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# ALGEBRA-VALUED YANG-MILLS-TYPE THEORIES: POLYNOMIAL OBSTRUCTIONS AND HOLONOMY REDUCTIONS

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Abstract. We study algebra-valued Yang-Mills-type ($A$-YMT) theories over a smooth $n$-manifold $M$, where connections take values in a general (not necessarily associative, commutative, or Lie) $\mathfrak{m}$-graded algebra $(A, *)$ and descend to an associated algebra bundle $E_A = P \times_\rho A \to M$. We derive the variational equations, infinitesimal gauge transformations, and generalized Bianchi identities: affine Ehresmann compatibility forces the vertical image of $\mathfrak{g}$ into the Lie nucleus $\mathcal{N}_{\mathrm{Lie}}(A)$ of the commutator algebra, and both the Bianchi identity and gauge covariance are obstructed exactly by the Jacobiator, principal gauge transformations being always anomaly-free. We then prove polynomial obstructions in three settings. First, internal-trace Chern-Weil and self-dual $A$-YMT functionals vanish in weak $(2, s)$-solvable algebras for $n \geq 2(s+1)$, invariantly under $(Q, S)$-splitting extensions. Second, for reductive Cartan-Yang-Mills theories valued in $A = \mathbb{R}^k \oplus \mathfrak{h}$ we compare quartic-block and recursive binary-tree parenthesizations, obtaining a cutoff at $n \geq 8$ for blocks and a Jacobsthal-type cutoff for binary trees; we show that the soldering product $e \curlywedge_\rtimes \Omega_\omega$ is the first-Bianchi defect of the curvature, so every term containing it vanishes on-shell for torsion-free connections, and we analyse the $G_2$ case at $n = k = 7$. Third, we prove an associator-quotient collapse and a Lie/non-Lie Jacobiator dichotomy for internal $A$-Chern-Simons theories, with applications to $A$-$BF$ theories, octonionic and reductive Proca mass spectra, and dualities.

## 1. Introduction

Classical Yang-Mills (YM) theory is formulated on a principal $G$-bundle $\pi : P \to M$ over an orientable $n$-dimensional semi-Riemannian manifold $(M, g)$, where $G$ is a compact semisimple Lie group with Lie algebra $\mathfrak{g}$ [22, 13]. Given a connection 1-form $D \in \mathrm{Conn}(P; \mathfrak{g}) \subset \Omega^1(P; \mathfrak{g})$, its curvature (or field strength) is the horizontal, $G$-equivariant $\mathfrak{g}$-valued 2-form $F_D = dD + \frac{1}{2} D[\wedge]_\mathfrak{g} D \in \Omega^2_{\mathrm{heq}}(P; \mathfrak{g})$, where $[\wedge]_\mathfrak{g}$ denotes the exterior product induced by the Lie bracket of $\mathfrak{g}$. Using the canonical Weil isomorphism $j : \Omega^2_{\mathrm{heq}}(P; \mathfrak{g}) \xrightarrow{\sim} \Omega^2(M; E_\mathfrak{g})$ onto 2-forms with values in the adjoint bundle $E_\mathfrak{g} = P \times_{\mathrm{Ad}} \mathfrak{g}$, the classical Yang-Mills action functional is given by

$$S_{\mathrm{YM}}[D] = \int_M \langle F_D, F_D \rangle_\mathfrak{g} \, \mathrm{dvol}_g = \int_M \mathrm{tr}\left( j(F_D) \wedge *_g j(F_D) \right), \tag{1.1}$$

where $*_g$ is the Hodge star operator of $g$ and tr represents the Killing form $B_\mathfrak{g}$ of $\mathfrak{g}$.

While physically foundational, the classical action (1.1) relies on three structural assumptions:

(1) The base manifold $M$ must admit a semi-Riemannian metric $g$ of prescribed signature (which is subject to topological obstructions, such as $\chi(M) = 0$ for compact Lorentzian manifolds [18, 5]);
(2) The internal symmetry algebra must be a Lie algebra $\mathfrak{g}$ admitting a non-degenerate Ad-invariant quadratic form (typically requiring $G$ to be semisimple or compact, or belong to specific classes of quadratic solvable Lie algebras [31]);
(3) The gauge potential $D$ takes values in a Lie algebra rather than a more general (possibly non-associative or $\mathfrak{m}$-graded) algebra $(A, *)$ modeling internal, worldsheet, supergeometric, Malcev, or reductive Cartan symmetries [3, 43, 33, 10].

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In the context of first-order tetradic gravity, it was shown in [24] that the Einstein-Hilbert-Palatini (EHP) action functional can be detached from Lorentzian Cartan geometry and formulated algebraically over an arbitrary $\mathfrak{m}$-graded algebra $(A,*)$ decomposed as $A \simeq A_0 \oplus A_1$. Doing so revealed that the non-triviality of the $A$-valued EHP functional is governed by the graded polynomial identities of $(A,*)$: specifically, if $A_0$ is weak $(k,s)$-solvable (in the indexing of [24], which differs from the one in Definition 2.2 below), the cosmological constant decouples for $n \geq k+s+1$ and the entire $A$-EHP functional vanishes identically for $n \geq k+s+3$ [24].

The purpose of the present paper is to formulate *algebra-valued Yang-Mills-type (A-YMT) theories* directly in a concrete differential-geometric and algebraic setting (without requiring category-theoretic functorial bundle systems), establish their variational and gauge-theoretic foundations, and determine the algebraic, parenthesization, and holonomy obstructions that govern them. Given a principal $G$-bundle $\pi : P \to M$ and a representation $\rho : G \to \mathrm{Aut}_{\mathrm{Alg}}(A)$ of $G$ by algebra automorphisms of an $\mathfrak{m}$-graded algebra $(A,*)$, horizontal $\rho$-equivariant $A$-valued differential forms $\Omega^k_{\mathrm{heq},\rho}(P;A)$ descend via the canonical Weil algebra isomorphism $j : \Omega^\bullet_{\mathrm{heq},\rho}(P;A) \xrightarrow{\sim} \Omega^\bullet(M;E_A)$ to forms valued in the associated algebra bundle $E_A = P \times_\rho A \to M$. We simultaneously decouple the quadratic curvature pairing from both the metric Hodge star $*_g$ and the Killing form $B_\mathfrak{g}$ by allowing a general $\mathbb{R}$-bilinear or $C^\infty(M)$-bilinear pairing $\langle\cdot,\cdot\rangle_A$ on $\Omega^2(M;E_A)$, in the spirit of the Yang-Mills-type theories of [25].

Gauge theories with non-associative or non-Lie internal structure have been considered before: octonionic and Moufang-loop gauge theories [29, 23], octonion-valued Yang-Mills connections with gauge group the automorphisms of $\mathbb{O}$ [34], gauge theory on smooth loop bundles [17], and Lie-algebroid and Leibniz-algebra gauge theories [39, 6]. In contrast with these works, here the structure group $G$ remains a Lie group acting by algebra automorphisms and only the coefficient algebra is allowed to be non-Lie; the failure of gauge covariance is then measured by the Jacobiator of the commutator algebra and localized outside its Lie nucleus.

A fundamental structural contrast immediately emerges between $A$-EHP gravity and $A$-YMT gauge theory:

- In $A$-EHP theory, the Hilbert-Palatini $n$-form $\alpha_{n,\Lambda} = \wedge_*^{n-2} e \wedge_* \Omega + \frac{\Lambda}{(n-1)!} \wedge_*^n e$ contains a number of factors of $e$ that grows linearly with $\dim M = n$, requiring either quotienting by associativity-like polynomial ideals or fixing a parenthesization scheme whenever $(A,*)$ is non-associative, and converting $(k,s)$-solvability directly into upper bounds on the spacetime dimension $n$ [24].
- In pure $A$-YMT theory, the curvature 2-form $F_D = dD + \frac{1}{2} D \wedge_* D = dD + \frac{1}{4} D \wedge_{[\cdot,\cdot]_*} D$ involves only a single binary product $D \wedge_* D$, which automatically projects onto the skew-symmetric commutator bracket $[x,y]_* = x*y - y*x$ while annihilating the symmetric Jordan product $x \circ_* y = \frac{1}{2}(x*y + y*x)$. Consequently, $F_D$ is unambiguously defined for *any* non-associative algebra $(A,*)$ without choosing parenthesizations or quotienting by associativity ideals, and the action $S^A[D] = \int_M \langle j(F_D), j(F_D)\rangle_A$ is universally quartic in $D$ across all spacetime dimensions $n \geq 2$. However, whenever higher curvature powers $\wedge_*^r F_D$, constitutive soldering powers $\wedge_*^{n-4} e$, or differential identities such as the Bianchi identity $dF_D + \frac{1}{2}[D, F_D]_*$ and Chern-Simons transgression are formed, the symmetric product $\circ_*$, the parenthesization scheme, the associator $\mathrm{Assoc}_*$, and the commutator Jacobiator $\mathrm{Jac}_{[\cdot,\cdot]_*}$ play decisive roles.

Specifically, the main results of this paper are organized as follows:

(1) **Ehresmann Admissibility, the Lie Nucleus Constraint, Variational Field Equations, and the Jacobiator Bianchi/Gauge Obstructions (Proposition 2.4 and Theorem 2.7):** On a principal $G$-bundle $\pi : P \to M$ with automorphism representation $\rho : G \to \mathrm{Aut}_{\mathrm{Alg}}(A)$, we prove that an affine Ehresmann connection $D \in \Omega^1_{\mathrm{eq},\rho}(P;A)$ with vertical map $D(\underline{X}) = \eta(X)$ has horizontal curvature $F_D \in \Omega^2_{\mathrm{heq},\rho}(P;A)$ if and only if $d\rho(X)(y) = \frac{1}{2}[\eta(X), y]_*$, which in turn forces the vertical image $\eta(\mathfrak{g})$ to lie in the *Lie nucleus* $\mathcal{N}_{\mathrm{Lie}}(A)$ of $(A,[\cdot,\cdot]_*)$. We then derive the Euler-Lagrange equations of a general $A$-YMT theory and prove that under an infinitesimal gauge transformation $\delta_\epsilon D = d\epsilon + \frac{1}{2} D \wedge_{[\cdot,\cdot]_*} \epsilon$, the failure of homogeneous curvature covariance $\delta_\epsilon F_D - \frac{1}{2} F_D \wedge_{[\cdot,\cdot]_*} \epsilon$

and the failure of the differential Bianchi identity $dF_D + \frac{1}{2}D \wedge_{[\cdot,\cdot]_*} F_D$ are governed purely by the horizontal Jacobiator 2-form $-\frac{1}{8}\mathrm{Jac}_{[\cdot,\cdot]_*}(D, D, \epsilon)$ and 3-form $-\frac{1}{24}\mathrm{Jac}_{[\cdot,\cdot]_*}(D, D, D)$. In particular, principal $G$-gauge transformations $\epsilon \in \Omega^0_{\mathrm{heq},\rho}(P; \eta(\mathfrak{g})) \subset \Omega^0_{\mathrm{heq},\rho}(P; \mathcal{N}_{\mathrm{Lie}}(A))$ are universally anomaly-free for any algebra $(A, *)$, whereas extended $A$-gauge transformations $\epsilon \in \Omega^0_{\mathrm{heq},\rho}(P; A)$ outside $\mathcal{N}_{\mathrm{Lie}}(A)$ are obstructed by $\mathrm{Jac}_{[\cdot,\cdot]_*}$. Using a rigorous pointwise localization argument relative to the basepoint connection $D_0 = \eta \circ \omega_0$, we also establish the exact converse when $\langle\cdot,\cdot\rangle_{A,\mathrm{sym}}$ is $C^\infty(M)$-bilinear (tensorial), pointwise $\mathrm{ad}_{[\cdot,\cdot]_*}$-invariant, and fiberwise non-degenerate.

(2) **Internal-Product Topological and Self-Dual $A$-YMT Obstructions (Theorem 3.2 and Proposition 3.3):** After recording the commutator projection, abelianization of weak $(1,1)$-solvable algebras, and derived-filtration localization (Proposition 3.1), we study internal-trace topological and self-dual pairings constructed purely from the internal multiplication $\wedge_*$ and a bundle linear functional $\mathrm{tr}_\xi : E_A \to M \times \mathbb{R}$ (as opposed to an external symmetric bilinear invariant form $\kappa \in \mathrm{inv}^2(\mathfrak{g})$). We prove that weak $(2,s)$-solvability forces the $r$-th degree internal Chern-Weil action $\int_M \mathrm{tr}_\xi(j(\wedge_*^r F_D))$ in dimension $n = 2r$ to vanish for $n \geq 2(s+1)$, and we show that these obstructions are preserved under $(Q,S)$-splitting extensions. For $s = 1$, this internal-product vanishing encompasses all skew-commutative algebras (including Lie algebras under $[\wedge]_{\mathfrak{g}}$, Berger holonomy algebras, Hitchin's generalized complex geometry, and Cayley-Dickson unitary algebras $\mathfrak{u}(k; \mathrm{CD}^l(A))$ under the skew-symmetric matrix product $\curlywedge$).

(3) **Parenthesization Schemes, the First-Bianchi Defect, and Berger Holonomy Reductions in Cartan-Yang-Mills Theories (Lemmas 2.1, 4.3, 4.4, Theorem 4.6, and Proposition 4.7):** For reductive connections $\nabla = e + \omega$ valued in $A = \mathbb{R}^k \oplus \mathfrak{h}$, $\mathfrak{h} \subset \mathfrak{so}(k)$, we formulate an internal product $\curlywedge_\rtimes$ via an $H$-equivariant soldering map $\sigma : \mathbb{R}^k \wedge \mathbb{R}^k \to \mathfrak{h}$ (and the $G_2$ cross product when $k = 7$). We compare two parenthesization schemes for powers of $e$: quartic blocks, for which $\curlywedge^m_{\rtimes,\mathrm{blk}} e = 0$ for all $m \geq 4$, and recursive binary trees, for which $\curlywedge^m_{\rtimes,\mathrm{bal}} e$ can be non-zero only if $m \leq k$ and $m \in \{1,2,3\} \cup \bigcup_{r\geq 3}\{2J_r, J_{r+1}\} = \{1,2,3,5,6,10,11,21,22,\dots\}$, where $J_r$ are the Jacobsthal numbers. Under quartic blocks the cosmological term vanishes for $n \geq 4$, the EHP term for $n \geq 6$, and the whole Cartan-Yang-Mills functional for $n \geq 8$; under binary trees the Yang-Mills sectors agree with the block ones up to $n = 8$, but the EHP and cosmological sectors can survive for $n \in \{5,6,7,8\}$ (we exhibit this for $U(3)$ and $G_2$). When $\times_H = 0$ we show that $e \curlywedge_\rtimes \Omega_\omega = (-\frac{1}{2}\Omega_\omega \cdot e, 0)$ is exactly the algebraic first-Bianchi defect of $\Omega_\omega$; together with the identity $e \curlywedge_\rtimes (e \curlywedge_\rtimes \Omega_\omega) = \frac{1}{2}(e \curlywedge_\rtimes e) \curlywedge_\rtimes \Omega_\omega$, this shows that every Lagrangian term containing this factor vanishes on-shell for every torsion-free connection, in every dimension and for every $\mathfrak{h}$ (in particular at $n = k = 4$ for $U(2)$, $SU(2)$ and at $n = k = 6$ for $U(3)$, $SU(3)$), while it survives off-shell. At $n = k = 7$ with $H = G_2$ we compute the cubic soldering form, $\curlywedge^3_{\rtimes,\mathrm{bal}} e = (-7\, i_{\partial_a}(*\varphi_0), 0)$, and show that the quadratic Yang-Mills 7-form is controlled by a Ricci-type contraction of the curvature, so it vanishes on-shell on torsion-free $G_2$-curvatures and is non-zero for suitable off-shell curvatures.

(4) **Associator-Quotient Collapse and Jacobiator Dichotomy in Internal Chern-Simons Theories, with Applications to $BF$, Higgs, and Dual Theories (Theorem 5.3 and Propositions 5.4–5.6):** We extend our framework to odd-dimensional internal $A$-valued Chern-Simons ($A$-CS) theories on linear connection spaces. We prove a three-way structural dichotomy for skew-commutative algebras $(A, \curlywedge)$: (a) in the associative quotient, all $\geq$ 3-fold products vanish identically; (b) in the unquotiented grouped formulation, if $(A, \curlywedge)$ is a Lie algebra ($\mathrm{Jac}_\curlywedge = 0$), the Jacobi identity forces $(D \curlywedge D) \curlywedge D = \frac{1}{3}\mathrm{Jac}_\curlywedge(D, D, D) = 0$, annihilating the 3D cubic vertex and killing all $\geq$ 5D internal $A$-CS theories under *both* quartic-block and recursive binary-tree parenthesizations; and (c) if $(A, \curlywedge)$ is a non-Lie skew-commutative algebra ($\mathrm{Jac}_\curlywedge \neq 0$), the 3D cubic and 5D quartic Jacobiator vertices survive under

quartic-block grouping (with an additional 5D quintic Jacobsthal vertex under recursive binary halving), while for $n \geq 7$ all block-grouped $A$-CS theories vanish identically and binary-tree $A$-CS theories vanish outside the sparse odd Jacobsthal pairs $n \in \bigcup_{m\geq 4}\{J_m, J_m+2\} \cap [7,\infty) = \{7\} \cup \{11,13\} \cup \{21,23\} \cup \dots$ (and for $n \geq \dim A + 3$ by rank). We conclude with applications to Cattaneo-Plebanski $A$-$BF$ equivalences, explicit octonionic and reductive Proca mass spectra in the $A$-valued Yang-Mills-Higgs mechanism, and reductive dualities.

## 2. Geometric-Algebraic Foundations of $A$-YMT Theories

2.1. **Non-Associative Algebras, Parenthesization, and $(k,s)$-Solvability.** Let $R$ be a commutative unital ring of characteristic zero (throughout, $R = \mathbb{R}$ or $\mathbb{C}$) and let $(A, *)$ be an $R$-algebra, i.e., an $R$-module $A$ equipped with an $R$-bilinear multiplication $* : A \times A \to A$, without assuming associativity, commutativity, or a unit element [38]. Every multiplication $*$ decomposes canonically into its skew-symmetric commutator bracket and its symmetric Jordan product:

$$x * y = \frac{1}{2}[x,y]_* + x \circ_* y, \quad \text{where} \quad [x,y]_* := x*y - y*x, \quad x \circ_* y := \frac{1}{2}(x*y + y*x). \tag{2.1}$$

We denote the *associator* of $(A,*)$ and the *Jacobiator* of $(A,[\cdot,\cdot]_*)$ for $x,y,z \in A$ by

$$\mathrm{Assoc}_*(x,y,z) := (x*y)*z - x*(y*z), \quad \mathrm{Jac}_{[\cdot,\cdot]_*}(x,y,z) := [[x,y]_*,z]_* + [[y,z]_*,x]_* + [[z,x]_*,y]_*. \tag{2.2}$$

Thus $(A,[\cdot,\cdot]_*)$ is a Lie algebra if and only if $\mathrm{Jac}_{[\cdot,\cdot]_*}(x,y,z) = 0$ for all $x,y,z \in A$ (in which case $(A,*)$ is called *Lie-admissible* [38]; every associative algebra is Lie-admissible since $\mathrm{Jac}_{[\cdot,\cdot]_*}(x,y,z) = \sum_{\sigma\in S_3} \mathrm{sign}(\sigma)\,\mathrm{Assoc}_*(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)})$). More generally, the *Lie nucleus* (or *Jacobi nucleus*) of $(A,[\cdot,\cdot]_*)$ is the Lie subalgebra

$$\mathcal{N}_{\mathrm{Lie}}(A) := \left\{ x \in A \;\middle|\; \mathrm{Jac}_{[\cdot,\cdot]_*}(x,y,z) = 0 \text{ for all } y,z \in A \right\}. \tag{2.3}$$

By definition of $\mathrm{Jac}_{[\cdot,\cdot]_*}$, the inner adjoint map $\mathrm{ad}_x^{[\cdot,\cdot]_*}(y) := [x,y]_*$ is a derivation of the commutator bracket $[y,z]_*$ if and only if $x \in \mathcal{N}_{\mathrm{Lie}}(A)$.

Given a smooth manifold $P$, let $\Omega(P;A) = \bigoplus_{k\geq 0}\Omega^k(P;A)$ denote the $\mathbb{N}$-graded $R$-module of smooth $A$-valued differential forms on $P$, where $\Omega^k(P;A) = \Gamma(\Lambda^k T^*P \otimes A)$. The bilinear multiplication $* : A \otimes A \to A$ induces a canonical exterior product

$$\wedge_* : \Omega^k(P;A) \times \Omega^l(P;A) \longrightarrow \Omega^{k+l}(P;A), \tag{2.4}$$

defined pointwise at $p \in P$ on tangent vectors $v_1,\dots,v_{k+l} \in T_pP$ by

$$(\alpha \wedge_* \beta)_p(v_1,\dots,v_{k+l}) = \frac{1}{k!\,l!}\sum_{\sigma\in S_{k+l}} \mathrm{sign}(\sigma)\,\alpha_p(v_{\sigma(1)},\dots,v_{\sigma(k)}) * \beta_p(v_{\sigma(k+1)},\dots,v_{\sigma(k+l)}). \tag{2.5}$$

In terms of the decomposition (2.1), for any $\alpha \in \Omega^k(P;A)$ and $\beta \in \Omega^l(P;A)$ we have

$$\alpha \wedge_{[\cdot,\cdot]_*} \beta = \alpha \wedge_* \beta - (-1)^{kl}\beta \wedge_* \alpha, \qquad \alpha \wedge_{\circ_*} \beta = \frac{1}{2}(\alpha \wedge_* \beta + (-1)^{kl}\beta \wedge_* \alpha). \tag{2.6}$$

Setting $\alpha = \beta \in \Omega^k(P;A)$ in (2.6) yields the basic parity dichotomy of exterior squares:

- **Odd-Degree Forms ($k$ odd, e.g., connection 1-forms $D \in \Omega^1(P;A)$):** $(-1)^{k^2} = -1$, so $\alpha \wedge_{\circ_*} \alpha = 0$ identically and

$$\alpha \wedge_* \alpha = \frac{1}{2}\alpha \wedge_{[\cdot,\cdot]_*} \alpha \quad (\alpha \in \Omega^{\mathrm{odd}}(P;A)). \tag{2.7}$$

  Hence the exterior square of any odd-degree form depends *solely* on the commutator bracket $[x,y]_*$ and vanishes identically for all odd forms (when $\dim P \geq k$) if and only if $(A,*)$ is commutative ($[x,y]_* = 0$).

- **Even-Degree Forms ($k$ even, e.g., curvature 2-forms $F_D \in \Omega^2(P;A)$):** $(-1)^{k^2} = +1$, so $\alpha \wedge_{[\cdot,\cdot]_*} \alpha = 0$ identically and
$$\alpha \wedge_* \alpha = \alpha \wedge_{\circ_*} \alpha \quad (\alpha \in \Omega^{\text{even}}(P;A)). \tag{2.8}$$
Hence the exterior square of any even-degree form depends *solely* on the symmetric Jordan product $x \circ_* y$ and vanishes identically whenever $(A,*)$ is skew-commutative ($x \circ_* y = 0$).

Notice that when passing from algebra elements $x_i \in A$ to differential forms $\alpha_i \in \Omega^{k_i}(P;A)$, permuting odd-degree forms introduces Koszul signs $(-1)^{k_i k_j}$ that twist symmetric algebra identities into alternating ones (and vice versa). Thus, whereas a non-permuting multilinear identity $p(x_1,\dots,x_m) = 0$ (such as a parenthesized nilpotency identity $x_1 * \cdots * x_m = 0$) or an identity evaluated on even-degree forms lifts directly by multilinearity [21, 14], polynomial nilpotency conditions on $k$-forms must be defined directly on $\Omega^k(P;A)$ [24].

When $(A,*)$ is non-associative, products of $m \geq 3$ forms depend essentially on the choice of parenthesization. To avoid any ambiguity in higher-degree powers $\wedge_*^m \alpha$, we distinguish three explicit parenthesization conventions for an $A$-valued form $\alpha \in \Omega^k(P;A)$:

(1) **Full parenthesization:** All non-associative monomial parenthesizations $p_m(\alpha,\dots,\alpha)$ of degree $m$.

(2) **Recursive binary-tree parenthesization ($\wedge^m_{*,\text{bal}}\alpha$):** Defined inductively for $m \geq 1$ by $\wedge^1_{*,\text{bal}}\alpha := \alpha$ and by exact binary halving
$$\wedge^m_{*,\text{bal}}\alpha := \left(\wedge^{\lceil m/2\rceil}_{*,\text{bal}}\alpha\right) \wedge_* \left(\wedge^{\lfloor m/2\rfloor}_{*,\text{bal}}\alpha\right) \quad (m \geq 2). \tag{2.9}$$
In particular, for $1 \leq m \leq 8$, recursive binary halving produces:
$$\begin{aligned}
&\wedge^2_{*,\text{bal}}\alpha = \alpha \wedge_* \alpha, \qquad \wedge^3_{*,\text{bal}}\alpha = (\alpha \wedge_* \alpha) \wedge_* \alpha, \qquad \wedge^4_{*,\text{bal}}\alpha = (\alpha \wedge_* \alpha) \wedge_* (\alpha \wedge_* \alpha),\\
&\wedge^5_{*,\text{bal}}\alpha = ((\alpha \wedge_* \alpha) \wedge_* \alpha) \wedge_* (\alpha \wedge_* \alpha), \qquad \wedge^6_{*,\text{bal}}\alpha = ((\alpha \wedge_* \alpha) \wedge_* \alpha) \wedge_* ((\alpha \wedge_* \alpha) \wedge_* \alpha),\\
&\wedge^7_{*,\text{bal}}\alpha = \left(\wedge^4_{*,\text{bal}}\alpha\right) \wedge_* \left(\wedge^3_{*,\text{bal}}\alpha\right), \qquad \wedge^8_{*,\text{bal}}\alpha = \left(\wedge^4_{*,\text{bal}}\alpha\right) \wedge_* \left(\wedge^4_{*,\text{bal}}\alpha\right).
\end{aligned} \tag{2.10}$$

(3) **Quartic-block parenthesization ($\wedge^m_{*,\text{blk}}\alpha$):** Defined so that it agrees with $\wedge^m_{*,\text{bal}}\alpha$ for $1 \leq m \leq 4$, and for $m \geq 5$ factors out leading quartic blocks $\wedge^4_{*,\text{bal}}\alpha$ recursively:
$$\wedge^m_{*,\text{blk}}\alpha := \begin{cases} \wedge^m_{*,\text{bal}}\alpha, & \text{if } 1 \leq m \leq 4,\\ \left(\wedge^4_{*,\text{bal}}\alpha\right) \wedge_* \left(\wedge^{m-4}_{*,\text{blk}}\alpha\right), & \text{if } m \geq 5. \end{cases} \tag{2.11}$$
By convention, when $m = 0$, $\left(\wedge^0_{*,\text{bal}}\alpha\right) \wedge_* \beta = \left(\wedge^0_{*,\text{blk}}\alpha\right) \wedge_* \beta := \beta$ omits the $\alpha$-factor.

**Lemma 2.1** (Comparison of Parenthesization Schemes and All-Orders Jacobsthal Characterization on 1-Forms in Skew-Commutative Algebras)**.** *Let $(A,*)$ be a skew-commutative algebra ($x * y = -y * x$, so $x \circ_* y = 0$), and let $e \in \Omega^1(P;A)$ be an $A$-valued 1-form.*

(1) ***Quartic-Block Vanishing:*** *Because $e \wedge_* e \in \Omega^2(P;A)$ is an even-degree form,* (2.8) *gives*
$$\wedge^4_{*,\text{bal}}e = (e \wedge_* e) \wedge_* (e \wedge_* e) = 0. \tag{2.12}$$
*Consequently, under* quartic-block parenthesization (2.11)*, $\wedge^m_{*,\text{blk}}e = 0$ identically for all $m \geq 4$.*

(2) ***Recursive Binary-Tree Halving and Jacobsthal Survivor Pairs:*** *Under* recursive binary-tree parenthesization (2.9)*, let $J_r := \frac{2^r-(-1)^r}{3}$ ($r \geq 1$) denote the $r$-th Jacobsthal number ($J_1 = 1, J_2 = 1, J_3 = 3, J_4 = 5, J_5 = 11, J_6 = 21, J_7 = 43, \dots$, satisfying $J_{r+1} = 2J_r + (-1)^r$), and define the set of binary-tree survivor degrees*
$$\mathcal{S}_{\text{bal}} := \{1,2,3\} \cup \bigcup_{r=3}^{\infty} \{2J_r,\, J_{r+1}\} = \{1,2,3\} \cup \{5,6\} \cup \{10,11\} \cup \{21,22\} \cup \{42,43\} \cup \dots \tag{2.13}$$

(a) *For every integer $m \notin \mathcal{S}_{\text{bal}}$ (in particular, for $m = 4$, $7 \leq m \leq 9$, $12 \leq m \leq 20$, $23 \leq m \leq 41$, etc.), $\wedge^m_{*,\text{bal}}e = 0$ holds identically in $\Omega^m(P;A)$. Moreover, if $\wedge^3_{*,\text{bal}}e = 0$ as well (which holds whenever $(A,*)$ is a Lie algebra), then $\wedge^m_{*,\text{bal}}e = 0$*

*for* all $m \geq 3$*; more generally, if* $\wedge^{J_r}_{*,\mathrm{bal}} e = 0$ *for some* $r \geq 3$*, then* $\wedge^{m}_{*,\mathrm{bal}} e = 0$ *for all* $m \geq 2J_r - 1$.

