A PROOF OF LUSZTIG'S CONJECTURES FOR EXTENDED AFFINE WEYL GROUPS

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Abstract. We prove Lusztig's conjectures P1–P15 for extended affine Weyl groups with positive integer-valued weight functions.

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1. Introduction

Kazhdan and Lusztig introduced the canonical basis of a Hecke algebra and the associated left, right, and two-sided cells in [17]. Their construction connects multiplication in the Hecke algebra with representations of Coxeter groups. Lusztig extended the cell construction to unequal parameters in [18], where the parameters are specified by a weight function on the simple reflections. The present paper proves properties P1–P15 of [24, §14.2] for all extended affine Weyl groups with positive integer weights.

The affine theory developed through Lusztig's papers [19, 20, 21]. The a-function bounds the degrees of the canonical structure constants. Extracting their leading coefficients gives the asymptotic algebra J, introduced in [20]. In the equal-parameter affine setting, the two-sided cells are related to unipotent classes in the corresponding complex reductive group, and their a-values are the dimensions of Springer fibers [21]. That paper also establishes a finite-parabolic intersection property: under its semisimple simply connected group hypothesis, each two-sided cell contains an element of a finite standard parabolic subgroup. These results relate the infinite affine cell decomposition to finite Weyl groups and to the geometry of unipotent classes.

Lusztig's monograph [24] formulates P1–P15 for unequal parameters. They govern the a-function, distinguished involutions, leading structure constants, and the original cell preorders. P12 compares the a-function with its parabolic counterpart. P15 asserts that left and right multiplication on an a-level commute with independent parameters. In the equal-parameter case these properties follow from positivity, as explained in [24, §15]. Canonical structure constants can have negative coefficients for unequal parameters, and that argument does not extend to arbitrary weights.

Let $(X, R_0, Y, R_0^\vee, F_0)$ be a based reduced root datum. Thus X, Y are free abelian groups in perfect duality, $R_0 \subset X$ and $R_0^\vee \subset Y$ are the roots and corresponding coroots, and F_0 is a set of simple roots, with positive system R_0^+. Put

$$W_0 = W(R_0), \qquad W_{\mathrm{aff}} = \mathbb{Z}R_0 \rtimes W_0, \qquad W_{\mathrm{ext}} = X \rtimes W_0.$$

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Let S_{aff} be the affine simple reflections and Ω the length-zero subgroup, so that $W_{\mathrm{ext}} = \Omega \ltimes W_{\mathrm{aff}}$. Let $L : S_{\mathrm{aff}} \longrightarrow \mathbb{Z}_{>0}$ be a weight function invariant under Ω. Its extension to W_{aff} is additive on length-additive products, and $L(\omega w) = L(w)$ for $\omega \in \Omega$.

Theorem 1.1. *Lusztig's properties P1–P15 hold for the Hecke algebra of W_{ext} with the weight function L.*

The theorem includes parameter walls, arbitrary rank, products of affine root systems, and root data with a central torus. Taking $X = \mathbb{Z}R_0$ gives the nonextended affine Weyl group. Rescaling the variable gives the corresponding assertion on every positive rational weight ray. The definitions and the extended-group conventions for P1–P15 are recorded in Section 2.

1.1. **Earlier results.** Work on unequal parameters has used finite cell representations, geometric parameters, and tempered representations of affine Hecke algebras. In finite type B_n, Bonnafé and Iancu described the left cells and their representations in the asymptotic parameter region by a generalized Robinson–Schensted correspondence [1]. Geck and Iancu computed the a-function there using leading matrix coefficients and proved most of the P-properties [11]. Geck subsequently obtained the full set of properties in this region and for finite F_4 with arbitrary positive weights [10]. These finite results provide cell filtrations and balanced matrix representations for the parabolic algebras used below.

For affine groups, Lusztig treats equal parameters, the quasisplit geometric setting, and infinite dihedral groups in [24, §§15–17]. Xie proves the P-properties on the lowest generalized two-sided cell of an extended affine Weyl group and describes its based ring [33]. Guilhot and Parkinson prove P1–P15 for arbitrary positive weights in $\widetilde{G}_2$ and $\widetilde{C}_2$ [13, 14]. Their proofs combine balanced cell representations with an asymptotic Plancherel formula. Together with the infinite dihedral case, these results settle affine root rank at most two. The finite B_n theorem for all positive integer weights and the nonextended affine B/C argument are given in [32]. We use its finite theorem in the precise form stated in Section 6, and give the affine attainment argument here, including every specified constituent at a singular parameter.

The Plancherel approach also describes the asymptotic algebra in settings where the P-properties are already known. Dawydiak studies its denominators and its realization in the Schwartz algebra in the equal-parameter Iwahori setting [7]. Chapelier-Laget, Guilhot, Little, and Parkinson construct balanced cell modules, an asymptotic Plancherel formula, and a relative Satake isomorphism in affine type A [2]. These works concern the structure of J as well as its relation to tempered representations.

1.2. **The proof.** We first construct a Laurent polynomial Fourier algebra from the positive Plancherel formula. Its integral orthonormal basis is indexed by the original group elements. A finite-support inverse identifies its algebraic image, and an integral analytic iteration produces its spectral coordinate projections. We then prove that finite parabolic representations of the same spectral degree generate each level. Relative canonical induction identifies these levels with the original Kazhdan–Lusztig filtration and proves the full two-parameter identity P15.

Finally, integral Galois packets determine the signed units and recover distinguished elements from the original trace.

The representation-theoretic requirement is finite parabolic attainment: each irreducible tempered constituent must contain a finite parabolic type whose Schur degree equals its Plancherel family degree. In affine B/C this follows from canonical finite types and a two-endpoint splitting argument. In affine F_4 we deform a prescribed constituent through its unitary central phase to an essential elliptic endpoint. Exact finite character calculations then separate the full central-character blocks from the span of strictly smaller-degree types. The separation uses all five finite facets and includes colliding blocks on parameter walls. Products and length-zero extensions complete the proof.

The analytic inputs are the Plancherel, induction, and reduction theorems [26, 8, 27, 30], the scaling and discrete-series classification [5], and the Dirac central-character theorem [6]. Their required forms are stated at the points of use. Section 8 specifies the exact finite data, algorithms, and identities used for F_4; the supplement contains the data and reproducible verifiers.

2. Conventions and the finite-attainment criterion

2.1. **The original basis.** Let (W, S) be an affine Weyl Coxeter system, let ℓ be its length function, and let L be a positive integer weight. Length-zero extensions are treated in Section 9. Put $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$, and let $\mathcal{H}$ be the associated Hecke algebra over $\mathcal{A}$. Its normalized standard basis $(N_w)_{w\in W}$ satisfies

$$N_x N_y = N_{xy} \quad \text{if } \ell(xy) = \ell(x) + \ell(y), \qquad (N_s - v^{L(s)})(N_s + v^{-L(s)}) = 0.$$

The bar involution sends v to v^{-1} and N_w to $N_{w^{-1}}^{-1}$. The canonical basis is characterized by

$$\overline{C_w} = C_w, \qquad C_w = N_w + \sum_{y<w} p_{y,w} N_y, \qquad p_{y,w} \in v^{-1}\mathbb{Z}[v^{-1}].$$

Here $<$ denotes the Bruhat order. Write

$$C_x C_y = \sum_z h_{x,y,z} C_z, \qquad a(z) = \max_{x,y} \deg_v h_{x,y,z}.$$

Boundedness for affine groups is available from [24, §13.4]; the maximum is finite. For $I \subseteq S$, put $W_I = \langle I \rangle$ and $\mathcal{H}_I = \mathrm{span}_{\mathcal{A}}\{N_w : w \in W_I\}$, and let a_I be its a-function. Set $\tau(N_w) = \delta_{w,1}$ and $N_w^* = N_{w^{-1}}$. On a positive numerical specialization, $*$ conjugates complex scalars and fixes v; it is distinct from the bar involution. For $\tau(C_w) \neq 0$ write

$$\tau(C_w) = n_w v^{-\Delta(w)} + \text{terms of smaller degree}, \qquad n_w \neq 0,$$

and set $\Delta(w) = +\infty$ if the trace is zero. Let

$$\mathcal{D} = \{w : a(w) = \Delta(w)\}, \qquad \gamma_{x,y,z} = [v^{a(z)}] h_{x,y,z^{-1}}.$$

The identity $a(z) = a(z^{-1})$ follows from the standard anti-involution. The relations $\leq_{\mathcal{L}}, \leq_{\mathcal{R}}, \leq_{\mathcal{LR}}$ are the preorders generated by nonzero coefficients of left, right, and two-sided canonical multiplication, respectively. Write $\sim_{\mathcal{L}}, \sim_{\mathcal{R}}, \sim_{\mathcal{LR}}$ for their associated equivalence relations.

2.2. **The fifteen properties.** For reference, we record the assertions in these conventions. The list is the one in [24, §14.2]; for a length-zero extension we use the zero-trace convention just specified.

P1. $a(w) \le \Delta(w)$.
P2. $d \in \mathcal{D}$ and $\gamma_{x,y,d} \neq 0$ imply $x = y^{-1}$.
P3. For every y there is a unique $d \in \mathcal{D}$ with $\gamma_{y^{-1},y,d} \neq 0$.
P4. $x \le_{\mathcal{LR}} y$ implies $a(x) \ge a(y)$.
P5. If $\gamma_{y^{-1},y,d} \neq 0$ and $d \in \mathcal{D}$, then $\gamma_{y^{-1},y,d} = n_d \in \{1, -1\}$.
P6. Every $d \in \mathcal{D}$ is an involution.
P7. $\gamma_{x,y,z} = \gamma_{y,z,x}$.
P8. $\gamma_{x,y,z} \neq 0$ implies $x \sim_{\mathcal{L}} y^{-1}$, $y \sim_{\mathcal{L}} z^{-1}$, and $z \sim_{\mathcal{L}} x^{-1}$.
P9. $x \le_{\mathcal{L}} y$ and $a(x) = a(y)$ imply $x \sim_{\mathcal{L}} y$.
P10. $x \le_{\mathcal{R}} y$ and $a(x) = a(y)$ imply $x \sim_{\mathcal{R}} y$.
P11. $x \le_{\mathcal{LR}} y$ and $a(x) = a(y)$ imply $x \sim_{\mathcal{LR}} y$.
P12. For $I \subseteq S$ and $w \in W_I$, $a_I(w) = a_W(w)$.
P13. Each left cell Γ contains one element $d \in \mathcal{D}$, and $\gamma_{x^{-1},x,d} \neq 0$ for every $x \in \Gamma$.
P14. $w \sim_{\mathcal{LR}} w^{-1}$.
P15. If $a(w) = a(y)$, then, for independent u, v,

$$\sum_z h_{w,x',z}(u) h_{x,z,y}(v) = \sum_z h_{x,w,z}(v) h_{z,x',y}(u). \tag{2.1}$$

All sums in (2.1) are finite. Once P4 is known, its contributing intermediate indices have the same a-value as w and y.

2.3. **Spectral degrees and finite types.** For $v > 1$, write $\mathcal{H}_v$ for the numerical specialization of $\mathcal{H}$. The trace and $*$ make it a Hilbert algebra. Choose a compact parabolic-induction family $\pi_{\alpha,v,z}$, with z on a compact torus $T_{\alpha,u}$, and include the finite association factors in its Plancherel measure. Write

$$d\mu_{\alpha,v}(z) = v^{-2A_\alpha} m_{\alpha,v}(z)\, dz, \qquad m_{\alpha,v} \longrightarrow m_{\alpha,0} \neq 0. \tag{2.2}$$

The degree A_α is constant on this family. The normalization of (2.2) is always relative to the trace τ.

If W_I is finite and σ is a simple finite Hecke type, let $s_{I,\sigma}(v)$ be its Schur element for $\tau|_{\mathcal{H}_I}$, and set

$$A_I(\sigma) = \tfrac{1}{2} \deg_v^{\max} s_{I,\sigma}(v). \tag{2.3}$$

This is the highest half-degree. For rank one the trivial type has $A = L(s)$ and the sign type has $A = 0$. These conventions agree with the positive canonical basis used throughout.

Definition 2.1. Finite parabolic attainment means the following assertion. For every compact tempered family α, every unitary z, and every irreducible constituent ρ of $\pi_{\alpha,v,z}$, there are a finite standard parabolic W_I and a simple type σ such that

$$\mathrm{Hom}_{\mathcal{H}_I}(\sigma, \rho) \neq 0, \qquad A_I(\sigma) = A_\alpha.$$

The assertion is along the fixed positive parameter ray.

Theorem 2.2 (Finite-attainment criterion)**.** *Let W be an irreducible affine Weyl group with a positive integer weight. Assume P1–P15 for its finite standard parabolic subgroups and finite parabolic attainment. Then P1–P15 hold for W.*

Sections 3–5 prove the criterion by constructing the original filtration and its independent-parameter commutation from the spectral algebra.

3. The integral asymptotic Fourier algebra

In this section W is an irreducible affine Weyl group. We fix the positive integral weight throughout; all limits are along this one ray. We construct a $\mathbb{Z}$-algebra on a basis indexed by W from the positive Plancherel formula.

3.1. **Bernstein notation.** Use the root-lattice datum $(X, R_0, Y, R_0^\vee, F_0)$, with $X = \mathbb{Z}R_0$, $Y = \mathrm{Hom}(X, \mathbb{Z})$, and $W = X \rtimes W_0$. Write $\langle x, y\rangle$ for its perfect pairing and t_x for translation by x. The dominant cone is

$$X^+ = \{x \in X : \langle x, \alpha^\vee\rangle \geq 0 \text{ for all } \alpha \in F_0\}.$$

For $x = x_+ - x_-$ with $x_\pm \in X^+$, set $\theta_x = N_{t_{x_+}} N_{t_{x_-}}^{-1}$. The Bernstein presentation gives independence of this choice, $\theta_x\theta_y = \theta_{x+y}$, and the center $\mathcal{A}[X]^{W_0}$ [27, Theorem 2.3 and Corollary 2.4]. Put $T = \mathrm{Hom}(X, \mathbb{C}^\times)$ and $T_u = \mathrm{Hom}(X, S^1)$; $\mathbb{C}[T] = \mathbb{C}[X]$ is its Laurent coordinate algebra. For $R \in Y \otimes \mathbb{Q}$, the character v^{2R} is defined by $x(v^{2R}) = v^{2\langle x,R\rangle}$. For $P \subseteq F_0$, put $R_P = R_0 \cap \mathbb{Z}P$ and let $R_P^\vee$ be its coroot system. Write $W_P = W(R_P)$ and let W^P be the minimal representatives of W_0/W_P. The semisimple Levi lattice is $X_P = X/(X \cap (\mathbb{R}R_P^\vee)^\perp)$, and $T^P \subseteq T$ is the connected unramified torus trivial on P. The notation Tr means ordinary matrix trace, a dagger means conjugate transpose, and $\|M\|_{\mathrm{HS}}^2 = \mathrm{Tr}(MM^\dagger)$. For a Laurent polynomial, CT denotes its constant term and Newt the convex hull of its exponent support. We write $r_+ = \max(r, 0)$ for $r \in \mathbb{R}$.

3.2. **Compact families and their leading coefficients.** Choose one compact torus of induction data from each association class. We index the resulting finite list by α. Its data consist of a semisimple Levi discrete series δ_α, a complex unramified torus T_α, and a finite association group G_α. Let $T_{\alpha,u}$ be its compact form, with Haar measure of mass one. The compact induced space has a Hermitian form independent of the unitary induction variable. Normalized intertwiners implement the association action. These statements and the exact Plancherel isomorphism are the compact-induction and Fourier theorems of [26, Propositions 4.19–4.20, Theorem 4.43 and Corollary 4.45]; unitary stabilizers are described in [8].

Lemma 3.1 (Algebraic compact models on a fixed ray)**.** *Fix the positive integral weight and let $v > 1$ vary on this ray. On a common tail $v > V$, all semisimple Levi discrete series admit simultaneous matrix models over one finite extension $E/\mathbb{Q}(v)$. The field may be realized as algebraic function germs at positive infinity and chosen stable under complex conjugation. The models have positive invariant Hermitian forms over E and algebraic orthonormalizing changes of basis. Their compact induced forms and these changes of basis are independent of the unramified variable z. For every fixed w, the induced matrices of N_w belong to* $\mathrm{Mat}(E[T_\alpha])$ *for the corresponding inducing torus.*

The finite association groupoid is constant on this tail. Its inducing-module isometries can be chosen over a finite enlargement of E, independently of z. With these choices, the Bernstein normalized induced intertwiners are rational in z, algebraic in v, and have only the standard Bernstein torus denominators. After substitution of the inducing weights, their factors are, up to invertible monomials,

$1 - \epsilon v^b z^\beta$, *where* ϵ *is a root of unity and* $b \in \mathbb{Q}$. *One finite ramification gives convergent Puiseux expansions for all scalar algebraic coefficients used in these constructions.*

Proof. The finite collection of generic residual points specializes onto all residual points, also at exceptional parameters. Its unitary parts are torsion, and its real logarithmic parts are rational linear functions of the logarithmic parameters [27, Theorem 2.47, Proposition 2.52, Corollary 2.54, Propositions 2.56 and 2.63]. After restriction to the chosen ray, all these candidates are therefore torsion monomials in rational powers of v. Include every such candidate for every semisimple Levi, identifying repeated central orbits on this ray.

Restrict the Hecke algebra to the ray before taking its generic central quotients. The Bernstein presentation makes each such quotient B finite-dimensional over a finite extension of $\mathbb{Q}(v)$ [27, Theorem 2.3 and Corollary 2.4]. Let J be its Jacobson radical. Over a further finite extension its semisimple quotient is split:

$$B/J \simeq \prod_j \mathrm{Mat}_{d_j}(E), \qquad J^m = 0.$$

Spread B, a basis of J, the quotient map and its matrix units over a coefficient curve with function field E. Inverting finitely many nonzero scalar functions makes the modules free and gives an exact sequence with nilpotent kernel and quotient $\prod_j \mathrm{Mat}_{d_j}(R)$ over the resulting ring R. Indeed, this requires only clearing the finitely many denominators and nonzero determinants in the multiplication table and these linear maps. In every remaining complex fiber the kernel is nilpotent and the quotient is semisimple, so that kernel is exactly the Jacobson radical. The displayed matrix models consequently remain simple and exhaust all simple modules in every such fiber. This step applies to the generic algebra of the fixed ray, even when that ray is contained in a parameter wall.

Every generalized Bernstein weight in these models belongs to the specified residual central orbit. After a further localization, the distinct weights and their multiplicities are constant: use the generalized spectral projectors of a linear combination of Bernstein generators separating the finitely many joint weights. Each weight has the form $\lambda_v(x) = \zeta(x) v^{b(x)}$ for $x \in X_P$, where ζ is torsion and $b : X_P \to \mathbb{Q}$ is additive. Casselman's strict inequalities are therefore independent of $v > 1$. Retaining exactly the models satisfying them gives all Levi discrete series at every point of a common tail [27, Theorem 2.25].

Realize the coefficient field in complex algebraic germs along the positive ray. Adjoin i and the conjugates of its finitely many generators. Then conjugation is an involution fixing v, its fixed field F consists of real-valued algebraic germs, and $E = F(i)$. For each model the equations

$$\delta(h)^\dagger H = H\delta(h^*), \qquad H^\dagger = H,$$

for a finite generating set form a finite linear system over F. At every retained real specialization the module is irreducible and unitarizable [26, Corollary 2.23]; hence its invariant Hermitian forms have real dimension one by Schur's lemma. After excluding the finitely many rank-changing specializations, the generic system likewise has dimension one. Choose a nonzero algebraic solution H. Its specializations are nonzero real multiples of positive forms, hence definite; continuity fixes the sign on a smaller tail. Choose the positive sign. Cholesky factorization then uses

only positive square roots of algebraic pivots. Adjoining these finitely many real algebraic germs gives an algebraic orthonormalizing matrix.

For the family induced from (P, δ), let V_δ be the space of δ. In the compact realization $\mathrm{span}\{N_w : w \in W^P\} \otimes V_\delta$, the induced form is the product of this form with $\tau(x^* y)$, independently of the unitary induction variable [26, Proposition 4.19]. The Bernstein twist multiplies θ_x by the Laurent character $z(x)$. Finite coset induction therefore gives Laurent polynomial matrices, and the change of basis just constructed is independent of z.

There are finitely many association arrows, including the finite torsion twists $K_P = T_P \cap T^P$, where T_P is the subtorus with tangent space $\mathbb{C}R_P^\vee$. For every possible pair of models the inducing-module intertwining equations are algebraic linear equations independent of z; their ranks are constant after another finite localization. Their nonzero solutions are isomorphisms by irreducibility. In orthonormalized models such a solution D satisfies $D^\dagger D = cI$ with c a positive real algebraic function. Replacing D by $c^{-1/2} D$ gives the required isometry, still independent of z. This also makes the finite association groupoid constant on the tail; see [8, (3.3)–(3.8)] for its action on inducing data.

For a Weyl association arrow n, let ι_n^0 be its normalized Bernstein intertwining element [26, (4.5)–(4.6)]. Use the constructed isometries in [26, Definition 4.32]: $h \otimes u \mapsto h\iota_{n^{-1}}^0 \otimes D(u)$. Theorem 4.33(i) there, and its formula (4.74), give rationality and the root-binomial denominators evaluated at the inducing weights. Substitution gives precisely the stated factors; formal half-root factors are paired into genuine character binomials as in [26, Remark 3.1]. Generalized weight spaces can increase the powers of these factors but introduce no new factors, as follows by taking the determinant of the corresponding commuting Bernstein operator. Theorem 4.33(ii) and Corollary 4.34 give unitarity and regularity near the compact torus. The inducing isometries contribute only scalar denominators in v.

All field enlargements and localizations are finite. Nonzero algebraic functions have only finitely many finite zeros and poles, and algebraic functions have only finitely many branch points. Thus one tail avoids every excluded specialization. Newton–Puiseux theory at the chosen place over infinity gives convergent expansions after one common finite ramification. Finitely many algebra generators and association arrows determine all the matrices, so the same field and ramification suffice for every fixed w. □

For a fixed w, the matrix $\pi_{\alpha,v,z}(N_w)$ has Laurent polynomial entries in z, with support independent of v. Balancing the Hermitian form conjugates by a matrix independent of z, so it preserves this assertion. The root product for the Plancherel measure gives

$$d\mu_{\alpha,v}(z) = v^{-2A_\alpha} m_{\alpha,v}(z)\, dz, \qquad m_{\alpha,v} \longrightarrow m_{\alpha,0} \tag{3.1}$$

uniformly on $T_{\alpha,u}$, where $m_{\alpha,0}$ is a nonzero nonnegative Laurent polynomial. The association constants are included in $m_{\alpha,v}$.

After substitution of a residual character, every nonconstant factor of the prime root product is a binomial $1 - \epsilon v^b z^\beta$. For $b \neq 0$, division by its leading power of v gives a Laurent unit uniformly on the compact torus. For $b = 0$, one has $|\epsilon| = 1$. Negative multiplicities of these stationary divisors cancel, since the exact density is regular on the unitary torus. Their compact points are Zariski dense, so this

cancellation is an algebraic identity. The remaining stationary product is Laurent polynomial. The residual scalar in the density is constant along the fixed ray by [26, Theorem 5.6]. This proves the stated uniformity. Section 5 factors the limiting relative density more precisely.

