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Class-two Lie algebras and quadratic-dual current constructions

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Abstract

For a reductive Lie algebra $\mathfrak{s}$, a finite-dimensional $\mathfrak{s}$-module R and an $\mathfrak{s}$-submodule $K \subseteq \bigwedge^2 R$, the class-two radical $R \oplus (\bigwedge^2 R/K)$ carries an $\mathfrak{s}$-action. For semisimple $\mathfrak{s}$ the semidirect products $\mathfrak{s} \ltimes (R \oplus (\bigwedge^2 R/K))$ are, up to isomorphism, all Lie algebras whose radical has class at most two. We study the isomorphisms that fix the acting algebra pointwise, including their central mixing and shear terms. On the locus where central mixing vanishes, the resulting groupoid admits a description by invariant Grassmannians and equivariant linear groups. We express the wild matrix-pair construction as a functor into class-two Lie algebras and give its equivariant square in $\mathfrak{s}$-modules. We work out single-isotypic slices, signed multiplicity-free tensor squares, and explicit low-dimensional multiplicity-space examples. The acting algebra may be zero and its central action need not be semisimple. Finally, we study pairs of Lie brackets on one object whose mixed composites vanish. Tensoring them with algebras carrying two independent commutative associative products, for instance pointwise multiplication and convolution exchanged by Fourier transform, produces further Lie algebras. For these pairs, in representations of $\mathfrak{s}$, we define the groupoid of objects that are isomorphic to their bracket exchange σX; we give the structure of the switches $X \to \sigma X$, the obstruction to an involutive switch, and its indecomposable objects in small dimension for semisimple $\mathfrak{s}$.

Let k be an algebraically closed field of characteristic zero, and let all representations below be finite-dimensional unless otherwise noted.

1 Introduction

In characteristic zero the radical of a finite-dimensional perfect Lie algebra is nilpotent; among the nonsemisimple perfect complex Lie algebras of dimension at most nine, every radical but one is abelian or of class two (Theorem 1). A Lie algebra with semisimple Levi factor $\mathfrak{s}$ and radical of class at most two is determined by the $\mathfrak{s}$-module R, the radical modulo its commutator, and the kernel $K \subseteq \bigwedge^2 R$ of its bracket. This reduction is standard and is recalled in Section 2.

Generator-and-relation approaches to two-step nilpotent Lie algebras go back to Gauger [4]; see also Scheuneman [5]. Unrestricted classification already contains wild matrix problems when the central commutator has dimension three as $\dim R$ grows [6]. That problem treats R only as a vector space. Here $R \oplus (\bigwedge^2 R/K)$ is a Lie algebra object in $\mathfrak{s}$-Rep, and isomorphisms are required to fix the acting algebra pointwise. For a nontrivial $\mathfrak{s}$, the corresponding orbit problems often live in much smaller multiplicity spaces, although wild behavior can return as those multiplicities grow.

The original motivating question came from the standard quadratic-duality construction of a Lie algebra on the tensor product of a $\mathcal{P}$-algebra and a $\mathcal{P}^!$-algebra applied with $\mathcal{P} = \mathrm{Com}\amalg\mathrm{Com}$. Those are two independent commutative associative products, such as pointwise multiplication and convolution. Its quadratic dual governs two Lie brackets whose mixed composites vanish individually. Class-two brackets on a common $R \oplus W$ provide a concrete family satisfying these relations.

Organization

Section 2 collects the background: small classifications, the construction and its converse. Section 3 types every block of an isomorphism fixing the acting algebra, describes the locus where central mixing vanishes (Section 3.1), and introduces the wedge multiplicity spaces (Section 3.2). Section 4 identifies the moduli on that locus, for a fixed representation and over all representations, as groupoids of invariant subspaces (Sections 4.1 and 4.2). It then expresses the wild matrix-pair construction as a functor and lifts it to $\mathfrak{s}$-modules (Section 4.3). Section 5 computes these groupoids in explicit cases. It specializes them to a single semisimple isotypic component (Section 5.1), compares compatible multiplicity-free tensor-square data across semisimple types (Section 5.2), and calculates explicit small binary multiplicity-space slices (Sections 5.3 and 5.4). It also treats the first ternary multiplicity space (Section 5.5) and lists the components of total dimension at most six for each semisimple acting algebra, including $\mathfrak{s} = 0$ (Section 5.6). Neither a polarization of R nor a trivialization of a central line is assumed.

Section 6 turns from one bracket to two. A pair of Lie brackets β_1, β_2 on one object is *mutually annihilating* when $\beta_1(\beta_2(x, y), z) = 0 = \beta_2(\beta_1(x, y), z)$; this is stronger than the compatibility of [13], which asks only that the cyclic sum of the mixed terms vanish. Such pairs are the algebras over the quadratic dual of Com $\amalg$ Com, so they tensor with algebras carrying two unrelated commutative associative products to give Lie algebras (Section 6.1). Exchanging the two brackets is an involution σ of the groupoid of pairs in $\mathfrak{s}$-Rep or $\mathfrak{s}$-sRep. Section 6.2 defines this groupoid and its full subgroupoid on the objects X that admit an arrow $X \to \sigma X$, with no such arrow chosen. The examples follow the definitions. Section 6.4 gives pairs of class-two brackets on a common $R \oplus W$, in the notation of the earlier sections, and semidirect pairs over a shared abelian algebra. Section 6.5 lists the indecomposable objects of the switchable subgroupoid of dimension at most four in Vect_k and in $\mathfrak{s}$-Rep for semisimple $\mathfrak{s}$, with their automorphism groups and switches.

Section 7.1 gives coefficient algebras, including pointwise multiplication and convolution exchanged by Fourier transform, and Section 7.2 the resulting Lie algebras.

2 From perfect Lie algebras to the class-two family

The radical of a finite-dimensional perfect Lie algebra in characteristic zero is nilpotent (Lemma 3 below). The following complete small-dimensional list shows how often it has class two. Let V_j be the irreducible $\mathfrak{sl}_2(\mathbb{C})$-module of highest weight j, let $\mathfrak{h}_d$ be the d-dimensional Heisenberg algebra, and let $\mathfrak{f}_{r,c}$ be the free nilpotent Lie algebra on r generators of class c. In direct sums inside a semidirect product, the brackets between the displayed summands vanish. In each occurrence of $\mathfrak{h}_3$, the specified $\mathfrak{sl}_2$-action induces V_1 on $\mathfrak{h}_3/Z(\mathfrak{h}_3)$ and is trivial on its center; its bracket is the invariant alternating pairing into that center. On $\mathfrak{f}_{2,3}$ and $\mathfrak{f}_{3,2}$ the action extends those on their generating spaces V_1 and V_2, respectively. There are two indicated actions on $\mathfrak{h}_5$, denoted $\mathfrak{h}_5^{(1,1)}$ and $\mathfrak{h}_5^{(3)}$: their symplectic quotients are $V_1 \oplus V_1$ and V_3, respectively, their centers are trivial $\mathfrak{sl}_2$-modules, and their brackets are the corresponding nondegenerate invariant alternating pairings. Finally, $\mathfrak{a}_{6,4}$ has nonzero brackets $[e_1, e_4] = e_5$, $[e_2, e_3] = -e_5$, and $[e_3, e_4] = e_6$; $\langle e_1, e_2\rangle$ and $\langle e_3, e_4\rangle$ are standard $\mathfrak{sl}_2$-modules, while e_5, e_6 are fixed.

Theorem 1 (Small complex perfect Lie algebras)**.** *Up to isomorphism, the nonsemisimple perfect complex Lie algebras of dimension at most nine are exactly the entries in the following table. The unique entry whose radical has nilpotency class three is $\mathfrak{sl}_2 \ltimes \mathfrak{f}_{2,3}$; every other radical is abelian or has class two.*

$\dim \mathfrak{g}$	*Isomorphism classes*
5	$\mathfrak{sl}_2 \ltimes V_1$
6	$\mathfrak{sl}_2 \ltimes V_2$, $\mathfrak{sl}_2 \ltimes \mathfrak{h}_3$
7	$\mathfrak{sl}_2 \ltimes V_3$, $\mathfrak{sl}_2 \ltimes (V_1 \oplus V_1)$
8	$\mathfrak{sl}_2 \oplus (\mathfrak{sl}_2 \ltimes V_1)$, $\mathfrak{sl}_2 \ltimes V_4$, $\mathfrak{sl}_2 \ltimes (V_1 \oplus V_2)$, $\mathfrak{sl}_2 \ltimes (V_1 \oplus \mathfrak{h}_3)$, $\mathfrak{sl}_2 \ltimes \mathfrak{f}_{2,3}$, $\mathfrak{sl}_2 \ltimes \mathfrak{h}_5^{(1,1)}$, $\mathfrak{sl}_2 \ltimes \mathfrak{h}_5^{(3)}$
9	$\mathfrak{sl}_2 \oplus (\mathfrak{sl}_2 \ltimes V_2)$, $\mathfrak{sl}_2 \oplus (\mathfrak{sl}_2 \ltimes \mathfrak{h}_3)$, $\mathfrak{sl}_2 \ltimes V_5$, $\mathfrak{sl}_2 \ltimes (V_1 \oplus V_3)$, $\mathfrak{sl}_2 \ltimes (V_2 \oplus V_2)$, $\mathfrak{sl}_2 \ltimes (V_1 \oplus V_1 \oplus V_1)$, $\mathfrak{sl}_2 \ltimes (V_2 \oplus \mathfrak{h}_3)$, $\mathfrak{sl}_2 \ltimes (\mathfrak{h}_3 \oplus \mathfrak{h}_3)$, $\mathfrak{sl}_2 \ltimes \mathfrak{f}_{3,2}$, $\mathfrak{sl}_2 \ltimes \mathfrak{a}_{6,4}$

The semisimple perfect algebras in this range are $\mathfrak{sl}_2$, $\mathfrak{sl}_2^{\oplus 2}$, $\mathfrak{sl}_3(\mathbb{C})$, *and* $\mathfrak{sl}_2^{\oplus 3}$ *(and the zero algebra, if dimension zero is included).*

Proof. The nonsemisimple classification is [1, Proposition 5.3]; the semisimple entries follow from the classification of complex simple Lie algebras. □

For comparison, the class-two nilpotent Lie algebras themselves can be listed in the same quotient-kernel language. For a vector space R and $K \subseteq \bigwedge^2 R$, write $\mathfrak{n}(R,K) = R \oplus (\bigwedge^2 R/K)$, with $[r,r'] = r \wedge r' \pmod K$ and the second summand central. Let R_q be a q-dimensional space with basis $u_1,\dots,u_q$ and dual basis $u^1,\dots,u^q$. The expression ij in the following table abbreviates the functional $u^i \wedge u^j : \bigwedge^2 R_q \to k$; in particular it is not a bracket. For forms $\omega_1,\dots,\omega_d$ put

$$K_q(\omega_1,\dots,\omega_d) = \bigcap_{a=1}^{d} \ker\bigl(\omega_a : \bigwedge^2 R_q \to k\bigr), \qquad K_q() = \bigwedge^2 R_q.$$

Theorem 2 (Class-two nilpotent algebras through dimension six)**.** *Over an algebraically closed field of characteristic zero, the following are all the nilpotent Lie algebras of class at most two and dimension* $1 \le n \le 6$*, without repetitions. Each displayed pair* (R,K) *denotes* $\mathfrak{n}(R,K)$*, and semicolons separate distinct isomorphism classes. The pair* $(R_n, K_n())$ *is abelian; omitting it from each row gives exactly the algebras of class two.*

n	*Isomorphism classes as* (R,K) *pairs*
1	$(R_1, K_1())$
2	$(R_2, K_2())$
3	$(R_3, K_3())$; $(R_2, K_2(12))$
4	$(R_4, K_4())$; $(R_3, K_3(12))$
5	$(R_5, K_5())$; $(R_4, K_4(12))$; $(R_4, K_4(12+34))$; $(R_3, K_3(12,13))$
6	$(R_6, K_6())$; $(R_5, K_5(12))$; $(R_5, K_5(12+34))$; $(R_4, K_4(12,13))$; $(R_4, K_4(12,34))$; $(R_4, K_4(12,13+24))$; $(R_3, K_3(12,13,23))$

Proof. This is the class-at-most-two part of the classification in [2, Table 1], with the lower-dimensional entries also obtained by deleting abelian direct summands. □

The complete eight-dimensional class-two list has 35 isomorphism classes [3, Theorem 1 and Corollary 2]; the table above is complete through dimension six, the range used in Section 5.6. For space reasons, we do not reproduce them all in this introduction.

Let $\mathfrak{g} = \mathfrak{s}_{\mathrm{ss}} \ltimes \mathfrak{r}$ be a Levi decomposition, with $\mathfrak{s}_{\mathrm{ss}}$ semisimple and $\mathfrak{r}$ the solvable radical. Recall that $\mathfrak{g}$ is *perfect* when $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$.

Lemma 3. *For every finite-dimensional Lie algebra in characteristic zero,* $[\mathfrak{g}, \operatorname{rad}\mathfrak{g}]$ *is a nilpotent ideal. Consequently the solvable radical of a perfect Lie algebra is nilpotent [1, Lemma 2.2].*

Proof. Write $\mathfrak{r} = \operatorname{rad}\mathfrak{g}$. By Lie's theorem, the operators $\operatorname{ad}(\mathfrak{r})$ on $\mathfrak{g}$ are upper triangular; let χ_i be their diagonal characters. For $x \in \mathfrak{g}$ and $z \in \mathfrak{r}$, conjugation by $e^{t \operatorname{ad} x}$ preserves $\operatorname{ad}(\mathfrak{r})$. The operator $\operatorname{ad}(e^{t\operatorname{ad}x} z) = e^{t\operatorname{ad}x} \operatorname{ad}(z) e^{-t\operatorname{ad}x}$ is conjugate to $\operatorname{ad} z$ and has the same characteristic polynomial, whose

coefficients do not depend on t. Its diagonal entries $\chi_i(e^{t\,\mathrm{ad}\,x}z)$, extended $k[[t]]$-linearly, are roots of that polynomial, so each is one of the finitely many constant roots $\chi_j(z)$ in $k[[t]]$. Its constant term is $\chi_i(z)$; therefore $\chi_i([x,z])=0$. Thus $\mathrm{ad}([\mathfrak{g},\mathfrak{r}])$ is strictly upper triangular, and Engel's theorem makes $[\mathfrak{g},\mathfrak{r}]$ nilpotent. If $\mathfrak{g}$ is perfect, its Levi decomposition gives $\mathfrak{r}=[\mathfrak{s}_{\mathrm{ss}},\mathfrak{r}]+[\mathfrak{r},\mathfrak{r}]=[\mathfrak{g},\mathfrak{r}]$. □

Proposition 4 (Perfectness criterion)**.** *For $\mathfrak{g}=\mathfrak{s}_{\mathrm{ss}}\ltimes\mathfrak{r}$ with $\mathfrak{s}_{\mathrm{ss}}$ semisimple, put $R=\mathfrak{r}/[\mathfrak{r},\mathfrak{r}]$. Then $\mathfrak{g}$ is perfect if and only if $R^{\mathfrak{s}_{\mathrm{ss}}}=0$.*

Proof. Since $[\mathfrak{s}_{\mathrm{ss}},\mathfrak{s}_{\mathrm{ss}}]=\mathfrak{s}_{\mathrm{ss}}$, the equality $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$ is equivalent to $\mathfrak{r}=[\mathfrak{s}_{\mathrm{ss}},\mathfrak{r}]+[\mathfrak{r},\mathfrak{r}]$. Passing to R, this says $\mathfrak{s}_{\mathrm{ss}}R=R$. Complete reducibility splits R into its trivial isotypic part $R^{\mathfrak{s}_{\mathrm{ss}}}$ and the sum of its nontrivial irreducible summands. The latter, and only the latter, lie in $\mathfrak{s}_{\mathrm{ss}}R$, giving the criterion. □

The following equivariant quotient construction refines the usual generator-and-relation description of two-step nilpotent Lie algebras [4, 5].

Proposition 5 (Construction)**.** *Let $\mathfrak{s}$ be any Lie algebra. For an $\mathfrak{s}$-module R and an $\mathfrak{s}$-submodule $K\subseteq\bigwedge^2 R$, put $W=\bigwedge^2 R/K$. The vector space $\mathfrak{r}_K=R\oplus W$, with bracket $[r_1,r_2]=r_1\wedge r_2 \pmod K$ and $[R,W]=[W,W]=0$, is a class-at-most-two nilpotent Lie algebra acted on by $\mathfrak{s}$ through derivations.*

Proof. Every triple bracket in $\mathfrak{r}_K$ vanishes, so Jacobi holds. The $\mathfrak{s}$-action on $\bigwedge^2 R$ is $x(r_1\wedge r_2)=(xr_1)\wedge r_2+r_1\wedge(xr_2)$. Since K is stable, the projection to W intertwines this action; this is precisely the derivation identity for the bracket. □

Corollary 6 (Radical and perfectness of the construction)**.** *If $\mathfrak{s}=\mathfrak{s}_{\mathrm{ss}}$ is semisimple in Proposition 5, the Lie algebra $\mathfrak{s}_{\mathrm{ss}}\ltimes\mathfrak{r}_K$ has radical $\mathfrak{r}_K$ and is perfect exactly when $R^{\mathfrak{s}_{\mathrm{ss}}}=0$.*

Proof. The quotient by $\mathfrak{r}_K$ is semisimple, so the nilpotent ideal $\mathfrak{r}_K$ is the radical. Its derived ideal is W; Proposition 4 gives the perfectness criterion. □

Theorem 7 (Converse for a class-two radical)**.** *Let $\mathfrak{s}_{\mathrm{ss}}$ be semisimple. Every $\mathfrak{s}_{\mathrm{ss}}\ltimes\mathfrak{r}$ whose radical $\mathfrak{r}$ has class at most two is isomorphic to one of these. The module R is $\mathfrak{r}/[\mathfrak{r},\mathfrak{r}]$ and K is the kernel of the induced surjection $\bigwedge^2 R\twoheadrightarrow[\mathfrak{r},\mathfrak{r}]$.*

Proof. $W=[\mathfrak{r},\mathfrak{r}]$ is central in $\mathfrak{r}$. Complete reducibility supplies an equivariant linear section of $\mathfrak{r}\twoheadrightarrow R$ and hence an equivariant vector-space identification $\mathfrak{r}\cong R\oplus W$. Because W is central, the bracket of lifts depends only on their images in R and defines an equivariant surjection $b:\bigwedge^2 R\twoheadrightarrow W$. With $K=\ker b$, the induced $\bigwedge^2 R/K\cong W$ identifies both brackets, giving the claimed Lie isomorphism. □

We study the resulting (R,K) systematically, and also allow a reductive algebra

$$\mathfrak{s}=\mathfrak{s}_{\mathrm{ss}}\oplus\bigoplus_{j=1}^{m}kt_j,$$

allowing $\mathfrak{s}_{\mathrm{ss}}=0$, $m=0$, and $\mathfrak{s}=0$. For $m>0$ this $\mathfrak{s}$ is a specified acting subalgebra, while $\mathfrak{s}_{\mathrm{ss}}$ is the Levi factor of the resulting Lie algebra. Let R be an $\mathfrak{s}$-representation, with $D_j^R=\rho_R(t_j)$. Choose an $\mathfrak{s}$-subrepresentation $K\subseteq\bigwedge^2 R$, put $W_K=\bigwedge^2 R/K$, and denote its quotient map by $b_K:\bigwedge^2 R\twoheadrightarrow W_K$. On the vector space

$$\mathfrak{g}_K=\mathfrak{s}\oplus R\oplus W_K$$

use the given $\mathfrak{s}$-actions and the bracket

$$[r_1,r_2]=b_K(r_1\wedge r_2),\qquad [R,W_K]=[W_K,W_K]=0.$$

Equivariance of b_K gives the Jacobi identity involving $\mathfrak{s}$; the other Jacobi identities follow because W_K is central in $R\oplus W_K$. This construction does not require $\mathfrak{s}$ to be semisimple.

A central line and the Heisenberg object. Suppose K is such that W_K is isomorphic in $\mathfrak{s}$-Rep to its monoidal unit $\mathbf{1}$ (the one-dimensional trivial module). The projection $b_K : \bigwedge^2 R \twoheadrightarrow W_K$ is then an invariant, line-valued alternating form, and $R \oplus W_K$ is a two-step central Lie algebra object in $\mathfrak{s}$-Rep. If the intrinsic map

$$b_K^\sharp : R \longrightarrow \mathrm{Hom}_k(R, W_K), \qquad b_K^\sharp(r)(v) = b_K(r \wedge v),$$

is an isomorphism, it is the Heisenberg Lie algebra object of the symplectic object (R, b_K), with its central line kept explicit. The condition $W_K \cong \mathbf{1}$ alone permits a degenerate form: $\ker b_K^\sharp$ is central and splits off as an abelian factor of the underlying Lie algebra. Such a splitting in $\mathfrak{s}$-Rep requires an equivariant complement. A presentation $R \cong X \oplus X^*$ with the evaluation pairing is a further $\mathfrak{s}$-equivariant polarization, not part of the construction.

3 General isomorphisms fixing the acting algebra

Write $\mathfrak{g}_K = \mathfrak{g}(\mathfrak{s}, R, K)$ and $\mathfrak{g}_{K'} = \mathfrak{g}(\mathfrak{s}, R', K')$ for two constructions, with $W = W_K$, $W' = W_{K'}$, and quotient maps b, b'. An isomorphism is required to fix the displayed $\mathfrak{s}$ pointwise; no preservation of R, W, or $R \oplus W$ is assumed. The central component below arises from an arbitrary such isomorphism.

Theorem 8 (General isomorphism criterion)**.** *Every isomorphism $\Phi : \mathfrak{g}_K \to \mathfrak{g}_{K'}$ fixing $\mathfrak{s}$ has the form*

$$\Phi(x, r, w) = \big(x + c(r),\ a(r) + p(w),\ h(r) + q(w)\big), \tag{1}$$

where the $\mathfrak{s}$-intertwiners have types

$$c : R \to Z(\mathfrak{s}), \quad a : R \to R', \quad p : W \to R', \quad h : R \to W', \quad q : W \to W'.$$

Here $Z(\mathfrak{s})$ has its trivial adjoint $\mathfrak{s}$-action. Conversely these maps define an isomorphism if and only if the linear block map

$$T = \begin{pmatrix} a & p \\ h & q \end{pmatrix} : R \oplus W \xrightarrow{\sim} R' \oplus W' \tag{2}$$

is invertible and, for $r, v \in R$ and $w, u \in W$, one has

$$p\, b(r \wedge v) = c(r) \cdot a(v) - c(v) \cdot a(r), \tag{3}$$
$$q\, b(r \wedge v) = b'(a(r) \wedge a(v)) + c(r) \cdot h(v) - c(v) \cdot h(r), \tag{4}$$
$$0 = c(r) \cdot p(w), \tag{5}$$
$$0 = b'(a(r) \wedge p(w)) + c(r) \cdot q(w), \tag{6}$$
$$0 = b'(p(w) \wedge p(u)). \tag{7}$$

Proof. The radical is $Z(\mathfrak{s}) \oplus R \oplus W$, since its quotient is $\mathfrak{s}_{\mathrm{ss}}$. Its characteristic property restricts the $\mathfrak{s}$-part of $\Phi(R \oplus W)$ to $Z(\mathfrak{s})$. This part vanishes on $W = [R, R]$, as the bracket of two elements with central $\mathfrak{s}$-parts has zero $\mathfrak{s}$-part. Thus it is the map c in (1). Bracketing with $\mathfrak{s}$, which Φ fixes, shows that all five maps are intertwiners. The triangular map Φ is invertible exactly when T is. Comparing respectively the R'- and W'-components of $[r, v]$, $[r, w]$, and $[w, u]$ gives (3)–(7). Conversely these identities check every bracket, including those with $\mathfrak{s}$. □

3.1 The locus where central mixing vanishes

First decompose R under $\mathfrak{s}_{\mathrm{ss}}$ and write the action of t_j on each multiplicity space as $D_{j,\lambda}$:

$$R = \bigoplus_\lambda V_\lambda \otimes M_\lambda, \qquad D_j^R|_{V_\lambda \otimes M_\lambda} = \mathrm{id}_{V_\lambda} \otimes D_{j,\lambda}.$$

Here λ runs through the finitely many occurring highest weights, V_λ is the corresponding simple $\mathfrak{s}_{\mathrm{ss}}$-module, and $V_0 = k$ is the trivial module. The operators $D_{j,\lambda}$ commute because the t_j do. Next take the joint generalized weight decomposition for the commuting $D_{j,\lambda}$:

$$M_\lambda = \bigoplus_{\mu \in k^m} M_{\lambda,\mu},$$

where $\mu = (\mu_1, \dots, \mu_m)$ and $M_{\lambda,\mu}$ consists of the vectors killed by sufficiently high powers of every $D_{j,\lambda} - \mu_j \,\mathrm{id}$. Only finitely many summands are nonzero. Write $\mathbf{0}_m = (0, \dots, 0)$; if $m = 0$, this is the empty tuple and $M_{\lambda,()} = M_\lambda$.

Set $R_{\mathbf{0}} = V_0 \otimes M_{0,\mathbf{0}_m}$ and let $R_{\neq}$ be the sum of the other summands, so $R = R_{\mathbf{0}} \oplus R_{\neq}$. These are generalized joint weight spaces; the D_j^R need not be diagonalizable. In particular, the $D_{j,0}$ may each have a nonzero generalized nullspace on M_0 while $M_{0,\mathbf{0}_m} = 0$. For instance, on $M_0 = k^2$, take

$$D_{1,0} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad D_{2,0} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The condition $M_{0,\mathbf{0}_m} = 0$ constrains only the trivial $\mathfrak{s}_{\mathrm{ss}}$-isotypic part; other types and multiplicities are arbitrary.

An $\mathfrak{s}$-invariant linear map $c : R \to Z(\mathfrak{s}) = \bigoplus_j k t_j$ vanishes on $R_{\neq}$ and factors through

$$R_{\mathbf{0}} \Big/ \sum_j D_j^R R_{\mathbf{0}} \longrightarrow Z(\mathfrak{s}).$$

Indeed it vanishes on $[\mathfrak{s}_{\mathrm{ss}}, R]$ and on $D_j^R R$; on any nonzero joint generalized weight, at least one D_j^R is invertible. Consequently,

$$M_{0,\mathbf{0}_m} = 0 \quad \Longrightarrow \quad \mathrm{Hom}_{\mathfrak{s}}(R, Z(\mathfrak{s})) = 0. \tag{8}$$

Proposition 9 (Sufficient, not necessary, vanishing)**.** *The condition $M_{0,\mathbf{0}_m} = 0$ forces $c = 0$ for every isomorphism fixing $\mathfrak{s}$, but it is not necessary. More generally, put*

$$Z_{\mathrm{nil}}(\mathfrak{s}; R') = \{z \in Z(\mathfrak{s}) : \rho_{R'}(z) \text{ is nilpotent}\}.$$

Every c arising in Theorem 8 has image in $Z_{\mathrm{nil}}(\mathfrak{s}; R')$. Thus $\mathrm{Hom}_{\mathfrak{s}}(R, Z_{\mathrm{nil}}(\mathfrak{s}; R')) = 0$ also forces $c = 0$.

Proof. The first assertion follows from (8). Since $R \oplus W$ is a nilpotent ideal, its image lies in the nilradical of $\mathfrak{g}_{K'}$. Every element of the nilradical acts nilpotently by adjoint on $\mathfrak{g}_{K'}$: its first bracket lands in that ideal, and subsequent brackets follow its lower central series. The action of $\mathrm{ad}\,\Phi(r)$ on $(R' \oplus W')/W' \cong R'$ is $\rho_{R'}(c(r))$, which must therefore be nilpotent.

For nonnecessity take $\mathfrak{s} = kt$, $R = ke_0 \oplus ke_1$ with $te_0 = 0$, $te_1 = e_1$, and $K = \bigwedge^2 R$. Here $M_{0,(0)} = ke_0 \neq 0$, whereas the only scalar multiple of $\rho_R(t) = \mathrm{diag}(0, 1)$ that is nilpotent is zero. Hence $c = 0$ for every automorphism of $\mathfrak{g}_K$ fixing t. □

Let $\mathcal{C}_{\mathfrak{s}}$ be the groupoid whose objects are pairs (R, K) as above and whose arrows are *all* Lie algebra isomorphisms $\mathfrak{g}(\mathfrak{s}, R, K) \to \mathfrak{g}(\mathfrak{s}, R', K')$ fixing $\mathfrak{s}$ pointwise. Define $\mathcal{C}_{\mathfrak{s}}^0$ to be its full subgroupoid on objects satisfying $M_{0,\mathbf{0}_m}(R) = 0$. This tests the representation R alone, independently of K and of any prospective arrow. Fullness means that no further restriction is placed on the isomorphisms between selected objects. Proposition 9 explains why this object condition simplifies every arrow in the subgroupoid. Let $\mathcal{B}_{\mathfrak{s}}^0$ have the same objects, with only the arrows $a_*(x, r, w) = (x, ar, \bar{a}w)$ induced by $\mathfrak{s}$-module isomorphisms $a : R \to R'$ carrying K onto K'; here $\bar{a}$ is the induced map of exterior-square quotients.

Theorem 10. *For $(R,K),(R',K') \in \mathcal{C}_{\mathfrak{s}}^0$, an arrow $\mathfrak{g}(\mathfrak{s},R,K) \to \mathfrak{g}(\mathfrak{s},R',K')$ exists if and only if there is an $\mathfrak{s}$-module isomorphism $a : R \xrightarrow{\sim} R'$ with*

$$(\bigwedge^2 a)K = K'. \tag{9}$$

For each such a, let $\bar{a} : W_K \to W_{K'}$ be the uniquely induced isomorphism, characterized by $\bar{a}\, b_K = b_{K'} \bigwedge^2 a$. All the isomorphisms above a are

$$\Phi_{a,H}(x,r,w) = \big(x, ar, \bar{a}(w) + H(ar)\big), \qquad H \in \mathrm{Hom}_{\mathfrak{s}}(R', W_{K'}). \tag{10}$$

Proof. Begin with an arbitrary isomorphism Φ fixing $\mathfrak{s}$ and apply Theorem 8. Equation (8) forces $c = 0$, so (3) and surjectivity of b_K force $p = 0$. Hence T is triangular, with invertible diagonal maps a and q; equation (4) becomes

$$q\, b_K = b_{K'} \bigwedge^2 a.$$

This implies (9) and $q = \bar{a}$. Finally $h = Ha$ for a unique $H \in \mathrm{Hom}_{\mathfrak{s}}(R', W_{K'})$. Conversely (10) preserves every bracket and is invertible. □

Proposition 11 (Composition and self-loops)**.** *For composable arrows $(a,H) : (R,K) \to (R',K')$ and $(d,J) : (R',K') \to (R'',K'')$, composition is*

$$(d,J) \circ (a,H) = \big(da,\ J + \bar{d}\, H\, d^{-1}\big), \tag{11}$$

where the second component belongs to $\mathrm{Hom}_{\mathfrak{s}}(R'', W_{K''})$. Thus at every *object (R,K) the group of all self-loops is*

$$\mathrm{Aut}_{\mathcal{C}_{\mathfrak{s}}^0}(R,K) = \mathrm{Aut}_{\mathfrak{s}}(\mathfrak{g}_K) \cong \mathrm{Hom}_{\mathfrak{s}}(R, W_K) \rtimes \mathrm{Aut}_{\mathcal{B}_{\mathfrak{s}}^0}(R,K), \tag{12}$$

where $\mathrm{Aut}_{\mathfrak{s}}(\mathfrak{g}_K)$ denotes the Lie algebra automorphisms of $\mathfrak{g}_K$ fixing $\mathfrak{s}$ pointwise, and a acts on a shear H by $\bar{a} H a^{-1}$.

