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# Joint marked moments and optimized zero-free half-planes for unitary Hecke $L$-functions

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**Abstract**

In this paper, we improve the uniform zero-free bound for the Quasi-Riemann Hypothesis and obtain the half-plane

$$L_K(s,\omega) \neq 0 \qquad (\operatorname{Re} s > B_* = 0.8733458561067185768\ldots)$$

for every continuous unitary Hecke character $\omega$ over every number field $K$, with the possible norm-character pole permitted. We prove joint moment estimates for inverse and ordinary character sums and apply them to zero detection and the averaged Mellin numerator. The proof uses the earlier number-field theta and moment theorems and the unitary estimates of Hu and Gao. The same half-plane holds for finite-dimensional continuous unitary Weil representations, with pole orders equal to the multiplicities of their norm-character constituents. We determine $B_*$ as an algebraic number of degree nine and as the infimum of the specified positive comparison, allowing unequal ordinary moments and variable comparison lengths.

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# 1 Introduction

In this paper, we continue the study of zero-free half-planes for Hecke $L$-functions over number fields and improve the uniform zero-free bound for the Quasi-Riemann Hypothesis. We obtain this half-plane for every continuous unitary Hecke character. The proof combines joint moment estimates for inverse and ordinary character sums with the unitary theta and moment estimates of Hu and Gao.

## 1.1 The main theorem

In this subsection, we state the zero-free theorem proved in Sections 8 and 9.

Let

$$B_* = 0.873345856106718576801184796065072356770 4\ldots. \tag{1}$$

The exact value is defined by (33) and (35). It is an algebraic number of degree nine; its minimal polynomial is given in Section 7.

**Theorem 1.1** (Main theorem)**.** *Let $K$ be a number field and let $\omega$ be a continuous unitary Hecke character of $K$. Then*

$$L_K(s,\omega) \neq 0 \qquad (\operatorname{Re} s > B_*),$$

*with the simple pole at $s = 1 + it$ permitted when $\omega = (N_K\cdot)^{it}$. The boundary $B_*$ is independent of $K$ and $\omega$.*

For the principal character, this gives $\zeta_K(s) \neq 0$ for $\operatorname{Re} s > B_*$, with the pole at $s = 1$ permitted.

The finite-order case and its unitary norm twists are proved in Theorem 8.14. Theorem 9.3 treats arbitrary continuous unitary characters. Constants and thresholds in the estimates may depend on the fixed field and character. No assertion is made on the line $\operatorname{Re} s = B_*$.

Corollary 9.5 gives the corresponding result for every finite-dimensional continuous unitary representation $r$ of the global Weil group $W_K$. In particular,

$$L_K(s, r) \neq 0 \quad (\operatorname{Re} s > B_*), \qquad \operatorname{ord}_{s=1+it} L_K(s, r) = -\dim \operatorname{Hom}_{W_K}((N_K\cdot)^{it}, r),$$

where poles are permitted in the nonvanishing statement.

### 1.2 Earlier bounds and the joint moments

In this subsection, we compare the earlier bounds with the joint moment estimates of Sections 2 and 8.

For finite-order Hecke characters over arbitrary number fields, [5] gives the boundary

$$B_2 = \frac{777 - \sqrt{1081}}{852} = 0.873381966613207519929\ldots.$$

Our method starts from the marked moment induction and Dirichlet comparison of [8], which gives the rational value

$$B_{137} = \frac{2183367}{2500000} = 0.8733468.$$

We adapt this induction to ideal-character sums over number fields and extend it to the paired unitary families. Sections 5–7 refine the finite construction and determine the infimum of the resulting positive comparison.

For the fixed construction in Section 5, Theorem 5.4 gives the comparison infimum $\theta_{240}$, where

$$\theta_{240} = 0.8733466968679481026286\ldots, \qquad B_* < \theta_{240} < B_{137} < B_2.$$

The limiting comparison in Section 7 then gives (1).

The resulting joint estimates treat inverse and ordinary sums simultaneously. For the normalized sums and character families of Section 2, each constructed pair $(R, z_R)$ gives

$$\sum_{Nu \ll Z} |M_P(u)Q(u)|^2 \ll Z^{1+\varepsilon},$$

$$\sum_{Nu \ll Z}^{*} |S_{n_1,1}(u)S_{n_2,2}(u)Q(u)|^2 \ll Z^{1+\varepsilon}, \qquad n_1 + n_2 = P,$$

$$P < R, \qquad S < (1 - R)/2 + z_R.$$

Here $S$ is the total exponent of the prime lengths in $Q$, and the star denotes the ordinary inducing-character exclusions. For each $\varepsilon > 0$, the original prime factors are sufficiently short; the mesh is fixed before the varying characters and actual primes. Test-function seminorms and polynomial height factors are suppressed here, with the conventions of Section 2. The inverse estimate is used in zero detection; the ordinary estimate improves both zero detection and the averaged Mellin numerator.

Theorem 8.12 proves the inductive enlargement of these two moment ranges. Its proof retains the original prime coefficients through complete-factor selection, Poisson summation and restoration of the removed Euler factors. Proposition 9.2 proves the same enlargement for the paired unitary character families.

### 1.3 The method and the comparison infimum

In this subsection, we describe the induction and the comparison used in Sections 6–9.

The induction begins with the moment estimates of [6, 5]. Given earlier joint moment bounds and a supporting line, put

$$H = r/R, \qquad u = (1-H)/6, \qquad Y(r) = Hz_R + (1/k-2)u, \qquad 2/3 < r < R \le 7/9, \quad 0 < k < 1/2.$$

Under the hypotheses of Theorem 8.12, including the strict inequalities of Subsection 8.7, the new bounds hold on

$$P < r - \delta_0, \qquad S < (1-r)/2 + Y(r) - \Delta - \delta_0, \qquad \delta_0, \Delta > 0.$$

The positive decrements are retained in each finite construction. Section 8 proves this step with arbitrarily small fixed positive margins; Proposition 9.2 gives the paired unitary version. The number of steps and the prime mesh are fixed before the asymptotic scale tends to infinity.

Section 6 determines an algebraic upper bound $G$ for the descending induction and approximates it by finite rational constructions. The tangent to $G$ at the optimizing point $r_*$ determines $B_*$. The exact comparison inequalities in Subsection 7.1 give every fixed boundary $B > B_*$ sufficiently close to $B_*$. The Mellin argument then proves Theorem 1.1 in the open half-plane stated above.

Theorem 7.2 identifies $B_*$ as the infimum for the comparison class of Definition 7.1. This class allows unequal ordinary moments, variable comparison lengths and repeated use of earlier moment estimates. The lower bound concerns these specified positive inequalities; it does not exclude stronger arithmetic estimates using additional cancellation.

We use the earlier theta, moment and continuation theorems of [6, 5], together with the unitary results of [3, 2]. Their normalizations and the estimates needed below are recalled in Sections 2 and 9. The new joint moment estimates and their compatibility with the paired unitary families are proved in Sections 8 and 9.

Here is the structure of the paper. Section 2 fixes the arithmetic notation and joint moment ranges. Section 3 describes the main steps of the proof, and Section 4 optimizes the zero-detection estimate. Section 5 gives a finite rational construction. Sections 6 and 7 determine the limiting comparison and its infimum. Section 8 proves the joint moments and the finite-order zero-free theorem. Section 9 extends the argument to continuous unitary characters, applies abelian base change, and proves the assertion for unitary Weil representations.

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## 2 Arithmetic notation and joint moment estimates

In this section, we fix the arithmetic notation, recall the earlier estimates, define the joint moment ranges and collect the comparison and unitary parameters. Section 3 then outlines the argument. Section 4 uses these ranges in zero detection, and Section 8 proves their inductive enlargement.

## 2.1 Arithmetic notation and earlier estimates

In this subsection, we fix the notation and recall the estimates used in Sections 5 and 8.

The letter $K$ denotes an arbitrary number field. For the moment and theta arguments we work over a fixed field $F \supset \mu_{12}$. Let $r_F$ be its number of complex places. Thus

$$\mu_m = \{z \in \mathbb{C} : z^m = 1\}, \qquad \zeta_m = e^{2\pi i/m}, \qquad n_F = [F : \mathbb{Q}] = 2r_F, \qquad F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{C}^{r_F}.$$

Write $\mathcal{O}_F$ for its ring of integers. Unless stated otherwise, $a, b, d, g, n, p$ denote nonzero integral ideals, with $p$ prime; fraktur letters are used interchangeably. Put

$$N\mathfrak{a} = |\mathcal{O}_F/\mathfrak{a}|, \qquad \mathrm{rad}(\mathfrak{a}) = \prod_{p|\mathfrak{a}} p, \qquad \tau(\mathfrak{a}) = \sum_{d|\mathfrak{a}} 1,$$

$$\mu_F(\mathfrak{a}) = \begin{cases} (-1)^j, & \mathfrak{a} = p_1 \cdots p_j,\ p_i \text{ distinct}, \\ 0, & p^2 \mid \mathfrak{a} \text{ for some } p. \end{cases}$$

The notation $(a, b)$ means the ideal greatest common divisor; $(a, b) = 1$ means coprimality. The symbol $1_E$ is the indicator of $E$, and $g \sqcup a$ means the product of coprime squarefree ideals when their prime factors are partitioned. Arithmetic indices in allowed-generator presentations are taken modulo the unit relations of [6]; their displayed norms are those of the corresponding ideals.

Let $\mathbb{A}_F$ be the adele ring, $C_F = F^\times \backslash \mathbb{A}_F^\times$ the idele class group, and $W_F$ the global Weil group. Hecke characters are continuous characters of $C_F$; they are unitary when their values have absolute value one. For a character $\chi$, $\mathfrak{f}_\chi$ denotes its finite conductor. Its ideal values are extended by zero:

$$(\mathfrak{a}, \mathfrak{f}_\chi) \neq 1 \quad \Longrightarrow \quad \chi(\mathfrak{a}) = 0.$$

The fixed finite set $S$ of excluded places contains the archimedean places and the fixed primes required by the theta, ray-class and Poisson constructions of [6]. It may be enlarged for a fixed coefficient character. Ramification of a varying character is retained through its zero extension.

We write $L_F(s, \chi)$ for the full Hecke $L$-function and $L_F^S(s, \chi)$ for the function with Euler factors at the finite places in $S$ omitted; $\zeta_F^S$ has the same convention. The superscript $(\mathfrak{r})$ omits the primes dividing an ideal $\mathfrak{r}$. Our norm-character convention is

$$L_F(s, \chi(N_F\cdot)^{it}) = L_F(s - it, \chi), \qquad t \in \mathbb{R}.$$

Thus the pole of $(N_F\cdot)^{it}$ is at $1 + it$. The order $\mathrm{ord}_{s=s_0}$ is positive at a zero and negative at a pole. We use the same norm character on $W_F$ through class field theory. The dual $\widehat{\mathrm{Gal}(F/K)}$ denotes the character group in abelian base change.

Fix the finite group $\Theta$ of coefficient ray-class characters from the earlier moment estimates, enlarged to contain the fixed finite characters arising in the transformations. Over $F \supset \mu_{12}$, $\chi_a(b)$ denotes the zero-extended sextic residue character in the ideal-class and ray presentations of [6]; $\chi_u$ is the corresponding varying character. In imprimitive residue presentations, zeros at primes of the original modulus are retained after primitive induction. A star on an ordinary moment sum retains the inducing-character exclusions of that estimate. In the paired unitary presentation $\lambda_u = \omega_0 \xi_u$, this exclusion is imposed on the finite-order factor $\xi_u$ as in (57).

For the finite-order comparison, write

$$\beta_*(F) = \sup(\{1/2\} \cup \{\mathrm{Re}\,\rho : L_F(\rho, \eta) = 0,\ 0 < \mathrm{Re}\,\rho < 1,\ \eta \text{ finite order}\}).$$

The fixed exponent is permitted since

$$13/18 < \kappa_1 = 374887/500000 < 3/4, \qquad \beta_*(F) \le B_2 < \frac{1+\kappa_1}{2} = 0.874887;$$

see [6, Lemma 12.7 and Theorems 10.7 and 11.2] for the corresponding finite-order prime and inverse estimates. We use reciprocity, the theta construction and radial reflection from [6, Sections 3–5], and Poisson summation and the transformed inverse families from [loc. cit., Sections 8 and 10–11]. The initial moments and common-divisor estimates are those of [loc. cit., Sections 10–12] and [5].

Zero detection, the Mellin comparison and continuation are treated in [6, Sections 13–14]. Their general-field formulations are [loc. cit., Sections 17–19]. The joint moment estimates are proved in Section 8; their paired unitary versions are proved in Section 9, using [3, 2].

## 2.2 Length exponents and test functions

In this subsection, we fix the scale and uniformity conventions used in Subsection 2.3 and in the induction of Section 8.

The asymptotic scale is $Z \ge 2$. An exponent $x$ denotes ideal norm length $Z^x$. For the original prime factors and their product, write

$$Np \asymp Z^{z_i} \quad (p \in \mathcal{P}_i), \qquad Q = \prod_i Q_i, \qquad S = \sum_i z_i, \qquad 0 < z_i \le \eta.$$

Here $\eta > 0$ is the prime mesh. The sets $\mathcal{P}_i$ may overlap; the original coefficients $b_i(p)$ are fixed before amplitude selection. A complete-factor selection retains or removes an entire $Q_i$. For a finite set $I$ of original positions, the injective mark is

$$M_I(n) = \sum_{\substack{(p_i)_{i\in I} \text{ distinct} \\ p_i \mid n \ (i\in I)}} \prod_{i\in I} b_i(p_i), \qquad M_\varnothing(n) = 1, \tag{2}$$

where $b_i$ is extended by zero outside $\mathcal{P}_i$.

Norm test functions $W, W_i$ lie in fixed smooth families, usually $C_c^\infty((0,\infty))$; Schwartz functions occur in Poisson summation. We use

$$\widehat{W}(s) = \int_0^\infty W(x) x^s \, \frac{dx}{x}, \qquad e(t) = e^{2\pi i t}, \qquad \widehat{f}(y) = \int_{F_\infty} f(x) e(-\operatorname{Tr}(xy)) \, dx,$$

with self-dual additive measure. At a complex place $|z|_v = |z|^2$. A fractional-ideal lattice $L \subset F_\infty$ has trace-dual $L^*$ and additive covolume $\operatorname{covol}(L)$. The notation $\mathcal{S}(F_\infty)$ denotes the Schwartz space.

The symbols $p_J(W)$ and $\mathfrak{S}_J$ denote finite homogeneous seminorms of the specified test functions; $H_{\mathrm{in}}$ denotes their incoming norm-twist heights. For these prescribed families, an abbreviated moment estimate suppresses factors of the form

$$\mathfrak{S}_J(\mathcal{W}, \mathbf{W})(1 + H_{\mathrm{in}})^{J'}, \qquad J, J' < \infty,$$

where $\mathcal{W} \ge 0$ is a common parameter weight. Implied constants may depend on $F$, the fixed excluded places and coefficient characters, the test supports, and the requested $\varepsilon > 0$.

Fix the field, numerical length margins and finite induction before choosing the prime mesh. The mesh precedes varying characters and actual prime factors. For each fixed character presentation, common Fourier and Mellin measures are chosen before the summation and divisor indices. Seminorm orders, polynomial height degrees and thresholds may depend on that presentation.

### 2.3 The joint moment ranges

In this subsection, we define the two moment bounds enlarged in Theorem 8.12 and used in the zero-detection estimate of Section 4.

With $Nu \ll Z$, the joint moment bounds at $(R, z_R)$ have polynomial length exponent $P$ and total prime length exponent $S$ in

$$P < R, \qquad S < \frac{1-R}{2} + z_R.$$

For the inverse sum, $P$ is its length exponent; for the ordinary product, $P$ is the sum of the two length exponents. For one common character presentation $\lambda_u$, extended by zero at ramified ideals, set

$$M_r(u) = Z^{-r/2} \sum_n \mu_F(n)\lambda_u(n)W(Nn/Z^r), \qquad S_{n,j}(u) = Z^{-n/2} \sum_a \lambda_u(a)W_j(Na/Z^n),$$

$$Q_i(u) = Z^{-z_i/2} \sum_{p \in \mathcal{P}_i} b_i(p)\lambda_u(p)W_i(Np/Z^{z_i}), \qquad Q = \prod_i Q_i, \qquad S = \sum_i z_i.$$

The joint moment bounds at $(R, z_R)$ are

$$\sum_{Nu \ll Z} |M_P(u)Q(u)|^2 \ll Z^{1+\varepsilon}, \qquad \sum_{Nu \ll Z}^{*} |S_{n_1,1}(u)S_{n_2,2}(u)Q(u)|^2 \ll Z^{1+\varepsilon}, \quad n_1 + n_2 = P,$$

with the ordinary inducing-character exclusions of the earlier theorem, finitely many test-function seminorms and polynomial height losses understood. Inverse positions may have the original bounded coefficients, independent of the summation parameter, and the original sets of primes. Ordinary positions are the original complete smooth prime ranges with fixed finite ray-character coefficients; in the unitary case they have the paired relative coefficient (56). Ray-class restrictions are expanded as finite linear combinations of characters. The original prime sets may overlap; their collision decomposition retains whole original factors and natural zeros before creating injective marked components.

No arbitrary new bounded ordinary prime coefficient is introduced. The Fourier and Mellin measures are fixed before the arithmetic indices.

The induction gives the recursion

$$H = \frac{r}{R}, \qquad u = \frac{1-H}{6}, \qquad Y(r) = Hz_R + \left(\frac{1}{k} - 2\right)u. \tag{3}$$

For $2/3 < r < R$ and $0 < k < 1/2$, we have $Y(r) > Hz_R$. Theorem 8.12 proves the two new moment estimates under the strict induction inequalities, using selection, Poisson summation and restoration of Euler factors.

### 2.4 Comparison and transformation parameters

In this subsection, we collect the parameters used in Sections 4–9. Their precise ranges and hypotheses are stated with the corresponding results.

| Notation | Meaning |
|---|---|
| $P, S$ | Polynomial length exponent and total prime length exponent. For an ordinary product, $P = n_1 + n_2$. |
| $(R, z_R)$ | Joint moment parameters, giving $P < R$ and $S < (1-R)/2 + z_R$. |
| $H, u, Y(r)$ | Relative parameter scale $H = r/R$, selected length $u = (1-H)/6$, and improvement in (3). |
| $k, \nu, e$ | Prime-amplitude cutoff and amplitude exponents in Subsection 8.6; there $k \le \nu \le 3/4$ and $0 \le e \le \nu$. |
| $g(r), \lambda(w)$ | Affine bounds for earlier moment improvements, written $g_0 - g_1 r$ and $a_0 - a_1 w$. |
| $R_o, R_n, R_g$ | Count exponents, with positive denominators $J_o, J_n, J_g$, in Subsection 5.2. The notation $R(\delta, x)$ denotes their permitted minimum. |
| $B_C, B_P, L_C, L_P$ | Moment parameters for the count and numerator comparisons, specified in Section 5. |
| $\delta, x$ | Amplitude parameters of the comparison in Section 5; $V = 5/6 - \delta$ in zero detection. |
| $\theta, B, h$ | Proposed zero-free boundary and outer parameter length; $Nu \ll Z^h$. The letters $\theta$ and $B$ are used in the fixed and limiting comparisons, respectively. |
| $l_x, l_y, \ell$ | Comparison length exponents and total exponent of the original prime factors in the Mellin comparison; see (7). |
| $B_2, B_{137}$ | Earlier comparison values recalled in the introduction. |
| $\theta_{240}, B_*$ | Fixed-comparison and full positive-comparison infima, defined by (26) and (35). |
| $G, \widehat{S}, \mathcal{G}$ | Limiting improvement, upper bound for all earlier prime lengths, and convex improvement majorant; see (27), (28) and (32). |
| $\mathcal{C}(P), C(P)$ | The initial prime-length bound $(1-P)/2$. These functions are distinct from the Gauss coefficient $C(n)$. |
| $M_{\mathrm{ref}}, \nu_{\mathrm{ref}}$ | Reference total-length bound and its cutoff in Proposition 6.3. The radii $R_c, R_+$ delimit the cases of $\widehat{S}$. |
| $L_0, L_K$ | Direct comparison bounds in Subsection 6.2; the additional numerator bounds are defined in Subsection 7.3. |
| $r_*, \delta_*, h_*$ | Optimizing point and limiting parameters in (33) and (36). |
| $m_D, c_*, R_*$ | Lower-bound parameters $m_D = r_*/2$, $c_* = m_D + G(r_*)$ and $R_* = 1 - \delta_*(1/2 + c_*)$. |
| $N, V, M, z$ | In a transformed inverse sum: total polynomial, auxiliary ideal, parameter and remaining prime length exponents. The free exponent is $N - z$ and $F_0 = N + V$. |
| $N', K', H_c$ | Exponents after selection and the second Poisson summation: $N' = N - u'$, $K' = M + 4u'$ and $H_c = 2(N' + V) - K' + \kappa_{\mathrm{P}}$. |
| $C(n), \Lambda, \gamma_2$ | Normalized cubic Gauss coefficient and its presentation $C = \bar{\Lambda}\gamma_2$, used in Sections 8 and 9. |

| Notation | Meaning |
|---|---|
| $p(P,s,M)$ | Excess exponent defined in Subsection 8.7 for the sixth-power summation and ordinary induction. |
| $a$ | Lower integration limit in Subsection 8.10, where $1/2 < a \le 51/100$. |
| $\beta, \Delta, \gamma$ | Locally specified positive endpoint decrements and exponent reductions. The zero supremum $\beta$ in Section 9 is separately defined below. |

The symbol $S$ denotes a set when it specifies excluded places and a real exponent in a length inequality. Likewise, $u$ and $k$ denote ideal parameters, in their fixed presentations, in $Nu, Nk$ and real parameters in the induction formulas. Letters such as $M, R, V, C$ are reused locally with the displayed definitions above; all auxiliary coefficients and margins are defined where they enter the argument. In the continuation argument, $\eta$ denotes an exponent reduction rather than the prime mesh, and $H$ denotes the external height cutoff rather than $r/R$. The polynomial $Q(\theta,\delta,x)$ in (25) is distinct from the prime product $Q$.

