A PROOF OF THE SCHEME-THEORETIC HUMPHREYS CONJECTURES

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Abstract. We prove the relative and traditional scheme-theoretic Humphreys conjectures for connected reductive groups over algebraically closed fields of positive characteristic exceeding the Coxeter number of every simple factor. We also prove that the self-extension support scheme of every finite-dimensional tilting module is reduced.

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1. Introduction

Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$, and let $\mathbf{G}$ be a connected reductive algebraic group. Put $G = \mathbf{G}^{(1)}$, let $\mathbf{G}_1$ be the kernel of the relative Frobenius morphism $\mathbf{G} \to G$, and let $\mathcal{N} \subset \mathrm{Lie}(G)$ be the nilpotent cone. Write h for the largest Coxeter number of the simple factors, with $h = 1$ for a torus; a subscript specifies the group whose Coxeter numbers are used. Throughout the paper, $p > h$.

The cohomology calculation of Andersen–Jantzen and Friedlander–Parshall identifies the graded algebra

$$R = H^\bullet(\mathbf{G}_1, \Bbbk) \cong \Bbbk[\mathcal{N}], \tag{1.1}$$

where linear functions have cohomological degree two [1, 9]. For any finite-dimensional $\mathbf{G}$-modules M, N, $\mathrm{Ext}^\bullet_{\mathbf{G}_1}(M, N)$ is a finitely generated graded R-module [11]. Its support scheme is $\mathrm{Spec}(R/\operatorname{Ann}_R \mathrm{Ext}^\bullet_{\mathbf{G}_1}(M, N))$. For a finitely generated R-module E, write $\operatorname{supp}_R E$ for its underlying closed support. All support annihilators and orbit-closure ideals I_Z are taken in the ring (1.1) on the Frobenius-twisted nilpotent cone.

1.1. **The statements.** Choose a maximal torus and a Borel subgroup of $\mathbf{G}$. Let $\mathbf{X}$ be the character lattice, Φ the root system, Φ^+ the positive roots, $\mathbf{X}_+$ the dominant weights, and W the finite Weyl group, with longest element w_0. Put $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ and

$$W_{\mathrm{aff}} = \mathbb{Z}\Phi \rtimes W, \qquad W_{\mathrm{ext}} = \mathbf{X} \rtimes W.$$

Here t_μ denotes translation by μ. Write ℓ for Coxeter length on W_{aff}, extended to W_{ext} by giving the stabilizer of the fundamental alcove length zero. The p-dilated dot action is $(t_\mu v)\bullet\nu = p\mu + v(\nu+\rho) - \rho$. We use ordinary Kazhdan–Lusztig two-sided cells: two elements lie in the same cell when their canonical-basis elements generate the same based two-sided ideal of the affine Hecke algebra. This relation is extended to W_{ext} by the length-zero subgroup. Lusztig's cell–orbit correspondence assigns to $w \in W_{\mathrm{ext}}$ a nilpotent G-orbit $C(w)$ [16, 5].

A module is tilting if it has both a Weyl filtration and a dual Weyl filtration. Denote the indecomposable tilting module of highest weight $\nu \in \mathbf{X}_+$ by $T(\nu)$. For $\lambda \in \mathbf{X}_+$, let w_λ be the unique shortest element of $Wt_\lambda W$ and put

$$T_\lambda = T(w_\lambda \bullet 0), \qquad Z_\lambda = \overline{C(w_\lambda)}_{\mathrm{red}}.$$

The subscript red specifies the reduced induced scheme structure. For a reduced closed subscheme $Z \subset \mathcal{N}$, write $I_Z \subset R$ for its defining ideal.

Theorem 1.1. *For every $\lambda \in \mathbf{X}_+$,*

$$\operatorname{Ann}_R H^\bullet(\mathbf{G}_1, T_\lambda) = I_{Z_\lambda}. \tag{1.2}$$

In particular, $I_{Z_\lambda} H^\bullet(\mathbf{G}_1, T_\lambda) = 0$.

This is the scheme-theoretic relative Humphreys conjecture, Conjecture 3.7 of [5]. The equality is stronger than the equality of the radicals of the two ideals: it excludes any nilpotent thickening of the predicted orbit closure.

Corollary 1.2. *For every finite-dimensional tilting $\mathbf{G}$-module T, the ideal $\operatorname{Ann}_R \mathrm{Ext}^\bullet_{\mathbf{G}_1}(T, T)$ is radical.*

Let $W^{\varnothing}_{\mathrm{ext}}$ denote the elements of minimal length in their cosets Ww. The traditional scheme-theoretic assertion is Conjecture 3.6 of [5].

Corollary 1.3. *Let $w \in W^{\varnothing}_{\mathrm{ext}}$ and $T = T(w \bullet 0)$. Then*

$$\operatorname{Ann}_R \mathrm{Ext}^\bullet_{\mathbf{G}_1}(T, T) = I_{\overline{C(w)}_{\mathrm{red}}}.$$

Corollary 1.3 combines the reducedness assertion of Corollary 1.2 with the ordinary support formula

$$\operatorname{supp}_R \operatorname{Ext}^\bullet_{\mathbf{G}_1}(T(w \bullet 0), T(w \bullet 0)) = \overline{C(w)}, \tag{1.3}$$

of [21, Theorem 1.1].

1.2. **Background and previous results.** Support varieties connect the cohomology of finite group schemes with the geometry of nilpotent elements. Friedlander–Parshall developed the theory for restricted Lie algebras in 1986 [10]. Restriction of a rational $\mathbf{G}$-module to $\mathbf{G}_1$ brings it into this setting. The calculations of Andersen–Jantzen in 1984 and Friedlander–Parshall in 1986 identify the cohomology ring with the coordinate ring of the nilpotent cone when $p > h$ [1, 9]. The support of a rational module is invariant under the connected group action. Since the nilpotent cone has finitely many orbits, each irreducible component of the support is an orbit closure. The problem is to determine the orbits attached to a representation.

The combinatorial orbit labels come from Lusztig's theory of cells. His 1989 paper *Cells in affine Weyl groups. IV* [16] relates affine two-sided cells to unipotent classes, and hence to nilpotent orbits. This connects the canonical basis of an affine Hecke algebra with orbit geometry. In 1997 Humphreys proposed, as Hypothesis 12 of [15], that this correspondence describes the supports of indecomposable tilting modules with regular highest weights. The proposal arose from comparing modular representations of semisimple groups with representations of their Lie algebras. Translation functors give the equivalent formulation using lower closures of alcoves; see [2, §8.2] and [5, Remark 3.4]. Including the lower walls gives each dominant weight a unique alcove label.

The quantum-group analogue was established first. Ginzburg–Kumar identified small-quantum-group cohomology with functions on a nilpotent cone in 1993 [12]. Ostrik computed the supports of quantum tilting modules in type A in 1998 [17]. In 2006 Bezrukavnikov identified the complexes computing quantum tilting cohomology with simple perverse-coherent sheaves [6]. These sheaves have the predicted reduced orbit closure as their scheme-theoretic support. This proves the relative quantum statement in every type; the passage to the traditional quantum statement is explained in [2, §8.4].

In positive characteristic, Cooper's 2010 work gives support formulas for several infinite families of GL_n tilting modules, including singular weights, characteristic two, and families associated with hook partitions [7]. Hardesty's 2018 paper proves the type-A conjecture for $p > h$ [14, Theorem 1.4.1]. His argument uses the partition description of affine cells for the upper support inclusion and comparison with quantum tilting modules for the lower inclusion. Achar–Hardesty–Riche's 2019 paper proves the predicted lower inclusion in every type for $p > h$, and equality for p larger than a bound depending on the root system [2, Theorems 1.1 and 9.3]. They introduced integral exotic parity objects and compared their supports with those of their characteristic-zero fibers. Their large-characteristic argument also yields the scheme-theoretic assertions, as explained in [5, §3.5].

The relative formulation concerns $H^\bullet(\mathbf{G}_1, T)$, while the traditional formulation concerns $\operatorname{Ext}^\bullet_{\mathbf{G}_1}(T, T)$. Achar–Hardesty formulated both scheme-theoretic versions in [3, Conjectures 7.1 and 7.2], first posted in 2020 and published in 2024. Their Theorem 7.6 proves the relative version for GL_N and $p > N$, and Corollary 7.7 gives the traditional version. The proof constructs co-t-structures on the nilpotent cone and parabolic Springer resolutions. In type A, every nilpotent orbit is Richardson, so these resolutions cover every orbit closure.

Their subsequent paper [4], first posted in 2021 and published in 2024, constructs the same supportive coheart directly from nilpotent orbits and tilting bundles on them. The resulting silting objects compute the nonzero cohomology of indecomposable tilting modules and prove the relative closed-support formula for all $p > h$. Their orbit construction gives the underlying

closed support. The scheme-theoretic question asks in addition whether the ideal of the reduced orbit closure annihilates every cohomology sheaf. The distinction appears in the 2026 account [5, Conjectures 3.6–3.7 and Remark 3.11]. Theorem 1.1 establishes this annihilation for every type throughout $p > h$. Tensor–Hom adjunction then gives Corollary 1.2, and the ordinary support theorem [21] identifies the orbit in Corollary 1.3.

1.3. **The proof.** The geometric assertion proved here is

$$I_{\overline{C}_{\mathrm{red}}}\,\mathcal{H}^q(\mathcal{S}_{\mathcal{N}}(C,\mathcal{T})) = 0 \qquad (q \in \mathbb{Z}), \tag{1.4}$$

where $\mathcal{S}_{\mathcal{N}}(C,\mathcal{T})$ is the supportive silting complex associated with a nilpotent orbit C and a tilting orbit bundle $\mathcal{T}$; $\mathcal{H}^q$ denotes ordinary cohomology sheaves. We construct a bounded coherent complex on the reduced closure $\overline{C}_{\mathrm{red}}$ whose pushforward has $\mathcal{S}_{\mathcal{N}}(C,\mathcal{T})$ as a direct summand.

The coefficient estimate needed for this construction is stable under arbitrary quotients. For the conormal bundle $\mathcal{W}$ to a closed orbit in a smooth Lie-algebra neighborhood, let $A = \mathrm{gr}_{I_{C/Y}}\mathcal{O}_Y$ be the normal-cone algebra of any intervening closed subscheme Y. Its ordinary degree r and its Kazhdan weight c are distinct. Here wfd is the least length of a resolution by Weyl-filtered modules. For simple G outside type A, every nonzero weight component $(A_r)_c$ satisfies

$$\mathrm{wfd}(A_r)_c \le c - 1 \qquad (r > 0), \qquad c \ge 2r.$$

For classical groups this follows from an estimate in natural tensor degree. For exceptional groups, strong linkage reduces the estimate to finite inequalities on affine faces, and a norm estimate treats the remaining low degrees. The comparison of centralizer weights uses the embedded reductive models supplied by Cotner [8].

An extended Rees construction converts these normal-cone estimates into a finite filtration of an adjunction cocone on Y. This yields the strict estimates required to correct a coherent extension at each boundary orbit. The corrections take place on the reduced closure throughout. The established classification of the ambient supportive coheart then identifies the desired direct summand. The finite root-system inequalities are verified with integer and rational arithmetic; their complete verification procedure and exact input data are specified in Appendix A.

