor. The pointwise part, (65), is proved in Lemma 8.11. Given (D2) and (D3), Lemma 8.15 gives $\#\mathcal{C}(\nu,e)\ll Z^{C+\varepsilon}(1+H_{\mathrm{in}})^{J'}$ for some $C\in\{C_s,C_l\}$, where

$$C_s=H-\nu Hw-e\min\{s,H\sigma(w)\},\qquad C_l=H/6+(5/6-\nu)Ht(w).\tag{70}$$

In C_s the prime product of length $\min\{s,H\sigma(w)\}$ consists of the complete factors of largest individual slope; its slope is at least e. At the heights (68), with $\|\mathfrak{t}\|\le H_{\mathrm{in}}$, (60) and $I\ll e_b^{-1}$,

$$(3+|v|)^{J'}\le(3+V_0+H_{\mathrm{in}})^{J'}(3I+2)^{J'}T_1^{J'}\ll_{e_b}(1+H_{\mathrm{in}})^{J'}Z^{\varepsilon/4};\tag{71}$$

the factor $(1+H_{\mathrm{in}})^{J'}$ is that of the abbreviated convention of Subsection 2.2. With the further factor $T_1^{J'}\le Z^{\varepsilon/4}$ of the supremum over $\mathfrak{B}_\mathfrak{d}$, $Z^{\varepsilon/2}$ enters the loss. The pointwise exponent on $\mathcal{C}(\nu,e)$ is at most $(\nu+14e_b)r+es$.

The bound $\nu\le 2B_2-1$ in (65) comes from the zero-free strip of the families (10).

Lemma 8.10 (Möbius and prime blocks in the zero-free strip)**.** *Let ω be a fixed continuous unitary Hecke character of F and let $\lambda=\theta\omega^j(N_{F^\cdot})^{it}\in\mathcal{F}_\omega$, extended by zero. Put $\beta=\beta_\omega$ and*

$$\mathfrak{q}_\lambda(y)=N\mathfrak{f}_\lambda(3+|t|+|y|),\qquad \mathfrak{q}_\lambda=\mathfrak{q}_\lambda(0).$$

Then $\beta\le B_2$ by [2], and $\beta_1=\beta_*(F)\le B_2$ by [5]. Let $\delta_\lambda=1$ if $\lambda(\mathfrak{a})=N\mathfrak{a}^{it}$ for all $\mathfrak{a}$ prime to $\mathfrak{f}_\lambda$, and $\delta_\lambda=0$ otherwise. For $W\in C_c^\infty((0,\infty))$, an integral ideal $\mathfrak{r}$, $X\ge X_0(\operatorname{supp}W)$ and $\varepsilon>0$, there is J such that

$$\sum_{(n,\mathfrak{r})=1}\mu_F(n)\lambda(n)W(Nn/X)\ll_\varepsilon X^{\beta+\varepsilon}(\mathfrak{q}_\lambda N\mathfrak{r})^\varepsilon p_J(W),\tag{72}$$

$$\sum_p\lambda(p)W(Np/X)-\delta_\lambda\int_0^\infty W(x/X)x^{it}\frac{dx}{\log x}\ll_\varepsilon X^{\beta+\varepsilon}\mathfrak{q}_\lambda^\varepsilon p_J(W),\tag{73}$$

and, if $\delta_\lambda=0$,

$$\sum_{(n,\mathfrak{r})=1}\lambda(n)W(Nn/X)\ll_\varepsilon X^{\beta+\varepsilon}(\mathfrak{q}_\lambda N\mathfrak{r})^\varepsilon p_J(W).\tag{74}$$

The implied constants depend only on F, ω, ε and $\operatorname{supp}W$.

Proof. Let λ^* be the primitive character inducing λ and $\mathcal{P}_0 = \{p \mid \mathfrak{f}_\lambda \mathfrak{r}\}$. Since $|\lambda^*(p) N p^{-s}| \le 2^{-1/2}$ on $\operatorname{Re} s \ge 1/2$,

$$\sum_{(n,\mathfrak{r})=1} \frac{\mu_F(n)\lambda(n)}{Nn^s} = \frac{E(s)}{L_F(s,\lambda^*)}, \qquad E(s) = \prod_{p \in \mathcal{P}_0} (1 - \lambda^*(p) N p^{-s})^{-1},$$

$$|E(s)| \le 4^{\#\mathcal{P}_0} \ll_\varepsilon (N\mathfrak{f}_\lambda N\mathfrak{r})^\varepsilon \qquad (\operatorname{Re} s \ge 1/2).$$

We first show, for $\operatorname{Re} s \ge \beta + \varepsilon$ and $|\operatorname{Im} s - t| > 2$ when $\delta_\lambda = 1$,

$$|\log L_F(s,\lambda^*)| \le \varepsilon \log \mathfrak{q}_\lambda(\operatorname{Im} s) + C_\varepsilon. \tag{75}$$

For $\operatorname{Re} s \ge 2$ this is trivial. Fix T and put $s_0 = 2 + iT$ and

$$\rho_1 = \tfrac{1}{2}, \quad \rho_2 = 2 - \beta - \varepsilon, \quad \rho_3 = 2 - \beta - \tfrac{\varepsilon}{2}, \quad \rho_4 = 2 - \beta - \tfrac{\varepsilon}{4}.$$

The disc $|s - s_0| \le \rho_4 < 3/2$ lies in $\operatorname{Re} s > \beta$ and avoids $1 + it$; thus $f = \log L_F(\cdot, \lambda^*)$ is holomorphic there. The convexity bound gives $\operatorname{Re} f \le A_F \log \mathfrak{q}_\lambda(T) + O(1)$ on it, and $|f(s_0)| \le \log \zeta_F(2)$. Borel–Carathéodory gives

$$\max_{|s-s_0|=\rho_3} |f(s)| \le \frac{2\rho_3}{\rho_4 - \rho_3}\{A_F \log \mathfrak{q}_\lambda(T) + O(1)\} + \frac{\rho_4 + \rho_3}{\rho_4 - \rho_3}|f(s_0)| \ll_\varepsilon \log \mathfrak{q}_\lambda(T).$$

On $|s - s_0| = \rho_1$ one has $\operatorname{Re} s \ge 3/2$ and $|f| \le \log \zeta_F(3/2)$. Hadamard's three-circle theorem gives, with $a = \log(\rho_2/\rho_1)/\log(\rho_3/\rho_1) < 1$,

$$\max_{|s-s_0|\le\rho_2} |f(s)| \ll_\varepsilon (\log \mathfrak{q}_\lambda(T))^a \le \varepsilon \log \mathfrak{q}_\lambda(T) + C_\varepsilon.$$

The segment $[\beta + \varepsilon, 2] + iT$ lies in $|s - s_0| \le \rho_2$; this is (75). If $\delta_\lambda = 1$, then $L_F(s,\lambda^*) = \zeta_F(s - it)$, and on the compact set $\beta + \varepsilon \le \operatorname{Re} s \le 2$, $|\operatorname{Im} s - t| \le 2$, the functions $\zeta_F(s-it)^{-1}$ and $\zeta_F'/\zeta_F(s-it) + (s - 1 - it)^{-1}$ are holomorphic and $O_{F,\varepsilon}(1)$, because $\beta_*(F) \le \beta$.

By Mellin inversion $W(x) = (2\pi i)^{-1} \int_{(2)} \widehat{W}(s) x^{-s}\, ds$,

$$\sum_{(n,\mathfrak{r})=1} \mu_F(n)\lambda(n) W(Nn/X) = \frac{1}{2\pi i} \int_{(2)} \widehat{W}(s) X^s \frac{E(s)}{L_F(s,\lambda^*)}\, ds.$$

The integrand is holomorphic on $\operatorname{Re} s > \beta$; the pole of L_F is a zero of $1/L_F$. Shift to $\operatorname{Re} s = \beta + \varepsilon$ and use (75) and $|\widehat{W}(\sigma + iy)| \ll p_J(W)(1 + |y|)^{-2}$ for $\sigma \in [1/2, 2]$:

$$\Big| \sum_{(n,\mathfrak{r})=1} \mu_F(n)\lambda(n) W(Nn/X) \Big| \ll X^{\beta+\varepsilon} (N\mathfrak{f}_\lambda N\mathfrak{r})^\varepsilon p_J(W) \int_{\mathbb{R}} \frac{\mathfrak{q}_\lambda(y)^\varepsilon}{(1 + |y|)^2}\, dy \ll X^{\beta+\varepsilon} (\mathfrak{q}_\lambda N\mathfrak{r})^{2\varepsilon} p_J(W).$$

This is (72) after renaming ε.

For (74), let $\delta_\lambda = 0$. Then $L_F(s,\lambda^*)$ is entire, (75) holds on all of $\operatorname{Re} s \ge \beta + \varepsilon$, and

$$\sum_{(n,\mathfrak{r})=1} \frac{\lambda(n)}{Nn^s} = E'(s) L_F(s,\lambda^*), \qquad E'(s) = \prod_{p \in \mathcal{P}_0} (1 - \lambda^*(p) N p^{-s}), \qquad |E'(s)| \le 2^{\#\mathcal{P}_0} \quad (\operatorname{Re} s \ge 1/2).$$

The same contour shift, with $|L_F(s,\lambda^*)| = \exp \operatorname{Re} \log L_F(s,\lambda^*) \le e^{C_\varepsilon} \mathfrak{q}_\lambda(\operatorname{Im} s)^\varepsilon$ on $\operatorname{Re} s = \beta + \varepsilon$, gives (74).

For (73), Cauchy's estimate on circles of radius $\varepsilon/2$ and (75) with $\varepsilon/2$ give $|L_F'/L_F(s,\lambda^*)| \ll_\varepsilon \log \mathfrak{q}_\lambda(\operatorname{Im} s)$ on $\operatorname{Re} s \ge \beta+\varepsilon$, away from $1+it$. Put $W_X(x) = W(x)\log X/\log(xX)$; then $p_J(W_X) \ll p_J(W)$ for $X \ge X_0$. The same contour shift, with the residue $\delta_\lambda \widehat{W}_X(1+it)X^{1+it}$ at $s = 1+it$, gives

$$\sum_n \Lambda_F(n)\lambda^*(n)W_X(Nn/X) = \delta_\lambda \widehat{W}_X(1+it)X^{1+it} + O_\varepsilon(X^{\beta+\varepsilon}\mathfrak{q}_\lambda^\varepsilon p_J(W)).$$

Prime powers p^m, $m \ge 2$, contribute $O(X^{1/2}\log X)$, and primes with $\lambda(p) \ne \lambda^*(p)$ contribute $O(\log N\mathfrak{f}_\lambda)$; here $\Lambda_F(p^m) = \log Np$ and $\Lambda_F = 0$ otherwise. Since $\log Np \cdot W_X(Np/X) = \log X \cdot W(Np/X)$ and $\widehat{W}_X(1+it)X^{1+it} = \log X \int_0^\infty W(x/X)x^{it}\,dx/\log x$, division by $\log X$ gives (73). □

By (6), the inverse blocks of this subsection have $\lambda \in \mathcal{F}_{\omega'}$, $\omega' = \omega_0^j\omega^{\pm j'}$ a fixed unitary character ($\omega' = 1$ for finite-order presentations), and $\beta_{\omega'} \le B_2$ by [5, 2]; Lemma 8.10 applies with ω' in place of ω. Their conductors are $\ll Z^{O(1)}$, their heights are $\le V_0 + \|\mathfrak{t}\| + (3I+1)T_1 \ll Z^{O(1)}$, and $N\mathfrak{r} \le Z^{\varrho_0}$. By (72) with $X = Z^{r'}$, for every parameter y and every $r' \ge 0$,

$$|M_{r'}^{(\mathfrak{r})}(y)|^2 \ll Z^{(2B_2-1)r'+\varepsilon}, \qquad 2B_2 - 1 = \frac{351-\sqrt{1081}}{426} = 0.74676\ldots < \kappa_1 < \frac{3}{4}, \tag{76}$$

with the height factor $(3+|v|)^\varepsilon$ absorbed by the abbreviated convention. The bound holds equally for $\delta_\lambda = 1$, in particular at sixth powers $y = B^6$. Hence, for every parameter, the slope of the Möbius amplitude is at most $2B_2 - 1 + \varepsilon/r < 3/4$ ($\varepsilon < (7/4 - 2B_2)r$), and an amplitude $\ge Z^{\nu' r'/2-\varepsilon}$ at length $Z^{r'}$ gives $\nu' \le 2B_2 - 1 + 3\varepsilon/r'$. Thus no parameter has inverse amplitude slope in $(3/4, 1]$. Parameters with $Ny < Z^{H/5}$ number $\ll Z^{H/5}$, and by (76) their second moment is

$$\sum_{Ny<Z^{H/5}} |M_r^{(\mathfrak{r})}(y)Q(y)|^2 \ll Z^{H/5+(2B_2-1)r+s+\varepsilon} \le Z^{H/6+1/15+3r/4+s+\varepsilon}, \tag{77}$$

since $H/5 \le H/6 + 1/15$ for $H \le 2$.

Lemma 8.11 (Zero bins)**.** *Assume (4), (5) with bound $j_{\max}$, and let the complete windows be smooth: $1_{\mathcal{P}_i}(p)W_i(Np/Z^{z_i}) = 1_{p\notin S}W_i(Np/Z^{z_i})$. Let $y \notin \mathcal{Y}_0$, $Ny \le Z^H$, with zero slope $\nu = \nu(y)$ as in (64). For every $\varepsilon > 0$ and $Z \ge Z_0(e_b, \varepsilon)$, uniformly in $N\mathfrak{r} \le Z^{\varrho_0}$:*

(a) $\nu \le 2B_2 - 1 < 3/4$;

(b) $|M_r^{(\mathfrak{r})}(y)|^2 \ll Z^{(\nu+14e_b)r+\varepsilon}$ and $|Q_i(y)|^2 \ll Z^{(\nu+14e_b)z_i+\varepsilon}$ for every i;

(c) if $|Q(y)|^2 \asymp Z^{es}$, then $e \le \nu + 16e_b$; if moreover $|Q(y)|^2 > Z^{ks}$, then $\nu \ge k - 16e_b$ and $a(y) > a_f$.

The implied constants depend on F, the class data, e_b, ε and the test functions, and on no other property of y.

Proof. No norm characters. Write $y = cB^6$ with c sixth-power-free. Since $y \notin \mathcal{Y}_0$, $Nc > Z^{\theta_0}$. On ideals prime to $6B$, $\chi_\cdot(y) = \chi_\cdot(c)$ is the Kummer character of $F(c^{1/6})/F$, whose conductor is divisible by every $p \mid c$ with $p \nmid 6$, since $1 \le v_p(c) \le 5$. Every $\psi \in \mathcal{X}_y$ is $\nu_\psi\chi_\bullet(c)^{-\epsilon_\psi}$ away from the natural holes. Its fixed factor ν_ψ belongs to the paired finite list of characters of conductor dividing a fixed $\mathfrak{m}_0$. If ψ were a norm character, $\chi_\cdot(c)$ would have conductor dividing $\mathfrak{m}_0$, whereas

$$N\Big(\prod_{p\mid c,\ p\nmid 6\mathfrak{m}_0} p\Big) \gg_{\mathfrak{m}_0} (Nc)^{1/5} > Z^{\theta_0/5}.$$

Hence for Z large no $\psi \in \mathcal{X}_y$ is a norm character, and $L_F(s,\psi^*)$ is entire.

(a) The chosen target ψ_0 belongs to $\mathcal{F}_\omega$, which is closed under conjugation and finite twists. The moving sextic character and the ray factors have finite order, so every member of $\mathcal{X}_y$ is in $\mathcal{F}_\omega$. The prior bound $\beta_\omega \le B_2$ of [2] (or [5] in the finite-order case) gives $a \le B_i \le B_2$ and $\nu = 2a-1 \le 2B_2-1$.

(b) Let $\psi \in \mathcal{X}_y$ and $q_\psi(v) = N\mathfrak{f}_\psi(3+|v|)$; by (6), $\log q_\psi(v) \ll \log Z$ for $|v| \le (3I+2)T_1+V_0+\|\mathbf{t}\|$. Let $v_b \in \mathcal{V}_y$ be the height of the block, or of the prime twist, under consideration. Fix v with $|v-v_b| \le (3i+2)T_1$ and put $s_0 = 2+iv$,

$$\rho_1 = \tfrac12, \quad \rho_2 = 2-a-6e_b, \quad \rho_3 = 2-a-4e_b, \quad \rho_4 = 2-a-2e_b.$$

A point of $|s-s_0| \le \rho_4$ has $\operatorname{Re} s \ge a+2e_b > a_f$ and $|\operatorname{Im} s - v_b| \le (3i+2)T_1+2 \le 3(i+1)T_1$, while every zero with $\operatorname{Re}\rho \ge a_f$ and $|\operatorname{Im}\rho - v_b| \le 3(i+1)T_1$ has $\operatorname{Re}\rho \le B_{i+1} < a+2e_b$ by (64) and $v_b \in \mathcal{V}_y$. So $f = \log L_F(\cdot,\psi^*)$ is holomorphic on $|s-s_0| \le \rho_4$, with $\operatorname{Re} f \le A_F \log q_\psi(v) + O(1)$ by convexity and $|f(s_0)| \le \log \zeta_F(2)$. As in the proof of Lemma 8.10, Borel–Carathéodory on (ρ_3,ρ_4) and Hadamard's three circles on (ρ_1,ρ_2,ρ_3) give, with $\alpha = \log(\rho_2/\rho_1)/\log(\rho_3/\rho_1) \le \alpha(e_b) < 1$ uniformly in $a \in [a_f,1]$,

$$\max_{|s-s_0|\le\rho_2} |f(s)| \ll_{e_b} (\log Z)^\alpha \le \varepsilon_1 \log Z + C_{e_b,\varepsilon_1}.$$

Points $\sigma+iv$ with $\sigma \ge a+7e_b$ lie at distance $\ge e_b$ from $|s-s_0| = \rho_2$, so Cauchy's estimate on circles of radius e_b gives, on the box $\mathcal{R} = \{a+7e_b \le \operatorname{Re} s \le 2,\ |\operatorname{Im} s - v_b| \le (3i+2)T_1\}$,

$$|\log L_F(s,\psi^*)| \le \varepsilon_1 \log Z + C_{e_b,\varepsilon_1}, \qquad |L_F'/L_F(s,\psi^*)| \ll_{e_b} e_b^{-1}(\log Z)^\alpha, \tag{78}$$

hence $|L_F(s,\psi^*)|^{\pm1} \ll_{e_b,\varepsilon_1} Z^{\varepsilon_1}$ on $\mathcal{R}$. For the Möbius block at height v_b, the Dirichlet series is

$$\sum_{(n,\mathfrak{r})=1} \mu_F(n)\psi_y(n)Nn^{-iv_b-s} = \frac{E(s+iv_b)}{L_F(s+iv_b,\psi_y^*)},$$

where E is the product over $p \mid \mathfrak{f}_{\psi_y}\mathfrak{r}B$ in Lemma 8.10, $|E(s)| \le 4^{\omega(\mathfrak{f}_{\psi_y}\mathfrak{r}B)} \ll Z^\varepsilon$ on $\operatorname{Re} s \ge 1/2$. In Mellin inversion on $\operatorname{Re} s = 2$, replace the segment $|\operatorname{Im} s| \le T_1$ by the other three sides of the rectangle $[a+7e_b,2]\times[-T_1,T_1]$. Its translate by iv_b lies in $\mathcal{R}$, since $T_1 \le (3i+2)T_1$, and $1/L_F(s+iv_b,\psi_y^*)$ is holomorphic on it. With $|\widehat{W}(\sigma+iy')| \ll p_J(W)(1+|y'|)^{-J}$ and $J\tau_1 > 100H+2 \ge 2r+2$ (the order fixed after (60)),

$$M_r^{(\mathfrak{r})}(y) \ll Z^{-r/2}\big(Z^{r(a+7e_b)}Z^{\varepsilon_1+\varepsilon} + Z^{2r+\varepsilon_1+\varepsilon}T_1^{1-J}\big) \ll Z^{(2a-1+14e_b)r/2+\varepsilon_1+\varepsilon}.$$

For Q_i, apply the logarithmic derivative bound to the finite components $\phi_\theta \in \mathcal{X}_y$ of the actual prime character, as in (59). A refined list uses the same upper bound up to the subpower discrepancy of its complete window. With $X = Z^{z_i}$ and $W_X(x) = W_i(x)\log X/\log(xX)$, the same contour, centred at the height v_b of the prime twist, with $-L_F'/L_F(s+iv_b,\phi_\theta^*)$ and (78), without a residue, gives

$$\sum_n \Lambda_F(n)\phi_\theta^*(n)W_X(Nn/X) \ll X^{a+7e_b}Z^{\varepsilon_1+\varepsilon}.$$

Prime powers contribute $O(X^{1/2}\log X)$; primes in S or dividing $\mathfrak{f}_{\phi_\theta}B$ contribute $O(\log Z)$. Division by $\log X$ and the normalization $X^{-1/2}$ give, since $a+7e_b > 1/2$,

$$|Q_i(y)| \le \sum_\theta |c_{i,\theta}| \Big|X^{-1/2}\sum_p \phi_\theta(p)W_i(Np/X)\Big| \ll Z^{(2a-1+14e_b)z_i/2+\varepsilon_1+\varepsilon}.$$

Take $\varepsilon_1 = \varepsilon/4$; this is (b).

(c) Apply (b) with $\varepsilon_0 = \min\{\varepsilon, e_b s/\#I_0\}$ in place of ε: $|Q|^2 = \prod_i |Q_i|^2 \ll Z^{(\nu+14e_b)s+\#I_0\,\varepsilon_0} \le Z^{(\nu+15e_b)s}$. At dyadic precision $es \le (\nu+15e_b)s + O(1/\log Z)$, so $e \le \nu + 16e_b$. If $|Q|^2 > Z^{ks}$, then $k < \nu + 16e_b$. Since $2a_f - 1 + 16e_b = 1/50 + 16e_b < 1/6 \le k$, this forces $a > a_f$. □

Lemma 8.12 (Variable-length detection)**.** *Assume (4), (5), and (H1), (H2) of Proposition 8.9, with $\Theta \supset \Xi$ as in (2). Then (D2) holds for every class $\mathcal{C}(\nu, e)$ of the large-value case. The datum $\mathfrak{d}$ consists of $(b, \theta, \epsilon_\psi)$ from a fixed finite set, a dyadic pair (r', m') from $O((\log Z)^2)$ pairs, and an interval $\mathcal{V}$ of length $2T_1$ from a fixed cover of $[-V_0 - \|\mathbf{t}\| - (3I+1)T_1,\ V_0 + \|\mathbf{t}\| + (3I+1)T_1]$; the real part $\sigma \in [a_f, 1]$ and the height $v \in \mathcal{V}$ are the coordinates of $\mathfrak{B}_\mathfrak{d}$.*

Proof. Unit twist. $\mathcal{O}_F^\times/\mathcal{O}_F^{\times 6}$ is finite and the Kummer characters $n \mapsto \chi_n(\varsigma)$ form the fixed group Ξ of S-supported characters. Since $\chi_n(\varsigma y) = \chi_n(\varsigma)\chi_n(y)$ and $\Theta\Xi = \Theta$, the two paired sets at y and ςy agree. On an ordinary side the bins use its actual parameter ςy.

The zero. Let $y \in \mathcal{C}(\nu, e)$, $i = i(y)$, $a = a(y)$. By (65), $a > a_f$, so $B_i(y) \ge a > a_f$ is attained: there are $\psi \in \mathcal{X}_y$, $v_0 \in \mathcal{V}_y$ and $\rho = \sigma + i\gamma$ with

$$L_F(\rho, \psi) = 0, \qquad a \le \sigma < a + e_b, \qquad |\gamma - v_0| \le 3iT_1,$$

while, by (64), for all $\psi' \in \mathcal{X}_y$,

$$L_F(s, \psi') \ne 0 \qquad (\operatorname{Re} s \ge a + 2e_b,\ |\operatorname{Im} s - v_0| \le 3(i+1)T_1). \tag{79}$$

Extension by zero changes only Euler factors that vanish on $\operatorname{Re} s = 0$, so ρ is a zero of $L_F(s, \psi^*)$. By the proof of Lemma 8.11 ψ is not a norm character, so $L_F(s, \psi^*)$ is entire. With $U = Ny$ and $|u| \le V_0 + \|\mathbf{t}\| + (3I+2)T_1 \le 2Z^{\varepsilon_h}$, (H1) and $\tau_1 \le \varepsilon_h/2$ give

$$\mathfrak{q}_\psi(u) = N\mathfrak{f}_\psi(3 + |u|)^{n_F} \ll UZ^{2n_F\varepsilon_h}, \qquad L := \log_Z \max_u \mathfrak{q}_\psi(u) \le H + 2n_F\varepsilon_h + o(1).$$

Upper bounds. Let $|v - v_0| \le (3i+1)T_1$ and $\varepsilon_2 = \varepsilon/100$. The proof of Lemma 8.11(b), with $v_b = v_0$, any length $r'' \ge 0$ and any test function of $\mathfrak{B}_\mathfrak{d}$, gives

$$|M_{r''}(\psi(N\cdot)^{-iv})|^2 \ll Z^{(\nu+14e_b)r''+\varepsilon_2}, \qquad |S_{m''}(\psi(N\cdot)^{-iv})|^2 \ll Z^{(\nu+14e_b)m''+\varepsilon_2},$$

without a pole for $S_{m''}$. Shifting the central segment of $S_{m''}$ to $\operatorname{Re} s = 1 - a - 7e_b$ instead, the functional equation, conjugation symmetry and (78) give

$$|L_F(1 - a - 7e_b + iu, \psi^*)| = \mathfrak{q}_\psi(u)^{a+7e_b-1/2+o(1)}|L_F(a + 7e_b + iu, \psi^*)| \ll \mathfrak{q}_\psi(u)^{a+7e_b-1/2}Z^{\varepsilon_2};$$

the tails on $\operatorname{Re} s = 2$ and the horizontal joins are $\ll Z^{3m''/2+1}T_1^{1-J} \le Z^{3m''/2-100H}$ by convexity and $J\tau_1 > 100H + 2$ in (60). This bound is used for all $m'' \le \log_Z(2U^{41}) \le 41H + o(1)$, the range of the dyadic pieces below, and $3m''/2 - 100H \le -(\nu + 14e_b)\max\{0, m'' - L\} - H$ there. Hence

$$|S_{m''}(\psi(N\cdot)^{-iv})|^2 \ll Z^{(\nu+14e_b)\min\{m'', L-m''\}+2\varepsilon_2}. \tag{80}$$

Detector. Put $D_* = Z^t$, $t = Ht(w)$, $Y_* = U^{40}$ and, with $V_\le$ as in [6, Lemma 13.7],

$$C_\psi(s) = \sum_l \mu_F(l)\psi(l)V_\le(Nl/D_*)Nl^{-s}, \qquad \mathcal{J} = \frac{1}{2\pi i}\int_{(2)} Y_*^z\Gamma(z)L_F(\rho + z, \psi)C_\psi(\rho + z)\,dz.$$

Shift to $\operatorname{Re} z = -1/4$; the residue at $z = 0$ vanishes. On $\operatorname{Re} z = -1/4$, $|L_F| \ll \mathfrak{q}_\psi^{3/8+o(1)}$. Since $w \le g(3/4)/H + \upsilon < 79/100 + \upsilon \le 1$ (Proposition 8.18, g increasing in ν), $t(w) \le 3/2 + \tau$; and

$\log_Z U \ge H/5$, since the large-value case has $U = Ny \ge Z^{H/5}$. Thus $t/\log_Z U \le 5(3/2+\tau) \le 8$ and $|C_\psi| \ll D_*^{5/4-\sigma} \log Z \le U^{6+o(1)}$. Hence

$$|\mathcal{J}| \ll U^{-10+3/8+6+o(1)} Z^{n_F(\varepsilon_h+\tau_1)} \le U^{-3} = o(1).$$

On $\operatorname{Re} z = 2$ the coefficient of n with $1 < Nn \le D_*$ is $\psi(n)\sum_{l|n}\mu_F(l) = 0$, so

$$\sum_{l,k} \mu_F(l)\psi(lk) V_{\le}\Big(\frac{Nl}{D_*}\Big)\Big(1 - V_{\le}\Big(\frac{2N(lk)}{D_*}\Big)\Big) N(lk)^{-\rho} e^{-N(lk)/Y_*} = -1 + o(1).$$

Truncate at $N(lk) \le 2U^{41}$ and separate dyadically as in [6, Lemma 13.7]. One pair $D = Z^{r'}$, $N = Z^{m'}$ and one frequency $|\nu'| \le T_1$ give the oscillatory factor $(N(lk))^{-i\gamma}(N(lk))^{i\nu'} = (N(lk))^{-iv}$ at $v = \gamma - \nu'$. Hence

$$r' \le t + O(\tfrac{1}{\log Z}), \quad r'+m' \ge t - O(\tfrac{1}{\log Z}), \quad |M_{r'} S_{m'}(\psi(N\cdot)^{-iv})|^2 \ge Z^{(2\sigma-1)(r'+m')-\varepsilon_2} \ge Z^{\nu(r'+m')-\varepsilon_2}, \tag{81}$$

with profiles $x^{-\sigma}$ times uniformly smooth cutoffs, and $|v| \le V_0 + \|\mathbf{t}\| + (3I+1)T_1$, which is (68).