We record the degree calculation, including the nonreduced case. Put $q(s) = v^{2L(s)}$. Label the affine coroots by $q_{wa} = q_a$ and $q_{a+1} = q(s_a)$ for a simple affine coroot a. For $\beta^\vee \in 2Y$, set $q_{\beta^\vee/2} = q_{1+\beta^\vee}/q_{\beta^\vee}$, and put

$$R_{\mathrm{nr}} = R_0 \cup \{2\beta : \beta^\vee \in 2Y\}, \qquad R_1 = \{\alpha \in R_{\mathrm{nr}} : 2\alpha \notin R_{\mathrm{nr}}\}.$$

These are the root labels of [26, §2.3]. For $\alpha \in R_1$ set $q_{2\alpha^\vee} = 1$ if $\alpha/2 \notin R_0$. On a residual coset $\mathcal{L}$ the prime product is

$$\mathscr{M}_{\mathcal{L}}(t) = q(w_0)\frac{\prod'_{\alpha\in R_1}(\alpha(t)-1)}{\prod'_{\alpha\in R_1}(q_{\alpha^\vee}^{1/2}\alpha(t)^{1/2}+1)\prod'_{\alpha\in R_1}(q_{\alpha^\vee}^{1/2}q_{2\alpha^\vee}\alpha(t)^{1/2}-1)}. \tag{3.2}$$

Here w_0 is the longest element of W_0, both root signs occur, and a prime deletes factors identically zero on $\mathcal{L}$; see [26, (3.47)]. If a half root is not a character, its two formal factors combine into a genuine character binomial [26, Remark 3.1].

Write $q_{\alpha^\vee}^{1/2} = v^{u_\alpha}$, $q_{2\alpha^\vee} = v^{b_\alpha}$, $t = szv^{2R}$ with sz unitary, and $r_\alpha = \langle \alpha, R\rangle$. Set $d_0 = \deg_v q(w_0)$. The generic degree of a nonzero binomial with exponent c is c_+. Therefore

$$-2A_R = d_0 + \sum_{\alpha\in R_1}\Big((2r_\alpha)_+ - (u_\alpha + r_\alpha)_+ - (u_\alpha + b_\alpha + r_\alpha)_+\Big). \tag{3.3}$$

Every deleted factor has exponent zero, so its contribution to this sum is zero. Formula [26, (4.96)] relates the actual induced measure to (3.2), and its residual scalar has degree zero by the cited base-change theorem. Hence (3.3) is the degree of (3.1). It is rational and depends only on the real exponent orbit. On a parameter wall the same calculation uses the residual families occurring on that wall.

Lemma 3.2. *For every α, w the limit*

$$c_{\alpha,w}(z) = \lim_{v\to\infty} v^{-A_\alpha}\pi_{\alpha,v,z}(N_w)$$

exists uniformly on $T_{\alpha,u}$ and is a Laurent polynomial matrix. Replacing N_w by C_w gives the same limit. Moreover,

$$\sum_\alpha \int_{T_{\alpha,u}} \mathrm{Tr}(c_{\alpha,x}c_{\alpha,y}^*)m_{\alpha,0}\,dz = \delta_{x,y}. \tag{3.4}$$

Proof. Exact Plancherel and the orthonormal standard basis give

$$1 = \tau(N_wN_w^*) = \sum_\alpha \int \|\pi_{\alpha,v,z}(N_w)\|_{\mathrm{HS}}^2\, v^{-2A_\alpha}m_{\alpha,v}(z)\,dz.$$

If a matrix entry had Puiseux degree larger than A_α, its nonzero leading Laurent coefficient would be bounded away from zero on some compact open subset where $m_{\alpha,0} > 0$. The displayed positive integral would then be unbounded. Thus every entry has degree at most A_α. There are only finitely many Laurent coefficients for this fixed w, so their limits give uniform convergence.

For each $y < w$, the product of $p_{y,w}$ with $v^{-A_\alpha}\pi_\alpha(N_y)$ tends uniformly to zero. The finite triangular expansion of C_w therefore gives the same limit. Finally apply exact Plancherel to $N_xN_y^*$ and pass to the limit. Uniform convergence of the matrices and densities justifies this passage and proves (3.4). □

Use the Bernstein normalized intertwiners $U_{\alpha,g,v}(z)$ of Lemma 3.1. They have uniform Laurent invertible limits $U_{\alpha,g,0}(z)$. Indeed their possible torus denominators are Bernstein binomials. On a stationary unitary divisor, unitarity makes each apparent pole removable. Its unitary points are Zariski dense in the divisor; hence the numerator is divisible by that binomial. Algebraicity in v makes the divisibility an identity before the limit is taken. After these factors are cancelled, the normalized nonstationary denominators have invertible monomial limits. Unitarity excludes a positive Puiseux degree. The same reasoning applies to the inverse intertwiner, proving invertibility of the Laurent limit. Scalar projective factors cancel in conjugation, so the limiting conjugation maps define an ordinary finite group action.

Let d_α be the dimension of the compact induced space. Set

$$\mathcal{B}_\alpha = \Big(\mathrm{Mat}_{d_\alpha}(\mathbb{C}[T_\alpha])\Big)^{G_\alpha}, \qquad \mathcal{B} = \bigoplus_\alpha \mathcal{B}_\alpha, \tag{3.5}$$

where the action combines its affine monomial action on the torus with conjugation by $U_{\alpha,g,0}$. The involution is matrix adjunction on the compact torus. In particular it preserves Laurent matrices. Let

$$\mathscr{H}_0 = \bigoplus_\alpha L^2\Big(T_{\alpha,u}, m_{\alpha,0}\, dz; \mathrm{Mat}_{d_\alpha}(\mathbb{C})\Big)^{G_\alpha}.$$

Lemma 3.2 defines an isometry $\mathcal{F}_0 : \ell^2(W) \longrightarrow \mathscr{H}_0$ by $\mathcal{F}_0(N_w) = (c_{\alpha,w})_\alpha$.

3.3. **Finite Bernstein expansions.** We prove that $\mathcal{F}_0$ maps finitely supported vectors onto $\mathcal{B}$, by first proving Hilbert surjectivity and then finite support of the algebraic inverse.

A finite collection of lattice charts covers W:

$$w = w(s,m), \qquad m \in \mathbb{N}^{r_s}. \tag{3.6}$$

Here s records two finite Weyl group factors, a rational simplicial cone in X^+, and a bounded lattice offset. To obtain these charts, subdivide each face of X^+ into rational simplicial cones and use integral generators of their rays. Their span has finite index in the lattice of the face, leaving finitely many fundamental parallelepiped offsets. Relative interiors are handled with coefficients in $(0,1]$ in the offset, followed by nonnegative integer multiples. Faces of smaller dimension are included.

In each chart, for every family, the exact normalized matrix has a finite expansion

$$v^{-A_\alpha}\pi_{\alpha,v,z}(N_{w(s,m)}) = \sum_j R_j(v,z)P_j(m)\epsilon_j(m)v^{\kappa_j(m)}z^{\nu_j(m)}. \tag{3.7}$$

In this formula P_j is polynomial, ϵ_j is a unitary character of the lattice, ν_j is an integral linear frequency map, and κ_j is a rational linear form nonpositive on $\mathbb{R}^{r_s}_{\geq 0}$. The matrices R_j are rational in z and algebraic in v; fixed offsets and fixed powers of v have been absorbed into them. Each coefficient in their Puiseux expansions can have poles only on divisors

$$1 - \omega z^\beta = 0, \qquad |\omega| = 1, \quad \beta \text{ a primitive torus character.} \tag{3.8}$$

We give the details underlying these assertions. The facetwise Bernstein expansion [26, (2.25)] writes $N_{ut_xu'}$ as a finite sum of fixed left and right finite factors around θ_x, with coefficients unchanged as x varies in one dominant face. On a

simultaneous generalized Bernstein weight space the commuting unipotent parts have commuting nilpotent logarithms. Therefore their powers are a polynomial of bounded degree in the lattice coordinates, multiplied by the weight character; compare [26, (2.28)–(2.29)]. All weights are torsion monomials times a real power of v and an unramified character.

For each lattice generator, use the Hermite interpolation projectors for its distinct eigenvalues; multiply these commuting projectors to obtain the joint generalized-weight projectors. Their denominators are powers of differences of monomial eigenvalues. Expanding at $v = \infty$, a denominator with unequal real exponents has a monomial leading term. Equal exponents give precisely the binomials (3.8), after primitive factorization. Taking logarithms of the unipotent parts is a finite polynomial operation and introduces no additional torus poles.

Temperedness gives $\kappa_j \le 0$ by the Bernstein-weight Casselman criterion. The strict version for the inducing discrete series gives the stronger statement

$$m \in \mathbb{R}^{r_s}_{\ge 0}, \quad \kappa_j(m) = 0, \quad \nu_j(m) = 0 \quad \Longrightarrow \quad m = 0. \tag{3.9}$$

To check it, write an induced weight as $u(rz)$, where u is a minimal representative for the relevant finite parabolic coset and r is a weight of the inducing discrete series. If x is dominant, then $u^{-1}x$ is Levi-dominant, because u takes positive Levi roots to positive ambient roots. Vanishing of the unramified frequency says that $u^{-1}x$ lies in the real Levi root span. Unless $x = 0$, it gives a nonzero semisimple Levi translation, on which the strict discrete-series inequality is $|r(u^{-1}x)| < 1$. This is the required strict negativity of κ_j. The relevant statements are [26, Lemmas 2.20, 2.22 and Proposition 4.20]. The real-cone assertion follows from the lattice assertion: the intersection of the dominant cone with the kernel of the integral frequency map is a rational polyhedral cone. Each nonzero extreme ray has a lattice multiple, and strict negativity on its rays implies strict negativity on every nonzero point.

Lemma 3.3 (Uniform cone margin)**.** *There is $c > 0$, common to all the finitely many terms and charts, such that*

$$-\kappa_j(m) + \|\nu_j(m)\| \ge c|m|_1 \qquad (m \in \mathbb{R}^{r_s}_{\ge 0}).$$

Consequently bounded frequency forces $\kappa_j(m) \le C - c|m|_1$, and on a face where $\kappa_j = 0$ one has $\|\nu_j(m)\| \ge c|m|_1$.

Proof. On the compact simplex $m \ge 0$, $|m|_1 = 1$, the continuous function $-\kappa_j(m) + \|\nu_j(m)\|$ is strictly positive by (3.9). Take its minimum, then take the minimum over the finite set of terms. Homogeneity gives the assertion. □

3.4. **Surjectivity in Hilbert space.** For the exact measure in (3.1), write

$$\mathscr{H}_v = \bigoplus_\alpha L^2(T_{\alpha,u}, m_{\alpha,v}\, dz; \mathrm{Mat}_{d_\alpha}(\mathbb{C}))^{G_{\alpha,v}}.$$

The normalized exact Fourier map $\mathcal{F}_v$, sending N_w to $v^{-A_\alpha}\pi_{\alpha,v,z}(N_w)$, is unitary from $\ell^2(W)$ to $\mathscr{H}_v$.

Lemma 3.4. *The isometry $\mathcal{F}_0 : \ell^2(W) \longrightarrow \mathscr{H}_0$ is surjective.*

Proof. Start with a Laurent matrix B on one family and average it under the exact finite conjugation action, obtaining B_v. The uniform limits of the intertwiners and

their inverses give $B_v \to B_0$ uniformly, where B_0 is the limiting average. If B was invariant for the limiting action, $B_0 = B$.

Clear the denominators of B_v, $m_{\alpha,v}$, and all $R_j(v,z)^*$ occurring in (3.7). Take the product of the finitely many conjugates of this clearing scalar under G_α, so it is invariant. Normalize its highest v-power to obtain an invariant Laurent polynomial f_v with

$$f_v \longrightarrow f_0 \neq 0$$

uniformly. A product of nonzero leading Laurent coefficients is nonzero, so this normalization introduces no cancellation. By construction f_v clears the exact denominators in the pairings below.

Put $f_v B_v$ on this family and zero on the others. Its inverse $x_v = \mathcal{F}_v^{-1}(f_v B_v)$ has, in a lattice chart, coordinates that are sums of terms

$$\overline{P_j(m)\epsilon_j(m)}\, v^{\kappa_j(m)}\, \mathrm{CT}_z\Big(f_v\ \mathrm{Tr}(B_v R_j(v,z)^*)\, m_{\alpha,v} z^{-\nu_j(m)}\Big)\,. \tag{3.10}$$

Before the final character $z^{-\nu_j(m)}$, the integrand is Laurent polynomial with support in a fixed finite set. Its coefficients lie in a fixed finite set of algebraic functions of v. Lemma 3.3 consequently gives constants $C, D, N, c > 0$ such that

$$|(x_v)_{w(s,m)}| \leq C(1+|m|_1)^N v^{D-c|m|_1} \tag{3.11}$$

for all sufficiently large v.

When $c|m|_1 > D$, the right side tends to zero and is dominated by its value at one fixed $v_* > 1$. Its squares are summable: the number of tuples with a specified value of $|m|_1$ grows polynomially. There are finitely many charts. On the remaining finite set of tuples boundedness follows from the norm of $f_v B_v$. Every fixed coordinate converges, since the exact pairing converges to $\langle f_0 B_0, \mathcal{F}_0(N_w)\rangle_{\mathscr{H}_0}$. It follows that $x_v \to x$ strongly and that x has finite support. Taking norms gives

$$x = \mathcal{F}_0^*(f_0 B_0), \qquad \|x\| = \|f_0 B_0\|\,.$$

For an isometry this equality says that $f_0 B_0$ lies in its closed range. The same proof, with any invariant Laurent scalar g inserted, gives $g f_0 B_0$ in the range.

For $\varepsilon > 0$ approximate the continuous invariant function

$$\frac{\overline{f_0}}{|f_0|^2 + \varepsilon}$$

uniformly on the compact torus by invariant Laurent polynomials. Multiplying by $f_0 B_0$ shows that $|f_0|^2 B_0/(|f_0|^2+\varepsilon)$ lies in the closed range. The zero set of a nonzero Laurent polynomial has Haar measure zero. Dominated convergence therefore gives B_0 in that range as $\varepsilon \downarrow 0$. Finally Laurent matrices are dense in the ambient matrix L^2-space. Averaging under the limiting finite unitary action is a bounded projection preserving Laurent matrices, so their invariant averages are dense in $\mathscr{H}_0$. Thus the range is all of $\mathscr{H}_0$. □

3.5. **The inverse preserves finite support.**

Lemma 3.5. *The leading Fourier map induces an isomorphism of vector spaces*

$$\mathcal{F}_0 : \mathbb{C}^{(W)} \xrightarrow{\sim} \mathcal{B}.$$

Proof. Its image is contained in $\mathcal{B}$ by Lemma 3.2. We prove that the inverse of every Laurent packet has finite support.

In (3.7) the top Puiseux exponent of each R_j is fixed. Whenever $\kappa_j(e_i) < 0$, there is a cutoff, independent of all other nonnegative coordinates, beyond which this term cannot contribute to the exponent-zero coefficient. Partition each lattice chart into finitely many strata by fixing every such small coordinate or allowing it to run above a common cutoff. After translation, the free coordinates run over $\mathbb{N}^r$. On a stratum every surviving term has $\kappa_j = 0$ on all free directions. Taking the exponent-zero coefficient therefore gives

$$c_m(z) = \sum_j Q_j(m, z)\chi_j(m, z), \qquad \chi_j(m, z) = \epsilon_j(m) z^{\nu_j(m)}. \tag{3.12}$$

Here Q_j is polynomial in the free coordinates with rational torus coefficients. Its poles are among (3.8). Terms of positive Puiseux degree, if present separately, cancel by the bounded limit already proved; extracting the degree-zero coefficient is thus legitimate. Combine identical lattice characters. The remaining characters are distinct for almost every unitary z. Fixed coordinate offsets affect only the fixed prefactors. For each remaining frequency, Lemma 3.3 gives

$$\|\nu_j(m)\| \geq c_j |m|_1 \tag{3.13}$$

on the free cone.

We first show that each Q_j has polynomial degree zero in m. Suppose the largest degree is d, and let $Q_{j,d}$ be its homogeneous part of degree d. For a generic fixed unitary point z, consider

$$A_M(z) = M^{-r-2d} \sum_{0 \leq m_i < M} \|c_m(z)\|_{\mathrm{HS}}^2 .$$

The diagonal terms converge by Riemann sums to

$$\sum_j \int_{[0,1]^r} \|Q_{j,d}(x, z)\|_{\mathrm{HS}}^2 \, dx. \tag{3.14}$$

Every cross term tends to zero. Indeed some coordinate of the quotient of two distinct characters has phase $\omega \neq 1$. For an integer $h \geq 0$, summation by parts gives

$$\sum_{\ell=0}^{M-1} \ell^h \omega^\ell = O_\omega(M^h).$$

In that coordinate one gains a factor M^{-1} compared with an ordinary power sum. Expansion into monomials then shows that the normalized cross term is $O_z(M^{-1})$.

The total leading norm of each $N_{w(s,m)}$ is one. Its contribution on this family is at most one, whence

$$\int A_M(z) m_{\alpha,0}(z) \, dz \leq M^{-2d}.$$

Apply Fatou's lemma to the nonnegative full expression A_M, after its pointwise limit has been taken. If $d > 0$, (3.14) must vanish almost everywhere where $m_{\alpha,0} > 0$. A nonzero nonnegative Laurent density is positive almost everywhere. Thus every $Q_{j,d}$ vanishes identically, a contradiction. We have proved $d = 0$. The same argument at $d = 0$ gives

$$\sum_j \int \|Q_j(z)\|_{\mathrm{HS}}^2 \, m_{\alpha,0}(z) \, dz \leq 1. \tag{3.15}$$

A zero-dimensional stratum is finite.

Let q be an entry of Q_j with a pole of order $k > 0$ on $D = \{1-\omega z^\beta = 0\}$. Choose a generic unitary point of D, away from all other divisors and from the zeros of the leading residue. Such points exist because the unitary points are Zariski dense in D. If $m_{\alpha,0}$ has vanishing order s on D, a real transverse coordinate u on the compact torus gives a positive lower bound for $|q|^2 m_{\alpha,0}$ by a constant times $|u|^{s-2k}$. Integrability in (3.15) implies $s - 2k > -1$ and hence $s \geq 2k$, because the orders are integers. The involution on Laurent functions takes the defining binomial of D to an associate of itself. Thus $m_{\alpha,0} q^*$ has no pole on D. All possible poles have this form, so unique factorization in the Laurent polynomial ring gives

$$m_{\alpha,0} Q_j^* \in \mathrm{Mat}_{d_\alpha}(\mathbb{C}[T_\alpha]). \tag{3.16}$$

For a fixed Laurent packet B, its pairing with (3.12) is consequently a sum of constant terms

$$\overline{\epsilon_j(m)}\,\mathrm{CT}_z\left(\mathrm{Tr}(B m_{\alpha,0} Q_j^*) z^{-\nu_j(m)}\right).$$

The Laurent coefficient before the final character has fixed finite support. A nonzero pairing permits only finitely many frequencies, and (3.13) then permits only finitely many m. There are finitely many strata, charts, and families. Therefore $\mathcal{F}_0^* B$ has finite support. Lemma 3.4 gives $\mathcal{F}_0 \mathcal{F}_0^* B = B$, as required. □

3.6. **An integral analytic projection lemma.** We identify the spectral summands by coordinate subsets of the original standard basis using norm-convergent integral projections.

Let $\mathscr{E}$ be the ring of bounded operators on $\ell^2(W)$ whose matrices in the basis (N_w) have integer entries and finite support in every row and every column. It is closed under sums, products, and adjoints. For products, the support of a row or column lies in a finite union of finite supports.

Lemma 3.6. *Suppose $Y(t)$ is analytic in operator norm near $t = 0$, its Taylor coefficients belong to $\mathscr{E}$, and $Y(0)$ is an orthogonal projection. Suppose that for small positive real t, $Y(t)$ is self-adjoint and its spectrum lies near 0 and 1. The spectral projection onto the cluster near 1 extends analytically to $t = 0$, and every Taylor coefficient of this projection belongs to $\mathscr{E}$.*

Proof. Use the polynomial $p(y) = 3y^2 - 2y^3$. Direct multiplication gives

$$p(Y) - Y = -(2Y - 1)(Y^2 - Y), \tag{3.17}$$

$$p(Y)^2 - p(Y) = (Y^2 - Y)^2(4Y^2 - 4Y - 3). \tag{3.18}$$

Shrink a closed complex disk about zero so that initially $\|Y\| \leq 5/4$ and $\|Y^2 - Y\| \leq 1/100$. As long as $\|Y\| \leq 2$, a defect bound δ gives an increment at most 5δ and a new defect at most $27\delta^2$. Since $\delta \leq 1/100$, the defects decrease at least geometrically with ratio $27/100$. The total increment is at most

$$\frac{5/100}{1 - 27/100} = \frac{5}{73} < \frac{3}{4}.$$

The iterates thus stay within the norm bound 2 and converge uniformly on the disk to an analytic idempotent $E(t)$. For real t, scalar iteration converges to zero on the spectral cluster near zero and to one on the cluster near one. Continuous functional calculus identifies $E(t)$ with the desired exact spectral projection.

The initial defect has a zero at $t = 0$. Equation (3.18) doubles its order at each iteration. By (3.17), each fixed Taylor coefficient of the iterates therefore stabilizes after finitely many steps. It is obtained by finitely many ring operations from the coefficients of Y. It belongs to $\mathscr{E}$, proving the assertion. □

We shall use two immediate facts. First, if an operator-meromorphic function with a pole of finite order is bounded for positive real $t \downarrow 0$, its negative Laurent coefficients vanish. Multiply by its highest possible pole power and take the operator-norm limit, then proceed successively. Second, an orthogonal projection with integral matrix entries is a coordinate projection. Indeed

$$0 \le E_{w,w} \le 1, \qquad \sum_x |E_{x,w}|^2 = E_{w,w}.$$

The diagonal entry is either zero or one. In the first case the column is zero, and in the second its diagonal entry exhausts the squared norm, so the column is N_w.

3.7. **Central projections in the original coordinates.** Let $\mathcal{R}$ be the finite set of real exponent orbits of the tempered central characters. We write such a character as $W_0(szv^{2R})$, with $R \in Y \otimes \mathbb{Q}$ and sz unitary. The degree in (3.1) will also be denoted A_R. Choose a rational W_0-invariant positive definite inner product on $Y \otimes \mathbb{R}$. We identify its metric dual with $X \otimes \mathbb{R}$.

Proposition 3.7. *After choosing $t = v^{-1/e}$ with a sufficiently divisible positive integer e, the exact central spectral projections $E_R(t)$ onto these real exponent orbits are analytic in operator norm at $t = 0$. Their Taylor coefficients belong to $\mathscr{E}$. The constant projections partition the standard basis:*

$$E_R(0)\ell^2(W) = \ell^2(S_R), \qquad W = \coprod_{R \in \mathcal{R}} S_R.$$

Under $\mathcal{F}_0$, this is exactly the partition into Fourier families with real exponent orbit R.