### 2.5 Unitary families and angular parameters

In this subsection, we fix the unitary-family notation used in Section 3 and in the continuation argument of Section 9.

For a fixed continuous unitary Hecke character $\omega$ of $F$, put

$$\mathcal{F}_\omega = \{\theta\omega^j(N_F\cdot)^{it} : j \in \{-1,0,1\},\ \theta \text{ finite order},\ t \in \mathbb{R}\}, \tag{4}$$

$$\beta_\omega = \sup\big(\{1/2\} \cup \{\operatorname{Re}\rho : L_F(\rho,\vartheta) = 0,\ \vartheta \in \mathcal{F}_\omega,\ 0 < \operatorname{Re}\rho < 1\}\big). \tag{5}$$

In the unitary contradiction argument, $\beta = \beta_\omega$. The finite-order supremum $\beta_*(F)$ above is distinct. The paired inverse class $\mathcal{X}_u$ is defined in (59); its fixed factors are $\xi \in \Theta \cup \psi\Theta$. For ordinary factors, $\lambda_u = \omega_0\xi_u$ has $\xi_u$ finite order and paired relative coefficients given by (56).

At a complex place, write

$$\psi_v(z) = (z/|z|)^{k_v}|z|_v^{i\tau_v}, \qquad k_v \in \mathbb{Z}, \quad \tau_v \in \mathbb{R}.$$

The fixed theta character has angular vector $\boldsymbol{\lambda} \in \{1,-1\}^{r_F}$; $\boldsymbol{k} = (k_v)_v$ is the angular vector of the target. The notation CT denotes a theta constant term and $w$ in $wv$ the Weyl action on a test vector. The multiplier $\mathcal{K}_{m,\tau}$ of (58) uses angular mode $m$ and radial unitary parameter $\tau$.

## 3 Outline of the proof

In this section, we describe the moment induction, the limiting comparison and the passage to unitary characters used to prove Theorem 1.1. The analytic estimates are proved in Sections 8 and 9. We first work over a fixed number field $F \supset \mu_{12}$.

**Joint inverse and ordinary moments.** The two estimates of Section 2 are proved together. Given their validity at $(R, z_R)$, an earlier supporting line and the strict inequalities of Subsection 8.7, put

$$H = r/R, \qquad u = (1-H)/6, \qquad Y(r) = Hz_R + (1/k-2)u, \qquad 2/3 < r < R \le 7/9, \quad 0 < k < 1/2.$$

For these parameters, we have $Y(r) > Hz_R$. Theorem 8.12, in the finite-order contradiction regime $\beta_*(F) \geq 51/100$, gives both moments for

$$P < r - \delta_0, \qquad S < (1-r)/2 + Y(r) - \Delta - \delta_0, \qquad \delta_0, \Delta > 0.$$

The positive decrements are fixed at each step. The unitary version is Proposition 9.2.

**Selection and Poisson summation.** The first inverse transformation gives $M \leq 2(N+V)-1$. Enlarging the positive parameter sum allows $M = 2(N+V) - 1 + O(\eta)$. Here $N, V, M$ are the exponents of the polynomial length, auxiliary ideal length and parameter range. Selecting complete prime factors of total length $u' = u + O(\eta)$ gives

$$\begin{aligned} N' &= N - u', & K' &= M + 4u', \\ H_0 &= 2(N'+V) - K' = 1 - 6u' + O(\eta), & H_c &= H_0 + \kappa_{\mathrm{P}}, \quad \kappa_{\mathrm{P}} > 0. \end{aligned}$$

The fourth power comes from sextic reciprocity on coprime squarefree ideals:

$$C(dn) = C(d)C(n)\chi_n(d)^4\omega(d,n), \qquad (d,n) = 1,$$

where $C$ is the normalized cubic Gauss coefficient and $\omega$ is a finite linear combination of ray-class characters.

The second Poisson summation converts the cubic Gauss coefficients to Möbius coefficients. Coprimality expansion and (46) retain the remaining prime coefficients. The normalization satisfies

$$(K' - N' - V) + (H_c + u') = N + V + \kappa_{\mathrm{P}}.$$

For a fixed reduction $\gamma > 0$ and requested loss $\epsilon > 0$, choose the auxiliary parameters so that

$$\kappa_{\mathrm{P}} + C\eta + 2\epsilon_1 < \min\{\gamma/4, \epsilon\}, \qquad \kappa_{\mathrm{P}}, \eta, \epsilon_1 > 0,$$

as in (47). If the prime length is insufficient for the selection, the initial inverse estimate applies.

For the ordinary moment, Proposition 8.10 gives the reference estimate. The unmarked Euler factors converge absolutely since

$$3(1/2 - 15/98) = 51/49 > 1.$$

Marked primes remain in their original normalized sums, with nonpositive local exponents. These estimates give a finite induction with the same coefficient families in both moments.

**The limiting comparison.** For descending sequences beginning with $z_0 = 0$ and $2/3 < R_0 < 7/9$, Proposition 6.2 gives the upper bound

$$G(R) = \frac{11R}{16} - \frac{7}{24} - \frac{5}{16}\{7R^4(R - 2/3)\}^{1/5}, \qquad 2/3 < R \leq 7/9.$$

It satisfies

$$G(7/9) = 0, \qquad G'(R) = \frac{(45R-24)G(R) + 9R - 7}{R(45R-30)}, \qquad G''(R) > 0.$$

Proposition 6.3 allows earlier estimates to be used in any order. Its bound $\widehat{S}$ agrees with $(1-R)/2 + G(R)$ for $R_+ \leq R \leq 7/9$, and

$$R_+ < 367/500 < r_* < 7/9.$$

Proposition 6.4 approximates $G$ by finite rational sequences on every compact subinterval of $(367/500, 7/9)$. Subsection 7.1 supplies the short extension below $367/500$ by truncating the last moment range.

For an affine bound $g(R) = g_0 - g_1 R$ valid for both moment classes, the reductions in the zero-detection exponents, on their respective moment ranges, are

$$\begin{aligned} &\delta(A + Kr), && \delta(A + 2Km), \\ A = x(1 + 2g_0), \qquad & && K = 1 - x(1 + 2g_1). \end{aligned}$$

They agree at $r = 2m = 2t/3$, where $r + m = t$, provided both moment ranges permit this split. For $\delta > 0$, $K > 0$ and $V = 5/6 - \delta > 0$, Proposition 4.1 minimizes

$$\max\{1 - \delta(A + 2Kt/3),\ 1/6 + Vt\}, \qquad a \le t \le b,$$

at

$$t = \min\left\{b, \max\left\{a, \frac{5/6 - \delta A}{V + 2\delta K/3}\right\}\right\}.$$

The tangent at $r_*$ gives

$$B_* = \frac{99r_* - 14}{12(9r_* - 1)}.$$

Subsection 7.1 proves the full comparison for each fixed nearby $B > B_*$. Subsections 7.2 and 7.3 prove the matching lower bound within Definition 7.1.

**Unitary characters and Mellin continuation.** For a fixed unitary twist $\omega_0$, write $\lambda_u = \omega_0 \xi_u$, where $\xi_u$ is finite order. Include the ramified places of $\omega_0$ in the fixed set $S$. For $p \notin S$, the paired coefficients satisfy

$$b_i(p) = \omega_0(p)^{-1} \sum_j c_{ij} \vartheta_{ij}(p), \qquad b_i(p)\lambda_u(p) = \xi_u(p) \sum_j c_{ij} \vartheta_{ij}(p).$$

The characters $\vartheta_{ij}$ belong to the fixed finite ray-class family. Thus the prime estimates still concern finite-order characters. In Gao's mixed ordinary moment (57), use $(\omega_1, \omega_2) = (\omega_0, \omega_0)$ for detection and $(1, 1)$ for the finite-order numerator. Subsection 9.2 checks the character orientations, vanishing theta constant terms and transformations needed for the new moments.

Fix $\omega$ and let

$$\mathcal{F}_\omega = \{\theta \omega^j (N\cdot)^{it} : j \in \{-1, 0, 1\},\ \theta \text{ finite order},\ t \in \mathbb{R}\}.$$

Let $\beta$ be the supremum of its nontrivial zero real parts, with $1/2$ adjoined. Gao's theorem gives $\beta \le B_2$. If $\beta > B_*$, choose $B_* < B' < B < \beta$, with $B'$ sufficiently close to $B_*$. The finite construction at $B'$ and the Mellin comparison give an $\eta > 0$, independent of $\psi \in \mathcal{F}_\omega$, such that

$$e^{(s-5/6)^2} \frac{H_\psi^S(s)}{L_F^S(s, \psi)} \quad \text{is holomorphic for } \operatorname{Re} s > \sigma_0, \qquad \sigma_0 = \max\{B, \beta - \eta\} < \beta.$$

The numerator is nonzero there. The omitted finite Euler factors have no zero or pole in $\operatorname{Re} s > 0$. Consequently

$$L_F(s, \psi) \ne 0 \ (\operatorname{Re} s > \sigma_0) \quad \text{for every } \psi \in \mathcal{F}_\omega \quad \Longrightarrow \quad \beta \le \sigma_0 < \beta.$$

This contradiction proves the unitary bound over $F$. Norm-character poles are permitted. Constants and thresholds may depend on $\psi$; the exponent reduction $\eta$ is common to the family.

**Descent and Weil representations.** For an arbitrary number field $K$, take $F = K(\zeta_{12})$. Theorem 9.4 applies abelian base change with all ramified local factors retained. For a finite-dimensional continuous unitary representation $r$ of $W_K$, unitary induction gives finite extensions $E_j/K$ and unitary Hecke characters $\omega_j$ such that

$$[r] = \sum_j n_j [\mathrm{Ind}_{W_{E_j}}^{W_K} \omega_j], \qquad L_K(s,r) = \prod_j L_{E_j}(s,\omega_j)^{n_j}, \qquad n_j \in \mathbb{Z}.$$

Each Hecke factor is zero-free away from its norm-character pole in $\mathrm{Re}\, s > B_*$. Frobenius reciprocity gives

$$\mathrm{ord}_{s=1+it} L_K(s,r) = -\dim \mathrm{Hom}_{W_K}((N_K \cdot)^{it}, r) \le 0.$$

This also excludes zeros at the possible poles and proves Corollary 9.5.

# 4 Zero detection from an affine moment bound

In this section, we optimize the zero-detection estimate supplied by the joint moments of Section 2. The resulting count is used in the comparisons of Sections 5 and 7.

Let $g(r) = g_0 - g_1 r$ and set

$$A = x(1+2g_0), \qquad K = 1 - x(1+2g_1), \qquad V = \frac{5}{6} - \delta, \qquad D = V + \frac{2\delta K}{3}.$$

The inverse and ordinary detector savings at widths $r$ and $m$ are respectively

$$\delta(A + Kr), \qquad \delta(A + 2Km).$$

At the common split $r + m = t$ they agree at $r = 2m = 2t/3$.

**Proposition 4.1** (Exact two-branch minimax)**.** *Suppose $\delta > 0$, $K > 0$, $V > 0$, and $a \le b$. Put*

$$f(t) = 1 - \delta\left(A + \frac{2K}{3}t\right), \qquad q(t) = \frac{1}{6} + Vt, \qquad t_* = \frac{5/6 - \delta A}{D}.$$

*Then*

$$\operatorname*{argmin}_{a \le t \le b} \max\{f(t), q(t)\} = \{\min(b, \max(a, t_*))\}.$$

*If $a \le t_* \le b$, the minimum equals*

$$C_* = \frac{1}{6} + V\frac{5/6 - \delta A}{D}. \tag{6}$$

*Proof.* $f' = -2\delta K/3 < 0$ and $q' = V > 0$. Their unique crossing is $t_*$. To its left the maximum is $f$; to its right it is $q$. □

**Corollary 4.2** (Sensitivity at an interior crossing)**.** *Under the hypotheses of Proposition 4.1, with $\delta, x$ fixed and $a < t_* < b$,*

$$\frac{\partial C_*}{\partial g_0} = -\frac{2\delta x V}{D}, \qquad \frac{\partial C_*}{\partial g_1} = \frac{4\delta x V t_*}{3D}.$$

*Thus $\partial C_*/\partial g_0 < 0$ when $x > 0$. These derivatives hold on a region where every moment and zero-detection hypothesis remains valid.*

*Proof.* Differentiate (6), using $\partial_{g_0} A = 2x$ and $\partial_{g_1} K = -2x$. □

## 5 A finite rational construction and its comparison

In this section, we give a finite rational moment construction and determine the infimum of its comparison inequalities. The construction uses Section 4; Sections 6 and 7 then allow the moment ranges and the comparison lengths to vary.[1]

Write $\theta$ for a proposed boundary and use $x$ for the amplitude ratio. The comparison lengths are denoted by $l_x, l_y$:

$$\ell = \frac{11}{3} - 4\theta, \quad l_x = 1 + \ell - h, \quad l_y = h - 2\ell, \quad \kappa = 2\theta - 1. \tag{7}$$

Subsection 7.1 uses the independently known parameter $\kappa_1$ in the analytic application. The comparison parameters and fixed losses are

$$B_C = B_P = 1 - 10^{-12}, \quad L_C = L_P = 2 + 10^{-11}, \quad c_C = \frac{3}{20}, \quad c_P = \frac{3}{25}, \quad c_N = \frac{3}{40}, \quad \rho = 10^{-5}.$$

These parameters agree with the finite comparison in [8]. The construction uses the coefficient families of Section 2.

### 5.1 The rational moment induction

In this subsection, we specify the finite rational induction used in Subsection 5.2. Its comparison infimum is computed in Subsection 5.4.

Set

$$R_j = \frac{311}{400} - j\Delta R \quad (0 \le j \le 240), \qquad R_{240} = \frac{1467}{2000}, \qquad \Delta R = \frac{11}{60000},$$
$$H_j(r) = r/R_j, \qquad u_j(r) = (1 - H_j(r))/6, \qquad \Delta_z = \frac{1}{25000000}, \quad p = 10^{-7}.$$

Start with $z_0 = a_{0,0} = a_{1,0} = 0$. At stage $j$ put

$$w_j = \frac{R_j/2 + z_j - a_{0,j}}{1/2 - a_{1,j}}, \tag{8}$$

$$k_j = \max\left\{\frac{45457}{125000},\ 10^{-6}\left\lceil 10^6\left(\frac{5w_j/4 - 5/6}{3w_j/2 - (1+R_j)/2 - z_j} + \frac{18}{10^5}\right)\right\rceil\right\}, \tag{9}$$

$$Y_j(r) = \frac{r}{R_j}z_j + \frac{1/k_j - 2}{6}\left(1 - \frac{r}{R_j}\right), \tag{10}$$

$$z_{j+1} = 10^{-10}\left\lfloor 10^{10}(Y_j(R_{j+1}) - \Delta_z)\right\rfloor, \tag{11}$$

$$a_{1,j+1} = -\frac{Y_j(R_j) - Y_j(R_{j+1})}{\Delta R}, \tag{12}$$

$$a_{0,j+1} = Y_j(R_{j+1}) - \Delta_z - 2p + a_{1,j+1}R_{j+1}. \tag{13}$$

All constants are rational. The explicit lower cutoff in (9) is essential: without it the final stage violates the lower cutoff in the induction inequalities.

[1] Exact rational calculations and verification programs for Sections 5–7 are supplied in the accompanying computational supplement.

**Proposition 5.1** (The finite moment sequence)**.** *The 240-stage chain satisfies all fourteen inequalities of Subsection 8.7, including both branches of its thirteenth condition, at both endpoints of every stage. It also satisfies*

$$u_j(R_{j+1}) \le 10^{-4}, \quad R_j \le w_j, \quad \frac{45457}{125000} \le k_j < \frac{1}{2}, \quad z_{j+1} \ge 0.$$

*Each incoming line is strictly below every preceding stage improvement with the required margin. The lowered line of every stage is a global minorant of all 240 stage improvements.*

*Proof.* Apply the rational recursion to the inequalities of Subsection 8.7. Each condition is evaluated at both endpoints. For each incoming or supporting line, evaluate its difference from $Y_j - \Delta_z - p$ at both endpoints of every required stage. These differences are affine in $r$, so endpoint positivity suffices. The analytic induction over a number field is proved in Theorem 8.12. □

## 5.2 The complete comparison inequalities

In this subsection, we combine the moment sequence of Subsection 5.1 with the count of Section 4. We verify the comparison at all four conductor boundaries; Subsection 5.3 gives the corresponding lower bound.

For a count exponent $R(\delta, x)$ and conductor exponent $d$, set

$$E(d, C; \delta, x) = \frac{1}{2} + \frac{13h}{75} - \frac{l_y}{2} - \frac{\ell}{2} - \frac{9d}{100} + \frac{\delta}{2} + \delta x \left(\ell - \frac{C}{2}\right), \tag{14}$$

$$C_{\text{old}}(d) = \frac{\min(d, 2l_y - d)}{6\kappa}, \tag{15}$$

$$C_{\text{new}}(d) = \min\left(\frac{3d}{40}, \frac{2l_y - (2 - B_P)d}{L_P}\right). \tag{16}$$

The four conductor edges are

$$(d, C) = \left(\frac{1}{200}, C_{\text{old}}(\frac{1}{200})\right), \quad (\frac{19}{25}, C_{\text{old}}(\frac{19}{25})), \quad (\frac{19}{25}, C_{\text{new}}(\frac{19}{25})), \quad (h + 10^{-10}, C_{\text{new}}(h + 10^{-10})).$$

Put $V = 5/6 - \delta$. The old and initial count functions are

$$\begin{aligned} D_o &= 3 - x - 2x/(3\kappa), & C_o &= 2(1 - x)(1 - x/(3\kappa)), \\ J_o &= VD_o + \delta C_o, & R_o &= 1 - \delta + \frac{\delta V C_o}{2J_o}, \\ D_n &= 3 - x - 4x/L_C, & A_n &= 2 - 4x/L_C, \\ C_n &= 1 - 2xB_C/L_C, & J_n &= VD_n + \delta(1 - x)A_n, \\ & & R_n &= 1 - \delta + \frac{\delta V(1 - x)C_n}{J_n}. \end{aligned}$$

For an affine improvement $g(r) = g_0 - g_1 r$, Section 4 gives

$$\begin{aligned} J_g &= 3V + 2\delta\{1 - x(1 + 2g_1)\}, \\ R_g &= 1 - \delta + \frac{\delta V\{1 - x(1 + 6g_0 - 4g_1)\}}{J_g}. \end{aligned}$$

The symbols $J_{\text{old}}$ and $R_{\text{old}}$ below mean $J_o$ and $R_o$. For each branch, retain the range hypotheses of the cited zero-detection estimate.

The count envelope is the minimum of the old count, the initial count range, and the improved ranges supplied by the lowered own lines of stages 132, 133 and 134. Each branch is used only inside its range of validity. The improved range is

$$[189/512, 15/32] \times [15/32, 1/2],$$

and the initial range is $[15/64, 9/16] \times [9/32, 1/2]$.

**Theorem 5.2** (A rational comparison bound)**.** *With*

$$\theta_+ = \frac{873346697}{10^9}, \qquad h_+ = \frac{81449508391}{10^{11}},$$

*all four edge inequalities*

$$E(d, C; \delta, x) + \frac{3d}{4} R(\delta, x) < \theta_+ - 10^{-12} \tag{17}$$

*hold throughout* $[0, 3/4] \times [0, 1/2]$*. All 46 denominator and range conditions hold strictly. The comparison geometry and its floor margin also hold.*

*Proof.* For a branch $R = N/J$, multiply the margin in (17) by its positive denominator $J$. This gives a bivariate polynomial over $\mathbb{Q}$. On a box, positivity of every tensor Bernstein coefficient implies positivity of the polynomial. Successive dyadic subdivisions use respectively $1, 2, 2, 57$ subrectangles for the four edges, with maximal depth 25. The conditions on the parameter ranges are strict without subdivision on their respective ranges.