Lengths. (81), the bound for $M_{r'}$ and (80) with $L - m'$ give $2\nu m' \le \nu L + 14e_b(r'+L) + 4\varepsilon_2$. Since $\nu \ge k - 16e_b \ge 1/12$, $r', L \le 2H$ and (60),

$$m' \le \frac{H}{2} + n_F\varepsilon_h + 336e_bH + 24\varepsilon_2 \le \frac{H}{2} + \varepsilon.$$

(81) and the upper bounds at r' and at m' give

$$|M_{r'}|^2 \ge Z^{\nu r' - 14e_b m' - 2\varepsilon_2} \ge Z^{\nu r' - \varepsilon}, \qquad |S_{m'}|^2 \ge Z^{\nu m' - 14e_b r' - 2\varepsilon_2} \ge Z^{\nu m' - \varepsilon}.$$

This is (69).

Data. $(b, \theta, \epsilon_\psi)$ and i range over fixed finite sets, (r', m') over $O((\log Z)^2)$ pairs, and the interval of the cover containing v is one of $\ll Z^{\varepsilon_h} T_1^{-1} + I$. Hence $\#\mathcal{T}(\nu, e) \ll Z^{\varepsilon}$. The argument uses only $|\psi| \le 1$, complete multiplicativity, the functional equation of the entire function $L_F(s, \psi^*)$, the conductor bound and (79); it applies verbatim when ω' has infinite order and to the paired presentations (110). □

Lemma 8.13 (Unmarked inverse moment for oriented presentations)**.** *Let $\psi \in \mathcal{X}_y$ have ω' of infinite order and satisfy (66). Then (67) holds for ν_ψ.*

Proof. By (66), $\omega^\natural$ is an admissible fixed character in the family [3, (4.1)]

$$\mathcal{X}_u = \{\theta\chi_u, \theta\bar{\chi}_u : \theta \in \Theta\} \cup \{\omega^\natural\theta\bar{\chi}_u, \overline{\omega^\natural}\theta\chi_u : \theta \in \Theta\}, \qquad \chi_u(n) = \chi_n(u),$$

with Θ enlarged to contain ours. After whole conjugation if needed, the fixed character has the form $\omega^\natural\theta''$ with θ'' finite. The norm twist enters the test function with polynomial height cost. Put $Z = C_0H'$, $m = 1$, $I = \varnothing$ and $r = \log_Z D \le 1/(1+c)$. Then

$$r \le 1 - \frac{c}{1+c}, \qquad 2r \le 3 - \frac{c}{1+c},$$

so [3, Theorem 4.1] with $c_1 = c_2 = c/(1+c)$ gives

$$\sum_{0<q_u\le C_0H'} \Big| D^{-1/2} \sum_n \mu_F(n)\lambda_u(n) W(Nn/D) \Big|^2 \ll H'^{1+\varepsilon}, \qquad \lambda_u \in \{\omega^\natural\theta\bar{\chi}_u, \overline{\omega^\natural}\theta\chi_u\},$$

uniformly in r. By [3, Proposition 3.1(ii)], every exterior ideal k has an allowed generator u with class in the isotropic complement B, unique modulo the finite-index subgroup Γ, and the rows are taken modulo Γ. The character $\chi_n(k)$ of [6, Theorem 10.7] is evaluated at this allowed generator, for the same S, $R = \mathcal{O}_{F,S}$ and B; another calibration multiplies $\chi_\cdot(k)$ by a unit character in $\Xi \subset \Theta$ (2), which is absorbed into Θ. Hence the ideal sum is at most the generator sum. The fixed exclusions of [3] are generators $u = \epsilon B^6$; there $|D^{-1/2} \sum \mu_F \lambda_u W|^2 \ll D^{3/4}$ by (76), and they number $\ll H'^{1/6}$, which contributes $\ll H'^{1/6} D^{3/4} \le H'$. □

Lemma 8.14 (Prime lengths on $[w, 1]$)**.** *Let $R_0 > R_1 > \dots$ and Y_i be the radii and stage improvements of a descending chain with $z_0 = 0$ (Proposition 5.1, Proposition 6.4, or the one-stage chain (89), for which only $j = 0$ occurs), and let stage j have supporting line $\lambda = a_0 - a_1 w$, $0 \le a_1 \le 2/5$; for $j = 0$, $\lambda = 0$. For $j \ge 1$ let $c_\lambda > 0$ be a margin with*

$$\lambda \le Y_i - c_\lambda \quad \text{on } [R_{i+1}, R_i] \quad (0 \le i < j). \tag{82}$$

Assume the joint moment estimates of the stages $i < j$, and let

$$0 < \delta_1 < \min\{2c_\lambda,\ \upsilon',\ 1 - R_0\}$$

(c_λ omitted when $j = 0$). Then for every $w' \in [R_j, 1-\delta_1]$ the inverse and the ordinary joint moment bounds hold at scale Z^H with

$$P = Hw', \qquad S \le H\sigma^*_{\upsilon'}(w'),$$

and $\sigma^*_{\upsilon'}(w') = 0$ for $w' \in [1 - \delta_1, 1]$. If $j \ge 1$, then moreover $\lambda(R_0) \le -c_\lambda$ and $\sigma^*(w') = 0$ on $[1 - \delta_1, 1]$.

For the chain of Proposition 5.1, $c_\lambda = \Delta_z + p$ (its last sentence); for the chains of Proposition 6.4, $c_\lambda = 2q^3$; for the one-stage chain (89) only $j = 0$ occurs and no c_λ enters.

The initial estimates used below are the following, at scale Z (exponents relative to the parameter range $Nu \ll Z$), with original coefficients in the Θ-character class, $\Theta \supset \Xi$:

(I1) inverse, $S = 0$: [6, Theorem 10.7], for every $P < 1$ (there $H = Z$, $D = Z^P$, $c = (1 - P)/P$);

(I2) inverse, $S > 0$: [6, Theorem 11.2], for $P + 2S < 1$ and $2P + 8S < 3$ with fixed margins;

(I3) ordinary, $S \ge 0$: the coefficient-two face [5, Theorem 4.9, (4.77)], $n_1 + n_2 + 2S \le 1$, starred over the inducing characters in Θ, uniform in the common extra coprimality conditions of [5, Definition 4.1];

(I4) ordinary, $S = 0$: the arbitrary-length no-prime estimate [5, (2.3)], the zero-prime case of [6, Theorem 12.16], for all bounded n_1, n_2.

Each has finite seminorm and polynomial height orders, so the supremum over $\mathfrak{B}_\mathfrak{d}$ is taken as in [6, Lemma 13.9, (13.15)].

Proof. Moment ranges at scale Z are invariant under $Z \mapsto Z^H$ with exponents divided by H; we work at scale Z.

Stage $j = 0$. Here $R_j = R_0$ and $\lambda = 0$, so for $w' \in [R_0, 1 - \delta_1]$

$$\sigma^*_{\upsilon'}(w') = \max\Big\{0, \frac{1 - w'}{2} - \frac{\upsilon'}{2}\Big\},$$

For a positive quota and $P = w'$ we have

$$P + 2S \le 1 - \upsilon', \qquad 2P + 8S \le 4 - 2w' - 4\upsilon' < 3 - 4\upsilon',$$

using $w' \ge R_0 > 2/3$. Apply (I2) and (I3). At zero quota use (I1) and (I4). For $w' \ge 1 - \delta_1 > 1 - \upsilon'$ the quota is zero by the formula.

Stages $j \ge 1$. By (82) with $i = 0$ at $w = R_0$ and $Y_0(R_0) = z_0 = 0$, $\lambda(R_0) \le -c_\lambda$; by $a_1 \ge 0$, $\lambda \le -c_\lambda$ on $[R_0, 1]$. Fix $0 < \beta < \min\{\upsilon', c_\lambda, R_0 - 2/3\}/8$.

(1) $w' \in [R_j, R_0 - 2\beta)$. Then $r := w' + 2\beta \in (R_j, R_0)$; choose $i < j$ with $r \in [R_{i+1}, R_i)$. Theorem 8.21 at stage i, with $\Delta = c_\lambda/2$, gives both moments for $P < r - \beta$ and $S < (1 - r)/2 + Y_i(r) - c_\lambda/2 - \beta$. By (82) at r and $\lambda(w') \le \lambda(r) + 2a_1\beta \le \lambda(r) + \beta$,

$$\frac{1 - r}{2} + Y_i(r) - \frac{c_\lambda}{2} - \beta \ge \frac{1 - w'}{2} + \lambda(w') + \frac{c_\lambda}{2} - 3\beta > \sigma^*(w') - \frac{\upsilon'}{2} \quad \text{if } \sigma^*(w') > 0.$$

If $\sigma^*_{\upsilon'}(w') = 0$, use (I1) and (I4).

(2) $w' \in [R_0 - 2\beta, 1 - \delta_1]$. Here $\lambda(w') \le -c_\lambda + \beta < 0$, so

$$\sigma^*_{\upsilon'}(w') \le \max\Big\{0, \frac{1 - w'}{2} - \frac{\upsilon'}{2}\Big\}, \qquad w' \ge R_0 - 2\beta > \frac{2}{3}.$$

Let $P = w'$ and $0 \le S \le \sigma^*_{\upsilon'}(w')$. If $S > 0$,

$$P + 2S \le 1 - \upsilon', \qquad 2P + 8S \le 4 - 2w' - 4\upsilon' < 3 - 4\upsilon',$$

so (I2) gives the inverse bound and (I3), with $n_1 + n_2 = P$, the ordinary bound. If $S = 0$, then $P \le 1 - \delta_1$, and (I1), (I4) apply.

(3) $w' \ge 1 - \delta_1$: $(1 - w')/2 + \lambda(w') \le \delta_1/2 - c_\lambda < 0$, so $\sigma^*(w') = \sigma^*_{\upsilon'}(w') = 0$.

In the paired unitary family the earlier joint estimates are used at stages $i < j$, as in the finite-order case. For the initial estimates replace (I1)–(I2) by [3, Theorem 4.1] on (110) with $r + 2z \le m - c_1$ and $2r + 8z \le 3m - c_2$, and replace (I3)–(I4) by (108). □

Proof of Proposition 8.9. (D2) is Lemma 8.12. Let $\mathfrak{d} \in \mathcal{T}(\nu, e)$. By the proof of Proposition 8.18, $w = w(\nu) > R = R_j$.

First bound. For $w' = r'/H \in [w, 1 - \delta_1]$, Lemma 8.14 applies to the inverse block and its marked product. If the detected character is $\vartheta\overline{\chi_\bullet(y)}$ with $\vartheta \in \Theta \cup \psi_0\Theta$, the actual prime factor is written with relative coefficient ϑ^{-1} times its finite coefficient. The witness norm mode is put in the ordinary or inverse test. For the conjugate sign conjugate the whole product and its profiles first, sending $(\psi, v, \mathcal{V})$ to $(\bar\psi, -v, -\mathcal{V})$. In either case every live prime character is the same finite twist used in its amplitude bin, up to this whole conjugation. Fixed masks are restored by (56).

Second bound. If $r' < Hw$, then by (69) $m' \ge t - r' - \varepsilon > H(w/2 + \tau) - \varepsilon$, so $Hw < 2m' \le H + 2\varepsilon$ for $\varepsilon < H\tau$, and $w' = \min\{2m'/H, 1\} \in (w, 1]$. For $w' \le 1 - \delta_1$, Lemma 8.14 (ordinary half, $n_1 = n_2 = m'$, one test function) applies, using the earlier paired joint moment wherever the width exceeds the native quota. The same rebasing retains the actual prime factors. For $w' > 1 - \delta_1$, $\sigma^*_{\upsilon'}(w') = 0$, so $K = \varnothing$, and the arbitrary-length estimate (I4) applies at the lengths $n_1 = n_2 = m'$, $2m' \le H + 2\varepsilon$; in the paired unitary case (108) at $z = 0$.

Supremum. At fixed (σ, v) both bounds hold with polynomial dependence on σ and v through the test functions; [6, Lemma 13.9, (13.15)] (Sobolev on $\mathfrak{B}_\mathfrak{d}$, $|\mathcal{V}| = 2T_1$) converts them into the stated form at a further cost $T_1^{J'}$; with the factor $T_1^{J'}$ of (71) the total is $T_1^{2J'} \le Z^{\varepsilon/2}$ by $\tau_1 \le \varepsilon/(4J')$.

Unmarked moment. For ω' of finite order (67) is [6, Theorem 10.7]; the infinite-order branch uses Lemma 8.13 in the orientation of (H4). □

Lemma 8.15 (Counts from detection)**.** *Assume the hypotheses of Proposition 8.9 and* $0 \le a_1 \le 2/5$. *Let* $\mathcal{C}(\nu, e)$ *be a class of the large-value case, restricted to* $Z^{H/5} \le Ny \le Z^H$, *and* $w = w(\nu)$. *Then*

$$\#\mathcal{C}(\nu, e) \ll Z^{\max\{C_s, C_l\}+8\varepsilon}(1 + H_{\mathrm{in}})^{J'},$$

with C_s, C_l *of* (70) *evaluated at this* ν *and* w. *Replacing* ε *by* $\varepsilon/8$, *with* $e_b, \varepsilon_h, J', \tau_1, J$ *chosen by* (60) *for* $\varepsilon/8$ *and* $\delta_1 = \varepsilon/(8H)$, *the loss is* Z^{ε}.

Proof. By (65), $\nu \ge k - 16e_b \ge (1 - 2/25)k$ and $e \le \nu + 16e_b$; with $e_b \le k/200$,

$$e\Big(\frac{1}{2} + a_1\Big) \le \frac{9}{10}(\nu + 16e_b) \le \nu. \tag{83}$$

Put $E(w') = \nu(r - Hw') + e[s - H\sigma^*_{v'}(w')]_+$ on $[w, 1]$. The function $H\sigma^*_{v'}$ is the maximum of 0 and an affine function of slope $-H(1/2 + a_1)$, so by (83) E is nonincreasing. For every $w' \in [w, 1]$,

$$H - \nu H w' - e\min\{s, H\sigma^*_{v'}(w')\} = H + E(w') - \nu r - es \le H + E(w) - \nu r - es \le C_s,$$

the last step by $\sigma^*_{v'} \ge \sigma$. By (D2) it suffices to bound $\#\mathcal{C}_{\mathfrak{d}}(\nu, e)$ for one datum, with pair (r', m') and height v satisfying (68); the height factor is (71). Let $K(y)$ be the complete factors of largest individual slope at y with $\min\{s, H\sigma^*_{v'}(w')\} - O(\eta) \le s_{K(y)} \le \min\{s, H\sigma^*_{v'}(w')\}$, w' fixed below. Partition $\mathcal{C}_{\mathfrak{d}}(\nu, e)$ into the $\le 2^{\#I_0}$ sets $\{y : K(y) = K\}$ and fix K. Since $e > k > 0$ and the average slope of K is at least e, $|Q_K(y)|^2 \ge Z^{e\min\{s, H\sigma^*_{v'}(w')\} - O(\eta)}$ on this set.

(i) $Hw \le r' \le H(1 - \delta_1)$, $w' = r'/H$. For $y \in \mathcal{C}_{\mathfrak{d}}$ the supremum over $\mathfrak{B}_{\mathfrak{d}}$ is at least the value at the detected (σ, v). Chebyshev's inequality in the first bound of (D3) and (69) give $\#\mathcal{C}_{\mathfrak{d}} \ll Z^{H - \nu H w' - e\min\{s, H\sigma^*_{v'}(w')\} + O(\eta) + 2\varepsilon} \le Z^{C_s + O(\eta) + 2\varepsilon}$.

(i′) $H(1 - \delta_1) < r' \le H$. The argument of case (ii) applies verbatim, since [6, Lemma 10.10], and (67) for ω' of infinite order, hold for all lengths $r'' \le r'$. It gives $\#\mathcal{C}_{\mathfrak{d}} \ll Z^{4\varepsilon}(Z^{H - \nu r'} + Z^{H/6 + (5/6 - \nu)r'})$, and

$$H - \nu r' - \Big(\frac{H}{6} + \Big(\frac{5}{6} - \nu\Big)r'\Big) = \frac{5}{6}(H - r') < \varepsilon.$$

Since $r' \le Ht(w) + \varepsilon$, this is $\le Z^{C_l + 6\varepsilon}$, and the supremum over $\mathfrak{B}_{\mathfrak{d}}$ is treated as in case (ii).

(ii) $r' > H$. Write $y = cB^6$, c sixth-power-free, $Nc > Z^{\theta_0}$, and $\nu_{\mathfrak{d}} = \nu_\psi(N\cdot)^{-iv}$. Since $\chi_n(cB^6) = \chi_n(c)1_{(n,B)=1}$, $M_{r'}(\psi(N\cdot)^{-iv})$ at y is the block at c with excluded ideal $\mathrm{rad}(B)$, and Lemma 8.7 gives

$$M_{r'}(y) = \sum_{\substack{\mathrm{rad}(d)\mid B \\ Nd \le b_W Z^{r'}}} \frac{\nu_{\mathfrak{d}}(d)\chi_d(c)^{-\epsilon_\psi}}{\sqrt{Nd}}\, M_{r' - \log_Z Nd}(c),$$

with $\ll_\varepsilon Z^{\varepsilon}$ terms and coefficient mass $\ll (NB)^{\varepsilon}$. So $|M_{r'}(y)|^2 \ge Z^{\nu r' - \varepsilon}$ implies $|M_{r''}(c)|^2 \ge Z^{\nu r' - 3\varepsilon}$ for some d, $r'' = r' - \log_Z Nd \le r'$. The detected (σ, v) depends on y; we bound $|M_{r''}(c)|^2$ at it by $\sup_{\mathfrak{B}_{\mathfrak{d}}} |M_{r''}(c)|^2$, and, as in [6, Lemma 13.9, (13.15)], the supremum is taken before [6, Lemma 10.10] is applied, at the cost $T_1^{J'}$ included above. For dyadic $Nc \in [U, 2U]$, $Z^{\theta_0} \le U \le Z^H$, the lengths satisfy $r''/\log_Z U \le (3H/2 + 1)/\theta_0$, a bounded range, and the implied constant of [6, Lemma 10.10] is uniform for $0 \le r''/\log_Z U \le (3H/2 + 1)/\theta_0$: its proof

bounds $\sup_{T_0\le T\le D}|M_k(T)|^2$ with $D=U^{(3H/2+1)/\theta_0}$, and chooses c,δ once for this range. Thus [6, Lemma 10.10] with this $\nu_{\mathfrak{d}}$ gives

$$\sum_{\substack{U\le Nc\le 2U\\ c \text{ sixth-power-free}}}|M_{r''}(c)|^2\ll U^{1+\varepsilon}+Z^{(\log_Z U+5r'')/6+\varepsilon}.$$

[6, Lemma 10.10] is stated for ν of finite order, with conductor supported in S, and polynomial norm-twist height cost; this covers finite-order presentations, where the fixed factor ν_ψ is finite order and the norm mode is in the test. Its proof uses [6, Theorem 10.7] for ν, the Euler identity [6, (10.16)] and the injectivity of $(u,a)\mapsto ua^6$; the last two use only complete multiplicativity and $|\nu|\le 1$. For presentations with ω' of infinite order, (67) replaces [6, Theorem 10.7], and the same proof gives the same bound. Chebyshev's inequality, $NB^6\le Z^H/U$ and $\sum_{NB\le(Z^H/U)^{1/6}}1\ll(Z^H/U)^{1/6}$ give

$$\#\mathcal{C}_{\mathfrak{d}}\ll Z^{4\varepsilon}\max_U\Big(\frac{Z^H}{U}\Big)^{1/6}(U+U^{1/6}Z^{5r'/6})Z^{-\nu r'}\ll Z^{4\varepsilon}(Z^{H-\nu r'}+Z^{H/6+(5/6-\nu)r'}).$$

Since $r'>H$, $H-\nu r'\le H/6+(5/6-\nu)r'$; since $\nu<5/6$ and $r'\le Ht(w)+\varepsilon$, this is $\le Z^{C_l+5\varepsilon}$.

(iii) $r'<Hw$. Then $m'\ge t-r'-\varepsilon>H(w/2+\tau)-\varepsilon$, so $Hw<2m'\le H+2\varepsilon$ for $\varepsilon<H\tau$. Put $w'=\min\{2m'/H,1\}$. Let $y=cB^6\notin\mathcal{Y}_0$. Every $\psi\in\mathcal{X}_y$, and every $\psi(N\cdot)^{-iv}$, equals $\nu_\psi\chi_\bullet(c)^{-\epsilon_\psi}$ times a norm character on the support, with ν_ψ from the fixed list of conductor dividing $\mathfrak{m}_0$. If it were induced from a character of conductor dividing $\mathfrak{m}_0$ times a norm character, $\chi_\cdot(c)$ would be unramified outside $\mathfrak{m}_0\infty$; but it ramifies at every $p\mid c$, $p\nmid 6\mathfrak{m}_0$, and these primes have total norm $\gg(Nc)^{1/5}>Z^{\theta_0/5}$. Hence for $Z\ge Z_1$ the starred and the unstarred sums agree on $\mathcal{C}_{\mathfrak{d}}\subset\{y\notin\mathcal{Y}_0\}$, and by positivity $\sum_{y\in\mathcal{C}_{\mathfrak{d}}}\le\sum^*_{Ny\le Z^H}$. The same holds with c replaced by ςc on an ordinary side, and for the finite exclusion for ξ_u in (108). Hence the second bound of (D3) with $|S_{m'}|^4\ge Z^{2\nu m'-2\varepsilon}$ gives $\#\mathcal{C}_{\mathfrak{d}}\ll Z^{H-\nu Hw'-e\min\{s,H\sigma^*_{v'}(w')\}+O(\eta)+4\varepsilon}\le Z^{C_s+O(\eta)+4\varepsilon}$, since $2m'\ge Hw'$.

The union over $\ll Z^\varepsilon$ data and $\le 2^{\#I_0}$ sets K, the height cost $Z^{\varepsilon/2}$ of (71) and the supremum, and the choice of the mesh η with $O(\eta)\le\varepsilon/2$ give

$$\#\mathcal{C}(\nu,e)\ll Z^\varepsilon\cdot Z^{\varepsilon/2}\cdot Z^{\varepsilon/2}\cdot Z^{\max\{C_s,C_l\}+6\varepsilon}(1+H_{\mathrm{in}})^{J'}=Z^{\max\{C_s,C_l\}+8\varepsilon}(1+H_{\mathrm{in}})^{J'}.$$

□

Remark 8.16 (Prime amplitudes). The class restriction in Proposition 8.9 is necessary. Fix y_0 with $Z^{H/5}\le Ny_0\le\min\{Z^H,Z^{H_c}\}$, $y_0\notin\mathcal{Y}_0$, let Ψ' be unitary with $|\Psi'(p)|=1$ for $p\notin S$, and put $b_i(p)=\overline{\psi_{y_0}(p)}$ on $\mathcal{P}_i$. Then $|b_i|\le 1$, b_i is independent of the summation parameter, $|\psi_{y_0}(p)|=1$ for $p\nmid y_0$, $p\notin S$, and

$$Q_i(y_0)=Z^{-z_i/2}\sum_{p\in\mathcal{P}_i,\ p\nmid y_0}W_i(Np/Z^{z_i})\gg Z^{z_i/2-\varepsilon}\implies e(y_0)=1>2B_2-1\ge\nu(y_0)$$

by Lemma 8.11(a). This b_i is not in the class (4): its conductor has norm $\gg(Nc_0)^{1/5}\ge Z^{\theta_0/5}\to\infty$ ($y_0=c_0B_0^6$), so it is not a fixed combination $\omega_0^{-j}\sum c_\theta\theta$. Likewise Ψ must be fixed: $b_i=1$ and $\Psi=\chi_\cdot(y_0)$ give ψ_{y_0} principal on ideals prime to y_0 and $e(y_0)=1$; this Ψ violates (5), since it depends on Z.

Remark 8.17 (Status of the detection step). (D1) (65) is Lemma 8.11; the counts (70) follow from (D2), (D3) by Lemma 8.15.

(D2) Lemma 8.12: [6, Lemma 13.7] recentred at $v_0 \in \mathcal{V}_y$, with $Y_* = U^{40}$ (tail U^{-3}), the unit twist through $\Xi \subset \Theta$ (bins at ςy), and the paired unitary presentations verbatim. Three statement changes are needed: $e_b \le \varepsilon/(10^3(H+1))$ is chosen after ε (60), with $\varepsilon_0 = \min\{\varepsilon, e_b s/\#I_0\}$ in Lemma 8.11(c); the height bound (63), which is no restriction by (62); and the Mellin order $J\tau_1 > 100H+2$, since (80) is used at lengths up to $41\log_Z U$.

(D3) The first bound follows from the inverse half, and the second from the ordinary half, of Theorem 8.21 (Proposition 9.2) at the earlier stages, together with the initial estimates [6, Theorems 10.7 and 11.2], [5, Theorem 4.9] and the arbitrary-length unmarked estimate [5, (2.3)], by Lemma 8.14; the end range $r' > H(1-\delta_1)$ is case (i′) of Lemma 8.15. The supremum over $\mathfrak{B}_\mathfrak{d}$ inside the sum is [6, Lemma 13.9, (13.15)], at cost $T_1^{J'}$; it is needed for Chebyshev's inequality, since σ and v vary with y.

(D3-u) The fixed inverse factors after canonical orientation are in $\Theta \cup \psi_0\Theta$. Use [6, Theorem 10.7] and Lemma 8.13 in the respective branches.

Heights: [6, Lemmas 13.7 and 13.9] detect a zero of height $\le 3iT_1$ through a twist of height $\le (3i+1)T_1$ and charge $T_1^{J'}$; the costs $T_1^{J'}$ of (71) and of the supremum total $T_1^{2J'} \le Z^{\varepsilon/2}$, with $\tau_1 \le \varepsilon/(4J')$ chosen after $\varepsilon < H\tau$, so the padding τ is unaffected. Heights above Z^{ε_h} are absorbed by (62), with ε_h chosen before J'. In Proposition 8.18 the inequality $e \le \nu$ is replaced by $e \le \nu + 16e_b$, with e_b chosen after v, v', τ and ε. Collision terms contribute bounded scalars and leave the live paired coefficients unchanged; see Lemma 2.2. The initial inverse rectangles at $S > 0$ in the paired unitary case are [3, Theorem 4.1] for the oriented presentations (110).

