hexagon:2611.00415v1

EVERY EVEN INTEGER GREATER THAN 2 IS A PRIME PLUS A PRODUCT OF AT MOST 18 PRIMES

WREN AQUILA

Abstract. We prove that every even integer $N \geq 4$ can be written as $N = p + \eta$ with p prime and $\eta \geq 2$ a product of at most 18 primes, counted with multiplicity. The best previously stated unconditional value we have located is due to Johnston and Starichkova: 395 in the latest version of their paper, which revised the 369 of its first two arXiv versions. Assuming that no Dirichlet L-function vanishes in the half-plane $\operatorname{Re} s > 7/8$, the same holds with at most 14 primes, and under the generalised Riemann hypothesis for Dirichlet L-functions with at most 9. The improvement comes from counting rather than bounding: an explicit lower bound for the number of primes p with $N - p$ free of small primes is played against explicit Hardy–Ramanujan-type upper bounds for the integers with many prime factors, both in the range below the sieve threshold and inside the sieved sets. These upper bounds weight by Chebyshev's identity $\log n = \sum_{d|n} \Lambda(d)$, use the exact power-series coefficients of a majorant of a finite Euler product and the Mertens density of rough primes, and pre-sieve modulo 1155, so that the integers to be excluded inherit the residue classes of the primes that produce them. All numerical steps are certified in interval or ball arithmetic, except some discrete convolutions with nonnegative terms, which are evaluated in binary64 under an explicit rounding-error bound and were also recomputed independently in ball arithmetic.

1. Introduction

In 1948 Rényi [35] proved that there is a constant K such that every even integer $N \geq 4$ is the sum of a prime and a number with at most K prime factors. For large N Chen [7] showed that one may take $K = 2$, but Chen's and Rényi's arguments are ineffective. An explicit version of Chen's theorem, valid for $N > \exp(\exp(32.7))$, was obtained by Bordignon, Johnston and Starichkova [4], and Johnston and Starichkova [23, Theorem 1.3] then stated that every even $N \geq 4$ is $p + \eta$ with $\Omega(\eta) \leq 395$, where Ω counts prime factors with multiplicity; arXiv versions v1 and v2 of [23] had stated 369, the value quoted in [19]. Under the generalised Riemann hypothesis they obtained 31, and Johnston's thesis [22] improved this to 16.

Theorem 1. *Every even integer $N \geq 4$ can be written as $N = p + \eta$, where p is prime and $\eta \geq 2$ satisfies $\Omega(\eta) \leq 18$.*

Theorem 2. *Assume that no Dirichlet L-function has a zero in $\operatorname{Re} s > 7/8$. Then every even integer $N \geq 4$ can be written as $N = p + \eta$ with p prime, $\eta \geq 2$ and $\Omega(\eta) \leq 14$.*

Theorem 3. *Assume the generalised Riemann hypothesis for Dirichlet L-functions: every nontrivial zero of every Dirichlet L-function, $\zeta(s)$ included, has real part $1/2$. Then every even $N \geq 4$ is $p + \eta$ with p prime, $\eta \geq 2$ and $\Omega(\eta) \leq 9$.*

Where the old bound came from. In arXiv v3 of [23], the proof of Theorem 1.3 combines a large-range sieve producing at most 39 prime factors with the small-range result of Hathi and Johnston [19] ($N = p + \eta$ with η squarefree and coprime to $2 \cdot 3 \cdots 13$). At its stated threshold $X_2 = \exp(e^{7.9})$, the latter argument gives 395 by counting consecutive primes starting at 17. Here we apply [23, Theorem 4.2] only for $N \geq X_2' = \exp(2736.86)$, where Appendix B verifies all its hypotheses. Our counting bounds improve both ranges.

Date: 10 November 2026.

The method. For N below the sieve threshold we count primes $p \le N-2$ with $N-p$ coprime to 1155, using a single modulus of the explicit prime number theorem for progressions of Bennett, Martin, O'Bryant and Rechnitzer [2]. Against this we bound the number of integers $n \le N$ coprime to 2310 with $\Omega(n) > K$, by a Hardy–Ramanujan-type count [17]: Chebyshev's identity $\log n = \sum_{d\mid n} \Lambda(d) = \sum_{p^a \| n} a \log p$, combined with a bound for θ as in Hardy and Wright [18, §22.18], followed by the exact coefficients of $\exp(zS'(X) + C_{11}(z))$, which majorises the finite Euler product $\prod_{13\le p\le N}(1-z/p)^{-1}$ coefficientwise for $N \le X$, in place of Rankin's trick [34] (Lemmas 4 and 6). For $N \ge e^{1800}$ we apply the weighting to the squarefree part of n only. An integer $n = N-P$ that is to be excluded inherits the residue class of P modulo 1155, so in $n = pn'$ the prime p lies in as many reduced classes as the primes we count, and the class factor cancels (Lemma 8).

Above the threshold we count inside the set sifted by Johnston and Starichkova: their lower bound for the number of p with $N-p$ free of primes below $N^{1/40}$, with the coefficient re-evaluated in Appendix B, is compared with the number of $N^{1/40}$-rough integers with many prime factors. The same weighting reduces this to reciprocal sums over rough cofactors. In logarithmic units these are bounded by products of the measure $\varepsilon\,\delta_1 + \mathbf{1}_{[1,\infty)}(u)\,du/u$, which keeps the Mertens density of the rough primes (Lemmas 11 and 12), and the pre-sieve modulo 1155 lowers the count again (Lemma 13). Theorem 2 replaces the sieve above the threshold by the explicit linear sieve of Bordignon, Johnston and Starichkova [4, Theorem 6], in the form of [22, Theorem 2.1], with the remainder controlled by an explicit form of the prime number theorem for characters under the 7/8 hypothesis, and uses the same counting inside the sieved set. Theorem 3 pre-sieves to larger moduli below e^{73}, using a prime number theorem for progressions under GRH [5], counts inside Johnston's GRH sieve [22, Proposition 4.18] up to $\exp(7.7\cdot 10^6)$, and uses the explicit Chen theorem under GRH of Bordignon and Starichkova [5] beyond.

Credits and prior art. None of the devices is new. Johnston's thesis [22, §6.2, after Question 6.2] already suggests that, for $N \le \exp(73)$ under GRH, the few $\eta = N-p$ with exactly 16 prime factors could be "carefully excluded" from the representation function to reach 15; we implement this idea systematically and unconditionally.

The good-count template, a log-weighted representation function fed by [2], is that of Hathi and Johnston [19]; its restriction $(N-p, 1155) = 1$ is their coprimality condition $(n-p, k) = 1$ with $k = 1155$, a pre-sieve. Recent explicit work on sums of two almost primes pre-sieves by 3 (an arXiv preprint [6]) and by 3 and 5 (a deposit, not peer reviewed [9]); Fan's explicit estimates for rough numbers pre-sieve by 2, 3 and 5 [14, §2]. Lemmas 8 and 13 pre-sieve the count of the integers to be excluded as well; the analogy is with Chen's switching, but the step here is elementary.

The weighting $\log n = \sum_{d\mid n}\Lambda(d)$ is Chebyshev's identity. Landau's count of integers by their number of prime factors [24, vol. I, §56] brings in θ through a lemma on sums $\sum_{p\le x} F(p,x)$ and weights each product by its number of prime factors: it counts the solutions of $pQ \le x$, with Q squarefree with a fixed number of prime factors, in two ways [24, vol. I, p. 208]. The logarithmic weight, with θ as the first case of an induction on the number of prime factors, is the device of the proof of Landau's theorem in Hardy and Wright [18, §22.18], which their notes attribute to Wright (1954); the lemmas of Hardy and Ramanujan [17] do not use it: they count each integer with $k \ge 2$ distinct prime factors $k-1$ times, once for each prime factor other than the largest [17, (2.23), (3.14)], and similarly with multiplicity [17, (4.23)]. Pollack [33, eq. (3)] states a Hardy–Ramanujan inequality for integers composed entirely of primes from a given set, rough integers included; it counts ω, and its constants are absolute but not numerical. The power-series identity behind Lemma 4, its recurrence, and the coefficientwise bound for the exponential of a nonnegative series are in Mathar [27] and Lichtman [25, Proposition 3.1, (3.3) and Lemma 5.4]; the closest explicit use in sieve work is Fan's evaluation of elementary symmetric sums by Newton's identities [14, §2]. The one-dimensional step of Lemma 11 is the explicit partial-summation lemma quoted from Greaves [16] in [4, Lemma 20] and used in [4, Lemma 52]; the split at $N^{1-\delta}$ is the usual

Chebyshev split. Splitting off the largest prime factor and counting rough cofactors with explicit integrals appears in the explicit Chen theorem of [4, Lemma 52] (after Nathanson [29]); explicit Hardy–Ramanujan inequalities that split off the largest prime are in Bayless, Kinlaw and Klyve [1]; Ford [15] combines the largest-prime split with Rankin's trick for sifted sets; explicit counts of rough products with a fixed number of prime factors are in Loebenberger and Nüsken [26]. Fan [13, Theorem 1.1] gives a weighted Hardy–Ramanujan inequality for sifted sets, including factor counts with multiplicity; its constants are not supplied numerically. The idea of counting the distribution of ω over shifted primes goes back to Erdős [11]. What is new here is the specialised explicit form of these bounds and the numerical results. Platt and Trudgian [32, §5] note that their results on zeros of quadratic Dirichlet L-functions would improve explicit results of this type, including those of [23].

Related large-range results. Recent public drafts give Chen-type results for very large N: under the hypothesis of Theorem 2, every even $N \geq \exp(51046)$ is $p + P_2$, and under GRH every even $N \geq \exp(24274)$ [20]; under GRH and published inputs of Bordignon and Starichkova [5], every even $N \geq \exp(\exp(13.375))$ [21]. They do not bridge to the range of numerical verification, and our proofs do not use them. Separately, a recent manuscript [31] claims that no Dirichlet L-function vanishes in $\operatorname{Re} s > 7/8$. We have not verified that claim and do not use it: Theorem 2 is stated under this hypothesis. If the claim holds, the conclusion of Theorem 2 holds unconditionally.

Certification. Every inequality whose truth depends on a numerical evaluation was checked in interval or ball arithmetic. The certificates of record make one exception: the discrete convolutions that bound the integrals J_m of Lemma 11, whose terms are nonnegative, are evaluated in binary64 under an explicit bound on the rounding error; an independent audit recomputed them in ball arithmetic. §8 says which implementation certifies which step. The scripts, their parameter files and outputs form the supplement.

2. Notation and inputs

We use p for primes and q for a modulus, except where q is explicitly a prime divisor. All unqualified logarithms are natural, and $\log_2 x = (\log x)/(\log 2)$. Further, $\Omega(n)$ and $\omega(n)$ denote the number of prime factors of n with and without multiplicity, $P^+(n)$ and $P^-(n)$ the largest and smallest prime factor of $n > 1$, Λ is von Mangoldt's function, and $L = \log N$. We write $\theta(x) = \sum_{p \leq x} \log p$, $\theta(x; q, a)$ for the same sum restricted to $p \equiv a \pmod q$, $\pi(x; q, a)$ for the number of primes $p \leq x$ with $p \equiv a \pmod q$, $\operatorname{li}(x) = \int_0^x dt/\log t$ (principal value), $B = 0.2614972128\ldots$ for Mertens' constant and $c_\theta := 1 + 1/36260$. For even N let

$$\mathcal{A} = \{N - p : p \leq N,\ p \nmid N\},$$

$$U_N = 2e^{-\gamma} \prod_{p>2} \Big(1 - \frac{1}{(p-1)^2}\Big) \prod_{\substack{p>2 \\ p \mid N}} \frac{p-1}{p-2} \ \geq\ U_{\mathrm{low}} := 2e^{-\gamma} \cdot 0.6601618158.$$

For $z > 1$ let $S(\mathcal{A}, z)$ be the number of elements of $\mathcal{A}$ with no prime factor below z. A prime dividing N cannot divide $N - p$ when $p \nmid N$, so these elements are z-rough. We use the following published inputs and the derived bound (I4′).

(I1) *Goldbach's conjecture holds for all even* $4 \leq N \leq 4 \cdot 10^{18}$ [30].

(I2) [2, Theorem 1.2 with (1.10)–(1.11)]: for $3 \leq q \leq 10^4$, $(a, q) = 1$ and $x \geq 8 \cdot 10^9$, $\big|\theta(x; q, a) - x/\varphi(q)\big| < x/(840 \log x)$.

