THE SHAPIRO–SHAPIRO CONJECTURE FOR ORDERED POSITIVE FLAGS

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Abstract. We prove the Shapiro–Shapiro conjecture for ordered totally positive flags in Grassmannians.

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1. Introduction

A Schubert problem asks for subspaces satisfying prescribed incidence conditions with several complete flags. Over the complex numbers, general flags and complementary total codimension give finitely many solutions, counted by the degree of the product of the corresponding Schubert classes. Over the real numbers, the solutions may occur in complex-conjugate pairs. The question of finding real data for which all solutions are real was emphasized by Fulton in the 1980s; Shapiro and Shapiro recount its relation to the qualitative theory of differential equations in [17]. The Shapiro–Shapiro conjectures propose explicit families of such data.

The osculating conjecture, formulated by Boris Shapiro and Michael Shapiro in the early 1990s, uses the flags of a real rational normal curve at distinct real parameters [18, 19]. In the polynomial model of a Grassmannian, these incidence conditions prescribe vanishing orders. The Wronskian is the determinant formed from a polynomial basis and its successive derivatives; a change of basis multiplies it by a nonzero scalar. Its roots record the Schubert divisor conditions [13]. Eremenko and Gabrielov proved the case corresponding to two-dimensional polynomial spaces in 2002 [3]. Their equivalent rational-function theorem says that a rational function whose critical points lie on a circle maps that circle into a circle. This established total reality for the corresponding Grassmannian problems.

Mukhin, Tarasov, and Varchenko proved the general osculating conjecture using the Gaudin model and its Bethe algebra [12]. Their companion paper identifies the algebra of the relevant Schubert intersection with a Bethe algebra and proves transversality for arbitrary Grassmannian Schubert conditions at distinct real parameters [13]. Levinson and Purbhoo later gave a topological proof through the Wronski map: its degree on suitable real strata is expressed by symmetric-group characters, and its restriction over polynomials with distinct real roots is a covering of the expected degree [11, Theorem 1.3 and Corollary 1.5].

Total positivity led to a second formulation. Shapiro's letter of 26 October 1997 [15] replaces osculating flags by flags ordered in an upper-unitriangular chart by the semigroup of strictly totally positive matrices. Sottile states this formulation as Conjecture 4.1 in [19]. He proves the four-divisor problem in $\mathrm{Gr}(2,\mathbb{C}^4)$ and gives a reduction from divisor problems to Pieri problems [19, Theorems 4.2–4.3]. A Pieri condition is indexed by a one-row partition; the reduction permits two further conditions at the ends of the ordered tuple. The conjecture itself permits arbitrary partitions at every flag. The precise row-frame convention is given in Definition 1.1.

The matrix formulation is connected with convex curves and disconjugate differential equations. A fundamental system of solutions of a linear differential equation gives the coordinate functions of a nondegenerate projective curve. Zeros of a nonzero solution are

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the intersections of this curve with the corresponding hyperplane, counted with multiplicity. Disconjugacy bounds their number by one less than the dimension of the solution space; the corresponding convexity condition bounds each hyperplane intersection by the dimension of the ambient projective space [17, Section 2]. Fock and Goncharov characterized their positive flag configurations by ordered osculating flags of smooth convex curves [4, Theorem 1.3].

Shapiro and Shapiro described the boundary of the totally positive triangular semigroup by corner minors in 1995 [16]. In their 2010 Gökova conference article [17], they formulated a local crossing conjecture for the osculating flag of a nondegenerate curve and a fixed Schubert boundary. Goulart and Saldanha developed triangular coordinates for locally convex curves and proved positive triangular interpolation [6, Lemma 5.3 and Remark 5.4]. They proved the crossing property, together with persistence of singular times in continuous families [7, Lemmas 3.1–3.2]. These results concern the motion of one flag curve relative to a fixed boundary.

The convex-curve history also contains a corrected counterexample. Sedykh and Shapiro's 2005 paper [14] asserted that four tangent lines to a convex curve in real projective three-space could have no common real transversal. Shapiro and Shapiro's 2022 corrigendum [18] identifies an incorrect multiplication order in the example and proves that every such four-tangent configuration has two distinct real transversals. The corrected argument uses the positive-discriminant calculation of Arkani-Hamed, Lam, and Spradlin [1, Section 4].

Eremenko related the divisor secant conjecture to disconjugacy of polynomial spaces with real-rooted Wronskians [2]. Recent work also relates total positivity to the solutions themselves. Karp characterized total nonnegativity of complete polynomial flags by nonvanishing of their Wronskians on $(0,\infty)$, and total positivity by nonvanishing on $[0,\infty)$ together with maximal Wronskian degrees [8, Theorem 1.1]. He also formulated positivity refinements of real Schubert problems [8]. Karp and Purbhoo constructed universal Plücker coordinates in the symmetric-group algebra, proving these positivity statements, the disconjugacy conjecture, and the divisor form of the secant conjecture [9, Theorems 1.8–1.11]. Their positive Shapiro–Shapiro theorem concerns the Plücker coordinates of solutions for rational-normal osculating flags [9, Corollary 5.30]. Their discussion separately formulates the general secant and convex-curve total-reality questions [9, Sections 5.7.2–5.7.3].

We prove the ordered-positive-flag conjecture for every Grassmannian and arbitrary Schubert conditions of complementary total codimension. The conclusion includes transversality, reducedness, and emptiness when the intersection number is zero.

1.1. **Flags and Schubert conditions.** Fix an integer $N \geq 2$. Vectors are rows, and ambient linear transformations act on the right. The notation $\mathrm{GL}_N(\mathbb{K})$ denotes the group of invertible $N \times N$ matrices over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A row frame $F \in \mathrm{GL}_N(\mathbb{R})$ represents the flag $F_\bullet = (F_k)_{k=0}^N$, where F_k is spanned by the first k rows of F. Let U be the real upper unitriangular group. Its strictly totally positive (TP) part $U_{>0}$ consists of the matrices for which every minor not identically zero on U is positive. The space of strictly lower triangular matrices is denoted by $\mathfrak{n}_-$. We write I for an identity matrix, with its size indicated by a subscript when needed.

Definition 1.1. An m-tuple of real complete flags, for an integer $m \geq 1$, is *ordered positive* if it has row representatives $F_1, \ldots, F_m$ satisfying

$$F_{i+1}F_i^{-1} \in U_{>0} \qquad (1 \leq i < m). \tag{1.1}$$

This is the upper-unitriangular flag-chart convention of [15, 19]. A common ambient change of coordinates $v \mapsto vF_1^{-1}$ replaces the frames by $F_iF_1^{-1}$. The identity

$$(F_{i+1}F_1^{-1})(F_iF_1^{-1})^{-1} = F_{i+1}F_i^{-1}$$

shows that it preserves all left quotients and makes the first frame I. All the resulting frames then belong to U. We use this normalization when working with matrices.