Proof. Order the remaining real exponent orbits by decreasing norm. For an orbit $R \neq 0$ of largest remaining norm, choose an integral lattice vector λ equal to a positive rational multiple of its metric dual: $\lambda = cR^\flat$, with $c > 0$. Put $D = 2\langle \lambda, R \rangle$ and use the integral central element

$$z_\lambda = \sum_{\mu \in W_0\lambda \text{ distinct}} \theta_\mu.$$

If R' is a remaining orbit, then

$$2\langle \mu, R' \rangle \le 2c \, \|R\| \, \|R'\| \le 2c \, \|R\|^2 = D. \tag{3.19}$$

Equality forces R' to belong to the orbit of R and μ to be its metric-dual multiple. There is exactly one such orbit element in the sum. Thus $v^{-D} z_\lambda$ has scalar limit of modulus one on the selected orbit and scalar limit zero on every other remaining orbit, uniformly in its unitary phase. Summing over distinct orbit elements is what makes the top coefficient one.

Identify all regular Hilbert spaces by their orthonormal standard basis. Write L_h for left multiplication by h. For an integral Laurent Hecke element h, this is a finite Laurent polynomial in t with coefficients in $\mathscr{E}$. For a simple generator this follows directly from its multiplication formula: the coefficient operators are the simple permutation of the basis and the diagonal indicator of its descent set. Finite

products give the assertion for all standard basis elements. Bernstein elements are integral as well, since $N_s^{-1} = N_s - (v^{L(s)} - v^{-L(s)})$. Enlarge e once to clear every exponent required in the finite construction.

Suppose the sum of already constructed central projections is $P(t)$. Form

$$B(t) = v^{-D}(1 - P(t))L_{z_\lambda}.$$

It is operator-meromorphic with a pole of finite order, and its Laurent coefficients belong to $\mathscr{E}$. On the positive real ray its norm is bounded by (3.19) and exact Plancherel. Its negative coefficients therefore vanish, so $B(t)$ is analytic.

For $B(t) = \sum_{n\geq 0} B_n t^n$, set $B^\sharp(t) = \sum_{n\geq 0} B_n^* t^n$ and $Y(t) = B^\sharp(t)B(t)$. This is analytic with coefficients in $\mathscr{E}$ and equals $B(t)^*B(t)$ on the real ray. Its scalar spectral values tend uniformly to one on the selected orbit and to zero elsewhere. Hence $\|Y^2 - Y\| \to 0$, and $Y(0)$ is an integral orthogonal projection. Lemma 3.6 constructs the exact selected projection $E_R(t)$. It vanishes on $P(t)$, and its constant term is a coordinate projection.

Continue over the finite set $\mathcal{R}$. If the last orbit is zero, use $z_\lambda = 1$ on the remaining subspace. The projections are pairwise orthogonal and sum to the identity for small positive real t; analyticity gives the same identities at $t = 0$. This proves the asserted partition.

For fixed x, y, exact Plancherel gives

$$\langle E_R(t)N_x, N_y\rangle = \sum_{\alpha: R_\alpha \in W_0 R} \int \mathrm{Tr}(\mathcal{F}_v(N_x)_\alpha \mathcal{F}_v(N_y)_\alpha^*) m_{\alpha,v}\, dz.$$

Pass to the uniform limit. The result is

$$\langle E_R(0)N_x, N_y\rangle = \sum_{\alpha: R_\alpha \in W_0 R} \int \mathrm{Tr}(c_{\alpha,x} c_{\alpha,y}^*) m_{\alpha,0}\, dz. \tag{3.20}$$

By Lemma 3.4, the right side is the Fourier projection onto those families. This proves the final assertion. □

Group the S_R with equal A_R and write their union as S_A. Matrix adjunction preserves a family cutoff, and $c_{\alpha,w^{-1}} = c_{\alpha,w}^*$. Therefore $S_A^{-1} = S_A$. Section 4 identifies the resulting filtration with canonical ideals.

3.8. **Integral multiplication and a finite integral unit.**

Proposition 3.8. *There is a unital associative $\mathbb{Z}$-algebra*

$$J = \bigoplus_{w\in W} \mathbb{Z}t_w$$

whose complexification is identified with $\mathcal{B}$ by $t_w \mapsto (c_{\alpha,w})_\alpha$. Its involution is $t_w^* = t_{w^{-1}}$, and its trace satisfies

$$\tau_J(t_x t_y^*) = \delta_{x,y}. \tag{3.21}$$

The unit has finite integral support. For each degree A, the coordinate span $J_A = \bigoplus_{w\in S_A} \mathbb{Z}t_w$ is a unital two-sided summand, and $J = \bigoplus_A J_A$.

Proof. Define the central operator families

$$P_-(v) = \sum_R v^{-A_R} E_R(v^{-1/e}), \qquad P_+(v) = \sum_R v^{A_R} E_R(v^{-1/e}).$$

They are meromorphic at $t = 0$ with poles of finite order and Laurent coefficients in $\mathscr{E}$. For fixed x, the operator $P_- L_{N_x}$ is bounded on the positive ray: in exact Fourier coordinates it acts by multiplication by $v^{-A_\alpha}\pi_{\alpha,v,z}(N_x)$, uniformly bounded by Lemma 3.2. Consequently its negative Laurent coefficients vanish. Its constant coefficient applied to N_y is a finite integer vector.

Pair this constant vector with N_z. The exact Plancherel expression whose limit is being taken is

$$\sum_\alpha \int \mathrm{Tr}\Big((v^{-A_\alpha}\pi_\alpha(N_x))(v^{-A_\alpha}\pi_\alpha(N_y))(v^{-A_\alpha}\pi_\alpha(N_z))^*\Big)\, m_{\alpha,v}\, dz.$$

It converges to the inner product of the Laurent packet $(c_{\alpha,x}c_{\alpha,y})_\alpha$ with $\mathcal{F}_0(N_z)$. Lemma 3.5 identifies the finite constant vector with the inverse Fourier transform of that packet. Thus pointwise multiplication in $\mathcal{B}$ has finite integral structure constants in this basis. This defines J and proves associativity.

The meromorphic vector $P_+(v)N_1$ has bounded norm, since

$$\|P_+(v)N_1\|^2 = \sum_\alpha d_\alpha \int m_{\alpha,v}(z)\, dz.$$

Its negative Laurent coefficients vanish and its constant coefficient is a finite integer vector. In normalized exact Fourier coordinates $P_+(v)N_1$ is the identity matrix on every family. Its constant coefficient therefore maps under $\mathcal{F}_0$ to the unit of $\mathcal{B}$. This supplies the finite integral unit of J.

The involution follows from the corresponding matrix identity. Define

$$\tau_J(b) = \sum_\alpha \int \mathrm{Tr}(b_\alpha(z))m_{\alpha,0}(z)\, dz.$$

It is a positive trace and (3.21) is (3.4). Finally (3.20) identifies each $J_A \otimes \mathbb{C}$ with a direct sum of family algebras. The coordinate cutoff has integral entries, so applying it to the finite integral unit gives a finite integral unit of J_A. Different summands multiply to zero. □

The algebra J, its trace, its integral unit, and its spectral levels use the original indexing set W. The next section identifies its multiplication with the leading coefficients of C_xC_y.

4. Finite sources and the original canonical filtration

We now use the finite parabolic hypothesis of Theorem 2.2. The sets S_A and the algebra J constructed in Section 3 are still spectral objects. This section identifies their multiplication and their filtration in the original canonical coordinates. Here W is irreducible affine, so every proper standard parabolic is finite. Direct products are treated in Section 9.

4.1. **The finite upper bound.**

Lemma 4.1. *If a finite type σ of $\mathcal{H}_I$ occurs in an irreducible constituent of $\pi_{\alpha,v,z}$ at a unitary point, then $A_I(\sigma) \le A_\alpha$.*

Proof. We may transport an occurrence at any numerical $v_1 > 1$ to the common tail of Lemma 3.1. Choose $v_2 > v_1$ on that tail and apply positive scaling from $q(v_2)$ with exponent $\epsilon = \log v_1/\log v_2$. This is an equivalence preserving temperedness and discrete series [30, Corollary 4.2.2]. Take its inverse on the

specified constituent. The weight formula $|t| \mapsto |t|^{\epsilon}$ preserves both the weak and strict Casselman inequalities in both directions. Along the analytic scaling family, the finite central idempotents have constant rank, so finite-type multiplicities are preserved. The normalized real exponents are unchanged, so (3.3) preserves the family degree. It therefore suffices to work on the common tail.

Let $e_{I,\sigma}(v)$ be the central primitive orthogonal idempotent of the finite algebra. Its trace is

$$\tau(e_{I,\sigma}(v)) = \frac{\dim \sigma}{s_{I,\sigma}(v)} \asymp v^{-2A_I(\sigma)}.$$

The rank of $\pi_{\alpha,v,z}(e_{I,\sigma}(v))$ is an integer-valued continuous function of z. It is constant on the connected compact induction torus. Occurrence at the specified point makes this rank positive throughout that torus. It is also constant on the connected common tail, where the inducing matrices and finite idempotents are continuous. Integrate the positive Plancherel trace over a fixed regular open subset on which $m_{\alpha,0}$ is positive. For large v this gives

$$\frac{\dim \sigma}{s_{I,\sigma}(v)} \geq c\, v^{-2A_\alpha}$$

with $c > 0$. Comparing exponents proves the assertion. This argument applies to a constituent at a singular point because the integration is over its whole inducing family. □

For a finite parabolic, P1–P15 give its original based cell ideals, finite asymptotic algebras with signed units, and balanced finite matrix blocks. Their degree convention can be read directly from [24, Theorem 18.9, §18.10, Propositions 20.6, 20.10, and Corollary 20.11]. Let E be a finite Tits label, put $F = E \otimes \mathrm{sgn}$, and let ρ_F be the associated simple module for the finite asymptotic algebra J_I. Let $\mathfrak{c}_F$ be its supporting two-sided cell and put $B = a_I(\mathfrak{c}_F)$. Lusztig's homomorphism, composed with $N_s \mapsto -N_s^{-1}$, gives a realization σ_E with Tits label E and

$$\sigma_E(C_p) = \sum_{\substack{z \in \mathfrak{c}_F,\ d \in \mathcal{D}_I \\ a_I(d) = B}} h_{p,d,z} n_d\, \rho_F(t_z),$$

where $\mathcal{D}_I$ is the finite distinguished set. Corollary 20.11 identifies B with the highest Schur half-degree (2.3), namely $a_{E\otimes\mathrm{sgn}}$ in Lusztig's notation. Each coefficient in the sum has degree at most $a_I(z) = B$. If $B < a_I(p)$, finite P4 forces every coefficient to vanish, since $h_{p,d,z} \neq 0$ implies $z \leq_{\mathcal{LR}} p$. Taking the coefficient of v^B and using the finite unit $\sum_{d \in \mathcal{D}_I} n_d t_d$ gives

$$\lim_{v \to \infty} v^{-B} \sigma_E(C_p) = \rho_F(t_p).$$

Thus the normalized matrices are bounded, their limits vanish unless $a_I(p) = B$, and every block of degree below $a_I(p)$ annihilates C_p exactly.

Lemma 4.2. *For $p \in W_I$ with W_I finite, one has $p \in S_{a_I(p)}$.*

Proof. Restrict a compact family of degree A to $\mathcal{H}_I$. All its simple types have degree at most A by Lemma 4.1. Blocks of smaller degree disappear after normalization by v^{-A}. In a block of degree A, the finite leading matrix of C_p is zero unless $a_I(p) = A$. These matrix statements are unchanged by passing to a compact unitary realization: the squared Hilbert–Schmidt error is the finite character of

the error times its adjoint. Thus the leading Fourier vector of p is supported only in degree $a_I(p)$. Its total squared norm is one, proving the claim. □

4.2. **Algebraic generation by finite source units.** Let $\mathfrak{c}$ be a two-sided cell of a finite parabolic, of degree A, and let $e_{\mathfrak{c}}$ be the unit of its finite asymptotic algebra. Its normalized canonical lift is

$$v^{-A} \sum_{p \in \mathfrak{c}} [t_p] e_{\mathfrak{c}}\, C_p.$$

In every finite simple block of degree at most A, this lift converges to the identity if the block belongs to $\mathfrak{c}$, and to zero otherwise. A smaller-degree block annihilates the source exactly; in degree A, the leading representation sends the cell unit to its identity or to zero. Lemma 4.1 therefore makes its limit in a compact family of degree A a Laurent polynomial idempotent $E_{\mathfrak{c}}(z)$. The convergence is in matrix norm. This follows from the finite balanced model and the character-of-the-square observation in the preceding proof.

Proposition 4.3. *Assume finite parabolic attainment. In the degree-A Laurent Fourier algebra, the ordinary two-sided ideals generated by the $E_{\mathfrak{c}}$ have sum equal to the whole algebra.*

Proof. Write one family algebra as $B = (\mathrm{Mat}_d \mathbb{C}[T])^G$, where G acts through Laurent invertible intertwiners. At a unitary point z, the exact stabilizer intertwiners describe the full commutant of the induced module [8, Theorem 5.5]. Work on a sufficiently large connected tail of the parameter ray, where the algebraic inducing models and the association data are fixed. Locally choose dth roots of their determinants and divide each intertwiner by that root. The resulting multiplier takes values in the finite group of dth roots of unity. It is constant along the parameter ray. The corresponding finite central extension has isotypic idempotents of constant rank. These commute with the finite parabolic central idempotents. For a fixed source cell let $P_{\mathfrak{c}}(v)$ be the sum of the exact finite central projectors onto its simple blocks. In the present degree-A family the normalized source unit differs from $P_{\mathfrak{c}}(v)$ by a matrix whose norm tends to zero: this follows block by block from the finite balanced assertion, and Lemma 4.1 excludes higher finite degrees. Thus $P_{\mathfrak{c}}(v)$ converges to $E_{\mathfrak{c}}(z)$. Attainment in a specified irreducible constituent makes the product of one isotypic projector with some $P_{\mathfrak{c}}(v)$ have positive rank. Its rank is constant on the tail and is preserved at the limit. Hence the finite sources meet every simple stabilizer block at every unitary point.

We next extend this assertion to complex points. For a subgroup $K \subseteq G$, every component of its fixed torus locus is a torsion-unitary translate of a subtorus. Points whose exact stabilizer is K form a relatively open subset, with dense unitary points when it is nonempty. The determinant of a Laurent invertible intertwiner is a Laurent unit; its absolute value is one on the compact torus. Adjoining its dth roots therefore gives a finite torus cover, with dense unitary points on each component. On the common cover for the elements of K, the normalized cocycle is constant on each connected component. The corresponding isotypic projectors are regular. They commute with $E_{\mathfrak{c}}$, so their products with it are idempotents. Their traces are integer-valued regular functions, and are therefore constant on a connected component. Coverage of a block at a unitary point gives coverage everywhere on that component.

Finally B is a finite module over $\mathbb{C}[T]^G$. At a maximal ideal of this invariant ring the coordinate algebra of T becomes an Artinian algebra supported on one finite orbit. Evaluation at that orbit has nilpotent kernel; taking invariants is exact because G is finite in characteristic zero. The resulting semisimple quotient of the fiber is the stabilizer evaluation algebra. If the source-generated ideal were proper, its quotient would have a nonzero fiber by Nakayama's lemma, and that fiber would have a simple quotient killed by every source. This contradicts the block coverage just proved. □

4.3. **Relative canonical source images.** We give the algebraic comparison needed for the descent. An *original based quotient* means a quotient by a two-sided ideal spanned by a subset of the original C_w. Its remaining C_w are its specified coordinates.

Lemma 4.4 (Relative canonical source image)**.** *Let Q be an original based quotient of $\mathcal{H}$ in which every product coefficient has degree at most A. Let $\mathfrak{c}$ be a two-sided cell of a finite standard parabolic W_I, of intrinsic degree A, whose strictly lower cell boundary is zero in Q. Write $Q_{\mathbb{Z},0}$ for the constant integral coordinate lattice of Q. There is a subgroup $\Lambda_{\mathfrak{c}} \subseteq Q_{\mathbb{Z},0}$ generated by constant integral columns and containing all the surviving source vectors C_p, $p \in \mathfrak{c}$, and their finite source unit. Put $V_{\mathfrak{c}} = \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_{\mathfrak{c}}$. Then the following assertions hold.*

(i) The natural map $\mathcal{A} \otimes_{\mathbb{Z}} \Lambda_{\mathfrak{c}} \longrightarrow Q$ is injective, and its image is the image of the ordinary two-sided induction map

$$\mathcal{H} \otimes_{\mathcal{H}_I} (\mathcal{A} \otimes_{\mathbb{Z}} J_{\mathfrak{c}}) \otimes_{\mathcal{H}_I} \mathcal{H} \longrightarrow Q, \qquad h \otimes t_p \otimes k \longmapsto hC_pk. \tag{4.1}$$

(ii) The rational Laurent extension $\mathbb{Q}[v, v^{-1}] \otimes_{\mathbb{Q}} V_{\mathfrak{c}}$ is the rationalization of that ordinary source image and is stable under the original left and right actions.

(iii) On $V_{\mathfrak{c}}$ these original left and right actions commute with independent parameters.

(iv) Every representation of Q that annihilates the source cell annihilates this source image, and the corresponding assertion holds after rationalization.

The middle factor in (4.1) carries the ordinary cell-subquotient actions, with t_p identified as a vector with C_p.

Proof. Let $J_{\mathfrak{c}}$ be the finite asymptotic cell algebra, write its multiplication as $\star$, and write e for its unit. Identify the finite cell quotient with the same symbol space. Its ordinary multiplication μ_v commutes, on opposite sides, with the constant $\star$-actions: this is the leading coefficient of finite P15. A map of a unital right regular $J_{\mathfrak{c}}$-module is left multiplication by its value at e, and the analogous statement holds on the other side. It follows that

$$\mu_v(p, r) = p \star d(v) \star r, \qquad d(v) = \mu_v(e, e). \tag{4.2}$$

The matrix $M(v)$ of right $\star$-multiplication by $d(v)$ is bar invariant and satisfies

$$\deg M \leq A, \qquad [v^A]M(v) = \mathrm{Id}\,. \tag{4.3}$$

In particular it is invertible over $\mathbb{Q}(v)$.

Let X_I be the minimal representatives of W/W_I, and put $Y_I = X_I^{-1}$. The leading coefficient of finite P15 also shows that the ordinary left action of every $h \in \mathcal{H}_I$ is left $\star$-multiplication by $h \cdot e$; the right action is right $\star$-multiplication by $e \cdot h$. Induce

the cell module using the standard symbols $N_x \otimes t_p$, $x \in X_I$. The bar operator is triangular for the relative Bruhat order. Its negative triangular correction is obtained recursively from $f - \overline{f} = g$: the antisymmetric Laurent polynomial g has a unique strictly negative solution after its prescribed leading term is fixed. Each recursion has finite support and integral coefficients. Its left relative canonical column has the form

$$\sum_{x' \in X_I} N_{x'} \otimes (a_{x',x}(v) \star t_p), \qquad a_{x',x}(v) \in \mathcal{A} \otimes_{\mathbb{Z}} J_{\mathfrak{c}}. \tag{4.4}$$

The right construction has correction coefficients $b_{y,y'}(u) \in \mathbb{Z}[u, u^{-1}] \otimes_{\mathbb{Z}} J_{\mathfrak{c}}$ acting by right $\star$-multiplication. The diagonal correction blocks are the identity and the off-diagonal blocks have strictly negative coefficients. Both correction matrices and their triangular inverses are finite on each input column, since the relevant Bruhat intervals are finite. The relative canonical expansion has the precise form

$$C_{xp} = N_x C_p + \sum_{\substack{x' < x,\ q \leq_{\mathcal{L},I} p \\ x'q < xp}} a_{x'q,xp}\, N_{x'} C_q, \qquad a_{x'q,xp} \in v^{-1}\mathbb{Z}[v^{-1}].$$

It follows by expanding $\overline{N_x} C_p$ in the hybrid basis $N_{x'} C_q$ and applying the preceding negative triangular recursion; see [12, Proposition 3.2, Lemma 3.4, Corollary 3.5]. In the cell quotient a term with q outside $\mathfrak{c}$ belongs to the strictly lower source boundary. Consequently multiplication sends (4.4) to the original C_{xp} in Q. The right-handed expansion gives C_{py} in the same way.

We justify these forms and the use of independent parameters. The left bar matrix uses left $\star$-multiplications on the middle coordinate, so it commutes with all constant right $\star$-multiplications. Group the coordinates into coset blocks, each with middle space $J_{\mathfrak{c}}$. The bar matrix is lower triangular in these blocks with identity diagonal. Its strictly negative correction is unique: a maximal differing coset block in the finite support of two corrections would be both bar invariant and strictly negative, hence zero. Constant right $\star$-multiplication preserves every block and its integral negative-coefficient lattice. Uniqueness therefore gives the same commutation for the correction. It is therefore a matrix of left $\star$-multiplications. The right correction is a matrix of right $\star$-multiplications. The two corrections commute. Moreover the right correction commutes with the uncorrected left ambient action, whose operations on the middle coordinate are left $\star$-multiplications; the converse is identical. Thus after both corrections the left matrix depends only on v, and the right matrix only on u. Put $J_{\mathfrak{c}}(v,u) = \mathbb{Z}[v^{\pm 1}, u^{\pm 1}] \otimes_{\mathbb{Z}} J_{\mathfrak{c}}$, with the ordinary cell actions at parameter v on the left and parameter u on the right. The underlying two-sided induced module is

$$\mathcal{H}(v) \otimes_{\mathcal{H}_I(v)} J_{\mathfrak{c}}(v,u) \otimes_{\mathcal{H}_I(u)} \mathcal{H}(u).$$

Finite P15 supplies its commuting middle actions. Hence the corrected left and right actions commute for independent u, v.

For $r \in \mathcal{A} \otimes J_{\mathfrak{c}}$, let C_r denote the same linear combination of the original source elements. The images of the double-corrected columns on the diagonal are the explicitly defined elements

$$Z_{x,p,y} = \sum_{\substack{x' \in X_I \\ y' \in Y_I}} N_{x'} C_{a_{x',x}(v) \star t_p \star b_{y,y'}(v)} N_{y'} \quad \text{in } Q. \tag{4.5}$$

This formula gives integral Laurent columns. Triangular invertibility of the two corrections shows that they span the image of (4.1). Equation (4.2) and one-sided canonical induction give

$$C_{xp}C_{e,y} = \sum_r M_{r,p}(v) Z_{x,r,y}. \tag{4.6}$$

Here $C_{e,y} = \sum_{p\in\mathfrak{c}}[t_p]e\, C_{py}$ in the quotient; its coefficients in this expression are constant integers. In particular the left side of (4.6) is a constant integral combination of products of original canonical elements, so the degree cap applies to it. For each fixed (x, y) all columns in (4.6) have finite support.