The floor margin is

$$\theta - \frac{51}{100} - \frac{h}{3} - \frac{53\ell}{100} - \frac{67}{100} 10^{-256} \geq \rho. \tag{18}$$

Indeed $h_+ = (234\theta_+ - 184)/25 - 3\rho - 10^{-11}$, giving margin $\rho + 10^{-11}/3 - (67/100)10^{-256} > \rho$. The geometry in (7) also gives

$$\begin{gathered} 1/6 < \ell < 1/5, \qquad \ell < l_x < l_y < 1/2, \\ 2l_x - l_y - \ell > 0, \qquad 2(l_x + l_y) - 1 - 3\ell > 0, \\ l_y/2 - 13h_+/75 - 1/50 > 7/100. \end{gathered}$$

□

### 5.3 A lower bound for the fixed comparison

In this subsection, we bound the infimum of the fixed comparison of Subsection 5.2 from below. The exact infimum is determined in Subsection 5.4.

Let $\mathcal{S}_{240}$ be precisely the preceding scalar system, allowing all 240 lowered supporting lines in its detector envelope and allowing $h$ to vary, but retaining the displayed parameter ranges, numerator estimate, four-edge requirements, floor margin, and $0.873 \leq \theta \leq 0.873347$. No other moment or signed comparison is included. Let $B(\mathcal{S}_{240})$ be its infimum.

**Theorem 5.3** (Bounds for the fixed comparison)**.**

$$0.87334669685 < B(\mathcal{S}_{240}) \leq 0.873346697. \tag{19}$$

*This interval has length* $1.5 \cdot 10^{-10}$*.*

*Proof.* The upper bound is Theorem 5.2. For the lower bound, fix $\delta_0 = 42767/10^5$, $x = 1/2$. Allow every supporting line without its restrictions on the parameters. Exact rational comparison gives the smallest supplied count $q_0$ at stage 133. Both the initial count and the old count at $\kappa = 0.746$ exceed $q_0$; the old count increases with $\kappa$ on the permitted range. Hence $R \geq q_0$ even in this optimistic relaxation. Put

$$s_0 = \frac{5}{12} + \frac{\delta_0}{8} - \frac{3q_0}{4} > 0.$$

Remove the positive additional error exponent and extension of the parameter range, and enlarge the numerator marked-length bound to $(2l_y - h)/2$. The resulting necessary endpoint inequality is

$$(3 + 4\delta_0)\theta - \frac{7}{3} - \frac{25\delta_0}{6} + hs_0 \geq 0.$$

The floor condition implies $h \leq (234\theta - 184)/25 - 3\rho$. Therefore

$$\theta \geq \frac{7/3 + 25\delta_0/6 + (184/25 + 3\rho)s_0}{3 + 4\delta_0 + (234/25)s_0} = 0.8733466968582541\ldots > 0.87334669685. \tag{20}$$

□

### 5.4 The exact algebraic infimum

In this subsection, we determine the algebraic infimum of the fixed comparison in Subsection 5.2. Section 7 treats the larger class with variable moment ranges and comparison lengths.

Let $(g_0, g_1)$ be the lowered supporting line of stage 133 in the 240-stage recursion. Define

$$H(\theta) = \frac{234\theta - 184}{25} - 3\rho - \frac{201}{100}10^{-256}, \tag{21}$$

$$J(\delta, x) = 3(5/6 - \delta) + 2\delta\{1 - x(1 + 2g_1)\}, \tag{22}$$

$$U(x) = 1 - x(1 + 6g_0 - 4g_1), \tag{23}$$

$$N(\delta, x) = (1 - \delta)J(\delta, x) + (5/6 - \delta)\delta U(x). \tag{24}$$

In the following definitions use $h = H(\theta)$ and the geometry (7). Put

$$d_\theta = H(\theta) + 10^{-10}, \qquad C_\theta = \frac{2l_y - (2 - B_P)d_\theta}{L_P},$$

and

$$Q(\theta, \delta, x) = J(\delta, x)\{\theta - 10^{-12} - E(d_\theta, C_\theta; \delta, x)\} - \frac{3d_\theta}{4}N(\delta, x). \tag{25}$$

At $x = 1/2$ this is a quadratic in $\delta$:

$$Q(\theta, \delta, 1/2) = A(\theta)\delta^2 + B(\theta)\delta + C(\theta),$$

whose coefficients are affine rational functions of $\theta$. The polynomial $\mathfrak{D}(\theta) = B(\theta)^2 - 4A(\theta)C(\theta)$ has degree two. Let $\theta_{240}$ be its unique root in

$$0.8733466968 < \theta_{240} < 0.8733466970. \tag{26}$$

The rational recursion and (25) define this algebraic number without a rounded input. Numerically,

$$\boxed{\theta_{240} = 0.873346696867948102628637768733828847 2\ldots.}$$

**Theorem 5.4** (The infimum of the fixed comparison)**.** *For the system $\mathcal{S}_{240}$ defined above,*

$$B(\mathcal{S}_{240}) = \theta_{240}.$$

*Every fixed $\theta > \theta_{240}$ sufficiently close to it supplies strict comparison inequalities, with $h = H(\theta)$. At the infimum, the only zero margin among the four comparison inequalities is a double contact at*

$$x = 1/2, \qquad \delta = \delta_{240} := -\frac{B(\theta_{240})}{2A(\theta_{240})} = 0.4276734\ldots.$$

*At $\theta = \theta_{240}$, the strict inequality (17) fails at this contact; the floor inequality (18) holds with equality.*

*Proof.* On the interval (26), $A(\theta) > 0$, the discriminant has one root by Sturm's theorem, and

$$27/64 < \delta_{240} < 57/128, \qquad (\delta_{240}, 1/2) \in \mathcal{C} := [27/64, 57/128] \times [31/64, 1/2].$$

Thus

$$Q(\theta_{240}, \delta, 1/2) = A(\theta_{240})(\delta - \delta_{240})^2.$$

Exact Bernstein bounds give, throughout $\mathcal{C}$ and the entire $\theta$ interval in (26),

$$\partial_x Q < 0, \qquad \partial_\theta Q > 0.$$

Hence the stage-133 branch is nonnegative on $\mathcal{C}$ at the endpoint, vanishing only at the displayed contact, and is strictly positive there whenever $\theta > \theta_{240}$.

Outside $\mathcal{C}$, a uniform three-variable Bernstein cover in $(\theta, \delta, x)$ proves strict positivity. It has respectively $1, 2, 2, 41$ ordinary subrectangles on the four edges, and one further subrectangle on the fourth edge treated by the contact argument. The old branch's denominators are cleared by the positive factor $(2\theta - 1)^2$. All range conditions remain the same as before. The branches of the marked-length bound and parameter geometry are valid on the whole $\theta$ interval. This proves attainability arbitrarily close above $\theta_{240}$.

For necessity, evaluate at $(\delta_{240}, 1/2)$. Exact interval polynomial comparisons show that all other 239 counts from the supporting lines, even with their restrictions on the parameters removed, exceed the stage-133 count $R_{133}$. The initial count does also. The old count at $\kappa = 0.746$ exceeds it, and its derivative for $x = 1/2$ is

$$\partial_\kappa R_{\text{old}} = \frac{\delta(5/6 - \delta)^2}{24\kappa^2 J_{\text{old}}^2} > 0.$$

Thus the full supplied count is at least $R_{133}$ for every $\theta \geq 0.873$. The affine upper bound for the numerator marked length grants at least as much as the actual minimum defining that bound. The corresponding margin has positive coefficient of $h$:

$$\frac{5}{12} + \frac{\delta_{240} B_P}{4L_P} - \frac{3}{4} R_{133} > 0.$$

The floor inequality gives $h \leq H(\theta)$. Therefore the actual top-comparison margin is no larger than $Q(\theta, \delta_{240}, 1/2)/J(\delta_{240}, 1/2)$. The latter is affine and strictly increasing in $\theta$, and is zero at $\theta_{240}$. No $\theta \leq \theta_{240}$ supplies the required strict comparison. □

## 6 The limiting moment range

In this section, we determine an algebraic upper bound for the moment recursion of Section 2. We also treat repeated use of earlier estimates and prove finite approximation on the interval needed in Section 7.

Define, for $2/3 < R \le 7/9$,

$$G(R) = \frac{11R}{16} - \frac{7}{24} - \frac{5}{16}\left(7R^4(R-2/3)\right)^{1/5}. \tag{27}$$

The fifth root is positive. We first bound the recursion from above. Subsection 6.1 constructs finite moment ranges approaching this bound on the required interval.

**Lemma 6.1** (Differential identity and convexity)**.** *We have* $G(7/9) = 0$ *and*

$$G'(R) = \frac{(45R-24)G(R) + 9R - 7}{R(45R-30)}, \qquad G''(R) > 0.$$

*Proof.* Put $H(R) = R^{4/5}(R-2/3)^{1/5}$. Then

$$\frac{H'}{H} = \frac{4}{5R} + \frac{1}{5(R-2/3)}, \qquad \frac{H''}{H} = -\frac{16}{25R^2(3R-2)^2}, \qquad G''(R) = \frac{7^{1/5}}{45R^{6/5}(R-2/3)^{9/5}} > 0.$$

Substitution gives the differential equation and $G(7/9) = 0$. □

**Proposition 6.2** (An upper bound for the moment recursion)**.** *Suppose a moment sequence starts at* $2/3 < R_0 \le 7/9$ *with* $z_0 \le G(R_0)$*. For a step* $R \mapsto r < R$*, assume*

$$w \ge R, \quad 0 < k < 1/2, \quad \frac{5}{6} - \left(\frac{5}{4} - \frac{3k}{2}\right) w - k\frac{1+R}{2} - kz \ge 0,$$

*and* $2/3 < r < R \le R_0 \le 7/9$*. If* $z \le G(R)$*, every resulting improvement no larger than* (3) *satisfies* $z_{\text{new}} \le G(r)$*.*

*Proof.* Since $5/4 - 3k/2 > 0$, replace $w$ by $R$ to obtain

$$k(R - 1/2 - z) \ge 5R/4 - 5/6 > 0.$$

Consequently $k \ge k_*(R,z) := (5R/4 - 5/6)/(R - 1/2 - z)$. The propagated improvement decreases with $k$. At $k = k_*$ it is

$$Y_*(r;R,z) = z + (r-R)\{a(R)z + b(R)\}, \quad a(R) = \frac{45R-24}{R(45R-30)}, \quad b(R) = \frac{9R-7}{R(45R-30)}.$$

The expression is increasing in $z$, since

$$(R-r)a(R) < (R-2/3)a(R) = 1 - \frac{8}{15R} < 1.$$

Lemma 6.1 and convexity therefore give

$$Y(r) \le Y_*(r;R,z) \le G(R) + (r-R)G'(R) \le G(r).$$

Positive decrements and downward rounding can only decrease the resulting improvement. □

**Proposition 6.3** (An upper bound for all earlier moment ranges)**.** *Allow arbitrary reuse of proved moment bounds, nonlinear reference boundaries, and every cutoff admitted by the short- and long-sum exponents for large values. Allow every positive finite step $0 < u < 1/6$, remove the cutoff floor, and enlarge the prime-length bound for the new estimate to $c = 1/6$. Start with all initial joint rectangles $0 < R \le 1$, $S \le C(R) := (1-R)/2$. Put*

$$M_{\mathrm{ref}} = \frac{48+\sqrt{909}}{90} = 0.86832918737069634234\ldots, \qquad \nu_{\mathrm{ref}} = \frac{1}{6M_{\mathrm{ref}}-1}, \qquad R_c = M_{\mathrm{ref}} - 1/6.$$

*There is a unique $R_+ \in (0.7286, 0.7287)$ on the increasing branch of $F_G(R) = (1+R)/2 + G(R)$ with $F_G(R_+) = M_{\mathrm{ref}}$. Every available joint marked length is at most*

$$\widehat{S}(R) = \begin{cases} C(R), & 0 \le R \le 2/3, \\ 1/6, & 2/3 \le R \le R_c, \\ M_{\mathrm{ref}} - R, & R_c \le R \le R_+, \\ C(R) + G(R), & R_+ \le R \le 7/9, \\ C(R), & 7/9 \le R \le 1. \end{cases} \tag{28}$$

*In particular the limiting moment bound is unchanged throughout the critical interval for zero detection $R \ge 0.734$. The conclusion is preserved by free-width truncations of any proved rectangle.*

*Proof.* The defining equations and rational interval arithmetic give

$$0.7016 < R_c < 0.7017, \quad 0.7286 < R_+ < 0.7287, \quad G'(0.728) > -1/4, \quad F_G(R_c) < M_{\mathrm{ref}}.$$

Thus $\widehat{S}$ is continuous and nonincreasing. Its total-length bound $F(R) = R + \widehat{S}(R)$ is nondecreasing, is constant $M_{\mathrm{ref}}$ on $[R_c, R_+]$, and has derivative at least $1/4$ above $R_+$. Convexity of $F_G$ gives $F_G(R) \le M_{\mathrm{ref}}$ on $[R_c, R_+]$.

Put $A_\nu = 5/4 - 3\nu/2$. The quadratic identity $180M_{\mathrm{ref}}^2 - 192M_{\mathrm{ref}} + 31 = 0$ gives

$$A_{\nu_{\mathrm{ref}}} R_c + \nu_{\mathrm{ref}} M_{\mathrm{ref}} = 5/6, \qquad 1 + 1/\nu_{\mathrm{ref}} = 6M_{\mathrm{ref}}. \tag{29}$$

For a stage write $r = HR$, $H = 1-6u$, let $S'$ be its proposed marked length, and put $s = S' - u$. If $S' < u$, then $S' < C(r)$ since $C(r) \ge 3u$, so the initial moment domain suffices. We may therefore assume $s \ge 0$.

Here is a general envelope argument which removes every small-step restriction. Suppose a previous total-length bound is $F(R)$ and $\nu_0 \in (0, 3/4]$ satisfies

$$A_{\nu_0} R + \nu_0 F(R) \ge 5/6. \tag{30}$$

Then every admitted stage obeys

$$r + S' \le HF(R) + u(1 + 1/\nu_0). \tag{31}$$

If $k > \nu_0$, this is immediate from propagation and the previous bound. If $k \le \nu_0$, use its class of large amplitudes and write $F' = r + S'$. Its short and long inequalities imply, since $\min\{s/H, \sigma\} \le \sigma$,

$$\nu_0 H(w+\sigma) \ge \nu_0 F' - u(1+\nu_0), \qquad A_{\nu_0} Hw + \nu_0 F' \le 5H/6 + u(1+\nu_0).$$

For $w \le R$, monotonicity of the total-length bound gives $w + \sigma \le F(R)$, proving (31) by the first inequality. For $w > R$, use the second inequality and (30) to obtain

$$\nu_0 F' \le H\{5/6 - A_{\nu_0} w\} + u(1+\nu_0) \le H\nu_0 F(R) + u(1+\nu_0).$$

Thus the same bound holds for all cutoffs. This argument also includes endpoint decrements, since they only decrease the advertised $S'$.

Induct on the number of completed stages. Initial rectangles satisfy (28). If $r \le R_c$, the enlarged mark cap proves the ceiling directly. For a previous $R \in [R_c, R_+]$, use $\nu_0 = \nu_{\mathrm{ref}}$. The identity $A_{\nu_{\mathrm{ref}}} R_c + \nu_{\mathrm{ref}} M_{\mathrm{ref}} = 5/6$ and $R \ge R_c$ prove (30), while (31) gives $r + S' \le M_{\mathrm{ref}}$. If $R \ge 7/9$, use $\nu_0 = 1/2$. Indeed

$$A_{1/2} R + F(R)/2 = 3R/4 + 1/4 \ge 5/6,$$

and the resulting bound is $H(1+R)/2 + 3u = (1+r)/2 \le r + \widehat{S}(r)$. For $R \in [R_+, 7/9]$, put

$$\nu_0 = k_*(R, G(R)) = \frac{5R/4 - 5/6}{R - 1/2 - G(R)} \in (0, 1/2].$$

The denominator is positive because $R - 1/2 - G(R) = R/2 - \widehat{S}(R) > 1/5$. The upper bound follows from $9R + 6G(R) \le 7$, since its derivative is positive on this interval and its value at $7/9$ is 7. Here (30) is an equality and the ODE gives

$$1 + 1/\nu_0 = 6\{F_G(R) - R F_G'(R)\}.$$

If $r < R_c$, the marked-length bound gives the conclusion directly. For $r \ge R_c > 2/3$, the finite-step inequality is

$$r + S' \le F_G(R) + (r - R) F_G'(R) \le F_G(r).$$

For $r \ge R_+$ this is the bound for the limiting moment range. For $R_c \le r \le R_+$, use $F_G(r) \le M_{\mathrm{ref}}$. This closes the induction for arbitrary finite step sizes. Finally $\widehat{S}$ is nonincreasing, which proves preservation under free-width truncation. □

For the simultaneous ordinary obstruction we need a convex improvement majorant rather than the exact ceiling. Define

$$\mathcal{G}(P) = \begin{cases} M_{\mathrm{ref}} - (1+P)/2, & 2/3 \le P \le R_+, \\ G(P), & R_+ \le P \le 7/9, \\ 0, & 7/9 \le P \le 1. \end{cases} \tag{32}$$

It is continuous and convex: its successive derivatives are $-1/2$, $G'(P) \ge G'(R_+) > -1/4$, and zero. Moreover $\widehat{S}(P) \le C(P) + \mathcal{G}(P)$ and $\widehat{S}(P) \le \max\{C(P), 1/6\}$. Thus the repeated use of the shorter moment ranges is included without changing the critical improvement at $r_*$.

For the descending chains used below, Proposition 6.2 applies at every finite step with $r > 2/3$. These chains start with $z_0 = 0$, while $G(311/400) > 0$. Thus $G$ is an upper bound for those chains. The upper bound for all earlier estimates above also covers arbitrary reuse and large steps, with the same $G$ on the critical interval for zero detection. The infimum of the resulting comparison is determined in Section 7, with a variable lower integration limit in place of the fixed choice of Section 5.

### 6.1 Finite approximation of the limiting moment range

In this subsection, we approximate the algebraic bound of Section 6 by finite rational moment sequences. Subsection 7.1 uses these sequences at each fixed boundary above the limiting infimum.

Here the margins in the scalar inequalities are parameters: the large-value exponent margin, supporting-line padding, endpoint decrement, and rounding mesh may be arbitrarily small positive numbers. This is a larger scalar system than $\mathcal{S}_{240}$.

**Proposition 6.4** (Finite approximation)**.** *Put $a = 367/500$. For every compact interval $I \subset (a, 7/9)$ and every $\epsilon > 0$, there is a finite rational chain with $R_0 = b < 7/9$, $z_0 = 0$, $R_n = a$, and $I \subset [a, b]$, such that*

$$|G(R) - z^{(n)}(R)| < \epsilon \qquad (R \in I),$$

*where $z^{(n)}$ is its polygonal interpolation. The first condition is $HF_{\mathrm{high}} > 0$; conditions 2–14 retain their displayed strict margins. The cutoff $k \geq 45457/125000$ is retained.*

*Proof.* Write $a_1, b_1$ for the coefficients in the differential equation of Lemma 6.1. On $I_0 = [a, 7/9]$, set

$$C_0 = 1 + \max_{I_0}(|a_1| + |a_1'| + |b_1'| + |G''|), \qquad c_0 = \min_{I_0} G'' > 0.$$

Choose rational $b \in (\max I, 7/9)$ and $q = (b-a)/n$. For $R_j = b - jq$, choose positive supporting-line separations, rounding errors and total endpoint decrements $d_j \in [q^3, 2q^3]$, and a positive large-value margin comparable to $q^2$. Choose the cutoff above its exact threshold with rounding $O(q^2)$. The incoming line then gives

$$w_j - R_j = O_{I_0}(q^3), \qquad k_j - k_*(R_j, z_j) = O_{I_0}(q^2).$$

The propagation formula therefore becomes

$$z_{j+1} = z_j - q\{a_1(R_j)z_j + b_1(R_j)\} + O_{I_0}(q^3).$$

Write $d_j$ for the total downward error, so that $z_{j+1} = Y_j(R_{j+1}) - d_j$. If $e_j = G(R_j) - z_j$, Taylor's formula gives

$$0 \leq e_{j+1} \leq (1 - qa_1(R_j))e_j + C_1 q^2, \qquad \max_j e_j \leq G(b) + C_1(b - 367/500)q.$$

Here $C_1$ is independent of $b$ and $n$. Choose

$$G(b) < \epsilon/3, \qquad C_1(b-a)q + C_0 q^2 < 2\epsilon/3.$$

Linear interpolation on each interval gives

$$\sup_{[a,b]} |G - z^{(n)}| \leq \max_j e_j + (q^2/8) \max_{I_0} |G'''| < \epsilon.$$

For completeness the supporting-line condition is stable under this construction. Put $v(R, z) = a_1(R)z + b_1(R)$. Along the graph of $G$,

$$v_R + v_z v = G'' \geq c_0.$$

In a sufficiently small neighbourhood it is at least $c_0/2$. Consequently, for sufficiently small $q$,

$$v(R_{j+1}, z_{j+1}) \leq v(R_j, z_j) - c_0 q/4.$$

Choose the rational supporting lines

$$\lambda_j(R) = Y_j(R) - d_j - 4q^3.$$

For small enough $q$, consecutive slopes differ by at least $c_0 q/8$, whereas $d_j \leq 2q^3$. Hence

$$Y_\ell(R) - d_\ell - \lambda_j(R) \geq 2q^3 \quad (R \in [R_{\ell+1}, R_\ell],\ 0 \leq j, \ell < n).$$

Indeed, at the next junction the difference is $4q^3 - d_{j+1}$; across each further interval it increases by at least $c_0q^2/8 - 2q^3 > 0$. In the opposite direction both the slope difference and the junction increment are positive. Affineness reduces each interval to its endpoints. At the next entry,

$$z_{j+1} - \lambda_j(R_{j+1}) = 4q^3, \qquad w_{j+1} - R_{j+1} = O_{I_0}(q^3).$$

Moreover,

$$\sup_{R\in[R_{j+1},R_j]} |G(R) - \lambda_j(R)| \le \max_{\ell} e_\ell + C_0q^2 + 6q^3.$$

This tends to zero after choosing $b \uparrow 7/9$ and then $q \downarrow 0$. The first incoming line is zero. Since $b < 7/9$, its cutoff is strictly below $1/2$ once $q$ is small enough.