Let $\theta_0 = 2/25$ and

$$\mathcal{Y}_0 = \{y = cB^6 : \ c \text{ sixth-power-free}, \ Nc \le Z^{\theta_0}, \ Ny \le Z^H\}.$$

Then

$$\#\mathcal{Y}_0 \le \sum_{Nc \le Z^{\theta_0}} \#\{B : NB \le (Z^H/Nc)^{1/6}\} \ll Z^{H/6} \sum_{Nc \le Z^{\theta_0}} (Nc)^{-1/6} \ll Z^{H/6+5\theta_0/6+\varepsilon}.$$

Write $\chi_{cB^6} = \chi_c 1_{(\cdot,B)=1}$. Then $M_r^{(\mathfrak{r})}(cB^6)$ is the block at c with excluded ideal $\mathfrak{r}\,\mathrm{rad}(B)$. By Lemma 8.7 it is a combination of blocks at c of lengths $Z^{r'} \le Z^r$ with absolute coefficient mass $\ll (N\mathfrak{r}\,NB)^\varepsilon \ll Z^{(\varrho_0+1)\varepsilon}$. By (76), uniformly in c, $|M_{r'}(c)|^2 \ll Z^{(2B_2-1)r'+\varepsilon} \le Z^{\kappa_1 r'+\varepsilon}$. With $|Q(y)|^2 \le Z^{s+O(\eta)}$,

$$\sum_{y\in\mathcal{Y}_0} |M_r^{(\mathfrak{r})}(y)Q(y)|^2 \ll Z^{H/6+5\theta_0/6+\kappa_1 r+s+\varepsilon} \le Z^{H/6+1/15+3r/4+s+\varepsilon}.$$

The same argument at a parameter exponent M, with $\mathcal{Y}_0(M) = \{y = cB^6 : \ Nc \le Z^{\theta_0}, \ Ny \le Z^M\}$ and polynomial length Z^P, gives $\#\mathcal{Y}_0(M) \ll Z^{M/6+5\theta_0/6+\varepsilon}$ and the total $Z^{M/6+1/15+3P/4+s+\varepsilon} = Z^{M+A_0+\varepsilon}$, with A_0 as in Subsection 8.7. On an ordinary side, a small core $y \in \mathcal{Y}_0$ whose presented character $\lambda = \psi(N\cdot)^{-iv}$ has $\delta_\lambda = 0$ satisfies $|S_m(y)|^2 \ll Z^{(2B_2-1)m+\varepsilon}$ by (74) with $X = Z^m$, and the same bound follows; the cores with $\delta_\lambda = 1$ are the irregular cores of Subsection 8.7, removed by the ordinary inducing-character exclusion of Θ, whose signed mean is estimated in Subsection 8.8.

Proposition 8.18 (The estimate with variable positive margins)**.** *Assume* $0 \le a_1 \le 2/5$, $\lambda(R) \le z_R$, $1/6 \le k < 1/2$, *and set*

$$w_b = \frac{R/2 + z_R - a_0}{D}, \qquad F_{\mathrm{high}} = \frac{5}{6} - \left(\frac{5}{4} - \frac{3k}{2}\right) w_b - \frac{k(1+R)}{2} - kz_R.$$

Assume the hypotheses of Proposition 8.9 and the remaining strict stage conditions listed in Subsection 8.7. Suppose $HF_{\mathrm{high}} > 0$, the unperturbed endpoints $g(k)/H$ and $g(3/4)/H$ lie strictly inside $[2/3, 79/100]$, and

$$\frac{r}{2} - s - \frac{3}{2}(5/6 - k)\frac{u}{k^2 D} > 0, \qquad H + u - H/6 - 1/15 - 3r/4 - s > 0.$$

Then $\upsilon, \upsilon', \tau$, and after them e_b, may be chosen positive so that every class $\mathcal{C}(\nu, e)$ of the large-value case, the parameters with $Ny < Z^{H/5}$, and the set $\mathcal{Y}_0$, have second moment at most $Z^{H+u+\varepsilon}$, uniformly over the finite character and test-function families. No parameter has zero slope or inverse amplitude slope in $(3/4, 1]$.

Proof. The last assertion is (65) and the remark after (76). By (65), $k - 16e_b \le \nu \le 3/4$ and $e \le \nu + 16e_b$. By Lemmas 8.11 and 8.15 the second moment of $\mathcal{C}(\nu, e) \cap \{Ny \ge Z^{H/5}\}$ is $\ll Z^{\max\{C_s, C_l\} + (\nu + 14e_b)r + es + \varepsilon}$.

First let $k \le \nu \le 3/4$ and put $g = g(\nu)$. Then

$$D(g - r) = H\{z_R - \lambda(R)\} + u(1/k - 1/\nu) \ge 0,$$
$$s - (H - g)/2 - Ha_0 + a_1 g = g - r + u/\nu \ge 0.$$

Since $H\sigma(w) \ge (H - g)/2 + Ha_0 - a_1 g - H\{(1/2 + a_1)\upsilon + \upsilon'\}$,

$$[s - H\sigma(w)]_+ \le g - r + u/\nu + H\{(1/2 + a_1)\upsilon + \upsilon'\}.$$

Consequently, using $e \le \nu + 16e_b$ from (65), $a_1 \le 2/5$ and $g - r \le H$,

$$\begin{aligned} C_s + \nu r + es &\le H + \nu(r - g) + e(g - r + u/\nu) \\ &\quad + H\{-\nu\upsilon + e[(1/2 + a_1)\upsilon + \upsilon']\} \\ &\le H + u - H\nu(\upsilon/10 - \upsilon') + 16e_b(2H + u/k) \\ &< H + u - Hk\upsilon/20 + 16e_b(2H + u/k). \end{aligned}$$

Moreover $w(\nu) = g/H + \upsilon > R$, so the short estimate uses an earlier moment estimate with the original prime coefficients.

For the long branch define

$$\mathfrak{G}(\nu) = H + u - H/6 - (5/6 - \nu)3g(\nu)/2 - \nu(r + s).$$

Then

$$\mathfrak{G}(k) = HF_{\mathrm{high}},$$

$$\mathfrak{G}(\nu) - \mathfrak{G}(k) = (\nu - k)\left\{\frac{3g(\nu)}{2} - \frac{3}{2}(5/6 - k)\frac{u}{k\nu D} - (r + s)\right\} \ge 0.$$

The displayed derivative condition and $g(\nu) \ge r$ prove the last inequality. The positive margin HF_{high} absorbs $(5/6 - \nu)H(3\upsilon/2 + \tau)$ uniformly. Choose $\upsilon, \tau > 0$ so that

$$(5/6 - k)H(3\upsilon/2 + \tau) < HF_{\mathrm{high}}/2.$$

Then, with $es \le \nu s + 16e_b s$, $C_l + \nu r + es \le H + u - HF_{\mathrm{high}}/2 + 16e_b H$.

Now let $k - 16e_b \le \nu < k$, so $w = w(k)$. Since $Hw > r$ and $Ht(w) > r$, the expressions $C_s + \nu r + es$ and $C_l + \nu r + es$ are nonincreasing in ν for fixed e, and $e \le k + 16e_b$. Hence they exceed the bounds at $\nu = k$ by at most $16e_b Ht(w) \le 24e_b H$.

The pointwise surplus is $14e_b r \le 14e_b H$. Thus every class has exponent at most

$$H + u - \min\{Hkv/20,\ HF_{\mathrm{high}}/2\} + e_b(70H + 16u/k).$$

Choose $e_b \le k/200$ after v, v', τ with $e_b(70H + 16u/k) < \min\{Hkv/40, HF_{\mathrm{high}}/4\}$. The small-core contribution and, by (77), the contribution of $Ny < Z^{H/5}$ are at most $H + u - \gamma_0$, where $\gamma_0 = H + u - H/6 - 1/15 - 3r/4 - s > 0$. Thus one may take

$$0 < \gamma < \min\{u/2, Hkv/40, HF_{\mathrm{high}}/4, \gamma_0\}.$$

The loss ε of (60) is taken with $8\varepsilon < \gamma/2$, in addition to (61). The finitely many amplitude subdivisions, zero bins, test-function integrals and selection errors are estimated after fixing γ. □

This proves that the fixed constants $3 \cdot 10^{-5}$, 10^{-6} and 10^{-7} in the earlier estimates for large values are auxiliary choices. They are not intrinsic positive lower bounds for the exponent for large values. The limiting scalar argument varies all of them together; it does not retain a fixed cutoff padding while sending the exponent margin to zero.

The contribution from small values is equally explicit. On a prime-product amplitude cell of slope at most k, write $|Q_i|^2 = Z^{a_i z_i + o(1)}$ and order the complete factors by $a_1 \le \cdots \le a_J$. Every initial block $D = \{1, \dots, j\}$, of width $s_D = \sum_{i \le j} z_i$, then satisfies

$$\sum_{i \le j} a_i z_i \le \frac{s_D}{\sum_{i \le J} z_i} \sum_{i \le J} a_i z_i \le k\, s_D + o(1), \qquad |Q_D|^2 \le Z^{k s_D + o(1)}$$

[5, Lemma 4.2]. Discard the shortest initial block after whose removal the previous moment estimate applies at the same excluded ideal $\mathfrak{r}$. By Subsection 2.3 that estimate holds for every $\mathfrak{r}$ with $N\mathfrak{r} \le Z^{\varrho_0}$, with loss Z^{ε}; the block $M_r^{(\mathfrak{r})}$ has $r = HR$, so only the prime length is restricted, by $s' < H[(1-R)/2 + z_R]$ after the endpoint decrement. This condition holds at $s' = 0$, so discarding stops at a nonempty range in every case. The discarded width is at most $s - H[(1-R)/2 + z_R] + O(\eta)$, and the pointwise exponent is

$$k\{s - H[(1-R)/2 + z_R]\} = u;$$

an endpoint decrement $\Delta > 0$ improves it to $u - k\Delta$. Errors from selecting complete factors are at most the chosen mesh. Thus small-value and large-value contributions have the same $H + u$ exponent bound; the strict decrements permit a common mesh for the finite induction.

8.7 The induction inequalities and sixth-power factors

In this subsection, we state the sufficient induction inequalities and justify summation over sixth-power factors. Together with Subsection 8.6, these estimates prove the inverse part of Theorem 8.21.

For reference the fourteen sufficient scalar conditions are displayed here. Use $S = (1-r)/2 + Y(r)$, $s = S - u$, $D = 1/2 - a_1$ and $w_+ = w(3/4)$. Every displayed margin is required to be positive.

Condition	Margin
1	$HF_{\mathrm{high}} - \rho_{\mathrm{high}}$
2	$137/1000 - S$
3	$7/50 - S/H$
4	$9/10 - r - S - 4u$
5	$91/100 - r - S - 4u$
6	$5/6 + 21/250 - r - S - 3u - 1/100$
7	$3H - 2r - 8s - 1/5$
8	$r/2 - s - 1/5$
9	$r/2 - s - \frac{3}{2}(5/6 - k)u/(k^2 D) - 1/10$
10	$H + u - H/6 - 1/15 - 3r/4 - s - 3/100$
11	$w(k) - 2/3$
12	$79/100 - w_+$
13	$5/6 - \frac{3}{4}w_+ - \frac{3}{4}\max\{0, (1 - w_+)/2 + \lambda(w_+)\}$
14	$79/100 - r - (2 + 10^{-11})u$

Here $0 < \rho_{\mathrm{high}}$ may be arbitrarily small, and the cutoff, supporting-line and endpoint paddings are chosen after it. The parameter restrictions are

$$45457/125000 \le k < 1/2, \quad 0 \le a_1 \le 2/5, \quad \lambda(R) \le z_R, \quad u \le \frac{1}{20},$$

together with the low-height margin

$$\frac{4}{15} - 3u - \frac{r}{3} - Y(r) > 0 \qquad (r \in \{R_{j+1}, R_j\}), \tag{84}$$

recorded in the one-stage calculation; it is affine in r on a stage. The fixed construction and the limiting sufficient construction keep all numerical margins in conditions 2–14. Condition 13 denotes both polynomial branches of its positive part. Each margin is affine in r, or becomes affine after multiplication by the positive H; endpoint verification therefore covers a whole stage.

The role of condition 13 is not merely numerical. It bounds the sixth-power cofactors. Let $\mathcal{I}$ be the fixed cutoff interval from $w(k)$ to w_+, with the small cutoff paddings included. Since $w + \sigma(w)$ is nondecreasing when $a_1 \le 1/2$, condition 13 gives

$$\nu w + e\sigma(w) < 5/6 \qquad (w \in \mathcal{I},\ k \le \nu \le 3/4,\ 0 \le e \le \nu),$$

and, the inequality being strict on a compact set, also for $k - 16e_b \le \nu$ and $e \le \nu + 16e_b$ once e_b is small. Define the excess exponent for large values by

$$\begin{aligned} A_s &= \nu P + es - \nu M w - e\min\{s, M\sigma(w)\}, \\ A_l &= -5M/6 + (5/6 - \nu)Mt(w) + \nu P + es, \\ A_0 &= -5M/6 + 1/15 + 3P/4 + s, \\ p(P, s, M) &= \max\left\{0, A_0, \sup_{\substack{k \le \nu \le 3/4 \\ 0 \le e \le \nu}} \inf_{w \in \mathcal{I}} \max\{A_s, A_l\}\right\}. \end{aligned}$$

Parameters with $Ny < Z^{M/5}$ need no term: by (77) at exponent M their excess is $M/5 - M + 3P/4 + s \le A_0$, since $M \le 2$. The classes of (65) with $k - 16e_b \le \nu < k$ or $\nu < e \le \nu + 16e_b$,

and the pointwise surplus $14e_bP$, change these expressions by $O(e_b(M+P+s))$; as in the proof of Proposition 8.18, this is absorbed by the endpoint decrements, e_b being chosen after them. For $0 \le M_1 \le M_2$,

$$p(P,s,M_1) \le p(P,s,M_2) + \frac{5}{6}(M_2 - M_1). \tag{85}$$

Indeed the short excess has M-slopes $-\nu w$ or $-\nu w - e\sigma(w) \ge -\nu(w+\sigma(w)) - 16e_b$, both at least $-5/6$ for e_b small; the long excess has slope $-5/6 + (5/6-\nu)t(w) \ge -5/6$. The small-core slope is exactly $-5/6$. Therefore each defining expression A satisfies

$$A(M_1) \le A(M_2) + (5/6)(M_2 - M_1).$$

The same inequality holds after taking the indicated infimum, supremum and maximum. Restricting the cutoff infimum to $\mathcal{I}$ is legitimate: every class still has its chosen cutoff $w(\nu)$ there.

Write a nonzero original parameter as cB^6, with c sixth-power-free, and let $d_B = M - 6\log_Z NB$. On a fixed B-class Proposition 8.9, Proposition 8.18 at parameter exponent d_B and the previous moment bounds give

$$\mathcal{E}_B \ll Z^{d_B + p(P,s,d_B) + \varepsilon}.$$

By (85),

$$d_B + p(P,s,d_B) \le M + p(P,s,M) - \log_Z NB.$$

Consequently, for every $\delta > 0$,

$$\begin{aligned}\sum_B \mathcal{E}_B &\ll Z^{M+p(P,s,M)+\varepsilon+\delta M} \sum_B (NB)^{-1-6\delta} \\ &\ll_{F,\delta} Z^{M+p(P,s,M)+\varepsilon+\delta M}.\end{aligned}$$

Natural zeros at B are restored before this summation; their divisor mass is a subpower and is absorbed by choosing the subsidiary loss smaller. Cores of norm at most Z^{θ_0} form the set $\mathcal{Y}_0(M)$ of Subsection 8.6; their contribution is $\ll Z^{M+A_0+\varepsilon}$ by the summed bound there. The finitely many irregular cores retain the original ordinary exclusion of Θ and are estimated by the same pointwise estimate for all parameters. For the other cores, their width is at least $2/25$, so the common prime mesh rescales uniformly. Condition 14 and the strict cutoff/factor bounds retain the set of original prime positions required in (D2). This proves the bound for large values on all original parameters, rather than only sixth-power-free parameters.

8.8 The reference ordinary moment estimate

In this subsection, we prove the reference ordinary moment estimate used in Subsection 8.9. The unmarked Euler factors converge absolutely with exponent $51/49 > 1$; the marked factors are retained in their original normalized sums.

The earlier ordinary theorem supplies, with strict margin, the face

$$P + \frac{49}{20}S \le \frac{99}{100}, \qquad S \le \frac{7}{50}.$$

It is a restriction of the coefficient-two face [5, Theorem 4.9, (4.77)], $P + 2S \le 1$, since $P + 2S \le P + \frac{49}{20}S \le \frac{99}{100}$. The new reference parameters are

$$l_{\mathrm{ref}} = 7/50, \qquad \alpha = 3/5, \qquad P_{\mathrm{ref}} = 2/3.$$

Ordinary second moments retain the exclusion ind $\notin \Theta$ of [5, Definition 4.1, (4.2)] for the fixed exceptional inducing-character group Θ. Its exact signed mean is combined with the diagonal before positivity and is estimated separately by the exceptional gap below. For a root with $P \ge P_{\mathrm{ref}}$, $S < 137/1000$ and an ordinary factor of width $x \le l_{\mathrm{ref}}$, reflect its partner. The resulting rectangle has widths x and $1 - P + x + o(1)$, and

$$x + (1 - P + x) + \frac{49}{20} S \le \frac{1}{3} + \frac{7}{25} + \frac{49}{20}\frac{137}{1000} < \frac{99}{100} - \frac{1}{25}. \tag{86}$$

The term estimated directly or after half-length reflection also has the strict condition

$$\frac{7}{50} + \frac{1}{2} + \frac{49}{20}\frac{137}{1000} = 0.97565 < \frac{99}{100}.$$

Its marked width has margin $7/50 - 137/1000 = 3/1000$. Reflection is the reflection identity [5, (4.4)] ([6, Lemma 12.6]) for the full character, with the redundant-radical expansion [5, (4.3)]; no natural zero is replaced by a new restriction.

The dual weight depends on the height. Let λ be the full character of the reflected factor, with conductor q_λ and archimedean ratio $G_\lambda(w) = \gamma_\lambda(1/2 - w)/\gamma_\lambda(1/2 + w)$ in the functional equation $L_F(1/2+w, \lambda) = \varkappa_\lambda q_\lambda^{-w} G_\lambda(w) L_F(1/2-w, \overline{\lambda})$, $|\varkappa_\lambda| = 1$. For a test function W, height v and length X, Mellin inversion gives

$$\sum_n \frac{\lambda(n)}{N n^{1/2+iv}} W\Big(\frac{Nn}{X}\Big) = \frac{1}{2\pi i} \int_{(1)} \widehat{W}(w) X^w L_F(1/2 + w + iv, \lambda)\, dw.$$

Shift to $\operatorname{Re} w = -1$; the polar term is absent for the nonprincipal characters of the starred sums. The functional equation and absolute convergence on $\operatorname{Re} w = -1$ give

$$S^{\mathrm{refl}}(y; v) = \varkappa_\lambda q_\lambda^{-iv} \sum_m \frac{\overline{\lambda}(m)}{N m^{1/2-iv}} W_v^\vee\Big(\frac{X\, Nm}{q_\lambda}\Big), \qquad W_v^\vee(\xi) = \frac{1}{2\pi i} \int_{(-1)} \widehat{W}(w) G_\lambda(w + iv) \xi^w\, dw.$$

By Stirling's formula, $|G_\lambda(\sigma + i\tau)| \asymp (1 + |\tau|)^{-n_F \sigma}$ for $|\sigma| \le A$, so moving the line to $\operatorname{Re} w = -A$ or $\operatorname{Re} w = 1/2 - \varepsilon$ gives

$$W_v^\vee(\xi) \ll_A \Big(\frac{\xi}{(1 + |v|)^{n_F}}\Big)^{-A} (1 + |v|)^{A_0}, \qquad W_v^\vee(\xi) \ll \xi^{1/2-\varepsilon} (1 + |v|)^{A_0}.$$

Insert a smooth dyadic partition $1 = \sum_{\ell \in \mathbb{Z}} V(\xi/2^\ell)$, $V \in C_c^\infty([1/2, 2])$, in the dual variable. The terms with $2^\ell > (1 + |v|)^{n_F} Z^\varepsilon$ are $O(Z^{-A})$, and those with $2^\ell < X/(2q_\lambda)$ are empty since $Nm \ge 1$; on $|v| \le Z^{O(1)}$ there remain $O(\log Z)$ dual lengths $Y_\ell = 2^\ell q_\lambda / X$. For each of them Mellin inversion in ξ gives

$$W_v^\vee(\xi) V(\xi/2^\ell) = \frac{1}{2\pi} \int_{\mathbb{R}} \varpi_{v,\ell}(\tau) \Big(\frac{\xi}{2^\ell}\Big)^{i\tau} V\Big(\frac{\xi}{2^\ell}\Big) d\tau, \qquad \int_{\mathbb{R}} |\varpi_{v,\ell}(\tau)| (1 + |\tau|)^J\, d\tau \ll_J (1 + |v|)^{A_\varpi},$$

by integration by parts in ξ and the bounds for the derivatives of $W_v^\vee$. Put $\varpi_v = \sum_\ell \varpi_{v,\ell}$, with the index ℓ carried by the measure. Then

$$S^{\mathrm{refl}}(y; v) = \int_{\mathbb{R}} \varpi_v(\tau)\, S_{W_\tau}^\vee(y; v + \tau)\, d\tau, \qquad W_\tau(x) = x^{i\tau} V(x),$$

up to $O(Z^{-A})$; the unimodular factors $(XNm/(q_\lambda 2^\ell))^{i\tau}$ and q_λ^{-iv} are absorbed into $\varpi_{v,\ell}$ and into the height $v + \tau$ of the dual sum, where $S^\vee$ is the dual sum of length Y_ℓ, with the same redundant-prime expansion, and

$$\int_{\mathbb{R}} |\varpi_v(\tau)| (1 + |\tau|)^J\, d\tau \ll_J (1 + |v|)^{A_\varpi} \log Z, \qquad \mathfrak{S}_J(W_\tau) \ll_J (1 + |\tau|)^J.$$

The dual lengths exceed the reflected length by at most $(1+|v|)^{n_F} Z^{\varepsilon}$. On heights $|v| \le Z^{\varepsilon_2}$, with $\varepsilon_2 > 0$ fixed after the strict margins and before the height order (heights $|v| > Z^{\varepsilon_2}$ are absorbed by the trivial bound as in (62)), this increases the length exponent by $O(\varepsilon_2 + \varepsilon)$, which the margin $1/25$ in (86) absorbs. By Cauchy's inequality with the weight $|\varpi_v|$,

$$|S^{\mathrm{refl}}(y;v)|^2 \le \Big(\int_{\mathbb{R}} |\varpi_v|\Big) \int_{\mathbb{R}} |\varpi_v(\tau)|\, |S^{\vee}_{W_\tau}(y; v+\tau)|^2 \, d\tau.$$

Proposition 8.19 (Reference-moment bound)**.** *The reflected reference second moment is $\ll Z^{1+\varepsilon}$, with a common finite Sobolev bound and polynomial height order, for every fixed $F \supset \mu_{12}$. The prime mesh is chosen before the factor data; constants may depend on the fixed coefficient characters and test function supports. The estimate remains valid after the forward cubic kernel and the original collision assignments.*

Proof. The forward cubic kernel is the local kernel of the reflected ordinary family, formed from the expansion and reflection [5, (4.3)–(4.4)] ([6, Lemma 12.6]) and the local Gauss formulas [6, Lemmas 12.9–12.10]. Of it we use only the degree restrictions in the table after (87) and the absolute local bounds proved below. If an extracted ordinary divisor has width t_2 and fixed marked factors have width f, its reflected factor grows by t_2. Erase complete remaining marked factors of width e until $\frac{49}{20}(f+e) \ge t_2$, or until all remaining marked factors are exhausted. In the exhausted case use the arbitrary-length unmarked ordinary estimate [5, (2.3)]; its erasure loss is smaller than the same weighted upper bound below. Otherwise the reflected face still satisfies (86). Removing these factors contributes at most $Z^{\kappa e}$ to the second moment, where $\kappa = 3/4$. Use the weighted Hilbert norm

$$r_0 = \frac{\kappa}{2(49/20)} = \frac{15}{98}, \qquad \kappa e \le 2r_0\{t_2 - (49/20)f\}_+ + O(\eta).$$

At Hilbert-norm level this gives the two-term bound

$$Z^{\kappa e/2} \le Z^{O(\eta)}\{1 + Z^{r_0(t_2-(49/20)f)}\}.$$

The positive part cannot be replaced by its signed argument when $(49/20)f \ge t_2$. The first term is the unweighted kernel and the second is the weighted kernel. Both ordinary-only products are summable, since

$$3/2 > 1, \qquad 3(1/2 - r_0) = 51/49 > 1.$$

For every fixed polynomial coefficient-growth order A and all sufficiently large Np,

$$\sum_{k\ge 3} (k+1)^A (Np)^{-k(1/2-r_0)} \ll_A (Np)^{-51/49}, \qquad \prod_p \{1 + O_A((Np)^{-51/49})\} < \infty.$$

The finite omitted primes are absorbed into S. For completeness, if an exterior prime has ordinary kernel degree k and m marked positions, the exponent after summing its original prime set is

$$E(k,m) = 1 - \frac{k+m}{2} + r_0\{k - (49/20)m\}. \tag{87}$$

The compensated cubic kernel has only the following possibilities:

term	degree restriction	upper bound for $E(k,m)$
ordinary only	$m = 0,\ k \ge 3$	$-2/49$
one original mark	$m = 1,\ k \ge 1$	$-87/392$
collision	$m \ge 2,\ k \ge 0$	$-3/4$

These follow by substituting $r_0 = 15/98$; the coefficient of k is $-17/49 < 0$. The unweighted term has

$$E_0(k,m) = 1 - (k+m)/2 \le \begin{cases} -1/2, & m = 0,\ k \ge 3, \\ 0, & m = 1,\ k \ge 1, \\ 0, & m \ge 2. \end{cases}$$

Thus both terms have nonpositive marked exponents, while the weighted term has the displayed strict margins. Their ordinary-only Euler products converge absolutely. Marked primes retain their original factor measures; they are not counted as a fresh unrestricted Euler product. There is no field-degree multiple in these norm exponents.

Apply the face $P + (49/20)S \le 99/100$ of Subsection 8.8, a restriction of [5, Theorem 4.9, (4.77)], to the dual sums $S^{\vee}_{W_\tau}(\cdot; v+\tau)$ of the reflected, shortened and erased rectangle, for each τ and uniformly in τ through the seminorms $\mathfrak{S}_J(W_\tau) \ll (1+|\tau|)^J$, and integrate against $|\varpi_v(\tau)|$. With the moment bound for ϖ_v at order $J + J'$ and $(1+|v|+|\tau|)^{J'} \le (1+|v|)^{J'}(1+|\tau|)^{J'}$, the resulting factor is

$$\Big(\int_{\mathbb{R}} |\varpi_v|\Big) \int_{\mathbb{R}} |\varpi_v(\tau)|(1+|\tau|)^J(1+|v|+|\tau|)^{J'}\, d\tau \ll (1+|v|)^{2A_\varpi + J'}(\log Z)^2,$$

a polynomial height factor of the abbreviated convention. Use one common conductor and kernel Sobolev measure on the whole product. The absolute local kernel bounds just proved allow Tonelli's theorem on that positive measure and its divisor sums. Choose the original prime mesh below the face and mark margins in (86); this choice is independent of the later prime lists. The finitely many collision patterns, finite ray-class character expansions and common test derivatives change constants and height orders only. The forward kernel tail has arbitrary decay and is chosen after those finite orders. This proves the stated second moment bound. □

8.9 The ordinary moment induction

The first correlation gives a side at cap $M_C = 1 + t - 6u' + O(\eta)$. Dividing the frequency at the second correlation can lower the cap of an individual label. Proposition A.4 groups the labels at each Kummer parameter and reduces the signed correlation to one regular estimate on the packed ball. We verify that estimate here using only the native estimates and earlier stages.

Proposition 8.20 (The packed ordinary stage)**.** *Let $H = r/R = 1 - 6u$, $u \le 1/100$, and impose the strict inequalities of Subsection 8.7. In the finite-order or oriented paired family, assume the earlier joint rectangles and the native estimates. Then the ordinary moment at*

$$P < r, \qquad S_* < S = \frac{1-r}{2} + Hz_R + (1/k - 2)u$$

holds with the prescribed fixed endpoint decrements and test uniformity.

Proof. Choose the endpoint decrement $0 < \beta < u/(2k)$, decreasing it as needed to accommodate any fixed strict target. Thus $P \le r - \beta$ and $S_* \le S - \beta$. The initial moments cover the strictly native region. Fix the smaller errors after the endpoint decrement. Let $\ell = 101/1000$.

First prove the short positive estimate on the unrestricted family. Put $A = P + S_*$. The original stage conditions give

$$A + 4u < 9/10, \qquad S/H < 7/50, \qquad H \ge 47/50.$$

Lemma A.2 applies at the root and every packed cap $M \ge H - O(\eta)$ because

$$4(\ell + S) < 4\ell + \tfrac{14}{25}H, \qquad \tfrac{11}{25}H - 4\ell \ge \tfrac{6}{625}.$$

Use the forward cube-free kernel of Lemma A.3 to obtain the short estimate on cube-free columns. Next apply the reverse kernel to express the unrestricted target as a bounded-mass combination of shortened cube-free roots. At the outset enlarge the fixed set of excluded places to contain the finitely many small primes needed for the local inverse expansions. Their fixed factors, including natural zero extensions, are restored in the finite packets. The kernels remove common prefixes, with their central scaling, and consume some of the marks. Root total ordinary width and prime width can only decrease. For each shortened root whose ordinary coordinates both exceed ℓ, choose a fresh reference with coordinates $(\ell, P' - \ell)$, where P' is that root's ordinary width. Retain the same two tests and surviving prime subproduct. The two scale products agree. Each norm twist is normalized in its own coordinate as in Lemma A.1. If either coordinate is already at most ℓ, apply the short estimate directly. This order preserves the lower cap ℓ for the fixed-inducing estimate on both long comparison arms.