2. Orbit categories and geometric inputs

Put $\Gamma = G \times \mathbb{G}_{\mathrm{m}}$, acting on $\mathrm{Lie}(G)$ by $(g,z)x = z^{-2}\,\mathrm{Ad}(g)x$. Write $\mathrm{Coh}^{\Gamma}(Y)$ for the category of Γ-equivariant coherent sheaves on a Γ-scheme Y. An internal shift $\langle k\rangle$ changes a torus weight by $-k$; $[n]$ denotes cohomological shift. Unless a different category is specified, Hom_Y denotes Hom in $D^b\,\mathrm{Coh}^{\Gamma}(Y)$. In particular, a module of Kazhdan weight c is written as a weight-zero module shifted by $\langle -c\rangle$.

Let C be a nilpotent orbit, choose $x \in C$, and choose a cocharacter $\tau : \mathbb{G}_{\mathrm{m}} \to G$ associated with x. Thus $\mathrm{Ad}(\tau(z))x = z^2 x$, and τ takes values in the derived group of a Levi subgroup in which x is distinguished. Here distinguished means that every torus in its centralizer lies in the center of that Levi. The homomorphism $z \mapsto (\tau(z), z)$ lies in the Γ-stabilizer of x and is called the Kazhdan torus. Write U_x for the unipotent radical of $Z_G(x)$. Then

$$Z_G(x) = L \ltimes U_x, \qquad L = Z_G(x) \cap Z_G(\tau), \qquad H = L^{\circ}.$$

The stabilizer $\Gamma_x = \mathrm{Stab}_{\Gamma}(x)$ is $(\mathbb{G}_{\mathrm{m}} \times L) \ltimes U_x$, where the first factor is the Kazhdan torus. Its action on U_x has strictly positive weights. The finite group L/H has order prime to p. Equivariant coherent sheaves on the homogeneous orbit are equivalent to finite-dimensional stabilizer representations; in particular they are vector bundles. We use this equivalence for orbit bundles and their fibers throughout. A bare Weyl or tilting orbit bundle will always mean the inflation of such a module for L, with trivial unipotent action and Kazhdan weight zero. All other weights are written as internal shifts.

2.1. **Filtration dimension and co-t-structures.** For a connected reductive group H, write $\Delta_H(\nu)$ and $\nabla_H(\nu)$ for its Weyl and dual Weyl modules, and $L_H(\nu)$ for its simple module of highest weight ν. Subscripts are omitted when the group is fixed. A Weyl-filtered module has a finite filtration with factors $\Delta_H(\nu)$. The number $\operatorname{wfd}_H M$ is the least length of a resolution of M by Weyl-filtered modules. Ext groups for H are computed in the category of rational H-modules. Equivalently, $\operatorname{wfd}_H M$ is the largest possibly nonzero degree of $\operatorname{Ext}^i_H(M, \nabla_H(\nu))$ as ν varies. The finite highest-weight subcategories give finite such resolutions for every finite-dimensional module. Tensor products of Weyl-filtered modules are Weyl filtered; consequently

$$\operatorname{wfd}(M \otimes N) \le \operatorname{wfd} M + \operatorname{wfd} N. \tag{2.1}$$

Finite extensions preserve a common upper bound. For a reductive group L with identity component H and $p \nmid |L/H|$, Weyl and costandard modules are obtained from their H-counterparts by finite induction and taking direct summands. We use the same resolution and Ext definition of wfd_L. Restriction of these modules and exactness of component-group invariants transfer the connected bounds to L.

We use co-t-structures with the convention

$$\mathcal{D}_{\ge b} = \mathcal{D}_{\ge 0}[-b], \qquad \mathcal{D}_{\le b} = \mathcal{D}_{\le 0}[-b].$$

The halves are closed under direct summands and satisfy $\mathcal{D}_{\ge 0}[-1] \subset \mathcal{D}_{\ge 0}$, $\mathcal{D}_{\le 0}[1] \subset \mathcal{D}_{\le 0}$, and $\operatorname{Hom}(\mathcal{D}_{\ge 1}, \mathcal{D}_{\le 0}) = 0$. For every E and every integer b there is a triangle $A \to E \to B \to A[1]$ with $A \in \mathcal{D}_{\ge b+1}$ and $B \in \mathcal{D}_{\le b}$. Boundedness means that every object lies in $\mathcal{D}_{\ge a} \cap \mathcal{D}_{\le b}$ for some $a \le b$. The coheart is $\mathcal{D}_{\ge 0} \cap \mathcal{D}_{\le 0}$. An object has pure co-$t$-degree b if it belongs to $\mathcal{D}_{\ge b} \cap \mathcal{D}_{\le b}$. The orbit co-$t$-structure is generated by the objects $\mathcal{T}[n]\langle -n\rangle$; explicitly, its nonnegative part is generated under finite extensions and summands by $\mathcal{T}[n]\langle k\rangle$ with $n + k \le 0$, and its nonpositive part by those with $n + k \ge 0$.

For a closed subset $Z \subset Y$, write $\mathcal{D}_Z(Y) = D^b_Z \operatorname{Coh}^\Gamma(Y)$ for complexes whose ordinary cohomology is set-theoretically supported on Z. For a closed immersion $C \hookrightarrow Y$, the symbol $I_{C/Y}$ denotes its ideal sheaf. The following are the geometric inputs from [4].

Proposition 2.1 (Ambient orbit and support results)**.** *Let X be a smooth invariant open in $\operatorname{Lie}(G)$ in which C is closed, and let $u : C \hookrightarrow X$.*

(1) *If $\mathcal{W} = I_{C/X}/I^2_{C/X}$, then $\mathcal{W}$ has Kazhdan weights at least two. For $i > 0$, every nonzero weight-c exterior coefficient satisfies*

$$\operatorname{wfd}_L(\Lambda^i \mathcal{W})_c \le i - 1, \qquad c \ge 2i.$$

(2) *The orbit co-t-structure induces a bounded co-t-structure on $\mathcal{D}_C(X)$ for which u_* is co-t-exact. Its coheart is generated by $u_*\mathcal{T}[n]\langle -n\rangle$.*

(3) *The supportive coheart on $D^b \operatorname{Coh}^\Gamma(\mathcal{N})$ is Krull–Schmidt. Its indecomposable objects are $\mathcal{S}_{\mathcal{N}}(C, \mathcal{T})[n]\langle -n\rangle$, where $\mathcal{T}$ ranges over indecomposable weight-zero tilting orbit bundles. Set $d_C = \operatorname{codim}_{\mathcal{N}} C$; this is even because both $\dim \mathcal{N}$ and $\dim C$ are even. The unshifted object has support $\overline{C}$. For the closed immersion $i_C : C \hookrightarrow \mathcal{N} \setminus (\overline{C} \setminus C)$, its restriction to this open is*

$$i_{C*}\mathcal{T}[-d_C/2]\langle d_C/2\rangle.$$

Membership in this coheart is characterized by the inequalities $Lu^* F_X \in D^b \operatorname{Coh}^\Gamma(C)_{\ge 0}$ and $u^! F_X \in D^b \operatorname{Coh}^\Gamma(C)_{\le 0}$ at every orbit, where F_X is the closed pushforward to a smooth ambient neighborhood.

(4) *For each $\lambda \in \mathbf{X}_+$, the complex associated with T_λ is one of these supportive complexes, with orbit label $C(w_\lambda)$. Its total graded hypercohomology is $H^\bullet(\mathbf{G}_1, T_\lambda)$, compatibly with the R-action, and the latter module has ordinary support Z_λ.*

Part (1) is the dual formulation of [4, Lemma 4.1], part (2) is Proposition 4.6 there, and part (3) follows from Lemmas 6.7, 6.10 and Theorem 7.2 there. Part (4) is its Theorem 8.1 and Remark 8.2. In applying that theorem we use its antidominant representative $w_0\lambda$; the double cosets $Wt_{w_0\lambda}W$ and $Wt_\lambda W$ coincide, so their shortest element is our w_λ. A Weyl-filtered orbit bundle belongs to the nonnegative part: a finite tilting coresolution expresses it using terms $\mathcal{T}^i[-i]$, $i \geq 0$; see also [4, Remark 2.3]. Consequently part (2) gives the vanishing from a pushed Weyl-filtered source to $u_*\mathcal{T}[n]\langle k\rangle$ whenever $n + k > 0$.

We prove (1.4) by constructing the required complex on the reduced orbit closure.

3. Filtration bounds for normal-cone coefficients

Lemma 3.1 (Finite coresolutions in the source variable)**.** *If* $0 \to M \to I^0 \to \cdots \to I^s \to 0$ *is exact and* $\operatorname{wfd}_H I^j \leq d$ *for every* j*, then* $\operatorname{wfd}_H M \leq d$*.*

Proof. Put $Q_0 = M$, $Q_s = I^s$, and use the exact sequences $0 \to Q_j \to I^j \to Q_{j+1} \to 0$ for $0 \leq j < s$. The contravariant Ext sequence gives, for $i > d$, $\operatorname{Ext}^i(Q_j, \nabla(\nu)) \cong \operatorname{Ext}^{i+1}(Q_{j+1}, \nabla(\nu))$. Iterating reaches $\operatorname{Ext}^{i+s}(I^s, \nabla(\nu)) = 0$. □

In this paragraph Φ and ρ refer to the root system and half sum of positive roots of H. Dominance means that $\mu \leq \lambda$ when $\lambda - \mu$ is a nonnegative integral combination of simple roots. For dominant weights μ, λ, write $\mu \uparrow \lambda$ if there is a sequence from μ to λ in which each step is a reflection in a hyperplane $\langle \nu + \rho, \alpha^\vee \rangle = mp$, with $\alpha \in \Phi$ and $m \in \mathbb{Z}$, and increases the weight in dominance order. This is the strong-linkage partial order. Let $b_H(\lambda)$ be the largest number of strict steps in a chain of dominant weights for this order ending at λ. The lower ideal is finite; central torus characters are fixed throughout. Parker's highest-weight-category bound [18, Lemma 2.9 and the discussion before §4] gives $\operatorname{wfd} \nabla(\lambda) \leq b_H(\lambda)$. The same bound holds for $L(\lambda)$. Indeed, in

$$0 \longrightarrow L(\lambda) \longrightarrow \nabla(\lambda) \longrightarrow Q \longrightarrow 0,$$

every constituent of Q has chain length at most $b_H(\lambda) - 1$. Induction and the long exact sequence prove the claim. Thus, for every nonzero finite-dimensional H-module M,

$$\operatorname{wfd}_H M \leq \max_{L(\lambda) \text{ a constituent of } M} b_H(\lambda). \tag{3.1}$$

In a smooth invariant open X of $\operatorname{Lie}(G)$ in which an orbit C is closed, let

$$C \xrightarrow{i} Y \xrightarrow{j} X, \quad \mathcal{W} = I_{C/X}/I_{C/X}^2, \quad A = \operatorname{gr}_{I_{C/Y}} \mathcal{O}_Y = \bigoplus_{r \geq 0} A_r.$$

Here Y is a Γ-stable closed subscheme of X containing C, and both displayed arrows are closed immersions. The canonical map $\operatorname{Sym} \mathcal{W} \twoheadrightarrow A$ is surjective. The index r is ordinary ideal-adic degree. Separately, c denotes the Kazhdan weight at a chosen orbit point. These are different gradings. As in Section 2, L denotes the full reductive centralizer and $H = L^\circ$ its identity component.

