Causal prediction of rare recovery responses and the optimal scale of observation time

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Abstract

We determine the information and physical waiting needed to predict a rare intervention response with error smaller than its occurrence probability in a one-dimensional growth process with renewed Poisson interval catastrophes. Deletions of initial sources in macroscopic intervals define a mixed response, with all intervention arms sharing the same stationary past and future forcing. A fixed number of growing submacroscopic gaps determines its rare-event probability scale. We identify the response probability and amplitude, characterize when terminal extinction bits lose predictive information, and construct a gate that recovers this information from the recovery trajectory. For absolute and squared loss, a deterministic-age Bayes-risk expansion holds uniformly with additive error smaller than the rare-event probability. Our main waiting theorem covers every honest causal stopping rule, allowing the complete realized catastrophe history. Its average-time benchmark is the sum of each ordered gap length weighted by the extinction probabilities of preceding gaps. Prediction error smaller than the rare-event probability requires mean waiting divided by this benchmark to diverge; every prescribed diverging multiplier is achievable using actual recovery records. With gap scales $n^{2/5}$ and $n^{4/5}$ and integrated catastrophe tail $A_x \sim C/x$, deterministic observation requires age larger than the $n^{4/5}$ scale, whereas a stopping rule achieves mean waiting $O(n^{2/5} \log n)$. The proofs preserve the rare-event scale through weighted response approximation, risk transfer over arbitrary information fields, and posterior lower bounds on rare history branches.

Keywords: interacting particle systems; spatial catastrophes; causal prediction; rare events; stopping times; sequential observation

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1. Introduction

Once an intervention effect has been defined on a realized stochastic history, predicting that effect becomes an information and observation-time problem. For a response whose occurrence probability tends to zero, we assess prediction error relative to that probability. We ask which observations identify the rare response, how long a fixed observation schedule must run, and how much average waiting an observer can save by stopping uninformative trials early.

We answer these questions for a binary growth process with independently renewed Poisson interval deletions. At each integer time an occupied site sends occupancy to itself and its right neighbour; the next catastrophe row removes occupied sites. A preparation deletes initial sources in selected macroscopic root intervals, separated by a fixed number of growing submacroscopic gaps. The full mixed response compares all deletion arms under the same stationary past and future

forcing and is integrated over a fixed macroscopic horizon. This graphical intervention target is fixed throughout the information and waiting comparisons.

The results connect three parts of the prediction problem. First, we identify the probability and conditional amplitude of the response. Second, we determine when terminal gap-extinction bits lose key historical information: local disappearances can produce the response, whereas a single catastrophe can extinguish those gaps and also remove the intervening sources needed for the response. A gate read from the recovery trajectory distinguishes these histories by checking survival of center sources. Third, we obtain a uniform deterministic-age risk expansion and an average-time theorem over all honest causal stopping rules, with an attaining strategy based on the actual recovery record.

Write $g_{(1)} \le \cdots \le g_{(q)}$ for the ordered gap lengths, $\pi_{(i)}$ for their extinction probabilities and $\Pi_n = \prod_i \pi_i$ for the response probability scale. The average waiting benchmark is

$$S_n = \sum_{k=1}^{q} g_{(k)} \prod_{i<k} \pi_{(i)}.$$

For absolute or squared prediction error $o(\Pi_n)$, every honest causal stopping rule satisfies $\mathbb{E}T_n/S_n \to \infty$. Conversely, every prescribed diverging multiplier of S_n is attainable by a finite-stage recovery-record rule. Thus the theorem determines a baseline scale with an arbitrarily slow necessary excess. A deterministic observation age instead must exceed the largest-gap scale by a diverging factor. Physical waiting is the cost; spatial sensing, storage and arithmetic are uncharged.

A two-gap example makes the distinction explicit. For tail exponent $\gamma = 1$, with integrated catastrophe tail $A_x \sim C/x$ and gap lengths $n^{2/5}$ and $n^{4/5}$, the response probability is of order $n^{-6/5}$. A deterministic age achieving error $o(\Pi_n)$ must be larger than the $n^{4/5}$ scale by a diverging factor. A recovery-record stopping rule achieves the same accuracy with mean waiting $O(n^{2/5} \log n)$. In this geometry the terminal extinction bits also lose all leading predictive benefit. Section 7.4 derives these comparisons for the same target.

Preserving the probability Π_n throughout the proof is essential. The weighted response approximation first controls the response uniformly over the rare finite-box configurations and then integrates at their product probability. A risk-transfer lemma carries this approximation to arbitrary information fields. The single-gap age law is established by a two-history expansion in the Technical Supplement; local $o(A_g)$ errors become $o(\Pi_n)$ after multiplication by the other gap probabilities. The stopping-time lower bound retains posterior uncertainty on a rare past branch and uses independent future forcing after an early cutoff. These links form the precision chain detailed in the supplement's dependency guide.

Spatial catastrophes in contact processes were studied by Lanchier (2011), who compares survival under cube and hyperplane deletions. Belitsky, Ferrari, Konno and Liggett (1997, Theorems 1.4–1.6) prove preservation of an all-zero lattice inequality for contact processes and the corresponding extinction inequality for independent oriented percolation. Their discrete-time induction uses independently included sites in the one-step reached set. Interval catastrophes couple site statuses through shared marks, so transferring that induction requires a representation satisfying its independence conditions. The directed-percolation and contact-process extensions of van den Berg, Häggström and Kahn (2006b, Section 3) make precise the relation between conditional association and all-zero lattice inequalities. Their graph and process assumptions require a separate representation check for interval catastrophes; here the needed inequalities are proved directly for the independent interval coordinates. Our probability estimates use the classical disjoint-occurrence inequality of van den Berg and Kesten (1985), applied to independent catastrophe coordinates, together with a model-specific separation by surviving source paths. The conditional extremal-path argument underlying the positive phase follows the strategy of van den Berg, Häggström and Kahn (2006a). Additive graphical representations and their pathwise duals are classical; see Sturm and Swart (2018, Section 2.6 and Section 3.2). The independent-site specialization is the Stavskaya automaton; Rakovszky, Gopalakrishnan and von Keyserlingk (2024, Section V) discuss its erosion and stable phases. The exponential mixing results for Toom probabilistic cellular automata of de Maere and Ponselet (2012) assume sitewise product

transition kernels, an assumption absent for interval catastrophes. Swart, Szabó and Toninelli (2025, Sections 1.2 and 1.6) develop random Toom-contour bounds for i.i.d. local update rules and explain the additional difficulty of defining effective edge speeds under intrinsic randomness. Here a single interval mark can close an arbitrarily long block; the front and history estimates in the supplement treat that spatial dependence directly.

Huang (2026a, Sections 3 and 5) establishes the survival-moment and stationary-density results. Huang (2026b, Sections 5–7 and Appendices A–B) develops the finite-preparation response, front and history estimates. Appendices A–D of the present Technical Supplement reproduce and adapt these foundation arguments, including proof passages retained from Huang (2026b). The zero-/one-root expansion also builds on Huang (2026b, Section 7). The present results concern the rare mixed response, its prediction from terminal bits or recovery records, the deterministic-age risk expansion and the lower bound over all honest causal stopping rules. Appendix E applies the expansion to a two-history extinction representation. The supplement identifies the shared tools and gives all estimates used here. Both companion citations refer to their public version 1.

Sequential testing already explains why a sum of costs weighted by preceding success probabilities appears. Ünlüyurt (2004), Deshpande, Hellerstein and Kletenik (2014), and Nielsen, Rohwedder and Schewior (2025) study evaluation under paid revelation of Boolean variables. Gkenosis, Grammel, Hellerstein and Kletenik (2022, Section 3) state the classical AND/OR ratio orders explicitly. In particular, a fixed testing order can stop when the function value is determined. Our finite-stage upper rule also has a predetermined order. Correlated distributions are studied explicitly by Ghuge, Moondra and Singh (2026, Sections 1.1–1.2) in stochastic function certification, where individual probes reveal component states and the objective is expected probe count. The contribution here is a lower bound over all honest causal stopping rules for the same dynamical response at error $o(\Pi_n)$, attained up to any prescribed diverging multiplier using recovery records. Its cost is elapsed physical time, with sensing and computation uncharged. A fixed physical observation age is a different comparator from a non-adaptive testing policy in stochastic Boolean function evaluation.

Cheng and Weare (2024) study relative-accuracy estimation of rare-event value functions from trajectory data for finite-state Markov chains, using least-squares temporal difference learning. Their resource is the data used to learn a conditional expectation. Here the law is known, and the observer waits along the realized history to predict its intervention response. The waiting lower bound concerns this fixed response and loss criterion; learning a rare-event probability is a separate statistical task.

Sections 2–3 define the model and state the results. Sections 4–7 prove the response representation, information comparison, deterministic-age expansion and stopping-time theorem. The accompanying Technical Supplement develops the positive-phase, front, history and two-field estimates in Appendices A–E. Appendix references throughout this manuscript refer to that supplement.

2. Model, intervention and information

2.1. Original catastrophe law and stationary preparation

For every $s, z \in \mathbb{Z}$ and integer $r \geq 2$, let $N_{s,z,r}$ be independent Poisson variables with mean $\rho\nu_r$, and set

$$D_s = \bigcup_{N_{s,z,r}>0} [z, z+r-1], \qquad O_{s+1} = (O_s \cup (O_s+1)) \setminus D_{s+1}. \tag{2.1}$$

Intervals and cardinalities are integer-valued. The hole field is $H_s = \mathbb{Z} \setminus O_s$. Define the erosion operator $E(H) = H \cap (H+1)$; then $H_{s+1} = E(H_s) \cup D_{s+1}$. The fixed nonzero length measure satisfies

$$\chi_0 = \sum_{r\geq 2} \nu_r \frac{r(r+1)}{2} < \infty, \quad 0 < c = \rho\chi_0 \leq \frac{1}{32}, \quad \nu_r \sim r^{-p}\ell(r), \quad p > 3, \tag{2.2}$$

where ℓ is positive and slowly varying. The catastrophe law is fixed as n grows. Since $\sum r\nu_r < \infty$, every finite observation cone is affected by finitely many active marks almost surely.

Starting all sites occupied at row $-M$ and letting $M \to \infty$ gives the upper stationary occupied field O_0^*. We use this unmodified field at time zero and its original future rows in every arm. Put

$$u = \Pr(0 \in O_0^*), \qquad \kappa = 2v - 1,$$

where v is the velocity of the left boundary of a right-infinite hole ray in an initially occupied exterior. Appendices A–C show that $u > 0$, that the stationary exterior has the same velocity, and that $\kappa > 0$. The symbol u denotes the open density, and ν_r denotes the length weights. Define

$$A_h = \sum_{r \ge \max\{2,h\}} \nu_r \frac{(r-h+1)(r-h+2)}{2}, \quad \gamma = p - 3, \quad C_A = \frac{1}{(p-1)(p-2)(p-3)}. \tag{2.3}$$

Thus $A_h \sim C_A h^{-\gamma} \ell(h)$.

2.2. Roots, gaps and the fixed response

Fix $q \ge 2$. For every n, arrange $q+1$ root intervals and q nonempty gap intervals consecutively in their spatial order:

$$I_{1,n}, G_{1,n}, I_{2,n}, \dots, G_{q,n}, I_{q+1,n}.$$

Assume

$$|I_{i,n}|/n \to a_i > 0, \qquad g_{j,n} = |G_{j,n}| \to \infty, \qquad g_{*,n} = \max_j g_{j,n} = o(n). \tag{2.4}$$

For $S \subset \{1, \dots, q+1\}$, start arm S from $O_0^* \setminus \bigcup_{i \in S} I_{i,n}$ and use the common future of (2.1). Let $X^S(x,s)$ be its hole indicator. Define

$$K_n(x,s) = \sum_{S \subset [q+1]} (-1)^{q+1-|S|} X^S(x,s), \quad Z_n = \frac{1}{n^2} \sum_{s=0}^{\lfloor T_0 n \rfloor} \sum_x K_n(x,s), \tag{2.5}$$

where $T_0 > \sum_i a_i/\kappa$. Throughout the paper Z_n is this same full compact-horizon target, including when the observation time is random or exceeds the target horizon.

For a deterministic finite initial set J, let $C_J(0) = J \cap O_0^*$ and let $C_J(s)$ follow (2.1) separately, with the same future rows. Each descendant process is determined by its own initial sources and the common damage. Write

$$\tau_j = \inf\{s \ge 0 : C_{G_j}(s) = \varnothing\}, \quad \eta_j = \{\tau_j < \infty\}, \quad \pi_j = \Pr(\eta_j), \quad \Pi_n = \prod_{j=1}^{q} \pi_j. \tag{2.6}$$

Initial extinction has age zero. Appendix E proves

$$\pi_g \sim \frac{2\rho}{\kappa} A_g. \tag{2.7}$$

Set $m = u a_1 a_{q+1}/\kappa > 0$ and

$$d_n = |\operatorname{span}(G_1, \dots, G_q)| = \sum_j g_j + \sum_{i=2}^{q} |I_i| \asymp n. \tag{2.8}$$

Subscripts n are suppressed when unambiguous.