Proof. For $H \in \mathrm{Hom}_{\mathfrak{s}}(R, W_K)$ write $S_H(x,r,w) = (x,r,w + H(r))$, so that $\Phi_{a,H} = S_H \circ a_*$ with S_H taken on the target. The identity $d_* S_H d_*^{-1} = S_{\bar{d} H d^{-1}}$ gives (11); restricting it to self-loops gives (12). □

Corollary 12 (Closure of the vanishing locus)**.** *The full subgroupoid $\mathcal{C}_{\mathfrak{s}}^0$ is closed under isomorphisms in $\mathcal{C}_{\mathfrak{s}}$.*

Proof. An isomorphism whose source satisfies $M_{0,\mathfrak{0}_m}(R) = 0$ has $c = 0$ by Proposition 9. The general bracket equations in Theorem 8 then force $p = 0$, as b_K is surjective. The resulting triangular isomorphism has an invertible $\mathfrak{s}$-intertwiner $a : R \to R'$ on its diagonal. Hence $R' \cong R$ as $\mathfrak{s}$-modules and the target satisfies the same condition. □

Equivalently, at each object $X = (R,K)$ put $\mathcal{H}_X = \mathrm{Hom}_{\mathfrak{s}}(R, W_K)$. Transport along $a : X \to X'$ is $H \mapsto \bar{a} H a^{-1}$. Formula (11) identifies $\mathcal{C}_{\mathfrak{s}}^0$ with the split semidirect product groupoid $\mathcal{H} \rtimes \mathcal{B}_{\mathfrak{s}}^0$. The $\mathcal{H}_X$ consist of the shear loops

$$S_H(x,r,w) = (x,r,w + H(r))$$

and every arrow factors uniquely as $\Phi_{a,H} = S_H \circ a_*$. For a fixed R, the restriction of $\mathcal{B}_{\mathfrak{s}}^0$ to objects (R,K) is the action groupoid of $\mathrm{Aut}_{\mathfrak{s}}(R)$ on the $\mathfrak{s}$-subrepresentations $K \subseteq \bigwedge^2 R$. The corresponding restriction of $\mathcal{C}_{\mathfrak{s}}^0$ adds the shear loops without changing its isomorphism classes.

3.2 The wedge multiplicity spaces

Fix an object (R, K) of $\mathcal{C}_{\mathfrak{s}}^0$ and keep its $\mathfrak{s}$-module R fixed while varying K. Write $R = \bigoplus_\lambda V_\lambda \otimes M_\lambda$, and fix a total order on the finitely many occurring highest weights. Define $N_\nu = \mathrm{Hom}_{\mathfrak{s}_{\mathrm{ss}}}(V_\nu, \bigwedge^2 R)$, so that $\bigwedge^2 R = \bigoplus_\nu V_\nu \otimes N_\nu$. For two distinct $\mathfrak{s}_{\mathrm{ss}}$-types $\lambda < \kappa$, the cross term is governed by the usual Clebsch–Gordan space $\mathrm{Hom}_{\mathfrak{s}_{\mathrm{ss}}}(V_\nu, V_\lambda \otimes V_\kappa)$. Within one type, the canonical wedge identity gives the two spaces

$$A_{\nu,\lambda} = \mathrm{Hom}_{\mathfrak{s}_{\mathrm{ss}}}(V_\nu, \bigwedge^2 V_\lambda), \qquad B_{\nu,\lambda} = \mathrm{Hom}_{\mathfrak{s}_{\mathrm{ss}}}(V_\nu, \mathrm{Sym}^2 V_\lambda).$$

Consequently

$$N_\nu \cong \bigoplus_\lambda \left(A_{\nu,\lambda} \otimes \mathrm{Sym}^2 M_\lambda \oplus B_{\nu,\lambda} \otimes \bigwedge^2 M_\lambda\right) \oplus \bigoplus_{\lambda<\kappa} \mathrm{Hom}_{\mathfrak{s}_{\mathrm{ss}}}(V_\nu, V_\lambda \otimes V_\kappa) \otimes M_\lambda \otimes M_\kappa. \tag{13}$$

This decomposition uses the entire M_λ, including all its joint generalized central weights.

4 Moduli problems

Section 3 describes the arrows between two objects. This section assembles the objects and arrows of $\mathcal{C}_{\mathfrak{s}}^0$ (Section 3.1) into moduli groupoids, first with the $\mathfrak{s}$-module R fixed (Section 4.1) and then over all R (Section 4.2). Section 4.3 embeds the matrix-pair problem into the categories of class-two Lie algebra objects and compares it with these equivariant moduli.

4.1 Moduli for an arbitrary fixed representation

For each j and λ, set $D_{j,\lambda}^{\{2\}} = D_{j,\lambda} \otimes \mathrm{id}_{M_\lambda} + \mathrm{id}_{M_\lambda} \otimes D_{j,\lambda}$, and write $C_{\nu,\lambda\kappa} = \mathrm{Hom}_{\mathfrak{s}_{\mathrm{ss}}}(V_\nu, V_\lambda \otimes V_\kappa)$. Under (13), the t_j-action on N_ν which is denoted $E_{j,\nu}$ has blocks

$$\begin{aligned} E_{j,\nu}|_{A_{\nu,\lambda}\otimes \mathrm{Sym}^2 M_\lambda} &= \mathrm{id}_{A_{\nu,\lambda}} \otimes D_{j,\lambda}^{\{2\}}|_{\mathrm{Sym}^2 M_\lambda}, \\ E_{j,\nu}|_{B_{\nu,\lambda}\otimes \bigwedge^2 M_\lambda} &= \mathrm{id}_{B_{\nu,\lambda}} \otimes D_{j,\lambda}^{\{2\}}|_{\bigwedge^2 M_\lambda}, \\ E_{j,\nu}|_{C_{\nu,\lambda\kappa}\otimes M_\lambda \otimes M_\kappa} &= \mathrm{id}_{C_{\nu,\lambda\kappa}} \otimes (D_{j,\lambda} \otimes \mathrm{id}_{M_\kappa} + \mathrm{id}_{M_\lambda} \otimes D_{j,\kappa}) \quad (\lambda < \kappa). \end{aligned} \tag{14}$$

Put $G_R = \mathrm{Aut}_{\mathfrak{s}}(R)$, the group acting in this fixed-R slice of $\mathcal{B}_{\mathfrak{s}}^0$. Its elements are $a_g = \bigoplus_\lambda \mathrm{id}_{V_\lambda} \otimes g_\lambda$, where

$$G_R = \prod_\lambda G_\lambda, \qquad G_\lambda = \{g_\lambda \in GL(M_\lambda) : g_\lambda D_{j,\lambda} = D_{j,\lambda} g_\lambda \text{ for all } j\}. \tag{15}$$

The action induced by $\bigwedge^2 a_g$ on N_ν has blocks

$$\mathrm{id}_{A_{\nu,\lambda}} \otimes \mathrm{Sym}^2 g_\lambda, \qquad \mathrm{id}_{B_{\nu,\lambda}} \otimes \bigwedge^2 g_\lambda, \qquad \mathrm{id}_{C_{\nu,\lambda\kappa}} \otimes g_\lambda \otimes g_\kappa. \tag{16}$$

Denote this entire operator by g_ν. It commutes with all $E_{j,\nu}$; the g_λ act simultaneously across every ν.

Lemma 13 (Invariant-subspace parameter space)**.** *Fix any $\mathfrak{s}$-module R occurring in an object of $\mathcal{C}_{\mathfrak{s}}^0$. In its decomposition $R = \bigoplus_\lambda V_\lambda \otimes M_\lambda$, the only condition is $M_{0,\mathbf{0}_m} = 0$; in particular R may have several isotypic types and several joint weights.*

With N_ν, $E_{j,\nu}$, and G_R as above, the possible $\mathfrak{s}$-subrepresentations $K \subseteq \bigwedge^2 R$ are precisely

$$K = \bigoplus_\nu V_\nu \otimes L_\nu, \qquad L_\nu \subseteq N_\nu, \quad E_{j,\nu} L_\nu \subseteq L_\nu \quad \text{for all } j. \tag{17}$$

For $\mathbf{d} = (\dim L_\nu)_\nu$, set

$$X_{\mathbf{d}}(R) = \prod_\nu \{ L \in \mathrm{Gr}_{d_\nu}(N_\nu) : E_{j,\nu} L \subseteq L \text{ for all } j \}. \tag{18}$$

Proof. Semisimple complete reducibility identifies a submodule of $\bigwedge^2 R = \bigoplus_\nu V_\nu \otimes N_\nu$ with its collection of multiplicity subspaces L_ν. Stability under each t_j is exactly $E_{j,\nu} L_\nu \subseteq L_\nu$. This proves (17) and (18). □

Theorem 14 (Fixed-representation moduli)**.** *Under the assumptions of Lemma 13, the isomorphism classes with this fixed R form the orbit set*

$$\coprod_{\mathbf{d}} X_{\mathbf{d}}(R)/G_R. \tag{19}$$

Proof. The commutant of $\mathfrak{s}_{\mathrm{ss}}$ on R consists of the g_λ; commuting also with every t_j gives (15). Taking exterior squares gives the three blocks in (16). Theorem 10 now identifies isomorphism classes with the G_R-orbits. □

Corollary 15 (Stabilizers and self-loops at fixed R)**.** *At $X = (R, K)$ the stabilizer is exactly the self-loop group in the base groupoid $\mathcal{B}^0_{\mathfrak{s}}$, and the full self-loop group in $\mathcal{C}^0_{\mathfrak{s}}$ is its split shear extension:*

$$G_K = \mathrm{Aut}_{\mathcal{B}^0_{\mathfrak{s}}}(R, K) = \{ g \in G_R : g_\nu L_\nu = L_\nu \text{ for all } \nu \}, \qquad \mathrm{Aut}_{\mathcal{C}^0_{\mathfrak{s}}}(R, K) \cong \mathrm{Hom}_{\mathfrak{s}}(R, W_K) \rtimes G_K. \tag{20}$$

For $g \in G_K$, the induced map $q_g : W_K \to W_K$ is determined by $q_g b_K = b_K \bigwedge^2 a_g$.

Proof. The stabilizer condition is exactly preservation of every L_ν under the action in (16). Theorem 10 and Proposition 11 identify the full self-loop group as its shear extension. The equation for q_g is the universal property of the quotient by K. □

The quotient in (19) denotes an orbit set; no existence claim for a geometric quotient is needed. A stabilizer preserves K as a subspace, not necessarily pointwise. If W_K is a line, then $q_g = \chi(g)\,\mathrm{id}_{W_K}$ for an induced scalar $\chi(g)$; this does not require an identification $W_K \cong k$.

4.2 The moduli of all objects

The fixed-R loci assemble by first choosing the isomorphism class of the underlying $\mathfrak{s}$-representation. This choice involves the full commuting actions on the multiplicity spaces, including their nilpotent parts. For a set X acted on by a group G, write $X//G$ for its *action groupoid*: its objects are points of X and an arrow $x \to x'$ is an element $g \in G$ with $gx = x'$.

Lemma 16 (Intertwiners between representation fibers)**.** *For decompositions*

$$R = \bigoplus_\lambda V_\lambda \otimes M_\lambda, \qquad R' = \bigoplus_\lambda V_\lambda \otimes M'_\lambda,$$

the representations R, R' are isomorphic precisely when, for every λ, there is a linear isomorphism $f_\lambda : M_\lambda \xrightarrow{\sim} M'_\lambda$ such that

$$f_\lambda D_{j,\lambda} = D'_{j,\lambda} f_\lambda \qquad \text{for every } j. \tag{21}$$

Equivalently, their multiplicity spaces are isomorphic as modules for the commuting central operators. If this fails for any λ, there are no arrows in $\mathcal{C}^0_{\mathfrak{s}}$ from an object over R to an object over R', regardless of K, K'. If it holds, transporting K along $\bigwedge^2(\bigoplus_\lambda \mathrm{id}_{V_\lambda} \otimes f_\lambda)$ reduces every arrow to the fixed-R' description.

Proof. Any $\mathfrak{s}_{\mathrm{ss}}$-intertwiner $R \to R'$ acts independently on each V_λ-isotypic part and has the form $\bigoplus_\lambda \mathrm{id}_{V_\lambda} \otimes f_\lambda$. It intertwines the t_j exactly when (21) holds, and is invertible exactly when every f_λ is. Theorem 10 gives the claim about arrows and transport of K. □

Theorem 17 (Global groupoid decomposition)**.** *Let $\mathcal{R}^0_{\mathfrak{s}}$ be the set of isomorphism classes of finite-dimensional $\mathfrak{s}$-modules R satisfying $M_{0,\mathbf{0}_m}(R) = 0$, and choose one representative of each class. The groupoid $\mathcal{C}^0_{\mathfrak{s}}$ is equivalent to the disjoint union, over these representatives R, of the split shear groupoids $\mathcal{H}_R \rtimes (\mathcal{K}(R)//G_R)$, where $\mathcal{K}(R)$ is the set of $\mathfrak{s}$-subrepresentations of $\bigwedge^2 R$ and $\mathcal{H}_R$ is the restriction of the shear system $\mathcal{H}$ to the objects over R, with fiber $\mathrm{Hom}_{\mathfrak{s}}(R, W_K)$ at K.*

Proof. Lemma 16 separates the representation classes. Within each class, Theorem 10 gives precisely the base arrows and shears. Proposition 11 gives their composition. Choosing one R in each class produces the asserted equivalence. □

Corollary 18 (Global orbit set)**.** *With $X_{\mathbf{d}}(R)$ from (18) and G_R from (15), the set of isomorphism classes of all objects of $\mathcal{C}^0_{\mathfrak{s}}$ is*

$$\pi_0(\mathcal{C}^0_{\mathfrak{s}}) \cong \coprod_{[R] \in \mathcal{R}^0_{\mathfrak{s}}} \coprod_{\mathbf{d}} X_{\mathbf{d}}(R)/G_R. \tag{22}$$

Proof. Take isomorphism classes in Theorem 17 and apply Theorem 14 in each representation fiber. □

Joint weight multiplicities alone need not establish (21): commuting operators can have different nilpotent parts on generalized weight spaces of the same dimension. When the $D_{j,\lambda}$ and the $D'_{j,\lambda}$ are each jointly diagonalizable, the dimensions of the ordinary joint weight spaces do suffice. When $m = 0$, the condition reduces to $\dim M_\lambda = \dim M'_\lambda$ for each λ.

4.3 Wildness in class-two Lie algebra objects

For any Lie algebra $\mathfrak{s}$, let $\mathrm{LieAlg}_{\leq 2}(\mathfrak{s}\text{-Rep})$ be the category of Lie algebra objects of nilpotence class at most two in $\mathfrak{s}$-Rep, with equivariant Lie homomorphisms. $\mathfrak{s} = 0$ is allowed. The following construction exhibits a matrix-pair problem inside these categories and compares it with the equivariant moduli above.

The word *wild* in this subsection refers to an explicit family of moduli problems, not to a fixed dimension. Put $\mathcal{A} = k\langle x, y\rangle\text{-mod}_{\mathrm{fd}}$; an object is a triple (V, T_1, T_2) with $T_i \in \mathrm{End}_k(V)$. Write $\mathrm{Core}(\mathcal{A})$ for its isomorphism groupoid. The target category has all Lie homomorphisms as arrows. We give the block construction from [6, Lemma 7 and Theorem 4] in functorial form.

Theorem 19 (The matrix-pair functor)**.** *There is a functor*

$$\mathcal{F}_0 : \mathrm{Core}(\mathcal{A}) \longrightarrow \mathrm{LieAlg}_{\leq 2}(\mathrm{Vect}_k)$$

such that for every $N > 0$ and $\dim V = N$ the Lie algebra $\mathcal{F}_0(V, T_1, T_2)$ has dimension $350N + 3$ and derived algebra of dimension three.

Proof. The entire object construction below is the alternating-form construction of [6, Lemma 7, formulas (22)–(24), and Lemma 6], expressed without a choice of basis; its fixed blocks are part of that construction. We supply the arrow map to make its functoriality explicit. On V set

$$E_V = V^{\oplus 100} \oplus V^{\oplus 50} \oplus V^{\oplus 20} \oplus V \oplus V^{\oplus 4}, \qquad M_V = E_V \oplus E_V^*.$$

Each $C_i \in \mathrm{End}_k(E_V)$ has five diagonal blocks in the displayed order of the summands of E_V; a zero denotes the zero endomorphism of its corresponding summand:

$$\begin{aligned} C_1 &= \mathrm{diag}\big(\mathrm{id}_{V^{\oplus 100}}, 0, 0, \mathrm{id}_V, \mathrm{id}_{V^{\oplus 4}}\big),\\ C_2 &= \mathrm{diag}\big(0, \mathrm{id}_{V^{\oplus 50}}, 0, \mathrm{id}_V, J_4(0) \otimes \mathrm{id}_V\big),\\ C_3 &= \mathrm{diag}\big(0, 0, \mathrm{id}_{V^{\oplus 20}}, \mathrm{id}_V, \mathrm{diag}(\mathrm{id}_V, T_1, T_2, 0)\big). \end{aligned}$$

For $e, e' \in E_V$ and $\alpha, \alpha' \in E_V^*$ put

$$h_i\big((e,\alpha),(e',\alpha')\big) = \alpha'(C_i e) - \alpha(C_i e'), \qquad q_V = (h_1, h_2, h_3) : \bigwedge^2 M_V \longrightarrow k^3.$$

The first three fixed blocks make q_V surjective for $V \neq 0$. Let $K_{T_1,T_2} = \ker q_V$ and give $\mathcal{F}_0(V, T_1, T_2) = M_V \oplus k^3$ the bracket q_V, with k^3 central. This has the asserted dimensions. Define $\mathcal{F}_0(0,0,0) = 0$.

For an intertwining isomorphism $S : V \to V'$, let $E(S) : E_V \to E_{V'}$ be the direct sum of its indicated copies and put $M(S) = E(S) \oplus (E(S)^{-1})^* : M_V \to M_{V'}$. The identities $E(S)C_i = C_i' E(S)$ give $q_{V'} \circ \bigwedge^2 M(S) = q_V$. Hence $M(S) \oplus \mathrm{id}_{k^3}$ is a Lie isomorphism; inverse duals and direct sums respect composition. This proves that $\mathcal{F}_0$ is a functor on the core. The inverse dual is the reason this arrow formula is stated only for isomorphisms of input triples. □

Theorem 20 (Recovering a matrix pair from an output isomorphism)**.** *For every Lie isomorphism $\phi : \mathcal{F}_0(V, T_1, T_2) \xrightarrow{\sim} \mathcal{F}_0(V', T_1', T_2')$ there exists an isomorphism of input modules $g_\phi : (V, T_1, T_2) \xrightarrow{\sim} (V', T_1', T_2')$. No equality $\mathcal{F}_0(g_\phi) = \phi$ or compatibility of g_ϕ with composition is asserted. Consequently $\mathcal{F}_0$ induces an injection on isomorphism classes of matrix pairs of every dimension.*

Proof. For nonzero inputs, ϕ induces an isomorphism $P : M_V \to M_{V'}$ on the quotients by their derived algebras and an isomorphism $\Gamma : k^3 \to k^3$ on those derived algebras, with $q_{V'} \circ \bigwedge^2 P = \Gamma \circ q_V$. Thus the three alternating forms above are related by simultaneous congruence and an arbitrary invertible substitution among the three forms. The fixed blocks of sizes $100N, 50N, 20N$ distinguish the three coordinate lines of the form space by rank. Indeed, the corresponding endomorphisms C_1, C_2, C_3 have ranks $105N$, $54N$, and $22N + \mathrm{rank}\, T_1 + \mathrm{rank}\, T_2 \leq 24N$; mixing two or three coordinates puts the rank in a disjoint range. Thus the substitution is diagonal. The common N-block $(\mathrm{id}_V, \mathrm{id}_V, \mathrm{id}_V)$ then forces its three diagonal scalars to agree by the cancellation argument. After absorbing their common scalar, the remaining block $(I_{4N}, J_4(0_N), \mathrm{diag}(I_N, T_1, T_2, 0_N))$ recovers a simultaneous intertwiner $S : V \to V'$. These are the block comparison and cancellation steps of [6, Theorem 1(a) and Lemma 7]; in particular they handle the freedom to change the basis of k^3. Taking $g_\phi = S$ proves the assertion. Different positive input dimensions have different output dimensions, while the zero input maps to zero, so the assertion also holds across dimensions. □

Corollary 21 (The equivariant square)**.** *Let $\mathfrak{s}$ be any finite-dimensional Lie algebra. Let $\mathcal{A}_\mathfrak{s}$ have as objects triples (V, T_1, T_2) with $V \in \mathfrak{s}$-Rep and $T_i \in \mathrm{End}_\mathfrak{s}(V)$, and as morphisms the $\mathfrak{s}$-linear maps intertwining both T_i. The construction of Theorem 19 defines a functor $\mathcal{F}_\mathfrak{s} : \mathrm{Core}(\mathcal{A}_\mathfrak{s}) \to \mathrm{LieAlg}_{\leq 2}(\mathfrak{s}\text{-Rep})$. With the vertical arrows forgetting the $\mathfrak{s}$-action, the square*

$$\begin{array}{ccc} \mathrm{Core}(\mathcal{A}_\mathfrak{s}) & \xrightarrow{\mathcal{F}_\mathfrak{s}} & \mathrm{LieAlg}_{\leq 2}(\mathfrak{s}\text{-Rep}) \\ \big\downarrow \text{forget} & & \big\downarrow \text{forget} \\ \mathrm{Core}(\mathcal{A}) & \xrightarrow{\mathcal{F}_0} & \mathrm{LieAlg}_{\leq 2}(\mathrm{Vect}_k) \end{array}$$

commutes up to the canonical identification of the underlying vector spaces and brackets.

Proof. All the summands in E_V inherit the $\mathfrak{s}$-action from V, its dual E_V^* has the dual action, and k^3 has the trivial action. Because T_1, T_2 intertwine $\mathfrak{s}$, each C_i does too. Evaluation $E_V^* \otimes E_V \to k$ is equivariant, so q_V is an $\mathfrak{s}$-morphism and its kernel is an $\mathfrak{s}$-submodule. The bracket is therefore equivariant. If S is $\mathfrak{s}$-linear, so are $E(S)$ and its inverse dual, proving functoriality and the square. □

Lemma 22 (Equivariant separation of the coding block)**.** *Let $\mathcal{D}_\mathfrak{s}$ be the category of diagrams consisting of two finite-dimensional $\mathfrak{s}$-modules and three $\mathfrak{s}$-linear maps between them; morphisms are pairs of $\mathfrak{s}$-linear maps commuting with the three arrows. For $T_1, T_2 \in \mathrm{End}_\mathfrak{s}(V)$ put*

$$\mathcal{Q}(V, T_1, T_2) = \big(\mathrm{id}_{V^{\oplus 4}}, J_4(0) \otimes \mathrm{id}_V, \mathrm{diag}(\mathrm{id}_V, T_1, T_2, 0)\big).$$

The diagram obtained from the three alternating forms defining $\mathcal{F}_\mathfrak{s}(V, T_1, T_2)$, viewed as maps $M_V \to M_V^*$, is isomorphic in $\mathcal{D}_\mathfrak{s}$ to a direct sum of $\mathcal{Q}(V, T_1, T_2)$ and a diagram with no nonzero direct summand isomorphic to a direct summand of any $\mathcal{Q}(V', T_1', T_2')$. Consequently, an isomorphism between two such diagrams induces an isomorphism between their $\mathcal{Q}$ summands.

Proof. The category $\mathcal{D}_\mathfrak{s}$ is abelian, its objects have finite length, and its Hom spaces are finite-dimensional over k. Idempotents split, and the Fitting decomposition of a diagram endomorphism is a decomposition by $\mathfrak{s}$-submodules. Hence Krull–Schmidt applies in this category; no semisimplicity of the $\mathfrak{s}$-action is required.

Reordering the target of $M_V \to M_V^*$ and changing the sign on one target summand identifies the forms with the direct sum of the triples (C_1, C_2, C_3) on E_V and (C_1^*, C_2^*, C_3^*) on E_V^*. These are equivariant identifications, where $*$ denotes the linear dual with its contragredient action. The four fixed blocks, on copies of V or V^*, have arrow triples

$$(\mathrm{id}, 0, 0), \quad (0, \mathrm{id}, 0), \quad (0, 0, \mathrm{id}), \quad (\mathrm{id}, \mathrm{id}, \mathrm{id}).$$

The remaining blocks are $\mathcal{Q}(V, T_1, T_2)$ and, after reversing the four dual copies,

$$\mathcal{Q}^\vee(V, T_1, T_2) = \big(\mathrm{id}_{(V^*)^{\oplus 4}}, J_4(0) \otimes \mathrm{id}_{V^*}, \mathrm{diag}(0, T_2^*, T_1^*, \mathrm{id}_{V^*})\big).$$

On a diagram whose first arrow is invertible, use it to identify the two vertex spaces and denote the other arrows by J, D. Every nonzero direct summand of $\mathcal{Q}$ has only size-four Jordan blocks for J: its underlying J-module is a direct summand of a sum of size-four nilpotent blocks. This separates it from the fixed blocks with first arrow invertible, where $J = 0$ or $J = \mathrm{id}$. The other fixed blocks have first arrow zero and cannot share a nonzero summand with it. For both $\mathcal{Q}$ and $\mathcal{Q}^\vee$, D preserves $\mathrm{im}\, J$. Its induced action on $\mathrm{coker}\, J$ is respectively zero and the identity. Every nonzero direct summand has nonzero $\mathrm{coker}\, J$, since J is nilpotent. Thus these two kinds of coding blocks cannot share a nonzero summand either. All these obstructions already hold after forgetting the $\mathfrak{s}$-action and hence also hold in $\mathcal{D}_\mathfrak{s}$. Krull–Schmidt now separates the $\mathcal{Q}$ summands in any isomorphism of the full diagrams. □

Corollary 23 (Equivariant recovery from an isomorphism)**.** *Under the assumptions of Corollary 21, if $\phi : \mathcal{F}_{\mathfrak{s}}(V, T_1, T_2) \xrightarrow{\sim} \mathcal{F}_{\mathfrak{s}}(V', T_1', T_2')$ is an $\mathfrak{s}$-equivariant Lie isomorphism, the block recovery produces an $\mathfrak{s}$-linear input isomorphism $g_\phi : V \xrightarrow{\sim} V'$ intertwining T_1, T_2.*

Proof. The zero-input case is immediate, so suppose $V, V' \neq 0$. The induced maps $P : M_V \to M_{V'}$ and $\Gamma : k^3 \to k^3$ satisfy $q_{V'} \bigwedge^2 P = \Gamma q_V$, with P equivariant. The ordinary rank and scalar-comparison argument in Theorem 20, equivalently [6, Lemma 7], gives $\Gamma = \gamma \,\mathrm{id}$ for some $\gamma \neq 0$. This conclusion concerns the original Γ; it does not require an equivariant choice of the changes of coordinates used to prove it. Writing $h_i^\sharp : M_V \to M_V^*$ for the three forms, we have

$$P^* h_i'^\sharp P = \gamma h_i^\sharp.$$

Thus P and $\gamma(P^*)^{-1}$ give an isomorphism of the three-arrow diagrams in $\mathcal{D}_{\mathfrak{s}}$. Lemma 22 gives an equivariant diagram isomorphism $\mathcal{Q}(V, T_1, T_2) \cong \mathcal{Q}(V', T_1', T_2')$. Its first-arrow equation identifies its two vertex maps with one invertible map $S : V^{\oplus 4} \to (V')^{\oplus 4}$. The equation $SJ = J'S$ makes it upper triangular Toeplitz:

$$S = \begin{pmatrix} S_0 & S_1 & S_2 & S_3 \\ 0 & S_0 & S_1 & S_2 \\ 0 & 0 & S_0 & S_1 \\ 0 & 0 & 0 & S_0 \end{pmatrix}.$$

Each S_i is $\mathfrak{s}$-linear and S_0 is invertible. Comparing the second and third diagonal blocks in $SD = D'S$ gives $S_0 T_i = T_i' S_0$ for $i = 1, 2$. Take $g_\phi = S_0$. □

Proposition 24 (Orbit criterion for trivial-action objects)**.** *Let $\mathfrak{s}$ be any finite-dimensional Lie algebra. Give a finite-dimensional vector space M the trivial $\mathfrak{s}$-action, and let $K \subseteq \bigwedge^2 M$ have codimension three. The object $M \oplus (\bigwedge^2 M/K)$ of $\mathrm{LieAlg}_{\leq 2}(\mathfrak{s}\text{-Rep})$ is determined up to isomorphism by the $\mathrm{GL}(M)$-orbit of K: for a second such pair (M', K'), the objects $M \oplus (\bigwedge^2 M/K)$ and $M' \oplus (\bigwedge^2 M'/K')$ are isomorphic if and only if some linear isomorphism $P : M \to M'$ satisfies $(\bigwedge^2 P)K = K'$.*

Proof. The derived object is $\bigwedge^2 M/K$ and its quotient in the class-two Lie algebra object is M. An isomorphism therefore induces $P : M \to M'$ with $(\bigwedge^2 P)(K) = K'$, and such a P induces an isomorphism. Since all the $\mathfrak{s}$-actions are trivial, the same condition applies in $\mathfrak{s}$-Rep. □

Corollary 25 (Matrix pairs in every equivariant category)**.** *Let $\mathfrak{s}$ be any finite-dimensional Lie algebra and let (V, T_1, T_2) be a finite-dimensional $k\langle x, y\rangle$-module, where $T_i \in \mathrm{End}_k(V)$ are the actions of x and y. Endowing V with the trivial $\mathfrak{s}$-action makes (V, T_1, T_2) an object of $\mathcal{A}_{\mathfrak{s}}$ in Corollary 21. For $V \neq 0$, its image $\mathcal{F}_{\mathfrak{s}}(V, T_1, T_2)$ has the form of Proposition 24 with $M = M_V$ and $K = K_{T_1,T_2} = \ker q_V$. These objects realize every matrix-pair isomorphism class of positive dimension, and distinct classes give nonisomorphic objects.*

Proof. The first claim is the object formula of Theorem 19 applied through the equivariant square. Corollary 23 lifts any isomorphism of outputs to one of input pairs. □

Proposition 26 (Symmetric-module amplification)**.** *Let $\mathfrak{s}$ be any finite-dimensional Lie algebra and let U be an $\mathfrak{s}$-module with an invariant symmetric bilinear form $\beta : U \otimes U \to k$. For a class-two Lie algebra object L in $\mathfrak{s}$-Rep, write $Q(L) = L/[L, L]$, $D(L) = [L, L]$, and let $b_L : \bigwedge^2 Q(L) \twoheadrightarrow D(L)$ be its induced bracket. The formulas*

$$\mathcal{P}_U(L) = (U \otimes Q(L)) \oplus D(L), \qquad [u \otimes q, v \otimes q']_{\mathcal{P}_U(L)} = \beta(u, v) b_L(q \wedge q'),$$

with $D(L)$ central, and $\mathcal{P}_U(f) = (\mathrm{id}_U \otimes Q(f)) \oplus D(f)$ on a Lie homomorphism f, define an endofunctor $\mathcal{P}_U$ of $\mathrm{LieAlg}_{\leq 2}(\mathfrak{s}\text{-Rep})$.