Proposition 3.2 (Coefficient estimate)**.** *For every non-type-A simple* G *with* $p > h_G$*, every nilpotent orbit* C*, every such* Y*, and every* $r > 0$*,*

$$\operatorname{wfd}_L(A_r)_c \leq c - 1 \quad \textit{whenever } (A_r)_c \neq 0. \tag{3.2}$$

Here $c \geq 2r \geq 2$*. The estimate applies to every equivariant quotient of* $\operatorname{Sym}^r \mathcal{W}$*.*

The positive-weight statement follows from [4, Lemma 4.1]: the ambient normal space has weights at most -2, so $\mathcal{W}$ has weights at least 2. Its other needed consequence, for each nonzero weight-c coefficient, is

$$\operatorname{wfd}_L(\Lambda^i \mathcal{W})_c \leq i - 1, \qquad c \geq 2i \quad (i > 0). \tag{3.3}$$

This is the dual normal-space estimate, using the centralizer good-filtration restriction theorem established in [4, Theorem 3.1].

4. Classical groups in arbitrary rank

Let H be Sp_{2n}, SO_{2n+1}, or SO_{2n} in characteristic $p > \max(2, n)$, and let V be its natural module. In the orthonormal coordinates $\lambda = \sum_{i=1}^{n} \lambda_i \varepsilon_i$, use positive roots $\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j$ for $i < j$, together with ε_i in type B and $2\varepsilon_i$ in type C. Thus dominant integer weights satisfy $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ in B, C, and $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n|$ in D. Set

$$\deg \lambda = \sum_i \lambda_i \quad (B, C), \qquad \deg \lambda = \sum_{i<n} \lambda_i + |\lambda_n| \quad (D).$$

For an arbitrary weight, natural degree means the Weyl-invariant quantity $\sum_i |\lambda_i|$. On dominant weights this agrees with the displayed formulas. Representations of these matrix groups have integer weights. The group SO_2 is a torus, and SO_4 is treated with its reducible root system D_2.

Theorem 4.1 (Natural-degree bound)**.** *Let $r \geq 0$ be an integer. If every constituent of a finite-dimensional H-module M has natural degree at most r, then*

$$\mathrm{wfd}_H M \leq \max(0, r - 1).$$

Proof. Exterior powers. Put $N = \dim V$. Since V is tilting, so is $V^{\otimes a}$. For $a = \min(i, N - i) \leq n < p$, the alternating idempotent is defined and exhibits $\Lambda^a V$ as a tilting summand. Exterior duality and triviality of $\det V$ give the same conclusion for every $\Lambda^i V$.

Polynomial restriction. The exterior-bar resolution in [22, §2] is

$$0 \to \Lambda^r V \to \bigoplus_{\substack{i+j=r \\ i,j>0}} \Lambda^i V \otimes \Lambda^j V \to \cdots \to V^{\otimes r} \to \mathrm{Sym}^r V \to 0.$$

Its length is $r-1$, and its terms restrict to tilting H-modules. Tensoring the analogous resolutions for a composition $r = a_1 + \cdots + a_t$ gives a Weyl-filtered resolution of $\bigotimes_j \mathrm{Sym}^{a_j} V$ of length at most $r - t$. Injectives in the finite category of degree-r polynomial $\mathrm{GL}(V)$-modules are summands of sums of these modules, and that category has finite global dimension [22]. Restrict a finite injective coresolution and apply Lemma 3.1. Therefore

$$\mathrm{wfd}_H \operatorname{Res} M \leq r - 1 \quad (r > 0) \tag{4.1}$$

for every degree-r polynomial $\mathrm{GL}(V)$-module M. The exterior-bar sequence is exact for every dimension. Alternatively, enlarge V by a space E until $\dim(V \oplus E) \geq r$, and take weight zero for the torus scaling E; this exact operation recovers the stated sequence on V.

Good restriction and maximal factors. The exterior-power criterion for a good-filtration subgroup [13, §2.4.1] applies to $\mathrm{SL}(V)$ and H. The determinant character of $\mathrm{GL}(V)$ is trivial on H, so the same restriction assertion applies to $\mathrm{GL}(V)$. Thus $F = \mathrm{Res}_H \nabla_{\mathrm{GL}(V)}(\lambda)$ has a good filtration for every partition λ of r with at most n parts. Every label in this filtration has natural degree at most r, since every torus weight of the polynomial module has that bound.

For B, C, use the dominant cocharacter $\tau_+ = (1, \ldots, 1)$. The weight-r character of F is the single Schur character $\mathrm{ch}\, \nabla_{\mathrm{GL}_n}(\lambda)$ on the weight-one subspace of V. The top-τ_+ character of a costandard H-module is the corresponding Levi Weyl character. Such characters are independent; therefore the only degree-r good-filtration factor is $\nabla_H(\lambda)$, once.

For D, use both $\tau_+ = (1, \ldots, 1)$ and $\tau_- = (1, \ldots, 1, -1)$. If $\lambda_n > 0$, the degree-r factors are exactly $\nabla_H(\lambda^+)$ and $\nabla_H(\lambda^-)$, each once, where $\lambda^\pm = (\lambda_1, \ldots, \lambda_{n-1}, \pm\lambda_n)$. They are incomparable. If $\lambda_n = 0$, there is one factor, once. The same character calculation on the two weight-one subspaces proves this, including rank two.

These factors are maximal in dominance among all labels of F. The directed Ext property of costandards says $\mathrm{Ext}^1(\nabla(\sigma), \nabla(\lambda)) = 0$ unless $\sigma > \lambda$; see [18, Proposition 2.1(i)] and [13, §2.2.4].

Thus maximal factors can be moved to the quotient of a good filtration. Extensions between the two incomparable D factors split. We obtain

$$0 \to K \to F \to Q \to 0,$$

where K is good filtered in degrees at most $r-1$, and Q is the desired costandard or the direct sum of the two desired costandards.

Induction. Degree-zero and degree-one costandards are tilting. For $r \geq 2$, induction gives $\operatorname{wfd} K \leq r-2$ and (4.1) gives $\operatorname{wfd} F \leq r-1$. The exact sequence

$$\operatorname{Ext}^{j-1}(K, \nabla(\nu)) \to \operatorname{Ext}^{j}(Q, \nabla(\nu)) \to \operatorname{Ext}^{j}(F, \nabla(\nu))$$

proves $\operatorname{wfd} Q \leq r-1$.

Finally, the integral dominant weights of degree at most r form a finite saturated ideal: degree is pairing with τ_+ in B, C and the maximum of pairing with $\tau_\pm$ in D. These cocharacters are dominant. Induct within this finite dominance ideal to pass from costandards to simples. In $0 \to L(\lambda) \to \nabla(\lambda) \to Q \to 0$, all constituents of Q have smaller highest weight. If $j > \max(0, r-1)$, the exact sequence

$$\operatorname{Ext}^{j}(\nabla(\lambda), \nabla(\nu)) \longrightarrow \operatorname{Ext}^{j}(L(\lambda), \nabla(\nu)) \longrightarrow \operatorname{Ext}^{j+1}(Q, \nabla(\nu))$$

has zero outer terms. Hence every simple in the ideal has the same bound. A composition series and extension closure prove the assertion for M. □

4.1. **The embedded classical centralizer.** For a nilpotent skew-adjoint endomorphism x, graded Jordan form gives

$$V = \bigoplus_d M_d \otimes J_d,$$

where J_d has weights $-d+1, -d+3, \ldots, d-1$ and x has weight two. The form on J_d has sign $(-1)^{d-1}$ and that on M_d has the complementary sign. The multiplicity-space isometry groups embed in G through this orthogonal decomposition.

The x-commuting endomorphism algebra has grading

$$E = E_0 \oplus \mathfrak{r}, \qquad E_0 = \prod_d \operatorname{End}(M_d),$$

where $\mathfrak{r}$ is a nilpotent positive-weight ideal. Indeed, $\operatorname{Hom}_{\Bbbk[x]}(J_d, J_f)$ has weights $d+f-2-2j$ for $0 \leq j < \min(d, f)$; zero occurs only when $d = f$. The adjoint involution preserves this grading. Projection of the isometry centralizer to E_0 has the multiplicity-space section. Its kernel is $\{g \in 1 + \mathfrak{r} : g^* g = 1\}$, scheme-isomorphic via the Cayley maps

$$x \longmapsto (1-x)(1+x)^{-1}, \qquad g \longmapsto (1-g)(1+g)^{-1}$$

to $\{x \in \mathfrak{r} : x^* = -x\}$. The inverses are finite series and 2 is invertible. The kernel is therefore smooth, connected and unipotent. The reductive identity component is a product of symplectic and special orthogonal multiplicity groups. In the special orthogonal case the only extra relation is $\prod_{d \text{ odd}} \det(g_d) = 1$ on components; the component group is a 2-group.

Under $p > h_G$, every allowed Jordan length is at most p. For Sp_{2n} and SO_{2n+1} this follows from $h_G = 2n$. For SO_{2n}, an even Jordan length has even multiplicity, so the largest possible length is $2n-1 \leq p$. Realize $J_d = \operatorname{Sym}^{d-1}(\Bbbk^2)$; all nonzero raising coefficients $1, \ldots, d-1$ are invertible. The diagonal SL_2 commutes with the multiplicity Levi and gives the displayed cocharacter. If S is a maximal torus of that Levi and $M = Z_G(S)$, then x is distinguished in M and $Z(M)^\circ = S$. The diagonal SL_2 lies in $[M, M]$. Thus its cocharacter is associated, as required by the ambient orbit construction.

Every weight of V has total natural degree at most one. Since $\operatorname{Lie}(\mathrm{Sp}(V)) \cong \operatorname{Sym}^2 V$ and $\operatorname{Lie}(\mathrm{SO}(V)) \cong \Lambda^2 V$, every weight of $\mathcal{W}$ has total natural degree at most two. Consequently every

constituent of $(A_r)_c$, as a quotient of $\operatorname{Sym}^r \mathcal{W}$, has total degree at most $2r \le c$. For a product of multiplicity groups, tensoring Weyl resolutions gives

$$\operatorname{wfd} \boxtimes_j L(\lambda_j) \le \sum_j \max(0, \deg \lambda_j - 1) \le c - 1.$$

If every degree is zero the left side is zero. Finite extensions and prime-to-p component invariants give (3.2). Under a central cover these matrix-group modules belong to the trivial-kernel-character summand, which is equivalent to the representation category of the quotient. This proves Proposition 3.2 in all classical non-type-A cases.

5. Linkage, residues, and finite affine-face certificates

We reduce the exceptional coefficient bounds to finite inequalities on the faces of a fundamental affine alcove.