2.3. Causal information and loss

Let U be independent jointly of the entire primitive catastrophe history. Use the nested filtration

$$\mathcal{F}_t = \sigma(U, N_{s,z,r} : s \le t, z \in \mathbb{Z}, r \ge 2), \qquad t \in \mathbb{Z}_{\ge 0}. \tag{2.9}$$

It gives the observer the infinite original past and all damage coordinates already realized. The recovery-record filtration is

$$\mathcal{H}_t = \sigma(U, H_s(x) : 0 \le s \le t, x \in \mathbb{Z}) \subset \mathcal{F}_t.$$

All laws, geometry and rules are known in advance. Auxiliary randomization is included in U. For any information field $\mathcal{I}$ define

$$R_1(\mathcal{I}) = \inf_{Y \ \mathcal{I}\text{-measurable}} \mathbb{E}|Z_n - Y|, \qquad R_2(\mathcal{I}) = \inf_{Y \ \mathcal{I}\text{-measurable}} \mathbb{E}(Z_n - Y)^2. \tag{2.10}$$

A stopping rule is an almost surely finite, nonnegative integer $\mathcal{F}_t$-stopping time T. Its information satisfies $\mathcal{I}_T \subset \mathcal{F}_T$, where

$$\mathcal{F}_T = \{A : A \cap \{T \le t\} \in \mathcal{F}_t \text{ for every integer } t \ge 0\}. \tag{2.11}$$

Its cost is the unconditional physical time $\mathbb{E}T$, allowing infinity. Spatial sensing, storage and arithmetic are uncharged.

3. Main results

Theorem 3.1 (rare response and terminal-bit criterion). Under (2.1)–(2.4),

$$\Pr(Z_n > 0) \sim \Pi_n, \quad \mathbb{E}Z_n \sim m\Pi_n, \quad \mathbb{E}Z_n^2 \sim m^2\Pi_n. \tag{3.1}$$

Let $M_n = \Pr(\bigcap_j \eta_j)$ and $\mathcal{G}_n = \sigma(1_{\eta_1}, \dots, 1_{\eta_q}, U)$. Then

$$M_n - \Pi_n = \Theta(A_{d_n}), \tag{3.2}$$

$$R_1(\mathcal{G}_n) = m \min\{\Pi_n, M_n - \Pi_n\} + o(\Pi_n), \quad R_2(\mathcal{G}_n) = m^2\Pi_n \left(1 - \frac{\Pi_n}{M_n}\right) + o(\Pi_n). \tag{3.3}$$

Consequently, for either loss,

$$R_i(\mathcal{G}_n) = o(\Pi_n) \quad \Longleftrightarrow \quad A_{d_n}/\Pi_n \to 0. \tag{3.4}$$

For every deterministic $w = w_n$ with $w/g_* \to \infty$ and $w/n \to 0$, there is a binary $\mathcal{H}_w$-measurable gate B_w such that

$$\mathbb{E}|Z_n - mB_w| = o(\Pi_n), \qquad \mathbb{E}(Z_n - mB_w)^2 = o(\Pi_n). \tag{3.5}$$

The gate works for every ratio A_{d_n}/Π_n. The response amplitude conditional on occurrence satisfies, for every fixed $\varepsilon > 0$,

$$\Pr(|Z_n - m| > \varepsilon \mid Z_n > 0) \longrightarrow 0. \tag{3.5a}$$

This follows from the weighted Bernoulli approximation in (4.10) and (3.1).

Theorem 3.2 (uniform Bayes-risk expansion at the rare-event scale). Set

$$f_{j,n}(L) = 1 - \frac{1}{2}(1 + \kappa L/g_j)^{-\gamma}, \qquad \Phi_n(L) = 1 - \prod_{j=1}^{q} f_{j,n}(L). \tag{3.6}$$

Uniformly over integers $L \ge 0$,

$$R_1(\mathcal{F}_L) = m\Pi_n\Phi_n(L) + o(\Pi_n), \qquad R_2(\mathcal{F}_L) = m^2\Pi_n\Phi_n(L) + o(\Pi_n). \tag{3.7}$$

Equivalently, with $c_1 = m$ and $c_2 = m^2$,

$$\sup_{L\in\mathbb{Z}_{\geq 0}} \left| \frac{R_i(\mathcal{F}_L)}{\Pi_n} - c_i\Phi_n(L) \right| \longrightarrow 0, \qquad i = 1, 2. \tag{3.7a}$$

The same statements hold with $\mathcal{H}_L$ in place of $\mathcal{F}_L$. The expansion gives relative leading asymptotics along sequences on which $\Phi_n(L)$ is bounded away from zero. For $L/g_* \to \infty$, it gives the vanishing-risk conclusion below at scale Π_n. Thus $R_i(\mathcal{H}_L) = o(\Pi_n)$, or $R_i(\mathcal{F}_L) = o(\Pi_n)$, exactly when $L/g_* \to \infty$. For arbitrary $\mathcal{I}_L \subset \mathcal{F}_L$, this condition is necessary.

Theorem 3.3 (optimal average waiting benchmark). Sort gap lengths increasingly, resolving ties by their original spatial index. The corresponding probabilities have the same rank labels. Set

$$P_0 = 1, \quad P_k = \prod_{i=1}^{k} \pi_{(i)}, \quad s_k = g_{(k)}P_{k-1}, \qquad S_n = \sum_{k=1}^{q} s_k. \tag{3.8}$$

For every sequence of stopping rules and stopped information fields satisfying either $R_1(\mathcal{I}_T) = o(\Pi_n)$ or $R_2(\mathcal{I}_T) = o(\Pi_n)$,

$$\frac{\mathbb{E}T}{g_{(k)}P_{k-1}} \to \infty \quad (1 \leq k \leq q), \qquad \frac{\mathbb{E}T}{S_n} \to \infty. \tag{3.9}$$

For every prescribed $r_n \to \infty$, a finite-stage $\mathcal{H}_t$-stopping rule and predictor achieve both errors $o(\Pi_n)$ with

$$\mathbb{E}T \leq r_n S_n + q \quad \text{for all sufficiently large } n. \tag{3.10}$$

Thus S_n is the average waiting benchmark with a necessary, arbitrarily slowly diverging excess at $o(\Pi_n)$ precision.

Corollary 3.4 (all power-law geometries). If $g_{(k)} = n^{\beta_{(k)}+o(1)}$ with $0 < \beta_{(1)} \leq \cdots \leq \beta_{(q)} < 1$, the infimum, over all strategies achieving either error $o(\Pi_n)$, of $\limsup_n \log(\mathbb{E}T)/\log n$ is

$$c_* = \max_{1\leq k\leq q} \left\{ \beta_{(k)} - \gamma \sum_{i<k} \beta_{(i)} \right\}. \tag{3.11}$$

The corresponding deterministic-age exponent is $\beta_{(q)}$.

Initial information and zero waiting. With no process observation, let $\mathcal{I}_\varnothing = \sigma(U)$. Since $Z_n \geq 0$ and $\Pr(Z_n = 0) \to 1$, zero is a median for all large n, and (3.1) gives

$$R_1(\mathcal{I}_\varnothing) \sim m\Pi_n, \qquad R_2(\mathcal{I}_\varnothing) = \mathrm{Var}(Z_n) \sim m^2\Pi_n. \tag{3.12}$$

At age zero, both observed fields already contain the initial state; $\mathcal{F}_0$ also contains the infinite original past. Theorem 3.2 gives

$$R_i(\mathcal{H}_0) = R_i(\mathcal{F}_0) + o(\Pi_n) = c_i\Pi_n(1 - 2^{-q}) + o(\Pi_n). \tag{3.13}$$

Thus initial-state observation has a leading benefit before any physical waiting occurs.

4. Source separation and local response

4.1. Exact source identity and finite certificates

For a target (x, s), let $V(x, s)$ be the set of its initial occupied ancestors in the unprepared process. It is finite, contained in $[x - s, x] \cap O_0^*$. Additivity of (2.1) gives

$$K_n(x, s) = 1_{\{\varnothing \neq V(x,s) \subset \bigcup_i I_i,\ V(x,s)\cap I_i \neq \varnothing\ \forall i\}}. \tag{4.1}$$

Indeed, if V is empty all arm indicators equal one and their contrast vanishes. If it contains an exterior source all indicators vanish. Otherwise, with root-label set J, the indicator in arm S is $1_{J\subset S}$, whose full Boolean difference is $1_{J=[q+1]}$. Thus $K_n \in \{0,1\}$. The first and last root paths imply $\min I_{q+1} \le x \le \max I_1 + s$, so

$$0 \le Z_n \le B_n 1_{\{Z_n>0\}}, \qquad B_n = \frac{(\lfloor T_0 n\rfloor + 1)(\lfloor T_0 n\rfloor + 2)}{2n^2} = O(1). \tag{4.2}$$

Each event $\{\tau_j \le s\}$ has a finite positive certificate in the independent Bernoulli coordinates $1_{N_{t,z,r}>0}$. To prove this for a fixed configuration, note that $O_0^{(-M)} \cap G_j$ decreases to $O_0^* \cap G_j$ and hence agrees with it for all sufficiently large M, since G_j is finite. If the actual source descendants are empty at s, the finite-history descendants are therefore empty for some M. The required past and future cones contain finitely many vertices. At each damaged vertex choose one active interval covering it. Keeping those finitely many coordinates active preserves every damaged vertex and certifies extinction from an all-open row $-M$, and hence from the stationary past for any assignment of the remaining coordinates. The same proof applies in every subconfiguration obtained by deleting atoms.

Lemma 4.1 (separated extinction). For any subset $J \subset [q]$ and age s, if every gap in J is extinct at s and every internal root has a surviving initial source at s, then the gap extinction events occur with disjoint positive certificates. Consequently

$$\Pr\left(\bigcap_{j\in J}\{\tau_j \le s\},\ C_{I_i}(s) \ne \varnothing\ (2 \le i \le q)\right) \le \prod_{j\in J} \Pr(\tau_j \le s) \le \prod_{j\in J} \pi_j. \tag{4.3}$$

The same bound holds when each internal root is replaced by any nonempty subinterval of that root.

Proof. Choose a surviving path from each internal root, extending to the infinite negative past through its stationary initial source. Infinite extension follows from finite branching. Sort their positions at each row. Each order statistic still has increments zero or one, and consists of original open vertices. At row zero it is in the corresponding spatially ordered internal root. Paths may later meet or coalesce.

Every active interval before row s avoids these paths. Connectedness puts it wholly in one of their q strips; coinciding boundaries leave an empty strip. Retain in a subconfiguration only the atoms of the strip associated with a selected gap. That gap must still be extinct at s under this subconfiguration's own stationary past. Otherwise an infinite-past path through the gap and surviving to s can be spliced into an original path: in negative rows use its last meeting with a boundary and follow the original boundary before that meeting; after row zero use its first boundary meeting and follow the boundary thereafter. If no meeting occurs it stays in the strip. Changing order requires a common vertex because the increments are zero or one. All relevant nonboundary vertices are protected by the retained strip atoms. The resulting original gap path contradicts extinction.

Take a finite certificate in each selected strip. The supports are disjoint. The finite Bernoulli BK inequality gives (4.3). To pass to countably many coordinates, enumerate them, set all but the first M to zero, and use continuity from below; every finite certificate is eventually included. Iteration gives the finite-event version. The BK bound is applied to finite certificates under the unconditional product law. □

If (4.1) holds, a surviving gap path up to age s would be trapped between a first-root and a last-root path ending at the same target, and would meet one and become a gap ancestor. All gaps are therefore extinct at s, and every internal root survives to that target. These certificates certify the same disjoint occurrence $\square_{j=1}^{q} \eta_j$ regardless of the positive target. Hence

$$\Pr\{\exists (x,s), s<\infty : K_n(x,s) > 0\} \le \Pi_n. \tag{4.4}$$

4.2. Independent finite boxes

For $G = [a, b]$ and integer $w \geq |G|$, retain the finite coordinate box

$$\mathcal{B}(G, w) = \{(s, z, r) : -w + 1 \leq s \leq w,\ 2 \leq r \leq w,\ a - 2w + 1 \leq z \leq b + w\}. \tag{4.5}$$

Start all sites occupied at row $-w$, set all coordinates outside this box to zero, and track the sources in G from row zero. Let $e_G(w)$ indicate their extinction by row w. This is increasing in its finite damage coordinates and satisfies

$$e_G(w) \leq 1_{\{\tau_G \leq w\}}, \qquad 0 \leq \pi_{|G|} - \mathbb{E}e_G(w) \leq CA_w. \tag{4.6}$$

For clarity, expose the future rows through age w first. They determine the random finite target

$$A_t = \{y \in G : \text{some allowed path from } (y, 0) \text{ reaches row } t\}, \qquad 0 \leq t \leq w. \tag{4.6a}$$

It depends only on the future damage, independently of the initial history. Extinction by t is exactly $A_t \subset H_0^*$. Replacing the initial stationary past by w all-open history rows therefore costs, conditionally on this target, $p_*(A_t) - p_w(A_t) \leq CA_w$ by (D6). For $A_t = \varnothing$ the difference is zero. This is a uniform finite-target bound, with no sum over its sites. For the separate late-extinction term, condition on the initial finite source set $C_G(0) = G \cap O_0^*$. It is determined by the original past and is independent of the positive-age damage rows. Applying the uniform ordinary-seed bound (D8) from age zero gives

$$\Pr(w < \tau_G < \infty) = \mathbb{E}\left[\Pr_{C_G(0)}(w < \tau^{\text{ordinary}} < \infty)\right] \leq CA_w. \tag{4.6b}$$

The empty initial-seed branch contributes zero. Deleting lengths $r > w$ costs at most

$$2w\rho \sum_{r>w} \nu_r(r + |G| + 2w - 1) = O(A_w). \tag{4.7}$$

All other relevant short atoms have origins in (4.5). Constants are independent of $|G| \leq w$. This proves (4.6) without a factor proportional to the gap length.