(b) *Conversely, at each dyadic level* $r \geq 3$*, the consecutive pair* $\{2J_r,\, J_{r+1}\} \subset \mathcal{S}_{\mathrm{bal}}$ *consists of the exterior square of the odd-degree* $J_r$*-form* $\wedge^{J_r}_{*,\mathrm{bal}} e \in \Omega^{J_r}(P; A)$,

$$\wedge^{2J_r}_{*,\mathrm{bal}} e = (\wedge^{J_r}_{*,\mathrm{bal}}\, e) \wedge_* (\wedge^{J_r}_{*,\mathrm{bal}}\, e) = \frac{1}{2}(\wedge^{J_r}_{*,\mathrm{bal}}\, e) \wedge_{[\cdot,\cdot]_*} (\wedge^{J_r}_{*,\mathrm{bal}}\, e),$$

*which by* (2.7) *projects onto the commutator bracket* $[x, y]_* = 2\, x * y$ *rather than the Jordan product, together with the odd-degree product of* $\wedge^{J_r}_{*,\mathrm{bal}} e$ *and its even neighbor* $\wedge^{J_r+(-1)^r}_{*,\mathrm{bal}} e = \wedge^{2J_{r-1}}_{*,\mathrm{bal}} e$*:*

$$\wedge^{J_{r+1}}_{*,\mathrm{bal}} e = \begin{cases} (\wedge^{J_r}_{*,\mathrm{bal}}\, e) \wedge_* (\wedge^{2J_{r-1}}_{*,\mathrm{bal}}\, e), & \text{if } r \text{ is odd } (J_{r+1} = 2J_r - 1), \\ (\wedge^{2J_{r-1}}_{*,\mathrm{bal}}\, e) \wedge_* (\wedge^{J_r}_{*,\mathrm{bal}}\, e), & \text{if } r \text{ is even } (J_{r+1} = 2J_r + 1). \end{cases}$$

(c) *Furthermore, if* $e$ *takes values in a* $k$*-dimensional subspace* $V \subset A$ *(or has pointwise rank* $\leq k$*, so that* $e = \sum_{a=1}^{k} e^a \otimes v_a$ *is spanned by* $k$ *scalar 1-forms* $e^1, \ldots, e^k \in \Omega^1(P)$*), then* $\wedge^{m}_{*,\mathcal{P}} e = 0$ *holds identically on* $P$ *for all* $m > k$ *and every parenthesization scheme* $\mathcal{P}$ *(since* $\Lambda^m(\mathbb{R}^k)^* = 0$ *for* $m > k$*). In particular, if* $\mathrm{rk}(e) \leq 9$*, then* $\wedge^{m}_{*,\mathrm{bal}} e = 0$ *for all* $m \geq 7$.

(3) ***Products in Lie Algebras and Left-Normed Products:*** *If* $(A, *)$ *is a Lie algebra (for instance* $\mathfrak{so}(k)$ *with* $X * Y = \frac{1}{2}[X, Y]$*), then every product of* $m \geq 3$ *copies of* $e$*, under* any *parenthesization, vanishes identically. In a non-Lie skew-commutative algebra, by contrast, left-normed products* $((e \wedge_* e) \wedge_* e) \wedge_* e$ *need not vanish at degree* 4.

*Proof.* Since $e \wedge_* e$ has even degree 2 and $(A, *)$ is skew-commutative, (2.8) yields $\wedge^{4}_{*,\mathrm{bal}} e = (e \wedge_* e) \wedge_* (e \wedge_* e) = 0$. By (2.11), $\wedge^{m}_{*,\mathrm{blk}} e = (\wedge^{4}_{*,\mathrm{bal}} e) \wedge_* (\wedge^{m-4}_{*,\mathrm{blk}} e) = 0$ for all $m \geq 4$, proving (1).

For (2a)–(2b), note that under recursive binary halving (2.9), an even degree $m = 2q$ satisfies $\wedge^{2q}_{*,\mathrm{bal}} e = (\wedge^{q}_{*,\mathrm{bal}} e) \wedge_* (\wedge^{q}_{*,\mathrm{bal}} e)$, which vanishes identically by (2.8) whenever $q$ is even, and can survive only when $q$ is odd and $\wedge^{q}_{*,\mathrm{bal}} e \neq 0$. Meanwhile, an odd degree $m = 2q + 1$ splits into $\lceil m/2 \rceil = q + 1$ and $\lfloor m/2 \rfloor = q$, so $\wedge^{2q+1}_{*,\mathrm{bal}} e = (\wedge^{q+1}_{*,\mathrm{bal}} e) \wedge_* (\wedge^{q}_{*,\mathrm{bal}} e)$ can survive only if the consecutive pair $\{q, q + 1\}$ both survive at the previous level. Let $\mathcal{S}$ be the set of degrees not forced to vanish by these two rules, i.e. $1 \in \mathcal{S}$, $2q \in \mathcal{S}$ if and only if $q \in \mathcal{S}$ is odd, and $2q + 1 \in \mathcal{S}$ if and only if $q, q + 1 \in \mathcal{S}$. Directly, $\mathcal{S} \cap [1, 4] = \{1, 2, 3\}$; in particular no power of two $2^j \geq 4$ lies in $\mathcal{S}$. We claim that $\mathcal{S} \cap I_r = \{2J_r, J_{r+1}\}$ for every $r \geq 3$, where $I_r := [2^{r-1} + 1,\, 2^r]$. For $r = 3$, 5 (halves $\{2, 3\}$) and $6 = 2 \cdot 3$ lie in $\mathcal{S}$, while 7 (halves $\{3, 4\}$) and $8 = 2 \cdot 4$ do not, and $\{5, 6\} = \{J_4, 2J_3\}$. Assume the claim for $r$ and let $m \in I_{r+1}$. The halves of $m$ lie in $[2^{r-1}, 2^r]$ and $2^{r-1} \notin \mathcal{S}$, so they must lie in $\mathcal{S} \cap I_r = \{2J_r, J_{r+1}\}$, a pair of consecutive integers with $2J_r$ even and $J_{r+1}$ odd. If $m = 2q$, then $q$ must be odd, so $q = J_{r+1}$ and $m = 2J_{r+1}$. If $m = 2q + 1$, then $\{q, q + 1\} = \{2J_r, J_{r+1}\}$ and $m = 2J_r + J_{r+1} = J_{r+2}$ by the recurrence $J_{r+2} = J_{r+1} + 2J_r$. Both values lie in $\mathcal{S} \cap I_{r+1}$, and they are consecutive since $J_{r+2} - 2J_{r+1} = (-1)^{r+1}$. This closes the induction, so $\mathcal{S} = \mathcal{S}_{\mathrm{bal}}$ and $\wedge^{m}_{*,\mathrm{bal}} e = 0$ identically for all $m \notin \mathcal{S}_{\mathrm{bal}}$. (These rules only prove vanishing; for a generic skew-commutative algebra of dimension at least 11, a direct computation shows that $\wedge^{m}_{*,\mathrm{bal}} e \neq 0$ for every $m \in \mathcal{S}_{\mathrm{bal}} \cap [1, 11]$.) Moreover, if $\wedge^{J_r}_{*,\mathrm{bal}} e = 0$ for some $r \geq 2$, then both $\wedge^{2J_r}_{*,\mathrm{bal}} e$ and $\wedge^{J_{r+1}}_{*,\mathrm{bal}} e$ vanish, which inductively annihilates all higher levels.

For (2c), expanding $e = \sum_{a=1}^{k} e^a \otimes v_a$ shows by multilinearity that every $m$-fold exterior product $\wedge^{m}_{*,\mathcal{P}} e$ is a linear combination of $m$-forms $e^{a_1} \wedge \cdots \wedge e^{a_m}$, all of which vanish identically for $m > k$.

For (3), evaluating $(e \wedge_* e) \wedge_* e$ on $v_1, v_2, v_3$ gives $2 \sum_{\mathrm{cyc}} (e(v_1) * e(v_2)) * e(v_3)$, twice the Jacobiator of the skew-commutative product $*$, which vanishes when $(A, *)$ is a Lie algebra; also $e \wedge_* (e \wedge_* e) = -(e \wedge_* e) \wedge_* e$. A binary parenthesization of $m \geq 3$ leaves always contains a sub-product with 3 or 4 leaves both of whose factors have at most 2 leaves, i.e. one of $(e \wedge_* e) \wedge_* e$, $e \wedge_* (e \wedge_* e)$, $(e \wedge_* e) \wedge_* (e \wedge_* e)$; all three vanish in a Lie algebra, hence so does the whole product.

In a non-Lie skew-commutative algebra the degree-4 left-normed product is generically non-zero, as a direct computation with random structure constants shows. □

**Definition 2.2** ($(k,s)$-Nil and $(k,s)$-Solvable Algebras [24])**.** Let $(A,*)$ be an $R$-algebra and let $k,s\geq 1$ be integers. Fix a parenthesization scheme $\mathcal{P}\in\{\text{full},\text{bal},\text{blk}\}$.

(1) We say that $(A,*)$ is $\mathcal{P}$-$(k,s)$-*nilpotent of index* $\leq s$ (or simply $(k,s)$-*nil* when $\mathcal{P}=\text{full}$, *binary-balanced* $(k,s)$-*nil* when $\mathcal{P}=\text{bal}$, and *block-balanced* $(k,s)$-*nil* when $\mathcal{P}=\text{blk}$) if for every smooth manifold $P$ and every $A$-valued $k$-form $\alpha\in\Omega^k(P;A)$, we have $\wedge^m_{*,\mathcal{P}}\alpha=0$ for all $m\geq s+1$. (For $s=1$, $\wedge^2_*\alpha=\alpha\wedge_*\alpha$ is a single binary product, so all three schemes coincide: $(1,1)$-nilpotency is equivalent to commutativity of $(A,*)$ and $(2,1)$-nilpotency is equivalent to skew-commutativity of $(A,*)$. By Lemma 2.1(1), every skew-commutative algebra is block-balanced $(1,3)$-nil since $\wedge^m_{*,\text{blk}}\alpha=0$ for all $\alpha\in\Omega^1(P;A)$ and $m\geq 4$.)

(2) A submodule $V\subset A$ is a $\mathcal{P}$-$(k,s)$-*nilpotent submodule* if every $V$-valued $k$-form $\alpha\in\Omega^k(P;V)$ satisfies $\wedge^m_{*,\mathcal{P}}\alpha=0$ for all $m\geq s+1$ in $\Omega(P;A)$.

(3) We say that $A$ (resp. $V\subset A$) is $\mathcal{P}$-$(k,s)$-*solvable*—denoted $(k,s)$-*solv*—if it decomposes as a finite direct sum $A=\bigoplus_{i=1}^r A_i$ of mutually orthogonal $\mathcal{P}$-$(k,s)$-nilpotent subalgebras satisfying $A_i*A_j=0$ for $i\neq j$ (so each $A_i$ is an ideal of $A$; the condition $A_i*A_i\subset A_i$ guarantees that iterated products of the components of a form remain in their summand). More generally, if $A=\bigoplus_{i=1}^r A_i$ is a vector space direct sum of $\mathcal{P}$-$(k,s)$-nilpotent subalgebras without $A_i*A_j=0$, a $k$-form $\alpha=\sum_{i=1}^r\alpha^{(i)}\in\Omega^k(P;A)$ is called *algebraically split-adapted* if $\alpha^{(i)}\wedge_*\alpha^{(j)}=0$ for all $i\neq j$; we denote the set of algebraically split-adapted $k$-forms on $P$ by $\mathcal{C}^k_{\text{split}}(P;A)$.

(4) Suppose $A=\bigoplus_{m\in\mathfrak{m}}A^m$ is an $\mathfrak{m}$-graded algebra for an abelian group (or monoid) $(\mathfrak{m},+)$. A graded submodule $V=\bigoplus_{m\in\mathfrak{m}}V^m$ is *homogeneous weak* $\mathcal{P}$-$(k,s)$-*nilpotent* if $\wedge^r_{*,\mathcal{P}}\alpha^m=0$ for all $r\geq s+1$ and each individual homogeneous degree $m\in\mathfrak{m}$ [24], and *(multidegree) weak* $\mathcal{P}$-$(k,s)$-*nilpotent* if $p_r(\alpha^{m_1},\dots,\alpha^{m_r})=0$ for all $\mathcal{P}$-parenthesizations of degree $r\geq s+1$, every $V$-valued $k$-form $\alpha=\sum_{m\in\mathfrak{m}}\alpha^m$, and all $m_1,\dots,m_r\in\mathfrak{m}$. (Notice that for finite-dimensional algebras or finitely supported graded sums, multidegree weak $\mathcal{P}$-$(k,s)$-nilpotency of $V$ is equivalent by multilinearity to ungraded $\mathcal{P}$-$(k,s)$-nilpotency of $V$ and differs only for infinite-dimensional $\mathfrak{m}$-gradings, whereas homogeneous weak $\mathcal{P}$-$(k,s)$-nilpotency requires vanishing only on single-degree forms $\alpha^m\in\Omega^k(P;V^m)$ and is strictly weaker even in finite dimensions.) We say that $V$ (and in particular $A$) is *weak* $\mathcal{P}$-$(k,s)$-*solvable* (resp. *homogeneous weak* $\mathcal{P}$-$(k,s)$-*solvable*) if it decomposes as a finite direct sum $V=\bigoplus_{i=1}^r V_i$ of mutually orthogonal weak (resp. homogeneous weak) $\mathcal{P}$-$(k,s)$-nilpotent graded subalgebras ($V_i*V_i\subset V_i$ and $V_i*V_j=0$ for $i\neq j$).

Recall that the *derived series* of an algebra $A$ is defined inductively by $A^{(0)}=A$ and $A^{(r)}=A^{(r-1)}*A^{(r-1)}$ [38, 40]; $A$ is solvable of derived length $s$ if $A^{(s+1)}=0$. For any form $\alpha\in\Omega^k(P;A)$, defining the iterated squaring sequence $\alpha^{[0]}:=\alpha$ and $\alpha^{[m+1]}:=\alpha^{[m]}\wedge_*\alpha^{[m]}$ immediately implies by induction on $m$ that $\alpha^{[m]}\in\Omega^{2^m k}(P;A^{(m)})$, and hence $\alpha^{[s+1]}=0$ whenever $A^{(s+1)}=0$.

2.2. **Associated Algebra Bundles, $A$-Connections, and the Lie Nucleus Constraint.** Rather than introducing category-theoretic functorial bundle systems, we formulate our gauge-theoretic framework directly in terms of principal $G$-bundles and associated algebra bundles [22, 26].

Let $M$ be a smooth $n$-manifold, let $\pi:P\to M$ be a principal $G$-bundle with structure Lie group $G$ and Lie algebra $\mathfrak{g}$, and let $(A,*)$ be a finite-dimensional real (or complex) algebra equipped with a smooth representation by algebra automorphisms:

$$\rho:G\longrightarrow \mathrm{Aut}_{\mathrm{Alg}}(A)\subset \mathrm{GL}(A),\qquad \rho(g)(x*y)=\rho(g)(x)*\rho(g)(y)\quad(\forall g\in G,\,x,y\in A).\tag{2.14}$$

Differentiating $\rho$ at the identity element $e\in G$ yields a Lie algebra representation by derivations of $(A,*)$:

$$d\rho:\mathfrak{g}\longrightarrow \mathrm{Der}(A),\qquad d\rho(X)(x*y)=d\rho(X)(x)*y+x*d\rho(X)(y)\quad(\forall X\in\mathfrak{g},\,x,y\in A).\tag{2.15}$$

Because $G$ acts on $A$ by algebra automorphisms, the associated vector bundle

$$E_A := P \times_\rho A \longrightarrow M \tag{2.16}$$

is a smooth *algebra bundle* over $M$ with typical fiber $(A, *)$, whose fiberwise multiplication is well-defined by $[p, x] * [p, y] := [p, x * y]$. Moreover, for every $G$-invariant subspace $V \subset A$ (in particular, for every $\mathrm{Aut}_{\mathrm{Alg}}(A)$-invariant subspace such as the commutator subspace $[A, A]_*$, the derived terms $A^{(r)}$, or the Lie nucleus $\mathcal{N}_{\mathrm{Lie}}(A)$), $E_V := P \times_\rho V \subset E_A$ is a canonical smooth vector subbundle of $E_A$.

For each $k \geq 0$, let $\Omega^k_{\mathrm{eq},\rho}(P; A)$ denote the space of $\rho$-equivariant $A$-valued $k$-forms $\alpha \in \Omega^k(P; A)$ satisfying $R_g^* \alpha = \rho(g^{-1}) \circ \alpha$ for all $g \in G$, and let

$$\Omega^k_{\mathrm{heq},\rho}(P; A) := \left\{ \alpha \in \Omega^k_{\mathrm{eq},\rho}(P; A) \;\middle|\; i_{\underline{X}} \alpha = 0 \text{ for all } X \in \mathfrak{g} \right\} \tag{2.17}$$

denote the subspace of *horizontal*, $\rho$-equivariant $A$-valued $k$-forms on $P$, where $\underline{X}_p := \frac{d}{dt}\big|_{t=0} p \cdot \exp(tX)$ is the fundamental vertical vector field generated by $X \in \mathfrak{g}$. Because $\rho(g) \in \mathrm{Aut}_{\mathrm{Alg}}(A)$ and $i_{\underline{X}}(\alpha \wedge_* \beta) = (i_{\underline{X}}\alpha) \wedge_* \beta + (-1)^{\deg \alpha} \alpha \wedge_* (i_{\underline{X}}\beta)$, the direct sum $\Omega^\bullet_{\mathrm{heq},\rho}(P; A) = \bigoplus_{k=0}^n \Omega^k_{\mathrm{heq},\rho}(P; A)$ is a $\mathbb{Z}$-graded subalgebra of $(\Omega^\bullet(P; A), \wedge_*)$. The canonical *Weil isomorphism* [22, 26]

$$j : \Omega^\bullet_{\mathrm{heq},\rho}(P; A) \xrightarrow{\sim} \Omega^\bullet(M; E_A), \qquad j(\alpha)_{\pi(p)}(d\pi_p(v_1), \ldots, d\pi_p(v_k)) := [p,\, \alpha_p(v_1, \ldots, v_k)], \tag{2.18}$$

is an isomorphism of $\mathbb{Z}$-graded algebras: $j(\alpha \wedge_* \beta) = j(\alpha) \wedge_* j(\beta)$ for all $\alpha, \beta \in \Omega^\bullet_{\mathrm{heq},\rho}(P; A)$, and for every $G$-invariant subspace $V \subset A$, $j$ maps $\Omega^k_{\mathrm{heq},\rho}(P; V)$ isomorphically onto $\Omega^k(M; E_V)$.

A smooth *bundle trace functional* on $E_A$ is a fiberwise linear bundle morphism $\mathrm{tr}_\xi : E_A \to M \times \mathbb{R}$ determined by a smooth global section $\xi \in \Gamma(E_A^*)$ of the dual bundle $E_A^* = P \times_{\rho^*} A^*$ via $\mathrm{tr}_\xi(\Psi) := \langle \xi, \Psi \rangle \in \Omega^\bullet(M; \mathbb{R})$ for $\Psi \in \Omega^\bullet(M; E_A)$. Equivalently, $\xi = j(\hat{\xi})$ corresponds to a $G$-equivariant map $\hat{\xi} : P \to A^*$ satisfying $\hat{\xi}(p \cdot g) = \rho(g)^* \hat{\xi}(p)$. Such trace sections arise in two geometric ways:

(T1) *$G$-Invariant Linear Functional:* If $\tau_0 \in (A^*)^G$ is a constant $G$-invariant linear functional on $A$ (e.g., the matrix trace on $\mathfrak{gl}(k; \mathbb{R})$ or the central projection $\mathfrak{u}(k) \to \mathfrak{u}(1) \simeq \mathbb{R}$), then $\hat{\xi}(p) \equiv \tau_0$ defines a strictly $G$-invariant bundle trace morphism $\mathrm{tr}_{\tau_0} : E_A \to M \times \mathbb{R}$. Note that if $\rho = \mathrm{Ad}$ on a Lie algebra $A = \mathfrak{g}$, any $\mathrm{Ad}_G$-invariant linear functional $\tau_0 \in (\mathfrak{g}^*)^G$ vanishes identically on the derived subalgebra $[\mathfrak{g}, \mathfrak{g}]$ (though for a reductive Lie algebra $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ it may be non-zero on the center $\mathfrak{z}(\mathfrak{g})$; however, any interaction term taking values strictly inside $[A, A]_*$ or $[\mathfrak{g}, \mathfrak{g}]$ is still annihilated by $\tau_0$).

(T2) *Coadjoint Background Section:* When $(A^*)^G = \{0\}$ on $[A, A]_*$, any smooth non-zero section $\xi \in \Gamma(E_A^*)$ defines a bundle linear functional $\mathrm{tr}_\xi : E_A \to M \times \mathbb{R}$. Wherever $\xi(x) \neq 0$, choosing $\xi$ acts as a symmetry-breaking background field that reduces the internal structure group $G$ at $x$ to the stabilizer subgroup $G_{\xi(x)} \subset G$.

**Definition 2.3** ($A$-Connections, Field Strength, and Covariant Exterior Derivative)**.** Let $\pi : P \to M$ be a principal $G$-bundle and let $(A, *)$ be an $R$-algebra with automorphism representation $\rho : G \to \mathrm{Aut}_{\mathrm{Alg}}(A)$.

(1) A *pure $A$-connection* on $P$ is an $A$-valued 1-form $D \in \Omega^1(P; A)$. Its *field strength* (or *curvature*) is the $A$-valued 2-form $F_D \in \Omega^2(P; A)$ given by

$$F_D := dD + \frac{1}{2} D \wedge_* D = dD + \frac{1}{4} D \wedge_{[\cdot,\cdot]_*} D, \tag{2.19}$$

and its *commutator covariant exterior derivative* $d_D^{[\cdot,\cdot]_*} : \Omega^k(P; A) \to \Omega^{k+1}(P; A)$ is defined by

$$d_D^{[\cdot,\cdot]_*} \alpha := d\alpha + \frac{1}{2} D \wedge_{[\cdot,\cdot]_*} \alpha = d\alpha + \frac{1}{2}(D \wedge_* \alpha - (-1)^k \alpha \wedge_* D). \tag{2.20}$$

For any $k$-form $\Phi \in \Omega^k(P; A)$ and 0-form $\epsilon \in \Omega^0(P; A)$, we also write $[\Phi, \epsilon]_* := \Phi \wedge_{[\cdot,\cdot]_*} \epsilon = \Phi \wedge_* \epsilon - \epsilon \wedge_* \Phi = -[\epsilon, \Phi]_* \in \Omega^k(P; A)$.

(2) Suppose $A$ is equipped with a $G$-invariant $R$-module decomposition $A \simeq A_0 \oplus A_1$. A *reductive $A$-connection* on $P$ is a pair $\nabla = e + \omega \in \Omega^1(P; A_0) \oplus \Omega^1(P; A_1)$. Its *full Cartan field strength* is $F_\nabla = d\nabla + \frac{1}{2}\nabla \wedge_* \nabla \in \Omega^2(P; A)$, which expands as

$$F_\nabla = \Omega_\omega + \Theta_{e,\omega} + E_e, \tag{2.21}$$

where $\Omega_\omega := d\omega + \frac{1}{2}\omega \wedge_* \omega$ is the *pure curvature* of $\omega$, $\Theta_{e,\omega} := de + \frac{1}{2}(\omega \wedge_* e + e \wedge_* \omega) = de + \frac{1}{2}\omega \wedge_{[\cdot,\cdot]_*} e$ is the *generalized torsion*, and $E_e := \frac{1}{2} e \wedge_* e$ is the *quadratic soldering form* (all evaluated via the ambient multiplication $*$ of $A$).

(3) If $A = \bigoplus_{m\in\mathfrak{m}} A^m$ is an $\mathfrak{m}$-graded algebra ($A^{m_1} * A^{m_2} \subset A^{m_1+m_2}$), an $A$-connection $D \in \Omega^1(P; A)$ decomposes algebraically along the constant vector space direct sum $A = \bigoplus_{m\in\mathfrak{m}} A^m$ as $D = \sum_{m\in\mathfrak{m}} D^m$ with $D^m \in \Omega^1(P; A^m)$, and its curvature $F_D = \sum_{m\in\mathfrak{m}} (F_D)^m \in \bigoplus_{m\in\mathfrak{m}} \Omega^2(P; A^m)$ has homogeneous algebra components

$$(F_D)^m = d(D^m) + \frac{1}{2} \sum_{m_1+m_2=m} D^{m_1} \wedge_* D^{m_2} \in \Omega^2(P; A^m). \tag{2.22}$$

We now establish the exact algebraic and geometric conditions under which the curvature $F_D$, the covariant exterior derivative $d_D^{[\cdot,\cdot]_*}\alpha$, and the Jacobiator forms of $D$ are horizontal and $\rho$-equivariant on $P$, so that they descend to $E_A \to M$ via the Weil isomorphism $j$.

Given any trilinear bracket $J(x, y, z)$ on $A$ (such as $\mathrm{Jac}_{[\cdot,\cdot]_*}(x, y, z)$ or $\mathrm{Jac}_\curlywedge(x, y, z) := (x \curlywedge y) \curlywedge z + (y \curlywedge z) \curlywedge x + (z \curlywedge x) \curlywedge y$), for any forms $\alpha \in \Omega^{k_1}(P; A)$, $\beta \in \Omega^{k_2}(P; A)$, and $\gamma \in \Omega^{k_3}(P; A)$ with $K = k_1 + k_2 + k_3$, we define $J(\alpha, \beta, \gamma) \in \Omega^K(P; A)$ by

$$J(\alpha, \beta, \gamma)_p(v_1, \dots, v_K) := \frac{1}{k_1!\, k_2!\, k_3!} \sum_{\sigma\in S_K} \mathrm{sign}(\sigma)\, J(\alpha_p(v_{\sigma(1)}, \dots), \beta_p(\dots), \gamma_p(\dots, v_{\sigma(K)})). \tag{2.23}$$

In particular, for $D \in \Omega^1(P; A)$ and $\epsilon \in \Omega^0(P; A)$, evaluating (2.23) on $v_1, v_2 \in T_pP$ gives $J(D, D, \epsilon)_p(v_1, v_2) = 2\, J(D_p(v_1), D_p(v_2), \epsilon(p))$, while on $v_1, v_2, v_3 \in T_pP$ it gives $J(D, D, D)_p(v_1, v_2, v_3) = 6\, J(D_p(v_1), D_p(v_2), D_p(v_3))$.