Proof. The induced bracket b_L is well-defined because $D(L)$ is central. Symmetry of β makes the displayed bracket alternating, and invariance of β and equivariance of b_L make it an $\mathfrak{s}$-morphism. Its image is central, so Jacobi holds. Every Lie homomorphism f induces $Q(f)$ and $D(f)$ with $D(f)b_L = b_{L'}\bigwedge^2 Q(f)$; multiplying this identity by β proves preservation of the bracket. These induced maps respect identities and composition, proving functoriality. □

Corollary 27 (Isomorphism classes under amplification)**.** *In Proposition 26, suppose $\beta \neq 0$ and U is absolutely irreducible. For objects L, L' with trivial $\mathfrak{s}$-action, every isomorphism $\phi : \mathcal{P}_U(L) \xrightarrow{\sim} \mathcal{P}_U(L')$ yields an isomorphism $g_\phi : L \xrightarrow{\sim} L'$. No equality $\mathcal{P}_U(g_\phi) = \phi$ is asserted. Thus $\mathcal{P}_U$ induces an injection on isomorphism classes of objects with trivial $\mathfrak{s}$-action.*

Proof. Suppose L, L' have trivial $\mathfrak{s}$-action and let $\phi : \mathcal{P}_U(L) \xrightarrow{\sim} \mathcal{P}_U(L')$ be arbitrary. Since $\beta \neq 0$, their derived objects are $D(L)$ and $D(L')$. The induced map $U \otimes Q(L) \to U \otimes Q(L')$ is $\mathrm{id}_U \otimes P$ for a linear isomorphism P, by absolute irreducibility. The induced map $H : D(L) \to D(L')$ then satisfies $Hb_L = b_{L'}\bigwedge^2 P$, by evaluating the bracket on a pair u, v with $\beta(u, v) \neq 0$. Linear splittings of the quotient maps $L \to Q(L)$ and $L' \to Q(L')$ identify each object with its quotient-plus-derived presentation, so (P, H) gives an isomorphism $g_\phi : L \xrightarrow{\sim} L'$. Its image under $\mathcal{P}_U$ need not be ϕ. The forward implication on isomorphism classes follows from functoriality. □

Corollary 28 (A matrix-pair family of perfect Lie algebras)**.** *Let $\mathfrak{s}$ be semisimple, and let V_λ be a nontrivial irreducible $\mathfrak{s}$-module with a nondegenerate invariant symmetric form β. Let the subscript* triv *denote the full subgroupoid or subcategory on objects with trivial $\mathfrak{s}$-action. The trivial-action functors in the first row of vertical arrows are equivalences, and the second row consists of inclusions:*

$$\begin{array}{ccc}
\mathrm{Core}(\mathcal{A}) & \xrightarrow{\mathcal{F}_0} & \mathrm{LieAlg}_{\leq 2}(\mathrm{Vect}_k) \\
\downarrow{\scriptstyle \mathrm{triv}} & & \downarrow{\scriptstyle \mathrm{triv}} \\
\mathrm{Core}(\mathcal{A}_\mathfrak{s})_{\mathrm{triv}} & \xrightarrow{\mathcal{F}_\mathfrak{s}|_{\mathrm{triv}}} & \mathrm{LieAlg}_{\leq 2}(\mathfrak{s}\text{-Rep})_{\mathrm{triv}} \\
\downarrow{\scriptstyle \mathrm{incl}} & & \downarrow{\scriptstyle \mathrm{incl}} \\
\mathrm{Core}(\mathcal{A}_\mathfrak{s}) & \xrightarrow{\mathcal{F}_\mathfrak{s}} & \mathrm{LieAlg}_{\leq 2}(\mathfrak{s}\text{-Rep}).
\end{array}$$

Both squares commute. For every $k\langle x, y\rangle$-module (V, T_1, T_2), follow the left vertical arrows to give V the trivial $\mathfrak{s}$-action, apply $\mathcal{F}_\mathfrak{s}$ and then $\mathcal{P}_{V_\lambda}$, and put

$$\mathfrak{g}_{V,T_1,T_2} = \mathfrak{s} \ltimes \mathcal{P}_{V_\lambda}(\mathcal{F}_\mathfrak{s}(V, T_1, T_2)).$$

These Lie algebras are perfect. Two of them are isomorphic by an isomorphism fixing $\mathfrak{s}$ pointwise if and only if their $k\langle x, y\rangle$-modules are isomorphic. For simple $\mathfrak{s}$, its adjoint module with the Killing form is one possible V_λ.

Proof. Giving a vector space the trivial $\mathfrak{s}$-action identifies the top row with the middle row, and restricting $\mathcal{F}_\mathfrak{s}$ to trivial-action inputs gives the indicated trivial-action outputs. Thus the diagram commutes. For any L with trivial $\mathfrak{s}$-action, $(V_\lambda \otimes Q(L))^\mathfrak{s} = 0$. Proposition 4 shows that $\mathfrak{s} \ltimes \mathcal{P}_{V_\lambda}(L)$ is perfect. Its radical is $\mathcal{P}_{V_\lambda}(L)$; hence an isomorphism fixing $\mathfrak{s}$ restricts to an isomorphism of these Lie algebra objects. Corollary 27 and Corollary 23 recover an isomorphism of the input matrix-pair modules. Conversely, an input isomorphism induces one of the displayed semidirect products. □

5 Explicit slices and low-dimensional components

This section computes the groupoids of Sections 3 and 4 in explicit cases: a single isotypic component (Sections 5.1–5.4), the first ternary multiplicity space (Section 5.5), and all objects of total dimension

at most six for semisimple $\mathfrak{s}$, including $\mathfrak{s} = 0$ (Section 5.6). Throughout this section V_λ denotes the irreducible module of highest weight λ for the displayed semisimple algebra. For a named simple factor, ω_i is its ith fundamental weight in the usual Dynkin numbering. An external tensor product $\boxtimes$ records the specified order of simple factors; their permutations are not arrows fixing the acting algebra pointwise.

5.1 One isotypic component with arbitrary multiplicity

The reduction to a single isotypic component is independent of $\mathfrak{sl}_2$. The following statement records exactly what that reduction gives for an arbitrary irreducible type.

Theorem 29 (Parameter spaces for one isotypic component)**.** *Let $\mathfrak{s} = \mathfrak{s}_{\mathrm{ss}} \oplus kt_1 \oplus \cdots \oplus kt_m$, and let $V = V_\lambda$ be an irreducible $\mathfrak{s}_{\mathrm{ss}}$-module. Put $R = V \otimes M$, where the t_j act by commuting D_j on M. Assume R belongs to $\mathcal{C}^0_{\mathfrak{s}}$ (automatically so when V is nontrivial). For each irreducible V_ν set*

$$A_\nu = \mathrm{Hom}_{\mathfrak{s}_{\mathrm{ss}}}(V_\nu, \bigwedge^2 V), \quad B_\nu = \mathrm{Hom}_{\mathfrak{s}_{\mathrm{ss}}}(V_\nu, \mathrm{Sym}^2 V), \quad N_\nu = (A_\nu \otimes \mathrm{Sym}^2 M) \oplus (B_\nu \otimes \bigwedge^2 M).$$

Let $E_{j,\nu}$ act as the restriction of $D_j \otimes \mathrm{id} + \mathrm{id} \otimes D_j$ on each multiplicity factor. The possible K are exactly $K_L = \bigoplus_\nu V_\nu \otimes L_\nu$, with $L_\nu \subseteq N_\nu$ invariant under every $E_{j,\nu}$. Their fixed-R base groupoid is the action groupoid of

$$G_R = \{g \in GL(M) : gD_j = D_j g \text{ for every } j\}$$

on these tuples, with action $\mathrm{id}_{A_\nu} \otimes \mathrm{Sym}^2 g \oplus \mathrm{id}_{B_\nu} \otimes \bigwedge^2 g$ on N_ν.

Proof. The exterior-square identity gives $\bigwedge^2(V \otimes M) = (\bigwedge^2 V \otimes \mathrm{Sym}^2 M) \oplus (\mathrm{Sym}^2 V \otimes \bigwedge^2 M)$. Decomposing its two $\mathfrak{s}_{\mathrm{ss}}$-factors yields the N_ν; stability under t_j is precisely stability under $E_{j,\nu}$. Schur's lemma gives $\mathrm{Aut}_{\mathfrak{s}}(R) = G_R$ and the displayed action on N_ν. Theorem 10 supplies the action groupoid. □

Corollary 30 (Self-loops for one isotypic component)**.** *Under the assumptions and notation of Theorem 29, put $U_L = N_\lambda / L_\lambda$ (zero if $N_\lambda = 0$), with t_j acting by the induced $E_{j,\lambda}$. The full self-loop group is*

$$\mathrm{Hom}_{k[t_1,\dots,t_m]}(M, U_L) \rtimes G_{K_L}, \qquad G_{K_L} = \{g \in G_R : gL_\nu = L_\nu \text{ for all } \nu\}.$$

Proof. Only the V_λ-isotypic quotient of $W_L = \bigwedge^2 R / K_L$ receives an $\mathfrak{s}_{\mathrm{ss}}$-map from R. Intertwining the t_j then identifies the shear maps $R \to W_L$ with $\mathrm{Hom}_{k[t_1,\dots,t_m]}(M, U_L)$. Proposition 11 gives the split self-loop group. □

5.2 Changing the semisimple algebra in a multiplicity-free slice

The labels of the simple summands in $V \otimes V$ belong to the representation theory of $\mathfrak{s}_{\mathrm{ss}}$. An equivalence with a slice for a different semisimple algebra therefore needs a *chosen* matching of those labels. The numbers of summands of each flip parity determine whether such a matching exists, but do not specify one.

Lemma 31 (The signed tensor-square data)**.** *Let $V = V_\lambda$ be a simple module for a semisimple algebra $\mathfrak{a}$, and suppose $V \otimes V$ is multiplicity-free. Denote its set of constituent highest weights by I_λ. The tensor flip acts on the constituent V_α by a sign $\epsilon_\alpha \in \{+, -\}$. Put*

$$I^+_\lambda = \{\alpha : \epsilon_\alpha = +\}, \quad I^-_\lambda = \{\alpha : \epsilon_\alpha = -\}, \qquad r_+(\lambda) = |I^+_\lambda|, \quad r_-(\lambda) = |I^-_\lambda|.$$

For a fixed multiplicity module $(M, D_1, \dots, D_m)$, the subrepresentation parameters in $\bigwedge^2(V \otimes M)$ are one $D_j^{\{2\}}$-invariant subspace of $\bigwedge^2 M$ for every $\alpha \in I^+_\lambda$ and one invariant subspace of $\mathrm{Sym}^2 M$ for

every $\alpha \in I_\lambda^-$. The same group $G = \{g \in GL(M) : gD_j = D_j g \ \forall j\}$ acts diagonally on all these choices, by $\bigwedge^2 g$ and $\mathrm{Sym}^2 g$, respectively. In particular, writing $\mathcal{I}(T)$ for the set of subspaces of T invariant under all the restricted $D_j^{\{2\}}$, the fixed-type base groupoid is

$$\left(\prod_{\alpha \in I_\lambda^-} \mathcal{I}(\mathrm{Sym}^2 M) \times \prod_{\alpha \in I_\lambda^+} \mathcal{I}(\bigwedge^2 M) \right) //G. \tag{23}$$

Proof. The flip commutes with $\mathfrak{a}$, so Schur's lemma makes it a scalar on each multiplicity-one constituent; its square is one. The Cauchy identity $\bigwedge^2(V \otimes M) = (\bigwedge^2 V \otimes \mathrm{Sym}^2 M) \oplus (\mathrm{Sym}^2 V \otimes \bigwedge^2 M)$ proves the assertion, including the interchange of symmetric and exterior powers of M. The central and G actions are those of Theorem 29. □

Theorem 32 (Equivalence across semisimple types)**.** *Let $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{z}$ and $\mathfrak{s}' = \mathfrak{a}' \oplus \mathfrak{z}'$ be reductive, with a chosen identification of their m-dimensional centers. Let $V_{\lambda,\mathfrak{a}}$ and $V_{\mu,\mathfrak{a}'}$ be simple modules whose tensor squares are each multiplicity-free in their respective representation categories. Let $\mathcal{C}^0_{\mathfrak{s},\lambda}$ be the full subgroupoid of $\mathcal{C}^0_{\mathfrak{s}}$ on objects (R, K) for which $R \cong_{\mathfrak{s}} V_{\lambda,\mathfrak{a}} \otimes M$ for some finite-dimensional $k[t_1, \dots, t_m]$-module M, subject to the object condition $\lambda \neq 0$ or $M_{\mathbf{0}} = 0$, where*

$$M_{\mathbf{0}} = \{u \in M : \text{there is an } e \geq 1 \text{ such that } t_j^e u = 0 \text{ for every } j\}.$$

This is the space $M_{0,\mathbf{0}_m}$ of Section 3.1 when $\lambda = 0$. Define $\mathcal{C}^0_{\mathfrak{s}',\mu}$ likewise, using μ and $\mathfrak{a}'$. Choose two bijections of constituent highest weights

$$\theta_+ : I_\lambda^+ \xrightarrow{\sim} I_\mu^+, \qquad \theta_- : I_\lambda^- \xrightarrow{\sim} I_\mu^-, \tag{24}$$

and require the pointed-subset equations

$$\theta_\pm(I_\lambda^\pm \cap \{\lambda\}) = I_\mu^\pm \cap \{\mu\}. \tag{25}$$

Then reindexing the K-components by θ_+ and θ_- gives an equivalence of the full *groupoids*

$$\mathcal{C}^0_{\mathfrak{s},\lambda} \simeq \mathcal{C}^0_{\mathfrak{s}',\mu}.$$

The equivalence retains the multiplicity module M on objects and the intertwining isomorphism $f : M \to M'$ on arrows; both modules vary over the allowed representations. Its construction uses the two chosen bijections. The numerical equalities

$$(r_+(\lambda), r_-(\lambda)) = (r_+(\mu), r_-(\mu)) \tag{26}$$

say exactly when bijections (24) exist; (25) is the additional condition on the chosen bijections for this full-groupoid construction.

Proof. Since $\mathfrak{a}$ is semisimple, $\lambda = 0$ exactly when $V_{\lambda,\mathfrak{a}}$ is one-dimensional, equivalently $I_\lambda^- = \varnothing$; the same holds for μ. The bijection θ_- therefore implies $\lambda = 0$ if and only if $\mu = 0$. Thus the two full subgroupoids admit precisely the same multiplicity modules: all finite-dimensional modules when the weights are nonzero, and those with $M_{\mathbf{0}} = 0$ when they are zero. For any object, the multiplicity module $M = \mathrm{Hom}_{\mathfrak{a}}(V_{\lambda,\mathfrak{a}}, R)$ has its induced $k[t_1, \dots, t_m]$-action, and evaluation identifies R with $V_{\lambda,\mathfrak{a}} \otimes M$.

Multiplicity-freeness makes every constituent multiplicity space in $V_{\lambda,\mathfrak{a}}^{\otimes 2}$ and $V_{\mu,\mathfrak{a}'}^{\otimes 2}$ a line. Choose an isomorphism of each such line with k. For each allowed M this presents the K-parameters as in (23). A plus slot uses $\bigwedge^2 M$, and a minus slot uses $\mathrm{Sym}^2 M$. The two bijections reindex these slots for every M.

An arrow between objects over M and M' has an underlying $k[t_1,\dots,t_m]$-module isomorphism $f : M \to M'$. The maps $\bigwedge^2 f$ and $\mathrm{Sym}^2 f$ act in the corresponding slots on both sides, so the reindexing carries base arrows bijectively to base arrows and respects their composition. If the input weight λ belongs to a sign set, the shear fiber at (M,K) is

$$\mathrm{Hom}_{k[t_1,\dots,t_m]}(M, T_\pm/L_\lambda), \qquad T_+ = \bigwedge^2 M, \quad T_- = \mathrm{Sym}^2 M, \tag{27}$$

with the sign of that set; otherwise the fiber is zero. Equation (25) makes the corresponding fiber on the μ side the same quotient construction, including the induced t_j actions. It also matches transport of a shear along f, namely $H \mapsto \bar{f} H f^{-1}$. The composition law (11) therefore agrees for every arrow, giving a fully faithful, essentially surjective functor of the full groupoids. The line identifications specify this functor. □

Proposition 33 (A rectangular $\mathfrak{sl}_2$ comparison)**.** *Let $V_a \boxtimes V_b$ be the simple module for $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, where $a, b \geq 0$, and put $h = (a+1)(b+1) - 1$. The simple constituents of its tensor square are indexed by (i,j) with $0 \leq i \leq a$, $0 \leq j \leq b$:*

$$V_{2a-2i} \boxtimes V_{2b-2j}, \qquad \epsilon_{i,j} = (-1)^{i+j}.$$

Those of $V_h \otimes V_h$ for one $\mathfrak{sl}_2$ are V_{2h-2r}, $0 \leq r \leq h$, with flip sign $(-1)^r$. The chosen *bijection*

$$r(i,j) = i(b+1) + \begin{cases} j, & i \text{ even}, \\ b-j, & i \text{ odd} \end{cases} \tag{28}$$

preserves these signs. It also matches the input constituents: they are absent on both sides unless a, b are both even; in that case $(a/2, b/2)$ maps to $r = h/2$. Thus (28) yields an equivalence of the full *groupoids $\mathcal{C}^0_{\mathfrak{s},(a,b)}$ and $\mathcal{C}^0_{\mathfrak{s}',h}$ of Theorem 32, where $\mathfrak{s} = (\mathfrak{sl}_2 \oplus \mathfrak{sl}_2) \oplus \mathfrak{z}$ and $\mathfrak{s}' = \mathfrak{sl}_2 \oplus \mathfrak{z}'$ have identified centers.*

Proof. Clebsch–Gordan gives the displayed rectangular index set. Interchanging the two copies of $V_a \boxtimes V_b$ interchanges the two V_a factors and the two V_b factors, so the signs on the two Clebsch–Gordan summands are $(-1)^i$ and $(-1)^j$, respectively, so their product is $(-1)^{i+j}$. Indeed the summand indexed by i is obtained by i contractions with the alternating form on the standard two-dimensional module; exchanging the tensor factors changes the sign of each contraction. Equation (28) lists the $b+1$ entries in each successive row alternately forward and backward. Its values are exactly $0,\dots,(a+1)(b+1)-1$, without repetition; in either kind of row $r(i,j) \equiv i + j \pmod 2$. An input constituent requires $2a - 2i = a$ and $2b - 2j = b$, so it occurs exactly when both a, b are even, at $(i,j) = (a/2, b/2)$. Then h is even and direct substitution gives $r(a/2, b/2) = h/2$, the input index on the other side. If at least one of a, b is odd, h is odd and neither input type occurs. Theorem 32 now applies, including its shear identification. □

For instance $a = b = 1$ gives $h = 3$ and the ordered indices $(0,0),(0,1),(1,1),(1,0)$ with signs $+,-,+,-$. For $a = b = 2$ one has $h = 8$ and the input index $(1,1) \mapsto 4$, illustrating the pointed condition when shears can occur.

Remark 34 (Counts can miss the shear constituent)**.** *Take no central factors and $M = k$. For $\mathfrak{sl}_2$, the module V_3 has $\mathrm{Sym}^2 V_3 = V_6 \oplus V_2$ and $\bigwedge^2 V_3 = V_4 \oplus V_0$; V_3 is absent. For the seven-dimensional simple G_2-module V_{ω_1}, one has $\mathrm{Sym}^2 V_{\omega_1} = V_{2\omega_1} \oplus V_0$ and $\bigwedge^2 V_{\omega_1} = V_{\omega_2} \oplus V_{\omega_1}$. Both signed counts are $(2,2)$, so their base groupoids are equivalent after a choice of signed bijection. At $K = 0$, however, the first full self-loop group is $k^\times$, whereas the second contains the nonzero shear group $\mathrm{Hom}_{\mathfrak{g}_2}(V_{\omega_1}, V_{\omega_1}) = k$ and is $k \rtimes k^\times$. In particular these full groupoids are not equivalent: all self-loop groups in the first slice are abelian, while the second slice has a nonabelian one.*

Theorem 35 (Stembridge: the tensor-square condition)**.** *Let $\mathfrak{a}$ be simple and λ a dominant weight. If $V_\lambda \otimes V_\lambda$ is multiplicity-free, then $\lambda = 0$ or $\lambda = d\omega_i$ for one fundamental weight ω_i and an integer $d \geq 1$. In type A_n this condition is also sufficient: the complete list is $\lambda = 0$ and $\lambda = d\omega_i$, where $1 \leq i \leq n$ and $d \geq 1$. For the remaining simple types the admissible pairs (i, d) are subject to the type and node restrictions in Stembridge's classification; the condition $\lambda = d\omega_i$ alone is not sufficient in general.*

Proof. This is the tensor-square specialization of [7, Theorem 1.1]. □

Corollary 36 (Semisimple tensor squares)**.** *Let $\mathfrak{a} = \bigoplus_{h=1}^{q} \mathfrak{a}_h$ be semisimple, where the $\mathfrak{a}_h$ are simple, and write $V_\lambda = \boxtimes_{h=1}^{q} V_{\lambda_h}$. Then $V_\lambda \otimes V_\lambda$ is multiplicity-free if and only if every $V_{\lambda_h} \otimes V_{\lambda_h}$ is multiplicity-free. This includes $q = 0$, for which $V_\lambda = k$.*

Proof. There is an isomorphism $V_\lambda \otimes V_\lambda \cong \boxtimes_{h=1}^{q} (V_{\lambda_h} \otimes V_{\lambda_h})$. The simple summands of an external tensor product are indexed by tuples of simple summands of its factors, and their multiplicities are the products of the corresponding multiplicities. Each factor has at least one summand, so a multiplicity exceeds one in the product exactly when one does in a factor. The empty product is the one-dimensional trivial module. □

5.3 The explicit $\mathfrak{sl}_2$ tensor squares

Let $\mathfrak{s}_{\mathrm{ss}} = \mathfrak{sl}_2$ and retain the m central generators t_j. Fix $n \geq 1$ and an ℓ-dimensional multiplicity space M carrying arbitrary commuting operators $D_1, \ldots, D_m$; thus $R = V_n \otimes M$, with $V_n = \mathrm{Sym}^n V_1$ for the standard two-dimensional $\mathfrak{sl}_2$-module V_1. The integer n denotes the highest weight, leaving m as the number of central generators. This is a single $\mathfrak{sl}_2$-isotypic component, with no restriction on ℓ or on the Jordan parts of the D_j. Since $n \geq 1$, it satisfies $M_{0,\mathbf{0}_m} = 0$. Write $C_\nu = \mathrm{Hom}_{\mathfrak{sl}_2}(V_\nu, V_n \otimes V_n)$.

Lemma 37 (Clebsch–Gordan reduction for $V_n \otimes M$)**.** *For $0 \leq i \leq n$ put $\nu_i = 2n - 2i$. The complete list of Clebsch–Gordan multiplicities and flip eigenspaces is*

$$\dim C_{\nu_i} = 1, \qquad (\dim A_{\nu_i,n}, \dim B_{\nu_i,n}) = \begin{cases} (1,0) & i \text{ odd}, \\ (0,1) & i \text{ even}, \end{cases} \tag{29}$$

with all three spaces zero for other ν. In particular,

$$N_{\nu_i} = \begin{cases} \mathrm{Sym}^2 M, & i \text{ odd}, \\ \bigwedge^2 M, & i \text{ even}, \end{cases} \qquad \dim N_{\nu_i} = \begin{cases} \binom{\ell+1}{2}, & i \text{ odd}, \\ \binom{\ell}{2}, & i \text{ even}. \end{cases} \tag{30}$$

Proof. Realize $V_n \otimes V_n$ as polynomials homogeneous of degree n in each of (x_1, y_1) and (x_2, y_2). The vectors

$$(x_1 y_2 - y_1 x_2)^i x_1^{n-i} x_2^{n-i} \qquad (0 \leq i \leq n)$$

are highest-weight vectors of distinct weights $2n-2i$. Their irreducible summands have total dimension $\sum_{i=0}^{n} (2n - 2i + 1) = (n+1)^2$, so they exhaust $V_n \otimes V_n$. The tensor flip sends the ith vector to $(-1)^i$ times itself, proving (29). The wedge identity then gives (30). □

Theorem 38 (Fixed-R groupoid for one $\mathfrak{sl}_2$ type)**.** *Write $D_j^{\{2\}} = D_j \otimes \mathrm{id}_M + \mathrm{id}_M \otimes D_j$. Let $\mathcal{P}(R)$ be the set of all tuples $(Q_0, \ldots, Q_n)$, where Q_i is a $D_j^{\{2\}}$-invariant subspace of the space N_{ν_i} in (30) for every j. Equivalently, $\mathcal{P}(R)$ is the product, over i, of the unions of the corresponding invariant Grassmannians of all dimensions; by (31) below it is identified with the set $\mathcal{K}(R)$ of Theorem 17. Then every K is uniquely*

$$K_Q = \bigoplus_{i=0}^{n} V_{\nu_i} \otimes Q_i, \qquad W_Q = \bigoplus_{i=0}^{n} V_{\nu_i} \otimes (N_{\nu_i}/Q_i). \tag{31}$$

The group and its action on these tuples are completely explicit:

$$G_R = \{g \in GL(M) : gD_j = D_j g \text{ for all } j\}, \qquad gQ_i = \begin{cases} (\mathrm{Sym}^2 g)Q_i, & i \text{ odd}, \\ (\bigwedge^2 g)Q_i, & i \text{ even}. \end{cases} \tag{32}$$

Thus the full fixed-R base groupoid is $\mathcal{P}(R)//G_R$, its orbit set is $\mathcal{P}(R)/G_R$, and $G_{K_Q} = \{g \in G_R : gQ_i = Q_i \text{ for every } i\}$.

Proof. The t_j act on the spaces in (30) by the restrictions of $D_j^{\{2\}}$. Since the V_{ν_i} are pairwise distinct, their submodules are precisely (31) with the stated invariance conditions. The commutant of the $\mathfrak{sl}_2$-action on R is $GL(M)$; requiring commutation with all t_j gives (32). Theorem 10 identifies the base arrows and orbits. □

Proposition 39 (Shears for one $\mathfrak{sl}_2$ type)**.** *The shear group is also determined by these data. If n is odd, $\mathrm{Hom}_{\mathfrak{s}}(R, W_Q) = 0$. If n is even, set $i_* = n/2$ and let $U_Q = N_{\nu_{i_*}}/Q_{i_*}$, equipped with the induced $D_j^{\{2\}}$-actions. Then*

$$\mathrm{Hom}_{\mathfrak{s}}(R, W_Q) \cong \mathrm{Hom}_{k[t_1,\dots,t_m]}(M, U_Q), \tag{33}$$

where t_j acts by D_j on M and by the induced $D_j^{\{2\}}$ on U_Q. At each tuple Q the full self-loop group is the semidirect product of this additive shear group with G_{K_Q}; g acts on a shear H by $H \mapsto \bar{g}Hg^{-1}$, where $\bar{g}$ is induced by $\mathrm{Sym}^2 g$ or $\bigwedge^2 g$ on U_Q. In particular the shear dimension is the nullity of the explicit linear map

$$\mathrm{Hom}_k(M, U_Q) \longrightarrow \bigoplus_{j=1}^{m} \mathrm{Hom}_k(M, U_Q), \qquad F \longmapsto (FD_j - \widehat{D}_j F)_j,$$

where $\widehat{D}_j$ is the induced action on U_Q. These data present the local groupoid as $\mathcal{H}_R \rtimes (\mathcal{P}(R)//G_R)$ for every n, ℓ and every commuting tuple (D_j); a concrete orbit calculation requires further analysis of this action.

Proof. A V_n constituent occurs in W_Q only if $2n - 2i = n$, or $i = n/2$, which requires n even. Its multiplicity module is U_Q, and the remaining intertwining condition is exactly (33). The action on shears is that of Theorem 10, completing the description. □

5.4 Two-dimensional multiplicity spaces

Throughout this subsection keep the setting of Section 5.3 with $\ell = 2$, and write M_2 for the two-dimensional multiplicity space, so that $R = V_n \otimes M_2$. The subscript records the dimension, not a highest weight as in M_λ.

Lemma 40 (Binary exterior-square choices)**.** *Suppose all $D_j = \mu_j \mathrm{id}_{M_2}$. For $R = V_n \otimes M_2$, each even index $i \in \{0, \dots, n\}$ has $N_{\nu_i} = \bigwedge^2 M_2$, a one-dimensional space. Its subspace Q_i is independently either 0 or $\bigwedge^2 M_2$. Thus every orbit configuration of the odd-indexed Q_i occurs in $2^{\lfloor n/2 \rfloor + 1}$ distinct copies. These choices do not affect its stabilizer in $G_R = GL(M_2)$. For $n = 1, 2, 3$, they do not affect the full self-loop group either.*

Proof. Lemma 37 gives one $\bigwedge^2 M_2$ at each even index. Both its subspaces are stable under the scalar $D_j^{\{2\}}$ and fixed by every $g \in GL(M_2)$. There are $\lfloor n/2 \rfloor + 1$ even indices. For $n = 1, 3$ the shear group vanishes; for $n = 2$ it depends only on $N_2/Q_1 = \mathrm{Sym}^2 M_2/Q_1$. Proposition 11, with the shear groups of Proposition 39, gives the claim for full loops. □

Lemma 41 (The intrinsic pole of a plane)**.** *For the two-dimensional space M_2, the discriminant is a canonical quadratic map $\Delta : \operatorname{Sym}^2 M_2 \longrightarrow (\det M_2)^{\otimes 2}$. Use the polarization*

$$\Delta(p,q) = \Delta(p+q) - \Delta(p) - \Delta(q), \qquad \Delta(p,p) = 2\Delta(p).$$

It induces an isomorphism

$$\psi : \operatorname{Sym}^2 M_2 \xrightarrow{\sim} (\operatorname{Sym}^2 M_2)^* \otimes (\det M_2)^{\otimes 2}, \qquad \psi(p)(q) = \Delta(p,q).$$

For a plane $U = \ker f \subset \operatorname{Sym}^2 M_2$, where $0 \neq f \in (\operatorname{Sym}^2 M_2)^*$, define its pole line *intrinsically by*

$$\varpi_U = \psi^{-1}\big(kf \otimes (\det M_2)^{\otimes 2}\big) \subset \operatorname{Sym}^2 M_2.$$

It is independent of the choice of f defining U. If $0 \neq p_f \in \varpi_U$, then $U = \{q : \Delta(p_f, q) = 0\}$, and $\varpi_{gU} = g\varpi_U$ for every $g \in GL(M_2)$. Neither an inner product nor a trivialization of $\det M_2$ is involved.

Proof. For $p \in \operatorname{Sym}^2 M_2$ let $\widetilde{p} : M_2^* \to M_2$ be its associated symmetric linear map. The determinant of this map belongs canonically to $\operatorname{Hom}(\det M_2^*, \det M_2) = (\det M_2)^{\otimes 2}$; put $\Delta(p) = -4 \det \widetilde{p}$. Choose any basis x, y of M_2 with dual e_1, e_2. If $p = ax^2 + bxy + cy^2$ and $\delta = (x \wedge y)^2$, then

$$\Delta(p) = (b^2 - 4ac)\delta, \qquad \psi(p) = (-4ce_1^2 + 4be_1e_2 - 4ae_2^2) \otimes \delta.$$

Here the natural symmetric-tensor pairing satisfies $e_1e_2(xy) = 1/2$. The displayed formula shows that ψ is invertible. Since $(\det M_2)^{\otimes 2}$ is one-dimensional, $kf \otimes (\det M_2)^{\otimes 2}$ is a line; its inverse image is ϖ_U. The equation $\psi(p_f) \in kf \otimes (\det M_2)^{\otimes 2}$ gives $\ker f = \ker \Delta(p_f, -)$ without identifying the target line with k. Finally $\Delta(gp) = (\det g)^2 \Delta(p)$, so its polarization and the construction of ϖ_U are $GL(M_2)$-equivariant. □

Lemma 42 (Divisors and the discriminant parameter)**.** *Each line $kp \subseteq \operatorname{Sym}^2 M_2$ has the effective zero divisor of p on $\mathbb{P}(M_2^*)$. A plane U has the zero divisor of any $p \in \varpi_U \setminus \{0\}$, with ϖ_U as in Lemma 41. Thus both kinds of subspaces give degree-two divisors on the same projective line. For two divisors with distinct roots, represented by nonzero $p, q \in \operatorname{Sym}^2 M_2$, the scalar*

$$\eta([p],[q]) = \frac{\Delta(p,q)^2}{\Delta(p)\Delta(q)} \in k \tag{34}$$

does not depend on their representatives or on an ordering of their roots. Equal divisors and divisors sharing exactly one root both have $\eta = 4$; they are separate incidence types. The divisors are disjoint exactly when $\eta \neq 4$. On the disjoint locus, η is a complete invariant of the ordered pair under $PGL(M_2)$; $\eta = 0$ is the harmonic locus.