Choose $w/g_* \to \infty$, $w = o(n)$, and put $e_j = e_{G_j}(w)$, $b_j = \mathbb{E}e_j$, $E_w = \bigcap_j \{e_j = 1\}$. The boxes have disjoint origin ranges for large n, so they are independent. Regular variation and (4.6) give

$$p_E = \Pr(E_w) = \prod_j b_j \sim \Pi_n, \qquad 0 \leq 1 - p_E/\Pi_n \leq C_q A_w / A_{g_*} \to 0. \tag{4.8}$$

4.3. Fluid replacement at the product probability scale

Lemma 4.2 (weighted response). For every such w,

$$\sup_{\xi : e_j(\xi_j) = 1\ \forall j} \mathbb{E}_{\text{outside}} |Z_n^\xi - m| \to 0. \tag{4.9}$$

Consequently

$$\mathbb{E}|Z_n - m1_{E_w}| = o(\Pi_n), \qquad \mathbb{E}(Z_n - m1_{E_w})^2 = o(\Pi_n). \tag{4.10}$$

Proof. Step 1. Uniform geometric envelopes. Fix any admissible configurations in the boxes and independently refresh those coordinates from their original law to form a reference history. Its row-w field has the unconditional stationary law; rows after w are common and independent. Changes made in box j affect row w only inside

$$J_j = [a_j - 2w + 1, b_j + 4w - 2], \qquad |J_j| = g_j + 6w - 2 = o(n).$$

For each arm merge consecutive selected roots into runs, filling their intervening gaps. All these gaps' sources have disappeared by w. If a run has span $[a, b]$, its interval $[a + w, b]$ is entirely holes at row w: any initial ancestor belongs to a deleted root or an extinct gap. The preparation's influence is contained in its deterministic forward cone. Writing A_S for the selected roots and C_S for these run cores gives the simultaneous bounds

$$(H_w^{\mathrm{ref}} \setminus \bigcup_j J_j) \cup C_S \subset H_w^{S,\xi} \subset H_w^{\mathrm{ref}} \cup \bigcup_j J_j \cup (A_S + [0, w]). \tag{4.11}$$

We make the uniformity in the box configurations explicit. Fix $\delta > 0$ small relative to all root proportions and the strict horizon margin. Cover the J_j by deterministic patches P_j of radius δn around the gaps, rounding patch endpoints outwards. Shrink each run core inward at both ends by $10\delta n/\kappa$, rounding core endpoints inwards, and denote their union by C_S^δ. Each rounding changes an endpoint by at most one lattice site. For large n, every patch lies either inside one of these cores or at distance at least $(10/\kappa - 2)\delta n$ from all of them. Put

$$D_S = H_w^{\mathrm{ref}} \cup C_S^\delta, \quad L_S = (H_w^{\mathrm{ref}} \setminus \bigcup_j P_j) \cup C_S^\delta, \quad U_S = H_w^{\mathrm{ref}} \cup \bigcup_j P_j \cup (A_S + [0, w]). \tag{4.11a}$$

Here L_S, D_S, U_S are hole sets at row w. Equation (4.11) gives $L_S \subset H_w^{S,\xi} \subset U_S$ and $L_S \subset D_S \subset U_S$, simultaneously for every admissible ξ.

Patch-erasure claim. Let a reference occupied field, a finite union of deterministic hole cores C, and finitely many patches $P = [l, r]$ be given at one row. Assume each patch is contained in C or its enlargement by $h + 2$ is disjoint from C. Use the same subsequent damage for all fields. For a patch outside C, let p_h be the rightmost descendant at age h of reference occupied sources strictly left of l, and q_h the leftmost descendant of reference occupied sources strictly right of r. Assume that both extreme descendant sets are nonempty and that their extrema are finite at age h for each such patch. If $p_h \geq q_h - 1$ for every such patch, then the fields started with holes $(H^{\mathrm{ref}} \setminus \bigcup P) \cup C$ and $H^{\mathrm{ref}} \cup C$ agree at age h and at every later age.

To prove the claim, a path to p_h starts in $[r - h, l - 1]$: indeed $q_h \geq r + 1$ and coverage implies $p_h \geq r$, while its increments are zero or one. Its whole prefix lies in $[l - h - 2, r + h + 2]$, since $p_h \leq l - 1 + h$. A path to q_h starts in $[r + 1, l + h]$ and has the same confinement, using $q_h \leq p_h + 1 \leq l + h$. These sources and prefixes avoid C. Any extra occupied source in P has an endpoint y at age h. The coverage condition implies either $y \leq p_h$ or $y \geq q_h$. In the first case its path crosses the left reference path; in the second it crosses the right reference path. Exchanging prefixes at the crossing makes y reachable from a reference source outside C. This proves that all extra descendants are already present in the field with holes $H^{\mathrm{ref}} \cup C$. The opposite inclusion is monotonicity. Patches inside C add no occupied sources. The argument treats all other patches simultaneously, without independence; equality is preserved by subsequent common updates.

Apply the claim with $h = \lceil 4\delta n/\kappa \rceil$. In this application the reference field is stationary with positive open density. Its sources on either half-line have nonempty descendants at each finite age almost surely: the unrestricted descendant field is spatially ergodic and has positive density, since a fixed source has positive probability of avoiding damage along its vertical path. Outside a distance h from the half-line boundary its descendants coincide with those of the unrestricted reference field. Finite propagation bounds the left-source descendants above and the right-source descendants below, so the finite extrema required in the claim exist almost surely. Since $0 < \kappa \leq 1$, for large n the patch-to-core separation exceeds $h + 2$. Let $F_{n,\delta}$ be the event that all the reference endpoint pairs cover at this age. Appendix C gives

$$\Pr(F_{n,\delta}^c) \longrightarrow 0, \qquad H_{w+h}^{L_S} = H_{w+h}^{D_S} \quad \text{on } F_{n,\delta} \quad \text{for every } S. \tag{4.11b}$$

The reference field and this event can be realized using the outside coordinates and an independent refresh of all boxes, with the same realization for every ξ. Their joint law does not depend on ξ.

From age $w+h$ onwards, on this single event, the actual arm is bracketed between the evolutions from D_S and U_S.

Step 2. Multi-interval fluid response. We justify the use of disconnected run triangles. For disjoint selected components $R_1, \dots, R_k$, write Δ_R for the baseline-minus-prepared occupied indicator, equivalently the prepared-minus-baseline hole indicator. The source identity gives

$$0 \le \Delta_{\cup_i R_i}(x,s) - \sum_i \Delta_{R_i}(x,s) \le 1. \tag{4.11c}$$

A nonzero defect requires baseline ancestors from at least two distinct components and no ancestor outside their union. Choose an omitted macroscopic root Q between those two components. Any surviving path from its row-zero occupied sources is trapped between the two ancestor paths to (x,s); it must meet one and become a Q ancestor, a contradiction. Thus the event of any defect, at any target or age, is contained in a union of finitely many events $\{\tau_Q < \infty\}$. Each Q has length at least $c_0 n$ for a fixed $c_0 > 0$, so Appendix E gives

$$\Pr\{\exists (x,s) : \Delta_{\cup_i R_i}(x,s) \ne \sum_i \Delta_{R_i}(x,s)\} \le \sum_Q \pi_{|Q|} = O(A_n). \tag{4.11d}$$

The same argument applies to the deterministic interval envelopes above. Merge overlapping cores, patches and selected-root cones into components. For fixed δ, distinct components are separated either by a subinterval of an omitted macroscopic root or by a buffer of length at least $c_\delta n$, where $c_\delta > 0$. These stationary reference source intervals give a defect probability $O_\delta(A_n) = o(1)$. Components consisting only of patches have length $O(\delta n)$; their individual compact fluid masses sum to $O(\delta)$, and moving the other component endpoints by $O(\delta n)$ changes their compact mass by $O(\delta)$.

Choose a deterministic spatial window Λ_n of width $O(n)$ containing all preparation and patch cones through $\lfloor T_0 n \rfloor$. Responses vanish outside this window. All normalized compact sums are bounded by a fixed constant. Before age $w+h$ their contribution is $O_q(w/n + \delta)$. Afterwards (4.11b) brackets each arm, and (4.11d) costs $O_\delta(A_n)$ in expected normalized mass. Appendix C.5 applies to each deterministic interval in D_S, U_S. Their scaled endpoints differ from the ideal run endpoints by $O_q(\delta + w/n)$; the resulting upper-minus-lower compact mass has expectation $O_q(\delta + w/n) + o_{n,\delta}(1)$. All reference estimates are taken under the unconditioned refreshed law, uniformly in ξ.

For an interval $[l,r]$ the fluid insertion is u times the triangle

$$\Gamma_{[l,r]} = \{(y,t) : t \ge 0, l + vt \le y \le r + (1-v)t\}.$$

In the full alternating sum of run triangles, all runs cancel except the full chain, the chain missing its first root, the chain missing its last root, and the chain missing both. Let $L = \sum_i a_i$, $B = \sum_{i=2}^{q} a_i$, $a = a_1$ and $b = a_{q+1}$. Their row-width difference is

$$W(t) = (L - \kappa t)_+ - (L - a - \kappa t)_+ - (L - b - \kappa t)_+ + (B - \kappa t)_+. \tag{4.12}$$

Its integral through $T_0 > L/\kappa$ equals

$$\frac{L^2 - (L-a)^2 - (L-b)^2 + B^2}{2\kappa} = \frac{ab}{\kappa}.$$

Combining the arm envelopes, the patch-erasure claim and the interval fluid limits gives the explicit uniform estimate

$$\sup_{\xi : e_j(\xi_j) = 1 \ \forall j} \mathbb{E}_{\text{outside}} |Z_n^\xi - m| \le C_{q,T_0}\delta + \varepsilon_n(\delta), \qquad \varepsilon_n(\delta) \longrightarrow 0 \quad (n \to \infty) \tag{4.12a}$$

for each fixed admissible δ. Here $\varepsilon_n(\delta)$ includes Cw/n, $C\Pr(F_{n,\delta}^c)$, $C_\delta A_n$ and the finitely many reference compact L^1 errors. Each is independent of ξ; the extra refresh used in the comparison does not change the left-hand expectation. Taking $n \to \infty$ and then $\delta \downarrow 0$ proves (4.9). **Step 3. Rare-event approximation.** Integrating over the boxes gives $\mathbb{E}[|Z_n - m|; E_w] = o(p_E)$. Equations

(4.4), (4.8) and this estimate imply that both $E_w \cap \{Z_n = 0\}$ and $E_w^c \cap \{Z_n > 0\}$ have probability $o(\Pi_n)$. Boundedness (4.2) proves (4.10) and (3.1). □

Lemma 4.3 (risk transfer over arbitrary information fields). Suppose bounded targets Z_n, V_n satisfy $\mathbb{E}|Z_n - V_n| = o(\Pi_n)$, $\mathbb{E}(Z_n - V_n)^2 = o(\Pi_n)$ and $\mathbb{E}Z_n^2 + \mathbb{E}V_n^2 = O(\Pi_n)$. Their optimal absolute and squared risks differ by $o(\Pi_n)$, uniformly over all information fields, including stopped information.

Proof. The absolute risks differ by at most $\mathbb{E}|Z_n - V_n|$. For squared loss, write the residual as $(1 - P_{\mathcal{I}})Z_n$, where $P_{\mathcal{I}}$ is conditional expectation. Its norm differs from the corresponding norm for V_n by at most $\|Z_n - V_n\|_2 = o(\sqrt{\Pi_n})$. Both residual norms are $O(\sqrt{\Pi_n})$, so their squares differ by $o(\Pi_n)$. These bounds do not depend on $\mathcal{I}$. □

4.4. Reading the gate from recovery states

For any initial source interval J, the actual movie gives its exact descendants through

$$C_J(0) = O_0^* \cap J, \qquad C_J(s+1) = (C_J(s) \cup (C_J(s) + 1)) \cap O_{s+1}. \tag{4.13}$$

The descendants are a subset of the baseline occupied set. On their possible offspring, avoidance of D_{s+1} is equivalent to membership in O_{s+1}. This proves (4.13) by induction, without observing damage labels.