The other conditions have uniform strict limiting margins. On $367/500 \le R \le 7/9$,

$$0 \le G(R) < 31/10^4, \quad -1/6 < G'(R) \le 0, \quad k_*(R, G(R)) > 0.3644 > 45457/125000,$$

Indeed, $G'' > 0$, $G'(7/9) = 0$, and exact endpoint bounds give $G(a) < 31/10^4$ and $G'(a) > -1/6$. Furthermore,

$$\frac{d}{dR}k_*(R, G(R)) = \frac{33R - 48G(R) - 14}{72R\{R - 1/2 - G(R)\}^2} > 0,$$

so the cutoff lower bound follows from its value at $a$. Also

$$S = (1 - R)/2 + G(R) < 0.1361 < 0.137, \quad R + S < 0.892, \quad R - 0.7 - G(R) > 0.0309.$$

The limiting expressions in conditions 2–14 have the following strict rational lower bounds:

| $j$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $L_j$ | 0.0009 | 0.0039 | 0.008 | 0.018 | 0.015 | 0.243 | 0.0309 |
| $j$ | 9 | 10 | 11 | 12 | 13 | 14 | |
| $L_j$ | 0.130 | 0.039 | 0.067 | 0.012 | 0.147 | 0.012 | |

These follow from the preceding bounds with $H = 1$, $u = 0$, $w(k) = w_+ = R$, and $s = S$. Continuity supplies all strict inequalities for sufficiently small $q$. Rational rounding preserves them. The analytic estimate permitting these variable margins is Proposition 8.9. □

## 6.2 An algebraic lower bound for the comparison

In this subsection, we derive a lower bound for the comparison from the algebraic function in (27). Section 7 extends the argument to unequal ordinary moments and variable comparison lengths.

Let $r_*$ be the unique root in

$$\frac{7523220538}{10^{10}} < r_* < \frac{7523220539}{10^{10}}$$

of

$$\left\{\frac{19(3r - 2)(9r + 2)}{3(45r - 14)}\right\}^5 = 7r^4(r - 2/3). \tag{33}$$

Define

$$\delta_* = \frac{45r_* - 14}{6 + 54r_*}, \qquad B_{\mathrm{rel}} = \frac{14 + 25\delta_*}{18 + 24\delta_*} = 0.8733458561067\ldots. \tag{34}$$

Clearing denominators in (33) and removing the factor $3r - 2$ gives a degree-nine polynomial. Sturm's theorem isolates $r_*$ in the displayed interval. The expression inside the fifth power is positive there.

**Proposition 6.5** (A lower-bound identity)**.** *Let $M, \ell \in \mathbb{R}$, set $\delta = \delta_*$, and put*

$$L_0 = \frac{5}{6} + \frac{M}{12} - \frac{\ell}{6}, \quad L_K = \frac{1}{3} + \frac{7M}{12} + \frac{\ell}{3}, \quad H = 1 + \frac{3\delta}{4} - \left(\frac{1}{2} + \frac{\delta}{4}\right) M + \frac{3\delta\ell}{4}.$$

*If a comparison requires $B \geq L_0, L_K, H$, then $B \geq B_{\mathrm{rel}}$.*

*Proof.* The positive weights

$$u = \frac{25\delta + 8}{6(4\delta + 3)}, \quad v = \frac{4 - \delta}{6(4\delta + 3)}, \quad w = \frac{1}{4\delta + 3}$$

satisfy $u + v + w = 1$ and the exact identity

$$uL_0 + vL_K + wH = \frac{14 + 25\delta}{18 + 24\delta}.$$

□

The connection with the relaxed detector is explicit. Put $F(r) = (1+r)/2 + G(r)$ and $t_* = 3r_*/2$. Equations (27)–(33) give

$$1 - \delta_* F(r_*) = \frac{1}{6} + \left(\frac{5}{6} - \delta_*\right) t_* = \frac{10 + 3\delta_*}{18} =: R_*.$$

Thus

$$\frac{5}{12} + \frac{\delta_*}{8} - \frac{3R_*}{4} = 0.$$

This is the amplitude at which the optimistic endpoint comparison loses its dependence on $h$. It produces the plane $H$ of Proposition 6.5 when the numerator retains its coefficient-two fourth-moment/Hölder estimate.

This obstruction has a specified scope: the limiting moment range of Proposition 6.2, the supplied short/long detector alternatives, and the unchanged positive numerator estimate. It does not exclude an improved long inverse estimate, a new mixed moment, a signed comparison, or a numerator estimate beyond those inequalities. Section 7 proves finite attainability and the lower bound allowing unequal ordinary moments.

# 7 The infimum of the positive comparison

In this section, we determine the infimum of the positive comparison using the moment ranges of Section 6. We prove the upper bound by finite constructions in Subsection 7.1 and the lower bound for unequal ordinary moments and variable lengths in Subsections 7.2 and 7.3.

We now allow the whole limiting moment range and all unequal two-ordinary moments, rather than retaining only the numerator estimate used in Section 6.

**Definition 7.1** (The positive comparison class)**.** Let $\mathcal{S}_{\mathrm{pos}}$ consist of the positive comparisons using the earlier direct majorant, pointwise amplitude bounds, ordinary reflection, Hölder's inequality and the long inverse exponent

$$e(r) = \max\{1, (1 + 5r)/6\}.$$

The comparison parameters satisfy

$$l_x, l_y > 0, \qquad \ell \geq 0, \qquad b = l_y - l_x \geq 0, \qquad 1/2 < a \leq 51/100.$$

Allow every finite joint moment construction satisfying the fourteen strict inequalities of Subsection 8.7, every permitted zero-detection cutoff and all unequal two-ordinary moments supplied by these constructions. Earlier moment estimates may be used in any order, with nonlinear or piecewise reference boundaries. The short- and long-sum estimates for large values and the bounds for small values are retained. Only the auxiliary positive margins, supporting-line and cutoff decrements, endpoint decrements and rounding errors may tend to zero, after fixing a strict target.

For the lower bound, enlarge the ordinary prime-length bound to $S < 1/6$ and the new reference prime-length bound to $1/6$. Also allow every finite step $0 < u < 1/6$, remove the reference cutoff floor, and omit the restrictions on the number of available prime factors and on reflection. Proposition 6.3 bounds these enlarged ranges by $\widehat{S}$ and the corresponding convex majorant $\mathcal{G}$. Allow the initial range

$$S \le \mathcal{C}(P) := (1-P)/2, \qquad 0 \le P \le 1,$$

and every improvement permitted by $\widehat{S}$ and $\mathcal{G}$, with the same bounds for independent fixed unitary twists.

**Theorem 7.2** (The comparison infimum)**.** *With $r_*$ defined by (33), the infimum for Definition 7.1 is*

$$\boxed{B_* = \frac{99r_* - 14}{12(9r_* - 1)} = 0.873345856106718576801184796065072356 77\ldots.} \tag{35}$$

*The lower bound allows arbitrary finite step sizes and repeated use of earlier estimates, reference mark cap $1/6$, removal of the cutoff floor, and all granted unequal ordinary moments. In particular $\theta_{240}$ is not the infimum of this larger comparison class. The obstruction includes positive interpolation of all unequal two-ordinary moments and arbitrary $b \ge 0$; it does not apply to a new mixed estimate using cancellation between inverse and ordinary sums.*

The number $B_*$ has algebraic degree nine. Its primitive minimal polynomial is

$$\begin{aligned}\mathcal{P}(B) = {} & 487371101862528B^9 - 4126408710053184B^8 \\ & + 15517896164111592B^7 - 34020380857037616B^6 \\ & + 47916748658170215B^5 - 44964545308854309B^4 \\ & + 28111661257098222B^3 - 11291220876239226B^2 \\ & + 2643837682856883B - 274959111914881.\end{aligned}$$

It is obtained from the degree-nine polynomial for $r_*$ by $r = (12B-14)/(108B-99)$. Exact bisection isolates the indicated root to more than 70 decimal places. Irreducibility follows modulo 11: if $f \in \mathbb{F}_{11}[X]$ is the monic reduction, then

$$X^{11^9} \equiv X \pmod{f}, \qquad \gcd(f, X^{11^3} - X) = 1.$$

These are the Rabin conditions for degree nine.

## 7.1 The limiting inequalities and finite constructions

In this subsection, we combine the finite approximation of Subsection 6.1 with the exact comparison inequalities. This proves the upper bound for the infimum in Theorem 7.2.

Set

$$\begin{aligned}\delta_* &= \frac{45r_* - 14}{6(9r_* + 1)}, \qquad h_* = \frac{2(9r_* + 2)(45r_* - 14)}{(9r_* - 1)(117r_* - 14)}, \\ g_0 &= \frac{17 - 18r_*}{3(45r_* - 14)}, \quad g_1 = \frac{135r_*^2 - 42r_* - 38}{6r_*(45r_* - 14)}.\end{aligned} \tag{36}$$

Then $g_0 - g_1 r$ is the tangent to $G$ at $r_*$, and

$$h_* = 0.81532450385864759077976559 75\ldots, \qquad \delta_* = 0.42583004744489334143211239 51\ldots.$$

Use the expressions of Theorem 5.2, with

$$\kappa = \kappa_1 = 374887/500000, \qquad B_C = B_P = 1, \qquad L_C = L_P = 2.$$

Here $\kappa$ is fixed independently of $B$ in the old count and old numerator estimate; all auxiliary losses tend to zero. Use the original initial range and also

$$[15/64, 17/32] \times [1/4, 1/2]. \tag{37}$$

All fifteen initial conditions are strictly positive on this larger range. The ideal tangent satisfies all ten improved-range conditions with initial radius $7/9$.

We verify the limiting comparison as follows. Clear positive denominators and write $Q_j$ for its numerator. At the top parameter and $x = 1/2$,

$$Q_3(\delta, 1/2) = A_*(\delta - \delta_*)^2, \qquad A_* > 0. \tag{38}$$

On the contact rectangle $[27/64, 57/128] \times [31/64, 1/2]$, $-\partial_x Q_3 > 0$. Away from this rectangle an exact rational Bernstein cover gives strict positivity on all four boundary segments:

| edge | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| ordinary subrectangles | 1 | 2 | 2 | 37 |
| contact subrectangles | 0 | 0 | 0 | 1 |

The maximum subdivision depth is 13. The coefficients are uniform on a rational interval of width less than $10^{-35}$ isolating $r_*$. Thus the limiting comparison margin is nonnegative, with its only possible zero at $(\delta_*, 1/2)$ on the top edge.

For $B > B_*$ close to $B_*$, fix $h = h_*$ and the tangent line. At the top parameter,

$$\partial_B\{J(B - E) - 3hN/4\} = J(3 + 8\delta x) > 0.$$

The denominator and range inequalities have strict margins. Compactness, the finite cover and Proposition 6.4 therefore give a finite rational chain with positive edge margins for every such fixed target. The direct inequalities are made strict by using an intermediate boundary $B' \in (B_*, B)$, as in Theorem 8.14. Put $a_- = 1467/2000$, $a = 367/500$, and $\lambda_*(R) = g_0 - g_1 R$. Exact endpoint bounds give

$$\lambda_*(a_-) < G(a) - (a - a_-)/2.$$

Since $R \mapsto (1 - R)/2 + \lambda_*(R)$ is decreasing,

$$\frac{1 - R}{2} + \lambda_*(R) < \frac{1 - a}{2} + G(a) \qquad (a_- \le R \le a).$$

Choose the finite approximation error smaller than this strict gap. Truncating the free length of its last moment range supplies this extension. On $[a, b]$, choose the supporting-line approximation with $|\lambda_j - G| < \varepsilon$. Then

$$\lambda_* - 2\varepsilon \le \lambda_j - \varepsilon \qquad (R \in [R_{j+1}, R_j]).$$

Choose rational coefficients for a line $\lambda_{\mathbb{Q}}$ with

$$\lambda_* - 3\varepsilon \le \lambda_{\mathbb{Q}} \le \lambda_* - 2\varepsilon \qquad (R \in [a_-, b]).$$

The denominators in the earlier estimates have a positive minimum on their compact ranges. All count functions are therefore uniformly continuous in the line coefficients. Choosing $\varepsilon$ after the strict comparison margin proves the finite rational assertion.

The floor imposes no limiting restriction after retuning. Its general exponent at scale $Nu \asymp Z^{v}$ is

$$E_a(v) = \frac{4}{25}l_x + a - \frac{33}{50} - al_y + (a + 4/25)v + (a - 33/50)\ell. \tag{39}$$

Let $C(B) = B - (4 + l_y - l_x)/6$ be the principal Mellin exponent. Using (7),

$$\begin{aligned} C(B) &= B - 1/2 - h/3 + \ell/2, \\ E_{51/100}(h) &= 1/100 + (103/100)\ell, \\ E_a(h) - E_{51/100}(h) &= (a - 51/100)(1 + 3\ell), \\ C(B) - E_a(h) &= B - 51/100 - h/3 - (53/100)\ell \\ &\quad + (51/100 - a)(1 + 3\ell). \end{aligned}$$

At the limiting parameters, $a = 509/1000$ gives a floor margin greater than $10^{-3}$ by rational interval bounds. Proposition 8.13 gives the corresponding integration path and the estimate for the compensated cubic product.

### 7.2 Amplitudes satisfying all unequal ordinary moments

In this subsection, we give simultaneous amplitudes satisfying the permitted unequal ordinary moments. Subsection 7.3 uses these amplitudes to prove the lower bound in Theorem 7.2.

For the necessary direction put

$$m_D = r_*/2, \quad z_* = G(r_*), \quad c_* = m_D + z_*, \quad R_* = 1 - \delta_*(1/2 + c_*).$$

Use the convex majorant for all earlier estimates $\mathcal{G}$ of (32), which agrees with $G$ throughout $[R_+, 7/9]$. For the necessary direction, allow every marked length up to $Z(P) := \widehat{S}(P)$, including the initial rectangles. Then

$$Z(P) \le \mathcal{C}(P) + \mathcal{G}(P) \quad (2/3 < P \le 1), \qquad Z(P) = \mathcal{C}(P) \quad (0 \le P \le 2/3). \tag{40}$$

For the enlarged class used in the obstruction, the granted marked cap gives the global bound

$$Z(P) \le \max\{\mathcal{C}(P), 1/6\} \le \mathcal{C}(P) + 1/6 \qquad (0 \le P \le 1). \tag{41}$$

This necessary direction grants mark cap $1/6$, removes the cutoff floor, and allows every finite step $0 < u < 1/6$. Proposition 6.3 includes arbitrary repeated use of earlier estimates and free-width truncation. It therefore grants more reference marked length than the original fourteen-condition construction.

On a set of $\lfloor U^{R_*} \rfloor \asymp U^{R_*}$ parameters, give every prime subproduct squared amplitude $U^{\delta_* z}$ and put

$$s(y) = \begin{cases} 0, & 0 \le y < 1/3, \\ \min\left\{ \dfrac{\delta_* y}{2}, \dfrac{\delta_*}{4}(y + c_* - \mathcal{G}(2y)) \right\}, & 1/3 \le y \le 1/2. \end{cases} \qquad |S_y| = U^{s(y)}. \tag{42}$$

On $[1/3, R_+/2]$, $y + \mathcal{G}(2y) = M_{\mathrm{ref}} - 1/2 < c_*$. On $[R_+/2, m_D]$,

$$\frac{d}{dy}\{y + G(2y)\} = 1 + 2G'(2y) > 0, \qquad m_D + G(2m_D) = c_*.$$

Therefore $s(y) = \delta_* y/2$ for $1/3 \le y \le m_D$. If both factors are at least $1/3$, convexity gives

$$\mathcal{G}(y_1 + y_2) \le \frac{\mathcal{G}(2y_1) + \mathcal{G}(2y_2)}{2},$$

and hence

$$R_* + 2s(y_1) + 2s(y_2) + \delta_* Z(y_1 + y_2) \le 1. \tag{43}$$

If one factor is below $1/3$, the excess is at most

$$-\delta_*\{c_*/2 + \min(y_1, y_2)/2 - 1/6\} < 0.$$

If both are below $1/3$, it is at most $-\delta_* c_* < 0$. Thus every unequal ordinary moment is respected simultaneously. Assigning the same length-dependent amplitude to every fixed unitary twist also respects all ordinary moments with two fixed unitary twists at these bounds on marked length. Pointwise ordinary bounds hold by construction. For $1/2 < y \le 1$ use the reflected amplitude at $1 - y$; for $y > 1$ assign zero amplitude to the empty-dual limiting model. For $0 \le y_i \le 1$, put $\widetilde{y}_i = \min(y_i, 1 - y_i)$. Then $\widetilde{y}_1 + \widetilde{y}_2 \le y_1 + y_2$, and

$$Z(y_1 + y_2) \le Z(\widetilde{y}_1 + \widetilde{y}_2).$$

Thus the inequalities already proved apply to every reflected pair. Set $Z(P) = 0$ for $P > 1$. The pointwise and unmarked estimates are also preserved. Apply the same extension to the small-numerator amplitude below.

The inverse amplitudes may be assigned as in the known obstruction: $|M_{r_*}|^2 = U^{\delta_* r_*}$, $|M_r|^2 = U^{\delta_* r}$ for $3r_*/2 \le r \le 3/2$, and 1 elsewhere. At the two critical splits,

$$R_* + \delta_* r_* + \delta_* Z(r_*) = 1, \qquad R_* + \delta_* \frac{3r_*}{2} = \frac{1 + 5(3r_*/2)}{6}.$$

The other short inverse inequalities are strict; the long inverse margin is $(5/6 - \delta_*)(r - 3r_*/2) \ge 0$. Put $t_0 = 3r_*/2$. For any split $r + m = t$,

$$t \le t_0 \implies r \le r_* \text{ or } m \le m_D, \qquad t \ge t_0 \implies \frac{1}{6} + (5/6 - \delta_*)t \ge R_*.$$

The short-moment endpoint equalities and the long inverse inequality therefore permit count exponent $R_*$ for every supplied cutoff. The selected operators inherited from the earlier paper have prime-amplitude thresholds above $1/2$; all positions in this model have squared amplitude slope $\delta_* < 1/2$, so those operators impose no additional saving. The same observation applies to the $\delta_0 = 37/100$ model below. These are models of inequalities, not assertions that arithmetic parameters realize the assigned amplitudes.

### 7.3 Variable comparison lengths and the numerator estimate

In this subsection, we allow the comparison lengths to vary and complete the lower bound in Theorem 7.2. The proof uses the amplitudes of Subsection 7.2 and separates the possible numerator lengths.