Let Φ be an irreducible root system of rank l, with simple roots $\alpha_1, \dots, \alpha_l$. Let ω_i be the fundamental weights, characterized by $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. In this section W, W_{aff} and h refer to Φ, and $b(\lambda)$ is the maximal number of strict steps in a dominant strong-linkage chain for Φ ending at λ. Write ρ and $\rho^\vee$ for the half sums of the positive roots and coroots, and $\vartheta^\vee$ for the highest root of the *coroot* system. Put $\vartheta^\vee = \sum_{i=1}^l m_i \alpha_i^\vee$, $m_0 = 1$; then $\sum_{i=0}^l m_i = h$. Define

$$n_p(\lambda) = \sum_{\alpha>0} \left\lfloor \frac{\langle \lambda + \rho, \alpha^\vee \rangle - 1}{p} \right\rfloor, \quad S_\rho = \sum_{\alpha>0} \langle \rho, \alpha^\vee \rangle.$$

Use positive residues $r_p(t) \in \{1, \dots, p\}$, so $r_p(0) = p$, and put $S_p^+(y) = \sum_{\alpha>0} r_p(\langle y, \alpha^\vee \rangle)$. For rational arguments, use the periodic extension $r_p(t) = t - p\lceil t/p \rceil + p \in (0, p]$. Then

$$p n_p(\lambda) = \langle \lambda, 2\rho^\vee \rangle + S_\rho - S_p^+(\lambda + \rho). \tag{5.1}$$

In this section affine Weyl elements act on $y = \lambda + \rho$ by the p-dilated linear action $(t_\mu v)y = p\mu + vy$. The fundamental closed p-alcove has affine coordinates $a_i = \langle y, \alpha_i^\vee \rangle$ for $1 \le i \le l$ and $a_0 = p - \langle y, \vartheta^\vee \rangle$, all nonnegative. Its relative open face F_J is given by $a_i = 0$ for $i \in J$ and $a_i > 0$ otherwise. Set

$$\ell_0(J) = \min\{\ell(u) : u \in W_{\mathrm{aff}},\ uF_J \text{ lies in the strict dominant chamber}\},$$
$$\kappa(J) = \max(1, \ell_0(J)) \qquad (J \neq \varnothing).$$

For a dominant λ, let y be the unique point of $W_{\mathrm{aff}}(\lambda + \rho)$ in the fundamental closed alcove, and put $J = \{i : a_i(y) = 0\}$. The weight is called regular if $J = \varnothing$, and singular otherwise. The strong-linkage/Bruhat comparison for shortest face-coset representatives is [19, Appendix I, (9.2) and Theorem 9.6]. The length of the shortest incident alcove at $\lambda + \rho$ is $n_p(\lambda)$: the hyperplanes through the endpoint are excluded. Since the point is strictly dominant, all incident alcoves are dominant. Thus

$$b(\lambda) \le n_p(\lambda) - \ell_0(J).$$

Let s_0 be the affine reflection in the wall $a_0 = 0$. There is also a one-step loss for every singular orbit with $n_p > 0$. A chain of n_p steps would have representative lengths $0, 1, \dots, n_p$. A dominant length-zero point has no finite-wall coordinate zero, so singularity puts it on the affine top wall. The sole length-one dominant alcove is s_0 of the fundamental alcove, and s_0 fixes that point. A strict second member is impossible. Consequently

$$b(\lambda) \le n_p(\lambda) - \kappa(J) \quad (n_p > 0), \qquad b(\lambda) = 0 \quad (n_p = 0). \tag{5.2}$$

5.1. **Regular residues.** For a root system Ψ, let $\mathrm{Inv}(w) = \{\beta \in \Psi^+ : w^{-1}\beta \in -\Psi^+\}$, and let $\mathrm{ht}_\Psi(\sum_i b_i\alpha_i) = \sum_i b_i$. Define $A_\Psi(w) = \sum_{\beta\in\mathrm{Inv}(w)} \mathrm{ht}_\Psi(\beta)$. For simply-laced Φ, normalize roots to squared length two. Then

$$2A_\Phi(w) \le h\ell(w). \tag{5.3}$$

The root-frame operator $\sum_{\alpha>0} \alpha\otimes\alpha$ is W-invariant, hence scalar on the irreducible reflection representation. Its trace is $2|\Phi^+| = lh$, so it equals $h\,\mathrm{id}$. For $v = \rho - w\rho = \sum_{\alpha\in\mathrm{Inv}(w)} \alpha$, Weyl invariance of the norm gives $\|v\|^2 = 2(\rho, v) = 2A_\Phi(w)$. Cauchy–Schwarz gives

$$\|v\|^4 = \Big(\sum_{\alpha\in\mathrm{Inv}(w)} (v,\alpha)\Big)^2 \le \ell(w)\sum_{\alpha>0}(v,\alpha)^2 = h\ell(w)\|v\|^2.$$

For $v \neq 0$, divide by $\|v\|^2$; for $v = 0$ the inequality is immediate.

The same inequality holds in B_n, C_n. For this calculation choose positive roots $e_j - e_i, e_j + e_i$ for $i < j$, together with e_i in B_n and $2e_i$ in C_n. Then $\rho_C = (1, 2, \dots, n)$ and $\rho_C^\vee = (\frac{1}{2}, \frac{3}{2}, \dots, n-\frac{1}{2})$. Write $a = (a_1, \dots, a_n) = w\rho_C$; its absolute entries are $1, \dots, n$. Since ρ_C is regular, this identifies signed tuples with Weyl elements; we evaluate ℓ and A_C on tuples through this identification. Put

$$I(a) = \#\{i < j : a_i > a_j\}, \quad N(a) = \sum_{a_i<0} |a_i|, \quad Q(a) = \sum_i i^2 - \sum_i i a_i.$$

Counting positive roots sent to negative roots gives $\ell = I + N$: the roots $e_j - e_i$ contribute I, and the negative coordinate roots together with the roots $e_j + e_i$ contribute N. Also $A_C = (\rho_C - w\rho_C, \rho_C^\vee) = Q - N$. Delete the entry of absolute value n, in position j, to obtain $b = (b_1, \dots, b_{n-1})$, and put $S = \sum_{i=j}^{n-1} b_i$. Insertion of $+n$ increases ℓ by $n - j$ and changes A_C by $n(n-j) - S$. Insertion of $-n$ increases ℓ by $n + j - 1$ and changes A_C by $n(n+j-1) - S$. In either case,

$$n\ell(a) - A_C(a) = (n-1)\ell(b) - A_C(b) + \ell(b) + \sum_{i=j}^{n-1} b_i.$$

The sum over the suffix is at least $-N(b)$: including all negative entries makes the sum no larger, and positive entries only increase it. Thus the last two terms are at least $I(b)$, proving the claim inductively. For the dual root systems, invariance of the pairing gives

$$A_B(w) = \langle \rho_B - w\rho_B, \rho_B^\vee\rangle = \langle \rho_B, \rho_B^\vee - w^{-1}\rho_B^\vee\rangle = A_C(w^{-1}).$$

This proves the other case as well.

An integral regular point y in the fundamental p-alcove has all affine coordinates at least one. Thus $y = \rho + (p-h)z$, with $\langle z, \alpha_i^\vee\rangle \ge 0$ and $\langle z, \vartheta^\vee\rangle \le 1$. For a positive coroot β, its contribution to $S_p^+(wy)$ is $\langle y, \beta\rangle$ if $w\beta > 0$ and $p - \langle y, \beta\rangle$ if $w\beta < 0$. Summing gives

$$S_p^+(wy) - S_\rho = h\ell(w) - 2A_{\Phi^\vee}(w^{-1}) + (p-h)\big(\ell(w) + \langle z, 2w^{-1}\rho^\vee\rangle\big).$$

The atomic-height term uses the coroot system, because the residues are coroot pairings; apply (5.3) to $\Phi^\vee$. The last bracket equals

$$\sum_{\substack{\beta\in(\Phi^\vee)^+\\ w\beta>0}} \langle z, \beta\rangle + \sum_{\substack{\beta\in(\Phi^\vee)^+\\ w\beta<0}} (1 - \langle z, \beta\rangle) \ge 0,$$

because every positive-coroot pairing with z lies in $[0, 1]$. This proves $S_p^+(y) \ge S_\rho$ for every regular integral point in these types. Translation by p times the root lattice does not change residues.

For F_4, exact enumeration of all 1152 Weyl elements in the coroot system verifies (5.3). For G_2, we verify the residue bound directly. At $p = 7$ the only integral point in the fundamental alcove is ρ, and all 12 inequalities $7\ell(w) - 2A_{\Phi^\vee}(w) \ge 0$ hold. At $p = 8$ all three vertices of the unit-alcove relaxation give nonnegative residue differences for every w; monotonicity of each

rootwise summand extends this to $p \geq 8$. These finite checks are performed by the accompanying verifier. Thus for all types and all regular λ,

$$\operatorname{wfd} L(\lambda) \leq b(\lambda) \leq n_p(\lambda) \leq \left\lfloor \frac{\langle \lambda, 2\rho^\vee \rangle}{p} \right\rfloor. \tag{5.4}$$

5.2. **All singular points reduce to a finite list.** For a nonempty proper J, put $I = \{0, \dots, l\} \setminus J$. An integral point of F_J has affine coordinates $a_i = 0$ on J and $a_i \geq 1$ on I. These inequalities define a rational simplex whose vertices are

$$y_{J,k}(p) = \sum_{i \in I,\ i>0} \omega_i + \frac{p - \sum_{i \in I} m_i}{m_k} \omega_k, \quad k \in I, \quad \omega_0 = 0. \tag{5.5}$$

For $p \geq h+1$ every vertex remains in the relative interior of the same original face. A positive-coroot pairing on that face is identically 0, identically p, or strictly between them. Endpoint residues are always p. For each finite Weyl element w, $S_p^+(wy)$ is therefore affine on this simplex. On a vertex ray, the nonendpoint slopes are u or $1-u$, where $0 \leq u \leq 1$; the endpoint slope is one. Hence $S_p^+(wy_{J,k}(p)) + p\kappa(J)$ is nondecreasing in p.

For $D \geq 0$, checking

$$S_{p_0}^+(wy_{J,k}(p_0)) + p_0\kappa(J) \geq S_\rho - D \tag{5.6}$$

at one $p_0 \geq h+1$, for all required faces, vertices and finite w, proves the same inequality for every $p \geq p_0$ and every integral singular point. Equations (5.1) and (5.2) imply

$$\operatorname{wfd} L(\lambda) \leq \left\lfloor \frac{\langle \lambda, 2\rho^\vee \rangle + D}{p} \right\rfloor \quad \text{for every dominant integral } \lambda. \tag{5.7}$$

5.3. **Exact computation of face corrections.** Only $\ell_0(J) < N_0 = \lceil S_\rho / p_0 \rceil$ needs to be determined; otherwise $p\kappa \geq S_\rho$ is automatic. Enumerate dominant alcoves by right-simple-reflection breadth-first search. Dominant alcoves correspond to minimal representatives in $W \backslash W_{\mathrm{aff}}$. Prefixes of a reduced expression of such a representative are again minimal: if finite left multiplication shortened a prefix, it would shorten the whole element. Thus the enumeration is complete.

At scale h, the regular point ρ identifies each alcove. In fundamental-weight coordinates write $u_p(y) = My + p\gamma$ and let $z = u_h(\rho)$. Then $\gamma = (z - M\rho)/h$. The ith finite chamber wall pulls back to

$$\gamma_i a_0 + \sum_{j=1}^{l} (M_{ij} + \gamma_i m_j) a_j.$$

On a dominant alcove all these coefficients are nonnegative integers. The face uF_J is strictly dominant exactly when, for every finite wall, the support of this coefficient vector is not contained in J. This checks every face. A face first meeting the strict dominant chamber at length at least N_0 is assigned that lower bound, which suffices. The checker also recomputes every edge length by the affine hyperplane count.