Suppose that some output coordinate of the vector of $Z_{x,r,y}$ has largest degree $j > 0$. By (4.3), its nonzero degree-j vector produces a nonzero degree-$A + j$ coefficient on the right of (4.6). Lower matrix coefficients cannot cancel it. The assumed degree cap in Q rules this out. Invertibility and bar invariance of M show that each Z is bar invariant. A bar-invariant Laurent vector of degree at most zero is constant. Thus all these columns are integral constant columns.

Let $\Lambda_{\mathfrak{c}}$ be the $\mathbb{Z}$-span of the columns $Z_{x,p,y}$ in $Q_{\mathbb{Z},0}$. The triangular inverses of the relative corrections show that their $\mathcal{A}$-span is precisely the ordinary source image. Since $\mathcal{A}$ is flat over $\mathbb{Z}$, the inclusion $\Lambda_{\mathfrak{c}} \subseteq Q_{\mathbb{Z},0}$ gives an injection $\mathcal{A} \otimes_{\mathbb{Z}} \Lambda_{\mathfrak{c}} \hookrightarrow Q$ with that image. Set $V_{\mathfrak{c}} = \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_{\mathfrak{c}}$. Let Z denote the constant rational comparison matrix on finitely supported input vectors. Its constant kernel extends by scalars to its Laurent kernel, since Laurent polynomial extension over $\mathbb{Q}$ is flat. Diagonal equivariance says that $\ker Z$ is preserved by the left corrected action $L(v)$ and the right corrected action $R(v)$. For a constant vector in $\ker Z$, compare the Laurent coefficients of $ZL(v)k = 0$. Each coefficient preserves the constant kernel. The same argument applies to R. Therefore the commuting actions $L(v)$ and $R(u)$ descend for independent parameters to the constant image. This image is $V_{\mathfrak{c}}$. With both coset indices equal to the identity the columns contain the finite source and its unit. All columns are obtained by multiplication through that source, proving the annihilation assertion. □

4.4. **Descending through the spectral degrees.**

Proposition 4.5. *The original canonical basis satisfies*

$$a(w) = A \quad (w \in S_A).$$

Its leading multiplication is the product of J. Moreover its threshold spans are two-sided based ideals, P12 holds, and the independent identity (2.1) holds.

Proof. There are finitely many spectral degrees. Descend through them. Suppose that the span of original C_w in levels strictly above A is a two-sided based ideal and is annihilated exactly by every family of degree at most A. The assertion is vacuous at the largest degree. Form the original quotient by this ideal.

On the quotient use the positive Plancherel form truncated to families of degree at most A. For any fixed finite set of surviving original labels, its Gram matrix tends to the identity. Indeed the limiting Fourier transform is an isometry, the removed spectral projections have zero constant coordinates on these labels, and the N-to-C transition is strictly negative triangular on this finite set. For fixed x, y, the matrix product in a family of degree B is $O(v^{2B})$; its squared norm after multiplication by the measure is $O(v^{2B})$. Summing over $B \le A$ bounds the squared

truncated norm of C_xC_y by $O(v^{2A})$. The product has fixed finite original support. The Gram matrix limit therefore forces every surviving coefficient to have degree at most A.

Dividing by v^A and taking the same finite Gram limit gives

$$[v^A]h_{x,y,z} = \sum_{\alpha:A_\alpha=A} \int_{T_{\alpha,u}} \mathrm{Tr}(c_{\alpha,x}c_{\alpha,y}c^*_{\alpha,z})m_{\alpha,0}\,dz \tag{4.7}$$

for surviving z. A smaller family has an extra factor v^{B-A} and contributes zero. The right side is zero unless x, y, z all belong to S_A.

By Lemma 4.2, a finite source of intrinsic degree A lies in S_A. Its strictly lower cell boundary has larger intrinsic degree, by finite P4 and P11, and has already been removed. Thus the two ambient hypotheses of Lemma 4.4 have now been proved in the original quotient. Applying that lemma gives constant source images. They annihilate every smaller family exactly: all finite types in such a family have degree below A by Lemma 4.1. Hence these images have no constant coordinates in smaller spectral levels and lie in S_A.

The source images are stable under the original left and right actions. Since they are constant subspaces, every Laurent coefficient of those actions preserves them. Equation (4.7) makes their images ideals for the leading multiplication. They contain the finite source units. Proposition 4.3 now shows that their sum spans the whole degree-A coordinate space over $\mathbb{C}$, and consequently over $\mathbb{Q}$.

This rational spanning assertion is enough over the original integral ring. For each coordinate vector clear the finitely many denominators in one rational expression in source columns. A nonzero integral multiple of that vector belongs to the integral source sum. Every outside coordinate of its original left or right action is zero. Torsion-freeness of the original coordinate lattice gives the same assertion for the vector itself. The identical argument, coefficient by coefficient in $\mathbb{Z}[u^{\pm1}, v^{\pm1}]$, gives the independent commutation on the whole level. Exact annihilation by smaller families also passes to each original C_w in the level. This completes the induction.

For $w \in S_A$, the quotient cap gives $a(w) \le A$. The finite integral unit of the degree-A leading algebra implies that at least one product has nonzero output coefficient at w. By (4.7) this is a degree-A original coefficient. Thus $a(w) \ge A$. The coordinate ideals give P4, and the independent commuting actions give P15. Finally Lemma 4.2 gives P12 for every proper standard parabolic; the assertion for $I = S$ is tautological. □

5. Integral packets and distinguished elements

The original comparison has established P4, P12, and P15. We determine the unit of J and recover the remaining properties. The packet construction below is needed because an individual inducing character need not be rational.

5.1. **The leading relative density.** Fix $P \subseteq F_0$ and an inducing discrete series δ of its semisimple Levi algebra, with unramified torus $T = T^P$. In the relative c-function product, the Levi factors have been removed. After a common ramification put $t = v^{-1/e}$. The nonconstant factors have the form

$$\left|1 - \zeta t^b z^\chi\right|^{2\epsilon}, \qquad \epsilon \in \{1, -1\}, \quad b \in \mathbb{Z}, \quad \chi \neq 0,$$

where ζ is a root of unity. A root whose restriction to the unramified torus is zero belongs to the Levi span and has already been removed. After its t-order is extracted, a factor with $b \neq 0$ has absolute-square limit one.

Lemma 5.1. *There are a reduced crystallographic root system $\Phi \subseteq X^*(T)$, possibly empty, and a torsion point $\eta \in T_u$, such that, for a positive system Φ^+, the normalized relative density is*

$$m^P_{\delta,0}(z) = \left|\Delta_\Phi(\eta^{-1}z)\right|^2, \qquad \Delta_\Phi(u) = \prod_{\alpha\in\Phi^+}(1-u^\alpha). \tag{5.1}$$

The scalar in this relative formula is one. The inducing formal-degree constant remains a separate factor.

Proof. The stationary factors $b = 0$ are unitary binomials. Factor each nonprimitive character into its connected cyclotomic factors. At a fixed positive t, a nonstationary factor cannot cancel one of these divisors, since its modulus is different from one. Regularity of the actual density excludes negative net multiplicities. The Plancherel-mirror theorem [8, Proposition 4.2 and Theorem 4.3] supplies an actual reflection in each mirror and a simple c-function pole there. Thus every nonzero density multiplicity is exactly two. All its complex prime divisors are these unitary binomials.

Write $T_u = V/\Lambda$ and lift the finite affine monomial inertia action to V. Its linear parts preserve a lattice, and its translation parts lie in a fixed rational enlargement of that lattice. Average a positive definite rational inner product over the finite linear holonomy group. The full lift group is then a discrete cocompact group of Euclidean isometries. The lifted mirrors form an invariant locally finite hyperplane arrangement. Each has its affine reflection: lift the corresponding torus reflection and adjust by a deck translation so that it fixes a point of the chosen hyperplane. It then fixes that hyperplane pointwise.

Let $\mathcal{H}$ be the lifted mirror arrangement and $W_\mathcal{H}$ the group generated by its reflections. It is a product of affine Weyl groups on its essential subspace [15, Lemma 7.3]. The given arrangement $\mathcal{H}$ is invariant under $W_\mathcal{H}$ and consists of its reflection hyperplanes. Suppose that one conjugacy class of reflections were absent. In a Coxeter presentation of $W_\mathcal{H}$, assign -1 to simple reflections in that class and 1 to all other simple reflections. This defines a character: every odd braid relation joins conjugate simple reflections, and an even relation places no restriction on these signs. All reflections generating $W_\mathcal{H}$ from $\mathcal{H}$ would have character 1, whereas a simple reflection in the missing class would have character -1, a contradiction. Thus $\mathcal{H}$ meets every reflection class. Its invariance makes it the full reflection arrangement.

Choose a special vertex x_0 in the essential affine arrangement. The initial normals and offsets are rational, so x_0 can be chosen rational, including its component in the fixed subspace. The hyperplanes are

$$\alpha(x - x_0) = n, \qquad \alpha \in \Phi, \quad n \in \mathbb{Z}.$$

Every deck translation preserves all levels in each parallel direction. Hence $\alpha(\Lambda) \subseteq \mathbb{Z}$, which puts Φ in the actual character lattice. The point $\eta = \exp(2\pi i x_0)$ is torsion.

The two sides of (5.1) have the same complex divisor. In the Laurent unique factorization domain their quotient is a Laurent unit. More precisely, factoring all stationary binomials with constant term one, and orienting them by a fixed regular linear functional, changes each product only by a root-of-unity monomial. Nonstationary absolute squares have limit one. The half-density binomial product therefore differs from $\Delta_\Phi(\eta^{-1}z)$ only by a monomial of unit modulus on T_u. This proves the equality with scalar one. □

Put $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$. The Weyl denominator identity gives

$$\Delta_\Phi(u) = \sum_{w\in W(\Phi)} \det(w)\, u^{\rho - w\rho}. \tag{5.2}$$

The exponents are integral and distinct, even when ρ itself is not in the character lattice. Thus the nonzero coefficients in (5.2) are 1 and -1.

5.2. **Equal weights on a full Galois packet.** Choose a finite normal extension $K'/\mathbb{Q}(t)$ containing arithmetic models for all inducing discrete series and all torsion points used above. A Galois conjugate of an inducing discrete series remains discrete series on a sufficiently small positive t-tail. Its generalized Bernstein weights have the same real exponents and conjugate torsion phases, so the strict Casselman inequalities are unchanged [27, Theorem 2.25]. Generic simplicity is preserved. Choose finitely many matrix words spanning the full endomorphism algebra at the generic point. A nonzero spanning minor stays nonzero on a common ray tail, so all conjugates remain irreducible there. Each is given its own positive compact form.

Lemma 5.2. *For $\sigma \in \mathrm{Gal}(K'/\mathbb{Q}(t))$,*

$$\mathrm{fdeg}(\delta^\sigma) = \mathrm{fdeg}(\delta)$$

as rational parameter functions.

Proof. The finite-face Euler–Poincare formula expresses the formal degree in terms of finite-facet multiplicities, orientation characters, and finite formal degrees [27, §4.1, (52)–(59)]. The finite algebras in that formula include their finite length-zero stabilizers. Their individual formal degrees are rational parameter functions, by [27, Lemma 4.1]. Galois permutes the finite simple types and the restriction multiplicities, fixes these rational degrees, and fixes the determinant orientation characters. Applying it to the formula proves the equality. □

The trace can be written using all discrete-series labels. Choose one P from each W_0-association class of based Levi subsets, and let N_P be its finite based normalizer. Then

$$\tau(h) = \sum_{[P]} \frac{1}{|N_P|} \sum_{\delta\in\mathcal{D}_P} \mathrm{fdeg}(\delta) \int_{T_u^P} \mathrm{Tr}(\pi_{\delta,z}(h)) m_\delta^P(z)\, dz, \tag{5.3}$$

where $\mathcal{D}_P$ is the set of all inequivalent inducing discrete series. To check this coefficient, in an association orbit let K_P be the finite twisting group, K_δ its stabilizer, and $W(\mathcal{O})$ the normalizer of the twisting orbit. There are

$$\frac{|K_P||N_P|}{|K_\delta||W(\mathcal{O})|}$$

component tori at this based P. The coefficient of an association representative in the Plancherel formula is $\mathrm{fdeg}(\delta)|K_P|/(|K_\delta||W(\mathcal{O})|)$ [26, (4.96)–(4.98)]. Dividing gives (5.3). Normalized Haar measure on a finite torus quotient equals normalized Haar integration on its cover for an invariant scalar trace. We now index the leading matrices by their inducing labels and write them as $c_{\delta,w}$.

Lemma 5.2 therefore gives equal positive leading scalars on every full Galois packet in (5.3). Under σ, equation (5.1) uses the torsion shift η^σ and the same integer root system Φ. The changes of variables $z = \eta^\sigma u$ put all packet members over the same unshifted alternant (5.2). Summing over every automorphism repeats each distinct inducing label with one common multiplicity, which is divided out when returning to (5.3). If the inducing label is fixed while its chosen torsion shift changes, the two changes of variables still give the same scalar norm and trace integrals. The common real exponents and equal formal degrees also give the same normalization degree A throughout the packet.

5.3. **Integral coefficient matrices.**

Lemma 5.3. *For a fixed w and a torus exponent ν, the matrix*

$$D_\nu = \bigoplus_{\sigma\in\mathrm{Gal}(K'/\mathbb{Q}(t))} [u^\nu]\Big(c_{\delta^\sigma,w}(\eta^\sigma u)\Delta_\Phi(u)\Big) \tag{5.4}$$

has characteristic polynomial in $\mathbb{Z}[T]$.

Proof. Put $R = \mathbb{Z}[t, t^{-1}]$. Let $\mathcal{H}_R^P$ be the parabolic Bernstein algebra and $\mathcal{H}_{P,R}$ its semisimple quotient-root-datum algebra. The induction homomorphism is

$$\begin{aligned} &\mathcal{H}_R^P \longrightarrow \mathcal{H}_{P,R} \otimes_R R[X^*(T^P)], \\ &N_s \longmapsto N_s, \qquad \theta_x \longmapsto \theta_{\bar{x}}\, z^{x|_{T^P}}, \end{aligned} \tag{5.5}$$

where s is a finite Levi simple reflection and $\bar{x}$ is the natural image in X_P; see [26, Proposition 4.16, (4.22)]. For $s = s_\alpha$, $\alpha \in P$, the torus restrictions of x and sx agree, so this formula preserves the integral Bernstein relations. Both exponent maps are natural lattice maps, including when their combined image has finite index. The ambient algebra is finite free as a right $\mathcal{H}_R^P$-module, with the finite Weyl coset basis. Left multiplication by C_w gives a finite matrix over $\mathcal{H}_R^P$. Applying (5.5) and extracting a torus coefficient gives a matrix over $\mathcal{H}_{P,R}$. Thus every fixed torus coefficient before balancing is the image of an integral Levi matrix.

The residual central character takes invariant values integral over $\mathbb{Z}[t, t^{-1}]$, after adjoining finitely many roots of unity. Every Bernstein lattice generator, and its inverse, satisfies its monic Weyl-orbit polynomial over that evaluated center. The central quotient is consequently finite over this integral ring. The determinant trick makes every matrix over it integral over $\mathbb{Z}[t, t^{-1}]$. The normalization power of t and the integral alternant preserve this property. More explicitly, enlarge a finite coefficient ring R' over $R = \mathbb{Z}[t, t^{-1}]$ to contain the finitely many central values and torsion phases. The quotient is finite as an R'-module, hence as an R-module. Left multiplication on this finite module supplies the monic annihilating equations. The factor $v^{-A} = t^{eA}$ is a Laurent unit of R, after the common ramification has also cleared A. Thus every exact coefficient after this normalization and the substitution $z = \eta u$ remains integral over R.

Take the product of the exact characteristic polynomials over the full Galois group before taking a limit. Its coefficients are Galois fixed, hence belong to $\mathbb{Q}(t)$, and are integral over $\mathbb{Z}[t, t^{-1}]$. Integral closedness puts them in $\mathbb{Z}[t, t^{-1}]$. The compact balancing changes are independent of the unramified torus variable. They conjugate each coefficient matrix, so leave this product unchanged. All balanced coefficient matrices have finite limits. A Laurent polynomial in t with integer coefficients and a finite limit at zero has an integer limit. Taking that limit proves the assertion. □

Lemma 5.4. *If a complex matrix D has characteristic polynomial in $\mathbb{Z}[T]$, then*

$$\operatorname{Tr}(DD^*) \geq |\operatorname{Tr} D|.$$

Equality holds precisely when $D = 0$ or D is a signed orthogonal projection.

Proof. Schur triangularization gives $\operatorname{Tr}(DD^*) \geq \sum_i |\lambda_i|^2$, with equality exactly when D is normal. Group the nonzero eigenvalues by complete irreducible factors of the integer characteristic polynomial, with multiplicity. For a factor of degree r, let $a_i = |\lambda_i|$. Its nonzero integral constant term gives $\prod_i a_i \geq 1$, so $S = \sum_i a_i \geq r$. Thus

$$\sum_i a_i^2 \geq S^2/r \geq S.$$

Summing and using the triangle inequality proves the claim. Equality requires normality, unit absolute values for all nonzero eigenvalues, and a common phase for them. The trace is an integer, so that common phase is 1 or -1. This is precisely the stated equality case. □

5.4. **The signed unit.** Set $c_\sigma(u) = c_{\delta^\sigma, w}(\eta^\sigma u)$ and write $\Delta_\Phi = \sum_{\nu \in S} \epsilon_\nu u^\nu$, where $\epsilon_\nu \in \{1, -1\}$. Apply Lemma 5.4 to (5.4). Parseval and the coefficients ± 1 of (5.2) give

$$\sum_\sigma \int \operatorname{Tr}(c_\sigma c_\sigma^*) |\Delta_\Phi|^2 \geq \left| \sum_\sigma \int \operatorname{Tr}(c_\sigma) |\Delta_\Phi|^2 \right|. \tag{5.6}$$

Indeed the full chain of inequalities is

$$\sum_\nu \operatorname{Tr}(D_\nu D_\nu^*) \;\geq\; \sum_\nu |\operatorname{Tr} D_\nu| \;\geq\; \left| \sum_{\nu \in S} \epsilon_\nu \operatorname{Tr} D_\nu \right|.$$

The first term and the final term are respectively the two sides of (5.6). All coefficient sums are finite.

If equality holds, then for $\nu \notin S$ the omitted absolute trace is zero and equality in the first inequality gives $\operatorname{Tr}(D_\nu D_\nu^*) = 0$. Thus every coefficient of $c_\sigma \Delta_\Phi$ outside that support is zero. For a nonzero Laurent entry f of c_σ this implies

$$\operatorname{Newt}(f) + \operatorname{Newt}(\Delta_\Phi) \subseteq \operatorname{Newt}(\Delta_\Phi).$$

Taking support functions in every direction gives $\operatorname{Newt}(f) = \{0\}$. Thus every c_σ is constant. The equality case of the matrix lemma then makes its nonzero members orthogonal projections with one common sign. For example the constant coefficient of the alternant is 1, so for constant c_σ its corresponding matrix in (5.4) is $\bigoplus_\sigma c_\sigma$ itself. Its equality case gives this common sign directly.

Proposition 5.5. *Write the finite integral unit of J as $e = \sum_w n'_w t_w$. Then $n'_w \in \{0, 1, -1\}$. For $n'_w \neq 0$, the elements $p_w = n'_w t_w$ are pairwise orthogonal selfadjoint idempotents summing to e. Their left and right multiplication operators are coordinate projections in the basis (t_w).*

Proof. Orthonormality and $e^* = e$ give $n'_w = \tau_J(t_w) \in \mathbb{Z}$. Sum (5.6) with the equal positive packet weights of (5.3). We obtain

$$1 = \tau_J(t_w t_w^*) \geq |\tau_J(t_w)| = |n'_w|.$$

If $n'_w \neq 0$, equality holds in every positive packet term and in the final triangle inequality. The preceding equality analysis says that p_w is an orthogonal projection in every Laurent Fourier family, and hence in J. The finitely many positive projections p_w sum to e, so they are pairwise orthogonal.

Their regular operators are orthogonal projections with integral entries in the orthonormal basis (t_w), hence coordinate projections by the integral-projection observation in Section 3. □

5.5. **Recovering the original trace and cells.** For $w \in S_A$, Proposition 4.5 gives exact annihilation by all families of degree below A. The trace contribution of a family of degree $B \geq A$ is $O(v^{-B})$, since its matrix is $O(v^B)$ and its measure is $O(v^{-2B})$. Consequently

$$\deg_v \tau(C_w) \leq -A, \qquad [v^{-A}]\tau(C_w) = \tau_J(t_w). \tag{5.7}$$

This proves P1. It also identifies the support of the signed unit with the original set $\mathcal{D}$, and gives $n'_d = n_d$ for $d \in \mathcal{D}$.

The product formula (4.7) is cyclic under the trace, proving P7. On a fixed degree, define a left multiplication graph by all nonzero coefficients of $t_x t_y$. Its arrows are reversible, because the adjoint of left multiplication by t_x is left multiplication by $t_{x^{-1}}$. Each arrow is an original left-preorder comparison. Conversely P15 says that the original left action on this level commutes with right leading multiplication. It is therefore left multiplication by its value on the finite unit. Every same-level original generator edge occurs in the leading graph. By P4 an original preorder path between equal-degree endpoints stays in that degree. This proves P9; its right and two-sided versions prove P10 and P11.

For $d \in \mathcal{D}$, selfadjointness of $p_d = n_d t_d$ gives $d = d^{-1}$ and $n_d = \pm 1$, proving P6. Every t_y has a unique right coordinate unit p_d. Orthonormality gives the explicit coefficient formula

$$[t_d](t_x t_y) = n_d \, \delta_{x,y^{-1}} \, \mathbf{1}_{\{t_y p_d = t_y\}}. \tag{5.8}$$

This proves P2, P3, and P5. Left multiplication preserves the right unit. The nonzero coefficient of t_d in $t_{y^{-1}} t_y$ connects y to d in the left graph, and $t_y t_d = n_d t_y$ gives the reverse connection. Thus each original left cell contains exactly that distinguished element, proving P13.

If $\gamma_{x,y,z} \neq 0$, the product formula (4.7) puts x, y, z in a common degree A. Moreover $h_{x,y,z^{-1}} \neq 0$, so $z^{-1} \leq_{\mathcal{L}} y$. Since $a(z^{-1}) = a(y) = A$, P9 gives $y \sim_{\mathcal{L}} z^{-1}$. Apply the same argument to the two cyclic permutations supplied by P7. This proves all three assertions of P8. Finally $\tau_J(t_w t_{w^{-1}}) = 1$ shows that a distinguished coordinate occurs in this product. It is two-sided below both w and w^{-1} and has the same degree. P11 gives $w \sim_{\mathcal{LR}} w^{-1}$, which is P14. Together with Proposition 4.5, this completes the proof of Theorem 2.2.

6. Finite parabolic attainment in affine types B and C

We prove the representation-theoretic input in every rank. The middle simple weight is first normalized to one. Write $b > 0, c \geq 0$ for the two endpoint weights of $\widetilde{C}_n$, with $c = 0$ used only for the passage to $\widetilde{B}_n$. Put

$$m = \frac{b+c}{2}, \qquad d = \frac{b-c}{2}. \tag{6.1}$$

For a positive middle weight a, divide b, c and all real exponents by a in the formulas below and multiply the resulting degrees by a.