Take a centered subinterval J_i of length w in every internal root, and set

$$B_w = 1_{\{C_{G_j}(w) = \varnothing \ \forall j\} \cap \{C_{J_i}(w) \neq \varnothing \text{ for } 2 \le i \le q\}}. \tag{4.14}$$

Eventually each internal root is wider than $12w + 4$. The complete center-source vector through age w is then unchanged by any modification of the boxes (4.5), including in their negative rows. It is jointly independent of their coordinate vector. Its failure probability is at most $(q-1)\pi_w = o(1)$. Lemma 4.1 yields

$$p_E[1 - (q-1)\pi_w] \le \Pr(B_w = 1) \le \Pi_n,$$

and hence $\Pr(\{B_w = 1\} \triangle E_w) = o(\Pi_n)$. Together with (4.10) this proves (3.5). The algorithm reads only the gap and center forward cones of the recovery movie. Their site count through age w is

$$(w+1)\sum_j g_j + \frac{4q-3}{2} w(w+1) = O_q(w^2) = o(n^2). \tag{4.15}$$

Equation (4.15) gives the reading cost of this gate.

5. Why the terminal extinction bits can fail

Choose $v_n = \lfloor \min_{2 \le i \le q} |I_i|/20 \rfloor \asymp n$. The boxes $\mathcal{B}(G_j, v_n)$ remain disjoint. Their joint event E^{mac} has probability p_{mac} with

$$0 \le \Pi_n - p_{\mathrm{mac}} \le C A_{d_n} \sum_j \prod_{i \neq j} \pi_i = o(A_{d_n}). \tag{5.1}$$

On the other hand $M_n - p_{\mathrm{mac}} \le \sum_j \Pr(\eta_j, e_j(v_n) = 0) \le C_q A_{d_n}$ by (4.6). This proves the upper half of (3.2).

For the lower half write G_1's left endpoint as a and G_q's right endpoint as b, so $d = b - a + 1 = d_n$. Consider the past coordinate family

$$s = -k, \quad 1 \le k \le d, \qquad a - 2d \le z \le a - d, \qquad 4d \le r \le 5d. \tag{5.2}$$

Any active atom in this family leaves, even without further damage, a row-zero hole interval $[z + k, z + r - 1]$ containing $[a, b]$. Additional damage only adds holes. Its occurrence event R makes

every gap and every internal root initially empty, so $R \subset \bigcap_j \eta_j$ and $Z_n = 0$ on R. The family has mean

$$\lambda_n = \rho d(d+1) \sum_{r=4d}^{5d} \nu_r \sim \rho d^{-\gamma} \ell(d) \int_4^5 y^{-p}\, dy \asymp A_{d_n}. \tag{5.3}$$

Its lengths exceed v_n, so it is independent of E^{mac}. Therefore $M_n \geq p_{\mathrm{mac}} + \Pr(R)(1 - p_{\mathrm{mac}})$. Equation (5.1) and $p_{\mathrm{mac}} \to 0$ prove $M_n - \Pi_n \geq cA_{d_n}$ eventually.

Put $V_n = m1_{\{Z_n > 0\}}$ and $p_n = \Pr(Z_n > 0) \sim \Pi_n$. Lemma 4.2 gives both $\mathbb{E}|Z_n - V_n| = o(\Pi_n)$ and $\mathbb{E}(Z_n - V_n)^2 = o(\Pi_n)$. By (4.1), Z_n vanishes on every extinction-bit cell except the all-one cell, whose probability is M_n. Conditional on this cell V_n/m is Bernoulli with success probability p_n/M_n. Its exact absolute and squared Bayes risks are

$$m \min(p_n, M_n - p_n), \qquad m^2 p_n (1 - p_n/M_n).$$

Independent U does not alter them. Lemma 4.3 transfers both risks from V_n to Z_n with error $o(\Pi_n)$. Replacing p_n by Π_n proves (3.3); the expressions are uniformly Lipschitz on the relevant ranges. Equations (3.2)–(3.3) give (3.4), including subsequences on which A_{d_n}/Π_n is bounded away from zero.

For $g_j \sim c_j n^{\beta_j}$, the terminal-bit ratio has power $n^{\gamma(\sum_j \beta_j - 1)}$ times

$$\frac{\ell(d_n)}{\prod_j \ell(g_j)}. \tag{5.4}$$

The sum of exponents decides the strict subcritical and supercritical regimes. At $\sum_j \beta_j = 1$, the slowly varying factor remains decisive.

6. The deterministic-age risk curve

6.1. A single gap's posterior

Let $\beta(C)$ be ordinary future extinction probability from a deterministic finite seed C. The propagation/kill conjugacy in Appendix D gives

$$\beta(C) = p_*(C \cup (C+1)), \qquad \sup_{C \neq \varnothing} \beta(C) \leq h_2 =: b < 1. \tag{6.1}$$

For a gap, conditional on $\mathcal{F}_L$, its eventual extinction probability is $\beta_L = \beta(C_G(L))$. Write $p_g(L) = \Pr(\tau_G \leq L)$ and $J_g(L) = \mathbb{E}[\beta_L^2; \tau_G > L]$. Uniform late-extinction control gives $f(K) = \sup_{C \neq \varnothing} \Pr_C(K < \tau < \infty) \to 0$. Split extinction of a nonempty current seed into the next K rows and its remaining tail. Pointwise,

$$\beta(C)^2 \leq \Pr_C(\tau \leq K) + f(K)\beta(C).$$

Averaging on $\{\tau_G > L\}$ gives

$$J_g(L) \leq p_g(L+K) - p_g(L) + f(K)[\pi_g - p_g(L)]. \tag{6.2}$$

Appendix E proves, for every $L/g \to \theta \in [0, \infty]$,

$$\frac{p_g(L)}{\pi_g} \to 1 - \frac{1}{2}(1 + \kappa\theta)^{-\gamma}, \tag{6.3}$$

with the last factor zero at infinity. Choose $K = \lfloor \sqrt{g} \rfloor$. Continuity of the limiting curve and compactness of $[0, \infty]$ show, by taking a subsequence of any failing supremum sequence, that

$$\sup_{L \geq 0} J_g(L) = o(\pi_g), \qquad \mathbb{E}\beta_L^k = p_g(L) + o(\pi_g) \quad (k = 2, 3), \tag{6.4}$$

uniformly over deterministic ages. The data-selected stopping analysis uses the posterior bounds proved in Section 7.3.

6.2. Product posterior moments

Use the boxes of Section 4 with any fixed sequence $w/g_* \to \infty$, $w = o(n)$. Put $Q_{j,L} = \mathbb{E}[e_j \mid \mathcal{F}_L]$, $Q_L = \mathbb{E}[1_{E_w} \mid \mathcal{F}_L]$. The unrevealed future coordinates in the boxes retain their independent product distribution. Each $Q_{j,L}$ is a measurable function only of the already revealed coordinates in box j; the independent auxiliary variable U does not change the conditional law. Thus the $Q_{j,L}$, as random variables, are independent and

$$Q_L = \prod_j Q_{j,L}, \qquad \mathbb{E}Q_L^k = \prod_j \mathbb{E}Q_{j,L}^k. \tag{6.5}$$

For $L \geq w$ the bits are already revealed and these identities still hold. From $e_j \leq 1_{\eta_j}$ and contraction,

$$\mathbb{E}|Q_{j,L} - \beta_{j,L}| \leq \pi_j - b_j \leq CA_w = o(\pi_j).$$

Combining (6.3)–(6.5), uniformly in L,

$$\mathbb{E}Q_L = \Pi_n + o(\Pi_n), \qquad \mathbb{E}Q_L^k = \Pi_n \prod_j f_{j,n}(L) + o(\Pi_n), \quad k = 2, 3. \tag{6.6}$$

For $k = 2, 3$, put $a_{j,L} = \mathbb{E}Q_{j,L}^k$ and $c_{j,L} = \pi_j f_{j,n}(L)$. The single-gap estimates give $a_{j,L}, c_{j,L} = O(\pi_j)$ and $|a_{j,L} - c_{j,L}| = o(\pi_j)$ uniformly in L. The telescoping product bound therefore yields

$$\left| \prod_j a_{j,L} - \prod_j c_{j,L} \right| \leq \sum_j |a_{j,L} - c_{j,L}| \prod_{i<j} a_{i,L} \prod_{i>j} c_{i,L} = o(\Pi_n), \tag{6.6a}$$

uniformly in L, since q is fixed. Each single-gap error retains the probability factors of all other gaps.

For $V = m1_{E_w}$ the squared risk is exactly $m^2\mathbb{E}[Q_L - Q_L^2]$. The absolute risk is $m\mathbb{E}\min(Q_L, 1-Q_L)$. For $0 \leq z \leq 1$,

$$0 \leq \min(z, 1-z) - z(1-z) \leq 2z^2(1-z).$$

Equation (6.6) gives $\mathbb{E}(Q_L^2 - Q_L^3) = o(\Pi_n)$, so both risks have the leading form (3.7). Lemma 4.3 transfers this expansion uniformly over information fields to Z_n.

6.3. Recovery-record attainment at every age

For $0 \leq t \leq w$, stop the finite-box replay at t and denote its extinction bit by $e_j^{\leq t}(w)$. This bit uses no rows after t, is at most $e_j(w)$, and is at most $1_{\{\tau_j \leq t\}}$. The history defect and long-atom count used in (4.6) give

$$0 \leq p_{g_j}(t) - \mathbb{E}e_j^{\leq t}(w) \leq CA_w, \qquad 0 \leq t \leq w. \tag{6.7}$$

Only initial-history truncation is needed here. The exact future-conditioned target is A_t from (4.6a), so its cost is $p_*(A_t) - p_w(A_t) \leq CA_w$, with zero cost when $A_t = \varnothing$. The long-atom count is bounded by (4.7), uniformly in t.

Let $E_t = \bigcap_j \{e_j^{\leq t}(w) = 1\}$ and define B_t as in (4.14), at age t, using the same length-w initial centers throughout. Center survival is independent of the box vector, while Lemma 4.1 at finite age gives $\Pr(B_t = 1) \leq \prod_j p_{g_j}(t)$. Hence

$$\Pr(\{B_t = 1\} \triangle E_t) = o(\Pi_n), \qquad \Pr(E_t) = \prod_j p_{g_j}(t) + o(\Pi_n), \tag{6.8}$$

uniformly for $t \leq w$. Since $E_t \subset E_w$, the predictor mB_t has absolute and squared errors

$$m[\Pi_n - \prod_j p_{g_j}(t)] + o(\Pi_n), \qquad m^2[\Pi_n - \prod_j p_{g_j}(t)] + o(\Pi_n).$$

For arbitrary L, use $t = \min(L, w)$. If $L > w$, both the risk of mB_w and $\Pi_n \Phi_n(L)$ are $o(\Pi_n)$. This attains the full-prefix lower bound in $\mathcal{H}_L$ and proves Theorem 3.2.

If $g_j/g_* \to c_j \in (0, 1]$, the first age reducing the initial Bayes risk to a fixed fraction $\varepsilon \in (0, 1)$ satisfies $L_\varepsilon/g_* \to \theta_\varepsilon$, where

$$1 - \prod_j \left[1 - \tfrac{1}{2}(1 + \kappa\theta_\varepsilon/c_j)^{-\gamma}\right] = \varepsilon(1 - 2^{-q}). \tag{6.9}$$

The limiting curve is continuous and strictly decreasing. The uniform error and monotonicity of Bayes risk under nested information prove this first-crossing assertion. When some ratios tend to zero, (3.7) remains valid, but (6.9) is not a relative first-crossing claim at a possible boundary plateau.

7. Honest stopping and the average waiting benchmark

7.1. Recovery-record upper rule

Choose $a_n \to \infty$ with $a_n g_*/n \to 0$ and set $t_k = \lceil a_n g_{(k)} \rceil$, $t = t_q$. Choose the same initial length-t center in each internal root for every stage. At stage $k < q$, at age t_k, continue only if the first k gaps in length order are empty and all centers survive. Otherwise stop and output zero. At age t_q output mB_t. Tied lengths can be checked at the same physical age.