5.4. **Exact lower-bound certificates.** Most vertices satisfy (5.6) by the rootwise bound

$$S_p^+(wy) \geq \sum_{\alpha>0} l_p(\langle y, \alpha^\vee \rangle), \qquad l_p(t) = \begin{cases} \min(t, p-t) & 0 < t < p, \\ p & t = 0 \text{ or } p. \end{cases}$$

For the remaining vertices, number the positive coroots $\gamma_1, \dots, \gamma_N$. For a finite Weyl element w, let $z_i = 1$ if $w\gamma_i$ is negative and $z_i = 0$ otherwise. Put $z = (z_1, \dots, z_N)$, and let e_i be the standard basis of $\mathbb{Q}^N$. Index by t the triples (i, j, k) with $\gamma_i + \gamma_j = \gamma_k$. Every addition $\gamma_i + \gamma_j = \gamma_k$ gives

$$v_t \cdot z \geq 0, \qquad v_t = e_i + e_j - e_k, \qquad 0 \leq z_i \leq 1.$$

Indeed, if both γ_i and γ_j remain positive under a Weyl element, then their sum remains positive. The residue objective has the form $c_0 + c \cdot z$. For any nonnegative rational multipliers q_t, put $r = c - \sum_t q_t v_t$. Then

$$c \cdot z = \sum_t q_t (v_t \cdot z) + r \cdot z \geq \sum_i \min(0, r_i). \tag{5.8}$$

Thus a list of exact rational multipliers is a directly checkable lower-bound certificate. The checker regenerates roots and vertices, validates each root addition, checks coverage, and recomputes this identity using rational arithmetic.

The finite bounds used below are

Type	p_0	Faces	Automatic	Vertices	Rational certificates	D
E_6	13	126	82	210	0	14
E_7	19	254	170	457	0	58
E_8	31	510	338	1034	37	128

Here "automatic" means that $p_0\kappa(J) \geq S_\rho$. The E_6 and E_7 rows use only the rootwise bound. In the E_8 row it suffices to certify the 37 vertices for which the rootwise lower bound for $S^+_{p_0}(wy) + p_0\kappa(J) - S_\rho$ is less than -128. Formula (5.8) verifies the required improvement at every one of them. Appendix A specifies all root data, the enumeration, and the rational certificates.

The same finite reduction gives $D = 0$ for E_6 at $p_0 = 17$ and for E_7 at $p_0 = 29$, using respectively 161 and 275 vertices and the rootwise bound alone. For singular F_4, G_2, enumeration of the respective finite Weyl groups at the 41 and 6 required vertices gives $D = 0$ for every $p > h$.

6. Exceptional origin and nonzero centralizers

6.1. **The origin: combining norm and height.**

Proposition 6.1. *Let θ be the highest root of an exceptional simple group G. Let $d \geq 1$ be an integer. For $p > h_G$, every finite-dimensional G-module M whose simple constituents have highest weights $\lambda \leq d\theta$ satisfies* $\operatorname{wfd}_G M \leq 2d - 1$.

Proof. For F_4, G_2, (5.7) with $D = 0$ gives $\operatorname{wfd} L(\lambda) \leq 2d - 1$, since $\langle \lambda, 2\rho^\vee \rangle \leq 2d(h-1) < 2dp$.

For E_6, E_7, E_8, dominant linked weights $\lambda > \nu$ satisfy

$$\|\lambda + \rho\|^2 - \|\nu + \rho\|^2 \in 2p\mathbb{Z}_{>0}.$$

Indeed, write $\lambda + \rho = w(\nu + \rho) + p\gamma$ with γ in the root lattice. The difference is

$$2p(w(\nu + \rho), \gamma) + p^2\|\gamma\|^2.$$

The first pairing is integral and the simply-laced root lattice is even, so the difference belongs to $2p\mathbb{Z}$. It is positive because it also equals $(\lambda - \nu, \lambda + \nu + 2\rho)$, where $\lambda - \nu$ is a nonzero sum of positive roots and $\lambda + \nu + 2\rho$ is strictly dominant. Summing over a linkage chain gives

$$b(\lambda) \leq \left\lfloor \frac{\|\lambda\|^2 + 2(\lambda, \rho)}{2p} \right\rfloor.$$

Since $\lambda \leq d\theta$, pairing the positive-root sum $d\theta - \lambda$ with the dominant vector $d\theta + \lambda + 2\rho$ gives $\|\lambda\|^2 + 2(\lambda, \rho) \leq \|d\theta\|^2 + 2(d\theta, \rho) = 2d^2 + 2d(h-1)$. Hence

$$\operatorname{wfd} L(\lambda) \leq \left\lfloor \frac{d(d+h-1)}{p} \right\rfloor \leq 2d - 1 \quad (1 \leq d \leq 2p - h). \tag{6.1}$$

For larger degrees, (5.7) gives $\operatorname{wfd} \leq 2d - 1$ whenever $D < 2d(p - h + 1)$. Taking the simpler sufficient constants $D = 14, 58, 128$, the two ranges at the smallest modulus are

Type	D	Norm covers $d \leq$	Height covers $d \geq$
E_6	14	14	4
E_7	58	20	15
E_8	128	32	33

As p increases the first range grows and the second starts earlier. The ranges overlap for E_6, E_7 and are adjacent for E_8. Thus every positive degree and every $p > h$ is covered. Apply (3.1) and finite-extension closure to finish. □

At the zero orbit, $\mathcal{W} = \mathrm{Lie}(G)^*$, the Kazhdan degree is $c = 2r$, and every constituent of a quotient of $\mathrm{Sym}^r \mathcal{W}$ has highest weight at most $r\theta$. Proposition 6.1 proves (3.2) there.

6.2. **An embedded weight-height bound.**

Lemma 6.2. *Let x be nilpotent in simple G with $p > h_G$, let τ be associated, and let $H = (Z_G(x) \cap Z_G(\tau))^\circ$. Choose a maximal torus and positive roots for H, and let $\rho_H^\vee$ be half the sum of its positive coroots. Every H-weight η of $\mathrm{Lie}(G)$ satisfies*

$$|\langle \eta, 2\rho_H^\vee \rangle| \leq 2(h_G - 1).$$

Proof. We use the embedded mixed-characteristic construction of Cotner [8, Corollary 5.6, Lemmas 5.1 and 5.7, Theorem 5.8], in version 3 dated 2 July 2026. Start with the special-fiber pair (x, τ). The Bala–Carter Levi and degree-two Richardson lifting in the proof of Corollary 5.6 give a pure fiberwise nilpotent section over a mixed-characteristic discrete valuation ring $\mathcal{R}$, after faithfully flat local extensions. Purity here means that the centralizer dimension is constant on the fibers. For a group K containing the image of τ, write $P_K(\tau)$ for the subgroup of $g \in K$ such that $\lim_{z\to 0} \tau(z) g \tau(z)^{-1}$ exists. More explicitly, lift a Bala–Carter Levi M, its distinguished cocharacter parabolic $P_M(\tau)$, and τ in the split root datum, denoting the lifted cocharacter by $\tau_{\mathcal{R}}$. Choose a lift $x_{\mathcal{R}}$ of x in the free degree-two module of $\mathrm{Lie}(M_{\mathcal{R}})$. The section $x_{\mathcal{R}}$ belongs to the Lie algebra of the unipotent radical of $P_M(\tau)$ in both fibers. Upper semicontinuity of stabilizer dimension shows that its generic parabolic orbit has dimension at least the special Richardson orbit, hence is Richardson as well. Distinguishedness of the parabolic persists with the root datum, so the lifted cocharacter is associated in both fibers. Write $\mathrm{Lie}(G)(\tau, a)$ for the subspace on which $\mathrm{Ad}\,\tau(z)$ acts by z^a. The associated-cocharacter formula

$$\dim Z_G(x) = \dim P_G(\tau) - \dim \bigoplus_{n \geq 2} \mathrm{Lie}(G)(\tau, n)$$

then makes the centralizer dimension constant and the section pure. The associated cocharacter lifts after the indicated strictly henselian extension. Theorem 5.8 then supplies the embedded reductive group scheme

$$H_{\mathcal{R}} = (Z_{G_{\mathcal{R}}}(x_{\mathcal{R}}) \cap Z_{G_{\mathcal{R}}}(\tau_{\mathcal{R}}))^\circ \subset G_{\mathcal{R}}.$$

Its special fiber is H, up to conjugacy. The center and fundamental-group hypotheses hold after the usual central cover when $p > h_G$.

Choose a split torus and Borel of $H_{\mathcal{R}}$ after a further allowed extension, and put $\psi = 2\rho_H^\vee$. The ψ-weight direct summands of the finite free module $\mathrm{Lie}(G_{\mathcal{R}})$ are finite free, so the occurring weights are the same in the two fibers. In characteristic zero, ψ is the diagonal principal SL_2 cocharacter of the simple factors of H (the map is trivial if H is a torus). Its composite into G has dominant weighted Dynkin labels in $\{0, 1, 2\}$. To check the upper bound, suppose the label at a simple root α_i exceeds two. Then the degree-two and degree-minus-two vectors e, f of the $\mathfrak{sl}_2$-triple belong to the derived Lie algebra of the Levi obtained by deleting α_i. Write

$[e,f] = \sum_{j\neq i} c_j \alpha_j^\vee$. Dominance and nonnegativity of the inverse Cartan matrix of each Levi component give $c_j \geq 0$. But then

$$\alpha_i([e,f]) = \sum_{j\neq i} c_j \langle \alpha_i, \alpha_j^\vee \rangle \leq 0,$$

contrary to the positive label at α_i. Integrality gives the three possible labels. Every G-root therefore has ψ-weight of absolute value at most $2(h_G - 1)$. Constancy of the weight summands transfers this to the special fiber. □

Since $\mathcal{W}$ is an H-subquotient of $\mathrm{Lie}(G)^*$, every highest weight η of a simple constituent of $(A_r)_c$ satisfies

$$\langle \eta, 2\rho_H^\vee \rangle \leq 2r(h_G - 1) \leq c(h_G - 1). \tag{6.2}$$

Every subquotient has the same weight bound.

6.3. **Every possible proper exceptional centralizer.** If $x \neq 0$, then $\mathrm{rk}(H_{\mathrm{der}}) \leq \mathrm{rk}(G) - 1$. In the adjoint quotient, the nontrivial cocharacter τ centralizes H. Were H_{der} full rank, its maximal torus would be a maximal torus of G, and its roots would span the rational character space. A cocharacter centralizing them all would be zero, contradicting its weight-two action on x.

Thus one may cover all possible factors by the larger list of root systems of ranks at most $1, 3, 5, 6, 7$ for ambient G_2, F_4, E_6, E_7, E_8. The face calculation of Section 5 gives the following all-weight bounds (5.7):

Factors and characteristic threshold	D
A_1, $p \geq 7$	0
A_n $(n \leq 5)$, B_n, C_n $(2 \leq n \leq 4)$, D_n $(4 \leq n \leq 5)$, $p \geq 13$	0
B_5, C_5, $p \geq 13$	3
A_6, C_6, D_6, $p \geq 19$	0
B_6, $p \geq 19$	3
A_7, B_7, C_7, D_7, $p \geq 31$	0
F_4, G_2, $p > h$; E_6, $p \geq 17$; E_7, $p \geq 29$	0

The classical rows comprise 24 type/threshold pairs and 1264 vertices. The rootwise lower bound for $S_p^+(wy) + p\kappa(J) - S_\rho$ proves all of them. Its only negative minima are -3 for B_5 at 13, $-3/2$ for C_5 at 13, and -2 for B_6 at 19; the common upper bound on the defects is therefore 3.

