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DOMINO INSERTION AND CELLS IN TYPE B

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Abstract. We prove Conjectures A, A$^+$, and B of Bonnafé, Geck, Iancu, and Lam on Kazhdan–Lusztig cells in type B.

1. Introduction

Kazhdan and Lusztig introduced the canonical basis of a Hecke algebra, its cells, and the associated representations in [20]. In the symmetric group, the Robinson–Schensted correspondence describes the one-sided cells by tableau fibres and the two-sided cells by shape. For a Hecke algebra with unequal parameters, the cells also depend on the weight function. Lusztig's work [22, 23] provides the general framework for this dependence.

Let W_n be the group of signed permutations on n letters, with simple reflections $t, s_1, \dots, s_{n-1}$. In one-line notation, right multiplication by t changes the sign of the first letter, and right multiplication by s_i interchanges positions i and $i+1$. Put $W_0 = \{1\}$. Let Γ be a totally ordered abelian group and let $L : W_n \to \Gamma$ be the weight function defined by

$$L(t) = b > 0, \qquad L(s_i) = a > 0.$$

Thus $L(xy) = L(x) + L(y)$ whenever the Coxeter lengths satisfy $\ell(xy) = \ell(x) + \ell(y)$. A standard parabolic subgroup is generated by a subset of these simple reflections.

Several ranges of positive integer weights were understood before the domino conjectures were formulated. Garfinkle's work on primitive ideals [8, 9, 10] gives the equal-parameter description by domino tableaux and open cycles; see [6, Theorem 4.4]. Lusztig's fixed-point comparison with type A gives the ratios $b/a = 1/2$ and $3/2$ [22, Theorem 11], as explained in [6, Theorem 3.14]. Bonnafé and Iancu [7, Theorem 7.7 and Proposition 7.9] described asymptotic left cells by recording bitableaux and proved that their representations are irreducible. Bonnafé [2, Corollary 3.6 and Corollary 5.2] determined the two-sided cells and established the specialization range $b > (n-1)a$. These descriptions suggested an interpolation by domino tableaux with varying staircase core.

For each nonnegative integer j, rank-j domino insertion is a bijection from W_n

$$w \longmapsto \bigl(P^j(w), Q^j(w)\bigr)$$

onto the pairs of standard domino tableaux of the same shape and core $\delta_j = (j, j-1, \dots, 1)$, with exactly n labelled dominoes each; $\delta_0 = \varnothing$. Here $P^j(w)$ is the insertion tableau and $Q^j(w)$ the recording tableau. Garfinkle's algorithms for ranks zero and one were extended to arbitrary staircase cores by van Leeuwen [28, §4.2]; Lam [21] developed the corresponding growth-diagram description. Write $\lambda_j(w)$ for their common shape. Define equivalence relations on W_n by

$$\begin{aligned} x \mathrel{\mathcal{L}_j} y &\iff Q^j(x) = Q^j(y), \\ x \mathrel{\mathcal{R}_j} y &\iff P^j(x) = P^j(y), \\ x \mathrel{\mathcal{S}_j} y &\iff \lambda_j(x) = \lambda_j(y). \end{aligned}$$

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For equivalence relations E, F, their join $E \vee F$ is the smallest equivalence relation containing both, or equivalently the transitive closure of $E \cup F$.

Geck [11, Conjecture 2.17] predicted the parameter chambers, the integer walls up to $n-1$, and the asymptotic range, and recorded an independent formulation due to Bonnafé. The joint left-cell conjecture of Bonnafé, Geck, Iancu, and Lam was already stated in Bonnafé's 2005 habilitation synthesis [1, §3.B.2, Conjecture 2]: for positive real ratios, recording fibres describe the open chambers, and the partition at an integer wall is the common coarsening generated by its two neighbouring partitions. Iancu's report on the March–April 2006 Oberwolfach meeting [19] describes the same joint project and its asymptotic, off-wall, and wall cases. The 2006 preprint of [6] was revised in 2007 and published in 2010. In the revised numbering, Conjecture A concerns $a = 2$, $b = 2r + 1$; Conjecture A^+ extends the rank-r description to the whole interval $ra < b < (r+1)a$ in an ordered abelian group. Conjecture B describes the wall $b = ra$ as the join of the two adjacent tableau-fibre relations. The paper proves the preceding special cases and records cell computations through rank six.

The choice $a = 2$, $b = 2r+1$ also has a representation-theoretic origin. The corresponding finite-field specializations of these Hecke algebras occur as endomorphism algebras of induced cuspidal representations of finite unitary groups. Their irreducible modules parametrize the corresponding unipotent representations labelled by partitions with core δ_r [6, Remark 1.1].

Pietraho [24, Theorem 3.6] identified the join of adjacent recording-fibre relations with moving through noncore open cycles. He subsequently expressed constructible characters as sums over the shapes in such cycle families [25, Corollary 4.6]. Geck's identification of constructible and cell characters [12, 14] supplies the representation-theoretic connection, subject to Lusztig's properties P1–P15. Under these properties, the integer-wall classification is already stated in [26, Corollary 5.7(2)]. The inclusion theorem used there was proved by Bonnafé [4] and corrected in [5]. The corrected Theorem 1 retains the inclusion of combinatorial cells in Kazhdan–Lusztig cells at every positive integer wall. The correction restricts the proof of the general inclusion to these walls and the previously known asymptotic range.

Further unconditional results concern the last wall and the asymptotic range. Howse [17, Theorem 8.8] characterized left cells by generalized Vogan classes for $b/a \geq n-1$. Howse and Pietraho [18, Theorem 4.12] obtained the predicted rank-$(n-2)$ cycle description at $b/a = n-1$. For arbitrary positive integer weights, the finite-type argument in [27, §4] proves P1–P15 and identifies the off-wall cells within the same induction on n. Together with the wall argument, this gives the following theorem.

