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A PROOF OF LUSZTIG'S CONNECTEDNESS CONJECTURE IN FINITE TYPE

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Abstract. We prove that every equal-parameter Kazhdan–Lusztig left cell in a finite Weyl group is connected by simple left multiplications.

1. Introduction

Kazhdan and Lusztig introduced cells in a Coxeter group through the multiplication of their distinguished basis of its Hecke algebra [12]. A left cell indexes a basis of a representation, and the corresponding W-graph records coefficients of Kazhdan–Lusztig polynomials. Lusztig's work on special representations and families relates these representations to the characters of reductive groups over finite fields [14]. Alongside this representation theory, cells raise a question about the original generators of the Coxeter group: can two elements of a left cell be joined by successive simple left multiplications inside that cell?

Let (W, S) be a finite Weyl group with its standard Coxeter generators, and let Γ be a Kazhdan–Lusztig left cell for the equal-parameter Hecke algebra. Give Γ the undirected graph with edges

$$\{w, sw\}, \qquad s \in S, \quad w, sw \in \Gamma.$$

The cell is *left-connected* if this graph is connected. Lusztig proposed that every left cell of a finite or affine Weyl group has this property in the problems of the 1983 Katata symposium [13, p. 14]. The simple-left formulation appears in [8, Conjecture 7.2]; its equivalent right-handed form appears in [11, Conjecture 3.19 of the 2019 version]. We prove the finite-type assertion.

Theorem 1.1. *Every equal-parameter Kazhdan–Lusztig left cell of a finite Weyl group is left-connected. More explicitly, if x, y belong to the same left cell Γ, there are $s_1, \dots, s_m \in S$ such that*

$$y = s_m \cdots s_1 x, \qquad s_j \cdots s_1 x \in \Gamma \quad (0 \le j \le m).$$

The proof is by classification. In type A, Kazhdan and Lusztig identify left cells with the fibers of the recording tableau in the Robinson–Schensted correspondence [12, §5]. Their star operations connect each fiber. A left star for two simple generators whose product has order three sends w to one of its simple left multiples, so these operations give the required paths. The dihedral case is also explicit: apart from the identity and the longest element, each left cell consists of alternating words with a fixed rightmost generator [10, §7.15].

For the other classical groups, Garfinkle developed domino insertion, cycles and generalized τ-invariants in three papers published in 1990, 1992 and 1993 [1, 2, 3]. Domino insertion associates a pair of standard domino tableaux to a signed permutation. Moving through open cycles brings a tableau to a special shape, and local tableau operations describe the changes induced by the Weyl-group operators. The resulting classification, together with the relation between primitive

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ideals and cells [14, §5.25], gives the equal-parameter types B and C; this consequence of [3, Theorem 3.5.9] is recorded explicitly in [8, Example 7.3]. McGovern subsequently described left-cell representations in terms of domino tableaux and the special symbols of the classical Weyl groups [15].

There was also progress for specified classes of cells in finite and affine groups. Shi proved connectedness for cells containing a generalized Coxeter element in 2002 [17]. In 2006 he extended this to cells containing a fully commutative element, that is, an element whose reduced expressions are connected using only commutations of adjacent commuting generators [18]. His proof gives a distinguished minimal element of such a cell and a length-additive factorization of every other element through it. Further criteria for left-connectedness in simply-laced finite and affine Weyl groups were obtained in [19].

Mi and Shi proved that every left cell in E_6 is left-connected in 2013 [16]. Geck and Halls then verified connectedness for all exceptional finite Coxeter types in their work on the cells of E_8 [8, Example 7.3]. Their 2015 account lists type D_n as the remaining finite family. Jensen's 2019 preprint records the same status [11, Conjecture 3.19 of the 2019 version]. Thus the classification proof reduces to the following statement.

Theorem 1.2. *For every $n \geq 4$, every equal-parameter Kazhdan–Lusztig left cell in $W(D_n)$ is left-connected.*

The type D operator theory begins with the branched-diagram operations of Garfinkle and Vogan [5]. The diagram has a subsystem of type D_4, with a central simple root adjacent to three outer roots. Its operators supplement the ordinary stars and can have one or two outputs. Garfinkle Johnson developed their Kazhdan–Lusztig edge-transport theory in 2019 [4]. Her Figures 7–9 display the relevant rank-four left cells, and Propositions 4.7–4.8 describe their translates in a parabolic coset. The edges marked by simple generators connect each displayed cell. Consequently every branched operator can be realized by a simple-left path inside its parabolic cell translate. We use the reciprocal four-word formulas of [4, Proposition 10.4].

Pietraho, McGovern and Garfinkle supplied the type D tableau transitivity theorem in their 2023 preprint, revised in 2026 [9]. In the notation developed below, it joins two tableau pairs with equal special recording tableau and equal total vertical count modulo four. This is Theorem 4.5.2 of versions 1 and 3, and Theorem 4.6.2 of version 2. Together with operator compatibility for the reciprocal formulas above, the simple-left paths in [4], and the classical special-representation count, this theorem yields Theorem 1.2. The proof presented here combines these ingredients and gives the tableau induction with explicit coordinates and finite verification data.

The induction has three statements: normalization of the second tableau, placement of the largest domino, and transitivity. When the largest first-tableau domino is deleted, reverse insertion in the second tableau recovers an insertion position and a sign. Retaining both makes the lift of a smaller-rank endpoint unique. For placement at an unboxed corner, the auxiliary first tableau has vertical count two modulo four above the reduced first tableau, as in [9, version 2, Lemma 4.6.8]; the subsequent cycle move cancels this difference. The coordinate tables give all normalization and placement cases, including their residue conditions.

To pass from tableaux to left cells, we count both independently. Every special irreducible representation occurs once in each left-cell representation in its two-sided cell. The regular representation therefore gives the number of left cells as the sum of the dimensions of the special irreducibles [7, §2 and Theorems 2.8, 4.6–4.7]. The special-symbol parametrization and the domino-tableau counting formula give the same sum for special tableaux. Transitivity places each nonempty special-recording fiber in one connected component of the simple-left graph. Equality of the two counts then gives exactly one component in each left cell.

Section 2 gives the group operations and their paths. Section 3 proves deletion and lifting, and Sections 4–6 give normalization, placement, transitivity and the count. Section 7 completes the finite-type classification. The finite inputs, verified in Section 8, consist of local operator compatibility, bases of rank at most four, two rank-seven normalization configurations, and four fixed corner-transfer shapes.

2. Cells and paths

2.1. **Conventions.** Write $\ell(w)$ for Coxeter length and $\le$ for Bruhat order: $x \le w$ means that a reduced expression for x occurs as a subword of a reduced expression for w. For $I \subseteq S$, put $W_I = \langle I \rangle$; for $a, b \in S$, let $m(a,b)$ be the order of ab. Put $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. The equal-parameter Hecke algebra has basis $\{T_w : w \in W\}$ and multiplication

$$T_s T_w = \begin{cases} T_{sw}, & \ell(sw) > \ell(w), \\ T_{sw} + (v - v^{-1})T_w, & \ell(sw) < \ell(w). \end{cases}$$

The bar involution sends v to v^{-1} and T_w to $T_{w^{-1}}^{-1}$. Its invariant basis with $C'_w - T_w \in \sum_{x<w} v^{-1}\mathbb{Z}[v^{-1}]T_x$ is the Kazhdan–Lusztig basis, with expansion

$$C'_w = \sum_{x \le w} v^{\ell(x)-\ell(w)} P_{x,w}(v^2) T_x.$$

Thus $P_{w,w} = 1$, and $\deg P_{x,w} \le (\ell(w) - \ell(x) - 1)/2$ for $x < w$. The left preorder is the reflexive transitive closure of the relations $x \leftarrow_L y$ for which C'_x has nonzero coefficient in $C'_s C'_y$ for some $s \in S$. Denote it by $\le_L$ and write $x \sim_L y$ when both $x \le_L y$ and $y \le_L x$ hold. Its equivalence classes are the left cells. The two-sided preorder is defined in the same way using coefficients in both $C'_s C'_y$ and $C'_y C'_s$; its associated equivalence classes are the two-sided cells. A subscript I means that the same definitions are taken in (W_I, I). Write $\mathrm{Des}_L(w) = \{s \in S : \ell(sw) < \ell(w)\}$.

Lemma 2.1. *Let $I \subseteq S$, let z be the minimum-length representative of a coset $W_I z$, and let $u_0, \dots, u_m$ be a simple-left path inside a left cell of W_I. Then $u_0 z, \dots, u_m z$ is a simple-left path inside a left cell of W.*

Proof. The length formula $\ell(uz) = \ell(u) + \ell(z)$ gives $\mathrm{Des}_L(uz) \cap I = \mathrm{Des}_L(u)$ for $u \in W_I$. Parabolic compatibility gives

$$u \le_{L,I} v \implies uz \le_L vz;$$

one may use [6, Lemma 4.3]. In both directions this implies $u \sim_{L,I} v \Rightarrow uz \sim_L vz$. Finally, multiplication of a vertex by a generator $s \in I$ commutes with appending the common right factor z. □

For $a, b \in S$ with $m(a,b) = 3$, define the ordered star $\sigma_{a,b}$ on

$$a \notin \mathrm{Des}_L(w), \qquad b \in \mathrm{Des}_L(w)$$

to be the unique member of $\{aw, bw\}$ whose descents contain a and not b. The inverse is $\sigma_{b,a}$. In $W(A_2)$ the nonsingleton left cells are $\{a, ba\}$ and $\{b, ab\}$, and the stars exchange the two elements of either set. Lemma 2.1 therefore puts w and $\sigma_{a,b}(w)$ in the same left cell in every rank. Each star is one edge of the required graph.

2.2. **The branched operations.** Realize $W(D_n)$ as the signed permutations of $1,\dots,n$ with an even number of negative entries. A signed permutation is extended by $w(-i) = -w(i)$. Writing $e_1,\dots,e_n$ for the standard basis of $\mathbb{R}^n$, use the simple roots

$$\alpha_1' = e_1 + e_2, \qquad \alpha_i = e_i - e_{i-1} \quad (2 \le i \le n).$$

For a simple root α, write s_α for its reflection; thus $s_1' = s_{\alpha_1'}$ and $s_i = s_{\alpha_i}$. For every signed permutation, including the odd-sign coset, define

$$\begin{aligned} s_i \in \mathrm{Des}_L(w) &\iff w^{-1}(i) < w^{-1}(i-1) \qquad (i \ge 2), \\ s_1' \in \mathrm{Des}_L(w) &\iff w^{-1}(1) + w^{-1}(2) < 0. \end{aligned} \tag{2.1}$$

The signed inverse positions are ordered as integers. On $W(D_n)$ these conditions are the Coxeter descents, since they say that $w^{-1}(\alpha_i)$ or $w^{-1}(\alpha_1')$ is a negative root. The same descent predicates define the stars and branched relations on both sign-parity classes. To see this, let t change the sign of e_1 and fix the other coordinates. It preserves the positive roots $e_j - e_i, e_j + e_i$ for $i < j$. Every odd-sign permutation is uniquely ut with $u \in W(D_n)$, and $\mathrm{Des}_L(ut) = \mathrm{Des}_L(u)$, while $s(ut) = (su)t$. Right multiplication by t therefore transports all the defining relations from the even-sign class.

The generator s_1' exchanges the values $1, -2$ and $2, -1$, while s_i exchanges $i-1, i$ and their negatives. All multiplications below are on values, hence on the left. In the standard D_4 subsystem let $c = s_3$ be the central generator. Choose any $b \in \{s_1', s_2, s_4\}$ and denote the other two outer generators by g, d. We use the relations for all three choices of b.

For this paragraph, descents are restricted to $I = \{b, c, g, d\}$. Define the following types; the stars in the last column are all defined under the descent condition in the middle column.

Type	$\mathrm{Des}_L(w) \cap I$	Additional condition
A_b	$\{b, c\}$	$\sigma_{g,c}(w) \ne \sigma_{d,c}(w)$
B_b	$\{g, d\}$	$\sigma_{c,g}(w) \ne \sigma_{c,d}(w)$
C	$\{b, g, d\}$	$\sigma_{c,b}(w) = \sigma_{c,g}(w) = \sigma_{c,d}(w)$
D	$\{c\}$	$\sigma_{b,c}(w) = \sigma_{g,c}(w) = \sigma_{d,c}(w)$

For an input of type A_b (respectively B_b), retain the elements of type D (respectively C) among

$$bw, \quad cgdw, \quad cdcw, \quad cgcw. \tag{2.2}$$

For an input of type D (respectively C), retain the elements of type A_b (respectively B_b) among

$$bw, \quad gdcw, \quad cdcw, \quad cgcw. \tag{2.3}$$

These are reciprocal relations with one or two values. The order gdc in (2.3) is part of the definition: it is the inverse of cgd. These are the formulas of [4, Proposition 10.4]. Exchanging g and d leaves the predicates and candidate sets unchanged.

Lemma 2.2. *For $w \in W(D_n)$, every arrow in (2.2)–(2.3) can be replaced by a path of simple left multiplications inside one left cell. The executing labels are as follows, where $e \in \{g, d\}$.*

Multiplier	*Input type*	*Executing labels*
b	*Any of the applicable types*	b
cgd	A_b, B_b	d, g, c
gdc	C, D	c, d, g
cec	A_b, D	c, e, c
cec	B_b, C	e, c, e

An executing list (r_1, r_2, r_3) means $w, r_1w, r_2r_1w, r_3r_2r_1w$.