Theorem 6.1. *Every specified irreducible tempered representation of $\widetilde{C}_n$, with $b > 0, c \geq 0$, has a finite standard parabolic constituent of degree equal to its Plancherel family degree. The same assertion holds for $\widetilde{B}_n$ with positive weights.*

6.1. **Phases, actual scaling, and ordinary restrictions.** For a unitary phase $z = (z_1, \dots, z_n)$, let $n_\pm = \#\{i : z_i = \pm 1\}$. Choose representatives $\zeta_j \neq \pm 1$ of its remaining inversion orbits, and put $r_j = \#\{i : z_i \in \{\zeta_j, \zeta_j^{-1}\}\}$. Its Weyl stabilizer W_z has the form

$$W_z = W(B_{n_+}) \times W(B_{n_-}) \times \prod_j \mathfrak{S}_{r_j}.$$

The graded parameters on the two signed-permutation factors are m, d, respectively; the other factors have type A. One checks (6.1) directly in the Bernstein relation. In the parameters of [4, Definition 1.1], put $q = \exp(\varepsilon)$, $q_0 = -q^d$, $q_1 = q^m$, $q_2 = q$. The endpoint numerator in that orientation is

$$(1 + q_0 q_1) - (q_0 + q_1) z_i e^{\varepsilon x_i} = -\varepsilon(b + z_i c) + O(\varepsilon^2), \qquad z_i = \pm 1.$$

The middle-root numerator has coefficient -1 in the same orientation. Comparing these coefficients with the denominator $e^{2\varepsilon x_i} - 1$ gives $(b + z_i c)/2$. Throughout this section, u denotes the parameter of the root $2e_n$ in the type-C graded presentation of [4, Definition 1.13]; the equivalent type-B presentation has short-root parameter $2u$.

Phase reduction preserves temperedness [4, Theorem 1.10 and Corollary 1.12]. The real central direction of a type A factor is zero. A negative d is changed to positive by the short-sign automorphism, which exchanges the two finite colors. For a prescribed irreducible tempered affine module π, let U be the actual tensor product of its irreducible real-central graded factors. For a character z of X, let $\mathbb{C}_z$ be the one-dimensional X-module on which x acts by $z(x)$. The scaling theorem gives an actual finite-dimensional group representation

$$\zeta^*(\pi) = \mathrm{Ind}_{X \rtimes W_z}^{X \rtimes W_0}\big(\mathbb{C}_z \otimes \mathrm{Res}_{W_z} U\big); \tag{6.2}$$

see [30, Theorem 2.3.1(d), Corollary 4.4.3]. This assertion concerns the full representation, although the representation on the right can be reducible. All positive numerical Hecke labels are allowed, including the label one at $c = 0$.

For a finite standard facet, the central primitive idempotents vary continuously along scaling, including its endpoint at the group algebra. Their ranks are integral and constant. Thus (6.2) computes the actual finite-parabolic multiplicities of π in the Tits labeling.

The facet with blocks of sizes $k, n-k$ is

$$F_k = \{(p_k - wp_k, w) : w \in W(B_k) \times W(B_{n-k})\}, \qquad p_k = (\underbrace{\tfrac{1}{2}, \dots, \tfrac{1}{2}}_{k}, 0, \dots, 0).$$

Choose permutation representatives distributing each phase packet between the two blocks. In the first block, a sign change on a -1 phase coordinate contributes -1 through the translation character. This is the left color exchange on that negative-phase packet. For nonreal phases the chosen intersections are permutation subgroups, with no translation factor. These are actual Mackey summands of (6.2); it is enough to use these summands.

The prime c-function product gives the degree $\mathscr{A}_{b,c}(R)$ in (6.4), where R is the list of real Bernstein exponents. This follows by pairing opposite roots: the pair with short size u contributes $K_u(x) = (u - |x|)_+$ to the half-degree. The nonreduced endpoint factors give the two terms with $|m|, |d|$. Thus the degree is independent of the unitary phase. Lemma 4.1, whose proof only uses positivity and finite Schur elements, gives the upper bound for every finite constituent.

6.2. **Generic canonical finite types.** Consider a signed-permutation graded factor with positive endpoint parameter $u \notin \frac{1}{2}\mathbb{Z}$. Its discrete-series real exponents are the boxes $u + j - i$ of a partition. Its general real-central tempered data add balanced Steinberg strings

$$R_k = ((k-1)/2, (k-3)/2, \dots, -(k-1)/2).$$

On an open half-integer chamber the residual mass factors are nonzero. The continuous discrete-series deformation of [27], and constant finite-idempotent ranks, make the full finite character constant there. For a repeated string size the signed-block mirror calculation gives the entire signed-permutation stabilizer as its reflection subgroup. The analytic R-group is trivial; the full commutant theorem [8, Theorem 5.5] makes the corresponding generic tempered induction irreducible. Its full finite character is likewise constant throughout the chamber. This is the classical mirror calculation described in [29, §5.1.1 and Propositions 5.3, 5.5]; it is used only away from the half-integer parameters.

Every such chamber contains a geometric quarter parameter. The ordinary generalized Springer description there gives a canonical finite label occurring with multiplicity one, and the full character matrix is nonnegative unitriangular [4, Theorems 4.12, 4.15, Algorithm 4.14, Claim E, Proposition 4.19]. We describe the mixed partition/string rule explicitly, so its use is not restricted to discrete series.

After the common full-sign twist, delete an extreme row or column of the partition according to its largest absolute box exponent. Interleave these with deletion of the balanced strings in order of their radii. A string of length k at center u contributes rows of lengths

$$C(k,u) = \max\left(0, \min\left(k, \left\lfloor \frac{k+1}{2} - u \right\rfloor\right)\right), \qquad k - C(k,u). \tag{6.3}$$

To verify the string part at a quarter parameter, the cuspidal $\mathfrak{sl}_2$ blocks in [22, §2.13(e)] assign the four weights $\pm(4r+1), \pm(4r-1)$ to a real exponent r. Across R_k these give two even Jordan blocks of size $2k$. They have defect zero and do not change the odd-deletion rule. For $u = h + \frac{1}{4}$ the upper row is $\lfloor k/2 \rfloor - h$; for

$u = h + \frac{3}{4}$ it is $\lceil k/2 \rceil - h - 1$, with clipping to $[0, k]$. Both are (6.3). The real-central tempered criterion [23, Theorem 1.21] identifies the exponents with the middle element of this $\mathfrak{sl}_2$ triple.

The cuspidal deletion causes no extra row. The surviving cuspidal odd part is at most $4u - 2$, whereas a nonempty discrete-series remainder supplies an odd part at least $4u + 2$. Thus a cuspidal deletion occurs only when that remainder is empty. If a string remains below it, then $2k < 4u - 2$, so its clipped upper row is zero both at u and at $u - 1$. The deletion records a zero and leaves every nonzero string row unchanged. Between half-integers none of these comparisons or capacities changes. Chamber constancy therefore proves that this canonical finite label occurs for every generic parameter; see also [4, Corollary 4.20]. The identity $C(k, -u) = k - C(k, u)$ off the walls supplies the short-sign color exchange.

6.3. **Each prescribed constituent at a critical parameter.**

Lemma 6.2. *Let U, V be nonnegative integral square matrices on the same finite label set, each unitriangular in some simultaneous row and column ordering. If $U = TV$ with T nonnegative integral, then $T_{ii} = 1$ for every i.*

Proof. Use the ordering in which U is upper triangular. For $i > j$, $0 = U_{ij} = \sum_k T_{ik} V_{kj} \geq T_{ij} V_{jj} = T_{ij}$. Thus T is upper triangular in that ordering. Simultaneous permutations preserve determinants; $\det T = \det U / \det V = 1$. Its nonnegative integral diagonal entries multiply to one, so each is one. □

At a positive half-integer u, the generalized Springer description gives a canonical label for each irreducible real-central tempered module. After the same full-sign twist, the full character matrix V is nonnegative and unitriangular: its own label occurs once, another label on the same orbit does not occur, and all remaining labels lie on strictly larger orbit closures. The parameter normalization and these support assertions are [16, Theorems 3.1, 3.5 and the proof of Theorem 4.12].

Choose an adjacent generic chamber, with full character matrix U. Realize its modules in an affine algebra at a fixed positive numerical base, with both endpoints equal to the varying endpoint parameter. In an orthonormal realization the simple generator matrices are selfadjoint, with two uniformly bounded eigenvalues bounded away from zero. A subsequence of the finite matrix tuple converges. It is a unitary representation at the critical parameter; Bernstein elements and their inverses also converge. The tempered inequalities and real center persist. Complete reducibility gives a nonnegative integral multiplicity matrix T, and constant finite-idempotent ranks give $U = TV$. Lemma 6.2 says that the generic label i has the *specified* critical module i in its limit. Each central Bernstein element is scalar in the generic module, so the limiting central character of that module is exactly that of the prescribed constituent.

The parameter zero is also included. In rank $r \geq 2$ the graded algebra is $\mathcal{H}(D_r, 1) \rtimes C_2$. Its extra sign-change reflection exchanges the two fork reflections. The type-D ordinary Springer matrix has the same unitriangular property. The Springer correspondence is equivariant for this fork involution, so a tempered module is fixed exactly when its canonical finite label is fixed. For two exchanged modules, induction gives one extended simple and induces its canonical finite label with multiplicity one. For a fixed module, rescale its intertwiner so that its square is one. Its multiplicity-one canonical isotypic space then selects one finite extension;

twisting by the quotient sign gives the other. All remaining finite types are strictly later in the orbit order. Thus the resulting type-B matrix is again unitriangular with every label, including the exchanged very-even pair. In rank one the two real-central tempered simples of $\mathbb{C}[x] \rtimes C_2$ have $x = 0$ and the two one-dimensional finite labels. Rank zero is immediate. The lemma therefore applies at zero from the positive side; the short-sign twist handles the negative side.

6.4. **The two-endpoint splitting lemma.** We normalize the middle parameter to one. In this subsection the endpoint parameters b, c may be real: intermediate deletions can change their signs. Use m, d from (6.1) and put $K_u(x) = (u - |x|)_+$ for $u \geq 0$. For a finite list $R = (R_1, \dots, R_N)$ define

$$\mathscr{A}_{b,c}(R) = \sum_{i<j} \Big(K_1(R_i - R_j) + K_1(R_i + R_j) \Big) + \sum_i \Big(K_{|m|}(R_i) + K_{|d|}(R_i) \Big). \tag{6.4}$$

For partitions α, β with zero tails, set

$$\begin{aligned} a_t(\alpha, \beta) &= n(\alpha) + n(\beta) \\ &\quad + \sum_{i,j \geq 1} \left(\min(\alpha_i - i + t, \beta_j - j) - \min(-i + t, -j) \right), \\ n(\alpha) &= \sum_i (i-1)\alpha_i. \end{aligned} \tag{6.5}$$

Only finitely many terms of the difference are nonzero. For $t \geq 0$ this is minus one half of the lowest exponent in the finite type-B Schur element, by [3, Theorem 3.2(2)] with $q = v^2$, $Q_0 = v^{2t}$, and $Q_1 = -1$. Consequently it is the highest Schur half-degree of the full-sign twist of the indicated bipartition. Formula (6.5) also defines a_t for signed t, and directly gives

$$a_{-t}(\beta, \alpha) = a_t(\alpha, \beta).$$

The input consists of two partitions λ, μ, whose box exponents are $m + j - i$ and $d + j - i$, together with an arbitrary finite list of balanced strings

$$R_k = \left(\frac{k-1}{2}, \frac{k-3}{2}, \dots, -\frac{k-1}{2} \right), \qquad k \geq 1.$$

Let R be the concatenation of these exponent lists. We first assume

$$b, c, b + c, b - c \notin \mathbb{Z}. \tag{6.6}$$

At each step choose the largest among the following candidates: for a nonempty partition of center u, width w, and height h, the row candidate $u + w - 1$ and the column candidate $h - 1 - u$; for a string of length k, its radius $(k-1)/2$. Delete the first row, the first column, or the entire string, respectively. Thus the chosen candidate is the largest absolute box exponent or string radius. Delete equal strings in any order. Condition (6.6) excludes all ties between a strip and a string and between strips in different partitions.

Call each deletion an event. A strip has a pair of signs, in left-right order,

strip	λ row	λ column	μ row	μ column
(s, t)	$(+, +)$	$(-, -)$	$(-, +)$	$(+, -)$.

After deleting a strip of signs (s, t) replace (c, b) by $(c - s, b - t)$. Reindexing the surviving boxes leaves their exponents unchanged. A string leaves the parameters unchanged. The genericity condition is preserved by all these updates.

We allocate a strip of length g as two entries, of lengths x and $g-x$, in its designated colors at the left and right endpoints. At the current parameters its standalone restrictions are

$$x \geq \min(g, \max(0, \lceil -sc \rceil)), \qquad g - x \geq \min(g, \max(0, \lceil -tb \rceil)). \tag{6.7}$$

For a string of length k, at its current centers put

$$e = C(k, m), \quad f = k - e, \quad E = C(k, d), \quad F = k - E, \tag{6.8}$$

where $C(k, u)$ is defined in (6.3). Its left upper and lower entries are x, y and its right upper and lower entries are $e - x, f - y$, where

$$0 \leq x \leq e, \qquad 0 \leq y \leq f, \qquad x - y = e - E. \tag{6.9}$$

Thus the four color-pair totals, in the order upper–upper, lower–lower, lower–upper, upper–lower, are e, f, E, F. In particular a string assigned to the negative phase is accommodated by exchanging the two left colors.

Number events in deletion order. At one endpoint of initial weight u let $r(i, s)$ be the index, beginning at one, of the color-s entry at event i. Include entries of length zero in this count; a string has one entry of each color. Write $z(i, s)$ for the entry length. For $i < j$ define

$$\delta((i, s), (j, t)) = \begin{cases} 0, & s = t, \\ \lceil r(i, s) - r(j, t) - u(\mathbf{1}_{s=+} - \mathbf{1}_{t=+}) \rceil, & s \neq t. \end{cases} \tag{6.10}$$

The ordering conditions are

$$z(i, s) \geq z(j, t) + \delta((i, s), (j, t)) \qquad (i < j, \ z(j, t) > 0). \tag{6.11}$$

They say that entries of one color form a partition and that each new symbol entry dominates all positive entries of the opposite color in the remaining suffix. Zero entries are harmless: after any zero in a color all later entries of that color must be zero.

Lemma 6.3 (Two-endpoint extension)**.** *Every allocation on a suffix of the deletion list satisfying* (6.7), (6.9), *and* (6.11) *extends through the preceding event. Consequently there are endpoint bipartitions* (α_L, β_L) *and* (α_R, β_R) *with total size* N *and*

$$a_c(\alpha_L, \beta_L) + a_b(\alpha_R, \beta_R) = \mathscr{A}_{b,c}(R). \tag{6.12}$$

The rows of each input phase's canonical deletion label are split rowwise between the two endpoint labels, with the left color exchange for the negative phase.

Proof. We prove the extension assertion first. At an endpoint only the first positive row of each color can give a strongest requirement on a new entry. Later rows are shorter, and their opposite-color shifts are smaller. Each of these first rows occurs no later than the first string, since a string contributes both colors. Thus if two retained requirements come from different events, the earlier event is a strip. At current endpoint weight u, a new color-s strip entry is bounded below by zero, the first same-color length, and the first opposite-color length plus $\lceil -su \rceil$. For a string entry the last shift is $\lceil -su - 1 \rceil$. These formulas follow by counting indices in (6.10); the string has also prepended an opposite-color entry.

We shall use the following elementary triangle inequality. If $i < j < k$ and k is the first entry after i having its color, then

$$\delta(i,k) \le \delta(i,j) + \delta(j,k). \tag{6.13}$$

If two successive signs agree, the needed index comparison is immediate. In the alternating case i, k have the same color and consecutive indices. Because u is nonintegral, the two opposite-color ceilings sum to $r(i,s) - r(k,s) + 1 = 0$. This proves (6.13) also in that case.

Independent changes of endpoint signs, accompanied by exchanges of the corresponding endpoint colors, preserve all constraints. To see this on the input data, let the sign changes be $l, r \in \{\pm 1\}$ on the left and right. Multiply phase labels by lr, transpose both residual partitions if $r = -1$, and multiply all box exponents by r. The four signed centers $(tb + sc)/2$ and the four capacities are permuted. Exchanging the endpoints and transposing μ is another symmetry. We can therefore treat a current strip as a row of λ of signs $(+,+)$; and, for a current string, it is enough to prove the upper-upper capacity inequality.

A current strip. Let its length be g, let h be the height of λ, and write $K = \lceil -c \rceil$, $J = \lceil -b \rceil$. Its radius is $M = m + g - 1$. The bound on the lower-left exponent of λ gives $h \le g + \lfloor b + c \rfloor$. The width and height of μ, if nonempty, are at most $g - K$ and $g - J$. Accordingly, later strips of signs $(+,+), (-,+), (+,-), (-,-)$ have lengths at most

$$g, \quad g - K, \quad g - J, \quad g + \lfloor b + c \rfloor - 1,$$

and the sums of their shifts from the current left and right entries are at most $0, K, J, K + J$. In every case length plus both shifts is at most g, since $\lfloor b + c \rfloor - \lfloor b \rfloor - \lfloor c \rfloor \le 1$.

Take a positive tail requirement at each endpoint. If their source events differ, suppose the first is on the left, at a strip j. Applying (6.11) at the right endpoint and then (6.13) bounds the right requirement by the right row of j plus its shift from the current event. The sum is therefore bounded by the length of j plus its two shifts, hence by g. The same reasoning works with left and right exchanged, or when both requirements come from one strip.

If both arise from a string of size k' with selected colors (s,t), all intervening strips have signs $(-s,-t)$. Let p be their number and let $M' = (k'-1)/2 < M$. The selected capacity is the clipping to $[0,k']$ of the floor of

$$M' + 1 - \frac{tb + sc}{2} + \frac{s+t}{2} - p.$$

In the four sign cases this floor is at most, respectively,

$$g - p, \quad g - K - p, \quad g - J - p, \quad g + \lfloor b + c \rfloor - 1 - p.$$

The selected capacity is positive, so clipping cannot increase these bounds. Adding the opposite-color shifts again bounds the sum by g.

We must also compare tail requirements with the standalone bound at the other endpoint. Every retained right requirement B satisfies

$$B \le g - \max(0, K). \tag{6.14}$$

Here are the complete strip cases. If the right sign of the source is $+$, count preceding strips of signs $(-,-), (+,-)$ by p, q; its current left weight is $c - 1 + p - q$.

If the right sign is $-$, count preceding $(+,+),(-,+)$ strips by p,q; its left weight is $c-1-p+q$, and its right shift is J. The source lengths, lower bounds for the companion left row, and the resulting two bounds for B are

source	length bound	left companion bound	bounds for B	
$(+,+)$	$g-p$	$K+1-p+q$	$g-p$	$g-K-1-q$
$(-,+)$	$g-K-q$	$-K+p-q$	$g-K-q$	$g-p$
$(+,-)$	$g-J-q$	$K+1+p-q$	$g-q$	$g-K-1-p$
$(-,-)$	$h-1-p$	$-K-p+q$	$g-K-p$	$g-q.$

For the last line use $h-1+K+J\le g$. A negative companion bound causes no difficulty. Clipping at the full strip length would force the retained right row to be zero, which has been excluded. If $K\ge 0$ one displayed bound is at most $g-K$; if $K\le 0$ one is at most g. This proves (6.14) in all four cases.

For a string source with right upper row, use the same counts p,q for $(-,-),(+,-)$ strips. Its centers are $m-1+p, d+q$, so the row is bounded by both $C(k', m-1+p)$ and $C(k', d+q)$. Since $M'<M$, the corresponding unclipped floors are at most $g-p, g-K-q$. For a right lower row, count $(+,+),(-,+)$ strips. The centers are $m-1-p, d-q$ and the relevant capacities are $C(k', -m+1+p)$ and $C(k', -d+q)$. Their floors are at most $g+\lfloor b+c\rfloor-1-p, g-J-q$. Adding the right shift J gives bounds $g-K-p, g-q$. These rows are positive, so clipping only decreases the floors. Equation (6.14) follows again. Endpoint exchange gives the corresponding left bound.

Finally the two standalone bounds are compatible. If $K,J>0$, the inequalities $h\ge 1$ and $h-1+K+J\le g$ give $K+J\le g$. If one is nonpositive its standalone bound is zero, and the other is at most g by clipping. The unary estimates handle every pair with one tail and one standalone bound, and the preceding two-target argument handles every pair of tail bounds. Their maxima are thus two nonnegative integers with sum at most g, and give the required split.

A current string. Let its length be k, and retain the notation e,f,E,F from (6.8). Let l_+,l_-,r_+,r_- be the nonnegative integer requirements supplied by the suffix. Under $x-y=e-E$, the feasible integers x are precisely those in

$$[\max(l_+, l_-+e-E),\ \min(e-r_+, f-r_-+e-E)]\,.$$

The interval is nonempty exactly when

$$l_++r_+\le e,\quad l_-+r_-\le f,\quad l_-+r_+\le E,\quad l_++r_-\le F. \tag{6.15}$$

The endpoint sign symmetries reduce these four inequalities to the first.

Suppose first that $0<e<k$. Since the string has largest radius, the width and height of λ are bounded by e,f, and those of μ by E,F. Indeed, for a nonempty partition of center u, its first box bounds $|u|$ by $(k-1)/2$, and its two extreme boxes give the unclipped width and height bounds $\lfloor(k+1)/2-u\rfloor, \lfloor(k+1)/2+u\rfloor$; these are already in $[0,k]$. Empty partitions satisfy the clipped bounds as well. Writing

$$E'=\left\lfloor\frac{k+1}{2}-d\right\rfloor,\quad F'=\left\lfloor\frac{k+1}{2}+d\right\rfloor,\quad K=\lceil -c\rceil,\quad J=\lceil -b\rceil,$$

we have

$$e-E'\in\{K-1,K\},\qquad e-F'\in\{J-1,J\},\qquad E'+F'=k.$$

Thus $E+K-1 \le e$ if $E>0$, $F+J-1 \le e$ if $F>0$, and $f+K+J-2 \le e$. The four strip lengths are at most e, E, F, f and their two shifts at most $0, K-1, J-1, K+J-2$. Hence their sum is at most e. Two positive retained requirements coming from different events, or from the same strip, are now handled by the preceding triangle-inequality argument. If both come from a later string, every intervening strip has both colors opposite to the selected colors. Its selected signed center has therefore increased by the number of those strips, while its length has weakly decreased. The function $C(k,u)$ increases with k and decreases with u; the same inequalities give a total at most e.