The exceptional factor thresholds suffice by rank: E_6 can first occur in ambient E_7, where $p \geq 19$, and E_7 can first occur in ambient E_8, where $p \geq 31$. An E_8 factor cannot occur. Each factor with nonzero defect has rank at least five. Since the total semisimple rank is at most seven, at most one such factor occurs. After a central isogeny, tensor Weyl resolutions across the factors; torus characters contribute filtration dimension zero. The exact character-summand decomposition for the diagonalizable kernel identifies the relevant Ext groups. Consequently

$$\mathrm{wfd}_H L(\eta) \leq \left\lfloor \frac{\langle \eta, 2\rho_H^\vee \rangle + 3}{p} \right\rfloor. \tag{6.3}$$

Combining (6.2) with $c \geq 2$ gives

$$cp - \left[c(h_G - 1) + 3\right] \geq 2c - 3 > 0.$$

Every constituent therefore has Weyl filtration dimension at most $c - 1$. A composition series gives the same bound for $(A_r)_c$.

Finally, the full reductive centralizer has prime-to-p component group by [8, Theorem 5.8(4)]. Its costandards restrict to sums of connected costandards and

$$\mathrm{Ext}_L^i(M,N) = \mathrm{Ext}_H^i(M,N)^{L/H}.$$

Exactness of component invariants transfers the bound. This completes the proof of Proposition 3.2 in exceptional types.

7. A finite Rees–Koszul filtration

Retain $C \xrightarrow{i} Y \xrightarrow{j} X$ and put $u = ji$, $B = \operatorname{Sym}_{\mathcal{O}_C} \mathcal{W}$ and $I = I_{C/Y}$. For an equivariant orbit vector bundle Q define

$$K_Q = \operatorname{Cone}(Lj^* u_* Q \longrightarrow i_* Q)[-1].$$

Lemma 7.1 (Finite Rees–Koszul filtration)**.** *The object K_Q has a finite extension filtration whose pieces are $i_* E_r(Q)$ for finitely many $r > 0$, where*

$$E_r(Q)^{-i} = A_{r-i} \otimes \Lambda^i \mathcal{W} \otimes Q \quad (0 \le i \le \operatorname{rk} \mathcal{W}), \qquad A_s = 0 \ (s < 0).$$

The differential is the normal-cone Koszul differential.

Proof. Use the canonical extended Rees deformations $D_C Y \hookrightarrow D_C X \to \mathbb{A}^1_t$, with rings $\mathcal{O}_X[t, I_{C/X} t^{-1}]$ and $\mathcal{O}_Y[t, I_{C/Y} t^{-1}]$. The parameter has Rees degree -1. Their zero fibers are $\operatorname{Spec}_C A \hookrightarrow \operatorname{Spec}_C B$, and their unit fibers are $Y \hookrightarrow X$. There is a closed immersion $C \times \mathbb{A}^1 \hookrightarrow D_C Y$.

Since $C \subset X$ is a smooth closed immersion, $D_C X$ is smooth over $\mathbb{A}^1$: in an étale coordinate chart it replaces the normal coordinates x_i by $t u_i$. Moreover $D_C X$ is affine over the quasi-affine X, so it is quasi-affine. Its trivial line bundle is ample and equivariantly linearized. Thomason's equivariant resolution criterion in [23, Theorem 2.1(1)] supplies vector-bundle surjections for Γ times the Rees torus. Iterating on kernels terminates in a locally free syzygy because the underlying scheme is regular of finite dimension. This gives a bounded equivariant vector-bundle resolution of the pushforward of $Q \boxtimes \mathcal{O}_{\mathbb{A}^1}$.

Tensor the resolution with $\mathcal{O}_{D_C Y}$ and take the augmentation cocone. Every term is flat over $\Bbbk[t]$: an extended Rees ring embeds in a Laurent extension and is torsion free over $\Bbbk[t]$, and the target is flat as well. Its unit fiber is a bounded coherent complex P representing K_Q.

A finite graded Rees module $\mathcal{M}$ with t injective embeds into $M[t, t^{-1}]$, where $M = \mathcal{M}/(t-1)\mathcal{M}$. Set $F^r M = \{m : m t^{-r} \in \mathcal{M}_r\}$. If homogeneous generators have degrees d_q and unit-fiber images m_q, then locally

$$F^r M = \sum_q I^{\max(r-d_q, 0)} m_q.$$

Thus the terms are coherent, $F^r M = M$ for $r \ll 0$, $I F^r M \subset F^{r+1} M$, and equality holds for $r \gg 0$. Termwise this gives a good filtration of P.

The target of the original resolution is $\Bbbk[t]$-flat. Induction through its short exact sequences of syzygies shows that every syzygy is $\Bbbk[t]$-flat. Specialization at $t = 0$ therefore remains exact. Its derived tensor and augmentation identify

$$\operatorname{gr}_F P \simeq \operatorname{Cone}(A \otimes^L_B Q \longrightarrow Q)[-1].$$

The canonical zero-section Koszul resolution of Q over B gives exactly the stated degree-r complex in the graded equivariant derived category. The augmentation cancels the degree-zero copy of Q; negative degrees vanish.

Because A is a finite graded B-module, each $\operatorname{Tor}_i^B(A, \mathcal{O}_C)$ is finite. Let d be the Koszul differential and let h_v denote exterior multiplication by the generator corresponding to a normal variable v. Then $d h_v + h_v d = v$. Thus every normal variable acts by zero on $\operatorname{Tor}_i^B(A, \mathcal{O}_C)$, which is finite graded over $\mathcal{O}_C$. There are finitely many homological degrees and a finite affine cover of C. Finite homogeneous generating sets on this cover give $N \ge 0$ above every nonzero ordinary degree, uniformly for all $i > 0$. Hence $\operatorname{gr}^r_F P$ is acyclic for $r > N$.

For $m > N+1$, put $Q_m = F^{N+1}P/F^mP$. Its finite filtration has acyclic graded pieces, so Q_m is acyclic. After forgetting equivariance and working on an affine open, the degreewise surjective transition maps give

$$0 \longrightarrow \varprojlim_m{}^1 H^{q-1}(Q_m) \longrightarrow H^q(\varprojlim_m Q_m) \longrightarrow \varprojlim_m H^q(Q_m) \longrightarrow 0.$$

Both outer terms vanish. Choose $r_0 \geq N+1$ so that $F^{r+1}P = IF^rP$ for $r \geq r_0$. For $q \geq 0$ one has

$$I^{q+r_0-N-1}F^{N+1}P \subset F^{q+r_0}P = I^qF^{r_0}P \subset I^qF^{N+1}P.$$

Thus this filtration and the ideal-adic filtration of $F^{N+1}P$ are cofinal. The inverse limit is its ideal-adic completion. Away from C, the ideal is the unit ideal and every filtration term equals P; its cohomology there is zero. Thus the coherent cohomology of the tail is supported on C and killed by a power of the ideal. Noetherian completion is exact on finite modules and acts as the identity on such torsion modules [20]. Vanishing of the completed tail forces the original tail to be acyclic.

Consequently $P/F^{N+1}P$ represents K_Q. Removing the finitely many acyclic lower-degree pieces gives the finite representative

$$F^1P/F^{N+1}P \simeq K_Q.$$

The exact faithful forgetful functor detects acyclicity, so the finite filtration is equivariant. Since C is homogeneous, its equivariant coherent coefficient sheaves A_r are vector bundles. □

8. Strict adjunction estimates and local corrections

In this section and Section 9, assume that G is simple and not of type A. Retain the notation of Section 2. The unipotent radical of the stabilizer has strictly positive Kazhdan weights. The decreasing filtration by Kazhdan weights at least c is stable, and the unipotent radical acts trivially on its associated graded.

Lemma 8.1 (The strict finite presentation)**.** *If Q is a weight-zero Weyl-filtered module of the reductive stabilizer, inflated with trivial unipotent action, then K_Q is a finite extension of objects*

$$i_*Q_s[\ell]\langle -c\rangle, \qquad \ell \geq 0, \qquad c \geq \ell + 1, \tag{8.1}$$

where each Q_s is Weyl filtered.

Proof. Apply the Kazhdan-weight filtration and the bounded-complex filtration to Lemma 7.1. A term is $M[i]\langle -c\rangle$, where

$$M = (A_a)_{c_1} \otimes (\Lambda^i\mathcal{W})_{c_2} \otimes Q, \quad a + i = r > 0, \quad c = c_1 + c_2.$$

By (3.2), (3.3), and tensor subadditivity, a Weyl resolution length m satisfies

	m at most	$c - i - m$ at least
$a > 0, i > 0$	$c_1 + i - 2$	2
$a = 0, i > 0$	$i - 1$	1
$a > 0, i = 0$	$c - 1$	1.

A left Weyl-filtered resolution places its terms at shifts $[s]$, $0 \leq s \leq m$. Thus $\ell = i + s \geq 0$ and $c \geq \ell + 1$. □

8.1. **Presilting and reflection.** Suppose

$$\mathrm{Hom}_Y(i_*Q, i_*T'[n]\langle k\rangle) \neq 0, \qquad n+k>0,$$

where Q is Weyl filtered and T' is weight-zero tilting, both with the stated trivial unipotent action. Choose n minimal among all such choices; $n \geq 0$ since both unshifted objects are sheaves. Ambient orbit vanishing [4, Proposition 4.6] and adjunction give

$$\mathrm{Hom}_Y(Lj^*u_*Q, i_*T'[n]\langle k\rangle) = 0.$$

The triangle $K_Q \to Lj^*u_*Q \to i_*Q \to$ and (8.1) then force a nonzero group

$$\mathrm{Hom}_Y(i_*Q_s, i_*T'[n-\ell-1]\langle k+c\rangle).$$

Here $n-\ell-1<n$ and $(n-\ell-1)+(k+c) \geq n+k>0$, a contradiction. In particular the objects $i_*T[n]\langle -n\rangle$ form a presilting family: the Hom from any member to a positive cohomological shift of any other member vanishes. Indeed,

$$\mathrm{Hom}(i_*T[a]\langle -a\rangle, i_*T'[b]\langle -b\rangle[m]) = \mathrm{Hom}(i_*T, i_*T'[b+m-a]\langle a-b\rangle),$$

whose two indices sum to $m>0$.

They generate $\mathcal{D}_C(Y)$ as follows. Each coherent cohomology sheaf of an object of $\mathcal{D}_C(Y)$ is killed by a power of $I_{C/Y}$. Its ideal-adic filtration has finitely many quotients i_*M, with M an equivariant orbit bundle. The Kazhdan filtration reduces M to modules for L. Their finitely many highest weights lie in a finite highest-weight subcategory, whose derived category is generated by its tilting objects. Ordinary truncation triangles then give the required generation of $\mathcal{D}_C(Y)$. By [4, §2], the presilting family defines a bounded co-t-structure. Weyl-filtered orbit bundles belong to its nonnegative part by finite tilting coresolutions, and their allowed shifts generate that part.

Equation (8.1) gives $K_Q \in \mathcal{D}_C(Y)_{\geq 1}$. To extend this estimate functorially, use the standard derived enhancement. The counit cocone on an object E is the derived tensor product of E with the bimodule

$$\mathcal{K}_j = \mathrm{Cone}(\mathcal{O}_Y \overset{L}{\otimes}_{\mathcal{O}_X} \mathcal{O}_Y \longrightarrow \mathcal{O}_Y)[-1].$$

It is an exact functor and preserves direct summands. Applying the estimate to the diagonal tilting generators, and then to their shifts, extensions and summands, proves

$$\mathrm{Cone}(Lj^*j_*E \to E)[-1] \in \mathcal{D}_C(Y)_{\geq b+1} \quad \text{if } E \in \mathcal{D}_C(Y)_{\geq b}. \tag{8.2}$$

Closed pushforward is co-t-exact on these supported categories, because it takes their explicit coheart generators to those in X.