Theorem 1.1. *Let $\sim_L, \sim_R, \sim_{LR}$ be the Kazhdan–Lusztig cell equivalence relations for (W_n, L).*

(1) *If $r \in \mathbb{Z}_{\geq 0}$ and $ra < b < (r+1)a$, then*

$$\sim_L = \mathcal{L}_r, \qquad \sim_R = \mathcal{R}_r, \qquad \sim_{LR} = \mathcal{S}_r.$$

(2) *If $r \in \mathbb{Z}_{\geq 1}$ and $b = ra$, then*

$$\sim_L = \mathcal{L}_{r-1} \vee \mathcal{L}_r, \qquad \sim_R = \mathcal{R}_{r-1} \vee \mathcal{R}_r, \qquad \sim_{LR} = \mathcal{S}_{r-1} \vee \mathcal{S}_r.$$

Part (1) is Conjecture A^+ of [6]; its specialization $a = 2$, $b = 2r + 1$ is Conjecture A. Part (2) is Conjecture B. All the inequalities in the theorem are interpreted in Γ.

For positive integer weights, part (1) is the off-wall result of [27, Theorem 4.1, Lemma 4.4, and §4.4]. We state that input, including its parabolic consequences, in Proposition 2.1. Our proof of part (2) revisits the character comparison in [26, Lemma 5.5 and Theorem 5.6]. We use the identity

$$\langle \chi_C, \chi_{C'} \rangle = |\mathcal{S}(C) \cap \mathcal{S}(C')|,$$

where χ_C is the cell character in the convention of Section 2.1, and $\mathcal{S}(C)$ is the set of rank-$(r-1)$ recording shapes of C on the wall $b = ra$. It identifies the shape sets of

cells with the same character. Truncated induction and a dimension count then identify their characters with the corresponding shape sums and show that each cell contains one cycle family. This gives the comparison required in the proof of [26, Lemma 5.5]. For ordered-group parameters we give the finite-support specialization argument explicitly; its antecedents include [11, Proposition 2.10] and [3, §5].

The additional dominance assertion in [6, Remark 1.2(c$^+$)] concerns the two-sided preorder. Geck–Iancu [15, Remark 7.12] reported a closely related example of Bonnafé for the induction preorder on irreducible characters. That preorder is related to the Kazhdan–Lusztig preorder by the implication in [15, Remark 2.6]. Subsequent work of Guilhot and Jacon [16] describes the induction order by symbols for integer ratios. Remark 5.1 gives a direct Kazhdan–Lusztig counterexample at $a = 2$, $b = 1$ in W_5. Its proof uses folding, explicit insertion, and the canonical-basis recursion.

2. Conventions and the finite-type input

2.1. **Hecke algebras and cells.** Let $\mathcal{A} = \mathbb{Z}[\Gamma]$, with basis elements e^γ for $\gamma \in \Gamma$. We use the normalized standard basis $(N_w)_{w\in W_n}$ of the Hecke algebra H. Its multiplication is determined by $N_xN_y = N_{xy}$ when $\ell(xy) = \ell(x) + \ell(y)$ and by

$$(N_s - e^{L(s)})(N_s + e^{-L(s)}) = 0.$$

The bar involution sends e^γ to $e^{-\gamma}$ and N_w to $N_{w^{-1}}^{-1}$. Our canonical basis is characterized by

$$\overline{C_w} = C_w, \qquad C_w = N_w + \sum_{x<w} p_{x,w}N_x, \qquad p_{x,w} \in \bigoplus_{\gamma<0} \mathbb{Z}e^\gamma. \tag{2.1}$$

Here $<$ is Bruhat order. Set $p_{w,w} = 1$ and $p_{x,w} = 0$ when $x \not\leq w$. In particular $C_s = N_s + e^{-L(s)}$. Write $x \leftarrow_L y$ if C_x has nonzero coefficient in C_sC_y for some simple reflection s. The reflexive transitive closure is $\leq_L$. Right multiplication defines $\leq_R$, and their joint transitive closure is $\leq_{LR}$. Mutual comparability gives the three cell equivalence relations. For a left cell C and $c \in C$, its left cell module is

$$M_C = \left(\bigoplus_{x\leq_L c} \mathcal{A}C_x\right) \Big/ \left(\bigoplus_{\substack{x\leq_L c\\ x\notin C}} \mathcal{A}C_x\right).$$

Both sums are left ideals, and the residue classes indexed by C form an $\mathcal{A}$-basis.

For integer weights work over $\mathbb{Z}[v, v^{-1}]$, with $e^1 = v$. The character χ_C is obtained from M_C by twisting with the $\mathcal{A}$-linear algebra involution

$$N_s^\dagger = -N_s^{-1} \tag{2.2}$$

and then specializing at $v = 1$. This is the ordinary character in the constructible-character convention of [12, 14]; in particular $\chi_C(1) = |C|$. The sign character sgn takes every simple reflection to -1. The common sign twist preserves character inner products, where

$$\langle \chi, \psi\rangle_V = \frac{1}{|V|}\sum_{w\in V} \chi(w)\overline{\psi(w)}$$

for characters of a finite group V; omit V when it is W_n. The longest-element singleton in a symmetric group affords the sign character in this convention.

2.2. **Domino tableaux and character labels.** A partition $p = (p_1, p_2, \ldots)$ is identified with its diagram $\{(i, j) : 1 \le j \le p_i\}$. Its size is $|p| = \sum_i p_i$, and p^{t} is its transpose. Its 2-core is the partition left after removing rim dominoes until none remains. A standard domino tableau of rank q has core δ_q, filled with zeroes, and n dominoes labelled $1, \ldots, n$. Each positive label occupies two adjacent squares, and labels weakly increase along rows and columns. Write $\mathrm{SDT}_q(p)$ for the set of these tableaux of shape p, so $|p| = 2n + q(q+1)/2$, and write $\operatorname{sh} T$ for a tableau's shape. We use the insertion conventions of [6, §3]. Inversion interchanges P^q and Q^q, and multiplication by the longest element $w_0 = (-1, -2, \ldots, -n)$ transposes both tableaux [6, Proposition 3.6].