Lemma A.1 gives the fixed-inducing saving 2/3 with independent normalized heights. The reference and volume inequalities required for packing hold with uniform reserves:

$$A + 3u < \frac{5}{6} + \frac{2}{3}\ell, \qquad A + 4u < 1, \qquad 6u < 1.$$

Indeed $\frac{5}{6} + \frac{2}{3}\ell = \frac{9}{10} + \frac{1}{1500}$. In particular the fixed-inducing face $A + 3u < 5/6 + \ell$ in Proposition A.4 holds. Its cube-free comparison and restoration use the original complete-factor positions. The extraction kernels shorten the ordinary factors in each arm separately and keep the same characters, heights and zero masks.

We now check the positive terminal at its packed radius. On each shortened signed root reframe A at its actual product scale, which is common to its two equal-product arms. After first common factors of widths c, d, let q be the first common residue and complementary divisor width, with $q \le \min(c, d)$, and let

$$k_1 = 2A - c - d + q - 1 - t, \qquad t \ge -O(\eta),$$

be the first frequency exponent. On the left use the actual selected width $u' = u + O(\eta)$ if the selection fills. As in Proposition A.4, the fourth-power shift gives

$$a = A - c - u' + O(\eta), \quad k_C = k_1 + 4u' + O(\eta), \quad q_C = q + c - d.$$

Here Z^a is the actual residual annular scale. Use $Z^{k_C - a}$ and the actual radius of this frame throughout the side estimate. The first Cauchy inequality is in the first squarefree seed and the nonzero frequency jointly. At a fixed raised frequency $\Delta^4 h$ only divisor-many selected tuples Δ can occur, since Δ divides its exterior ideal. The two second common ideals are equal after cube-free reduction; denote their common value by G. If $j = A_2 z$ is the divided second frequency on its local support, the parameter in the surviving character is $y = V_C A_2 z$, where $\log_Z \mathrm{N} V_C = q_C$. The packed ball has exponent cap

$$M = 2a - k_C + q_C + O(\eta) = 1 + t - 6u' + O(\eta). \tag{88}$$

Take the same smooth positive ball majorant for all its labels. Every actual divided frequency maps to a point in this ball. The lower bound on the cap is $H - O(\eta)$. For a nonzero first frequency, $k_1 \ge -O(\eta)$ and $q \le \min(c, d)$ give $t \le 2A - 1 + O(\eta)$, hence $M \le 2(r + S) + O(\eta) < 2$. The right side has the identical bound with $q_D = q + d - c$. An induction root has one moving Kummer parameter

and zero extra displayed radical width. The substitution absorbs $V_C A_2$ into that parameter, so the earlier-stage calls have the same form. An extra pure coprimality mask only requires exact Möbius shortening.

This enlargement is paid by summing the labels at a fixed y. The label weight is $\rho(G)/\mathrm{N}D_2$, with $\rho(G) = 2^{\omega(G)}$, with D_2 the artificial coprimality divisor of the two residual columns. Each residual column retains its coprimality with G, so $(D_2, G) = 1$ and their normalization can charge $\mathrm{N}G\,\mathrm{N}D_2$. Both G and A_2 divide the exterior ideal of y. The fiber estimate in Proposition A.4 is

$$\sum_{\text{labels at } y} \frac{\rho(G)}{\mathrm{N}D_2} \ll_F \tau_F((y))^3 \sum_{\mathrm{N}D_2 \ll Z^C} \frac{1}{\mathrm{N}D_2} \ll_{\varepsilon,F} Z^{\varepsilon}.$$

This uses the reciprocal weight also when $D_2 \nmid y$. Before grouping by y, open the label-dependent coprimality masks by the central Möbius identities of Proposition A.4. Their fixed prefixes shorten the ordinary scales, with subpower divisor mass. A forced mark contributes its scalar and drops out of the live list. Natural zeros in the remaining character are retained. After the common Sobolev separation, the positive terminal therefore has coefficient $Z^{k_C - a}$ and one ordinary second moment at scale Z^M. Each surviving pair has total ordinary width $P' \le r - \beta + O(\eta)$ and live prime width $s' \le S - u - \beta + O(\eta)$. These bounds hold also for the reference arm, whose initial total ordinary width equals P. In fact the reference arm keeps one coordinate at most ℓ; Lemma A.2 pays $Z^{M+\varepsilon}$ for that arm throughout. The following price estimates are needed only for the original arm.

Write $q_R = (1 - R)/2 + z_R$ and $s = S - u$. If the squared response of the live prime product on a row has slope at most k, sort the live slots by their individual squared slopes and remove the smallest initial subproduct that leaves the remaining width strictly below the preceding ordinary quota Mq_R. Its weighted average slope is at most k up to a bin loss, and its width is at most $[s' - Mq_R + \delta_{\mathrm{w}}]_+ + O(\eta)$, where $\delta_{\mathrm{w}} > 0$ includes the strict width buffer. If all slots are removed, the earlier unmarked ordinary moment applies. Since $MR = r + R(M - H)$, the preceding ordinary length cap admits P', even for $M = H - O(\eta)$. Also

$$s - Hq_R = u/k, \qquad s' - Mq_R \le u/k - \beta + O(\eta).$$

Choose the mesh, bin errors and δ_{w} below the fixed endpoint margin. The earlier ordinary bound on the whole packed ball then gives $O(Z^{M+u-k\beta/2+\varepsilon})$.

For the high-response part use the buffered detector in Lemmas 8.11, 8.12, 8.14–8.15 and Proposition 8.18, in the physical family above. On a regular zero bin of slope $\nu = 2a_{\mathrm{bin}} - 1$, the squared product of two ordinary blocks with total width P' is bounded by $Z^{\nu P' + O(e_b) + \varepsilon}$. Indeed the same buffered logarithm estimate as for the inverse block bounds L, and Mellin inversion for each centered ordinary block gives the factor $Z^{(a_{\mathrm{bin}} - 1/2)n_i + O(e_b) + \varepsilon}$. The pure mask factors have subpower mass. Thus the short, long and small-core excess prices are those of the inverse stage with the same P' and s':

$$\begin{aligned} A_s &= \nu P' + es' - \nu M w - e \min\{s', M\sigma(w)\}, \\ A_l &= \nu P' + es' - 5M/6 + (5/6 - \nu) M t(w), \\ A_0 &= 3P'/4 + s' + 1/15 - 5M/6. \end{aligned}$$

The strict stage inequalities pay these prices at $M = H$, with the allowed cutoffs $w = w(\nu)$ and small bin losses. At the packed cap M, run the same detector in scale Z^M with the cutoffs $w, t(w)$ fixed in relative units. Both arms' height windows are included in its buffered zero classes, and the test order in (60) is chosen for the largest packed cap. The earlier inverse rectangles and the earlier balanced ordinary rectangles give the short counts, including the enhanced widths above

the coefficient-two face. The unmarked amplified long count uses the same detector. The possible radial slopes, with P', s', ν, e, w fixed, are

$$-\nu w, \quad -\nu w - e\sigma(w), \quad -5/6 + (5/6 - \nu)t(w), \quad -5/6.$$

They belong to $[-5/6, 0]$. The short lower bound is condition 13 with the allocated bin errors. For the long slope use $5/18 < k - 16e_b \le \nu < 3/4$ and $1 < t(w) < 3/2$. Maxima, minima over the admissible cutoff interval, and suprema over the buffered response bins preserve both slope inequalities. The strict bound at H thus holds for $M \ge H$ and, by its buffer, at $H - O(\eta)$. If $y = cB^6$, the bound on the sixth-power-free core at $M - 6\log_Z \mathrm{N}B$ restores with the harmonic weight

$$Z^{M+p(M)-\log_Z \mathrm{N}B}$$

before the cofactor sum. Here p is the supremal excess from these prices, with the permitted cutoff on each bin. At fixed B, carry the extra radical of B in the live prime and ordinary slots and in the short counts. The long count restores the same mask by the reciprocal identity. Use A_0 on tiny regular cores and keep the finite inducing exceptions in the signed fixed-inducing term. It follows that the high-response part is $O(Z^{M+u-\gamma+\varepsilon})$ for some fixed $\gamma > 0$.

For the inverse witness and its transforms, the fixed angular factors are 1 and the chosen ψ, up to finite twists. The choice of ψ or its conjugate gives Hu's orientation; later whole conjugation transports the theta coefficient as well. The transformations preserve the fixed factor and only multiply it by finite characters, so the buffered maximum is taken over this same paired class. This is the restriction of Subsection 9.2. The short count and refinements of the original positions use Lemmas A.5–A.7. For overlapping original slots these results expand each position into a disjoint refinement plus a subpower error. Fix the refinement depth for the finite induction. A later amplification can take distinct descendants of an existing slot. The complete parents supply the pointwise upper bounds; amplitude selection uses the actual refined descriptors. The same earlier moment estimates apply to any fixed finite disjoint refinement of these parents. In particular no inverse estimate for a resonant presentation outside the paired class is used.

Combining the two response ranges with the packed coefficient gives the filled side allowance

$$Z^{\,k_C - a + M + u' - \rho + O(\eta+\varepsilon)} = Z^{A - c + q_C - \rho + O(\eta+\varepsilon)}$$

for some $\rho > 0$. If there are too few live slots to fill the selection, exhaust them, with the actual selection width $u' \le u + O(\eta)$. The residual leaf is unmarked and the same calculation uses Z^M in place of $Z^{M+u'-\rho}$. It gives the required allowance also at $u' = 0$. The fixed-inducing terms use the centered difference and the same weight bound in Proposition A.4. At the first origin a trivial quotient character forces equality of the two cube-free columns. The central diagonal mass is subpower. At the second origin the correlation is diagonal with value at most $\mathrm{N}I$ on column I; its total contribution on the side is $O(Z^{k_C+\varepsilon})$. Its excess over the first-side allowance is

$$k_C - (A - c + q_C) = A - c - 1 - t + 4u' + O(\eta) < 0.$$

The direct first-origin bound is $O(Z^{1+\varepsilon})$. Thus each signed side meets its full allowance. The one common first family count and the two Cauchy factors combine to give $O(Z^{1+\varepsilon})$ for the original root. □

The one-stage chain. We also use the chain with $n = 1$,

$$R_0 = \frac{191}{250}, \quad R_1 = \frac{37}{50}, \quad z_0 = 0, \quad \lambda = a_0 = a_1 = 0, \quad D = \frac{1}{2}, \quad k = \frac{460859}{10^6}. \tag{89}$$

Here $w_b = R_0$, $k_*(R_0,0) = \frac{5R_0/4-5/6}{R_0-1/2} = \frac{365}{792}$, $k - k_*(R_0,0) = \frac{41}{99000000}$, $u(R_1) = \frac{1}{191}$, $H(R_1) = \frac{185}{191}$, and

$$Y(r) = c\Big(1 - \frac{r}{R_0}\Big), \qquad c = \frac{1/k-2}{6} = \frac{13047}{460859} \qquad (R_1 \le r \le R_0).$$

Only the stage $j = 0$ occurs, so $\theta_0 = 0$ is the only stage parameter, there is no incoming line and (82) is vacuous on the displayed interval. The step satisfies $u \le 1/191 < 1/100$ there. In Theorem 8.21, the same initial moment and zero incoming line apply to any lower endpoint $r < R_0$ at which the strict stage inequalities and $u \le 1/100$ hold.

Theorem 8.21 (The joint moment induction)**.** *Let $F \supset \mu_{12}$ be fixed. Let $2/3 < r < R \le 7/9$. Given joint moment bounds at (R, z_R), take a supporting line of a descending chain of Proposition 6.4 satisfying (82), with the joint estimates of its earlier stages, or the line $\lambda = 0$ with $(R, z_R) = (R_0, 0)$ and k in (89). In both cases impose the strict stage conditions and $u \le 1/100$ at the chosen endpoint r. For the finite-order or oriented paired family of (59), assume (H4) with the original coefficients of (4), including after the finite transformations. The propagation (9) produces both moment estimates at every rectangle*

$$P < r - \beta, \qquad S < \frac{1-r}{2} + Y(r) - \Delta - \beta, \qquad \beta, \Delta > 0.$$

Both the initial and earlier moment inputs and the resulting moments are taken uniformly for the fixed disjoint refinements of Lemmas A.5–A.7. The tree is fixed with enough generations for the finite induction. The large-value, cutoff and supporting-line margins may be chosen arbitrarily small positive numbers, in their required order. The additional upper bound on u holds in the one-stage construction and in Proposition 6.4 by taking the finite step size small.

Proof. For the inverse half, use the first canonical reduction of Subsection 8.5 ([6, proofs of Theorems 10.7 and 11.2]), Proposition 8.8, Proposition 8.6 and Lemma 8.7. The exact exponent identity is (58). Consider a block with remaining marks I_1 and remaining width $z = \sum_{i \in I_1} z_i$.

If $z \ge u$, select width $u' = u + O(\eta)$ by Proposition 8.8. Then the small-value estimate reduces to the previous moment estimate, and the large-value estimate is Proposition 8.18.

If $0 \le z < u$, apply Proposition 8.8 with u replaced by z and select all remaining factors: $u' = z$, $z' = 0$. With $M_0 = 2F_0 - 1 + O(\eta)$,

$$N' = N - z, \qquad K'(z) = M_0 + 4z, \qquad H_0(z) = 2(N' + V) - K'(z) = 1 - 6z + O(\eta),$$

$$(K'(z) - N' - V) + H_c(z) = F_0 - z + \kappa_{\mathrm{P}}, \qquad H_c(z) = H_0(z) + \kappa_{\mathrm{P}}.$$

These are affine in z with slopes $4, -6, -1$; only their monotonicity in $z \in [0, u)$ is used. The remaining polynomial is unmarked, of free length $N' = N - z \le r - \beta + O(\eta)$, at parameter exponent $H_c(z) \ge H_c(u) \ge H + \kappa_{\mathrm{P}} - O(\eta)$. Hence

$$\frac{N'}{H_c(z)} \le \frac{r - \beta + O(\eta)}{H} = R - \frac{\beta}{H} + O(\eta) < R,$$

the earlier moment estimate applies with no prime factor and no excess, $\mathcal{B} \ll Z^{H_c(z)+\varepsilon_1}$, and the off-diagonal term is $\ll Z^{F_0 - z + \kappa_{\mathrm{P}} + 2\varepsilon_1} \le Z^{F_0 + \kappa_{\mathrm{P}} + 2\varepsilon_1}$. The diagonal requires $K'(z) \le F_0$, i.e. $F_0 + 4z + O(\eta) < 1$; by $F_0 \le r + S$ and condition 4 of Subsection 8.7,

$$1 - F_0 - 4z \ge 1 - r - S - 4u - O(\eta) > \frac{1}{10} - O(\eta) > 0.$$

Thus the margins used for $z < u$ are β/H, independent of z, and $1/10$; the z-dependent terms enter only through the displayed inequalities.

Proposition 8.9 applies: (H1) holds by the height reduction (62) with $\varepsilon_h = \varepsilon/(8n_F)$ chosen before J'; (H2) is the order of choices (60); (H3) is the hypothesis on the supporting line, with the loss ε taken so small that (61) holds, in particular $\delta_1 = \varepsilon/H < \min\{2c_\lambda, \upsilon', 1 - R_0\}$ (for the chains of Proposition 6.4, $c_\lambda = 2q^3$; at $j = 0$, in particular for the one-stage chain (89), c_λ is omitted, as in Lemma 8.14). The pairing input is (H4) in the statement. Since ε is arbitrary, these restrictions cost nothing. Since $g(k)/H = w_b \geq R > 2/3$ and the unperturbed upper cutoff is no larger than the perturbed one, the cutoff hypotheses of Proposition 8.18 follow from the stage conditions. Choose κ_{P} and η as in (58); the endpoint decrements absorb the remaining errors.

Apply Proposition 8.20 for the ordinary half. For both halves the short witness moments use only the initial bounds and stages $i < j$, since the detected short cutoffs exceed R_j. In particular, an amplified prime position uses disjoint descendants by Lemma A.7; the moment invoked is the same earlier estimate on those refined lists. The native estimates on such lists are the inputs of Lemma A.6 and [3, Theorem 4.1]. For the small amplitudes the earlier moment is applied at the respective packed cap. At a stage with $R = R_j$ its length allowance is MR_j, which exceeds the shortened ordinary width at $M \geq H - O(\eta)$ after the endpoint decrement. There is no current stage ordinary moment in either application.

At stage zero the initial range at $(R_0, 0)$ follows from

$$P < R_0, \qquad S < \frac{1 - R_0}{2} \quad \Longrightarrow \quad P + 2S < 1, \quad 2P + 8S < 3 \qquad (R_0 > 2/3).$$

For finite-order factors the inputs are (I1)–(I4); in the paired unitary family they are the oriented inverse estimates of [3] and the coefficient-two and unmarked ordinary estimates of [2], with the disjoint refinements of Lemma A.6. Rescaling a fixed earlier rectangle to the parameter exponent M divides both of its input lengths by M.

Fix the strict numerical target rectangles and desired loss first. Choose the finite recursion, endpoint margins, amplitude bins and a common prime mesh using only their numerical length margins. The field is fixed first; the mesh precedes the varying target characters and actual prime factors. These arithmetic data then determine the common measures for norm test functions and finite height and tail orders, in the order (60): the height exponent ε_h precedes the height order J', and heights above Z^{ε_h} are absorbed by (62). All separated modes here are ideal-norm modes.

For $\omega_0 = \omega = 1$ the infinity type of every presentation is trivial, and neither the fixed-type reflection of [3] nor Lemma 9.1 is used; for unitary presentations of infinite order their initial bounds retain the fixed infinity type, as in those results. The induction has finite depth, so the minimum of its finitely many numerical meshes works. This establishes both moments with polynomial height losses over each fixed field. □

8.10 The lower integration limit

In this subsection, we vary the lower integration limit in the Mellin comparison. Subsection 8.12 uses the choice $a = 509/1000$ together with a strict comparison inequality.

Proposition 8.22 (The choice of the lower integration limit)**.** *Let $1/2 < a \leq 51/100$. Use the finite-order family with supremum $\beta = \beta_*(F)$, or Hu's paired unitary family (59) with supremum $\beta = \beta_\omega$. Assume $\beta \geq a$. With floor $a_f = a$ and $\delta_f = 2a - 1$ in the detector, the fixed-cube floor-bin estimate of [6, Subsection 13.5] holds for the compensated quotient of the corresponding family. With the same ordering of the tests, arithmetic data and contours, the lowest bin has leading*

exponent $E_a(v)$ of (48) at scale $Nu \asymp Z^v$. With $B = 11/12 - \ell/4$ (by (14)), $b = l_y - l_x = 2h - 1 - 3\ell$ and $C_0^\flat = -1/4 + b/6 + 5\ell/4$, the constant C_0 of [5, (6.37)],

$$E_a(h) = \ell + (a - \tfrac{1}{2})(1 + 3\ell), \qquad C(B) - E_a(h) = -C_0^\flat - (a - \tfrac{1}{2})(1 + 3\ell). \tag{90}$$

Constants depend on the fixed distance $a - 1/2$.

Proof. The buffered family. Use the maxima and rounding of [6, Lemma 13.7] with $a' \in a + e\mathbb{Z}_{\ge 0}$, $I = \lceil 1/e \rceil + 2$ and a common height $T_1 = Z^\tau$ for small $\tau > 0$. In the unitary case take the maximum over (59). Include the actual denominator $\psi\overline{\chi_\bullet(u)}$, the finite numerator $\chi_\bullet(u)$ and the compensated finite prime characters, with their conjugates and actual height windows. If $B_i(u)$ denotes the buffered maximum including a, then $B_i(u) \le \beta$ and $a' \le B_i(u) < a' + e$, $B_{i+1}(u) < a' + 2e$. Multiplication by a finite-order character or a norm power preserves $\mathcal{F}_\omega$, and conjugation reverses the exponent of ω. Omitted Euler factors have no zeros on the positive strips. The lowest bin uses the trivial row count.

All fixed finite conductors are included in S. At an exterior prime of a sixth-power-free row with valuation $1 \le j \le 5$, the moving inertia has order $6/(6, j) > 1$. No fixed twist can remove it. Thus the retained primitive presentations are nonnorm and have entire L-functions. The bounded exterior-unit rows stay in the principal and small-row branches of [3, Section 6]. On the retained dyads $U = Z^v \asymp Nu$ with v bounded below by a fixed positive constant. The finite conductors are $O(U)$, the infinity types are fixed, and T_1 is a fixed power of U up to a bounded exponent. On disks centered at $2 + it$ the Euler logarithm is bounded on radius $1/2$, and the fixed-type Hecke strip bound and Borel–Carathéodory give $O_e(\log U)$ on radius $2 - a' - 4e$ for $|t| \le (3i + 2)T_1$ in the centered height windows. Three circles on radius $2 - a' - 6e$ then give $\log L = O_e((\log U)^\alpha)$ for some $\alpha < 1$. In particular, the upper and reciprocal bounds $|L| + |1/L| \ll_\varepsilon U^\varepsilon$ hold for $\operatorname{Re} s \ge a' + 6e$. Cauchy's inequality with a radius at most $e/2$ gives the L'/L bound at $\operatorname{Re} s = a' + 7e$. The retained height windows leave room for this radius, and moving the prime contour costs $O(e)$ in its exponent. The deleted Euler factors and their inverses cost U^ε on buffered positive strips. This is the proof of [6, Corollary 13.8(1)–(2)] for the fixed-type family. The reflected functional equation is applied only to the finite numerator; on $\operatorname{Re} w = 1 - a' - 6e$ its bound is $U^{a' - 1/2 + O(e) + \varepsilon}$ times a fixed height polynomial. More precisely, put $A = a' - 1/2 + 6e$. The tame conductor exponent is one at each strict position $\mathfrak{p}^j \parallel u$, so

$$(\mathrm{N}\mathfrak{f}_{\chi_u})^A \prod_{\text{strict } \mathfrak{p}} (\mathrm{N}\mathfrak{p})^{z_0 - \operatorname{Re} w} \ll_{F,S} U^A \prod_{\text{strict } \mathfrak{p}} (\mathrm{N}\mathfrak{p})^{z_0 - 1/2 - (j-2)A}.$$

Here $j \ge 2$. This is the simultaneous conductor compensation of [6, (13.17)]. The reflected line stays in a buffered positive strip since $a' \le 11/12$ and $e < 10^{-3}$. Its deleted Euler factors also cost U^ε.

The contours. Choose $0 < e < \min\{10^{-3}, (a - 1/2)/100, (1 - a)/100\}$ ($e < 9 \cdot 10^{-5}$ for $a = 509/1000$). Every retained bin $a \le a' \le 11/12$ lies in the region $1/2 \le a' \le 11/12$, $0 < e \le 1/1000$ of [6, Lemma 13.10], and the compensated cube is moved to the lines

$$\operatorname{Re} x = a' + 16e, \qquad \operatorname{Re} z = 17/50, \qquad \operatorname{Re} w = 1 - a' - 6e$$

of [6, Subsection 13.5]. With $z_0 = 17/50$, its local bounds are

$$z_0 + \tfrac{1}{2} - 2a' - 32e \le z_0 - \tfrac{1}{2} - 32e, \quad H_p - 1 \ll (Np)^{-1-10e}, \quad \operatorname{Re}(6z) = \tfrac{51}{25} > 1, \quad \operatorname{Re}(x + w) \ge 1 + 10e.$$

These local estimates hold coefficientwise for arbitrary unitary phases, including the ramified valuation terms, by the table and connecting region of [6, Lemma 13.10 and Subsection 13.5]. The

actual quotient and its fixed ramified factors are those of [3, Section 6], as used in [2, Section 3]. For each original zero bin keep the whole compensated product through the moves. Move z and the reciprocal first to $\operatorname{Re} z = z_0$ and $\operatorname{Re} x = \beta + 20e$ on the global reciprocal lines, then the nonprincipal numerator to $\operatorname{Re} w = 1 - a' - 6e$, where $\operatorname{Re}(x + w) \geq 1 + 14e$. The global reciprocal line has polynomial height growth by comparison of the zero-free logarithm with its Euler-line value, using the fixed gap to β. The untruncated reciprocal integrals use this line. After truncating the common heights, move only the retained x segment to $a' + 16e$ in the buffered rectangle. Normal convergence of the unselected product and the full local correction bounds apply throughout this connecting box. Fix the tuple expansion before splitting by prime amplitudes. The principal residue, the small rows and the height tails use their original branches.

The lowest bin. Its row exponent is the piecewise row sum [6, (13.17)], with exponent [6, (13.18)] at $a' = a$ and trivial count. The free-parameter factor is $(Nu)^{a-1/2+O(e)+\varepsilon(N+8)}$; each complete prime block has slope at most $a - 1/2 + O(e)$, so the prime positions give $\prod_i (Z^{\ell_i})^{-4/25+g_i+\eta} \leq Z^{(a-33/50)\ell+O(e\ell)+\eta\ell}$, $\sum_i \ell_i = \ell$; the Mellin weight is $(Nu)^{-17/50}$, so ideal counting on $Nu \asymp Z^v$ gives $v(1 + a - 1/2 - 17/50) = v(a + 4/25)$; the common test function gives $Z^{(4/25)l_x + a - 33/50 - al_y + O(e)}$. The product is $Z^{E_a(v)}$. By (14), $l_x = 1 + \ell - h$ and $l_y = h - 2\ell$, so

$$E_a(h) = \tfrac{4}{25} + a - \tfrac{33}{50} + \ell(\tfrac{4}{25} + 3a - \tfrac{33}{50}) = \ell + (a - \tfrac{1}{2})(1 + 3\ell),$$

and $C(B) = B - \frac{1}{2} - \frac{h}{3} + \frac{\ell}{2} = \frac{1}{4} - \frac{b}{6} - \frac{\ell}{4} = -C_0^\flat + \ell$. This proves (90). Since $a + 4/25 > 0$, $E_a(v) \leq E_a(h + \zeta) = E_a(h) + (a + \frac{4}{25})\zeta$ for $v \leq h + \zeta$.

The replaced inequality. For $a = 51/100$ the second identity in (90) gives the lowest-bin test $C_0^\flat + (1/2 + 3\ell/2)/50 < 0$ of [6, Subsection 13.5], the first inequality of [5, (6.38)]. At floor a use

$$C_0^\flat + (a - \tfrac{1}{2})(1 + 3\ell) + (a + \tfrac{4}{25})\zeta < 0, \tag{91}$$

where $\zeta \geq 0$ is the upper parameter buffer, $v \leq h + \zeta$. All remaining losses are fixed multiples of e, $\varepsilon(N + 8)$, the prime mesh and the height cutoff. Choose them after the strict margin in (91), with the test-function and arithmetic-data order of [6, Subsection 13.5]. □

8.11 The reflected numerator estimate

In this subsection, we prove the numerator marked-length bound C_{new} of Subsection 5.2, used at the conductor edges of Theorem 5.2 and in the limiting comparison of Subsection 7.1. The approximate functional equation is displaced block by block as in [5, Subsection 6.3]. The block of reflected length Z^{d-l_y} receives the ordinary moment with prime-length coefficient two [5, Theorem 4.9]. Subsection 8.12 applies the result.