Fix an integer $1 \le p < N$, put $q = N - p$, and write $G = \mathrm{Gr}(p, \mathbb{C}^N)$ for the Grassmannian of complex p-planes, of dimension $d = pq$. Real flags are complexified when used in G, and dimensions in the Schubert conditions are complex dimensions. A partition in the $p \times q$ rectangle means an integer sequence $q \ge \lambda_1 \ge \cdots \ge \lambda_p \ge 0$; write $|\lambda| = \sum_j \lambda_j$. For a complete flag $F_\bullet$, set $a_j = q + j - \lambda_j$ and define

$$\Omega_\lambda(F_\bullet) = \{W \in G : \dim(W \cap F_{a_j}) \ge j,\ 1 \le j \le p\}. \tag{1.2}$$

The variety has its reduced determinantal structure. Its open Schubert cell is

$$\Omega^\circ_\lambda(F_\bullet) = \{W \in G : \dim(W \cap F_k) = \#\{j : a_j \le k\} \text{ for every } 0 \le k \le N\}.$$

In the basis given by the rows of the flag frame, row-echelon coordinates with rightmost pivots in columns a_j give $\sum_j (a_j - j) = d - |\lambda|$ free entries. Thus this cell is an affine space of that dimension; its closure is (1.2). The boundary is the union of the cells indexed by partitions $\mu \supsetneq \lambda$, where containment means $\mu_j \ge \lambda_j$ for every j. In particular, each boundary cell has larger codimension. These cells are the orbits of the stabilizer of $F_\bullet$, and are smooth. We abbreviate $\Omega_\lambda(F_i)$ when a frame represents the flag. Let $\sigma_\lambda = [\Omega_\lambda(F_\bullet)]$ be its class in the Chow ring $A^*(G)$; this class is independent of the flag.

Theorem 1.2. *Let $N \ge 2$ and $1 \le p < N$. Let $F_1, \dots, F_m$ represent an ordered positive tuple, and let $\lambda_1, \dots, \lambda_m$ be partitions in the $p \times q$ rectangle. Suppose that*

$$\sum_{i=1}^m |\lambda_i| = d, \qquad D = \int_G \prod_{i=1}^m \sigma_{\lambda_i}.$$

Then the scheme-theoretic intersection

$$\bigcap_{i=1}^m \Omega_{\lambda_i}(F_i)$$

consists of exactly D reduced real points. Each point belongs to every prescribed open Schubert cell, and the Schubert varieties meet transversely there. In particular, the intersection is empty if $D = 0$.

Here the integral is the degree of a zero-dimensional Chow class. The Schubert classes form an integral basis of $A^*(G)$; in particular, $A^d(G) = \mathbb{Z}[\text{point}]$, generated by the full-rectangle class $\sigma_{(q^p)}$ [5]. A real point is a complex plane obtained by complexifying a real p-plane. Transversality means that the sum map from the individual conormal spaces to $T^*_W G$ is injective; in the displayed top-degree problem it is an isomorphism.

1.2. **The argument.** The main step is transversality at every real common point, for arbitrary Schubert cells at arbitrary total codimension. A cotangent vector at a real plane W is represented by an endomorphism η with $\operatorname{im}\eta \subset W \subset \ker\eta$. Consequently any two cotangent vectors at W have zero product. A dependence among Schubert conormals therefore gives matrices η_i satisfying

$$\eta_i\eta_j = 0, \qquad \sum_i \eta_i = 0, \qquad F_i\eta_i F_i^{-1} \in \mathfrak{n}_-.$$

We prove that all these matrices must vanish.

For this purpose we interpolate the flags by a positive simple-root flow and deform its row frame by $I - s\sum_{t_i<t}\eta_i$. The lower triangular jumps disappear after Gram–Schmidt orthogonalization, leaving a continuous family of positive Frenet curves with fixed endpoints. The local Bruhat-boundary persistence mechanism of Goulart and Saldanha [7, Lemma 3.1] prevents the family from acquiring a boundary intersection. We prove the needed finite-dimensional version in Section 2, using triangular factorization and the intermediate value theorem. A differential elimination then forces the conormal dependence to vanish.

The osculating theorem of Mukhin–Tarasov–Varchenko [13, Corollary 6.3] supplies an initial configuration with the expected number of real solutions. Proper continuation

preserves these solutions throughout the positive parameter space. Irreducibility of the universal complex incidence excludes additional nonreal points or positive-dimensional components. A separate intersection-theoretic argument treats zero intersection number.

2. Total positivity and local persistence

The corner-minor boundary description is due to Shapiro and Shapiro [16]. The local persistence theorem below is the C^1 matrix form of the crossing mechanism in [7, Lemmas 3.1–3.2]. We prove both statements using triangular coordinates. For a matrix M and increasing index sets I, J, write $M_{I,J}$ for the submatrix with those rows and columns. Empty determinants and empty products equal one. Write $[1,k] = \{1,\dots,k\}$. For a square matrix M of size n, set

$$\delta_k(M) = \det M_{\{n-k+1,\dots,n\},[1,k]} \qquad (0 \le k \le n).$$

Thus δ_k is the southwest corner minor. We use column frames in $\mathrm{SO}_N = \{Q \in \mathrm{GL}_N(\mathbb{R}) : Q^{\mathsf{T}}Q = I,\ \det Q = 1\}$. A *positive Frenet curve* is a C^1 curve satisfying

$$Q^{-1}Q' = \sum_{j=1}^{N-1} \kappa_j(t)(E_{j+1,j} - E_{j,j+1}), \qquad \kappa_j(t) > 0, \tag{2.1}$$

where the coefficients are continuous and E_{ij} is the matrix with its only nonzero entry, equal to one, in position (i,j). The boundary under consideration is

$$\Sigma = \{Q \in \mathrm{SO}_N : \delta_k(Q) = 0 \text{ for at least one } 1 \le k < N\}.$$

Let U_- be the lower-unitriangular group and put $\mathcal{P} = (U_{>0})^{\mathsf{T}} \subset U_-$. For increasing index sets $I = \{i_1 < \cdots < i_k\}$ and $J = \{j_1 < \cdots < j_k\}$ in $\{1,\dots,N\}$, write $I \ge J$ if $i_r \ge j_r$ for every r, and call such a pair *allowable*. Other minors vanish identically on U_-. A matrix is *totally nonnegative* (TNN) if all its minors are nonnegative. Let $\mathcal{C} \subset U_-$ consist of such matrices. Thus $\mathcal{P}$ is defined by strict positivity of the allowable minors.

Lemma 2.1 (Positive transitions)**.** *One has $\mathcal{C}\mathcal{P} \cup \mathcal{P}\mathcal{C} \subset \mathcal{P}$ and $\overline{\mathcal{P}} = \mathcal{C}$. Let $L : I \to U_-$ be a C^1 curve on a real interval satisfying*

$$L' = LB(t), \qquad B(t) = \sum_{j=1}^{N-1} \beta_j(t) E_{j+1,j}, \qquad \beta_j(t) > 0 \tag{2.2}$$

with continuous coefficients. Then $L(u)^{-1}L(v) \in \mathcal{P}$ for every $u, v \in I$ with $u < v$.

Proof. Cauchy–Binet gives nonnegative summands for every minor of a product of TNN matrices. For AC with $A \in \mathcal{C}$, $C \in \mathcal{P}$, the summand with intermediate index I in its I,J minor is $\det A_{I,I} \det C_{I,J} = \det C_{I,J} > 0$. Use intermediate index J for CA.