(I3) Explicit Mertens and Chebyshev bounds of Dusart [10, Theorem 6.10 and Proposition 5.1]. With the function $d(x) := 1/(10\log^2 x) + 4/(15\log^3 x)$ one has

$$-d(x) \le \sum_{p\le x}\frac{1}{p} - \log\log x - B \quad (x>1), \qquad \sum_{p\le x}\frac{1}{p} - \log\log x - B \le d(x) \quad (x \ge 10372),$$

and $\theta(x) < c_\theta x$ for $x > 0$. We write $D(\ell) := 1/(10\ell^2) + 4/(15\ell^3)$, so that $d(x) = D(\log x)$.

(I4) Johnston and Starichkova [23, §4, proof of Theorem 1.3]: with their parameters $Y_0 = 7.9$, $\alpha_1 = 7$, $\alpha_2 = 2$, $C = 3.98$, $u_0 = 400$, $\delta = 1.3$, $\alpha = 1/4$ and $M = 40$, their Theorem 4.2 gives $S(\mathcal{A}, N^{1/40}) > 5\,U_N N/\log^2 N$ for the even N to which it applies. That theorem assumes $\log\log x_1(X_2) \ge Y_0$ with $x_1(N) = N/\log^5 N$. Applied with $X_2 = N$, this holds exactly when $\log N - 5\log\log N \ge e^{7.9}$, that is from $\log N = 2736.8551\ldots$ on; the threshold stated with these parameters in the proof of [23, Theorem 1.3] (arXiv version v3, 11 April 2025) is $X_2 = \exp(e^{7.9})$, which is smaller; we have not consulted the journal version, which may differ at this point. We apply [23, Theorem 4.2] only for $N \ge X_2' := \exp(2736.86)$, where all its hypotheses hold (Appendix B), and only through (I4′); the coefficient 5 above is not used. Commented-out lines of the arXiv v3 source record a different admissible tuple, $(Y_0, \alpha_1, \alpha_2, C) = (7.8, 7, 2, 431.57)$, whose sieve output is not evaluated here; no error in [23, Theorem 1.3] is asserted.

(I4′) For every even $N \ge X_2' = \exp(2736.86)$, $S(\mathcal{A}, N^{1/40}) > c(L)\,U_N N/\log^2 N$, where $c(L) = 9.8036$ for $2736.86 \le L \le 3000$ and $c(L) = 26.218$ for $L \ge 3000$. This is a derived consequence of Johnston–Starichkova, [23, Theorem 4.2], applied separately with $X_2 = N$, using the parameters of (I4) and the corrected definition of $a(X_2)$ with denominator $\log^\delta N$. This definition is the one supported by the proof of [4] imported in case (b); the exponent $1+\delta$ printed in [23] (arXiv v3), §2.1, is a misprint. Appendix B proves the two coefficients, using interval arithmetic on $2736.86 \le \log N \le 3000$ and decreasing majorants for $\log N \ge 3000$.

(I5) [4, Corollary 4]: every even $N > \exp(\exp(32.7))$ is the sum of a prime and a squarefree number $\eta > 1$ with at most two prime factors (their proof counts representations with $\eta > 1$: [4, proof of Corollary 4]).

For Theorem 2 we use in addition the explicit linear sieve of Bordignon, Johnston and Starichkova [4, Theorem 6, Table 1], in the form of [22, Theorem 2.1, Table 2.1], the lemmas of [22, Chapter 4 and Appendix A], and an explicit bound for $\psi(t,\chi)$ under the 7/8 hypothesis (§6). For Theorem 3 we use, under GRH:

(I6) [5, Lemma 3], from [12, Theorem 1]: if $3 \le d \le \sqrt{X_2}$ and $X_2 \ge 4\cdot 10^{18}$, then for all real $x \ge X_2$ and all integers a with $(a,d) = 1$, $\left|\pi(x;d,a) - \mathrm{li}(x)/\varphi(d)\right| \le 0.640\sqrt{x}\log x$.

(I7) [22, Proposition 4.18]: for even $N \ge e^{73}$ and $z = N^{1/11}$, the set of elements of $\mathcal{A}$ free of primes below z has more than $0.146\,U_N N/\log^2 N$ elements.[1]

(I8) [5, Corollary 1]: every even $N > \exp(\exp(15.85))$ is the sum of a prime and a squarefree number $\eta > 1$ with at most two prime factors.

3. The range below the sieve threshold

Let $\sigma_{11} = \frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}$, and for $X \ge 10372$ put

$$S'(X) := \log\log X + B + d(X) - \sigma_{11}, \qquad P_{13}(k) := \sum_{p\ge 13} p^{-k}, \qquad C_{11}(z) := \sum_{k\ge 2}\frac{P_{13}(k)}{k}z^k,$$

[1] In [22, (4.35)] the factor $M|\mathcal{A}|U_N/(\log N)^2$ should read $M|\mathcal{A}|U_N/\log N$, as (4.37) in its proof shows; with this factor and $|\mathcal{A}| > N/\log N$, (4.35) has the order $U_N N/\log^2 N$ of [22, (4.39)], which is the form used here.

$$G_0 := 135\Big(\frac{1}{480} - \frac{1}{840\log(4\cdot 10^{18}-2)}\Big) = 0.27749787\ldots, \qquad A_a := \sum_{p\ge 13}\frac{a\log p}{p^a} \quad (a\ge 2),$$

$$L_{\mathrm{low}} := \log(4\cdot 10^{18}), \qquad R_j^* := [z^j]\exp\big(zS'(X) + C_{11}(z)\big) \quad (j\ge 0).$$

Here $[z^j]$ denotes the coefficient of z^j in a power series.

Lemma 4 (Exact coefficients)**.** *For $X \ge 10372$, $N \le X$ and $j \ge 0$,* $\displaystyle\sum_{\substack{m\le N,\ (m,2310)=1\\ \Omega(m)=j}} \frac{1}{m} \le R_j^*.$

Proof. The m in the sum are products of primes in $[13, N]$, so the sum is at most $[z^j]\prod_{13\le p\le N}(1-z/p)^{-1}$, the coefficient of a power series with nonnegative coefficients. Its logarithm is $\sum_{k\ge 1} b_k z^k$ with $b_k = k^{-1}\sum_{13\le p\le N} p^{-k} \ge 0$. Here $b_1 \le \sum_{p\le X} 1/p - \sigma_{11} \le S'(X)$ by (I3), since the prime sum increases with N and $X \ge 10372$; and $b_k \le P_{13}(k)/k$ for $k \ge 2$. The coefficients of $\exp(\sum b_k z^k)$ are polynomials with nonnegative coefficients in the b_k, so they do not decrease when each b_k is replaced by an upper bound. □

Remark 5. For any $0 < \tau < 13$ the coefficient R_j^* is at most $\tau^{-j}\exp(\tau S'(X) + C_{11}(\tau))$, a single term of a nonnegative series. This is Rankin's bound [34]. Keeping the coefficient can give a sharper estimate; no quantitative estimate for the ratio of these two bounds is used below. The coefficients are computed exactly from the recurrence $nR_n^* = S'(X)R_{n-1}^* + \sum_{k=2}^n P_{13}(k)R_{n-k}^*$, $R_0^* = 1$, whose terms are nonnegative.

Lemma 6. *Let $X \ge 4\cdot 10^{18}$, $L_X = \log X$, $k_{\max} = \lfloor L_X/\log 13\rfloor$, let $0 < \delta < 1$ with $\delta L_{\mathrm{low}} \ge 1$, and let $K \ge 1$ be an integer with*

$$L_{\mathrm{low}}\, e^{-\delta L_{\mathrm{low}}} + \frac{1}{1-\delta}\sum_{k=K+1}^{k_{\max}}\Big(c_\theta R_{k-1}^* + \sum_{a=2}^{k} A_a R_{k-a}^*\Big) \;<\; G_0(1-10^{-18}) - 10^{-18}. \tag{1}$$

Then every even N with $4\cdot 10^{18} \le N \le X$ can be written as $N = p + \eta$ with p an odd prime, $\eta \ge 3$ odd, $(\eta, 1155) = 1$ and $\Omega(\eta) \le K$.

Proof. Fix N, put $y = N - 2$ and $L = \log N$.

Good primes. Let $\mathcal{R}$ be the set of residues b mod 1155 with $(b, 1155) = 1$ and $(N - b, 1155) = 1$. For each prime $q \mid 1155$ the condition allows $q-1$ residues modulo q if $q \mid N$ and $q-2$ otherwise, so $|\mathcal{R}| \ge 1\cdot 3\cdot 5\cdot 9 = 135$. By (I2) with $q = 1155$ and $x = y \ge 8\cdot 10^9$, $\theta(y; 1155, b) > y/480 - y/(840\log y)$ for each $b \in \mathcal{R}$. Summing and discarding the prime 2,

$$\mathcal{G} := \sum_{\substack{p\le y,\ p\text{ odd}\\ p \bmod 1155\in\mathcal{R}}} \log p \;>\; G_0\, y - \log 2.$$

Every such p has $(N - p, 1155) = 1$ and $N - p$ odd.

Bad primes. If p is counted in $\mathcal{G}$ and $\Omega(N-p) > K$, then $n = N - p$ lies in $E := \{n \le N : (n, 2310) = 1,\ \Omega(n) > K\}$, and distinct p give distinct n. Hence the bad part of $\mathcal{G}$ is at most $L\cdot|E|$. At most $N^{1-\delta}$ elements of E are at most $N^{1-\delta}$; they contribute at most $LN^{1-\delta} = NLe^{-\delta L} \le NL_{\mathrm{low}}e^{-\delta L_{\mathrm{low}}}$, since $t \mapsto te^{-\delta t}$ decreases for $t \ge 1/\delta$ and $L \ge L_{\mathrm{low}} \ge 1/\delta$. An element $n > N^{1-\delta}$ has $\log n > (1-\delta)L$; so with $S_k := \sum \log n$ over $n \le N$, $(n, 2310) = 1$, $\Omega(n) = k$, the remaining contribution is at most $(1-\delta)^{-1}\sum_{k>K} S_k$, where $k \le k_{\max}$ since $13^{\Omega(n)} \le n$. By Chebyshev's identity $\log n = \sum_{p^a\|n} a\log p$,

$$S_k \le \sum_{a\ge 1}\sum_{p\ge 13} a\log p\cdot \#\big\{n' \le N/p^a : (n', 2310) = 1,\ \Omega(n') = k-a\big\}.$$

For $a=1$, exchanging the sums gives $\sum_{n'}\theta(N/n')\le c_\theta N\sum_{n'}1/n'$ over $n'\le N$ coprime to 2310 with $\Omega(n')=k-1$, by (I3); this is at most $c_\theta N R^*_{k-1}$ by Lemma 4. For $a\ge 2$, the number of admissible n' is at most $(N/p^a)\sum 1/n'$ over the same kind of n' with $\Omega(n')=k-a$, so the contribution is at most $NA_aR^*_{k-a}$.

Conclusion. The bad part is at most N times the left side of (1), hence less than $N\big(G_0(1-10^{-18})-10^{-18}\big)\le G_0y-\log 2$, since $y\ge N(1-5\cdot10^{-19})$ and $\log 2<10^{-18}N$ for $N\ge 4\cdot10^{18}$. So some odd $p\le N-2$ counted in $\mathcal{G}$ has $\Omega(N-p)\le K$. □

Lemma 7. *With $\delta=1/5$, condition (1) holds for $(X,K)=(e^{1800},18)$, $(e^{350},14)$ and $(e^{62},9)$: the left side is <0.209801, <0.149761 and <0.241386 respectively, against $G_0(1-10^{-18})-10^{-18}>0.2774978$. For $(X,K)=(e^{1800},17)$ the left side exceeds 0.5646.*

Proof. Interval computation, §8. The last statement is a lower bound for the left side at $\delta=1/5$; it says nothing about other values of δ. □

3.1. **Pre-sieving the integers to be excluded.** Put $Y:=8\cdot10^9$ and $\varepsilon_Y:=480/(840\log Y)$.

Lemma 8. *Let $4\cdot10^{18}\le Y_i\le X$ with $L_i:=\log Y_i>\log Y$, let $\delta,\delta_2>0$ with $\delta+\delta_2<1$, $\delta L_i\ge1$ and $\delta_2L_i\ge2$, let $K\ge1$ be an integer and $k_{\max}:=\lfloor\log X/\log 13\rfloor$, and put*

$$W_k:=\frac{c_\theta(1+\log Y)R^*_{k-2}+\sum_{a=2}^{k-1}A_aR^*_{k-1-a}}{L_i-\log Y}.$$

If

$$\frac{480}{135}\Big[L_ie^{-\delta L_i}+2\zeta(3/2)L_ie^{-\delta_2L_i/2}+\frac{c_\theta}{1-\delta-\delta_2}\sum_{k=K+1}^{k_{\max}}W_k\Big]+\frac{1+\varepsilon_Y}{1-\delta-\delta_2}\sum_{k=K+1}^{k_{\max}}R^*_{k-1}$$
$$<\ \Big(1-\frac{480}{840\log(4\cdot10^{18}-2)}\Big)(1-10^{-18})-\frac{480}{135}\cdot10^{-18},\tag{2}$$

then every even N with $Y_i\le N\le X$ can be written as $N=p+\eta$ with p an odd prime, $\eta\ge3$ odd, $(\eta,1155)=1$ and $\Omega(\eta)\le K$.