Fix (θ, h) and the lengths ℓ, l_x, l_y of (14). Fix κ with

$$\frac{13}{18} \leq \kappa \leq 1, \qquad \beta_*(F) \leq \frac{1 + \kappa}{2}.$$

In the analytic application $\kappa = \kappa_1$, and the second condition is [6, Theorem 17.1]: $\beta_*(F) \leq 0.87488664704 < (1 + \kappa_1)/2$. Let $\mathscr{B}$ be an amplitude class of parameters u with $Nu \asymp U = Z^d$ and amplitude parameters (δ, x),

$$\#\mathscr{B} \ll U^{R+\varepsilon}(1 + H)^A,$$

where R is the count exponent and H the height cutoff. Assume $l_y < d < 2l_y$. Put

$$\begin{gathered} \varphi_0 = \frac{\delta x}{6\kappa}, \qquad n_0(d) = 1 - \frac{l_y}{d}, \qquad N_{\mathrm{ref}}(R) = \frac{3R + 1}{4} - \frac{\varphi_0}{2}, \\ \mathcal{E}_0(d) = E(d, C_{\mathrm{old}}(d); \delta, x) + \frac{3d}{4}R - \theta, \end{gathered} \tag{92}$$

$$\Sigma_N(d)=\delta x\Big(\frac14-\frac{1}{12\kappa}\Big)(2l_y-d),\quad \Sigma_L(d)=\varphi_0\Big(\frac{7d}{10}-l_y\Big),$$
$$\Sigma_R(d)=\Big(\frac{\delta}{2}-\varphi_0\Big)\Big(l_y-\frac{53d}{100}\Big),\qquad T_{\mathrm{sh}}(d)=d\Big(\frac{R}{4}-\frac{17}{80}-\frac{\varphi_0}{2}\Big)+\varphi_0 l_y. \tag{93}$$

Since $\partial_C E=-\delta x/2$ and $C_{\mathrm{old}}(d)=(2l_y-d)/(6\kappa)$,

$$\mathcal{E}_0(d)-\Sigma_N(d)=E\Big(d,\frac{2l_y-d}{2};\delta,x\Big)+\frac{3d}{4}R-\theta. \tag{94}$$

For $B_P\le 1$ and $L_P\ge 2$, $C_{\mathrm{new}}(d)\le(2l_y-(2-B_P)d)/L_P\le(2l_y-d)/2$. Hence

$$\mathcal{E}_0(d)-\Sigma_N(d)\le E(d,C_{\mathrm{new}}(d);\delta,x)+\frac{3d}{4}R-\theta. \tag{95}$$

Lemma 8.23 (The contour range)**.** *The assertions of [5, Lemma 5.2] hold for every* $1/50\le\delta\le 3/4$*, with the same classwise shifts* $0\le\varphi\le\delta/2$*,* $\operatorname{Re} w=1/2-\varphi$*, and the same change*

$$(\varphi-\varphi_0)\{d(1-n)-l_y\}$$

of the comparison exponent for a dyadic sum of length U^n*.*

Proof. We re-run the proof of [5, Lemma 5.2] with $a=a_{\mathrm{det}}=(1+\delta)/2$. The parameter δ enters it only at the following places. For each we give the range used in [5] ($11/50\le\delta\le 13/20$) and the range for $1/50\le\delta\le 3/4$, $0\le\varphi\le\delta/2$.

(P1) The connecting region of [6, Lemma 13.10], $1/2\le a\le 11/12$, $0<e\le 1/1000$: there $a\in[61/100,33/40]$; here $a\in[51/100,7/8]$.

(P2) The swept range of $\operatorname{Re} w=1/2-\varphi$: there $[7/40,1/2]$; here $[1/8,1/2]$.

(P3) [5, Lemma 5.1] at w and $1-w$, with $\sigma_0\le\operatorname{Re} w,\operatorname{Re}(1-w)\le 1-\sigma_0$: there $\sigma_0=1/7$; here $\sigma_0=1/9$, since $1/9<1/8$ and $7/8<8/9$.

(P3′) The Stirling factor $T^{-2\varphi}$, with $2\varphi\le 1-2\sigma_0$: there $2\varphi\le 13/20<5/7$; here $2\varphi\le 3/4<7/9$.

(P4) $\operatorname{Re} w\ge 1-a\Rightarrow\operatorname{Re}(s+w)\ge 1+16e$: in both cases from $\varphi\le\delta/2$, i.e. $\frac12-\varphi\ge\frac12-\frac{\delta}{2}=1-a$.

(P5) The amplitude class [5, (6.6)]: $0<\delta\le 5/6$ in both cases.

(P6) The s-segment on $\operatorname{Re} s\ge a+16e$ [6, Corollary 13.8(1)]: $a>a_f$ in both cases; here $a\ge 51/100>a_f=509/1000$.

The following steps of the proof do not involve δ: the bound $q_p^{\varphi}q_p^{-(j_p-1)\varphi}=q_p^{-(j_p-2)\varphi}\le 1$ ($j_p\ge 2$, $\varphi\ge 0$); the fixed-place majorant $\prod_{p\in S_{\mathrm{fin}}}(1-q_p^{-1/6})^{-1}$; the order of moves $c_w\to\frac12$ (unsplit), insertion of the two-part approximate functional equation on $\operatorname{Re} w=1/2$, $\frac12\to\frac12-\varphi$ (each dyadic sum), then the local s-contour; removal of horizontal joins by Mellin decay; and

$$q_u^{\varphi-\varphi_0}D^{-(\varphi-\varphi_0)}Y^{-(\varphi-\varphi_0)}\asymp Z^{(\varphi-\varphi_0)\{d(1-n)-l_y\}}\qquad(D=U^n,\ Y=Z^{l_y}).$$

The only explicit range in [5, Lemma 5.2] is the remark after its proof, $a\in[61/100,33/40]$; it serves (P1). Hence every step holds on the stated range. □

Proposition 8.24 (Reflected numerator saving)**.** *Let* $1/50 \le \delta \le 3/4$ *and* $0 \le x \le 1/2$. *Assume*

$$\frac{3}{10} < n_0(d) < \frac{47}{100}, \tag{96}$$

$$\frac{\ell}{d} \ge \frac{21}{100}, \qquad \frac{21}{100}(6\kappa - 2\delta x) > R. \tag{97}$$

Let $\mathfrak{N}_d(\mathscr{B})$ *be the contribution of* $\mathscr{B}$ *to the averaged Mellin numerator, divided by the comparison main term. Then*

$$\mathfrak{N}_d(\mathscr{B}) \ll Z^{\mathfrak{E}(d)+\varepsilon}(1+H)^{A'}, \qquad \mathfrak{E}(d) = \mathcal{E}_0(d) - \min\{\Sigma_N(d), \Sigma_L(d), \Sigma_R(d)\}. \tag{98}$$

In particular, by (95)*, if* $\Sigma_L(d) > \mathcal{E}_0(d)$ *and* $\Sigma_R(d) > \mathcal{E}_0(d)$*, then*

$$\mathfrak{E}(d) < 0 \quad \Longleftarrow \quad E(d, C_{\mathrm{new}}(d); \delta, x) + \frac{3d}{4}R < \theta.$$

Proof. The comparison exponent. With $b = 2h - 1 - 3\ell$ and $p_1 = 6\kappa$, the geometry (14) is that of [5, (6.5)], with θ in place of B:

$$l_x + l_y = 1 - \ell, \qquad h = l_y + 2\ell, \qquad C_0^\flat = -\frac{1}{4} + \frac{b}{6} + \frac{5\ell}{4} = -\frac{5}{12} + \frac{h}{3} + \frac{3\ell}{4}.$$

Here $C_0^\flat$ is the constant written C_0 in [5, (6.5)]. Let $E^\flat(R)$ and $S^\flat(R)$ be the functions of [5, (6.11)] with $p_1 = 6\kappa$. Using $\theta = 11/12 - \ell/4$ and $l_y = h - 2\ell$,

$$\begin{aligned} C_0^\flat - \frac{3h}{4} - \frac{9(d-h)}{100} &= -\frac{5}{12} + \frac{3\ell}{4} - \frac{49h}{150} - \frac{9d}{100} \\ &= \frac{1}{2} + \frac{13h}{75} - \frac{l_y}{2} - \frac{\ell}{2} - \frac{9d}{100} - \theta, \\ \delta x\Big\{\ell + \frac{h/2 - l_y}{6\kappa}\Big\} + (d-h)\frac{\varphi_0}{2} &= \delta x\Big(\ell - \frac{C_{\mathrm{old}}(d)}{2}\Big). \end{aligned}$$

The R-terms combine to $3hR/4 + 3(d-h)R/4 = 3dR/4$. Therefore

$$E^\flat(R) + (d-h)S^\flat(R) = \mathcal{E}_0(d). \tag{99}$$

Block exponents. Apply the approximate functional equation [6, Lemma 14.2] to the whole numerator factor and decompose both halves dyadically. This gives $O(\log U)$ blocks of length U^n. Suppose a block satisfies

$$\sum_{u \in \mathscr{B}} |S_n(u)| \ll U^{X(n)+\varepsilon}(1+H)^{A_1},$$

and is displaced to $\operatorname{Re} w = 1/2 - \varphi$ with $0 \le \varphi \le \delta/2$. By [5, Proposition 6.1 (3)] and (99), its exponent is

$$\mathcal{E}_0(d) + d\{X(n) - \varphi_0 n - N_{\mathrm{ref}}(R)\} + (\varphi - \varphi_0)\{d(1-n) - l_y\}. \tag{100}$$

Here [5, (6.12)] is used with $d\gamma = d\{N_{\mathrm{ref}}(R) + \varphi_0 n - X(n)\}$; this is exponent bookkeeping, valid for either sign of γ. The contour step is Lemma 8.23, applied per block. Note that

$$d(1-n) - l_y = d\{n_0(d) - n\}.$$

First moments. By (97) and [5, Lemma 6.3], for every numerator length,

$$X(n) \le \varphi_0 n + N_{\mathrm{ref}}(R) \qquad (n \ge 0). \tag{101}$$

For $3/10 \le n \le 47/100$, put $z_2 = (1-2n)/2$. Then $z_2 \le 1/5 < 21/100 \le \ell/d$. Select whole prime factors of positive amplitude in decreasing order, up to relative width z_2. With the prime mesh η, this gives $|Q_{\mathrm{sel}}(u)|^2 \ge U^{2\delta x z_2 - O(\eta)}$ on $\mathscr{B}$. Since $2n + 2z_2 = 1$, [5, Theorem 4.9 and (6.4)] and Hölder's inequality give

$$\sum_{u\in\mathscr{B}} |S_n(u)|^4 \le U^{-2\delta x z_2 + O(\eta)} \sum_{u} |S_n(u)^2 Q_{\mathrm{sel}}(u)|^2 \ll U^{1-2\delta x z_2 + O(\eta)+\varepsilon},$$

$$\sum_{u\in\mathscr{B}} |S_n(u)| \le (\#\mathscr{B})^{3/4}\Big(\sum_{u\in\mathscr{B}} |S_n(u)|^4\Big)^{1/4} \ll U^{(3R+1)/4 - \delta x(1-2n)/4 + O(\eta)+\varepsilon}.$$

Hence, with $\Sigma_2(n) = \delta x(1/4 - 1/(12\kappa))(1-2n)$,

$$X(n) - \varphi_0 n - N_{\mathrm{ref}}(R) \le (1-2n)\Big(\frac{\varphi_0}{2} - \frac{\delta x}{4}\Big) = -\Sigma_2(n) \qquad \Big(\frac{3}{10} \le n \le \frac{47}{100}\Big). \tag{102}$$

Choice of contours. Take $\varphi = 0$ for $n \le n_0(d)$ and $\varphi = \delta/2$ for $n > n_0(d)$. By (96) and $\varphi_0 \le \delta/(12\kappa) < \delta/2$, the exponent (100) is bounded as follows.

- For $0 \le n \le 3/10$, by (101):

$$\mathcal{E}_0 - \varphi_0 d(n_0 - n) \le \mathcal{E}_0 - \varphi_0 d\Big(n_0 - \frac{3}{10}\Big) = \mathcal{E}_0 - \Sigma_L(d).$$

- For $n \ge 47/100$, by (101):

$$\mathcal{E}_0 - \Big(\frac{\delta}{2} - \varphi_0\Big)d(n - n_0) \le \mathcal{E}_0 - \Big(\frac{\delta}{2} - \varphi_0\Big)d\Big(\frac{47}{100} - n_0\Big) = \mathcal{E}_0 - \Sigma_R(d).$$

  This includes the tails $n > 1/2 + r_F/100 + \varepsilon$ of the approximate functional equation [6, Proposition 14.3]: there (101) needs no prime selection, and the bound decreases in n.

- For $3/10 \le n \le 47/100$, by (102), the exponent is $\mathcal{E}_0 - \Sigma^{\mp}(n)$, where

$$\Sigma^-(n) = d\Sigma_2(n) + \varphi_0 d(n_0 - n) \qquad (n \le n_0),$$
$$\Sigma^+(n) = d\Sigma_2(n) + \Big(\frac{\delta}{2} - \varphi_0\Big)d(n - n_0) \quad (n \ge n_0).$$

  These satisfy

$$\frac{d\Sigma^-}{dn} = -\frac{d\,\delta x}{2} \le 0, \qquad \frac{d\Sigma^+}{dn} = \frac{d\,\delta(1-x)}{2} \ge 0.$$

  Hence

$$\min_{3/10\le n\le 47/100} \Sigma^{\mp}(n) = d\Sigma_2(n_0) = \delta x\Big(\frac{1}{4} - \frac{1}{12\kappa}\Big)d(1-2n_0) = \Sigma_N(d),$$

  since $d(1-2n_0) = 2l_y - d$.

Assembly. The block count, the selection loss $O(\eta)$, the Sobolev step [5, (6.14)] and the height factors contribute $Z^{\varepsilon}(1+H)^{A'}$ after the mesh is chosen. Taking the maximum over blocks proves (98). The last assertion follows from (95). □

Remark 8.25 (Short blocks). For $0 \le n < 3/40$ the trivial bound $X(n) \le R + n/2$ and $\varphi = 0$ give, in (100),

$$\mathcal{E}_0 + d\Big(R + \frac{n}{2} - N_{\mathrm{ref}}(R)\Big) - \varphi_0(d - l_y) \le \mathcal{E}_0(d) + T_{\mathrm{sh}}(d),$$

without (97). The shift $\varphi = \varphi_0$ gives instead $\mathcal{E}_0(d) + dD_{\mathrm{sh}}(R)$, where

$$D_{\mathrm{sh}}(R) = \frac{R}{4} - \frac{17}{80} + \frac{17\varphi_0}{40}$$

is the short-block quantity of [5, (6.26)]. The two bounds differ by

$$dD_{\mathrm{sh}}(R) - T_{\mathrm{sh}}(d) = \varphi_0\Big(\frac{37d}{40} - l_y\Big) > 0.$$

At the parameters of Proposition 8.26 and $(\delta, x) = (9/40, 1/2)$, $\mathcal{E}_0(d_{\mathrm{top}}) + d_{\mathrm{top}}D_{\mathrm{sh}}(R) > 0$ in both configurations, whereas $\mathcal{E}_0 + T_{\mathrm{sh}} < 0$ by Proposition 8.26 (iii). Thus the short blocks are covered by the bound with T_{sh}, and independently by (101).

Proposition 8.26 (The numerator inequalities at the comparison parameters)**.** *Consider the two configurations:*

(A) $\theta = \theta_+$, $h = h_+$, $\kappa = 2\theta_+ - 1$, $B_P = 1 - 10^{-12}$, $L_P = 2 + 10^{-11}$, $d_{\mathrm{top}} = h_+ + 10^{-10}$, *and the count envelope of Theorem 5.2; in* $\mathcal{E}_0$, θ_+ *is replaced by* $\theta_+ - 10^{-12}$, *as in* (26)*;*

(B) $\theta = B_*$, $h = h_*$, $\kappa = \kappa_1$, $B_P = 1$, $L_P = 2$, $d_{\mathrm{top}} = h_*$, *and the count envelope of Subsection 7.1, including the range* (46) *and the ideal tangent.*

Let R *be the count envelope. Put*

$$\mathcal{D}_{\mathrm{new}} = \{(\delta, x) \in [0, 3/4] \times [0, 1/2] : \mathcal{E}_0(d_{\mathrm{top}}) \geq 0\}.$$

Then the following hold.

(i) $\mathcal{E}_0(19/25) < 0$ *on* $[0, 3/4] \times [0, 1/2]$, *and* $\mathcal{D}_{\mathrm{new}} \subset (21/100, 3/4] \times [0, 1/2]$.

(ii) For $d \in \{19/25, d_{\mathrm{top}}\}$, (96) *holds and* $\ell/d > 21/100$*:*

	(A)		(B)	
	$d = 19/25$	$d = d_{\mathrm{top}}$	$d = 19/25$	$d = d_{\mathrm{top}}$
$n_0(d)$	$0.38429\ldots$	$0.42549\ldots$	$0.38321\ldots$	$0.42506\ldots$
$\ell/d - 21/100$	$0.01799\ldots$	$0.00274\ldots$	$0.01800\ldots$	$0.00253\ldots$

(iii) On $\mathcal{D}_{\mathrm{new}}$, *the supply inequality* (97) *holds. For* $d \in \{19/25, d_{\mathrm{top}}\}$,

$$\Sigma_N(d) - \mathcal{E}_0(d) \geq 0, \quad \Sigma_L(d) - \mathcal{E}_0(d) > 0, \quad \Sigma_R(d) - \mathcal{E}_0(d) > 0, \quad -\mathcal{E}_0(d) - T_{\mathrm{sh}}(d) > 0.$$

In (A) the first inequality is strict. In (B) its only zero is the double contact (47) *at* $(\delta_*, 1/2)$, $d = h_*$.

Proof. For a count branch $R_j = N_j/J_j$, with $J_j > 0$ on its range, each margin is a bivariate polynomial over $\mathbb{Q}$ after multiplication by J_j. The margins $\Sigma_\bullet - \mathcal{E}_0$, $-\mathcal{E}_0 - T_{\mathrm{sh}}$ and the supply margin decrease in R, and $-\mathcal{E}_0$ decreases in R. Hence any branch $R_j \geq R$ that is valid on a box may be used.

A box is accepted if one of two polynomials has positive tensor Bernstein coefficients for some such branch: either the required inequality, or $-J_j\mathcal{E}_0(d_{\mathrm{top}})$, which places the box outside $\mathcal{D}_{\mathrm{new}}$.

The successive dyadic subdivisions use the following numbers of subrectangles:

	(A)		(B)	
inequality	19/25	d_{top}	19/25	d_{top}
$\Sigma_L - \mathcal{E}_0$	2	19	2	17
$\Sigma_R - \mathcal{E}_0$	1	10	1	10
$-\mathcal{E}_0 - T_{\mathrm{sh}}$	1	5	1	5
(97)	5	5	5	5

The maximal depth is 7. Positivity of $-\mathcal{E}_0(19/25)$ uses 2 subrectangles in each configuration. Exact Bernstein bounds give $\mathcal{E}_0(d_{\mathrm{top}}) < 0$ on $[0, 213/1000] \times [0, 1/2]$ in (A) and on $[0, 21/100] \times [0, 1/2]$ in (B). This proves (i).

In (B) all coefficients are Bernstein coefficients in r as well. They are uniform on a rational interval of width less than 10^{-40} isolating r_*. The values in (ii) are exact rational evaluations, rounded down.

The inequality $\Sigma_N - \mathcal{E}_0 \geq 0$ is, by (95), implied by the edge inequalities at $(19/25, C_{\mathrm{new}}(19/25))$ and $(d_{\mathrm{top}}, C_{\mathrm{new}}(d_{\mathrm{top}}))$: (26) in (A), and (47) together with its Bernstein cover in (B).

The proof uses only the certified positivity above. For orientation, approximate minima over $\mathcal{D}_{\mathrm{new}}$, from evaluation on a grid, are:

	(A)		(B)	
	19/25	d_{top}	19/25	d_{top}
$\Sigma_N - \mathcal{E}_0$	0.0202	$6.5 \cdot 10^{-9}$	0.0206	0
$\Sigma_L - \mathcal{E}_0$	0.0174	$5.8 \cdot 10^{-4}$	0.0176	$7.8 \cdot 10^{-4}$
$\Sigma_R - \mathcal{E}_0$	0.0276	$2.1 \cdot 10^{-3}$	0.0281	$2.1 \cdot 10^{-3}$
$-\mathcal{E}_0 - T_{\mathrm{sh}}$	0.0325	$4.2 \cdot 10^{-3}$	0.0326	$3.7 \cdot 10^{-3}$

The supply margin is approximately 0.074 in (A) and 0.0757 in (B). □

Configuration (A) is a statement about the scalar system of Theorem 5.2, where $\kappa = 2\theta_+ - 1$ as in (14), exactly as for C_{old}. With this κ the hypothesis $\beta_*(F) \leq (1+\kappa)/2 = \theta_+$ of Proposition 8.24 is not available in a contradiction argument at θ_+, so (A) is not used analytically. The analytic application, Theorem 8.29, uses only configuration (B), with $\kappa = \kappa_1$ and the unconditional input [6, Theorem 17.1]. The inequalities (i)–(iii) are verified at (A) and (B) only. For $\theta_{240} < \theta < \theta_+$, Theorem 5.4 concerns the scalar system $\mathcal{S}_{240}$, and (i)–(iii) are not asserted there.

Lemma 8.27 (The gate for C_{old})**.** *In configurations (A) and (B) of Proposition 8.26, let R be the count envelope, let $\theta' = \theta_+ - 10^{-12}$ in (A) and $\theta' = B_*$ in (B), and put*

$$C_{\mathrm{aff}}(d) = \frac{\ell}{d_{\mathrm{top}}}\,(2l_y - d).$$

(i) $19/25 < 6\kappa\ell$, *and* (25) *holds for* $6\kappa\ell \leq d \leq d_{\mathrm{top}}$ *and* $(\delta, x) \in [1/8, 3/4] \times [0, 1/2]$.

(ii) For $19/25 \leq d \leq d_{\mathrm{top}}$, *case (1) of [6, Proposition 14.3] with* $v = \ell/d_{\mathrm{top}}$ *applies and gives the numerator estimate with* $C = C_{\mathrm{aff}}(d)$.

(iii) For $(\delta, x) \in [0, 1/8] \times [0, 1/2]$ *and* $19/25 \leq d \leq d_{\mathrm{top}}$,

$$E(d, C_{\mathrm{aff}}(d); \delta, x) + \frac{3d}{4} R(\delta, x) - \theta' \leq -\frac{1}{200}.$$

Proof. (i) Exact rational evaluation gives

$$6\kappa\ell = \begin{cases} 0.77632\ldots & \text{(A)},\\ 0.77953\ldots & \text{(B)}. \end{cases}$$

Since $R \le R_o$, [6, (14.4)] gives $R \le 1 - 2\delta/3$; with $\delta x \le \delta/2$,

$$(6\kappa - 2\delta x)\frac{\ell}{d} - R \ge \Gamma_{\mathrm{g}}(d,\delta) := (6\kappa - \delta)\frac{\ell}{d} - 1 + \frac{2\delta}{3}.$$

For $d \ge 6\kappa\ell$, $\ell/d \le 1/(6\kappa) < 2/3$, so Γ_{g} increases in δ and decreases in d. Exact rational evaluation gives

$$\Gamma_{\mathrm{g}}(d_{\mathrm{top}}, 1/8) = \begin{cases} 0.00987\ldots & \text{(A)},\\ 0.01287\ldots & \text{(B)}. \end{cases}$$

Hence $\Gamma_{\mathrm{g}} > 0$ on $[6\kappa\ell, d_{\mathrm{top}}] \times [1/8, 3/4]$.

(ii) The hypotheses of [6, Proposition 14.3] other than the gate are those of case (2). For $d \le d_{\mathrm{top}}$,

$$\frac{\ell}{d} \ge \frac{\ell}{d_{\mathrm{top}}} > \frac{1}{5}, \qquad \frac{1}{6\kappa} - \frac{\ell}{d_{\mathrm{top}}} = \begin{cases} 0.01046\ldots & \text{(A)},\\ 0.00975\ldots & \text{(B)}, \end{cases}$$

so $v = \ell/d_{\mathrm{top}} \le \min\{\ell/d, 1/(6\kappa)\}$, and $\delta x \le 3/8 < 5/12$. The shift $\delta x v$ contributes $-l_y \delta x v + d\delta x v/2 = -\frac{\delta x}{2}\, v(2l_y - d) = -\frac{\delta x}{2} C_{\mathrm{aff}}(d)$.

(iii) Both C_{aff} and the left side are affine in d, so it suffices to take $d \in \{19/25, d_{\mathrm{top}}\}$. On $[0, 1/8] \times [0, 1/2]$ only the old count is in range, so $R = R_o = N_o/J_o$ there, with $J_o > 0$. After multiplication by J_o, the margin

$$J_o\Big(\theta' - \frac{1}{200} - E(d, C_{\mathrm{aff}}(d); \delta, x)\Big) - \frac{3d}{4} N_o$$

is a bivariate polynomial over $\mathbb{Q}$. For each of the two values of d and each configuration, all its tensor Bernstein coefficients on $[0, 1/8] \times [0, 1/2]$ are positive, without subdivision; the same holds for J_o. In (B) the coefficients are uniform on the rational interval of width less than 10^{-40} isolating r_*. □

Since $C_{\mathrm{old}}(d) - C_{\mathrm{aff}}(d) = (2l_y - d)\{1/(6\kappa) - \ell/d_{\mathrm{top}}\}$ and $\partial_C E = -\delta x/2 \ge -\delta/4$, the replacement costs at most

$$\frac{\delta}{4}\Big(2l_y - \frac{19}{25}\Big)\Big(\frac{1}{6\kappa} - \frac{\ell}{d_{\mathrm{top}}}\Big) \le \begin{cases} 4.6 \cdot 10^{-4}\delta & \text{(A)},\\ 4.4 \cdot 10^{-4}\delta & \text{(B)} \end{cases}$$

in the comparison exponent, hence at most $5.8 \cdot 10^{-5}$ for $\delta \le 1/8$. For orientation, a grid evaluation gives $\max(E(d, C_{\mathrm{aff}}) + 3dR/4 - \theta') \approx -0.0055$ in (A) and -0.0053 in (B) over $[0, 1/8] \times [0, 1/2] \times [19/25, d_{\mathrm{top}}]$, attained at $\delta = 1/8$; on the set where (25) fails it is ≈ -0.0090 in (A) and -0.0091 in (B).

Corollary 8.28 (The bound C_{new} at the conductor edges)**.** *In configuration (B), for every amplitude class with $(\delta, x) \in [0, 3/4] \times [0, 1/2]$ and $19/25 \le d \le h_*$, the numerator $\mathfrak{N}_d(\mathscr{B})$ has exponent at most 0, and strictly negative exponent unless $(d, \delta, x) = (h_*, \delta_*, 1/2)$. For every $B > B_*$ sufficiently close to B_*, with $\theta = B$ and the perturbed lines of Subsection 7.1, there are fixed $c_B, \zeta_B > 0$ with*

$$\mathfrak{N}_d(\mathscr{B}) \ll Z^{-c_B}(1 + H)^{A'} \qquad (19/25 \le d \le h_* + \zeta_B)$$

for every such amplitude class. This covers the conductors above $h = h_*$ within the fixed buffer used at the fourth edge.

Proof. All quantities in (92) and (93) are affine in d on $[19/25, h_*]$, since $19/25 > l_y$. Also n_0 and ℓ/d are monotone in d.

If $(\delta, x) \notin \mathcal{D}_{\mathrm{new}}$, then $\mathcal{E}_0(19/25) < 0$ and $\mathcal{E}_0(h_*) < 0$. Hence $\mathcal{E}_0(d) < 0$ on the interval. If $\delta \geq 1/8$, then Lemma 8.27 (i) gives (25) for $d > 6\kappa\ell$, so this is the comparison inequality for the estimate with C_{old} (case (2) of [6, Proposition 14.3]), as at the edge $(19/25, C_{\mathrm{old}}(19/25))$. If $\delta < 1/8$, Lemma 8.27 (ii)–(iii) gives the estimate with $C_{\mathrm{aff}}(d)$ and comparison exponent at most $-1/200$.

If $(\delta, x) \in \mathcal{D}_{\mathrm{new}}$, then $\delta > 21/100$, and Proposition 8.26 gives (96), (97) and the inequalities in (iii) at both endpoints, hence for every d in the interval, with $\Sigma_N - \mathcal{E}_0 > 0$ for $d < h_*$. Proposition 8.24 then gives the first assertion.