Proof. In $W(D_4)$ the finite verification of Proposition 8.1(1) checks the cell of every intermediate vertex for every arrow. There are 192 directed arrows, of which 84 use one edge and 108 use three. The last row has the required endpoint because $cec = ece$; its choice of braid orientation is necessary.

For arbitrary rank write $w = uz$ with $u \in W_I$ and z the minimum-length element of $W_I w$. The type predicates and the candidate multipliers use only I, and $\mathrm{Des}_L(uz) \cap I = \mathrm{Des}_L(u)$. Consequently the selected arrows are precisely the corresponding rank-four arrows with z appended. Apply Lemma 2.1 to their fixed paths. □

3. Domino operators and deletion

We use the rank-zero domino Robinson–Schensted correspondence, with the convention that left multiplication acts on values. Thus

$$\mathcal{R}(w) = (P(w), Q(w)),$$

where P is the insertion tableau and Q the recording tableau. The signed-pair input to Garfinkle's insertion is $\{(|w(i)|, i, \mathrm{sgn}(w(i))) : 1 \le i \le n\}$: the first coordinate is a value and the second a position [1, Definitions 1.1.3 and 1.2.7]. This is the signed input assigned to w^{-1} in Garfinkle's original convention. Thus the first-tableau operators below correspond to left multiplication.

3.1. **Tableaux and cycles.** Identify a partition $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots)$ with the Young diagram $\{(r, c) : 1 \le c \le \lambda_r\}$. Its size is $|\lambda| = \sum_r \lambda_r$, and its conjugate has parts $\lambda^t_c = \#\{r : \lambda_r \ge c\}$. Parts beyond the diagram are zero. A standard domino tableau of size n is a Young diagram of size $2n$ tiled by dominoes labelled $1, \ldots, n$, each label occurring in its two squares, with entries weakly increasing along rows and columns. Equivalently, the squares with labels at most k form a Young diagram for every k. Write $D_k(T)$ for the position of label k, $\mathrm{sh}(T)$ for the shape, and $\mathrm{v}(T)$ for the number of vertical dominoes. In one-based coordinates put

$$H(r, c) = \{(r, c), (r, c + 1)\}, \qquad V(r, c) = \{(r, c), (r + 1, c)\}.$$

A position is *removable*, or *extremal*, if its removal leaves a Young diagram. In the D grid a square (r, c) is fixed when $r \equiv c \pmod 2$. Every domino has one fixed square. A horizontal domino $H(r, c)$ is boxed when c is even; a vertical domino $V(r, c)$ is boxed when r is odd. The other dominoes are unboxed.

Here is an explicit form of the cycle rule. Let $a_T(r, c)$ be the entry in (r, c), with value 0 if $r \le 0$ or $c \le 0$, and value $+\infty$ at a missing square with positive coordinates. Suppose that the fixed square of $D_k(T)$ is (r, c). Its alternate position $D'_k(T)$ contains (r, c) and the square specified by the following table.

other square of $D_k(T)$	condition	other square of $D'_k(T)$
$(r + 1, c)$ or $(r, c - 1)$	$k < a_T(r - 1, c + 1)$	$(r - 1, c)$
$(r + 1, c)$ or $(r, c - 1)$	$k > a_T(r - 1, c + 1)$	$(r, c + 1)$
$(r - 1, c)$ or $(r, c + 1)$	$k < a_T(r + 1, c - 1)$	$(r, c - 1)$
$(r - 1, c)$ or $(r, c + 1)$	$k > a_T(r + 1, c - 1)$	$(r + 1, c)$

(3.1)

Equality in the middle column cannot occur. A cycle is an equivalence class of labels for the equivalence relation generated by $D'_i(T) \cap D_j(T) \ne \varnothing$. Moving through a cycle replaces $D_k(T)$ by $D'_k(T)$ for all its labels. Denote this operation by $E(T, c)$, and write $E(T, \mathcal{C})$ for simultaneous movement through a set $\mathcal{C}$ of cycles. A cycle is closed if it leaves the shape unchanged. Otherwise it is open: it removes one square $b(c)$ of the old shape and adds one square $f(c)$ outside that shape. Moves through distinct cycles commute and are involutions [1, Propositions 1.5.28–1.5.33].

For a pair $X = (P, Q)$ of equal shape, the extended cycle through a label k in P is defined as follows. If its cycle in P is closed, move that cycle and leave Q fixed. Otherwise take the

smallest collection of open cycles in the two tableaux which contains the cycle through k and is closed under sharing a back or forward boundary square. Move all these cycles simultaneously. The two resulting shapes agree. Write E_kX for this pair operation and $\mathrm{ec}_P(k;Q)$ for its set of labels on the first side. This boundary description is the extended-cycle definition of [2, Section 2.3].

A D-special partition has an even number of nonzero rows and, for every a, the pair of lengths $(\lambda_{2a-1},\lambda_{2a})$ is either equal or consists of two odd integers. A tableau is special when its shape is special. Every cycle consists entirely of boxed or entirely of unboxed dominoes. Moving through all open unboxed cycles of T gives its special representative $\mathsf{S}(T)$ [1, Propositions 1.5.22, 1.5.24 and Theorem 1.5.35]. In particular, $\mathsf{S}(T)=T$ for a special tableau. We shall use the classical open-cycle description

$$\mathsf{S}(U)=\mathsf{S}(V) \quad\Longleftrightarrow\quad V=E(U,\mathcal{C}) \text{ for a set of open cycles } \mathcal{C} \text{ of } U, \tag{3.2}$$

proved for types B,C,D in [3, Proposition 3.1.9].

Lemma 3.1. *If* $\mathsf{S}(U)=\mathsf{S}(V)$ *and* $\mathrm{sh}(U)=\mathrm{sh}(V)$*, then* $U=V$*.*

Proof. Use (3.2). Each moved open cycle removes a square of the original shape and adds a square outside it. Distinct cycles have distinct removed squares. A nonempty set of such moves therefore changes the shape. The set must be empty. □

3.2. **Insertion data.** The correspondence $\mathcal{R}$ is a bijection from signed permutations of size n to equal-shape pairs of standard domino tableaux of size n, with inverse given by reverse insertion [1, Proposition 1.2.12 and Theorem 1.2.13]. Its symmetry under inversion is

$$\mathcal{R}(w^{-1})=(Q(w),P(w))$$

by [1, Proposition 1.2.3]. We retain the full reverse-insertion data when deleting a tableau pair.

For $X=(P,Q)$ let $D=D_n(P)$, set $\bar{P}=P\setminus D$, and reverse-insert from Q at the extremal position D:

$$\Delta_nX=(\bar{P},\bar{Q};v,\epsilon), \qquad (\bar{Q},v,\epsilon)=\beta(Q,D), \qquad v\in\{1,\dots,n\}, \quad \epsilon\in\{1,-1\}. \tag{3.3}$$

Under inverse Robinson–Schensted this deletes the entry ϵn at position v. The remaining recording labels are order-standardized; v remembers the omitted label. Reconstruction reverses this standardization, inserts (v,ϵ) into $\bar{Q}$, and returns both Q and the added extremal position D. Consequently

$$(\bar{Q},v,\epsilon) \text{ determines } (Q,D) \text{ uniquely.} \tag{3.4}$$

Together with $\bar{P}$, it determines the full pair X.

Deletion of the largest label does not change $D'_k(P)$ for $k<n$: the only comparison in (3.1) that can change is with an entry n, replaced after deletion by $+\infty$; both are larger than k. Moreover, every extended cycle in the reduced pair is the nonempty restriction of an extended cycle of the full pair. This lift is unique, since the full extended cycles partition the first-side labels. If an extended cycle is not the singleton $\{n\}$, moving through it commutes with (3.3) and preserves both v and ϵ. These are [2, Proposition 2.2.4 and Proposition 2.3.3(b),(c)]; their statements include type D. The same insertion correspondence gives

$$\mathrm{v}(P(w))+\mathrm{v}(Q(w))\equiv 2\#\{i:w(i)<0\} \pmod 4. \tag{3.5}$$

Thus both congruence classes 0 and 2 occur in the induction, although $W(D_n)$ itself has class 0; see [9, Corollaries 4.1.4–4.1.6].

We also record the insertion–transpose identity

$$\mathcal{R}(-w)=(P(w)^t,Q(w)^t), \tag{3.6}$$

where $-w$ negates every signed value and T^t is the tableau obtained by interchanging row and column coordinates [2, proposition following Notation 2.1.2].

Lemma 3.2. *Let $n \geq 1$, and let $\iota_1 w$ be obtained from w by changing the sign of value 1. Then $\mathcal{R}(\iota_1 w) = E_1 \mathcal{R}(w)$ in every rank. This operation preserves the special recording tableau and changes the total vertical count by two modulo four.*

Proof. In rank one the identity is the simultaneous exchange of a horizontal domino and a vertical domino in the two tableaux. For $n > 1$, the extended cycle through 1 is not the singleton $\{n\}$. The cycle–deletion identity therefore retains the datum (v, ϵ) and induces the extended cycle through 1 on the reduced pair. On the signed-permutation side, changing the sign of value 1 commutes with deleting ϵn and retains its position and sign. Induction and (3.4) prove the identity. The second tableau is unchanged or moves through open cycles, so its special representative is preserved. Formula (3.5) gives the residue change. □

3.3. **The tableau operators.** For $n \geq 2$, define the tableau descent set $\tau(T) \subseteq \{\alpha'_1, \alpha_2, \ldots, \alpha_n\}$ as follows; for $n \leq 1$ it is empty. For $i \geq 2$ the condition is

$$\alpha_i \in \tau(T) \quad \Longleftrightarrow \quad \max\{r : (r, c) \in D_{i-1}(T)\} < \min\{r : (r, c) \in D_i(T)\}.$$

For α'_1 the condition is that the ordered positions of labels $1, 2$ are one of

$$(V(1,1), V(1,2)), \qquad (H(1,1), V(2,1)), \qquad (V(1,1), V(3,1)). \tag{3.7}$$

For adjacent simple roots α, β, the ordered star $T_{\alpha\beta}$ has domain $\alpha \notin \tau(P)$, $\beta \in \tau(P)$, and reverses these two conditions. With the insertion convention above, tableau and group descents satisfy

$$\alpha \in \tau(P(w)) \quad \Longleftrightarrow \quad s_\alpha \in \mathrm{Des}_L(w)$$

in both sign-parity classes [2, Proposition 2.1.18][9, Proposition 4.3.4].

For adjacent roots $\{\alpha_i, \alpha_{i+1}\}$, $i \geq 2$, use the ordinary three-domino rule of [2, Definitions 2.1.9–2.1.11]. It acts on P and fixes Q. Outside the following local configurations and their transposes, it is the unique interchange of two consecutive labels among $i-1, i, i+1$ which is standard and lies in the opposite descent domain. In the exceptional configurations, the three positions, in increasing label order, are replaced as follows:

$$\begin{aligned}(V(r,c), H(r,c+1), H(r+1,c+1)) \\ \longleftrightarrow (H(r,c), H(r+1,c), V(r,c+2)).\end{aligned} \tag{3.8}$$

The transposed local replacement is included in the rule.

For the fork star on $\{\alpha'_1, \alpha_3\}$ the following rules, together with reversal and simultaneous transposition, specify the operation. If $(2,2) \notin D_2(P) \cup D_3(P)$, interchange labels $2, 3$ in P. If the first three positions are

$$(V(1,1), V(1,2), H(1,3)),$$

first apply E_3 to the pair and then interchange the entries in squares $(2,2)$ and $(1,3)$ of the first tableau. If the first three positions are

$$(V(1,1), V(1,2), V(1,3)),$$

perform that square interchange first and then apply E_3 to the resulting pair. On all remaining domain elements define the operation by

$$T_{\alpha\beta}^{-1} = T_{\beta\alpha}, \qquad (T_{\alpha\beta} X)^t = T_{\beta\alpha}(X^t).$$

Here $X^t = (P^t, Q^t)$. These are the local rules in [9, Definition 4.3.5].

We next specify the D_4 correspondences. The central root is α_3, and the outer roots are $\alpha'_1, \alpha_2, \alpha_4$. For $\beta \in \{\alpha'_1, \alpha_2, \alpha_4\}$, define the tableau-pair types $\mathcal{A}_\beta, \mathcal{B}_\beta, \mathcal{C}, \mathcal{D}$ by the predicates of

Pattern	1	2	3	4
C^1	$V(1,1)$	$V(3,1)$	$H(1,2)$	$H(2,2)$
C^2	$H(1,1)$	$V(2,1)$	$H(1,3)$	$V(2,2)$
C^3	$V(1,1)$	$V(3,1)$	$H(1,2)$	$V(2,2)$
C^4	$H(1,1)$	$V(2,1)$	$H(1,3)$	$H(2,2)$
$B^1_{\alpha'_1}$	$H(1,1)$	$H(2,1)$	$V(1,3)$	$V(3,1)$
$B^2_{\alpha'_1}$	$H(1,1)$	$H(2,1)$	$H(1,3)$	$H(3,1)$
$B^1_{\alpha_2}$	$V(1,1)$	$V(1,2)$	$V(1,3)$	$V(3,1)$
$B^2_{\alpha_2}$	$V(1,1)$	$V(1,2)$	$H(1,3)$	$H(3,1)$
$B^1_{\alpha_4}$	$V(1,1)$	$V(3,1)$	$V(1,2)$	$V(1,3)$
$B^2_{\alpha_4}$	$H(1,1)$	$V(2,1)$	$V(2,2)$	$H(1,3)$

TABLE 1. The local patterns defining the D_4 tableau relations.