Proof. Fix N, $L:=\log N\ge L_i$ and $y:=N-2$, and let $\mathcal{R}$ be as in the proof of Lemma 6. The good primes are the odd primes $P\le y$ with $P \bmod 1155\in\mathcal{R}$, and $\mathcal{G}:=\sum_{\text{good}}\log P>|\mathcal{R}|(y/480-y/(840\log y))-\log 2$ by (I2), the term $\log 2$ allowing us to exclude $P=2$. Every good complement $N-P$ is odd, at least 3 and coprime to 1155. The bad ones have $n:=N-P$ with $\Omega(n)\ge K+1$; these n are distinct, and the bad part of $\mathcal{G}$ is at most $L\cdot\#E'$, where $E':=\{n=N-P:P\text{ good},\ \Omega(n)\ge K+1\}$. We split E' into three parts.

(i) $n\le N^{1-\delta}$: at most $N^{1-\delta}$ elements.

(ii) $n>N^{1-\delta}$ whose squarefull part $s(n)$ exceeds N^{δ_2}: at most $N\sum_{s\text{ squarefull},\,s>N^{\delta_2}}1/s\le 2\zeta(3/2)N^{1-\delta_2/2}$ elements. Indeed every squarefull s is u^2v^3 with v squarefree, and $\sum_{u>A}u^{-2}\le 2/A$ for every $A>0$ (for $A<1$ use $\zeta(2)<2<2/A$); hence $\sum_{s>T}1/s\le\sum_v v^{-3}\cdot2\sqrt{v^3/T}\le 2\zeta(3/2)T^{-1/2}$.

(iii) $n>N^{1-\delta}$ with $s(n)\le N^{\delta_2}$. Write $n=sr$ with r squarefree and coprime to s; then $\log r>(1-\delta-\delta_2)L$. So the number of such n with $\Omega(n)=k$ is at most $\sum\log r/((1-\delta-\delta_2)L)$, and $\sum\log r$ counts each prime p of r once: $\sum\log r=\sum_{(p,n')}\log p$ over the factorisations $n=pn'$ with $p\parallel n$ and $p\mid r$. For fixed n', which has $\Omega(n')=k-1$, is coprime to 2310 and determines the decomposition, the prime p satisfies $pn'=n\equiv N-b\pmod{1155}$ for some $b\in\mathcal{R}$, so p lies in the $|\mathcal{R}|$ reduced classes $(N-b)\bar n'$ mod 1155. If $N/n'\ge Y$, (I2) gives

$$\sum_{\substack{p\le N/n'\\ p\text{ in those classes}}}\log p\ \le\ |\mathcal{R}|\frac{N/n'}{480}\Big(1+\frac{480}{840\log(N/n')}\Big)\ \le\ \frac{|\mathcal{R}|}{480}(1+\varepsilon_Y)\frac{N}{n'}.$$

If $n' > N/Y$, the sum is at most $\theta(N/n') \le c_\theta N/n'$ by (I3). Hence the part (iii) with $\Omega(n) = k$ contributes at most $N\big[(|\mathcal{R}|/480)(1+\varepsilon_Y)R^*_{k-1}+c_\theta \mathcal{W}_k\big]/((1-\delta-\delta_2)L)$ elements, where $\mathcal{W}_k := \sum 1/n'$ over $N/Y < n' \le N$ with $\Omega(n') = k-1$ and $(n', 2310) = 1$.

The window. For $n' > N/Y$ we have $\log n' > L - \log Y \ge L_i - \log Y$, so $\mathcal{W}_k \le (L_i - \log Y)^{-1} \sum \log n'/n'$ over the same n'. By Chebyshev's identity again, writing $n' = p^a m$ with $p^a \parallel n'$: for $a = 1$ the prime p runs over $(N/(Ym), N/m]$, and partial summation with $\theta(t) \le c_\theta t$ gives $\sum_{A<p\le B} \log p/p \le \theta(B)/B + \int_A^B \theta(t)t^{-2}\,dt \le c_\theta(1 + \log(B/A))$, so this part is at most $c_\theta(1+\log Y)R^*_{k-2}$; the terms with $a \ge 2$ contribute at most $\sum_{a\ge 2} A_a R^*_{k-1-a}$. So $\mathcal{W}_k \le W_k$.

Assembly. The bad part of $\mathcal{G}$ is at most

$$LN^{1-\delta} + 2\zeta(3/2)LN^{1-\delta_2/2} + \frac{N}{1-\delta-\delta_2}\sum_{k=K+1}^{k_{\max}} \Big[\frac{|\mathcal{R}|}{480}(1+\varepsilon_Y)R^*_{k-1} + c_\theta W_k\Big],$$

and $Le^{-\delta L} \le L_i e^{-\delta L_i}$, $Le^{-\delta_2 L/2} \le L_i e^{-\delta_2 L_i/2}$, since these decrease for $L \ge 1/\delta$ and $L \ge 2/\delta_2$. Divide the upper bound for the bad part and the lower bound for $\mathcal{G}$ by $N\kappa$, where $\kappa := |\mathcal{R}|/480 \ge 135/480$. The terms that do not carry $|\mathcal{R}|$ acquire a factor $1/\kappa \le 480/135$, and the factor $|\mathcal{R}|/480$ of the main term cancels. The good side becomes $(1 - 480/(840\log y))(y/N) - (\log 2)/(N\kappa)$, which is at least the right side of (2) since $y/N \ge 1 - 5\cdot 10^{-19}$ and $\log 2 < 10^{-18}N$. So (2) implies that some good P has $\Omega(N-P) \le K$. □

Remark 9. The gain over Lemma 6 is the factor $480/|\mathcal{R}| \le 480/135$ in the main term: the integers to be excluded are confined to the residue classes of the primes that produce them.

Lemma 10. *With $Y_i = e^{1800}$, $X = X_2' = \exp(2736.86)$, $\delta = \delta_2 = 0.02$ and $K = 18$, the left side of (2) is < 0.406804, against a right side > 0.986659.*

Proof. Interval computation, §8. □

4. Rough numbers with many prime factors

Lemma 11 (The Mertens measure)**.** *Let $z \ge 2$, $\ell = \log z$, $S(z) := \sup_{x\ge z} \max\big(0,\ \sum_{p\le x} 1/p - \log\log x - B\big)$ and $\varepsilon \ge S(z) + D(\ell) + 1/z$; for $z \ge 10372$, (I3) gives $S(z) \le D(\ell)$, and we take $\varepsilon = 2D(\ell) + 1/z$. Let ν be the measure $\varepsilon\,\delta_1 + \mathbf{1}_{[1,\infty)}(u)\,du/u$ on $[1,\infty)$, where δ_1 is the unit mass at 1. For integers $j \ge 1$ and real v let*

$$F_j(v) := \sum_{\substack{p_1,\dots,p_j \ge z \\ p_1\cdots p_j \le z^v}} \frac{1}{p_1\cdots p_j} \quad (\textit{ordered tuples of primes}), \qquad J_m(w) := \int_{\substack{u_1,\dots,u_m\ge 1 \\ u_1+\cdots+u_m\le w}} \frac{du_1\cdots du_m}{u_1\cdots u_m},$$

with $J_0(w) = 1$ for $w \ge 0$, $J_0(w) = 0$ for $w < 0$, and $J_m(w) = 0$ for $w < m$. Then

$$F_j(v) \le N_j(v) := \nu^{\otimes j}\big\{u \in [1,\infty)^j : u_1 + \cdots + u_j \le v\big\} = \sum_{i=0}^{j} \binom{j}{i} \varepsilon^i J_{j-i}(v-i).$$

Proof. Let $\phi(u) := \sum_{z\le p\le z^u} 1/p$ for $u \ge 1$. By the definition of $S(z)$ at z^u, the lower bound of (I3) at z, and the possible term $p = z$, $\phi(u) \le \log u + S(z) + D(\ell) + 1/z \le \log u + \varepsilon = \nu([1,u])$. For $j = 1$, $F_1(v) = \phi(v) \le N_1(v)$. For $j \ge 2$, with $u_p := \log p/\ell$,

$$F_j(v) = \sum_{p\ge z} \frac{1}{p} F_{j-1}(v - u_p) \le \int_{[1,\infty)} g\,d\phi, \qquad g(u) := N_{j-1}(v-u),$$

by induction, where $d\phi$ is the measure with mass $1/p$ at each u_p. The function g is nonnegative and nonincreasing, and it vanishes for $u > v - j + 1$. By the layer-cake formula, $\int g\,d\phi = \int_0^\infty \phi(\{g > t\})\,dt$, and each set $\{g > t\}$ is an initial segment $[1, u_t)$ or $[1, u_t]$ of $[1,\infty)$. The $d\phi$-measure of an initial segment is at most its ν-measure: for closed segments this is the bound on ϕ, and for

half-open ones it follows by a limit, since ν has no atom beyond 1. Hence $\int g\,d\phi \le \int g\,d\nu = N_j(v)$. The closed form follows by choosing which i coordinates lie on the atom at 1. □

Lemma 12 (A Chebyshev-weighted count of rough integers)**.** *Let $M \ge 2$ be an integer, $N = z^M$ with $z \ge 2$, ε as in Lemma 11, $0 < \delta < 1$ and $1 \le k \le M-1$. Put $A_a(z) := \sum_{p \ge z} a \log p/p^a$ for $a \ge 2$, $\mu_0 := 1$, $\mu_1 := N_1(M-1)$, and for $j \ge 2$*

$$\mu_j := \frac{N_j(M-1)}{j!} + \frac{\mu_{j-2}}{z-1} \qquad \text{or, if } \frac{1}{z-1} \le \varepsilon, \qquad \mu_j := \frac{N_j(M-1)}{j!} + \frac{(M-1)e^{2\varepsilon}}{z-1}.$$

Then

$$\#\big\{n \in (N^{1-\delta}, N) : P^-(n) \ge z,\ \Omega(n) = k\big\} \le \frac{N}{(1-\delta)\log N}\Big[c_\theta \mu_{k-1} + \sum_{a=2}^{k} A_a(z)\,\mu_{k-a}\Big].$$

Proof. Each counted n has $\log n > (1-\delta)\log N$, so the count is at most $\sum_n \log n/((1-\delta)\log N)$, and by Chebyshev's identity $\sum_n \log n = \sum_n \sum_{p^a \| n} a \log p$. For $a = 1$ write $n = pn'$ with $p \nmid n'$; then n' is z-rough with $\Omega(n') = k-1$ and $n' \le N/z = z^{M-1}$, and $p \le N/n'$, so these terms contribute at most $\sum_{n'} \theta(N/n') \le c_\theta N \sum_{n'} 1/n'$ by (I3). The terms with $a \ge 2$ contribute at most $N \sum_{p \ge z} a \log p\, p^{-a} \sum_{n''} 1/n''$ over z-rough $n'' \le z^{M-1}$ with $\Omega(n'') = k-a$.

It remains to show that $\sum 1/m$ over z-rough $m \le z^{M-1}$ with $\Omega(m) = j$ is at most μ_j. The squarefree m contribute at most $F_j(M-1)/j! \le N_j(M-1)/j!$ by Lemma 11. A non-squarefree m is $q^2 m''$ for some prime $q \ge z$ and some z-rough $m'' \le z^{M-1}$ with $\Omega(m'') = j-2$; so these m contribute at most $\sum_{q \ge z} q^{-2}\,\mu_{j-2} \le \mu_{j-2}/(z-1)$, by induction on j. Alternatively, their contribution is at most $\sum_{q \ge z} q^{-2}$ times $\prod_{z \le p \le z^{M-1}} (1-1/p)^{-1} \le \exp(\log(M-1) + \varepsilon + 1/(z-1))$, by (I3) as in Lemma 11, and this is at most $(M-1)e^{2\varepsilon}$ when $1/(z-1) \le \varepsilon$. □

The tails satisfy $A_a(z) \le a\big[\log z/z^a + \log z/((a-1)z^{a-1}) + 1/((a-1)^2 z^{a-1})\big]$, by comparison with the integral over all integers $n \ge z$.