For $B > B_*$, every accepted box other than the contact box has a strictly positive polynomial lower bound. These bounds, the strict conditions (O), (N), (I) and the uniform continuity of the counts in the line coefficients give $\Sigma_L - \mathcal{E}_0$, $\Sigma_R - \mathcal{E}_0$, the supply margin and the outside-$\mathcal{D}_{\mathrm{new}}$ margin uniformly positive (the supply, Σ_L and Σ_R margins are uniformly positive on the closure of $\mathcal{D}_{\mathrm{new}}$ minus a neighbourhood of the contact $(\delta_*, 1/2)$, hence on an open neighbourhood $\mathcal{N}$ of it; on the compact set $[0, 3/4] \times [0, 1/2] \setminus \mathcal{N}$, $-\mathcal{E}_0(d_{\mathrm{top}}) \geq c > 0$ by continuity), together with the strict margins of Lemma 8.27 (i)–(iii), as in Subsection 7.1. By (94), the margin $\Sigma_N - \mathcal{E}_0$ is affine in d, and by (95) it is at least the comparison margin at $(d, C_{\mathrm{new}}(d))$. At $d = 19/25$ and $d = h_*$ these are the third and fourth edge margins, which are strictly positive for $B > B_*$ by Subsection 7.1; at the top edge this is the transverse derivative $J(3 + 8\delta x) > 0$. All these margins are affine or continuous in d on $[19/25, h_* + 1/100]$, since $l_y < 19/25$; strict positivity at $d = h_*$ therefore persists on $[h_*, h_* + \zeta_B]$ for some fixed $\zeta_B > 0$, taken with $\zeta_B < \ell/(21/100) - h_* = 0.0098338$, so that $\ell/d \geq 21/100$ persists. □

8.12 The finite-order zero-free theorem

In this subsection, we apply Theorem 8.21 and the Mellin estimate of Subsection 8.10 to finite-order characters. Section 9 extends the result to arbitrary continuous unitary characters.

Theorem 8.29. *For every number field K, every finite-order Hecke character η and every $t \in \mathbb{R}$,*

$$L_K(s, \eta(N\cdot)^{it}) \neq 0 \qquad (\Re s > B_*), \tag{103}$$

with its possible pole at $1 + it$ permitted. The infimum B_* is optimal within Definition 7.1.

Proof. First fix $F \supset \mu_{12}$. The known prior bound supplies the fixed parameter for the earlier pointwise and moment estimates

$$\kappa_1 = 374887/500000 = 0.749774 < 3/4.$$

The earlier count and numerator bounds in Subsection 7.1 use this independently known parameter. The new initial and joint-moment counts use the earlier ordinary moment with prime-length coefficient two and the joint moments constructed above.

Fix a target $B > B_*$. Choose $B_* < B' < B$ sufficiently close to B_*. If $\beta_*(F) < 509/1000$, the conclusion is immediate. Otherwise $\beta_*(F) \geq 509/1000$, which is the hypothesis of [6, Lemma 13.7] with $a_f = 509/1000$ used in Proposition 8.22.

The limiting inequalities, their strictly positive transverse derivative, and Proposition 6.4 give a finite rational moment construction with strict comparison margin at B'; its supporting lines satisfy (82) with $c_\lambda = 2q^3$ and $0 \leq a_1 \leq 2/5$. Apply Theorem 8.21 successively, using only earlier moment estimates at each step. All presentations have $\omega_0 = \omega = 1$. Set $a = 509/1000$ for the floor and apply Proposition 8.22. Its limiting floor margin is at least 0.0012507528 and remains positive after the fixed perturbations. At conductor exponents $19/25 \leq d \leq h_*$, apply Corollary 8.28. At

(B_*, h_*, κ_1) the inequalities of [5, Lemma 6.5] hold with the following lower bounds, obtained by exact rational evaluation at both ends of a rational interval of width 10^{-30} containing r_* (all printed digits agree at the two ends):

floor (91), $a=\frac{509}{1000}$, $\zeta=0$:	$-C_0^\flat-\frac{9}{1000}(1+3\ell)=0.0012507528$,
	at $a=\frac{51}{100}$ the floor margin is $-2.691\cdot10^{-4}$,
geometry [5, (6.33)]:	$b=0.110799$, $\ell-\frac16=0.0066166$, $\frac15-\ell=0.026717$,
	$l_x-\ell=0.18468$, $l_y-\ell-\frac{11b}{6}=0.092343$,
	$6\kappa_1\ell-\frac34=0.029540$, $2l_y-h=0.12219$,
	$\frac12-l_y=0.031242$, $B_*-0.87=0.0033459$,
	$\frac{\ell}{h}-\frac{21}{100}=0.0025329$, $\frac29-\frac{\ell}{h-1/40}=0.0029664$,
	$1-\frac{l_y}{h}=0.42507$, $\frac{l_y}{h}=0.57493$,
	$2l_x-l_y-\ell=0.073876$, $2(l_x+l_y)-1-3\ell=0.13358$,
direct [5, Proposition 6.1(4)]:	$k=0$, $\frac{1+3\ell-2M}{4}=-0.033396$, $M+\ell-1=0$,
	$\frac{3\ell-1}{2}=-0.24008$,
	$G=\frac{b}{12}$ $(l_y>\frac{11b}{6})$, $\frac56+\frac{l_x}{6}-\frac{\ell}{6}+\frac{b}{12}=B_*$,
stock and supply:	Proposition 8.26 (ii),(iii),
small amplitude:	Lemma 8.27 (iii) (margin 1/200),
small parameters [5, (6.38)]:	$\frac{l_y}{2}-\frac{13h}{75}-\frac{1}{50}=0.073056>\frac{1}{20}$,
principal [5, (6.39)]:	$B_*>0.87$, $l_y/20=0.023438$, $h/600=0.0013589$.

The margins exceed the interval width. For small amplitudes use Lemma 8.27 (iii) at $d\ge19/25$ and the C_{old} edges below 19/25; the constant inequality $C_0^\flat+(1-h)/16<-1/250$ of [5, (6.37)] is not used (at (B_*,h_*) only $C_0^\flat+(1-h_*)/16<0$ holds). The upper parameter buffer ζ of (91) and the buffer ζ_B of Corollary 8.28 are chosen with

$$0<\zeta<\frac{0.0012507528}{509/1000+4/25}=1.8696\cdot10^{-3},\qquad 0<\zeta_B<\frac{\ell}{21/100}-h_*=0.0098338.$$

The bound $\beta_*(F)\le0.87488664704<(1+\kappa_1)/2$ has margin $3.52\cdot10^{-7}$. All of these are strict at B_*; for B' close to B_* they persist by continuity.

For the geometry at B', put $M=l_x+l_y=1-\ell$ and $b=l_y-l_x=2h-1-3\ell$. Then

$$L_0=L_K=\frac{11}{12}-\frac{\ell}{4}=B',\qquad G_0=\frac{2b}{3}-\frac{M}{4}<-\frac{13}{100},\qquad \max\{L_0,L_K,L_0+G_0\}=B'<B.$$

The bound on G_0 holds near the limiting parameters in (45). The finitely many endpoint, test function, height, integration path and tail losses can therefore be chosen to fit all the margins simultaneously. The comparison output is [5, (6.40)] with these margins: for $\beta=\beta_*(F)>B'$ and fixed $0<\varepsilon_{\mathrm{low}}<(\beta-B')/4$,

$$|J_\eta(Z)|\ll Z^{C(B')+\varepsilon_{\mathrm{low}}},\qquad |J_\eta(Z)-f_\eta(Z)|\ll Z^{C(\beta)-m_{\mathrm{save}}/2}$$

for some $m_{\mathrm{save}}>0$. The Mellin continuation of [5, Proposition 7.2, (7.4)–(7.5)], with [6, (13.13) and Lemma 14.7], then shows that $e^{(s-5/6)^2}H_\eta(s)/L_F^S(s,\eta)$ is holomorphic for $\operatorname{Re}s>\sigma:=\max\{B'+\varepsilon_{\mathrm{low}},\beta-m_{\mathrm{save}}/2\}$. Since $\sigma>B_*>0.87$, the principal box [5, (6.39)] and [5, Lemma 7.1] make H_η holomorphic and nonzero there. Hence $\beta_*(F)\le\sigma<\beta_*(F)$, a contradiction; so $\beta_*(F)\le B'<B$, which is nonvanishing in $\Re s>B$ for every finite-order target over F. As $B>B_*$ was arbitrary, this proves the half-plane $\Re s>B_*$. No mesh or positive saving uniform at the infimum is required.

For a general K, take the finite abelian extension $F = K(\zeta_{12})$. The abelian identity (114), with its ramified Euler factors retained, descends nonvanishing to K. Away from $s = 1$ every factor is holomorphic in this half-plane; at $s = 1$ classical nonvanishing handles the possible principal pole. Finally use the vertical translation (106). Optimality in the specified positive system is Theorem 7.2. □

Remark 8.30 (Order of analytic operations). Fix the presentation, complex variables and common test functions; form the whole numerator and its functional-equation counterpart; move the holomorphic expression; then use absolute values, common Sobolev measures and positive moments. The earlier papers provide the Mellin continuation theorem with this order of operations.

8.13 The one-stage half-plane

Here the comparison uses the line of the one-stage chain (89). The joint moments follow from the initial ranges and one application of Theorem 8.21. Put

$$B_1' = \frac{3493459}{4000000} = 0.87336475, \qquad h_1 = \frac{4061871}{5000000}, \qquad d_{\mathrm{top}} = h_1 + 10^{-10}, \tag{104}$$

and let ℓ, l_x, l_y be given by (14) at $(\theta, h) = (B_1', h_1)$.

Theorem 8.31 (One-stage half-plane). *For every number field K, every modulus $\mathfrak{m}$ of K, every ray class character χ modulo $\mathfrak{m}$ and every $t \in \mathbb{R}$,*

$$L_K(s, \chi(N_K\cdot)^{it}) \neq 0 \qquad (\mathrm{Re}\, s > B_1' = 0.87336475),$$

with the simple pole at $s = 1 + it$ permitted when χ is principal. Moreover

$$B_* < B_1' < B_2, \qquad B_2 - B_1' > 1.72 \cdot 10^{-5}.$$

Proof. Fix $F \supset \mu_{12}$. All values below are exact rational evaluations, rounded down, or lower bounds from exact tensor Bernstein covers, as in the proofs of Theorem 5.2 and Proposition 8.26.

(a) The stage. For (89), at $r \in \{R_0, R_1\}$, with $S = \frac{1-r}{2} + Y(r)$, $s = S - u$, $w(k) = w_b = R_0$, $w_+ = (r + 2u(1/k - 4/3))/H$, $\lambda = 0$, the margins of Subsection 8.7 are

cond.	$r = R_0$	$r = R_1$	cond.	$r = R_0$	$r = R_1$
1 (HF_{high})	$\frac{41}{375000000}$	$\frac{1517}{14325000000}$	8	0.064	0.04434
2	0.019	0.00611	9	0.164	0.11680
3	0.022	0.00486	10	0.04566	0.03507
4	0.018	0.00816	11	0.09733	0.09733
5	0.028	0.01816	12	0.026	0.01695
6	0.02533	0.02073	13	0.26033, 0.17183	0.25355, 0.16844
7	0.328	0.22052	14	0.026	0.03952

(condition 13: both branches), with $\rho_{\mathrm{high}} = \frac{1517}{28650000000}$. Also

$$\tfrac{1}{2} - k > 0.039, \quad k - \tfrac{45457}{125000} > 0.097, \quad a_1 = 0, \quad \lambda(R_0) = 0 = z_0, \quad u \le \tfrac{1}{191} < \tfrac{1}{20},$$

and (84) has the values $\frac{3}{250}$ and $\frac{14981119}{4401203450}$. Each margin is affine in r, or affine after multiplication by $H > 0$, so it is positive on $[R_1, R_0]$. The input of stage 0 is (I1)–(I4) at $(R_0, 0)$ (proof of

Theorem 8.21), the line is $\lambda = 0$, and Lemma 8.14 is used at $j = 0$, with c_λ omitted. Theorem 8.21 gives both joint moments on

$$P < r - \beta, \qquad S < \frac{1-r}{2} + c\Big(1 - \frac{r}{R_0}\Big) - \Delta - \beta \qquad (R_1 \le r < R_0;\ \beta, \Delta > 0). \tag{105}$$

(b) The line and its domain. Put

$$g(r) = c\Big(1 - \frac{r}{R_0}\Big) - 10^{-9} = g_0 - g_1 r, \qquad g_0 = \frac{13046999539141}{460859000000000}, \qquad g_1 = \frac{c}{R_0} = \frac{3261750}{88024069},$$

with domain $[a_-, b] = [R_1, 7/9]$. Then $g_1 > -1/2$. On $[R_1, R_0)$, (105) with $\Delta + \beta < 10^{-9}$ gives the moments at width $z(r) = \frac{1-r}{2} + g(r)$, decreased by an arbitrarily small amount. On $[R_0, 7/9]$, $g(r) \le g(R_0) = -10^{-9}$, and for $P \le r$, $S < z(r)$,

$$P + 2S < 1 + 2g(r) \le 1 - 2 \cdot 10^{-9}, \qquad 2P + 8S < 4 - 2r + 8g(r) < 3 \quad (r \ge R_0 > \tfrac{1}{2}),$$

so the inverse moments are (I2), or (I1) if $S = 0$, and the ordinary moments are (I3), or (I4) if $S = 0$, with the fixed margin $2 \cdot 10^{-9}$. Thus the base estimates supply the line beyond R_0, and the hypothesis of Lemma 4.3 holds on $[R_1, 7/9]$. Condition (S3) holds for $d \le d_{\text{top}}$:

$$\frac{\ell}{d_{\text{top}}} - \Big(\frac{1 - a_-}{2} + g(a_-)\Big) > 0.0823.$$

(c) Count envelope and range conditions. Use the expressions of Theorem 5.2 with $\kappa = \kappa_1$, $B_C = B_P = 1$, $L_C = L_P = 2$, $c_C = 3/20$, $a_- = 37/50$, $b = 7/9$. The count envelope is the minimum of the old count on $[0, 3/4] \times [0, 1/2]$, the initial count on $[15/64, 9/16] \times [9/32, 1/2]$ and on (46), and the line count R_g on

$$W = \Big[\frac{99}{256}, \frac{117}{256}\Big] \times \Big[\frac{31}{64}, \frac{1}{2}\Big].$$

Condition (O) on the whole box, (N) on each initial range and (I1)–(I10) on W hold strictly, each without subdivision. The smallest Bernstein lower bounds of the denominator-multiplied margins are $5.85 \cdot 10^{-4}$ (the R_n-condition of (N)), $4.79 \cdot 10^{-3}$ ((I6), (I7)) and $4.81 \cdot 10^{-3}$ ((I5)). The geometry (14) gives

$$\ell - \tfrac{1}{6} = 0.006541, \quad \tfrac{1}{5} - \ell = 0.02679, \quad l_x - \ell = 0.18762, \quad \tfrac{1}{2} - l_y = 0.03404,$$
$$2l_x - l_y - \ell = 0.08250, \quad 2(l_x + l_y) - 1 - 3\ell = 0.13396, \quad \tfrac{l_y}{2} - \tfrac{13h_1}{75} - \tfrac{1}{50} - \tfrac{7}{100} = 0.00216,$$
$$\tfrac{234B_1' - 184}{25} - h_1 = 0.00231, \qquad B_1' - \tfrac{51}{100} - \tfrac{h_1}{3} - \tfrac{53\ell}{100} = 0.000773 > \rho = 10^{-5}.$$

(d) The four edges. With d_{top} of (104) at the fourth edge,

$$E(d, C; \delta, x) + \frac{3d}{4} R(\delta, x) < B_1' - 10^{-12} \qquad ((\delta, x) \in [0, 3/4] \times [0, 1/2])$$

at all four edges of Subsection 5.2. A box is accepted when J times the margin has positive Bernstein coefficients for a branch valid on it:

edge	0	1	2	3
subrectangles	1	2	2	39
lower bound of the margin	$3.37 \cdot 10^{-2}$	$2.08 \cdot 10^{-3}$	$3.42 \cdot 10^{-3}$	$1.61 \cdot 10^{-8}$

with maximal depth 19. The smallest bound occurs on $[1743/4096, 873/2048] \times [511/1024, 1/2]$, in the line branch. Since all margins are positive, no contact identity as in (47) is needed.

(e) The numerator inequalities. In the configuration $\theta = B_1'$, $h = h_1$, $\kappa = \kappa_1$, $B_P = 1$, $L_P = 2$, d_{top} as in (104), the count envelope of (c), and $B_1' - 10^{-12}$ in place of θ in $\mathcal{E}_0$, the assertions (i)–(iii) of Proposition 8.26 hold, with T_{sh} as in Proposition 8.26, all inequalities strict:

(i) $-\mathcal{E}_0(19/25) \geq 2.08 \cdot 10^{-3}$ on $[0, 3/4] \times [0, 1/2]$ (2 subrectangles), and $-\mathcal{E}_0(d_{\mathrm{top}}) \geq 3.8 \cdot 10^{-4}$ on $[0, 21/100] \times [0, 1/2]$ (1 subrectangle); hence $\mathcal{D}_{\mathrm{new}} \subset (21/100, 3/4] \times [0, 1/2]$;

(ii) $n_0(19/25) = 0.38690\ldots$, $n_0(d_{\mathrm{top}}) = 0.42642\ldots$, $\ell/d - 21/100 = 0.01790\ldots$, $0.00321\ldots$ at $d = 19/25, d_{\mathrm{top}}$, and $l_y < d < 2l_y$;

(iii) on $\mathcal{D}_{\mathrm{new}}$, with numbers of subrectangles and lower bounds,

	$d = 19/25$		$d = d_{\mathrm{top}}$	
$\Sigma_L - \mathcal{E}_0$	2	$4.17 \cdot 10^{-3}$	14	$1.99 \cdot 10^{-4}$
$\Sigma_R - \mathcal{E}_0$	1	$1.73 \cdot 10^{-3}$	9	$7.29 \cdot 10^{-5}$
$-\mathcal{E}_0 - T_{\mathrm{sh}}$	1	$1.51 \cdot 10^{-2}$	5	$9.11 \cdot 10^{-4}$
(97)	5	$1.39 \cdot 10^{-3}$	5	$1.39 \cdot 10^{-3}$
$\Sigma_N - \mathcal{E}_0$	2	$5.47 \cdot 10^{-3}$	39	$1.61 \cdot 10^{-8}$

A box is also accepted when $-J\mathcal{E}_0(d_{\mathrm{top}}) - \frac{3}{4} d_{\mathrm{top}} N$ has positive Bernstein coefficients, which places it outside $\mathcal{D}_{\mathrm{new}}$. The last row is also implied by (d) through (95). The covers of (c)–(e) pass also at $B_C = B_P = 1 - 10^{-12}$, $L_C = L_P = 2 + 10^{-11}$, with the same numbers of subrectangles; the lower bounds change by less than 10^{-13}; hence they hold with $B_C < 1 < L_C/2$, as the count requires.

(f) The gate for C_{old}. Lemma 8.27 holds with $\theta' = B_1' - 10^{-12}$ and the present data:

$$6\kappa_1 \ell - \tfrac{19}{25} = 0.01919, \quad \Gamma_{\mathrm{g}}(d_{\mathrm{top}}, \tfrac{1}{8}) = 0.01584, \quad \tfrac{\ell}{d_{\mathrm{top}}} - \tfrac{1}{5} = 0.01321, \quad \tfrac{1}{6\kappa_1} - \tfrac{\ell}{d_{\mathrm{top}}} = 0.00907,$$

and in (iii), for $d \in \{19/25, d_{\mathrm{top}}\}$, all Bernstein coefficients on $[0, 1/8] \times [0, 1/2]$ of $J_o(\theta' - \frac{1}{200} - E(d, C_{\mathrm{aff}}(d))) - \frac{3d}{4} N_o$ and of J_o are positive without subdivision; the margins are at least 0.0198 and $6.8 \cdot 10^{-4}$.

(g) Conductors up to d_{top}. Since $19/25 > l_y$, all quantities in (92) and (93) are affine in d on $[19/25, d_{\mathrm{top}}]$, and n_0, ℓ/d are monotone. By (e), (f) at both endpoints, the proof of Corollary 8.28 gives, for every amplitude class,

$$\mathfrak{N}_d(\mathscr{B}) \ll Z^{-c_1}(1 + H)^{A'} \qquad (19/25 \leq d \leq d_{\mathrm{top}}),$$

with a fixed $c_1 > 0$; here the buffer is $\zeta_B = 10^{-10}$, and there is no contact.

(h) The inequalities of [5, Lemma 6.5]. At (B_1', h_1, κ_1), with $b = l_y - l_x$, $M = l_x + l_y$,

floor (91), $a = \frac{509}{1000}$, $\zeta = 0$:	$-C_0^{\flat} - \frac{9}{1000}(1 + 3\ell) = \frac{6878729}{3000000000} = 0.0022929\ldots$,
geometry [5, (6.33)]:	$b = 0.105125$, $l_y - \ell - \frac{11b}{6} = 0.10002$,
	$6\kappa_1\ell - \frac{3}{4} = 0.02919$, $2l_y - h_1 = 0.11954$,
	$B_1' - 0.87 = 0.00336$, $\frac{\ell}{h_1} - \frac{21}{100} = 0.00321$,
	$\frac{2}{9} - \frac{\ell}{h_1 - 1/40} = 0.00224$, $\frac{l_y}{h_1} = 0.57357$,
direct [5, Proposition 6.1(4)]:	$k = 0$, $\frac{1+3\ell-2M}{4} = -0.03349$, $M + \ell - 1 = 0$,
	$\frac{3\ell-1}{2} = -0.24018$,
	$G = \frac{b}{12}$ $(l_y > \frac{11b}{6})$, $\frac{5}{6} + \frac{l_x}{6} - \frac{\ell}{6} + \frac{b}{12} = B_1'$,
stock, supply, small amplitude:	(e), (f),
small parameters [5, (6.38)]:	$\frac{l_y}{2} - \frac{13h_1}{75} - \frac{1}{50} = 0.07216 > \frac{1}{20}$,
principal [5, (6.39)]:	$B_1' > 0.87$, $l_y/20 = 0.02329$, $h_1/600 = 0.00135$,

and $G_0 = \frac{2b}{3} - \frac{M}{4} = -0.13661\ldots < -\frac{13}{100}$, $L_0 = L_K = \frac{11}{12} - \frac{\ell}{4} = B_1'$. The buffers are taken with

$$0 < \zeta < \frac{0.0022929}{509/1000 + 4/25} = 3.42 \cdot 10^{-3}, \qquad \zeta_B = 10^{-10} < \frac{\ell}{21/100} - h_1 = 0.01242,$$

and $\beta_*(F) \le 0.87488664704 < (1+\kappa_1)/2$ holds with margin $3.52 \cdot 10^{-7}$.

(i) Conclusion. All margins in (a)–(h) are strict at $\theta = B_1'$. Run the proof of Theorem 8.29 with $B' = B_1'$, with the finite construction there replaced by the one-stage chain of (a) and the line g of (b), and with (d)–(h) in place of the limiting inequalities, Corollary 8.28 and the table there. If $\beta_*(F) < 509/1000$ there is nothing to prove. Otherwise Proposition 8.22 applies with $a = 509/1000$; the finitely many endpoint, test-function, height, integration-path and tail losses are chosen after the strict margins; and for $\beta = \beta_*(F) > B_1'$, $0 < \varepsilon_{\mathrm{low}} < (\beta - B_1')/4$,

$$|J_\eta(Z)| \ll Z^{C(B_1')+\varepsilon_{\mathrm{low}}}, \qquad |J_\eta(Z) - f_\eta(Z)| \ll Z^{C(\beta)-m_{\mathrm{save}}/2}$$

for some $m_{\mathrm{save}} > 0$. The Mellin continuation, the principal box and [5, Lemma 7.1] give $\beta_*(F) \le \max\{B_1' + \varepsilon_{\mathrm{low}}, \beta - m_{\mathrm{save}}/2\} < \beta_*(F)$, a contradiction. Hence $\beta_*(F) \le B_1'$. All presentations have $\omega_0 = \omega = 1$. Descent from $F = K(\zeta_{12})$ by (114) and the twist (106) are as in Theorem 8.29. Finally $B_1' - B_* < 1.9 \cdot 10^{-5}$ and $B_2 - B_1' > 1.72 \cdot 10^{-5}$ by (1) and $B_2 = (777 - \sqrt{1081})/852$. □

Corollary 8.32 (Artin L-functions and Dedekind zeta functions)**.** *Let K be a number field, ρ a finite-dimensional continuous representation of $\mathrm{Gal}(\bar{K}/K)$ and $t \in \mathbb{R}$. Then $L_K(s, \rho \otimes (N_K \cdot)^{it})$ has no zeros in $\mathrm{Re}\, s > B_1'$, and*

$$\mathrm{ord}_{s=1+it}\, L_K(s, \rho \otimes (N_K \cdot)^{it}) = -\dim \rho^{\mathrm{Gal}(\bar{K}/K)}.$$

In particular $\zeta_L(s) \neq 0$ for $\mathrm{Re}\, s > B_1'$ for every number field L.

Proof. By Brauer's induction theorem,

$$[\rho] = \sum_j n_j [\mathrm{Ind}_{E_j}^K \chi_j], \qquad L_K(s, \rho \otimes (N_K \cdot)^{it}) = \prod_j L_{E_j}(s, \chi_j (N_{E_j} \cdot)^{it})^{n_j}, \qquad n_j \in \mathbb{Z},$$

with χ_j ray class characters of finite order of finite extensions E_j/K. By Theorem 8.31 and Hecke's theorem, each factor is holomorphic and nonzero in $\mathrm{Re}\, s > B_1'$ away from $s = 1 + it$, where it has a simple pole if $\chi_j = 1$ and is holomorphic and nonzero otherwise. Frobenius reciprocity gives

$$\sum_j n_j \dim \mathrm{Hom}_{E_j}(1, \chi_j) = \dim \mathrm{Hom}_K(1, \rho) \ge 0,$$

which is the order formula; hence there is no zero at $1+it$ either. For ζ_L take $K = \mathbb{Q}$ and $\rho = \mathrm{Ind}_L^{\mathbb{Q}} 1$, or apply Theorem 8.31 with $K = L$, $\chi = 1$. □

Remark 8.33 (The envelope for one stage). For a stage from $(R_0, 0)$, $\lambda(R_0) \le 0$ gives $w_b \ge R_0$. Condition 1 forces $k > k_*(R_0, 0) = (5R_0/4 - 5/6)/(R_0 - 1/2)$, so that

$$Y(r) < Y_{R_0}(r) = \frac{(7 - 9R_0)(R_0 - r)}{R_0(45R_0 - 30)} \qquad (r < R_0).$$

The one-stage affine envelope is bounded by $\sup_{2/3<R_0<7/9} Y_{R_0}(r)$. For the line of Theorem 8.31 the fixed comparison gives $0.87336474286\ldots$, and the strict checks also hold at $\theta = 0.8733647429$, $h = 81237421/10^8$ (fourth-edge bound $3.7 \cdot 10^{-11}$).

9 Unitary Hecke characters

In this section, we prove Theorem 1.2 for continuous unitary Hecke characters. We first extend the joint moment induction of Section 8 to the paired families of [2, 3]. Subsection 9.4 then applies abelian base change and unitary induction to obtain the assertion for Weil representations.

We use the unitary results of [2, 3]. Their precise paired coefficient family resolves the angular vanishing needed for the theta constant terms. No reduction of a general unitary character to a finite-order norm twist is made.

Ideal norm twists are normalized by

$$L_K(s, \eta(N\cdot)^{it}) = L_K(s - it, \eta). \tag{106}$$

Thus their zeros are translated vertically and a norm-character pole is at $1 + it$.

9.1 Earlier unitary estimates and their normalization

In this subsection, we recall the unitary moment estimates of [2, 3] and fix the paired prime coefficients. Subsection 9.2 checks their compatibility with the new joint moment induction.

Proposition 3.2 and Lemma 3.3 of [3] supply fixed-type theta reflection and its gamma-strip estimates. Theorem 4.1, Lemma 4.3 and Proposition 4.10 of [3] supply the marked inverse moment, its coefficient identity and its canonical energy bound. Lemma 4.4 of [3] supplies the ordinary phase identity. Theorem 2 and Lemmas 1–2 of [2] supply the coefficient-two ordinary mixed moment, complete extraction and exceptional estimates. We use these analytic theorems in their stated character families. The compatibility with the new joint moment induction is proved below.

For a fixed unitary character ω_0, enlarge S to contain its ramified primes. For a fixed finite group of ray-class characters Θ, containing the finite transformation characters, write

$$\lambda_u = \omega_0 \xi_u, \qquad \xi_u \text{ finite order.}$$

The ordinary prime positions have relative coefficients

$$b_i(p) = \omega_0(p)^{-1} \sum_j c_{ij} \vartheta_{ij}(p), \qquad \vartheta_{ij} \in \Theta. \tag{107}$$

Thus the actual prime character is finite order:

$$b_i(p)\lambda_u(p) = \xi_u(p) \sum_j c_{ij} \vartheta_{ij}(p).$$

The original full prime ranges, coefficients, coprimality indicators and natural zeros are retained. The fixed exponent $\kappa_1 = 374887/500000$ therefore uses only the known finite-order prime estimate. Its use requires no inductive hypothesis for an infinite-order prime character.