For a partition p of size $2n + q(q+1)/2$ with core δ_q, fix its two-quotient as follows. Pad p to a length k with $k \equiv q + 1 \pmod 2$ and form the beta numbers $p_i + k - i$. Separate the even and odd numbers; divide the even ones by two and subtract one from the odd ones before dividing by two. If the resulting decreasing lists are $a_1 > \cdots > a_e$ and $b_1 > \cdots > b_o$, put

$$\alpha_i = a_i - e + i, \qquad \beta_i = b_i - o + i,$$

omitting zero parts. The even runner is first. For fixed q, this construction is a bijection with the bipartitions of n, namely the ordered pairs (α, β) of partitions such that $|\alpha| + |\beta| = n$ [28, §2.4]. Under $W_n = C_2 \wr \mathfrak{S}_n$, where $C_2 = \{1, -1\}$, α uses the trivial character of C_2 and β the nontrivial character. Denote the associated ordinary irreducible character by E_p. Increasing k by two adds one zero on each runner and shifts its other entries by one; the subtracted staircases change by the same amount. Thus the label is unchanged. In these conventions [26, Theorem 4.2 and Corollary 4.4],

$$E_{p^{\mathrm{t}}} = E_p \otimes \mathrm{sgn}, \qquad \dim E_p = \binom{n}{|\alpha|} f^\alpha f^\beta = |\,\mathrm{SDT}_q(p)|, \tag{2.3}$$

where f^α counts ordinary standard tableaux of shape α. The bipartition representation is induced from $C_2^n \rtimes (\mathfrak{S}_{|\alpha|} \times \mathfrak{S}_{|\beta|})$, whose index is $\binom{n}{|\alpha|}$; its two Specht factors have dimensions f^α and f^β. For the last equality, adding one domino moves one bead by one position on its runner and adds a box to the corresponding quotient partition. A standard domino tableau is a chain of these additions. Such a chain is specified by a standard tableau on each quotient and an interleaving of their $|\alpha|$ and $|\beta|$ additions, giving the binomial factor. Thus a fixed recording tableau of shape p has exactly $\dim E_p$ preimages under domino insertion.

Proposition 2.1 (Finite-type input)**.** *For every n and all positive integers a, b, (W_n, L) satisfies Lusztig's properties P1–P15* [23, §14.2]*, as do all its standard parabolic subgroups. If $r \in \mathbb{Z}_{\ge 0}$ and $ra < b < (r+1)a$, its left cells are the fibres of Q^r, and its right cells are the fibres of P^r.*

Proof. The assertions for W_n are [27, Theorem 4.1, Lemma 4.4, and §4.4]. The off-wall classification is established in the same induction as P1–P15, before the integer-wall specialization step.

Every standard parabolic is a product of a group W_m with parameters (a, b) and symmetric groups with equal parameter a. The first factor is covered by the finite-type theorem and the other factors by the equal-parameter theorem [23, Chapter 15]. P1–P15 pass to these products [27, Lemma 4.14]: the canonical bases and structure constants factor, the a-values add, and the preorders are componentwise. □

For integer weights, write $C_x C_y = \sum_z h_{x,y,z} C_z$ and set $\mathbf{a}(z) = \max_{x,y} \deg_v h_{x,y,z}$, taking the degree of zero to be $-\infty$.

Lemma 2.2. *For positive integer weights,*

$$\sim_{LR} = \sim_L \vee \sim_R .$$

For every rank j, independently of parameters, $\mathcal{L}_j \vee \mathcal{R}_j = \mathcal{S}_j$.

Proof. If $x \sim_{LR} y$, concatenate directed mixed left/right paths in both directions. Proposition 2.1 supplies P4, P9, and P10. By P4, **a** is monotone along the resulting closed path and hence constant there. P9 and P10 turn each left or right edge into the corresponding equivalence. The reverse containment follows from the definitions. This is also [14, Lemma 1]. For the second equality, every pair of same-shape tableaux occurs. Thus (P, Q) and (P', Q') of the same shape can be connected through (P', Q). Each step preserves one tableau. □

3. The integer wall

Throughout this section a, b are positive integers, $b = ra$ with $r \geq 1$, and $q = r - 1$. The tableau rank q remains fixed when the group rank varies.

3.1. **Cycle families.** Let T be a standard domino tableau of rank q with n dominoes. A square (i, j) is variable if $i + j \equiv q \pmod 2$, and fixed otherwise. Let $T(i, j)$ be its label. Put the label 0 outside the positive quadrant, and $+\infty$ at positive-coordinate squares outside the diagram. For the domino $D(k)$ with fixed square (i, j), its alternative domino $D'(k)$ retains (i, j) and chooses its other square by

other square of $D(k)$	comparison	other square of $D'(k)$
$(i+1, j)$ or $(i, j-1)$	$k < T(i-1, j+1)$	$(i-1, j)$
	$k > T(i-1, j+1)$	$(i, j+1)$
$(i-1, j)$ or $(i, j+1)$	$k < T(i+1, j-1)$	$(i, j-1)$
	$k > T(i+1, j-1)$	$(i+1, j)$

A cycle is a connected component of labels under intersection of an old domino with an alternative domino. Moving through a cycle replaces its $D(k)$ by $D'(k)$; denote the result by $\mathrm{MT}(T, c)$. A cycle is open when the move removes one square from the tableau or adds it to the core, and adds one square to the diagram. Call these its beginning and ending squares. It is noncore if both endpoints lie beyond the core boundary, namely $i + j > q + 2$. Let $\mathrm{OC}^*(T)$ be the noncore open cycles. Their moves are independent involutions and preserve rank [25, §2.3]. For $U \subseteq \mathrm{OC}^*(T)$, $\mathrm{MT}(T, U)$ denotes their simultaneous, commuting moves.

Set

$$\mathcal{F}_q(T) = \{\mathrm{MT}(T, U) : U \subseteq \mathrm{OC}^*(T)\}, \qquad D_q(T) = \{w : Q^q(w) \in \mathcal{F}_q(T)\}. \tag{3.1}$$

The sets $D_q(T)$ are the combinatorial left cells of rank q. For any $X \subseteq W_n$, put $\mathcal{S}(X) = \{\mathrm{sh}\, Q^q(w) : w \in X\}$. Define $x \; D_q \; y$ by $Q^q(x) \in \mathcal{F}_q(Q^q(y))$. Then [24, Theorem 3.6] gives

$$D_q = \mathcal{L}_q \vee \mathcal{L}_{q+1}. \tag{3.2}$$

We also need the boundary description of the cycle families [25, §§2.2–2.3 and Proposition 3.3]. For a partition p with core δ_q, let $\mathrm{HC}(p)$ be the set of addable or removable squares (i, j) with $i + j \equiv q \pmod 2$ and $i + j > q + 1$, and let κ_p count those already in p. The beginning squares of the noncore open cycles are exactly these κ_p filled squares; any additional core-adjacent squares in $\mathrm{HC}(p)$ are empty. Each beginning is paired with a distinct empty ending square. Choosing the filled member of each pair gives the shapes in the family. The moves are independent, and distinct choices change distinct endpoint squares. Consequently

$$|\mathcal{F}_q(T)| = |\{\mathrm{sh}\, U : U \in \mathcal{F}_q(T)\}| = 2^{\kappa_p}, \qquad p = \mathrm{sh}\, T. \tag{3.3}$$

Moving through a noncore cycle exchanges one filled endpoint with one empty endpoint and preserves $\mathrm{HC}(p)$. Thus κ_p is constant on the family.