Let K_k be the stage-k continuation event and $A_k = \bigcap_{h \le k} K_h$. Lemma 4.1 applied to the checked subset gives $\Pr(A_k) \le P_k$. Thus

$$\mathbb{E}T = t_1 + \sum_{k<q} (t_{k+1} - t_k) \Pr(A_k) \le a_n \sum_{k=1}^{q} g_{(k)} P_{k-1} + q. \tag{7.1}$$

The event $B_t = 1$ implies every center survived at all previous stages. If $B_t = 1$ but an early stage failed, some already checked gap j was still alive at that stage's age t_h. Its short box $e_j(t_h)$ uses a subset of its long box coordinates. Independence of the long boxes gives

$$\Pr(E_t^{\text{long}}, \tau_j > t_h) \le \Pr(e_j(t) = 1, e_j(t_h) = 0) \prod_{i \ne j} \mathbb{E}e_i(t) \le C A_{t_h} \prod_{i \ne j} \pi_i = o(\Pi_n), \tag{7.2}$$

where $E_t^{\text{long}} = \bigcap_j e_j(t)$, and $t_h/g_j \ge a_n \to \infty$. Potter bounds control any slowly varying factors. Transfer B_t to E_t^{long} by the event symmetric-difference estimate in Section 4.4 and sum over finitely many stages. The multi-stage predictor differs from mB_t on a set of probability $o(\Pi_n)$, and hence achieves both errors $o(\Pi_n)$.

For a prescribed $r_n \to \infty$, choose

$$a_n = \min\{r_n, \sqrt{n/g_*}\}. \tag{7.3}$$

Then $a_n \to \infty$, $a_n g_*/n \to 0$, and (7.1) proves (3.10).

7.2. A rare branch in the original past

For the lower bound fix a length rank k. Throughout Sections 7.2–7.3 write $G_i = [a_i, b_i]$, g_i, and π_i for the length-ordered gaps, retaining their original spatial locations. Define, for $i < k$, the finite past family

$$-g_i \le s \le -1, \qquad a_i - 2g_i \le z \le a_i - g_i, \qquad 4g_i \le r \le 5g_i. \tag{7.4}$$

Let R_i^- be its occurrence event and $B_{k-1}^- = \bigcap_{i<k} R_i^-$, with B_0^- the whole space. Each such atom leaves a row-zero hole remnant containing G_i, so the gap is initially empty and its bit in

every sufficiently large comparison box is one. The families are disjoint. With $I_p = \int_4^5 y^{-p}\,dy$ and $\alpha = \kappa I_p/(2C_A) > 0$,

$$\Pr(R_i^-) \sim \alpha\pi_i, \qquad \Pr(B_{k-1}^-) \sim \alpha^{k-1}P_{k-1}. \tag{7.5}$$

There is an exact independence needed below. Changing any coordinates of family i affects row-zero states only in $[a_i - 2g_i, a_i + 5g_i - 1]$. For large n this interval misses every remaining gap $j \geq k$, since intervening roots are macroscopic. In finite all-open-past constructions this is deterministic finite propagation; taking their decreasing stationary limit preserves it. Thus the remaining initial source vector is a function of coordinates outside these families. All positive-age rows and U are also independent of them. Consequently B_{k-1}^- is jointly independent of the remaining initial sources and their entire ordinary future descendant processes. Dependence among the remaining extinction events is retained in the bounds below.

7.3. Future forcing after an early cutoff

Fix $B \geq 0$ and $L = \lfloor Bg_k \rfloor$. Choose a deterministic long-box age $w/g_* \to \infty$, $w = o(n)$, and let E_w be the fixed comparison event from Section 4. For each $j \geq k$ put $h_j = g_j + L$ and choose the future family

$$L + 1 \leq s \leq L + h_j, \qquad a_j - h_j \leq z \leq a_j, \qquad 4h_j \leq r \leq 5h_j. \tag{7.6}$$

The families are disjoint, lie strictly after L, and are eventually contained in their respective long boxes: $5h_j \leq w$ and $L + h_j \leq w$ because $h_j \leq (1 + B)g_*$; their origin lower bound also satisfies $a_j - h_j \geq a_j - 2w + 1$ eventually. Any such atom covers the whole gap-source forward cone in its row, killing those sources. If R^+ means occurrence in every remaining family, then

$$B_{k-1}^- \cap R^+ \subset E_w, \qquad p_R := \Pr(R^+) \geq c_B \prod_{j \geq k} \pi_j, \qquad R^+ \perp \mathcal{F}_L. \tag{7.7}$$

The lower bound follows from the same mean count as (5.3) and uniform regular variation on $h_j/g_j \in [1, 1 + B]$.

Let T be any admissible stopping time and put $Q_T = \Pr(E_w \mid \mathcal{F}_T)$. The target approximation (4.10) and Lemma 4.3 apply uniformly over information fields. Information monotonicity gives $R_i(\mathcal{F}_T) \leq R_i(\mathcal{I}_T)$. If either actual risk is $o(\Pi_n)$, the same field-uniform target transfer therefore gives

$$\mathbb{E}[Q_T(1 - Q_T)] = o(\Pi_n). \tag{7.8}$$

For squared risk this is its binary conditional variance. For absolute risk use $Q(1 - Q) \leq \min(Q, 1 - Q)$.

If $A \in \mathcal{F}_T$ and $A \subset B_{k-1}^- \cap \{T \leq L\}$, (2.11) gives $A \in \mathcal{F}_L$. Equation (7.7) therefore gives $\Pr(E_w \cap A) \geq p_R \Pr(A)$. As this holds for every such A,

$$Q_T \geq p_R \quad \text{on } B_{k-1}^- \cap \{T \leq L\}. \tag{7.9}$$

Let D_T^{rem} be the event that at least one remaining gap has a nonempty actual source population at T. The ordinary future after T has the original law, proved by splitting into the countably many events $\{T = s\}$ and using independence after deterministic rows. Since $E_w \subset \eta_j$ globally, (6.1) gives

$$Q_T \leq b < 1 \quad \text{on } D_T^{\mathrm{rem}}. \tag{7.10}$$

This applies also when $T > w$. In that case E_w is already revealed, and on $\{T \geq w\} \cap D_T^{\mathrm{rem}}$ its posterior is actually zero. The argument covers all almost surely finite admissible stopping times.

The complement of D_T^{rem} implies eventual extinction of every remaining gap. The exact independence in Section 7.2 yields

$$\Pr(B_{k-1}^-, (D_T^{\mathrm{rem}})^c) \leq \Pr(B_{k-1}^-) \min_{j \geq k} \pi_j = o(\Pr(B_{k-1}^-)). \tag{7.11}$$

The inclusion bounds the stopped event by eventual extinction; independence is then used for that terminal event and the past branch. Equations (7.9)–(7.11) imply

$$\mathbb{E}[Q_T(1-Q_T)] \geq (1-b)c_B \left(\prod_{j\geq k} \pi_j\right) \left[\Pr(B_{k-1}^-, T \leq L) - \Pr(B_{k-1}^-) \min_{j\geq k} \pi_j\right]_+ . \tag{7.12}$$

Divide by $\Pi_n = P_{k-1} \prod_{j\geq k} \pi_j$ and use (7.5), (7.8). For every fixed B,

$$\Pr(T \leq B g_k \mid B_{k-1}^-) \to 0.$$

Therefore $\liminf_n \mathbb{E}T/(g_k P_{k-1}) \geq B\alpha^{k-1}$. Letting $B \to \infty$ proves each lower bound in (3.9). As q is fixed,

$$\frac{\mathbb{E}T}{S_n} \geq \frac{1}{q} \min_k \frac{\mathbb{E}T}{s_k} \to \infty.$$

Infinite expectations satisfy the assertion directly. This completes Theorem 3.3.

7.4. Exponents and examples

Regular variation gives $\pi_{(i)} = n^{-\gamma\beta_{(i)}+o(1)}$, hence $S_n = n^{c_*+o(1)}$ with c_* in (3.11). The lower bound in (3.9) gives the exponent lower bound. Use, for example, $a_n = 1 + \log(n/g_*)$ in Section 7.1 to attain that exponent, proving Corollary 3.4.

For $\gamma = 1$, the gap exponents $(1/5, 1/2, 7/10)$ give stage-cost exponents $1/5, 3/10, 0$ and optimal average exponent $3/10$. A deterministic observation age has exponent $7/10$. If additionally $A_x \sim C/x$ and $g_i = \lfloor n^{\beta_i} \rfloor$, then $\Pi_n = \Theta(n^{-7/5})$, $S_n = \Theta(n^{3/10})$, and logarithmic excess gives $\mathbb{E}T = O(n^{3/10} \log n)$. For general $A \in RV_{-1}$ these exact orders retain their slowly varying factors. For example $\ell(x) = (\log x)^2$ contributes $(\log n)^6$ to Π_n and $(\log n)^2$ to its dominant stage cost.

A two-gap example combining information failure and waiting gains. Let $q = 2$, $\gamma = 1$, $A_x \sim C/x$ and $g_1 = \lfloor n^{2/5} \rfloor$, $g_2 = \lfloor n^{4/5} \rfloor$, with fixed positive root proportions. Then

$$\Pi_n = \Theta(n^{-6/5}), \quad A_{d_n}/\Pi_n = \Theta(n^{1/5}) \to \infty, \quad S_n = g_1 + g_2\pi_1 = \Theta(n^{2/5}). \tag{7.13}$$

The terminal bits have risks $R_1(\mathcal{G}_n) \sim m\Pi_n$ and $R_2(\mathcal{G}_n) \sim m^2\Pi_n$. A deterministic age achieving either error $o(\Pi_n)$ requires $L/n^{4/5} \to \infty$. A recovery-record stopping rule achieves both errors with $\mathbb{E}T = O(n^{2/5} \log n)$. This example compares all three results for the same intervention target. When all gaps have comparable length g, fixed q instead gives $S_n = \Theta(g)$, so the deterministic and average benchmarks have the same power exponent.

8. Discussion

A defined intervention response creates a second mathematical problem: the information and waiting needed to predict it on its own probability scale. The three results separate occurrence and amplitude, historical information, and physical waiting for the same graphical target. The average-time theorem bounds every honest causal stopping rule, even with the complete realized catastrophe prefix, and attains the benchmark up to any prescribed diverging multiplier using recovery records. The two-gap example shows that this distinction can change the waiting-time exponent.

The terminal-bit criterion identifies the historical information that matters. The ratio A_{d_n}/Π_n measures contamination by catastrophes that remove both the gaps and the intervening sources. Observed center-source survival lets the recovery gate distinguish such histories from response-producing disappearances.

The deterministic-age expansion measures unresolved risk at the response probability scale. Its uniform additive remainder allows leading risk comparison at bounded scaled ages and identifies exactly when risk becomes $o(\Pi_n)$. The factor one half comes from the original stationary preparation: among eventual extinctions of a large gap, asymptotically half are already present at age zero and

half occur through later recovery histories. Independent finite Poisson boxes yield the product curve while the actual extinction events retain dependence through catastrophes spanning several gaps.

The finite-stage rule uses predetermined checkpoints and stops most trials early. Its weighted cost structure connects the result to sequential testing. The lower bound covers the full class of honest stopping rules with access to the infinite past and all realized spatial damage; the recovery-record construction attains every prescribed diverging multiplier in the interacting process.

For a fixed number of gaps, catastrophe law and compact response horizon, the results characterize physical waiting at absolute or squared error $o(\Pi_n)$. A quantitative bound on the fluid remainder in the weighted bridge would provide finite-n error certificates and support waiting-time optimization at a specified finite error budget.

Statements and declarations

Author and responsibility. HUANG Man is the sole author. The mathematical statements concern the model defined in Section 2. No empirical dataset is used.

Data and code availability. The main manuscript and accompanying Technical Supplement contain the mathematical definitions and proofs. The original code provides deterministic finite diagnostics for the source identities, Bayes-loss algebra, update conjugacy and stage costs. It is archived with the bilingual edition and sources at doi:10.5281/zenodo.24138927, under the MIT license. The archive README specifies the scope and execution command.

Competing interests. The author is employed by a public fund management institution in China. This research was conducted independently and does not represent the views of the author's employer, any fund, or any client. The study did not use non-public employer or client information and does not constitute investment advice.

Declaration of generative AI and AI-assisted technologies in the writing process. During the preparation of this work, the author used OpenAI GPT-6.1 Sol and xAI Grok 4.7 to assist with literature synthesis, development of argumentative and theoretical structure, expansion of propositions into manuscript drafts, mathematical checking, and language revision. OpenAI Codex also assisted with proof review, manuscript restructuring, preparation of diagnostic code and package consistency checks. The author retains responsibility for the research question, analytical strategy, interpretation, conclusions, and the content of the manuscript. No AI tool or service is credited as an author or cited as an author.

Technical Supplement

This supplement accompanies the main manuscript. Model notation, assumptions (2.1)–(2.4), and main-section equation numbers refer to that manuscript. Appendices A–E establish the auxiliary results used in its proofs. Both documents form the complete mathematical argument.

Appendices A–D reproduce and adapt foundation arguments from the public version 1 of Huang (2026a, 2026b), including proof passages retained from Huang (2026b). The following table specifies the reuse and its role in this paper. The local arguments supply the stated uniform estimates for the present model.