For the transition equation put $T(t) = L(u)^{-1}L(t)$, so $T' = TB$ and $T(u) = I$. On the kth exterior power, the matrix $T^{(k)} = (\det T_{I,J})_{I,J}$ satisfies $(T^{(k)})' = T^{(k)}B^{[k]}$. The entry of $B^{[k]}$ from J to I is $\beta_j(t)$ if I is obtained from J by replacing j by $j+1$, with $j \in J$ and $j+1 \notin J$, and is zero otherwise. The sign is positive because the replaced index retains its position in the increasing list. Successive integration gives the finite formula

$$T^{(k)}(v) = I + \sum_{\ell=1}^{k(N-k)} \int_{u<t_1<\cdots<t_\ell<v} B^{[k]}(t_1)\cdots B^{[k]}(t_\ell)\, dt_1 \cdots dt_\ell. \tag{2.3}$$

Each factor raises the sum of the indices by one, so longer products vanish. Every summand has nonnegative entries. If $I \ge J$, move the largest index of J to its target in I, then the next largest, and continue downwards. This gives a permitted chain from J to I. In the corresponding term of (2.3), its steps occur from the rightmost factor to the leftmost factor. Its integrand is a product of strictly positive coefficients, so its integral is positive. For $I = J$, the identity term gives the value one. Finally, with $H = \sum_j E_{j+1,j}$, the transition

result gives $e^{\epsilon H} \in \mathcal{P}$ for $\epsilon > 0$. For $A \in \mathcal{C}$, the matrices $Ae^{\epsilon H} \in \mathcal{P}$ converge to A. The reverse closure inclusion follows from continuity of minors. □

Lemma 2.2 (Triangular coordinates and the boundary)**.** *The factorization* (2.4) *gives smooth coordinates* $(a_{r,j})_{1\le j\le r<N} \in \mathbb{R}_{>0}^{N(N-1)/2}$ *on* $\mathcal{P}$*, with smooth inverse. If* $L \in \mathcal{C} \setminus \mathcal{P}$*, then* $\delta_k(L) = 0$ *for some* $1 \le k < N$*.*

Proof. Set $u_j(x) = I + xE_{j+1,j}$. Its minors are 0, 1, or x, so it is TNN for $x \ge 0$. We first prove the factorization

$$\begin{aligned} L &= C_1 \cdots C_{N-1}, \qquad a_{r,j} > 0, \\ C_r &= u_r(a_{r,r}) \cdots u_1(a_{r,1}) \end{aligned} \tag{2.4}$$

for $L \in \mathcal{P}$. The induction starts at size one, where $L = (1)$ and the factor list is empty. In fact, positivity of the allowable minors with consecutive columns suffices. Put $b_j = L_{N,j}$, so $b_N = 1$, and take $a_{N-1,j} = b_j/b_{j+1} > 0$. For $r > c$, multiplication of the elementary factors gives $(C_{N-1})_{r,c} = \prod_{h=c}^{r-1} a_{N-1,h}$. The inverse factors occur in increasing index order; all products of two or more of their off-diagonal terms vanish. Hence

$$C_{N-1}^{-1} = I - \sum_{j=1}^{N-1} a_{N-1,j}E_{j+1,j}, \qquad LC_{N-1}^{-1} = \operatorname{diag}(K, 1).$$

For $I \subset \{1, \dots, N-1\}$ of size ℓ and $J = \{j, \dots, j+\ell-1\} \subset \{1, \dots, N-1\}$, the exact identity is

$$\det K_{I,J} = \frac{\det L_{I\cup\{N\},\{j,\dots,j+\ell\}}}{b_{j+\ell}}. \tag{2.5}$$

Indeed, simultaneously subtract (b_h/b_{h+1}) times original column $h+1$ from column h, for $j \le h < j+\ell$, in the numerator matrix. This determinant-one column operation leaves only $b_{j+\ell}$ in its last row. Expansion along that row proves the identity. An allowable minor on the left has an allowable numerator on the right, so K has the required consecutive-column positivity. Induction gives (2.4); the bottom-row ratios also give uniqueness. The recursion expresses the inverse coordinates by rational functions with positive denominators. The product map is polynomial, so both maps are smooth.

Conversely, a product in (2.4) belongs to $\mathcal{P}$. Induct on N and use $L = \operatorname{diag}(K, 1)C_{N-1}$. In an allowable I, J minor, if $N \notin I$, allowability implies $N \notin J$, and the Cauchy–Binet term with intermediate index J equals $\det K_{I,J} > 0$. If $N \in I$ and $J = \{j_1 < \cdots < j_k\}$, use $H = \{j_1, \dots, j_{k-1}, N\}$. The first factor is the allowable minor $\det K_{I\setminus\{N\},\{j_1,\dots,j_{k-1}\}} > 0$. In the second factor, the first $k-1$ rows form a lower triangular unit-diagonal block and vanish in the last column. Thus $\det(C_{N-1})_{H,J} = (C_{N-1})_{N,j_k} > 0$. All other terms are nonnegative.

The corner minors in these coordinates are the monomials

$$\delta_k(L) = \prod_{r=N-k}^{N-1} \prod_{j=r-N+k+1}^{r} a_{r,j}, \qquad 1 \le k < N. \tag{2.6}$$

To see this, the first k entries of the rows of C_{N-1} with index at least k are proportional. In the Cauchy–Binet expansion for $\delta_k(\operatorname{diag}(K, 1)C_{N-1})$, intermediate indices must contain N; the only nonzero term therefore uses $\{1, \dots, k-1, N\}$. Thus, with $\delta_0 = 1$,

$$\delta_k(L) = \delta_{k-1}(K) \prod_{j=k}^{N-1} a_{N-1,j}.$$

Iteration proves (2.6), which also holds for nonnegative parameters by polynomial continuity.

Now approximate $L \in \mathcal{C}$ by $L_\nu \in \mathcal{P}$. Their factor parameters are bounded, since

$$(L_\nu)_{i+1,i} = \sum_{r=i}^{N-1} a_{r,i}^{(\nu)}.$$

A subsequence of all parameters converges to nonnegative parameters whose product is L. If every $\delta_k(L)$ were nonzero, all would be positive. Formula (2.6) would then make every limiting parameter positive: the entire block $a_{r,1}, \dots, a_{r,r}$ occurs in δ_{N-r}. The converse factorization assertion would give $L \in \mathcal{P}$, a contradiction. □

For a matrix $M \in \mathrm{GL}_N^+(\mathbb{R}) := \{M \in \mathrm{GL}_N(\mathbb{R}) : \det M > 0\}$, column Gram–Schmidt gives the unique decomposition $M = QR$, where $Q \in \mathrm{SO}_N$ and R is upper triangular with positive diagonal. Write $\mathrm{QR}_{\mathrm{SO}}(M) = Q$.