3.2. **Characters and truncated induction.** Let $\mathrm{Irr}(V)$ be the ordinary irreducible characters of a standard parabolic subgroup V, identified with its generic Hecke characters by deformation. For the canonical symmetrizing trace $\tau(N_w) = \delta_{w,1}$, define the Schur elements $s_E(v)$ by

$$\tau(h) = \sum_{E \in \mathrm{Irr}(V)} s_E(v)^{-1} \operatorname{tr}_E(h), \qquad a_E = -\tfrac{1}{2} \min \deg_v s_E(v).$$

Here $\min \deg_v$ is the least exponent with nonzero coefficient. These are Lusztig's a-values [23, Corollary 20.7]. Let W be W_n or a standard parabolic subgroup of W_n. For an irreducible character F of a standard parabolic $V \subseteq W$, truncated induction is

$$J_V^W(F) = \sum_{\substack{E \in \mathrm{Irr}(W) \\ a_E = a_F}} \langle \mathrm{Ind}_V^W F, E \rangle_W \, E, \tag{3.4}$$

extended additively; Ind_V^W denotes ordinary character induction. The source and target Schur elements use their respective weights.

Constructible characters are defined recursively as follows [12, §4.1]. The trivial group has its unit character. For a nontrivial W, take $J_V^W(\phi)$ and $J_V^W(\phi) \otimes \mathrm{sgn}$ for proper standard parabolics V and constructible characters ϕ of V. For a constituent E of $\mathrm{Ind}_V^W F$, one has $a_E \geq a_F$ [23, Proposition 20.14(a)]. Along a standard parabolic chain $V \subseteq U \subseteq W$, any nonzero induction path $F \to G \to E$ therefore satisfies $a_F \leq a_G \leq a_E$. If $a_F = a_E$, equality holds at both steps, proving transitivity of truncated induction [23, 20.15(a)]. It also factors over direct products: Schur a-values add [23, 20.19], and equality of the sum forces equality in each factor. Here $\boxtimes$ denotes the external tensor product, whose character on (g, h) is the product of the two factor characters.

The constructible characters of a product are the external products of constructible characters of its factors. This follows by induction from the recursion. In the direction of constructing a product, a final sign twist in one factor is realized by first twisting the unchanged source factors by their signs and then taking the full product sign. In the other direction, a proper parabolic and its final induction factor componentwise, and a full sign twists each factor. Thus the recursion gives both inclusions.

For $n > 0$, every constructible character of W_n has one of the forms

$$J_{W_m \times \mathfrak{S}_\ell}^{W_n}(\phi \boxtimes \mathrm{sgn}_\ell), \qquad J_{W_m \times \mathfrak{S}_\ell}^{W_n}(\phi \boxtimes \mathrm{sgn}_\ell) \otimes \mathrm{sgn}, \quad m + \ell = n, \quad \ell \geq 1, \tag{3.5}$$

where ϕ is constructible and sgn_ℓ is the sign of $\mathfrak{S}_\ell$; $m = 0$ is allowed. Indeed, write a proper source as $W_d \times \prod_i \mathfrak{S}_{k_i}$, including trivial blocks so that $d + \sum_i k_i = n$ and $d < n$. The constructible characters of its symmetric-group factors are irreducible [23, Lemma 22.5]. The character indexed by a partition is truncated-induced from the signs of the Young subgroup whose block sizes are its column lengths [23, 22.5(a)]. Refine each symmetric factor accordingly and use transitivity. Absorb the resulting sign blocks successively into the type-B factor. Every intermediate character is constructible; the last absorption has the form (3.5). All earlier sign choices stay inside the smaller-rank constructible characters. This is the reduction of [26, §4.2].

Proposition 2.1 supplies P1–P15 for all groups just used. The corrected inclusion theorem [5, Theorem 1] places each $D_q(T)$ inside one left cell. By [12, Proposition 6.3], with [14], the left-cell characters are precisely the constructible characters for $n \geq 2$. For $n = 0$ both consist of the unit character; for $n = 1$ the two singleton cells afford the trivial and sign characters, which are exactly the two constructible characters. At $b = (q+1)a$, every constructible character is

$$\sum_{U \subseteq \mathrm{OC}^*(T)} E_{\mathrm{sh}\,\mathrm{MT}(T,U)} \tag{3.6}$$

for a standard rank-q tableau T [25, Corollary 4.6]. The distinct shapes in (3.3) give distinct irreducibles, so these characters are multiplicity-free. The cell intersection formulas [26, Lemma 2.4 and Corollary 2.5] therefore give

$$\langle \chi_C, \chi_{C'} \rangle = |C \cap (C')^{-1}|, \qquad C \cap C^{-1} \text{ consists of involutions.} \tag{3.7}$$

3.3. **The Pieri rule and its coordinates.** Fix $n = m + \ell$, $\ell \geq 1$. In this subsection J has source $W_m \times \mathfrak{S}_\ell$ and target W_n. For a shape p of rank q with m dominoes, pad by zeroes and put

$$p^I = (p_1 + 2, \ldots, p_\ell + 2, p_{\ell+1}, p_{\ell+2}, \ldots).$$

When $p_\ell = p_{\ell+1}$ and $p_\ell + q - \ell$ is even, also put

$$p^{II} = (p_1 + 2, \ldots, p_{\ell-1} + 2, p_\ell + 1, p_{\ell+1} + 1, p_{\ell+2}, \ldots). \tag{3.8}$$