The mixed ordinary estimate of [2, Theorem 2] is

$$\sum_u \mathcal{W}(u) |S_{\omega_1 \xi_u, n_1} S_{\omega_2 \xi_u, n_2} Q|^2 \ll Z^{M+\varepsilon} \mathfrak{S}_J (1 + H_{\mathrm{in}})^{J'}, \qquad n_1 + n_2 + 2z \le M. \tag{108}$$

The two ω_j are arbitrary fixed unitary characters; at $z = 0$ the ordinary lengths may be arbitrary in the stated bounded ranges. The excluded inducing-character family is the finite one for ξ_u. For detection use $(\omega_1, \omega_2) = (\omega_0, \omega_0)$; for the numerator use $(1, 1)$. This distinction retains the actual finite numerator in the Euler quotient of [3].

9.2 Angular factors and character orientations

In this subsection, we verify the angular Hankel transform and the character orientations required by the joint moment induction. Subsection 9.3 applies the resulting estimates to the zero-free theorem.

At a complex place write

$$\psi_v(z) = (z/|z|)^{k_v}|z|_v^{i\tau_v}.$$

The fixed theta coefficient has angular vector $\boldsymbol{\lambda} \in \{1,-1\}^{r_F}$. Choose once between ψ and $\bar{\psi}$ so that $\boldsymbol{k}-\boldsymbol{\lambda} \neq 0$. This is always possible: $\boldsymbol{k}-\boldsymbol{\lambda}$ and $-\boldsymbol{k}-\boldsymbol{\lambda}$ cannot both be zero. Finite twists preserve this choice.

Lemma 9.1 (Angular Hankel multiplier)**.** *For the complex constituent with normalized inducing exponents* $\{1/6,-1/6\}$*, angular mode* $m \in \mathbb{Z}$ *and radial unitary parameter* $\tau \in \mathbb{R}$*, the normalized multiplier is*

$$\mathcal{K}_{m,\tau}(t) = \frac{\Gamma(1/3+|m|/2+t+i\tau)\Gamma(2/3+|m|/2+t+i\tau)}{\Gamma(1/3+|m|/2-t-i\tau)\Gamma(2/3+|m|/2-t-i\tau)}. \tag{109}$$

It is holomorphic for $\operatorname{Re} t > -1/3$*, satisfies the meromorphic identity*

$$\mathcal{K}_{m,\tau}(t)\mathcal{K}_{-m,-\tau}(-t) = 1,$$

and has polynomial growth, with every fixed finite number of t*-derivatives, on every closed vertical strip in that half-plane. The polynomial degree can be chosen independently of* m,τ*; the bound is in* $1+|m|+|\tau|+|\Im t|$*. For* $m=\pm1$*,* $\tau=0$*, this is the earlier ratio with arguments* $5/6$ *and* $7/6$*.*

Proof. The complex Tate factor, in the normalization of [7, (2.6)–(2.7), Section 9.2], is

$$G_m(s) = i^{|m|}(2\pi)^{1-2s}\frac{\Gamma(s+|m|/2)}{\Gamma(1-s+|m|/2)}.$$

The two inducing characters give $G_m(1/2+t+i\tau-1/6)G_m(1/2+t+i\tau+1/6)$. Removing $(-1)^m(2\pi)^{-4(t+i\tau)}$ gives (109). Its numerator poles satisfy

$$\operatorname{Re} t = -1/3-|m|/2-j \quad \text{or} \quad \operatorname{Re} t = -2/3-|m|/2-j, \qquad j \in \mathbb{N}\cup\{0\}.$$

The reciprocal gamma factors are entire. The displayed inversion is an identity of meromorphic functions. On a fixed closed strip, Stirling's formula bounds each gamma ratio by a fixed power of $1+|m|+|\tau+\Im t|$; its exponential factors cancel. The same argument after logarithmic differentiation, or Cauchy's formula on a slightly wider strip, proves the derivative bounds. The compact region not covered by Stirling is bounded because the numerator poles have a fixed positive separation from the strip. □

The multiplier is realized by actual Weyl-related vectors. Indeed, $C_c^\infty(\mathbb{C}^\times)$ lies in the smooth Kirillov model of this infinite-dimensional generic constituent. Its Weyl action is the Hankel transform characterized by the local functional equation; see [7, (9.1)–(9.3), Theorem 9.5]. Thus one may first prescribe a compact radial test function in angular mode m and then use its actual Weyl transform. Division by the Mellin transform of a single Bessel vector is unnecessary.

Applying the multiplier to the rapidly decaying Mellin transform of the prescribed test, and shifting within $\operatorname{Re} t > -1/3$, gives the earlier compact-strip and discarded-tail bounds, with polynomial height orders. For several complex places one takes products before the common integral

over norm fibers. The original relative radial parameters remain fixed; the common Mellin parameters of the test functions used in the new Poisson estimate are ideal-norm modes only. The orders depend on the field and finite recursion, whereas their norm exponents are unchanged.

The inverse presentations form the paired class of [3]

$$\mathcal{X}_u = \{\theta\chi_u, \theta\bar{\chi}_u : \theta \in \Theta\} \cup \{\psi\theta\bar{\chi}_u, \bar{\psi}\theta\chi_u : \theta \in \Theta\}. \tag{110}$$

After conjugating the entire expression in the initial inverse transformation, the fixed character factors are

$$\xi \in \Theta \cup \psi\Theta. \tag{111}$$

Consequently the theta period character $\bar{\Lambda}\xi$ has angular vector $-\boldsymbol{\lambda}$ or $\boldsymbol{k} - \boldsymbol{\lambda}$, each nonzero at some place. The Jacquet characters at infinity are radial. Both weighted theta constant terms vanish by rotation, for the chosen vector and its actual Weyl transform. The conjugate canonical family conjugates its coefficient, theta representation and character orientation together; it does not introduce $\bar{\psi}$ as a new fixed character factor of C. This is the role of the orientation convention in [3].

Proposition 9.2 (Joint moments for the paired unitary family)**.** *Let $\mathcal{F}_\omega$ and β_ω be as in (10)–(11), and suppose $\beta_\omega \geq 51/100$. The moment conclusion of Theorem 8.21, with arbitrarily small fixed positive margins, holds for the paired presentations (110) associated with $\psi \in \mathcal{F}_\omega$. The ordinary half uses (107). All numerical length conditions are unchanged; constants and finite seminorm orders may depend on the fixed infinity type.*

Proof. Write $C = \bar{\Lambda}\gamma_2$ and $\xi \in \Theta \cup \psi\Theta$. The initial inverse transformation of [3] has coefficient

$$C(n)\xi(n)\chi_n(k)\chi_n(f)^4 M_I(n) W(Nn/X).$$

On the coprime squarefree support, complete-factor selection gives

$$\begin{aligned} C(dn)\xi(dn) &= C(d)\xi(d)C(n)\xi(n)\chi_n(d)^4\omega(d,n), \\ N' = N - u', \qquad K' &= M + 4u', \qquad N' - z' = N - z. \end{aligned}$$

Here ω is a finite ray-class character expansion. Thus the remaining fixed character is $\xi(n)$ and every original prime coefficient is retained.

Lemma 8.4 and the inverse calibration of [3, (4.8)–(4.9) and Lemma 4.3] give

$$C\xi \longmapsto \mu_F \xi G_0 \longmapsto C\xi \quad \text{up to fixed finite ray-class characters,}$$

where G_0 is a fixed finite ray-class character expansion. In particular,

$$\xi\bar{\chi}_y, \quad \bar{\xi}\chi_y \in \mathcal{X}_y.$$

The common-divisor and Euler-restoration factors satisfy

$$|\xi(d)|^2 = 1_{(d,\mathfrak{f}_\xi)=1} \leq 1, \qquad \sum_{Nd \leq X} \frac{1}{Nd} \ll_F 1 + \log X, \qquad \sum_{J \subseteq I} \sum_{\mathrm{rad}(d)|g} \frac{|a(d)B_J(d)|}{\sqrt{Nd}} \ll_{F,\varepsilon,I} (Ng)^\varepsilon.$$

Hence Proposition 8.6 and (56) preserve the same paired family and unrestricted restored quotient. The angular calculation preceding this proposition gives

$$\mathrm{ang}(\bar{\Lambda}\xi) \in \{-\boldsymbol{\lambda}, \boldsymbol{k} - \boldsymbol{\lambda}\} \subset \mathbb{Z}^{r_F} \setminus \{0\}, \qquad \mathrm{CT}(v) = \mathrm{CT}(wv) = 0,$$

for the chosen vector and its actual Weyl transform.

For the ordinary factors, put $\lambda_j = \omega_j \xi_u$ and define

$$\mathcal{C}^{u}_{\omega_1,\omega_2}(m; X, Y) = \sum_{ab=m} \frac{\lambda_1(a)\lambda_2(b)}{\sqrt{Nm}} W_1(Na/X) W_2(Nb/Y).$$

The normalized ordinary product is $\sum_m \sqrt{Nm/(XY)}\, \mathcal{C}^{u}_{\omega_1,\omega_2}(m; X, Y)$. The allocation $D_1 = (a, D)$, $D_2 = D/D_1$ gives the exact extraction identity of [2, Lemma 1], with all zero extensions retained:

$$\mathcal{C}^{u}_{\omega_1,\omega_2}(Dn; X, Y) = \frac{\xi_u(D)}{\sqrt{ND}} \sum_{D_1 D_2 = D} \omega_1(D_1)\omega_2(D_2) \mathcal{C}^{u;(a,D_2)=1}_{\omega_1,\omega_2}(n; X/ND_1, Y/ND_2).$$

Expand its restriction by Möbius inversion before positivity. For equal twists the fixed factor is $\omega_0(D)$; for unequal twists retain both displayed factors. Lemma 4.4 of [3] preserves

$$b_i(p)\omega_0(p)\xi_u(p) = \xi_u(p) \sum_j c_{ij} \vartheta_{ij}(p) \tag{112}$$

under extraction, the two finite transforms and whole-product conjugation. In the coefficient-two range apply (108) with equal twists for detection and with $(1, 1)$ for the numerator. An enhanced ordinary width in the short count uses the preceding joint rectangle.

For exceptional parameters, absorb norm powers into the tests and use the finite-order estimate. In the remaining case λ is not a norm character. Its Hecke function is entire, and Mellin inversion on $\operatorname{Re} s = 0$ gives

$$X^{-1/2} \sum_{(a,\mathfrak{r})=1} \lambda(a) W(Na/X) = \frac{1}{2\pi i} \int_{(0)} L_F^{(\mathfrak{r})}(s, \lambda) \widehat{W}(s) X^{s-1/2}\, ds,$$

$$\left| X^{-1/2} \sum_{(a,\mathfrak{r})=1} \lambda(a) W(Na/X) \right| \ll X^{-1/2} (N\mathfrak{r})^{\varepsilon} p_J(W)(1 + |t|)^A.$$

Here $L_F^{(\mathfrak{r})}$ omits the primes dividing $\mathfrak{r}$. The exceptional estimate of [2, Lemma 2] therefore applies with the same normalized exponent.

The live prime characters are finite twists of the common moving character. Take the buffered maximum over (59) and the actual height windows. Canonical conjugation sends either sign to a fixed factor in $\Theta \cup \psi\Theta$. A witness in the other anchor is rebased with its actual prime factors as in Proposition 8.9. In the packed ordinary step the second correlation gives the phases $\chi_I(j)$ and $\overline{\chi_J(-j)}$ in the two sides. On taking the second absolute value, whole conjugation aligns the sign, since $\chi_J(-1) = 1$. The fixed unit and ray factors are in Θ and the packed parameter is still a single moving sextic parameter. The paired transformations, (112) and the packed phase identify the family required by (H4). The short counts invoke only preceding-stage rectangles in this family. Choose the common test measures before the arithmetic indices and apply Theorem 8.21. □

Corollary 9.3 (The strict curved quota in the paired family)**.** *Fix $F \supset \mu_{12}$, a paired family as in Proposition 9.2, and a compact interval $I \subset [367/500, 7/9)$. In the same contradiction regime, the inverse and the equal-anchor ordinary moments of Subsection 2.3 hold for*

$$r \in I, \qquad P < r, \qquad S < \frac{1 - r}{2} + G(r).$$

The estimates hold for the original complete prime lists and each fixed finite disjoint refinement as in Lemmas A.5–A.7. They are uniform in common excluded ideals of any fixed polynomial norm cap and have the stated polynomial height and fixed-type uniformity, for fixed positive margins to the two boundaries.

Proof. Approximate G by Proposition 6.4, taking the finite step small enough for $u \le 1/100$. Proposition 9.2 and Theorem 8.21 supply each strict child range in succession. Choose the approximation error and endpoint decrements below the prescribed width margin. At the left endpoint $367/500$, a slightly larger radius in the finite construction supplies the required bound by shortening, using continuity of $(1-r)/2 + G(r)$. A finite cover of I with the same margins yields the uniform statement. □

9.3 The improved half-plane

In this subsection, we prove the unitary zero-free theorem by combining Subsections 9.1 and 9.2 with the finite comparisons of Section 7.

Theorem 9.4 (Unitary Hecke characters)**.** *For every number field K and every continuous unitary Hecke character ω,*

$$\boxed{L_K(s,\omega) \neq 0 \qquad (\operatorname{Re} s > B_*).} \tag{113}$$

The only possible pole in this half-plane is the simple pole at $1+it$ when $\omega = (N\cdot)^{it}$.

Proof. Work first over $F \supset \mu_{12}$ and fix a unitary ω. Use the family (10). Put $\beta = \beta_\omega$. The result of [2] gives $\beta \le B_2$. For each fixed target orient its canonical class as in (111). This changes no zero real part. The finite numerator and the paired finite prime characters are in the finite-order subfamily. Their fixed exponent is κ_1.

Suppose $\beta > B_* > 51/100$. Choose $B_* < B' < B < \beta$, with B' sufficiently close to B_*. The inequalities of Subsection 7.1 and Proposition 6.4 give a finite rational moment construction with strict margins at B'. Apply Proposition 9.2 to every stage. The native ordinary moments up to coefficient two are those of [2], with equal unitary twists for detection and the finite instance for the numerator. The larger marked widths use the preceding joint rectangles supplied by the repaired induction. The Euler quotient of [3] is

$$\mathscr{F}_{u,\psi}(s,w,z) = \zeta_F^S(6z)\frac{L_F^S(w,\chi_u)}{L_F^S(s,\psi\bar{\chi}_u)}H_{u,\psi}(s,w,z).$$

Thus the new scalar counts and numerator estimates apply to the actual characters, including all local correction terms. The fixed floor $a = 509/1000$ is supplied by the paired-unitary case of Proposition 8.22, with $\beta = \beta_\omega$. The limiting floor margin is greater than 0.00125075. Direct, reference, tail and principal-factor margins are strict after replacing B' by B. Lemma 6.1 of [3] gives the nonzero principal factor with unitary target phases; Section 3 of [2] verifies compatibility with this comparison.

Fix the numerical margins, finite depths and original prime partition before the target and its finite twists. The later arithmetic data determine finite seminorm orders and height degrees. For an error $Z^{C(\beta)-2\eta}(1+H)^{A_\psi} + Z^{D_\psi}H^{-N}$, choose

$$H = Z^{t_\psi}, \quad 0 < t_\psi < \frac{\eta}{2(1+A_\psi)}, \quad N > \max\left\{0, \frac{D_\psi - C(\beta) + \eta}{t_\psi}\right\}.$$

Then both error terms are $\ll_\psi Z^{C(\beta)-\eta}$. The numerical saving $\eta > 0$ is common to the family, whereas constants and sufficiently large thresholds may depend on the fixed target. The Mellin identity of [3] for the whole principal term therefore continues a nowhere-vanishing holomorphic numerator divided by $L_F^S(s,\psi)$ to

$$\operatorname{Re} s > \max\{B, \beta - \eta\} < \beta$$

for every $\psi \in \mathcal{F}_\omega$. More precisely, the continued function is

$$e^{(s-5/6)^2}\frac{H_\psi^S(s)}{L_F^S(s,\psi)}, \qquad H_\psi^S(s) \neq 0,$$

by the principal-factor estimate of [3, Lemma 6.1]. Thus a zero of the denominator would give a pole of a holomorphic function. Omitted finite Euler zeros lie on $\operatorname{Re} s = 0$; norm poles are zeros of the reciprocal. This contradicts the definition of β. Hence $\beta \leq B_*$. For general K descend from $K(\zeta_{12})$ by the next theorem. □

9.4 Abelian base change and Weil representations

In this subsection, we apply abelian base change to descend the unitary bound of Subsection 9.3. We then use unitary induction and Frobenius reciprocity to deduce the result for finite-dimensional unitary Weil representations.

Theorem 9.5 (Unitary descent)**.** *Let F/K be finite abelian and $0 < B < 1$. If every unitary Hecke L-function over F is nonzero for $\operatorname{Re} s > B$, apart from its possible norm-character pole, the same statement holds over K.*

Proof. For a unitary character χ of K, put $\chi_F = \chi \circ N_{F/K}$. Artin formalism for the induced one-dimensional representation gives

$$L_F(s,\chi_F) = \prod_{\omega \in \widehat{\mathrm{Gal}(F/K)}} L_K(s,\chi\omega). \tag{114}$$

This identity includes the ramified local factors. If χ is not a finite-order character times a norm power, none of the factors on the right is a norm character; all are entire. Hence a zero of $L_K(s,\chi)$ would give a zero of the left-hand side. Moreover,

$$\chi \circ N_{F/K} = (N_F\cdot)^{it} \implies \chi(N_K\cdot)^{-it}|_{N_{F/K}C_F} = 1 \implies \chi = \eta(N_K\cdot)^{it}, \quad \eta \text{ finite order},$$

since $N_{F/K}C_F$ has finite index in C_K. Thus the left-hand side has no pole that could cancel the zero.

If $\chi = \eta(N\cdot)^{it}$ with η finite order, use (106) and the finite-order version of (114). For $B < \operatorname{Re} s < 1$ no factor has a pole. On $\operatorname{Re} s = 1$, use the classical nonvanishing theorem for Hecke L-functions, with the principal pole permitted. For $\operatorname{Re} s > 1$, absolute convergence of the Euler product gives nonvanishing. □

In particular $F = K(\zeta_{12})$ is sufficient for descent. This reduces the field-generality issue to the analytic argument over totally complex fields containing μ_{12}; it does not eliminate the infinite-order part of a target character.

Corollary 9.6 (Unitary Weil representations)**.** *For every finite-dimensional continuous unitary representation r of the global Weil group W_K, $L_K(s,r)$ has no zeros in $\operatorname{Re} s > B_*$. Its pole order at $1+it$ equals $\dim \mathrm{Hom}_{W_K}((N\cdot)^{it}, r)$.*

Proof. Use the unitary induction and full local-factor identities of Sections 9.1–9.2 of [3] to obtain an identity of virtual representations

$$[r] = \sum_j n_j[\mathrm{Ind}_{W_{E_j}}^{W_K}\omega_j], \qquad n_j \in \mathbb{Z}, \quad \omega_j \text{ unitary},$$

and hence

$$L_K(s,r) = \prod_j L_{E_j}(s,\omega_j)^{n_j}.$$

Theorem 9.4 makes every factor nonzero away from its norm pole. Since $(N_K\cdot)^{it}|_{W_{E_j}} = (N_{E_j}\cdot)^{it}$, Frobenius reciprocity gives

$$\sum_j n_j \dim \mathrm{Hom}_{W_{E_j}}((N_{E_j}\cdot)^{it},\omega_j) = \dim \mathrm{Hom}_{W_K}((N_K\cdot)^{it},r) \geq 0.$$

Equivalently,

$$\mathrm{ord}_{s=1+it}\, L_K(s,r) = -\dim \mathrm{Hom}_{W_K}((N_K\cdot)^{it},r) \leq 0.$$

Hence no zero occurs at a norm-character pole either. If r factors through a finite quotient, Brauer's induction theorem gives the identity with ω_j of finite order (times $(N\cdot)^{it}$), and Theorem 8.29 replaces Theorem 9.4. By Hecke's theorem, $L_{E_j}(s,\eta_j(N\cdot)^{it})$ with η_j of finite order is entire unless $\eta_j = 1$, when it equals $\zeta_{E_j}(s-it)$, holomorphic except for a simple pole at $s = 1+it$. Thus each factor is holomorphic and nonzero in $\mathrm{Re}\, s > B_*$ away from $1+it$, and the order computation above applies. □

A Analytic estimates for the ordinary assembly

A.1 The two reference estimates

Let ψ_1, ψ_2 be fixed unitary Hecke characters of a fixed number field F. Absorb their norm powers into independent real heights v_1, v_2, so that each ψ_j is principal or has an entire L-function. Put $V_j(x) = W_j(x)x^{iv_j}$ for fixed compactly supported smooth W_j. In the sum

$$\mathcal{R}(X_1, X_2) = \sum_{\substack{\mathfrak{a}_1,\mathfrak{a}_2 \\ \mathfrak{a}_1\mathfrak{a}_2 \text{ cube-free}}} \psi_1(\mathfrak{a}_1)\psi_2(\mathfrak{a}_2)V_1(\mathrm{N}\mathfrak{a}_1/X_1)V_2(\mathrm{N}\mathfrak{a}_2/X_2),$$

allow at each prime dividing a common ideal $\mathfrak{R}$ any prescribed set of residual valuation pairs (i, j) with $i + j \le 2$. All these sets and finite ramification data are the same in the two arms. At a ramified prime the local polynomial has the actual coefficients of the two zero-extended characters.

Lemma A.1 (Centered cube-free reference)**.** *Suppose $X_1X_2 = Y_1Y_2 = T$ and write*

$$L = \min(X_1, X_2, Y_1, Y_2) \ge 1, \qquad D_0 = \max(X_1, X_2, Y_1, Y_2).$$

For $p_J(W) = \max_{0\le j\le J} \|W^{(j)}\|_\infty$, one has

$$|\mathcal{R}(X_1, X_2) - \mathcal{R}(Y_1, Y_2)| \ll_\varepsilon TL^{-2/3}(\mathrm{N}\mathfrak{R})^\varepsilon(1 + \log(2D_0))^C p_J(W_1)p_J(W_2)(1 + |v_1| + |v_2|)^J.$$

The constants are uniform in the prescribed residual sets. Division by $\sqrt{T}$ gives the corresponding assertion for the centrally normalized sums.

Proof. Write

$$F_j(Q) = \sum_{\mathfrak{a}} \psi_j(\mathfrak{a})V_j(\mathrm{N}\mathfrak{a}/Q).$$

Mellin inversion and the shift to $\operatorname{Re} s = 1/3$ give $F_j(Q) = m_jQ + O(Q^{1/3})$. Here

$$m_j = \operatorname{Res}_{s=1} L_F(s, \psi_j) \int_0^\infty V_j(x)\, dx.$$

The polynomial vertical bound for the fixed Hecke L-function and integration by parts in V_j give the stated polynomial height and seminorm dependence. Also $F_j(Q) \ll Q$ by direct ideal counting and compact support, including $Q < 1$. Thus for $Q_1, Q_2 > 0$,

$$|F_1(Q_1)F_2(Q_2) - m_1m_2Q_1Q_2| \ll Q_1Q_2 \sum_{j=1}^{2} \min(1, Q_j^{-2/3})$$

with the same test factors.

At a prime outside $\mathfrak{R}$, use $x = \psi_1(\mathfrak{p})(\mathrm{N}\mathfrak{p})^{-s_1}$ and $y = \psi_2(\mathfrak{p})(\mathrm{N}\mathfrak{p})^{-s_2}$. The convolution multiplier after dividing by both ordinary Euler factors is

$$(1 - x)(1 - y)(1 + x + y + x^2 + xy + y^2) = 1 - x^3 - x^2y - xy^2 - y^3 + x^3y + x^2y^2 + xy^3.$$

At a prime of $\mathfrak{R}$ multiply the prescribed local polynomial by the inverses of the ordinary local Euler factors, with the actual ramified conventions. Call the resulting global convolution coefficients $h(\mathfrak{d}_1, \mathfrak{d}_2)$, and put $D_j = \mathrm{N}\mathfrak{d}_j$. In both arms one can restrict the convolution to $D_j \le C_W D_0$ by the compact support. The subtracted products $m_1m_2T/(D_1D_2)$ cancel term by term in this common

truncation. For both arms the remaining error is bounded by $TL^{-2/3}(D_1D_2)^{-1/3}$ times the test factors. Finally,

$$\sum_{D_1,D_2\le C_W D_0} \frac{|h(\mathfrak{d}_1,\mathfrak{d}_2)|}{(D_1D_2)^{1/3}} \ll_\varepsilon (\mathrm{N}\mathfrak{R})^\varepsilon \prod_{\mathrm{N}\mathfrak{p}\le C_W D_0} (1+O((\mathrm{N}\mathfrak{p})^{-1})) \ll_\varepsilon (\mathrm{N}\mathfrak{R})^\varepsilon (1+\log(2D_0))^C.$$

Indeed the first nonconstant term outside $\mathfrak{R}$ has degree three. At $\mathfrak{p} \mid \mathfrak{R}$ the absolute local norm is bounded by a fixed constant times $1+O((\mathrm{N}\mathfrak{p})^{-1/3})$, whose product is $O_\varepsilon((\mathrm{N}\mathfrak{R})^\varepsilon)$ after absorbing the finitely many small primes. The first log tests are admissible since $W_j(x)\log x$ has the same smooth compact support. □

Lemma A.2 (The positive short reference)**.** *Let $F \supset \mu_{12}$, let the ordinary scales have exponents x, y, and let W be the total prime width. On the regular inducing rows of a smooth ball of exponent V, the unrestricted mixed energy with $x \le \ell$ is $O_\varepsilon(U^{V+\varepsilon})$ whenever $4(\ell+W) < V$ with fixed margin. This estimate has the stated norm-test and height dependence, and permits the original pure coprimality masks.*

Proof. For the centrally normalized ordinary sums A_x, A_y and the prime product Q, use Cauchy in the starred row set:

$$\sum^{*} |A_xA_yQ|^2 \le \Big(\sum^{*} |A_x^2Q^2|^2\Big)^{1/2} \Big(\sum^{*} |A_y|^4\Big)^{1/2}.$$

Use the unmarked fourth of [5, 2] with the actual fixed anchor and height in the second factor. In the first, apply the positive row lift of [6, Lemma 12.1] in the finite unit and ray charts, then expand the labeled ideals of $|A_x^2Q^2|^2$. The Kummer period on a fixed chart divides the radical of their product times a fixed ideal. Its norm is $\ll U^{4(x+W)}$. Absolute coefficient bounds allow the fixed column masks to remain in the expanded sum. The natural Kummer zeros are part of its local periodic functions. Orbit Poisson in these charts is as in [6, Section 8 and Lemmas 12.9–12.10]. For a nonzero dual vector, the norm of its nonzero algebraic frequency is $\gg_F (\mathrm{N}\mathfrak{m})^{-1}$ for a period ideal $\mathfrak{m}$. Compact norm-one charts give scaled archimedean frequency $\gg_F (U^V/\mathrm{N}\mathfrak{m})^{1/[F:\mathbb{Q}]}$ in at least one coordinate. The strict gap allows enough integrations by parts to pay all polynomial column counts and the sum over the dual lattice.

At zero frequency the signed exponents at a good prime must sum to zero modulo six. One occurrence in an ordinary or a marked slot has nonzero exponent. Each nonconstant local contribution thus has at least two occurrences, and the normalized geometric sum of its absolute values is $O((\mathrm{N}\mathfrak{p})^{-1})$. Euler multiplication over primes of norm $U^{O(1)}$ bounds the diagonal mass by $O_\varepsilon(U^\varepsilon)$. Fixed bad-prime packets have bounded local mass. The zero term contributes $O_\varepsilon(U^{V+\varepsilon})$. This argument includes repeated labels in the short fourth. □

A.2 Cube-free kernels and the second correlation

Let $F \supset \mu_{12}$ and enlarge the fixed set S as in [6, Lemma 3.4]. Write $R = \mathcal{O}_{F,S}$ and $\Gamma = (R^\times)^6$. An exterior ideal has one allowed generator modulo Γ. These generators multiply modulo Γ and give symmetric sextic reciprocity. Each nonzero frequency class modulo Γ is an allowed generator times a member of $R^\times/\Gamma$. We keep this finite packet as a finite collection of coefficient characters. In particular multiplication of a frequency by the allowed generator of an exterior ideal is an injection on classes modulo Γ and multiplies its exterior norm by the norm of the ideal.