Material here	Public source and treatment	Role here
A.1–A.3: conditional correlation, insertion and phase/flux estimates	Huang (2026a, Sections 3 and 5); Huang (2026b, Appendix A.1–A.3). Proofs are reproduced and adapted to the present notation.	Positive stationary density and the small-intensity bounds.
A.4–A.5 and B: spatial tails, constrained paths and compression	Huang (2026b, Appendices A.4–A.5 and B). Foundation arguments are retained with the current conventions and cutoff estimates.	Uniform finite-target and boundary control.
C–D: fronts, interval confinement and history/late-extinction control	Huang (2026b, Sections 5–6 and 7.1). Proofs are reproduced or adapted, with explicit history/cutoff uniformity and restart cases.	The uniform bounds used by the rare-box bridge and extinction expansion.
E: two-history representation, classification and extinction age law	Uses the zero-/one-root expansion framework of Huang (2026b, Section 7), applied here to the stationary-source extinction target and its two histories.	The single-gap input to the rare-response and information theorems.

The contribution statements in the main manuscript concern its mixed-response and information/waiting results. The shared phase, front and history tools retain their companion-paper attribution.

Precision and dependency guide

The main results compare prediction error with $\Pi_n = \prod_j \pi_j$, the response probability scale. The following route identifies where that scale is preserved. Main-section equation numbers and lemmas refer to the accompanying manuscript; lettered equations refer to this supplement.

Link	Estimate and uniformity	Role in the main results
Original process to finite history	(D6) and (D8) bound history and late-extinction defects by CA_w, uniformly over nonempty finite targets or seeds, including dispersed sets.	Conditioning on the future-generated target in (4.6a) costs CA_w without multiplying by the gap length.

Link	Estimate and uniformity	Role in the main results
Single-gap amplitude and age law	Candidate classification (E2)–(E6), finite-history control, and escape estimate (E12) give the amplitudes (E7), (E13) and age law (E15). Classification is uniform over the older baseline history.	Each gap supplies its own $\pi_j \asymp A_{g_j}$ and deterministic-age extinction profile.
Independent comparison boxes	For $g_* \ll w \ll n$, (4.6) gives $0 \le \pi_j - b_j \le CA_w = o(\pi_j)$ and the finite boxes are independent.	Product errors are relative $o(1)$, so their intersection has probability $\Pi_n + o(\Pi_n)$ by (4.8).
Weighted response approximation	Lemma 4.2 controls conditional response error uniformly over every admissible rare box configuration, before integrating that configuration. Equations (4.9)–(4.10) give target errors $o(\Pi_n)$ in both losses.	Uniformity over rare box configurations carries the fluid limit to the product probability scale.
Arbitrary observed information	Lemma 4.3 transfers absolute and squared Bayes risks with error $o(\Pi_n)$ uniformly over information fields.	The same target approximation applies to deterministic information and stopped information.
Deterministic-age risk	Single-gap posterior errors in (6.4) are multiplied by the other gap moments in (6.5)–(6.6).	The total error is $o(\Pi_n)$, uniformly in deterministic age. Recovery-record gates attain the same expansion using (6.8).
Honest stopping lower bound	The past branch has probability of order P_{k-1} in (7.5); post-cutoff forcing has probability at least $c_B \prod_{j \ge k} \pi_j$ in (7.7). Their product is of order Π_n.	Equations (7.9)–(7.12) retain conditional uncertainty on early-stopped branches, yielding every stage lower bound and then the S_n bound.
Recovery-record upper rule	Continuation probabilities are at most P_k by source separation. The missed late-extinction error in (7.2) is $o(\Pi_n)$.	The actual dependent recovery events give the waiting bound (7.1) and attain every prescribed diverging multiplier.

For the candidate expansion, fix the length and history caps first. Take $g \to \infty$ and then remove the parameter-boundary width; release the caps in the order specified in Sections E.3–E.5. The outside-fluid estimate (E4)–(E5) controls suprema over deterministic histories and cutoffs before summing ages. For the weighted bridge, fix the macroscopic patch width, take $n \to \infty$, and then shrink that width. These limit orders specify which uniform estimates are used at each link.

A direct reading route is D.3, E.2–E.5, main Lemmas 4.2–4.3, Section 6 and Section 7. The front and compression arguments needed for D and E are proved in A–C. Deterministic posterior moment estimates are used only at deterministic ages; the stopped lower bound is proved separately in Section 7.3. The diagnostic code checks finite identities and algebra, while this proof route carries the infinite-model precision estimates.

Appendix A. Conditional correlation, the positive phase and spatial tails

This appendix uses a fixed intensity λ and writes $c = \lambda\chi_0$, $u = u_\lambda$, $\beta = 1 - u$. Write $A_* = \chi_0$. Define $h_{j,t}$ as the probability of a hole block of length j after t rows from all-open sites, and $h_j = \lim_t h_{j,t}$. Set $V(r) = r(r+1)/2$, so $\sum_r \nu_r V(r) = A_* = \chi_0$. Lemma A.1 allows independent interval types with nonidentical absence probabilities, as specified there. The phase, stationary flux and tail calculations in Sections A.2–A.4 use spatially homogeneous rows with the stated intensity $\lambda\nu_r$, and a homogeneous last-row interpolation when needed. For nonstationary fields dominated by this model, the corresponding upper bounds are transferred by monotone coupling.

A.1. Conditional negative association

Lemma A.1. At every finite time, conditional on the origin being open, bounded increasing functions depending on finitely many hole coordinates strictly to its left and strictly to its right are negatively correlated. In a finite model, the statement allows arbitrary independent type-dependent probabilities. In the infinite model used here, the type means in a finite history are bounded by $\lambda\nu_r$, including a final row of intensity $\theta\nu_r$ with $0 \le \theta \le \lambda$.

Proof. First restrict to finitely many interval types in a finite slab containing the relevant backward cones. Set $Y_e = 1$ when disaster type e is absent. The variables are independent with parameters strictly between zero and one. The bottom row is fully occupied. Let Γ be the event that a path reaches terminal site 0, and let μ be the product law conditioned on Γ.

The pointwise minimum and maximum of two allowed paths are still allowed: their increments remain in $\{0, 1\}$. For open paths these operations choose an already open site in each row. Every configuration in Γ therefore has a leftmost path L and a rightmost path R to 0. For a specified path π, a connected disaster interval in its row is strictly left of π, intersects π, or is strictly right of π. Denote these disjoint type sets by $\mathcal{L}_\pi, \mathcal{C}_\pi, \mathcal{R}_\pi$.

Define a right update by resampling the variables in $\mathcal{R}_L$ from their original independent laws. This leaves L unchanged. Indeed, if a new path p appeared partly to the left of L, then $\min(p, L)$ would have been open before the update, because strictly right types cannot affect it. This contradicts leftmostness. Moreover, the event $L = \pi$ depends only on $\mathcal{L}_\pi \cup \mathcal{C}_\pi$: existence of a path further left can be tested among paths everywhere at or left of π. Conditional on the remaining variables and L, the right variables retain their product distribution. The right update thus preserves μ. Resampling $\mathcal{L}_R$ analogously preserves R and μ.

Alternating these kernels gives a finite irreducible, aperiodic chain. To verify irreducibility, set all resampled variables to 1 in a complete right-then-left step, an event of positive probability. After the right step the vertical path at 0 is open: any interval hitting it either hits the old leftmost path and was already absent, or is strictly to its right and has just been removed. The vertical path is the largest allowed path to 0. The left step then removes every remaining disaster strictly to its left; all other types are already absent. Thus every state can reach the all-1 state. Since μ is invariant and has full support on Γ, there can be no transient states or another closed class. Self-loops have positive probability.

Define a comparison between two configurations in Γ by $\widehat{\omega} \preceq \omega$ if (i) $\widehat{L} \le L$ and $\widehat{R} \le R$ rowwise, (ii) $\widehat{Y}_e \ge Y_e$ for types strictly left of $\widehat{L}$, and (iii) $\widehat{Y}_e \le Y_e$ for types strictly right of R. No ordering is imposed on the remaining variables. This is the interval-type version of the comparison in van den Berg, Häggström and Kahn (2006a, Section 3.1). For a fixed absence vector y, write the updates explicitly as

$$(T_R^y Y)_e = \begin{cases} y_e, & e \in \mathcal{R}_{L(Y)}, \\ Y_e, & e \notin \mathcal{R}_{L(Y)}, \end{cases}$$

$$(T_L^y Y)_e = \begin{cases} y_e, & e \in \mathcal{L}_{R(Y)}, \\ Y_e, & e \notin \mathcal{L}_{R(Y)}. \end{cases}$$

The same vector y is used for the two configurations; each configuration selects its own resampling

set.

A common right update preserves the comparison. Write $\alpha = \widehat{L} \le L = \gamma$ and let p be open after the update in $\widehat{\omega}$. We check $q = \max(p, \gamma)$ row by row in the updated ω. At a site of γ, every covering type is absent because γ remains open. At a site $p_t > \gamma_t$, a covering interval either intersects γ_t, in which case it is absent in ω, or has its entire interval strictly right of γ_t. In the latter case it is selected in both configurations and receives the same entry y_e; openness of p forces that entry to be 1. Connectedness of the interval exhausts these cases. Thus q is open, proving the order of the new rightmost paths. The leftmost paths are unchanged. Types strictly left of α are unchanged in both configurations. A type strictly right of the new rightmost path in ω is also strictly right of γ and α, so it was selected in both configurations and now has equal values. These observations verify all three comparison conditions after the right update.

For a common left update, set $\alpha = \widehat{R} \le R = \gamma$. Given a path p open after the update in ω, check $q = \min(p, \alpha)$ in the updated $\widehat{\omega}$. At a site of α, all covering types are absent. At a site $p_t < \alpha_t$, a covering interval either meets α_t and is absent in $\widehat{\omega}$, or is strictly left of both α_t and γ_t and receives the same entry y_e. This proves that q is open and orders the new leftmost paths. The rightmost paths are unchanged. Every type strictly left of the new $\widehat{L}$ is strictly left of both old rightmost paths, so it was identically resampled. Types strictly right of γ are unchanged in both configurations. Hence this update also preserves all three conditions, including when the two selected type sets differ.

For a single right update from identical input configurations, decreasing the resampled absence variables adds only right-side disasters. The old leftmost path stays open, and the new rightmost path can only move left; all types beyond the larger rightmost path have the required absence order, while types left of the common leftmost path are unchanged. This establishes the comparison after that update. For a single left update, increasing the resampled absences adds only left-side open paths; the rightmost path stays fixed and the leftmost path can only move left, with the required order on the two outer type sets. The common-update preservation just proved propagates either comparison through subsequent updates. Finally, if $\widehat{\omega} \preceq \omega$, every path to a terminal $x < 0$ open in ω yields an open path in $\widehat{\omega}$ by taking its minimum with $\widehat{L}$. An interval covering a strictly left point either intersects $\widehat{L}$ or belongs to its strictly left type set, where condition (ii) applies. Terminal reachability on the left therefore increases, while reachability on the right decreases by the corresponding maximum construction and condition (iii).

Realize the resampling chain on a product space by sampling arrays $Y^R_{k,e}$ and $Y^L_{k,e}$ in advance for every update index k and every retained interval type e. Use an entry only when its type belongs to the currently selected resampling set; otherwise ignore it. All these entries are independent with the original type-specific absence probabilities. If one entry is changed, the two chains have identical histories before its update. At that update either the entry is ignored in both chains or the single-update comparison established above applies. Every subsequent common update preserves that comparison, even when the two resampling sets differ. Thus, after any fixed number of updates, every left increasing hole functional is decreasing in the left-update entries and increasing in the right-update entries. Right increasing functionals have the opposite signs. Complementing the right-update entries places the first functional and the negative of the second in the same coordinatewise decreasing class. The product-measure association inequality gives their nonnegative covariance, equivalently the desired negative correlation. A direct proof for the type-dependent parameters used here is given below. Start at the all-1 configuration and let the finite conditional chain converge to μ to obtain the asserted conditional inequality.

For the infinite model, first fix the time and the finite terminal coordinates of the functions. Their backward cones are finite. Remove types of length greater than K; the total mean of omitted types intersecting the cones is at most a constant times $\lambda T \sum_{r>K} \nu_r(r+C)$, which tends to zero by $S_1 < \infty$. The conditional event has probability at least that of an open vertical path, $e^{-\lambda T S_1} > 0$. Thus the conditional inequality passes to the limit. Deterministic absence variables can be removed or obtained by parameter limits. The same argument applies to the stated type-dependent parameters and last-row interpolation. □

The historical correlation argument is due to Harris (1960); van den Berg and Kesten (1985,

Section 3) state the independent-coordinate inequality used here. In the finite independent-coordinate form used above, it follows directly by induction over the independent Bernoulli coordinates. For bounded increasing F, G, condition on the last coordinate Y: the total covariance is $\mathbb{E}\,\mathrm{Cov}(F, G \mid Y) + \mathrm{Cov}(\mathbb{E}[F \mid Y], \mathbb{E}[G \mid Y])$. The first term is nonnegative by induction; the second is $p(1-p)\Delta F \Delta G \geq 0$, where $p = \Pr(Y = 1)$ and $\Delta F = \mathbb{E}[F \mid Y = 1] - \mathbb{E}[F \mid Y = 0]$, with ΔG defined similarly. Negating one function gives the opposite-monotonicity form. The argument permits different parameters at different coordinates.