Lemma 13 (Pre-sieving modulo 1155)**.** *In the setting of Lemma 12, let N be even and $z \ge 8 \cdot 10^9$, and let $\mathcal{B}_k$ be the set of the counted n for which $N - n$ is a prime other than 3, 5, 7 and 11. Then*

$$\#\mathcal{B}_k \le \frac{N}{(1-\delta)\log N}\Big[\Pi(N)\,G_P\Big(1 + \frac{480}{840\log z}\Big)\mu_{k-1} + \sum_{a=2}^{k} A_a(z)\,\mu_{k-a}\Big],$$

where $G_P := 9/32$ and $\Pi(N) := \prod_{q \mid (N, 1155)} (q-1)/(q-2)$.

Proof. Let $n \in \mathcal{B}_k$ and let $q \in \{3, 5, 7, 11\}$. Then $q \nmid n$, since n is z-rough, and $q \nmid N - n$, since $N - n$ is a prime other than q. So n avoids the class 0 and the class of N modulo q: two classes if $q \nmid N$, one if $q \mid N$. In the terms $a = 1$ of the proof of Lemma 12, $n = pn'$ with n' coprime to 1155, so p lies in one of $A(N) := \prod_{q \nmid N}(q-2) \prod_{q \mid N}(q-1)$ reduced classes modulo 1155. By (I2) with $q = 1155$ and $x = N/n' \ge z \ge 8 \cdot 10^9$, the sum of $\theta(x; 1155, b)$ over these classes b is at most $(A(N)/480)\,x\,(1 + 480/(840\log z))$, and $A(N)/480 = G_P\,\Pi(N)$. The terms $a \ge 2$ are bounded as before. □

Remark 14. Lemma 13 is used inside a comparison between a good count and a bad count. Every good count below has the form $\ge U_N N m - N r$ with $m > 0$ and $r \ge 0$ ($r = 0$ in Lemma 15; $m = \mathrm{main}/U_{\mathrm{low}}$ and $r = \mathrm{rem}$ in Lemma 20), and $U_N \ge U_{\mathrm{low}}\,\Pi(N)$. Since $\Pi(N) \ge 1$, every term of the bad count without the factor $\Pi(N)$, and Nr, may be multiplied by it; so if the bad count is at most $\Pi(N) N(\Sigma_1 + \Sigma_2)$, where Σ_1 is the presieved $a = 1$ expression and Σ_2 collects the other terms, including the at most four excluded $n = N - q$ with $q \in \{3, 5, 7, 11\}$ and the possible element 1, then $U_{\mathrm{low}} m - r - \Sigma_1 - \Sigma_2 > 0$ implies that the good count exceeds the bad count, whatever $\Pi(N)$ is. This is a statement about the combined comparison, not a bound for all rough integers.

5. The range of the Johnston–Starichkova sieve, and the proof of Theorem 1

Put $X_3 := \exp(\exp(32.7))$, $L_0 := 2736.86$, $L_3 := e^{32.7}$ and $z_0 := e^{L_0/40}$.

Lemma 15. *Let $K \geq 1$ be an integer, and let ε, μ_j (in either form of Lemma 12) and A_a be evaluated at $z = z_0$, $M = 40$. Put*

$$\Phi_P := \sum_{k=K+1}^{39} \Big[G_P\Big(1 + \frac{480}{840 \log z_0}\Big)\mu_{k-1} + \sum_{a=2}^{k} A_a(z_0)\,\mu_{k-a} \Big], \qquad G_P = \frac{9}{32}.$$

For each piece $(L_a, L_b, c) \in \{(L_0, 3000, 9.8036),\ (3000, L_3, 26.218)\}$ choose $0 < \delta < 1$ with $\delta L_a \geq 2$, and suppose

$$\frac{L_b\,\Phi_P}{(1-\delta)\,c\,U_{\mathrm{low}}} + \frac{L_a^2 e^{-\delta L_a} + 5L_b^2 e^{-L_a}}{c\,U_{\mathrm{low}}} < 1. \tag{3}$$

Then every even N with $X_2' \leq N \leq X_3$ is $p + \eta$, where p is prime, $\eta \geq 2$ has no prime factor below $N^{1/40}$, and $\Omega(\eta) \leq K$.

Proof. Fix N in a piece, $L := \log N \in [L_a, L_b]$ and $z := N^{1/40} \geq z_0 > 8 \cdot 10^9$. By (I4′), more than $c\,U_N N/L^2$ primes $p \leq N$ with $p \nmid N$ have $N-p$ z-rough; the elements $N-p$ are distinct, and at most one of them equals 1. A z-rough $n < z^{40}$ has $\Omega(n) \leq 39$. The bad elements, those with $\Omega(N-p) \geq K+1$, are: at most $N^{1-\delta}$ elements $n \leq N^{1-\delta}$; at most four elements $N - q$ with $q \in \{3, 5, 7, 11\}$; and the elements of the sets $\mathcal{B}_k$ of Lemma 13, $K < k \leq 39$. Since ε, μ_j, $A_a(z)$ and $1/\log z$ decrease as z grows, Lemma 13 at z is bounded by its value at z_0, so $\sum_k \#\mathcal{B}_k \leq \Pi(N)\,N\Phi_P/((1-\delta)L)$, after multiplying the terms $a \geq 2$ by $\Pi(N) \geq 1$. As $U_N \geq U_{\mathrm{low}}\Pi(N)$ (Remark 14), the number of bad elements plus one is less than the number of good ones as soon as

$$\frac{L\,\Phi_P}{(1-\delta)\,c\,U_{\mathrm{low}}} + \frac{L^2 N^{-\delta} + 5L^2/N}{c\,U_{\mathrm{low}}} < 1.$$

Here $L\Phi_P \leq L_b\Phi_P$; $L^2 e^{-\delta L} \leq L_a^2 e^{-\delta L_a}$, since $t^2 e^{-\delta t}$ decreases for $t \geq 2/\delta$; and $L^2/N \leq L_b^2 e^{-L_a}$. So (3) gives a good element $\eta = N - p \neq 1$ with $\Omega(\eta) \leq K$, and $\eta \geq z \geq 2$. □

Lemma 16. *With $K = 18$ and $\delta = 0.01$, the left side of (3) is $< 1.34 \cdot 10^{-6}$ on the first piece and < 0.4481 on the second.*

Proof. Interval computation, §8, using the crude form of μ_j ($1/(z_0 - 1) \leq \varepsilon$ holds at $z_0 = e^{68.4\ldots}$) and the conservative replacements described there. □

Proof of Theorem 1. For $4 \leq N \leq 4 \cdot 10^{18}$ use (I1). For $4 \cdot 10^{18} \leq N \leq e^{1800}$ use Lemma 6 with $\delta = 1/5$ and $K = 18$ (Lemma 7). For $e^{1800} \leq N \leq X_2'$ use Lemma 8 with Lemma 10 ($K = 18$). For $X_2' \leq N \leq X_3$ use Lemma 15 with Lemma 16 ($K = 18$); the two pieces meet at $\log N = 3000$. For $N > X_3$ use (I5). □

Remark 17. The stated parameters do not give $K = 17$: at e^{1800}, the left side of (1) with $\delta = 1/5$ exceeds 0.5646 (Lemma 7); at $K = 17$ the computed upper bound in Lemma 8 is about 1.07, and on the second piece of Lemma 15 about 14.3, so neither is established. These calculations do not prove that every choice of parameters fails, and no global optimisation is claimed. Lemma 6 alone cannot give $K = 18$ up to $X = X_2'$ for any δ: there the sum over k in (1), before the factor $(1-\delta)^{-1}$, already exceeds $0.4961 > G_0$ (from the enclosure of the left side at $\delta = 1/5$, which is > 0.628287); this is why Lemma 8 is needed.

6. The 7/8 hypothesis: proof of Theorem 2

Throughout this section assume that no Dirichlet L-function has a zero with real part $> 7/8$. By the functional equation, every nontrivial zero $\rho = \beta + i\gamma$ of a primitive $L(s,\chi)$ then satisfies $1/8 \le \beta \le 7/8$. The hypothesis is used only for $L(s,\chi)$ with χ primitive modulo $q > 1$; no zero-free half-plane for ζ is needed.

6.1. An explicit prime number theorem for characters.

Proposition 18. *Let χ be a primitive character modulo $q > 1$, $t \ge 16$ and $3 \le h \le t$, and put $T' = (t+h)/h$. Then*

$$|\psi(t,\chi)| \le E(t,h,q) := \frac{2h\log(t+h)}{\log h} + \log(t+h)\Big(\frac{0.65h}{\sqrt{t}} + \log\log_2(t+h) + 1\Big) + \frac{h}{2} + \frac{3}{2} + (t+h)^{7/8} Z_1(q,T') + 4.67\, Z_2(q) + \log(t+h) + 3.1,$$

where, with $a = 0.298$, $b = 4.358$, $c_0 = \log(2\pi e)$ and $N^+(1) = (\log q - c_0)/\pi + a\log q + b$,

$$Z_1(q,T') \le 7N^+(1) + \tfrac{1}{\pi}\big[(\log q - c_0)\log T' + \tfrac{1}{2}\log^2 T'\big] + (a\log q + b)(1 - 1/T') + a\big(1 - (\log T' + 1)/T'\big) + \tfrac{4}{\pi}\big(\log(qT') - c_0 + 1\big) + 2\big(a\log(qT') + a/2 + b\big)/T',$$

$$Z_2(q) \le 6.12\, N^+(1) + \tfrac{2}{\pi}(\log q - c_0 + 1) + a\log q + a/2 + b.$$

Outline. Smooth $\psi(t,\chi)$ by the ramp weight equal to 1 on $[2,t]$ and linear on $[t,t+h]$; the difference is bounded by the Brun–Titchmarsh inequality and the prime powers (the first terms). The explicit formula for the ramp sum runs over all zeros; the trivial zeros give the last terms; for the nontrivial zeros the transform W_t of the upper ramp satisfies $|W_t(\rho)| \le (t+h)^{\beta}\min(1/|\rho|, 2T'/(|\rho||\rho+1|))$ with $\beta \le 7/8$ and $|\rho| \ge 1/8$, the fixed lower ramp contributes at most $4.67/(|\rho||\rho+1|)$ per zero, hence $4.67\,Z_2(q)$ in all, and the zero counts of Bennett, Martin, O'Bryant and Rechnitzer [3] bound the resulting sums by Z_1 and Z_2. The complete proof is in Appendix A. □

6.2. Distribution in progressions.

Lemma 19. *Let $N = e^L$ with $L \ge 300$, $\lambda > 0$, $H := \lambda N^{1/8}/L^3 < N$, and let $3 \le d \le H$ be odd squarefree with $(d,N) = 1$. Then*

$$\Big|\pi(N;d,N) - \frac{\pi(N)}{\varphi(d)}\Big| \le \frac{c_E}{L}\Big(1 + \frac{32}{7L}\Big)N^{7/8} + 3\sqrt{N} + \frac{2\log H}{\log 3},$$

where c_E is any upper bound for $E(t,t^{7/8},q)/t^{7/8}$ over $e^{L/2} \le t \le e^L$ and $1 < q \le H$.