The character rule [26, Corollary 4.9] is

$$J(E_p \boxtimes \operatorname{sgn}_\ell) = \begin{cases} E_{p^I} + E_{p^{II}}, & p_\ell = p_{\ell+1} \text{ and } p_\ell + q - \ell \text{ is even,} \\ E_{p^I}, & \text{otherwise.} \end{cases} \tag{3.9}$$

To derive the formula, pad p with zeroes to a length $k > \ell$ with $k \equiv q + 1 \pmod 2$, and set $c_i = p_i + k - i$. The divided even beta row has $q + 1$ more entries than the odd row. Indeed, the core at length $q + 1$ has beta numbers $2q, 2q - 2, \ldots, 0$; moving a bead by two preserves its runner, and increasing the padding length by two adds one bead on each runner. These are the two rows of the symbol for $b = (q + 1)a$, with its entries divided by a. The symbol induction rule [23, 22.15 and Lemma 22.17] raises the ℓ largest entries by one, with multiplicity one for each choice at a tied cutoff. Their weakly decreasing multiset is $\lfloor c_1/2 \rfloor, \ldots, \lfloor c_k/2 \rfloor$. Since the c_i are strictly decreasing, a value occurs at most twice, and a tie at the cutoff is equivalent to

$$\lfloor c_\ell/2 \rfloor = \lfloor c_{\ell+1}/2 \rfloor \quad \Longleftrightarrow \quad p_\ell = p_{\ell+1}, \quad p_\ell + q - \ell \equiv 0 \pmod 2.$$

The first choice raises $c_1, \ldots, c_\ell$ by two and gives p^I. At a tie, write $(c_\ell, c_{\ell+1}) = (2d+1, 2d)$. The second choice raises $c_1, \ldots, c_{\ell-1}, c_{\ell+1}$ by two. Sorting replaces the boundary pair by $(2d + 2, 2d + 1)$. The preceding raised entry, if present, is at least $2d + 4$, and the following untouched entry, if present, is at most $2d - 1$, so no other entries change order. Subtracting $k - i$ gives exactly p^{II}. In either case, the size increases by 2ℓ and the runner lengths are unchanged. Every shape used in (3.9) therefore has core δ_q.

Embed $W_m \times \mathfrak{S}_\ell$ as the standard subgroup acting on the first m and last ℓ positions, respectively. Its element $(w, w_{0,\mathfrak{S}_\ell})$, with $w_{0,\mathfrak{S}_\ell}$ the longest permutation on the last block, is

$$\widehat{w} = (w(1), \ldots, w(m), n, n - 1, \ldots, m + 1). \tag{3.10}$$

If K is the shape set of $\mathcal{F}_q(Q^q(w))$, the shape set of $\mathcal{F}_q(Q^q(\widehat{w}))$ is

$$\{p^I : p \in K\} \ \cup \ \{p^{II} : p \in K,\ p_\ell = p_{\ell+1},\ p_\ell + q - \ell \text{ even}\}. \tag{3.11}$$

We give the local calculation underlying this identity [26, Lemma 4.10]. Put $T' = Q^q(w)$, $p = \operatorname{sh} T'$, and $T = Q^q(\widehat{w})$. The appended decreasing positive block adds to row i, for $1 \leq i \leq \ell$, a horizontal domino H_i with label $m + i$ and squares

$$A_i = (i, p_i + 1), \qquad B_i = (i, p_i + 2).$$

The alternative domino of every old label is unchanged: any comparison square formerly outside the diagram either remains outside or receives a label greater than every old label.

If $p_i + q - i$ is even, A_i is fixed. When $p_i > p_{i+1}$, the comparison at $(i + 1, p_i)$ is with a later new label or with $+\infty$, so the alternative of H_i uses (i, p_i). This is an old noncore beginning square; H_i joins that cycle and moves its beginning to B_i. When $p_i = p_{i+1}$, the comparison is with an old label or zero, so the alternative uses $(i + 1, p_i + 1)$. If $i < \ell$, H_i and H_{i+1} form a closed cycle. If $i = \ell$, H_ℓ forms a singleton open cycle with endpoints

$(\ell, p_\ell + 2)$ and $(\ell + 1, p_\ell + 1)$. Both lie beyond the core boundary, since $p_\ell + \ell > q$ for a partition containing δ_q.

If $p_i + q - i$ is odd, B_i is fixed. When $i > 1$ and $p_{i-1} = p_i$, the alternative uses $(i-1, p_i+2)$, giving the second member of the closed pair. Otherwise the comparison at $(i-1, p_i+3)$ is with an old label, an earlier new label, or zero, so the alternative uses $(i, p_i + 3)$. Here A_i is a variable addable square of p. By the boundary description in [25, §2.3], it is either the ending square of an old open cycle or a vacant square with $i + p_i + 1 = q + 2$. In the first case H_i extends the old cycle, moving its ending two columns to the right. In the second case H_i is a new singleton core open cycle.

Each attachment meets one old cycle, while the new closed pairs and core singletons are isolated, so distinct old cycles remain distinct. An old core cycle retains its beginning on the core boundary; an old noncore cycle retains both endpoints beyond it. Thus the noncore cycles of T are the transported noncore cycles of T', together with the singleton $\{n\}$ exactly when $p_\ell = p_{\ell+1}$ and $p_\ell + q - \ell$ is even.

This splitting condition is constant on the source family. Suppose it holds and write $p_\ell = p_{\ell+1} = h$. Noncore moves preserve all fixed squares. The absent square $(\ell, h + 1)$ is fixed, so row ℓ cannot grow. If $h > 0$, the square $(\ell + 1, h)$ is fixed and occupied, so row $\ell + 1$ cannot shrink; the partition inequalities prevent row ℓ from shrinking or row $\ell + 1$ from growing. If $h = 0$, neither row can shrink, and the fixed absent square $(\ell, 1)$ keeps row ℓ empty; the next row is then empty too. Both row lengths therefore stay unchanged. Involutivity shows that the condition also stays false when it starts false.

Every old noncore endpoint in the first ℓ rows moves two columns to the right; the other endpoints stay fixed. Moving through any subset of the transported cycles therefore produces exactly the shapes p^I for $p \in K$. When the additional singleton exists, moving through it replaces $(\ell, p_\ell + 2)$ by $(\ell + 1, p_\ell + 1)$ and produces the corresponding p^{II}. Core and closed cycles give no choices in $\mathcal{F}_q(T)$. This proves (3.11).