Lemma 2.3 (Positive coordinates and normalization)**.** *Every $Q_* \in \mathrm{SO}_N$ admits a fixed upper-triangular $A \in \mathrm{GL}_N^+(\mathbb{R})$ such that the positive-diagonal QR action $\Phi_A(Q) = \mathrm{QR}_{\mathrm{SO}}(AQ)$ preserves Σ and positive Frenet curves. Moreover, $\Phi_A(Q_*)$ lies in the chart with positive leading principal minors, and its lower unitriangular LU factor belongs to $\mathcal{C}$.*

Proof. First write $AQ = \widetilde{Q}R$, where R is upper triangular with positive diagonal. If $Q' = Q\sum_j \kappa_j(E_{j+1,j} - E_{j,j+1}) = QK$, differentiation gives

$$\widetilde{Q}^{\mathsf{T}}\widetilde{Q}' = RKR^{-1} - R'R^{-1}, \qquad \widetilde{\kappa}_j = \frac{R_{j+1,j+1}}{R_{jj}}\kappa_j > 0. \tag{2.7}$$

Below the diagonal only the stated first-subdiagonal entries occur; skew symmetry supplies the upper part. For upper-triangular A,

$$\delta_k(\widetilde{Q}) = \delta_k(Q) \prod_{i=N-k+1}^{N} A_{ii} \prod_{j=1}^{k} R_{jj}^{-1}.$$

Every diagonal factor in this identity is nonzero, so Φ_A preserves Σ. It is a smooth diffeomorphism with inverse $\Phi_{A^{-1}}$.

On the open chart where the northwest leading minors of Q are positive, write $Q = LV$, with $L \in U_-$ and V upper triangular with positive diagonal. The nonzero leading minors give the unique LU factorization, and their positive successive ratios make the diagonal of V positive. This chart identifies with all of U_-; its inverse takes the orthogonal QR factor of L. Differentiation and triangularity give

$$\begin{aligned} L^{-1}L' = VKV^{-1} - V'V^{-1} &= \sum_j \beta_j E_{j+1,j}, \\ \beta_j = \frac{V_{j+1,j+1}}{V_{jj}}\kappa_j > 0, \qquad \delta_k(Q) &= \delta_k(L)\prod_{j=1}^{k} V_{jj}. \end{aligned} \tag{2.8}$$

It remains to normalize Q_*. Gaussian elimination assigns to every invertible matrix M a permutation $w(M) \in S_N$, the symmetric group on $\{1, \dots, N\}$: its suffix-row, prefix-column ranks are

$$\operatorname{rank} M_{\{r,\dots,N\},\{1,\dots,c\}} = \#\{j \le c : w(M)(j) \ge r\}, \qquad 1 \le r, c \le N. \tag{2.9}$$

Let $w = w(Q_*)$ and $s_i = (i, i+1)$. Choose a reduced word $w = s_{i_1} \cdots s_{i_\ell}$, that is, an expression with the fewest adjacent transpositions, and set $L_w = u_{i_1}(1) \cdots u_{i_\ell}(1) \in \mathcal{C}$. We check that its permutation is w. Multiplication on the right by $u_i(x)$, $x \neq 0$, adds x times column $i+1$ to column i; all ranks in (2.9) with $c \neq i$ are unchanged. If the current permutation τ satisfies $\tau(i) = a < b = \tau(i+1)$, restrict to rows $r, \dots, N$ and let u, v be the classes of columns $i, i+1$ modulo the first $i-1$ columns. The rank formula gives

$$\dim\langle u\rangle = \mathbf{1}_{a \ge r}, \qquad \dim\langle u, v\rangle = \mathbf{1}_{a \ge r} + \mathbf{1}_{b \ge r},$$

where $\mathbf{1}$ is the indicator of the stated inequality. For $r \le a$ the classes are independent; for $a < r \le b$ one has $u = 0$ and $v \neq 0$; for $r > b$ both vanish. Therefore the rank with $c = i$ increases exactly for $a < r \le b$. This is precisely the rank pattern of τs_i. Every step of a reduced word increases length, so induction proves the claim.

To obtain the required triangular factors, apply signed elimination. Choose the lowest nonzero entry in the leftmost remaining column c, and denote its row by r_* and its value by p. Subtract $M_{r,c}/p$ times its row from each remaining row r above it, and then subtract $M_{r_*,c'}/p$ times the pivot column from each remaining column c' to its right. These are upper triangular row and column operations. They preserve the suffix-row, prefix-column ranks: an upper triangular row operation acts invertibly on every suffix of rows, and an upper triangular column operation acts invertibly on every prefix of columns. Delete the pivot row and column and repeat in the inherited order. Earlier pivot rows and columns remain clear, since the active rows and columns have zero entries in those positions. All these elementary matrices have diagonal one and positive-diagonal inverses. Positive diagonal rescalings normalize the absolute values of the pivots, leaving their signs in a signed permutation matrix. For a determinant-one matrix this signed permutation has determinant one. Applied to Q_* and L_w, the procedure gives

$$Q_* = U_0 P U_1, \qquad L_w = V_0 P' V_1,$$

with upper-triangular positive-diagonal factors and determinant-one signed permutation matrices P, P' having the same underlying permutation. Thus $P'P^{-1}$ is diagonal and

$$A = V_0 P' P^{-1} U_0^{-1} \in \mathrm{GL}_N^+(\mathbb{R}), \qquad AQ_* = L_w(V_1^{-1}U_1).$$

The matrix A is upper triangular and the last factor has positive diagonal. Therefore AQ_* and L_w have the same orthogonal QR factor, whose triangular coordinate is L_w, as required. □

Theorem 2.4 (Local persistence)**.** *Let $Q : I \to \mathrm{SO}_N$ be a C^1 positive Frenet curve on a real interval, with continuous strictly positive coefficients. If $Q(t_*) \in \Sigma$ at an interior time, then every sufficiently uniformly close continuous SO_N-valued path on I has a hit of Σ in any prescribed neighborhood of t_*.*

Proof. Apply Lemma 2.3 at $Q(t_*)$ and use a short closed interval $[t_-, t_+]$ around t_* contained in its triangular chart. The normalized hit satisfies $L_* = L(t_*) \in \mathcal{C} \setminus \mathcal{P}$, since both coordinate changes preserve the vanishing of corner minors. By (2.8) and Lemma 2.1,

$$L(t_+) \in \mathcal{P}, \qquad L(t_-) \notin \mathcal{C}.$$

Indeed, the second assertion follows because otherwise $L_* = L(t_-)\big(L(t_-)^{-1}L_*\big)$ would belong to $\mathcal{P}$.

Define the continuous function

$$g(L) = \min_{\substack{1 \le k \le N \\ |I|=|J|=k,\ I \ge J}} \det L_{I,J}.$$

Then $g(L(t_-)) < 0 < g(L(t_+))$. A sufficiently close continuous path stays in the same chart and retains these strict endpoint inequalities. It has a point with $g = 0$, hence a point in $\mathcal{C} \setminus \mathcal{P}$; Lemma 2.2 forces a zero corner minor there. The coordinate identities carry this hit back to Σ. The times t_-, t_+ can be chosen arbitrarily close to t_*. □

Corollary 2.5 (Connected-family conclusion)**.** *Let S be a real interval, $a < b$, and let $Q : S \times [a, b] \to \mathrm{SO}_N$ be continuous. Write $Q_s(t) = Q(s, t)$, with each time curve as in Theorem 2.4. Suppose $Q_s(a), Q_s(b) \notin \Sigma$ for every s. If one curve avoids Σ on $[a, b]$, then every curve does.*

Proof. The set of parameters having a hit is closed because Σ is closed and $[a, b]$ is compact. All hits are interior, so Theorem 2.4 makes this parameter set open as well. Connectedness proves the assertion. □

3. Positive interpolation and conormal relations

The following version of positive triangular interpolation [6, Lemma 5.3 and Remark 5.4] has constant controls near the prescribed times.