Proof. By orthogonality, $\pi(N;d,N) - \pi(N)/\varphi(d) = -\omega(d)/\varphi(d) + \varphi(d)^{-1}\sum_{\chi\neq\chi_0}\bar\chi(N)\pi(N,\chi)$ with $\pi(N,\chi) = \sum_{p\le N}\chi(p)$, and $|\pi(N,\chi) - \pi(N,\chi^*)| \le \omega(d)$ for the primitive χ^* inducing χ; here $\omega(d) \le \log H/\log 3$. By partial summation, $\pi(N,\chi^*) = \theta(N,\chi^*)/L + \int_2^N \theta(t,\chi^*)\,dt/(t\log^2 t)$ and $|\theta(t,\chi^*)| \le |\psi(t,\chi^*)| + \psi(t) - \theta(t)$. For $t \le \sqrt{N}$ use $|\theta(t,\chi^*)| \le \theta(t) < t\log 4$ (by (I3)), contributing less than $(2/\log 2)\sqrt{N} < 2.885391\sqrt{N}$. For $t \ge \sqrt{N}$, Proposition 18 with $h = t^{7/8}$ gives $|\psi(t,\chi^*)| \le c_E t^{7/8}$, and $\int_{\sqrt{N}}^N t^{-1/8}\,dt/\log^2 t \le (32/(7L^2))N^{7/8}$. The prime powers contribute at most $[\log 4\,(1/L + 8/L^2) + (2 + 12/L)e^{-L/6}]\sqrt{N} < 0.004745\sqrt{N}$ for $L \ge 300$, using $\psi(t) - \theta(t) = \sum_{k\ge2}\theta(t^{1/k}) \le \log 4\,(\sqrt{t} + t^{1/3}\log_2 t)$, again by (I3). □

6.3. **The sieve.** We use the explicit linear sieve of Bordignon, Johnston and Starichkova [4, Theorem 6, Table 1], in the form [22, Theorem 2.1, Table 2.1] (whose Table 2.1 is their Table 1), in the setting of Johnston's Chapter 4: $\mathcal{A}$ as above, $\mathcal{P}$ the primes not dividing N, $g(d) = 1/\varphi(d)$, $V(z) = \prod_{p<z,\,p\in\mathcal{P}}(1-g(p))$, and $Q = \{3,5,7,11\} \cap \mathcal{P}$. By [22, Proposition A.5] the hypothesis of his Theorem 2.1 holds with $\varepsilon_J = 4.77\cdot 10^{-3}$ for every $z \ge 694$ (for $1 < u < 13$ the relevant product

equals its value at $u = 13$), and $C_2(\varepsilon_J) \le 154$ by Table 2.1. His error term is $\varepsilon_J C_2(\varepsilon_J) e^2 h(s)$ with $h(s) = e^{-s}$ on $[2,3]$ and $h(s) = 3s^{-1}e^{-s}$ for $s \ge 3$ [22, (2.4)], so $h(s) \le e^{-s}$ on $[2,4]$. Thus for $D_s \ge z^2$ and $s = \log D_s/\log z \in (2,4]$,

$$S(\mathcal{A},z) > |\mathcal{A}|\,V(z)\,\big(f(s) - \varepsilon_J\,C_2(\varepsilon_J)\,e^2e^{-s}\big) - \sum_{\substack{d\mid P(z)\\ d<1155D_s}} |r(d)|, \qquad f(s) = \frac{2e^{\gamma}\log(s-1)}{s},$$

with $r(d) = |\mathcal{A}_d| - |\mathcal{A}|/\varphi(d)$. We also use $|\mathcal{A}| > N/\log N$, $V(z) > \frac{U_N}{\log z}\big(1 - \frac{1.002M}{z-1}\big)\big(1 - \frac{1}{2\log^2 z}\big)$ for $z = N^{1/M} \ge 285$, $\omega(N) \le 1.3841 \log N/\log\log N$, $\#\{d \le x \text{ odd squarefree}\} \le 0.45x$ ($x \ge 25$), and $|r(d)| \le |\pi(N;d,N) - \pi(N)/\varphi(d)| + \omega(N)$ [22, Lemmas 4.1–4.4 and the proof of Lemma 4.16].

Lemma 20. *Let N be even with $L = \log N \in [L_1, L_2]$, $L_1 \ge 300$, and fix integers $M \ge 2$ and $K \ge 1$, $0 < \lambda \le 100$ and $0 < \delta < 1$. Put $z = N^{1/M}$, $H = \lambda N^{1/8}/L^3$, $D_s = H/1155$, $s = \log D_s/\log z$, and let z_1, s_1 be the values at L_1 and s_2 the value at L_2. Suppose $H(L_1) \ge 25$ (then $H < N$, as $\lambda \le 100$ and $L \ge 300$), $z_1 \ge 10372$, $2 < s_1 \le s_2 \le 4$, $F(s_1) := f(s_1) - 4.77 \cdot 10^{-3} \cdot 154\, e^{2-s_1} > 0$, $1 - 1.002M/(z_1 - 1) > 0$, and*

$$\text{main} - \text{rem} - e^{-L_1} - \text{bad} > 0,$$

where $\text{main} = \frac{U_{\text{low}}M}{L_2^2}\big(1 - \frac{1.002M}{z_1-1}\big)\big(1 - \frac{M^2}{2L_1^2}\big)F(s_1)$, $\text{rem} = \frac{0.45\lambda}{L_1^3}\big[\frac{c_E}{L_1}(1 + \frac{32}{7L_1}) + 3e^{-3L_1/8} + (\frac{2\log H(L_2)}{\log 3} + \frac{1.3841L_2}{\log L_2})e^{-7L_1/8}\big]$ *with c_E valid on the whole interval, and*

$$\text{bad} = e^{-\delta L_1} + \frac{1}{(1-\delta)L_1}\sum_{k=K+1}^{M-1}\Big[c_\theta \mu_{k-1} + \sum_{a=2}^{k} A_a(z_1)\,\mu_{k-a}\Big],$$

with ε and the recursive μ_j of Lemma 12 taken at $z = z_1$. Then $N = p + \eta$ with $\eta \ge 2$, $P^-(\eta) \ge z$ and $\Omega(\eta) \le K$. If moreover $z_1 \ge 8 \cdot 10^9$, the same holds with bad *replaced by*

$$\text{bad}_P = e^{-\delta L_1} + 4e^{-L_1} + \frac{1}{(1-\delta)L_1}\sum_{k=K+1}^{M-1}\Big[G_P\Big(1 + \frac{480}{840\log z_1}\Big)\mu_{k-1} + \sum_{a=2}^{k} A_a(z_1)\,\mu_{k-a}\Big], \qquad G_P = \frac{9}{32}.$$

Proof. The sieve inequality, the lower bounds for $|\mathcal{A}|$ and $V(z)$, and $h(s) \le e^{-s}$ on $[2,4]$ give at least $(U_N/U_{\text{low}})\,\text{main} \cdot N$ for the main term; Lemma 19 and the counting bounds give at most $\text{rem} \cdot N$ for the remainder, since $1155D_s \le H$ and the d are odd squarefree. One sifted element may be 1, and $1 \le e^{-L_1}N$. A z-rough $n < z^M$ has $\Omega(n) \le M - 1$. Those with $\Omega(n) > K$ number at most $N^{1-\delta}$ with $n \le N^{1-\delta}$, plus, by Lemma 12, at most $N \cdot (\text{bad} - e^{-\delta L_1})$ others, since every term of the bound decreases as L grows: $1/L \le 1/L_1$, and ε, μ_j and $A_a(z)$ decrease as z grows. Every factor of main and rem has the stated monotonicity on $[L_1, L_2]$: s increases with L for $\lambda \le 100$, F increases on $(2,4]$ and z increases. As $U_N \ge U_{\text{low}}$, the first statement follows. For the second, Lemma 13 bounds the elements with $\Omega(n) > K$, apart from the at most four $n = N - q$, $q \in \{3,5,7,11\}$, by $\Pi(N)$ times the corresponding expression; with $U_N \ge U_{\text{low}}\Pi(N)$ and $\Pi(N) \ge 1$, the good count exceeds the bad count plus one when $\text{main} - \text{rem} - e^{-L_1} - \text{bad}_P > 0$ (Remark 14). □

Proof of Theorem 2. For $N \le 4 \cdot 10^{18}$ use (I1); for $4 \cdot 10^{18} \le N \le e^{350}$ use Lemma 6 with $\delta = 1/5$ and $K = 14$ (Lemma 7). For $350 \le \log N \le 51046$ use Lemma 20 with $K = 14$, $\delta = 0.05$ and bad, on the 501 intervals of §8. For $51046 \le \log N \le 159011874857757$ (the right end exceeds $e^{32.7}$) use Lemma 20 with $K = 14$, $\delta = 0.05$ and bad_P, on the 449 intervals of §8; there $z_1 \ge e^{51046/32} > 8 \cdot 10^9$. For $N > \exp(\exp(32.7))$ use (I5). □

Remark 21. Without Lemma 13 the top range gives 15 but 14 is not established for $\log N \ge 7.6 \cdot 10^{13}$; with it, 13 is not established on 103 of the 449 top intervals. Since the parameters are chosen by a search, these statements do not prove that 13 or 14 fails.

7. The generalised Riemann hypothesis: proof of Theorem 3

Throughout this section assume the generalised Riemann hypothesis (GRH) for Dirichlet L-functions: every nontrivial zero of every Dirichlet L-function, $\zeta(s)$ included, has real part $1/2$.

7.1. **Pre-sieving to larger moduli.** For a prime $w \ge 3$ let $q_w := \prod_{3\le p\le w} p$, $g_w := \prod_{3\le p\le w}(p-2)/(p-1)$, $\sigma_w := \sum_{p\le w} 1/p$, and let w^+ be the least prime above w. For $X \ge 10372$ put $S'_w(X) := \log\log X + B + d(X) - \sigma_w$,

$$P_w(k) := \sum_{p>w} p^{-k}, \qquad C_w(z) := \sum_{k\ge 2} \frac{P_w(k)}{k} z^k,$$

$$R_j^{(w)} := [z^j]\exp\bigl(zS'_w(X) + C_w(z)\bigr), \qquad A_a^{(w)} := \sum_{p>w} \frac{a\log p}{p^a}.$$

For $w = 11$ these are $S'(X)$, C_{11}, R_j^* and A_a of §3. For $X \ge \max(10372, w)$, Lemma 4 holds, with the same proof, for m coprime to $2q_w$, with σ_w, $P_w(k)$ and $R_j^{(w)}$ in place of σ_{11}, $P_{13}(k)$ and R_j^*.

Lemma 22. *Assume GRH. Let $w \ge 3$ be prime, $4\cdot 10^{18} \le Y_{\text{low}} \le X$ with $q_w^2 \le Y_{\text{low}} - 2$, $L_Y := \log Y_{\text{low}}$, $L_X := \log X$, $k_{\max} := \lfloor L_X/\log w^+\rfloor$, let $0<\delta<1$ with $\delta L_Y \ge 1$, and let $K \ge 1$ be an integer with*

$$L_Y\, e^{-\delta L_Y} + \frac{1}{1-\delta}\sum_{k=K+1}^{k_{\max}} \Bigl(c_\theta R_{k-1}^{(w)} + \sum_{a=2}^{k} A_a^{(w)} R_{k-a}^{(w)}\Bigr) \tag{4}$$
$$< \; g_w\bigl(1-4e^{-L_Y}\bigr) - 1.01\cdot 0.64\,\varphi(q_w)\,L_Y^2 e^{-L_Y/2} - L_Y e^{-L_Y} - 10^{-12}.$$

Then every even N with $Y_{\text{low}} \le N \le X$ and $N > 4\cdot 10^{18}$ is $p+\eta$ with p an odd prime, $\eta \ge 3$, $(\eta, 2q_w) = 1$ and $\Omega(\eta) \le K$.

Proof. Fix N, $L := \log N$ and $y := N-2$, and put $X_2 := \max(4\cdot 10^{18}, Y_{\text{low}} - 2)$. Then $y \ge X_2$, and $q_w^2 \le Y_{\text{low}} - 2$ gives $3 \le q_w \le \sqrt{X_2}$, so (I6) applies with $d = q_w$ and $x = y$. The classes $a \bmod q_w$ with $(a, q_w) = 1 = (N-a, q_w)$ number at least $\prod_{3\le p\le w}(p-2) = g_w\varphi(q_w)$ and at most $\varphi(q_w)$. Summing (I6) over them, the number of odd primes $p \le y$ in these classes is at least $g_w \operatorname{li}(y) - 0.64\,\varphi(q_w)\sqrt{y}\log y - 1$. Each gives $\eta = N - p \ge 3$, odd, with $(\eta, q_w) = 1$.

The bad ones have $n = N - p$ in $E := \{n \le N : (n, 2q_w) = 1,\ \Omega(n) \ge K+1\}$. As in the proof of Lemma 6, with the primes up to w removed (every such n has $P^-(n) \ge w^+$, so $\Omega(n) \le k_{\max}$), and with this form of Lemma 4,

$$\#E \le N^{1-\delta} + \frac{N}{(1-\delta)L}\sum_{k=K+1}^{k_{\max}} \Bigl(c_\theta R_{k-1}^{(w)} + \sum_{a=2}^{k} A_a^{(w)} R_{k-a}^{(w)}\Bigr).$$

Multiply both counts by L/N. Since $\operatorname{li}(y) \ge (y-2)/\log y$ and $\log y \le L$, $L\operatorname{li}(y)/N \ge 1 - 4/N \ge 1 - 4e^{-L_Y}$. The error term is at most $0.64\,\varphi(q_w)L^2e^{-L/2}$, which decreases in L, and $L/N \le L_Y e^{-L_Y}$; also $Le^{-\delta L} \le L_Y e^{-\delta L_Y}$. So (4) implies that the good primes outnumber the bad ones. □

Lemma 23. *Condition* (4) *holds with $K = 9$ and $\delta = 0.15$ for $(w, Y_{\text{low}}, X) = (19, e^{62}, e^{70})$, the left side being < 0.147535 against a right side > 0.2281295, and for $(w, Y_{\text{low}}, X) = (23, e^{65}, e^{73})$, with < 0.139061 against > 0.2171299.*

Proof. Interval computation, §8; here $q_{19}^2 = 4849845^2 < e^{62} - 2$ and $q_{23}^2 = 111546435^2 < e^{65} - 2$. □

7.2. **The sieve range.**

Lemma 24. *Assume GRH. Let $K \geq 1$ be an integer, $73 \leq L_1 < L_2$, $0 < \delta < 1$ with $\delta L_1 \geq 2$, $z_1 := e^{L_1/11}$, and let ε and the recursive μ_j of Lemma 12 be taken at $z = z_1$ and $M = 11$, with $\varepsilon = S(z_1) + D(\log z_1) + 1/z_1$. Put $\Phi := \sum_{k=K+1}^{10}[c_\theta \mu_{k-1} + \sum_{a=2}^{k} A_a(z_1)\mu_{k-a}]$. If*

$$\frac{L_2\,\Phi}{(1-\delta)\cdot 0.146\,U_{\text{low}}} + \frac{L_1^2 e^{-\delta L_1} + 5L_2^2 e^{-L_1}}{0.146\,U_{\text{low}}} < 1,$$

then every even N with $L_1 \leq \log N \leq L_2$ is $p + \eta$ with $\eta \geq 2$, $P^-(\eta) \geq N^{1/11}$ and $\Omega(\eta) \leq K$.