A single retained left requirement is also at most e, even when the right requirement vanishes. For a left upper source, count preceding $(-,+),(-,-)$ strips by p,q. Its centers are $m+q, d-p$, and its right endpoint weight is $b-p+q$. A $(+,+)$ strip has length at most $e-q$. A $(+,-)$ strip has length at most $F-p$, but its positive left row forces a right companion at least $\lceil b-p+q \rceil = 1-J-p+q$, leaving at most $F+J-1-q \le e$. A string upper row is bounded by $C(k', m+q) \le e$. For a left lower source, count $(+,+),(+,-)$ strips by p,q; the centers are $m-p, d+q$, with left shift $K-1$. A $(-,+)$ strip gives at most $E-q+K-1 \le e$. A $(-,-)$ strip has length at most $f-p$ and a right companion at least $1-J-p+q$, giving $f+K+J-2-q \le e$. A string lower row is bounded by $C(k', d+q)+K-1 \le E+K-1 \le e$. Endpoint exchange proves the right version. This completes the first cut when $0<e<k$.

If $e=0$, then $m>(k-1)/2$ and λ is empty. Its center remains fixed during all later deletions. A later row of μ, of length g at center d', satisfies $d'+g-1 \le (k-1)/2 < m$. Its standalone left lower bound is therefore g, forcing its right upper row to vanish. A later column similarly has zero left upper row; a later string has zero upper-upper capacity. A first positive left lower row can only be preceded by columns of μ. After q such columns its center is $d+q$ and its length is at most $\lceil c \rceil - q$, for strips by the radius bound and for strings by the unclipped capacity. Its requirement on the current upper row is at most $\lceil c \rceil + \lceil -c-1 \rceil = 0$. The right upper requirement is zero by endpoint exchange. Hence $l_+ + r_+ = 0 = e$.

If $e=k$, simultaneous sign and color reversal gives that λ is empty and every lower tail row vanishes. The first positive left upper row can only be preceded by rows of μ. After p such rows a column source has length at most $F-p$; a string source has length at most $C(k', -d+p) \le F$, since its right lower row is zero. Thus $l_+ \le F$, and endpoint exchange gives $r_+ \le E$. Their sum is at most $F+E=k$. This exhausts the clipped cases and proves all four cuts.

The degree equality. Construct the allocation backwards from the empty suffix. Conditions (6.11) give the endpoint partitions and the dominance conditions for the following recursions. If an upper row of length x is prepended to (α,β) and

$$x \ge \alpha_1, \qquad x+t-1 \ge \beta_j - j \quad \text{whenever } \beta_j > 0,$$

then direct telescoping in (6.5) gives

$$a_t((x,\alpha),\beta) = a_{t-1}(\alpha,\beta) + |\alpha| + |\beta| + \sum_{j=0}^{x-1} (-t-j)_+. \tag{6.16}$$

The same formula applies with $x=0$ under its stated inequalities. For a current $(+,+)$ strip of length g, the largest-radius condition and the identity

$$H(m-t) + H(m+t) + K_{|m|}(t) - K_{|m-1|}(t) = 1, \qquad H(x) = (1-x)_+ - (-x)_+,$$

show that deleting the strip lowers the root degree by the length of the remaining list plus $\sum_{j=0}^{g-1}((-b-j)_+ + (-c-j)_+)$. One way to verify the identity is to use $(x-t)_+ + (x+t)_+ = x + \max(|x|,|t|)$; it is valid for every real m, t. For the standalone contribution write $r_j = m+j$ and $\Psi(u) = \sum_{j\geq 0}(u-j)_+$. Difference-root contributions vanish, and

$$(-r_j - d)_+ + (-r_j + d)_+ = K_{|d|}(r_j) + 2(-r_j)_+.$$

For $m \geq 0$ the remaining sum-root and $|m|$-potential terms are zero. For $m = -t < 0$, the radius hypothesis gives $g - 1 \geq 2t$. The $|m|$-potential is $\Psi(2t) - 2\Psi(t)$. If $t = h+\theta$, with $h \in \mathbb{Z}_{\geq 0}$ and $0 \leq \theta < 1$, the sum-root terms are $h+2\theta$ for $\theta \leq 1/2$ and $h+1$ for $\theta \geq 1/2$: only pairs $j < l$ with $j + l = \lfloor 2t \rfloor$ or $\lfloor 2t \rfloor + 1$ contribute, and there are $\lfloor (q+1)/2 \rfloor$ pairs with sum q. This is $4\Psi(t) - \Psi(2t)$. The two terms therefore sum to $2\Psi(t)$, as required. Since $r_j + d = b + j$ and $r_j - d = c + j$, the stated standalone formula follows. The two standalone bounds (6.7) retain exactly all positive terms of those progression sums after splitting g. Thus (6.16) gives precisely the same change at the endpoints. Sign and color symmetries handle the other three strips.

For a string, every allocation (6.9) satisfies

$$a_c((x),(y)) + a_b((e-x),(f-y)) = \left\lfloor \frac{k}{2} \right\rfloor + \sum_{r \in R_k} \Big(K_{|m|}(r) + K_{|d|}(r) \Big). \tag{6.17}$$

Here zero parts are omitted. An elementary verification is as follows. By the sign and endpoint symmetries take $b \geq c \geq 0$, so $0 \leq e \leq E \leq \lfloor k/2 \rfloor$. Put $P(u) = \{u\}(1 - \{u\})$, where $\{u\} = u - \lfloor u \rfloor$. For $t \geq 0$ and $0 \leq z \leq w$,

$$a_t((z),(w)) = z + \Psi(t) - \Psi(t-w-1) - (t-w+z)_+.$$

Use $2\Psi(u) = u^2 + u + P(u)$ for $u \geq -1$, and

$$\tfrac{1}{2}\Big(P(U+V) + P(U-V) \Big) = \max(|U|,|V|) - U^2 - V^2 \quad (|U|,|V| \leq \tfrac{1}{2}).$$

With $M = m + e - k/2$ and $D = d + E - k/2$, if $e > 0$ then $|M|, |D| < 1/2$. The two $\Psi(t-w-1)$ terms vanish, and the sum reduces to

$$e + \Psi(b) + \Psi(c) - (M+D)_+ - (M-D)_+ = k/2 + m^2 - M^2 + d^2 - D^2.$$

If $e = 0 < E$, the allocation is $x = 0, y = E$, and the sum is

$$\Psi(c) + \Psi(b) - \Psi(M-D) - \Psi(M+D) = k/2 + m^2 - M^2 + d^2 - D^2.$$

Here $M > -1/2$, $|D| < 1/2$, $c = M - D + E$, and $b = M + D + k - E$; periodicity of P proves the equality. If $e = E = 0$, the sum is $\Psi(b) - \Psi(b-k) = kb - k(k-1)/2$. To check the root side directly, let ϵ be the parity of k. For $u \geq (k-1)/2$,

$$\sum_{r \in R_k} K_u(r) = ku - (k^2 - \epsilon)/4.$$

For $0 \leq u \leq (k+1)/2$ the same sum is $(2j+1)u - j(j+1)$ when k is odd and $j = \lfloor u \rfloor$, and $2ju - j^2$ when k is even and $j = \lfloor u + 1/2 \rfloor$. These formulas follow by pairing entries $r, -r$; a missing term at the upper endpoint has coefficient zero. Substitution gives precisely the three displayed expressions. This proves (6.17), including both clipped cases.

The difference-root terms within a balanced string vanish, and its distinct opposite pairs supply $\lfloor k/2 \rfloor$ sum-root terms. Against each remaining exponent, its

two root sums are both one, because that exponent and its negative lie between the extreme string entries. The total cross increment is therefore twice the remaining length. On an endpoint, prepending both colors increases each within-color sum and the new cross sums in (6.5) by the sizes of the appropriate tails; the two endpoints give exactly twice that remaining length. Equation (6.17) supplies the standalone contribution. Starting from the empty list proves (6.12).

Finally, in the $+1$ phase the allocated upper and lower rows add rowwise to its canonical deletion label. In the -1 phase the left color exchange changes the row totals to E, F, which are precisely its canonical row totals. This proves the last assertion. □

6.5. **Occurrence and passage to every wall.** Apply Lemma 6.3 to the two signed phase partitions and all balanced strings of the prescribed generic tempered module. The allocated rows in each phase add rowwise to its canonical bipartition, with the left color exchange in the negative phase. The rowwise-sum Littlewood–Richardson coefficient is one. A type-A Steinberg string becomes trivial after the full-sign twist, and each allocated two-row bipartition contains the required trivial symmetric-group type. At an endpoint, sorted union of the allocated rows is a Littlewood–Richardson constituent, by transposing the Cartan constituent. The ordering inequalities in the lemma make these precisely the constructed endpoint partitions. The Mackey summands of (6.2) therefore place the corresponding finite type in the actual affine representation.

The highest finite Schur half-degree of that type is the left side of (6.12), since the canonical finite labels were full-sign twisted. The right side is the family degree. This proves attainment at generic parameters.

For $b, c > 0$, perturb the two centers independently inside $m > |d|$, avoiding the integer hyperplanes $b, c, b + c, b - c \in \mathbb{Z}$. At a critical center fix the canonical label of the specified target factor and use Lemma 6.2 to choose that same label in an adjacent chamber. At zero use the type-D extension argument. For fixed rank there are only finitely many endpoint labels and branching choices; pass to a subsequence making them constant. The canonical finite label occurs in the prescribed target itself, so the same Littlewood–Richardson construction gives actual occurrence there. Both the charged-symbol degree and the root energy are continuous. Equality of degrees therefore passes to the wall.

At $c = 0$ and critical $b/2$, choose $b_\varepsilon = b + \varepsilon/2$, $c_\varepsilon = \varepsilon$. Then $m_\varepsilon = b/2 + 3\varepsilon/4$ and $d_\varepsilon = b/2 - \varepsilon/4$, so the two prescribed factors are approached from the required opposite chambers. When $b/2$ is noncritical both remain in its chamber. Finite restrictions stay semisimple at the label-one endpoint. This proves the $\widetilde{C}_n$ assertion of Theorem 6.1.

6.6. **The zero endpoint and affine B.** Let U be the zero-weight endpoint generator of $\widetilde{C}_n$ and write s_1 for its neighbor. The positive fork generators $t_0 = Us_1U$, $t_i = s_i$ give the affine B_n system, and

$$\mathcal{H}_C^0 = \mathcal{H}_B \rtimes \langle U \rangle, \qquad U^2 = 1. \tag{6.18}$$

This is an identification in the standard bases. One direct verification uses galleries: a reduced B gallery crosses each positive-weight B wall once; expanding a fork letter adds only the zero-weight C walls. Coxeter deletion in its C expression can

therefore remove only pairs of zero-weight letters. The quadratic relation for such a letter is $N_U^2 = 1$, so deletion preserves the standard element. Products and the coefficient-of-identity trace are the ones in (6.18).

Let ρ be a prescribed irreducible tempered $\mathcal{H}_B$-module. Choose an irreducible module π above it in (6.18). Induction and restriction across this finite crossed product preserve temperedness. The restriction of π is either ρ or $\rho \oplus \gamma\rho$, where γ interchanges the fork nodes. The Bernstein lattice for B is the even-sum sublattice of $\mathbb{Z}^n$. Its central invariant algebra is contained in the larger center, and the two lifts of a torus character differ by simultaneous sign. Their real exponents are identical. The endpoint identity

$$K_b(2r) = 2K_{b/2}(r)$$

gives identical family degrees in B and zero-endpoint C.

Intersect an attaining C facet with B. For block size $k = 0$ this is a fork facet; for $k = 1$ it is B_{n-1}; for $k \geq 2$ it is the standard $D_k \times B_{n-k}$ facet. For $k = 0$ the finite algebra is unchanged, so the Schur elements agree directly. For $k \geq 1$ the C-facet algebra is the index-two extension of its intersection with B. For a simple finite type σ above a constituent η, finite Clifford theory and the restricted trace give

$$s_\sigma = \begin{cases} s_\eta, & \eta \not\simeq \gamma\eta, \\ 2s_\eta, & \eta \simeq \gamma\eta. \end{cases} \tag{6.19}$$

In the second case the two extensions have equal Schur elements because the quotient-sign automorphism preserves the trace. In both cases the degree is unchanged. An attaining constituent lies in ρ or $\gamma\rho$. In the latter case conjugate its type and its finite subgroup by γ. This preserves the standard facet, or sends one fork facet to the other, and puts the attaining type in the prescribed ρ. This proves Theorem 6.1. The fork argument is used for $n \geq 3$; the rank-two overlap is $\widetilde{C}_2$, and rank one is infinite dihedral.

6.7. **The finite input and the affine conclusion.** The finite-type assertion used here is the following precise input from [32, Theorem 4.1]: the original P1–P15 hold for B_r for every $r \geq 1$ and every positive integer weight, with the independent-parameter interpretation of P15. Finite A and D factors are equal-parameter cases of [24, §15]. Every proper standard parabolic of affine C is a product of finite A/B factors; in affine B, finite D factors can also occur. Tensor products give their finite properties.

Thus all finite hypotheses of Theorem 2.2 hold for these affine types. Theorem 6.1 supplies its remaining hypothesis and proves P1–P15 for positive integer weights in $\widetilde{B}_n$ and $\widetilde{C}_n$.

7. Tempered attainment in affine type F_4

Number the affine simple generators by $0, 1, 2, 3, 4$, with

$$m_{01} = m_{12} = m_{34} = 3, \qquad m_{23} = 4,$$

and all other off-diagonal Coxeter exponents equal to 2. Thus

$$L(s_0) = L(s_1) = L(s_2) = b, \qquad L(s_3) = L(s_4) = a, \qquad a, b > 0.$$

Let $J_i = S \setminus \{s_i\}$, $0 \le i \le 4$. The subgroups W_{J_i} are the five maximal finite standard parabolics. Throughout this section the parameters may lie on any fixed positive real ray; the integer-weight hypothesis enters the integral transfer theorem.

Theorem 7.1. *Every irreducible tempered representation of the affine F_4 Hecke algebra contains, on restriction to some $\mathcal{H}_{J_i}$, a simple finite type σ satisfying*

$$A_{J_i}(\sigma) = A_\alpha,$$

where α is any tempered inducing family containing the representation. This holds at all positive parameter values and for each constituent at a reducibility point.

We prove the theorem by reducing a specified irreducible representation to one with a nonzero elliptic character in its phase centralizer. The required separation of this character from finite types of smaller degree is the exact finite calculation in Section 8.

7.1. **Root coordinates and the spectral degree.** Let $e_1, \dots, e_4$ be the standard orthonormal basis of $\mathbb{R}^4$. Use the following simple roots for the reduced Bernstein root system:

$$\alpha_1 = e_2 - e_3, \quad \alpha_2 = e_3 - e_4, \quad \alpha_3 = e_4, \quad \alpha_4 = \tfrac{1}{2}(e_1 - e_2 - e_3 - e_4).$$

The roots are

$$R_0 = \{\pm e_i\} \ \cup \ \{\pm e_i \pm e_j : i < j\} \ \cup \ \{\tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}.$$

All 16 choices of signs occur in the last set. This is the root convention of the residual tables in [5]. Its long-root parameter is a and its short-root parameter is b. The relation with the displayed affine Coxeter numbering is duality followed by reversal of the four finite nodes. We denote the corresponding root parameter by k_α. Write a central character in the form

$$W_0(sv^{2\lambda}), \qquad s \text{ unitary}, \quad \lambda \text{ real},$$

where $\alpha(v^{2\lambda}) = v^{2\lambda(\alpha)}$.

Lemma 7.2. *The spectral degree of every tempered inducing family with real central exponent λ is*

$$A(\lambda) = \sum_{\alpha \in R_0^+} \max\{k_\alpha - |\lambda(\alpha)|, 0\}. \tag{7.1}$$

In particular it depends only on the real Weyl orbit, and is unchanged by varying the unitary phase.

Proof. For this reduced root datum the rank-one factor is

$$c_\alpha(t) = \frac{1 - v^{-2k_\alpha}\alpha(t)^{-1}}{1 - \alpha(t)^{-1}}.$$

For induction from a discrete series of a Levi P, the relative Plancherel product has the prefactor $v^{-2(L(w_0)-L(w_P))}$. Multiplying by the actual Levi formal-degree product gives the full root product with prefactor $v^{-2L(w_0)}$; see [26, Proposition 3.27 and Theorems 4.43 and 5.6] and [27, (40), Theorem 4.6 and (62)]. The residual scalar is constant along the ray, and the finite association factor is positive and independent of v. Identically zero residual factors are removed before evaluating the product.

Put $x = \lambda(\alpha)$. At a generic unitary point the pair $\alpha, -\alpha$, including its share of the prefactor, has degree

$$-2k_\alpha + 2|x| - \max\{-2k_\alpha - 2x, 0\} - \max\{-2k_\alpha + 2x, 0\} = -2\max\{k_\alpha - |x|, 0\}.$$

An identically zero numerator has $x = 0$; an identically zero denominator has $x = \pm k_\alpha$. Each identically vanishing binomial has exponent zero in the corresponding numerator or denominator. Removing that factor therefore changes none of the degree contributions displayed above. Summing proves (7.1). The normalized density has a nonzero positive leading coefficient on a regular compact subset of positive measure. This calculation uses the measure of the entire inducing family, and therefore remains valid when a particular point of that family is reducible. □

By Lemma 4.1, every finite type occurring in a tempered constituent with real exponent λ satisfies

$$A_{J_i}(\sigma) \le A(\lambda). \tag{7.2}$$

This includes the facets containing the affine generator.

7.2. **Compact phases and actual finite restrictions.** For a compact phase s put

$$R_s = \{\alpha \in R_0 : \alpha(s) = 1\}, \qquad W_s = \mathrm{Stab}_{W_0}(s).$$

In F_4 the root and weight lattices agree. The semisimple centralizers are connected, so $W_s = W(R_s)$. This equality can also be checked directly in the root model just given, for all faces of the phase alcove, as in Section 8. The highest long root is

$$\theta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4.$$

The proper subsets of $\{-\theta, \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ describe the 31 open faces of the compact phase alcove. The full-rank possibilities are

$$F_4, \quad B_4, \quad C_3 \times A_1, \quad A_2 \times A_2, \quad A_3 \times A_1. \tag{7.3}$$

Every other phase subsystem has rank at most three.

The phase alcove and the affine Coxeter alcove have different roles. In the present Bernstein coordinates the translation part of the actual affine generator uses the highest short root, whose coordinates are $(1, 2, 3, 2)$. The long-root marks $(2, 3, 4, 2)$ classify phases. Both are retained in the finite Mackey calculation below.

Let V be an ordinary W_s-module. Its associated affine group representation is

$$\mathcal{M}(s, V) = \mathrm{Ind}_{X \rtimes W_s}^{X \rtimes W_0}(\mathbb{C}_s \otimes V). \tag{7.4}$$

For a fixed open phase face and a fixed V, the multiplicity of each irreducible type of any finite facet in $\mathcal{M}(s, V)$ is constant as s varies in that face. Indeed its character matrices vary continuously, while each finite multiplicity is an integer. The face is connected. Thus it suffices to compute these multiplicities at one exact rational interior point of each face.

The use of (7.4) is at the level of actual representations. For a tempered affine module, the analytic scaling theorem identifies the representation at scaling parameter zero with (7.4), where V is the actual ordinary restriction of the graded module obtained by phase reduction [30, Theorem 2.3.1(d) and Corollary 4.4.3]. Finite Hecke central idempotents vary continuously throughout this scaling, including the zero endpoint. Their ranks are constant. Consequently every finite facet character, with its nonnegative multiplicities, is preserved.

7.3. **Temperedness under local phase reduction.** Fix a numerical value $v_0 > 1$. The analytic versions of the first and second reduction theorems give a matrix algebra equivalence on a neighborhood of $W_0 s v_0^{2\lambda}$, followed by a logarithmic equivalence with the graded Hecke algebra of R_s [30, Theorems 2.1.2 and 2.1.4]. Put $c_0 = 2 \log v_0$. The logarithmic algebra initially has parameters $c_0 k_\alpha$ and central character $W_s(c_0\lambda)$. Divide the polynomial action by c_0 to obtain the normalized algebra $\mathcal{H}_s = \mathcal{H}(R_s, k|_{R_s})$ with central character $W_s\lambda$. Explicitly, for $E_s = \mathbb{R}R_s$, this algebra is generated by $\mathbb{C}[W_s]$ and $S(E_s \otimes \mathbb{C})$, with cross-relations

$$x s_\alpha - s_\alpha s_\alpha(x) = k_\alpha \langle x, \alpha^\vee \rangle \qquad (x \in E_s \otimes \mathbb{C}).$$

We use the Euclidean pairing to identify dual spaces and specialize the polynomial algebra on $E_s^\perp$ at zero. All graded parameters and centers below refer to this normalization. The reduction isomorphisms are defined on neighborhoods and therefore carry analytic families of finite-dimensional modules.

Lemma 7.3. *The graded module obtained from a tempered affine module by phase reduction is tempered for R_s. Its real central component perpendicular to $\mathbb{R}R_s$ is zero.*

Proof. Let $R_s^+ = R_s \cap R_0^+$ and consider its dominant chamber $\mathscr{C}_s$, retaining both directions of its central subspace. It is convex and contains the ambient dominant chamber. The ambient chambers contained in $\mathscr{C}_s$ are connected by galleries in $\mathscr{C}_s$. All walls crossed in such a gallery belong to roots outside R_s.

Take a generalized weight $s \exp \mu$, with μ real, in the first-reduction corner. At a wall belonging to $\beta \notin R_s$ one has $\beta(s) \neq 1$. The rank-one normalized intertwiner and its inverse are regular and invertible on that generalized weight space: in this reduced positive-parameter datum the possible zeros and poles have positive real root value. Following the gallery supplies all the Weyl-translated weights required by those ambient chambers. Temperedness of the original module therefore gives $\langle \mu, x \rangle \leq 0$ for every x in their union, hence for every $x \in \mathscr{C}_s$ by continuity. The polar of $\mathscr{C}_s$ is the negative cone generated by its positive coroots. This is the graded tempered criterion, and both directions in every central line force the central component to vanish. □

Lemma 7.4. *At fixed numerical Hecke parameters, let M_t be an analytic finite-dimensional family whose central character is $W_0(s_t c)$, where s_t is unitary and c is fixed positive real. Then the multiplicity of every absolute generalized Bernstein weight is locally constant. Temperedness and all finite parabolic multiplicities are locally constant.*

Proof. The Bernstein algebra is finite over its invariant center, so all generalized weights belong to $W_0(s_t c)$. Their absolute characters lie in the fixed finite set $W_0 c$. Choose an integral character x separating these positive characters. The eigenvalues of $M_t(\theta_x)$ then lie on finitely many distinct fixed circles. Riesz projections onto disjoint annuli around these circles vary analytically and have constant rank. These are exactly the sums of generalized joint weight spaces with a specified absolute character. This argument permits nilpotent parts. The tempered inequalities depend only on those absolute weights. For a finite parabolic, use instead its primitive central idempotents, whose ranks are again continuous integers. □

An analytic graded family whose central characters stay in the reduction neighborhood extends to the analytic localization by functional calculus on its generalized weight spaces. The inverse matrix algebra equivalence therefore gives an actual affine family. Lemma 7.4 proves temperedness of this family from its absolute Bernstein weights. Positive-ray scaling then applies to each fiber. The reduction neighborhood is chosen at the fixed numerical value v_0.