Lemma 3.1 (Exact positive interpolation)**.** *After the normalization in Section 1, the matrices F_i can be interpolated at distinct interior times $t_1 < \cdots < t_m$ by a smooth curve $F : [a, b] \to U$ satisfying*

$$F' = XF, \qquad X(t) = \sum_{j=1}^{N-1} w_j(t) E_{j,j+1}, \qquad w_j(t) > 0. \tag{3.1}$$

The controls can be constant on collars of the interpolation times.

Proof. Put $J = \sum_{j=1}^{N-1} E_{j,j+1}$. If $m = 1$, the curve $F(t) = e^{(t-t_1)J} F_1$ has the required properties. Assume $m \geq 2$. Transposing the coordinates in Lemma 2.2 gives a factorization of each element of $U_{>0}$ as

$$T(a) = \prod_{\nu=1}^{\ell} (I + a_\nu E_{j_\nu, j_\nu+1}), \qquad a_\nu > 0, \qquad \ell = N(N-1)/2,$$

where the sequence of indices j_ν is fixed and the product is written from left to right in increasing ν. This is a smooth coordinate map with smooth inverse. On a given interval $[u, v]$, choose smooth nonnegative functions ρ_ν of integral one, with pairwise disjoint supports in its interior, ordered chronologically as $\rho_\ell, \ldots, \rho_1$. Set

$$w_j(t) = \epsilon + \sum_{\nu : j_\nu = j} a_\nu \rho_\nu(t).$$

Let $H' = XH$, $H(u) = I$, and denote its endpoint $H(v)$ by $\mathcal{E}(a, \epsilon)$. At $\epsilon = 0$, each pulse multiplies the current matrix on the left by $e^{a_\nu E_{j_\nu, j_\nu+1}} = I + a_\nu E_{j_\nu, j_\nu+1}$. Thus $\mathcal{E}(a, 0) = T(a)$. The endpoint depends smoothly on (a, ϵ): successive integration gives finite iterated integrals, since all coefficient matrices are strictly upper triangular. The derivative with respect to the ℓ weights is invertible at $(a, 0)$ in the strictly upper triangular coordinates of U. The implicit function theorem therefore supplies weights $a(\epsilon)$ with $\mathcal{E}(a(\epsilon), \epsilon) = T(a)$ for all sufficiently small $\epsilon > 0$. They remain positive by continuity, so every w_j is strictly positive. Near both endpoints all controls equal ϵ.

Apply this construction to each $F_{i+1} F_i^{-1}$ and write the path on that interval as $H(t) F_i$. A single sufficiently small background works on the finitely many intervals. On a collar of each t_i, both adjacent paths equal $e^{\epsilon(t-t_i)J} F_i$, so they glue smoothly with all derivatives. Extend by the same positive controls before the first and after the last specified time. The resulting matrices remain upper unitriangular.

□

Lemma 3.2 (Vanishing of a closed conormal relation)**.** *Let $F : [a, b] \longrightarrow U$ be a smooth curve satisfying* (3.1)*, and let $a < t_1 < \cdots < t_m < b$. Suppose real $N \times N$ matrices η_i satisfy*

$$\eta_i \eta_j = 0 \quad \text{for all } i, j, \qquad \sum_i \eta_i = 0, \qquad \Delta_i := F(t_i) \eta_i F(t_i)^{-1} \in \mathfrak{n}_-. \tag{3.2}$$

Then $\eta_i = 0$ for every i.

Proof. A family with fixed endpoints. On the intervals between the times t_i, define the piecewise smooth matrices below, using their one-sided limits at each t_i:

$$S(t) = \sum_{t_i < t} \eta_i, \qquad Z(t) = F(t) S(t) F(t)^{-1}, \qquad M_s(t) = F(t)(I - sS(t)), \quad s \in \mathbb{R}. \tag{3.3}$$

All products in (3.2) vanish, so $S^2 = 0$ and $(I - sS)^{-1} = I + sS$ for every real s. All eigenvalues of S are zero, so $\det(I - sS) = 1$ and $\det M_s = 1$. Away from the times t_i,

$M_s' = XM_s$. At such a time,

$$\begin{aligned} M_s(t_i+) &= F(t_i)\big(I - s(S(t_i-) + \eta_i)\big) \\ &= (I - s\Delta_i)M_s(t_i-). \end{aligned} \tag{3.4}$$

Indeed, $(I - s\eta_i)(I - sS(t_i-)) = I - s(\eta_i + S(t_i-))$ because $\eta_i S(t_i-) = 0$, which proves the identity for every s. The jump factor is lower unitriangular. It preserves the oriented flag of initial row spans. Also, $S = 0$ on collars of both endpoints.

Preservation of the positive Frenet equation. Use row Gram–Schmidt with positive diagonal:

$$M_s = B_s Q_s, \qquad B_s \text{ lower triangular with positive diagonal}, \qquad Q_s \in \mathrm{SO}_N\,.$$

At a time t_i, uniqueness of this factorization gives

$$Q_s(t_i+) = Q_s(t_i-), \qquad B_s(t_i+) = (I - s\Delta_i)B_s(t_i-). \tag{3.5}$$

In particular the diagonal entries of B_s agree on both sides. Between these times,

$$Q_s' Q_s^{-1} = B_s^{-1} X B_s - B_s^{-1} B_s'. \tag{3.6}$$

The first term has no entries above the first superdiagonal: its (i,j) entry vanishes when $j > i+1$, since a contributing index r must satisfy $r \le i$ and $j \le r+1$. At $(j, j+1)$ the only possible index is $r = j$, giving

$$(B_s^{-1} X B_s)_{j,j+1} = w_j \frac{(B_s)_{j+1,j+1}}{(B_s)_{jj}} =: \kappa_j(s,t) > 0.$$

The second term in (3.6) is lower triangular. Its left side is skew symmetric, so it is exactly the skew Jacobi matrix with these positive superdiagonal entries. Consequently $\Gamma_s = Q_s^{\mathsf{T}}$ satisfies (2.1).

On each of the finitely many closed time pieces, Gram–Schmidt depends smoothly on (s,t). Define $Q_s(t_i)$ by its common one-sided limit. The values match at each t_i, so $(s,t) \mapsto Q_s(t)$ is continuous on $\mathbb{R} \times [a,b]$. The coefficients κ_j match by (3.5) and continuity of w_j. Therefore the skew matrices $Q_s' Q_s^{-1}$ have matching one-sided limits. Since Q_s itself matches, its one-sided derivatives agree. Each time curve is thus C^1 with continuous positive Frenet coefficients. The endpoint frames are fixed because S vanishes on endpoint collars.