Section 2, replacing $s_\alpha \in \mathrm{Des}_L(w)$ by $\alpha \in \tau(P)$ and $\sigma_{s_\alpha,s_\gamma}$ by $T_{\alpha\gamma}$. It suffices to give $\mathcal{C} \to \mathcal{B}_\beta$; simultaneous transposition gives $\mathcal{D} \to \mathcal{A}_\beta$, and the opposite directions are the inverse relations.

Table 1 lists the required first-four-label patterns. Each entry in column k is the position of label k. No unspecified part of the tableau is changed by a replacement in this table.

Let $X = (P, Q)$ have type $\mathcal{C}$, and let C^j be its first-four pattern. For $j = 1, 2$ let Y be the pair obtained by replacing C^j by B^j_β in P and keeping Q. This replacement uses exactly the same eight squares. Put $\delta = 0$ if $\mathrm{v}(C^j) = \mathrm{v}(B^j_\beta)$ and $\delta = 1$ otherwise. The output set is

	$\delta = 0$	$\delta = 1$
$4 \in \mathrm{ec}_P(3; Q)$	$\{Y, E_4E_3Y\}$	$\{E_3Y, E_4Y\}$
$4 \notin \mathrm{ec}_P(3; Q)$	$\{Y\}$	$\{E_3Y\}$.

(3.9)

In the first row the two extended cycles in Y are distinct. If $j = 3, 4$, first apply E_l according to the next table, and then replace the resulting C^k by B^k_β:

	$\beta = \alpha'_1$	$\beta = \alpha_2$ or α_4
$j = 3$	$(l, k) = (3, 2)$	$(l, k) = (4, 1)$
$j = 4$	$(l, k) = (4, 2)$	$(l, k) = (3, 1)$.

There is one output in these cases. The preliminary cycle move makes the following replacement a same-region replacement.

The rules fix $\mathsf{S}(Q)$. Indeed, a local replacement fixes Q; an extended-cycle move either fixes Q or moves it through open cycles, which preserves $\mathsf{S}(Q)$ by (3.2). The same argument applies to the transposed and inverse relations. The tableau stars also have this property. All admissible branches are retained.

An *allowed operation* is a tableau star for adjacent simple roots, including the fork pair, or a branch of one of the D_4 relations. An *allowed sequence* is a succession of such operations. An operation *avoids label* n if its defining local rule uses only labels smaller than n; its extended cycles may contain n.

Lemma 3.3. *Let a tableau operator use only labels smaller than n. It commutes with Δ_n, preserves the datum (v, ϵ) in* (3.3)*, and preserves $\mathsf{S}(\bar{Q})$. Every branch on the reduced pair lifts to a branch on the full pair.*

Proof. For an ordinary star this is the usual insertion identity [2, Theorem 2.1.19]. More explicitly, its local change fixes the position of label n and the full second tableau. Its reverse-insertion input $(Q, D_n(P))$ is therefore unchanged, even if the reverse-insertion path passes through small labels.

For the fork and D_4 rules, every extended cycle which is moved contains one of the labels $3, 4$ (only 3 for the fork rule). It is not the singleton $\{n\}$. Proposition 2.3.3(b),(c) of [2] therefore applies, including its assertion that both v and ϵ are retained. The test whether 3 and 4 belong to the same extended cycle is unchanged on deleting n. The first-four patterns and their vertical counts are unchanged as well. Each same-region replacement fixes Q and $D_n(P)$, hence fixes the reverse-insertion data directly. The preliminary move for $j = 3, 4$ has already been included before the replacement. These observations apply in both directions to each branch, so they give lifting as well as commutation. They also apply to the transposed rules, using the insertion–transpose identity [2, proposition following Notation 2.1.2]. The type predicates survive deletion as well. Their descent tests involve surviving labels; equality of two reduced star outputs is equivalent to equality of the full outputs by their common retained datum and (3.4). This also shows, by successive deletion, that the listed first-four patterns exhaust the type $\mathcal{C}$ inputs.

The induced reduced operator is again one of the displayed rules. Its second tableau changes only through open cycles, so its special representative is unchanged. This proves preservation of $\mathsf{S}(\bar{Q})$. □

Proposition 3.4. *The tableau stars and D_4 relations above correspond under $\mathcal{R}$ to the group stars and D_4 correspondences. This holds for all ranks and in both signed-permutation parity classes. Every such operator preserves* $\mathrm{v}(P) + \mathrm{v}(Q) \pmod 4$.

Proof. The adjacent-root stars with $i \geq 2$ are covered by [2, Theorem 2.1.19]. The fork rule has a rank-three base; the D_4 rules have a rank-four base. The exact local checks are included in Proposition 8.1. They compare the coordinate and cycle rules with the group formulas. The rank-four comparison has 60 input–outer-root pairs of type $\mathcal{C}$ and 96 output arrows, among all 384 signed permutations. Inverse reciprocity and the insertion–transpose identity give the other directions. The same calculation separately checks, on all 384 inputs, that changing the sign of value 1 corresponds to E_1.

Suppose now that $n > 4$. On the group side a D_4 operator changes only the signed values $1, 2, 3, 4$. Deleting ϵn therefore preserves its position v, its sign ϵ, and every relevant inverse-order comparison. The domain tests and the candidate-output tests consequently commute with this deletion. On the tableau side the corresponding assertion is Lemma 3.3. Apply deletion to any output of either construction. By induction the reduced output sets agree, and the retained (v, ϵ) agree. Unique reconstruction in (3.4) gives equality of the full output sets. Conversely, every reduced branch lifts, so no full output is lost. This proves the assertion for the D_4 correspondences. The identical argument, deleting labels above 3, proves the fork assertion from its rank-three base.

All group multipliers belong to the even signed-permutation subgroup. They preserve the parity of the number of negative entries. The last assertion follows from (3.5). □

3.4. **Persistence of the largest domino.** The following statements apply to allowed sequences avoiding the largest first-tableau label n.

Lemma 3.5. *Removing a boxed extremal domino from a D-special shape leaves a D-special shape.*

Proof. A boxed horizontal corner $H(r, c)$ has c even. Its row has odd length $c+1$, and its paired row has odd length as well. Removing the corner decreases this row length by two, so both stay odd. No row disappears. A boxed vertical corner $V(r, c)$ has r odd. Extremality forces rows $r, r+1$ to have the same length c; both lengths decrease by one. If $c = 1$, the whole paired block disappears. All other row pairs are unchanged. □

Lemma 3.6. *Let $X = (P, Q)$ and let $X' = (P', Q')$ be obtained by a sequence avoiding label n. Then*

$$D_n(P') \in \{D_n(P), D'_n(P)\}.$$

If the reduced first tableaux have the same shape, then $D_n(P') = D_n(P)$ and $Q' = Q$.

Proof. Each local replacement fixes the largest domino. A cycle move either fixes it or replaces it by its alternate position. Its fixed square never changes, and the set of fixed squares of the shape is unchanged. In the alternate-position rule for label n, a diagonal entry is either smaller than n or is absent. Thus the choice depends only on the presence of that diagonal fixed square. The two positions form the same unordered pair throughout the sequence. This proves the first assertion.

For the second, write $\Delta_n X = (\bar{P}, \bar{Q}; v, \epsilon)$ and $\Delta_n X' = (\bar{P}', \bar{Q}'; v, \epsilon)$. Lemma 3.3 gives $\mathsf{S}(\bar{Q}') = \mathsf{S}(\bar{Q})$. Their shapes agree, so Lemma 3.1 gives $\bar{Q}' = \bar{Q}$. Equation (3.4) now gives both $Q' = Q$ and equality of the added domino positions. □

Lemma 3.7. *Suppose that $D_n(P)$ is boxed. If a sequence avoiding label n takes $X = (P, Q)$ to $X' = (P', Q')$ and $\mathrm{sh}(P' \setminus D_n(P'))$ is special, then $D_n(P') = D_n(P)$ and the full shape of X' is special.*

Proof. Put $D = D_n(P)$ and $\Delta_n X = (\bar{P}, \bar{Q}; v, \epsilon)$. We first construct a second full pair having the same enriched data. Move all open unboxed cycles simultaneously in the pair X. They form a union of entire extended cycles. Indeed, the back square of an unboxed open cycle has odd row and even column, and its forward square has even row and odd column. For a boxed open cycle the two parity types are reversed [1, Proposition 1.5.24]. Matching boundary squares between equal-shape tableaux therefore joins cycles of the same kind. The resulting pair is therefore

$$X^* = (\mathsf{S}(P), \mathsf{S}(Q)).$$

No moved cycle contains n, because a cycle has uniform boxed status and D is boxed. In particular, $D_n(\mathsf{S}(P)) = D$.

The cycle–deletion identity preserves (v, ϵ) in this construction. Moreover, the induced first-tableau cycles after deletion remain open: deletion of n changes neither the positions nor the alternate positions of their smaller labels, and hence changes neither boundary square. The induced second-tableau operation is thus through open cycles, preserving $\mathsf{S}(\bar{Q})$. Deleting the unchanged boxed domino from $\mathrm{sh}(\mathsf{S}(P))$ leaves a special shape by Lemma 3.5. The reduced second tableau is consequently special and has special representative $\mathsf{S}(\bar{Q})$; it must equal $\mathsf{S}(\bar{Q})$ itself. We have shown that

$$\Delta_n X^* = (\bar{P}^*, \mathsf{S}(\bar{Q}); v, \epsilon), \qquad D_n(\mathsf{S}(P)) = D. \tag{3.10}$$

At the actual endpoint X', Lemma 3.3 gives the same (v, ϵ) and the same reduced special representative. Its reduced shape is special by assumption, so its reduced second tableau is again $\mathsf{S}(\bar{Q})$. Insertion into this tableau with (v, ϵ) has a unique result. Comparing with (3.10), we obtain $Q' = \mathsf{S}(Q)$ and $D_n(P') = D$. The full shape is the shape of X^* and is special. □

4. Special shapes and normalization

Use the row lengths λ_i, column lengths λ^t_j, and boxed-domino convention above. The special-shape row condition is equivalent to the following column condition: the first column has even height, and each pair of columns $(2a, 2a+1)$ has equal heights or two even heights. Indeed, for each positive even integer m, the rows of length m occupy the interval $[\lambda^t_{m+1} + 1, \lambda^t_m]$. The row condition says that the number of rows is even and each nonempty such interval starts in an odd row and ends in an even row: every even-length row is then paired with a row of the same

length, and an unequal pair has two odd lengths. An empty interval gives equal column heights; a nonempty one gives two even heights. These implications also prove the converse.

Lemma 4.1. *For a domino shape λ, put*

$$A_a = \left\lceil \frac{\lambda_{2a-1}}{2} \right\rceil, \qquad B_a = \left\lfloor \frac{\lambda_{2a}}{2} \right\rfloor.$$

The special shape with the same fixed squares as λ has row pairs

$$\begin{cases} (2A_a, 2A_a), & A_a = B_a, \\ (2A_a - 1, 2B_a + 1), & A_a > B_a. \end{cases} \tag{4.1}$$

In particular, it is the shape of $\mathsf{S}(T)$ for every tableau T of shape λ.

Proof. The fixed squares in an odd row occupy its odd columns; those in an even row occupy its even columns. Hence their numbers in the indicated row pair are A_a, B_a, with $A_a \geq B_a$. The special-shape row condition determines the row pair uniquely and gives (4.1).

The new odd rows weakly shorten and the new even rows weakly lengthen. Consequently inequalities between consecutive row pairs are preserved; the inequality inside a pair follows from $A_a \geq B_a$. The total area is unchanged: each original domino has one fixed square, whereas the new row pair has area $2(A_a + B_a)$. Thus the new shape has twice as many squares as there are fixed squares in λ, namely $|\lambda|$. Moving through cycles preserves fixed squares, so specialization has the displayed shape.

□

Lemma 4.2. *Let λ be a special shape containing the square $(2, 2)$. It can be reduced to the 2-by-2 square by removing boxed dominoes. Its standard domino tableaux realize both possible residues of the number of vertical dominoes modulo 4. More generally, prescribed successive largest-domino positions can be filled in either shape-compatible residue whenever their removal leaves a special shape containing $(2, 2)$.*

Proof. Remove the last row pair until only the first pair remains. If its lengths are unequal, they are odd; repeatedly remove a horizontal domino from the longer row until they become equal. Then remove vertical dominoes from their common right end. All these removals are boxed and preserve the Young diagram. For the first row pair, stop when both lengths are two. This constructs the required chain.

The two tilings of the 2-by-2 square have zero and two vertical dominoes. Append the same labelled dominoes along the reverse chain. The resulting standard tableaux have vertical counts differing by two. These are the only two possible residues: the parity of the vertical count is the parity of the number of squares in the odd rows, since a vertical domino meets those rows in one square and a horizontal domino in zero or two squares. Appending any fixed list of largest dominoes proves the last assertion. □

We now prove the normalization step used in the simultaneous induction. For a pair $X = (P, Q)$, its residue means $\mathrm{v}(P) + \mathrm{v}(Q) \pmod 4$. In the following lemma, the transitivity and normalization statements in ranks smaller than n are induction hypotheses.

Lemma 4.3. *Let $X = (P, Q)$ be a pair of standard domino tableaux with n dominoes in each tableau and with $(2, 2) \in \mathrm{sh}(P)$. There is an allowed sequence taking X to a pair $(P', \mathsf{S}(Q))$. The sequence preserves the residue of the pair.*

Proof. The ranks at most four are included in Proposition 8.1. Assume $n \geq 5$. If the full shape is special, then $Q = \mathsf{S}(Q)$ and there is nothing to do. Denote the position of the largest domino by D, and delete that domino with its enriched insertion datum as in Lemma 3.3. We write

$\overline{X} = (\overline{P}, \overline{Q})$ for the reduced pair. Every use of a smaller-rank sequence below retains the enriched datum and is lifted by that lemma.