Proof. The proof of Lemma 15 applies without Lemma 13, with (I7) in place of (I4′) and $M = 11$: $z = N^{1/11} \geq z_1$, a z-rough $n < z^{11}$ has $\Omega(n) \leq 10$, the bad elements number at most $N^{1-\delta}$ plus the count of Lemma 12, which is bounded by its value at z_1 because $S(z)$, $D(\log z)$, $1/z$, $1/(z-1)$ and $A_a(z)$ decrease in z, and one sifted element may be 1. The term $5L_2^2 e^{-L_1}$ is more than this needs. □

Lemma 25. *With $K = 9$, the condition of Lemma 24 holds on each of the 238 intervals of §8, which cover $73 \leq \log N \leq 7.7 \cdot 10^6$; the largest left side is < 0.000265.*

Proof. Interval computation, §8. Here $z_1 \geq e^{73/11} > 761$, and $S(z_1)$ is bounded by enclosed prime sums below 10372 and by $D(\log x)$ above, using that $\sum_{p \leq x} 1/p - \log\log x - B$ decreases between consecutive primes. □

Proof of Theorem 3. For $N \leq 4\cdot 10^{18}$ use (I1). For $4\cdot 10^{18} \leq N \leq e^{62}$ use Lemma 6 with $\delta = 1/5$ and $K = 9$ (Lemma 7); this step is unconditional. For $e^{62} \leq N \leq e^{73}$ use Lemma 22 with Lemma 23: the pieces $[e^{62}, e^{70}]$ and $[e^{65}, e^{73}]$ overlap. For $73 \leq \log N \leq 7.7 \cdot 10^6$ use Lemma 24 with Lemma 25. For $N > \exp(\exp(15.85))$ use (I8); since $e^{15.85} < 7.65 \cdot 10^6 < 7.7 \cdot 10^6$, the ranges overlap. In every range $\eta \geq 2$ and $\Omega(\eta) \leq 9$. □

Remark 26. The value 8 is not established by these steps. With $K = 8$, $w = 29$, $Y_{\text{low}} = e^{65}$, $X = e^{73}$ and $\delta = 0.15$, the left side of (4) is > 0.363 and the right side is < 0.189, and the left side of (1) at $X = e^{73}$, $K = 8$ and $\delta = 1/5$ is about 1.004. These are failures of sufficient conditions at these parameters, not proofs that 8 cannot be reached. The sieve range alone passes $K = 8$ when Lemma 13 is added for $z_1 \geq 8 \cdot 10^9$. Larger w lowers the bad count but costs $\varphi(q_w)$ in the error term, which is why the middle range is cut into pieces with growing w.

8. Computations

Apart from the discrete convolutions bounding the integrals J_m, described below, every numerical step was carried out in outward-rounded interval arithmetic (mpmath `iv`) or ball arithmetic (Arb, via python-flint), with decimal inputs treated as exact rationals and Euler's constant and Mertens' constant enclosed. Each positive comparison was certified by at least two separately written implementations, one of which is, for some steps, an independent audit (separately written scripts, in the supplement's `audit/` directory, that recompute the same quantities in Arb); the details follow. The scripts named below, with their parameter files and outputs, form the supplement to this paper, which is the `anc/` directory of the paper's record, and each is named by its path in it. Its `run_all.sh.txt` (a bash script) reruns every certificate; `compare.py` compares each output with the stored one and checks every bound that this section quotes from the outputs; `SHA256SUMS.txt` and `REPRODUCTION.md` record the files' hashes, the software versions and a complete run.

The middle range. For Lemmas 7, 10 and 23, the coefficients R_j^* and $R_j^{(w)}$ are computed from the recurrence in the remark after Lemma 4, all of whose terms are nonnegative. The prime sums $P_{13}(k)$, $P_w(k)$, A_a and $A_a^{(w)}$ are summed exactly over the primes up to 10^5 (or 1000–2000 for large k), and every truncation is carried as an enclosure: the tails $\sum_{n>P} n^{-k} \leq P^{1-k}/(k-1)$, the integral

tails of A_a for $a \le 80$, a geometric bound for $a > 80$, and an enclosure of Mertens' constant. So for `author/cert_C3s.py` and `third/t2_check/c3_grh.py` the computed interval encloses the left side itself; its upper end proves a condition and its lower end disproves it at the stated parameters. `second/cert_C3.py` and `second/cert_C4.py` compute upper bounds.

- Lemma 7: at $(e^{1800}, 18)$, the enclosure is $[0.209698, 0.209801]$ in mpmath `iv` (`author/cert_C3s.py`) and the Arb upper bound is $0.2098006747\ldots$ (`second/cert_C3.py`); the independent audit's bound (`audit/t2v6/audit_c4.py`) is < 0.2097564. At $(e^{1800}, 17)$ the enclosure is $[0.564615, 0.564888]$. At $(e^{350}, 14)$ the Arb upper bound is $0.1497600658\ldots$, and `author/cert_C3s_v7.py`, which evaluates `author/cert_C3s.py`, gives the enclosure $[0.149686, 0.149761]$. At $(e^{62}, 9)$ the mpmath enclosure is $[0.241321, 0.241386]$ (the case $w = 11$ of `third/t2_check/c3_grh.py`, which is Lemma 6), and the independent audit's Arb bound (`audit/t2v6/audit_grh9.py`) is < 0.2413750.
- Lemma 10: the Arb upper bound is $0.4068036291\ldots$ (`second/cert_C4.py`), of which the term with $(1 + \varepsilon_Y)$ contributes at most $0.360897\ldots$ and the rest at most $0.045906\ldots$; the independent audit's bound (`audit/t2v6/audit_c4.py`) is < 0.4067970. The right side is enclosed as $0.98665909713\ldots$.
- Lemma 23: `third/t2_check/c3_grh.py` (mpmath `iv`) gives < 0.147535 and < 0.139061 against right sides enclosed above 0.2281295 and 0.2171299; the independent audit's Arb bounds (`audit/t2v6/audit_grh9.py`) are < 0.1475264 and < 0.1390529.

The integrals J_m. For Lemmas 16, 25 and the grids of Theorem 2, $J_m(w)$ is bounded above by cell sums: the region $\{u_i \ge 1, \sum u_i \le w\}$ is covered by the cubes $\prod[1+n_i/H_c, 1+(n_i+1)/H_c)$ whose lower corners lie in the region, and each cube carries the weight $\prod 1/(H_c+n_i)$ or $\prod \log(1+1/(H_c+n_i))$, which bounds $\int du/u$ over the cell. So $J_m(w) \le \sum_{n \le (w-m)H_c} D_m(n)$, where D_m is the m-fold discrete convolution of the cell weights. The convolutions are direct (not by FFT), in binary64 with all terms nonnegative; every weight is multiplied by $1 + 10^{-15}$ (the logarithmic weights by $1 + 10^{-14}$) and every convolution and extracted sum by $1 + 10^{-10}$, which exceeds the accumulated rounding error ($< 3.38 \cdot 10^{-12}$ relative for the lengths used). The assembly of Lemma 12 and of the conditions is in interval or ball arithmetic. In every application of Lemma 11 the certificates bound $N_j(w)$ by $\sum_{i=0}^{j} \binom{j}{i} \varepsilon^i J_{j-i}(w)$, with ε exactly as stated there. This is at least the closed form of Lemma 11, because $J_{j-i}(w - i) \le J_{j-i}(w)$, and the same holds for the cell sums. So the values quoted below are upper bounds for the left sides as printed. The independent audit checked the cell bounds, every inflated weight and every extracted sum used against Arb enclosures (`audit/t2v6/check_float_cells.py`), and recomputed the convolutions wholly in 256-bit Arb for Lemma 16 (both cell choices), Lemma 25 and all 950 intervals of Theorem 2; every condition holds there as well.

The Johnston–Starichkova range. Lemma 16: at $H_c = 400$ and $\delta = 0.01$, the certificates also use the common remainder $(L_0^2 e^{-\delta L_0} + 5L_3^2 e^{-L_0})/(c\,U_{\mathrm{low}})$ on both pieces. It dominates the remainder in (3), since $t^2 e^{-\delta t}$ decreases for $t \ge 2/\delta$. In addition, `third/t2_check/cert_S3P_K18.py` replaces 0.6601618158 in U_{low} by the smaller certified lower bound $0.66016175369\ldots$ of the twin-prime constant (`third/js_cert/tp_bound.py`). Each of these changes can only increase the left side. With them, `third/t2_check/cert_S3P_K18.py` (logarithmic cells; the certificate of record) gives $1.339973\ldots \cdot 10^{-6}$ and $0.448050660\ldots$ on the two pieces, and `second/cert_S4.py` (rational cells, U_{low} as defined) gives $1.339973\ldots \cdot 10^{-6}$ and $0.4540790\ldots$. The independent audit's bounds (`audit/t2v6/audit_s4.py`), made in the same way with the twin-prime bound 0.6601617536, are < 0.448050660231 and < 0.454079084257. The coefficients 9.8036 and 26.218 are read from the certificate of (I4′), Appendix B, which has two interval implementations: one in mpmath `iv` (`third/js_cert/js_cert.py`, with its twin-prime constant `third/js_cert/tp_bound.py`) and one in 256-bit Arb (`audit/audit_js.py`).

The interval grids for Theorem 2. The range $350 \le \log N \le 51046$ is covered by 501 intervals and $51046 \le \log N \le 159011874857757$ by 449 intervals, generated with successive endpoint ratios 1.01 and 1.05, the last interval of each grid shortened to reach its endpoint. On each interval the parameters (M, λ) are chosen by a floating-point search, and then every hypothesis of Lemma 20 (integrality, $L_1 \ge 300$, $\lambda \le 100$, $H \ge 25$, $z_1 \ge 10372$ as an interval comparison, $s_1 > 2$ and $s_1, s_2 \le 4$ ($s_1 \le s_2$ because s increases with L), positivity of $F(s_1)$ and of the V factors, and $z_1 \ge 8 \cdot 10^9$ on the second grid) and the final inequality are checked in interval or ball arithmetic, with $\delta = 0.05$ and rational cells with $H_c = 200$; the selected (M, λ) of every interval are in the supplement's parameter files. The certificates are `second/cert_qrhB3.py` and `second/cert_qrhB3p.py`. Both call the author's `margin_iv` from `author/cert_qrh_range.py` with its own K set to $M-1$, so that it returns $\mathrm{main} - \mathrm{rem} - e^{-L_1}$; `margin_iv` applies the gates on M, L_1, λ, H, s_1, s_2, $F(s_1)$ and the V factors, but of the gates on z_1 it checks only $z_1 \ge 694$ and $z_1^{M-1} \ge 10372$. The gate $z_1 \ge 10372$ is applied by the two scripts themselves, which search only over $\log z_1 \ge \log 10372 + 0.01$ and assert $z_1 > 10372$ in ball arithmetic on every interval, and the gate $z_1 \ge 8 \cdot 10^9$ by `second/cert_qrhB3p.py`, also in ball arithmetic. Over the two grids, $z_1 > 10493$ (with $M \le 48$) and $\log z_1 \ge 1595.1875$ (with $M \le 32$). No interval fails; the least relative margins $(\mathrm{main} - \mathrm{rem} - e^{-L_1} - \mathrm{bad})/\mathrm{main}$ (with bad_P on the second grid) are $0.9030\ldots$ and $0.4237\ldots$, and the independent audit's Arb calculation of the second (`audit/t2v6/audit_v6.py`, all 950 intervals) is > 0.42371750647. The constant c_E (`cE_i` in `author/cert_qrh_range.py`) is the term-by-term majorant of Proposition 18 on the rectangle $t \in [e^{L_1/2}, e^{L_2}]$, $q \le H(L_2)$: the terms decreasing in L are evaluated at L_1 and the increasing ones (the Z_1 term, with $T' \le 1 + e^{L_2/8}$ and $q = H(L_2)$) at L_2.