Nonvanishing of leading minors. Let $P \in \mathrm{SO}_N$ be the signed reversal

$$P_{i,N+1-i} = (-1)^{N-i}, \qquad P_{ij} = 0 \text{ otherwise}. \tag{3.7}$$

The row-reversal sign cancels the product of these signs, so for every matrix A,

$$\delta_k(PA) = \det A_{[1,k],[1,k]}.$$

Left multiplication by the constant P preserves the logarithmic derivative of Γ_s. Moreover,

$$\delta_k(P\Gamma_s) = \det(Q_s)_{[1,k],[1,k]} = \frac{\det(M_s)_{[1,k],[1,k]}}{\prod_{j=1}^{k}(B_s)_{jj}}. \tag{3.8}$$

At $s = 0$, $M_0 = F$ is upper unitriangular, so every numerator in (3.8) is one. Both endpoints have the same property for all s. Corollary 2.5, applied on each compact parameter interval containing 0, implies that every leading minor of $M_s(t)$ is nonzero for every finite real s and every t between the times t_i, including the one-sided limits there. At t_i the minors on the two sides agree, since its jump factor is lower unitriangular.

Since $M_s = (I - sZ)F$ and F is upper unitriangular,

$$(M_s)_{[1,k],[1,k]} = (I_k - sZ_{[1,k],[1,k]})F_{[1,k],[1,k]}.$$

The second factor has determinant one, so

$$\det\big(I_k - sZ(t)_{[1,k],[1,k]}\big) \neq 0 \qquad (s \in \mathbb{R},\ 1 \le k \le N). \tag{3.9}$$

The case $k = N$ also follows directly from $S^2 = 0$.

Triangular elimination. Between consecutive times t_i, $Z' = [X, Z] = XZ - ZX$, or entrywise,

$$Z'_{ij} = w_i Z_{i+1,j} - w_{j-1} Z_{i,j-1}, \tag{3.10}$$

where the first term is omitted for $i = N$ and the second for $j = 1$. Equation (3.9) for $k = 1$ says $1 - sZ_{11} \neq 0$ for every real s, so $Z_{11} = 0$ identically on the interval. Differentiating $Z_{11} = 0$ gives $w_1 Z_{21} = 0$, and hence $Z_{21} = 0$. Repeating the same argument down the first column gives $Z_{21} = \cdots = Z_{N1} = 0$.

For $2 \leq k \leq N$, suppose the first $k-1$ columns vanish on and below the diagonal. The leading $k \times k$ block is upper triangular with diagonal $0, \ldots, 0, Z_{kk}$. Thus the determinant in (3.9) is $1 - sZ_{kk}$, and $Z_{kk} = 0$. For $i \geq k$, the entry $Z_{i,k-1}$ is zero by induction. Differentiating $Z_{kk} = 0$ in (3.10) therefore gives $w_k Z_{k+1,k} = 0$ when $k < N$. Once $Z_{ik} = 0$ is known, the same equation gives $w_i Z_{i+1,k} = 0$ for $i < N$. All w_i are positive, so the entire kth column vanishes on and below the diagonal. Induction makes Z strictly upper triangular on every interval. Its one-sided limits at the times t_i have the same property. Their difference is Δ_i, which by assumption is strictly lower triangular. Hence $\Delta_i = 0$, and therefore $\eta_i = 0$. □

4. Real Schubert transversality

Proposition 4.1. *For any ordered positive tuple and any Schubert-cell labels, the real cells meet simultaneously transversely at every real common point. This assertion allows arbitrary codimension sums.*

Proof. For a real row p-plane $W \subset V = \mathbb{R}^N$, the tangent space of $\mathrm{Gr}(p, V)$ is $\mathrm{Hom}(W, V/W)$. Its dual is $\mathrm{Hom}(V/W, W)$, paired by the trace of the composition. Thus the cotangent space is represented by matrices

$$\eta \in \mathrm{End}(V), \qquad \operatorname{im} \eta \subset W \subset \ker \eta. \tag{4.1}$$

Here image and kernel refer to the right action on row vectors. An ambient infinitesimal right action A is paired with η by $\mathrm{tr}(\eta A)$. Any two matrices in (4.1) have zero product in either order.

The stabilizer of the standard initial-row flag for the right action is the invertible lower triangular group. Its orbits in the Grassmannian are the Schubert cells, and the orbit tangent space is the image of its Lie algebra under the infinitesimal action. A cotangent vector is conormal precisely when it annihilates every lower triangular infinitesimal action Y. Since

$$\mathrm{tr}(\eta Y) = \sum_{i,j} \eta_{ij} Y_{ji},$$

this annihilator consists of strictly lower triangular matrices. For the flag represented by F_i, use the coordinate change $v \mapsto vF_i^{-1}$. The cotangent representative becomes $F_i \eta F_i^{-1}$. Thus the conormal space to its Schubert cell is

$$\{\eta : \operatorname{im} \eta \subset W \subset \ker \eta,\ F_i \eta F_i^{-1} \in \mathfrak{n}_-\}.$$

If simultaneous transversality failed at W, the sum map from the direct sum of the cell conormal spaces into $T^*_W \mathrm{Gr}(p, V)$ would have a nonzero kernel. It would supply matrices η_i, not all zero, with $\sum_i \eta_i = 0$. They satisfy every hypothesis of Lemma 3.2, after exact interpolation by Lemma 3.1. This is impossible. □

Corollary 4.2. *If the sum of cell codimensions exceeds d, there is no real common point. In a closed Schubert intersection of total prescribed codimension d, every real point belongs to every prescribed open cell. At such a point the complex scheme-theoretic intersection is a reduced point.*

Proof. Independent conormal spaces have total dimension at most d. At a boundary point, at least one actual cell has codimension strictly larger than the prescribed one. This would make the total exceed d. Otherwise all varieties are smooth there. Their real conormal

spaces complexify to the corresponding complex conormal spaces, so the complex intersection is transverse of dimension zero, with local ring $\mathbb{C}$. □

5. The universal incidence and exhaustion of the complex fiber

The universal complex incidence will show that D transverse real points exhaust a fiber when $D > 0$.

Let $\mathrm{Fl}(\mathbb{C}^N)$ denote the variety of complete flags. Set $B = \mathrm{Fl}(\mathbb{C}^N)^m$ and, initially with its reduced structure, let

$$\mathcal{X} = \{(W, E_1, \ldots, E_m) \in G \times B : W \in \Omega_{\lambda_i}(E_i)\ \forall i\}.$$

Write $\pi : \mathcal{X} \to B$ for the projection.

Lemma 5.1. *The variety $\mathcal{X}$ is irreducible and proper over B, and*

$$\dim \mathcal{X} = d + m \dim \mathrm{Fl}(\mathbb{C}^N) - \sum_i |\lambda_i|.$$

For top-degree data with $D > 0$, π is dominant and generically finite of degree D.