For the high-label operations, use the abbreviations

$$\tau_+ = T_{\alpha_{n-1},\alpha_n}, \qquad \tau_- = T_{\alpha_n,\alpha_{n-1}}. \tag{4.2}$$

Thus the descent pair (α_{n-1}, α_n) is $(0,1)$ on the domain of τ_+ and $(1,0)$ on the domain of τ_-. Here and below the first root in $T_{\alpha,\beta}$ is absent and the second is present in the descent set.

The largest domino meets $(2,2)$. Suppose first that $(2,2) \in D$ and that the full shape F is not already special. Put $r = F_1$, $k = F_1^t$. Removability gives precisely the following possibilities:

D	additional condition	choice of arm
$H(2,1)$	$k = 2$	top
$V(2,2)$	$r = 2$	column
$V(2,2)$	$r > 2$	top if $r \geq 5$, column otherwise
$H(2,2)$	r even	top if $r \geq 5$, column otherwise

A removable domino containing $(2,2)$ can only be $H(2,1)$, $H(2,2)$, $V(1,2)$, or $V(2,2)$. The choice $V(1,2)$ gives a special shape, as does $H(2,2)$ with odd first-row length.

The reduced shape is a hook. Every tiling of a given domino hook has the same number of vertical dominoes. On the top arm choose

$$D_1 = H(1, r-1), \qquad D_2 = H(1, r-3),$$

and on the column choose

$$D_1 = V(k-1, 1), \qquad D_2 = V(k-3, 1).$$

The chosen arm has room for these two dominoes when $n \geq 5$. Place labels $n-2, n-1$ at D_2, D_1 in an auxiliary reduced tableau. It has the required vertical residue. Smaller-rank transitivity reaches this auxiliary tableau with $\overline{Q}$ restored at the endpoint. Since its reduced shape has not changed, the lifted largest domino is still at D.

Apply τ_+ for the top arm or τ_- for the column. The three largest dominoes have the required opposite descents, and the ordinary rule interchanges $n-1, n$. Thus the largest domino moves to D_1, and its deletion now leaves a shape containing $(2,2)$. Normalize that reduced pair by induction. If D_1 is boxed, Lemma 3.7 finishes the argument. If $D_1 = H(1, r-1)$ is unboxed, r is even; after specialization the reduced first row has length $r-3$. The old position would leave a gap, so Lemma 3.6 forces the alternative $H(1, r-2)$. If $D_1 = V(k-1, 1)$ is unboxed, k is odd; the specialized reduced shape has height $k-1$. The old position overlaps it, and the only alternative is $V(k, 1)$. In each case the resulting full shape is special by (4.1).

The reduced shape contains $(2,2)$. We may now assume that deleting D leaves $(2,2)$. Normalize the reduced pair first. If the lifted full shape is special, we are done. Otherwise its largest domino is unboxed: a boxed one would give a special full shape by Lemma 3.7. Put $\lambda = \mathrm{sh}(\overline{P})$; this is special. The possibilities are

$$D = H(i,j), \quad i,j \text{ odd}; \qquad D = V(j,i), \quad i,j \text{ even}.$$

These follow by adding an unboxed removable domino to the paired rows, or to the shifted paired columns, of λ.

The next two tables prescribe removable positions D_1, D_2 in λ, in that removal order. Place $n-1$ at D_1 and $n-2$ at D_2. The column "new n" gives the position just after the indicated high star, before the final reduced normalization. All row and column lengths in the tables belong to λ.

Table 2: Horizontal largest domino $D = H(i,j)$, i,j odd.

Condition	D_1	D_2	Star	New n
$i \geq 3$: put $r = \lambda_{i-2},\ s = \lambda_{i-1}$.				
$r > s$, and $i \geq 5$ or $s > 3$	$H(i-2, r-1)$	$H(i-1, s-1)$	τ_+	D_1
$i = 3, s = 3, r \geq 7$	$H(1, r-1)$	$H(1, r-3)$	τ_+	D_1
$i = 3, s = 3, r = 5$ (hence $j = 1$)	$H(1,4)$	$V(1,3)$	τ_+	D_1
$r = s > j+1$, and $i \geq 5$ or $r > 3$	$V(i-2, r)$	$V(i-2, r-1)$	τ_+	D_1
$r = s = j+1$, and $i \geq 5$ or $j \geq 3$	$V(i-2, j+1)$	$V(i-2, j)$	τ_+	$V(i-1, j+1)$
$i = 1$: the first two rows of λ have length $j-1$.				
$\lambda_3 \geq j-1$, $r = \lambda^t_{j-1}$, and $j \geq 5$ or $r \geq 6$	$V(r-1, j-1)$	$V(r-3, j-1)$	τ_-	D_1
$j = 3, \lambda^t_2 = 4, k = \lambda^t_1 \geq 6$	$V(k-1, 1)$	$V(3,2)$	τ_-	D_1
$j = 3, \lambda^t_2 = \lambda^t_1 = 4$	$H(4,1)$	$H(3,1)$	τ_-	D_1
$\lambda_3 = j-2,\ j \geq 5,\ r = \lambda^t_{j-2}$ odd	$H(r, j-3)$	$V(1, j-1)$	τ_-	D_1
$\lambda_3 = j-2,\ j \geq 5,\ r = \lambda^t_{j-2}$ even	$V(r-1, j-2)$	$V(1, j-1)$	τ_-	D_1
$j = 3, \lambda_3 = 1, k = \lambda^t_1 \geq 6$	$V(k-1, 1)$	$V(k-3, 1)$	τ_-	D_1
$\lambda_3 \leq j-3,\ j \geq 5$	$H(2, j-2)$	$H(1, j-2)$	τ_-	D_1

The horizontal cases excluded from the table have rank at most four. They are $i = 3, r = s = 3, j = 1$; $i = 3, r = s = 2, j = 1$; $i = 1, j = 3, \lambda = (2,2,1,1)$; and $i = 1, j = 3, \lambda = (2,2)$. To make the residue choices explicit, in $\lambda = (3,3)$ take $D_1 = V(1,3)$ and take $D_2 = V(1,2)$ for residue 3, or $D_2 = H(2,1)$ for residue 1. In $\lambda = (2,2,1,1)$, take $D_1 = V(3,1)$ and take $D_2 = H(2,1)$ for residue 1, or $D_2 = V(1,2)$ for residue 3. The remaining one domino gives the required residue in each case.

Table 3: Vertical largest domino $D = V(j,i)$, i,j even.

Condition	D_1	D_2	Star	New n
$i \geq 4$: put $r = \lambda^t_{i-2},\ s = \lambda^t_{i-1}$.				
$r > s$	$V(r-1, i-2)$	$V(s-1, i-1)$	τ_-	D_1
$r = s > j+1$	$H(r, i-2)$	$H(r-1, i-2)$	τ_-	D_1
$r = s = j+1$, and $i \geq 6$ or $j \geq 4$	$H(j+1, i-2)$	$H(j, i-2)$	τ_-	$H(j+1, i-1)$
$i = 4, j = 2, \lambda = (t,3,3,1^{k-3}),\ k > 4$	$V(k-1, 1)$	$H(3,2)$	τ_-	D_1
$i = 4, j = 2, \lambda = (t,3,3,1),\ t \geq 7$	$H(1, t-1)$	$H(1, t-3)$	τ_+	D_1
$i = 4, j = 2, \lambda = (5,3,3,1)$	the first rank-seven construction below			
$i = 2, j \geq 4$: put $k = \lambda^t_1$.				
$t = \lambda_{j-1} \geq 5$	$H(j-1, t-1)$	$H(j-1, t-3)$	τ_+	D_1
$\lambda_{j-1} = 3,\ t = \lambda_{j-2} \geq 4$ even	$V(j-3, t)$	$H(j-1, 2)$	τ_+	D_1
$\lambda_{j-1} = 3,\ t = \lambda_{j-2} \geq 5$ odd	$H(j-2, t-1)$	$H(j-1, 2)$	τ_+	D_1
$\lambda_{j-2} = \lambda_{j-1} = 3,\ j \geq 6$	$V(j-2, 3)$	$V(j-2, 2)$	τ_+	D_1
$j = 4, \lambda_2 = \lambda_3 = 3,\ k \geq 8$	$V(k-1, 1)$	$V(k-3, 1)$	τ_-	D_1

Condition	D_1	D_2	Star	New n
$j = 4, k = 6, \lambda = (t, 3, 3, 1, 1, 1),\ t \geq 5$	$H(1, t-1)$	$H(3, 2)$	τ_+	D_1
$j = 4, \lambda = (3, 3, 3, 1, 1, 1)$	the second rank-seven construction below			

We verify the shape and residue assertions for all rows of these tables. In the horizontal case with $i \geq 3$, the alternatives $r > s$, $r = s > j + 1$, and $r = s = j + 1$ exhaust the row condition. If $r > s$, then r, s are odd with $r \geq s + 2$ and $s \geq j + 2$. The first construction shortens both members of this odd pair by two. At $i = 3, s = 3$, the next two constructions leave $(r - 4, 3)$, or $(2, 2)$ when $r = 5$. When $r = s > j + 1$, the two vertical removals change (r, r) to $(r - 2, r - 2)$. When $r = s = j + 1$, they change it to $(j - 1, j - 1)$, matching the following pair. In these cases the first 2 by 2 square survives unless we are in one of the listed small ranks.

For $i = 1$, split according to $\lambda_3 \geq j - 1$, $\lambda_3 = j - 2$, or $\lambda_3 \leq j - 3$. In the first case the two vertical removals shorten two equal row pairs. At $j = 3, \lambda_2^t = 4$, the displayed bottom-corner construction leaves $(2, 2, 1^{k-4})$ if $k \geq 6$, and the two horizontal removals leave $(2, 2)$ if $k = 4$. In the second case, $\lambda_{j-1}^t = 2$. If $r = \lambda_{j-2}^t$ is odd, the shifted column condition forces $\lambda_{j-3}^t = r$, so $H(r, j - 3)$ is removable and boxed. If r is even, $V(r - 1, j - 2)$ is removable and boxed. Removing this domino and then $V(1, j - 1)$ preserves specialness. At $j = 3$, the two bottom vertical removals have the same effect. In the third case the two removals form a complete 2 by 2 block, changing the first equal row pair from $(j - 1, j - 1)$ to $(j - 3, j - 3)$. Thus every nonexceptional residual is special and contains $(2, 2)$.

For the vertical table with $i \geq 4$, apply the shifted column condition to columns $(i - 2, i - 1)$. If $r > s$, both heights are even and decrease by two. If $r = s > j + 1$, both decrease by two together. If $r = s = j + 1$, the same removal leaves the special residual except at $i = 4, j = 2$. There the shape is $(t, 3, 3, 1^{k-3})$, with t odd and k even. For $k > 4$ the residual is $(t, 3, 1^{k-4})$; for $k = 4, t \geq 7$ it is $(t - 4, 3, 3, 1)$. Only $(5, 3, 3, 1)$ remains.

For $i = 2$, the rows numbered $j, \ldots, k$ have length one. If $t = \lambda_{j-1} \geq 5$, it is odd, and the two horizontal removals shorten it by four. If $\lambda_{j-1} = 3$, split according to $t = \lambda_{j-2} \geq 4$ or $t = 3$. In the former case, even t forces $\lambda_{j-3} = t$, allowing the displayed vertical removal; odd t allows the displayed horizontal removal. Next remove $H(j - 1, 2)$. In the latter case, $j \geq 6$ permits the two vertical removals in columns three and two. For $j = 4$, the bottom pair works when $k \geq 8$; when $k = 6, t \geq 5$, the residual is $(t - 2, 3, 1, 1, 1, 1)$. The remaining shape is $(3, 3, 3, 1, 1, 1)$. This proves exhaustiveness and specialness of the residuals.

Lemma 4.2 now gives an auxiliary reduced tableau with labels $n - 2, n - 1$ in the specified positions and with the same vertical residue as $\overline{P}$. Smaller-rank transitivity reaches it with $\overline{Q}$ prescribed at the endpoint. The reduced shape and the enriched datum are unchanged, so Lemma 3.3 reconstructs the same position D for the largest full domino.

Here is a direct check of the high stars. For two domino positions set

$$d(A, B) = \begin{cases} 1, & \max\{r : (r, c) \in A\} < \min\{r : (r, c) \in B\}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

The two high descents are $d(D_2, D_1), d(D_1, D)$. The displayed coordinates give $(0, 1)$ in every τ_+ row and $(1, 0)$ in every τ_- row. Except for the two rows with output different from D_1, interchanging the labels at D_1, D is standard. In these rows the four values

$$\big(d(D_2, D_1), d(D_1, D), d(D_2, D), d(D, D_1)\big)$$

are respectively $(0,1,1,0)$ and $(1,0,0,1)$. Hence the proposed interchange reverses both high descents and is exactly the ordinary star rule. These equalities follow from the row inequalities used above; the other labels are all smaller than the three displayed ones. The two remaining configurations are the three-domino rotations:

(D_2, D_1, D)	after the star	after normalization
$(V(i-2,j), V(i-2,j+1), H(i,j))$	$V(i-1,j+1)$	$V(i-2,j+1)$
$(H(j,i-2), H(j+1,i-2), V(j,i))$	$H(j+1,i-1)$	$H(j+1,i-2)$

The ordinary star fixes the second tableau. Normalize the reduced pair once more. For the first rotation, deleting the current largest domino leaves the four row lengths at indices $i-2, i-1, i, i+1$

$$(j+1, j, j, j-1).$$

Formula (4.1) changes them to (j,j,j,j). The current position $V(i-1,j+1)$ would add a square beyond the end of row $i-1$ while leaving row $i-2$ shorter; its alternative $V(i-2,j+1)$ gives the special row pairs $(j+1,j+1)$, (j,j). For the second rotation, the four column heights at indices $i-2, i-1, i, i+1$ are again $(j+1,j,j,j-1)$. The shifted column pairing changes them to (j,j,j,j). The current position $H(j+1,i-1)$ would make column $i-1$ taller than column $i-2$; its alternative $H(j+1,i-2)$ gives the allowed column pairs $(j+1,j+1)$, (j,j). Lemma 3.6 therefore forces the displayed alternatives.