The sieve range under GRH. Lemma 25: `third/t2_check/cert_grh_sieve.py` covers $[73, 7.7 \cdot 10^6]$ by 238 intervals with endpoint ratio 1.05, takes $H_c = 400$, chooses δ among 0.30, 0.25, 0.20, 0.15, 0.10, 0.05 and 0.01 with $\delta L_1 \ge 2$ on each interval, and bounds $S(z_1)$ for $z_1 < 10372$ by the values at primes of $\sum_{p \le x} 1/p - \log\log x - B$, which decreases between consecutive primes, together with $D(\log x)$ above 10372. The largest left side is $0.0002645896\ldots$, on the last interval. The independent audit's Arb grid (`audit/t2v6/audit_grh9.py`; exact rational endpoints, $H_c = 400$) gives every row below 0.000265.

What limits the method. For Theorem 1 the binding steps at $K = 17$ are Lemma 6 at e^{1800}, Lemma 8 (upper bound about 1.07) and the second piece of Lemma 15 (about 14.3). For Theorem 2, $K = 13$ would need the sieve range to start lower than e^{350}, which needs Lemma 20 with $z_1 < 10372$ (possible through the general form of Lemma 11, with $\varepsilon = S(z_1) + D(\log z_1) + 1/z_1$) and Lemma 19 below $L = 300$, and the top range, where 13 is not established on 103 of 449 intervals at the parameters found by the search. Under GRH the binding step for $K = 8$ is the middle range at e^{73}.

Appendix A. Proof of Proposition 18

Throughout, χ is primitive modulo $q > 1$, $t \ge 16$ and $3 \le h \le t$. We use the Brun–Titchmarsh inequality of Yamada [36, Theorem 2], which sharpens Montgomery–Vaughan [28]: for $x > 0$ and $y > 1$, $\pi(x+y) - \pi(x) < 2y/(\log y + 0.8601)$. We apply it only with the integer length $H = \lceil h \rceil$: $\pi(t+h) - \pi(t) \le \pi(t+H) - \pi(t) < 2H/(\log H + 0.8601) \le 2(h+1)/(\log h + 0.8601) \le 2h/\log h$, the last step because $\log h \le 0.8601h$. We also use the zero count of [3, Corollary 1.2]: for χ of conductor $q > 1$ and $T \ge 5/7$, the number $N(T, \chi)$ of zeros $\rho = \beta + i\gamma$ with $0 < \beta < 1$ and $|\gamma| \le T$ satisfies $|N(T,\chi) - \frac{T}{\pi}\log\frac{qT}{2\pi e}| \le 0.298\log(qT) + 4.358$. Put $a = 0.298$, $b = 4.358$, $N^+(u) := \frac{u}{\pi}\log\frac{qu}{2\pi e} + a\log(qu) + b$.

The weight. Let $r_t(u) = 1$ on $[0, t]$, $(t+h-u)/h$ on $[t, t+h]$, 0 for $u \ge t+h$, let r_1 be the same with $(t, h) = (1, 1)$, and $w = r_t - r_1$. Then w is continuous, supported on $[1, t+h]$, and since $\Lambda(1) = 0$ and $r_1(n) = 0$ for $n \ge 2$, $\psi_w := \sum_n \Lambda(n)\chi(n)w(n) = \sum_n \Lambda(n)\chi(n)r_t(n)$. Hence $|\psi(t,\chi) - \psi_w| \le$

$\sum_{t<n<t+h}\Lambda(n)$. By Brun–Titchmarsh the primes contribute at most $\log(t+h)\cdot 2h/\log h$; for $k\ge 2$ there are at most $(t+h)^{1/k}-t^{1/k}+1\le h/(k\sqrt{t})+1$ prime powers p^k in $(t,t+h]$, each weighted $\log p\le\log(t+h)/k$, so they contribute at most $\log(t+h)(0.65h/\sqrt{t}+\log\log_2(t+h)+1)$.

The transform. $W(s):=\int_0^\infty w(u)u^{s-1}\,du = W_t(s)-W_1(s)$ with $W_t(s)=\frac{(t+h)^{s+1}-t^{s+1}}{hs(s+1)} = \frac{1}{hs}\int_t^{t+h}u^s\,du$ and $W_1(s)=\frac{2^{s+1}-1}{s(s+1)}$. Both have a simple pole at $s=0$ with residue 1, so W is entire, $W(s)=O(|s|^{-2})$ in vertical strips, and $|W(s)|\le 2/|s|^2$ on $\operatorname{Re}s=-U$ for every half-odd $U\ge 7/2$.

The explicit formula. By Mellin inversion (absolutely convergent on $\operatorname{Re}s=2$),

$$\psi_w=\frac{1}{2\pi i}\int_{(2)}\Big(-\frac{L'}{L}(s,\chi)\Big)W(s)\,ds.$$

Shift the contour to $\operatorname{Re}s=-U$ along horizontal segments at heights $\pm T_k\to\infty$ chosen with $\big||\gamma|-T_k\big|\gg 1/\log(qT_k)$ for all zeros; there $|L'/L(s,\chi)|\ll\log^2(qT_k)$ for $-1\le\sigma\le 2$ and $\ll\log(q|s|)$ for $\sigma\le -1$, while on $\operatorname{Re}s=-U$, $|L'/L|\ll\log(q(U+|\tau|))$ [8, Chapters 17 and 19]. Only the existence of such heights and these growth rates are used, so no constant is lost. As χ is nonprincipal there is no pole at $s=1$, and $\psi_w=-\sum_\rho W(\rho)$ over all zeros of $L(s,\chi)$, trivial and nontrivial, with multiplicity, the sum converging absolutely.

Nontrivial zeros. By the hypothesis and the functional equation, $1/8\le\beta\le 7/8$, so $|\rho|\ge 1/8$ and $|\rho+1|\ge|\rho|$. From the two forms of W_t, $|W_t(\rho)|\le(t+h)^\beta\min\big(1/|\rho|,\ 2T'/(|\rho||\rho+1|)\big)$, so $\sum_\rho|W_t(\rho)|\le(t+h)^{7/8}Z_1(q,T')$ with $Z_1(q,T'):=\sum_\rho\min(1/|\rho|,2T'/(|\rho||\rho+1|))$. Also $|W_1(\rho)|\le(2^{\beta+1}+1)/(|\rho||\rho+1|)\le 4.67/(|\rho||\rho+1|)$, and we put $Z_2(q):=\sum_\rho 1/(|\rho||\rho+1|)$. Split the zeros with $|\gamma|\le T'$ into $|\gamma|\le 1$ (each term ≤ 8, at most $N(1)\le N^+(1)$ of them) and $1<|\gamma|\le T'$ (each term $\le 1/|\gamma|$); for $|\gamma|>T'$ use $2T'/\gamma^2$. Stieltjes integration by parts against $N(u)\le N^+(u)$ ($u\ge 5/7$), dropping a nonpositive boundary term, gives

$$Z_1(q,T')\le 7N^+(1)+\int_1^{T'}\frac{N^+(u)}{u^2}\,du+4T'\int_{T'}^\infty\frac{N^+(u)}{u^3}\,du,$$

and evaluating the integrals gives the bound for Z_1 in Proposition 18. Similarly $Z_2(q)\le\frac{64}{9}N(1)+\sum_{|\gamma|>1}\gamma^{-2}\le(\frac{64}{9}-1)N^+(1)+2\int_1^\infty N^+(u)u^{-3}\,du$, which gives the bound for Z_2 (as $55/9<6.12$).

Trivial zeros. They are at $0,-2,-4,\dots$ for even χ and at $-1,-3,\dots$ for odd χ. One has $|W(0)|\le\log(t+h)+1$, $|W(-1)|\le 1/t+\log 2\le 1$, and $|W(-k)|\le t^{-k}/k+1/(k(k-1))$ for $k\ge 2$, summing to at most 1.01; in all, at most $\log(t+h)+3.1$.

Collecting the terms gives $|\psi(t,\chi)|\le E(t,h,q)$; the term $h/2+3/2$ in E (which comes from the principal character in the general statement) is not needed here and is kept only as a harmless upper bound. □

Appendix B. The coefficient in (I4′)

We use the parameters of (I4): $(Y_0,\alpha_1,\alpha_2,C)=(7.9,7,2,3.98)$, $u_0=400$, $\delta=1.3$, $\alpha=1/4$, and $M=40$. We may take $\varepsilon=0.0005543$, $\xi\le 801$, and $C_1(\varepsilon),C_2(\varepsilon)\le 114$ [23, proof of Theorem 1.3]. For each even $N\ge X_2'$, apply [23, Theorem 4.2] separately with $X_2=N$. Write

$$L=\log N\ge L_0:=2736.86,\quad w=L-5\log L,$$
$$j(w)=\frac{1.3841\cdot 7\log w}{\log(7\log w)},\quad s_J(L)=40\left(\frac14-\frac{14\log L}{L}\right).$$

Set $\mathfrak{S}_2=\prod_{p>2}(1-(p-1)^{-2})$ and $T=2e^{-\gamma}\mathfrak{S}_2$.

Correction to the definition of a. In [23, §2.1, p. 5] (arXiv v3), the denominator $\log^{1+\delta} n$ in $a(X_2)$ should read $\log^{\delta} n$, as in the definition (138) of [4]. We use

$$a(X_2) = a_1(X_2) \max_{n \ge X_2} \left[\frac{\log\log\log n}{\log^{\delta} n} \mathfrak{S}_2^{-1} \left(e^{\gamma} \log\log(\log^{\alpha_1} x_1(n)) + \frac{2.5}{\log\log(\log^{\alpha_1} x_1(n))} \right) \right].$$

This is supported by the proof of [4, Theorem 44(b)] imported in [23, Theorem 4.2(b)]: the error bound $a_1 N V^{(\ell)} \log\log\log N / \log^{1+\delta} N$ becomes $a_1 |A| V^{(\ell)} \log\log\log N / \log^{\delta} N$ using $|A| > N/\log N$. The theorem expresses the error relative to $|A|V(z)$. Neither a_1 nor the outside factor changes. The authors' Mathematica notebook for the unconditional case (`https://github.com/DJmath1729/prime_and_almost_prime`, commit `b2330c99d8`, definition of a) also uses δ.

Hypotheses. The condition $\log\log x_1(X_2) \ge 7.9$ reads $w \ge e^{7.9}$. At $L = L_0$, $w - e^{7.9} > 0.00483894$, and the other conditions have margins

$$\frac{w}{2} - 7\log w - 7\log L > 1237.94, \quad 7\log L - \log 10^9 > 34.67, \quad 1 - 801/L^2 > 0.999893,$$
$$\frac{1}{74} - \varepsilon > 0.012959, \quad \frac{L}{4} - 7\log w - 2.5\log L - 400(1 + 9\cdot 10^{-7}/\log 400) > 209.12,$$
$$0.2L - 14\log L > 436.56, \qquad w^{1.3} > 28853.95 > 3022.$$

These margins persist: w increases, $\log L/L$ decreases, and the support and level expressions have positive derivatives. Also $N \ge 4\cdot 10^{18}$, $z_0 = N^{1/40} > 10^{12}$, and $w^{1.3} > 15015$, so $\varepsilon_1(N,\delta) \le 1/11$. Finally $s_J(L) \ge 8.3805685 > 7$, giving $f(s_J(L)), f(10) \ge 1-\rho$ and $\overline{m} \le \rho$, where $\rho = 0.000105$. Here f increases and F decreases to 1 (see [23, §2]), so these follow from $1 - f(6) \le 1.05\cdot 10^{-4}$ and $F(7) - 1 \le 5\cdot 10^{-6}$; ball arithmetic on the closed forms quoted there gives $1 - f(6) = 1.0493998\ldots \cdot 10^{-4}$ and $F(7) - 1 = 4.9945225\ldots \cdot 10^{-6}$ (`author/fF_enclose.py`). We use $h(s_J(L)) < 0.000082075421$; the bound $h \le 10^{-5}$ printed in the proof of [23, Theorem 1.3] is too small ($h(s_J(L)) > 8.4\cdot 10^{-5}$ at their X_2).