Proof. Let $H = \mathrm{GL}_N(\mathbb{C})$ act on row subspaces on the right. Fix a flag E and a p-plane W_0, and denote their stabilizers by Q and P. These groups are irreducible: in adapted bases they are triangular or block triangular groups with invertible diagonal blocks. Put

$$Z_i = \{E' : W_0 \in \Omega_{\lambda_i}(E')\}.$$

The map $a : H \to G$, $a(g) = W_0 g^{-1}$, is a Zariski locally trivial bundle whose fiber through $g_0 \in H$ is $g_0 P$, isomorphic to P. For local sections, choose a basis adapted to W_0. On a Grassmannian graph chart the map $g \mapsto W_0 g$ has the section

$$\mathrm{rowspan}(I_p\ Z) \longmapsto \begin{pmatrix} I_p & Z \\ 0 & I_q \end{pmatrix}.$$

Inverting this matrix gives a section of a; the same construction applies on the other graph charts. Consequently $a^{-1}(\Omega_{\lambda_i}(E))$ is irreducible of dimension $\dim H - |\lambda_i|$. Indeed, $W_0 g^{-1} \in \Omega_{\lambda_i}(E)$ is equivalent to $W_0 \in \Omega_{\lambda_i}(Eg)$. Its image under $g \mapsto Eg$ is Z_i, and the fiber over $Eg_0 \in Z_i$ is Qg_0, isomorphic to Q. Hence Z_i is irreducible of dimension $\dim \mathrm{Fl}(\mathbb{C}^N) - |\lambda_i|$.

The map

$$H \times \prod_i Z_i \longrightarrow \mathcal{X}, \qquad (g, E_1', \ldots, E_m') \longmapsto (W_0 g, E_1' g, \ldots, E_m' g)$$

is surjective. Indeed, for a target $(W, E_1, \ldots, E_m)$ choose g_0 with $W_0 g_0 = W$. Its full fiber is parametrized isomorphically by

$$h \longmapsto (hg_0, E_1 g_0^{-1} h^{-1}, \ldots, E_m g_0^{-1} h^{-1}), \qquad h \in P.$$

The source is irreducible, and subtracting $\dim P = \dim H - d$ gives the dimension formula. Properness follows because G is projective. Kleiman's transversality theorem in characteristic zero [10], applied successively to the finitely many smooth Schubert cells, makes their intersections transverse for general flags. Boundary strata in a top-degree problem have negative expected dimension and hence are absent for general flags. Thus there is a nonempty Zariski open subset B° whose fibers contain exactly D distinct transverse points. Their intersection cycle is the Schubert product. For $D > 0$ this proves the last assertion. □

Lemma 5.2 (Analytic exhaustion)**.** *Let $\pi : X \to B$ be a dominant morphism of irreducible complex algebraic varieties, with exactly D points in every fiber over a nonempty Zariski open B°. Suppose a fiber over b contains D distinct points where π is a local complex analytic isomorphism. Those points exhaust the entire underlying fiber over b.*

Proof. Suppose there is an additional point x. Choose pairwise disjoint analytic neighborhoods $U_1, \dots, U_D, U_x$ of the specified points and x. The first D points give analytic inverse sections s_i; shrink their common base neighborhood to $V \ni b$ so that every $s_i(V)$ is contained in its designated U_i.

The set $U_x \cap \pi^{-1}(V)$ is a nonempty analytic open subset of X. The nonempty Zariski open $\pi^{-1}(B^\circ)$ is analytically dense in irreducible X, so the former set contains a point x' over $b' \in V \cap B^\circ$. That fiber contains the D distinct points $s_i(b')$ and also x', a contradiction. Analytic density holds at singular points as well: a proper algebraic subset has smaller dimension and cannot contain a nonempty analytic open subset. □

At a real solution W_* covered by Corollary 4.2, this local analytic isomorphism holds over the full complex base B. To see it, fix the flag tuple $(E_1, \dots, E_m)$ and choose local holomorphic sections $g_i(E)$ of the flag action with $E_i g_i(E) = E$ and $g_i(E_i) = I$. Such sections are given by the usual triangular flag charts. Since W_* lies in the open cell, the germ of $\Omega_{\lambda_i}(E_i)$ at W_* is smooth of codimension $|\lambda_i|$. Choose local holomorphic equations $f_{i,a}$, $1 \le a \le |\lambda_i|$, generating its ideal and having independent differentials. Near $(W_*, E_1, \dots, E_m)$ the moving conditions are

$$f_{i,a}(W g_i(E_i')^{-1}) = 0 \qquad (1 \le i \le m,\ 1 \le a \le |\lambda_i|).$$

There are d equations. Their derivative in the d coordinates of G is invertible by transversality. The holomorphic implicit function theorem makes their common zero scheme the graph of a unique holomorphic function of all flag coordinates. Thus π is a local analytic isomorphism at W_*.

6. Positive intersection number

Theorem 6.1 (Mukhin–Tarasov–Varchenko [13, Corollary 6.3])**.** *Let $t_1, \dots, t_m$ be distinct real numbers, and let $F_\bullet(t_i)$ be the osculating complete flags of the rational normal curve in $\mathbb{R}^N$. For nonzero partitions λ_i in the $p \times q$ rectangle, of total size d, suppose $D = \int_G \prod_i \sigma_{\lambda_i} > 0$. The intersection $\bigcap_i \Omega_{\lambda_i}(F_\bullet(t_i))$ contains D distinct real points in the prescribed open cells, and the conditions meet transversely at those points.*

Conversion from the polynomial formulation. Let $\mathbb{R}[u]_{<N}$ be the vector space of polynomials of degree less than N, and let $e_1, \dots, e_N$ be the standard row basis. The real linear isomorphism

$$\Phi(e_{k+1}) = \frac{(N-1)!}{(N-1-k)!}\, u^{N-1-k} \qquad (0 \le k < N)$$

sends $\gamma(t) = (1, t, \dots, t^{N-1}/(N-1)!)$ to $(u+t)^{N-1}$. The first j rows of e^{tJ}, $J = \sum_{j=1}^{N-1} E_{j,j+1}$, are $\gamma(t), \dots, \gamma^{(j-1)}(t)$. Their images span

$$(u+t)^{N-j}\mathbb{R}[u]_{<j}.$$

Thus the flag becomes the polynomial osculating flag at $b = -t$. For $a_j = q + j - \lambda_j$, its a_jth subspace consists of polynomials divisible by $(u-b)^{p-j+\lambda_j}$. The open Schubert condition is therefore the condition that the increasing vanishing orders of a basis are

$$\lambda_p,\ \lambda_{p-1}+1,\ \dots,\ \lambda_1 + p - 1.$$

These are the finite labels in [13, Corollary 6.3], with their plane rank equal to p and polynomial ambient dimension equal to N. Choose their highest-weight partition to be the full rectangle (q^p). Its complementary partition is zero, so their extra condition at infinity is an open cell of codimension zero. Write $\mathfrak{gl}_p$ for the Lie algebra of complex $p \times p$ matrices. The corollary gives as many distinct real transverse points as the multiplicity of the irreducible $\mathfrak{gl}_p$-module of highest weight (q^p) in the tensor product with highest weights λ_i. By the Littlewood–Richardson rule, this multiplicity is the coefficient of the Schur polynomial $s_{(q^p)}$ in $\prod_i s_{\lambda_i}$, where s_λ denotes the Schur polynomial in p variables. The same

rule computes the Schubert product, so this coefficient is D [5]. The real isomorphism Φ^{-1} carries these points and their transverse intersections to the stated flags. □

Proposition 6.2. *Theorem 1.2 holds when $D > 0$.*

Proof. After normalizing the first frame to I, the positive parameter space is $\mathcal{B}_+ = (U_{>0})^{m-1}$, parametrized by the independent increments $G_i = F_{i+1}F_i^{-1}$. The frames are recovered recursively from $F_1 = I$ and $F_{i+1} = G_iF_i$. This base is a connected smooth manifold by the coordinates of Lemma 2.2; for $m = 1$ it is a point. It contains an osculating tuple

$$F_i^0 = e^{t_iJ}, \qquad J = \sum_{j=1}^{N-1} E_{j,j+1}, \qquad 0 = t_1 < t_2 < \cdots < t_m.$$

These are row flags of the rational normal curve $(1, t, t^2/2!, \ldots, t^{N-1}/(N-1)!)$. The increments $e^{(t_{i+1}-t_i)J}$ are strictly totally positive.