The proof adapts the extremal-path resampling strategy of van den Berg, Häggström and Kahn (2006a) to connected interval errors. For a disconnected error closing $\{-1, 1\}$ with probability $1/2$ while leaving 0 open, the two side-hole indicators have joint probability $1/2$ and product of marginals $1/4$. The conclusion is therefore false for arbitrary spatially correlated row errors.

A.2. Finite-history phase construction

Lemma A.3 (full insertion count). In a translation-invariant hole field F, insert $[0, r-1]$. Let L_z be the number of consecutive holes immediately left of z, and R_r the number beginning at r, with values in $\mathbb{Z}_{\geq 0} \cup \{\infty\}$. A summand in (A1) is defined to be zero whenever $z \in F$, including configurations with infinite closed runs. The count and identity are interpreted as nonnegative extended quantities. The number of newly closed, nonempty finite intervals is

$$\Delta_r(F) = \sum_{z=0}^{r-1} \mathbf{1}_{\{z \notin F\}} (L_z + 1)(r - z + R_r). \tag{A1}$$

Every newly closed interval has a unique first previously open point z. Its left endpoint has $L_z + 1$ choices and its right endpoint has $r - z + R_r$ choices.

Assume $u_F = \Pr(0 \notin F) > 0$ and the conditional inequality of Lemma A.1. If $h_k(F)$ is the k-interval closure probability and $\zeta = \lim_k h_k(F)$, boundary tail sums give

$$\mathbb{E}[\mathbf{1}_{\{z \notin F\}}(L_z + 1)] = u_F + \beta_F - \zeta \leq 1.$$

For $d = r - z$, split $\{z \notin F, [r, r + b - 1] \subset F\}$ by the last open site in $[z, r-1]$, discard other restrictions, and sum over $b \geq 1$. Each of the d possible boundary sites contributes at most $\beta_F - \zeta$, so

$$\mathbb{E}[\mathbf{1}_{\{z \notin F\}} R_r] \leq d\beta_F, \qquad \mathbb{E}[\mathbf{1}_{\{z \notin F\}}(d + R_r)] \leq d.$$

Conditional negative association, followed by bounded truncation and monotone convergence, now yields

$$\mathbb{E}\Delta_r(F) \leq V(r)/u_F. \tag{A2}$$

Lemma A.4 (finite-history bound). If $c \leq 1/32$, then

$$\sup_t \beta_t \leq \frac{1 - \sqrt{1 - 8c}}{2} < \frac{1}{2}. \tag{A3}$$

Proof. Use the last-row interpolation field

$$F_{t,\theta} = E(H_t) \cup D^{(\theta)}, \qquad 0 \leq \theta \leq \lambda,$$

where the first t rows have intensities $\lambda\nu_r$ and the last row has intensities $\theta\nu_r$. It is exactly a finite-history process of Lemma A.1 with non-identical type parameters; its open density is at least $u_{t+1} > 0$.

Work first on the infinite integer line with only lengths $r \leq K$ retained. This model remains translation invariant. A fixed finite observation and its derivative involve finitely many retained types in a backward cone of this spatially translation-invariant model. Define $\phi_M(F, x) = \sum_{j=1}^{M} \mathbf{1}_{\{[x, x+j-1] \subset F\}}$.

Poisson differentiation of the last row, followed by translation mass counting on this infinite-line law, gives

$$\frac{d}{d\theta}\mathbb{E}\phi_M(F_{t,\theta},0)=\sum_r \nu_r\,\mathbb{E}\Delta_{r,\le M}(F_{t,\theta})\le A_*/u_{t+1}. \tag{A4}$$

Here $\Delta_{r,\le M}$ counts new intervals of length at most M. The same upper bound holds for all K since removal of long disasters only increases openness. Integrate over θ and let $K\to\infty$ by bounded local convergence. Then let $M\to\infty$ in the nonnegative insertion increments. Starting from zero, induction proves that $\Phi_t:=\sum_{j\ge1}h_{j,t}<\infty$ for each finite t: erosion does not increase it, and each row adds at most c/u_{t+1}. Since erosion replaces $h_{j,t}$ by $h_{j+1,t}$,

$$\Phi_{t+1}-\Phi_t\le c/u_{t+1}-\beta_t. \tag{A5}$$

Adding past rows makes the hole marginals larger, so β_t is nondecreasing. A single new row gives $\beta_{t+1}\le\beta_t+q$ with $q=1-e^{-\lambda S_1}\le c$. Independent time-block concatenation gives $u_{a+b}\ge u_au_b$: choose, by a fixed rule using only the last a rows, an open path to 0 and its initial site z when one exists. The first b rows reach that site with probability u_b, independently. The shared boundary row is checked by the first block only. Thus $\beta_{a+b}\le\beta_a+\beta_b$.

Summing this inequality over complementary time indices, and summing (A5) while retaining $\Phi_T\ge0$, gives

$$\frac{T-1}{2}\beta_T\le\sum_{t<T}\beta_t\le Tc/u_T,\qquad \beta_T(1-\beta_T)\le\frac{2T}{T-1}c\le4c\quad(T\ge2). \tag{A6}$$

If T is the first crossing of $1/2$, then $1/2\le\beta_T\le1/2+c$. Its product with $1-\beta_T$ is at least $1/4-c^2\ge255/1024>1/8\ge4c$, contradicting (A6). Hence the crossing never occurs. Taking $T\to\infty$ in the sharper version of (A6) gives $\beta(1-\beta)\le2c$ on the small branch, proving (A3). The constant $1/32$ supplies a uniform positive-density regime. □

A.3. Stationary flux

The stationary field is a translation-equivariant measurable function of the independent columns of marked Poisson variables, hence is spatially ergodic. Equation (A3) implies $u>0$, so it has no all-closed component and $h_j\to0$. A stationary field cannot have a closed half-line with a finite endpoint at positive probability: the events specifying its unique endpoint are disjoint translates with equal probabilities. Thus closed runs are finite almost surely. The interpolation fields $F_\theta=E(H^*)\cup D^{(\theta)}$ are coupled below F_λ, which has law H^*; they inherit this property and have open densities $u_\theta\ge u$.

Lemma A.5 (local stationary interpolation). Let $\Delta_{r,j}(F)$ count new closed intervals of length j after inserting $[0,r-1]$. Only the $r+j-1$ windows intersecting the insertion can change, so

$$0\le\Delta_{r,j}(F)\le r+j-1.$$

For fixed j, this bound is integrable under ν by $S_1<\infty$. Approximate H^* by finite histories and truncated lengths. Each local insertion difference converges and is bounded by the same integrable envelope. Dominated convergence passes the last-row interpolation to stationarity, giving

$$q_j:=h_j-h_{j+1}=\int_0^\lambda\sum_r\nu_r\mathbb{E}\Delta_{r,j}(F_\theta)\,d\theta. \tag{A7}$$

The conditional inequality also passes to these fields from Lemma A.1, since their open densities are bounded below by $u>0$.

In a stationary field with $h_j(F)\to0$, the boundary tail identity sharpens to $\mathbb{E}[\mathbf{1}_{\{z\notin F\}}L_z]=\beta_F$. Substituting it in (A1) yields

$$\mathbb{E}\Delta_r(F)=V(r)+\sum_{z=0}^{r-1}\mathbb{E}[\mathbf{1}_{\{z\notin F\}}(L_z+1)R_r]\ge V(r).$$

For the interpolation fields under consideration, $h_j(F) \to 0$. Combining the identity with (A2) and summing (A7) by Tonelli gives

$$\beta = \int_0^\lambda \sum_r \nu_r \mathbb{E}\Delta_r(F_\theta)\, d\theta, \qquad c \le \beta \le c/u. \tag{A8}$$

The inequality $\beta < 1/2$ selects the small root of $\beta(1-\beta) \le c$, giving the density bound (C1). Also $\beta \le 2c$ and $\beta - c \le \beta^2 \le 4c^2$. The phase bound (A3) determines the branch used in the stationary flux inequality.

A.4. Branching comparison and the spatial tail bound

Put $q_0 = u$ and $q_j = h_j - h_{j+1}$ for $j \ge 1$. This is a probability mass function on the nonnegative integers; let Q have this law, so $\Pr(Q \ge k) = h_k$. For an interpolation field F, define $q_a^F = h_a^F - h_{a+1}^F$. This is also the probability that a site is open and the closed run immediately to its left has length at least a.

Assign each new interval of length j in (A1) to its first old open point z. Let its left and right endpoint distances from z be a, b with $a + b = j - 1$, and put $d = r - z$. If $b < d$, its right requirement is automatic. If $b \ge d$, partition by the last old open site in $[z, r-1]$ and discard additional restrictions. Both cases give

$$\Pr(z \notin F,\ d + R_r \ge b + 1) \le \sum_{i=0}^{\min(d-1,b)} q_{b-i}^F.$$

Conditional negative association separates the left run event from this right requirement. Summing over endpoints and z gives

$$\mathbb{E}\Delta_{r,j}(F) \le \frac{1}{u_F} \sum_{i=0}^{r-1} (r-i)(q^F * q^F)_{j-1-i}, \tag{A9}$$

with negative subscripts zero. In the coupling $F_\theta \subset F_\lambda$, Q_θ is stochastically bounded by Q. Applying this stochastic domination to tails, sum (A7) over $j \ge k$ and use (A9) to obtain

$$h_k \le \frac{\lambda}{u} \sum_r \nu_r \sum_{i=0}^{r-1} (r-i) \Pr(Q_1 + Q_2 + i \ge k - 1). \tag{A10}$$

Here Q_1, Q_2 are independent auxiliary copies of Q. Define I by

$$\Pr(I = i) = \frac{\sum_r \nu_r (r-i)_+}{A_*}, \qquad \Pr(I \ge m) = A_{m+1}/A_*.$$

With $a = c/u$, (A10) says $Q \le_{\mathrm{st}} B(1 + I + Q_1 + Q_2)$, all variables on the right independent and B Bernoulli with parameter a. Iterate the distributional upper map on a binary tree. The chance of any active leaf at depth n is at most $(2a)^n$. On its complement the depth-n comparison no longer depends on the leaf copies of Q. Since $2a < 1$, this proves

$$Q \le_{\mathrm{st}} X = \sum_{i=1}^{J} W_i, \qquad W = 1 + I,$$

where J is the number of successful nodes, success having probability a and producing two children. The costs are independent of the tree, and

$$\Pr(J = j) = \mathrm{Catalan}(j) a^j (1-a)^{j+1} \le (4a)^j, \qquad \mathbb{E}J = \frac{a}{1-2a}. \tag{A11}$$

A direct bound suffices for this tail. For $j \le k/K$, the union bound and Potter's inequality give

$$\Pr(W_1 + \cdots + W_j \ge k) \le j \Pr(W \ge k/j) \le C_\epsilon j^{1+\gamma+\epsilon} A_k.$$

Choose a fixed large K so that the Potter estimate applies to $k/j \geq K$. The exponential tail of J makes the sum of these bounds finite. For $j > k/K$, use $\Pr(J > k/K)$, which is exponentially small and hence $o(A_k)$. Thus $\Pr(X \geq k) \leq CA_k$ without any finite-mean assumption on W.

It follows that $h_k \leq CA_k$ for the regularly varying law in Section 2. The constant is finite for each fixed law and intensity. For every length truncation, monotone coupling gives $h_k^{(m)} \leq h_k$, so the same upper envelope applies uniformly in m. Independently eroded old intervals are contained in the actual hole field and give $\beta \geq 1 - e^{-c}$; the stronger bound $\beta \geq c$ follows from the stationary flux calculation above.

A.5. Constrained paths and boundary observables

Lemma A.6 (deterministic checkpoint and half-line edge comparison). Fix a finite time slab with rows $0, \ldots, n$, allowed forward increments $0, 1$, and initially open sites at row 0. At every later row each retained disaster type is a connected integer interval; its absence indicator is independent of all other retained types and has its own deterministic parameter. First take finitely many types and a finite spatial slab large enough to contain every backward cone under consideration. Fix a deterministic checkpoint row $r \in \{0, \ldots, n\}$ and integer c. A constrained path must have coordinate at most c at row r. Write H^0 and H^c for the terminal hole sets with unrestricted and constrained paths, respectively, coupled using the same types. Thus $r = 0$ is the source restriction $(-\infty, c]$.

For a deterministic endpoint z, let

$$A_z = \{z \notin H^0\}, \qquad O_z = \{z \notin H^c\}.$$

Whenever $\Pr(O_z) > 0$, the following assertions hold.

(i) On O_z, the two terminal fields agree strictly to the left of z.

(ii) Given O_z, bounded increasing functions of constrained hole coordinates strictly left and strictly right of z are negatively correlated.