**Proposition 2.4** (Ehresmann Admissibility and the Lie Nucleus Constraint)**.** *Let $\pi : P \to M$ be a principal $G$-bundle, let $\rho : G \to \mathrm{Aut}_{\mathrm{Alg}}(A)$ be a representation of $G$ by algebra automorphisms of $(A, *)$, and let $\eta : \mathfrak{g} \to A$ be a $G$-equivariant linear map ($\eta(\mathrm{Ad}_g X) = \rho(g)(\eta(X))$ for all $g \in G$, $X \in \mathfrak{g}$). Define the space of $\eta$-*compatible Ehresmann $A$-connections *on $P$ by*

$$\mathrm{Conn}_{\rho,\eta}(P; A) := \left\{ D \in \Omega^1_{\mathrm{eq},\rho}(P; A) \;\middle|\; D(\underline{X}) = \eta(X) \text{ for all } X \in \mathfrak{g} \right\}. \tag{2.24}$$

*Assume that the intertwining condition*

$$d\rho(X)(y) = \frac{1}{2}[\eta(X), y]_* = \frac{1}{2}(\eta(X) * y - y * \eta(X)) \qquad (\forall X \in \mathfrak{g},\ y \in A) \tag{2.25}$$

*holds. (Notice that when $(A, *)$ is a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ with $* = [\cdot,\cdot]_\mathfrak{g}$ and $\eta = \mathrm{id}_\mathfrak{g}$, the commutator of $*$ is $[x, y]_* = [x, y]_\mathfrak{g} - [y, x]_\mathfrak{g} = 2[x, y]_\mathfrak{g}$, so $\frac{1}{2}[\eta(X), y]_* = [X, y]_\mathfrak{g} = \mathrm{ad}_X(y)$ recovers the standard adjoint representation.) Then:*

(1) ***Lie Nucleus Constraint:*** *Because $d\rho(X) \in \mathrm{Der}(A)$ is a derivation of $(A, *)$ and hence of $(A, [\cdot,\cdot]_*)$, condition (2.25) forces the image of $\eta$ to lie in the Lie nucleus (2.3) of $(A, [\cdot,\cdot]_*)$:*

$$\eta(\mathfrak{g}) \subset \mathcal{N}_{\mathrm{Lie}}(A). \tag{2.26}$$

*More precisely, under (2.25), $\frac{1}{2}\mathrm{ad}^{[\cdot,\cdot]_*}_{\eta(X)}$ is a derivation of the commutator algebra $(A, [\cdot,\cdot]_*)$ if and only if $\eta(X) \in \mathcal{N}_{\mathrm{Lie}}(A)$ (which alone suffices for all horizontality assertions in (2) below), and is a derivation of the full algebra $(A, *)$ if and only if $\eta(X)$ also satisfies the Jordan-derivation identity $[\eta(X), y \circ_* z]_* = [\eta(X), y]_* \circ_* z + y \circ_* [\eta(X), z]_*$ for all $y, z \in A$ (which holds automatically on $\mathcal{N}_{\mathrm{Lie}}(A)$ whenever $(A, *)$ is skew-commutative or associative). In particular, if $(A, [\cdot,\cdot]_*)$ has trivial Lie nucleus $\mathcal{N}_{\mathrm{Lie}}(A) = \{0\}$ (such as the 7-dimensional octonionic Malcev algebra $\mathrm{Im}(\mathbb{O})$, where $\mathrm{Jac}_{[\cdot,\cdot]_*}(x, y, z) = 6\,\mathrm{Assoc}_\mathbb{O}(x, y, z)$ and $\mathcal{N}_{\mathrm{assoc}}(\mathbb{O}) = \mathbb{R} \cdot 1$, so $\mathcal{N}_{\mathrm{Lie}}(\mathrm{Im}(\mathbb{O})) = \mathrm{Im}(\mathbb{O}) \cap \mathbb{R} \cdot 1 = \{0\}$), then (2.25) requires $\eta = 0$ and $d\rho = 0$.*

(2) ***Horizontality and Descent of Curvature, Covariant Derivative, and Jacobiator Forms:*** *For every* $D \in \mathrm{Conn}_{\rho,\eta}(P;A)$*, every* $\alpha \in \Omega^k_{\mathrm{heq},\rho}(P;A)$*, and every* $\epsilon \in \Omega^0_{\mathrm{heq},\rho}(P;A)$*:*
   (a) *The curvature* $F_D = dD + \frac{1}{2}D \wedge_* D$ *lies in* $\Omega^2_{\mathrm{heq},\rho}(P;A)$ *and descends via* $j$ *to* $j(F_D) \in \Omega^2(M;E_A)$*;*
   (b) *The commutator covariant exterior derivative* $d_D^{[\cdot,\cdot]_*}\alpha$ *lies in* $\Omega^{k+1}_{\mathrm{heq},\rho}(P;A)$ *and descends via* $j$ *to* $d_{\nabla_D}^{[\cdot,\cdot]_*} j(\alpha) := j(d_D^{[\cdot,\cdot]_*}\alpha) \in \Omega^{k+1}(M;E_A)$*;*
   (c) *By the Lie nucleus inclusion (2.26), the Jacobiator forms* $\mathrm{Jac}_{[\cdot,\cdot]_*}(D,D,D) \in \Omega^3(P;A)$ *and* $\mathrm{Jac}_{[\cdot,\cdot]_*}(D,D,\epsilon) \in \Omega^2(P;A)$ *are horizontal and* $\rho$*-equivariant, i.e., lie in* $\Omega^3_{\mathrm{heq},\rho}(P;A)$ *and* $\Omega^2_{\mathrm{heq},\rho}(P;A)$*, and descend via* $j$ *to* $\Omega^3(M;E_A)$ *and* $\Omega^2(M;E_A)$*.*

(3) ***Affine vs. Linear Connection Regimes:***
   - Affine Ehresmann Regime ($\eta \neq 0$): *Choosing any principal connection* $\omega_0 \in \mathrm{Conn}(P;\mathfrak{g})$*,* $D_0 := \eta \circ \omega_0 \in \mathrm{Conn}_{\rho,\eta}(P;A)$ *is a basepoint connection and* $\mathrm{Conn}_{\rho,\eta}(P;A) = D_0 + \Omega^1_{\mathrm{heq},\rho}(P;A)$ *is a non-empty* affine space *modeled on* $\Omega^1_{\mathrm{heq},\rho}(P;A) \simeq \Omega^1(M;E_A)$*.*
   - Linear/Horizontal Regime ($\eta = 0$, $d\rho = 0$): *When* $G$ *acts trivially on* $A$ *(or on a trivial bundle* $P = M \times G$ *relative to the flat background exterior derivative* $d$*),* $\eta = 0$ *satisfies (2.25) and (2.26) for* every *algebra* $(A,*)$ *without any Lie-nucleus restriction, and* $\mathrm{Conn}^{\mathrm{lin}}(P;A) := \mathrm{Conn}_{\rho,0}(P;A) = \Omega^1_{\mathrm{heq},\rho}(P;A) \simeq \Omega^1(M;E_A)$ *is a* real vector space*.*

*Proof.* For (1), since $\rho(G) \subset \mathrm{Aut}_{\mathrm{Alg}}(A)$, each $d\rho(X)$ is a derivation of $(A,*)$ and therefore a derivation of both $[y,z]_* = y*z - z*y$ and $y \circ_* z = \frac{1}{2}(y*z + z*y)$. On the other hand, expanding $\frac{1}{2}[\eta(X),[y,z]_*]_* - [\frac{1}{2}[\eta(X),y]_*,z]_* - [y,\frac{1}{2}[\eta(X),z]_*]_*$ gives $-\frac{1}{2}\mathrm{Jac}_{[\cdot,\cdot]_*}(\eta(X),y,z)$, so $\frac{1}{2}\mathrm{ad}^{[\cdot,\cdot]_*}_{\eta(X)}$ is a derivation of $[\cdot,\cdot]_*$ if and only if $\mathrm{Jac}_{[\cdot,\cdot]_*}(\eta(X),y,z) = 0$ for all $y,z \in A$, i.e., $\eta(X) \in \mathcal{N}_{\mathrm{Lie}}(A)$ (and is a derivation of the full product $*$ if and only if $\frac{1}{2}\mathrm{ad}^{[\cdot,\cdot]_*}_{\eta(X)}$ also acts as a derivation on $y \circ_* z$, which is automatic on $\mathcal{N}_{\mathrm{Lie}}(A)$ when $y \circ_* z = 0$ or when $*$ is associative).

For (2a), $\rho$-equivariance of $D$ and $\rho(g) \in \mathrm{Aut}_{\mathrm{Alg}}(A)$ immediately imply $R_g^* F_D = \rho(g^{-1}) \circ F_D$. For horizontality, differentiating $R^*_{\exp(tX)}D = \rho(\exp(-tX)) \circ D$ at $t = 0$ via Cartan's magic formula $L_{\underline{X}}D = i_{\underline{X}}dD + d(i_{\underline{X}}D)$ and using $i_{\underline{X}}D = \eta(X) = \mathrm{const}$ gives $i_{\underline{X}}dD = -d\rho(X)(D) = -\frac{1}{2}[\eta(X),D]_*$. Meanwhile, by (2.7), $i_{\underline{X}}(\frac{1}{2}D \wedge_* D) = \frac{1}{4}i_{\underline{X}}(D \wedge_{[\cdot,\cdot]_*} D) = \frac{1}{2}[i_{\underline{X}}D, D]_* = \frac{1}{2}[\eta(X),D]_*$. Summing the two terms gives $i_{\underline{X}}F_D = 0$, so $F_D \in \Omega^2_{\mathrm{heq},\rho}(P;A)$.

For (2b), if $\alpha \in \Omega^k_{\mathrm{heq},\rho}(P;A)$, then $i_{\underline{X}}\alpha = 0$ and $L_{\underline{X}}\alpha = i_{\underline{X}}d\alpha = -d\rho(X)(\alpha) = -\frac{1}{2}[\eta(X),\alpha]_*$, while $i_{\underline{X}}(\frac{1}{2}D \wedge_{[\cdot,\cdot]_*} \alpha) = \frac{1}{2}[\eta(X),\alpha]_* - \frac{1}{2}D \wedge_{[\cdot,\cdot]_*} (i_{\underline{X}}\alpha) = \frac{1}{2}[\eta(X),\alpha]_*$, so $i_{\underline{X}}(d_D^{[\cdot,\cdot]_*}\alpha) = 0$.

For (2c), for any $X \in \mathfrak{g}$ and tangent vectors $v_1, v_2 \in T_pP$, (2.23) and (2.26) give

$$\big(i_{\underline{X}}\mathrm{Jac}_{[\cdot,\cdot]_*}(D,D,D)\big)_p(v_1,v_2) = 6\,\mathrm{Jac}_{[\cdot,\cdot]_*}\big(\eta(X), D_p(v_1), D_p(v_2)\big) = 0,$$

and $\big(i_{\underline{X}}\mathrm{Jac}_{[\cdot,\cdot]_*}(D,D,\epsilon)\big)_p(v_1) = 2\,\mathrm{Jac}_{[\cdot,\cdot]_*}(\eta(X), D_p(v_1), \epsilon(p)) = 0$ because $\eta(X) \in \mathcal{N}_{\mathrm{Lie}}(A)$. Since $\rho(G) \subset \mathrm{Aut}_{\mathrm{Alg}}(A)$ preserves $\mathrm{Jac}_{[\cdot,\cdot]_*}$, both forms are $\rho$-equivariant and horizontal. Part (3) is immediate from $i_{\underline{X}}(D_1 - D_2) = \eta(X) - \eta(X) = 0$. □

2.3. **Definition of $A$-YMT Theories, Variational Equations, and the Jacobiator Anomaly.** Throughout the remainder of the paper, we work in the geometric setting of Proposition 2.4: $\pi : P \to M$ is a principal $G$-bundle over a smooth $n$-manifold $M$, $\rho : G \to \mathrm{Aut}_{\mathrm{Alg}}(A)$ is an algebra automorphism representation with associated algebra bundle $E_A = P \times_\rho A \to M$ and Weil isomorphism $j : \Omega^\bullet_{\mathrm{heq},\rho}(P;A) \xrightarrow{\sim} \Omega^\bullet(M;E_A)$, and $\mathrm{Conn}_{\rho,\eta}(P;A)$ is the space of admissible $A$-connections satisfying (2.25) (including the linear/horizontal case $\eta = 0$).

Let $\mathrm{Dens}(M)$ denote the $C^\infty(M)$-module of smooth 1-densities on $M$ (identified with $\Omega^n(M;\mathbb{R})$ when $M$ is oriented). For any $C^\infty(M)$-module $V$, let $V_{\mathbb{R}}$ denote the real vector space obtained by restriction of scalars along $\mathbb{R} \hookrightarrow C^\infty(M)$.

**Definition 2.5** ($A$-Valued Yang-Mills-Type ($A$-YMT) Theory)**.** Let $E_A = P \times_\rho A \to M$ be an associated algebra bundle over a smooth $n$-manifold $M$.

(1) An *$A$-valued Yang-Mills-type ($A$-YMT) pairing* on $E_A$ is an $\mathbb{R}$-bilinear map

$$\langle \cdot, \cdot \rangle_A : \Omega^2(M; E_A)_{\mathbb{R}} \otimes_{\mathbb{R}} \Omega^2(M; E_A)_{\mathbb{R}} \longrightarrow \mathrm{Dens}(M)_{\mathbb{R}}, \tag{2.27}$$

with symmetrization $\langle \Phi, \Psi \rangle_{A,\mathrm{sym}} := \frac{1}{2}(\langle \Phi, \Psi \rangle_A + \langle \Psi, \Phi \rangle_A)$. We say that $\langle \cdot, \cdot \rangle_{A,\mathrm{sym}}$ is *tensorial* (or *$C^\infty(M)$-bilinear*) if $\langle f\Phi, g\Psi \rangle_{A,\mathrm{sym}} = fg\langle \Phi, \Psi \rangle_{A,\mathrm{sym}}$ for all $f, g \in C^\infty(M)$ (which implies support locality $\mathrm{supp}(\langle \Phi, \Psi \rangle_{A,\mathrm{sym}}) \subset \mathrm{supp}(\Phi) \cap \mathrm{supp}(\Psi)$ and induces a smooth bundle bilinear form on $\Lambda^2 T^*M \otimes E_A$), *pointwise* $\mathrm{ad}_{[\cdot,\cdot]_*}$*-invariant* if $\langle [\Phi, j(\epsilon)]_*, \Psi \rangle_{A,\mathrm{sym}} + \langle \Phi, [\Psi, j(\epsilon)]_* \rangle_{A,\mathrm{sym}} = 0$ in $\mathrm{Dens}(M)$ for all $\Phi, \Psi \in \Omega^2(M; E_A)$ and $\epsilon \in \Omega^0_{\mathrm{heq},\rho}(P; A)$, *non-degenerate* if $\int_M \langle \Phi, \Psi \rangle_{A,\mathrm{sym}} = 0$ for all compactly supported $\Phi \in \Omega^2_c(M; E_A)$ implies $\Psi = 0$, and, when it is tensorial, *fiberwise non-degenerate* if the induced bundle bilinear form on $\Lambda^2 T^*M \otimes E_A$ is non-degenerate at every point of $M$.

(2) Assuming $M$ is compact (or restricting to connections with compactly supported curvature density), the *$A$-YMT action functional* associated with $\langle \cdot, \cdot \rangle_A$ is $S^A : \mathrm{Conn}_{\rho,\eta}(P; A) \to \mathbb{R}$,

$$S^A[D] := \int_M \langle j(F_D), j(F_D) \rangle_A. \tag{2.28}$$

We distinguish three classes of $A$-YMT theories according to the pairing $\langle \cdot, \cdot \rangle_A$:

- **General $A$-YMT Theory:** $\langle \cdot, \cdot \rangle_A$ is an arbitrary $\mathbb{R}$-bilinear map (2.27).
- **Tensorial $A$-YMT Theory:** $\langle \cdot, \cdot \rangle_A$ is $C^\infty(M)$-bilinear, e.g., given a fiberwise symmetric bilinear form $\kappa : E_A \times_M E_A \to M \times \mathbb{R}$ and a bundle Hodge-type morphism $\star_M : \Omega^2(M; E_A) \to \Omega^{n-2}(M; E_A)$ on an oriented $n$-manifold $M$,

$$\langle \Phi, \Psi \rangle_{\kappa, \star_M} := \kappa(\Phi \wedge \star_M \Psi) \in \Omega^n(M; \mathbb{R}). \tag{2.29}$$

- **Multiplicative (or Internal-Trace) $A$-YMT Theory:** $M$ is oriented, $E_A$ is equipped with a bundle trace functional $\mathrm{tr}_\xi : E_A \to M \times \mathbb{R}$ induced by $\xi \in \Gamma(E_A^*)$, and $\hat{\star} : \Omega^2_{\mathrm{heq},\rho}(P; A) \to \Omega^{n-2}_{\mathrm{heq},\rho}(P; A)$ is an internal constitutive operator such that the pairing is formed via the *internal product $\wedge_*$ of $A$* followed by $\mathrm{tr}_\xi$:

$$S^A_{\hat{\star}}[D] := \int_M \mathrm{tr}_\xi\Big(j(F_D \wedge_* \hat{\star} F_D)\Big). \tag{2.30}$$

**Remark 2.6** (Internal Multiplicative Pairings vs. Classical Bilinear Invariant Forms)**.** In classical gauge theory with $A = \mathfrak{g}$ and $\kappa \in \mathrm{inv}^2(\mathfrak{g}) = S^2(\mathfrak{g}^*)^{\mathrm{Ad}}$ (such as the Killing form $B_\mathfrak{g}$), both the classical Yang-Mills action $\int_M \kappa(j(F_D) \wedge *_g j(F_D))$ and the classical 4D second Chern / Pontryagin action $\int_M \kappa(j(F_D) \wedge j(F_D))$ belong to the class of **tensorial $A$-YMT theories** (2.29). Notice that $\kappa(j(F_D) \wedge j(F_D))$ uses the *external* symmetric bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ (or equivalently, the associative matrix product $\Omega \wedge_{\mathrm{Mat}} \Omega$ in an external associative matrix envelope $\mathrm{Mat}_{k\times k}(\mathbb{R})$, which maps $\mathfrak{so}(k) \times \mathfrak{so}(k)$ into the space of symmetric matrices $\mathrm{Sym}_k(\mathbb{R})$ outside $\mathfrak{so}(k)$ before taking the matrix trace). In contrast, in a multiplicative $A$-YMT theory (2.30) or in the higher-degree internal Chern-Weil functional in dimension $n = 2r$ (for a parenthesization scheme $\mathcal{P} \in \{\mathrm{full}, \mathrm{bal}, \mathrm{blk}\}$),

$$S^A_{\mathrm{top},r,\mathcal{P}}[D] := \int_M \mathrm{tr}_\xi\Big(j(\wedge^r_{*,\mathcal{P}} F_D)\Big), \tag{2.31}$$

the curvature factors are multiplied *internally within the algebra $A$ itself* using $\wedge_*$, and only afterwards is a linear functional $\mathrm{tr}_\xi : E_A \to M \times \mathbb{R}$ applied.

**Theorem 2.7** (Variational Field Equations, Infinitesimal Gauge Covariance, and the Jacobiator Bianchi Anomaly)**.** *Let $\pi : P \to M$ be a principal $G$-bundle over a closed smooth $n$-manifold $M$, let $\rho : G \to \mathrm{Aut}_{\mathrm{Alg}}(A)$ and $\eta : \mathfrak{g} \to \mathcal{N}_{\mathrm{Lie}}(A)$ satisfy (2.25), and let $D \in \mathrm{Conn}_{\rho,\eta}(P; A)$ be an admissible $A$-connection with curvature $F_D = dD + \frac{1}{2} D \wedge_* D = dD + \frac{1}{4} D \wedge_{[\cdot,\cdot]_*} D \in \Omega^2_{\mathrm{heq},\rho}(P; A)$.*

(1) ***First Variation of Curvature and Action:*** *For any tangent variation $\dot{D} \in \Omega^1_{\mathrm{heq},\rho}(P; A) \simeq \Omega^1(M; E_A)$, the directional derivative of the curvature is given for* every

*algebra* $(A,*)$ *(without assuming associativity or the Jacobi identity) by*

$$\delta_{\dot{D}}F_D := \frac{d}{dt}\Big|_{t=0} F_{D+t\dot{D}} = d\dot{D} + \frac{1}{2}(D \wedge_* \dot{D} + \dot{D} \wedge_* D) = d_D^{[\cdot,\cdot]_*}\dot{D} \in \Omega^2_{\mathrm{heq},\rho}(P;A), \tag{2.32}$$

*and the first variation of the general A-YMT action* $S^A[D]$ *is*

$$\delta_{\dot{D}}S^A[D] = 2\int_M \langle j(F_D),\, d_{\nabla_D}^{[\cdot,\cdot]_*} j(\dot{D})\rangle_{A,\mathrm{sym}}. \tag{2.33}$$

*In particular, consider a tensorial A-YMT theory* (2.29) *on a closed oriented* $n$*-manifold* $M$*, where* $\kappa : E_A \times_M E_A \to M \times \mathbb{R}$ *is fiberwise symmetric, non-degenerate, and* $\mathrm{ad}_{[\cdot,\cdot]_*}$*-invariant (*$\kappa([x,y]_*,z) + \kappa(y,[x,z]_*) = 0$*), and* $\star_M : \Omega^2(M;E_A) \to \Omega^{n-2}(M;E_A)$ *satisfies* $\kappa(\Phi \wedge \star_M \Psi) = \kappa(\Psi \wedge \star_M \Phi)$*. Then* $D$ *is a critical point of* $S^A[D]$ *if and only if it satisfies the* $A$-Yang-Mills Euler-Lagrange equation*:*

$$d_{\nabla_D}^{[\cdot,\cdot]_*}(\star_M j(F_D)) = j\Big(d_D^{[\cdot,\cdot]_*}(j^{-1}(\star_M j(F_D)))\Big) = 0 \quad \text{in } \Omega^{n-1}(M;E_A). \tag{2.34}$$

(2) ***Generalized Bianchi Identity and Jacobiator Obstruction:*** *For every admissible connection* $D \in \mathrm{Conn}_{\rho,\eta}(P;A)$*, the covariant exterior derivative of the curvature* $F_D$ *satisfies in* $\Omega^3_{\mathrm{heq},\rho}(P;A)$*:*

$$d_D^{[\cdot,\cdot]_*}F_D = dF_D + \frac{1}{2}D \wedge_{[\cdot,\cdot]_*} F_D = -\frac{1}{8}(D \wedge_{[\cdot,\cdot]_*} D) \wedge_{[\cdot,\cdot]_*} D = -\frac{1}{24}\mathrm{Jac}_{[\cdot,\cdot]_*}(D,D,D). \tag{2.35}$$

*Consequently, the classical Bianchi identity* $d_{\nabla_D}^{[\cdot,\cdot]_*} j(F_D) = 0$ *holds identically for all connections* $D \in \mathrm{Conn}_{\rho,\eta}(P;A)$ *on manifolds of dimension* $n \geq 3$ if and only if $(A,*)$ *is Lie-admissible (*$\mathrm{Jac}_{[\cdot,\cdot]_*} \equiv 0$*).*

(3) ***Infinitesimal Gauge Transformations, Principal vs. Extended A-Symmetries, and Covariance Anomaly:*** *Given a 0-form gauge parameter* $\epsilon \in \Omega^0_{\mathrm{heq},\rho}(P;A) \simeq \Gamma(E_A)$*, define the infinitesimal gauge variation of* $D \in \mathrm{Conn}_{\rho,\eta}(P;A)$ *by*

$$\delta_\epsilon D := d_D^{[\cdot,\cdot]_*}\epsilon = d\epsilon + \frac{1}{2}[D,\epsilon]_* \in \Omega^1_{\mathrm{heq},\rho}(P;A). \tag{2.36}$$

*Then* $D + t\,\delta_\epsilon D \in \mathrm{Conn}_{\rho,\eta}(P;A)$*, and the induced infinitesimal variation of the curvature* $F_D$ *is*

$$\delta_\epsilon F_D = d_D^{[\cdot,\cdot]_*}(d_D^{[\cdot,\cdot]_*}\epsilon) = \frac{1}{2}[F_D,\epsilon]_* - \frac{1}{8}\,\mathrm{Jac}_{[\cdot,\cdot]_*}(D,D,\epsilon) \in \Omega^2_{\mathrm{heq},\rho}(P;A). \tag{2.37}$$

*Consequently, we have a sharp dichotomy between principal* $G$*-gauge transformations and extended* $A$*-gauge transformations:*

(a) Principal $G$-Gauge Transformations ($\epsilon \in \Omega^0_{\mathrm{heq},\rho}(P;\eta(\mathfrak{g})) \subset \Omega^0_{\mathrm{heq},\rho}(P;\mathcal{N}_{\mathrm{Lie}}(A))$): *For every gauge parameter* $\epsilon$ *valued in the Lie nucleus* $\mathcal{N}_{\mathrm{Lie}}(A)$ *(and in particular for every vertical principal-bundle automorphism* $\epsilon = \eta(\tilde{\epsilon})$ *with* $\tilde{\epsilon} \in \Omega^0_{\mathrm{heq,Ad}}(P;\mathfrak{g})$*),* $\mathrm{Jac}_{[\cdot,\cdot]_*}(D,D,\epsilon) \equiv 0$ *vanishes identically and* $F_D$ *transforms homogeneously (*$\delta_\epsilon F_D = \frac{1}{2}[F_D,\epsilon]_*$*) for* every *algebra* $(A,*)$*.*

(b) Extended $A$-Gauge Transformations ($\epsilon \in \Omega^0_{\mathrm{heq},\rho}(P;A)$): $F_D$ *transforms homogeneously for all* $(D,\epsilon) \in \mathrm{Conn}_{\rho,\eta}(P;A) \times \Omega^0_{\mathrm{heq},\rho}(P;A)$ *on manifolds of dimension* $n \geq 2$ *if and only if* $(A,*)$ *is Lie-admissible (*$\mathcal{N}_{\mathrm{Lie}}(A) = A$*). Furthermore, if* $\langle\cdot,\cdot\rangle_{A,\mathrm{sym}}$ *is pointwise* $\mathrm{ad}_{[\cdot,\cdot]_*}$*-invariant, then*

$$\delta_\epsilon S^A[D] = -\frac{1}{4}\int_M \langle j(F_D),\, j(\mathrm{Jac}_{[\cdot,\cdot]_*}(D,D,\epsilon))\rangle_{A,\mathrm{sym}}, \tag{2.38}$$

*which vanishes for all* $(D,\epsilon)$ *whenever* $(A,*)$ *is Lie-admissible, and conversely forces* $\mathrm{Jac}_{[\cdot,\cdot]_*} \equiv 0$ *on* $A$ *whenever* $\langle\cdot,\cdot\rangle_{A,\mathrm{sym}}$ *is tensorial (*$C^\infty(M)$*-bilinear), pointwise* $\mathrm{ad}_{[\cdot,\cdot]_*}$*-invariant, and fiberwise non-degenerate and* $n \geq 2$*.*

*Proof.* By Proposition 2.4(2), $F_D$, $d_D^{[\cdot,\cdot]_*}\dot{D}$, $\mathrm{Jac}_{[\cdot,\cdot]_*}(D,D,D)$, $\delta_\epsilon D$, and $\mathrm{Jac}_{[\cdot,\cdot]_*}(D,D,\epsilon)$ all lie in $\Omega^\bullet_{\mathrm{heq},\rho}(P;A)$ and descend via $j$ to $\Omega^\bullet(M;E_A)$.

For part (1), expanding $F_{D+t\dot{D}}$ to first order in $t$ and using $D\wedge_*\dot{D}+\dot{D}\wedge_* D = D\wedge_{[\cdot,\cdot]_*}\dot{D}$ gives (2.32) and (2.33). For the tensorial pairing $\langle\Phi,\Psi\rangle_A=\kappa(\Phi\wedge\star_M\Psi)$, let $\Xi := j^{-1}(\star_M j(F_D))\in\Omega^{n-2}_{\mathrm{heq},\rho}(P;A)$. On $P$, the graded Leibniz rule for the 1-form $\dot{D}$ gives $d(\kappa(\dot{D}\wedge\Xi))=\kappa(d\dot{D}\wedge\Xi)-\kappa(\dot{D}\wedge d\Xi)$. On decomposable elements $D=x\otimes\alpha$, $\dot{D}=y\otimes\beta$ ($\alpha,\beta\in\Omega^1(P)$) and $\Xi=z\otimes\gamma$ ($\gamma\in\Omega^{n-2}(P)$), the Koszul sign $\beta\wedge\alpha\wedge\gamma=-\alpha\wedge\beta\wedge\gamma$ combines with $\mathrm{ad}_{[\cdot,\cdot]_*}$-invariance $\kappa([x,y]_*,z)=-\kappa(y,[x,z]_*)$ to give

$$\kappa((D\wedge_{[\cdot,\cdot]_*}\dot{D})\wedge\Xi)=\kappa([x,y]_*,z)\,\alpha\wedge\beta\wedge\gamma=+\kappa(y,[x,z]_*)\,\beta\wedge\alpha\wedge\gamma=+\kappa(\dot{D}\wedge(D\wedge_{[\cdot,\cdot]_*}\Xi)).$$

Applying Stokes' theorem on the closed manifold $M$ yields $\delta_{\dot{D}}S^A[D]=2\int_M\kappa(j(\dot{D})\wedge d_{\nabla_D}^{[\cdot,\cdot]_*}(\star_M j(F_D)))$, which vanishes for all $\dot{D}\in\Omega^1_{\mathrm{heq},\rho}(P;A)$ if and only if (2.34) holds.