Since $8741/10000 > B_*$, it suffices to consider $B \le 8741/10000$. Let $l_x, l_y > 0$ and $\ell \ge 0$, and put

$$M = l_x + l_y, \quad b = l_y - l_x \ge 0, \quad h = 1 - (M - b)/2 + \ell, \quad n = 1 - l_y/h.$$

The known direct majorant requires

$$B \ge L_0, \quad B \ge L_K, \quad B \ge L_0 + G_0, \qquad G_0 = 2b/3 - M/4.$$

For $\mu = 37/100$, the known direct-plane identity is

$$h - l_y - \mu h = \frac{11(4-3\mu)}{6} - \left(\frac{22}{3} - \frac{25\mu}{4}\right) L_0 - \left(\frac{2}{3} + \frac{\mu}{4}\right) L_K - \frac{3\mu}{4}(L_0 + G_0).$$

All three subtracted coefficients are positive. If $B \le 0.8741$, the right side is at least $41671/12000000 > 0$. Also $M \le (12/7)(B - 1/3) < 1$ and $h \ge 1 - M + \ell > 0$. Hence $n > 0.37$. If $7/18 \le n < 1/2$, then $\mathcal{G}(2n) = 0$ and $c_* < 19/50 < 7/18 \le n$. Consequently (42) gives

$$R_* + s(n) = \frac{3R_* + 1 - \delta_* \mathcal{C}(2n)}{4}.$$

Thus all the additional unequal moments permit the same adverse numerator value as the earlier fourth-moment estimate. The plane $H$ in Proposition 6.5 is necessary, proving $B \ge B_*$. For $n \ge 1/2$ the estimate from the approximate functional equation uses its half-length endpoint. The same model has $R_* + 4s(1/2) = 1$ and therefore permits exactly the known unmarked endpoint value $(3R_* + 1)/4$. The unmarked endpoint obstruction gives the stronger bound. Explicitly, put $\delta = \delta_*$,

$$H_{1/2} = 1 + 5\delta/8 - (1/2 + \delta/16)M + \delta b/16 + 5\delta\ell/8, \qquad D_e = h - 2l_y \ge 0,$$

$$q_e = 19 - 21\delta/2, \quad v_e = 13/q_e, \quad u_e = (6 - 21\delta/2)/q_e, \quad w_e = u_e/3 + 5\delta v_e/8, \quad z_e = \delta v_e/16 + w_e/2.$$

The exact dual identity is

$$B - v_e = u_e(B - L_K) + v_e(B - H_{1/2}) + w_e D_e + z_e b.$$

All weights are positive since $21/50 < \delta_* < 1/2$. Therefore

$$B \ge \frac{13}{19 - (21/2)\delta_*} > 0.89.$$

It remains to exclude $0.37 < n < 7/18$, where the new numerator moments may genuinely be stronger. Set $c = 1/250$. At $\delta_0 = 37/100$ its balanced contact is rational:

$$r_0 = \frac{24257}{33000}, \quad m_0 = r_0/2 < 0.37, \quad R_0 = \frac{4471723}{6600000}.$$

Use $\lfloor U^{R_0} \rfloor$ parameters, prime squared amplitudes $U^{\delta_0 z}$, and the inverse assignment above with $(r_*, \delta_*, R_*)$ replaced by $(r_0, \delta_0, R_0)$. The endpoint identities are

$$R_0 + \delta_0\{r_0 + \mathcal{C}(r_0) + c\} = 1, \qquad R_0 + \delta_0(3r_0/2) = \{1 + 5(3r_0/2)\}/6.$$

To respect even the stronger limiting moment estimates at shorter lengths, use the convex majorant

$$\widetilde{G}_0(P) = \begin{cases} \mathcal{G}(P) + c - \mathcal{G}(r_0), & 2/3 \le P \le r_0, \\ c, & r_0 \le P \le 1. \end{cases}$$

Here $0 < \mathcal{G}(r_0) = G(r_0) < c$, and $\mathcal{G}'(r_0) < 0$, so the function is convex and majorizes the whole available improvement. Use the amplitude function

$$s_0(y) = \min\{\delta_0 y/2, \delta_0[y + m_0 + c - \widetilde{G}_0(2y)]/4\}$$

for $1/3 \le y \le 1/2$, and zero below $1/3$; extend by reflection for $1/2 < y \le 1$ as in Subsection 7.2. The same convexity calculation as in (43) proves all unequal moment bounds. For a single shorter factor, the inequality is strict because $(m_0 + c)/2 > 1/6$. The endpoint inequality $m_0 + G(r_0) > M_{\mathrm{ref}} - 1/2$ ensures $s_0(y) = \delta_0 y/2$ for $1/3 \le y \le m_0$. At the detector length $s_0(m_0) = \delta_0 m_0/2$; at every numerator length $n > 0.37 > m_0$, $s_0(n) = \delta_0(n + m_0)/4$. Thus its adverse numerator value is precisely that obtained from the marked-length bound $\mathcal{C}(2n) + c$. Its necessary numerator plane is

$$H_c = \frac{1+\delta_0}{2} - \frac{l_y}{2} + h\left(\frac{3R_0+1}{4} - \frac{1}{6}\right) + \frac{\delta_0 - 1}{2}\ell - \frac{\delta_0(2l_y - h)}{8} - \frac{h\delta_0 c}{4}.$$

Set $q = 7/18$ and define the nonnegative differences

$$A = B - L_0, \quad B' = B - L_K, \quad C = B - H_c, \quad D = qh - (h - l_y).$$

Exact coefficient comparison gives

$$B - \frac{72703411}{83204812} = \frac{57303411}{83204812}A + \frac{10501401}{83204812}B' + \frac{3850000}{20801203}C + \frac{3553203}{166409624}D.$$

All coefficients are positive. Hence this entire range forces

$$B \ge \frac{72703411}{83204812} = 0.873788537614867755 48\ldots > B_*.$$

Together with the preceding cases, this proves the lower bound in Theorem 7.2. The limiting comparison inequalities and finite approximation prove the upper bound.

Theorem 7.2 concerns the stated positive inequalities. Section 8 proves the analytic induction and the choice of the lower integration limit, yielding Theorem 8.14.

# 8 The joint moment induction over number fields

In this section, we prove the joint moment induction underlying the constructions of Sections 5–7. The argument combines Gauss-sum identities, Poisson summation, complete-factor selection and the reference ordinary moment estimate. Subsection 8.11 applies these estimates to the finite-order zero-free theorem.

Let $F \supset \mu_{12}$ be fixed and totally complex. All implied constants below may depend on $F$, the excluded places $S$, and fixed coefficient characters. Polynomial height exponents must remain finite and the prime mesh must precede the varying arithmetic data.

## 8.1 Gauss sums and ideal sums

In this subsection, we record the Gauss-sum and ideal-sum identities used in Subsections 8.2–8.5. The identities hold over the fixed number field without a restriction on its residue degrees.

**Lemma 8.1** (Fourth-power Gauss identity)**.** *Let $k$ be a finite field of order $q$, $\psi$ a nontrivial additive character, and $\chi$ a character of exact order three. Extend $\chi$ by zero and put*

$$\tau = \sum_{a \in k} \chi(a)\psi(a), \qquad J = \sum_{a \in k} \chi(a)\chi(1-a), \qquad g = \tau/\sqrt{q}.$$

*Then*

$$\tau^3 = qJ, \quad |J| = \sqrt{q}, \quad g^4 = \frac{J}{\sqrt{q}}\, g.$$

*Proof.* The change of variables $a+b=t$ and Gauss conjugation give

$$\tau(\chi)^2 = J\tau(\chi^2), \qquad \tau(\chi)\tau(\chi^2) = \chi(-1)q = q, \qquad |\tau(\chi)| = |\tau(\chi^2)| = \sqrt{q}.$$

Consequently

$$\tau^3 = qJ, \qquad |J| = \sqrt{q}, \qquad g^4 = \tau^4/q^2 = (J/\sqrt{q})g.$$

□

This is the local power identity used to select remaining prime positions. For a number field, its phase $J/\sqrt{q}$ must be retained in the actual coefficient character; it cannot be replaced by its absolute value before the transformed polynomial is identified.

**Lemma 8.2** (Ideal sums and injective marking)**.** *Let $\{\mathfrak{p}_i\}$ be distinct nonzero prime ideals outside $S$. The map*

$$U \longmapsto \prod_{i\in U} \mathfrak{p}_i$$

*is injective on finite subsets. For $X \geq 1$, its image of norm at most $X$ has $O_F(X)$ elements, and*

$$\sum_{N\mathfrak{a}\leq X} \frac{1}{N\mathfrak{a}} \ll_F 1 + \log X.$$

*Fixed-degree divisor multiplicities are $O_{F,\varepsilon}(X^\varepsilon)$.*

*Proof.* Unique factorization of nonzero ideals proves injectivity. The ideal counting bound and partial summation prove the second and third assertions. The last is the usual fixed-degree ideal-divisor bound. □

**Lemma 8.3** (Norm-scaled archimedean Poisson homogeneity)**.** *Let $n = [F:\mathbb{Q}]$, $Y > 0$, and let $f$ be a Schwartz function on $F \otimes \mathbb{R}$. For $f_Y(v) = f(Y^{-1/n}v)$, with self-dual additive measure,*

$$\widehat{f}_Y(\xi) = Y\widehat{f}(Y^{1/n}\xi).$$

*Proof.* Substitute $v = Y^{1/n}w$ in the $n$-dimensional Fourier integral. The real Jacobian is $(Y^{1/n})^n = Y$. □

The lemma explains why the leading norm-volume exponent need not grow with the degree. It does not by itself prove the required dual-parameter estimate. In higher degree, unit orbits and the full archimedean shape must be included. Replacing $N\alpha$ by one complex embedding would be invalid when the unit rank is positive.

## 8.2 Conversion of cubic Gauss coefficients

In this subsection, we convert cubic Gauss coefficients to Möbius coefficients in every residue degree. Subsection 8.4 uses the identity in the transformed inverse second moment.

Let $\mathbb{F}_q$ have odd characteristic, $q \equiv 1 \pmod 6$, and let $\chi$ have order six. Fix a nontrivial additive character and put $g_j = q^{-1/2}\sum_x \chi(x)^j\psi(x)$ for $1 \leq j \leq 5$, with every nontrivial multiplicative character zero at zero. Define

$$A = -g_2^3, \qquad C = \overline{A}\, g_2, \qquad G = \overline{\chi(4)}g_3.$$

**Lemma 8.4** (Cubic-to-Möbius conversion)**.** *For every such finite field,*

$$|A| = |C| = |G| = 1, \quad g_1 g_2 = G g_2^3, \quad -g_1 \overline{G} = \overline{C}, \quad -g_5 G = \chi(-1) C.$$

*Proof.* The Gauss norm and conjugation identities give $|g_j| = 1$ and $g_4 = \overline{g_2}$. The duplication identity is

$$g_1 g_4 = \overline{\chi(4)} g_2 g_3.$$

For clarity it follows directly from

$$J(\chi, \chi) = \sum_y \chi(y)\{1 + \chi^3(1 - 4y)\} = \overline{\chi(4)} J(\chi, \chi^3)$$

and $J(\alpha, \beta) = \sqrt{q}\, g(\alpha) g(\beta) / g(\alpha\beta)$. Thus $g_1 = G g_2^2$. Since $A = -g_2^3$, $C = -\overline{g_2}^2$, which proves the first conversion. Finally $g_5 = \chi(-1)\overline{g_1}$ proves the other orientation. No degree-one residue-field assumption occurs. □

For squarefree coprime ideal supports, CRT gives the cross factor $\chi_a(b)\chi_b(a)$, whose sixth power is one. Consequently the multiplicative extension of $A = -g_2^3$ is compatible with the Jacobi character in the known number-field normalization. After the conversion of Lemma 8.4, the only cross factor is

$$\Omega(a, b) = \chi_a(b)^3 \chi_b(a)^3.$$

The known allowed-generator reciprocity and quadratic Gauss identities express $G(a)$ and $\Omega(a, b)$ in finite linear combinations of $S$-ray-class characters. Explicitly, on a fixed finite ray group $\mathcal{R}$ one has finite expansions

$$G(a) = \sum_{\xi \in \widehat{\mathcal{R}}} c_\xi \xi(a), \qquad \Omega(a, b) = \sum_{\xi, \eta \in \widehat{\mathcal{R}}} c_{\xi,\eta} \xi(a) \eta(b).$$

The sums of absolute coefficients are constants depending on $F, S$. The cubic Gauss factor itself is never replaced by an arbitrary bounded coefficient or by a finite ray phase.

### 8.3 Poisson summation with coprimality restrictions

In this subsection, we impose the coprimality conditions before applying lattice Poisson summation. The resulting identity is used in the second-moment estimate of Subsection 8.4.

Let $L$ be a fixed fractional-ideal lattice in $F_\infty$, and let $\mathfrak{q}, \mathfrak{e}$ be integral ideals prime to its fixed presentation denominators. Write $L^*$ for the trace-dual lattice, and use the Fourier convention $\widehat{f}(y) = \int f(x) e(-\operatorname{Tr}(xy))\, dx$. For $x \in L$, put $\mathfrak{a}_x = (x)L^{-1} \subseteq \mathcal{O}_F$. Thus $\mathfrak{e} \mid \mathfrak{a}_x$ if and only if $x \in \mathfrak{e}L$. On the fixed ideal-class presentation, let $\vartheta$ be a character modulo $\mathfrak{q}$, and define

$$\mathcal{G}_{\mathfrak{e},L}(y) = \sum_{a \in \mathfrak{e}L/\mathfrak{q}\mathfrak{e}L} \vartheta(a) e(\operatorname{Tr}(ay)).$$

All sums below use the fixed ideal-class and ray charts of the known unit-orbit Poisson formula.

**Lemma 8.5** (Coprimality-restricted pair Poisson)**.** *For $f \in \mathcal{S}(F_\infty)$ and squarefree $\mathfrak{g}$ prime to $\mathfrak{q}$,*

$$\sum_{x \in L} f(x) \vartheta(x) 1_{(\mathfrak{a}_x, \mathfrak{g}) = 1} = \sum_{\mathfrak{e} \mid \mathfrak{g}} \frac{\mu_F(\mathfrak{e})}{\operatorname{covol}(\mathfrak{q}\mathfrak{e}L)} \sum_{y \in (\mathfrak{q}\mathfrak{e}L)^*} \widehat{f}(y) \mathcal{G}_{\mathfrak{e},L}(y).$$

*If $\vartheta$ is primitive, every nonzero Gauss coefficient has absolute value $\sqrt{N\mathfrak{q}}$. For two squarefree sextic ideal variables $U_1 = ga$, $U_2 = gb$, with $g = (U_1, U_2)$, so that $(a, b) = (g, ab) = 1$, this identity applies with $\vartheta = \overline{\chi_a}\chi_b$ and $\mathfrak{q} = ab$.*

*Proof.* Expand the coprimality indicator as $\sum_{\mathfrak{e}|\mathfrak{g}} \mu_F(\mathfrak{e}) 1_{x\in\mathfrak{e}L}$, split each lattice into its $\mathfrak{q}\mathfrak{e}L$ cosets, and apply the lattice Poisson formula on each coset. All series converge absolutely. The primitive Gauss norm follows from finite additive orthogonality. The shared-prime factor in $\overline{\chi_{U_1}(x)}\chi_{U_2}(x)$ is precisely the coprimality indicator at $g$; the difference supports are disjoint. Fixed charts and the finite family of $S$-unit characters affect only the constant. □

### 8.4 The second Poisson estimate after selection

In this subsection, we estimate the inverse second moment after complete-factor selection. We combine the conversion identity of Subsection 8.2 with the Poisson formula of Subsection 8.3; Subsection 8.5 then gives the required length exponents.

We apply the pair identity to the second moment. Let $X = Z^{N'}$, $T = Z^{K'}$ and $F' = N' + V$. For a squarefree ideal $n$ let $C(n)$ be the canonical normalized cubic Gauss coefficient, and let $M_I(n)$ be an injective mark on the original, possibly overlapping, prime lists, with the convention (2) and $|b_i(p)| \le 1$. The fixed completely multiplicative coefficient $\Psi$ satisfies $|\Psi(n)| \le 1$. Put

$$c_{f,m}(n) = C(n)\Psi(n)1_{(n,m)=1}\chi_n(f)^4,$$

with the same natural zero extensions as in the known canonical inverse family. For one compactly supported norm test function $W$, define

$$\mathcal{E}_{\mathrm{sel}} = Z^{-F'} \sum_f D_f \sum_{0<Nk\le T} \left| \sum_{n \text{ squarefree}} c_{f,m}(n)\chi_n(k) M_I(n) W(Nn/X) \right|^2,$$

$$\mathcal{D} = X^{-1} \sum_n |M_I(n)W(Nn/X)|^2.$$

The $k$-sum is on allowed unit orbits. Assume $Nf \le Z^V$, $0 \le D_f \le D_{\max}$, $N', V, K' \ge 0$, $N' + V \le 2$ and $K' \le 4$.

A bound from an earlier moment estimate $\mathcal{B}$ means the following uniform bound for the actual Möbius polynomial and its whole conjugate:

$$\sum_{0<Ny\le Z^{H_c}} \left| \sum_n \frac{\mu_F(n)\Psi'(n)\overline{\chi_n(y)}}{\sqrt{Nn}} 1_{(n,m')=1} M_{I\setminus J}(n)\Phi(Nn/X_c) \right|^2 \le \mathcal{B}(1 + \|\mathbf{t}\|)^{2d}.$$

Here $\Psi'$ ranges over the finite family of ray-class characters times $\Psi$; the whole conjugate conjugates this coefficient and its character orientation together. Here $J \subseteq I$ consists of removed original positions. Their extracted prime product $d_J$ satisfies $X_c N d_J \ll X$. The remaining mark $M_{I\setminus J}$ is injective; repeated primes in removed positions occur only in the scalar restoration coefficients. Common independent norm Fourier modes are included in $\mathbf{t}$. The test functions belong to a finite family chosen from $W$ before the arithmetic indices.

**Proposition 8.6** (Selected second Poisson)**.** *For every $\epsilon, \kappa_{\mathrm{P}} > 0$, there are finite derivative and height orders and a constant $C$ such that the bound from an earlier moment estimate at*

$$H_c = 2(N' + V) - K' + \kappa_{\mathrm{P}}$$

*implies*

$$\mathcal{E}_{\mathrm{sel}} \le CZ^{K'-F'+\epsilon} D_{\max}(1+\|\mathbf{t}_0\|)^A \mathcal{B} + CZ^{K'+\epsilon} D_{\max}\left(Z^{-V}\#\{f\}\right)\mathcal{D}. \tag{44}$$

*This implication is valid over every fixed $F \supset \mu_{12}$. It does not assume finite order for $\Psi$; the earlier moment bounds must still be supplied for its character family.*

*Proof.* First majorize the whole positive second moment by the nonnegative average over fibers of the norm map described below, and only then expand its square. Separate its diagonal pairs; they are estimated by the diagonal below. For every other pair put $U_1 = ga$, $U_2 = gb$; the primitive parameter modulus is $ab \neq 1$. Use the known unit-orbit Poisson formula on norm fibers [6, Proposition 8.1], equivalently the orbit average of Lemma 8.5. A fixed nonnegative Gaussian test function has integral over norm fibers $\Psi_\Phi(x) \gg 1$ for $0 < x \le 1$: keep all but one archimedean radius in fixed positive intervals and let the last radius supply the norm $x$. Hence it majorizes the whole set of ideal parameters, after fixed normalization. The logarithmic small-norm bound, rapid large-norm decay and ideal counting also give

$$\sum_{[k]\neq 0} \Psi_\Phi(Nk/T) \ll_{F,\Phi} T.$$

Thus the contribution from identical ideal variables is bounded by the same diagonal estimate. The primitive character is already zero at the additive origin, so the convention $\Phi(0) = 0$ is unnecessary on these nontrivial-character terms. Both origin terms vanish before absolute values. The resulting nonzero norm kernel is

$$K_\Phi\left(\frac{TNk}{NeNaNb}\right), \qquad K_\Phi = \Psi_{\widehat{\Phi}},$$

with the known logarithmic small-norm bounds and arbitrary large-norm decay for every fixed Euler derivative. The Poisson factor is

$$\frac{T\mu_F(e)}{Ne\sqrt{NaNb}}$$

and the dual kernel, with $e \mid g$, $g = eg'$. The canonical Gauss coefficients are converted before absolute values, using Lemma 8.4 on each exterior prime and CRT. Apart from the fixed finite family of quadratic characters, the two surviving factors are exactly

$$\mu_F(a)\Psi'(a)1_{(a,m')=1}\overline{\chi_a(y)}, \qquad \mu_F(b)\Psi''(b)1_{(b,m')=1}\overline{\chi_b(y)}, \qquad y = f^2eg'^2k.$$

The cubic Gauss coefficients have canceled; the Möbius signs have not been discarded. The factors $1/\sqrt{NaNb}$ are exactly the paired normalizations at the earlier moment estimate.

The radius computation is especially useful. On the original ideal sum annulus $N(ga), N(gb) \asymp X$,

$$\frac{TNk}{NeNaNb} \asymp \frac{T\,Ny}{(Nf)^2X^2}, \qquad y = f^2eg'^2k. \tag{45}$$

Thus outside $Ny \le Z^{2(N'+V)-K'+\kappa_{\mathrm{P}}}$ the Fourier kernel has argument at least a fixed multiple of $Z_{\mathrm{P}}^{\kappa}$.

For the marks use the literal finite identity

$$M_I(g \sqcup a) = \sum_{J\subseteq I} M_J(g)M_{I\setminus J}(a).$$

It is valid on overlapping lists because every tuple is injective. A fixed tuple divides $g$, so its contribution is included through the smaller summation scale $X_c \asymp X/Ng$. The scalar factors

depending only on $g$ are bounded by fixed divisor powers. They introduce no factor coefficient depending on the summation parameter.

The exterior coprimality is retained until it is separated exactly. After evaluating the primitive Gauss identities on the original coprime pairs, extend their finite reciprocity character expansion by its ray-character formula to all pairs and insert

$$1_{(a,b)=1} = \sum_{d|a,\ d|b} \mu_F(d), \qquad a = da', \quad b = db'.$$

No primitive Gauss identity is applied on the extended noncoprime locus. The paired central normalization is

$$\frac{1}{\sqrt{NaNb}} = \frac{1}{Nd}\frac{1}{\sqrt{Na'Nb'}}.$$

The resulting sums have scale $X_c \asymp X/(NgNd)$ and retain $1_{(a',gd)=1}$ and $1_{(b',gd)=1}$. Split the original marks between $gd$ and the exterior ideal sum before freezing a tuple. The outer character factors have modulus at most one. Their common factor in the ideal parameter $|\chi_d(y)|^2$ is its natural coprimality indicator; it may be bounded by one after forming the positive resulting second moment. The ideal parameter remains $y = f^2eg'^2k$, and the new divisor sum is harmonic:

$$\sum_{Nd \ll X/Ng} \frac{1}{Nd} \ll_F 1 + \log(2X).$$

Thus $d$ introduces no additional power from the multiplicity of the ideal parameter. Restore coprimality conditions in the resulting sum by (46). The remaining marks have their original coefficients; all factors at the extracted ideals are in the scalar coefficients $B_J(d)$. No new restriction is imposed on the quotient.