Proof. The pole construction is equivariant by Lemma 41. Scaling p and q cancels from the ratio, and numerator and denominator both lie in $(\det M_2)^{\otimes 4}$, so their quotient is a scalar without any trivialization. The expression is symmetric in p, q. For the incidence claims, use an arbitrary basis to put $p = xy$, $q = ax^2 + bxy + cy^2$. Then, suppressing the common determinant factors only in this calculation,

$$\Delta(p) = 1, \quad \Delta(p,q) = 2b, \quad \Delta(q) = b^2 - 4ac, \quad 4\Delta(p)\Delta(q) - \Delta(p,q)^2 = -16ac.$$

The roots of p are $[x = 0]$ and $[y = 0]$; q vanishes at one of them exactly when $ac = 0$. For distinct-root q, equality of divisors is the case $a = c = 0$. When $ac \neq 0$, the span of p, q is nondegenerate for Δ. Two indexed pairs with the same η can be rescaled to have the same Gram matrix, and an isometry between their spans extends to $\operatorname{Sym}^2 M_2$; a reflection on the orthogonal complement adjusts its determinant. The resulting projective orthogonal transformation comes from $PGL(M_2)$. Over algebraically closed k, varying $4b^2/(b^2 - 4ac)$ with $ac \neq 0$ attains every value except 4. This proves completeness and the harmonic assertion. □

Lemma 43 (The four typed formulas for η)**.** *Put $V = \operatorname{Sym}^2 M_2$ and $H = (\det M_2)^{\otimes 2}$. The inverse of $\psi \otimes \mathrm{id}_{H^{-1}} : V \otimes H^{-1} \to V^*$ is a map $\gamma : V^* \to V \otimes H^{-1}$. Define* separate *maps on this twisted space by*

$$\Delta_{\mathrm{tw}} : V \otimes H^{-1} \longrightarrow H^{-1}, \qquad \Delta_{\mathrm{tw}}(p \otimes \lambda) = \Delta(p) \otimes \lambda^{\otimes 2},$$
$$\Delta_{\mathrm{tw}} : (V \otimes H^{-1})^2 \longrightarrow H^{-1}, \quad \Delta_{\mathrm{tw}}(p \otimes \lambda, q \otimes \mu) = \Delta(p, q) \otimes \lambda \otimes \mu.$$

The targets use the canonical contraction $H \otimes H^{-2} \cong H^{-1}$. The first map is quadratic and the second is bilinear; since H^{-1} is a line, every element of $V \otimes H^{-1}$ is a pure tensor, and the formulas are unchanged by rescaling its two factors inversely. Now define the dual-domain maps

$$\Delta^\vee = \Delta_{\mathrm{tw}} \circ \gamma : V^* \to H^{-1}, \qquad \Delta^\vee(g, h) = \Delta_{\mathrm{tw}}(\gamma(g), \gamma(h)) \quad (g, h \in V^*).$$

Let $0 \neq f_1, f_2 \in V$ and $0 \neq g_1, g_2 \in V^*$ represent lines $\mathsf{L}(f_i)$ and planes $\mathsf{D}(g_i)$. Whenever both associated divisors have distinct roots, the parameter of Lemma 42 is

$$\eta(\mathsf{L}(f_1), \mathsf{L}(f_2)) = \frac{\Delta(f_1, f_2)^2}{\Delta(f_1)\Delta(f_2)},$$
$$\eta(\mathsf{D}(g_1), \mathsf{L}(f_2)) = \frac{g_1(f_2)^2}{\Delta^\vee(g_1)\Delta(f_2)},$$
$$\eta(\mathsf{L}(f_1), \mathsf{D}(g_2)) = \frac{g_2(f_1)^2}{\Delta(f_1)\Delta^\vee(g_2)},$$
$$\eta(\mathsf{D}(g_1), \mathsf{D}(g_2)) = \frac{\Delta^\vee(g_1, g_2)^2}{\Delta^\vee(g_1)\Delta^\vee(g_2)}.$$

Every quotient is a scalar: the mixed denominators lie in $H^{-1} \otimes H = k$, and the two other ratios cancel their respective determinant twists. In particular $g_i(f_j)$ is the natural evaluation $V^* \times V \to k$.

Proof. For $g \in V^*$ choose a nonzero $d \in H$ and write $\gamma(g) = p \otimes d^{-1}$. Then $\psi(p) = g \otimes d$, so the pole line of $\mathsf{D}(g)$ is kp. Moreover $\Delta^\vee(g) = \Delta(p)d^{-2}$ and $g(f) = \Delta(p, f)d^{-1}$. Substitution in the line–line formula cancels d and gives both mixed formulas. With two functionals, write $\gamma(g_i) = p_i \otimes d_i^{-1}$; then $\Delta^\vee(g_1, g_2) = \Delta(p_1, p_2)d_1^{-1}d_2^{-1}$, which gives the last formula. The calculation is independent of the chosen d_i and of scaling the representatives. □

Lemma 44 (Stabilizers of ordered divisor pairs)**.** *For two doubled divisors, equal roots have projective point stabilizer and distinct roots have two-point stabilizer. For a doubled divisor and a two-root divisor, the stabilizer is a two-point stabilizer when the double root is on the two-root divisor, and C_2 when it is off it. For two two-root divisors, the stabilizer is an unordered-pair stabilizer when they are the same divisor, trivial when they share one root, C_2 when disjoint with $\eta \neq 0$, and V_4 when disjoint with $\eta = 0$. The same list applies whichever slot is a line or a plane, using the pole line of a plane.*

Proof. The projective action on degree-two divisors is the standard $PGL(M_2)$-action on unordered pairs of points of $\mathbb{P}(M_2^*)$. Fixing one or two points gives the first claims. A point off an unordered pair leaves its transposition, while three distinct marked points have trivial stabilizer. For disjoint two-root divisors, their nondegenerate span in $\operatorname{Sym}^2 M_2$ has projective stabilizer C_2 if $\Delta(p, q) \neq 0$: the normalized quadratics have the same sign. If $\Delta(p, q) = 0$, their signs vary independently, giving V_4. Lemma 42 identifies the latter case with $\eta = 0$. The polarity is equivariant, so a plane slot has the same stabilizer as its pole divisor. □

Notation for representatives. For the following tables alone, choose a basis e_1, e_2 of M_2^* and write $x = e_1^*$, $y = e_2^*$ for its dual basis in M_2. Thus $x^2 = e_1^* e_1^*$ and $xy = e_1^* e_2^*$ are elements of $\operatorname{Sym}^2 M_2$. For a nonzero quadratic $p \in \operatorname{Sym}^2 M_2$ and a nonzero $f \in \operatorname{Sym}^2 M_2^*$, write

$$\mathsf{L}(p) = kp, \qquad \mathsf{D}(f) = (k\langle f\rangle)^{\perp} \subseteq \operatorname{Sym}^2 M_2.$$

The perpendicular is the annihilator under the natural $\operatorname{Sym}^2 M_2^*$–$\operatorname{Sym}^2 M_2$ pairing, so each $\mathsf{D}(f)$ is a plane dual to the displayed line $k\langle f\rangle$. For example, $\mathsf{D}(e_2^2) = \langle x^2, xy\rangle$ and $\mathsf{D}(e_1 e_2) = \langle x^2, y^2\rangle$. These basis-dependent representatives do not enter the classification or its invariants.

Theorem 45 (The discrete $V_1 \otimes M_2$ slice)**.** *Let every central generator act on M_2 as $D_j = \mu_j \operatorname{id}_{M_2}$, with arbitrary $\mu_j \in k$. For $R = V_1 \otimes M_2$, every K is uniquely*

$$K = (V_2 \otimes L) \oplus (V_0 \otimes U), \qquad L \in \{0, \textstyle\bigwedge^2 M_2\}, \quad U \subseteq \operatorname{Sym}^2 M_2.$$

The following table gives all six $GL(M_2)$-orbits of U. Each row represents 2^1 distinct fixed-R isomorphism classes as L varies, yielding twelve classes in total. The indicated stabilizer is the full self-loop group for every scalar tuple (μ_j). Write $B \subseteq GL(M_2)$ for the stabilizer of a line in M_2, and $N(T)$ for the stabilizer of an unordered pair of distinct lines in M_2, where T stabilizes each line separately. The discriminant polarity identifies a plane U with its pole in $\mathbb{P}(\operatorname{Sym}^2 M_2)$.

$\dim U$	*orbit of U*	*representative U*	$\operatorname{Aut}_{\mathfrak{s}}(\mathfrak{g}_K)$
0	$U = 0$	0	$GL(M_2)$
1	*double-root line*	$\mathsf{L}(x^2)$	B
1	*distinct-root line*	$\mathsf{L}(xy)$	$N(T)$
2	*tangent plane (double-root pole)*	$\mathsf{D}(e_2^2)$	B
2	*secant plane (distinct-root pole)*	$\mathsf{D}(e_1 e_2)$	$N(T)$
3	$U = \operatorname{Sym}^2 M_2$	$\operatorname{Sym}^2 M_2$	$GL(M_2)$

Proof. Lemma 37 gives $\bigwedge^2 R \cong (V_2 \otimes \bigwedge^2 M_2) \oplus (V_0 \otimes \operatorname{Sym}^2 M_2)$. The scalar actions induce $2\mu_j \operatorname{id}$ on both multiplicity spaces, so all subspaces are stable and $G_R = GL(M_2)$. The line $\bigwedge^2 M_2$ has exactly two subspaces, each fixed by $GL(M_2)$. The discriminant polarity identifies lines and planes in $\operatorname{Sym}^2 M_2$ equivariantly; each has two orbits, according to whether its quadratic has a double root or two distinct roots. The stabilizers are respectively B and $N(T)$. The zero and full subspaces are fixed by $GL(M_2)$. Proposition 39 gives no shear since $n = 1$ is odd; Theorem 10 makes the listed groups full. □

Proposition 46 (Shears for scalar central actions on $V_2 \otimes M_2$)**.** *Let $R = V_2 \otimes M_2$, and let the central generators act on M_2 by $D_j = \mu_j \operatorname{id}_{M_2}$, with arbitrary scalars $\mu_j \in k$. For*

$$Q = (Q_0, Q_1, Q_2), \qquad Q_0, Q_2 \subseteq \textstyle\bigwedge^2 M_2, \quad Q_1 \subseteq \operatorname{Sym}^2 M_2$$

put $U_Q = \operatorname{Sym}^2 M_2 / Q_1$. Regard M_2 and U_Q as modules over $A = k[t_1, \dots, t_m]$, with t_j acting on them by $\mu_j \operatorname{id}_{M_2}$ and $2\mu_j \operatorname{id}_{U_Q}$, respectively. Then the additive group of all shears at (R, K_Q) is

$$\mathcal{H}_Q = \operatorname{Hom}_A(M_2, U_Q) = \{F \in \operatorname{Hom}_k(M_2, U_Q) : F(\mu_j u) = 2\mu_j F(u) \text{ for every } j, u\}. \tag{35}$$

The full self-loop group is $\mathcal{H}_Q \rtimes G_{K_Q}$, with $g \in G_{K_Q}$ acting by $F \mapsto \bar{g} F g^{-1}$, where $\bar{g}$ is induced by $\operatorname{Sym}^2 g$ on U_Q. Using the V_2-isotypic summand $V_2 \otimes U_Q \subseteq W_Q$ from Lemma 37, each $F \in \mathcal{H}_Q$ is the automorphism

$$S_F(x, v \otimes u, w) = \big(x, v \otimes u, w + \iota_Q(v \otimes F(u))\big) \tag{36}$$

on decomposable vectors, extended linearly. Here ι_Q includes that summand in W_Q. The shears satisfy $S_F^{-1} = S_{-F}$ and $S_F S_{F'} = S_{F+F'}$.

Proof. The V_2-isotypic component of W_Q is $V_2 \otimes U_Q$, since the other two components have types V_4 and V_0. An $\mathfrak{sl}_2$-intertwiner $R \to W_Q$ is therefore uniquely given by a linear map $F : M_2 \to U_Q$ under the fixed isotypic identification. It intertwines every t_j exactly when $F(\mu_j u) = 2\mu_j F(u)$, proving (35). Because W_Q is central in $R \oplus W_Q$, formula (36) preserves the bracket of two elements of R; the module condition preserves brackets with $\mathfrak{s}$. Its inverse and composition follow by addition. The factorization and conjugation action are Proposition 11. □

Theorem 47 (The twenty-four discrete $V_2 \otimes M_2$ classes)**.** *Keep $D_j = \mu_j \,\mathrm{id}_{M_2}$ for arbitrary $\mu_j \in k$. For $R = V_2 \otimes M_2$, write*

$$K = (V_4 \otimes L_0) \oplus (V_2 \otimes U) \oplus (V_0 \otimes L_2), \quad L_0, L_2 \in \{0, \textstyle\bigwedge^2 M_2\}, \quad U \subseteq \operatorname{Sym}^2 M_2.$$

The two $\bigwedge^2 M_2$ summands contribute 2^2 distinct choices of (L_0, L_2) for every orbit type of U. They do not change the stabilizer or the shear group. The table lists the six orbit types of U once each; each row therefore represents four distinct isomorphism classes, giving twenty-four classes in total. The labels $\mathrm{d}, \mathrm{e}, \mathrm{t}, \mathrm{s}$ denote a double-root line, a distinct-root line, a tangent plane, and a secant plane in $\operatorname{Sym}^2 M_2$, respectively. The column h is the dimension of the shear group when all $\mu_j = 0$.

U	$\dim U$	*representative* U	G_K	h
0	0	0	$GL(M_2)$	6
d	1	$\mathsf{L}(x^2)$	B	4
e	1	$\mathsf{L}(xy)$	$N(T)$	4
t	2	$\mathsf{D}(e_2^2)$	B	2
s	2	$\mathsf{D}(e_1 e_2)$	$N(T)$	2
$\operatorname{Sym}^2 M_2$	3	$\operatorname{Sym}^2 M_2$	$GL(M_2)$	0

For at least one $\mu_j \neq 0$, the full self-loop group in every row is G_K.

Proof. Lemma 37 gives the displayed decomposition; again $G_R = GL(M_2)$ and all its multiplicity subspaces are invariant under the scalar $D_j^{\{2\}} = 2\mu_j \,\mathrm{id}$. The two $\bigwedge^2 M_2$ summands each contribute a fixed binary choice. The six orbits of U and their stabilizers were proved in Theorem 45. This proves the $2^2 \cdot 6 = 24$ distinct classes. Proposition 46 identifies the shears with $\operatorname{Hom}_{k[t_1,\dots,t_m]}(M_2, \operatorname{Sym}^2 M_2/U)$. If some $\mu_j \neq 0$, its intertwining equation $\mu_j F = 2\mu_j F$ forces $F = 0$. □

Corollary 48 (Scalar weights and the shears in the table)**.** *For the arbitrary scalar tuple (μ_j) of Theorem 47, the shear group in each row is the polynomial-module Hom space in (35), with $U_Q = \operatorname{Sym}^2 M_2/U$. If at least one $\mu_j \neq 0$, it is zero. If every $\mu_j = 0$, the full self-loop group is*

$$\operatorname{Hom}_k(M_2, \operatorname{Sym}^2 M_2/U) \rtimes G_K, \tag{37}$$

and its additive factor has dimension $h = 2(3 - \dim U)$, as recorded in the table. In particular, the rows with $U = 0$ have a six-dimensional group of nonidentity shears, explicitly given by (36).

Proof. The module-intertwining equation in (35) is $\mu_j F = 2\mu_j F$ for every j. A nonzero μ_j forces $F = 0$; when all are zero, every linear $F : M_2 \to \operatorname{Sym}^2 M_2/U$ intertwines. Its dimension is $2(3 - \dim U)$, and Proposition 46 supplies the stated action and explicit automorphisms. □

Proposition 49 (Parameter space for $V_3 \otimes M_2$)**.** *Take $n = 3$ and $D_j = \mu_j \,\mathrm{id}_{M_2}$ for all central generators, where $\mu_j \in k$ are arbitrary, including the case $m = 0$. Write*

$$\mathcal{S}(M_2) = \{0\} \sqcup \mathbb{P}(\operatorname{Sym}^2 M_2) \sqcup \mathbb{P}((\operatorname{Sym}^2 M_2)^*) \sqcup \{\operatorname{Sym}^2 M_2\}.$$

The two projective planes parameterize respectively lines and planes $U \subseteq \operatorname{Sym}^2 M_2$. The complete parameter space of objects with this fixed R is the disjoint union of standard algebraic varieties

$$\mathcal{P}(R) = \{0, \bigwedge^2 M_2\}^2 \times \mathcal{S}(M_2)^2. \tag{38}$$

Its points (L_0, U_1, L_2, U_3) give exactly

$$K = (V_6 \otimes L_0) \oplus (V_4 \otimes U_1) \oplus (V_2 \otimes L_2) \oplus (V_0 \otimes U_3). \tag{39}$$

Every orbit of the pair (U_1, U_3) occurs in 2^2 distinct choices of (L_0, L_2), all fixed separately by $GL(M_2)$. There is no shear at any object, so the full group of self-loops is the stabilizer of the ordered pair (U_1, U_3) in $GL(M_2)$.

Proof. Lemma 37 gives $N_6 = N_2 = \bigwedge^2 M_2$ and $N_4 = N_0 = \operatorname{Sym}^2 M_2$. These have dimensions one and three, respectively. All D_j are scalar, so $D_j^{\{2\}} = 2\mu_j \,\mathrm{id}$ on every N_{ν_i}; every subspace is invariant and $G_R = GL(M_2)$. Theorem 38 gives (38) and (39). Since $n = 3$ is odd, Proposition 39 gives zero shear. The groupoid factorization gives the claimed full self-loop group. □

Proposition 50 (Extreme slots in the $V_3 \otimes M_2$ slice)**.** *Under the assumptions of Proposition 49, fix a choice of (L_0, L_2). If both U_1, U_3 belong to $\{0, \operatorname{Sym}^2 M_2\}$, their four ordered choices are distinct orbits, each with full self-loop group $GL(M_2)$. If exactly one belongs to $\{0, \operatorname{Sym}^2 M_2\}$, its slot and its value are fixed. The other slot has four orbits: the line or plane associated to a doubled divisor, each with self-loop B, and the line or plane associated to a two-root divisor, each with self-loop $N(T)$. Thus these cases have twenty orbits. Representatives for the nonextreme slot are $\mathsf{L}(x^2)$, $\mathsf{D}(e_2^2)$, $\mathsf{L}(xy)$, and $\mathsf{D}(e_1 e_2)$, respectively. Here $B \subseteq GL(M_2)$ is a point stabilizer, T fixes two distinct points separately, and $N(T)$ fixes their unordered pair.*

Proof. The extreme subspaces are fixed by all of $GL(M_2)$. Lemma 42 and Lemma 44 give the two single-divisor orbits and their stabilizers for each of the four choices of extreme slot and value. Proposition 39 gives zero shears for $n = 3$; hence the listed stabilizers are the full self-loop groups. □

Theorem 51 (The line–plane orbit tables for $V_3 \otimes M_2$)**.** *Under the assumptions of Proposition 49, fix any of the 2^2 choices of (L_0, L_2). When both U_1, U_3 are lines or planes, their fixed-R isomorphism classes, representatives, orbit parameters, and full self-loop groups are exactly the forty rows of the four tables below. Each row occurs for every choice of (L_0, L_2), without changing its self-loop group. Together with Proposition 50, these are all the objects and arrows in this fixed-R groupoid.*

In each table, a line has the divisor of its quadratic, and a plane has the divisor of its pole from Lemma 41. Both divisors lie on $\mathbb{P}(M_2^*)$, as in Lemma 42. "Same divisor" means equality as effective divisors; "one common root" and "disjoint" compare their supports. Under $\pi : GL(M_2) \twoheadrightarrow PGL(M_2)$, write $\widetilde{C}_2$ and $\widetilde{V}_4$ for the inverse images of the indicated projective stabilizers. Each group in the last column is the full self-loop group. For the generic rows set $q_\eta = 4\eta x^2 + 4\eta xy + (\eta - 4)y^2$ with $\eta \notin \{0, 4\}$, and for the harmonic rows set $q_0 = x^2 + y^2$. In the examples using planes, put $f_\eta = (4 - \eta)e_1^2 + 4\eta e_1 e_2 - 4\eta e_2^2$. It is the discriminant polar of q_η, up to a nonzero scalar, so $\mathsf{D}(f_\eta)$ has pole $[q_\eta]$. The parameter η was defined in (34). Lemma 43 gives its formula for each of the four line–plane slot types.

U_1 a line; U_3 a line

representative (U_1, U_3)	relation of divisors	orbit parameter	$\mathrm{Aut}_{\mathcal{C}_s^0}(R, K)$
$(\mathsf{L}(x^2), \mathsf{L}(x^2))$	same doubled divisor	point	B
$(\mathsf{L}(x^2), \mathsf{L}(y^2))$	distinct doubled divisors	point	T
$(\mathsf{L}(x^2), \mathsf{L}(xy))$	exactly one divisor doubled; support intersection nonempty	point	T
$(\mathsf{L}(x^2), \mathsf{L}(y(x+y)))$	exactly one divisor doubled; support intersection empty	point	$\widetilde{C}_2$
$(\mathsf{L}(xy), \mathsf{L}(x^2))$	exactly one divisor doubled; support intersection nonempty	point	T
$(\mathsf{L}(xy), \mathsf{L}((x+y)^2))$	exactly one divisor doubled; support intersection empty	point	$\widetilde{C}_2$
$(\mathsf{L}(xy), \mathsf{L}(xy))$	same divisor	point	$N(T)$
$(\mathsf{L}(xy), \mathsf{L}(x(x+y)))$	one common root	point	$k^\times$
$(\mathsf{L}(xy), \mathsf{L}(q_\eta))$	disjoint, generic	$\eta \in \mathbb{A}^1 \setminus \{0, 4\}$	$\widetilde{C}_2$
$(\mathsf{L}(xy), \mathsf{L}(q_0))$	disjoint, harmonic	$\eta = 0$	$\widetilde{V}_4$

U_1 a line; U_3 a plane

representative (U_1, U_3)	relation of divisors	orbit parameter	$\mathrm{Aut}_{\mathcal{C}_s^0}(R, K)$
$(\mathsf{L}(x^2), \mathsf{D}(e_2^2))$	same doubled divisor	point	B
$(\mathsf{L}(x^2), \mathsf{D}(e_1^2))$	distinct doubled divisors	point	T
$(\mathsf{L}(x^2), \mathsf{D}(e_1e_2))$	exactly one divisor doubled; support intersection nonempty	point	T
$(\mathsf{L}(x^2), \mathsf{D}(e_1e_2 - e_1^2))$	exactly one divisor doubled; support intersection empty	point	$\widetilde{C}_2$
$(\mathsf{L}(xy), \mathsf{D}(e_2^2))$	exactly one divisor doubled; support intersection nonempty	point	T
$(\mathsf{L}(xy), \mathsf{D}(e_1^2 - 2e_1e_2 + e_2^2))$	exactly one divisor doubled; support intersection empty	point	$\widetilde{C}_2$
$(\mathsf{L}(xy), \mathsf{D}(e_1e_2))$	same divisor	point	$N(T)$
$(\mathsf{L}(xy), \mathsf{D}(e_1e_2 - e_2^2))$	one common root	point	$k^\times$
$(\mathsf{L}(xy), \mathsf{D}(f_\eta))$	disjoint, generic	$\eta \in \mathbb{A}^1 \setminus \{0, 4\}$	$\widetilde{C}_2$
$(\mathsf{L}(xy), \mathsf{D}(e_1^2 + e_2^2))$	disjoint, harmonic	$\eta = 0$	$\widetilde{V}_4$

U_1 a plane; U_3 a line

representative (U_1, U_3)	relation of divisors	orbit parameter	$\mathrm{Aut}_{\mathcal{C}_s^0}(R, K)$
$(\mathsf{D}(e_2^2), \mathsf{L}(x^2))$	same doubled divisor	point	B
$(\mathsf{D}(e_2^2), \mathsf{L}(y^2))$	distinct doubled divisors	point	T
$(\mathsf{D}(e_2^2), \mathsf{L}(xy))$	exactly one divisor doubled; support intersection nonempty	point	T
$(\mathsf{D}(e_2^2), \mathsf{L}(y(x+y)))$	exactly one divisor doubled; support intersection empty	point	$\widetilde{C}_2$
$(\mathsf{D}(e_1e_2), \mathsf{L}(x^2))$	exactly one divisor doubled; support intersection nonempty	point	T
$(\mathsf{D}(e_1e_2), \mathsf{L}((x+y)^2))$	exactly one divisor doubled; support intersection empty	point	$\widetilde{C}_2$
$(\mathsf{D}(e_1e_2), \mathsf{L}(xy))$	same divisor	point	$N(T)$
$(\mathsf{D}(e_1e_2), \mathsf{L}(x(x+y)))$	one common root	point	$k^\times$
$(\mathsf{D}(e_1e_2), \mathsf{L}(q_\eta))$	disjoint, generic	$\eta \in \mathbb{A}^1 \setminus \{0, 4\}$	$\widetilde{C}_2$
$(\mathsf{D}(e_1e_2), \mathsf{L}(q_0))$	disjoint, harmonic	$\eta = 0$	$\widetilde{V}_4$

U_1 **a plane;** U_3 **a plane**

representative (U_1, U_3)	relation of divisors	orbit parameter	$\mathrm{Aut}_{\mathcal{C}_5^0}(R, K)$
$(\mathsf{D}(e_2^2), \mathsf{D}(e_2^2))$	same doubled divisor	point	B
$(\mathsf{D}(e_2^2), \mathsf{D}(e_1^2))$	distinct doubled divisors	point	T
$(\mathsf{D}(e_2^2), \mathsf{D}(e_1e_2))$	exactly one divisor doubled; support intersection nonempty	point	T
$(\mathsf{D}(e_2^2), \mathsf{D}(e_1e_2 - e_1^2))$	exactly one divisor doubled; support intersection empty	point	$\widetilde{C}_2$
$(\mathsf{D}(e_1e_2), \mathsf{D}(e_2^2))$	exactly one divisor doubled; support intersection nonempty	point	T
$(\mathsf{D}(e_1e_2), \mathsf{D}(e_1^2 - 2e_1e_2 + e_2^2))$	exactly one divisor doubled; support intersection empty	point	$\widetilde{C}_2$
$(\mathsf{D}(e_1e_2), \mathsf{D}(e_1e_2))$	same divisor	point	$N(T)$
$(\mathsf{D}(e_1e_2), \mathsf{D}(e_1e_2 - e_2^2))$	one common root	point	$k^\times$
$(\mathsf{D}(e_1e_2), \mathsf{D}(f_\eta))$	disjoint, generic	$\eta \in \mathbb{A}^1 \setminus \{0, 4\}$	$\widetilde{C}_2$
$(\mathsf{D}(e_1e_2), \mathsf{D}(e_1^2 + e_2^2))$	disjoint, harmonic	$\eta = 0$	$\widetilde{V}_4$

The orbit-parameter column uses the intrinsic invariant (34). The generic and harmonic rows partition the disjoint locus $\mathbb{A}^1 \setminus \{4\}$; the table separates coincident divisors from those sharing one root even though both have $\eta = 4$. No arrow joins different rows, different values of η, or different (L_0, L_2).

Proof of Theorem 51. By Lemma 41, a line and a plane each determine a projective quadratic, equivariantly for $GL(M_2)$. For two doubled divisors and for a doubled divisor paired with a two-root divisor, their equality or incidence gives precisely the first six rows in each table. Two two-root divisors can be the same, share exactly one root, or be disjoint. Lemma 42 classifies the disjoint pairs by η and separates the other two incidence types. In the chosen basis, $\Delta(q_\eta) = 64\eta(x \wedge y)^2$ and $\Delta(xy, q_\eta) = 8\eta(x \wedge y)^2$, so the displayed q_η realizes its stated value of η. The formula for ψ in Lemma 41 checks the displayed plane annihilators; in particular $\psi(q_\eta)$ is proportional to f_η. Lemma 44 gives the projective stabilizers in each row. Their inverse images in $GL(M_2)$ are the groups displayed, with $k^\times$ for the trivial projective stabilizer. Proposition 39 gives zero shears, and Lemma 40 shows that the binary L_i do not change these groups. Theorem 38 identifies the rows and parameters as all the arrows and self-loops. □

Corollary 52 (Trivial central action)**.** *If every $\mu_j = 0$ in Proposition 49, then the t_j act trivially on R and W_K, and*

$$\mathfrak{g}(\mathfrak{s}, V_3 \otimes M_2, K) \cong \big(\mathfrak{sl}_2 \ltimes ((V_3 \otimes M_2) \oplus W_K)\big) \oplus \bigoplus_{j=1}^{m} k t_j.$$

Its fixed-$\mathfrak{s}$ isomorphism classes and full self-loop groups are precisely those of Proposition 50 and Theorem 51.

Proof. The induced action on $\bigwedge^2 R$ is $D_j^{\{2\}} = 2\mu_j \,\mathrm{id} = 0$, and so the action on W_K vanishes. The stated direct sum follows from the brackets. Proposition 49 and Theorem 51 apply at these scalar values. □

5.5 A ternary multiplicity space: conics with a marked point or line

The binary tables above do not describe the first ternary multiplicity space. Here take $R = V_1 \otimes M_3$, where M_3 is a three-dimensional multiplicity space; the subscript records the dimension, not a highest weight as in M_λ. Suppose that $D_j = \mu_j \,\mathrm{id}_{M_3}$ for every central generator t_j. Put $P = \mathbb{P}(M_3^*)$, so an element of $\mathrm{Sym}^2 M_3$ is a conic equation on P. The scalar-action assumption makes every subspace below an $\mathfrak{s}$-subrepresentation, for arbitrary $\mu_j \in k$.

Lemma 53 (The two independent pieces)**.** *Every allowed K has a unique expression*

$$K = (V_2 \otimes L) \oplus (V_0 \otimes U), \qquad L \subseteq \bigwedge^2 M_3, \quad U \subseteq \operatorname{Sym}^2 M_3.$$

Write $a = \dim L$ and $d = \dim U$. Under the canonical, determinant-twisted isomorphism $\bigwedge^2 M_3 \simeq M_3^* \otimes \det M_3$, the choices $a = 0, 1, 2, 3$ correspond respectively to no mark, a point $p \in P$, a line $\ell \subset P$, and the whole plane. An isomorphism in the fixed-$\mathfrak{s}$ groupoid exists precisely when one $g \in GL(M_3)$ carries both L to L' and U to U'. Its self-loop group is the common stabilizer of L and U in $GL(M_3)$.