Consider a root of free row width one and total column width A. The two ordinary profiles may have separate fixed characters and norm heights. Both ordinary factors have the same moving

sextic feature. The prime slots are those of (4), with their actual bounded coefficients. At the cube-free stage the full product moduli have valuations in $\{0,1,2\}$. We keep the equal-product difference of the original and reference arms, including their scale phases, on each full column. For the two long arms let ℓ_{ref} be the common lower bound on the four scale exponents at the cube-free signed root.

Lemma A.3 (Cube-free restoration over exterior ideals)**.** *Both cube-free comparisons use central kernels of bounded absolute mass, with absolute mass depending on the fixed slot list and coefficient bounds. The kernels preserve the two independent profiles, the equal-product identity and the natural zero extensions.*

Proof. Give each original prime slot one formal variable w_i with $w_i^2 = 0$. This relation is imposed across the whole Euler product, before taking the coefficient in which each prescribed slot occurs once. At a good prime write x, y for its two ordinary positions and set $w_i = 0$ in the local polynomial if the prime is outside slot i's pool. The two local series are

$$U_{\mathfrak{p}} = (1-x)^{-1}(1-y)^{-1}\prod_i (1+w_i), \qquad R_{\mathfrak{p}} = 1 + x + y + x^2 + xy + y^2 + \sum_i w_i.$$

Thus the restricted series allows a total ordinary valuation at most two and permits at most one mark, disjoint from both ordinary positions. The identities $R_{\mathfrak{p}} = (R_{\mathfrak{p}}/U_{\mathfrak{p}})U_{\mathfrak{p}}$ and $U_{\mathfrak{p}} = (U_{\mathfrak{p}}/R_{\mathfrak{p}})R_{\mathfrak{p}}$ are formal power series identities. In both kernels the unmarked part starts in degree three beyond its constant. All nonconstant terms containing a mark have total degree at least two.

In the absolute central norm, $|x|, |y| \le (\mathrm{N}\mathfrak{p})^{-1/2}$ and $|w_i| \ll (\mathrm{N}\mathfrak{p})^{-1/2}$ on its annulus. The unmarked inverse is expanded geometrically after a fixed small-prime cutoff. Enlarge the excluded set once to include those primes. To recover their moment terms, prescribe the two ordinary valuations v_1, v_2 at each newly excluded prime $\mathfrak{p}$. Keep the literal character factors of the extracted powers as row scalars, including their zeros, and divide the two scales by $(\mathrm{N}\mathfrak{p})^{v_1}$ and $(\mathrm{N}\mathfrak{p})^{v_2}$ in both rectangles. The absolute central mass at those primes is bounded by $\prod_{\mathfrak{p}}(1-(\mathrm{N}\mathfrak{p})^{-1/2})^{-2}$ over this fixed finite set. Thus the exact mask and equal-product identity survive the restoration. The marked inverse is a finite nilpotent expansion. Every live prime window of fixed positive width eventually lies above the cutoff, so the geometric majorants there are uniform. If $m_{\mathfrak{p}}$ slots are active at $\mathfrak{p}$, the absolute local mass of either kernel is therefore

$$1 + O\big((\mathrm{N}\mathfrak{p})^{-3/2} + m_{\mathfrak{p}}^2(\mathrm{N}\mathfrak{p})^{-1}\big).$$

Here the constants may depend on the fixed coefficient bounds and the finite slot list. The second term has bounded sum, by $m_{\mathfrak{p}}^2 \le (\#\text{slots})m_{\mathfrak{p}}$ and $\sum_{\mathfrak{p}\text{ in one pool}}(\mathrm{N}\mathfrak{p})^{-1} \ll_F 1$. The ordinary sum converges absolutely. The finite excluded set is retained as its actual deletion or bounded local factors.

After coefficient extraction, a monomial fixes some of the original slots and shifts each ordinary coordinate by its own ideal factor. These shifts divide the scales of the two arms by the same respective norms. Their Mellin weights retain the central normalization and the original height in each coordinate. The common character factor on the extracted ideals has modulus at most one, including its literal zeros. Extra zero masks are imposed before the local identities, or expanded exactly by Möbius inversion, with subpower divisor mass. Thus the common prime coefficients or natural zeros do not require enlarging the local support. Minkowski's inequality with the above absolute masses completes both comparisons. □

Apply Lemma A.2 first to unrestricted products. Its positive row lift retains the column masks. The short reference for cube-free products then follows from the kernel $R_{\mathfrak{p}}/U_{\mathfrak{p}}$ of Lemma A.3: all

remaining ordinary scales decrease. For the main moment use the opposite kernel $U_{\mathfrak{p}}/R_{\mathfrak{p}}$ before making any signed comparison. On each resulting cube-free family, use the short reference if one ordinary scale is at most $U^{\ell_{\mathrm{ref}}}$. Otherwise form a fresh reference of the same scale product, with minimum scale $U^{\ell_{\mathrm{ref}}}$. The lower bound on the four scales holds at the start of each new signed correlation.

Proposition A.4 (The packed regular terminal)**.** *Let $u > 0$ be the width selected from the live prime positions when the selection fills, with mesh error $O(\eta)$. Put $H_0 = 1 - 6u$. Suppose the regular positive leaves from the original arm, with their actual componentwise shortened scales and surviving subproducts, satisfy the following bound over a ball of exponent M:*

$$\mathcal{L}(M) \ll_{\varepsilon} U^{M+u-\rho+(5/6)(H_0-M)_+ +\varepsilon}, \qquad 0 < \rho \le u. \tag{115}$$

Assume $4(\ell_{\mathrm{ref}} + W) < H_0$, where W is the total live prime width, for Lemma A.2. The constants are uniform in the leaves' fixed profile seminorms and polynomially in their independent heights. For an exhausted selection the unmarked estimate suffices. In the signed correlation, the two regular sides then meet their original exponent allowances. For the fixed-inducing sides and the origins it suffices that

$$A + 3u < \frac{5}{6} + \ell_{\mathrm{ref}}, \qquad A + 4u < 1, \qquad 6u < 1. \tag{116}$$

Use Lemma A.1 with $\delta = 2/3$ on the fixed packets. Lemma A.3 supplies both cube-free restorations, with the same fixed unit and ray packets.

Proof. Use the unrestricted positive row lift of [6, Lemma 12.1]. Its integration over the compact norm-one quotient precedes orbit Poisson. After cube-free restoration, frame each signed root with its actual component scales and retained prime positions. In this proof A is the full-product dyadic exponent on that root, bounded by the original cap. The compact profile supports put the two rectangles of the signed root in the same fixed-factor product annulus. The gaps (116) persist under this shortening. The first complete-product formula is [6, (12.2)]. Write C, D for the complete common parts of the two column moduli, $c = \log_U \mathrm{N}C$, $d = \log_U \mathrm{N}D$, and p for their common radical width. Let r be the squarefree product of the primes where the two valuations differ, and e the divisor in the common complementary coprimality expansion. Put $q = \log_U \mathrm{N}(er)$. The retained first frequency has width

$$k_1 = 2A - c - d + q - 1 - t, \qquad t \ge -O(\eta),$$

where the loss includes the logarithmic dyadic enlargements and the fixed cutoff padding. Complete the squarefree Möbius expansion before positivity. The artificial divisor occurs through its Möbius sign and one divisibility indicator in each column. Apply Cauchy's inequality jointly in this seed and the nonzero first frequency after the common kernel separation. Each resulting squared norm carries the seed with weight one.

Let ξ_r denote the actual primitive common quotient character. By the CRT in [6, (12.2)], the additional column factors on the left and right are respectively $\chi_I(er)\xi_r(I)$ and $\chi_I(er)\overline{\xi_r(I)}$. On supported columns these are $\chi_I(V_C)$ and $\chi_I(V_D)$, where the exponents at a prime of r are $(1+i-j) \bmod 6$ and $(1+j-i) \bmod 6$ for the left and right common valuations $i, j \in \{1, 2\}$. Both include the e-part. Thus active pairs $(2,1), (1,2)$ have exponents $(2,0), (0,2)$ and

$$q_C := \log_U \mathrm{N}V_C = q + c - d, \qquad q_D := \log_U \mathrm{N}V_D = q + d - c. \tag{117}$$

The exponent-zero primes remain genuine natural holes on the corresponding columns. All products on supported columns use symmetric reciprocity with the finite unit character of the frequency

included. The presentation on the supported ideals and its packed presentation have the same primitive inducing character after common finite deletions.

In each first Cauchy norm separately, allocate its common factor among the two ordinary positions and the marked positions. There are divisor-many genuine allocations. Select a complete subproduct of the slots common to that signed pair of actual width $u' = u + O(\eta)$, or take all of them at exhaustion. For a selected tuple of ideal product Δ, CRT gives the fourth-power Gauss shift on the whole residual column, with its fixed finite unit characters. This is the tame identity $\mathcal{G}(\Delta I, h) = \mathcal{G}(\Delta, h)\mathcal{G}(I, \Delta^4 h)$ on $(I, \Delta) = 1$, using allowed generators. Indeed the CRT cross factor is $\chi_I(\Delta)^2$ and the fourth-power frequency shift contributes $\overline{\chi_I(\Delta)}^4 = \chi_I(\Delta)^2$ on units. On the original forbidden locus the transferred Gauss factor is zero: a prime of Δ dividing the residual cube-free column has modulus valuation one or two, whereas its shifted frequency valuation is at least four. Only the genuine supported assignments are used before this identity. The residual seed is obtained by removing the primes of Δ from the first squarefree seed. For fixed extracted tuple and retained seed, the original seed has at most $2^{\#\{\text{selected slots}\}}$ preimages.

In the first bridge the central factor $1/\sqrt{\mathrm{N}I}$ on the full column contributes $1/\sqrt{\mathrm{N}\Delta}$ to each extracted tuple. The sum of the squared extracted coefficients is $U^{O(\varepsilon)}$ by the fixed prime-position counting bounds. Its remaining frequency is $\Delta^4 h$. At any nonzero resulting frequency the number of possible selected tuples is divisor-many, because each selected prime divides its exterior ideal. The positive sum over the residual seeds may now be enlarged to contain every squarefree divisor needed in the next correlation. In the filled case the new product cap and raised frequency bound on the C-side have logarithms

$$a = A - c - u' + O(\eta), \qquad k = k_1 + 4u' + O(\eta), \qquad X = U^a, \quad K = U^k.$$

Here X is the scale of the actual residual product annulus. At a shorter root both k_1 and a are recomputed with its actual A before the first correlation. The compact factors and the dyadic roundings contribute only $O(\eta)$ to the exponents.

For the second Poisson step, retain first the two actual product norms $\mathrm{N}I_1, \mathrm{N}I_2 \asymp X$ (after absorbing the frozen prefixes into the scales). The bridge from the two factors $1/\sqrt{\mathrm{N}I_i}$ and finite Fourier inversion has scalar $K/(\mathrm{N}I_1\mathrm{N}I_2)$ multiplying $\mathcal{F}(I_1, I_2; j)$ of [6, Lemma 12.10]. Before imposing the last Möbius restriction, use that lemma on the genuine coprime residual pairs. In the nonzero correlation, unequal common multiplicities vanish since the full columns are cube-free. The two second common ideals therefore equal one ideal G. Write $A_2 = GB_2$, where B_2 is the squarefree product of the common primes at which the frequency divided by G is a nonunit. The row has $j = A_2 z$ on this partition, and $|\mathcal{F}(G, G; j)| \le \mathrm{N}G$. The completed seed appears in both full product moduli. On their genuine coprime residual locus this restricts the seed precisely to the squarefree divisors of G, giving the multiplicity $\rho(G) = 2^{\omega(G)}$ after the first Cauchy inequality.

In either side the phase on the residual ideal is now exactly $\chi_I(V_C j)$, up to the fixed anchor and the finite unit packet. Apply the injection $y = V_C A_2 z$, with the sublattice and natural holes retained. For the right first side use V_D and its own ball. There is no change of real measure in this discrete substitution. For example with a radial weight the row factor after substitution is $\mathbf{1}_{V_C A_2 | y} \Phi(\mathrm{N}y/(\mathrm{N}(V_C A_2) K_z))$, where K_z is the corresponding divided-frequency scale. The kernel has the same norm argument $K\mathrm{N}y/(\mathrm{N}V_C \mathrm{N}I_1 \mathrm{N}I_2)$ on a first-side frame. The common logarithmic Fourier measures of [6, Section 12 and the appendix on weighted moment estimates] apply to the same compact norm-one integral and the full row and column norm variables. The fixed ratio dilations are absorbed with their actual norm phases. A power of the whole column norm adds the same Mellin mode to both normalized ordinary profiles, with the surviving prime powers in their coefficients. The scale factor of this mode is the same on the two arms since the products

of their two scales agree. This retains both independent heights and their signed equal-product subtraction. The padded positive row envelope has exponent

$$M = 2a - k + q_C + O(\eta) = 1 + t - 6u' + O(\eta). \tag{118}$$

The permissible lower endpoint is $M \geq H_0 - O(\eta)$ in the filled case. Use this actual radius with its $O(\eta)$ padding, and one smooth positive ball majorant. All exponents including the terminal price are evaluated at the radius of the same frame.

Extend the genuine second CRT expression to artificial noncoprime residual pairs with its full norm kernel. Insert the complete Möbius sum over a second squarefree divisor D_2 before splitting the sides. The residual columns are coprime to G, so $(D_2, G) = 1$. At a prime $\mathfrak{p} \mid D_2$, open the divisibility of the full residual product with

$$1_{\mathfrak{p} \mid \prod_i I_i} = \sum_{\varnothing \neq J} (-1)^{|J|+1} \prod_{i \in J} 1_{\mathfrak{p} \mid I_i}.$$

Here the indices are the labeled positions in one column. Extract each prescribed prefix in both signed arms. A shortened ordinary variable is unrestricted at that prime unless a prescribed or natural zero remains. The squarefree ideal D_2 need not divide the packed row. In each side the assigned column prefixes therefore remove norm at least $\mathrm{N}G\mathrm{N}D_2$. Central normalization in each ordinary shortening supplies the inverse square root of the extracted norm. A fixed prime position has squared mass $O(1/\mathrm{N}\mathfrak{p})$ for its fixed supplying prime. Thus the squared raw row from one block is bounded, on regular rows after separating its two arms, by a positive combination of shortened centrally normalized leaves with total mass

$$U^{O(\varepsilon)} \frac{X}{\mathrm{N}G\mathrm{N}D_2}.$$

This uses the forward expansion of Lemma A.3. The ordinary-only local terms start in degree three, so their central absolute masses have a summable Euler product. Terms involving original marked positions are summed on those positions; a collision uses at least two factors of $P_i^{-1/2}$ on its one prime. The exact common-hole expansions shorten the actual ideal sums and have subpower mass. Bounded and empty coordinates retain their stated annular supports. Every normalized leaf therefore uses an original or reference pair with componentwise shortenings and a surviving subproduct.

Multiplying by the second scalar $K\mathrm{N}G/X^2$ leaves K/X times the label weight $\rho(G)/\mathrm{N}D_2$. At each nonzero packed row, both A_2 and G divide its exterior ideal. For a fixed frame and D_2, the possible pairs (G, A_2) and their unit packets are divisor-many. In particular

$$\sum_{\text{labels at } y} \frac{\rho(G)}{\mathrm{N}D_2} \ll_F \tau_F((y))^3 \sum_{\mathrm{N}D_2 \ll U^C} \frac{1}{\mathrm{N}D_2} \ll_{F,\varepsilon} U^{\varepsilon}. \tag{119}$$

The reciprocal weight is retained also when $D_2 \nmid y$.

For regular rows, first open any remaining label-dependent pure coprimality masks at fixed y. Their exact divisor expansions shorten the ordinary scales, fix any selected prime slots, and give bounded row scalars and total absolute mass $U^{O(\varepsilon)}$ per label. These operations retain the natural zeros of the character on the surviving variables. Weighted Cauchy in these expansions costs another factor $U^{O(\varepsilon)}$ and preserves the harmonic weight $\rho(G)/\mathrm{N}D_2$. Partition the surviving scale parameters into logarithmically many dyads, and the prime subproducts and finite anchors into fixed finite families. For an ordinary scale $X = X_0\lambda$ in one such dyad with $1 \leq \lambda \leq 2$, put

$x = \mathrm{N}\mathfrak{a}/X_0$. If $V(t) = W(t)t^{iv}$, introduce a fixed smooth Ω equal to one on the union of the resulting compact supports. Then

$$X^{-1/2}V(\mathrm{N}\mathfrak{a}/X) = X_0^{-1/2}\Omega(x)x^{iv}\int_{\mathbb{R}}\widehat{W_{\log}}(\tau)\lambda^{-1/2-i(v+\tau)}x^{i\tau}\,d\tau.$$

Here $W_{\log}(t) = W(e^t)$ and the Fourier measure has its fixed normalization. An identical separation is made in the other coordinate. For fixed Fourier variables the normalized tests and the character $\chi_\bullet(y)$ are now common to the labels in a dyad. Twists already present in the whole-product kernels are included in the common Mellin measure and use the same inverse-norm dilation in the two signed arms. The factors depending on λ have modulus at most one. Cauchy's inequality with the absolute Fourier measure produces a positive integral of squared moments with common tests. The label bound (119) can therefore be inserted at each y inside that integral. Rapid decay of the fixed logarithmic transforms integrates the polynomial seminorm and height factors. Neither a count of polynomially many shortened scales nor a sum of their separate moment bounds is needed. Use this positive argument for regular rows. On the fixed-inducing packets keep the signed difference through the same extractions and use the centered estimate before taking absolute values.

On a regular reference arm one coordinate remains at most $U^{\ell_{\mathrm{ref}}}$ after every shortening. Its positive moment on the full packed ball is $O(U^{M+\varepsilon})$ by the short fourth of Lemma A.2, since $M \geq H_0 - O(\eta)$ and the surviving prime width is at most the original W. Use (115) on the original arm with its shortened scales and surviving prime subproduct.

By (115), the regular side is at most

$$U^{k-a+M+u'-\rho+O(\eta+\varepsilon)} = U^{A-c+q_C-\rho+O(\eta+\varepsilon)}.$$

The unmarked branch at exhaustion uses exponent M, and the side allowance has a reserve of u'. On a fixed-inducing row, freeze the same tuple of live marked primes in both arms before estimating the signed difference. Their holes, together with the frozen prefixes and other exact masks, give the same allowed residual valuation pairs in the two ordinary sums. Apply Lemma A.1 with $\delta = 2/3$ to this pair. Sum the live tuples by the ideal count in each prime annulus, with their actual bounded coefficients. Those prime lengths are already included in X, and they carry the original common central normalization in the scalar $K/(\mathrm{N}I_1\mathrm{N}I_2)$. If L_f is the pre-extraction minimum scale and D is the full extracted norm, the raw signed value before the square is

$$\ll U^{\varepsilon}(X/D)\max(1, L_f/D)^{-\delta}.$$

Here if a surviving scale is less than one the bound follows from the trivial direct ideal count. After squaring, for $D \geq \mathrm{N}G\mathrm{N}D_2$ and $2\delta \geq 1$ this is at most $U^{\varepsilon}X^2/(\mathrm{N}G\mathrm{N}D_2\max(1, L_f))$. The row count for a finite inducing packet is $O(U^{M/6+\varepsilon})$, from its prescribed exterior valuations modulo six and the finite unit packet. Thus the fixed side has exponent at most $k + M/6 - (\ell_{\mathrm{ref}} - c)_+ + O(\eta+\varepsilon)$. Its excess over $a + q_C + u'$ is

$$A + 3u' - \frac{5}{6} - \frac{5}{6}t - c - (\ell_{\mathrm{ref}} - c)_+ + O(\eta+\varepsilon) \leq A + 3u' - \frac{5}{6} - \ell_{\mathrm{ref}} + O(\eta+\varepsilon).$$

At the second origin the exact correlation is supported on $I_1 = I_2 = I$, with value $\varphi(I) \leq \mathrm{N}I$. Counting the completed seeds on the origin gives $\rho(I)$ and the squared column mass on $\mathrm{N}I \asymp X$ is $O(XU^{\varepsilon})$ by the ideal-divisor bound. The volume factor at this origin is the compact quotient integral $O(K)$. The resulting $O(KU^{\varepsilon})$ fits the side allowance by $A + 4u < 1$ and (118). The first origin is diagonal for cube-free columns and is $O(U^{1+\varepsilon})$. A positive enlargement in the raised frequency may include zero. The normalized Gauss sum $\mathcal{G}(I, 0)$ vanishes for nonunit cube-free I;

the unit column has bounded support. The omitted Poisson dyads are estimated on the original genuine supports by the rapid decay of the norm kernel. At both steps all column, seed and modulus caps are polynomial. A fixed high derivative order pays these counts and the remaining infinite dual tail by ideal counting. No zero term is assigned to the nonzero divisor fibers.

The resulting two first-side allowances have exponents $A - c + q_C$ and $A - d + q_D$. With $R_1 = \log_U \mathrm{N}r$ and $E_1 = \log_U \mathrm{N}e$, the original first scalar and the single common family count then give total exponent

$$1 - A - R_1/2 - E_1 + p + \tfrac{1}{2}(2A - c - d + q_C + q_D) = 1 + p - (c+d)/2 + R_1/2 \le 1,$$

since $c + d \ge 2p + R_1$ prime by prime. This accounts for the two orientations independently. The inverse cube-free and short-reference restorations of the initial signed comparison use their actual fixed local coefficients and the original row exclusion. □

A.3 Disjoint refinement of a repeated prime factor

A short inverse count may require several copies of one prime factor. We refine its prime list at fixed density inside its original annulus.

Lemma A.5 (Prime lists with bounded discrepancy)**.** *Fix the field, a bound U^C on the norms of the moving conductors and excluded ideals, and a finite depth and branching number. Exterior prime ideals admit a fixed partition by the leaves of a tree with these parameters. For each node of density b and each original prime factor Q at norm scale P, its restriction Q_b to that node satisfies*

$$Q_b(\lambda) = bQ(\lambda) + O((\log U)^2) \tag{120}$$

uniformly for the finite prime characters and coprimality masks in the prescribed norm range. The implied factor includes the supremum and first variation of the compact norm test. Norm twists of polynomial height are allowed. Its constants may depend on the fixed tree. A child of relative density b' therefore satisfies $Q_{bb'} = b'Q_b + O((\log U)^2)$.

Proof. At each prime choose independent uniform labels at each depth, with equal branching. At a fixed node the centered indicator has mean zero and modulus at most one. In a norm annulus of scale P there are $O_F(P)$ eligible prime ideals. For a fixed interval endpoint, finite ray character and fixed exclusion, the unnormalized centered sum is $O_F(P^{1/2}(\log U)^2)$ outside an event of probability $O(\exp(-c(\log U)^4))$, by the bounded-variables inequality applied to its real and imaginary parts.

Use dyadic U, dyadic annuli up to U^C, and all prefixes of the primes ordered by norm in each annulus. An endpoint at an equality of norms is included as one prefix. The number of finite ray characters of conductor norm at most U^C is polynomial in U by the ray class number bound. The number of ideals and allowed masks is also polynomial. For a norm twist $\mathrm{N}\mathfrak{p}^{it}$ with $|t| \le U^C$, include a grid of spacing U^{-C_1} with $C_1 > C + 2$. Differentiation in t gives an unnormalized error at most $O_F(P \log U\, U^{-C_1})$ between grid points. The union over the fixed tree and these tests has summable exceptional probability. Thus a single choice of labels works at all sufficiently large dyadic U. Subtraction gives subintervals. Partial summation gives the smooth windows with their first variation. Finite coefficient combinations, central normalization and dyadic enlargement of U give (120). □

The complete sums before refinement are called the parent sums. Refinement does not change their norm scales.

Lemma A.6 (Coefficient-two estimate on the refined lists)**.** *The ordinary mixed estimate of [2, Theorem 2] holds for products of disjoint refined lists from Lemma A.5. For finite-order characters this is the refined form of [5, Theorem 4.9]. The lengths are those of the parent annuli, with the same height and mask conventions.*

Proof. Run the two native inductions in [5, 2] with the actual coefficients $b_i(\mathfrak{p})1_{\mathrm{color}(\mathfrak{p})\in A_i}$, where A_i is its node. We verify the coefficient uses in these inductions.

Coefficient identities. The convolution of the two ordinary factors and the gcd allocation in [2, Lemma 1] involve prime weights only at the actual primes of a marked tuple. Distinct positions still have disjoint prime lists, so their tuples are injective even when their norm scales agree. First and second common divisors, upper and lower Gauss valuations, correction factors and pure coprimality restrictions are local at these primes. They are unchanged when each coefficient is multiplied by its color indicator. A consumed position contributes its actual scalar. A live position retains its indicator. Fourier separation is of the norm tests, so the indicator does not enter their derivatives or common measure. In the unitary case the two fixed ordinary character factors are extracted separately as in [2].

Positive and signed bounds. The selected operator and complete Gauss estimates of [5, Theorem 3.10 and Lemma 4.5] work with arbitrary common ideal coefficients and their divisor square-mass bounds. Each restricted prime weight still has bounded modulus. Fiber multiplicity is bounded by the same divisor power because tuples are injective and indexed by ideals. The high-amplitude term with a proper selected block uses the pointwise bound on the retained prime factors, as distinguished in [5, Theorem 3.13] from the whole-product terminal. The unmarked two-factor mean, the Gauss small-parameter terminal with its natural zeros, and the fixed-inducing ordinary estimate use bounded local coefficients. The signed comparison and its exceptional part have the same colored coefficients in both rectangles before absolute values. The auxiliary prime average retains the complete pool outside the original parent annuli [5, Section 4.1].

Prime upper bounds. For each live position (120) compares it to the complete Hecke-compatible sum with the same finite twists, mask and norm window. Thus the squared prime exponent κ_1 of [6, Lemma 12.7] holds for each refined factor with an arbitrary fixed subpower loss. Use the actual refined descriptor and common source coefficients in both the whole-product and proper-block estimates [5, (3.27)–(3.28)]. Both amplitude bounds of a selected bin refer to the actual refined factors. The coprimality conditions in [5, Lemma 3.12] are imposed on the actual tuples. The complete comparison supplies only the prime upper bounds and the buffered zero-band upper bound. This supplies the small-width pointwise stripping, the sixth-power block-selected bound, and the final stripping from coefficient $L > 2$ to coefficient two. The ordering and removal of low-amplitude factors use their actual amplitudes. The length inequalities depend on the norms of the primes and on the sum of their widths. The disjointness of the original intervals is used through the injectivity and coprimality of the marked tuples, both of which hold for the refined lists.

At zero prime width the earlier estimate is the same unmarked theorem. These coefficient, signed and pointwise bounds supply the coefficient-three and finite coefficient-two inductions of [5, 2]. The discrepancy loss is absorbed in the original prime loss. □

Lemma A.7 (Amplification on disjoint descendants)**.** *Fix a finite list of complete parent prime windows and a finite maximum amplification depth. In every positive-response class, an amplified position can be tested by distinct disjoint descendants at the same norm scale. The moment bound for fixed disjoint refined lists implies the same bound for the original lists and their subproducts.*

Proof. If the squared response slope e of a live position on its scale P has a fixed positive lower bound, then $|Q| \gg P^{e/2-o(1)}$. By (120), all descendants of fixed relative density in the chosen

generation have modulus $\gg |Q|$ on that class. Choose a distinct descendant for each copy. This keeps all norm lengths and total amplification quota, up to rounding by one original width in the integer number of copies. If an input was already refined, use descendants of that node. Disjointness from the other positions is preserved. Hu's native marked inverse estimate [3, Theorem 4.1] allows disjoint prime lists with bounded coefficients, independently of their norm annuli.

For the original product assign distinct nodes at the first level to the original positions, whether or not their parent windows overlap. Write $Q = b^{-1}Q_b + O((\log U)^2)$ for each position, expand, and use finite Cauchy before summing the rows. Every surviving subproduct is supported on disjoint refined lists and has no larger total width. The uniform pointwise errors contribute $U^{O(\varepsilon)}$. □

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