If C' is a left cell of W_m, its product with the longest-element singleton of $\mathfrak{S}_\ell$ is a parabolic left cell, since the canonical basis factors over the two groups. The ambient left cell C containing that entire source cell satisfies

$$\chi_C = J(\chi_{C'} \boxtimes \mathrm{sgn}_\ell), \qquad \widehat{w} \in C \quad (w \in C'). \tag{3.12}$$

This is [12, Lemma 5.5], also stated in [26, Lemma 4.7]. The source singleton has character sgn_ℓ by the convention in Section 2.1.

Proposition 3.1. *At $b = ra$, each left cell of W_n is one combinatorial cell $D_{r-1}(T)$.*

Proof. Shape sets. The corrected inclusion writes a left cell as a disjoint union $C = \coprod_i D_i$ of rank-q combinatorial cells. Each D_i is a union of full recording fibres. There cannot be two distinct recording tableaux Q, Q' of the same shape in C: the element with insertion pair (Q', Q) and its inverse, with pair (Q, Q'), would both belong to C, and that element would not be an involution, contrary to (3.7). For any left cell C' of W_n, each common shape of C, C' therefore determines exactly one element of $C \cap (C')^{-1}$. Consequently

$$\langle \chi_C, \chi_{C'} \rangle = |\mathcal{S}(C) \cap \mathcal{S}(C')|. \tag{3.13}$$

Two self-pairings and the cross-pairing give

$$\chi_C = \chi_{C'} \quad \Longleftrightarrow \quad \mathcal{S}(C) = \mathcal{S}(C'). \tag{3.14}$$

For the forward implication, each shape set has cardinality equal to that of their intersection. For the reverse implication, the squared norm of $\chi_C - \chi_{C'}$ is zero.

Characters. We prove by strong induction on n that

$$\chi_C = \sum_{p \in \mathcal{S}(C)} E_p. \tag{3.15}$$

For $n = 0$ the sole tableau has shape δ_q and $E_{\delta_q} = 1$. For $n > 0$, choose the normal form (3.5) for a constructible character. First treat the form without its final sign. Choose a left cell C' of W_m affording ϕ, and take the particular target C in (3.12). Since $m < n$, the induction hypothesis applies to C'.

Write $C' = \coprod_{i\in I} D'_i$, choose $w_i \in D'_i$, and let $\widehat{D}_i = D_q(Q^q(\widehat{w}_i))$. The source-cell inclusion gives $\widehat{w}_i \in C$, and the integer-wall inclusion gives $\widehat{D}_i \subseteq C$. Expand $\chi_{C'} = \sum_{p\in\mathcal{S}(C')} E_p$ by induction and apply (3.9) linearly, retaining the multiplicities. Its sum is χ_C, which is multiplicity-free. All coefficients in the expansion are nonnegative integers, and distinct fixed-core shapes label distinct irreducibles. Thus no target shape occurs twice. The source components have disjoint shape sets by the first paragraph of the proof. Applying (3.11) to each component now gives

$$\chi_C = \sum_{i\in I} \sum_{p\in\mathcal{S}(\widehat{D}_i)} E_p, \tag{3.16}$$

with pairwise disjoint displayed shape sets.

The sets $\widehat{D}_i$ are therefore disjoint and contained in C. Each has one recording tableau of every shape in its shape set. Using (2.3) and (3.16), we obtain

$$\left|\coprod_{i\in I} \widehat{D}_i\right| = \sum_{i\in I} \sum_{p\in\mathcal{S}(\widehat{D}_i)} |\operatorname{SDT}_q(p)| = \sum_{i\in I} \sum_{p\in\mathcal{S}(\widehat{D}_i)} \dim E_p = \chi_C(1) = |C|.$$

The union exhausts C, proving (3.15) for this representative.

If the normal form has a final sign, use Cw_0. Longest-element duality [26, Lemma 4.6] gives $\chi_{Cw_0} = \chi_C \otimes \operatorname{sgn}$, and its recording shapes are the transposes of those of C. Equation (2.3) proves (3.15) for this representative as well. In either case (3.14) transfers the identity to every cell with that character. Every cell character is constructible, completing the induction.

The number of components. By (3.6), the irreducible support of χ_C, identified with shapes, is the shape set K of one family $\mathcal{F}_q(T)$. By (3.15), it is also the disjoint union of the shape sets of the components D_i of C. The integer κ_p has one value κ on K. Equation (3.3) gives $|K| = 2^\kappa$. Every component contains a shape in K, so the same equation gives 2^κ shapes for each component. Disjointness forces there to be exactly one component. □

4. Proof of the theorem and ordered-group parameters

For positive integer weights, Proposition 2.1 gives part (1) for left and right cells, and Lemma 2.2 gives the two-sided assertion. On the wall, Proposition 3.1 and (3.2) give

$$\sim_L = \mathcal{L}_{r-1} \vee \mathcal{L}_r.$$

Inversion gives the right-hand statement. Joining the relations and applying Lemma 2.2 at each rank gives

$$\sim_{LR} = (\mathcal{L}_{r-1} \vee \mathcal{L}_r) \vee (\mathcal{R}_{r-1} \vee \mathcal{R}_r) = \mathcal{S}_{r-1} \vee \mathcal{S}_r.$$

This proves both parts for integer weights.

Lemma 4.1. *Let Λ be a finitely generated ordered abelian group and $S \subset \Lambda_{>0}$ a finite set. There is a homomorphism $f : \Lambda \to \mathbb{Z}$ such that $f(s) > 0$ for every $s \in S$.*

Proof. For $S = \varnothing$ take the zero homomorphism. Otherwise Λ is torsion-free, since a nonzero element and any positive multiple have the same sign. Thus identify Λ with $\mathbb{Z}^d$.

Suppose that zero belongs to the convex hull of S in $\mathbb{R}^d$. Choose a relation $0 = \sum_i c_i s_i$, $\sum_i c_i = 1$, $c_i > 0$, with the smallest possible number of terms. The columns $(s_i, 1)$ are linearly independent: a dependence would permit a perturbation of the c_i, preserving both equations, until one coefficient became zero and all others stayed nonnegative. Consequently the coefficients are the unique solution of a full-column-rank rational linear system and are

rational. Clearing denominators expresses zero as a nonempty sum of positive elements of Λ, a contradiction.