The unit-orbit average over norm fibers has already integrated the archimedean shape coordinates. Separate only the actual independent ideal-norm ratios and $K_\Phi$, with the common Sobolev–Mellin measure of [6, Section 8], before fixing $g, e, f, k, d$ or a fixed tuple. Its density is integrable after every required fixed height weight and contributes only a fixed power of $\log Z$. No extra angular or relative-radial character is introduced. Group terms by the subset of remaining marks, the finite ray characters, and dyadic summation scales before applying the moment bound. Within each such group, separation of the bounded norm ratios leaves polynomials $\mathcal{M}_{\alpha,\mathbf{t}}(y)$ independent of the divisor indices. Use

$$2|\mathcal{M}_{\alpha,\mathbf{t}}(y)\overline{\mathcal{M}_{\alpha',\mathbf{t}'}(y)}| \le |\mathcal{M}_{\alpha,\mathbf{t}}(y)|^2 + |\mathcal{M}_{\alpha',\mathbf{t}'}(y)|^2$$

within the original earlier character families. For fixed nonzero ideal parameter $y$, the ideals $f, g', e$ divide $y$ in the displayed powers; there are $O_{\epsilon,I}((Ny)^\epsilon)$ presentations. The finite unit classes and ray-class character expansions have fixed absolute mass. The $d$-sum has the harmonic bound above. Since $1/Ne \le 1$ and the restoration coefficients have the preceding small-power bound, the off-diagonal contribution is at most

$$Z^{K'-F'+\epsilon} D_{\max} \sum_\alpha \int w_\alpha(\mathbf{t}) \sum_{0<Ny\le Z^{H_c}} |\mathcal{M}_{\alpha,\mathbf{t}}(y)|^2 \, d\mathbf{t},$$

where $w_\alpha \ge 0$ and

$$\sum_\alpha \int w_\alpha(\mathbf{t})(1+\|\mathbf{t}\|)^{2d}\, d\mathbf{t} \ll Z^\epsilon (1+\|\mathbf{t}_0\|)^A.$$

Apply the stated earlier moment bound and decrease the subsidiary $\epsilon$ to obtain

$$\mathcal{E}_{\mathrm{off}} \ll Z^{K'-F'+\epsilon} D_{\max}(1+\|\mathbf{t}_0\|)^A \mathcal{B}.$$

This also explains why the number of auxiliary ideals is not counted again on this term.

The diagonal term separated at the start has total at most

$$CTZ^{-F'} \sum_f D_f \sum_n |M_I(n) W(Nn/X)|^2 \le CZ^{K'} D_{\max}(Z^{-V}\#\{f\})\mathcal{D}.$$

For the off-diagonal pairs the primitive Gauss sum at dual frequency zero is zero, so there is no further principal contribution. For the tail in (45), use the bound $(1+Z_{\mathrm{P}}^{\kappa})^{-J}$ after common separation of test functions. All untransformed polynomial, auxiliary-ideal and divisor counts are polynomial in $Z$ on the stated bounded exponent box. Choose $J > (C_0+1)/\kappa_{\mathrm{P}}$, where $C_0$ is their fixed total degree, and use Cauchy's inequality on the two original ideal sums. The tail is bounded by the same diagonal estimate, with an arbitrarily small $Z^{\epsilon}$ loss.

Every scaling above uses the field norm: the real Jacobian of multiplication by $x \in F^{\times}$ is $|N_{F/\mathbb{Q}}x|$. Ideal counting on a fixed compact unit-orbit shape is $O_F(H)$ up to powers of $\log H$. Hence increasing $[F:\mathbb{Q}]$ changes only finite derivative orders and constants, not any displayed power of $Z$. This proves (44) with the required order of quantifiers. □

**Lemma 8.7** (Reciprocal Euler restoration)**.** *Let $a(n)$ be a completely multiplicative coefficient with $|a(p)| \le 1$, including all natural zero extensions. Let $W \in C_c^{\infty}((0,\infty))$. For a squarefree ideal $g$, put*

$$M_y(X) = X^{-1/2} \sum_n \mu_F(n) a(n) W(Nn/X).$$

*Put*

$$M_y^{(g)}(X) = X^{-1/2} \sum_{(n,g)=1} \mu_F(n) a(n) W(Nn/X).$$

*Then*

$$M_y^{(g)}(X) = \sum_{\mathrm{rad}(d)|g} \frac{a(d)}{\sqrt{Nd}} M_y(X/Nd).$$

*Its absolute coefficient mass is*

$$\prod_{p|g} (1-(Np)^{-1/2})^{-1} \ll_{F,\epsilon} (Ng)^{\epsilon}.$$

*For the injective marks $M_I(n)$, put*

$$\mathcal{M}_I(X) = X^{-1/2} \sum_n \mu_F(n) a(n) M_I(n) W(Nn/X),$$

*and let $\mathcal{M}_I^{(g)}$ impose $(n,g)=1$. Write $\mathbf{T}^J = \prod_{i\in J} T_i$. In the ring $\mathbb{C}[\mathbf{T}]/(T_i^2 : i \in I)$, define*

$$B_J(d) = [\mathbf{T}^J] \prod_{p|d} \left(1 + \sum_{i\in I} b_i(p) T_i\right)^{v_p(d)}.$$

*Then*

$$\mathcal{M}_I^{(g)}(X) = \sum_{J\subseteq I} \sum_{\mathrm{rad}(d)|g} \frac{a(d) B_J(d)}{\sqrt{Nd}} \mathcal{M}_{I\setminus J}(X/Nd), \tag{46}$$

*and, for* $|b_i(p)| \le 1$,

$$\sum_{J\subseteq I}\sum_{\mathrm{rad}(d)\mid g}\frac{|a(d)B_J(d)|}{\sqrt{Nd}} \ll_{F,\epsilon,I} (Ng)^{\epsilon}.$$

*The coefficients on the remaining prime factors are unchanged.*

*Proof.* At each $p \mid g$ multiply the full inverse Euler factor $1-a(p)(Np)^{-s}$ by its geometric reciprocal; elsewhere leave it unchanged. This is a coefficient identity, including $a(p) = 0$. Compact support makes the smoothed sum locally finite. For the mass, exclude the finitely many primes for which $-\log(1-(Np)^{-1/2}) > \epsilon \log Np$ and absorb them into the constant. The remaining product is at most $(Ng)^{\epsilon}$.

For the marked identity, first work on $\mathrm{Re}\, s > 1$. The generating Dirichlet series is

$$\mathscr{D}(\mathbf{T}, s) = \prod_p \left\{1 - a(p)\left(1 + \sum_{i\in I} b_i(p)T_i\right)(Np)^{-s}\right\}.$$

Its coefficient of $\mathbf{T}^I$ is $\sum_n \mu_F(n)a(n)M_I(n)(Nn)^{-s}$. Removing the factors at $p \mid g$ gives

$$\mathscr{D}^{(g)}(\mathbf{T}, s) = \mathscr{D}(\mathbf{T}, s)\sum_{\mathrm{rad}(d)\mid g}\frac{a(d)}{(Nd)^s}\prod_{p\mid d}\left(1 + \sum_i b_i(p)T_i\right)^{v_p(d)}.$$

Coefficient extraction and smoothing prove (46). For a fixed integer $A_I$,

$$\sum_{J\subseteq I}|B_J(d)| \le \tau(d)^{A_I}, \qquad \prod_{p\mid g}\sum_{j\ge 0}(j+1)^{A_I}(Np)^{-j/2} \ll_{F,\epsilon,I} (Ng)^{\epsilon}.$$

Here $\tau(d) = \#\{e : e \mid d\}$. The last estimate follows by the same large-prime argument as above. Every individual nonzero contribution to $B_J(d)$ extracts a product $d_J \mid d$, with multiplicities included. Hence, if $z_J$ is its total marked length,

$$\log_Z(X/Nd) - (z - z_J) \le \log_Z X - z + O(\eta).$$

Thus restoration cannot increase the remaining free length. Repeated primes occur only in the scalar coefficient; the remaining mark is still injective.

The quotient in each restored Möbius polynomial is unrestricted; adding another coprimality condition would invalidate the identity. □

## 8.5 Fourth-power selection and the exact exponent identity

In this subsection, we select complete prime factors and retain their original character coefficients. The resulting exponent identity is used in the estimates of Subsections 8.6 and 8.9.

**Proposition 8.8** (Complete-factor selection)**.** *Consider one canonical inverse block over $F$, with polynomial length exponent $N$, auxiliary ideal exponent $V$, parameter exponent $M$ and remaining marked length $z$. Its free length exponent is $N - z$. Write $F_0 = N + V$. Suppose $z \ge u \ge 0$. Whole original factors can be selected with total width $u' = u + O(\eta)$ so that their removal gives*

$$N' = N - u', \qquad K' = M + 4u', \qquad z' = z - u'.$$

*Up to $Z^{O(\eta)+\epsilon}$ and one common test function measure, the original normalized second moment is bounded by the selected second moment at these new scales. Its residual free width is unchanged:*

$$N' - z' = N - z.$$

*Proof.* Select whole factors greedily, stopping just below $u$; each factor width is at most $\eta$. Fix an injective selected tuple with product $d$. CRT and allowed-generator sextic reciprocity give, on the natural coprime support,

$$C(dn) = C(d)C(n)\chi_n(d)^4\omega(d,n),$$

where $\omega$ belongs to a fixed finite family of ray-class characters. Thus the remaining Gauss parameter is evaluated at $kd^4$, not at $kd$ or $kd^2$. All terms where $(d,n) \neq 1$ already vanish. Restoring the old prime lists after removing the selected tuple is exact, since any restored term reusing a prime of $d$ is killed by $\chi_n(kd^4) = 0$.

Separate the smooth ratios $Nd/Z^{u'}$ and their compact shapes before tuple selection, with a common measure. The central Cauchy inequality is

$$\sum_k \left|\sum_d b_d X(kd^4)\right|^2 \leq \left(\sum_d |b_d|^2\right) \max_y \#\{(d,k) : kd^4 = y\} \sum_y |X(y)|^2.$$

Here $\sum_d |b_d|^2 \ll Z^{u'+\epsilon}$, and the fibre is bounded by a fixed divisor power, hence by $Z^{\epsilon}$. The norm of the new parameter is at most $Z^{M+4u'+O(\eta)}$. The normalization changes by

$$Z^{-F_0}Z^{u'} = Z^{-(N'+V)}.$$

The remaining coefficients are still the original factor coefficients, the same $\Psi$, and the canonical Gauss coefficient. No arbitrary coefficient moment has been used. □

The known first canonical reduction supplies, before small endpoint errors,

$$N - z \leq r, \qquad F_0 \leq r + S, \qquad M \leq 2F_0 - 1.$$

The overlap of original prime sets decreases both $N$ and the remaining width by the same amount; it cancels from the first inequality. Enlarging a positive sum over ideal parameters, when needed, lets us use the upper parameter cap $M = 2F_0 - 1 + O(\eta)$. The selected sum then satisfies

$$\begin{aligned} H_0 &= 2(N'+V) - K' = 1 - 6u' + O(\eta), \\ H_c &= H_0 + \kappa_{\mathrm{P}}, \\ (K' - N' - V) + (H_c + u') &= F_0 + \kappa_{\mathrm{P}}, \\ K' &\leq F_0 \quad \text{if } F_0 + 4u' + O(\eta) < 1. \end{aligned} \tag{47}$$

Thus $\mathcal{B} \ll Z^{H_c+u'+\epsilon_1}$ gives

$$Z^{K'-N'-V+\epsilon_1}\mathcal{B} \ll Z^{F_0+\kappa_{\mathrm{P}}+2\epsilon_1}.$$

For a fixed strict margin $\gamma > 0$ and a prescribed loss $\epsilon > 0$, choose

$$0 < \kappa_{\mathrm{P}}, \qquad \kappa_{\mathrm{P}} + C\eta + 2\epsilon_1 < \min\{\gamma/4, \epsilon\}.$$

The diagonal is $\ll Z^{F_0+\epsilon}$ under the last condition in (47). Apply Lemma 8.7 with the same subsidiary losses.

### 8.6 The estimate for large values

In this subsection, we prove the estimate for large values with arbitrarily small fixed positive margins. Subsection 8.7 verifies the remaining induction inequalities and sums the sixth-power factors.

Let $\lambda(w) = a_0 - a_1 w$ be a supporting line from earlier moment estimates, $D = 1/2 - a_1 > 0$, and put

$$0 < H < 1, \quad r = HR, \quad u = (1-H)/6, \quad s = H\{(1-R)/2 + z_R\} + u/k.$$

The remaining marked length is $s = S - u$ before the endpoint decrement. For $k \le \nu \le 3/4$, define

$$g(\nu) = \frac{r + s - H/2 - Ha_0 - u/\nu}{D}, \quad w(\nu) = g(\nu)/H + \omega,$$

where $\omega > 0$ is a padding. Let

$$\sigma(w) = \max\{0, (1-w)/2 + \lambda(w) - \pi\}, \quad t(w) = 3w/2 + \tau,$$

with $0 < \pi < \omega/20$ and $\tau > 0$ chosen sufficiently small. On a class of large amplitudes $(\nu, e)$, $k \le \nu \le 3/4$ and $0 \le e \le \nu$, the known detector argument, applied only to complete retained prime products, gives the two counts

$$C_s = H - \nu H w - e \min\{s, H\sigma(w)\}, \qquad C_l = H/6 + (5/6 - \nu) H t(w). \tag{48}$$

The pointwise exponent for the family is $\nu r + es$. Small sixth-power cores contribute at most

$$H/6 + 1/15 + 3r/4 + s.$$

Indeed a core of norm at most $Z^{2/25}$ has count $Z^{H/6+5(2/25)/6+o(1)}$; its sixth-power cofactor is counted once.

**Proposition 8.9** (The estimate with variable positive margins)**.** *Assume $0 \le a_1 \le 2/5$, $\lambda(R) \le z_R$, $1/6 \le k < 1/2$, and set*

$$w_b = \frac{R/2 + z_R - a_0}{D}, \qquad F_{\text{high}} = \frac{5}{6} - \left(\frac{5}{4} - \frac{3k}{2}\right) w_b - \frac{k(1+R)}{2} - k z_R.$$

*Assume the remaining strict stage conditions listed in Subsection 8.7. Suppose $HF_{\text{high}} > 0$, the unperturbed endpoints $g(k)/H$ and $g(3/4)/H$ lie strictly inside $[2/3, 79/100]$, and*

$$\frac{r}{2} - s - \frac{3}{2}(5/6 - k)\frac{u}{k^2 D} > 0, \qquad H + u - H/6 - 1/15 - 3r/4 - s > 0.$$

*Then $\omega, \pi, \tau$ may be chosen positive so that every class of large amplitudes has second moment at most $Z^{H+u+\epsilon}$, uniformly over the finite character and test-function families.*

*Proof.* Put $g = g(\nu)$. Then

$$D(g - r) = H\{z_R - \lambda(R)\} + u(1/k - 1/\nu) \ge 0,$$
$$s - (H - g)/2 - Ha_0 + a_1 g = g - r + u/\nu \ge 0.$$

Since $H\sigma(w) \ge (H-g)/2 + Ha_0 - a_1 g - H\{(1/2 + a_1)\omega + \pi\}$,

$$[s - H\sigma(w)]_+ \le g - r + u/\nu + H\{(1/2 + a_1)\omega + \pi\}.$$

Consequently, using $e \le \nu$ and $a_1 \le 2/5$,

$$\begin{aligned} C_s + \nu r + es &\le H + \nu(r-g) + e(g - r + u/\nu) \\ &\quad + H\{-\nu\omega + e[(1/2 + a_1)\omega + \pi]\} \\ &\le H + u - H\nu(\omega/10 - \pi) \\ &< H + u - Hk\omega/20. \end{aligned}$$

Moreover $w(\nu) = g/H + \omega > R$, so the short estimate uses an earlier moment estimate with the original prime coefficients.

For the long branch define

$$\mathfrak{G}(\nu) = H + u - H/6 - (5/6 - \nu)3g(\nu)/2 - \nu(r+s).$$

Then

$$\mathfrak{G}(k) = HF_{\text{high}},$$

$$\mathfrak{G}(\nu) - \mathfrak{G}(k) = (\nu - k)\left\{\frac{3g(\nu)}{2} - \frac{3}{2}(5/6 - k)\frac{u}{k\nu D} - (r+s)\right\} \ge 0.$$

The displayed derivative condition and $g(\nu) \ge r$ prove the last inequality. The positive margin $HF_{\text{high}}$ absorbs $(5/6 - \nu)H(3\omega/2 + \tau)$ uniformly. Choose $\omega, \tau > 0$ so that

$$(5/6 - k)H(3\omega/2 + \tau) < HF_{\text{high}}/2.$$

Then $C_l + \nu r + es \le H + u - HF_{\text{high}}/2$. The small-core contribution is at most $H + u - \gamma_0$, where $\gamma_0 = H + u - H/6 - 1/15 - 3r/4 - s > 0$. Thus one may take

$$0 < \gamma < \min\{u/2, Hk\omega/20, HF_{\text{high}}/2, \gamma_0\}.$$

The finitely many amplitude subdivisions, test-function integrals and selection errors are estimated after fixing $\gamma$. □

This proves that the fixed constants $3 \cdot 10^{-5}$, $10^{-6}$ and $10^{-7}$ in the earlier estimates for large values are auxiliary choices. They are not intrinsic positive lower bounds for the exponent for large values. The limiting scalar argument varies all of them together; it does not retain a fixed cutoff padding while sending the exponent margin to zero.

The contribution from small values is equally explicit. On a prime-product amplitude cell of slope at most $k$, discard complete factors of total excess width until the previous moment estimate applies. The pointwise exponent is

$$k\{s - H[(1 - R)/2 + z_R]\} = u.$$

An endpoint decrement $\Delta > 0$ improves this to $u - k\Delta$. Errors from selecting complete factors are at most the chosen mesh. Thus small-value and large-value contributions have the same $H + u$ exponent bound; the strict decrements permit a common mesh for the finite induction.

### 8.7 The induction inequalities and sixth-power factors

In this subsection, we state the sufficient induction inequalities and justify summation over sixth-power factors. Together with Subsection 8.6, these estimates prove the inverse part of Theorem 8.12.

For reference the fourteen sufficient scalar conditions are displayed here. Use $S = (1-r)/2 + Y(r)$, $s = S-u$, $D = 1/2-a_1$ and $w_+ = w(3/4)$. Every displayed margin is required to be positive.

| Condition | Margin |
|---|---|
| 1 | $HF_{\mathrm{high}} - \rho_{\mathrm{high}}$ |
| 2 | $137/1000 - S$ |
| 3 | $7/50 - S/H$ |
| 4 | $9/10 - r - S - 4u$ |
| 5 | $91/100 - r - S - 4u$ |
| 6 | $5/6 + 21/250 - r - S - 3u - 1/100$ |
| 7 | $3H - 2r - 8s - 1/5$ |
| 8 | $r/2 - s - 1/5$ |
| 9 | $r/2 - s - \frac{3}{2}(5/6 - k)u/(k^2 D) - 1/10$ |
| 10 | $H + u - H/6 - 1/15 - 3r/4 - s - 3/100$ |
| 11 | $w(k) - 2/3$ |
| 12 | $79/100 - w_+$ |
| 13 | $5/6 - \frac{3}{4}w_+ - \frac{3}{4}\max\{0, (1-w_+)/2 + \lambda(w_+)\}$ |
| 14 | $79/100 - r - (2 + 10^{-11})u$ |

Here $0 < \rho_{\mathrm{high}}$ may be arbitrarily small, and the cutoff, supporting-line and endpoint paddings are chosen after it. The parameter restrictions are

$$45457/125000 \le k < 1/2, \quad 0 \le a_1 \le 2/5, \quad \lambda(R) \le z_R, \quad u \le 10^{-4}.$$

The fixed construction and the limiting sufficient construction keep all numerical margins in conditions 2–14. Condition 13 denotes both polynomial branches of its positive part. Each margin is affine in $r$, or becomes affine after multiplication by the positive $H$; endpoint verification therefore covers a whole stage.