Proof. The Cauchy decomposition is

$$\bigwedge^2(V_1 \otimes M_3) = (\operatorname{Sym}^2 V_1 \otimes \bigwedge^2 M_3) \oplus (\bigwedge^2 V_1 \otimes \operatorname{Sym}^2 M_3) = (V_2 \otimes \bigwedge^2 M_3) \oplus (V_0 \otimes \operatorname{Sym}^2 M_3).$$

The two simple $\mathfrak{sl}_2$ factors are inequivalent, and the central action on both multiplicity spaces is scalar, proving the first assertion. The determinant-twisted identification sends a decomposable $u \wedge v$ to the covector $w \mapsto u \wedge v \wedge w$. It sends a one-dimensional subspace to a point of P and a two-dimensional subspace to a projective line of P, equivariantly. By Theorem 38, arrows are exactly the indicated simultaneous changes of basis. Proposition 39 eliminates shears because V_1 does not occur in $\bigwedge^2 R$. □

The following table covers all possible dimensions, including a single conic. Its entries specify which geometric data have to be classified *together*. The complete orbit test immediately after the table also handles continuous marked families. In the last three rows $U^\perp$ means the annihilator under the contraction pairing $\operatorname{Sym}^2 M_3 \times \operatorname{Sym}^2 M_3^* \to k$. Its conics live on the *dual* plane $P^* = \mathbb{P}(M_3)$; the point $p \in P$ becomes a line of P^*, and a line $\ell \subset P$ becomes a point of P^*.

d	linear-system invariant	$a = 1$	$a = 2$
0	empty system	point of P	line of P
1	one conic, rank 1, 2, or 3	conic with point	conic with line
2	pencil of conics	pencil with point	pencil with line
3	net of conics	net with point	net with line
4	dual pencil $U^\perp \subset \operatorname{Sym}^2 M_3^*$	dual pencil with line	dual pencil with point
5	dual single conic $U^\perp$	dual conic with line	dual conic with point
6	complete system	point of P	line of P

For $a = 0$ and $a = 3$ the mark is unique, so the invariant is just the system in the second column. In particular, the $d = 0, 6$ rows each have exactly one class for every a.

Theorem 54 (The complete single-conic incidence table)**.** *For $d = 1$, the following are all classes for $a = 1, 2$. Coordinates $[X : Y : Z]$ are on P. For $d = 5$ the same table applies to the conic $U^\perp$ on P^*, with the $a = 1$ and $a = 2$ columns interchanged. Thus it gives seven point classes and seven line classes for each of $d = 1$ and $d = 5$.*

rank	*representative*	*point classes*	*line classes*
1	X^2	*on $X = 0$; off $X = 0$*	*$X = 0$; any other line*
2	XY	*node; on one component away from node; off both*	*a component; through node but not a component; avoids node*
3	$XZ - Y^2$	*on conic; off conic*	*tangent; secant*

The self-loop group of any entry is the inverse image in $GL(M_3)$ of the projective stabilizer of the displayed conic and indicated point or line. For $a = 0, 3$ there are exactly the three rank classes.

Proof. Every ternary quadratic has one of the displayed ranks. A change of coordinates preserving X^2 is transitive on and off its support line and on the lines other than that support. A change preserving XY preserves or exchanges the two component lines: it is transitive separately on the node, their smooth points, and the complement. Dually, lines are either a component, pass through the node without being a component, or avoid the node, and the same coordinate transformations act transitively in each case. The stabilizer of a smooth conic is PGL_2 acting on its Veronese plane; it is transitive on the conic and its complement and, on dual lines, on tangents and secants. These incidence conditions are mutually exclusive and invariant. The contraction pairing is $GL(M_3)$ equivariant, so annihilating U converts $d = 5$ to the same classification in P^*; the point/line interchange follows from projective duality. Lemma 53 gives the loop groups. □

For pencils and nets, incidence is not exhausted by the simple predicates in Theorem 54. Here is a finite, explicit equivalence test for *every* row of the dimension table. It is useful when the stabilizer of a pencil or net has a continuous family of marked orbits.

Fix a basis (x, y, z) of M_3 and the ordered quadratic basis $\mathcal{B} = (x^2, y^2, z^2, xy, xz, yz)$. Let B be the $6 \times d$ full-rank matrix of a basis of U in $\mathcal{B}$, and let H be the $3 \times a$ full-rank matrix of a basis of L in $(x \wedge y, x \wedge z, y \wedge z)$. Empty matrices are allowed. Write $S_2(G)$ and $E_2(G)$ for the matrices of $\operatorname{Sym}^2 G$ and $\bigwedge^2 G$ in these ordered bases; their entries are explicit quadratic polynomials in the nine entries of G.

Theorem 55 (Polynomial orbit criterion)**.** *For two entries (B, H) and (B', H') with the same (d, a), form the polynomial system in $G = (g_{ij})$, an auxiliary variable u, and square matrices C of size d and J of size a:*

$$S_2(G)B = B'C, \qquad E_2(G)H = H'J, \qquad u \det(G) \det(C) \det(J) = 1. \tag{40}$$

For a matrix of size zero its determinant is 1*. The two Lie algebra objects are isomorphic in the fixed-$\mathfrak{s}$ groupoid if and only if (40) has a solution over k. Equivalently, its generated ideal is proper. For a self-loop, the solutions projected to G give the* full *loop group* $\operatorname{Stab}_{GL(M_3)}(L, U)$*. This is a necessary-and-sufficient isomorphism criterion for all pencils, nets, and their marked incidences, including nonclosed orbits. If the input coefficients lie in a specified computable subfield $k_0 \subseteq k$, with effective field operations and equality testing, it is a decision procedure: a Gröbner basis over k_0 tests whether the generated ideal contains* 1*.*

Proof. The first two equations say exactly that G maps the two subspaces onto their primed counterparts; the invertibility equation makes G, C, J isomorphisms. Conversely, every simultaneous isomorphism gives unique C, J relative to the chosen column bases and a suitable u. Lemma 53 identifies these simultaneous isomorphisms with all arrows. Since k is algebraically closed, the weak Nullstellensatz says the finite polynomial system has a solution exactly when its ideal is proper. For coefficients in k_0, extension from k_0 to k is faithfully flat, so the ideal contains 1 over k exactly when it does over k_0. A Gröbner basis computes this membership using effective arithmetic in k_0. Substituting the same input twice gives the stated self-loop equations. □

Proposition 56 (The eight unmarked pencil types)**.** *When $d = 2$, the underlying pencil U belongs to exactly one row of the following table. The partition records multiplicities in the base scheme when the pencil contains a smooth conic. In the last three rows every member is singular. In each row the* marked *classes are its points or lines modulo the stabilizer of the displayed pencil, tested exactly by (40) above.*

type	*pencil* $U \subset \operatorname{Sym}^2 M_3$	*intrinsic description*
A	$\langle x^2 - z^2, y^2 - z^2 \rangle$	*four simple base points*
B	$\langle x^2 - z^2, yz \rangle$	*base multiplicities* $2, 1, 1$
C	$\langle xy, z^2 \rangle$	*base multiplicities* $2, 2$
D	$\langle xy, xz - y^2 \rangle$	*base multiplicities* $3, 1$
E	$\langle x^2, xz - y^2 \rangle$	*one base point of length* 4
F	$\langle xy, xz \rangle$	*common line, no common singular point*
G	$\langle x^2, xy \rangle$	*common line and singular point*
H	$\langle x^2, y^2 \rangle$	*common singular point, no common line*

For $d = 4$, take annihilators of these eight pencils in $\operatorname{Sym}^2 M_3$ after placing their displayed equations in $\operatorname{Sym}^2 M_3^*$: point and line marks exchange roles in the dual plane.

Proof. Suppose a smooth member F exists and let G be an independent member. Regard their polarizations as symmetric bilinear forms on M_3^*. Then F is nondegenerate and $T = F^{-1}G$ is F-self-adjoint. Simultaneous congruence of (F, G) is classified by the Jordan blocks of T: on a block of size r and eigenvalue λ the pair is $(\mathsf{E}_r, \mathsf{E}_r J_r(\lambda))$, where E_r has ones on its antidiagonal and $J_r(\lambda)$ has ones on its superdiagonal [17]. This form is valid over our field k. Indeed, generalized eigenspaces of a self-adjoint operator are orthogonal. On one of them write $T = \lambda \operatorname{id} + N$ and choose a maximal chain generated by v with $F(v, N^{r-1}v) \neq 0$; such a vector exists because $F(-, N^{r-1}-)$ is a nonzero symmetric form. The cyclic span is nondegenerate. Replacing v by an invertible polynomial in N applied to v normalizes its Gram matrix to E_r: the required square root of a unit exists in $k[t]/(t^r)$ since k is algebraically closed and 2 is invertible. Its orthogonal complement is N-stable, so induction gives the asserted blocks.

A change of the two pencil generators replaces T by

$$(a \operatorname{id} + bT)^{-1}(c \operatorname{id} + dT), \qquad ad - bc \neq 0,$$

provided the new first form is nondegenerate. This changes eigenvalues by a fractional linear transformation and preserves the sizes of their Jordan blocks. There are at most three distinct eigenvalues, so projective transformations normalize their positions without a parameter. The complete list of block patterns is therefore

A	$J_1(\lambda) \oplus J_1(\mu) \oplus J_1(\nu), \quad \lambda, \mu, \nu$ distinct
B	$J_2(\lambda) \oplus J_1(\mu), \quad \lambda \neq \mu$
C	$J_1(\lambda)^{\oplus 2} \oplus J_1(\mu), \quad \lambda \neq \mu$
D	$J_3(\lambda)$
E	$J_2(\lambda) \oplus J_1(\lambda)$.

The omitted scalar operator would make F, G dependent. Computing these patterns for the displayed representatives identifies them with A–E. Their base schemes have, respectively, local lengths $(1, 1, 1, 1)$, $(2, 1, 1)$, $(2, 2)$, $(3, 1)$ and (4), as substitution in their two equations shows. This argument uses changes of the smooth member as well as projective coordinates; it does not fix a conic and discard the cross-ratio of four points on it.

Now suppose every member is singular. Some member has rank two: two independent rank-one quadratics are squares of independent linear forms and their sum has rank two. Choose coordinates in which this member is xy, and subtract a multiple of it from the other generator to write

$$G = ax^2 + cy^2 + dxz + eyz + fz^2.$$

The coefficient of t in the determinant of the symmetric matrix of $xy + tG$ forces $f = 0$. With $f = 0$ that determinant is

$$\tfrac{1}{4}\big(t^2 de - t^3(ae^2 + cd^2)\big).$$

Thus $de = 0$ and $ae^2 + cd^2 = 0$. If $d \neq 0$, then $e = c = 0$ and the pencil is $\langle xy, x(ax + dz)\rangle$; replacing the third coordinate gives $\langle xy, xz\rangle$. The case $e \neq 0$ is symmetric. This is type F, with a common line and no common singular point. If $d = e = 0$, both quadratics depend only on x, y and have the common singular point $[0 : 0 : 1]$. A plane in the three-dimensional space of binary quadratics has two orbits, by the discriminant polarity of Lemma 41: it is equivalent to $\langle x^2, xy\rangle$ or $\langle x^2, y^2\rangle$. These are G and H, distinguished by the presence or absence of a common line. This proves both exhaustion and separation in the singular case. Contraction is equivariant, proving the $d = 4$ statement. The marked assertion follows from Lemma 53 and Theorem 55. □

Lemma 57 (Why marked-pencil and marked-net families remain)**.** *Let U be a pencil through four points in general position. Its projective stabilizer is the finite group S_4 permuting those four points. Consequently the point-marked classes above this one pencil are the points of P/S_4, and the line-marked classes are those of P^*/S_4; both families have dimension two. Likewise, any net with finite projective stabilizer has two-dimensional families of point and line marks. For a fixed system U in any row, the precise marked moduli are the orbits of $\mathrm{Stab}_{PGL(M_3)}(U)$ on P or P^*, and (40) is an explicit test for equality in those families.*

Proof. The base scheme of the generic pencil consists of four points in general position. A projectivity preserving the pencil permutes the base scheme. Conversely any permutation of a projective frame of four points is realized by a unique projectivity, and it preserves the two-dimensional space of conics through them. Thus its stabilizer is S_4. A finite group has finite orbits, so quotienting a plane of possible marks leaves dimension two. The same argument applies to the point and dual-line planes for any net with finite stabilizer. For any U, fixing a representative of its orbit leaves precisely its stabilizer acting on the mark; this is also immediate by specializing (40) to $B = B'$. □

Proposition 58 (A net with genuinely moving marks)**.** *Set*

$$U_{\mathrm{He}} = \langle x^2 - 3yz,\ y^2 - 3xz,\ z^2 - 3xy\rangle \subset \mathrm{Sym}^2 M_3.$$

Its projective stabilizer $\Gamma \subset PGL(M_3)$ is finite. Hence the marked point and line classes above this one net are, respectively, P/Γ and P^*/Γ, each a two-dimensional family. These families occur for every choice of the central weights μ_j. For any net U, the canonical evaluation map $\mathrm{ev}_p : U \to \mathcal{O}_P(2)|_p$ takes values in the fiber of the quadratic line bundle at p. It is zero exactly when the point mark lies in the base scheme; no trivialization of that fiber is needed. The restriction map $U \to H^0(\ell, \mathcal{O}_\ell(2))$ records the conics containing a line mark. The full marked position, subject to the stabilizer action, is retained by (40).

Proof. An infinitesimal projectivity preserving U_{He} is induced by $T \in \mathfrak{gl}(M_3)$ for which $Tq \in U_{\mathrm{He}}$ for its three displayed generators. Expanding coefficients of $x^2, y^2, z^2, xy, xz, yz$ gives eight independent linear constraints on the nine entries of T: its off-diagonal entries are zero and its three diagonal entries are equal. Thus the projective stabilizer has zero Lie algebra, and is finite in characteristic zero. Its orbits on P and P^* are finite, leaving two-dimensional marked families. Evaluation and restriction commute with projective change of coordinates, and their stated kernels have the claimed geometric meanings. The scalar t_j-actions add no restriction to L or U. □

5.6 Low-dimensional components of the isomorphism groupoids

In this subsection $\mathfrak{s}$ is semisimple, including $\mathfrak{s} = 0$; in particular there are no kt_j factors. Write $\mathcal{C}_{\mathfrak{s}}^{\leq 6}$ for the full subgroupoid of $\mathcal{C}_{\mathfrak{s}}$ on objects with $\dim \mathfrak{n}(R, K) = \dim R + \dim(\bigwedge^2 R/K) \leq 6$. No condition on $M_{0,\mathbf{0}_0}(R)$ is imposed, so the zero-action objects are included.

Theorem 59 (Bounded isomorphism components)**.** *Choose an $\mathfrak{s}$-module R in each isomorphism class $[R]$. Then*

$$\pi_0(\mathcal{C}_{\mathfrak{s}}^{\leq 6}) \;\cong\; \coprod_{\substack{[R]\\ \dim R\leq 6}} \;\coprod_{d=0}^{6-\dim R} \mathrm{Gr}^{\mathfrak{s}}_{\binom{\dim R}{2}-d}\Big(\bigwedge^2 R\Big)\big/\operatorname{Aut}_{\mathfrak{s}}(R). \tag{41}$$

Here $\mathrm{Gr}^{\mathfrak{s}}_q(\bigwedge^2 R)$ is the set of invariant q-planes, with an empty Grassmannian interpreted as an empty summand. The quotient is the orbit set for the induced exterior-square action.

Proof. The total dimension is invariant under isomorphism. Since $Z(\mathfrak{s}) = 0$, the central mixing map $c : R \to Z(\mathfrak{s})$ in Theorem 8 vanishes for every arrow in $\mathcal{C}_{\mathfrak{s}}$, including when $\mathfrak{s} = 0$. The bracket identities and surjectivity of the canonical projection $\bigwedge^2 R \to \bigwedge^2 R/K$ then give $p = 0$ and show that an arrow exists precisely when an $\mathfrak{s}$-module isomorphism $a : R \to R'$ satisfies $(\bigwedge^2 a)K = K'$. Conversely this a induces an arrow. Partitioning the objects by $[R]$ and $d = \dim(\bigwedge^2 R/K)$ gives the displayed orbit sets. □

Theorem 59 computes components separately for each fixed acting algebra $\mathfrak{s}$. The table uses patterns for that algebra: a label $\mathfrak{a}+\text{junk}$ means $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{j}$ for any chosen semisimple $\mathfrak{j}$, which acts trivially on R and $\bigwedge^2 R/K$. Substituting a particular $\mathfrak{j}$ makes the row a point of $\pi_0(\mathcal{C}_{\mathfrak{a}\oplus\mathfrak{j}}^{\leq 6})$. The same row and orbit condition work for *every* such $\mathfrak{j}$: its trivial action imposes no additional condition on K or on an intertwiner of R. In each application the chosen $\mathfrak{s}$ stays fixed during the isomorphism test. The plus sign belongs only to the *acting-algebra label*; it does not adjoin $\mathfrak{s}$ to $\mathfrak{n}(R, K)$.

Theorem 60 (Small-dimensional components for specified actions)**.** *Let k be algebraically closed of characteristic zero and let $\dim \mathfrak{n}(R, K) \leq 6$. The table lists every component in this dimension range for each active-algebra pattern, with the acting algebra fixed pointwise in its isomorphisms. For each row labelled $\mathfrak{a}+\text{junk}$ and each chosen semisimple $\mathfrak{j}$, the row gives a point of $\pi_0(\mathcal{C}_{\mathfrak{s}}^{\leq 6})$ for the fixed algebra $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{j}$. Taking $\mathfrak{j} = 0$ in the rows labelled $0+\text{junk}$ exhausts $\pi_0(\mathcal{C}_0^{\leq 6})$. The two binary-form ranks for $R \cong 2V_1$ are distinct for each fixed $\mathfrak{s} = \mathfrak{sl}_2 \oplus \mathfrak{j}$. Each row specifies R, the quotient dimension d, its orbit condition, and a representative kernel K. The acting algebra is never counted in the first column's dimension. Within each dimension the rows are ordered by $\dim \mathfrak{a}$ from smallest to largest; the wildcard $\mathfrak{j}$ does not enter this ordering. The proof gives the case reduction for all representation types of total dimension at most six.*

Here triv *is the one-dimensional trivial module. For $\mathfrak{sl}_2$, V_n is its irreducible module of highest weight n (dimension $n+1$). For every other simple factor, ω_i denotes its ith fundamental weight in the usual Dynkin numbering, and V_λ is the irreducible module of highest weight λ; thus $V_{2\omega_1}$ has highest weight twice the first fundamental weight. In particular, V_{ω_1} is the standard module of each displayed $\mathfrak{sl}_n$ ($n \geq 3$) or $\mathfrak{sp}_{2n}$. The acting algebra in each row specifies which module is meant; $V_{\omega_2} \subset \bigwedge^2 V_{\omega_1}$ in the $\mathfrak{sp}_4$ rows. The symbol $\boxtimes$ denotes an external tensor product over the displayed simple factors. For each isotypic summand $V_a \otimes M_a$ of R, choose a basis $e_{a,1}, \ldots, e_{a,m_a}$ of the multiplicity space M_a. Write $e_{a,ij} = e_{a,i} \wedge e_{a,j}$ in $\bigwedge^2 M_a$, and write $x_{a,i}$ for the same basis vector when it occurs in $\operatorname{Sym} M_a$; thus $x_{a,i}x_{a,j}$ belongs to $\operatorname{Sym}^2 M_a$. The label $a = 0$ denotes the trivial representation; in a row with one nontrivial constituent we label it a, and for two constituents we use a, b in displayed order. A cross term $V_a \otimes V_b \otimes M_a \otimes M_b$ uses $e_{a,i} \otimes e_{b,j}$ in its multiplicity factors. Consequently every example in the "Example K" column is a submodule of the stated decomposition of $\bigwedge^2 R$, rather than a span requiring a basis of R. In particular the same-isotypic term is*

$$\bigwedge^2 (V_a \otimes M_a) = \big(\operatorname{Sym}^2 V_a \otimes \bigwedge^2 M_a\big) \oplus \big(\bigwedge^2 V_a \otimes \operatorname{Sym}^2 M_a\big).$$

For $K = 0$ or $K = \bigwedge^2 R$ the "Example K" column says precisely that.

For a one-dimensional quotient, the table writes $K = \ker \omega_{\mathbf{q}}$. *List the nontrivial symplectic irreducible types in* R *in the order displayed, and let* $\mathbf{q} = (q_0, q_1, \dots, q_t)$: q_0 *is the even rank of an alternating form on the trivial multiplicity space, while* q_i *is the rank of a symmetric form on the multiplicity space of the* i*th symplectic type. Zeros are retained, including* $q_0 = 0$ *if there is no trivial summand. Choose standard forms of these ranks and fixed nonzero invariant alternating forms on the symplectic irreducibles. Their direct sum* $\omega_{\mathbf{q}}$ *is an invariant two-form on* R; $\ker \omega_{\mathbf{q}}$ *is its kernel as a map* $\bigwedge^2 R \to k$. *Over* k, *these ranks determine the orbit under* $\mathrm{Aut}_{\mathfrak{s}}(R)$. *Explicitly, on the trivial multiplicity space choose the form* $\sum_{h=1}^{q_0/2} e^*_{0,2h-1} \wedge e^*_{0,2h}$. *On the* i*th symplectic isotypic summand* $V_{\lambda_i} \otimes M_i$, *use* $\beta_i \otimes \sum_{h=1}^{q_i} (x^*_{i,h})^2$, *where* $(x^*)^2(u,v) = x^*(u)x^*(v)$; *all cross terms vanish. When* $t = 0$, *the rank tuple is written simply* (q_0).

For the three trivial-action rows with $\dim R = 4$ *and* $\dim K = 4$, *the* annihilator pencil *is* $\mathbb{P}(K^\perp) \subset \mathbb{P}(\bigwedge^2 R^*)$. *The Klein quadric* $Q \subset \mathbb{P}(\bigwedge^2 R^*)$ *consists of decomposable two-forms. Thus "contained," "secant," and "tangent" below describe how this annihilator pencil meets* Q. *For* $R = V_1 \otimes M$ *with* $\dim M = 2$, *invariant alternating quotient forms correspond to* $\mathrm{Sym}^2 M^*$; *the rank of the annihilator line in the two dimension-five* $2V_1$ *rows is the rank of its binary quadratic form. The rank-one form gives a three-dimensional Heisenberg summand and two central generators; the rank-two form is the evaluation pairing after choosing an identification* $V_1 \cong V_1^*$ *between two copies of* V_1. *Thus* $\mathrm{Heisen}_\omega(V_1)$ *has* $R = V_1$ *and bracket given by its invariant alternating form, and* $\mathrm{Heisen}(V_1)$ *has* $R = 2V_1$ *with bracket* $b((u,\alpha),(v,\beta)) = \beta(u) - \alpha(v)$. *For the hyperbolic form* $x^*_{a,1}x^*_{a,2}$, *which over* k *is equivalent to the standard form above, the kernel of* $\mathrm{Heisen}(V_1)$ *is* $(V_2 \otimes e_{a,12}) \oplus \langle x^2_{a,1}, x^2_{a,2}\rangle$. *In the two rows with* $R = 2V_1$ *and* $d = 2$, $M = M_a$ *is a two-dimensional multiplicity space and* $U \subset \mathrm{Sym}^2 M$ *is the kernel line, as in Theorem 45;* $\bigwedge^2(V_1 \otimes M) = V_2 \otimes \bigwedge^2 M \oplus \mathrm{triv} \otimes \mathrm{Sym}^2 M$. *A rank-one line in* $\mathrm{Sym}^2 M$ *is generated by a square* m^2 *for some* $0 \neq m \in M$. *Apart from the zero-dimensional row, the* $0+\mathrm{junk}$ *rows in the final column use the* $(R_q, K_q(\cdots))$ *labels of Theorem 2, including its abelian rows. For the other rows, "abelian" means* $K = \bigwedge^2 R$. *Write* $\mathrm{Heisen}_\omega(S) = S \oplus k$ *when* S *has a nondegenerate invariant alternating form* ω, *with* $[u,v] = \omega(u,v)$ *in its trivial central line; write* $\mathrm{Heisen}(V) = V \oplus V^* \oplus k$ *for the evaluation bracket. The Heisenberg labels in the final column refer to rows in which* $\omega_{\mathbf{q}}$ *is nondegenerate on all of* R. *A dash leaves that column unfilled; it does not assert that no decomposition or other description exists.*

n	$\mathfrak{a}+\mathrm{junk}$	R	d	W and orbit	Example K	Special form
0	$0+\mathrm{junk}$	0	0	$W=0$	0	zero algebra
1	$0+\mathrm{junk}$	triv	0	$W=0$	$\bigwedge^2 R$	$(R_1, K_1())$
2	$0+\mathrm{junk}$	2triv	0	$W=0$	$\bigwedge^2 R$	$(R_2, K_2())$
2	$\mathfrak{sl}_2+\mathrm{junk}$	V_1	0	$W=0$	$\bigwedge^2 R$	abelian
3	$0+\mathrm{junk}$	2triv	1	$W=\mathrm{triv}$; $\mathbf{q}=(2)$	$\ker \omega_{(2)}$	$(R_2, K_2(12))$
3	$0+\mathrm{junk}$	3triv	0	$W=0$	$\bigwedge^2 R$	$(R_3, K_3())$
3	$\mathfrak{sl}_2+\mathrm{junk}$	V_1	1	$W=\mathrm{triv}$; $\mathbf{q}=(0,1)$	$\ker \omega_{(0,1)}$	$\mathrm{Heisen}_\omega(V_1)$
3	$\mathfrak{sl}_2+\mathrm{junk}$	V_2	0	$W=0$	$\bigwedge^2 R$	abelian
3	$\mathfrak{sl}_2+\mathrm{junk}$	$\mathrm{triv} \oplus V_1$	0	$W=0$	$\bigwedge^2 R$	abelian
3	$\mathfrak{sl}_3+\mathrm{junk}$	V_{ω_1}	0	$W=0$	$\bigwedge^2 R$	abelian
3	$\mathfrak{sl}_3+\mathrm{junk}$	V_{ω_2}	0	$W=0$	$\bigwedge^2 R$	abelian
4	$0+\mathrm{junk}$	3triv	1	$W=\mathrm{triv}$; $\mathbf{q}=(2)$	$\ker \omega_{(2)}$	$(R_3, K_3(12))$
4	$0+\mathrm{junk}$	4triv	0	$W=0$	$\bigwedge^2 R$	$(R_4, K_4())$
4	$\mathfrak{sl}_2+\mathrm{junk}$	$2V_1$	0	$W=0$	$\bigwedge^2 R$	abelian
4	$\mathfrak{sl}_2+\mathrm{junk}$	$2\mathrm{triv} \oplus V_1$	0	$W=0$	$\bigwedge^2 R$	abelian
4	$\mathfrak{sl}_2+\mathrm{junk}$	V_3	0	$W=0$	$\bigwedge^2 R$	abelian
4	$\mathfrak{sl}_2+\mathrm{junk}$	$\mathrm{triv} \oplus V_1$	1	$W=\mathrm{triv}$; $\mathbf{q}=(0,1)$	$\ker \omega_{(0,1)}$	—
4	$\mathfrak{sl}_2+\mathrm{junk}$	$\mathrm{triv} \oplus V_2$	0	$W=0$	$\bigwedge^2 R$	abelian
4	$(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2)+\mathrm{junk}$	$V_1 \boxtimes V_1$	0	$W=0$	$\bigwedge^2 R$	abelian
4	$(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2)+\mathrm{junk}$	$V_1 \boxtimes \mathrm{triv} \oplus \mathrm{triv} \boxtimes V_1$	0	$W=0$	$\bigwedge^2 R$	abelian
4	$\mathfrak{sl}_3+\mathrm{junk}$	$\mathrm{triv} \oplus V_{\omega_1}$	0	$W=0$	$\bigwedge^2 R$	abelian

n	$\mathfrak{a}+\text{junk}$	R	d	W and orbit	Example K	Special form
4	$\mathfrak{sl}_3+\text{junk}$	$\text{triv}\oplus V_{\omega_2}$	0	$W=0$	$\bigwedge^2 R$	abelian
4	$\mathfrak{sp}_4+\text{junk}$	V_{ω_1}	0	$W=0$	$\bigwedge^2 R$	abelian
4	$\mathfrak{sl}_4+\text{junk}$	V_{ω_1}	0	$W=0$	$\bigwedge^2 R$	abelian
4	$\mathfrak{sl}_4+\text{junk}$	V_{ω_3}	0	$W=0$	$\bigwedge^2 R$	abelian
5	$0+\text{junk}$	3triv	2	$W=2\text{triv}$; $\dim K=1$	$\langle e_{0,23}\rangle$	$(R_3, K_3(12,13))$
5	$0+\text{junk}$	4triv	1	$W=\text{triv}$; $\mathbf{q}=(2)$	$\ker\omega_{(2)}$	$(R_4, K_4(12))$
5	$0+\text{junk}$	4triv	1	$W=\text{triv}$; $\mathbf{q}=(4)$	$\ker\omega_{(4)}$	$(R_4, K_4(12+34))$
5	$0+\text{junk}$	5triv	0	$W=0$	$\bigwedge^2 R$	$(R_5, K_5())$
5	$\mathfrak{sl}_2+\text{junk}$	$2V_1$	1	$W=\text{triv}$; $\mathbf{q}=(0,1)$	$\ker\omega_{(0,1)}$	—
5	$\mathfrak{sl}_2+\text{junk}$	$2V_1$	1	$W=\text{triv}$; $\mathbf{q}=(0,2)$	$\ker\omega_{(0,2)}$	$\text{Heisen}(V_1)$
5	$\mathfrak{sl}_2+\text{junk}$	$2\text{triv}\oplus V_1$	1	$W=\text{triv}$; $\mathbf{q}=(0,1)$	$\ker\omega_{(0,1)}$	—
5	$\mathfrak{sl}_2+\text{junk}$	$2\text{triv}\oplus V_1$	1	$W=\text{triv}$; $\mathbf{q}=(2,0)$	$\ker\omega_{(2,0)}$	—
5	$\mathfrak{sl}_2+\text{junk}$	$2\text{triv}\oplus V_1$	1	$W=\text{triv}$; $\mathbf{q}=(2,1)$	$\ker\omega_{(2,1)}$	$\text{Heisen}_\omega(R)$
5	$\mathfrak{sl}_2+\text{junk}$	$2\text{triv}\oplus V_2$	0	$W=0$	$\bigwedge^2 R$	abelian
5	$\mathfrak{sl}_2+\text{junk}$	$3\text{triv}\oplus V_1$	0	$W=0$	$\bigwedge^2 R$	abelian
5	$\mathfrak{sl}_2+\text{junk}$	$V_1\oplus V_2$	0	$W=0$	$\bigwedge^2 R$	abelian
5	$\mathfrak{sl}_2+\text{junk}$	V_3	1	$W=\text{triv}$; $\mathbf{q}=(0,1)$	$\ker\omega_{(0,1)}$	$\text{Heisen}_\omega(R)$
5	$\mathfrak{sl}_2+\text{junk}$	V_4	0	$W=0$	$\bigwedge^2 R$	abelian
5	$\mathfrak{sl}_2+\text{junk}$	$\text{triv}\oplus 2V_1$	0	$W=0$	$\bigwedge^2 R$	abelian
5	$\mathfrak{sl}_2+\text{junk}$	$\text{triv}\oplus V_1$	2	$W=V_1$	$\bigwedge^2 V_1$	—
5	$\mathfrak{sl}_2+\text{junk}$	$\text{triv}\oplus V_3$	0	$W=0$	$\bigwedge^2 R$	abelian
5	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$V_1\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_1$	1	$W=\text{triv}$; $\mathbf{q}=(0,0,1)$	$\ker\omega_{(0,0,1)}$	—
5	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$V_1\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_1$	1	$W=\text{triv}$; $\mathbf{q}=(0,1,0)$	$\ker\omega_{(0,1,0)}$	—
5	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$V_1\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_1$	1	$W=\text{triv}$; $\mathbf{q}=(0,1,1)$	$\ker\omega_{(0,1,1)}$	$\text{Heisen}_\omega(R)$
5	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$V_1\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_2$	0	$W=0$	$\bigwedge^2 R$	abelian
5	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$V_2\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_1$	0	$W=0$	$\bigwedge^2 R$	abelian
5	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$\text{triv}\boxtimes\text{triv}\oplus V_1\boxtimes V_1$	0	$W=0$	$\bigwedge^2 R$	abelian
5	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$\text{triv}\boxtimes\text{triv}\oplus V_1\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_1$	0	$W=0$	$\bigwedge^2 R$	abelian
5	$\mathfrak{sl}_3+\text{junk}$	$2\text{triv}\oplus V_{\omega_1}$	0	$W=0$	$\bigwedge^2 R$	abelian
5	$\mathfrak{sl}_3+\text{junk}$	$2\text{triv}\oplus V_{\omega_2}$	0	$W=0$	$\bigwedge^2 R$	abelian
5	$\mathfrak{sp}_4+\text{junk}$	V_{ω_1}	1	$W=\text{triv}$; $\mathbf{q}=(0,1)$	$\ker\omega_{(0,1)}$	$\text{Heisen}_\omega(R)$
5	$\mathfrak{sp}_4+\text{junk}$	V_{ω_2}	0	$W=0$	$\bigwedge^2 R$	abelian
5	$\mathfrak{sp}_4+\text{junk}$	$\text{triv}\oplus V_{\omega_1}$	0	$W=0$	$\bigwedge^2 R$	abelian
5	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_3)+\text{junk}$	$V_1\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_{\omega_1}$	0	$W=0$	$\bigwedge^2 R$	abelian
5	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_3)+\text{junk}$	$V_1\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_{\omega_2}$	0	$W=0$	$\bigwedge^2 R$	abelian
5	$\mathfrak{sl}_4+\text{junk}$	$\text{triv}\oplus V_{\omega_1}$	0	$W=0$	$\bigwedge^2 R$	abelian
5	$\mathfrak{sl}_4+\text{junk}$	$\text{triv}\oplus V_{\omega_3}$	0	$W=0$	$\bigwedge^2 R$	abelian
5	$\mathfrak{sl}_5+\text{junk}$	V_{ω_1}	0	$W=0$	$\bigwedge^2 R$	abelian
5	$\mathfrak{sl}_5+\text{junk}$	V_{ω_4}	0	$W=0$	$\bigwedge^2 R$	abelian
6	$0+\text{junk}$	3triv	3	$W=\bigwedge^2 R$	0	$(R_3, K_3(12,13,23))$
6	$0+\text{junk}$	4triv	2	$W=2\text{triv}$; pencil contained in Q	$\langle e_{0,14}, e_{0,23}, e_{0,24}, e_{0,34}\rangle$	$(R_4, K_4(12,13))$
6	$0+\text{junk}$	4triv	2	$W=2\text{triv}$; pencil secant to Q	$\langle e_{0,13}, e_{0,14}, e_{0,23}, e_{0,24}\rangle$	$(R_4, K_4(12,34))$
6	$0+\text{junk}$	4triv	2	$W=2\text{triv}$; pencil tangent to Q	$\langle e_{0,14}, e_{0,23}, e_{0,34}, e_{0,13}-e_{0,24}\rangle$	$(R_4, K_4(12,13+24))$
6	$0+\text{junk}$	5triv	1	$W=\text{triv}$; $\mathbf{q}=(2)$	$\ker\omega_{(2)}$	$(R_5, K_5(12))$
6	$0+\text{junk}$	5triv	1	$W=\text{triv}$; $\mathbf{q}=(4)$	$\ker\omega_{(4)}$	$(R_5, K_5(12+34))$
6	$0+\text{junk}$	6triv	0	$W=0$	$\bigwedge^2 R$	$(R_6, K_6())$
6	$\mathfrak{sl}_2+\text{junk}$	$2V_1$	2	$W=2\text{triv}$; $\operatorname{rk} U=1$	$(V_2\otimes e_{a,12})\oplus\langle x_{a,1}^2\rangle$	—
6	$\mathfrak{sl}_2+\text{junk}$	$2V_1$	2	$W=2\text{triv}$; $\operatorname{rk} U=2$	$(V_2\otimes e_{a,12})\oplus\langle x_{a,1}x_{a,2}\rangle$	—
6	$\mathfrak{sl}_2+\text{junk}$	$2V_2$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sl}_2+\text{junk}$	$2\text{triv}\oplus 2V_1$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sl}_2+\text{junk}$	$2\text{triv}\oplus V_1$	2	$W=2\text{triv}$	$V_1\otimes M_a\otimes M_0$	—
6	$\mathfrak{sl}_2+\text{junk}$	$2\text{triv}\oplus V_1$	2	$W=V_1$	$(V_1\otimes e_{a,1}\otimes e_{0,2})\oplus(\bigwedge^2 V_1\otimes x_{a,1}^2)\oplus\langle e_{0,12}\rangle$	—
6	$\mathfrak{sl}_2+\text{junk}$	$2\text{triv}\oplus V_2$	1	$W=\text{triv}$; $\mathbf{q}=(2)$	$\ker\omega_{(2)}$	—