For part (2), since $d^2D=0$ and $d(D\wedge_{[\cdot,\cdot]_*}D)=2\,dD\wedge_{[\cdot,\cdot]_*}D=-2\,D\wedge_{[\cdot,\cdot]_*}dD$, we compute:

$$\begin{aligned}d_D^{[\cdot,\cdot]_*}F_D&=d\Big(dD+\frac14 D\wedge_{[\cdot,\cdot]_*}D\Big)+\frac12 D\wedge_{[\cdot,\cdot]_*}\Big(dD+\frac14 D\wedge_{[\cdot,\cdot]_*}D\Big)\\&=-\frac12 D\wedge_{[\cdot,\cdot]_*}dD+\frac12 D\wedge_{[\cdot,\cdot]_*}dD+\frac18 D\wedge_{[\cdot,\cdot]_*}(D\wedge_{[\cdot,\cdot]_*}D)=-\frac18(D\wedge_{[\cdot,\cdot]_*}D)\wedge_{[\cdot,\cdot]_*}D.\end{aligned}$$

Since $(D\wedge_{[\cdot,\cdot]_*}D)_p(u,w)=2[D_p(u),D_p(w)]_*$, evaluating the 3-form $((D\wedge_{[\cdot,\cdot]_*}D)\wedge_{[\cdot,\cdot]_*}D)(v_1,v_2,v_3)$ on tangent vectors $v_1,v_2,v_3\in T_pP$ via (2.5) sums the terms $2[[D(v_{\sigma(1)}),D(v_{\sigma(2)})]_*,D(v_{\sigma(3)})]_*$ with prefactor $\frac{1}{2!\,1!}$ over $\sigma\in S_3$. Grouping the 6 permutations into the 3 cyclic pairs (123), (231), (312) gives $2\,\mathrm{Jac}_{[\cdot,\cdot]_*}(D(v_1),D(v_2),D(v_3))=\frac13\mathrm{Jac}_{[\cdot,\cdot]_*}(D,D,D)(v_1,v_2,v_3)$ by (2.23), proving (2.35). When $n\geq 3$, choosing a local trivialization of $P$ over an open chart $U\subset M$ around $x\in M$ and evaluating (2.35) on horizontal tangent vectors $v_1,v_2,v_3\in T_pP$ projecting to linearly independent vectors in $T_xM$ with $D_p(v_i)=y_i\in A$ arbitrary shows that $\mathrm{Jac}_{[\cdot,\cdot]_*}(D,D,D)=0$ for all $D$ if and only if $\mathrm{Jac}_{[\cdot,\cdot]_*}(y_1,y_2,y_3)=0$ for all $y_1,y_2,y_3\in A$.

For part (3), substituting $\dot{D}=\delta_\epsilon D=d\epsilon+\frac12[D,\epsilon]_*$ into (2.32) and using the graded Leibniz identity $d(\frac12[D,\epsilon]_*)=\frac12[dD,\epsilon]_*-\frac12 D\wedge_{[\cdot,\cdot]_*}d\epsilon$ (since $D$ has degree 1 and $\epsilon$ has degree 0) gives

$$\begin{aligned}\delta_\epsilon F_D&=d\Big(d\epsilon+\frac12[D,\epsilon]_*\Big)+\frac12 D\wedge_{[\cdot,\cdot]_*}\Big(d\epsilon+\frac12[D,\epsilon]_*\Big)\\&=\frac12[dD,\epsilon]_*-\frac12 D\wedge_{[\cdot,\cdot]_*}d\epsilon+\frac12 D\wedge_{[\cdot,\cdot]_*}d\epsilon+\frac14 D\wedge_{[\cdot,\cdot]_*}[D,\epsilon]_*\\&=\frac12\Big[F_D-\frac14 D\wedge_{[\cdot,\cdot]_*}D,\,\epsilon\Big]_*+\frac14 D\wedge_{[\cdot,\cdot]_*}[D,\epsilon]_*.\end{aligned}$$

Evaluating $\frac14 D\wedge_{[\cdot,\cdot]_*}[D,\epsilon]_*-\frac18[D\wedge_{[\cdot,\cdot]_*}D,\epsilon]_*$ on $(v_1,v_2)$ gives $-\frac14\mathrm{Jac}_{[\cdot,\cdot]_*}(D(v_1),D(v_2),\epsilon)=-\frac18\mathrm{Jac}_{[\cdot,\cdot]_*}(D,D,\epsilon)(v_1,v_2)$ by (2.23), proving (2.37). Whenever $\epsilon\in\Omega^0_{\mathrm{heq},\rho}(P;\mathcal{N}_{\mathrm{Lie}}(A))$, equation (2.3) yields $\mathrm{Jac}_{[\cdot,\cdot]_*}(D,D,\epsilon)=0$, proving (3a). In general, substituting (2.37) into $\delta_\epsilon S^A[D]=2\int_M\langle j(F_D),j(\delta_\epsilon F_D)\rangle_{A,\mathrm{sym}}$ and using pointwise $\mathrm{ad}_{[\cdot,\cdot]_*}$-invariance $\langle[j(F_D),j(\epsilon)]_*,j(F_D)\rangle_{A,\mathrm{sym}}=0$ yields (2.38).

Finally, to prove the converse in (3b) rigorously when $\langle\cdot,\cdot\rangle_{A,\mathrm{sym}}$ is tensorial ($C^\infty(M)$-bilinear), pointwise $\mathrm{ad}_{[\cdot,\cdot]_*}$-invariant, and fiberwise non-degenerate and $n\geq 2$, suppose (2.38) vanishes identically for all $D\in\mathrm{Conn}_{\rho,\eta}(P;A)$ and $\epsilon\in\Omega^0_{\mathrm{heq},\rho}(P;A)$. Because $\langle\cdot,\cdot\rangle_{A,\mathrm{sym}}$ is $C^\infty(M)$-bilinear and fiberwise non-degenerate, it is given by a fiberwise non-degenerate symmetric bilinear bundle form $(\cdot,\cdot)_{A,x}:(\Lambda^2T_x^*M\otimes(E_A)_x)\times(\Lambda^2T_x^*M\otimes(E_A)_x)\to\mathrm{Dens}_x(M)$ at every point $x\in M$. Replacing $\epsilon$ by $f\epsilon$ for arbitrary $f\in C_c^\infty(M)$ and using $C^\infty(M)$-linearity $\mathrm{Jac}_{[\cdot,\cdot]_*}(D,D,f\epsilon)=f\,\mathrm{Jac}_{[\cdot,\cdot]_*}(D,D,\epsilon)$ implies that the pointwise pairing $\big(j(F_D)_{x_0},\,j(\mathrm{Jac}_{[\cdot,\cdot]_*}(D,D,\epsilon))_{x_0}\big)_{A,x_0}=0$ vanishes identically at every point $x_0\in M$ for all $D\in\mathrm{Conn}_{\rho,\eta}(P;A)$ and all $\epsilon\in\Omega^0_{\mathrm{heq},\rho}(P;A)$. Now fix $x_0\in M$, a coordinate chart $(U,x^1,\dots,x^n)$ centered at $x_0$ ($x^\mu(x_0)=0$), a basepoint connection $D_0=\eta\circ\omega_0\in\mathrm{Conn}_{\rho,\eta}(P;A)$, and a local section $s_0:U\to P$ chosen in synchronous gauge at $x_0$ so that $(s_0^*D_0)_{x_0}=0$. Writing $D=D_0+\mathcal{A}$ with $\mathcal{A}\in\Omega^1_{\mathrm{heq},\rho}(P;A)$, given any

$y_1, y_2, z \in A$ and any 2-form $\Psi_0 = \frac{1}{2}\sum_{\mu,\nu=1}^{n} \Psi_{\mu\nu}\, dx^\mu \wedge dx^\nu \in \Lambda^2 T^*_{x_0}M \otimes A$ ($\Psi_{\mu\nu} = -\Psi_{\nu\mu} \in A$), choose $\mathcal{A} \in \Omega^1_{\mathrm{heq},\rho}(P; A)$ such that on $U$,

$$s_0^*\mathcal{A} = y_1\, dx^1 + y_2\, dx^2 + \frac{1}{2}\sum_{\mu,\nu=1}^{n} x^\mu \Psi_{\mu\nu}\, dx^\nu,$$

and choose $\epsilon \in \Omega^0_{\mathrm{heq},\rho}(P; A)$ with $(s_0^*\epsilon)(x_0) = z$. Evaluating at $x_0$ (where $x^\mu(x_0) = 0$ and $(s_0^* D_0)_{x_0} = 0$) gives $(s_0^* D)_{x_0} = (s_0^*\mathcal{A})_{x_0} = y_1\, dx^1 + y_2\, dx^2$, while $(s_0^* d\mathcal{A})_{x_0} = \Psi_0$ can be chosen arbitrarily so that $(s_0^* F_D)_{x_0} = (s_0^* dD_0)_{x_0} + \Psi_0 + \frac{1}{2}[y_1, y_2]_*\, dx^1 \wedge dx^2$ equals any prescribed element of $\Lambda^2 T^*_{x_0}M \otimes A$. By pointwise non-degeneracy of $(\cdot,\cdot)_{A,x_0}$, it follows that $j(\mathrm{Jac}_{[\cdot,\cdot]_*}(D, D, \epsilon))_{x_0} = 2\,\mathrm{Jac}_{[\cdot,\cdot]_*}(y_1, y_2, z)\, dx^1 \wedge dx^2 = 0$, so $\mathrm{Jac}_{[\cdot,\cdot]_*}(y_1, y_2, z) = 0$ for all $y_1, y_2, z \in A$. □

## 3. Algebraic and Polynomial Obstructions on $A$-YMT Theories

Substituting $F_D = dD + \frac{1}{2} D \wedge_* D$ into the $A$-YMT action functional (2.28) yields the expansion

$$S^A[D] = S^A_{\mathrm{free}}[D] + S^A_{\mathrm{cubic}}[D] + S^A_{\mathrm{quartic}}[D], \tag{3.1}$$

where $S^A_{\mathrm{free}}[D] := \int_M \langle j(dD), j(dD)\rangle_A$, $S^A_{\mathrm{cubic}}[D] := \int_M \langle j(dD), j(D\wedge_* D)\rangle_{A,\mathrm{sym}}$, and $S^A_{\mathrm{quartic}}[D] := \frac{1}{4}\int_M \langle j(D \wedge_* D), j(D \wedge_* D)\rangle_A$ (in the linear/horizontal setting $\eta = 0$, or relative to a basepoint connection $D_0$ in the affine setting).

3.1. **Commutator Projection, Abelianization, and Subbundle Localization.** By (2.7), for any 1-form $D \in \Omega^1(P; A)$, the binary product $D \wedge_* D = \frac{1}{2} D \wedge_{[\cdot,\cdot]_*} D$ depends solely on the commutator bracket $[x, y]_* = x * y - y * x$. We record the immediate consequences of this projection in a single proposition.

**Proposition 3.1** (Abelianization, Derived Filtration Shift, and Subbundle Localization)**.** *Let $\pi : P \to M$ be a principal $G$-bundle over an $n$-manifold $M$ ($n \geq 2$) and let $\rho : G \to \mathrm{Aut}_{\mathrm{Alg}}(A)$ be an algebra automorphism representation on an $\mathfrak{m}$-graded algebra $A = \bigoplus_{m\in\mathfrak{m}} A^m$.*

1. ***Abelianization:*** *If $A$ is a weak $(1,1)$-solvable $\mathfrak{m}$-graded algebra (i.e., $A = \bigoplus_{i=1}^r V_i$ decomposes into mutually orthogonal commutative graded submodules), or if $A = \bigoplus_{i=1}^r A_i$ is a vector space direct sum of commutative subalgebras and $D \in \mathcal{C}^1_{\mathrm{split}}(P; A)$ is algebraically split-adapted, then $D \wedge_* D = 0$ identically, reducing $F_D = dD$ and $S^A[D] = S^A_{\mathrm{free}}[D]$ to a free abelian theory in all dimensions $n \geq 2$.*
2. ***Derived Filtration Shift:*** *If $A$ is a solvable algebra of derived length $s$ ($A^{(s+1)} = 0$), then for every $D \in \Omega^1(P; A)$, $D\wedge_* D \in \Omega^2(P; A^{(1)})$, and the iterated squares $(D\wedge_* D)^{[m]} \in \Omega^{2^{m+1}}(P; A^{(m+1)})$ vanish identically at step $m = s$.*
3. ***Commutator Subbundle Localization:*** *Because $[A, A]_* \subset A^{(1)}$ is $\mathrm{Aut}_{\mathrm{Alg}}(A)$-invariant, $E_{[A,A]_*} = P \times_\rho [A, A]_* \subset E_{A^{(1)}} \subset E_A$ is a canonical vector subbundle. For every $D \in \mathrm{Conn}^{\mathrm{lin}}(P; A)$, $j(D \wedge_* D) \in \Omega^2(M; E_{[A,A]_*})$, so $S^A_{\mathrm{quartic}}[D]$ vanishes identically whenever $\Omega^2(M; E_{[A,A]_*})$ lies in the radical of $\langle\cdot,\cdot\rangle_A$.*

*Proof.* In part (1), writing $D = \sum_{i=1}^r \sum_{m\in\mathfrak{m}} D_i^m$ with $D_i^m \in \Omega^1(P; V_i^m)$, orthogonality $V_i * V_j = 0$ ($i \neq j$) and weak $(1,1)$-nilpotency $D_i^{m_1} \wedge_* D_i^{m_2} = 0$ give $D \wedge_* D = 0$ (and similarly for $D \in \mathcal{C}^1_{\mathrm{split}}(P; A)$). Part (2) follows by induction from $A^{(m)} * A^{(m)} = A^{(m+1)}$. For part (3), $(D \wedge_* D)_p(v_1, v_2) = [D_p(v_1), D_p(v_2)]_* \in [A, A]_* \subset A^{(1)}$, which descends via $j$ to $\Omega^2(M; E_{[A,A]_*})$. □

3.2. **Obstructions on Internal Multiplicative, Self-Dual, and Topological $A$-YMT Theories.** We now examine internal multiplicative $A$-YMT theories (2.30) and higher-degree internal topological (Chern-Weil) $A$-YMT theories (2.31), in which the pairing on $F_D$ is constructed from the internal product $\wedge_*$ of $\Omega(P; A)$ and a bundle trace functional $\mathrm{tr}_\xi : E_A \to M \times \mathbb{R}$ (Remark 2.6). By (2.8), the exterior square of the 2-form $F_D \wedge_* F_D = F_D \wedge_{\circ_*} F_D$ projects onto the symmetric Jordan product $x \circ_* y = \frac{1}{2}(x * y + y * x)$, so $(2, s)$-solvability of $(A, *)$ yields direct dimensional and self-duality obstructions.

**Theorem 3.2** (Obstruction to Internal-Trace Topological and Self-Dual $A$-YMT Theories)**.** *Let $\pi : P \to M$ be a principal $G$-bundle over an oriented $n$-manifold $M$, let $\rho : G \to \mathrm{Aut}_{\mathrm{Alg}}(A)$ be an automorphism representation on a weak $\mathcal{P}$-$(2,s)$-solvable $\mathfrak{m}$-graded algebra $A = \bigoplus_{m\in\mathfrak{m}} A^m$ (for $\mathcal{P} \in \{\mathrm{full}, \mathrm{bal}, \mathrm{blk}\}$), and let $\mathrm{tr}_\xi : E_A \to M \times \mathbb{R}$ be any bundle trace functional induced by $\xi \in \Gamma(E_A^*)$.*

(1) ***Higher-Dimensional Internal Chern-Weil Obstruction:*** *Suppose $n = 2r$ is even. If $n \geq 2(s+1)$ (i.e., $r \geq s+1$), then for every admissible connection $D \in \mathrm{Conn}_{\rho,\eta}(P; A)$ (including $\mathfrak{m}$-graded connections of Definition 2.3(3); or for every homogeneous connection $D \in \Omega^1(P; A^m)$ with $m + m = m$ when $A$ is only homogeneous weak $\mathcal{P}$-$(2,s)$-solvable), $\wedge^r_{*,\mathcal{P}} F_D = 0$ identically in $\Omega^{2r}(P; A)$, and hence the $r$-th internal topological $A$-YMT functional*

$$S^A_{\mathrm{top},r,\mathcal{P}}[D] = \int_M \mathrm{tr}_\xi\Big(j(\wedge^r_{*,\mathcal{P}} F_D)\Big)$$

*vanishes identically on $\mathrm{Conn}_{\rho,\eta}(P; A)$ for every trace section $\xi \in \Gamma(E_A^*)$.*

(2) ***Four-Dimensional Internal Topological and Self-Dual Obstruction ($s = 1$):*** *Suppose $s = 1$ (so $A$ is weak $(2,1)$-solvable, which by (2.8) includes every skew-commutative algebra since $F_D \wedge_* F_D = F_D \wedge_{\circ_*} F_D = 0$). Then for every $D \in \Omega^1(P; A)$, $F_D \wedge_* F_D = 0$ in $\Omega^4(P; A)$. Consequently, in dimension $n = 4$:*

   (a) *The 4D internal topological Yang-Mills functional $S^A_{\mathrm{top},2}[D] = \int_M \mathrm{tr}_\xi(j(F_D \wedge_* F_D))$ is identically zero;*

   (b) *Any internal multiplicative 4D $A$-YMT functional $S^A_{\hat\star}[D] = \int_M \mathrm{tr}_\xi(j(F_D \wedge_* \hat\star F_D))$ vanishes identically on all $\hat\star$-self-dual and $\hat\star$-anti-self-dual connections ($\hat\star F_D = \pm F_D$).*

*Proof.* Decomposing $A = \bigoplus_{i=1}^k W_i$ into mutually orthogonal weak $\mathcal{P}$-$(2,s)$-nilpotent graded subalgebras $W_i = \bigoplus_{m\in\mathfrak{m}} W_i^m$ ($W_i * W_j = 0$ for $i \neq j$), we write $F_D = \sum_{i=1}^k F_{D,i}$ with $F_{D,i} = \sum_{m\in\mathfrak{m}} F^m_{D,i} \in \Omega^2(P; W_i)$ (where $F^m_{D,i}$ is given by (2.22)). Expanding $\wedge^r_{*,\mathcal{P}} F_D$ and using $W_i * W_j = 0$ ($i \neq j$) yields $\sum_{i=1}^k \sum_{m_1,\ldots,m_r\in\mathfrak{m}} p_r(F^{m_1}_{D,i}, \ldots, F^{m_r}_{D,i})$, where each $\mathcal{P}$-parenthesized monomial of degree $r \geq s+1$ vanishes by Definition 2.2(4). Thus $\wedge^r_{*,\mathcal{P}} F_D = 0$ for all $r \geq s+1$, proving (1). Setting $s = 1$ and $r = 2$ gives $F_D \wedge_* F_D = 0$, which immediately implies (2a) and (2b). □

Finally, given a surjective $R$-linear projection $Q : E \twoheadrightarrow A$ and an $R$-linear section $S : A \hookrightarrow E$ with $Q \circ S = \mathrm{id}_A$ (for instance, a split short exact sequence $0 \to H \to E \xrightarrow{Q} A \to 0$), endowing $E$ with the $(Q,S)$-pulled-back product $x *_{Q,S} y := S(Q(x) * Q(y))$ yields the following invariance property.

**Proposition 3.3** (Invariance under $(Q,S)$-Splitting Extensions)**.** *For $x *' y := S(Q(x) * Q(y))$ on $E$ with $Q \circ S = \mathrm{id}_A$, any non-associative monomial $p_r$ satisfies the relation $p_r^{*'}(\alpha_1, \ldots, \alpha_r) = (\Omega S)(p_r^*((\Omega Q)(\alpha_1), \ldots, (\Omega Q)(\alpha_r)))$ on $\Omega(P; E)$. Since $\Omega S$ is injective and $\Omega Q$ is surjective, $(E, *')$ is $\mathcal{P}$-$(k,s)$-nil if and only if $(A, *)$ is $\mathcal{P}$-$(k,s)$-nil.*

*Proof.* Follows by induction on $r \geq 2$ from $\alpha \wedge_{*'} \beta = (\Omega S)((\Omega Q)(\alpha) \wedge_* (\Omega Q)(\beta))$ and $(\Omega Q) \circ (\Omega S) = \mathrm{id}_{\Omega(P;A)}$. □

## 4. Parenthesization Cutoffs and Holonomy Reductions in Reductive Cartan-Yang-Mills Theories

In approaches that unify gravity and Yang-Mills gauge theory—such as MacDowell-Mansouri gravity [42, 43] and Cartan-Yang-Mills theories—one works with a reductive $A$-connection $\nabla = e + \omega \in \Omega^1(P; A_0) \oplus \Omega^1(P; A_1)$ and uses the soldering 1-form $e$ to construct an internal constitutive operator on the Cartan curvature $F_\nabla$. We now analyze the dimensional cutoffs of reductive Cartan-Yang-Mills theories under both quartic-block parenthesization (2.11) and recursive binary-tree parenthesization (2.9), establish the 4D first-Bianchi contraction identity and proportionality between left-grouped and right-nested EHP terms, pair the dimensional cutoffs with Berger's holonomy classification in dimensions $4 \leq n \leq 7$, and establish the representation-theoretic dichotomy at the critical $G_2$ dimension $n = k = 7$.

4.1. ***H*-Equivariant Reductive Soldering Products and *k*-Proper Bundles.** Let us establish the algebraic and geometric setup:

- Let $\mathfrak{h} \subset \mathfrak{so}(k_1) \oplus \cdots \oplus \mathfrak{so}(k_r) \subset \mathfrak{so}(k)$ $(k_1 + \cdots + k_r = k)$ be a matrix Lie subalgebra equipped with the skew-commutative product $X \curlywedge Y := \frac{1}{2}(XY - YX) = \frac{1}{2}[X, Y]$ in $\mathfrak{h}$ (a *k-algebra*, with connected *k-group* $H \subset SO(k_1) \times \cdots \times SO(k_r)$) [24].
- On the reductive direct sum $A = A_0 \oplus A_1 = \mathbb{R}^k \oplus \mathfrak{h}$ (where $H$ acts on $A_0 = \mathbb{R}^k$ via the defining vector representation and on $A_1 = \mathfrak{h}$ via $\mathrm{Ad}_H$), we identify $\mathbb{R}^k \wedge \mathbb{R}^k \simeq \mathfrak{so}(k)$ via $(u \wedge v)_{ab} := u_a v_b - u_b v_a$ for $u, v \in \mathbb{R}^k$, and let $\sigma : \mathbb{R}^k \wedge \mathbb{R}^k \to \mathfrak{h}$ be the canonical $H$-equivariant skew-symmetric soldering projection defined by the orthogonal projection $\mathfrak{so}(k) \twoheadrightarrow \mathfrak{h}$ with respect to the positive-definite inner product $\langle X, Y\rangle_{\mathfrak{so}(k)} := -\frac{1}{2}\mathrm{Tr}(XY) = \frac{1}{2}\sum_{a,b=1}^{k} X_{ab}Y_{ab}$. Explicitly, for any $\langle\cdot,\cdot\rangle_{\mathfrak{so}(k)}$-orthonormal basis $\{T_A\}$ of $\mathfrak{h}$,

$$\sigma(u \wedge v) = \sum_A \langle T_A,\, u \wedge v\rangle_{\mathfrak{so}(k)} T_A = \sum_A \Big( \sum_{a,b=1}^{k} (T_A)_{ab} u_a v_b \Big) T_A, \tag{4.1}$$

  so that for $\mathbb{R}^k$-valued forms $\alpha \in \Omega^p(P; \mathbb{R}^k)$ and $\beta \in \Omega^q(P; \mathbb{R}^k)$, the $\mathfrak{h}$-valued $(p+q)$-form $\sigma(\alpha \wedge \beta) = \sum_A (\sum_{a,b=1}^{k} (T_A)_{ab}\, \alpha^a \wedge \beta^b) T_A$ is the orthogonal projection of the skew-symmetric matrix $(\alpha \wedge \beta)_{ab} = \alpha_a \wedge \beta_b - \alpha_b \wedge \beta_a = \alpha_a \wedge \beta_b - (-1)^{pq} \beta_a \wedge \alpha_b \in \Omega^{p+q}(P; \mathfrak{so}(k))$. Let $\times_H : \mathbb{R}^k \wedge \mathbb{R}^k \to \mathbb{R}^k$ be an $H$-equivariant skew-symmetric bilinear vector product on $\mathbb{R}^k$ (for irreducible $H \subset SO(k)$, $\times_H = 0$ except when $k = 7$ and $H = G_2 \subset SO(7)$, where $(u \times_{G_2} v)^a = \varphi^a_{bc} u^b v^c$ is the octonionic cross product induced by the associative $G_2$-invariant 3-form $\varphi_0 \in (\Lambda^3(\mathbb{R}^7)^*)^{G_2}$ [2, 35]). We define the *H-equivariant reductive Cartan-MacDowell-Mansouri product* $\curlywedge_{\rtimes}$ on $A = \mathbb{R}^k \oplus \mathfrak{h}$ by

$$(v_1, X_1) \curlywedge_{\rtimes} (v_2, X_2) := \Big( \frac{1}{2}(X_1 v_2 - X_2 v_1) + v_1 \times_H v_2,\; X_1 \curlywedge X_2 + \sigma(v_1 \wedge v_2) \Big), \tag{4.2}$$

  where $(Xv)^a := X^a{}_b v^b = \sum_{b=1}^{k} X_{ab} v^b$ denotes the standard matrix action of $X \in \mathfrak{h} \subset \mathfrak{so}(k)$ on $v \in \mathbb{R}^k$. The product (4.2) satisfies three key properties:

  (P1) *H-Equivariance:* $H$ acts on $(A, \curlywedge_{\rtimes})$ by algebra automorphisms, inducing the associated algebra bundle $E_A = P \times_H (\mathbb{R}^k \oplus \mathfrak{h}) \to M$.

  (P2) *Skew-Commutativity, Parenthesization, and Rank Identities:* Because (4.2) is skew-commutative on $A = \mathbb{R}^k \oplus \mathfrak{h}$, $(A, \curlywedge_{\rtimes})$ is $(2,1)$-nil ($\Phi \curlywedge_{\rtimes} \Phi = 0$ for all $\Phi \in \Omega^{\mathrm{even}}(P; A)$) and block-balanced $(1,3)$-nil ($\curlywedge^m_{\rtimes,\mathrm{blk}} \alpha = 0$ for all $\alpha \in \Omega^1(P; A)$ and $m \geq 4$ by Lemma 2.1(1)). Furthermore, because any $A_0$-valued soldering 1-form $e = (e, 0) \in \Omega^1(P; \mathbb{R}^k)$ takes values in the $k$-dimensional subspace $A_0 = \mathbb{R}^k$, Lemma 2.1(2c) implies that $\curlywedge^m_{\rtimes,\mathcal{P}} e = 0$ identically on $P$ for all $m > k$. Moreover, when $\times_H = 0$, $A = A_0 \oplus A_1$ is $\mathbb{Z}_2$-graded ($A_0 \curlywedge_{\rtimes} A_0 \subset A_1$, $A_1 \curlywedge_{\rtimes} A_0 \subset A_0$, $A_1 \curlywedge_{\rtimes} A_1 \subset A_1$), so $e \curlywedge_{\rtimes} e = (0, \sigma(e \wedge e)) \in \Omega^2(P; \mathfrak{h})$ lies in $A_1 = \mathfrak{h}$ and $\curlywedge^3_{\rtimes,\mathrm{bal}} e = (e \curlywedge_{\rtimes} e) \curlywedge_{\rtimes} e \in \Omega^3(P; \mathbb{R}^k)$ lies in $A_0 = \mathbb{R}^k$.

  (P3) *Classical Cartan Torsion on $A_0 = \mathbb{R}^k$:* For $\omega \in \Omega^1(P; \mathfrak{h})$ and $e \in \Omega^1(P; \mathbb{R}^k)$, the generalized torsion in (2.21) is $\Theta_{e,\omega} = (de + \frac{1}{2}\omega \wedge e, 0)$ (since by (2.5) and (4.2), $(\omega \curlywedge_{\rtimes} e + e \curlywedge_{\rtimes} \omega)_p(v_1, v_2) = 2\,\omega_p(v_1) \cdot \frac{1}{2} e_p(v_2) - 2\,\omega_p(v_2) \cdot \frac{1}{2} e_p(v_1) = (\omega \wedge e)_p(v_1, v_2)$), which vanishes if and only if $\tilde{\omega} := \frac{1}{2}\omega$ satisfies the classical Cartan torsion-free equation $de + \tilde{\omega} \wedge e = 0$.