The role of condition 13 is not merely numerical. It bounds the sixth-power cofactors. Let $\mathcal{I}$ be the fixed cutoff interval from $w(k)$ to $w_+$, with the small cutoff paddings included. Since $w + \sigma(w)$ is nondecreasing when $a_1 \le 1/2$, condition 13 gives

$$\nu w + e\sigma(w) < 5/6 \qquad (w \in \mathcal{I},\ k \le \nu \le 3/4,\ 0 \le e \le \nu).$$

Define the excess exponent for large values by

$$\begin{aligned} A_s &= \nu P + es - \nu M w - e\min\{s, M\sigma(w)\}, \\ A_l &= -5M/6 + (5/6 - \nu)Mt(w) + \nu P + es, \\ A_0 &= -5M/6 + 1/15 + 3P/4 + s, \end{aligned}$$

$$p(P,s,M) = \max\left\{0, A_0, \sup_{\substack{k \le \nu \le 3/4 \\ 0 \le e \le \nu}} \inf_{w \in \mathcal{I}} \max\{A_s, A_l\}\right\}.$$

For $0 \le M_1 \le M_2$,

$$p(P,s,M_1) \le p(P,s,M_2) + \frac{5}{6}(M_2 - M_1). \tag{49}$$

Indeed the short excess has $M$-slopes $-\nu w$ or $-\nu w - e\sigma(w)$, both at least $-5/6$; the long excess has slope $-5/6 + (5/6 - \nu)t(w) \geq -5/6$. The small-core slope is exactly $-5/6$. Therefore each defining expression $A$ satisfies

$$A(M_1) \leq A(M_2) + (5/6)(M_2 - M_1).$$

The same inequality holds after taking the indicated infimum, supremum and maximum. Restricting the cutoff infimum to $\mathcal{I}$ is legitimate: every class still has its chosen cutoff $w(\nu)$ there.

Write a nonzero original parameter as $cB^6$, with $c$ sixth-power-free, and let $d_B = M - 6\log_Z NB$. On a fixed $B$-class the known detector and the previous moment bounds give

$$\mathcal{E}_B \ll Z^{d_B + p(P,s,d_B) + \epsilon}.$$

By (49),

$$d_B + p(P, s, d_B) \leq M + p(P, s, M) - \log_Z NB.$$

Consequently, for every $\delta > 0$,

$$\begin{aligned}\sum_B \mathcal{E}_B &\ll Z^{M + p(P,s,M) + \epsilon + \delta M} \sum_B (NB)^{-1-6\delta} \\ &\ll_{F,\delta} Z^{M + p(P,s,M) + \epsilon + \delta M}.\end{aligned}$$

Natural zeros at $B$ are restored before this summation; their divisor mass is a subpower and is absorbed by choosing the subsidiary loss smaller. Cores of norm at most $Z^{2/25}$ are estimated directly by their count $Z^{M/6 + 1/15 + o(1)}$. The finitely many irregular cores retain the original ordinary exclusion of $\Theta$ and are estimated by the same pointwise estimate for all parameters. For the other cores, their width is at least $2/25$, so the common prime mesh rescales uniformly. Condition 14 and the strict cutoff/factor bounds retain the set of original prime positions required by the known detector. This proves the bound for large values on all original parameters, rather than only sixth-power-free parameters.

### 8.8 The reference ordinary moment estimate

In this subsection, we prove the reference ordinary moment estimate used in Subsection 8.9. The unmarked Euler factors converge absolutely with exponent $51/49 > 1$; the marked factors are retained in their original normalized sums.

The earlier ordinary theorem supplies, with strict margin, the face

$$P + \frac{49}{20} S \leq \frac{99}{100}, \qquad S \leq \frac{7}{50}.$$

It is a restriction of the known coefficient-two face; this assertion is taken as known over each fixed $F$. The new reference parameters are

$$l_{\mathrm{ref}} = 7/50, \qquad \alpha = 3/5, \qquad P_{\mathrm{ref}} = 2/3.$$

Ordinary second moments retain the known exclusion of the fixed exceptional inducing-character group $\Theta$. Its exact signed mean is combined with the diagonal before positivity and is estimated separately by the exceptional gap below. For a root with $P \geq P_{\mathrm{ref}}$, $S < 137/1000$ and an ordinary factor of width $x \leq l_{\mathrm{ref}}$, reflect its partner. The resulting rectangle has widths $x$ and $1 - P + x + o(1)$, and

$$x + (1 - P + x) + \frac{49}{20} S \leq \frac{1}{3} + \frac{7}{25} + \frac{49}{20} \frac{137}{1000} < \frac{99}{100} - \frac{1}{25}. \tag{50}$$

The term estimated directly or after half-length reflection also has the strict condition

$$\frac{7}{50}+\frac{1}{2}+\frac{49}{20}\frac{137}{1000}=0.97565<\frac{99}{100}.$$

Its marked width has margin $7/50-137/1000=3/1000$. Reflection is the known reflection identity for the full character, with the exact redundant-prime expansions; no natural zero is replaced by a new restriction.

**Proposition 8.10** (Reference-moment bound)**.** *The reflected reference second moment is $\ll Z^{1+\epsilon}$, with a common finite Sobolev bound and polynomial height order, for every fixed $F\supset\mu_{12}$. The prime mesh is chosen before the factor data; constants may depend on the fixed coefficient characters and test function supports. The estimate remains valid after the forward cubic kernel and the original collision assignments.*

*Proof.* The forward cubic kernel and its coefficient normalization are those of the known ordinary identity. If an extracted ordinary divisor has width $t_2$ and fixed marked factors have width $f$, its reflected factor grows by $t_2$. Erase complete remaining marked factors of width $e$ until $\frac{49}{20}(f+e)\ge t_2$, or until all remaining marked factors are exhausted. In the exhausted case use the known arbitrary-length unmarked ordinary estimate; its erasure loss is smaller than the same weighted upper bound below. Otherwise the reflected face still satisfies (50). Removing these factors contributes at most $Z^{\kappa e}$ to the second moment, where $\kappa=3/4$. Use the weighted Hilbert norm

$$r_0=\frac{\kappa}{2(49/20)}=\frac{15}{98},\qquad \kappa e\le 2r_0\{t_2-(49/20)f\}_+ +O(\eta).$$

At Hilbert-norm level this gives the two-term bound

$$Z^{\kappa e/2}\le Z^{O(\eta)}\{1+Z^{r_0(t_2-(49/20)f)}\}.$$

The positive part cannot be replaced by its signed argument when $(49/20)f\ge t_2$. The first term is the unweighted kernel and the second is the weighted kernel. Both ordinary-only products are summable, since

$$3/2>1,\qquad 3(1/2-r_0)=51/49>1.$$

For every fixed polynomial coefficient-growth order $A$ and all sufficiently large $Np$,

$$\sum_{k\ge3}(k+1)^A(Np)^{-k(1/2-r_0)}\ll_A (Np)^{-51/49},\qquad \prod_p\{1+O_A((Np)^{-51/49})\}<\infty.$$

The finite omitted primes are absorbed into $S$. For completeness, if an exterior prime has ordinary kernel degree $k$ and $m$ marked positions, the exponent after summing its original prime set is

$$E(k,m)=1-\frac{k+m}{2}+r_0\{k-(49/20)m\}. \tag{51}$$

The compensated cubic kernel has only the following possibilities:

| term | degree restriction | upper bound for $E(k,m)$ |
|---|---|---|
| ordinary only | $m=0,\ k\ge3$ | $-2/49$ |
| one original mark | $m=1,\ k\ge1$ | $-87/392$ |
| collision | $m\ge2,\ k\ge0$ | $-3/4$ |

These follow by substituting $r_0 = 15/98$; the coefficient of $k$ is $-17/49 < 0$. The unweighted term has

$$E_0(k,m) = 1 - (k+m)/2 \le \begin{cases} -1/2, & m = 0,\ k \ge 3, \\ 0, & m = 1,\ k \ge 1, \\ 0, & m \ge 2. \end{cases}$$

Thus both terms have nonpositive marked exponents, while the weighted term has the displayed strict margins. Their ordinary-only Euler products converge absolutely. Marked primes retain their original factor measures; they are not counted as a fresh unrestricted Euler product. There is no field-degree multiple in these norm exponents.

Apply the known previous moment estimate to the reflected, shortened and erased rectangle. Use one common conductor and kernel Sobolev measure on the whole product. The absolute local kernel bounds just proved allow Tonelli's theorem on that positive measure and its divisor sums. Choose the original prime mesh below the face and mark margins in (50); this choice is independent of the later prime lists. The finitely many collision patterns, finite ray-class character expansions and common test derivatives change constants and height orders only. The forward kernel tail has arbitrary decay and is chosen after those finite orders. This proves the stated second moment bound. □

### 8.9 The ordinary moment induction

In this subsection, we prove the ordinary part of the joint moment induction. The proof combines the reference estimate of Subsection 8.8 with the inverse estimates of Subsections 8.5–8.7.

We specify the use of the known ordinary raw identities, so the new reference moment domain is not silently replaced by the initial centered estimate. On a positive side let $f$ be the fixed original marked width, $c_g \ge f$ the common-divisor width, $P_C \ge 0$ the remaining ordinary width, and $t$ the first-frequency defect. Write $S_*$ for the actual root prime length and $S$ for its target upper bound. For $P \le r - \beta$, $S_* \le S - \beta$, greedy selection gives

$$\begin{aligned} P_C + c_g - f \le r - \beta + O(\eta), \qquad & 0 \le s_C \le S_* - f - u_C, \\ u_C = \min\{u, S_* - f\} + O(\eta), \qquad & M_C = 1 + t - 6u_C, \quad t \ge -O(\eta). \end{aligned} \tag{52}$$

For clarity the complete side normalization is explicit. With root free summation exponent $A$, common widths $c, d$, common radical $q$, and first frequency $K = K_0 - t$, the known raw identities give

$$K_0 = 2A - c - d + q - 1, \quad a = A - c - u_C, \quad k_C = K + 4u_C, \quad q_C = q + c - d.$$

Thus

$$2a - k_C + q_C = 1 + t - 6u_C = M_C.$$

A final term of excess $p_C \le u_C$ has side exponent

$$a + q_C + p_C \le A - c + q_C.$$

On the other side $q_D = q + d - c$, so the paired exponents satisfy

$$(A - c + q_C) + (A - d + q_D) = 2(A + q) - c - d.$$

This is the exact cancellation with the once-counted first common-divisor normalization. The tuple Cauchy factor $Z^{u_C}$ and its normalization $Z^{-u_C}$ cancel before this calculation. Its first-common and second-common valuation inequalities are the known ones from [5]; they concern integral valuations and norm exponents and therefore are unchanged over $F$.

**Proposition 8.11** (The ordinary moment estimate)**.** *Suppose the small-value and large-value estimates for the final terms above hold at $(r, S-u, H)$, $H = 1-6u$. Suppose also that the reference moment domain of Proposition 8.10 applies and the following gaps are positive:*

$$1-r-S-4u>0, \qquad 5/6+(3/5)(7/50)-r-S-3u>0,$$
$$5/6-2/3-S-3u>0, \qquad 3H-2r-8(S-u)>0, \qquad r/2-(S-u)>0, \qquad S/H<7/50. \tag{53}$$

*Then the ordinary moment rectangle $P<r$, $S'<S$ holds after any fixed positive endpoint decrements.*

*Proof.* Use a strict known initial moment estimate $P+L_0S\le B_0$ with $B_0<1$, $L_0>2$ arbitrarily close to $1,2$. Roots below the crossing $S_0=(B_0-r+\beta)/L_0$ are inherited directly. On the other roots, apply the known first common-divisor and second-divisor identities, retaining the full signed equal-product difference through reflection. They give (52). Their positive sides retain two ordinary factors with a common character and original prime subproducts. In particular the operation has not introduced arbitrary ordinary coefficients.

If $S_*-f<u$, all remaining marked factors are selected and $s_C=0$. The known arbitrary-length unmarked ordinary estimate applies. If $S_*-f\ge u$, then $u_C=u+O(\eta)$,

$$P_C\le r-\beta+O(\eta), \qquad s_C\le S-u-\beta+O(\eta), \qquad M_C\ge H-O(\eta).$$

Let $p$ be the excess exponent defined in Subsection 8.7. For fixed $w\in\mathcal{I}$, the expressions in Subsection 8.7 satisfy

| | $\partial_P$ | $\partial_s$ | $\partial_M$ |
|---|---|---|---|
| $A_s$ | $\nu$ | $0$ or $e$ | $-\nu w$ or $-\nu w-e\sigma(w)$ |
| $A_l$ | $\nu$ | $e$ | $-5/6+(5/6-\nu)t(w)$ |
| $A_0$ | $3/4$ | $1$ | $-5/6$ |

The $M$-derivatives are nonpositive since $w>0$, $\sigma(w)\ge 0$, $\nu\ge 1/6$ and $t(w)<5/4$. Thus $p$ is nondecreasing in $P,s$ and nonincreasing in $M$; the small-value estimate has the same monotonicities. They are uniformly Lipschitz on the fixed exponent rectangle. The strict small-value and large-value margins therefore give

$$p(P_C,s_C,M_C)\le u-\gamma+L\eta\le u_C-\gamma/2$$

when $\eta$ and $\kappa_{\mathrm{P}}$ are sufficiently small relative to $\beta,\gamma$. Thus $p_C\le u_C$, as required by the side normalization.

The first gap in (53) bounds the full diagonal and the volume terms. The second is the exceptional inducing-character gap after the reference exponent improvement $\alpha l_{\mathrm{ref}}$. The third guarantees that a high root needing the reference estimate has $P\ge P_{\mathrm{ref}}$. The last three are the strict parameter inequalities for the initial and reference estimates for the final terms.

Whenever the original or comparison rectangle has a factor of width at most $l_{\mathrm{ref}}$, invoke Proposition 8.10. This is the reference estimate required beyond the corner $P+2S=1$; the initial centered interval is not being assumed nonempty there.

The known raw identities count each common-divisor family once. Their local increments are nonpositive on each Cauchy side; the auxiliary-prime errors retain their normalized average over prime ideals. Thus the final-term estimate satisfies the original exponent bound. The zero-frequency terms and the inducing-character exception are estimated directly under the stated strict inequalities. Common finite measures for separated test functions are chosen before all subsequent divisor and prime data, as in the earlier ordinary moment induction. A finite backward induction on the decreasing induction parameter for the ideal sums and their norm ranges then proves the target rectangle. The number of steps depends on the fixed strict margins, not on $Z$. □

**Theorem 8.12** (The joint moment induction). *Let $F \supset \mu_{12}$ be fixed and use the contradiction hypothesis $\beta_*(F) \geq 51/100$ of the known comparison. Let $2/3 < r < R \leq 7/9$. Given joint moment bounds at $(R, z_R)$, a supporting line supplied by earlier moment estimates and the strict stage conditions, the propagation (3) produces both resulting moment estimates at every rectangle*

$$P < r - \beta, \qquad S < \frac{1-r}{2} + Y(r) - \Delta - \beta, \qquad \beta, \Delta > 0.$$

*The large-value, cutoff and supporting-line margins may be chosen arbitrarily small positive numbers, in their required order.*

*Proof.* For the inverse half, use the known first canonical reduction, Proposition 8.8, Proposition 8.6 and Lemma 8.7. The exact exponent identity is (47). If the remaining marked length is below the requested selection, use the known initial inverse margins; reducing the remaining width only improves them. Otherwise the small-value estimate reduces to the previous moment estimate, and the large-value estimate is Proposition 8.9. Since $g(k)/H = w_b \geq R > 2/3$ and the unperturbed upper cutoff is no larger than the perturbed one, the cutoff hypotheses of Proposition 8.9 follow from the stage conditions. Choose $\kappa_{\mathrm{P}}$ and $\eta$ as in (47); the endpoint decrements absorb the remaining errors.

For the ordinary half, the same estimates for small and large values give the final term in Proposition 8.11. The strict stage conditions imply (53), while Proposition 8.10 supplies its reference estimate. Both estimates retain the original coefficient families with finite-order ray-class twists and common Mellin parameters of the test functions. Thus every application of the earlier estimate is to an earlier moment estimate; there is no circular use of the new moment estimate.

Fix the strict numerical target rectangles and desired loss first. Choose the finite recursion, endpoint margins, amplitude bins and a common prime mesh using only their numerical length margins. The field is fixed first; the mesh precedes the varying target characters and actual prime factors. These arithmetic data then determine the common measures for norm test functions and finite height and tail orders. Choose the external height cutoff after those orders. All separated modes here are ideal-norm modes.

Their initial bounds retain the fixed infinity type, as in Hu's fixed-type reflection and Lemma 9.1. The induction has finite depth, so the minimum of its finitely many numerical meshes works. This establishes both moments with polynomial height losses over each fixed field. □

## 8.10 The lower integration limit

In this subsection, we vary the lower integration limit in the Mellin comparison. Subsection 8.11 uses the choice $a = 509/1000$ together with a strict comparison inequality.

**Proposition 8.13** (The choice of the lower integration limit). *For every fixed $1/2 < a \leq 51/100$, the known fixed-cube argument holds with floor parameter $a$, and its leading exponent at scale $Nu \asymp Z^v$ is $E_a(v)$ in (39). It has the same ordering of common test functions, arithmetic data and positivity as the original argument. Constants may depend on the fixed distance $a - 1/2$.*

*Proof.* Choose the integration path buffer $e$ with

$$0 < e < \min\{10^{-3}, (a - 1/2)/100, (1 - a)/100\}.$$

The whole compensated cube is moved to the known lines

$$\Re x = a + 16e, \qquad \Re z = 17/50, \qquad \Re w = 1 - a - 6e.$$

The zero-free buffered parameter rectangle is fixed before moving the whole product. The correction estimates of [6, Lemma 13.10 and Subsection 13.5] apply for

$$1/2 \le a \le 11/12, \qquad 0 < e \le 1/1000.$$

In particular, with $z_0 = 17/50$,

$$\begin{aligned} z_0 + 1/2 - 2a - 32e &\le z_0 - 1/2 - 32e, \\ H_p - 1 &\ll (Np)^{-1-10e}, \\ \mathrm{Re}(6z) &= 51/25 > 1, \\ \mathrm{Re}(x+w) &= 1 + 10e > 1. \end{aligned}$$

The floor-bin definition makes the reciprocal holomorphic on the chosen rectangle. Thus the same ordered deformation is permitted; no extra pole is crossed when $51/100$ is replaced by $a$.

The known fixed-cube estimate, before summing the parameters, has free-parameter factor

$$(Nu)^{a-1/2+O(e)+\epsilon(N+8)}.$$

In the floor amplitude bin each complete prime block has slope at most $a - 1/2 + O(e)$. The normalized prime positions therefore contribute

$$\prod_i (Z^{\ell_i})^{-4/25+g_i+\eta} \le Z^{(a-33/50)\ell+O(e\ell)+\eta\ell}, \qquad \sum_i \ell_i = \ell.$$

The Mellin weight in the ideal parameter is $(Nu)^{-17/50}$. On a norm annulus $Nu \asymp Z^v$, ideal counting gives the leading parameter exponent

$$v\{1 + a - 1/2 - 17/50\} = v(a + 4/25).$$

The common Mellin test function supplies the scale

$$Z^{(4/25)l_x+a-33/50-al_y+O(e)}.$$

Multiplying these three displayed powers gives exactly $E_a(v)$. All remaining losses are fixed multiples of $e$, $\epsilon(N+8)$, the prime mesh and the height cutoff. Choose them after the strict floor margin, with the same uniform test function and arithmetic-data order as in the known argument. This proves the assertion throughout the stated interval, including $509/1000$. □

### 8.11 The finite-order zero-free theorem

In this subsection, we apply Theorem 8.12 and the Mellin estimate of Subsection 8.10 to finite-order characters. Section 9 extends the result to arbitrary continuous unitary characters.

**Theorem 8.14.** *For every number field $K$, every finite-order Hecke character $\eta$ and every $t \in \mathbb{R}$,*

$$L_K(s, \eta(N\cdot)^{it}) \neq 0 \qquad (\Re s > B_*), \tag{54}$$

*with its possible pole at $1 + it$ permitted. The infimum $B_*$ is optimal within Definition 7.1.*

*Proof.* First fix $F \supset \mu_{12}$. The known prior bound supplies the fixed parameter for the earlier pointwise and moment estimates

$$\kappa_1 = 374887/500000 = 0.749774 < 3/4.$$

The earlier count and numerator bounds in Subsection 7.1 use this independently known parameter. The new initial and joint-moment counts use the earlier ordinary moment with prime-length coefficient two and the joint moments constructed above.

Fix a target $B > B_*$. Choose $B_* < B' < B$ sufficiently close to $B_*$. If $\beta_*(F) < 51/100$, the conclusion is immediate. Otherwise work in the contradiction hypothesis of Theorem 8.12.

The limiting inequalities, their strictly positive transverse derivative, and Proposition 6.4 give a finite rational moment construction with strict comparison margin at $B'$. Apply Theorem 8.12 successively, using only earlier moment estimates at each step. Set $a = 509/1000$ for the floor and apply Proposition 8.13. Its limiting floor margin is at least 0.0012507528 and remains positive after the fixed perturbations. All other geometry and the earlier estimates have fixed positive margins.

For the geometry at $B'$, put $M = l_x + l_y = 1 - \ell$ and $b = l_y - l_x = 2h - 1 - 3\ell$. Then

$$L_0 = L_K = \frac{11}{12} - \frac{\ell}{4} = B', \qquad G_0 = \frac{2b}{3} - \frac{M}{4} < -\frac{13}{100}, \qquad \max\{L_0, L_K, L_0 + G_0\} = B' < B.$$

The bound on $G_0$ holds near the limiting parameters in (36). The finitely many endpoint, test function, height, integration path and tail losses can therefore be chosen to fit all the margins simultaneously. The known parametric Mellin comparison and its continuation argument apply to the whole numerator. They give nonvanishing in $\Re s > B$ for every finite-order target over $F$. As $B > B_*$ was arbitrary, this proves the half-plane $\Re s > B_*$. No mesh or positive saving uniform at the infimum is required.