Let p be the point of that compact convex hull closest to zero. Then $p \neq 0$. For $s \in S$, the segment $p + t(s - p)$, $0 \leq t \leq 1$, lies in the hull. Differentiating its squared norm at $t = 0$ gives $p \cdot (s - p) \geq 0$, so $p \cdot s \geq \|p\|^2 > 0$. A sufficiently close rational vector retains these finitely many strict inequalities. Multiplying it by a common denominator gives an integer linear functional positive on S. □

Lemma 4.2. *For a finite Coxeter group with positive weights in a finitely generated ordered abelian group Λ, a specialization to positive integer weights can preserve the left, right, and two-sided preorders, together with any specified finite list of strict inequalities between elements of Λ.*

Proof. Construct the canonical basis over $\mathbb{Z}[\Lambda]$. After extending scalars, bar invariance and triangular uniqueness identify it with (2.1). Thus all its coefficients and structure coefficients lie in $\mathbb{Z}[\Lambda]$.

Choose a finite set $S \subset \Lambda_{>0}$ containing the generator weights, the specified positive differences, and the negatives of all exponents occurring in the lower coefficients $p_{x,w}$. For each nonzero coefficient of C_z in a product $C_s C_w$, also include $\gamma - \delta$ whenever $\gamma > \delta$ are distinct exponents in its support. There are finitely many such data.

Apply Lemma 4.1 and specialize $e^\gamma \mapsto v^{f(\gamma)}$. The Hecke relations and bar involution are preserved, and the lower terms in (2.1) still have strictly negative degree. Uniqueness identifies each specialized C_w with the canonical basis element for the weights $f \circ L$. In each nonzero generator coefficient, distinct exponents remain distinct, so that coefficient stays nonzero. Zero coefficients stay zero. Hence the left generator graphs are identical. The anti-involution $C_w \mapsto C_{w^{-1}}$ gives the right graphs; their respective transitive closures give the three preorders. □

Completion of the proof of Theorem 1.1. Take $\Lambda = \mathbb{Z}a + \mathbb{Z}b \subseteq \Gamma$. In part (1), preserve $a, b, b - ra, (r+1)a - b > 0$ in Lemma 4.2. The specialized weights satisfy $rf(a) < f(b) < (r+1)f(a)$. In part (2), preserving $a, b > 0$ retains positive weights, and the homomorphism preserves $f(b) = rf(a)$. The integer-weight conclusions transfer along the identical generator graphs. □

5. Dominance and the two-sided preorder

For partitions of the same integer, write $\lambda \trianglelefteq \mu$ when $\sum_{i \leq k} \lambda_i \leq \sum_{i \leq k} \mu_i$ for every $k \geq 1$. Write $\lambda \triangleleft \mu$ if $\lambda \trianglelefteq \mu$ and $\lambda \neq \mu$.

Remark 5.1. The dominance-order assertion in [6, Remark 1.2(c$^+$)] is false. In W_5, take $a = 2$, $b = 1$ and

$$x = (1, -2, -3, 5, 4), \qquad y = (-1, -4, 3, -2, 5).$$

Then

$$\lambda_0(x) = (4, 3, 1, 1, 1) =: \lambda \ \triangleleft \ \mu := (5, 2, 2, 1) = \lambda_0(y), \qquad x \not\leq_{LR} y. \tag{5.1}$$

Proof. A necessary shape order. Realize $\mathfrak{S}_{10}$ on the ordered alphabet $-5 < -4 < \cdots < -1 < 1 < \cdots < 5$, and W_5 as the fixed subgroup of its diagram involution. The central simple reflection has orbit length one and the other four generators have orbit length two, giving exactly the weights $b = 1$, $a = 2$. After the conventional sign adjustment, Lusztig's folding formulas [22, §9, (9.1)–(9.2)] express fixed-group structure coefficients as graded traces on spaces whose graded dimensions are the ambient structure coefficients. A nonzero trace has nonzero ambient dimension. The two canonical normalizations are related by $\widetilde{C}_w = (-1)^{\ell(w)} C_w^\dagger$; their structure coefficients differ by $(-1)^{\ell(x)+\ell(y)-\ell(z)}$. Here each Coxeter length is taken in the group under consideration. Thus vanishing is unchanged, and the preorder implication in loc. cit. (9.2) applies in our convention. A left or right preorder

relation in W_5 therefore implies the corresponding ambient two-sided relation. By [6, Theorem 3.3], rank-zero domino shape agrees with ordinary Robinson–Schensted shape under the embedding

$$w \longmapsto (-w(5), \dots, -w(1), w(1), \dots, w(5)). \tag{5.2}$$

The type-A preorder is dominance on these shapes [13, Theorem 5.1 and Corollary 5.6]. Geck indexes the cells by the transposes of their Robinson–Schensted shapes; conjugation reverses dominance. In our shape convention this gives

$$u \leq_{LR} w \quad \Longrightarrow \quad \lambda_0(u) \trianglelefteq \lambda_0(w). \tag{5.3}$$

The two embedded words are

$$(-4, -5, 3, 2, -1, 1, -2, -3, 5, 4), \qquad (-5, 2, -3, 4, 1, -1, -4, 3, -2, 5).$$

Their row-insertion calculations are given below. A chain $(a_1, \dots, a_k)$ starts with the letter inserted in row one; for $j > 1$, a_j is bumped into row j, and the final a_k is appended at the end of row k.

step	chain for x	chain for y
1	(-4)	(-5)
2	$(-5, -4)$	(2)
3	(3)	$(-3, 2)$
4	$(2, 3)$	(4)
5	$(-1, 2, 3)$	$(1, 4)$
6	(1)	$(-1, 1, 2)$
7	$(-2, -1, 2, 3)$	$(-4, -3, 1, 2)$
8	$(-3, -2, -1, 2, 3)$	(3)
9	(5)	$(-2, -1, 4)$
10	$(4, 5)$	(5)

The resulting insertion tableaux are

$$\begin{array}{cccc} -5 & -3 & 1 & 4 \\ -4 & -2 & 5 & \\ -1 & & & \\ 2 & & & \\ 3 & & & \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} -5 & -4 & -2 & 3 & 5 \\ -3 & -1 & & & \\ 1 & 4 & & & \\ 2 & & & & \end{array} .$$

This verifies the two shapes in (5.1). Their prefix sums are $(4, 7, 8, 9, 10)$ and $(5, 7, 9, 10, 10)$. The full dominance interval between them is

$$(4, 3, 1, 1, 1), \quad (4, 3, 2, 1), \quad (5, 2, 1, 1, 1), \quad (5, 2, 2, 1). \tag{5.4}$$

Indeed the first two row lengths are forced to be $(4, 3)$ or $(5, 2)$; the remaining three boxes can only give $(2, 1)$ or $(1, 1, 1)$. A tail (3) would exceed the third prefix sum of μ after $(4, 3)$, and would violate the partition condition after $(5, 2)$. The second partition in (5.4) is the staircase δ_4; the third has core δ_3, after removing the last two boxes of its first row and the last two boxes of its first column. Thus neither has empty 2-core. Among rank-zero shapes, $\lambda \lhd \mu$ is a cover.