For every other boxed output D_1, finish with Lemma 3.7. The only remaining unboxed outputs in the tables are as follows; formula (4.1) forces the indicated alternative:

position before normalization	obstruction	forced position
$H(2,j-2)$	overlaps the row pair $(j,j-2)$	$H(2,j-1)$
$H(4,1)$ in $(4,2,2,2)$	overlaps $(3,3,1,1)$	$V(3,2)$
$V(j-2,3)$ $(j \geq 6)$	overlaps the specialized reduced shape	$V(j-1,3)$

The resulting full shape is special in every case.

The two rank-seven configurations. For $\lambda = (5,3,3,1)$, choose the reduced auxiliary tableau with positions

label	1	2	3	4	5	6
position	$V(1,1)$	$H(1,2)$	$V(3,1)$	$H(1,4)$	$H(2,2)$	$H(3,2)$

and pair it with the current $\overline{Q}$. Its first tableau has vertical count two. If the required reduced residue differs by two, choose instead the pair obtained by changing the sign of value one in its inverse insertion parameter. By Lemma 3.2, this is the extended-cycle operation through one; it preserves the special representative of the second tableau and changes the pair residue by two. This sign change only specifies the auxiliary endpoint. The path to that endpoint is supplied by smaller-rank transitivity and preserves the required residue.

The first fixed calculation in Proposition 8.1 says that, after lifting this choice, the largest domino is at $V(2,4)$ and $\tau_- = T_{\alpha_7,\alpha_6}$ puts it at $H(3,3)$. The reduced special shape is $(5,5,1,1)$. The old position leaves a gap in its third row, so the two-position lemma forces $H(3,2)$. The full shape is $(5,5,3,1)$.

For $\lambda = (3,3,3,1,1,1)$, use the auxiliary tableau

label	1	2	3	4	5	6
position	$V(1,1)$	$V(3,1)$	$H(1,2)$	$V(5,1)$	$V(2,2)$	$V(2,3)$.

Its vertical count is five. Make the same auxiliary-endpoint correction when the required count is three modulo four. The second fixed calculation gives the lifted largest position $V(4,2)$ and the high star $\tau_+ = T_{\alpha_6,\alpha_7}$, whose output position is $V(2,3)$. The reduced special shape is

$(3,3,2,2,1,1)$, which overlaps that position. Its only alternative is $V(3,3)$, and the full shape is $(3,3,3,3,1,1)$.

These are fixed rank-seven inputs. Each has thirty possible reduced first tableaux and thirty-five full second tableaux, hence 1050 input pairs; the finite verification covers both residue sectors. The auxiliary-endpoint paths and subsequent normalizations use rank-six transitivity and normalization.

Every step of the construction is a lifted smaller-rank operation or an ordinary high star, and therefore preserves the pair residue and $\mathsf{S}(Q)$. The endpoint second tableau is special, so it is exactly $\mathsf{S}(Q)$, as required. □

Lemma 4.4. *Suppose two domino pairs (P,Q) and (P',Q') have the same special second tableau and the same residue. If their shapes do not contain $(2,2)$, they are connected by ordinary first-tableau stars.*

Proof. The empty pair is immediate; assume $n \geq 1$. The square $(2,2)$ is fixed. Since specialization preserves fixed squares, the hypothesis that it is absent holds for both pairs. Their shapes are hooks with the same fixed-square set. Let A and B be the numbers of fixed squares in the first row and the first column. The only possible even hook shapes are

$$\lambda_+ = (2A-1, 1^{2B-1}), \qquad \lambda_- = (2A, 1^{2B-2}), \qquad n = A+B-1.$$

The first is special. Its first column must be tiled vertically, and the rest of its first row horizontally; thus every tableau has B vertical dominoes. For the second shape, the first row is tiled horizontally and the remaining column vertically, giving $B-1$ vertical dominoes. The two pair residues are consequently $2B$ and $2B-2$, which differ modulo four. The two given pairs must have the same actual shape.

The second tableaux now have equal shape and equal special representative, so they are equal by Lemma 3.1. The domino containing $(1,1)$ has label one and fixed orientation. Every other domino lies on one of the two arms. Its labels are therefore encoded by a word in H, V for labels $2, \ldots, n$, with fixed numbers of the two letters and increasing labels along each arm.

For $i \geq 2$, the descent α_i is present precisely when domino i is vertical. If dominoes $i, i+1$ belong to different arms, the ordinary star for $\{\alpha_i, \alpha_{i+1}\}$ interchanges their labels. The three-domino rotation alternatives cannot occur in a hook, as each would contain $(2,2)$. Adjacent exchanges of unequal letters connect all words with these fixed multiplicities. These stars leave the second tableau unchanged, proving the assertion. □

5. Placement of the largest domino

Throughout this section, transitivity and normalization in ranks smaller than n are induction hypotheses. The recording tableau Q is special, and $F = \mathrm{sh}(Q)$ contains $(2,2)$. A removable position C in F is *feasible for the residue r* if there is a standard tableau A of shape F with

$$D_n(A) = C, \qquad \mathrm{v}(A) \equiv r \pmod 4.$$

This condition concerns the first tableau. Since Q will be restored exactly, it is equivalent to feasibility for the corresponding pair residue.

Lemma 5.1. *Let (P,Q) be a standard domino pair of size n, with Q special and $(2,2) \in F$. If C' is feasible for $r = \mathrm{v}(P) \pmod 4$, an allowed sequence takes (P,Q) to a pair (P',Q) with $D_n(P') = C'$.*

We prove the lemma in several steps. The finite bases $n \leq 4$ are included in Proposition 8.1. All other uses of transitivity below concern rank $n-1$.

5.1. **Fixed positions and two local transfers.** Suppose first that the source and target largest positions are boxed. Fix $r = \mathrm{v}(P) \pmod 4$. For each boxed corner C feasible for r, consider the pairs (A, Q) satisfying

$$\mathrm{sh}(A) = F, \qquad D_n(A) = C, \qquad \mathrm{v}(A) \equiv r \pmod 4.$$

Any two such pairs can be joined using smaller-rank operations, with the exact second tableau and largest position restored at the endpoint. Indeed, reverse insertion from the common Q at C gives the same $(\bar{Q}, v, \epsilon)$. Deleting the same oriented domino from the first tableaux leaves equal vertical residues. Apply rank-$(n-1)$ transitivity and Lemma 3.3. The endpoint smaller shape is the special shape $F \setminus C$. Lemma 3.7 keeps the largest position at C, and unique reconstruction restores the exact endpoint pair. Since every allowed operation is reciprocal, a transfer from one feasible boxed corner to another can consequently be reversed: first join to the endpoint of a reverse-direction transfer by this smaller-rank argument, then reverse its path. This uses rank-$(n-1)$ transitivity.

For domino positions A, B, use $d(A, B)$ from (4.3). The following rule is the general transfer used in the coordinate tables. Let C, C' be disjoint boxed corners of F, and prescribe the three largest labels at the positions D_k in the prepared tableau:

$$D_{n-2} = R, \qquad D_{n-1} = C', \qquad D_n = C.$$

Assume that these positions can be filled in the required residue and that

$$\big(d(R, C'), d(C', C), d(R, C)\big) = (0, 1, 1) \quad \text{or} \quad (1, 0, 0). \tag{5.1}$$

Rank-$(n-1)$ transitivity prepares the prescribed smaller tableau, and boxed persistence restores C at the endpoint. The high star τ_+ in the first case, or τ_- in the second, interchanges labels $n-1, n$ and fixes Q. Here $\tau_\pm$ have the meanings in (4.2).

To verify the last assertion directly, swapping the two largest labels is standard because both corners were removable in the original shape. The two adjacent descents are opposite and are both reversed by this swap. Swapping $n-2, n-1$ instead would leave the second high descent equal to $d(R, C) = d(C', C)$, so it cannot enter the opposite descent domain. None of the exceptional configurations in (3.8) can occur: in each of them the penultimate domino is blocked by the largest domino and is not removable from the original shape. Thus the ordinary-star definition selects precisely the claimed swap.

We also use the two three-domino rotations

(D_{n-2}, D_{n-1}, D_n)	position of n after τ_+
$(V(i-1, j-1), H(i-1, j), H(i, j))$	$V(i-1, j+1)$
$(V(i-2, j), V(i-2, j+1), H(i, j))$	$V(i-1, j+1)$

(5.2)

whenever the displayed positions are successively removable and have the required residue. The first row is (3.8); the second is its transposed local replacement. These assertions involve only the three indicated dominoes and hold for every recording tableau.

5.2. **The complete generic coordinate tables.** The next tables prescribe an auxiliary rim domino R_0. Replace it by its boxed version R in the special residual shape $F \setminus (C \cup C')$:

$$R = \begin{cases} R_0, & R_0 \text{ is boxed,} \\ V(a-1, b+1), & R_0 = H(a, b) \text{ is unboxed,} \\ H(a+1, b-1), & R_0 = V(a, b) \text{ is unboxed.} \end{cases} \tag{5.3}$$

These replacements are removable and have the same bottom-right square. For the horizontal case, specialness forces a even, b odd, and equal lengths $b+1$ in rows $a-1, a$. For the vertical case, it forces a even, b odd, and equal heights $a+1$ in columns $b-1, b$. This proves (5.3).

All row and column lengths below belong to F, except when a residual shape is explicitly indicated. The substitutions following the tables are part of the prescriptions.

Table 4: Horizontal source $C = H(i, j)$.

Case	Condition	Auxiliary R_0
Horizontal target $C' = H(k, l)$.		
A	$k = i - 1$	$H(i-1, l-2)$
B	$k < i - 1$, put $u = F_{i-1}$	$H(i-1, u-1)$
C	$k = i + 1$	$H(i, j-2)$
D	$k > i + 1$, put $u = F_{k-1}$	$H(k-1, u-1)$
Vertical target $C' = V(k, l)$.		
E	$k + 1 = i - 1$	$V(i-2, l-1)$
F	$k + 1 < i - 1$, put $u = F_{i-1}$	$H(i-1, u-1)$
G	$k = i + 1,\ l < j - 1$	$H(i, j-2)$
H(a)	$k > i + 1,\ F_{i+1} \le j - 3$	$H(i, j-2)$
H(b)	$k > i + 1,\ F_{i+1} = j - 1$; put $u = (F \setminus C')^t_{j-1}$	$V(u-1, j-1)$
H(c)	$k > i + 1,\ F_{i+1} = j - 2$; put $u = (F \setminus C')^t_{j-2}$	$V(u-1, j-2)$
I	$k = i - 1$, necessarily $l = j + 1$	case I below
J	$k = i + 1,\ l = j - 1$	case J below

Table 5: Vertical source $C = V(j, i)$.

Case	Condition	Auxiliary R_0
Vertical target $C' = V(l, k)$.		
A′	$k = i - 1$	$V(l-2, k)$
B′	$k < i - 1$, put $u = F^t_{i-1}$	$V(u-1, i-1)$
C′	$k = i + 1$	$V(j-2, i)$
D′	$k > i + 1$, put $u = F^t_{k-1}$	$V(u-1, k-1)$
Horizontal target $C' = H(l, k)$.		
E′	$k + 1 = i - 1$	$V(l-2, i-1)$
F′	$k + 1 < i - 1$, put $u = F^t_{i-1}$	$V(u-1, i-1)$
G′	$k = i + 1,\ l < j - 1$	$V(j-2, i)$
H′(a)	$k > i + 1,\ F^t_{i+1} \le j - 3$	$V(j-2, i)$
H′(b)	$k > i + 1,\ F^t_{i+1} = j - 1$; put $u = (F \setminus C')_{j-1}$	$H(j-1, u-1)$
H′(c)	$k > i + 1,\ F^t_{i+1} = j - 2$; put $u = (F \setminus C')_{j-2}$	$H(j-2, u-1)$
I′	$k = i - 1$	reverse of I
J′	$k = i + 1,\ l = j - 1$	reverse of J

Here is the full list of small-coordinate substitutions and exclusions. Parameters such as u retain their meanings in the corresponding row. A row marked as an exception is handled by the subsequent routing argument.

Table 6: Substitutions in the generic tables.