For a general $K$, take the finite abelian extension $F = K(\zeta_{12})$. The abelian identity (62), with its ramified Euler factors retained, descends nonvanishing to $K$. Away from $s = 1$ every factor is holomorphic in this half-plane; at $s = 1$ classical nonvanishing handles the possible principal pole. Finally use the vertical translation (55). Optimality in the specified positive system is Theorem 7.2. □

*Remark* 8.15 (Order of analytic operations). Fix the presentation, complex variables and common test functions; form the whole numerator and its functional-equation counterpart; move the holomorphic expression; then use absolute values, common Sobolev measures and positive moments. The earlier papers provide the Mellin continuation theorem with this order of operations.

# 9 Unitary Hecke characters

In this section, we prove Theorem 1.1 for continuous unitary Hecke characters. We first extend the joint moment induction of Section 8 to the paired families of Hu and Gao. Subsection 9.4 then applies abelian base change and unitary induction to obtain the assertion for Weil representations.

We use the unitary results of Hu and Gao. Their precise paired coefficient family resolves the angular vanishing needed for the theta constant terms. No reduction of a general unitary character to a finite-order norm twist is made.

Ideal norm twists are normalized by

$$L_K(s, \eta(N\cdot)^{it}) = L_K(s - it, \eta). \tag{55}$$

Thus their zeros are translated vertically and a norm-character pole is at $1 + it$.

### 9.1 Earlier unitary estimates and their normalization

In this subsection, we recall the unitary moment estimates of Hu and Gao and fix the paired prime coefficients. Subsection 9.2 checks their compatibility with the new joint moment induction.

Hu, Proposition 3.2 and Lemma 3.3, supply fixed-type theta reflection and its gamma-strip estimates. Hu, Theorem 4.1, Proposition 4.10 and Lemma 4.4, supply the marked inverse moment, transformed inverse sum and paired-character identity. Gao, Theorem 2 and Lemmas 3–4, supply the coefficient-two ordinary mixed moment, complete extraction and exceptional estimates [2]. We use these analytic theorems in their stated character families. The compatibility with the new joint moment induction is proved below.

For a fixed unitary character $\omega_0$, enlarge $S$ to contain its ramified primes. For a fixed finite group of ray-class characters $\Theta$, containing the finite transformation characters, write

$$\lambda_u = \omega_0 \xi_u, \qquad \xi_u \text{ finite order.}$$

The ordinary prime positions have relative coefficients

$$b_i(p) = \omega_0(p)^{-1} \sum_j c_{ij} \vartheta_{ij}(p), \qquad \vartheta_{ij} \in \Theta. \tag{56}$$

Thus the actual prime character is finite order:

$$b_i(p)\lambda_u(p) = \xi_u(p) \sum_j c_{ij} \vartheta_{ij}(p).$$

The original full prime ranges, coefficients, coprimality indicators and natural zeros are retained. The fixed exponent $\kappa_1 = 374887/500000$ therefore uses only the known finite-order prime estimate. Its use requires no inductive hypothesis for an infinite-order prime character.

Gao's mixed ordinary estimate is

$$\sum_u \mathcal{W}(u) |S_{\omega_1 \xi_u, n_1} S_{\omega_2 \xi_u, n_2} Q|^2 \ll Z^{M+\varepsilon} \mathfrak{S}_J (1 + H_{\mathrm{in}})^{J'}, \qquad n_1 + n_2 + 2z \le M. \tag{57}$$

The two $\omega_j$ are arbitrary fixed unitary characters; at $z = 0$ the ordinary lengths may be arbitrary in the stated bounded ranges. The excluded inducing-character family is the finite one for $\xi_u$. For detection use $(\omega_1, \omega_2) = (\omega_0, \omega_0)$; for the numerator use $(1, 1)$. This distinction retains the actual finite numerator in Hu's Euler quotient.

### 9.2 Angular factors and character orientations

In this subsection, we verify the angular Hankel transform and the character orientations required by the joint moment induction. Subsection 9.3 applies the resulting estimates to the zero-free theorem.

At a complex place write

$$\psi_v(z) = (z/|z|)^{k_v} |z|_v^{i\tau_v}.$$

The fixed theta coefficient has angular vector $\boldsymbol{\lambda} \in \{1, -1\}^{r_F}$. Choose once between $\psi$ and $\bar{\psi}$ so that $\boldsymbol{k} - \boldsymbol{\lambda} \neq 0$. This is always possible: $\boldsymbol{k} - \boldsymbol{\lambda}$ and $-\boldsymbol{k} - \boldsymbol{\lambda}$ cannot both be zero. Finite twists preserve this choice.

**Lemma 9.1** (Angular Hankel multiplier)**.** *For the complex constituent with normalized inducing exponents* $\{1/6, -1/6\}$*, angular mode* $m \in \mathbb{Z}$ *and radial unitary parameter* $\tau \in \mathbb{R}$*, the normalized multiplier is*

$$\mathcal{K}_{m,\tau}(t) = \frac{\Gamma(1/3 + |m|/2 + t + i\tau)\Gamma(2/3 + |m|/2 + t + i\tau)}{\Gamma(1/3 + |m|/2 - t - i\tau)\Gamma(2/3 + |m|/2 - t - i\tau)}. \tag{58}$$

*It is holomorphic for* $\operatorname{Re} t > -1/3$*, satisfies the meromorphic identity*

$$\mathcal{K}_{m,\tau}(t)\mathcal{K}_{-m,-\tau}(-t) = 1,$$

*and has polynomial growth, with every fixed finite number of* $t$*-derivatives, on every closed vertical strip in that half-plane. The polynomial degree can be chosen independently of* $m, \tau$*; the bound is in* $1 + |m| + |\tau| + |\Im t|$*. For* $m = \pm 1$*,* $\tau = 0$*, this is the earlier ratio with arguments* $5/6$ *and* $7/6$*.*

*Proof.* The complex Tate factor, in the normalization of [7, (2.6)–(2.7), Section 9.2], is

$$G_m(s) = i^{|m|}(2\pi)^{1-2s}\frac{\Gamma(s + |m|/2)}{\Gamma(1 - s + |m|/2)}.$$

The two inducing characters give $G_m(1/2 + t + i\tau - 1/6)G_m(1/2 + t + i\tau + 1/6)$. Removing $(-1)^m(2\pi)^{-4(t+i\tau)}$ gives (58). Its numerator poles satisfy

$$\operatorname{Re} t = -1/3 - |m|/2 - j \quad \text{or} \quad \operatorname{Re} t = -2/3 - |m|/2 - j, \qquad j \in \mathbb{N} \cup \{0\}.$$

The reciprocal gamma factors are entire. The displayed inversion is an identity of meromorphic functions. On a fixed closed strip, Stirling's formula bounds each gamma ratio by a fixed power of $1+|m|+|\tau+\Im t|$; its exponential factors cancel. The same argument after logarithmic differentiation, or Cauchy's formula on a slightly wider strip, proves the derivative bounds. The compact region not covered by Stirling is bounded because the numerator poles have a fixed positive separation from the strip. □

The multiplier is realized by actual Weyl-related vectors. Indeed, $C_c^\infty(\mathbb{C}^\times)$ lies in the smooth Kirillov model of this infinite-dimensional generic constituent. Its Weyl action is the Hankel transform characterized by the local functional equation; see [7, (9.1)–(9.3), Theorem 9.5]. Thus one may first prescribe a compact radial test function in angular mode $m$ and then use its actual Weyl transform. Division by the Mellin transform of a single Bessel vector is unnecessary.

Applying the multiplier to the rapidly decaying Mellin transform of the prescribed test, and shifting within $\operatorname{Re} t > -1/3$, gives the earlier compact-strip and discarded-tail bounds, with polynomial height orders. For several complex places one takes products before the common integral over norm fibers. The original relative radial parameters remain fixed; the common Mellin parameters of the test functions used in the new Poisson estimate are ideal-norm modes only. The orders depend on the field and finite recursion, whereas their norm exponents are unchanged.

The inverse presentations are precisely Hu's paired class

$$\mathcal{X}_u = \{\theta\chi_u, \theta\bar{\chi}_u : \theta \in \Theta\} \cup \{\psi\theta\bar{\chi}_u, \bar{\psi}\theta\chi_u : \theta \in \Theta\}. \tag{59}$$

After conjugating the entire expression in the initial inverse transformation, the fixed character factors are

$$\xi \in \Theta \cup \psi\Theta. \tag{60}$$

Consequently the theta period character $\bar{\Lambda}\xi$ has angular vector $-\boldsymbol{\lambda}$ or $\boldsymbol{k} - \boldsymbol{\lambda}$, each nonzero at some place. The Jacquet characters at infinity are radial. Both weighted theta constant terms vanish by rotation, for the chosen vector and its actual Weyl transform. The conjugate canonical family conjugates its coefficient, theta representation and character orientation together; it does not introduce $\bar{\psi}$ as a new fixed character factor of $C$. This is the role of Hu's orientation convention.

**Proposition 9.2** (Joint moments for the paired unitary family)**.** *Let $\mathcal{F}_\omega$ and $\beta_\omega$ be as in (4)–(5), and suppose $\beta_\omega \geq 51/100$. The moment conclusion of Theorem 8.12, with arbitrarily small fixed positive margins, holds for the paired presentations (59) associated with $\psi \in \mathcal{F}_\omega$. The ordinary half uses (56). All numerical length conditions are unchanged; constants and finite seminorm orders may depend on the fixed infinity type.*

*Proof.* Write $C = \bar{\Lambda}\gamma_2$ and $\xi \in \Theta \cup \psi\Theta$. Hu's initial inverse transformation has coefficient

$$C(n)\xi(n)\chi_n(k)\chi_n(f)^4 M_I(n)W(Nn/X).$$

On the coprime squarefree support, complete-factor selection gives

$$\begin{aligned} C(dn)\xi(dn) &= C(d)\xi(d)C(n)\xi(n)\chi_n(d)^4\omega(d,n), \\ N' = N - u', \qquad K' &= M + 4u', \qquad N' - z' = N - z. \end{aligned}$$

Here $\omega$ is a finite ray-class character expansion. Thus the remaining fixed character is $\xi(n)$ and every original prime coefficient is retained.

Lemma 8.4 and Hu's paired-character identity give

$$C\xi \longmapsto \mu_F \xi G_0 \longmapsto C\xi \quad \text{up to fixed finite ray-class characters,}$$

where $G_0$ is a fixed finite ray-class character expansion. In particular,

$$\xi\bar{\chi}_y, \quad \bar{\xi}\chi_y \in \mathcal{X}_y.$$

The common-divisor and Euler-restoration factors satisfy

$$|\xi(d)|^2 = 1_{(d,\mathfrak{f}_\xi)=1} \leq 1, \qquad \sum_{Nd\leq X} \frac{1}{Nd} \ll_F 1 + \log X, \qquad \sum_{J\subseteq I} \sum_{\mathrm{rad}(d)|g} \frac{|a(d)B_J(d)|}{\sqrt{Nd}} \ll_{F,\varepsilon,I} (Ng)^\varepsilon.$$

Hence Proposition 8.6 and (46) preserve the same paired family and unrestricted restored quotient. The angular calculation preceding this proposition gives

$$\mathrm{ang}(\bar{\Lambda}\xi) \in \{-\boldsymbol{\lambda}, \boldsymbol{k} - \boldsymbol{\lambda}\} \subset \mathbb{Z}^{r_F} \setminus \{0\}, \qquad \mathrm{CT}(v) = \mathrm{CT}(wv) = 0,$$

for the chosen vector and its actual Weyl transform.

For the ordinary factors, put $\lambda_j = \omega_j\xi_u$ and define

$$\mathcal{C}^u_{\omega_1,\omega_2}(m; X, Y) = \sum_{ab=m} \frac{\lambda_1(a)\lambda_2(b)}{\sqrt{Nm}} W_1(Na/X)W_2(Nb/Y).$$

The normalized ordinary product is $\sum_m \sqrt{Nm/(XY)}\,\mathcal{C}^u_{\omega_1,\omega_2}(m; X, Y)$. The allocation $D_1 = (a, D)$, $D_2 = D/D_1$ gives Gao's exact extraction identity, with all zero extensions retained:

$$\mathcal{C}^u_{\omega_1,\omega_2}(Dn; X, Y) = \frac{\xi_u(D)}{\sqrt{ND}} \sum_{D_1D_2=D} \omega_1(D_1)\omega_2(D_2)\mathcal{C}^{u;(a,D_2)=1}_{\omega_1,\omega_2}(n; X/ND_1, Y/ND_2).$$

Expand its restriction by Möbius inversion before positivity. For equal twists the fixed factor is $\omega_0(D)$; for unequal twists retain both displayed factors. Hu's Lemma 4.4 preserves

$$b_i(p)\omega_0(p)\xi_u(p) = \xi_u(p)\sum_j c_{ij}\vartheta_{ij}(p)$$

under extraction, the two finite transforms and whole-product conjugation. Apply (57) with equal twists for detection and with $(1,1)$ for the numerator.

For exceptional parameters, absorb norm powers into the tests and use the finite-order estimate. In the remaining case $\lambda$ is not a norm character. Its Hecke function is entire, and Mellin inversion on $\operatorname{Re} s = 0$ gives

$$X^{-1/2} \sum_{(a,\mathfrak{r})=1} \lambda(a) W(Na/X) = \frac{1}{2\pi i} \int_{(0)} L_F^{(\mathfrak{r})}(s,\lambda) \widehat{W}(s) X^{s-1/2}\, ds,$$

$$\left| X^{-1/2} \sum_{(a,\mathfrak{r})=1} \lambda(a) W(Na/X) \right| \ll X^{-1/2} (N\mathfrak{r})^{\varepsilon} p_J(W) (1+|t|)^A.$$

Here $L_F^{(\mathfrak{r})}$ omits the primes dividing $\mathfrak{r}$. Gao's exceptional estimate therefore applies with the same normalized exponent.

Choose the common test measures before the arithmetic indices. The parameter $y = f^2 e g'^2 k$, (47), and the reference exponent $51/49 > 1$ are unchanged. Propositions 8.9 and 8.11 now give the two estimates by the finite induction of Theorem 8.12. □

### 9.3 The improved half-plane

In this subsection, we prove the unitary zero-free theorem by combining Subsections 9.1 and 9.2 with the finite comparisons of Section 7.

**Theorem 9.3** (Unitary Hecke characters)**.** *For every number field $K$ and every continuous unitary Hecke character $\omega$,*

$$\boxed{L_K(s,\omega) \neq 0 \qquad (\operatorname{Re} s > B_*).} \tag{61}$$

*The only possible pole in this half-plane is the simple pole at $1+it$ when $\omega = (N\cdot)^{it}$.*

*Proof.* Work first over $F \supset \mu_{12}$ and fix a unitary $\omega$. Use the family (4). Put $\beta = \beta_\omega$. Gao's known result gives $\beta \le B_2$. For each fixed target orient its canonical class as in (60). This changes no zero real part. The finite numerator and the paired finite prime characters are in the finite-order subfamily. Their fixed exponent is $\kappa_1$.

Suppose $\beta > B_* > 51/100$. Choose $B_* < B' < B < \beta$, with $B'$ sufficiently close to $B_*$. The inequalities of Subsection 7.1 and Proposition 6.4 give a finite rational moment construction with strict margins at $B'$. Apply Proposition 9.2 to every stage. Use Gao's mixed moment with equal unitary twists for detection and its finite instance for the numerator. Hu's actual Euler quotient is

$$\mathscr{F}_{u,\psi}(s,w,z) = \zeta_F^S(6z) \frac{L_F^S(w,\chi_u)}{L_F^S(s,\psi\bar{\chi}_u)} H_{u,\psi}(s,w,z).$$

Thus the new scalar counts and numerator estimates apply to the actual characters, including all local correction terms. The fixed floor $a = 509/1000$ is supplied by Proposition 8.13; its proof is phase uniform and uses the same whole compensated product. The limiting floor margin is greater than 0.00125075. Direct, reference, tail and principal-factor margins are strict after replacing $B'$ by $B$. Hu's Lemma 6.1 gives the nonzero principal factor with unitary target phases; Gao's Section 3 verifies compatibility with this comparison.

Fix the numerical margins, finite depths and original prime partition before the target and its finite twists. The later arithmetic data determine finite seminorm orders and height degrees. For

an error $Z^{C(\beta)-2\eta}(1+H)^{A_\psi} + Z^{D_\psi}H^{-N}$, choose

$$H = Z^{t_\psi}, \quad 0 < t_\psi < \frac{\eta}{2(1+A_\psi)}, \quad N > \max\left\{0, \frac{D_\psi - C(\beta) + \eta}{t_\psi}\right\}.$$

Then both error terms are $\ll_\psi Z^{C(\beta)-\eta}$. The numerical saving $\eta > 0$ is common to the family, whereas constants and sufficiently large thresholds may depend on the fixed target. Hu's Mellin identity for the whole principal term therefore continues a nowhere-vanishing holomorphic numerator divided by $L_F^S(s,\psi)$ to

$$\operatorname{Re} s > \max\{B, \beta - \eta\} < \beta$$

for every $\psi \in \mathcal{F}_\omega$. More precisely, the continued function is

$$e^{(s-5/6)^2}\frac{H_\psi^S(s)}{L_F^S(s,\psi)}, \qquad H_\psi^S(s) \neq 0,$$

by Hu's principal-factor estimate. Thus a zero of the denominator would give a pole of a holomorphic function. Omitted finite Euler zeros lie on $\operatorname{Re} s = 0$; norm poles are zeros of the reciprocal. This contradicts the definition of $\beta$. Hence $\beta \leq B_*$. For general $K$ descend from $K(\zeta_{12})$ by the next theorem. □

### 9.4 Abelian base change and Weil representations

In this subsection, we apply abelian base change to descend the unitary bound of Subsection 9.3. We then use unitary induction and Frobenius reciprocity to deduce the result for finite-dimensional unitary Weil representations.

**Theorem 9.4** (Unitary descent)**.** *Let $F/K$ be finite abelian and $0 < B < 1$. If every unitary Hecke L-function over $F$ is nonzero for $\operatorname{Re} s > B$, apart from its possible norm-character pole, the same statement holds over $K$.*

*Proof.* For a unitary character $\chi$ of $K$, put $\chi_F = \chi \circ N_{F/K}$. Artin formalism for the induced one-dimensional representation gives

$$L_F(s,\chi_F) = \prod_{\omega \in \widehat{\mathrm{Gal}(F/K)}} L_K(s,\chi\omega). \tag{62}$$

This identity includes the ramified local factors. If $\chi$ is not a finite-order character times a norm power, none of the factors on the right is a norm character; all are entire. Hence a zero of $L_K(s,\chi)$ would give a zero of the left-hand side. Moreover,

$$\chi \circ N_{F/K} = (N_F\cdot)^{it} \implies \chi(N_K\cdot)^{-it}|_{N_{F/K}C_F} = 1 \implies \chi = \eta(N_K\cdot)^{it}, \quad \eta \text{ finite order},$$

since $N_{F/K}C_F$ has finite index in $C_K$. Thus the left-hand side has no pole that could cancel the zero.

If $\chi = \eta(N\cdot)^{it}$ with $\eta$ finite order, use (55) and the finite-order version of (62). For $B < \operatorname{Re} s < 1$ no factor has a pole. On $\operatorname{Re} s = 1$, use the classical nonvanishing theorem for Hecke $L$-functions, with the principal pole permitted. For $\operatorname{Re} s > 1$, absolute convergence of the Euler product gives nonvanishing. □

In particular $F = K(\zeta_{12})$ is sufficient for descent. This reduces the field-generality issue to the analytic argument over totally complex fields containing $\mu_{12}$; it does not eliminate the infinite-order part of a target character.

**Corollary 9.5** (Unitary Weil representations)**.** *For every finite-dimensional continuous unitary representation $r$ of the global Weil group $W_K$, $L_K(s,r)$ has no zeros in $\operatorname{Re} s > B_*$. Its pole order at $1+it$ equals $\dim \operatorname{Hom}_{W_K}((N\cdot)^{it}, r)$.*

*Proof.* Use the unitary induction and full local-factor identities of Hu, Sections 9.1–9.2, to obtain an identity of virtual representations

$$[r] = \sum_j n_j [\operatorname{Ind}_{W_{E_j}}^{W_K} \omega_j], \qquad n_j \in \mathbb{Z}, \quad \omega_j \text{ unitary},$$

and hence

$$L_K(s,r) = \prod_j L_{E_j}(s,\omega_j)^{n_j}.$$

Theorem 9.3 makes every factor nonzero away from its norm pole. Since $(N_K\cdot)^{it}|_{W_{E_j}} = (N_{E_j}\cdot)^{it}$, Frobenius reciprocity gives

$$\sum_j n_j \dim \operatorname{Hom}_{W_{E_j}}((N_{E_j}\cdot)^{it}, \omega_j) = \dim \operatorname{Hom}_{W_K}((N_K\cdot)^{it}, r) \geq 0.$$

Equivalently,

$$\operatorname{ord}_{s=1+it} L_K(s,r) = -\dim \operatorname{Hom}_{W_K}((N_K\cdot)^{it}, r) \leq 0.$$

Hence no zero occurs at a norm-character pole either. □

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