7.4. **Preserving a prescribed irreducible constituent.** Work in the normalized essential algebra $\mathcal{H}_s$. A real-central tempered irreducible is a constituent of induction from $\xi = (P, \delta, 0)$, where P is a subset of the chosen simple roots of R_s and δ is a discrete series of its semisimple graded Levi quotient. Put

$$V = E_s \cap (\mathbb{R}P)^{\perp}.$$

Write λ_δ for the real central exponent of δ and $I(P, \delta, i\nu)$ for normalized graded parabolic induction with real parameter $\nu \in V$. A graded discrete series has real center, and $W_s(\lambda_\delta + i\nu)$ is real only when $\nu = 0$ [31, Theorems 7.2 and 7.5]. Let G_ξ be the stabilizer of ξ in the based Weyl normalizer: its elements preserve P and the isomorphism class of δ. The local codimension-one zero hyperplanes of the relative Plancherel density through ξ are its mirrors. Let W_{mir} be the group generated by their reflections, and let $\mathfrak{R}$ preserve a chosen positive mirror chamber. Then

$$G_\xi = W_{\mathrm{mir}} \rtimes \mathfrak{R} \tag{7.5}$$

by [8, Proposition 4.8]. We use the regular normalized intertwiners and the full commutant theorem [8, Theorems 3.5 and 5.5], transported by logarithmic reduction.

Lemma 7.5. *For every proper inducing Levi in a phase subsystem of F_4, its discrete series extends as a representation to its based normalizer stabilizer. Consequently the intertwiner cocycle in (7.5) can be chosen trivial, at every positive parameter value.*

Proof. Such a Levi has rank at most three. Its factors are of type A, B_2, B_3, or C_3. The discrete series of a type A factor is the one-dimensional Steinberg module. It has its natural invariant action under based diagram automorphisms and under permutations of identical factors. The length-preserving based automorphism group of a B_2, B_3, or C_3 factor is trivial. A based Weyl normalizer therefore acts identically on that factor's semisimple graded algebra; take the identity on its discrete-series space. Tensoring these actions gives an extension for every positive parameter value.

Normalized intertwining elements satisfy the Weyl multiplication relations. Their remaining scalar composition defect is precisely the defect in the chosen isomorphisms of the inducing module [8, (3.10)–(3.12)]. The honest extension removes it. When one uses an auxiliary affine realization to apply the intertwiner theorem, a nonidentity unitary lattice twist cannot fix an everywhere positive real inducing central orbit. Thus no additional twist stabilizer is lost in passing to this graded setting. For a full-rank inducing Levi the based stabilizer is trivial. □

Lemma 7.6. *If $V^{\mathfrak{R}} \neq 0$, a specified irreducible constituent can be deformed to an irreducible tempered affine representation with the same real central exponent and*

the same five finite facet characters, whose phase root system has strictly smaller rank.

Proof. Choose $0 \neq \eta \in V^{\mathfrak{R}}$ and consider induction at the imaginary parameter $it\eta$. Shrink the neighborhood so that no mirror absent at zero can enter. For small generic nonzero t the isotropy is

$$(G_\xi)_\eta = (W_{\mathrm{mir}})_\eta \rtimes \mathfrak{R}.$$

The point stabilizer $(W_{\mathrm{mir}})_\eta$ is generated by the old mirror roots vanishing on η. Since $\mathfrak{R}$ fixes η and preserves the old positive mirror system, it preserves the inherited positive system. The analytic R-group is therefore the same subgroup $\mathfrak{R}$ at every such point, including zero.

The full-commutant theorem gives

$$\mathbb{C}[\mathfrak{R}] \simeq \mathrm{End}_{\mathcal{H}_s} I(P, \delta, it\eta)$$

with regular intertwiner matrices. Choose a primitive idempotent e whose image at zero is one copy of the prescribed irreducible. Its image $E(t)$ is an analytic idempotent of constant rank. An explicit local trivialization is

$$S(t) = E(t)E(0) + (1 - E(t))(1 - E(0)); \qquad S(t)E(0) = E(t)S(t), \quad S(0) = 1.$$

For small t it is invertible. Complete reducibility and the full commutant give

$$I(P, \delta, it\eta) = \bigoplus_{\rho \in \mathrm{Irr}(\mathfrak{R})} \pi_{t,\rho} \otimes \rho, \qquad \mathrm{End}_{\mathcal{H}_s} I(P, \delta, it\eta) = \bigoplus_{\rho \in \mathrm{Irr}(\mathfrak{R})} \mathrm{End}(\rho).$$

A primitive idempotent selects a one-dimensional subspace in one multiplicity space, hence exactly one copy of the simple $\pi_{t,\rho}$. This preserves the specified simple even when $\dim \rho > 1$.

Transfer the family through the analytic reductions. Its center becomes

$$W_0(s \exp(ic_0 t\eta) v_0^{2\lambda}).$$

Lemma 7.4 preserves temperedness and finite facet multiplicities, and Lemma 7.2 preserves the target degree. For generic small t,

$$R_{s \exp(ic_0 t\eta)} = \{\alpha \in R_s : \alpha(\eta) = 0\}.$$

It contains P and has smaller rank because $\eta \neq 0$ lies in $\mathbb{R}R_s$. Positive-ray scaling transports this conclusion along the parameter ray. □

7.5. **Nonzero elliptic class at the endpoint.** Repeated application of Lemma 7.6 terminates after at most four rank decreases. At its endpoint $V^{\mathfrak{R}} = 0$. The mirror system is empty: otherwise the sum of its positive coroots is a nonzero $\mathfrak{R}$-fixed vector. Hence $G_\xi = \mathfrak{R}$. The action on V is faithful. A Weyl element fixing the parabolic orthogonal complement pointwise belongs to W_P, and the based normalizer complement has trivial intersection with W_P.

Lemma 7.7. *Every endpoint constituent has a nonzero elliptic class for the semisimple phase root system.*

Proof. For the ordinary R-group type ρ labeling that constituent, let χ_ρ be its character. The Euler–Poincaré formula is

$$\frac{1}{|\mathfrak{R}|} \sum_{r \in \mathfrak{R}} \det(1 - r \mid V) |\chi_\rho(r)|^2. \tag{7.6}$$

This is the actual essential normal tangent space in [28, Theorem 6.5]. Compatibility of the Euler–Poincaré pairing with scaling identifies it with the elliptic norm of the ordinary phase-Weyl restriction [6, Corollary 2.11]. The orthogonal central polynomial directions have already been removed. Every summand in (7.6) is nonnegative.

If $\dim V = 0$ the sum is positive. In dimension one, essentiality gives -1, at which both ordinary characters are nonzero. In dimension two a faithful finite orthogonal group is cyclic or dihedral. Essentiality supplies a nonidentity rotation. If its rotation subgroup has even order, its central half-turn has nonzero character in every irreducible. If that order n is odd, a two-dimensional dihedral character at a rotation generator is $2\cos(2\pi j/n) \neq 0$. One-dimensional characters and cyclic characters never vanish there. Thus the sum is positive in these cases.

Suppose $\dim V = 3$ and $P \neq \varnothing$. The phase system has rank four and P rank one. For its positive root α the source is Steinberg, so

$$G_\xi = \mathrm{Stab}_{W_s}(\alpha) = W(R_s \cap \alpha^\perp).$$

The following elementary perpendicular-root list is obtained in the root realization above.

Phase system	Position of α	Perpendicular roots
F_4	long; short	C_3; B_3
B_4	long; short	$B_2 \times A_1$; B_3
$C_3 \times A_1$	long in C_3	$C_2 \times A_1$
	short in C_3	A_1^3
	isolated A_1	C_3
$A_2 \times A_2$	either factor	rank 2
$A_3 \times A_1$	in A_3	rank 2
	isolated A_1	A_3

A perpendicular system of rank below three gives a common fixed direction and is not an endpoint. All the rank-three systems in the table contain $-I_V$, except the last A_3. At $-I_V$ every ordinary irreducible character has absolute value $\dim \rho$, giving a positive summand. In the exception the entire A_3 factor is positive-parameter principal induction at zero imaginary parameter. Its density has all the A_3 root mirrors: the numerator vanishes at a root value one, whereas its positive-parameter denominator does not. This contradicts the absence of mirrors.

Finally, an empty inducing Levi at positive phase rank has the full principal-series mirror system and trivial R-group. It still has a fixed normal direction and is not an endpoint. At phase rank zero its class is the nonzero rank-zero class. This also treats the possible normal dimension four. □

7.6. **Full central-character support.** For a phase Weyl group W_s, let $\overline{R}_{\mathbb{Z}}(W_s)$ be the lattice of ordinary characters modulo its elliptic radical. Its pairing is

$$\langle f, g\rangle_{\mathrm{ell}} = \frac{1}{|W_s|} \sum_{w \in W_s} \det(1 - w \mid \mathbb{R}R_s) f(w) \overline{g(w)}.$$

Section 8 supplies an integral orthonormal basis $\beta_1, \dots, \beta_e$ of this lattice, and a residual central character $W_s\lambda_j(a, b)$ for each basis vector. For the principal F_4 sector two basis vectors have the same generic center. They are kept together.

For $\dim E_s > 0$, let $p : \mathrm{Pin}(E_s) \to O(E_s)$ be the pin covering and put

$$\widehat{W}_s = p^{-1}(W_s), \qquad \widehat{W}'_s = \begin{cases} \widehat{W}_s, & \dim E_s \text{ odd}, \\ p^{-1}(W_s \cap SO(E_s)), & \dim E_s \text{ even}. \end{cases}$$

A representation is genuine if the nontrivial element of $\ker p$ acts by -1. In even dimension, S^+, S^- are the two half-spin modules of the even Clifford algebra; in odd dimension they are the two simple Clifford modules. Define

$$i(f) = \mathrm{infl}_p\big(f|_{p(\widehat{W}'_s)}\big) \otimes (S^+ - S^-).$$

The character inner product satisfies $\langle i(f), i(g)\rangle = 2\langle f, g\rangle_{\mathrm{ell}}$. Associate genuine characters are related by determinant twist in odd dimension and by conjugation with a reflection lift in even dimension [6, §4.1].

Lemma 7.8. *Let U be an irreducible graded phase module with nonzero elliptic class and central character $W_s\lambda$. Its elliptic class is supported on the allowed coordinates*

$$B(\lambda) = \{j : W_s\lambda_j(a,b) = W_s\lambda\}.$$

That is, coordinates outside $B(\lambda)$ vanish. The assertion holds on parameter walls.

Proof. For a rank-zero phase system the assertion is immediate; assume its rank is positive. The Dirac index maps an integral elliptic norm-one class to the difference of an associate pair of irreducible genuine spin characters [6, Theorem 4.2]. The pairs for distinct β_j are disjoint: a common member would make their index inner product nonzero.

On a nonempty generic open cone in the positive (a, b)-plane, the residual classification realizes the listed centers by discrete series [27, 5]. Their integral elliptic norm is one [6, Corollary 2.10], so each is one of the signed β_j. The computed quadratic Dirac polynomials match the residual squared norms. Except for the specified F_4 double family these polynomials identify the basis vector on a generic open cone. The double family has the same full center and is retained as one two-dimensional block.

The universal Dirac central map is defined over the polynomial parameter ring [6, Theorem 3.2 and Corollary 3.3]; its image is in the even part of the pin-group algebra. For every invariant central polynomial f, its scalar on a fixed genuine spin character is polynomial in a, b. On the preceding open cone it equals $f(\lambda_j(a, b))$. Polynomial identity extends this equality to every parameter value. Thus the scalar is known for every invariant central polynomial at every wall.

Write $[U]_{\mathrm{ell}} = \sum_j c_j\beta_j$. If $c_j \neq 0$, one member of the disjoint spin pair for β_j occurs in the Dirac index of U. The index is the difference of its Dirac cohomologies [6, Lemma 4.1], so one of these genuine types has nonzero cohomology. The full central-character theorem [6, Corollary 4.6] forces its scalar for every invariant f to equal $f(\lambda)$. Therefore $W_s\lambda_j = W_s\lambda$, as asserted. □

7.7. **Positive characters and finite separation.** For $\sigma \in \mathrm{Irr}(W_s)$ let $B_{i,\sigma,\tau}$ be the multiplicity of the finite facet type τ in $\mathcal{M}(s, \sigma)$. Define

$$M_s(\sigma) = \max\{A_{J_i}(\tau) : B_{i,\sigma,\tau} > 0\}, \qquad \mathcal{L}_{s,A} = \mathrm{span}_{\mathbb{Q}}\{[\sigma]_{\mathrm{ell}} : M_s(\sigma) < A\}. \quad (7.7)$$

Let $\mathcal{E}_{s,\lambda}$ be the coordinate subspace spanned by β_j with $j \in B(\lambda)$. The finite calculation proves

$$\mathcal{L}_{s,A(\lambda)} \cap \mathcal{E}_{s,\lambda} = 0 \tag{7.8}$$

for every relevant phase, residual central block, and positive parameter value.

Proof of Theorem 7.1. Start with the specified irreducible tempered representation. Apply phase descent until it has an endpoint. The endpoint has a nonzero essential elliptic class by Lemma 7.7, supported in $\mathcal{E}_{s,\lambda}$ by Lemma 7.8. Its ordinary phase-Weyl restriction is an actual representation $V = \bigoplus_\sigma n_\sigma \sigma$ with $n_\sigma \geq 0$. By the actual scaling and Mackey comparison, its finite restrictions are those of $\bigoplus_\sigma n_\sigma \mathcal{M}(s,\sigma)$.

If no finite type attained $A(\lambda)$, the upper bound (7.2) and nonnegativity would imply $M_s(\sigma) < A(\lambda)$ for every $n_\sigma > 0$. Thus the nonzero elliptic class of V would lie in $\mathcal{L}_{s,A(\lambda)}$, contradicting (7.8). Some finite facet type therefore has degree exactly $A(\lambda)$. Phase descent preserved its entire finite facet character and its real central exponent. The same finite type attains the degree in the original specified representation. □

8. The finite F_4 calculation

This section specifies the exact calculation used in (7.8) and supplies the finite parabolic hypothesis for affine F_4. The supplementary data contain the character tables, integral elliptic bases, residual centers, branching matrices, and rational separating matrices. The algorithms below construct these objects from finite root and partition data and give finite checks of all their required identities.

8.1. **The affine facets and their characters.** In the simple-coroot lattice of the finite Coxeter numbering, put

$$H = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \qquad q = (2,3,2,1)^{\mathsf{t}}.$$

The finite generators act by $s_i = I - e_i(H^{\mathsf{t}})_i$, where $(H^{\mathsf{t}})_i$ is the ith row. The affine generator is the pair

$$s_0 = (q, I - q(1,0,0,0)). \tag{8.1}$$

Pairs multiply by $(t,w)(t',w') = (t+wt', ww')$. Reversing the four finite coordinates identifies this lattice with the Bernstein root lattice used in Section 7. In particular the translation in (8.1) becomes $(1,2,3,2)$.

Enumeration from these generators gives the following finite groups.

Omitted node	Coxeter type	Order	Simple types
0	F_4	1152	25
1	$A_1 \times C_3$	96	20
2	$A_2 \times A_2$	36	9
3	$A_3 \times A_1$	48	10
4	B_4	384	20

There are 84 finite types in total.

For each finite group, conjugacy classes and the integral multiplication matrices of their class sums give its ordinary character table. Write G for the group and C

for its character table, with columns indexed by conjugacy classes $\mathcal{C}$. A complete table is checked by the class-sum eigenvalue equations, the positive integral degrees, and exact row orthogonality

$$C\ \mathrm{diag}(|\mathcal{C}|)\, C^{\mathsf{t}} = |G| I.$$

The principal F_4 rows are also matched to the official CHEVIE table [25]. Each official class representative word is evaluated in the integral reflection model. Matching entire character rows after this class permutation removes the ambiguities between characters of equal degree. For each proper facet, evaluate representatives as signed permutations or products of permutations. The Murnaghan–Nakayama rule matches every ordinary partition label to the corresponding complete class-algebra row.

For a phase s, the character needed at an affine facet element (t, w) is

$$\chi_{s,\sigma}(t,w) = \sum_{\substack{r \in W_0/W_s \\ r^{-1}wr \in W_s}} s(r^{-1}t)\chi_\sigma(r^{-1}wr). \tag{8.2}$$

This is the ordinary induced-character formula for (7.4). The translation factor is included even when s is nontrivial. Inner products with the facet character tables give $B_{i,\sigma,\tau}$. All these coefficients are nonnegative integers; their weighted dimensions equal $[W_0 : W_s] \dim \sigma$.

Here is a uniform exact choice of phase representatives. Let I be a proper subset of $\{0,1,2,3,4\}$ and let $m = (2,3,4,2)$. Set

$$x_j = \begin{cases} 0, & j \in I, \\ 1/((5-|I|)m_j), & j \notin I, \end{cases} \qquad 1 \le j \le 4, \qquad \alpha_j(s) = e^{2\pi i x_j}.$$

The remaining alcove coordinate is $1 - \sum_j m_j x_j$; it is zero exactly when $0 \in I$. The subsystem consists of roots with integral evaluation on x. Generate its reflection group and compare it directly with the stabilizer of the phase among all 1152 ambient Weyl elements. They agree for all 31 choices of I. Cyclotomic reduction evaluates (8.2) exactly at these representatives. Constancy on a face, proved above, then covers every compact phase.

8.2. **Finite Schur degrees.** The degree input uses Schur elements directly. For a type A factor of weight r and partition λ it is $r\, n(\lambda')$, where $n(\lambda) = \sum_i (i-1)\lambda_i$; primes denote conjugate partitions. For a type B factor with regular weight r and exceptional weight s, write its ordinary label as (λ, μ) and put

$$h_{ij}^{\lambda,\mu} = \lambda_i - i + \mu_j' - j + 1.$$

Let $\overline{(\lambda,\mu)}$ be the partition obtained by sorting all parts of the two partitions. For a box $x = (i,j)$, write $h_x = h_{ij}$ with the same superscripts. The cancellation-free Schur formula [3, Theorem 3.2(2)], specialized at $q = v^{2r}$, $Q_0 = v^{2s}$, $Q_1 = -1$, gives

$$\begin{aligned} A_+(\lambda,\mu;r,s) = r\left(-n(\overline{(\lambda,\mu)}) + \sum_{x\in\lambda}(h_x^{\lambda,\lambda}-1) + \sum_{x\in\mu}(h_x^{\mu,\mu}-1)\right) \\ + \sum_{x\in\lambda}\max\{rh_x^{\lambda,\mu}+s,0\} + \sum_{x\in\mu}\max\{rh_x^{\mu,\lambda}-s,0\}. \end{aligned} \tag{8.3}$$

The cross factors are $1+v^{2(rh+s)}$ and $1+v^{2(rh-s)}$. At zero exponent they equal 2, so this formula remains valid on walls. Degrees add for direct products.

For the B_4 facet, $(r,s)=(b,a)$ and the signed-permutation node order is $(3,2,1,0)$. For the C_3 factor, $(r,s)=(a,b)$ with node order $(2,3,4)$; its A_1 factor is node 0. The two A_2 factors use $(0,1)$ and $(3,4)$. The A_3 and A_1 factors use $(0,1,2)$ and node 4. These conventions attach the degree formulas to the labelled facet characters.

For finite F_4 use the CHEVIE cyclotomic Schur products with eigenvalue list $(v^b,-v^{-b},v^a,-v^{-a})$. A factor $\Phi_d(\varepsilon v^e)$ contributes $\varphi(d)\max(e,0)$ to the highest degree. Here Φ_d is the dth cyclotomic polynomial, φ is Euler's function, and ε is the factor's root-of-unity coefficient. Whenever $e=0$, the specialized value $\Phi_d(\varepsilon)$ is checked nonzero. The initial Laurent monomial contributes its own exponent. Together with the full class matching, this determines all 25 degrees on every cone and wall. The trivial degree is $L(w_{J_i})$ and the full-sign degree is zero, fixing the normalization. At $v=1$ the complete Schur elements equal $|W_{J_i}|/\dim\sigma$.

8.3. **The integral elliptic lattice and the Dirac calculation.** For each phase group retain its elliptic classes in its essential reflection space. Choose a representative $w_c\in c$ of each class and put

$$d_c=\frac{|c|}{|W_s|}\det(1-w_c\mid \mathbb{R}R_s).$$

If the subgroup is represented on $\mathbb{R}^4$, the essential determinant is the coefficient of $t^{4-\operatorname{rank}R_s}$ in $\det((1+t)I-w_c)$. Take the integral row lattice of the ordinary character restrictions to these classes. Hermite normal form gives an integral basis; unimodular integral row operations then give vectors β_j satisfying

$$\Big(\beta_j(c)\Big)\operatorname{diag}(d_c)\Big(\beta_j(c)\Big)^{\mathsf{t}}=I. \tag{8.4}$$

The supplementary algorithms reconstruct these unimodular row operations and verify the resulting basis and Gram identity. Thus these vectors form an orthonormal basis of the entire integral lattice. Its ranks in the five full-rank sectors are $9,5,3,1,1$.

The residual-center labels are determined by the Dirac Casimir. In the Clifford convention $xy+yx=-2\langle x,y\rangle$, choose the reflection lifts $\widetilde{s}_\alpha=\alpha/|\alpha|$. The central element used is [6, (3.2.5)]

$$\Omega_{\widehat{W}_s}=\frac{z}{4}\sum_{\substack{\alpha,\beta\in R_s^+\\ s_\alpha(\beta)<0}}k_\alpha k_\beta|\alpha^\vee||\beta^\vee|\widetilde{s}_\alpha\widetilde{s}_\beta,$$

where z is the nontrivial kernel element of the pin cover. The Dirac index is obtained by multiplying a Weyl character by the spin difference. It may be evaluated using the top exterior component of a Clifford lift in the essential root span. More explicitly, for an elliptic w with lift $\widetilde{w}$, a pair (α,β) in the displayed sum contributes the rational factor

$$-\frac{\operatorname{top}(\alpha\beta\widetilde{w})}{|\alpha|^2|\beta|^2\operatorname{top}(\widetilde{w})}. \tag{8.5}$$

Any nonzero component of the top exterior volume can be used in the ratio. If the resulting group element is nonelliptic, its numerator is zero. Thus this constructs

the operator directly on elliptic class functions, including the lower-rank embedded systems.

The coefficients of a^2, ab, b^2 give three rational matrices, self-adjoint for (8.4). Every β_j is checked to be a simultaneous eigenvector. Its eigenvalue is the squared norm of the residual center assigned below. This computes the quadratic matching used on the generic open cone in Lemma 7.8; the polynomial central-map argument in that lemma supplies the full central character at all specializations.

8.4. **Residual centers and the double family.** The five isolated phase sectors can be taken as follows; unlisted simple-root values are one.