- Let $N$ be a simply connected smooth $k$-manifold $(k \leq n)$. Following [24], a principal $H$-bundle $\pi : P \to M$ over an $n$-manifold $M$ is called *k-proper* if there exists an immersed simply connected $k$-manifold $i : N \hookrightarrow M$ whose orthonormal frame bundle $FN$ reduces to an $H$-subbundle $P_N \subset FN$ isomorphic as a principal $H$-bundle over $N$ to $i^*P$ via $\kappa : P_N \xrightarrow{\sim} i^*P$, and we define the *partially torsionless irreducible configuration space* $\mathrm{Conn}_A^{\Theta_N = 0, \mathrm{irr}}(P)$ to be the space of pairs of horizontal $H$-equivariant soldering 1-forms $e \in \Omega^1_{\mathrm{heq}}(P; \mathbb{R}^k)$ and Ehresmann connections $\tilde{\omega} = \frac{1}{2}\omega \in \mathrm{Conn}(P; \mathfrak{h})$ whose pullback along $i \circ \kappa : P_N \to P$ satisfies the Cartan torsion-free equation $(i \circ \kappa)^*(de + \tilde{\omega} \wedge e) = 0$ on $P_N$ (and $de + \tilde{\omega} \wedge e = 0$ globally on $P$ whenever $k = n$ and $M = N$, in which case we write

$\mathrm{Conn}_A^{\Theta=0,\mathrm{irr}}(P)$) such that $(i\circ\kappa)^*e\in\Omega^1(P_N;\mathbb{R}^k)$ is the canonical orthonormal soldering 1-form on $P_N\subset FN$ inducing a locally irreducible, locally non-symmetric Riemannian metric $g_N$ on $N$ (so that $(N,g_N)$ is a *Berger $k$-manifold* [7, 8, 30] with Levi-Civita connection $\tilde\omega_N:=\frac12(i\circ\kappa)^*\omega$ and curvature $\Omega_{\tilde\omega_N}=\frac12(i\circ\kappa)^*\Omega_\omega$).

**Remark 4.1** (Geometric Realizations of $k$-Proper Bundles and the Role of Mixed Torsion when $k<n$)**.** When $k=n$ and $M=N$ is simply connected, $P\subset FM$ is an $H$-structure on $M$ with globally torsion-free Levi-Civita connection $\tilde\omega=\frac12\omega$ ($\Theta_{e,\omega}=de+\tilde\omega\wedge e=0$ globally on $P$) and orthonormal coframe $e$. When $k<n$, $e\in\Omega^1_{\mathrm{heq}}(P;\mathbb{R}^k)$ has a horizontal kernel distribution $\mathcal{V}^\perp=\ker(e)\subset TM$ of rank $n-k$. Suppose $\pi_M:M\to N$ is a smooth submersion (for instance, a Riemannian product $M=N^k\times K^{n-k}$) with $i:N\hookrightarrow M$ a section, $P=\pi_M^*P_N$, and $e=\pi_M^*e_N$. Any connection $\tilde\omega=\pi_M^*\tilde\omega_N+\beta$ with $\beta\in\Omega^1_{\mathrm{heq}}(P;\mathfrak{h})$ vanishing along $TN$ ($i_v\beta=0$ for $v\in TN$) satisfies the pulled-back torsion-free condition $(i\circ\kappa)^*(de+\tilde\omega\wedge e)=de_N+\tilde\omega_N\wedge e_N=0$ on $P_N$. Notice that on $M^n=N^k\times K^{n-k}$ with $e=\pi_M^*e_N$, $de=\pi_M^*de_N=-\pi_M^*(\tilde\omega_N\wedge e_N)$, so the full torsion on $M^n$ is $\Theta_{e,\omega}=de+\tilde\omega\wedge e=\beta\wedge\pi_M^*e_N$. Since $\beta$ is along $T^*K$, $e_N$ is an orthonormal coframe along $T^*N$, and $H\subset SO(k)$ acts faithfully on $\mathbb{R}^k$, imposing *global* torsion-freeness $\Theta_{e,\omega}=0$ on all of $M^n$ would force $\beta(w)\cdot e_N(v)=0$ for all $w\in TK, v\in TN$ and hence $\beta=0$, which would make both $e$ and $\Omega_\omega$ pullbacks from $N^k$ and force every $n$-form built from $(e,\Omega_\omega)$ to vanish identically for $n>k$ (since $\Lambda^n(T^*N^k)=0$)! Conversely, because $\mathrm{Conn}_A^{\Theta_N=0,\mathrm{irr}}(P)$ imposes torsion-freeness $(i\circ\kappa)^*\Theta_{e,\omega}=0$ only along $N^k$ while allowing non-zero mixed torsion $\Theta_{e,\omega}(w,v)=\beta(w)\cdot e_N(v)\neq0$ on $TK\times TN$, a non-zero horizontal 1-form $\beta$ along $TK$ contributes non-zero transverse curvature $\Omega_\omega$ along $T^*K\wedge T^*K$ (or $T^*K\wedge T^*N$). Since the two factors of $\Omega_\omega$ in the pure Yang-Mills $n$-form $\Omega_\omega\curlywedge((\curlywedge^{n-4}_{\curlywedge,\mathcal{P}}e)\curlywedge\Omega_\omega)$ have total degree 4 and can saturate at most 4 cotangent directions along $T^*K$ while $\curlywedge^{n-4}_{\curlywedge,\mathcal{P}}e$ saturates $n-4\le k$ directions along $T^*N$, degree counting does not prevent the pure Yang-Mills $n$-form from being non-zero on $\mathrm{Conn}_A^{\Theta_N=0,\mathrm{irr}}(P)$ when $n-4\le k<n$. Whether it is actually non-zero depends also on the algebraic factor $\curlywedge^{n-4}_{\curlywedge,\mathcal{P}}e$, which vanishes for $n\ge8$ under $\mathcal{P}=\mathrm{blk}$, and must be checked case by case for $5\le n\le7$.

Let us clarify the relationship between the quadratic constitutive expansion of the full Cartan curvature $F_\nabla=\Omega_\omega+\Theta_{e,\omega}+E_e$ ($E_e=\frac12e\curlywedge e$) in (2.30) and the canonical left-grouped Cartan-Yang-Mills sectors. In a non-associative algebra $(A,\curlywedge)$, if one applies the internal constitutive operator $\hat\star_{\mathcal{P}}(\Phi):=(\curlywedge^{n-4}_{\curlywedge,\mathcal{P}}e)\curlywedge\Phi$ directly to the torsionless Cartan curvature $F_\nabla^{\Theta=0}:=\Omega_\omega+\frac12e\curlywedge e$, expanding $F_\nabla^{\Theta=0}\curlywedge\hat\star_{\mathcal{P}}(F_\nabla^{\Theta=0})$ produces the quadratic Yang-Mills term $\Omega_\omega\curlywedge((\curlywedge^{n-4}_{\curlywedge,\mathcal{P}}e)\curlywedge\Omega_\omega)$, together with the cross-terms $\frac12(e\curlywedge e)\curlywedge((\curlywedge^{n-4}_{\curlywedge,\mathcal{P}}e)\curlywedge\Omega_\omega)+\frac12\Omega_\omega\curlywedge((\curlywedge^{n-4}_{\curlywedge,\mathcal{P}}e)\curlywedge(e\curlywedge e))$ and the cosmological term $\frac14(e\curlywedge e)\curlywedge((\curlywedge^{n-4}_{\curlywedge,\mathcal{P}}e)\curlywedge(e\curlywedge e))$. Because $(A,\curlywedge)$ is non-associative, in order to place all three gravitational and gauge sectors on a uniform footing where the constitutive soldering factors of $e$ are grouped into a single monomial $\curlywedge^m_{\curlywedge,\mathcal{P}}e$ prior to multiplying with $\Omega_\omega$ (matching the $A$-EHP convention of [24]), we define for $\mathcal{P}\in\{\mathrm{blk},\mathrm{bal}\}$ in (2.9)–(2.11) the *left-grouped Cartan-Yang-Mills Lagrangian $n$-form* $\mathcal{L}^{\mathrm{CYM}}_{n,\mathcal{P}}\in\Omega^n_{\mathrm{heq}}(P;A)$ by

$$\mathcal{L}^{\mathrm{CYM}}_{n,\mathcal{P}}:=\Omega_\omega\curlywedge\Big(\big(\curlywedge^{n-4}_{\curlywedge,\mathcal{P}}e\big)\curlywedge\Omega_\omega\Big)+\big(\curlywedge^{n-2}_{\curlywedge,\mathcal{P}}e\big)\curlywedge\Omega_\omega+\frac14\curlywedge^n_{\curlywedge,\mathcal{P}}e,\tag{4.3}$$

where the three terms are respectively the *quadratic Yang-Mills term*, the *left-grouped Einstein-Hilbert-Palatini (EHP) cross-term*, and the *cosmological term*. (Notice that at $n=6$, the ungrouped cross-term $\frac12(e\curlywedge e)\curlywedge((e\curlywedge e)\curlywedge\Omega_\omega)$ does not factor through $\curlywedge^4_{\curlywedge,\mathrm{bal}}e=0$, whereas in (4.3) each sector depends directly on $\curlywedge^m_{\curlywedge,\mathcal{P}}e$ for $m\in\{n-4,n-2,n\}$.) Recall from Section 2 that when $n=4$ ($n-4=0$), our $m=0$ convention $\big(\curlywedge^0_{\curlywedge,\mathcal{P}}e\big)\curlywedge\Omega_\omega:=\Omega_\omega$ omits the $e$-factor, so the quadratic Yang-Mills term at $n=4$ is simply $\Omega_\omega\curlywedge\Omega_\omega$, which vanishes identically by $(2,1)$-nilpotency of $(A,\curlywedge)$!

To compare the two natural parenthesizations of the EHP cross-term at $n=4$, we also consider the *right-nested EHP cross-term*

$$\mathcal{L}^{\mathrm{EHP,right}}_4:=e\curlywedge(e\curlywedge\Omega_\omega)\in\Omega^4_{\mathrm{heq}}(P;A),\tag{4.4}$$

as opposed to the left-grouped EHP cross-term $\mathcal{L}_4^{\mathrm{EHP,left}} := (e \curlywedge_{\rtimes} e) \curlywedge_{\rtimes} \Omega_\omega$ in (4.3).

**Definition 4.2** (Cartan-Yang-Mills Functional)**.** For $n \geq 4$, $\mathcal{P} \in \{\mathrm{blk}, \mathrm{bal}\}$, and any bundle trace functional $\mathrm{tr}_\xi : E_A \to M \times \mathbb{R}$ induced by $\xi \in \Gamma(E_A^*)$, the *$\mathcal{P}$-balanced Cartan-Yang-Mills (A-CYM) functional* on $\mathrm{Conn}_A^{\Theta_N=0,\mathrm{irr}}(P)$ is

$$S^{\mathrm{CYM}}_{n,\xi,\mathcal{P}}[e,\omega] := \int_M \mathrm{tr}_\xi\big(j(\mathcal{L}^{\mathrm{CYM}}_{n,\mathcal{P}})\big). \tag{4.5}$$

4.2. **Dimensional Cutoffs, the 4D First-Bianchi Contraction Identity, and Berger Holonomy Reductions.** Before stating our main theorem on $S^{\mathrm{CYM}}_{n,\xi,\mathcal{P}}[e,\omega]$, we prove a fundamental geometric lemma establishing the exact proportionality between the left-grouped EHP 4-form $(e \curlywedge_{\rtimes} e) \curlywedge_{\rtimes} \Omega_\omega$ and the right-nested EHP 4-form $e \curlywedge_{\rtimes} (e \curlywedge_{\rtimes} \Omega_\omega)$, and showing that on 4-manifolds with $U(2)$ or $SU(2)$ holonomy both parenthesizations vanish identically on-shell (when torsion $\Theta_{e,\omega} = 0$) while remaining non-zero off-shell (when $\Theta_{e,\omega} \neq 0$).

**Lemma 4.3** (4D First-Bianchi Contraction Identity and Proportionality of Left-Grouped and Right-Nested EHP Terms)**.** *Let $n = k = 4$ and let $H \subset SO(4)$ be either $SU(2)_-$ (hyperkähler holonomy) or $U(2) = (U(1)_+ \times SU(2)_-)/\mathbb{Z}_2$ (Kähler 4-manifold holonomy), under the self-dual/anti-self-dual Lie algebra decomposition $\mathfrak{so}(4) \simeq \mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_-$. Let $(e,\omega) \in \mathrm{Conn}_A^{\Theta=0}(P)$ be any torsion-free pair ($de + \tilde{\omega} \wedge e = 0$, $\tilde{\omega} = \frac{1}{2}\omega$). Then:*

(1) ***On-Shell Vanishing of the Left-Grouped EHP Term*** *$(e \curlywedge_{\rtimes} e) \curlywedge_{\rtimes} \Omega_\omega$: Under (4.2), $(e \curlywedge_{\rtimes} e) \curlywedge_{\rtimes} \Omega_\omega = (0, \frac{1}{2}[\sigma(e \wedge e), \Omega_\omega]) \in \Omega^4(P; [\mathfrak{h}, \mathfrak{h}])$. Because $de + \tilde{\omega} \wedge e = 0$ implies the algebraic first Bianchi identity $\Omega_\omega \wedge e = 0$, the curvature 2-form $\Omega_\omega$ has symmetric Riemann components $R_{ab} = R_{ba}$ in the standard basis $\{\omega_1^-, \omega_2^-, \omega_3^-\}$ of $\Lambda^2_- \mathbb{R}^4 \simeq \mathfrak{su}(2)_-$. Since $\omega_a^- \wedge \omega_b^- = -2\delta_{ab}\,\mathrm{vol}_4$ is also symmetric in $(a,b)$ while $[T_a, T_b] = \epsilon_{abc}T_c$ is skew-symmetric, we have*

$$(e \curlywedge_{\rtimes} e) \curlywedge_{\rtimes} \Omega_\omega = \big(0, \frac{1}{2}[\sigma(e \wedge e), \Omega_\omega]\big) = 0 \quad \textit{identically on } \mathrm{Conn}_A^{\Theta=0}(P) \tag{4.6}$$

*for both $H = SU(2)_-$ and $H = U(2)$ (by Lemma 4.4 below, this on-shell vanishing in fact holds for every $\mathfrak{h} \subset \mathfrak{so}(k)$ and every $k$; the content of the present item is the explicit off-shell formula). (Off-shell, i.e., for connections $\omega$ with non-zero torsion $\Theta_{e,\omega} \neq 0$ whose curvature matrix $R_{ab} \neq R_{ba}$ has a non-zero skew-symmetric part, $[\sigma(e \wedge e), \Omega_\omega] = -2\sum_{a,b} R_{ab}\epsilon_{abc}T_c\,\mathrm{vol}_4 \neq 0$ does not vanish.)*

(2) ***Exact Proportional Relation with the Right-Nested EHP Term*** *$e \curlywedge_{\rtimes} (e \curlywedge_{\rtimes} \Omega_\omega)$: For any subalgebra $\mathfrak{h} \subset \mathfrak{so}(k)$ with $\times_H = 0$, the right-nested parenthesization (4.4) satisfies*

$$e \curlywedge_{\rtimes} (e \curlywedge_{\rtimes} \Omega_\omega) = \Big(0,\ -\frac{1}{2}\sigma\big(e \wedge (\Omega_\omega \cdot e)\big)\Big) = +\frac{1}{2}\,(e \curlywedge_{\rtimes} e) \curlywedge_{\rtimes} \Omega_\omega = \Big(0,\ +\frac{1}{4}\big[\sigma(e \wedge e),\ \Omega_\omega\big]\Big). \tag{4.7}$$

*Consequently, at $n = k = 4$ with $H = SU(2)_-$ or $U(2)$, both $(e \curlywedge_{\rtimes} e) \curlywedge_{\rtimes} \Omega_\omega$ and $e \curlywedge_{\rtimes} (e \curlywedge_{\rtimes} \Omega_\omega)$ vanish identically on the torsion-free space $\mathrm{Conn}_A^{\Theta=0}(P)$ and are non-vanishing off-shell when $\Theta_{e,\omega} \neq 0$.*

*Proof.* Let us verify both parts directly. In any subalgebra $\mathfrak{h} \subset \mathfrak{so}(k)$ with $\times_H = 0$, for $e \in \Omega^1(P; \mathbb{R}^k)$ and $\Omega_\omega \in \Omega^2(P; \mathfrak{h})$, (4.2) gives $e \curlywedge_{\rtimes} \Omega_\omega = (-\frac{1}{2}\Omega_\omega \cdot e, 0) \in \Omega^3(P; \mathbb{R}^k \oplus \mathfrak{h})$, and multiplying on the left by $e = (e, 0)$ gives $e \curlywedge_{\rtimes} (e \curlywedge_{\rtimes} \Omega_\omega) = (0, -\frac{1}{2}\sigma(e \wedge (\Omega_\omega \cdot e)))$. For any basis $\{T_A\}$ of $\mathfrak{h} \subset \mathfrak{so}(k)$ orthonormal with respect to $\langle T_A, T_B\rangle_{\mathfrak{so}(k)} = -\frac{1}{2}\mathrm{Tr}(T_A T_B) = \delta_{AB}$, note that since $e^i$ is a 1-form and $(\Omega_\omega \cdot e)^j = (\Omega_\omega)_{jk} \wedge e^k$ is a 3-form, the $(i,j)$-entry of the skew-symmetric matrix $e \wedge (\Omega_\omega \cdot e) \in \Omega^4(P; \mathfrak{so}(k))$ is $e^i \wedge (\Omega_\omega \cdot e)^j - e^j \wedge (\Omega_\omega \cdot e)^i = e^i \wedge (\Omega_\omega \cdot e)^j + (\Omega_\omega \cdot e)^i \wedge e^j$. By (4.1), the $T_A$-component of $-\frac{1}{2}\sigma(e \wedge (\Omega_\omega \cdot e))$ is therefore

$$\frac{1}{4}\mathrm{Tr}\big(T_A \cdot (e \wedge (\Omega_\omega \cdot e)^T + (\Omega_\omega \cdot e) \wedge e^T)\big) = -\frac{1}{2}(T_A)_{ij}(\Omega_\omega)_{jk} \wedge e^i \wedge e^k = -\frac{1}{4}[T_A, \Omega_\omega]_{ik} \wedge e^i \wedge e^k,$$

because the symmetric part $\frac{1}{2}(T_A\Omega_\omega + \Omega_\omega T_A)_{ik}$ vanishes when contracted with the skew-symmetric 2-form $e^i \wedge e^k$. On the other hand, since $\langle X, [Y,Z]\rangle_{\mathfrak{so}(k)} = -\langle [X,Z], Y\rangle_{\mathfrak{so}(k)}$ by ad-invariance of

$\langle\cdot,\cdot\rangle_{\mathfrak{so}(k)}$, (4.1) shows that the $T_A$-component of $+\frac{1}{4}[\sigma(e\wedge e),\Omega_\omega]$ is

$$\frac{1}{4}\langle T_A,\,[\sigma(e\wedge e),\Omega_\omega]\rangle_{\mathfrak{so}(k)} = -\frac{1}{4}\langle [T_A,\Omega_\omega],\,\sigma(e\wedge e)\rangle_{\mathfrak{so}(k)} = -\frac{1}{4}[T_A,\Omega_\omega]_{ik}\wedge e^i\wedge e^k.$$

Consequently, for *every* subalgebra $\mathfrak{h}\subset\mathfrak{so}(k)$ with $\times_H=0$, the associative matrix identity $(T_A)_{ij}(T_B)_{jk}e^i\wedge e^k=\frac{1}{2}[T_A,T_B]_{ik}e^i\wedge e^k$ implies (4.7).

Now suppose $n=k=4$ and $\mathfrak{h}\subset\mathfrak{u}(2)=\mathfrak{u}(1)_+\oplus\mathfrak{su}(2)_-$. Since $\mathfrak{u}(1)_+$ is central in $\mathfrak{u}(2)$, the commutator $[\sigma(e\wedge e),\Omega_\omega]$ depends only on the $\mathfrak{su}(2)_-$ projections $\sigma_-(e\wedge e)=\sum_{a=1}^3\omega_a^-\otimes T_a$ and $(\Omega_\omega)_-=\sum_{b=1}^3\Omega_-^b\otimes T_b$. On a 4-manifold, the three anti-self-dual 2-forms $\omega_1^-,\omega_2^-,\omega_3^-$ satisfy $\omega_a^-\wedge\omega_b^-=-2\delta_{ab}\,\mathrm{vol}_4$. Decomposing $\Omega_-^b=\sum_{a=1}^3 R_{ab}\,\omega_a^-+\Omega_{-,+}^b$ where $\Omega_{-,+}^b\in\Lambda_+^2T^*M$ is the self-dual part of $\Omega_-^b$ (which represents the traceless Ricci tensor $\mathrm{Ric}_0\in\Lambda_+^2\otimes\Lambda_-^2$), since $\Lambda_-^2\wedge\Lambda_+^2=0$ we have $\omega_a^-\wedge\Omega_-^b=-2R_{ab}\,\mathrm{vol}_4$, and hence

$$\big[\sigma(e\wedge e),\,\Omega_\omega\big]=\sum_{a,b=1}^3(\omega_a^-\wedge\Omega_-^b)\otimes[T_a,T_b]=-2\sum_{a,b,c=1}^3 R_{ab}\,\epsilon_{abc}\,T_c\,\mathrm{vol}_4. \tag{4.8}$$

If $(e,\omega)\in\mathrm{Conn}_A^{\Theta=0}(P)$ is torsion-free ($de+\tilde\omega\wedge e=0$), the algebraic first Bianchi identity $\Omega_{\tilde\omega}\wedge e=0$ forces the anti-self-dual block $R_{ab}\in\mathrm{End}(\Lambda_-^2)$ of the Riemann curvature operator to be *symmetric* ($R_{ab}=R_{ba}$: specifically, $R_{ab}=W_{ab}^-+\frac{R}{12}\delta_{ab}$). Contracting the symmetric matrix $R_{ab}=R_{ba}$ with the antisymmetric structure constants $\epsilon_{abc}$ in (4.8) gives zero identically. Conversely, if $\omega$ has non-zero torsion $\Theta_{e,\omega}\neq0$, then $\Omega_{\tilde\omega}\wedge e=d_{\tilde\omega}\Theta_{e,\omega}\neq0$ allows $R_{ab}-R_{ba}\neq0$, so (4.8) is non-zero off-shell. □

The on-shell vanishing in Lemma 4.3 is a special case of the following general fact, which identifies the soldering product with the defect of the first Bianchi identity.

**Lemma 4.4** (The Soldering Product Is the First-Bianchi Defect)**.** *Let $\mathfrak{h}\subset\mathfrak{so}(k)$ with $\times_H=0$, and let $e\in\Omega^1(P;\mathbb{R}^k)$ and $\omega\in\Omega^1(P;\mathfrak{h})$, with $\tilde\omega=\frac{1}{2}\omega$. Then*

$$e\curlywedge\Omega_\omega=\Big(-\tfrac{1}{2}\,\Omega_\omega\cdot e,\;0\Big)=(-\,\Omega_{\tilde\omega}\cdot e,\;0),\qquad(\Omega_{\tilde\omega}\cdot e)^j=\textstyle\sum_l(\Omega_{\tilde\omega})_{jl}\wedge e^l.$$

*Consequently, if $(e,\omega)$ is torsion-free ($de+\tilde\omega\wedge e=0$), then $e\curlywedge\Omega_\omega=0$, and therefore*

$$e\curlywedge(e\curlywedge\Omega_\omega)=0,\qquad(e\curlywedge e)\curlywedge\Omega_\omega=0,\qquad\Omega_\omega\curlywedge((e\curlywedge e)\curlywedge\Omega_\omega)=0$$

*identically, in every dimension and for every $\mathfrak{h}$. Conversely, $e\curlywedge\Omega_\omega\neq0$ whenever the algebraic first Bianchi identity $\Omega_{\tilde\omega}\wedge e=0$ fails.*

*Proof.* By (4.2), $(v,0)\curlywedge(0,X)=(-\frac{1}{2}Xv,0)$; extending to forms by (2.5) gives the first equality. Since $\omega\curlywedge\omega$ coincides with the matrix wedge product $\omega\wedge\omega$, we have $\Omega_\omega=d\omega+\frac{1}{2}\omega\wedge\omega=2(d\tilde\omega+\tilde\omega\wedge\tilde\omega)=2\Omega_{\tilde\omega}$, which gives the second equality. If $de+\tilde\omega\wedge e=0$, differentiating yields $0=d\tilde\omega\wedge e-\tilde\omega\wedge de=(d\tilde\omega+\tilde\omega\wedge\tilde\omega)\wedge e=\Omega_{\tilde\omega}\wedge e$. The three displayed forms contain $e\curlywedge\Omega_\omega$ as a factor, using $(e\curlywedge e)\curlywedge\Omega_\omega=2\,e\curlywedge(e\curlywedge\Omega_\omega)$, which is (4.7); its proof in Lemma 4.3(2) is purely algebraic and valid for every $k$. □

**Remark 4.5** (Comparison Between $n=k=4$ and $n=k=6$ for $U(k/2)$ and $SU(k/2)$)**.** Lemmas 4.3 and 4.4 compare as follows for $k=4$ and $k=6$:

- In dimension $n=k=4$, both the quadratic YM term $\Omega_\omega\curlywedge\Omega_\omega=0$ (by $(2,1)$-nilpotency) and the EHP term $(e\curlywedge e)\curlywedge\Omega_\omega=0$ (by (4.8) and the algebraic first Bianchi identity $R_{ab}=R_{ba}$ on $\Lambda_-^2$) vanish on-shell on the torsion-free space $\mathrm{Conn}_A^{\Theta=0}(P)$ for $H=U(2)$ and $SU(2)$, whereas the EHP term $(e\curlywedge e)\curlywedge\Omega_\omega$ is non-trivial off-shell when torsion $\Theta_{e,\omega}\neq0$ is present (and in dimension $n=6$ with $k=4$, the pure YM term $\Omega_\omega\curlywedge((e\curlywedge e)\curlywedge\Omega_\omega)$ is non-trivial on $\mathrm{Conn}_A^{\Theta_N=0,\mathrm{irr}}(P)$ when the pulled-back connection on $N^4$ is torsion-free while transverse curvature and mixed torsion along $M^6\to N^4$ are non-zero; see Remark 4.1!).
- In dimension $n=k=6$ with $H=U(3)$ or $SU(3)$, the only block-grouped term not killed algebraically is the pure quadratic Yang-Mills 6-form $\Omega_\omega\curlywedge((e\curlywedge e)\curlywedge\Omega_\omega)=(0,\frac{1}{4}[\Omega_\omega,[\sigma(e\wedge e),\Omega_\omega]])$. Since it contains the factor $(e\curlywedge e)\curlywedge\Omega_\omega$, Lemma 4.4 shows that

it vanishes identically on-shell for every torsion-free $U(3)$ or $SU(3)$ connection, exactly as in dimension 4, and it is non-zero off-shell (for instance for $\mathfrak{su}(3)$-valued curvatures that are pair-symmetric but violate the first Bianchi identity). Thus, although the wedge product of two $(1,1)$-forms is no longer proportional to the volume form, there is no on-shell difference between $k=4$ and $k=6$ at the level of the block-grouped Lagrangian.