Derived pullback satisfies

$$Lj^*(\mathcal{D}_C(X)_{\geq b}) \subset \mathcal{D}_C(Y)_{\geq b}.$$

For a generator u_*Q of the nonnegative part, the triangle $K_Q \to Lj^*u_*Q \to i_*Q \to$ has outside terms in $\mathcal{D}_C(Y)_{\geq 1}$ and $\mathcal{D}_C(Y)_{\geq 0}$, so its middle term lies in $\mathcal{D}_C(Y)_{\geq 0}$. Exactness, finite extensions, summands and shifts give the displayed inclusion.

For a class of objects $\mathcal{A}$, the symbols ${}^{\perp}\mathcal{A}$ and $\mathcal{A}^{\perp}$ mean vanishing of every Hom into, respectively out of, that class, with the shifts already included in the class.

We next prove reflection of perpendicular conditions for every $F \in D^b\,\mathrm{Coh}^{\Gamma}(Y)$. A failure of the right perpendicular at index b is witnessed by

$$\mathrm{Hom}_Y(i_*Q, F[n]\langle k\rangle) \neq 0, \qquad n+k \geq b. \tag{8.3}$$

If $\mathcal{H}^q(F)=0$ for $q<m$, then $n \geq m$ in every nonzero group (8.3), by ordinary cohomological orthogonality. If j_*F lies in the ambient perpendicular, ambient Hom vanishes, and the same

triangle lowers n while preserving $n+k \geq b$. This contradicts the lower bound. Conversely, for $E \in \mathcal{D}_C(X)_{\geq b}$, derived adjunction gives

$$\mathrm{Hom}_X(E, j_*F) = \mathrm{Hom}_Y(Lj^*E, F) = 0 \quad \text{if } F \in \mathcal{D}_C(Y)^{\perp}_{\geq b}.$$

This proves the right-perpendicular equivalence.

8.2. **Duality and lifting for arbitrary sources.** Put $n = \dim \mathrm{Lie}(G)$ and $r = \mathrm{rk}\, G$, and use the dualizing complexes

$$\omega_X = \mathcal{O}_X[r]\langle -n-r\rangle, \qquad \omega_Y = j^!\omega_X, \qquad \mathbb{D}_Y = R\mathcal{H}om_Y(-, \omega_Y).$$

These give bounded biduality on the coherent derived categories. We compute the normalization on C. Put $s = \dim C$, so $d_C = n - r - s$. The determinant of the adjoint representation of G is trivial; consequently the Kazhdan torus acts on $\det T_xX$ with weight $-2n$. Choose a nondegenerate invariant form $(\ ,\ )$ on $\mathrm{Lie}(G)$, which exists for $p > h$. The Kirillov symplectic form on C is

$$\beta_x([a,x],[b,x]) = (x,[a,b]).$$

For $\sigma_z = (\tau(z), z)$, invariance of the form gives $\beta_x(\sigma_z v, \sigma_z w) = z^{-2}\beta_x(v,w)$. Nondegeneracy pairs a torus weight a with weight $-2-a$, so the weight of $\det T_xC$ is $-s$. The G-stabilizer preserves β_x, and its determinant on T_xC is one. Thus these are equalities of full stabilizer characters. For the normal bundle $N_{C/X} = \mathcal{W}^\vee$ this gives

$$\det N_{C/X} = \mathcal{O}_C\langle 2n-s\rangle, \qquad u^!\mathcal{O}_X = \mathcal{O}_C[-n+s]\langle 2n-s\rangle.$$

The second formula is the regular-immersion duality formula. Transitivity now gives

$$i^!\omega_Y = u^!\omega_X = \mathcal{O}_C[-d_C]\langle d_C\rangle. \tag{8.4}$$

In particular, $\mathbb{D}_Y(i_*T[a]\langle -a\rangle) = i_*T^*[-d_C-a]\langle d_C+a\rangle$. Hence duality preserves the local coheart, exchanges $\geq b$ and $\leq -b$, and commutes with closed pushforward. It gives left-perpendicular reflection.

For $G' \in \mathcal{D}_C(Y)_{\leq a+1}$, put $E = \mathbb{D}_Y G'$. Then $E \in \mathcal{D}_C(Y)_{\geq -a-1}$, so its counit cocone lies in $\mathcal{D}_C(Y)_{\geq -a}$ by (8.2). Dualizing its triangle gives

$$\mathrm{Cone}(G' \to j^!j_*G') \in \mathcal{D}_C(Y)_{\leq a}.$$

Therefore

$$F \in {}^{\perp}\mathcal{D}_C(Y)_{\leq a}, \quad G' \in \mathcal{D}_C(Y)_{\leq a+1} \implies \mathrm{Hom}_Y(F, G') \twoheadrightarrow \mathrm{Hom}_X(j_*F, j_*G'). \tag{8.5}$$

The source F is any bounded coherent object: apply $\mathrm{Hom}(F,-)$ to the right-adjunction triangle, whose third term lies in $\mathcal{D}_C(Y)_{\leq a}$. Both Lj^* and $j^!$ are bounded on the ambient bounded coherent category: locally, every coherent $\mathcal{O}_X$-module has a finite locally free resolution because X is regular of finite dimension.

8.3. **A finite correction along one closed orbit.** Let $F \in D^b\,\mathrm{Coh}^\Gamma(Y)$ be arbitrary. The objects Lu^*j_*F and $u^!j_*F$ are bounded, because X is regular. Boundedness of the orbit co-t-structure, adjunction and reflection give integers $\alpha \leq -1$, $\beta \geq 1$ such that

$$F \in {}^{\perp}\mathcal{D}_C(Y)_{\leq \alpha} \cap \mathcal{D}_C(Y)^{\perp}_{\geq \beta}. \tag{8.6}$$

Suppose first that $\alpha < -1$. Put $A_F = Lu^*j_*F$. The lower bound in (8.6) says $A_F \in D^b\,\mathrm{Coh}^\Gamma(C)_{\geq \alpha+1}$. Choose a co-$t$-decomposition

$$G_1 \longrightarrow A_F \longrightarrow G_2 \longrightarrow, \qquad G_1 \geq \alpha+2, \quad G_2 \leq \alpha+1.$$

The rotated triangle and extension closure show that $G_2 \geq \alpha+1$ as well. Thus G_2 is pure of co-t-degree $\alpha+1$. The morphism

$$j_*F \longrightarrow u_*Lu^*j_*F \longrightarrow u_*G_2$$

lifts, by (8.5), to a morphism $F \to i_*G_2$ on Y. Define F' by the triangle

$$F' \longrightarrow F \longrightarrow i_*G_2 \longrightarrow . \tag{8.7}$$

It agrees with F on the entire invariant open $Y \setminus C$. The object $i_*G_2[-1]$ has pure co-t-degree $\alpha + 2$. Since $\alpha + 2 \leq \beta - 1$, the rotated triangle preserves both bounds in (8.6).

It remains to improve the lower bound at its new boundary. Let $H_C = \mathcal{T}[n]\langle k\rangle$ have pure degree $\alpha + 1$, so $n + k = -\alpha - 1$. The truncation triangle gives the exact sequence

$$\mathrm{Hom}_C(G_2, H_C) \longrightarrow \mathrm{Hom}_C(A_F, H_C) \longrightarrow \mathrm{Hom}_C(G_1, H_C) = 0.$$

Every morphism $j_*F \to u_*H_C$ therefore factors through the chosen map $j_*F \to u_*G_2$. Indeed, adjunction identifies it with a map $A_F \to H_C$, which lifts to a map $G_2 \to H_C$. After pushing the latter forward, one obtains the factorization. Consequently the first arrow in the following exact segment is surjective:

$$\begin{aligned}\mathrm{Hom}_X(u_*G_2, u_*H_C) \longrightarrow \mathrm{Hom}_X(j_*F, u_*H_C) \longrightarrow \mathrm{Hom}_X(j_*F', u_*H_C)\\ \longrightarrow \mathrm{Hom}_X(u_*G_2[-1], u_*H_C) = 0.\end{aligned}$$

The last vanishing is co-t-orthogonality, since its source has degree $\alpha + 2$ and its target degree $\alpha + 1$. Thus $\mathrm{Hom}_X(j_*F', u_*H_C) = 0$ for every boundary generator. Together with the old lower bound, this gives the improved ambient perpendicular condition; reflection brings it back to Y:

$$F' \in {}^{\perp}\mathcal{D}_C(Y)_{\leq \alpha+1} \cap \mathcal{D}_C(Y)^{\perp}_{\geq \beta}.$$

Repeat until $\alpha = -1$. Duality changes the remaining bounds to lower bound $-\beta$ and upper bound 1. The same construction raises the former one step at a time; dualizing back lowers β to 1 while preserving $\alpha = -1$. Each operation decreases $(-1 - \alpha) + (\beta - 1)$, so the process terminates. We obtain an object, still on Y, with

$$F^{\mathrm{corr}} \in {}^{\perp}\mathcal{D}_C(Y)_{\leq -1} \cap \mathcal{D}_C(Y)^{\perp}_{\geq 1}, \qquad F^{\mathrm{corr}}|_{Y\setminus C} \cong F|_{Y\setminus C}. \tag{8.8}$$

9. Constructing a reduced source for each ambient silting object

Let $Z = \overline{C_0}_{\mathrm{red}}$ and let $\mathcal{T}$ be an indecomposable tilting bundle on C_0. Start with $\mathcal{T}[-d_{C_0}/2]\langle d_{C_0}/2\rangle$ on the open orbit. Choose an increasing chain of invariant opens from C_0 to Z, adding one orbit at each step so the new orbit is closed in the new open.

A bounded equivariant coherent complex extends from an invariant open of a noetherian quasi-affine scheme. Indeed, let $h : V \hookrightarrow Y$ be that open immersion and let $F^\bullet$ be the complex on V. The terms h_*F^q are quasicoherent. Quasi-affineness supplies finitely many global sections whose restrictions generate F^q; rationality of the group action places them in finite-dimensional stable spaces. Let $A^q \subset h_*F^q$ be the coherent equivariant subobject they generate. Define

$$E^q = A^q + d(A^{q-1}) \subset h_*F^q.$$

Then $d(E^q) \subset E^{q+1}$ since $d^2 = 0$. These terms form a bounded equivariant coherent complex restricting to $F^\bullet$. Extend in this way at each step and apply the correction of Section 8 along the new orbit.

The ambient neighborhood can always be chosen smooth. If the new open is $U \subset Z$, let $D = Z \setminus U$, closed in $\mathrm{Lie}(G)$, and take $X = \mathrm{Lie}(G) \setminus D$. Then U is closed in X. If C is closed in U, its boundary is in D, so C is closed in X as required. Write $j : U \hookrightarrow X$ for this closed immersion. The normal-cone coefficients are unchanged by this restriction.

Each correction preserves the complex on the entire previous invariant open, and therefore preserves all derived orbit conditions already obtained there. Write F for the corrected complex on U. For the new orbit write $u_C : C \hookrightarrow X$. Equation (8.8), reflection and ambient adjunction give

$$Lu_C^* j_*F \in D^b \mathrm{Coh}^\Gamma(C)_{\geq 0}, \qquad u_C^! j_*F \in D^b \mathrm{Coh}^\Gamma(C)_{\leq 0}.$$

At the initial open orbit, its pushed balanced tilting bundle is in the ambient local coheart. By adjunction, Lu^* sends the nonnegative supported part to the nonnegative orbit part, while $u^!$ sends the nonpositive supported part to the nonpositive orbit part. These two single-sided consequences of co-t-exactness give the initial inequalities.