Sector	Phase condition	Order	Elliptic rank	Labels
F_4	$s = 1$	1152	9	1–9
B_4	$\alpha_4(s) = -1$	384	5	10–14
C_3A_1	$\alpha_1(s) = -1$	96	3	15–17
A_2A_2	$\alpha_2(s) = e^{2\pi i/3}$	36	1	18
A_3A_1	$\alpha_3(s) = i$	48	1	19

The center tuples below give $(\lambda(\alpha_1), \dots, \lambda(\alpha_4))$. Entries 1–9 and 11–19 are the residual families in [5, Table 2]. Entry 10 is the B_4 Steinberg family, determined by the root equations below.

j	λ_j	j	λ_j
1	(a, a, b, b)	2	$(a, a, b-a, b)$
3	$(a, a, b-a, a)$	4	$(a, a, b-2a, b)$
5	$(a, a, b-2a, 2a)$	6	$(a, a, b-2a, a)$
7	$(a, a, b-2a, -2b)$	8	$(0, a, 0, b-a)$
9	$(0, a, 0, b-a)$	10	$(a, a, b, -3a-2b)$
11	$(2a, -a, b, -a-2b)$	12	$(0, a, b-a, -2b)$
13	$(a, a, b-2a, a-2b)$	14	$(a, a, b-3a, 3a-2b)$
15	$(-2a-3b, a, b, b)$	16	$(-2a, a, -b, 2b)$
17	$(-2a+3b, a, -b, -b)$	18	$(a, -a-2b, b, b)$
19	$(a, a, -(3a+b)/2, b)$		

For label 10, the B_4 simple system is $\{-\theta, \alpha_1, \alpha_2, \alpha_3\}$. Its Steinberg equations are

$$\lambda(\alpha_1) = \lambda(\alpha_2) = a, \quad \lambda(\alpha_3) = b, \quad \lambda(-\theta) = a.$$

Since θ has marks $(2, 3, 4, 2)$, these give $\lambda(\alpha_4) = -3a - 2b$ and hence the displayed tuple. Substitution in the prime root product gives residual pole excess four.

Labels 8 and 9 give distinct integral elliptic classes with the same generic full central character. They are always grouped as a two-dimensional central block. For every other collision the block is computed by equality of actual Weyl orbits of the center tuples.

For the other 26 phase faces, a uniform enumeration replaces a table. In a subsystem of rank r, choose r independent roots and solve $\lambda(\alpha) = \pm k_\alpha$. The solution is required to lie in the essential span. Evaluate every root on it and retain precisely the solutions with residual pole excess r. Divide by the actual embedded Weyl group. Every generic residual point is obtained: the pole hyperplanes through an isolated residual point span the essential dual space, so a choice of r independent

ones appears in the enumeration. The resulting 32 orbits over these faces match the 32 integral elliptic basis vectors, each once, by their formal quadratic polynomials. The rank-zero face has the single center zero. The classical generic residual classification [27, §6] identifies these centers with the generic discrete-series families used in Lemma 7.8.

8.5. **The exact separation assertion.** Let C_s denote the matrix expressing ordinary W_s-characters in the integral elliptic basis:

$$(\mathsf{C}_s)_{\sigma,j} = \sum_c d_c\, \chi_\sigma(c)\beta_j(c).$$

All its entries are integers. Compute $M_s(\sigma)$ from (7.7), using the full phase-dependent branching and the direct Schur degrees. For a given center λ, let L consist of the rows of C_s for which $M_s(\sigma) < A(\lambda)$. Let E_B be the rows of the identity matrix selecting its full central block B.

Proposition 8.1. *For all the phase systems and central blocks described above, and every $a, b > 0$,*

$$\operatorname{rank}\begin{pmatrix} L \\ E_B \end{pmatrix} = \operatorname{rank} L + |B|. \tag{8.6}$$

Consequently (7.8) *holds.*

Proof. Here are the finite identities and the exhaustive parameter decomposition proving the assertion. Normalize $b = 1$ and put $r = a/b$. Each Schur degree is a sum of linear forms and positive parts of linear forms, by (8.3) and the cyclotomic products. Each root energy has the same form by (7.1). Insert every positive zero of a factor linear form, every zero of $\lambda(\alpha)$ and $\lambda(\alpha) \pm k_\alpha$, and every possible equality of two actual Weyl-orbit center tuples. On each resulting interval all degrees are linear. Insert every positive zero in that interval of $A_{J_i}(\tau) - A(\lambda)$. The ordinary low-row set and the central block are constant on every final open interval. Every endpoint is checked separately by exact rational evaluation.

The base positive degree breakpoints are

$$\tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{2}{3}, 1, \tfrac{6}{5}, \tfrac{4}{3}, \tfrac{3}{2}, \tfrac{5}{3}, 2, 3, 4, 6.$$

For the full-rank sectors the final set is

$$\begin{aligned}\{&\tfrac{1}{12}, \tfrac{1}{8}, \tfrac{1}{6}, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{3}{11}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{4}{9}, \tfrac{1}{2}, \tfrac{4}{7}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{5}{7}, \tfrac{3}{4}, \tfrac{4}{5},\\ &\tfrac{6}{7}, \tfrac{7}{8}, \tfrac{8}{9}, \tfrac{11}{12}, 1, \tfrac{10}{9}, \tfrac{9}{8}, \tfrac{13}{11}, \tfrac{6}{5}, \tfrac{5}{4}, \tfrac{9}{7}, \tfrac{4}{3}, \tfrac{3}{2}, \tfrac{14}{9}, \tfrac{8}{5}, \tfrac{5}{3}, \tfrac{9}{5},\\ &2, \tfrac{9}{4}, \tfrac{7}{3}, \tfrac{5}{2}, 3, \tfrac{7}{2}, 4, \tfrac{9}{2}, 5, 6, 7, 10, 15\}.\end{aligned}$$

Thus there are 47 open intervals and 46 exact points. The additional full-orbit collision ratios are $1/2, 2/3, 1, 2, 3, 4$, all already in this set. Exact rational row reduction verifies (8.6) for all 1656 whole blocks. The 1767 individual residual-family checks also have rational separating vectors.

For the lower-rank phase faces, the corresponding final set is

$$\begin{aligned}\{&\tfrac{1}{12}, \tfrac{1}{9}, \tfrac{1}{8}, \tfrac{1}{6}, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{3}{11}, \tfrac{1}{3}, \tfrac{3}{8}, \tfrac{2}{5}, \tfrac{3}{7}, \tfrac{4}{9}, \tfrac{5}{11}, \tfrac{1}{2}, \tfrac{7}{13}, \tfrac{5}{9}, \tfrac{4}{7}, \tfrac{3}{5},\\ &\tfrac{8}{13}, \tfrac{5}{8}, \tfrac{2}{3}, \tfrac{5}{7}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{6}{7}, \tfrac{7}{8}, \tfrac{8}{9}, \tfrac{11}{12}, 1, \tfrac{10}{9}, \tfrac{9}{8}, \tfrac{7}{6}, \tfrac{13}{11}, \tfrac{6}{5}, \tfrac{5}{4}, \tfrac{9}{7},\\ &\tfrac{13}{10}, \tfrac{4}{3}, \tfrac{11}{8}, \tfrac{18}{13}, \tfrac{7}{5}, \tfrac{3}{2}, \tfrac{14}{9}, \tfrac{8}{5}, \tfrac{13}{8}, \tfrac{5}{3}, \tfrac{12}{7}, \tfrac{7}{4}, \tfrac{9}{5}, 2, \tfrac{13}{6}, \tfrac{9}{4}, \tfrac{7}{3}, \tfrac{5}{2},\\ &\tfrac{21}{8}, \tfrac{8}{3}, 3, \tfrac{10}{3}, \tfrac{7}{2}, \tfrac{11}{3}, 4, \tfrac{9}{2}, 5, 6, \tfrac{13}{2}, 7, 8, 9, 10, 12, 15\}.\end{aligned}$$

It gives 72 open intervals and 71 exact points. For all 4574 whole-block cases the supplementary matrices D satisfy the explicit rational identities

$$LD = 0, \qquad E_B D = I_{|B|}. \tag{8.7}$$

These identities imply (8.6): multiplying an equality between a vector in the row space of L and one in the row space of E_B by D forces the latter's coordinates to vanish.

All entries are checked as rational numbers. Since the low sets, blocks, and comparison signs are fixed on an open interval, its single matrix identity proves the assertion on that entire interval. The endpoint checks cover all remaining positive ratios. Finally all formulas are homogeneous, so restoring $b > 0$ proves the assertion on the whole positive parameter plane. □

Remark 8.2. At 25 wall entries among the 1767 family checks the specialized discrete formal degree vanishes. The linear separation still holds for the whole central block. Theorem 7.1 therefore applies at these centers whenever an elliptic tempered constituent occurs, using its actual tempered-family degree.

8.6. **The original finite B_4 source.** We verify the original finite P1–P15 for the B_4 facet.

Proposition 8.3. *Every positive weight function on finite B_4 satisfies P1–P15.*

Proof. Compute the original canonical basis by bar-invariant triangular recursion, using normalized standard multiplication and exact Laurent coefficients. On an open two-parameter cone, compare each exponent linear form with zero in that cone and record the comparison. The finite comparison set yields the six cones separated by $1/2, 1, 3/2, 2, 3$. At each of the five walls recompute the canonical basis in the single-parameter algebra, collecting coincident exponents before the negative triangular correction.

The left and right generator matrices give the original preorders and cells. Specializing each left cell matrix at $v = 1$ gives its ordinary character, which is checked against the finite Coxeter relations and matched to the character tables above. Its irreducible constituents have a common degree from (8.3); call it a_0 on that cell. The exact checks are the following:

(i) every left or right generator edge $y \longrightarrow x$ satisfies $a_0(x) \geq a_0(y)$;
(ii) an edge of equal degree lies inside one two-sided cell;
(iii) a left edge inside a two-sided cell lies inside one left cell;
(iv) Δ has a unique minimum on each left cell, equal to a_0, with leading trace coefficient 1 or -1.

These are the finite E1–E4 checks in Geck's criterion [10, Proposition 4.4]. Geck's canonical-basis convention is the image of the positive basis under the usual dagger automorphism. Its structure constants and preorders have the same indices; its ordinary characters are full-sign twists. Consequently its Schur valuation is precisely our highest half-degree. The inequalities above therefore have the required direction. All proper parabolics of B_4 have rank at most three and satisfy the required P-properties [9, Theorem 9.1]. The criterion proves P1–P14 and identifies the original a with a_0.

For P15 form the actual left and right generator matrices on each two-sided cell subquotient. On an open cone compute

$$L_s(v_a, v_b)R_t(u_a, u_b) - R_t(u_a, u_b)L_s(v_a, v_b)$$

in the Laurent ring on four independent variables. On a wall use the two independent variables v, u. Every column is zero. There are

$$11 \cdot 384 \cdot 4 \cdot 4 = 67584$$

columns. Checking the generators gives the identity for all left and right algebra elements. The equal-degree levels are direct sums of the corresponding two-sided cell subquotients by the second check above. This is the original independent identity P15.

The cell counts are

regime	open cone	1/2	1	3/2	2	3
left cells	76	76	50	76	58	68
two-sided cells	20	20	10	20	14	16.

The certificate records the comparison constraints and check counts. The accompanying code reconstructs every exact matrix used in these checks. □

Corollary 8.4. *P1–P15 hold in affine type $\widetilde{F}_4$ for every positive integer-valued weight.*

Proof. Finite F_4 satisfies P1–P15 by [10, Proposition 5.2]. The other maximal finite parabolics have classical factors of rank at most three, apart from B_4, handled by Proposition 8.3. Their parabolics and direct products supply all finite sources. Apply Theorem 2.2 and Theorem 7.1. □

8.7. **Reproduction of the finite proof.** The accompanying certificate archive contains the finite inputs and their generators, together with an independent rational-matrix verifier. The principal assertions checked are:

Assertion	Number of checks
Lower-rank whole central blocks	4574
Full-rank whole central blocks	1656
Individual full-rank residual families	1767
Independent-parameter B_4 commutator columns	67584
Reconstructed B_4 left-cell characters	784
Wall generator products by direct basis conversion	7680

The 59 classical Schur models also give 354 open-cone degree identities and 295 wall identities against the independently computed finite cell characters.

The independent verifier constructs the 48 roots and 1152 Weyl elements anew, recovers every phase stabilizer and every full-orbit collision, and rebuilds all piecewise linear degrees and strictly low sets. Its Gaussian elimination and the identities (8.7) use exact fractions in a separate implementation. Complete regeneration also reconstructs the ordinary character, phase, Dirac, residual, and separation tables and compares their mathematical fields.

The archive includes the official finite F_4 Schur source, parsed as data. The two reproduction commands, from the extracted archive directory, are

```
python3 verify_and_replay.py
python3 verify_and_replay.py --full-regeneration
```

The first checks the independent matrices, the Schur extraction, the cell characters, and the finite B_4 identities. The second regenerates the intermediate finite tables as well. Python, NumPy, and SymPy suffice; dependency versions and file hashes are recorded in the archive.

9. Products, length-zero extensions, and the main theorem

9.1. **Products.**

Lemma 9.1. *P1–P15 are preserved under finite direct products of positively weighted Coxeter systems.*

Proof. For $W = W_1 \times \cdots \times W_r$, the original standard and canonical bases are tensor products. Thus the structure constants factor and

$$a(w_1, \dots, w_r) = \sum_i a_i(w_i), \qquad \Delta(w_1, \dots, w_r) = \sum_i \Delta_i(w_i).$$

The maxima defining a_i are attained because they are finite integral suprema; products of attaining coefficients give the displayed equality. P1 makes equality $a = \Delta$ equivalent to equality in every factor, so $\mathcal{D} = \prod_i \mathcal{D}_i$. Leading coefficients factor, and the three preorders are coordinatewise. This proves P1–P14.

For P15 there is one further point. Its hypothesis only equates the total a-values. In any nonzero summand of either side, the structure constants give $y_i \leq_{\mathcal{LR}} w_i$ in each factor. P4 implies $a_i(y_i) \geq a_i(w_i)$. Equality of the sums forces equality in every factor. If these coordinatewise conditions fail, both sums vanish; otherwise factorwise P15 applies and its products give (2.1). The variables remain independent. □

9.2. **Length-zero multiplication.**

Let any group Ω act on (W, S, L) by based, weight-preserving automorphisms, and set $W_e = \Omega \ltimes W$. Write $\phi_\omega(x) = \omega x \omega^{-1}$ and use the normal form $g = \omega x$. The Hecke algebra is $\mathcal{H}_e = \mathbb{Z}[\Omega] \ltimes \mathcal{H}$ and its canonical basis is

$$C_{\omega x} = N_\omega C_x.$$

Indeed the bar fixes N_ω, and the negative triangular characterization on each coset ωW is exactly the one in W. Multiplication gives

$$h^e_{\omega x, \eta y, \nu z} = \mathbf{1}_{\{\nu = \omega\eta\}}\, h_{\phi_{\eta^{-1}} x, y, z}. \tag{9.1}$$

In particular

$$\begin{aligned} a_e(\omega x) &= a(x), & (\omega x)^{-1} &= \omega^{-1}\phi_\omega(x^{-1}), \\ \tau_e(C_{\omega x}) &= \mathbf{1}_{\{\omega=1\}}\tau(C_x), & \mathcal{D}_e &= \mathcal{D}. \end{aligned} \tag{9.2}$$

Here $\Delta_e(\omega x) = +\infty$ when $\omega \neq 1$. The value cannot create a distinguished element, since a_e is finite. All formulas use only finite-support products; Ω need not be finite or act faithfully.

The full leading-coefficient formula is

$$\gamma^e_{\omega x, \eta y, \theta z} = \begin{cases} \gamma_{\phi_{\eta^{-1}} x, y, \phi_\theta z}, & \omega\eta\theta = 1, \\ 0, & \omega\eta\theta \neq 1. \end{cases} \tag{9.3}$$

In particular, for $g = \omega x$ and $d \in \mathcal{D}$,

$$\gamma^e_{g^{-1},g,d} = \gamma_{x^{-1},x,d}. \tag{9.4}$$

9.3. **Cells and all fifteen properties.** The preorders in $\mathcal{H}_e$ include multiplication by the invertible N_ω. From (9.1) one obtains

$$\begin{aligned} \omega x \leq^e_{\mathcal{L}} \eta y &\iff x \leq_{\mathcal{L}} y, \\ \omega x \leq^e_{\mathcal{R}} \eta y &\iff \phi_\omega x \leq_{\mathcal{R}} \phi_\eta y, \\ \omega x \leq^e_{\mathcal{LR}} \eta y &\iff x \leq_{\mathcal{LR}} \phi_\kappa y \text{ for some } \kappa \in \Omega. \end{aligned} \tag{9.5}$$

For the first identity, a product with last factor $C_{\eta y}$ has affine coordinate left-below y; every original left multiplication is recovered by choosing the left length-zero coordinate. The right normal form gives the second. For the third, push all length-zero multipliers in a finite two-sided path to one side. The remaining path is an original path from an automorphic image of y. Conversely such an image and such a path are both available in the extended algebra.

Proposition 9.2. *If P1–P15 hold for (W, S, L), they hold for $W_e = \Omega \ltimes W$.*

Proof. P1 follows from (9.2) and the zero-trace convention. For P2, a nonzero $\gamma^e_{\omega x,\eta y,d}$ requires $\eta = \omega^{-1}$ and $\phi_{\eta^{-1}} x = y^{-1}$ by original P2. This is precisely $\omega x = (\eta y)^{-1}$. Equation (9.4) proves P3 and P5, with the unchanged coefficient $n_d = \pm 1$. Since $\mathcal{D}_e = \mathcal{D}$, P6 follows.

The third preorder formula and invariance of a under Ω prove P4. The first two prove P9 and P10; the third, followed by original P11, proves P11. For P12, if $y \in W_I$ then $a_e(y) = a_W(y) = a_I(y)$. More generally, if $\Omega_I \leq \Omega$ preserves I setwise, the same coefficient formula inside $\Omega_I \ltimes W_I$ proves the extended-parabolic version.

For P7, when $\omega\eta\theta = 1$,

$$\begin{aligned} \gamma_{\phi_{\eta^{-1}} x, y, \phi_\theta z} &= \gamma_{y,\phi_\theta z,\phi_{\eta^{-1}} x} \\ &= \gamma_{\phi_{\theta^{-1}} y, z, \phi_\omega x}. \end{aligned}$$

The second line applies $\phi_{\theta^{-1}}$ to all indices and uses $\theta^{-1}\eta^{-1} = \omega$. This is the cyclic extended coefficient. When the product is not one all three cyclic coefficients vanish.

For P8, original P8 applied to (9.3), followed by the appropriate automorphisms, gives

$$x \sim_{\mathcal{L}} \phi_\eta(y^{-1}), \qquad y \sim_{\mathcal{L}} \phi_\theta(z^{-1}), \qquad z \sim_{\mathcal{L}} \phi_\omega(x^{-1}).$$

The first preorder formula and the inverse formula in (9.2) are exactly the required extended relations.

The extended left cells are $\Omega\Gamma$ for the original left cells Γ. Their intersections with W are exactly Γ, so each has the one original distinguished element. Equation (9.4) proves P13 for every element in that extended cell. Original P14, the automorphism action, and the third preorder formula prove P14.

It remains to prove P15 with its two parameters. Write the four extended inputs as

$$X = \alpha x, \quad W' = \beta w, \quad X' = \gamma x', \quad Y = \delta y.$$

Both sums are zero unless $\delta = \alpha\beta\gamma$. In that case the left intermediate index has the form $\beta\gamma z$, and the left side becomes

$$\sum_z h_{\phi_{\gamma^{-1}}w,x',z}(u)h_{\phi_{\gamma^{-1}\beta^{-1}}x,z,y}(v). \tag{9.6}$$

The right intermediate index has the form $\alpha\beta z'$. Substitute $z' = \phi_\gamma z$ in that sum and apply $\phi_{\gamma^{-1}}$ to its v-coefficient. The result is

$$\sum_z h_{\phi_{\gamma^{-1}\beta^{-1}}x,\phi_{\gamma^{-1}}w,z}(v)h_{z,x',y}(u). \tag{9.7}$$

The equality of $a_e(W')$ and $a_e(Y)$ gives $a(\phi_{\gamma^{-1}}w) = a(y)$. Original P15 equates (9.6) and (9.7), proving the independent-parameter assertion. □

The extended right cells are $\Upsilon\Omega$, where Υ ranges over the original right cells. The two-sided cells are indexed by the Ω-orbits of original two-sided cells, and have the form $\Omega\mathfrak{c}\Omega$. For the last statement, opposite extended preorder relations give equal a-values by P4; then original P11 applied to (9.5) identifies one original cell with an automorphic image of the other. The asymptotic algebra has the corresponding based isomorphism

$$J_e \simeq \mathbb{Z}[\Omega] \ltimes J_W.$$

9.4. **Root data and exhaustion of the affine types.** For a based root datum $(X, R_0, Y, R_0^\vee, F_0)$, allowing a central torus, one has

$$W_{\mathrm{aff}} = \mathbb{Z}R_0 \rtimes W_0, \qquad W_{\mathrm{ext}} = X \rtimes W_0.$$

Write

$$X \otimes \mathbb{R} = V_{\mathrm{ss}} \oplus V_z, \qquad V_{\mathrm{ss}} = \mathbb{R}R_0, \quad V_z = \{x : \langle x, \alpha^\vee\rangle = 0 \text{ for all } \alpha \in R_0\}.$$

Let $\mathfrak{a}_0$ be the base alcove in the semisimple root span. Define the discrete group

$$\Omega = \mathrm{Stab}_{W_{\mathrm{ext}}}(\mathfrak{a}_0 \times V_z).$$

The affine Coxeter group acts simply transitively on alcoves, so

$$W_{\mathrm{ext}} = \Omega \ltimes W_{\mathrm{aff}}, \qquad \Omega \simeq X/\mathbb{Z}R_0.$$

Every element of Ω has length zero; the pure central translations $X \cap V_z$ form a subgroup of finite index. A root-datum parameter function is constant on simple affine reflections conjugate in W_{ext}, and is therefore Ω-invariant. The Iwahori–Matsumoto presentation is the exact crossed product used above [30, §§1.1–1.2].

The auxiliary nonreduced system $R_{\mathrm{nr}} = R_0 \cup \{2\alpha : \alpha^\vee \in 2Y\}$ is a Bernstein parameter convention for this same algebra. A quotient parameter may have zero or negative exponent even when all affine simple weights are positive. The Coxeter system remains unchanged. In type C this signed parameter is the $d = (b - c)/2$ already treated in Section 6.

Proof of Theorem 1.1. For an irreducible simply laced affine diagram of rank at least two, the odd-edge graph is connected and its weight function is constant. These are the equal-parameter A, D, E cases of [24, §15]. The rank-one affine case is infinite dihedral: use [24, §15] for equal weights and [24, §17] for unequal weights, exchanging the two generators if necessary. The remaining irreducible diagrams are B, C, F_4, G_2. Section 6 proves the B/C cases. Sections 7 and 8, followed by Theorem 2.2, prove F_4. The arbitrary-parameter G_2 theorem is [13, Theorem 8.12

and §9], with P15 proved in Theorem 9.5 there. This exhausts the irreducible affine Coxeter classification. Lemma 9.1 treats finite direct products, and Proposition 9.2 treats every allowed length-zero extension. A rank-zero central lattice is included by taking $W = \{1\}$. □

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