n	$\mathfrak{a}+\text{junk}$	R	d	W and orbit	Example K	Special form
6	$\mathfrak{sl}_2+\text{junk}$	$2\text{triv}\oplus V_3$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sl}_2+\text{junk}$	$3V_1$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sl}_2+\text{junk}$	$3\text{triv}\oplus V_1$	1	$W=\text{triv}$; $\mathbf{q}=(0,1)$	$\ker\omega_{(0,1)}$	—
6	$\mathfrak{sl}_2+\text{junk}$	$3\text{triv}\oplus V_1$	1	$W=\text{triv}$; $\mathbf{q}=(2,0)$	$\ker\omega_{(2,0)}$	—
6	$\mathfrak{sl}_2+\text{junk}$	$3\text{triv}\oplus V_1$	1	$W=\text{triv}$; $\mathbf{q}=(2,1)$	$\ker\omega_{(2,1)}$	—
6	$\mathfrak{sl}_2+\text{junk}$	$3\text{triv}\oplus V_2$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sl}_2+\text{junk}$	$4\text{triv}\oplus V_1$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sl}_2+\text{junk}$	$V_1\oplus V_2$	1	$W=\text{triv}$; $\mathbf{q}=(0,1)$	$\ker\omega_{(0,1)}$	—
6	$\mathfrak{sl}_2+\text{junk}$	$V_1\oplus V_3$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sl}_2+\text{junk}$	V_2	3	$W=\bigwedge^2 R$	0	—
6	$\mathfrak{sl}_2+\text{junk}$	V_5	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sl}_2+\text{junk}$	$\text{triv}\oplus 2V_1$	1	$W=\text{triv}$; $\mathbf{q}=(0,1)$	$\ker\omega_{(0,1)}$	—
6	$\mathfrak{sl}_2+\text{junk}$	$\text{triv}\oplus 2V_1$	1	$W=\text{triv}$; $\mathbf{q}=(0,2)$	$\ker\omega_{(0,2)}$	—
6	$\mathfrak{sl}_2+\text{junk}$	$\text{triv}\oplus V_1$	3	$W=\bigwedge^2 R$	0	—
6	$\mathfrak{sl}_2+\text{junk}$	$\text{triv}\oplus V_1\oplus V_2$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sl}_2+\text{junk}$	$\text{triv}\oplus V_3$	1	$W=\text{triv}$; $\mathbf{q}=(0,1)$	$\ker\omega_{(0,1)}$	—
6	$\mathfrak{sl}_2+\text{junk}$	$\text{triv}\oplus V_4$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$2(V_1\boxtimes\text{triv})\oplus\text{triv}\boxtimes V_1$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$2(\text{triv}\boxtimes\text{triv})\oplus V_1\boxtimes V_1$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$2(\text{triv}\boxtimes\text{triv})\oplus V_1\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_1$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$V_1\boxtimes V_1\oplus V_1\boxtimes\text{triv}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$V_1\boxtimes V_1\oplus\text{triv}\boxtimes V_1$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$V_1\boxtimes V_2$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$V_1\boxtimes\text{triv}\oplus 2(\text{triv}\boxtimes V_1)$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$V_1\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_1$	2	$W=2\text{triv}$	$V_1\boxtimes V_1$	—
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$V_1\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_2$	1	$W=\text{triv}$; $\mathbf{q}=(0,1)$	$\ker\omega_{(0,1)}$	—
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$V_1\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_3$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$V_2\boxtimes V_1$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$V_2\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_1$	1	$W=\text{triv}$; $\mathbf{q}=(0,1)$	$\ker\omega_{(0,1)}$	—
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$V_2\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_2$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$V_3\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_1$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$\text{triv}\boxtimes\text{triv}\oplus V_1\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_1$	1	$W=\text{triv}$; $\mathbf{q}=(0,0,1)$	$\ker\omega_{(0,0,1)}$	—
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$\text{triv}\boxtimes\text{triv}\oplus V_1\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_1$	1	$W=\text{triv}$; $\mathbf{q}=(0,1,0)$	$\ker\omega_{(0,1,0)}$	—
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$\text{triv}\boxtimes\text{triv}\oplus V_1\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_1$	1	$W=\text{triv}$; $\mathbf{q}=(0,1,1)$	$\ker\omega_{(0,1,1)}$	—
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$\text{triv}\boxtimes\text{triv}\oplus V_1\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_2$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$\text{triv}\boxtimes\text{triv}\oplus V_2\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_1$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sl}_3+\text{junk}$	$2V_{\omega_1}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sl}_3+\text{junk}$	$2V_{\omega_2}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sl}_3+\text{junk}$	$2\text{triv}\oplus V_{\omega_1}$	1	$W=\text{triv}$; $\mathbf{q}=(2)$	$\ker\omega_{(2)}$	—
6	$\mathfrak{sl}_3+\text{junk}$	$2\text{triv}\oplus V_{\omega_2}$	1	$W=\text{triv}$; $\mathbf{q}=(2)$	$\ker\omega_{(2)}$	—
6	$\mathfrak{sl}_3+\text{junk}$	$3\text{triv}\oplus V_{\omega_1}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sl}_3+\text{junk}$	$3\text{triv}\oplus V_{\omega_2}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sl}_3+\text{junk}$	$V_{2\omega_1}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sl}_3+\text{junk}$	$V_{2\omega_2}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sl}_3+\text{junk}$	V_{ω_1}	3	$W=\bigwedge^2 R$	0	—
6	$\mathfrak{sl}_3+\text{junk}$	$V_{\omega_1}\oplus V_{\omega_2}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sl}_3+\text{junk}$	V_{ω_2}	3	$W=\bigwedge^2 R$	0	—
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$V_1\boxtimes V_1\boxtimes\text{triv}\oplus\text{triv}\boxtimes\text{triv}\boxtimes V_1$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$V_1\boxtimes\text{triv}\boxtimes V_1\oplus\text{triv}\boxtimes V_1\boxtimes\text{triv}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$V_1\boxtimes\text{triv}\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_1\boxtimes V_1$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2\oplus\mathfrak{sl}_2)+\text{junk}$	$V_1\boxtimes\text{triv}\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_1\boxtimes V_1$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sp}_4+\text{junk}$	$2\text{triv}\oplus V_{\omega_1}$	0	$W=0$	$\bigwedge^2 R$	abelian

n	$\mathfrak{a}+\text{junk}$	R	d	W and orbit	Example K	Special form
6	$\mathfrak{sp}_4+\text{junk}$	$\text{triv}\oplus V_{\omega_1}$	1	$W=\text{triv}$; $\mathbf{q}=(0,1)$	$\ker\omega_{(0,1)}$	—
6	$\mathfrak{sp}_4+\text{junk}$	$\text{triv}\oplus V_{\omega_2}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_3)+\text{junk}$	$V_1\boxtimes V_{\omega_1}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_3)+\text{junk}$	$V_1\boxtimes V_{\omega_2}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_3)+\text{junk}$	$V_1\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_{\omega_1}$	1	$W=\text{triv}$; $\mathbf{q}=(0,1)$	$\ker\omega_{(0,1)}$	—
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_3)+\text{junk}$	$V_1\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_{\omega_2}$	1	$W=\text{triv}$; $\mathbf{q}=(0,1)$	$\ker\omega_{(0,1)}$	—
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_3)+\text{junk}$	$V_2\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_{\omega_1}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_3)+\text{junk}$	$V_2\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_{\omega_2}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_3)+\text{junk}$	$\text{triv}\boxtimes\text{triv}\oplus V_1\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_{\omega_1}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_3)+\text{junk}$	$\text{triv}\boxtimes\text{triv}\oplus V_1\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_{\omega_2}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sp}_4)+\text{junk}$	$V_1\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_{\omega_1}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sl}_4+\text{junk}$	$2\text{triv}\oplus V_{\omega_1}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sl}_4+\text{junk}$	$2\text{triv}\oplus V_{\omega_3}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sl}_4+\text{junk}$	V_{ω_2}	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_3\oplus\mathfrak{sl}_3)+\text{junk}$	$\text{triv}\boxtimes V_{\omega_1}\oplus V_{\omega_1}\boxtimes\text{triv}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_3\oplus\mathfrak{sl}_3)+\text{junk}$	$\text{triv}\boxtimes V_{\omega_1}\oplus V_{\omega_2}\boxtimes\text{triv}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_3\oplus\mathfrak{sl}_3)+\text{junk}$	$\text{triv}\boxtimes V_{\omega_2}\oplus V_{\omega_1}\boxtimes\text{triv}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_3\oplus\mathfrak{sl}_3)+\text{junk}$	$\text{triv}\boxtimes V_{\omega_2}\oplus V_{\omega_2}\boxtimes\text{triv}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_4)+\text{junk}$	$V_1\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_{\omega_1}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$(\mathfrak{sl}_2\oplus\mathfrak{sl}_4)+\text{junk}$	$V_1\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_{\omega_3}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sp}_6+\text{junk}$	V_{ω_1}	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sl}_5+\text{junk}$	$\text{triv}\oplus V_{\omega_1}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sl}_5+\text{junk}$	$\text{triv}\oplus V_{\omega_4}$	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sl}_6+\text{junk}$	V_{ω_1}	0	$W=0$	$\bigwedge^2 R$	abelian
6	$\mathfrak{sl}_6+\text{junk}$	V_{ω_5}	0	$W=0$	$\bigwedge^2 R$	abelian

Proof. Fix the acting algebra $\mathfrak{s}$ throughout an orbit calculation. If $R=0$, then $K=0$ and $\mathfrak{n}(R,K)=0$, giving the first row for every $\mathfrak{s}$. Henceforth suppose $R\neq 0$. If R is a sum of trivial $\mathfrak{s}$-modules, the action on $\bigwedge^2 R/K$ is also trivial. These are exactly the cases represented by the $0+\text{junk}$ rows, taking $\mathfrak{j}=\mathfrak{s}$; Theorem 2 gives their complete kernel-orbit list through dimension six. Thus an additional row for a nonzero displayed $\mathfrak{a}$ must have at least one nontrivial irreducible constituent in R. Remove from $\mathfrak{s}$ the direct sum $\mathfrak{j}$ of simple ideals acting trivially on R. The remaining algebra $\mathfrak{a}$ acts faithfully; this is the active part in the row label.

Here is the dimension reduction behind the possible active parts. Every nontrivial irreducible constituent of R has dimension at most six. For a simple factor the possibilities in this range are $V_1,\dots,V_5$ for $\mathfrak{sl}_2$; $V_{\omega_1},V_{\omega_2},V_{2\omega_1},V_{2\omega_2}$ for $\mathfrak{sl}_3$ (dimensions $3,3,6,6$); $V_{\omega_1},V_{\omega_3},V_{\omega_2}$ for $\mathfrak{sl}_4$ (dimensions $4,4,6$); $V_{\omega_1},V_{\omega_4}$ for $\mathfrak{sl}_5$; $V_{\omega_1},V_{\omega_5}$ for $\mathfrak{sl}_6$; $V_{\omega_1},V_{\omega_2}$ for $\mathfrak{sp}_4$ (dimensions $4,5$); and V_{ω_1} for $\mathfrak{sp}_6$. For an irreducible on more than one factor, the only possibilities of dimension at most six are $V_1\boxtimes V_1$, $V_1\boxtimes V_2$, and $V_2\boxtimes V_1$ for two $\mathfrak{sl}_2$ factors, and $V_1\boxtimes V_{\omega_1}$ or $V_1\boxtimes V_{\omega_2}$ for $\mathfrak{sl}_2\oplus\mathfrak{sl}_3$. This follows from multiplying the dimensions of the nontrivial factors and the Weyl dimension formula for the simple types; in particular all other simple types have least nontrivial irreducible dimension above six. Take direct sums of these constituents and trivial modules with total dimension at most six, retaining every active factor. This bounds all possible pairs $(\mathfrak{a},R)$, even when a displayed row abbreviates a family of choices of $\mathfrak{j}$.

Put $r=\dim R$ and $d=\dim(\bigwedge^2 R/K)$. Since $r+d\leq 6$, the remaining orbit calculations break down as follows. For $d=0$, $K=\bigwedge^2 R$: every allowed module R gives its own abelian component, with no extra kernel choice. For $d=1$, the quotient is trivial as an $\mathfrak{s}$-module, and K is the kernel of a nonzero invariant alternating form, considered up to scalar and $\mathrm{Aut}_{\mathfrak{s}}(R)$. Its radical T and a nondegenerate equivariant complement S give $R\cong S\oplus T$. For $r\leq 5$ its rank is two or four. In rank two, S is either 2triv or the standard V_1 of an $\mathfrak{sl}_2$ factor; T is any complementary module of dimension $r-2$. In rank four, S is one of 4triv, $V_1\oplus 2\text{triv}$, $2V_1$ for one $\mathfrak{sl}_2$, $V_1\boxtimes\text{triv}\oplus\text{triv}\boxtimes V_1$ for two $\mathfrak{sl}_2$ factors, V_3 for $\mathfrak{sl}_2$, or V_{ω_1} for $\mathfrak{sp}_4$; here T has dimension $r-4$. In each case the nondegenerate invariant form

is unique up to an $\mathfrak{s}$-module automorphism and scalar. Keeping the isomorphism types of S and T, with the same fixed $\mathfrak{s}$, is therefore the complete rank-one-quotient orbit test. For $r = 5$ only $d = 0, 1$ occur, so this finishes that dimension.

For $r = 4$, the only further value is $d = 2$. Complete reducibility makes the two-dimensional quotient either 2triv or a standard V_1 for an $\mathfrak{sl}_2$ factor. A quotient V_1 occurs precisely for $R = V_1 \oplus 2\mathrm{triv}$ and uses one of the two cross-term copies of V_1; GL_2 on the trivial multiplicity space is transitive on its nonzero quotient lines. A quotient 2triv requires at least two independent invariant alternating forms. Apart from the purely trivial case, this leaves $R = 2V_1$ for one factor, $R = V_1 \oplus 2\mathrm{triv}$, or the sum of the two standards for two distinct $\mathfrak{sl}_2$ factors. For $2V_1$ the kernel choice is a line in $\mathrm{Sym}^2 M$, $\dim M = 2$, with the two rank orbits described above; in the other active cases the two trivial summands of $\bigwedge^2 R$ give a unique quotient. For $R = 4\mathrm{triv}$, the annihilator pencil is contained in, secant to, or tangent to the Klein quadric; these are the three classes $(R_4, K_4(12, 13))$, $(R_4, K_4(12, 34))$ and $(R_4, K_4(12, 13 + 24))$ of Theorem 2.

For $r = 3$, a nontrivial R is V_2 or $V_1 \oplus \mathrm{triv}$ for $\mathfrak{sl}_2$, or a standard module or its dual for $\mathfrak{sl}_3$. Their exterior squares are respectively V_2, $V_1 \oplus \mathrm{triv}$, and the opposite standard. The first and last admit only $K = 0$ or the full exterior square; the middle admits in addition its two individual summands as kernels. For $r = 2$, the only active R is V_1 for $\mathfrak{sl}_2$, with $K = 0$ or the full exterior square. There is no active R of smaller dimension. Finally $r = 6$ forces $d = 0$, already covered. This exhausts the dimension budget and gives every row of the displayed table.

For each displayed row, direct exterior-square decomposition and substitution of its example K verify the stated orbit condition. The equality $\mathrm{Gr}_q^{\mathfrak{a}\oplus\mathfrak{j}}(\bigwedge^2 R) = \mathrm{Gr}_q^{\mathfrak{a}}(\bigwedge^2 R)$ and $\mathrm{Aut}_{\mathfrak{a}\oplus\mathfrak{j}}(R) = \mathrm{Aut}_{\mathfrak{a}}(R)$ when $\mathfrak{j}$ acts trivially makes each such calculation uniform in the wildcard $\mathfrak{j}$. □

Remark 61 (No continuous moduli below the cutoff)**.** *The cutoff* 6 *is small enough that there are no continuous moduli: by Theorem 60, each* $\pi_0(\mathcal{C}_{\mathfrak{s}}^{\leq 6})$ *is finite. Larger cutoffs bring in continuous moduli, for instance the invariant* η *of* (34)*. For semisimple* $\mathfrak{s}$*, the slice of Theorem 51 is* $\mathfrak{s} = \mathfrak{sl}_2$ *with* $R = V_3 \otimes M_2$*, the case* $m = 0$ *of Proposition 49; already* $\dim R = 8$*. By* (39)*,*

$$\dim\big(\textstyle\bigwedge^2 R/K\big) = 7(1 - \dim L_0) + 5(3 - \dim U_1) + 3(1 - \dim L_2) + (3 - \dim U_3).$$

The rows of Theorem 51 with parameter η *have* U_1 *and* U_3 *each a line or a plane, so their total dimension is at least* $8 + 5 + 1 = 14$*. This bound is attained, with* U_1, U_3 *planes and* $L_0 = L_2 = \bigwedge^2 M_2$*, for every* $\eta \neq 4$*. Within the cutoff,* V_3 *occurs only with a one-dimensional multiplicity space.*

6 Mutually annihilating pairs of Lie brackets

The class-two objects above supply one source of pairs of Lie brackets on the same object. Pairing these brackets with two separately retained commutative associative products produces a further Lie algebra (Section 6.1). The rest of this section studies the pairs themselves: their groupoid and the exchange of the two brackets, the assembly of pairs in vector spaces from ordinary Lie algebras, and examples.

The operadic statements below are over the characteristic-zero field k. The algebraic tensor construction allows infinite-dimensional objects, so the finite-dimensional convention is relaxed in Section 6.1; it is restored from Section 6.2 to the end of this section. Operations in the super setting are even, and $|x|$ denotes the parity of a homogeneous element.

6.1 The quadratic-dual pair and its tensor construction

Definition 62 (Two independent commutative products)**.** *Write* $\mathrm{CommComm} = \mathrm{Com} \amalg \mathrm{Com}$ *for the coproduct of the two reduced operads governing commutative associative algebras without a specified*

unit. An algebra over it has products μ_1, μ_2 satisfying, separately for $i = 1, 2$,

$$\mu_i(a, b) = \mu_i(b, a), \qquad \mu_i(\mu_i(a, b), c) = \mu_i(a, \mu_i(b, c)).$$

There are no relations involving both products. In super vector spaces, commutativity means $\mu_i(a, b) = (-1)^{|a||b|}\mu_i(b, a)$. Units may exist in particular examples, but are not part of the operadic data.

Definition 63 (Mutually annihilating brackets)**.** *Write* LieLieZero *for the operad with two skew-symmetric binary operations β_1, β_2, each satisfying Jacobi, and with*

$$\beta_1(\beta_2(x, y), z) = 0, \qquad \beta_2(\beta_1(x, y), z) = 0. \tag{42}$$

In super vector spaces, skew-symmetry and Jacobi mean

$$\beta_i(x, y) = -(-1)^{|x||y|}\beta_i(y, x), \qquad \sum_{\text{cyc}} (-1)^{|x||z|}\beta_i(x, \beta_i(y, z)) = 0.$$

Proposition 64 (Quadratic duality)**.** *With the sign-twisted dual convention for binary quadratic operads,*

$$(\text{Com} \amalg \text{Com})^! \cong \text{LieLieZero}.$$

Proof. The binary generator space consists of two symmetric generators. Its sign-twisted dual consists of two skew-symmetric generators. In arity three, split the free-operad space according to the two vertex labels. On each monochromatic summand, orthogonality of relations gives the usual $\text{Com}^! = \text{Lie}$ calculation. The original relation space has zero component in the mixed summands, so its orthogonal contains those entire summands. Every mixed composite is therefore zero in the dual operad. This is the quadratic-dual construction of [12, Theorem 7.6.5]. □

The tensor construction uses only the quadratic dual $\mathcal{P}^!$. For a binary quadratic operad $\mathcal{P}$ with finite-dimensional generator space, the canonical morphism

$$\text{Lie} \longrightarrow \mathcal{P} \otimes_{\text{H}} \mathcal{P}^!$$

uses the coevaluation of the generator space and its sign-twisted dual; $\otimes_{\text{H}}$ denotes the arity-wise tensor product of operads. It gives a Lie algebra on the tensor product of a $\mathcal{P}$-algebra and a $\mathcal{P}^!$-algebra [14, Corollary 2.2.9(b)], [12, Proposition 7.6.10]. For Proposition 64 this becomes the following explicit formula.

Theorem 65 (The paired tensor bracket)**.** *Let (A, μ_1, μ_2) be a* CommComm*-algebra and (B, β_1, β_2) a* LieLieZero*-algebra. The formula*

$$[a \otimes x, b \otimes y] = \mu_1(a, b) \otimes \beta_1(x, y) + \mu_2(a, b) \otimes \beta_2(x, y) \tag{43}$$

defines a Lie algebra $L(A, B)$ on $A \otimes B$. In super vector spaces the right-hand side is multiplied by $(-1)^{|x||b|}$, the sign for interchanging the middle tensor factors. Morphisms preserving the two operations on each factor induce Lie algebra morphisms on their tensor products. No scalar weights are needed: rescaling β_1 or β_2 by a scalar gives another LieLieZero*-algebra, and likewise for μ_1, μ_2.*

Proof. Commutativity and skew-symmetry give skew-symmetry of the tensor bracket. In the ordinary case, its Jacobiator on pure tensors is the sum, over $i, j \in \{1, 2\}$, of

$$\sum_{\text{cyc}} \mu_j(\mu_i(a, b), c) \otimes \beta_j(\beta_i(x, y), z).$$

If $i \neq j$, each summand vanishes by (42). If $i = j$, associativity and commutativity give a common coefficient multiplying the Jacobiator of β_i. In the super case the symmetry morphisms give the displayed Koszul sign and the same argument uses super Jacobi. □

Remark 66 (Compatibility is weaker)**.** *The operad* Lie_2 *of [13] requires each bracket to be Lie and only the mixed cyclic identity*

$$\sum_{\mathrm{cyc}} \big(\beta_1(\beta_2(x,y),z) + \beta_2(\beta_1(x,y),z)\big) = 0$$

in the ordinary case. It guarantees that every scalar linear combination is Lie. In (43), however, mixed nested brackets carry different coefficients from the two products on A*. Their cyclic cancellation alone does not justify the construction with unrestricted* μ_1, μ_2*. There is a quotient* $\mathrm{Lie}_2 \twoheadrightarrow \mathrm{LieLieZero}$*; the latter imposes the individual mixed vanishings. In order to have the capability to do two different multiplications with no law between them, the bracket side becomes stricter as* LieLieZero *vs just* Lie_2*.*

6.2 The groupoid of pairs and the bracket exchange

From here to the end of Section 6 we return to the global conventions of the earlier sections: k is algebraically closed of characteristic zero, all representations are finite-dimensional, and $\mathfrak{s}$ is reductive, possibly zero, with a central action that need not be semisimple. Let $\mathcal{M}$ be $\mathfrak{s}$-Rep or its super analogue $\mathfrak{s}$-sRep; the case $\mathfrak{s} = 0$ gives Vect_k and sVect_k, which are included as this case. Skew-symmetry and Jacobi use the symmetry of $\mathcal{M}$; in particular an odd element need not have zero self-bracket. All subobjects, centers and isomorphisms are taken in $\mathcal{M}$, so arrows are even and $\mathfrak{s}$-equivariant. Since the central action may be nonsemisimple, equivariant splittings are not automatic; none of the arguments below uses one.

Definition 67 (The pair groupoid)**.** *Let* $\mathcal{G}_{\mathcal{M}} = \mathrm{Core}\,\mathrm{Alg}_{\mathrm{LieLieZero}}(\mathcal{M})$ *be the groupoid of* LieLieZero*-algebras in* $\mathcal{M}$ *(Definition 63) and their isomorphisms. Its objects are triples* $X = (B, \beta_1, \beta_2)$ *with* $B \in \mathcal{M}$*, each* $\beta_i \colon B \otimes B \to B$ *a Lie bracket in* $\mathcal{M}$*, and*

$$\beta_1 \circ (\beta_2 \otimes \mathrm{id}) = 0, \qquad \beta_2 \circ (\beta_1 \otimes \mathrm{id}) = 0.$$

The case $\beta_1 = \beta_2$ *is allowed, in which case it satisfies the mixed identities exactly when the common bracket has nilpotency class at most two. An arrow* $X \to X'$ *is an isomorphism* $g \colon B \to B'$ *in* $\mathcal{M}$ *with* $g\,\beta_i = \beta_i'\,(g \otimes g)$ *for* $i = 1, 2$*; the bracket labels are preserved. If both brackets have class at most two, each* (B, β_i) *is an object of* $\mathrm{LieAlg}_{\leq 2}(\mathcal{M})$*, defined as in Section 4.3, with quotient* $B/\operatorname{im}\beta_i$ *and derived object* $\operatorname{im}\beta_i$ *playing the roles of* R *and* $\bigwedge^2 R/K$ *in Proposition 5.*

The mixed identities are equivalent to $\operatorname{im}\beta_1 \subseteq Z(\beta_2)$ and $\operatorname{im}\beta_2 \subseteq Z(\beta_1)$.