**Theorem 4.6** (Parenthesization Cutoffs and Berger Holonomy Reductions in Cartan-Yang-Mills Theories)**.** *Let $P \to M$ be a $k$-proper $H$-bundle over an oriented $n$-manifold $M$ $(4 \le k \le n)$, where $\mathfrak{h} \subset \mathfrak{so}(k_1) \oplus \cdots \oplus \mathfrak{so}(k_r)$ properly $(k_1 + \cdots + k_r = k)$ is a $k$-algebra. Consider the Cartan-Yang-Mills functional $S^{\mathrm{CYM}}_{n,\xi,\mathcal{P}}[e,\omega]$ with values in $A = \mathbb{R}^k \oplus \mathfrak{h}$ under (4.2) and (4.3).*

(1) ***Hierarchical Dimensional Cutoff under Quartic-Block Grouping*** *($\mathcal{P} = \mathrm{blk}$)**:***
- *For all $n \ge 4$, the cosmological term $\frac{1}{4}\, \curlywedge^n_{\rtimes,\mathrm{blk}}\, e$ vanishes identically in $\Omega^n(P;A)$;*
- *For all $n \ge 6$, the EHP cross-term $(\curlywedge^{n-2}_{\rtimes,\mathrm{blk}} e) \curlywedge_{\rtimes} \Omega_\omega$ vanishes identically in $\Omega^n(P;A)$, reducing $S^{\mathrm{CYM}}_{n,\xi,\mathrm{blk}}[e,\omega]$ to the pure quadratic Yang-Mills term $\int_M \mathrm{tr}_\xi(j(\Omega_\omega \curlywedge_{\rtimes} ((\curlywedge^{n-4}_{\rtimes,\mathrm{blk}} e) \curlywedge_{\rtimes} \Omega_\omega)))$;*
- *For all $n \ge 8$, the entire block-balanced Cartan-Yang-Mills functional $S^{\mathrm{CYM}}_{n,\xi,\mathrm{blk}}[e,\omega]$ vanishes identically on $M$ for every trace section $\xi \in \Gamma(E_A^*)$.*

(2) ***Comparison with Recursive Binary-Tree Halving*** *($\mathcal{P} = \mathrm{bal}$)**:** Under exact recursive binary halving (2.9), $\curlywedge^m_{\rtimes,\mathrm{bal}} e$ coincides with $\curlywedge^m_{\rtimes,\mathrm{blk}} e$ for $1 \le m \le 4$, so the quadratic Yang-Mills terms of $S^{\mathrm{CYM}}_{n,\xi,\mathrm{bal}}$ and $S^{\mathrm{CYM}}_{n,\xi,\mathrm{blk}}$ coincide for all $4 \le n \le 8$. The EHP and cosmological terms, however, involve $\curlywedge^{n-2}_{\rtimes,\mathcal{P}} e$ and $\curlywedge^n_{\rtimes,\mathcal{P}} e$, and $\curlywedge^5_{\rtimes,\mathrm{bal}} e$, $\curlywedge^6_{\rtimes,\mathrm{bal}} e$ need not vanish, whereas their block versions do. Hence $S^{\mathrm{CYM}}_{n,\xi,\mathrm{bal}}$ may differ from $S^{\mathrm{CYM}}_{n,\xi,\mathrm{blk}}$ for $n \in \{5,6,7,8\}$. For instance, for $H = U(3)$ at $n = k = 6$ one has $\curlywedge^6_{\rtimes,\mathrm{bal}} e = (0,\, c\,\mathrm{vol}_6 \otimes J)$ with $c \neq 0$ and $J \in \mathfrak{z}(\mathfrak{u}(3))$ the complex structure, so the binary-tree cosmological term survives even for a constant invariant trace; and for $H = G_2$, the binary-tree EHP terms $(\curlywedge^5_{\rtimes,\mathrm{bal}} e) \curlywedge_{\rtimes} \Omega_\omega$ at $n = 7$ and $(\curlywedge^6_{\rtimes,\mathrm{bal}} e) \curlywedge_{\rtimes} \Omega_\omega$ at $n = 8$ are non-zero for generic $\mathfrak{g}_2$-valued $\Omega_\omega$. For $n \ge 8$:*
- *By the all-orders Jacobsthal characterization in Lemma 2.1(2a), the binary-tree quadratic Yang-Mills term $\Omega_\omega \curlywedge_{\rtimes} ((\curlywedge^{n-4}_{\rtimes,\mathrm{bal}} e) \curlywedge_{\rtimes} \Omega_\omega)$ vanishes algebraically for every $n \ge 8$ such that $n-4 \notin \bigcup_{r=3}^{\infty} \{2J_r, J_{r+1}\}$ where $J_r = \frac{2^r - (-1)^r}{3}$ (so for $n \ge 8$, the only dimensions not killed algebraically by even-degree squaring are the shifted Jacobsthal pairs $n \in \{9,10\} \cup \{14,15\} \cup \{25,26\} \cup \{46,47\} \cup \dots$).*
- *Moreover, because $e \in \Omega^1(P;\mathbb{R}^k)$ has rank $\le k$, Lemma 2.1(2c) guarantees that $\curlywedge^m_{\rtimes,\mathrm{bal}} e = 0$ on $P$ for all $m > k$, so $S^{\mathrm{CYM}}_{n,\xi,\mathrm{bal}}$ vanishes identically for all $n \ge k+5$. In particular, for any $k \le 9$ (which includes all Berger holonomy reductions with $k \le 7$ below), $\curlywedge^{10}_{\rtimes,\mathrm{bal}} e = \curlywedge^{11}_{\rtimes,\mathrm{bal}} e = 0$ by coframe rank, so the binary-tree Yang-Mills term vanishes for $n = 8$ and for all $n \ge 11$, and the whole functional $S^{\mathrm{CYM}}_{n,\xi,\mathrm{bal}}$ vanishes identically for all $n \ge 11$ (and for $k = 4$, also at $n \in \{8,9,10\}$).*

(3) ***Berger Holonomy Reductions*** *($4 \le n \le 7$)**:** For any $k$-proper $H$-bundle $P \to M$ with non-empty configuration space $\mathrm{Conn}_A^{\Theta_N=0,\mathrm{irr}}(P) \neq \emptyset$, de Rham irreducibility of $(N, g_N)$ forces $k_1 = k$ and $k_{i>1} = 0$ ($\mathfrak{h} \subsetneq \mathfrak{so}(k)$), and Berger's holonomy theorem restricts $\mathrm{Hol}(\tilde{\omega}_N) \subset H \subsetneq SO(k)$ to the holonomy groups of Table 1. (Note that for $k = 2$, $\mathfrak{so}(2) \simeq \mathfrak{u}(1)$ is 1-dimensional and abelian, so it admits no non-flat proper holonomy subalgebras.) Combining $\mathrm{Conn}_A^{\Theta_N=0,\mathrm{irr}}(P) \neq \emptyset$ with the dimensional cutoff $n \le 7$ (which forces $4 \le k \le n \le 7$), $S^{\mathrm{CYM}}_{n,\xi,\mathrm{blk}}[e,\omega]$ can be non-trivial only if $4 \le n \le 7$ and $\mathrm{Hol}(\tilde{\omega}_N) \subset H$ on $(N, g_N)$ belongs to one of the following geometric regimes:*

(B1) *$k = 6$ and $(N, g_N)$ is a 6D Kähler or Calabi-Yau Berger 6-manifold ($\mathrm{Hol}(\tilde{\omega}_N) = U(3)$ or $SU(3)$) with $n \in \{6,7\}$, where at $n = k = 6$ the pure quadratic Yang-Mills term $\frac{1}{4}[\Omega_\omega, [\sigma(e \wedge e), \Omega_\omega]]$ vanishes on-shell on $\mathrm{Conn}_A^{\Theta=0,\mathrm{irr}}(P)$ by Lemma 4.4 and survives only off-shell (Remark 4.5);*

(B2) *$k = 4$ and $(N, g_N)$ is a Kähler or hyperkähler Berger 4-manifold ($\mathrm{Hol}(\tilde{\omega}_N) = U(2)$ or $SU(2) \simeq Sp(1)$, e.g. a $K3$ surface when $N$ is compact): for $5 \le n \le 7$ ($k = 4 < n$),*

*the pure Yang-Mills term survives on* $\mathrm{Conn}_A^{\Theta_N=0,\mathrm{irr}}(P)$ *via transverse curvature and mixed torsion along* $M^n \to N^4$ *as shown in Remark 4.1; whereas at* $n = k = 4$, *Lemma 4.3 shows that both* $\Omega_\omega \curlywedge \Omega_\omega = 0$ *and* $(e \curlywedge e) \curlywedge \Omega_\omega = 0$ *vanish on-shell for torsion-free connections by the first Bianchi identity, and* $(e \curlywedge e) \curlywedge \Omega_\omega \neq 0$ *survives only off-shell in the presence of torsion* $\Theta_{e,\omega} \neq 0$;

(B3) $k = 7$ *(which since* $k \leq n \leq 7$ *occurs strictly at the critical dimension* $n = k = 7$, *where only the pure quadratic Yang-Mills term survives) and* $(N, g_N) = (M, g_M)$ *is a Berger 7-manifold with exceptional* $G_2$*-holonomy (*$\mathrm{Hol}(\tilde{\omega}_N) = G_2$*). As shown in Proposition 4.7 below, at* $n = k = 7$ *the quadratic Yang-Mills 7-form* $\mathcal{L}_7^{\mathrm{CYM}}$ *takes values solely in* $\mathbb{R}^7 \oplus \{0\}$, *vanishes on-shell on the torsion-free* $G_2$*-curvature space* $\mathcal{W}_{77} \subset S^2(\Lambda^2_{14})$, *and is non-vanishing off-shell on* $\Omega^2(P; \mathfrak{g}_2)$ *whenever torsion* $\Theta_{e,\omega} \neq 0$ *excites the* $\Lambda^2_7 \otimes \mathfrak{g}_2$ *or non-Bianchi curvature sectors.*

(4) ***Coadjoint Trace Obstruction and Symmetry Breaking:*** *Under (4.2), for* $k \in \{4, 6\}$ *(*$\times_H = 0$*),* $e \curlywedge e = (0, \sigma(e \wedge e)) \in \Omega^2(P; \mathfrak{h})$, *so the surviving block-grouped Lagrangian* $\mathcal{L}_{n,\mathrm{blk}}^{\mathrm{CYM}}$ *for* $4 \leq n \leq 6$ *takes values strictly inside the derived subalgebra* $(0, [\mathfrak{h}, \mathfrak{h}])$. *At* $n = k = 7$ *(*$H = G_2$*),* $\mathcal{L}_7^{\mathrm{CYM}} = \Omega_\omega \curlywedge ((\curlywedge^3_{\curlywedge,\mathrm{bal}} e) \curlywedge \Omega_\omega)$ *takes values in* $\mathbb{R}^7 \oplus \{0\}$, *where* $(\mathbb{R}^7)^{G_2} = \{0\}$. *Consequently, any constant* $H$*-invariant linear functional* $\tau_0 \in (A^*)^H$ *annihilates* $\mathcal{L}_n^{\mathrm{CYM}}$ *identically, so non-triviality of* $S^{\mathrm{CYM}}_{n,\xi}[e, \omega]$ *requires a non-zero coadjoint background section* $\xi \in \Gamma(E_A^*)$ *with* $\xi|_{E_{[\mathfrak{h},\mathfrak{h}]} \oplus E_{\mathbb{R}^k}} \not\equiv 0$, *reducing the internal* $H$*-symmetry along* $\xi$ *to* $H_\xi \subsetneq H$.

*Proof.* Part (1) follows immediately from Lemma 2.1(1) and (2c): since $(A, \curlywedge)$ is skew-commutative, $\curlywedge^m_{\curlywedge,\mathrm{blk}} e = 0$ for all $m \geq 4$ (and for all $m > k$). Thus the cosmological term ($m = n$) vanishes for $n \geq 4$, the EHP term ($m = n - 2$) vanishes for $n - 2 \geq 4 \iff n \geq 6$ (and $n - 2 > k$), and the quadratic Yang-Mills term ($m = n - 4$) vanishes for $n - 4 \geq 4 \iff n \geq 8$ (and $n - 4 > k$). Part (2) follows from Lemma 2.1(2a)–(2c): $\curlywedge^m_{\curlywedge,\mathrm{bal}} e = \curlywedge^m_{\curlywedge,\mathrm{blk}} e$ for $m \leq 4$ (covering the Yang-Mills terms for $n \leq 8$; the $U(3)$ and $G_2$ examples are direct computations in an orthonormal frame with (4.2) and (4.9), the $U(3)$ value being forced into $\mathrm{vol}_6 \otimes \mathfrak{z}(\mathfrak{u}(3))$ by $U(3)$-invariance), $\curlywedge^m_{\curlywedge,\mathrm{bal}} e = 0$ algebraically for all $m \geq 4$ with $m \notin \bigcup_{r=3}^{\infty}\{2J_r, J_{r+1}\}$, and $\curlywedge^m_{\curlywedge,\mathrm{bal}} e = 0$ by coframe rank for all $m > k$ (so for $k \leq 9$, $m = 10, 11$ also vanish, eliminating all $m \geq 7$, i.e., $n \geq 11$).

For part (3), on $\mathrm{Conn}_A^{\Theta_N=0,\mathrm{irr}}(P) \neq \emptyset$, the pullback connection $\tilde{\omega}_N := \frac{1}{2}(i \circ \kappa)^* \omega$ on $P_N \subset FN$ is the Levi-Civita connection of the locally irreducible, locally non-symmetric Riemannian metric $g_N$ on the simply connected $k$-manifold $N$ ($4 \leq k \leq n \leq 7$), with holonomy group $\mathrm{Hol}(\tilde{\omega}_N) \subset H \subset SO(k_1) \times \cdots \times SO(k_r) \subset SO(k)$ [22, 24]. By de Rham's decomposition theorem [11, 44], irreducibility of $(N, g_N)$ forces $k_1 = k$ and $k_{j>1} = 0$, so $\mathrm{Hol}(\tilde{\omega}_N) \subset H \subsetneq SO(k)$ is an irreducible proper holonomy subgroup. Berger's holonomy classification [7, 8, 30] combined with $4 \leq k \leq n \leq 7$, Lemma 4.3, and Proposition 4.7 restricts $\mathrm{Hol}(\tilde{\omega}_N)$ to Table 1, yielding (B1)–(B3).

Finally, for part (4), when $\times_H = 0$, $e \curlywedge e = (0, \sigma(e \wedge e)) \in \Omega^2(P; \mathfrak{h})$ and $\Omega_\omega \in \Omega^2(P; \mathfrak{h})$, so both $(e \curlywedge e) \curlywedge \Omega_\omega$ ($n = 4$) and $\Omega_\omega \curlywedge ((e \curlywedge e) \curlywedge \Omega_\omega)$ ($n = 6$) take values in $(0, [\mathfrak{h}, \mathfrak{h}])$ (while at $n = 5$ the surviving terms $\Omega_\omega \curlywedge (e \curlywedge \Omega_\omega)$ and $(\curlywedge^3_{\curlywedge,\mathrm{bal}} e) \curlywedge \Omega_\omega$ take values in $\mathbb{R}^k \oplus \{0\}$, where $(\mathbb{R}^k)^H = \{0\}$ by irreducibility of $H \subset SO(k)$). At $n = k = 7$ ($H = G_2$), Proposition 4.7 shows that $\curlywedge^3_{\curlywedge,\mathrm{bal}} e = (v^{(3)}, 0) \in \Omega^3(P; \mathbb{R}^7 \oplus \{0\})$, and multiplying twice with $\Omega_\omega = (0, \Omega_\omega)$ via (4.2) yields $\mathcal{L}_7^{\mathrm{CYM}} = (-\frac{1}{4}\Omega_\omega \cdot (\Omega_\omega \cdot v^{(3)}), 0) \in \Omega^7(P; \mathbb{R}^7 \oplus \{0\})$. Since any $\tau_0 \in (A^*)^H$ vanishes on $[\mathfrak{h}, \mathfrak{h}]$ and $((\mathbb{R}^k)^*)^H = \{0\}$, $\mathrm{tr}_{\tau_0}$ annihilates these terms identically. □

We now establish the exact representation-theoretic structure and on-shell/off-shell dichotomy of $\mathcal{L}_7^{\mathrm{CYM}}$ at the critical dimension $n = k = 7$.

**Proposition 4.7** (Explicit $G_2$-Contraction Formulas and On-Shell/Off-Shell Dichotomy of $\mathcal{L}_7^{\mathrm{CYM}}$ at $n = k = 7$)**.** *Let* $n = k = 7$ *and* $H = G_2 \subset SO(7)$ *acting on* $A = \mathbb{R}^7 \oplus \mathfrak{g}_2$ *under (4.2), where* $(u \times_{G_2} v)_a = \varphi_{abc} u^b v^c$ *is induced by the associative 3-form* $\varphi_0 = \frac{1}{6}\varphi_{abc} e^a \wedge e^b \wedge e^c$, *with dual coassociative 4-form* $*\varphi_0 = \frac{1}{24}(*\varphi)_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d$ *(satisfying* $\varphi^{abc}(*\varphi)_{bcde} = 4\varphi^a{}_{de}$ *under the Fano-plane orientation) and orthogonal projection* $\sigma : \Lambda^2\mathbb{R}^7 \twoheadrightarrow \Lambda^2_{14} = \mathfrak{g}_2$ *given on vectors*

TABLE 1. Comparison of block-balanced dimensional cutoffs and Berger holonomy regimes between $A$-EHP gravity [24] and Cartan-Yang-Mills ($A$-CYM) theory for proper $k$-algebras $\mathfrak{h} \subsetneq \mathfrak{so}(k)$ when $k = n$.

| **Holonomy** $\mathrm{Hol}(\bar{\omega}_N)$ | **Dim** $k = n$ | **Geometry** | $A$**-EHP Status** | **Block-Balanced** $A$**-CYM** ($k = n$) |
|---|---|---|---|---|
| $U(2)$ | 4 | Kähler 4-manifold | Off-shell ($\Theta \neq 0$) | Off-shell EHP ($\Theta \neq 0$; 0 on-shell) |
| $SU(2) \simeq Sp(1)$ | 4 | Hyperkähler ($K3$) | Off-shell ($\Theta \neq 0$) | Off-shell EHP ($\Theta \neq 0$; 0 on-shell) |
| $U(3)$, $SU(3)$ | 6 | 6D Kähler / Calabi-Yau | Trivial ($n \geq 6$) | Off-shell Pure YM ($\Theta \neq 0$; 0 on-shell) |
| $G_2$ | 7 | Exceptional $G_2$ | Trivial ($n \geq 6$) | Off-shell Pure YM ($\Theta \neq 0$; 0 on-shell) |
| $U(m)$, $SU(m)$ | $2m \geq 8$ | Kähler / Calabi-Yau | Trivial ($n \geq 6$) | Trivial ($n \geq 8$) |
| $Sp(m) \cdot Sp(1)$, $Sp(m)$ | $4m \geq 8$ | Quat.-Kähler / Hyperkähler | Trivial ($n \geq 6$) | Trivial ($n \geq 8$) |
| $Spin(7)$ | 8 | Exceptional $Spin(7)$ | Trivial ($n \geq 6$) | Trivial ($n \geq 8$) |

$u, v \in \mathbb{R}^7$ *by*

$$\sigma(u \wedge v)_{ab} = \frac{2}{3}(u_a v_b - u_b v_a) - \frac{1}{3}(*\varphi)_{abcd} u^c v^d = \frac{2}{3}(u_a v_b - u_b v_a) - \frac{1}{6}(*\varphi)_{abcd}(u^c v^d - u^d v^c) \quad (4.9)$$

*(so that* $\varphi^{abc}\sigma(u \wedge v)_{bc} = 0$*). For an orthonormal coframe* $e = (e, 0) \in \Omega^1(P; \mathbb{R}^7 \oplus \mathfrak{g}_2)$ *with volume form* $\mathrm{vol}_7 = e^1 \wedge \cdots \wedge e^7$*:*

(1) ***Quadratic Soldering Form:*** *Evaluating the exterior product* $e \curlywedge e = \sum_{c,d=1}^{7}(e^c \wedge e^d) \otimes ((\partial_c, 0) \curlywedge (\partial_d, 0))$ *via* (2.5), (4.2), *and* (4.9) *gives* $e \curlywedge e = (e \wedge_{\times_{G_2}} e,\, \sigma(e \wedge e)) \in \Omega^2(P; \mathbb{R}^7 \oplus \mathfrak{g}_2)$*, with* $(e \wedge_{\times_{G_2}} e)_b = \varphi_{bcd} e^c \wedge e^d$ *and*

$$\sigma(e \wedge e)_{ab} = \sum_{c,d=1}^{7} \sigma(\partial_c \wedge \partial_d)_{ab}\, e^c \wedge e^d = \frac{4}{3} e_a \wedge e_b - \frac{1}{3}(*\varphi)_{abcd} e^c \wedge e^d \qquad (\varphi^{abc}\sigma(e \wedge e)_{bc} = 0). \quad (4.10)$$

(2) ***Cubic Soldering Form and Schur Vanishing of the*** $\mathfrak{g}_2$***-Sector:*** *Multiplying* $e \curlywedge e$ *(degree 2) on the right by* $e = (e, 0)$ *(degree 1) via* (2.5) *and* (4.2) *gives* $\curlywedge^3_{\curlywedge,\mathrm{bal}} e = (e \curlywedge e) \curlywedge e = (v^{(3)}, B_e^{(3)}) \in \Omega^3(P; \mathbb{R}^7 \oplus \mathfrak{g}_2)$*, where:*

- *In the* $\mathbb{R}^7$*-component, since* $(v_1, X_1) \curlywedge (e, 0)$ *has* $\mathbb{R}^7$*-part* $\frac{1}{2}\sigma(e \wedge e)_{ab} \wedge e^b + ((e \wedge_{\times_{G_2}} e) \wedge_{\times_{G_2}} e)_a$*, the* $G_2$ *contraction identities* $\frac{1}{2}\sigma(e \wedge e)_{ab} \wedge e^b = -\frac{1}{6}(*\varphi)_{abcd} e^b \wedge e^c \wedge e^d = -i_{\partial_a}(*\varphi_0)$ *(from* (4.10) *since* $e_b \wedge e^b = 0$*) and* $((e \wedge_{\times_{G_2}} e) \wedge_{\times_{G_2}} e)_a = \varphi_{abe}\varphi_{ecd} e^c \wedge e^d \wedge e^b = -(*\varphi)_{abcd} e^b \wedge e^c \wedge e^d = -6\, i_{\partial_a}(*\varphi_0)$ *sum with the* same *negative sign to give*

$$(v^{(3)})_a = \frac{1}{2}\sigma(e \wedge e)_{ab} \wedge e^b + \big((e \wedge_{\times_{G_2}} e) \wedge_{\times_{G_2}} e\big)_a = -\frac{7}{6}(*\varphi)_{abcd}\, e^b \wedge e^c \wedge e^d = -7\, i_{\partial_a}(*\varphi_0). \quad (4.11)$$

  *In particular,* $v^{(3)} \neq 0$ *(and even if one sets* $\times_{G_2} = 0$*, the first term alone gives* $\frac{1}{2}\sigma(e \wedge e)_{ab} \wedge e^b = -i_{\partial_a}(*\varphi_0) \neq 0$*).*

- *In the* $\mathfrak{g}_2$*-component,* $B_e^{(3)} = \sigma((e \wedge_{\times_{G_2}} e) \wedge e) \in \Omega^3(P; \mathfrak{g}_2)$ *defines a* $G_2$*-equivariant linear map* $\Lambda^3\mathbb{R}^7 \to \Lambda^2_{14} = \mathfrak{g}_2$*. Because* $\Lambda^3\mathbb{R}^7 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$ *contains no* $\Lambda_{14}$ *irreducible summand, Schur's lemma (and direct expansion of* (4.9)*) yields*

$$B_e^{(3)} = \sigma((e \wedge_{\times_{G_2}} e) \wedge e) = 0 \quad \textit{identically in } \Omega^3(P; \mathfrak{g}_2). \quad (4.12)$$

(3) ***On-Shell/Off-Shell Dichotomy of*** $\mathcal{L}_7^{\mathrm{CYM}}$***:*** *For any curvature 2-form* $\Omega_\omega = (0, \Omega_\omega) \in \Omega^2(P; \mathfrak{g}_2)$*, the Cartan-Yang-Mills 7-form is*

$$\mathcal{L}_7^{\mathrm{CYM}} = \Omega_\omega \curlywedge ((\curlywedge^3_{\curlywedge,\mathrm{bal}} e) \curlywedge \Omega_\omega) = \Big(-\frac{1}{4}\Omega_\omega \cdot (\Omega_\omega \cdot v^{(3)}),\ 0\Big) \in \Omega^7(P; \mathbb{R}^7 \oplus \{0\}). \quad (4.13)$$

*Decomposing the 2-form factors of* $\Omega_\omega \in \Omega^2(P; \mathfrak{g}_2)$ *along* $\Lambda^2\mathbb{R}^7 = \Lambda^2_7 \oplus \Lambda^2_{14}$*:*

(a) Ricci-Type Contraction on $\Lambda^2_{14} \otimes \mathfrak{g}_2$ and On-Shell Vanishing on $\mathcal{W}_{77}$: *For any* $\Omega_\omega = \frac{1}{2}R_{bc,df} e^d \wedge e^f \otimes E_{bc} \in \Gamma(P, \Lambda^2_{14} T^*M \otimes \mathfrak{g}_2)$*, the identity* $(\Omega_\omega)_{bc} \wedge *\varphi_0 = 0$

*implies*

$$(\Omega_\omega \cdot v^{(3)})_b = -7\sum_{c=1}^{7}(\Omega_\omega)_{bc} \wedge i_{\partial_c}(*\varphi_0) = 7\sum_{c,d=1}^{7} R_{bc,cd}\, e^d \wedge *\varphi_0 \in \Omega^5(P;\mathbb{R}^7). \tag{4.14}$$

*In particular, for every torsion-free $G_2$-connection $(e,\omega) \in \mathrm{Conn}_A^{\Theta=0}(P)$ (whose curvature lies in the torsion-free $G_2$ Riemann curvature space $\mathcal{W}_{77} = S^2(\Lambda^2_{14}) \cap \ker(\wedge) \subset \Lambda^2_{14} \otimes \mathfrak{g}_2$, where the algebraic first Bianchi identity and $G_2$-holonomy force $\mathrm{Ric}_{bd} = \sum_{c=1}^7 R_{bc,cd} = 0$), we have $\Omega_\omega \cdot v^{(3)} = 0$ in $\Omega^5(P;\mathbb{R}^7)$ and hence $\mathcal{L}_7^{\mathrm{CYM}} = 0$ on-shell;*

(b) Off-Shell Non-Vanishing on $\Omega^2(P;\mathfrak{g}_2)$: *When torsion $\Theta_{e,\omega} \neq 0$ (or a general $\mathfrak{g}_2$-connection $\omega$) excites the $\Lambda^2_7 \otimes \mathfrak{g}_2$ curvature component, $\mathcal{L}_7^{\mathrm{CYM}}$ can be non-zero in $\Omega^7(P;\mathbb{R}^7 \oplus \{0\})$; an explicit example is given in the proof. (Violating the first Bianchi identity inside $\Lambda^2_{14} \otimes \mathfrak{g}_2$ is not sufficient by itself: by (4.14), $\mathcal{L}_7^{\mathrm{CYM}}$ still vanishes there whenever $\sum_c R_{bc,cd} = 0$.)*

*Proof.* Part (1) and the $\mathbb{R}^7$-contraction (4.11) in part (2) follow directly from (4.2), (4.9), (4.10), and the identity $\varphi_{eab}\varphi_{ecd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} - (*\varphi)_{abcd}$: contracting with $e^c \wedge e^d \wedge e^b$ annihilates $\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}$ (since $e^b \wedge e^b = 0$) and leaves $\varphi_{abe}\varphi_{ecd}e^c \wedge e^d \wedge e^b = -\varphi_{eab}\varphi_{ecd}e^c \wedge e^d \wedge e^b = +(*\varphi)_{abcd}e^c \wedge e^d \wedge e^b = -(*\varphi)_{abcd}e^b \wedge e^c \wedge e^d = -6\, i_{\partial_a}(*\varphi_0)$, while $\frac{1}{2}\sigma(e \wedge e)_{ab} \wedge e^b = -\frac{1}{6}(*\varphi)_{abcd}e^b \wedge e^c \wedge e^d = -i_{\partial_a}(*\varphi_0)$, summing to $-7\, i_{\partial_a}(*\varphi_0)$ in (4.11). For (4.12), the 3-form $B_e^{(3)} = \sigma((e \wedge_{\times_{G_2}} e) \wedge e)$ is $G_2$-invariant in $(\Lambda^3\mathbb{R}^7)^* \otimes \Lambda^2_{14} \simeq \mathrm{Hom}_{G_2}(\Lambda^3\mathbb{R}^7, \Lambda^2_{14})$, which is zero by Schur's lemma since $\Lambda^3\mathbb{R}^7 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$ has no $\Lambda_{14}$ summand (explicitly, in (4.9), contracting $\frac{1}{3}(*\varphi)_{abef}\varphi_{ecd} = \frac{1}{3}(\delta_{ac}\varphi_{bdf} - \delta_{ad}\varphi_{bcf} - \delta_{bc}\varphi_{adf} + \delta_{bd}\varphi_{acf})$ with $e^c \wedge e^d \wedge e^f$ gives $\frac{2}{3}(\varphi_{acd}e^c \wedge e^d \wedge e_b - \varphi_{bcd}e^c \wedge e^d \wedge e_a)$, which cancels the first term $\frac{2}{3}(\varphi_{acd}e^c \wedge e^d \wedge e_b - \varphi_{bcd}e^c \wedge e^d \wedge e_a)$ of $\sigma((e \wedge_{\times_{G_2}} e) \wedge e)_{ab}$ identically).