After finitely many steps we have $F_Z \in D^b \operatorname{Coh}^\Gamma(Z)$. For $q : Z \hookrightarrow \mathcal{N}$, its pushforward satisfies the ambient orbit inequalities at every orbit of Z, and both restrictions vanish at orbits outside Z. Hence [4, Lemma 6.10 and Theorem 7.2] put q_*F_Z in the established supportive coheart.

Decompose q_*F_Z into indecomposables $\mathcal{S}_{\mathcal{N}}(C, \mathcal{T}')[a]\langle -a\rangle$ in the ambient coheart. Each summand has support contained in Z. On the invariant open $V = \mathcal{N} \setminus (Z \setminus C_0)$, every summand labelled by a boundary orbit vanishes. A summand with label C_0 restricts to

$$i_{0*}\mathcal{T}'[-d_{C_0}/2 + a]\langle d_{C_0}/2 - a\rangle, \qquad i_0 : C_0 \hookrightarrow V.$$

This object is nonzero and indecomposable: its degree-zero endomorphism ring is that of the indecomposable orbit bundle $\mathcal{T}'$. The total restriction is the single indecomposable $i_{0*}\mathcal{T}[-d_{C_0}/2]\langle d_{C_0}/2\rangle$. Hence exactly one summand survives, with multiplicity one. Equality of its unique nonzero ordinary cohomology degree gives $a = 0$, and equality of the remaining sheaves gives $\mathcal{T}' = \mathcal{T}$. The surviving summand is therefore $\mathcal{S}_{\mathcal{N}}(C_0, \mathcal{T})$.

Every ordinary cohomology sheaf of q_*F_Z is annihilated by I_Z, since q_* is exact and F_Z is defined on the reduced scheme Z. Its direct summands inherit annihilation. This proves (1.4).

Apply this construction to the silting complex corresponding to T_λ, whose reduced support is Z_λ. Since $\mathcal{N}$ is affine, the cohomology of its global sections is the global sections of its ordinary cohomology sheaves. The comparison [4, Theorem 8.1] is compatible with the R-module action, so $H^\bullet(\mathbf{G}_1, T_\lambda)$ is annihilated by I_{Z_λ}. Its known ordinary support is Z_λ. Therefore

$$I_{Z_\lambda} \subset \operatorname{Ann}_R H^\bullet(\mathbf{G}_1, T_\lambda) \subset \sqrt{\operatorname{Ann}_R H^\bullet(\mathbf{G}_1, T_\lambda)} = I_{Z_\lambda},$$

proving Theorem 1.1 for the simple types treated above.

10. Type A, reductive groups, and the traditional conclusion

For GL_N and $p > N = h_{A_{N-1}}$, the relative reduced-support statement is [3, Theorem 7.6]. To pass to SL_N, use $\mathrm{SL}_N \times \mathbb{G}_\mathrm{m} \to \mathrm{GL}_N$ with kernel μ_N. An SL_N tilting can be tensored with a torus character chosen to have trivial kernel action. When considering nonzero first-kernel cohomology, choose this character also divisible by p; the two congruence conditions are compatible because $p \nmid N$. Torus first kernels are diagonalizable and have exact invariants. This identifies the relevant cohomology with the GL_N case and preserves the nilpotent cone and its reduced orbit ideals.

For a general connected reductive group, take the central isogeny from a torus times the simply connected simple factors. Its finite kernel can be chosen of order prime to p under $p > h$. The induced map on first Frobenius kernels is an isomorphism. Pullback of an indecomposable tilting is a torus character times an external tensor product of indecomposable tiltings. Künneth gives the tensor product of their cohomology modules (or zero if the torus character has no first-kernel invariants). The tensor product is annihilated by the ideal of the product of the reduced factor orbit closures. That product is reduced over the algebraically closed field. Ordinary relative support identifies this closed set with the orbit label for the original root datum, and hence identifies its unique reduced ideal. This proves Theorem 1.1 in the full stated scope.

To prove Corollary 1.2, let T be any finite-dimensional tilting module and decompose

$$T^* \otimes T \cong \bigoplus_i D_i$$

into indecomposable tiltings. Tensor–Hom adjunction gives a natural R-module isomorphism

$$\operatorname{Ext}^\bullet_{\mathbf{G}_1}(T, T) \cong \bigoplus_i H^\bullet(\mathbf{G}_1, D_i).$$

The R-action on the left is given by tensoring a unit extension with T; tensor–Hom adjunction carries it to cup product on the right, proving the asserted R-linearity. Write $D_i = T(\mu_i)$. The vanishing statement preceding [4, Theorem 8.1] gives $H^\bullet(\mathbf{G}_1, D_i) = 0$ unless $\mu_i = w_\lambda \bullet 0$ for a dominant λ. Thus Theorem 1.1 covers every nonzero summand, and its annihilator is radical. The annihilator of a finite direct sum is the intersection of the annihilators. If a power of f lies in every one of these radical ideals, then f lies in every one of them. Their intersection is therefore radical, so

$$\operatorname{Ann}_R \operatorname{Ext}^\bullet_{\mathbf{G}_1}(T,T) = \sqrt{\operatorname{Ann}_R \operatorname{Ext}^\bullet_{\mathbf{G}_1}(T,T)}.$$

This proves Corollary 1.2. By the ordinary support theorem [21, Theorem 1.1], namely (1.3), the radical is $I_{\overline{C(w)}_{\mathrm{red}}}$, proving Corollary 1.3. The orbit in (1.3) is expressed on $\mathcal{N}$ using the root-datum identification of reduced nilpotent orbit closures for G and $\mathbf{G}$. Thus both annihilator ideals are taken in the ring (1.1). The finite-sum reduction above is the argument of [3, Corollary 7.7].

Appendix A. Exact finite verification

The calculations in Section 5 involve finitely many root systems and rational vertices. They establish bounds for every highest weight and every larger characteristic because of (5.5)–(5.7). This appendix specifies the finite verification. The complete program `verify_scheme_bounds.py` and all 37 rational certificates in `e8_dual_certificates.json` are embedded in the two `filecontents*` blocks at the beginning of this article's LaTeX source. Compiling that source extracts both files into the working directory, replacing any existing copies with the embedded versions. No separate data download is needed.

These files form part of the finite proof. The program regenerates all other finite inputs from the root data below and uses only the standard library of Python 3.10 or later. All inputs are supplied; the program requires no interactive input. In a directory containing the source, run

```
pdflatex Humphreys_scheme_theoretic.tex
python3 verify_scheme_bounds.py > finite_verification.json
```

The first command extracts the files; the second verifies every finite inequality and records the results. A successful run reports `"status": "PASS"`. Run Python without `-O`, since the exact checks use assertions. The LaTeX compilation does not execute Python and requires no shell escape. The preceding sections prove the representation-theoretic and geometric consequences of these finite checks.

A.1. **Root data and affine faces.** Use a Cartan matrix $A = (\langle \alpha_j, \alpha_i^\vee \rangle)$ with diagonal entries 2. Unlisted off-diagonal entries are zero. For A_n, consecutive vertices have entries $(-1,-1)$. For B_n and C_n, use the same chain, with final ordered pair respectively $(-1,-2)$ and $(-2,-1)$. For D_n, the chain $1,\dots,n-2$ has two final neighbors $n-1, n$ at $n-2$. For E_n, the central node 1 has arms

$$(1,2), \qquad (1,3,4), \qquad (1,5,6,\dots,n).$$

For F_4, the chain has ordered pairs $(-1,-1),(-1,-2),(-1,-1)$; for G_2 use $(-1,-3)$. All simple edges not specified otherwise are symmetric. Finite nodes are numbered $1,\dots,n$, and the affine node is 0. The face mask uses bit i for node i; the record's vertex-node field uses the same numbering. Cartan-matrix and numerator-vector arrays store finite node i at index $i-1$.

Generate the roots by closing the simple roots under all simple reflections; generate the coroots by using A^{T}. Keep the vectors with nonnegative simple-root coordinates. This produces the entire positive system, since every positive nonsimple root is lowered by a simple reflection. In particular the positive-coroot counts for E_6, E_7, E_8, F_4, G_2 are $36, 63, 120, 24, 6$. The highest coroot gives the marks m_i and $h = 1 + \sum_i m_i$.

For each nonempty proper face J, calculate the truncated $\ell_0(J)$ by the dominant-alcove enumeration in Section 5. The cutoff is $\lceil S_\rho / p_0 \rceil$; a larger cutoff may be used without changing

the result. Every enumerated alcove is checked both by its wall-support coefficients and by its hyperplane length. For every nonautomatic face and every $k \notin J$, construct $y_{J,k}(p_0)$ exactly. A common denominator is m_k. All subsequent residue operations are performed on integer numerators.

A.2. **Certificates for E_8.** For each vertex let $q_\gamma = \langle y, \gamma \rangle$ for a positive coroot γ. If $0 < q_\gamma < p_0$, its contribution after a finite Weyl action is either q_γ or $p_0 - q_\gamma$. If q_γ is 0 or p_0, the contribution is p_0 in both cases. Thus the corrected residue sum is a constant plus a linear function of inversion indicators.

Order the positive coroots lexicographically. List all triples (i, j, k) with $i < j$ and $\gamma_i + \gamma_j = \gamma_k$ in lexicographic order of (i, j), using zero-based array indices. There are 1120 such triples for E_8. A certificate entry $[t, a, b]$ assigns the positive multiplier a/b to $z_i + z_j - z_k \geq 0$ for triple t. The verifier checks the root addition, reconstructs the objective from the vertex, and applies (5.8). All supplied multipliers use these one-sided inequalities.

At $p_0 = 31$, exactly 37 of the 1034 required E_8 vertices have rootwise corrected margin less than -128. The rational file contains exactly those vertices. Each record includes the affine face, its vertex index, the common denominator, the numerator vector, the face correction, and the nonzero rational multipliers. The verifier checks equality of this record set with the independently generated exceptional set, proving completeness of the certificate list. The 37 records contain 3281 nonzero rational multipliers. Every certified margin is at least -128; every other vertex already satisfies this bound rootwise. This supplies $D = 128$ in Section 6.1: at $p = 31$ the height estimate covers $d \geq 33$ and the norm estimate covers $d \leq 32$. The embedded certificate data, extracted as `e8_dual_certificates.json`, have SHA-256 digest

`3f8f41f5522385b769a7fa300d8d9751b5dd70c2c483bb73d9f48ace7bbacdc1`

A.3. **The remaining checks.** For E_6 at 13 and E_7 at 19, the minimum rootwise corrected margins are -14 and -58. At 17 and 29 the respective margins are nonnegative. The 24 proper-centralizer classical inputs are: A_1 at 7; A_n for $1 \leq n \leq 7$; B_n, C_n for $2 \leq n \leq 7$; and D_n for $4 \leq n \leq 7$, where the threshold is 13 for ranks at most five, 19 for rank six, and 31 for rank seven. All 1264 required vertices are checked rootwise. The only negative minima are the three specified in Section 6.3.

The F_4 and G_2 Weyl groups are generated by exact reflection matrices. Their orders are checked to be 1152 and 12. For F_4, the atomic-height inequality is checked on every element. For G_2, the regular residue inequality is checked at $p = 7$ and at the vertices of the $p = 8$ relaxation, as described in the proof. The singular check evaluates all Weyl images of every one of the 41 and 6 required vertices. These are exhaustive finite checks, and the monotonicity proved above extends their conclusions to all larger values of p.

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