Definition 68 (Bracket exchange and underlying object)**.** *Let* $\sigma \colon \mathcal{G}_{\mathcal{M}} \to \mathcal{G}_{\mathcal{M}}$ *be the strict involution given by* $\sigma(B, \beta_1, \beta_2) = (B, \beta_2, \beta_1)$ *on objects and by the same underlying isomorphism on arrows. Let* Fgt *send an object to its underlying vector space, or super vector space, and an arrow to its underlying linear map, so that* $\mathrm{Fgt} \circ \sigma = \mathrm{Fgt}$*. Every* Hom*-set of* $\mathcal{G}_{\mathcal{M}}$ *is thereby a subset of the linear maps between underlying spaces, cut out by* $\mathfrak{s}$*-equivariance and the bracket equations. In particular* σ *restricts to a group isomorphism* $\mathrm{Aut}(X) \to \mathrm{Aut}(\sigma X)$*,* $g \mapsto \sigma(g)$*, and* $\mathrm{Fgt}(\sigma(g)) = \mathrm{Fgt}(g)$*.*

Definition 69 (Switch; the switchable groupoid)**.** *A* switch *of* X *is an arrow* $F \colon X \to \sigma X$ *of* $\mathcal{G}_{\mathcal{M}}$*. Its underlying linear isomorphism* $\mathrm{Fgt}(F) \colon B \to B$ *is* $\mathfrak{s}$*-equivariant and satisfies*

$$\mathrm{Fgt}(F)\beta_1 = \beta_2\,(\mathrm{Fgt}(F) \otimes \mathrm{Fgt}(F)), \qquad \mathrm{Fgt}(F)\beta_2 = \beta_1\,(\mathrm{Fgt}(F) \otimes \mathrm{Fgt}(F)).$$

Let $\mathcal{G}_{\mathcal{M}}^{\mathrm{sw}}$ *be the full subgroupoid of* $\mathcal{G}_{\mathcal{M}}$ *on the objects* X *with* $\mathrm{Hom}(X, \sigma X) \neq \varnothing$*. No switch is chosen as part of an object. This subgroupoid is replete, and* σ *maps it to itself. At each object* X *we record the two hom-sets*

$$\mathrm{Hom}(X, X) = \mathrm{Aut}(X) \qquad \textit{and} \qquad \mathrm{Hom}(X, \sigma X).$$

An arrow $h \colon X \to Y$ *transports them to* Y *by* $g \mapsto hgh^{-1}$ *and* $F \mapsto \sigma(h)\,F\,h^{-1}$*.*

Proposition 70 (The set of switches). *Let $X \in \mathcal{G}_{\mathcal{M}}^{\mathrm{sw}}$ and $F_0 \in \mathrm{Hom}(X, \sigma X)$.*

1. *Precomposition $g \mapsto F_0 \circ g$ is a bijection $\mathrm{Aut}(X) \to \mathrm{Hom}(X, \sigma X)$, and postcomposition $h \mapsto h \circ F_0$ is a bijection $\mathrm{Aut}(\sigma X) \to \mathrm{Hom}(X, \sigma X)$. Thus $\mathrm{Hom}(X, \sigma X)$ is a right $\mathrm{Aut}(X)$-torsor and a left $\mathrm{Aut}(\sigma X)$-torsor.*

2. *The composite $c = \sigma(F_0) \circ F_0 \colon X \to \sigma X \to X$ lies in $\mathrm{Aut}(X)$.*

3. *The map $\theta(g) = F_0^{-1} \circ \sigma(g) \circ F_0$ is an automorphism of the group $\mathrm{Aut}(X)$ with $\theta^2(g) = c^{-1} g c$. Replacing F_0 by $F_0 \circ h$ replaces θ by $h^{-1}\theta(-)h$. Hence the class $[\theta] \in \mathrm{Out}(\mathrm{Aut}\, X)$ depends only on X, and $[\theta]^2 = 1$.*

Proof. For switches $F, F' \colon X \to \sigma X$, the composite $F^{-1} \circ F'$ is an arrow $X \to X$ and $F' \circ F^{-1}$ is an arrow $\sigma X \to \sigma X$, which gives (1). Since σ is a strict involution, $\sigma(F_0) \colon \sigma X \to X$, which gives (2). For (3), $\sigma(g) \in \mathrm{Aut}(\sigma X)$, so $\theta(g) \in \mathrm{Aut}(X)$ and θ is multiplicative. Using $\sigma(F_0) = c\, F_0^{-1}$ and $\sigma^2 = \mathrm{id}$,

$$\theta^2(g) = F_0^{-1}\, \sigma(F_0)^{-1}\, g\, \sigma(F_0)\, F_0 = c^{-1} g c.$$

Finally $(F_0 h)^{-1} \sigma(g) (F_0 h) = h^{-1} \theta(g) h$. □

Definition 71 (Homotopy fixed points). *Let $\mathcal{G}_{\mathcal{M}}^{h\mathbb{Z}/2}$ have objects (X, F) with $F \colon X \to \sigma X$ and $\sigma(F) \circ F = \mathrm{id}_X$; such an F is an* involutive switch. *An arrow $a \colon (X, F) \to (Y, F')$ is an arrow $a \colon X \to Y$ of $\mathcal{G}_{\mathcal{M}}$ with $\sigma(a) \circ F = F' \circ a$ as arrows $X \to \sigma Y$. The forgetful functor $\mathcal{G}_{\mathcal{M}}^{h\mathbb{Z}/2} \to \mathcal{G}_{\mathcal{M}}^{\mathrm{sw}}$ is faithful and in general not full, since arrows of $\mathcal{G}_{\mathcal{M}}^{h\mathbb{Z}/2}$ must commute with the chosen switches. An object X lies in its essential image exactly when some $F \in \mathrm{Hom}(X, \sigma X)$ satisfies $\sigma(F) \circ F = \mathrm{id}_X$.*

Proposition 72 (Lifting and fibers). *Let F_0, c and θ be as in Proposition 70. For $g \in \mathrm{Aut}(X)$,*

$$\sigma(F_0 g) \circ (F_0 g) = c\, \theta(g)\, g,$$

so $F_0 g$ is involutive exactly when $c\, \theta(g)\, g = 1$. An element $a \in \mathrm{Aut}(X)$ is an arrow $(X, F_0 g) \to (X, F_0 g')$ exactly when $g' = \theta(a)\, g\, a^{-1}$. Hence, if F_0 is itself involutive, the objects of $\mathcal{G}_{\mathcal{M}}^{h\mathbb{Z}/2}$ with underlying object X, up to the arrows over $\mathrm{Aut}(X)$, are classified by

$$H^1(\mathbb{Z}/2, \mathrm{Aut}\, X) = \{g : \ \theta(g)\, g = 1\} \big/ \big(g \sim \theta(a)\, g\, a^{-1}\big),$$

where the generator of $\mathbb{Z}/2$ acts on $\mathrm{Aut}(X)$ through θ, and $\mathrm{Aut}(X, F_0 g) = \{a \in \mathrm{Aut}(X) : \ \theta(a)\, g = g\, a\}$.

Proof. $\sigma(F_0 g)\, F_0 g = \sigma(F_0)\, \sigma(g)\, F_0\, g = c\, F_0^{-1} \sigma(g) F_0\, g = c\, \theta(g)\, g$. The arrow condition $\sigma(a)\, F_0 g = F_0 g'\, a$ is $g' = F_0^{-1} \sigma(a) F_0\, g\, a^{-1} = \theta(a)\, g\, a^{-1}$. □

Remark 73 (Parity in super categories). *In $\mathfrak{s}$-sRep the parity map $(-1)^{|\cdot|}$ of B commutes with all even maps and preserves both brackets, since they are even. It therefore underlies a central element $\mathsf{P}_X \in \mathrm{Aut}(X)$, with $\sigma(\mathsf{P}_X) = \mathsf{P}_{\sigma X}$. It underlies an arrow $X \to \sigma X$ only when the two brackets of X agree. If $F \colon X \to \sigma X$ is an involutive switch, so is $F \circ \mathsf{P}_X$. Their classes in $H^1(\mathbb{Z}/2, \mathrm{Aut}\, X)$ need not coincide.*

Definition 74 (Decomposability). *Direct sums are formed blockwise in $\mathcal{G}_{\mathcal{M}}$, and $\sigma(X \oplus X') = \sigma X \oplus \sigma X'$. Hence $\mathcal{G}_{\mathcal{M}}^{\mathrm{sw}}$ is closed under them, using $F \oplus F' \colon X \oplus X' \to \sigma X \oplus \sigma X'$. An object X of $\mathcal{G}_{\mathcal{M}}^{\mathrm{sw}}$ is* decomposable *if $X \cong X_1 \oplus X_2$ with X_1, X_2 nonzero objects of $\mathcal{G}_{\mathcal{M}}^{\mathrm{sw}}$, so that each summand is itself switchable. No switch is specified. In $\mathcal{G}_{\mathcal{M}}^{h\mathbb{Z}/2}$ the summands must be objects (X_i, F_i) with $F = F_1 \oplus F_2$.*

Proposition 75 (Symmetrization). *For every object X of $\mathcal{G}_{\mathcal{M}}$, the object $X \oplus \sigma X$ lies in $\mathcal{G}_{\mathcal{M}}^{\mathrm{sw}}$, and the exchange of the two summands underlies an involutive switch $X \oplus \sigma X \to \sigma(X \oplus \sigma X)$.*

Proof. Write $X = (B, \beta_1, \beta_2)$. Then $X \oplus \sigma X = (B \oplus B,\ \beta_1 \oplus \beta_2,\ \beta_2 \oplus \beta_1)$ and $\sigma(X \oplus \sigma X) = (B \oplus B,\ \beta_2 \oplus \beta_1,\ \beta_1 \oplus \beta_2)$. The exchange $\tau(b, b') = (b', b)$ satisfies $\tau(\beta_1 \oplus \beta_2) = (\beta_2 \oplus \beta_1)(\tau \otimes \tau)$ and $\tau(\beta_2 \oplus \beta_1) = (\beta_1 \oplus \beta_2)(\tau \otimes \tau)$, and $\tau^2 = \mathrm{id}$. □

When X itself is not switchable, $X \oplus \sigma X$ is a direct sum of two nonzero objects of $\mathcal{G}_{\mathcal{M}}$ but need not be decomposable in $\mathcal{G}^{\mathrm{sw}}_{\mathcal{M}}$; the object S of Proposition 82 is an example.

6.3 Assembly from ordinary Lie algebras

The following structural description is for ordinary vector spaces over k. Its construction is meaningful in other abelian symmetric monoidal categories, but the converse below uses a linear splitting.

Theorem 76 (Fiber product and central identification)**.** *Take ordinary Lie algebras L_1, L_2, surjections $p_i \colon L_i \to H$ onto an abelian Lie algebra, and injections*

$$j_i \colon C \hookrightarrow Z(L_i) \cap \ker p_i.$$

The vector space

$$B = (L_1 \times_H L_2)/\Delta_C^-, \qquad \Delta_C^- := \{(j_1 c, -j_2 c) : c \in C\}, \tag{44}$$

has mutually annihilating brackets induced by

$$\beta_1((x_1, x_2), (y_1, y_2)) = ([x_1, y_1], 0), \qquad \beta_2((x_1, x_2), (y_1, y_2)) = (0, [x_2, y_2]).$$

Every LieLieZero*-algebra in vector spaces admits such a presentation with $\ker p_i = [L_i, L_i]$. In these normalized presentations, isomorphism classes of pairs are in bijection with isomorphism classes of the diagrams $(L_1, L_2, H, C, p_1, p_2, j_1, j_2)$. The two bracket labels are preserved.*

Proof. Since H is abelian, the displayed brackets take values in the fiber product. The subspace Δ_C^- is central for both, so they descend. Jacobi holds in each coordinate and every mixed composition is zero.

Conversely, put $I_i = \operatorname{im} \beta_i$, $C = I_1 \cap I_2$, and $H = B/(I_1 + I_2)$. Mixed vanishing gives $I_1 \subseteq Z(B, \beta_2)$ and $I_2 \subseteq Z(B, \beta_1)$. Choose a linear section $s \colon H \to B$ and set $L_i = s(H) + I_i$, using β_i on L_i. Because the other derived subspace is central for that bracket, $[L_i, L_i] = I_i$. The map

$$(s(h) + u,\ s(h) + w) \longmapsto s(h) + u + w$$

from $L_1 \times_H L_2$ onto B preserves both brackets and has kernel $\{(c, -c) : c \in C\}$. This proves the asserted presentation.

To check independence of the section, write $s' - s = u + w$ with linear maps $u \colon H \to I_1$, $w \colon H \to I_2$. The map $s(h) + a \mapsto s(h) + w(h) + a$ identifies the first L_1 with the one constructed from s': $w(H)$ is central for β_1. It fixes I_1 and induces the identity on H. For L_2 use the corresponding addition of $u(h)$. Both maps fix C. An isomorphism of pairs preserves I_1, I_2, C, H and therefore induces an isomorphism of the resulting diagrams, after these section changes. Conversely, a diagram isomorphism induces an isomorphism of (44). Recovering a diagram from its assembled pair returns an isomorphic diagram: choose sections of both p_i and use their paired section. □

6.4 Examples of pairs

Example 77 (Two class-two brackets with common quotient and center)**.** *Let R and W be $\mathfrak{s}$-modules and let $b_1, b_2 \colon \bigwedge^2 R \to W$ be $\mathfrak{s}$-module maps. On $B = R \oplus W$ put*

$$\beta_i((r, w), (r', w')) = (0, b_i(r \wedge r')).$$

Both brackets have central image, so all same-label and mixed nested brackets vanish, and $X = (B, \beta_1, \beta_2)$ *is an object of* $\mathcal{G}_{\mathfrak{s}\text{-Rep}}$. *No surjectivity or nondegeneracy is required. If* b_i *is surjective with kernel* K_i, *then* (B, β_i) *is the class-two radical* $R \oplus (\bigwedge^2 R/K_i)$ *of Proposition 5, with* W *identified with* $W_{K_i} = \bigwedge^2 R/K_i$ *through* b_i.

Conversely, start from two objects (R_1, K_1) *and* (R_2, K_2) *of the earlier sections. Choose* $\mathfrak{s}$*-module isomorphisms* $\phi_R \colon R_1 \to R_2$ *and* $\phi_W \colon W_{K_1} \to W_{K_2}$. *The second bracket is transported to* $R_1 \oplus W_{K_1}$ *by*

$$\widetilde{b}_2(r \wedge r') = \phi_W^{-1}\, b_{K_2}(\phi_R r \wedge \phi_R r').$$

These identifications are data; the two brackets are subsequently classified under a simultaneous change of coordinates, that is, by arrows of $\mathcal{G}_{\mathfrak{s}\text{-Rep}}$. *Two unrelated class-two brackets on the same space do not automatically have zero mixed compositions: their central subspaces must interact as required.*

Proposition 78 (Arrows between common-quotient pairs)**.** *In Example 77, put* $\operatorname{rad} b_i = \{r \in R : b_i(r \wedge R) = 0\}$ *and suppose* $\operatorname{rad} b_1 \cap \operatorname{rad} b_2 = 0$. *Then* Fgt *identifies* $\mathrm{Hom}(X, \sigma X)$ *with the set of maps*

$$(r, w) \longmapsto \big(a(r),\ h(r) + q(w)\big), \qquad a \in \mathrm{Aut}_{\mathfrak{s}}(R),\ q \in \mathrm{Aut}_{\mathfrak{s}}(W),\ h \in \mathrm{Hom}_{\mathfrak{s}}(R, W),$$

subject to

$$q\, b_1 = b_2 \circ \textstyle\bigwedge^2 a, \qquad q\, b_2 = b_1 \circ \bigwedge^2 a,$$

with h *unrestricted. The same holds for* $\mathrm{Hom}(X, X)$ *with the conditions* $q\, b_i = b_i \circ \bigwedge^2 a$. *In particular* $X \in \mathcal{G}^{\mathrm{sw}}_{\mathfrak{s}\text{-Rep}}$ *exactly when such a pair* (a, q) *exists, and a switch is involutive exactly when* $a^2 = \mathrm{id}_R$, $q^2 = \mathrm{id}_W$ *and* $ha + qh = 0$*; for instance when* $h = 0$. *When* b_1 *and* b_2 *are surjective with kernels* K_1, K_2, *the conditions imply* $(\bigwedge^2 a)K_1 = K_2$ *and* $(\bigwedge^2 a)K_2 = K_1$.

Proof. An element (r, w) is central for β_i exactly when $r \in \operatorname{rad} b_i$, so the common center $Z(\beta_1) \cap Z(\beta_2)$ is $(\operatorname{rad} b_1 \cap \operatorname{rad} b_2) \oplus W = W$. An arrow $X \to \sigma X$ carries $Z(\beta_1)$ to $Z(\beta_2)$ and $Z(\beta_2)$ to $Z(\beta_1)$, hence preserves W. It therefore has the displayed form, where a is the induced automorphism of $B/W = R$ and q its restriction to W, both $\mathfrak{s}$-equivariant. Both brackets depend only on the R-components and take values in W, so the arrow conditions $g\beta_1 = \beta_2(g \otimes g)$ and $g\beta_2 = \beta_1(g \otimes g)$ reduce to the two displayed equations, and impose nothing on h. Conversely every such map satisfies them. The case of $\mathrm{Hom}(X, X)$ is identical, and the square of the displayed map is $(r, w) \mapsto \big(a^2 r,\ (ha + qh)(r) + q^2 w\big)$. The statement about kernels follows by applying the equations to $K_i = \ker b_i$. □

In the notation of Theorem 8, $p = 0$ and the maps a, h, q are the blocks of the arrow; the shear is $H = ha^{-1}$ as in Proposition 11, and the condition on kernels is the condition of Theorem 10 with the two kernels exchanged.

Example 79 (Shared abelian acting algebra)**.** *Let* H *be an abelian Lie algebra, allowed to be zero, and let* U_1, U_2 *be arbitrary Lie algebras. Choose Lie algebra homomorphisms*

$$\delta_1 \colon H \to \mathrm{Der}(U_1), \qquad \delta_2 \colon H \to \mathrm{Der}(U_2).$$

Thus the derivations in each action commute. On $B = H \oplus U_1 \oplus U_2$ *set*

$$\begin{aligned}\beta_1((h, u, w), (h', u', w')) &= (0,\ [u, u']_{U_1} + \delta_1(h)u' - \delta_1(h')u,\ 0),\\ \beta_2((h, u, w), (h', u', w')) &= (0,\ 0,\ [w, w']_{U_2} + \delta_2(h)w' - \delta_2(h')w).\end{aligned}$$

Each bracket is a semidirect-product bracket with an additional central summand. The first has image in U_1, *which is central for the second; the second has image in* U_2, *which is central for the first. Internal brackets on* U_1, U_2 *are retained, and neither algebra need be nilpotent. The choice* $H = 0$ *includes brackets supported on the separate summands of* $U_1 \oplus U_2$. *In the presentation of Theorem 76, this example uses the split extensions* $H \ltimes U_1$ *and* $H \ltimes U_2$ *over the displayed* H, *without requiring these quotient maps to be their full abelianizations.*

Proposition 80 (Switches of semidirect pairs)**.** *In Example 79, let $\phi\colon U_1 \to U_2$ be a Lie algebra isomorphism and α a linear automorphism of H with*

$$\phi\,\delta_1(h)\,\phi^{-1} = \delta_2(\alpha h), \qquad \phi\,\delta_1(\alpha h)\,\phi^{-1} = \delta_2(h) \qquad (h \in H).$$

Then $(h,u,w) \mapsto (\alpha h,\ \phi^{-1}w,\ \phi u)$ is Fgt *of an arrow $X \to \sigma X$. Its square is $\alpha^2 \oplus \mathrm{id}$, so it is an involutive switch when $\alpha^2 = \mathrm{id}_H$; in particular when $\alpha = \mathrm{id}_H$ and ϕ intertwines δ_1 with δ_2.*

Proof. Write g for the displayed map. Applied to β_1,

$$g\beta_1((h,u,w),(h',u',w')) = \big(0,0,\ \phi[u,u']_{U_1} + \phi\delta_1(h)u' - \phi\delta_1(h')u\big),$$

and the first condition rewrites the last entry as $[\phi u,\phi u']_{U_2} + \delta_2(\alpha h)\phi u' - \delta_2(\alpha h')\phi u$, which is the U_2-entry of $\beta_2(g(h,u,w),g(h',u',w'))$. Applied to β_2, the second condition in the form $\phi^{-1}\delta_2(h)\phi = \delta_1(\alpha h)$ gives $g\beta_2 = \beta_1(g\otimes g)$ in the same way. The square is $(h,u,w) \mapsto (\alpha^2 h,u,w)$. □

6.5 Small examples and equivariant switches

As in Definition 68, every Hom-set below is described as a subset of the linear maps of $\mathrm{Fgt}(X)$.

Proposition 81 (Total dimension at most two)**.** *Let $\mathfrak{s} = 0$.*

1. *In superdimensions $1\,|\,0$, $0\,|\,1$ and $0\,|\,2$ both brackets vanish.*
2. *In superdimension $2\,|\,0$ every class of $\mathcal{G}_{\mathcal{M}}$ is represented by exactly one of*

$$(\beta_1,\beta_2) = (0,0), \qquad \big([x,y]_1 = y,\ 0\big), \qquad \big(0,\ [x,y]_2 = y\big).$$

   Only $(0,0)$ lies in $\mathcal{G}^{\mathrm{sw}}_{\mathcal{M}}$.

3. *In superdimension $1\,|\,1$, with x even and ϑ odd, every class of $\mathcal{G}_{\mathcal{M}}$ is represented by exactly one of the zero pair,*

$$\big([x,\vartheta]_1 = \vartheta,\ 0\big), \qquad \big(0,\ [x,\vartheta]_2 = \vartheta\big), \qquad \big(0,\ [\vartheta,\vartheta]_2 = x\big),$$
$$O_\gamma = \big([\vartheta,\vartheta]_1 = x,\ [\vartheta,\vartheta]_2 = \gamma x\big) \qquad (\gamma \in k).$$

   The objects of $\mathcal{G}^{\mathrm{sw}}_{\mathcal{M}}$ among them are the zero pair and the objects $O_{\pm 1}$ with $\gamma = \pm 1$. The identity map underlies an involutive switch $O_{+1} \to \sigma O_{+1}$, and $x \mapsto -x$, $\vartheta \mapsto \vartheta$ underlies one for O_{-1}.

Proof. (1) An even bracket on $1\,|\,0$ vanishes by skew-symmetry. On $0\,|\,1$ and $0\,|\,2$ every bracket of two odd vectors lies in the even part, which is zero.

(2) A nonabelian Lie bracket on k^2 has zero center and image of dimension one. If $\beta_1 \neq 0$, then $\operatorname{im}\beta_1 \subseteq Z(\beta_2)$ forces $\beta_2 = 0$, and symmetrically. If $\beta_2 = 0$, every element of GL_2 preserves β_2, so the arrows act on β_1 alone and bring it to $[x,y]_1 = y$; likewise with the labels exchanged. The two nonzero representatives are not isomorphic, since arrows preserve labels. A switch carries β_1 to β_2, so a switchable object has both brackets zero or both nonzero; only $(0,0)$ remains.

(3) Write $[x,\vartheta]_i = \alpha_i\vartheta$ and $[\vartheta,\vartheta]_i = \gamma_i x$. Jacobi for each bracket is $\alpha_i\gamma_i = 0$, and mutual annihilation is $\alpha_1\alpha_2 = \alpha_1\gamma_2 = \alpha_2\gamma_1 = 0$. The even linear automorphisms are $x \mapsto ax$, $\vartheta \mapsto b\vartheta$, and they act on both brackets at once by $(\alpha_i,\gamma_i) \mapsto (a\alpha_i,\ a^{-1}b^2\gamma_i)$ with the same a,b. If some $\alpha_i \neq 0$, the equations force the other three coefficients to vanish, and a normalizes α_i to 1. If $\alpha_1 = \alpha_2 = 0$, both γ_i are rescaled by the single factor $a^{-1}b^2$, so the ratio $[\gamma_1 : \gamma_2] \in \mathbb{P}^1(k)$ is a complete invariant, and the displayed representatives realize each value once. The exchange σ sends $[\gamma_1 : \gamma_2]$ to $[\gamma_2 : \gamma_1]$ and exchanges the two semidirect classes, so the switchable classes are the zero pair and $\gamma^2 = 1$. The displayed maps satisfy $g\beta_1 = \beta_2(g\otimes g)$ and $g\beta_2 = \beta_1(g\otimes g)$ and square to the identity. □

Write $\mathfrak{r}_2$ for the nonabelian two-dimensional Lie algebra, $[e_1, e_2] = e_2$. For the next proposition, recall Theorem 2 and its notation $ij = u^i \wedge u^j$. The objects A_ϵ, N, D below are instances of Example 77, with R spanned by the noncentral basis vectors and W by the central ones: A_ϵ has $R = R_2$, $W = k$, $b_1 = 12$, $b_2 = \epsilon \cdot 12$; N has $R = R_3$, $W = k$, $b_1 = 23$, $b_2 = 13$; and D has $R = R_2$, $W = k^2$, $b_1 = (12, 0)$, $b_2 = (0, 12)$. The objects B_ϵ and S are instances of Example 79: for B_ϵ take $H = ke_1$, $U_1 = ke_2$, $U_2 = ke_3$ abelian, $\delta_1(e_1) = 1$, $\delta_2(e_1) = \epsilon$; for S take $H = 0$ and $U_1 = \langle e_1, e_2 \rangle \cong U_2 = \langle e_3, e_4 \rangle \cong \mathfrak{r}_2$. Equivalently, $\mathsf{S} = X \oplus \sigma X$ for $X = (\mathfrak{r}_2, [\ ,\ ], 0)$, as in Proposition 75.

Proposition 82 (Switchable pairs in Vect_k through dimension four)**.** *Let $\mathfrak{s} = 0$, so that $\mathcal{M} = \mathrm{Vect}_k$. Every object of $\mathcal{G}^{\mathrm{sw}}_{\mathcal{M}}$ of dimension at most four is isomorphic to a direct sum of the following objects, which are indecomposable in $\mathcal{G}^{\mathrm{sw}}_{\mathcal{M}}$ and pairwise nonisomorphic. All unlisted brackets are zero, $\epsilon = \pm 1$, and F_0 gives* Fgt *of an involutive switch $F_0 \colon X \to \sigma X$, fixing unlisted basis vectors.*

	β_1	β_2	F_0
$\mathbf{1}$	0	0	id
A_ϵ	$[e_1, e_2]_1 = e_3$	$[e_1, e_2]_2 = \epsilon e_3$	$e_3 \mapsto \epsilon e_3$
B_ϵ	$[e_1, e_2]_1 = e_2$	$[e_1, e_3]_2 = \epsilon e_3$	$e_1 \mapsto \epsilon e_1,\ e_2 \leftrightarrow e_3$
N	$[e_2, e_3]_1 = e_4$	$[e_1, e_3]_2 = e_4$	$e_1 \leftrightarrow e_2$
D	$[e_1, e_2]_1 = e_3$	$[e_1, e_2]_2 = e_4$	$e_3 \leftrightarrow e_4$
S	$[e_1, e_2]_1 = e_2$	$[e_3, e_4]_2 = e_4$	$e_1 \leftrightarrow e_3,\ e_2 \leftrightarrow e_4$

In every row $\mathrm{Hom}(X, \sigma X) = F_0 \circ \mathrm{Hom}(X, X)$, and $\mathrm{Fgt}(\mathrm{Hom}(X, X)) \subseteq GL(\mathrm{Fgt}\, X)$ consists of the following maps:

- A_ϵ*:* $\begin{pmatrix} M & 0 \\ w^T & \det M \end{pmatrix}$ *with $M \in GL_2$ and $w \in k^2$;*
- B_ϵ*:* $e_1 \mapsto e_1 + pe_2 + qe_3$, $e_2 \mapsto be_2$, $e_3 \mapsto de_3$*;*
- N*:* $e_1 \mapsto ae_1 + xe_4$, $e_2 \mapsto ae_2 + ye_4$, $e_3 \mapsto de_3 + pe_1 + qe_2 + re_4$, $e_4 \mapsto ade_4$*;*
- D*:* $\begin{pmatrix} M & 0 \\ N & \det M \cdot I_2 \end{pmatrix}$ *with $M \in GL_2$;*
- S*:* $\mathrm{Aut}(\mathfrak{r}_2) \times \mathrm{Aut}(\mathfrak{r}_2)$ *acting blockwise.*

The decomposable objects of dimension at most four are $\mathbf{1}^{\oplus n}$ with $n \geq 2$, $\mathsf{A}_\epsilon \oplus \mathbf{1}$ and $\mathsf{B}_\epsilon \oplus \mathbf{1}$. Every object in this range has an involutive switch, so here $\mathcal{G}^{\mathrm{sw}}_{\mathcal{M}}$ coincides with the essential image of $\mathcal{G}^{h\mathbb{Z}/2}_{\mathcal{M}}$.

Proof. Let $X = (B, \beta_1, \beta_2)$ be switchable with $n = \dim B \leq 4$, and put $I_i = \mathrm{im}\, \beta_i$. A switch carries β_1 to β_2, so either both brackets vanish, giving $X \cong \mathbf{1}^{\oplus n}$, or both are nonzero; assume the latter. A switch carries I_1 onto I_2, so $d = \dim I_1 = \dim I_2 \geq 1$. Mutual annihilation gives $I_j \subseteq Z(\beta_i)$ for $j \neq i$, so β_i factors through $\bigwedge^2(B/I_j)$ and $d \leq \binom{n-d}{2}$. For $n \leq 4$ this forces $n \geq 3$ and $d = 1$. Write $\beta_i = \omega_i \otimes v_i$ with $I_i = kv_i$ and ω_i a nonzero two-form vanishing on I_j. Since $\dim B/I_j \leq 3$, the form ω_i is decomposable, so $Z_i = \mathrm{rad}\, \omega_i = Z(\beta_i)$ has codimension two, and mutual annihilation reads $I_2 \subseteq Z_1$, $I_1 \subseteq Z_2$. Every decomposable $\omega \otimes v$ is a Lie bracket, since the Jacobiator is $v \otimes (\omega \wedge \iota_v \omega)$ and $\omega \wedge \iota_v \omega = 0$ for decomposable ω. Call β_i of central type if $v_i \in Z_i$ and of noncentral type otherwise. A switch carries (I_1, Z_1) to (I_2, Z_2), so both brackets have the same type.

Central type, $I_1 = I_2 = I$, $Z_1 = Z_2 = Z$. Two decomposable forms with the same radical are proportional, so $\beta_2 = t\beta_1$ for a scalar $t \neq 0$ that every arrow preserves; σ replaces t by t^{-1}, so switchability forces $t = \pm 1$. Choosing e_1, e_2 spanning a complement of Z and $e_3 = v_1$ gives A_t when $n = 3$. When $n = 4$, a vector $e_4 \in Z \setminus I$ is central for both brackets and outside their images, and $X \cong \mathsf{A}_t \oplus \mathbf{1}$.

Central type, $I_1 = I_2 = I$, $Z_1 \neq Z_2$. Since $n = 3$ would force $Z_1 = Z_2 = I$, we have $n = 4$ and $Z_1 \cap Z_2 = I$. Take $e_4 = v_1$, $e_1 \in Z_1 \setminus I$, $e_2 \in Z_2 \setminus I$ and e_3 completing a basis. Then ω_1 is a multiple of $e_2^* \wedge e_3^*$ and ω_2 of $e_1^* \wedge e_3^*$, and rescaling e_1, e_2 gives N.

Central type, $I_1 \neq I_2$. Then $I_1 + I_2 \subseteq Z_1 \cap Z_2$, so $n = 4$ and $Z_1 = Z_2 = I_1 + I_2 = Z$. Both forms lie in the line $\bigwedge^2 (B/Z)^*$, and rescaling v_2 gives $\beta_i = \omega \otimes v_i$, which is D with $e_3 = v_1$, $e_4 = v_2$.

Noncentral type. Here $v_i \notin Z_i$, so $I_1 \neq I_2$ and $Z_1 \neq Z_2$. Define linear forms c_i by $\beta_i(x, v_i) = c_i(x) v_i$; then $\ker c_i = Z_i + I_i$. An arrow $g\colon X \to X'$ carries v_i into I_i', so $c_i' \circ g = c_i$, and σ exchanges c_1 and c_2. If $n = 3$, then $Z_1 = I_2$ and $Z_2 = I_1$, so $\ker c_1 = \ker c_2 = I_1 + I_2$ and $c_2 = \lambda c_1$. The scalar λ is preserved by arrows and inverted by σ, so $\lambda = \pm 1$. With $e_2 = v_1$, $e_3 = v_2$ and e_1 normalized by $c_1(e_1) = 1$, the object is B_λ. If $n = 4$, then $\dim(Z_1 \cap Z_2) \in \{0, 1\}$. If it is one, write $Z_1 \cap Z_2 = k e_4$; the same normalization gives $\mathsf{B}_\lambda \oplus \mathbf{1}$ with $\lambda = \pm 1$, since e_4 is central for both brackets and $\ker c_1 = \ker c_2 = \langle e_2, e_3, e_4 \rangle$. If it is zero, then $B = Z_1 \oplus Z_2$, β_1 is a nonabelian bracket on Z_2 vanishing when one argument lies in Z_1, and symmetrically, so $X \cong \mathsf{S}$.