Case	Additional condition	Substitution or exception
A	$i=3, l=4$ (hence $j=2$)	A/C exception
C	$i=2, j=4$ (hence $l=2$)	reverse A/C exception
B	$i=3, u=3, l\geq 6$	$R_0 = H(1, l-2)$
B	$i=3, u=3, l=4$	$R_0 = V(1,3)$
D	$k=3, u=3, j\geq 6$	$R_0 = H(1, j-2)$
D	$k=3, u=3, j=4$	$R_0 = V(1,3)$
G	$i=2, j=4$	G/H′ exception
H(a)	$i=2, j=4, l=1, k\geq 5$	$R_0 = V(k-2,1)$
A′	$i=3, l=3$	A′/C′ exception
C′	$i=2, j=3$	reverse A′/C′ exception
B′	$i=3, u=2, l=3$	fixed shape $(3,3,1,1)$
B′	$i=3, u=2, l\geq 5$	$R_0 = V(l-2,1)$
B′	$i=4, u=3$	$R_0 = H(3,2)$
D′	$k=3, u=2, j=3$	fixed shape $(3,3,1,1)$
D′	$k=3, u=2, j\geq 5$	$R_0 = V(j-2,1)$
D′	$k=4, u=3$	$R_0 = H(3,2)$
G′	$i=1, l=1$	impossible since $(2,2)\in F$
H′(b)	$j=3, u=3, l=1, k\geq 6$	$R_0 = H(1, k-2)$
H′(b)	$j=3, u=3, l=1, k=4$	$R_0 = V(1,3)$
H′(b)	$j=3, u=3, l=2$	reverse G/H′ exception

We verify both completeness and the assertions used with these tables. Two horizontal rim corners have different rows, giving exactly A–D. For a horizontal and a vertical corner, the possibilities are E–J. The case $k=i$ would force equal lengths in rows $i, i+1$, making the horizontal domino nonremovable. If $k=i-1$, equality of the two affected row lengths forces $l=j+1$; if $k=i+1$, horizontal removability gives $l\leq j-1$. In H, the next row has length at most $j-1$, so its three displayed subcases exhaust the possibilities. Two vertical rim corners have different columns, giving A′–D′. The shifted column condition gives the displayed vertical-to-horizontal cases. Their exhaustiveness also follows by reversing E–J.

In every nonexceptional row, R_0 is a rim domino after removing C, C'. For A and C its row ends two squares before the higher corner; for B, D and F it lies at the end of the intervening row. In E it lies immediately to the left of the upper vertical corner. In G and H(a) it lies immediately before the source horizontal corner. For H(b),(c), the indicated residual column height is exactly its lower end. The primed formulas express the corresponding statements in terms of residual column heights and row lengths; the listed small substitutions are the cases in which that end lies in the first two columns or rows. Thus the displayed lengths prove successive removability. Removing boxed corners preserves specialness, and (5.3) makes the third removal boxed as well.

The same coordinates give the following descent triples after boxification:

cases	$(d(R,C'), d(C',C), d(R,C))$
$A, B, E, F, C', D', G', H'$	$(0,1,1)$
$C, D, G, H, A', B', E', F'$	$(1,0,0)$.

(5.4)

For example, in the first line the auxiliary domino is weakly below or alongside C', but strictly above C. In A its row is the row of C'; in B and F it is the immediately preceding row of C; E uses the rows of C'. In the primed cases the terminal row of the indicated intervening column gives the same inequality. The second line reverses these inequalities. Boxification retains the bottom-right square and changes the upper row by at most one, within the separation allowed in the corresponding case. These observations verify (5.4), including every substitution.

If the residual after the three removals contains $(2,2)$, Lemma 4.2 supplies the required residue. The only generic cases where it does not are the following hook boundaries and their reverses. The counts in the last column are exact vertical counts of the prescribed full tableau.

Shape	Two prescribed corners	Auxiliary R	Count
$(a,3,1^{2b})$, $a\geq 5$ odd	$H(1,a-1), H(2,2)$	$H(1,a-3)$	$b+1$
$(2,2,1^{2b})$, $b\geq 1$	$V(1,2), V(2b+1,1)$	$V(2b-1,1)$	$b+2$
$(a,3,1^{2b})$, $a\geq 3$ odd, $b\geq 2$	$H(2,2), V(2b+1,1)$	$V(2b-1,1)$	$b+1$

In the first and third rows, an endpoint with largest domino at $H(2,2)$ leaves a hook and forces the full count $b+1$. In the second, an endpoint at $V(1,2)$ leaves a column and forces $b+2$. The prescribed triples have precisely these counts: the residual hook contributes respectively $b+1, b-1, b-1$, and the triples contribute $0,3,2$ vertical dominoes. Hence target feasibility supplies exactly the needed residue. To see that the list is exhaustive, a boxed corner can meet $(2,2)$ only at $H(2,2)$ or $V(1,2)$; inserting those coordinates in the two tables gives the three displayed families. We have therefore proved every generic transfer by the pure-swap rule.

5.3. **Exceptional families and cases I and J.** We now treat the excluded parameter families. An intermediate corner specified below is boxed and avoids $(2,2)$. Its removal therefore leaves a special shape containing $(2,2)$, so Lemma 4.2 makes it feasible for the original residue. Each composed transfer is consequently legitimate.

The A/C family. Its exceptional coordinates force

$$F=(a,5,3,1^{m-3}),\qquad a\geq 5 \text{ odd},\quad m\geq 4 \text{ even},\qquad C=H(3,2),\quad C'=H(2,4).$$

If $a\geq 7$, route through $H(1,a-1)$, using B and then C. If $m\geq 6$, route through $V(m-1,1)$, using H and then H$'$. The latter transfer is H$'$(c) when $m=6$, and H$'$(a) when $m\geq 8$; it is not exceptional. Only $(5,5,3,1)$ remains. Reversal handles the excluded direction in C.

The A$'$/C$'$ family. Here

$$F=(3,3,2,2,1^{2b}),\qquad C=V(1,3),\qquad C'=V(3,2).$$

For $b\geq 1$, use $V(2b+3,1)$, giving B$'$ and then C$'$. The intermediate column in the first transfer has height four, so the B$'$ exception does not apply. Only $(3,3,2,2)$ remains. The exceptional B$'$ and D$'$ coordinates themselves force $F=(3,3,1,1)$.

The G/H$'$ family. The source is $H(2,4)$, the target is $V(3,l)$, where $l=1$ or 2, and $a=F_1$ is odd and at least five. For $a\geq 7$, use $H(1,a-1)$, giving A and H. For $a=5$, the first two rows have length five and the third has length at most two. The first rotation in (5.2), with $i=2, j=4$, moves the largest domino to $V(1,5)$. Its residual after the three removals has top row pair $(3,3)$, so both residues are available. Now use B$'$ from column five to the target; its exceptional source column is three. This settles the entire family, and reversal settles H$'$.

Case I. Write

$$C = H(i,j), \qquad C' = V(i-1,j+1), \qquad i,j \text{ even}.$$

First suppose $i \geq 4$, and put $u = F_{i-2}$. If $u = j+1$ and either $i \geq 6$ or $j \geq 4$, use the second rotation in (5.2). Removing its first two vertical dominoes after C removes a full two-by-two block. The resulting shape is special and retains $(2,2)$; the two vertical dominoes need not be individually boxed. If $i = 4, j = 2$ and the height is four, use instead

$$(D_{n-2}, D_{n-1}, D_n) = (V(3,1), H(3,2), H(4,2)).$$

The first rotation applies and its residual is $(F_1, 3)$. The remaining family with $u = j+1$ is

$$F = (a, 3, 3, 3, 1^{2b}), \qquad a \geq 3 \text{ odd}, \quad b \geq 1.$$

For $a \geq 5$, use the top corner $H(1, a-1)$, giving B then H. For $b \geq 2$, use $V(2b+3, 1)$, giving H then D′. Only $(3,3,3,3,1,1)$ remains.

If $F_{i-3} = u > j+1$, the corner $V(i-3, u)$ gives F followed by A′ or B′. Its column is at least four, excluding both exceptional coordinates. If $F_{i-3} > u$, specialness makes both lengths odd, with $F_{i-3} \geq u+2$. The corner $H(i-3, F_{i-3}-1)$ gives B then H.

Next let $i = 2, j \geq 4$. If $F_3 \leq j-2$, the first rotation in (5.2) applies. Its residual top row pair is $(j-2, j-2)$, so both residues are available. Otherwise $F_3 = j-1$. Let m be the even height. If the bottom row pair has equal length L and $m \geq 6$, use $V(m-1, L)$, giving H then D′; the latter has target column $j+1 \geq 5$, excluding its exception. If $m = 4$ and the bottom pair is equal, its length is $j-1$; use $H(4, j-2)$, giving D then F. If the bottom pair is unequal, both lengths are odd; use $H(m-1, F_{m-1}-1)$. For $m \geq 6$ this gives D then F. For $m = 4$ it gives C then E; C is exceptional only when $j = 4$, forcing the already retained shape $(5,5,3,1)$.

Finally let $i = j = 2$. Then $F = (3, 3, 1^{2b})$. For $b = 0$, the first rotation has empty residual and realizes the unique common feasible residue. For $b \geq 2$, route through $V(2b+1, 1)$, giving H then D′. For $b = 1$ we retain $(3,3,1,1)$.

Case J and the reverse cases. Here $C = H(i,j)$, $C' = V(i+1, j-1)$. Put $u = F_{i-1}$. If $u = j+1$, first use the now established case I to reach $V(i-1, j+1)$, then B′ to reach C'. Its sole exceptional possibility is $(3,3,1,1)$. If $u > j+1$, specialness gives u odd and $u \geq j+3$. Use $H(i-1, u-1)$, giving A and H. Reversal supplies I′ and J′.

5.4. **The four fixed residual shapes.** The preceding reductions leave four fixed shapes. In the next two tables, the residue is $\mathrm{v}(P) \pmod 4$. We first record the local connections; in the swap rows, R is the position of label $n-2$.

Shape	Residue	Connection	Auxiliary or rule
$(3,3,1,1)$	0	$V(1,3) \leftrightarrow V(3,1)$	$R = V(1,2)$
$(3,3,2,2)$	1	$V(1,3) \leftrightarrow V(3,2)$	$R = V(1,2)$
$(5,5,3,1)$	0,2	$V(1,5) \leftrightarrow H(3,2)$	$R = V(1,4)$
$(5,5,3,1)$	2	$H(2,4) \leftrightarrow H(3,2)$	$R = H(2,2)$
$(3,3,3,3,1,1)$	1,3	$V(3,3) \leftrightarrow V(5,1)$	$R = V(3,2)$
$(3,3,3,3,1,1)$	1	$H(4,2) \leftrightarrow V(3,3)$	second rotation in (5.2)

The residuals in the first two rows are $(1,1)$ and (1^4), with one and two vertical dominoes. The fourth row has residual $(5,1,1,1)$, with two vertical dominoes. The last rotation has residual $(3,1^5)$, with three vertical dominoes, and its prescribed triple contributes two more. In the third and fifth rows the residual is $(3,3,1,1)$, so Lemma 4.2 supplies both residues. These observations verify all the stated residue conditions directly.

The remaining connections are the fixed finite inputs of Proposition 8.1:

Shape	Residue	Directed connection	Input pairs
$(3,3,1,1)$	2	$V(1,3) \longrightarrow H(2,2)$	8
$(3,3,1,1)$	2	$H(2,2) \longrightarrow V(3,1)$	16
$(3,3,2,2)$	3	$V(1,3) \longrightarrow V(3,2)$	60
$(5,5,3,1)$	0	$H(3,2) \longrightarrow H(2,4)$	1995
$(3,3,3,3,1,1)$	3	$H(4,2) \longrightarrow V(3,3)$	1470

Every source tableau with that position and residue, and every recording tableau of the shape, is included. The paths restore the exact recording tableau. To check that these transfers exhaust the boxed corners, the complete connection chains in each shape and residue are

F	$\mathrm{v}(P) \bmod 4$	boxed-corner chain
$(3,3,1,1)$	0	$V(1,3) \leftrightarrow V(3,1)$
	2	$V(1,3) \leftrightarrow H(2,2) \leftrightarrow V(3,1)$
$(3,3,2,2)$	$1,3$	$V(1,3) \leftrightarrow V(3,2)$
$(5,5,3,1)$	$0,2$	$V(1,5) \leftrightarrow H(3,2) \leftrightarrow H(2,4)$
$(3,3,3,3,1,1)$	$1,3$	$H(4,2) \leftrightarrow V(3,3) \leftrightarrow V(5,1)$.

The rim-removal conditions and the boxing convention give exactly the positions listed. In $(3,3,1,1)$, deletion at $H(2,2)$ leaves the hook $(3,1,1,1)$, so that corner is feasible only in residue two. Deleting any other listed corner leaves a special shape containing $(2,2)$, which gives both shape-compatible residues by Lemma 4.2. In the first shape the residue-zero chain is local and the residue-two chain uses the two finite transfers. For $(3,3,2,2)$ the connection is local in residue one and finite in residue three. For $(5,5,3,1)$ the first link is local in both residues; the second is finite in residue zero and local in residue two. For $(3,3,3,3,1,1)$ the second link is local in both residues; the first is local in residue one and finite in residue three. This proves every feasible boxed-corner transfer.

5.5. **Unboxed source and target positions.** An unboxed rim position in a special shape has one of the forms

unboxed corner C	conditions	associated boxed corner B
$H(i,j)$	i even, j odd	$V(i-1,j+1)$
$V(i,j)$	i even, j odd, $j \geq 3$	$H(i+1,j-1)$.

(5.5)

These follow from the paired-row and shifted-column criteria. If the initial largest position is unboxed, normalize the reduced pair by rank-$(n-1)$ normalization. Formula (4.1) gives the new reduced shape $F \setminus B$. The old largest position overlaps that shape, and Lemma 3.6 forces the boxed alternative B. The full shape is again F. The special representative of the second tableau is still Q, so Lemma 3.1 restores Q itself. The reduced shape contains the square $(2,2)$, except when the full shape is the rank-two partition $(2,2)$, already a finite base. We may henceforth assume that the source is boxed.