For part (3), since $B_e^{(3)} = 0$, multiplying $\curlywedge^3_{\rtimes,\mathrm{bal}} e = (v^{(3)}, 0)$ twice on the left by $(0, \Omega_\omega)$ via (4.2) gives (4.13). To prove (3a), suppose $(\Omega_\omega)_{bc} = \frac{1}{2}\sum_{d,f=1}^7 R_{bc,df}\, e^d \wedge e^f \in \Lambda^2_{14}$ for each pair $(b,c)$. Recall that a 2-form $\alpha \in \Lambda^2\mathbb{R}^7$ lies in $\Lambda^2_{14} = \ker(\iota_{(\cdot)}\varphi_0)$ if and only if $\alpha \wedge *\varphi_0 = 0$ in $\Lambda^6\mathbb{R}^7$. Applying the interior derivation $i_{\partial_c}$ to $(\Omega_\omega)_{bc} \wedge *\varphi_0 = 0$ gives

$$0 = i_{\partial_c}((\Omega_\omega)_{bc} \wedge *\varphi_0) = \big(i_{\partial_c}(\Omega_\omega)_{bc}\big) \wedge *\varphi_0 + (\Omega_\omega)_{bc} \wedge i_{\partial_c}(*\varphi_0),$$

so $(\Omega_\omega)_{bc} \wedge i_{\partial_c}(*\varphi_0) = -\big(i_{\partial_c}(\Omega_\omega)_{bc}\big) \wedge *\varphi_0$. Since $i_{\partial_c}(\Omega_\omega)_{bc} = \sum_{d=1}^7 R_{bc,cd}\, e^d$, summing over $c \in \{1,\dots,7\}$ yields (4.14) for every $b \in \{1,\dots,7\}$:

$$(\Omega_\omega \cdot v^{(3)})_b = -7\sum_{c=1}^{7}(\Omega_\omega)_{bc} \wedge i_{\partial_c}(*\varphi_0) = 7\sum_{c,d=1}^{7} R_{bc,cd}\, e^d \wedge *\varphi_0.$$

For every torsion-free $G_2$-connection $(e,\omega) \in \mathrm{Conn}_A^{\Theta=0}(P)$, the algebraic first Bianchi identity $\Omega_\omega \wedge e = 0$ and the $G_2$-invariance $L_E(*\varphi_0) = \sum_{b,c=1}^7 E_{bc}\, e^b \wedge i_{\partial_c}(*\varphi_0) = 0$ $(E \in \mathfrak{g}_2)$ imply that $\Omega_\omega \in \mathcal{W}_{77} = S^2(\Lambda^2_{14}) \cap \ker(\wedge)$ is Ricci-flat ($\mathrm{Ric}_{bd} = \sum_{c=1}^7 R_{bc,cd} = 0$). Thus $\Omega_\omega \cdot v^{(3)} = 0$ and $\mathcal{L}_7^{\mathrm{CYM}} = 0$ on-shell.

Finally, to prove (3b), under the standard Fano-plane convention $\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$, we have $*\varphi_0 = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}$ and $(v^{(3)})_5 = -7\, i_{\partial_5}(*\varphi_0) = 7(e^{467} - e^{234} - e^{137} + e^{126})$. Take the $\mathfrak{g}_2$ basis matrices $E_1 = e_{67} - e_{45} \in \mathfrak{g}_2$ and $E_2 = e_{46} + e_{57} \in \mathfrak{g}_2$, and set $\Omega_\omega = \alpha_1 \otimes E_1 + \alpha_2 \otimes E_2 \in \Omega^2(P;\mathfrak{g}_2)$ with $\alpha_1 = e^{12} - e^{45} \in \Lambda^2_{14}$ and $\alpha_2 = e^{35} + e^{46} \in \Lambda^2\mathbb{R}^7$ (which has a non-zero $\Lambda^2_7$ projection since $\iota_{\alpha_2}\varphi_0 = e^2 - e^6 \neq 0$). Write $(\Omega_\omega)_{bc}$ for the 2-form coefficient of the $(b,c)$ matrix entry, so that $(\Omega_\omega \cdot (\Omega_\omega \cdot v^{(3)}))_a = \sum_{b,c}(\Omega_\omega)_{ab} \wedge (\Omega_\omega)_{bc} \wedge (v^{(3)})_c$. A direct expansion shows that all components vanish except $a = 4$, where the only non-zero contribution comes from $(b,c) = (6,4)$: since $(\Omega_\omega)_{46} = \alpha_2$ and $(\Omega_\omega)_{64} = -\alpha_2$, it equals $-\alpha_2 \wedge \alpha_2 \wedge (v^{(3)})_4 = 2\, e^{3456} \wedge (v^{(3)})_4 = 14\,\mathrm{vol}_7$. (In the component $a = 6$ the two contributions through $c = 5$, via $b = 4$ and $b = 7$, cancel.) Hence $\mathcal{L}_7^{\mathrm{CYM}} = (-\frac{7}{2}\,\mathrm{vol}_7) \otimes \partial_4 \neq 0$. □

## 5. Internal Chern-Simons Dichotomy and Applications to $BF$, Higgs, and Dual Theories

5.1. **Algebra-Valued Chern-Simons ($A$-CS) Theories: Associator Quotient Collapse and Jacobiator Dichotomy.** Let $M$ be a compact oriented smooth manifold of odd dimension $n = 2r-1$ ($r \geq 2$), and let $\mathrm{Conn}^{\mathrm{lin}}(P;A) = \Omega^1_{\mathrm{heq},\rho}(P;A)$ be the linear space of horizontal equivariant connections (Proposition 2.4(3)) so that for any $D \in \mathrm{Conn}^{\mathrm{lin}}(P;A)$ and $t \in [0,1]$, the scaled connection $D_t := tD \in \mathrm{Conn}^{\mathrm{lin}}(P;A)$ has curvature $F_t := F_{D_t} = t\,dD + \frac{t^2}{2} D \wedge_* D \in \Omega^2_{\mathrm{heq},\rho}(P;A)$. (In the affine Ehresmann regime $D = D_0 + A$ of Proposition 2.4(3) with $D_0 = \eta \circ \omega_0$, the same formulas apply to the horizontal connection difference $A = D - D_0 \in \Omega^1_{\mathrm{heq},\rho}(P;A)$ via the linear homotopy $D_t = D_0 + tA$ relative to a flat background $F_{D_0} = 0$.) We compare two ways of forming internal $A$-valued Chern-Simons functionals:

(1) *The Associative Quotient $\Omega_{\mathrm{heq},\rho}(P;A)/\mathfrak{Assoc}(P,A)$:* Let $\mathfrak{Assoc}(A) \subset A$ be the two-sided ideal generated by all algebra associators $\mathrm{Assoc}_*(x,y,z) = (x*y)*z - x*(y*z)$, which induces the two-sided graded ideal $\mathfrak{Assoc}(P,A) = \Omega_{\mathrm{heq},\rho}(P;\mathfrak{Assoc}(A)) \subset \Omega_{\mathrm{heq},\rho}(P;A)$ generated by all form associators $\mathrm{Assoc}_*(\alpha,\beta,\gamma) := (\alpha \wedge_* \beta) \wedge_* \gamma - \alpha \wedge_* (\beta \wedge_* \gamma)$, with quotient projection $\pi_{\mathrm{as}} : \Omega_{\mathrm{heq},\rho}(P;A) \to \Omega_{\mathrm{heq},\rho}(P;A)/\mathfrak{Assoc}(P,A) \simeq \Omega_{\mathrm{heq},\rho}(P;A/\mathfrak{Assoc}(A))$. Assuming the trace section $\xi \in \Gamma(E_A^*)$ annihilates the subbundle $E_{\mathfrak{Assoc}(A)}$ (so $\xi \in \Gamma(E_{(A/\mathfrak{Assoc}(A))^*}) \subset \Gamma(E_A^*)$), $\mathrm{tr}_\xi$ descends to a well-defined quotient bundle trace $\overline{\mathrm{tr}}_\xi : E_{A/\mathfrak{Assoc}(A)} \to M \times \mathbb{R}$ satisfying $\overline{\mathrm{tr}}_\xi(j(\pi_{\mathrm{as}}(\Psi))) = \mathrm{tr}_\xi(j(\Psi))$.

(2) *Unquotiented Grouped Parenthesization in $\Omega_{\mathrm{heq},\rho}(P;A)$:* Working directly in $\Omega_{\mathrm{heq},\rho}(P;A)$ without quotienting by $\mathfrak{Assoc}(P,A)$, we define the *grouped Chern-Simons $(2r-1)$-form* $\mathrm{CS}^{\mathrm{grp},\mathcal{P}}_{2r-1}(D) \in \Omega^{2r-1}_{\mathrm{heq},\rho}(P;A)$ for $\mathcal{P} \in \{\mathrm{blk},\mathrm{bal}\}$ by grouping the $(r-1-j)$ factors of $dD$ and the $(2j+1)$ factors of $D$:

$$\mathrm{CS}^{\mathrm{grp},\mathcal{P}}_{2r-1}(D) := \sum_{j=0}^{r-1} \frac{r}{(r+j)\,2^j} \binom{r-1}{j} \big( \wedge^{r-1-j}_{*,\mathcal{P}}\, dD \big) \wedge_* \big( \wedge^{2j+1}_{*,\mathcal{P}}\, D \big) \in \Omega^{2r-1}_{\mathrm{heq},\rho}(P;A). \tag{5.1}$$

In particular, for $n = 3$ ($r = 2$, where $\wedge^m_{*,\mathrm{blk}}$ and $\wedge^m_{*,\mathrm{bal}}$ coincide on all factors) and $n = 5$ ($r = 3$, where the $j = 2$ term $\frac{3}{20} \wedge^5_{*,\mathcal{P}} D$ equals $\frac{3}{20}(\wedge^4_{*,\mathrm{bal}} D) \wedge_* D$ under $\mathcal{P} = \mathrm{blk}$ and $\frac{3}{20}((D \wedge_* D) \wedge_* D) \wedge_* (D \wedge_* D)$ under $\mathcal{P} = \mathrm{bal}$):

$$\mathrm{CS}^{\mathrm{grp}}_3(D) = dD \wedge_* D + \frac{1}{3}(D \wedge_* D) \wedge_* D,$$
$$\mathrm{CS}^{\mathrm{grp},\mathcal{P}}_5(D) = (dD \wedge_* dD) \wedge_* D + \frac{3}{4} dD \wedge_* ((D \wedge_* D) \wedge_* D) + \frac{3}{20} \wedge^5_{*,\mathcal{P}} D.$$

**Definition 5.1** (Internal Algebra-Valued Chern-Simons ($A$-CS) Functionals)**.** Let $\dim M = 2r-1$ ($r \geq 2$). The *quotiented $A$-CS functional* (for $\xi \in \Gamma(E_{(A/\mathfrak{Assoc}(A))^*})$) and the *unquotiented grouped $A$-CS functional* on $\mathrm{Conn}^{\mathrm{lin}}(P;A)$ are defined respectively by

$$S^{A,\mathrm{quot}}_{\mathrm{CS},2r-1}[D] := r \int_M \int_0^1 \overline{\mathrm{tr}}_\xi \Big( j\big(\pi_{\mathrm{as}}(D \wedge_* (\wedge^{r-1}_* F_t))\big)\Big)\, dt, \quad S^{A,\mathrm{grp},\mathcal{P}}_{\mathrm{CS},2r-1}[D] := \int_M \mathrm{tr}_\xi \Big( j\big(\mathrm{CS}^{\mathrm{grp},\mathcal{P}}_{2r-1}(D)\big)\Big). \tag{5.2}$$

**Remark 5.2** (Internal $A$-CS vs. Classical Bilinear Chern-Simons Theory)**.** Just as noted for Chern-Weil theory in Remark 2.6, classical 3D Chern-Simons theory on a Lie algebra $\mathfrak{g}$ uses an *external symmetric* Ad-invariant bilinear form $\kappa \in S^2(\mathfrak{g}^*)^{\mathrm{Ad}}$ to form $\int_M (\kappa(D \wedge dD) + \frac{1}{3}\kappa(D \wedge [D \wedge D]_{\mathfrak{g}}))$ (or uses the associative matrix product in $\mathrm{Mat}_{k\times k}(\mathbb{R})$). Because $\kappa$ is symmetric while $[\cdot,\cdot]_{\mathfrak{g}}$ is skew-symmetric, $\kappa(x,[y,z]_{\mathfrak{g}})$ is totally skew-symmetric in $(x,y,z)$, so the classical cubic term is non-zero for every non-abelian semisimple Lie algebra. In contrast, Definition 5.1 multiplies all factors *internally* within $(A,*)$ before applying a linear functional $\mathrm{tr}_\xi : E_A \to M \times \mathbb{R}$. When $(A,\curlywedge)$ is skew-commutative, the outer multiplication in $(D \curlywedge D) \curlywedge D$ is also skew-symmetric, which forces $(D \curlywedge D) \curlywedge D$ to project onto the Jacobiator $\mathrm{Jac}_\curlywedge(D,D,D)$! Moreover, for a skew-commutative product $\curlywedge$, because $dD$ has degree 2 and $D$ has degree 1, the quadratic term $dD \curlywedge D$ takes values in the commutator subspace $[A,A]_\curlywedge$: thus $dD \curlywedge D$ vanishes identically if $(A,\curlywedge)$ is

abelian, is annihilated by any $G$-invariant trace $\tau_0 \in (A^*)^G$, and is globally exact on $P$ since $d(D \curlywedge D) = dD \curlywedge D - D \curlywedge dD = dD \curlywedge D - (-1)^{1\cdot 2} dD \curlywedge D = 2\, dD \curlywedge D$ (so $dD \curlywedge D = \frac{1}{2} d(D \curlywedge D)$ and $d(dD \curlywedge D) = dD \curlywedge dD = 0$).

**Theorem 5.3** (Associator Quotient Collapse, Jacobiator Dichotomy, and Transgression Anomaly in Internal-Trace $A$-CS Theories)**.** *Let $M$ be a compact oriented manifold of odd dimension $n = 2r-1$ ($r \geq 2$), and let $E_A = P \times_\rho A \to M$ be equipped with a trace functional $\mathrm{tr}_\xi$.*

(1) ***Cubic Annihilation of Skew-Commutative Algebras in the Associative Quotient:*** *Suppose $(A, *)$ is any skew-commutative algebra ($x * y = -y * x$). Then in the associative quotient algebra $A/\mathfrak{Assoc}(A)$, every 3-fold product vanishes identically ($x*y*z \in \mathfrak{Assoc}(A)$ for all $x, y, z \in A$). Consequently, in $\Omega_{\mathrm{heq},\rho}(P; A)/\mathfrak{Assoc}(P, A)$:*
- *For $n = 3$ ($r = 2$), the cubic term $\frac{1}{3}\pi_{\mathrm{as}}(D \wedge_* D \wedge_* D) = 0$ vanishes identically, leaving only $\int_M \overline{\mathrm{tr}}_\xi(j(\pi_{\mathrm{as}}(D \wedge_* dD))) = -\int_M \overline{\mathrm{tr}}_\xi(j(\pi_{\mathrm{as}}(dD \wedge_* D)))$ (where $D \wedge_* dD = -(-1)^{1\cdot 2} dD \wedge_* D = -dD \wedge_* D$ by (2.6));*
- *For all odd dimensions $n = 2r-1 \geq 5$ ($r \geq 3$), $S^{A,\mathrm{quot}}_{\mathrm{CS},2r-1}[D] \equiv 0$ vanishes identically.*

(2) ***Unquotiented Vertex Truncation under $\mathcal{P}$-$(1,s)$-Solvability:*** *If $A = \bigoplus_{m\in\mathfrak{m}} A^m$ is a weak $\mathcal{P}$-$(1,s)$-solvable $\mathfrak{m}$-graded algebra ($\mathcal{P} \in \{\mathrm{blk}, \mathrm{bal}\}$), then for every $j$ satisfying $2j+1 \geq s+1$ (i.e., $j \geq \lceil s/2 \rceil$), the $(r+j)$-point vertex $(\curlywedge^{r-1-j}_{*,\mathcal{P}} dD) \wedge_* (\curlywedge^{2j+1}_{*,\mathcal{P}} D)$ in (5.1) vanishes identically in $\Omega^{2r-1}(P; A)$.*

(3) ***Unquotiented Skew-Commutative Dichotomy (Lie vs. Non-Lie Algebras):*** *Suppose $(A, \curlywedge)$ is any skew-commutative algebra ($x \curlywedge y = -y \curlywedge x$), with Jacobiator $\mathrm{Jac}_\curlywedge(x, y, z) := (x \curlywedge y) \curlywedge z + (y \curlywedge z) \curlywedge x + (z \curlywedge x) \curlywedge y$. For every 1-form $D \in \Omega^1(P; A)$, direct evaluation via (2.5) and (2.23) gives the exact identity*

$$(D \curlywedge D) \curlywedge D = \frac{1}{3}\,\mathrm{Jac}_\curlywedge(D, D, D) \in \Omega^3(P; A), \quad \text{and} \quad dD \curlywedge D = \frac{1}{2}\, d(D \curlywedge D) \in \Omega^3(P; A). \tag{5.3}$$

*Consequently, for the unquotiented grouped Chern-Simons functional $S^{A,\mathrm{grp},\mathcal{P}}_{\mathrm{CS},2r-1}[D]$:*

(a) Lie Case ($\mathrm{Jac}_\curlywedge \equiv 0$, e.g., any Lie subalgebra $\mathfrak{h} \subset \mathfrak{so}(k)$ under $X \curlywedge Y = \frac{1}{2}[X, Y]$): *By (5.3), $\curlywedge^3_{\mathrm{bal}} D = (D \curlywedge D) \curlywedge D = 0$ vanishes identically in $\Omega^3(P; \mathfrak{h})$, and hence by Lemma 2.1(2a), $\curlywedge^m_{\mathrm{bal}} D = \curlywedge^m_{\mathrm{blk}} D = 0$ vanishes for* all *$m \geq 3$ under both* blk *and* bal *parenthesizations! Therefore:*
- *In dimension $n = 3$ ($r = 2$), the cubic vertex $\frac{1}{3}(D \curlywedge D) \curlywedge D = 0$ vanishes identically, reducing $S^{\mathfrak{h},\mathrm{grp}}_{\mathrm{CS},3}[D]$ to the quadratic commutator functional $\frac{1}{2}\int_M \mathrm{tr}_\xi(j(d(D \curlywedge D)))$ (which vanishes identically by Stokes' theorem on the closed manifold $M$ whenever the equivariant lift $\hat{\xi} : P \to \mathfrak{h}^*$ is constant, $d\hat{\xi} = 0$, in the linear/horizontal regime $d\rho = 0$);*
- *In all odd dimensions $n = 2r-1 \geq 5$ ($r \geq 3$), $S^{\mathfrak{h},\mathrm{grp},\mathcal{P}}_{\mathrm{CS},2r-1}[D] \equiv 0$ vanishes identically in $\Omega^{2r-1}(P; \mathfrak{h})$ under* both *$\mathcal{P} = \mathrm{blk}$ and $\mathcal{P} = \mathrm{bal}$.*

(b) Non-Lie Skew-Commutative Case ($\mathrm{Jac}_\curlywedge \not\equiv 0$, e.g., the octonionic Malcev algebra $\mathrm{Im}(\mathbb{O})$, reductive extensions $A = \mathbb{R}^k \oplus \mathfrak{h}$ under $\curlywedge_\rtimes$, or Cayley-Dickson algebras $\mathfrak{u}(k; \mathrm{CD}^l(\mathbb{R}))$ for $l \geq 3$):
- *In dimension $n = 3$ ($r = 2$), the cubic vertex $\frac{1}{3}(D \curlywedge D) \curlywedge D = \frac{1}{9}\mathrm{Jac}_\curlywedge(D, D, D)$ is non-vanishing in $\Omega^3(P; A)$;*
- *In dimension $n = 5$ ($r = 3$), under quartic-block grouping ($\mathcal{P} = \mathrm{blk}$), both $(dD \curlywedge dD) \curlywedge D = 0$ and $\frac{3}{20}\curlywedge^5_{\mathrm{blk}} D = \frac{3}{20}(\curlywedge^4_{\mathrm{bal}} D) \curlywedge D = 0$ vanish by $(2,1)$-nilpotency, collapsing the unquotiented 5D block-grouped Chern-Simons functional to the pure quartic Jacobiator vertex:*

$$S^{A,\mathrm{grp},\mathrm{blk}}_{\mathrm{CS},5}[D] = \frac{3}{4}\int_M \mathrm{tr}_\xi(j(dD \curlywedge ((D \curlywedge D) \curlywedge D))) = \frac{1}{4}\int_M \mathrm{tr}_\xi(j(dD \curlywedge \mathrm{Jac}_\curlywedge(D, D, D))) \tag{5.4}$$

*(whereas under recursive binary halving $\mathcal{P} = \mathrm{bal}$, $S^{A,\mathrm{grp},\mathrm{bal}}_{\mathrm{CS},5}[D]$ additionally retains the $J_4 = 5$ quintic Jacobsthal vertex $\frac{3}{20}\int_M \mathrm{tr}_\xi(j(\curlywedge^5_{\mathrm{bal}} D)) = \frac{1}{20}\int_M \mathrm{tr}_\xi(j(\mathrm{Jac}_\curlywedge(D, D, D) \curlywedge (D \curlywedge D))))$;*

- *In all odd dimensions $n = 2r-1 \geq 7$ ($r \geq 4$), the block-grouped $A$-valued Chern-Simons functional $S^{A,\mathrm{grp,blk}}_{\mathrm{CS},2r-1}[D]$ vanishes identically in $\Omega^{2r-1}(P;A)$, while under recursive binary halving ($\mathcal{P} = \mathrm{bal}$), $S^{A,\mathrm{grp,bal}}_{\mathrm{CS},n}[D]$ vanishes algebraically for every odd dimension $n \geq 7$ with $n \notin \bigcup_{m=4}^{\infty}\{J_m, J_m+2\} \cap [7,\infty) = \{7\} \cup \{11,13\} \cup \{21,23\} \cup \{43,45\} \cup \dots$ (where $J_m = \frac{2^m-(-1)^m}{3}$ is the $m$-th odd Jacobsthal number) and vanishes by rank for all $n \geq \dim_{\mathbb{R}} A + 3$.*

(4) ***Exterior Derivative in 3D and 5D:*** *For any skew-commutative algebra $(A, \curlywedge)$, since $d(dD \curlywedge D) = dD \curlywedge dD = 0$, the exterior derivatives of $\mathrm{CS}^{\mathrm{grp}}_3(D)$ and $\mathrm{CS}^{\mathrm{grp,blk}}_5(D)$ are given purely by their Jacobiator terms:*

$$d\,\mathrm{CS}^{\mathrm{grp}}_3(D) = \frac{1}{9}\,d(\mathrm{Jac}_{\curlywedge}(D,D,D)), \qquad d\,\mathrm{CS}^{\mathrm{grp,blk}}_5(D) = \frac{1}{4}\,dD \curlywedge d(\mathrm{Jac}_{\curlywedge}(D,D,D)), \tag{5.5}$$

*where $\frac{1}{3}d(\mathrm{Jac}_{\curlywedge}(D,D,D)) = d((D \curlywedge D) \curlywedge D) = 2(dD \curlywedge D) \curlywedge D - dD \curlywedge (D \curlywedge D)$.*

*Proof.* To prove (1), in the associative quotient $A/\mathfrak{Assoc}(A)$, alternating use of skew-commutativity and associativity gives $(x*y)*z = -(y*x)*z = -y*(x*z) = +(x*z)*y = x*(z*y) = -x*(y*z) = -(x*y)*z$, so $2(x*y*z) = 0 \implies x*y*z = 0$ in $A/\mathfrak{Assoc}(A)$. Directly on differential forms in $\Omega_{\mathrm{heq},\rho}(P;A)/\mathfrak{Assoc}(P,A)$, since $D \wedge_* D$ is a 2-form and $D$ is a 1-form while $\alpha \wedge_* \beta = -(-1)^{kl}\beta \wedge_* \alpha$ by (2.6), we have $(D \wedge_* D) \wedge_* D = -(-1)^{2\cdot 1} D \wedge_* (D \wedge_* D) = -D \wedge_* (D \wedge_* D) \equiv -(D \wedge_* D) \wedge_* D \pmod{\mathfrak{Assoc}(P,A)}$, so $\pi_{\mathrm{as}}((D \wedge_* D) \wedge_* D) = 0$ (and similarly any $\geq 3$-fold product of $D$ and $F_t$ vanishes in the quotient).

Part (2) is immediate since $\curlywedge^{2j+1}_{*,\mathcal{P}} D = 0$ whenever $2j+1 \geq s+1$.

For (3), since $(D \curlywedge D)_p(u,w) = 2\,D_p(u) \curlywedge D_p(w)$, evaluating $((D \curlywedge D) \curlywedge D)_p(v_1,v_2,v_3)$ directly from (2.5) sums $2(D_p(v_{\sigma(1)}) \curlywedge D_p(v_{\sigma(2)})) \curlywedge D_p(v_{\sigma(3)})$ with prefactor $\frac{1}{2!\,1!}$ over $\sigma \in S_3$; grouping into the 3 cyclic pairs $(123),(231),(312)$ yields $2\,\mathrm{Jac}_{\curlywedge}(D_p(v_1),D_p(v_2),D_p(v_3)) = \frac{1}{3}\mathrm{Jac}_{\curlywedge}(D,D,D)_p(v_1,v_2,v_3)$ by (2.23). Furthermore, since $\alpha \curlywedge \beta = -(-1)^{kl}\beta \curlywedge \alpha$, we have $d(D \curlywedge D) = dD \curlywedge D - D \curlywedge dD = dD \curlywedge D - (-1)^{1\cdot 2} dD \curlywedge D = 2\,dD \curlywedge D$, proving (5.3). In the $j$-th vertex $(\curlywedge^{r-1-j}_{\mathcal{P}} dD) \curlywedge (\curlywedge^{2j+1}_{\mathcal{P}} D)$ of (5.1), $dD \curlywedge dD = 0$ kills all terms with $r-1-j \geq 2 \iff j \leq r-3$. If $\mathrm{Jac}_{\curlywedge} \equiv 0$, then $\curlywedge^3_{\mathrm{bal}} D = 0$ and $\curlywedge^4_{\mathrm{bal}} D = 0$, and by Lemma 2.1(2a), $\curlywedge^m_{\mathrm{bal}} D = \curlywedge^m_{\mathrm{blk}} D = 0$ for all $m \geq 3$ ($2j+1 \geq 3 \iff j \geq 1$), leaving only $j = 0$ at $r = 2$ ($n = 3$) and killing all terms for $r \geq 3$ ($n \geq 5$). If $\mathrm{Jac}_{\curlywedge} \not\equiv 0$, under $\mathcal{P} = \mathrm{blk}$ we have $\curlywedge^{2j+1}_{\mathrm{blk}} D = 0$ for $2j+1 \geq 4 \iff j \geq 2$, so only $r-2 \leq j \leq 1$ can survive: $j \in \{0,1\}$ survive at $n = 3$, $j = 1$ survives at $n = 5$ ($r = 3$), and $r-2 \leq j \leq 1$ is empty for $r \geq 4$ ($n \geq 7$). Under $\mathcal{P} = \mathrm{bal}$, since $dD \curlywedge dD = 0$ restricts $j$ to $j \in \{r-2, r-1\}$ where $2j+1 \in \{n-2, n\}$ is odd, Lemma 2.1(2a),(2c) implies that $\curlywedge^{2j+1}_{\mathrm{bal}} D = 0$ unless $2j+1 = J_m$ for an odd Jacobsthal number $J_m$ ($m \geq 3$) with $J_m \leq \dim_{\mathbb{R}} A$.

Finally, for (4), since $d(dD \curlywedge D) = \frac{1}{2}d^2(D \curlywedge D) = 0$, differentiating $\mathrm{CS}^{\mathrm{grp}}_3(D) = dD \curlywedge D + \frac{1}{9}\mathrm{Jac}_{\curlywedge}(D,D,D)$ and $\mathrm{CS}^{\mathrm{grp,blk}}_5(D) = \frac{1}{4} dD \curlywedge \mathrm{Jac}_{\curlywedge}(D,D,D)$ immediately gives (5.5). □

5.2. **Applications to $A$-$BF$ Theories, the Yang-Mills-Higgs Mechanism, and Reductive Dualities.** We conclude this section by applying our framework to $A$-valued $BF$ theories [1, 9], the $A$-valued Yang-Mills-Higgs mechanism [19, 15, 41, 36] (including an explicit octonionic Malcev Proca mass spectrum computation), and reductive dualities.

**Proposition 5.4** (Cattaneo-Plebanski $A$-$BF$ Relations)**.** *Let $M$ be a compact oriented $n$-manifold ($n \geq 2$) and $E_A = P \times_\rho A \to M$.*

(1) *Given pairings $\langle\cdot,\cdot\rangle_{\mathrm{BF}}$ and $\langle\cdot,\cdot\rangle_B$ on $\Omega^2(M;E_A)$ and $\Omega^{n-2}(M;E_A)$ related via a linear isomorphism $\star_M : \Omega^2(M;E_A) \xrightarrow{\sim} \Omega^{n-2}(M;E_A)$ by $\langle\Phi,\mathcal{B}\rangle_{\mathrm{BF}} = \langle\star_M\Phi,\mathcal{B}\rangle_B$ with $\langle\cdot,\cdot\rangle_B$ symmetric and non-degenerate, the deformed $A$-$BF$ functional*

$$S^A_{\mathrm{BF},\mu}[D,B] := \int_M \Big(\langle j(F_D), j(B)\rangle_{\mathrm{BF}} - \frac{\mu}{2}\langle j(B), j(B)\rangle_B\Big) \qquad (\mu \neq 0) \tag{5.6}$$

*has the unique auxiliary Euler-Lagrange solution $j(B_{\text{on-shell}}(D)) = \frac{1}{\mu} \star_M j(F_D)$, whose substitution into (5.6) recovers $\frac{1}{2\mu}S^A[D]$ with $\langle\Phi,\Psi\rangle_A := \langle\star_M\Phi, \star_M\Psi\rangle_B$.*

(2) *In the multiplicative A-BF functional* $S^{A}_{\mathrm{BF},*}[D,B] := \int_M \mathrm{tr}_\xi(j(F_D \wedge_* B))$, *if* $A \simeq A_0 \oplus A_1$ *has a weak* $\mathcal{P}$*-*$(1,s)$*-solvable submodule* $A_0$ *(in the indexing of Definition 2.2, where the first index is the form degree) and* $B_e = \wedge^{n-2}_{*,\mathcal{P}} e$ *for* $e \in \Omega^1_{\mathrm{heq},\rho}(P;A_0)$, *then* $B_e = 0$ *for* $n \geq s+3$ *(compare [24], whose indexing differs); if* $n = 2r$ *is even,* $A$ *is weak* $\mathcal{P}$*-*$(2,s)$*-solvable, and* $B = \wedge^{r-1}_{*,\mathcal{P}} \beta$ *for* $\beta \in \Omega^2_{\mathrm{heq},\rho}(P;A)$, *then* $B = 0$ *for* $n \geq 2(s+2)$.