These cases exhaust the switchable objects. Each listed row is switchable by the displayed map, which one checks satisfies $f\beta_1 = \beta_2(f \otimes f)$ and $f\beta_2 = \beta_1(f \otimes f)$ for $f = \mathrm{Fgt}(F_0)$, together with $\mathrm{Fgt}(\sigma(F_0) \circ F_0) = f^2 = \mathrm{id}$. The rows are pairwise nonisomorphic: n, the type, $\dim(I_1 + I_2)$ and the invariants t and λ separate them. For indecomposability, a nonzero switchable summand of dimension at most two has both brackets zero by Proposition 81, so it lies in $Z(\beta_1) \cap Z(\beta_2)$ and meets $I_1 + I_2$ trivially. In every listed row with nonzero brackets the common center is contained in $I_1 + I_2$, so no such summand exists, and for $n \leq 4$ every decomposition has a summand of dimension at most two. Finally, an arrow preserves I_i and Z_i; for each row, solving $g\beta_i = \beta_i(g \otimes g)$ on these flags gives the displayed groups. For example, for N one finds $g(e_1) \in Z_1$, $g(e_2) \in Z_2$, and the two bracket equations give $g(e_4) = a_1 d e_4 = a_2 d e_4$, so the scalars on e_1 and e_2 agree. By Proposition 70, $\mathrm{Hom}(X, \sigma X) = F_0 \circ \mathrm{Hom}(X, X)$. □

Proposition 83 (Supertranslation pairs in dimension $1 \,|\, N$)**.** *Let* $\mathfrak{s} = 0$ *and* $B = kH \oplus \Pi E$ *in* sVect_k, *with* H *even.*

1. *For symmetric forms* $b_1, b_2\colon \mathrm{Sym}^2 E \to k$, *the brackets* $[H, x]_i = 0$ *and* $[x, y]_i = b_i(x, y) H$ *define an object* $X(b_1, b_2)$ *of* $\mathcal{G}_{\mathcal{M}}$.

2. Fgt *identifies the arrows* $X(b_1, b_2) \to X(b_1', b_2')$ *with the pairs* (a, S) *with* $a \in k^\times$ *and* $S \in GL(E)$, *acting by* $H \mapsto aH$, $x \mapsto Sx$, *such that* $b_i'(Sx, Sy) = a\, b_i(x, y)$ *for* $i = 1, 2$.

3. *If* b_1 *is nondegenerate, put* $T = b_1^{-1} b_2$. *The classes are represented by*

$$b_1 = \bigoplus_j \mathsf{E}_{r_j}, \qquad b_2 = \bigoplus_j \mathsf{E}_{r_j} J_{r_j}(\lambda_j),$$

   where E_r *has ones on its antidiagonal,* $J_r(\lambda)$ *is the Jordan block with ones on its superdiagonal, the* $J_{r_j}(\lambda_j)$ *are the Jordan blocks of* T, *and the blocks are unordered.*

4. $X(b_1, b_2)$ *lies in* $\mathcal{G}_{\mathcal{M}}^{\mathrm{sw}}$ *exactly when the elementary divisors of the pencil* (b_1, b_2), *regarded as points of* $\mathbb{P}^1$ *with their block sizes, form a multiset invariant under* $\lambda \mapsto \lambda^{-1}$.

5. *For a general pair in dimension* $1 \,|\, N$ *with* $[H, x]_i = A_i x$ *and* $[x, y]_i = b_i(x, y) H$, *the Jacobi identities are*

$$b_i(A_i x, y) + b_i(x, A_i y) = 0, \qquad b_i(x, y) A_i z + b_i(y, z) A_i x + b_i(z, x) A_i y = 0,$$

   and for $i \neq j$ *mutual annihilation is*

$$A_j A_i = 0, \qquad b_j(A_i x, y) = 0, \qquad b_i(x, y) A_j z = 0.$$

Proof. (1) All brackets land in kH, which is central for both, so every nested bracket of either label vanishes. (2) Even maps of $1 \mid N$ are block diagonal, and the arrow condition is the displayed equation. (3) The shared factor a is absorbed by replacing S with $a^{-1/2}S$, so the classes are those of pairs of symmetric forms under simultaneous congruence; for b_1 nondegenerate these are given by the stated canonical form [17]. Thompson works over $\mathbb{C}$; the canonical forms have rational entries and the invariants (elementary divisors and minimal indices) are defined algebraically, so the same statements hold over every algebraically closed field of characteristic zero by the Lefschetz principle. (4) The exchange σ sends (b_1, b_2) to (b_2, b_1), which changes the pencil parameter by $\lambda \mapsto \lambda^{-1}$. Congruence classes of symmetric pencils are determined by the elementary divisors and the minimal indices [17], and minimal indices do not depend on the choice of generators of the pencil. (5) Expand the Jacobi identity and the mixed compositions on H and on odd vectors. □

Proposition 84 (Supertranslation pairs with an involutive switch)**.** *Let $\mathfrak{s} = 0$ and $N = 2$ in Proposition 83, with even generator H and odd basis ϑ_1, ϑ_2, and write forms as Gram matrices in this basis.*

1. *For $\mu \in k^\times$ with $\mu^2 \neq 1$, take $r_1 = r_2 = 1$, $\lambda_1 = \mu^2$ and $\lambda_2 = \mu^{-2}$:*

$$b_1 = \mathsf{E}_1 \oplus \mathsf{E}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad b_2 = \mathsf{E}_1 J_1(\mu^2) \oplus \mathsf{E}_1 J_1(\mu^{-2}) = \begin{pmatrix} \mu^2 & 0 \\ 0 & \mu^{-2} \end{pmatrix}.$$

   Then $H \mapsto H$, $\vartheta_1 \mapsto \mu\vartheta_2$, $\vartheta_2 \mapsto \mu^{-1}\vartheta_1$ is Fgt *of an involutive switch $X \to \sigma X$.*

2. *Take one block with $r_1 = 2$ and $\lambda_1 = 1$:*

$$b_1 = \mathsf{E}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad b_2 = \mathsf{E}_2 J_2(1) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

   Then $H \mapsto -H$, $\vartheta_1 \mapsto \vartheta_1$, $\vartheta_2 \mapsto \frac{1}{2}\vartheta_1 - \vartheta_2$ is Fgt *of an involutive switch $X \to \sigma X$.*

In both cases $b_1 \neq b_2$. The elementary divisors are the points μ^2 and μ^{-2} of $\mathbb{P}^1$, each with block size one, in (1), and the point 1 with block size two in (2); each multiset is invariant under $\lambda \mapsto \lambda^{-1}$, as Proposition 83(4) requires.

Proof. By Proposition 83(2), applied with target $\sigma X = X(b_2, b_1)$, a pair (a, S) underlies an arrow $F\colon X \to \sigma X$ exactly when

$$b_2(Sx, Sy) = a\, b_1(x, y), \qquad b_1(Sx, Sy) = a\, b_2(x, y) \qquad (x, y \in E).$$

Since $\mathrm{Fgt} \circ \sigma = \mathrm{Fgt}$ and Fgt is faithful, $\sigma(F) \circ F = \mathrm{id}_X$ exactly when $\mathrm{Fgt}(F)^2 = \mathrm{id}$, that is, $a^2 = 1$ and $S^2 = \mathrm{id}_E$.

(1) Here $a = 1$. Since S exchanges the two b_1- and b_2-orthogonal lines $k\vartheta_1$ and $k\vartheta_2$, all mixed values vanish on both sides. On the diagonal,

$$b_1(S\vartheta_1, S\vartheta_1) = \mu^2 = b_2(\vartheta_1, \vartheta_1), \qquad b_1(S\vartheta_2, S\vartheta_2) = \mu^{-2} = b_2(\vartheta_2, \vartheta_2),$$
$$b_2(S\vartheta_1, S\vartheta_1) = \mu^2\mu^{-2} = b_1(\vartheta_1, \vartheta_1), \qquad b_2(S\vartheta_2, S\vartheta_2) = \mu^{-2}\mu^2 = b_1(\vartheta_2, \vartheta_2).$$

Finally $S^2\vartheta_1 = \mu\mu^{-1}\vartheta_1$ and $S^2\vartheta_2 = \mu^{-1}\mu\vartheta_2$.

(2) Here $a = -1$. With $S\vartheta_1 = \vartheta_1$ and $S\vartheta_2 = \frac{1}{2}\vartheta_1 - \vartheta_2$,

$$\begin{aligned} b_1(S\vartheta_1, S\vartheta_1) &= 0 = -b_2(\vartheta_1, \vartheta_1), & b_2(S\vartheta_1, S\vartheta_1) &= 0 = -b_1(\vartheta_1, \vartheta_1), \\ b_1(S\vartheta_1, S\vartheta_2) &= -1 = -b_2(\vartheta_1, \vartheta_2), & b_2(S\vartheta_1, S\vartheta_2) &= -1 = -b_1(\vartheta_1, \vartheta_2), \\ b_1(S\vartheta_2, S\vartheta_2) &= -1 = -b_2(\vartheta_2, \vartheta_2), & b_2(S\vartheta_2, S\vartheta_2) &= -1 + 1 = 0 = -b_1(\vartheta_2, \vartheta_2). \end{aligned}$$

Moreover $S^2\vartheta_2 = \frac{1}{2}\vartheta_1 - (\frac{1}{2}\vartheta_1 - \vartheta_2) = \vartheta_2$, and $a^2 = 1$. □

By contrast, the pair of Proposition 85 also has an inversion-invariant multiset of elementary divisors, but none of its switches is involutive.

Proposition 85 (Switchable without an involutive switch)**.** *Let $\mathfrak{s} = 0$ and $N = 2$ in Proposition 83, with even generator H, odd basis ϑ_1, ϑ_2, $b_1 = \operatorname{diag}(1,1)$, $b_2 = \operatorname{diag}(1,-1)$, that is,*

$$\beta_1 : \ [\vartheta_1, \vartheta_1] = H, \ [\vartheta_2, \vartheta_2] = H; \qquad \beta_2 : \ [\vartheta_1, \vartheta_1] = H, \ [\vartheta_2, \vartheta_2] = -H.$$

Then X lies in $\mathcal{G}^{\mathrm{sw}}_{\mathcal{M}}$ but not in the essential image of $\mathcal{G}^{h\mathbb{Z}/2}_{\mathcal{M}}$.

Proof. In the canonical form of Proposition 83(3), $b_1 = \mathsf{E}_1 \oplus \mathsf{E}_1$ and $b_2 = \mathsf{E}_1 J_1(1) \oplus \mathsf{E}_1 J_1(-1)$, so $T = b_1^{-1} b_2 = \operatorname{diag}(1,-1)$. The elementary divisors of the pencil (b_1, b_2) are therefore $\lambda - 1$ and $\lambda + 1$, that is, the points 1 and -1 of $\mathbb{P}^1$, each with block size one. Inversion $\lambda \mapsto \lambda^{-1}$ fixes each of these points, so it fixes the multiset, and Proposition 83(4) puts X in $\mathcal{G}^{\mathrm{sw}}_{\mathcal{M}}$. Explicitly, the map $H \mapsto H$, $\vartheta_1 \mapsto \vartheta_1$, $\vartheta_2 \mapsto -i\vartheta_2$ satisfies the switch equations. Every arrow $X \to X$ or $X \to \sigma X$ conjugates $T = b_1^{-1} b_2 = \operatorname{diag}(1,-1)$ to T or to $T^{-1} = T$, so it preserves the eigenlines $k\vartheta_1$ and $k\vartheta_2$. Hence $\mathrm{Fgt}(\mathrm{Hom}(X,X))$ consists of $H \mapsto cH$, $\vartheta_1 \mapsto h_1\vartheta_1$, $\vartheta_2 \mapsto h_2\vartheta_2$ with $h_1^2 = h_2^2 = c$, and $\mathrm{Fgt}(\mathrm{Hom}(X,\sigma X))$ of $H \mapsto cH$, $\vartheta_1 \mapsto h_1\vartheta_1$, $\vartheta_2 \mapsto -ih_2\vartheta_2$. For such a switch F, $\sigma(F) \circ F$ acts by c^2, c and $-c$ on H, ϑ_1 and ϑ_2, so it is never id_X. □

Proposition 86 (Equivariant switches with a symplectic module)**.** *Let V be a nontrivial irreducible $\mathfrak{s}$-module with a nonzero invariant alternating form ω, and let z, z_1, z_2 span trivial modules. Let $X_\epsilon = (V \oplus kz,\ \omega \otimes z,\ \epsilon\,\omega \otimes z)$ for $\epsilon = \pm 1$ and $Y = (V \oplus kz_1 \oplus kz_2,\ \omega \otimes z_1,\ \omega \otimes z_2)$. Then:*

1. *$\mathrm{Fgt}(\mathrm{Hom}(X_\epsilon, X_\epsilon))$ consists of $v \mapsto mv$, $z \mapsto m^2 z$, and $\mathrm{Fgt}(\mathrm{Hom}(X_\epsilon, \sigma X_\epsilon))$ of $v \mapsto mv$, $z \mapsto \epsilon m^2 z$, with $m \in k^\times$;*
2. *$\mathrm{Fgt}(\mathrm{Hom}(Y,Y))$ consists of $v \mapsto mv$, $z_i \mapsto m^2 z_i$, and $\mathrm{Fgt}(\mathrm{Hom}(Y, \sigma Y))$ of $v \mapsto mv$, $z_1 \mapsto m^2 z_2$, $z_2 \mapsto m^2 z_1$.*

In both cases $m = 1$ gives an involutive switch.

Proof. An arrow is $\mathfrak{s}$-equivariant, so it preserves the V-isotypic and trivial isotypic parts. By Schur's lemma it acts by a scalar m on V and by some C on the trivial part. The bracket equations read $C(\omega(v,v')z) = m^2\omega(v,v')z$ for $X_\epsilon \to X_\epsilon$ and $C(\omega(v,v')z) = \epsilon m^2 \omega(v,v')z$ for $X_\epsilon \to \sigma X_\epsilon$, and similarly with z_1, z_2 for Y. Since $\omega \neq 0$, they determine C as stated. □

Proposition 87 (Semisimple acting algebra through dimension four)**.** *Let $\mathfrak{s}$ be semisimple and $\mathcal{M} = \mathfrak{s}$-Rep. The indecomposable objects of $\mathcal{G}^{\mathrm{sw}}_{\mathcal{M}}$ of dimension at most four are the objects of Proposition 82 with trivial $\mathfrak{s}$-action, the nontrivial simple modules with both brackets zero, and the objects X_ϵ and Y of Proposition 86 with V the standard module of a simple ideal of $\mathfrak{s}$ isomorphic to $\mathfrak{sl}_2$.*

Proof. Restricting along $0 \to \mathfrak{s}$ gives a faithful functor $\mathcal{G}^{\mathrm{sw}}_{\mathcal{M}} \to \mathcal{G}^{\mathrm{sw}}_{\mathrm{Vect}_k}$, so the restriction X_0 of X is one of the objects of Proposition 82, and X is determined by a homomorphism $\rho\colon \mathfrak{s} \to \mathfrak{d} = \mathrm{Der}(\beta_1) \cap \mathrm{Der}(\beta_2)$, the Lie algebra of $G = \mathrm{Aut}(X_0)$, up to conjugation by G. By the theorems of Levi–Malcev and Malcev–Harish-Chandra [19, Ch. I, §6, no. 8], $\rho(\mathfrak{s})$ is contained in a Levi subalgebra of $\mathfrak{d}$, and any two Levi subalgebras are conjugate by $\exp(\operatorname{ad} z)$ with $z \in [\mathfrak{d}, \operatorname{rad}\mathfrak{d}]$. The argument of Lemma 3, applied to the representation of $\mathfrak{d}$ on $\mathrm{Fgt}\, X_0$ instead of the adjoint representation, shows that such z acts nilpotently, so $\exp(z)$ lies in G. From the groups in Proposition 82, together with those of $\mathsf{A}_\epsilon \oplus \mathbf{1}$ and $\mathsf{B}_\epsilon \oplus \mathbf{1}$ (which add a solvable factor), a Levi subalgebra of $\mathfrak{d}$ is $\mathfrak{sl}_n$ for $\mathbf{1}^{\oplus n}$, $\mathfrak{sl}_2$ acting on $\langle e_1, e_2\rangle$ for A_ϵ, $\mathsf{A}_\epsilon \oplus \mathbf{1}$ and D, and zero for B_ϵ, $\mathsf{B}_\epsilon \oplus \mathbf{1}$, N and S. In the last cases $\mathfrak{s}$ acts trivially. For $\mathbf{1}^{\oplus n}$, B is a semisimple $\mathfrak{s}$-module with zero brackets, and it is indecomposable exactly when it is simple. For the remaining cases, a nonzero ρ maps onto $\mathfrak{sl}_2$, so $\langle e_1, e_2\rangle$ is the standard module of a simple ideal isomorphic to $\mathfrak{sl}_2$ and the central vectors are trivial. This gives X_ϵ and Y, and $X_\epsilon \oplus \mathbf{1}$, which is decomposable. Finally, X_ϵ and Y are indecomposable because their restrictions A_ϵ and D are. □

The restriction to $\mathfrak{sl}_2$ comes from the dimension bound alone. The constructions of Proposition 86 apply to every nontrivial irreducible symplectic module, and multiplicity spaces permit the two brackets to use distinct invariant forms. There just isn't enough room yet with the cutoff imposed.

7 Paired current algebras

This section applies Theorem 65 to particular coefficient algebras, among them those whose two products are pointwise multiplication and convolution, exchanged by Fourier transform. Most of these coefficient algebras are infinite-dimensional, so the finite-dimensional convention is relaxed throughout this section. For the analytic examples take $k = \mathbb{C}$. All tensor products are algebraic unless a topology is explicitly specified.

If we do not demand the existence of such an exchange, then there are many more examples. In addition we want the two commutative algebras not to be algebras for $\mathrm{Lie}_2^!$ which would have some relation between the two products rather than them being completely independently specified (at least a priori). In the former case, we would have just done the Lie algebra with $\mathrm{Lie}_2^! \otimes_H \mathrm{Lie}_2$.

7.1 Coefficient algebras supplied by Fourier analysis

Example 88 (Rescaled products)**.** *For a commutative associative algebra (A, μ_1) and an element $c \in A$, define $\mu_2(a, b) = c\,\mu_1(a, b)$. In the super case take c even. Associativity and commutativity of the second product follow from those of the first. This is an example of* CommComm*, even though this particular algebra satisfies additional identities. A scalar rescaling $\mu_2 = r\mu_1$ makes (43) the single-product current bracket for $\beta_1 + r\beta_2$. An algebra element c need not be a scalar. This reduces to the very well studied cases.*

Example 89 (Schwartz functions on the line)**.** *Take $A = \mathcal{S}(\mathbb{R}, \mathbb{C})$, defined by the seminorms*

$$p_{m,n}(f) = \sup_{x \in \mathbb{R}} |x^m f^{(n)}(x)| < \infty \qquad (m, n \geq 0).$$

With Lebesgue measure fixed, put

$$\mu_1(f, g)(x) = f(x)g(x), \qquad \mu_2(f, g)(x) = (f * g)(x) = \int_{\mathbb{R}} f(y)g(x - y)\, dy.$$

Both products are jointly continuous and preserve $\mathcal{S}$. Leibniz's rule proves the pointwise assertion; for convolution one differentiates under the integral and expands $x^m = (y + (x - y))^m$, bounding the resulting convolutions by an L^1 norm times a supremum norm. Associativity of convolution follows from Fubini. Neither product has a unit in this space: the pointwise unit would be 1, and the convolution unit would be the Dirac distribution. Standard Fourier transform with usual normalizations interchanges the products and squares to reflection.

Example 90 (Feichtinger algebra)**.** *Fix $g(t) = e^{-\pi t^2}$ and set*

$$V_g f(x, \xi) = \int_{\mathbb{R}} f(t)\, g(t - x)\, e^{-2\pi i \xi t}\, dt, \qquad S_0(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : V_g f \in L^1(\mathbb{R}^2)\}.$$

The norm $\|V_g f\|_1$ makes S_0 a Banach space, continuously embedded in $L^1 \cap C_0$. It is closed under both pointwise multiplication and convolution, with continuous product maps, and Fourier transform exchanges them. Replacing g by any nonzero window in S_0 gives an equivalent norm; this assertion is not for an arbitrary L^2 window. The same two-product construction is available on $S_0(G)$ for locally compact abelian groups, with Haar measure and the dual Haar measure fixed. See [15] for these analytic facts.

Example 91 (Laurent polynomials and smooth loops)**.** *Let $\mathbb{T}_\ell = \mathbb{R}/\ell\mathbb{Z}$, $\ell > 0$, with normalized Haar measure dx/ℓ. Put $e_n(x) = e^{2\pi i n x/\ell}$. On the trigonometric polynomials, identified with $\mathbb{C}[t, t^{-1}]$ by $t^n \mapsto e_n$, the products are*

$$t^m \cdot t^n = t^{m+n}, \qquad t^m \circ t^n = \delta_{mn} t^n.$$

The latter is circular convolution with the normalized Haar measure. Thus both products close on the Laurent polynomials, which are dense in $C^\infty(\mathbb{T}_\ell)$ in its smooth topology. Fourier series identify the latter with rapidly decreasing sequences on $\mathbb{Z}$: pointwise multiplication of functions becomes Cauchy convolution of sequences, and circular convolution becomes componentwise multiplication. However, this example has no map to itself exchanging its two products (Proposition 92) unlike the previous.

Proposition 92 (No product exchange on the Laurent model)**.** *There is no linear isomorphism $\Psi\colon k[t, t^{-1}] \to k[t, t^{-1}]$ with*

$$\Psi(a \cdot b) = \Psi(a) \circ \Psi(b), \qquad \Psi(a \circ b) = \Psi(a) \cdot \Psi(b).$$

Proof. The first identity makes Ψ an isomorphism of commutative algebras from $\cdot$ to $\circ$. Since Ψ is surjective, $\Psi(1) \circ \Psi(b) = \Psi(1 \cdot b) = \Psi(b)$ shows that $\Psi(1)$ is a unit for $\circ$. If $u = \sum_n u_n t^n$ were such a unit, then $u \circ t^n = u_n t^n = t^n$ would give $u_n = 1$ for every $n \in \mathbb{Z}$, which is impossible for a Laurent polynomial. □

7.2 The resulting Lie algebras and Fourier transport

For the Schwartz or Feichtinger coefficients, write $L = L(A, B)$. Explicitly,

$$[f \otimes x, g \otimes y] = fg \otimes \beta_1(x, y) + (f * g) \otimes \beta_2(x, y). \tag{45}$$

The two coefficient products remain separate structures. For finite-dimensional B these are continuous brackets on the corresponding spaces of B-valued functions. With infinite-dimensional B, Theorem 65 supplies the algebraic tensor product; a completed tensor product needs a specified topology.

When $\beta_2 = 0$ this is the usual current bracket for (A, μ_1) and (B, β_1), including nonunital A. When $\beta_1 = 0$, only the second current bracket remains. For the Laurent model,

$$[t^m \otimes x, t^n \otimes y] = t^{m+n} \otimes \beta_1(x, y) + \delta_{mn} t^n \otimes \beta_2(x, y). \tag{46}$$

The first term is the Laurent current algebra; the second separately is the direct sum of the Lie algebras (B, β_2) over the modes.

Suppose $\Psi\colon A \to A'$ exchanges the two products, and $F\colon X \to \sigma X'$ is an arrow from $X = (B, \beta_1, \beta_2)$ to the exchange of $X' = (B', \beta_1', \beta_2')$, so that $\mathrm{Fgt}(F)$ carries β_1 to β_2' and β_2 to β_1'. Then the tensor formula gives

$$\Psi \otimes \mathrm{Fgt}(F)\colon L(A, B) \xrightarrow{\sim} L(A', B').$$

This applies with $A = A'$ equal to $\mathcal{S}(\mathbb{R}^n)$ or $S_0(\mathbb{R}^n)$ and Ψ equal to FT followed by the coordinate identification of the dual space through $x \cdot \xi$, the formulas of Example 89 written on $\mathbb{R}^n$; for $X' = X$ it gives an automorphism of $L(A, B)$. An involutive switch $F\colon X \to \sigma X$, so that $\mathrm{Fgt}(F)^2 = \mathrm{Fgt}(\sigma(F) \circ F) = \mathrm{id}_B$, gives $(\mathrm{FT} \otimes \mathrm{Fgt}(F))^2 = J \otimes \mathrm{id}$, where $Jf(x) = f(-x)$. Consequently the combined automorphism has fourth power the identity; an involution on the bracket factor does not remove Fourier reflection.

Proposition 93 (A simultaneous modified Yang–Baxter operator)**.** *In the Laurent model (46), the subspaces $L_+ = k[t] \otimes B$ and $L_- = t^{-1}k[t^{-1}] \otimes B$ are subalgebras. Let $P_\pm$ be the projections of $L = L_+ \oplus L_-$ and $\mathcal{R} = P_+ - P_-$. Then*

$$[\mathcal{R}\xi, \mathcal{R}\zeta] - \mathcal{R}\big([\mathcal{R}\xi, \zeta] + [\xi, \mathcal{R}\zeta]\big) = -[\xi, \zeta],$$

so $\mathcal{R}$ solves the modified classical Yang–Baxter equation. The associated bracket

$$[\xi,\zeta]_{\mathcal{R}} = [\mathcal{R}\xi,\zeta] + [\xi,\mathcal{R}\zeta] = 2\big([\xi_+,\zeta_+] - [\xi_-,\zeta_-]\big)$$

is a Lie bracket on the same space: it is the bracket of $L_+ \times L_-$ with the bracket of the second factor negated, rescaled by 2.

Proof. When $\beta_2 = 0$, the algebra $L(k[t,t^{-1}],B)$ is the loop algebra $B[t,t^{-1}]$ of the Lie algebra (B,β_1), and $\mathcal{R}$ is its splitting into $B[t]$ and $t^{-1}B[t^{-1}]$. This is the situation of [18, Proposition 5]: for a Lie algebra that is a direct sum of two subalgebras, $\mathcal{R} = P_+ - P_-$ satisfies the modified classical Yang–Baxter equation [18, Section 3, equation (12)], and the $\mathcal{R}$-bracket [18, Section 1, equation (1)] equals $2[x_+,y_+] - 2[x_-,y_-]$. The loop-algebra case appears in the example following that proposition, for the associative algebra $\mathrm{Mat}(n,\mathbb{C}) \otimes \mathbb{C}[\lambda,\lambda^{-1}]$ with the operator [18, equation (13)]. This follows the same argument.

□

For a single pair, Golubchik and Sokolov [16] construct a nonconstant solution of the classical Yang–Baxter equation from any pair of compatible Lie brackets that share a nondegenerate invariant symmetric form. Mutually annihilating pairs are compatible, so their construction applies to them. The next proposition shows how restrictive a common invariant form is in the mutually annihilating case. An invariant symmetric bilinear form $(\ ,\ )$ on B is *β-invariant* if $(\beta(x,y),z) = (x,\beta(y,z))$ for all x,y,z.

Proposition 94 (Common invariant forms)**.** *Let β_1,β_2 be Lie brackets on a finite-dimensional vector space B, and let $(\ ,\ )$ be a nondegenerate symmetric form that is β_1- and β_2-invariant. Put $I_i = \operatorname{im}\beta_i$.*

1. *$Z(\beta_i) = I_i^{\perp}$ for $i = 1,2$.*
2. *The pair is mutually annihilating if and only if $I_1 \perp I_2$.*
3. *If the pair is mutually annihilating and each β_i has class at most two, then $I_1 + I_2$ is isotropic. In particular $\dim(I_1 + I_2) \le \frac{1}{2}\dim B$.*
4. *If the pair is mutually annihilating and $\beta_1 + \beta_2$ is perfect, then $B = I_1 \oplus I_2$ is an orthogonal decomposition, β_1 vanishes on I_2 and β_2 vanishes on I_1. Thus $(B,\beta_1,\beta_2) \cong (\mathfrak{g}_1,0) \oplus (0,\mathfrak{g}_2)$ with $\mathfrak{g}_i = (I_i,\beta_i)$ perfect.*

Proof. (1) Fix i and $z \in B$. By definition $z \in Z(\beta_i)$ means $\beta_i(z,y) = 0$ for all $y \in B$. Since the form is nondegenerate, this holds exactly when $(\beta_i(z,y),w) = 0$ for all $y,w \in B$. By invariance $(\beta_i(z,y),w) = (z,\beta_i(y,w))$, so the condition becomes $(z,\beta_i(y,w)) = 0$ for all y,w. The elements $\beta_i(y,w)$ span I_i, so this says $z \in I_i^{\perp}$.

(2) The identity $\beta_1(\beta_2(x,y),z) = 0$ for all x,y,z says that every element of I_2 is central for β_1, that is, $I_2 \subseteq Z(\beta_1)$. Likewise $\beta_2(\beta_1(x,y),z) = 0$ for all x,y,z says $I_1 \subseteq Z(\beta_2)$. By (1), the first inclusion is $I_2 \subseteq I_1^{\perp}$ and the second is $I_1 \subseteq I_2^{\perp}$. Since the form is symmetric, both inclusions say $(a,b) = 0$ for all $a \in I_1$ and $b \in I_2$, that is, $I_1 \perp I_2$. So the two mixed identities hold together exactly when $I_1 \perp I_2$.

(3) Class at most two for β_i means $\beta_i(\beta_i(x,y),z) = 0$ for all x,y,z, that is, $I_i \subseteq Z(\beta_i)$. By (1) this is $I_i \subseteq I_i^{\perp}$, so $(a,a') = 0$ for all $a,a' \in I_i$. Now let $a_1 + a_2$ and $b_1 + b_2$ lie in $I_1 + I_2$, with $a_i,b_i \in I_i$. Then

$$(a_1 + a_2,\ b_1 + b_2) = (a_1,b_1) + (a_1,b_2) + (a_2,b_1) + (a_2,b_2).$$

The first and last terms vanish because each I_i is isotropic, and the middle terms vanish by (2). So $V = I_1 + I_2$ is isotropic, that is, $V \subseteq V^{\perp}$. Nondegeneracy gives $\dim V^{\perp} = \dim B - \dim V$, hence $\dim V \le \dim B - \dim V$, which is the stated bound.

(4) Every value $(\beta_1+\beta_2)(x,y)=\beta_1(x,y)+\beta_2(x,y)$ lies in I_1+I_2, so perfectness of $\beta_1+\beta_2$ gives $B=I_1+I_2$. Let $c\in I_1\cap I_2$. For $a\in I_1$ we have $(c,a)=0$ because $c\in I_2$ and $I_2\perp I_1$; for $b\in I_2$ we have $(c,b)=0$ because $c\in I_1$. Hence $c\perp I_1+I_2=B$, and $c=0$ by nondegeneracy. So $B=I_1\oplus I_2$, and the summands are orthogonal by (2); in particular the form restricts to a nondegenerate form on each I_i.

Write $x=x_1+x_2$ and $y=y_1+y_2$ with $x_i,y_i\in I_i$. Since $I_2\subseteq Z(\beta_1)$, every term of $\beta_1(x,y)$ involving x_2 or y_2 vanishes, so $\beta_1(x,y)=\beta_1(x_1,y_1)$, which lies in I_1. Thus β_1 restricts to a Lie bracket on I_1 and is zero on every pair with an argument in I_2. Symmetrically, $\beta_2(x,y)=\beta_2(x_2,y_2)\in I_2$. With $\mathfrak{g}_i=(I_i,\beta_i|_{I_i})$, the identity map of B is therefore an isomorphism $(B,\beta_1,\beta_2)\cong(\mathfrak{g}_1,0)\oplus(0,\mathfrak{g}_2)$ in $\mathcal{G}_{\mathcal{M}}$. Finally $(\beta_1+\beta_2)(x,y)=\beta_1(x_1,y_1)+\beta_2(x_2,y_2)$, so $B=\mathrm{im}(\beta_1+\beta_2)=\beta_1(I_1,I_1)\oplus\beta_2(I_2,I_2)$. Comparing with $B=I_1\oplus I_2$ gives $\beta_i(I_i,I_i)=I_i$, so each $\mathfrak{g}_i$ is perfect. □

Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During preparation of this manuscript, the author used OpenAI ChatGPT-6 Astra and Anthropic Claude Opus 5.5 to assist with drafting and revising exposition, exploring calculations, and preparing the LaTeX source and PDF. The choices of definitions, which morphisms to allow in which categories, and which special cases to consider were specified in the author's prompts during this session. The author reviewed and edited the generated material. The author remains responsible for the mathematical statements and the final contents of the released report.

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