Let X and Y be the full shape fibres of λ and μ. By (5.3), every intermediate shape on a mixed left/right path from Y to X lies between λ and μ. The empty-core cover forces each to be one of the endpoints. The first entrance into X would therefore be a single generator edge from Y to X. We show that no such edge exists.

Descents and parity on the two fibres. The shape λ has a unique domino tiling. Its vertical dominoes are $\{(4, 1), (5, 1)\}$ and $\{(2, 1), (3, 1)\}$; the other three are horizontal, in positions $\{(1, 1), (1, 2)\}$, $\{(1, 3), (1, 4)\}$, and $\{(2, 2), (2, 3)\}$. The bottom square forces the first vertical domino, the third-row square then forces the second, and all remaining pairs are forced

successively by $(1,4)$, $(2,3)$, and $(1,1)$. The tiling of $\mu = \lambda^{\mathrm{t}}$ is its transpose, with three vertical dominoes.

Let $\mathrm{vert}(T)$ denote the number of vertical dominoes of a tableau. The color-to-spin identity [21, Lemma 8] is

$$\mathrm{neg}(w) := |\{i : w(i) < 0\}| = \tfrac{1}{2}(\mathrm{vert}(P^0(w)) + \mathrm{vert}(Q^0(w))).$$

It follows that $\mathrm{neg}(u) = 2$ for $u \in X$ and $\mathrm{neg}(w) = 3$ for $w \in Y$. The northwest domino of $P^0(w)$ is horizontal for $w \in X$ and vertical for $w \in Y$. It is labelled 1, whose orientation records the sign of the letter with absolute value 1: that letter is inserted horizontally if positive and vertically if negative, and larger letters do not move it. The criterion $tw < w \iff w^{-1}(1) < 0$ for a left t-descent therefore gives

$$tu > u \quad (u \in X), \qquad tw < w \quad (w \in Y). \tag{5.5}$$

Also

$$L(w) \equiv \mathrm{neg}(w) \pmod 2, \tag{5.6}$$

because the long generators have even weight and each occurrence of t changes the parity of the number of negative letters.

Vanishing of all crossing coefficients. Fix $u \in X$ and $w \in Y$. The descent eigenvalue equation is $N_t C_w = vC_w$. Comparing the coefficient of N_u, using $tu > u$, gives

$$p_{u,w} = v^{-1} p_{tu,w}. \tag{5.7}$$

The case $tu = w$ cannot occur: the leading term of $C_t C_u$ would give $w \leq_{LR} u$, contrary to (5.3). Therefore

$$\deg p_{u,w} \leq -2 \tag{5.8}$$

whenever the coefficient is nonzero.

Multiplication by C_t on C_w is scalar, as is multiplication by C_s if $sw < w$. Suppose $s = s_i$ and $sw > w$. The generator formula is

$$C_s C_w = C_{sw} + \sum_{\substack{z<w \\ sz<z}} M^s_{z,w} C_z. \tag{5.9}$$

The leading term cannot belong to X, since s_i preserves neg. The remaining coefficients are bar invariant. Their defining recursion [22, §§3–4 and 6] is

$$M^s_{u,w} + \sum_{\substack{u<z<w \\ sz<z}} p_{u,z} M^s_{z,w} - v^{L(s)} p_{u,w} \ \in \ v^{-1}\mathbb{Z}[v^{-1}] \tag{5.10}$$

when $u < w$ and $su < u$; otherwise the lower coefficient is zero. For fixed s, w, the same recursion gives $\deg M^s_{z,w} \leq L(s) - 1$ for every nonzero lower coefficient, by downward induction in $z < w$. Indeed, each middle term $p_{z,z'} M^s_{z',w}$ with $z < z' < w$ has degree at most $-1 + (L(s) - 1) = L(s) - 2$ by induction, while $v^{L(s)} p_{z,w}$ has degree at most $L(s) - 1$. The recursion determines the nonnegative-degree part of $M^s_{z,w}$ and gives the bound, including the initial case of an empty sum. Bar invariance bounds its least exponent from below by $1 - L(s)$.

Return to $u \in X$. Here $L(s) = 2$, so every summand in the middle of (5.10) has degree at most $-1 + 1 = 0$. By (5.8), the last term has degree at most zero as well. The coefficient of v in $M^s_{u,w}$ is therefore zero, and bar invariance eliminates v^{-1}.

To determine the constant term, consider the ring automorphism

$$\sigma(v) = -v, \qquad \sigma(N_z) = (-1)^{L(z)} N_z.$$

The quadratic relation is preserved because $\sigma(v^{L(s)} - v^{-L(s)}) = (-1)^{L(s)}(v^{L(s)} - v^{-L(s)})$. The braid relations have equal total weight on both sides, and σ commutes with bar.

Canonical-basis uniqueness gives $\sigma(C_z) = (-1)^{L(z)}C_z$. Applying σ to (5.9) and comparing the coefficient of C_u yields

$$M^s_{u,w}(-v) = (-1)^{L(s)+L(w)-L(u)} M^s_{u,w}(v).$$

By (5.6) the exponent on the right is odd. Thus the constant coefficient vanishes. The only possible exponents were $-1, 0, 1$, so $M^s_{u,w} = 0$.

There is no left generator edge from Y to X. Inversion preserves both shapes, so there is no right generator edge either. The required path cannot exist, proving (5.1). □

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