Suprema and decreasing majorants. In the supremum defining p_2, $Q(x_1(N))$ and $\nu(x_1(N))$ are fixed while $t = \log y \ge w$ varies. The C contribution is constant, and $t^2 e^{-\nu t}$ decreases since $\nu w > 11.92 > 2$. The remaining terms decrease, so the supremum is attained at $y = x_1(N)$. The arXiv v3 definition of p_2 in [23, §2.1] contains a stray line "$+18.78(\log y)^{2.515}\exp(-0.8274\sqrt{\log y}) + c_1 p_3(y)$" with p_3 undefined. It is not in the authors' notebook, and we omit it; its first term is below $2\cdot 10^{-9}$ here. For c_1, the exponent $E_P(t) = t/(2\cdot 2.0452\cdot 7\log t)$ satisfies $dE_P/d\log t > 10.41497 > 2$; its denominator improves and its other terms decrease.

Put $r(L) = \log\log L$ and $q(L) = \log(7\log w)$, and define

$$M_3(L) = \left(\frac{1.5}{L} + \frac{L + 7\log L}{L - 7\log L}\right)\left(\frac{L}{w}\right)^{1.3}\left[e^{\gamma}\left(1 + \frac{\log 1.3}{r(L)}\right) + \frac{2.5}{\log(1.3\log w) r(L)}\right] + \frac{1.3841 L^{3.3} e^{-L}}{\log L\, r(L)},$$
$$\Gamma(L) = \frac{1.3841\cdot 7\log L}{L^{0.7} r(L) q(L)},$$
$$A_*(L) = \frac{r(L)}{\mathfrak{S}_2 L^{1.3}}\left[e^{\gamma}(\log 7 + r(L)) + \frac{2.5}{q(L)}\right].$$

Then $c_3(N) \le M_3(L)$, $a_1(N) \le c_2(N)\Gamma(L) + M_3(L)$, and $a(N) \le [c_2(N)\Gamma(L) + M_3(L)]A_*(L)$. All three majorants decrease for $L \ge L_0$. For M_3, use the decrease of $\log L/L$ and the displayed positive factors. For Γ, use the decrease of $\log L/L^{0.7}$ and the increasing denominators. For the first term of A_*, the numerator's logarithmic derivative is

$$\frac{1}{\log L}\left(\frac{1}{r(L)} + \frac{1}{\log 7 + r(L)}\right) < 1.3;$$

for its second term, $r(L)/L^{1.3}$ decreases and $q(L)$ increases. The terms of c_2 decrease as well. In particular $a(N) < 0.00379644$ for $L \ge L_0$.

Finite interval computation. Let $E_i(L) = \varepsilon C_i(\varepsilon) e^2 h(s_J(L))$ and set

$$\mathfrak{B}(L) = 1 - \rho - \rho/11 - (1 + 1/11)E_2(L) - (3/11 + a(N))(\rho + E_1(L)) - a(N) - 1602/L^2.$$

The factor $40(1+\xi/L^2)$ in case (b) cancels its reciprocal in the final remainder term. Once $\mathfrak{B}(L) > 0$, dropping the excess positive factor gives

$$C_b(L) = 40\mathfrak{B}(L) - \frac{c_4^*(N) j(w)}{LT}.$$

We enclose this expression on $[L_0, 3000]$, retaining the denominator L. For each interval, evaluate M_3, Γ, A_* at its left endpoint and enclose $c_2(N)$ on the whole interval.

The certificate `audit/audit_js.py`, with output `expected/audit/audit_js.json`, uses 256-bit Arb arithmetic and exact rational endpoints. It verifies a cover by twelve adjacent intervals with no gaps. With $C_1 = C_2 = 114$ and the certified bound $\mathfrak{S}_2 > 0.6601617536$, the minimum lower enclosures are > 38.41700825 in case (a) and > 9.80367826 in case (b). The brace is positive on every interval. The output retains every interval and its bounds; the first is $[136843/50, 280267621/102400] = [2736.86, 2736.988486328125]$. The certificate also checks $C_1 = 113$, as given in [23]. An mpmath interval implementation, `third/js_cert/js_cert.py`, confirms the conservative conclusions $C_a > 38.417$ and $C_b > 9.8036$.

Infinite tail. For $L \geq 3000$, replace w/L by 1 in the positive remainder terms and freeze $1/\beta$ at its upper bound at $L = 3000$. Including j, the delicate exponential terms have adverse logarithmic derivatives at most $1 + 2/\log w_0 < 1.254$. The exponent derivatives satisfy

$$\frac{dE_P}{d\log w} > 10.41497, \qquad \frac{dE_S}{d\log w} > 1.45786, \qquad E_S(w) = \frac{100 w^{0.35}}{1.3^2 \log^2 w},$$

and both increase. Thus these terms decrease. The C term and remaining power and exponential terms decrease even after multiplication by j. The partial-summation multiplier decreases. For the deletion term times j/L, an upper logarithmic derivative is $-1/2 + 0.5/L + 1/(w \log w) < 0$. Also M_3, Γ, A_* decrease, while $s_J(L)$ increases and $h(s_J(L))$ decreases. Evaluation at 3000 gives coefficients > 39.28 in case (a) and > 26.218 in case (b): both implementations give more than 26.2182 in case (b), and `audit/audit_js.py`, with $C_1 = C_2 = 114$, encloses it above 26.21829064.

The coefficients therefore exceed 38.417 and 9.8036 on $[L_0, 3000]$, and 39.28 and 26.218 for $L \geq 3000$. Their positivity and $|A| > N/L$ give $S(\mathcal{A}, N^{1/40}) > 9.8036\, U_N N/L^2$ for every even N with $L_0 \leq \log N \leq 3000$, and $S(\mathcal{A}, N^{1/40}) > 26.218\, U_N N/L^2$ for every even N with $\log N \geq 3000$, proving (I4′).

References

[1] J. Bayless, P. Kinlaw and D. Klyve, *Sums over primitive sets with a fixed number of prime factors*, Math. Comp. **88** (2019), 3063–3077.

[2] M. A. Bennett, G. Martin, K. O'Bryant and A. Rechnitzer, *Explicit bounds for primes in arithmetic progressions*, Illinois J. Math. **62** (2018), 427–532; arXiv:1802.00085.

[3] M. A. Bennett, G. Martin, K. O'Bryant and A. Rechnitzer, *Counting zeros of Dirichlet L-functions*, Math. Comp. **90** (2021), 1455–1482.

[4] M. Bordignon, D. R. Johnston and V. Starichkova, *An explicit version of Chen's theorem and the linear sieve*, Int. J. Number Theory **21** (2025), no. 10, 2497–2572; arXiv:2207.09452. Theorem, lemma, corollary, table and equation numbers here are those of arXiv v6.

[5] M. Bordignon and V. Starichkova, *An explicit version of Chen's theorem assuming the Generalized Riemann Hypothesis*, Ramanujan J. **64** (2024), no. 4, 1213–1242; arXiv:2211.08844v1 (2022). Lemma, theorem and corollary numbers here are those of arXiv v1: we cite Lemma 3 and Corollary 1, which follows from Theorem 3. The published version keeps Theorem 3 and Corollary 1 and states them for $N > \exp(\exp(14))$; the threshold $\exp(\exp(15.85))$ of arXiv v1 suffices here.

[6] P. J. Campbell, *A sharper explicit result for the sum of two almost primes*, arXiv:2608.09488v1 (2026).

[7] J.-R. Chen, *On the representation of a larger even integer as the sum of a prime and the product of at most two primes*, Sci. Sinica **16** (1973), 157–176.

[8] H. Davenport, *Multiplicative Number Theory*, 3rd ed., Springer, 2000.
[9] G. de Morais, *Goldbach-type sums $a + b = N$ with $\Omega(ab) \leq 32$*, Zenodo deposit, doi:10.5281/zenodo.23456153 (2026); submitted, not peer reviewed.
[10] P. Dusart, *Estimates of some functions over primes without R.H.*, arXiv:1002.0442 (2010); published as *Explicit estimates of some functions over primes*, Ramanujan J. **45** (2018), 227–251. We cite the numbering of the arXiv version.
[11] P. Erdős, *On the normal number of prime factors of $p-1$ and some related problems concerning Euler's φ-function*, Quart. J. Math. **6** (1935), 205–213.
[12] A.-M. Ernvall-Hytönen and N. Palojärvi, *Explicit bound for the number of primes in arithmetic progressions assuming the generalized Riemann hypothesis*, Math. Comp. **91** (2022), no. 335, 1317–1365.
[13] K. (S.) Fan, *The Hardy–Ramanujan inequality for sifted sets and its applications*, arXiv:2508.06005v3 (2025).
[14] K. (S.) Fan, *Numerically explicit estimates for the distribution of rough numbers*, J. Number Theory **260** (2024), 120–150.
[15] K. Ford, *A Hardy–Ramanujan type inequality for shifted primes and sifted sets*, Lithuanian Math. J. **61** (2021), no. 3, 323–329; arXiv:2101.03440.
[16] G. Greaves, *Sieves in Number Theory*, Springer-Verlag, Berlin, 2001.
[17] G. H. Hardy and S. Ramanujan, *The normal number of prime factors of a number n*, Quart. J. Math. **48** (1917), 76–92.
[18] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 4th ed., Clarendon Press, Oxford, 1960.
[19] S. Hathi and D. R. Johnston, *On the sum of a prime and a square-free number with divisibility conditions*, J. Number Theory **256** (2024), 354–372.
[20] C. Huang (maintainer), *Explicit Chen bounds: current research drafts* (GPT-generated drafts), GitHub repository `woznickiainslie141-dot/explicit-chen-c249`, commit 15368990 (7 October 2026), files `qrh/qrh_chen_51046.tex` and `grh/conditional_chen_grh_bridge.tex`; `https://github.com/woznickiainslie141-dot/explicit-chen-c249/tree/15368990`.
[21] S. Hubbard, *A lower explicit threshold for Chen's theorem under GRH*, Hexagon preprint hexagon:2610.00560v4 (2026); `https://hexagonmath.org/2610.00560v4`.
[22] D. R. Johnston, *New explicit and asymptotic results in sieve theory*, PhD thesis, UNSW Canberra (UNSW Sydney), November 2025, doi:10.26190/unsworks/31766; abstract in Bull. Aust. Math. Soc. **113** (2026), no. 2, 372–374.
[23] D. R. Johnston and V. V. Starichkova, *Some explicit results on the sum of a prime and an almost prime*, Acta Math. Hungar. **177** (2025), no. 1, 163–192; arXiv:2208.01229v3. Theorem, lemma, equation and page references here use arXiv v3.
[24] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, 2 vols., Teubner, Leipzig, 1909.
[25] J. D. Lichtman, *Almost primes and the Banks–Martin conjecture*, J. Number Theory **211** (2020), 513–529; arXiv:1909.00804 (numbering of arXiv v2).
[26] D. Loebenberger and M. Nüsken, *Coarse-grained integers – smooth? rough? both!*, arXiv:1003.2165 (2010, revised 2012).
[27] R. J. Mathar, *Series of reciprocal powers of k-almost primes*, arXiv:0803.0900 (2008).
[28] H. L. Montgomery and R. C. Vaughan, *The large sieve*, Mathematika **20** (1973), 119–134.
[29] M. B. Nathanson, *Additive Number Theory: The Classical Bases*, Springer, 1996.
[30] T. Oliveira e Silva, S. Herzog and S. Pardi, *Empirical verification of the even Goldbach conjecture and computation of prime gaps up to $4 \cdot 10^{18}$*, Math. Comp. **83** (2014), 2033–2060.
[31] OpenAI, *The Quasi-Riemann Hypothesis: A Zero-Free Half-Plane* $\operatorname{Re} s > 7/8$, preprint (30 September 2026; fixed revision adc7f124 of the repository `openai/math` on GitHub).
[32] D. Platt and T. Trudgian, *Zeros of quadratic Dirichlet L-functions*, arXiv:2610.29018 (2026).
[33] P. Pollack, *A generalization of the Hardy–Ramanujan inequality and applications*, J. Number Theory **210** (2020), 171–182.
[34] R. A. Rankin, *The difference between consecutive prime numbers*, J. London Math. Soc. **13** (1938), 242–247.
[35] A. Rényi, *On the representation of an even number as the sum of a single prime and single almost-prime number*, Izv. Akad. Nauk SSSR Ser. Mat. **12** (1948), no. 1, 57–78 (in Russian).
[36] T. Yamada, *Explicit improvements of the Brun–Titchmarsh theorem for arbitrary intervals*, arXiv:2312.16090 (2023).

Email address: `wren.aquila7@gmail.com`