Let the desired target C' be unboxed, and let B be its associated boxed corner in (5.5). The same normalization applied to a feasible tableau with largest position C' shows that B is feasible for the required residue. The proved boxed transfers put the largest domino at B. Delete it, writing $(\bar{P}, \bar{Q}; v, \epsilon)$ for the enriched deletion. We shall choose an auxiliary smaller tableau A of the same shape as $\bar{P}$ with

$$\mathrm{v}(A) \equiv \mathrm{v}(\bar{P}) + 2 \pmod 4. \tag{5.6}$$

The extended-cycle move below cancels this difference of two.

For a horizontal target $C' = H(i,j)$, prescribe

$$B = V(i-1, j+1), \qquad D_{n-1}(A) = V(i-1, j).$$

Removing these two positions from F removes a two-by-two block, changing the equal row lengths $(j+1, j+1)$ to $(j-1, j-1)$. The residual is special and contains $(2,2)$, unless $F = (2,2)$. Lemma 4.2, followed by appending the prescribed vertical domino, gives (5.6). In the excluded rank-two case, feasibility forces the horizontal tiling and the largest domino is already at the desired position.

For a vertical target $C' = V(i,j)$, prescribe

$$B = H(i+1, j-1), \qquad D_{n-1}(A) = H(i, j-1).$$

The two removals lower the equal heights in columns $j-1, j$ by two. Their residual is special. If it contains $(2,2)$, the same residue argument applies. Otherwise $i = 2, j = 3$, and specialness and removability force the whole family

$$F = (2k+1, 3, 3, 1^{2l+1}), \qquad F \setminus (B \cup D_{n-1}(A)) = (2k+1, 1^{2l+3}), \quad k \geq 1,\ l \geq 0. \tag{5.7}$$

Every tiling of the residual hook has $l+2$ vertical dominoes, so appending the horizontal $D_{n-1}(A)$ gives $\mathrm{v}(A) = l+2$. On the other hand, removing the desired target from F leaves

$$\mu = (2k+1, 2, 2, 1^{2l+1}).$$

Its tiling is forced geometrically. The rightmost square in its first row has no square below it; successive forced horizontal pairs fill columns $2, \ldots, 2k+1$ of that row. The remaining first column is forced vertical from its top, and column two in rows two and three is one vertical domino. Thus every tiling of μ has $l+3$ vertical dominoes. A feasible full tableau with largest domino at C' has $l+4$ vertical dominoes. Therefore the prescribed residue is $l \pmod 4$. Since B is horizontal, $\mathrm{v}(\bar{P})$ has that residue, and $\mathrm{v}(A) = l+2$ proves (5.6) for the entire boundary family.

To obtain the required pair residue, move through the extended cycle in $(A, \bar{Q})$ whose first-side component is $\{n-1\}$, and denote the result by (A^*, Q^*). The endpoints are

target	b	f	change in the pair's vertical count
$H(i,j)$	(i,j)	$(i-1, j+1)$	-2
$V(i,j)$	(i,j)	$(i+1, j-1)$	$+2$.

(5.8)

The alternate position of the first-tableau domino $n-1$ replaces b by f. No smaller label has an alternate position meeting b. For the horizontal target the only possible smaller fixed neighbor is $(i, j-1)$; entering b would require its label to exceed the entry $n-1$ at $(i-1, j)$. For the vertical target the only possible smaller fixed neighbor is $(i-1, j)$, and the same comparison is with the entry $n-1$ at $(i, j-1)$. The other neighbors are absent by target removability or by deletion of B. Thus (3.1) gives the singleton cycle $\{n-1\}$. For the smaller recording tableau obtained by reverse insertion from the special Q at B, apply [2, Theorem 2.2.3(2)(b)]. For the vertical target, $B = H(i+1, j-1)$ has fixed right square $(i+1, j)$, and target removability gives $(i, j+1) \notin F$. The theorem gives a singleton recording-side cycle with back square (i,j) and forward square $(i+1, j-1)$. For the horizontal target its transposed statement applies, using $(i+1, j) \notin F$, and gives back square (i,j) and forward square $(i-1, j+1)$. Thus the two singleton cycles have exactly the same endpoints, and no further first-side cycle is added. The cycles in the first row are up cycles: the row of f is smaller than that of b. Each reduces the vertical count by one. Those in the second row are down cycles, with the row of f larger than that of b, and each increases the vertical count by one [3, Propositions 3.3.7–3.3.8]. This proves (5.8). It also gives $\mathsf{S}(Q^*) = \mathsf{S}(\bar{Q})$.

By (5.6) and (5.8), the smaller pairs $(\bar{P}, \bar{Q})$ and (A^*, Q^*) have the same total vertical residue. Rank-$(n-1)$ transitivity connects them. Lift that sequence with the original (v, ϵ) retained. The smaller endpoint shape has acquired the square f, which lies in the old boxed position B.

Thus B is impossible at the full endpoint. Lemma 3.6 forces its alternative, precisely C'. The full shape returns to F, and its second tableau has special representative Q; hence it equals Q by Lemma 3.1. This completes the proof of Lemma 5.1.

6. Tableau transitivity and the number of cells

Theorem 6.1 ([9, Theorem 4.5.2])**.** *Let (P, Q) and (P', Q') be standard domino tableau pairs of size n, with*

$$\mathsf{S}(Q) = \mathsf{S}(Q'), \qquad \mathrm{v}(P) + \mathrm{v}(Q) \equiv \mathrm{v}(P') + \mathrm{v}(Q') \pmod 4.$$

Then an allowed sequence takes (P, Q) to (P', Q').

Proof. We use simultaneous induction. Let H_n be the present statement, let N_n be the normalization statement of Lemma 4.3, and let L_n be the largest-position statement of Lemma 5.1. The finite bases $n \le 4$ are verified in Proposition 8.1. Both congruence sectors, zero and two modulo four, are included.

Assume H_{n-1} and N_{n-1}. The proof of Lemma 4.3 gives N_n using only these smaller-rank statements and its two fixed rank-seven configurations. The proof of Lemma 5.1 gives L_n using smaller-rank transitivity, smaller-rank normalization, and the fixed corner-transfer inputs. In its coordinate argument, case I precedes case J. The induction dependencies are

$$(H_{n-1}, N_{n-1}) \Longrightarrow (N_n, L_n) \Longrightarrow H_n.$$

First suppose $Q = Q'$ is special and the shape contains $(2, 2)$. By L_n, put the largest domino of the first pair at $D_n(P')$ while restoring the exact tableau Q. Reverse insertion from that common Q at that common position produces the same smaller second tableau $\bar{Q}$ and the same insertion data (v, ϵ) for both pairs. The two smaller total vertical counts agree modulo four: the removed first-tableau domino has the same orientation, and the reverse-insertion operation on the second tableau is identical. Apply H_{n-1} to the smaller pairs. Lemma 3.3 lifts the sequence with (v, ϵ) fixed; unique reconstruction gives the exact desired full pair (P', Q).

For arbitrary pairs whose shapes contain $(2, 2)$, apply N_n to each. Their second tableaux become the same $\mathsf{S}(Q)$, and their total residues remain equal. Connect the normalized pairs by the preceding paragraph and reverse the second normalization. If $(2, 2)$ is absent, apply Lemma 4.4. That square is fixed in the D grid, so the two branches cannot be mixed by equality of special representatives. This proves H_n and completes the simultaneous induction. □

6.1. **An independent count of left cells.** Let $\mathrm{Irr}(W)$ be the isomorphism classes of irreducible complex W-representations, and let $\mathcal{S}(W) \subseteq \mathrm{Irr}(W)$ be Lusztig's special representations [14].

For a left cell Γ, take the quotient of the $\mathbb{Z}[v, v^{-1}]$-span of the C'_x with $x \le_L y$ for some $y \in \Gamma$ by the span of those with $x \notin \Gamma$. Both spans are left ideals. Specialize at $v = 1$ over $\mathbb{C}$ and tensor with the sign character $w \mapsto (-1)^{\ell(w)}$; denote the resulting representation by $[\Gamma]$. This is the cell-representation convention of [7, §2]. Indeed, the assignment $T_s \mapsto -T_s^{-1}$ preserves the quadratic and braid relations and is an involution. Bar invariance and triangularity give its image of C'_w as $(-1)^{\ell(w)} C_w$, where C_w is the positive-power KL basis used there, with $C_s = T_s - v$. The multiplication coefficients of the two bases have the same supports, so their cell preorders agree. The involution specializes to $w \mapsto (-1)^{\ell(w)} w$, giving the stated tensor factor.

Lemma 6.2. *The number of equal-parameter left cells of a finite Weyl group is*

$$N(W) = \sum_{E \in \mathcal{S}(W)} \dim E.$$

Proof. Order the left cells as $\Gamma_1, \dots, \Gamma_N$ by a linear extension of the left-cell order, with lower cells first. The spans of successive unions $\Gamma_1 \cup \cdots \cup \Gamma_i$ in the C'-basis form a filtration of the regular Hecke module. Each quotient is the free module with basis Γ_i described above, so specialization at $v = 1$ is exact. Semisimplicity over $\mathbb{C}$ splits the filtration. Tensoring the whole decomposition with the sign character leaves the regular representation isomorphic to itself; explicitly, the basis map is $w \otimes 1 \mapsto (-1)^{\ell(w)} w$. Hence

$$\mathbb{C}[W] \cong \bigoplus_{i=1}^{N} [\Gamma_i].$$

Every two-sided cell has a unique associated special irreducible E. It occurs with multiplicity one in $[\Gamma]$ for every left cell Γ in that two-sided cell, and occurs in none of the other left-cell representations [7, §2 and Theorems 2.8, 4.6–4.7]. Its multiplicity in the regular representation is $\dim E$, so its two-sided cell contains exactly $\dim E$ left cells. Summing over the special representations proves the formula. □

Lemma 6.3. *For $n \geq 4$, the number of standard special D-domino tableaux with n dominoes equals $N(W(D_n))$.*

Proof. Let λ be a special D-shape of size $2n$, padded with zero rows to length $2r$. Put $b_i = \lambda_i + 2r - i$. Each pair of rows contributes one even and one odd beta number. Write these as

$$2p_1 > \cdots > 2p_r, \qquad 2q_1 + 1 > \cdots > 2q_r + 1.$$

The special row-pair condition is equivalent to

$$p_1 \geq q_1 \geq p_2 \geq q_2 \geq \cdots \geq p_r \geq q_r. \tag{6.1}$$

Indeed, equal even rows give $p_i = q_i$, whereas two odd rows give $p_i > q_i$. The inequality between neighboring row pairs gives $q_i \geq p_{i+1}$. Conversely, merging the two lists under (6.1) recovers exactly these row pairs.

Padding λ by two further zero rows changes the two lists by

$$(p_1, \dots, p_r;\ q_1, \dots, q_r) \longmapsto (p_1 + 1, \dots, p_r + 1, 0;\ q_1 + 1, \dots, q_r + 1, 0).$$

The unordered pair of lists modulo this shift is the defect-zero symbol used to parametrize type D representations. Equal lists are called degenerate and index two representations. The special-symbol condition is (6.1); see [15, §2, Propositions 2.1–2.2 and (2.3)–(2.5)]. A nondegenerate special symbol has exactly one ordering satisfying this condition. If both orderings satisfy it, $p_i \geq q_i$ and $q_i \geq p_i$ for every i, so $p = q$. Thus special shapes count the nondegenerate special symbols once and the degenerate symbols once.

Set $\alpha_i = p_i - (r - i)$ and $\beta_i = q_i - (r - i)$, deleting terminal zeros. The strict decrease of each list makes α, β partitions. Moreover,

$$2\sum_i (p_i + q_i) + r = \sum_{i=1}^{2r} b_i = 2n + r(2r - 1), \qquad |\alpha| + |\beta| = n.$$

These two partitions are the two-quotient of λ.

We count domino tableaux directly from the beta set $\{b_1, \dots, b_{2r}\}$. Removing a rim domino replaces one beta number b by $b-2$, with $b-2 \geq 0$ and $b-2$ absent from the set. For a horizontal corner this shortens one row by two. For a vertical corner in two equal rows, their consecutive beta numbers $b, b-1$ become $b-1, b-2$, giving the same replacement of sets. Conversely, if $b-1$ is absent, the replacement gives a horizontal corner: the next beta number is at most $b-3$, so the row-length difference is at least two. If $b-1$ is present, the two corresponding rows are

equal, and absence of $b-2$ makes their common end a removable vertical corner. These are all possible beta-number moves.

On the even beta numbers this move decreases a single p_i by one without collision, hence removes a corner from α; on the odd beta numbers it removes a corner from β. It follows that a domino-removal chain is uniquely an interleaving of a corner-removal chain of α and one of β. When both partitions are empty, $p = q = (r-1, \dots, 0)$ and the beta set is $\{0, \dots, 2r-1\}$, which corresponds to the empty diagram. Thus every such interleaving reverses to a standard domino tableau. Choosing the $|\alpha|$ labels of the first chain and then the two chains gives

$$d(\lambda) = \binom{n}{|\alpha|} f^\alpha f^\beta, \tag{6.2}$$

where $d(\lambda)$ is the number of standard domino tableaux of shape λ and f^α is the number of standard Young tableaux of shape α.