Theorem 6.1 supplies D distinct real points at this tuple. Zero labels may be omitted in that application, since $\Omega_0 = G$ and $\sigma_0 = 1$. Corollary 4.2 places the resulting points in every prescribed open cell of the full tuple. They give D complex local inverse branches over B, so Lemmas 5.1 and 5.2 show that they exhaust the starting complex fiber.

The real incidence over the positive parameter space is closed in its product with the compact real Grassmannian. Its projection is proper. By Corollary 4.2 and the relative implicit function theorem, this projection is a local diffeomorphism. Each fiber is discrete and compact, hence finite. Choose disjoint local inverse neighborhoods of its points. Properness allows the base neighborhood to be shrunk so that its entire inverse image lies in their union: otherwise additional preimages over a sequence converging to the base point would have a limiting preimage outside that union. Thus the projection is a finite covering of its image, which is open by local invertibility and closed by properness. The image contains the osculating tuple and the base is connected. Thus every positive tuple has exactly D real solutions.

At any target tuple those D real points provide D complex local sections of π. Lemmas 5.1 and 5.2 exclude all other complex points, including every possible excess component. Finally, Corollary 4.2 identifies the original scheme-theoretic local ring at each point with $\mathbb{C}$. Hence the entire intersection scheme is exactly D reduced real points. □

7. Zero intersection number

Positive-degree completions first control intersections with a nonzero Schubert product. For an m-tuple $\alpha = (\alpha_1, \ldots, \alpha_m)$ of partitions in the $p \times q$ rectangle, write

$$P_\alpha = \prod_i \sigma_{\alpha_i}, \qquad Y_\alpha = \bigcap_i \Omega_{\alpha_i}(F_i).$$

Let h be the Plücker hyperplane class, and for an irreducible subvariety $T \subset G$ set $\deg_h(T) = \int_T h^{\dim T}$.

Lemma 7.1. *Suppose $P_\beta \neq 0$, with total codimension $c \leq d$, and set $r = d - c$. For ordered positive flags the support of Y_β is pure of dimension r, and every irreducible component contains a point in all prescribed open Schubert cells.*

Proof. For $r = 0$, the assertion follows from Proposition 6.2. Suppose $r > 0$. For general flags, Kleiman's theorem represents P_β by an effective cycle of dimension r. Because $P_\beta \neq 0$, this cycle is nonzero. Its degree for the ample Plücker class is positive, so

$$D' = \int_G P_\beta h^r > 0.$$

Append r new ordered positive flags after the last original flag, for example by repeated left multiplication by e^J. At each impose a simple Schubert divisor H_j, which is a hyperplane

section of the Plücker embedding. Proposition 6.2 makes

$$Y_\beta \cap H_1 \cap \cdots \cap H_r$$

a finite reduced set of real points.

Every component C of Y_β has dimension at least r by the intersection dimension inequality in the smooth variety G. In the Plücker projective space, a projective variety of positive dimension meets every hyperplane, and intersecting with one hyperplane decreases dimension by at most one. Applying this statement r times shows that $C \cap H_1 \cap \cdots \cap H_r$ is nonempty and has dimension at least $\dim C - r$. Finiteness of the completed intersection therefore implies $\dim C = r$.

Every such C contains a point of the completed intersection, and that point is real. It lies in all the original open cells by Corollary 4.2 applied to the completed top-degree problem. Thus C contains a point in all the prescribed open cells. □

Proposition 7.2. *If $\prod_i \sigma_{\lambda_i} = 0$ and $\sum_i |\lambda_i| \le d$, then the corresponding ordered-positive closed Schubert intersection is empty.*

Proof. Starting with empty diagrams, build the target diagrams by increasing one part by one at each step, preserving the partition inequalities and the rectangle bound. Let γ be the first tuple whose product is zero, and let β be the preceding tuple. Exactly one condition, say the jth, changes; its Schubert variety is replaced by a codimension-one boundary

$$\Omega_{\gamma_j}(F_j) \subsetneq \Omega_{\beta_j}(F_j).$$

Put $c = \sum_i |\beta_i|$ and $r = d - c \ge 1$. Lemma 7.1 says that the support of Y_β is pure r-dimensional, with no component contained in this boundary. Every component of Y_γ therefore has dimension at most $r - 1$. The ordinary intersection dimension inequality gives the reverse inequality for every nonempty component. Thus, if Y_γ is nonempty, the γ conditions meet properly in dimension $r - 1$.

Work in the smooth variety G^m, with the small diagonal map $\delta : G \to G^m$, $\delta(W) = (W, \ldots, W)$. Its image is a regular embedding of codimension $(m-1)d$. The irreducible variety

$$A = \prod_i \Omega_{\gamma_i}(F_i)$$

has dimension $md - (c+1)$ and meets this diagonal properly in dimension $r - 1$, with intersection identified with Y_γ. By positivity of proper intersection multiplicities for a regular embedding [20, Lemma 43.16.3], every component T has a strictly positive coefficient e_T in the intersection cycle. The regular sequence is supplied by the small diagonal: locally in smooth coordinates on G, it consists of the differences of each coordinate in the last $m - 1$ factors with that in the first factor. These $(m-1)d$ coordinate differences form a regular sequence in the local ring of the smooth ambient variety. The diagonal formula for intersection products gives, in the Chow group $A_{r-1}(G^m)$ of $(r-1)$-dimensional cycles modulo rational equivalence,

$$[\delta(G)] \cdot [A] = \sum_T e_T [\delta(T)] = \delta_* \left(\left(\prod_i \sigma_{\gamma_i} \right) \cap [G] \right) = 0.$$

If the intersection were nonempty, pair its effective cycle with the appropriate power of the ample class $\sum_i \mathrm{pr}_i^* h$, where $\mathrm{pr}_i : G^m \to G$ is the ith projection. Since $\delta^*(\sum_i \mathrm{pr}_i^* h) = mh$, its degree would be

$$m^{r-1} \sum_T e_T \deg_h(T) > 0,$$

contradicting the zero class. Therefore Y_γ is empty. Every later tuple imposes stronger conditions, so the target intersection is empty. An empty underlying support also means an empty intersection scheme. □

In top degree, $\prod_i \sigma_{\lambda_i} = D[\text{point}]$ in $A^d(G)$. Thus $D = 0$ gives the hypothesis of Proposition 7.2. Together with Proposition 6.2, this proves Theorem 1.2.

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