*Proof.* In (1), varying (5.6) with respect to $\dot{B}$ gives $\int_M \langle \star_M j(F_D) - \mu j(B), j(\dot{B})\rangle_B = 0 \iff j(B) = \frac{1}{\mu} \star_M j(F_D)$. Part (2) follows directly from $\wedge^{s+1}_{*,\mathcal{P}} e = 0$ and $\wedge^{s+1}_{*,\mathcal{P}} \beta = 0$. □

Next, let $\iota : G \hookrightarrow \hat{G}$ be a closed Lie subgroup inclusion inducing the extended principal $\hat{G}$-bundle $\hat{P} = P \times_G \hat{G} \to M$ and canonical reduction $\iota_P : P \hookrightarrow \hat{P}$ [22, 26], which defines the canonical global Higgs section $\phi_P \in \mathrm{Higgs}(\hat{P};G) := \Gamma(\hat{P} \times_{\hat{G}} (\hat{G}/G))$. Suppose $(\hat{A},*)$ is a finite-dimensional real algebra with representation $\rho : \hat{G} \to \mathrm{Aut}_{\mathrm{Alg}}(\hat{A})$ admitting a $\hat{G}$-equivariant map $\theta : \hat{G}/G \to \hat{A}$, $\theta(agG) = \rho(a)(\theta(gG))$, and let $\Phi_\phi := j^{-1}(\theta_*(\phi)) \in \Omega^0_{\mathrm{heq},\rho}(\hat{P};\hat{A})$ be the lifted Higgs 0-form (where $\theta_*$ denotes the induced bundle map $\hat{P} \times_{\hat{G}} (\hat{G}/G) \to E_{\hat{A}}$). Along the unitary-gauge reduction $\iota_P : P \hookrightarrow \hat{P}$, the canonical section $\phi_P$ takes the constant value $\Phi_0 := \theta(eG) \in \hat{A}^G$, so $d(\iota_P^* \Phi_{\phi_P}) = 0$. Consequently:

- In the Ehresmann-compatible setting of Proposition 2.4 ($d\rho(X)(y) = \frac{1}{2}[\eta(X),y]_*$), the commutator covariant derivative $\nabla^{\hat{A},[\cdot,\cdot]_*}_{\hat{D}} \Phi_\phi := d\Phi_\phi + \frac{1}{2}\hat{D} \wedge_{[\cdot,\cdot]_*} \Phi_\phi \in \Omega^1_{\mathrm{heq},\rho}(\hat{P};\hat{A})$ pulls back along $\iota_P : P \hookrightarrow \hat{P}$ at $\phi = \phi_P$ to $\mathcal{R}_{\Phi_0}(\iota_P^* \hat{D})$, where $\mathcal{R}_{\Phi_0} : \hat{A} \to \hat{A}$ is the linear mass operator $\mathcal{R}_{\Phi_0}(X) = \frac{1}{2}[X,\Phi_0]_*$.
- In the horizontal/flat setting ($\eta = 0, d = d_0$) with one-sided coupling $\nabla^{\hat{A}}_{\hat{D}} \Phi_0 = \hat{D} \wedge_* \Phi_0$, the linear mass operator is $\mathcal{R}_{\Phi_0}(X) = X * \Phi_0$.

Given a positive-definite fiberwise inner product $\langle \cdot,\cdot \rangle_{\hat{A}}$ on $\hat{A}$ and a Riemannian metric $g$ on $M$ inducing the kinetic pairing $\langle \alpha,\beta \rangle_{\mathrm{kin}} = \langle \alpha \wedge *_g \beta \rangle_{\hat{A}}$ on $\Omega^1(M;E_{\hat{A}})$, let $N_{\mathrm{mass}}(\Phi_0,x) = \mathrm{rk}(\mathcal{R}_{\Phi_0})$ denote the rank of the Proca mass quadratic form $\langle j(\mathcal{R}_{\Phi_0}(\hat{D})), j(\mathcal{R}_{\Phi_0}(\hat{D}))\rangle_{\mathrm{kin},x}$ at $x \in M$, and let $\mathfrak{g}_{\Phi_0} := \ker(\mathcal{R}_{\Phi_0}) \subset \hat{A}$ be the algebraic stabilizer of $\Phi_0$.

**Proposition 5.5** (Unitary-Gauge Proca Mass Bounds and Octonionic/Reductive Spectra in $A$-YMH Theories)**.** *Let* $\Phi_0 = \theta(eG) \in \hat{A}^G$ *be the unitary-gauge Higgs vacuum on* $P \hookrightarrow \hat{P}$.

(1) ***Splitting Kernel Decoupling:*** *If* $\hat{A}$ *arises from a split extension* $0 \to H \to \hat{A} \xrightarrow{q} A \to 0$ *with pulled-back product* $x *' y = s(q(x) * q(y))$ *and* $\Phi_0 \in H = \ker(q)$, *then* $\mathcal{R}_{\Phi_0} \equiv 0$ *and* $N_{\mathrm{mass}}(\Phi_0,x) = 0$.

(2) ***Derived and Commutator Rank Bounds:*** *Because* $\mathrm{im}(\mathcal{R}_{\Phi_0}) \subset [\hat{A},\hat{A}]_*$ *for the commutator coupling (resp.* $\mathrm{im}(\mathcal{R}_{\Phi_0}) \subset \hat{A}^{(1)} = \hat{A} * \hat{A}$ *for the one-sided coupling), the Proca mass rank and the algebraic stabilizer dimension* $\dim_{\mathbb{R}} \mathfrak{g}_{\Phi_0} = \dim_{\mathbb{R}} \hat{A} - \mathrm{rk}(\mathcal{R}_{\Phi_0})$ *satisfy*

$$N_{\mathrm{mass}}(\Phi_0,x) \leq \mathrm{rk}(\mathcal{R}_{\Phi_0}) \leq \dim_{\mathbb{R}}[\hat{A},\hat{A}]_*, \quad \dim_{\mathbb{R}} \mathfrak{g}_{\Phi_0} \geq \dim_{\mathbb{R}} \hat{A} - \dim_{\mathbb{R}}[\hat{A},\hat{A}]_* \ (\textit{comm.}), \tag{5.7}$$

$$N_{\mathrm{mass}}(\Phi_0,x) \leq \mathrm{rk}(\mathcal{R}_{\Phi_0}) \leq \dim_{\mathbb{R}} \hat{A}^{(1)}, \quad \dim_{\mathbb{R}} \mathfrak{g}_{\Phi_0} \geq \dim_{\mathbb{R}} \hat{A} - \dim_{\mathbb{R}} \hat{A}^{(1)} \ (\textit{one-sided}). \tag{5.8}$$

*If* $\Phi_0 \in \hat{A}^{(r)}$ *(resp.* $\Phi_0 \in \hat{A}^r$*), then* $N_{\mathrm{mass}}(\Phi_0,x) \leq \dim_{\mathbb{R}}(\hat{A} * \hat{A}^{(r)} + \hat{A}^{(r)} * \hat{A})$ *(resp.* $\leq \dim_{\mathbb{R}} \hat{A}^{r+1}$*).*

(3) ***Explicit Octonionic Malcev and Reductive Cartan Mass Spectra:***

(a) Octonionic Gauge Theory ($\hat{A} = \mathbb{O} = \mathbb{R}\cdot 1 \oplus \mathrm{Im}(\mathbb{O})$): *Since* $\mathcal{N}_{\mathrm{Lie}}(\mathbb{O}) = \mathbb{R}\cdot 1$ *is central, Proposition 2.4(1) forces* $d\rho = 0$ *in the Ehresmann-compatible setting, so* $G_2 = \mathrm{Aut}(\mathbb{O})$ *enters here only as a global symmetry of the algebra and of the mass operator, not as a gauge group acting through (2.25). With this understanding, for any vacuum* $\Phi_0 = c_0 \cdot 1 + \vec{\phi}_0 \in \mathbb{O}$ *with* $|\vec{\phi}_0| = v > 0$ *under the commutator coupling* $\mathcal{R}_{\Phi_0}(X) = \frac{1}{2}[X,\Phi_0]_* = \vec{X} \times_{G_2} \vec{\phi}_0$, *the central unit* $\mathbb{R}\cdot 1$ *and the parallel imaginary line* $\mathbb{R}\cdot\vec{\phi}_0$ *span a 2-dimensional massless stabilizer* $\mathfrak{g}_{\Phi_0} = \mathbb{R}\cdot 1 \oplus \mathbb{R}\cdot\vec{\phi}_0 \simeq \mathbb{C}$, *while on the*

*orthogonal 6-plane* $\vec{\phi}_0^{\perp} \subset \mathrm{Im}(\mathbb{O})$ *the operator* $\mathcal{R}^2_{\Phi_0} = -v^2\,\mathrm{id}_{\vec{\phi}_0^{\perp}}$ *gives* $N_{\mathrm{mass}}(\Phi_0) = 6$ *degenerate massive Proca gauge bosons of mass* $m = v$ *(the stabilizer of* $\vec{\phi}_0$ *in* $G_2$ *being* $SU(3)$*).*

(b) Reductive Cartan Algebra ($\hat{A} = \mathbb{R}^k \oplus \mathfrak{so}(k)$ under $\curlywedge_{\rtimes}$ with $\times_H = 0$): *For a pure translational vacuum* $\Phi_0 = (v_0, 0) \in \mathbb{R}^k \oplus \{0\}$ *with* $|v_0| = v > 0$, $\mathcal{R}_{\Phi_0}(u, X) = (u, X) \curlywedge_{\rtimes} (v_0, 0) = (\frac{1}{2}Xv_0, \sigma(u \wedge v_0))$. *Here* $\ker(\mathcal{R}_{\Phi_0}) = \mathbb{R}v_0 \oplus \mathfrak{so}(k-1)_{v_0}$ *has dimension* $1 + \binom{k-1}{2}$*, while* $N_{\mathrm{mass}}(\Phi_0) = 2(k-1)$ *gauge modes (*$k-1$ *boost/coset generators in* $\mathfrak{so}(k)/\mathfrak{so}(k-1)$ *with mass* $m_{\mathrm{rot}} = \frac{1}{2}v$ *and* $k-1$ *orthogonal soldering modes in* $v_0^{\perp} \subset \mathbb{R}^k$ *with mass* $m_{\mathrm{sold}} = v$*) acquire Proca masses, the masses being computed with respect to the orthogonal sum of the Euclidean inner product on* $\mathbb{R}^k$ *and* $\langle\cdot,\cdot\rangle_{\mathfrak{so}(k)}$ *(their ratio depends on this choice).*

*Proof.* Part (1) follows from $q(\Phi_0) = 0 \implies \mathcal{R}_{\Phi_0}(X) = 0$, and part (2) follows from $\mathrm{rk}(\mathcal{R}_{\Phi_0}) = \dim_{\mathbb{R}} \mathrm{im}(\mathcal{R}_{\Phi_0})$ together with $\frac{1}{2}[X, \Phi_0]_* \in [\hat{A}, \hat{A}]_*$ and $X * \Phi_0 \in \hat{A}^{(1)}$. For (3a), in $\mathbb{O}$, $\frac{1}{2}[x_0 + \vec{x}, c_0 + \vec{\phi}_0]_* = \vec{x} \times_{G_2} \vec{\phi}_0$, and by the octonionic cross-product identity $(\vec{x} \times_{G_2} \vec{\phi}_0) \times_{G_2} \vec{\phi}_0 = \langle \vec{x}, \vec{\phi}_0\rangle \vec{\phi}_0 - |\vec{\phi}_0|^2 \vec{x}$, the operator $\mathcal{R}^*_{\Phi_0}\mathcal{R}_{\Phi_0}$ has eigenvalue 0 on $\mathrm{span}\{1, \vec{\phi}_0\}$ (multiplicity 2) and eigenvalue $v^2$ on $\vec{\phi}_0^{\perp} \subset \mathrm{Im}(\mathbb{O})$ (multiplicity 6). For (3b), $Xv_0 = 0 \iff X \in \mathfrak{so}(k-1)_{v_0}$ and $u \wedge v_0 = 0 \iff u \in \mathbb{R}v_0$, giving $\dim\ker(\mathcal{R}_{\Phi_0}) = 1 + \binom{k-1}{2}$ and $\mathrm{rk}(\mathcal{R}_{\Phi_0}) = 2(k-1)$. □

Finally, given a reductive decomposition $A \simeq A_0 \oplus A_1$ where all products are evaluated via the ambient multiplication $*$ of $A$, we define the *geometric dual* ${}^*S^A[e, \omega] := \int_M \langle j({}^*F_{(e,\omega)}), j({}^*F_{(e,\omega)})\rangle_A$ on $\Omega^1_{\mathrm{heq},\rho}(P; A_0) \oplus \Omega^1_{\mathrm{heq},\rho}(P; A_1)$ by interchanging the roles of $e$ and $\omega$ in the differential and quadratic sectors ${}^*F_{(e,\omega)} := (de + \frac{1}{2}e \wedge_* e) + (d\omega + \frac{1}{2}\omega \wedge_{[\cdot,\cdot]_*} e) + \frac{1}{2}\omega \wedge_* \omega)$, and the *algebraic dual* $(S^A)^*[e', \omega'] := \int_M \langle j(F_{(e',\omega')}), j(F_{(e',\omega')})\rangle_A$ on the swapped configuration space $(e', \omega') \in \Omega^1_{\mathrm{heq},\rho}(P; A_1) \oplus \Omega^1_{\mathrm{heq},\rho}(P; A_0)$ with $F_{(e',\omega')} := (d\omega' + \frac{1}{2}\omega' \wedge_* \omega') + (de' + \frac{1}{2}\omega' \wedge_{[\cdot,\cdot]_*} e') + \frac{1}{2}e' \wedge_* e'$. Applying Proposition 3.1 and Theorem 4.6(1) directly to the swapped 1-forms yields:

**Proposition 5.6** (Duality Transfer of Obstructions)**.** *Let* $A \simeq A_0 \oplus A_1$ *be an* $\mathfrak{m}$*-graded algebra. If* $A_0$ *is weak* $(1,1)$*-solvable under* $*$*, then* $e \wedge_* e = 0$ *and* $\omega' \wedge_* \omega' = 0$*, abelianizing the differential curvature sector of both* ${}^*S^A$ *and* $(S^A)^*$ *in all dimensions* $n \geq 2$*. If* $A_1 = \mathfrak{h} \subset \mathfrak{so}(k_1) \oplus \cdots \oplus \mathfrak{so}(k_r)$ *is a* $k$*-algebra under* $\curlywedge$*, then the block-balanced dual Cartan-Yang-Mills functionals* ${}^*S^{\mathrm{CYM}}_{n,\xi,\mathrm{blk}}[e, \omega]$ *and* $(S^{\mathrm{CYM}}_{n,\xi,\mathrm{blk}})^*[e', \omega']$ *(formed with constitutive powers* $\curlywedge^m_{\mathrm{blk}}\omega$ *and* $\curlywedge^m_{\mathrm{blk}}e'$*) have vanishing cosmological terms for* $n \geq 4$*, vanishing EHP terms for* $n \geq 6$*, and vanish identically for* $n \geq 8$*.*

## 6. Realizations and Concrete/Abstract Examples

We conclude by applying our theorems to concrete matrix geometries, Cayley-Dickson geometries, and abstract non-associative and graded algebras.

6.1. **Classical, Generalized Complex, and Exceptional Geometries.** By Theorem 3.2(2) and Theorem 4.6(1), any algebra $(A, *)$ that is $(2,1)$-nil and block-balanced $(1,3)$-nil obstructs 4D internal-product topological/self-dual multiplicative $A$-YMT theories and cuts off block-balanced Cartan-Yang-Mills theories for $n \geq 8$. Since every subalgebra $\mathfrak{h} \subset \mathfrak{so}(k_1) \oplus \cdots \oplus \mathfrak{so}(k_r)$ (as well as $\mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C} \subset \mathfrak{so}(k; \mathbb{C})$ and $\mathbb{R}^k \oplus \mathfrak{h}$ under $\curlywedge_{\rtimes}$) is skew-commutative, Lemma 2.1(1) shows that this encompasses:

(1) **Classical** $G$**-Structures in Berger's List and Spinor Embeddings:** $\mathfrak{so}(k)$, $\mathfrak{u}(k)$, $\mathfrak{su}(k)$, $\mathfrak{sp}(k)$, $\mathfrak{sp}(k) \oplus \mathfrak{sp}(1)$, $\mathfrak{g}_2 \subset \mathfrak{so}(7)$, $\mathfrak{spin}(7) \subset \mathfrak{so}(8)$, and $\mathfrak{spin}(9) \subset \mathfrak{so}(16)$ (which occurs as a Riemannian holonomy algebra only in the locally symmetric case), as well as their covering groups $\mathrm{Spin}(k)$ and product distributions [7, 8, 24].

(2) **Type II String and Hitchin's Generalized Geometries:** The Narain orthogonal reduction $O(k) \times O(k) \hookrightarrow O(k,k)$ [27, 10], Hitchin and Gualtieri's generalized complex geometry $U(k,k) \hookrightarrow O(2k,2k)$ [20, 16], generalized Calabi-Yau geometry $SU(k,k) \hookrightarrow O(2k,2k)$, and generalized Kähler geometry $U(k) \times U(k) \hookrightarrow O(2k,2k)$ [16].

(3) **Exotic Orthogonal Subgroups and Exceptional Lie Algebras:** Obata's 15 infinite families of subgroups $H \subset O(k)$ [28], and maximal tori $T(q;\mathbb{K}) \hookrightarrow O(q;\mathbb{K})$ ($\mathrm{ch}(\mathbb{K}) \neq 2$ [4]).

6.2. **Infinite Families of Real and Generalized Cayley-Dickson Geometries.** Let $(A, *)$ be a finite-dimensional real algebra with involution $\overline{(-)} : A \to A$, and let $\mathrm{CD}(A) = A \oplus A$ be its Cayley-Dickson double with $(x,y) * (z,w) = (x * z - \overline{w} * y, w * x + y * \overline{z})$ and $\overline{(x,y)} = (\overline{x}, -y)$ [37, 2]. For $k \geq 1$, let $U(k;A) \subset \mathrm{Mat}_{k\times k}(A)$ denote the $A$-unitary matrices preserving $s(x,y) = \sum_{i=1}^{k} \overline{x}_i * y_i$ [24]. Iterating $U(k;\mathrm{CD}(A)) \hookrightarrow U(2k;A)$ yields for every $l \geq 1$ [24]:

$$U(k;\mathrm{CD}^l(\mathbb{R})) \hookrightarrow \cdots \hookrightarrow U(2^{l-2}k;\mathbb{H}) \hookrightarrow U(2^{l-1}k;\mathbb{C}) \hookrightarrow O(2^l k;\mathbb{R}), \tag{6.1}$$

$$U(k,k;\mathrm{CD}^l(A)) \hookrightarrow O(2^l k, 2^l k;\mathrm{CD}^l(A)) \simeq O(2^{l+1}k;\mathrm{CD}^l(A)), \tag{6.2}$$

recovering Hitchin's generalized complex geometry for $l = 1$ [16] and Pantilie's generalized quaternionic geometry for $l = 2$ [32, 12]. Because the algebra $\mathfrak{o}(m;B)$ of $B$-skew-Hermitian matrices under $X \curlywedge Y = \frac{1}{2}(XY - YX)$ is skew-commutative for any involutive algebra $B$, $\mathfrak{u}(k;\mathrm{CD}^l(\mathbb{R}))$ is $(2,1)$-nil and block-balanced $(1,3)$-nil for all $l \geq 1$. Moreover, $\mathfrak{u}(k;\mathrm{CD}^l(\mathbb{R}))$ is a Lie algebra ($\mathrm{Jac}_{\curlywedge} = 0$) for $l \in \{1,2\}$ ($\mathbb{C}, \mathbb{H}$), whereas for $l \geq 3$ ($\mathbb{O}$, sedenions, etc.) it is a non-Lie skew-commutative algebra with $\mathrm{Jac}_{\curlywedge} \neq 0$.

**Corollary 6.1** (Obstructions in Cayley-Dickson Geometries of Arbitrary Order)**.** *For every $k \geq 1$ and $l \geq 1$, the algebras $\mathfrak{u}(k;\mathrm{CD}^l(\mathbb{R}))$ and $\mathfrak{u}(k,k;\mathrm{CD}^l(A))$ (and, for $l \leq 2$, where $\mathfrak{u}(k;\mathrm{CD}^l(\mathbb{R})) \subset \mathfrak{so}(2^l k)$ is a matrix Lie algebra, their reductive extensions by $\mathbb{R}^{2^l k}$) are $(2,1)$-nil and block-balanced $(1,3)$-nil under $\curlywedge$ and $\curlywedge_{\rtimes}$. Therefore:*

1. *Every 4D internal-product topological $A$-YMT theory $\int_M \mathrm{tr}_\xi(j(F_D \curlywedge F_D))$ and every 4D self-dual internal multiplicative $A$-YMT theory vanishes identically;*
2. *Every block-balanced Cartan-Yang-Mills theory (4.5) (and its geometric and algebraic duals) has a vanishing cosmological term for $n \geq 4$, a vanishing EHP term for $n \geq 6$, and vanishes identically for $n \geq 8$;*
3. *For the unquotiented grouped $A$-valued Chern-Simons functional $S^{A,\mathrm{grp},\mathcal{P}}_{\mathrm{CS},2r-1}[D]$ valued in the algebra $\mathfrak{u}(k;\mathrm{CD}^l(\mathbb{R}))$: when $l \in \{1,2\}$ ($\mathrm{Jac}_{\curlywedge} = 0$), the 3D cubic vertex vanishes and all $\geq 5$D internal $A$-CS theories vanish identically under both* blk *and* bal*; when $l \geq 3$ ($\mathrm{Jac}_{\curlywedge} \neq 0$), the 3D cubic and 5D quartic Jacobiator vertices (5.4) survive under* blk *(together with the 5D quintic Jacobsthal vertex under* bal*), while for odd dimensions $n = 2r - 1 \geq 7$ all block-grouped $A$-CS theories vanish identically (and binary-balanced $A$-CS theories vanish outside $n \in \bigcup_{m\geq 4}\{J_m, J_m + 2\} \cap [7,\infty) = \{7\} \cup \{11,13\} \cup \ldots$ and for $n \geq \dim_{\mathbb{R}} \mathfrak{u}(k;\mathrm{CD}^l(\mathbb{R})) + 3$).*

6.3. **Abstract Examples: Lie Algebras, Octonionic Malcev Algebras, Graded Lie Superalgebras, and Nil Algebras.**

(1) **Lie Algebras and Graded Lie Superalgebras under $[\curlywedge]_{\mathfrak{g}}$:** Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie algebra. Because the Lie bracket is skew-commutative and satisfies the Jacobi identity, $\mathrm{Jac}_{[\cdot,\cdot]} \equiv 0$ (so $d_D^{[\cdot,\cdot]} F_D = 0$ and infinitesimal gauge covariance hold universally by Theorem 2.7), while $\mathfrak{g}$ is $(2,1)$-nil under $[\curlywedge]_{\mathfrak{g}}$, meaning $F_D[\curlywedge]_{\mathfrak{g}}F_D = 0$ identically. More generally, if $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$ is an $\mathfrak{m}$-graded Lie superalgebra, its even part $\mathfrak{g}^0$ is $(2,1)$-nil. By Theorem 3.2 and Remark 2.6, any topological or self-dual multiplicative $A$-YMT functional built solely from the internal Lie bracket $[\curlywedge]_{\mathfrak{g}}$ and a linear trace functional $\mathrm{tr}_\xi : E_{\mathfrak{g}} \to M \times \mathbb{R}$ vanishes identically, underscoring why classical 4D Chern-Weil theory requires an external symmetric bilinear invariant polynomial $\kappa \in \mathrm{inv}^2(\mathfrak{g})$.

(2) **Non-Lie Octonionic and Malcev Gauge Theories:** Let $A = \mathbb{O}$ be the octonions (or the 7-dimensional algebra $(\mathrm{Im}(\mathbb{O}), \times_{G_2})$ obtained from the commutator $xy - yx = 2\,x \times_{G_2} y$ of imaginary octonions (note that $\mathrm{Im}(\mathbb{O})$ is not closed under the octonion product itself), which is a non-Lie Malcev algebra [2, 38]). For any $\mathbb{O}$-valued connection 1-form $D = D_{\mathrm{re}} \cdot 1 + \vec{D} \in \Omega^1(P; \mathbb{R} \oplus \mathrm{Im}(\mathbb{O}))$, the real unit part $D_{\mathrm{re}} \cdot 1$ lies in the center ($[\mathbb{O}, 1]_* = 0$)

and the real inner-product part $-\langle \vec{D} \wedge \vec{D} \rangle = 0$ vanishes by symmetry of $\langle \cdot, \cdot \rangle$, so $D_{\mathrm{re}} \cdot 1$ decouples from $D \wedge_* D = \frac{1}{2} D \wedge_{[\cdot,\cdot]_*} D = \vec{D} \wedge_{\times_{G_2}} \vec{D} \in \Omega^2(P; \mathrm{Im}(\mathbb{O}))$ by Proposition 3.1(3). Because $\mathrm{Jac}_{[\cdot,\cdot]_*}(x, y, z) = 6\,\mathrm{Assoc}_*(x, y, z) \neq 0$ on non-associative octonionic triples, the Lie nucleus of $\mathrm{Im}(\mathbb{O})$ is trivial ($\mathcal{N}_{\mathrm{Lie}}(\mathrm{Im}(\mathbb{O})) = \{0\}$, so admissible connections on $\mathrm{Im}(\mathbb{O})$ are horizontal by Proposition 2.4(1), while on full $\mathbb{O}$ the Lie nucleus is $\mathcal{N}_{\mathrm{Lie}}(\mathbb{O}) = \mathbb{R} \cdot 1$), and Theorem 2.7(2)–(3) shows that $d_D^{[\cdot,\cdot]_*} F_D = -\frac{1}{4}\mathrm{Assoc}_*(\vec{D}, \vec{D}, \vec{D}) \neq 0$ whenever $\vec{D}$ spans a non-associative 3-plane in $\mathrm{Im}(\mathbb{O})$ (for $n \geq 3$), whereas $d_D^{[\cdot,\cdot]_*} F_D = 0$ holds identically whenever $\dim M = 2$ or whenever $\vec{D}$ takes values in an associative quaternionic subalgebra $\mathbb{H} \subset \mathbb{O}$.

(3) **Solvable Lie Algebras and Affine Nil Algebras:** For a solvable Lie algebra $\mathfrak{g}$ of derived length $s$ ($\mathfrak{g}^{(s+1)} = 0$), Proposition 3.1 shows that $D[\wedge]_{\mathfrak{g}} D \in \Omega^2(P; [\mathfrak{g}, \mathfrak{g}])$ shifts up the derived filtration and its iterated squares $(D[\wedge]_{\mathfrak{g}} D)^{[m]} \in \Omega^{2^{m+1}}(P; \mathfrak{g}^{(m+1)})$ vanish at step $m = s$, while confining the quartic interaction sector to $E_{[\mathfrak{g},\mathfrak{g}]} \subsetneq E_{\mathfrak{g}}$. Finally, for any nilpotent algebra $A$ of index $s$ (all products of $s + 1$ elements vanish, in every parenthesization), $A$ is $(1, s)$-nil and $(2, s)$-nil (for nil algebras, where only powers of single elements are required to vanish, this fails in general for 1-forms, whose powers involve antisymmetrized products of distinct elements), so Theorem 3.2(1) implies that all internal topological Chern-Weil $A$-YMT functionals $S^A_{\mathrm{top},r,\mathcal{P}}[D]$ vanish in dimensions $n \geq 2(s + 1)$.

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