Write S_m for the symmetric group on m letters. The full signed-permutation group is $\{\pm 1\}^n \rtimes S_n$. The character of $\{\pm 1\}^n$ that is trivial on the first $|\alpha|$ coordinates and nontrivial on the remaining $|\beta|$ coordinates has stabilizer $\{\pm 1\}^n \rtimes (S_{|\alpha|} \times S_{|\beta|})$. Extend this character trivially across the symmetric-group factors, tensor it with their Specht representations indexed by α, β, and induce to the full group. This gives its irreducible representation indexed by (α, β), of dimension $[S_n : S_{|\alpha|} \times S_{|\beta|}] f^\alpha f^\beta = d(\lambda)$ [15, Proposition 2.1]. Tensoring with the character $w \mapsto (-1)^{\#\{i : w(i) < 0\}}$ exchanges α and β. The index-two restriction rule therefore gives one irreducible $W(D_n)$-representation of dimension $d(\lambda)$ if $\alpha \neq \beta$, and two irreducibles of dimension $d(\lambda)/2$ if $\alpha = \beta$ [15, Propositions 2.1–2.2]. In the latter case both are special. Thus the contribution of each special symbol to Lemma 6.2 is $d(\lambda)$ in either case. Summing over the special shapes proves the assertion. □

Proof of Theorem 1.2. Let G be the graph on $W(D_n)$ containing precisely the simple-left edges whose endpoints belong to the same Kazhdan–Lusztig left cell. Let $\pi_0(G)$ be its set of connected components. Every component lies in one left cell, so

$$\#\pi_0(G) \geq N(W(D_n)).$$

Associate to w the special recording tableau $f(w) = \mathsf{S}(Q(w))$. All elements of $W(D_n)$ have total vertical count zero modulo four under domino insertion. Theorem 6.1 therefore joins any two elements of a nonempty fiber of f by the allowed operators. Proposition 3.4, the ordinary-star property, and Lemma 2.2 replace this sequence by a path in G. Hence every nonempty fiber of f is contained in a single component, and

$$\begin{aligned} \#\pi_0(G) &\leq \#\{\text{nonempty fibers of } f\} \\ &\leq \#\{\text{standard special } D\text{-tableaux of size } n\} \\ &= N(W(D_n)). \end{aligned}$$

Equality follows. Since every left cell is nonempty and is a union of components, it consists of exactly one component. □

7. Completion by finite type

Proof of Theorem 1.1. First assume that W is irreducible. For type A_n, left cells are the fibers of the recording-tableau map in the Robinson–Schensted correspondence and are connected by the ordinary left stars [12, §5]. Each such star is a simple-left edge by Section 2.

For types B_n and C_n, Garfinkle's result [3, Theorem 3.5.9], together with the correspondence between its primitive-ideal classes and KL left cells [14, §5.25], yields connectedness. This application, with the same simple-left convention used here, is recorded explicitly in [8, Example 7.3]. The two types have the same equal-parameter Coxeter system.

For the dihedral group $I_2(m) = \langle s, t : s^2 = t^2 = (st)^m = 1\rangle$, let w_0 be its longest element, of length m. The cells are $\{1\}$, $\{w_0\}$, and the two sets of alternating words of lengths $1, \dots, m-1$ with a fixed rightmost generator [10, §7.15]. Appending a simple generator on the left connects successive words in either set. This includes G_2. The rank-one case consists of singleton cells.

Type D_n is Theorem 1.2. Finally, all left cells in F_4, E_6, E_7, E_8 are left-connected by the exhaustive verification in [8, Example 7.3]. This is the exceptional-type input to the classification proof.

For a reducible finite Weyl group $W = W_1 \times \cdots \times W_r$, its Hecke algebra and KL basis are tensor products. Multiplication by a simple generator changes only one factor, and the left cells are exactly $\Gamma_1 \times \cdots \times \Gamma_r$, where Γ_i is a left cell of W_i. Join the first coordinates inside Γ_1, then the second coordinates inside Γ_2, and continue. Every intermediate tuple remains in the original product cell. The trivial group is immediate. □

8. The finite verifications

The finite inputs to the type D induction are the following bounded configurations. The accompanying verification files contain the exact integer algorithms, finite path data and a single execution script.

Proposition 8.1. *The following finite assertions hold.*

(1) *In $W(D_4)$, every arrow of (2.2)–(2.3) has the internal simple-left path specified in Lemma 2.2.*
(2) *Domino insertion intertwines the three-label fork stars and the four-label branched operations with their tableau rules in the respective initial ranks. The extended-cycle operation through label one corresponds to changing the sign of value one.*
(3) *The transitivity assertion of Theorem 6.1 holds in ranks at most four, in both total-vertical-count sectors.*
(4) *The two rank-seven normalization constructions in Lemma 4.3 have the stated lifted positions, high-star outputs and reduced special shapes, for every possible input pair.*
(5) *For each row of Table 7, every pair of the specified shape, first-tableau residue and initial largest position admits an allowed-operator path to the final largest position with its exact second tableau restored.*

8.1. **Exact cells and simple-left paths in rank four.** Compute the cells directly from the Kazhdan–Lusztig recurrence. Enumerate the signed permutations with an even number of negative entries and obtain their lengths by breadth-first search in the four simple generators. For each nonidentity y, choose s with $sy < y$. With $[1, y] = \{x : x \le y\}$, its lower Bruhat interval is obtained from that of sy by the subword recursion

$$[1, y] = [1, sy] \cup s[1, sy].$$

Starting with $P_{x,x} = 1$ and $P_{x,y} = 0$ when $x \not\le y$, compute the Kazhdan–Lusztig polynomials over $\mathbb{Z}[q]$ by

$$P_{x,y} = q^{1-c} P_{sx,sy} + q^c P_{x,sy} - \sum_{\substack{z<sy \\ sz<z}} \mu(z, sy) q^{(\ell(y)-\ell(z))/2} P_{x,z}, \qquad c = \begin{cases} 1, & sx < x, \\ 0, & sx > x. \end{cases}$$

Here $\mu(x,y)$ is the coefficient of $q^{(\ell(y)-\ell(x)-1)/2}$ in $P_{x,y}$, and is zero unless this exponent is a nonnegative integer. The multiplication rule for $C'_sC'_y$ gives the directed edges

$$y \longrightarrow sy \quad (sy > y), \qquad y \longrightarrow x \quad (\mu(x,y) \neq 0,\ sx < x,\ sy > y).$$

Its strongly connected components are exactly the left cells.

Enumerate the four-root type predicates and all four candidate words for each applicable outer root. For each resulting arrow, execute the corresponding row of the path table and compare the cell of every vertex with that of its source. This gives

$$|W(D_4)| = 192, \qquad N(W(D_4)) = 36, \qquad \#\text{arrows} = 192,$$

with 84 paths of length one and 108 paths of length three. Every path has the required endpoint and stays in its source cell. An independent check uses the defining Hecke bar involution: all 192 computed C'_w are bar-invariant and have the required strict triangular expansion. This proves assertion (1).

8.2. **Insertion, local rules and the initial ranks.** Represent a domino by (a, r, c, h), where a is its label, (r, c) its top-left square, and $h = 0$ for a horizontal domino and $h = 1$ for a vertical domino. All coordinates in the mathematical description and in the exported tableau lists are one-based. In the programs, generator indices $0, 1, 2, 3, \ldots$ mean $s'_1, s_2, s_3, s_4, \ldots$. An array of generator indices denoting a group word is applied from right to left; an executing path is applied in the order in which it is listed.

The initial-rank checks compare tableau descents with (2.1) before selecting any operator domain: there are 144 comparisons in rank three and 1536 in rank four. The transpose identity is checked on all $48 + 384 = 432$ inputs. For the fork rule enumerate all $2^3 3! = 48$ signed permutations. There are 32 elements in the star domain. The direct three-domino replacement and its cycle, inverse and transpose cases give the same 32 arrows as the group star. For the four-root rule enumerate all $2^4 4! = 384$ signed permutations. The extended-cycle identity through one holds for all 384. The direct $C \longrightarrow B_b$ rule has 60 input–outer-root pairs and 96 output images, all agreeing with the group definition. Reciprocity and the insertion–transpose identity give the reverse and complementary types. This proves assertion (2); the deletion argument of Proposition 3.4 is what extends it to arbitrary rank.

For the induction bases, form the undirected graph of all stars and four-root arrows on all signed permutations. Label a vertex by its negative-sign parity and the exact special recording tableau. Enumerating the connected components and the fibers of this label gives the following table. Every fiber is one component, and every inverse insertion recovers its original signed permutation.

Rank	Signed permutations	Directed operator arrows	Components	Label fibers
2	8	0	8	8
3	48	64	20	20
4	384	1152	72	72

In every case the total vertical count is twice the negative-sign parity modulo four. Rank zero and rank one are immediate. Thus assertion (3) includes both congruence sectors needed after deletion.

For normalization in these ranks, put $r \equiv \mathrm{v}(P) + \mathrm{v}(Q) \pmod 4$. The square $(2, 2)$ is fixed, so $\mathrm{sh}(\mathsf{S}(Q))$ contains it whenever $\mathrm{sh}(Q)$ does. By Lemma 4.2, choose a tableau P^* on the special shape with

$$\mathrm{v}(P^*) \equiv r - \mathrm{v}(\mathsf{S}(Q)) \pmod 4.$$

Since r is even, $r - \mathrm{v}(\mathsf{S}(Q))$ has the same parity as $\mathrm{v}(\mathsf{S}(Q))$, which is the vertical-count parity of this shape. The verified transitivity joins (P, Q) to $(P^*, \mathsf{S}(Q))$. For largest-position transfer, feasibility supplies the target pair, and the same verified transitivity joins it to the source. This proves both base statements.

8.3. **The two normalization configurations.** The first reduced shape is $(5, 3, 3, 1)$ and its full shape before the construction is $(5, 4, 4, 1)$. The second reduced shape is $(3, 3, 3, 1, 1, 1)$ and its full shape is $(3, 3, 3, 2, 2, 1)$. Each reduced shape has 30 standard domino tableaux and each full shape has 35, so each check has 1050 inputs. For each input the program performs the following finite operations:

(1) reverse insert the largest domino, retaining its position and sign;
(2) use the explicit six-domino auxiliary tableau in Lemma 4.3, making the sign-one correction exactly when the reduced residues differ by two;
(3) reinsert the retained datum and apply the prescribed high star;
(4) delete again, choose a first tableau of the required residue on the now computed special reduced shape, and reinsert.

The endpoint in the last step is supplied by rank-six induction. For every input, the verification asserts the largest positions before and after the high star and the special shape after the next deletion. In each configuration the sign-one correction occurs for 490 inputs. The final largest positions are respectively $H(3, 2)$ and $V(3, 3)$, on the special shapes $(5, 5, 3, 1)$ and $(3, 3, 3, 3, 1, 1)$. The final second tableau is the exact special representative of the input second tableau in every case. This proves assertion (4).

8.4. **The four corner-transfer shapes.** Table 7 lists the five required transfers on four shapes. The residue column is $\mathrm{v}(P) \pmod 4$, and Q ranges over all tableaux of the shape. Both signed parity sectors occur among the inputs. The path lengths count allowed operators; each four-root arrow is subsequently replaced by its path from Lemma 2.2.

Shape	Residue	Initial n	Final n	$\#P$	$\#Q$	Pairs	Max. length
$(3, 3, 1, 1)$	2	$V(1, 3)$	$H(2, 2)$	1	8	8	1
$(3, 3, 1, 1)$	2	$H(2, 2)$	$V(3, 1)$	2	8	16	1
$(3, 3, 2, 2)$	3	$V(1, 3)$	$V(3, 2)$	3	20	60	3
$(5, 5, 3, 1)$	0	$H(3, 2)$	$H(2, 4)$	19	105	1995	10
$(3, 3, 3, 3, 1, 1)$	3	$H(4, 2)$	$V(3, 3)$	14	105	1470	9

TABLE 7. Complete finite inputs for the residual corner transfers.

Here is an exhaustive enumeration procedure independent of the stored cardinalities. For a partition λ of $2n$, remove either the last two squares of row i when $\lambda_i - \lambda_{i+1} \geq 2$, or the final square of each of rows $i, i + 1$ when $\lambda_i = \lambda_{i+1} > \lambda_{i+2}$. Recurse and attach the label n to the removed domino. Start with the unique empty tableau. A largest domino in a standard tableau is removable, so this recursion gives every standard domino tableau exactly once. Select the first tableaux with the required largest position and residue, take every second tableau of the shape, and inverse insert every resulting pair. The resulting sets of signed permutations coincide exactly with the initial vertices of the stored traces, giving exactly 3549 distinct inputs in their specified cases.

Replay each trace using signed permutations and the four-word definition. At every step check the adjacent-root condition and star domain, or the four-root type predicates; also check the exact endpoint, reciprocity, sign parity and the special recording tableau. At the end check

the required largest position and the exact original recording tableau. There are 25259 operator steps, comprising 17445 stars and 7814 four-root arrows. All these assertions hold at every step. This proves assertion (5). The generic corner rules in Lemma 5.1 cover the other possible residue on each of these four shapes, as explicitly recorded there.

8.5. **Reproduction.** The verification directory is part of the supplementary material. Run

```
python3 certificates/run_all.py
```

from the extracted article directory. The script requires Python 3 and Node.js and runs ten verification jobs. Polynomial arithmetic is over the integers. The manifest records the results and file hashes. Path replay checks each listed operator using its definition; the independent input enumerator checks the complete input sets. A second implementation of specialization moves the open unboxed cycles given by (3.1); it agrees with the insertion runtime on all 607 distinct tableaux in the local checks.

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