(iii) For any bounded increasing function f of unrestricted terminal hole coordinates strictly left of z,

$$\mathbb{E}[f(H^0) \mid O_z] \leq \mathbb{E}[f(H^0) \mid A_z]. \tag{A6.1}$$

Consequently, if $Z = \sup\{y : y \notin H^c\}$ is the constrained rightmost open site, then, for every z with $\Pr(Z = z) > 0$,

$$\mathbb{E}[f(H^0) \mid Z = z] \leq \mathbb{E}[f(H^0) \mid A_z]. \tag{A6.2}$$

Here the edge event uses only the finite interval $z + 1, \ldots, c + n - r$, since no constrained path can end to its right.

(iv) Let X be any independent spatially stationary random open set. Define the inward union-hole run

$$R_z(H, X) = \inf\{j \geq 0 : z - j \notin H,\ z - j \in X\},$$

with value infinity if the set is empty. If H^0 is spatially stationary and $p = \Pr(A_z) > 0$, then

$$\mathbb{E}\Big[R_z(H^0, X) \mid Z = z\Big] \leq \frac{\mathbb{E}R_0(H^0, X)}{p}. \tag{A6.3}$$

On the left of (A6.3), H^0 can be replaced by H^c in the interior, and the coordinate z is open. The reflected statements hold for a lower-half-line constraint and its leftmost edge, with inward

runs directed right. The independent terminal field enters through its independence and spatial stationarity.

For the infinite interval model, the conclusions hold at each finite time provided the type means are bounded by $\bar{a}\nu_\ell$, $\sum_\ell \ell\nu_\ell < \infty$, and every conditioning event used has positive probability. Nonnegative observables are obtained by monotone truncation.

Proof. We first prove the finite-type assertion with parameters strictly between zero and one. All required paths are taken in a finite slab containing their backward cones, so its outer walls do not remove any of them. Minima and maxima of two allowed paths remain allowed, and the checkpoint restriction is preserved by both operations. If a full-source path ends at $y < z$ and a constrained path ends at z, their pointwise minimum is a constrained open path ending at y. The opposite inclusion is automatic. This proves (i).

For (ii), condition the independent type law on the existence of a constrained path to z. Let L, R be its leftmost and rightmost paths. Resample types strictly right of L from their independent original laws, then types strictly left of the resulting R. The right update leaves L fixed: a new path partly left of L would give, by minimum with L, an old constrained path contradicting leftmostness. Moreover, the event specifying L reads only types intersecting or left of it. Thus the right variables retain their product law conditional on that event; the update preserves the conditional measure. The symmetric assertion holds for the left update.

The greatest all-absent constrained path to z is

$$\pi_s = \min\{z, c + (s - r)_+\}, \qquad 0 \le s \le n. \tag{A6.4}$$

Positive probability of O_z ensures $z \le c + n - r$, so $\pi_n = z$; the increments are allowed and $\pi_r \le c$. Every constrained path is below π: before the checkpoint its coordinate is at most both its terminal coordinate and c; after the checkpoint it is at most both z and $c + s - r$. Force all selected absence variables to one in a right-then-left update. After the right update, π is open. Indeed, an interval hitting π_s either also hits L_s, in which case it was already absent, or is strictly right of L_s, in which case it was just cleared. The new rightmost constrained path is therefore π. The left update clears all types strictly left of π. Types intersecting π are absent; types strictly right of π were cleared in the first update. Thus the all-absent configuration is reachable with positive probability. Invariance with full support, and this reachability from every state, imply irreducibility; self-loops give aperiodicity.

For completeness, the common-update comparison used here retains the exact three conditions of A.1: $\widehat{L} \le L$, $\widehat{R} \le R$, absence variables ordered upward strictly left of $\widehat{L}$, and downward strictly right of R. The checkpoint adds no type to be resampled; it only restricts the class of paths. For a common right update, map an open path p in the hatted configuration to $\max(p, L)$ in the other configuration. At a point lying on L, covering types are absent. At a point strictly right of L, a covering interval either hits L, hence is absent, or is strictly right of both leftmost paths, hence receives the same resampling entry. This proves the order of rightmost paths. Leftmost paths are fixed. Outer types left of $\widehat{L}$ are unchanged, and those beyond the new R were resampled equally. For a common left update the same reasoning uses $\min(p, \widehat{R})$; rightmost paths are fixed and leftmost paths retain their order. All these minima and maxima satisfy the checkpoint. These checks prove preservation of the entire comparison, including the outer type orders.

From identical inputs, increasing a left-update absence entry puts the changed state on the left of this comparison; decreasing a right-update entry has the same effect. Subsequent common updates preserve it. Minimum with $\widehat{L}$ transfers terminal reachability on the left to the hatted state, while maximum with R transfers reachability on the right in the reverse direction. Consequently, a left increasing hole observable decreases in left-update absence entries and increases in right-update entries; a right increasing hole observable has the opposite signs. Realize all updates by independent pre-sampled arrays and start from the all-absent state. Complementing the right-update entries makes a left observable and the negative of a right observable both decreasing functions of one product vector. Harris' inequality proves their negative correlation. Convergence of the finite chain proves (ii).

To prove (iii), run the *unrestricted* conditional chain given A_z, and let P be its leftmost open path to z. Set

$$C = \{P_r \le c\}.$$

Existence of any constrained path is equivalent to C on A_z, so $O_z = A_z \cap C$. The common-update comparison orders P at every row, including row r. Hence $\mathbf{1}_C$ increases in left-update absence entries and decreases in right-update entries. Its signs are opposite those of f. The product-measure correlation inequality proved above, applied on the same product arrays and followed by chain convergence, gives

$$\mathbb{E}[f\mathbf{1}_C \mid A_z] \le \mathbb{E}[f \mid A_z]\Pr(C \mid A_z).$$

Division by $\Pr(C \mid A_z) > 0$ proves (A6.1). By (i), on O_z a left unrestricted observable equals its constrained version. Apply (ii) to this observable and the right increasing hole event $\{[z+1, c+n-r] \subset H^c\}$. Combining with (A6.1) proves (A6.2).

For (iv), force the coordinate z of H open, denoting this clamped field by $H^{[z]} = H \setminus \{z\}$. For an integer M, put

$$f_{z,M}(H) = \mathbb{E}_X[\min\{R_z(H^{[z]}, X), M\}].$$

This is a bounded increasing function only of the hole coordinates $z-1, \dots, z-M+1$; it reads no hole coordinate at z or outside the strict interior. Independence of X is used before conditioning, and spatial stationarity makes the translated function the same at every deterministic z. On both A_z and $\{Z = z\}$, clamping has no effect. Therefore (A6.2) gives

$$\begin{aligned}\mathbb{E}[R_z(H^0, X) \wedge M \mid Z = z] &\le \mathbb{E}[f_{z,M}(H^0) \mid A_z] \\ &= p^{-1}\mathbb{E}[\mathbf{1}_{A_z}(R_z(H^0, X) \wedge M)] \\ &\le p^{-1}\mathbb{E}R_0(H^0, X).\end{aligned}$$

Monotone convergence proves (A6.3). Reflection reverses both path direction and the inward side, preserving connected intervals and independence.

Finally, for fixed terminal coordinates their backward cones contain finitely many sites. Removing interval lengths greater than K changes their law with probability at most the total mean of omitted types meeting those cones, bounded by a finite constant times $\bar{a}\sum_{\ell>K}(\ell + C)\nu_\ell$, which tends to zero. A fixed feasible allowed path has positive probability of being open, giving a positive lower bound for each relevant conditioning probability in the approximation. This passes the bounded inequalities to the infinite-type model. Deterministic parameters follow by parameter limits, and nonnegative observables by monotone truncation. □

Finite-slab convention. In the finite proof, the assertion about the rightmost site is made for the set of reachable sites in the relevant terminal cone, with all deterministic sites to its right declared unreachable. The infinite-lattice half-line edge is then recovered by increasing the finite slab. For an observable at finitely many strict-interior sites and a fixed boundary value, the necessary terminal and backward cones are finite. This avoids treating a finite artificial outer wall as an additional random disaster.

Appendix B. Spatial compression and a subthreshold cutoff estimate

B.1. Compression of a finite target

For a finite target C, write $p_t(C) = \Pr(C \subset H_t)$ for the all-open initial field after t rows, and $p_*(C) = \lim_{t\to\infty} p_t(C)$, with $p_t(\varnothing) = p_*(\varnothing) = 1$. Intensity subscripts are used only when the intensity differs from the fixed original law.

For a finite target $U = \{x_1 < \cdots < x_q\}$ and a gap $x_{i+1} - x_i \geq 2$, let U' shift the right suffix one site to the left. We prove $p_t(U') \geq p_t(U)$ and hence

$$p_t(U) \leq p_t(\{0, \ldots, q-1\}), \qquad p_*(U) \leq h_{|U|}. \tag{B1}$$

The one-row statement needed for the induction is an order on survivor ranks. A set function f is supermodular if $f(A \cup C) + f(A \cap C) \geq f(A) + f(C)$. On a finite rank set, the disaster row is a pure-death chain with generator

$$(\mathcal{L}_U f)(S) = \lambda \sum_r \nu_r \sum_{\ell \in \mathbb{Z}} [f(S \setminus J_{r,\ell}(U)) - f(S)], \quad J_{r,\ell}(U) = \{j : \ell \leq x_j < \ell + r\}.$$

Its total rate on every nonempty state is at most $\lambda |S| S_1 < \infty$.

Fix S and expand $F(A) = f(S \setminus A)$ in Boolean Möbius coefficients. The number of length-r origins covering every member of a nonempty rank subset T is $(r - x_{\max T} + x_{\min T})_+$. Compression changes this number by one exactly when T crosses the gap and its original diameter is at most r. Group the coefficients by their extreme ranks $j \leq i < k$. The intermediate-rank Möbius sum becomes a mixed difference on the survivor set outside $[j,k]$. Thus

$$\begin{aligned}(\mathcal{L}_{U'} - \mathcal{L}_U) f(S) = \lambda \sum_{\substack{j,k \in S \\ j \leq i < k}} \overline{\nu}_{x_k - x_j} \big[& f(T_{jk}) + f(T_{jk} \cup \{j,k\}) \\ & - f(T_{jk} \cup \{j\}) - f(T_{jk} \cup \{k\}) \big] \geq 0,\end{aligned} \tag{B2}$$

where $T_{jk} = S \setminus [j,k]$ in rank coordinates and $\overline{\nu}_d = \sum_{r \geq d} \nu_r$. Deleting a fixed set of ranks preserves supermodularity, so each row semigroup preserves it. Duhamel's formula therefore transfers (B2) to the one-row survivor laws: every supermodular function has larger expectation after compression.

Now $p_t(A) = \Pr(A \subset H_t)$ is decreasing and supermodular. Put $\Psi(A) = A \cup (A-1)$. The function $F_U(S) = p_t(\Psi(U_S))$ is supermodular, because Ψ preserves unions and $\Psi(U_{S \cap T}) \subset \Psi(U_S) \cap \Psi(U_T)$. For fixed surviving ranks, the compressed predecessor target is a translate, a compression of the old predecessor target, or, when predecessor pieces meet, the old target with a duplicated requirement removed followed by compression. The induction hypothesis and decreasingness therefore compare these predecessor probabilities pointwise. Applying the one-row order to the exact recursion

$$p_{t+1}(U) = \mathbb{E}_D p_t(\Psi(U \setminus D))$$

closes the induction. Its base $t = 0$ is immediate. Iterated gap closure and the increasing remote-past limit prove (B1). The same proof applies to every length truncation and to a single Boolean row.

B.2. Marks shorter than the target by a fixed factor

Lemma B.1. Fix $0 < \delta < 1$, let $m_k = \lfloor \delta k \rfloor$, and write $h_k^{(m_k)}$ for the stationary block probability after deleting all marks longer than m_k. Under the assumptions of Section 2,

$$h_k^{(m_k)} = o(A_k). \tag{B3}$$

Proof. The branching comparison of Appendix A applies to the truncated law. Write $\chi_{0,m} = \sum_{r \leq m} \nu_r r(r+1)/2$ and $A_j^{(m)}$ for its tail sum. It gives

$$h_k^{(m)} \leq \Pr\left(\sum_{i=1}^{J_m} W_{m,i} \geq k \right), \qquad 1 \leq W_{m,i} \leq m, \quad \Pr(W_{m,i} \geq j) = A_j^{(m)} / \chi_{0,m}.$$

The costs are iid and independent of the tree; $\Pr(J_m = j) \leq (4\alpha_m)^j$ with $\alpha_m = \lambda \chi_{0,m} / u_m \leq 1/16$. For large m, $\chi_{0,m} \geq \chi_0/2$, so the cost tail is bounded by $C A_j$ uniformly in m.

Choose $\gamma < \alpha < \gamma/\delta$ and $s_k = \alpha \log k/k$. We claim $M_{m_k}(s_k) = \mathbb{E}e^{s_k W_{m_k,1}} \to 1$. The integer tail-sum identity gives

$$M_m(s) - 1 = (e^s - 1) \sum_{j=0}^{m-1} e^{sj} \Pr(W_{m,1} > j).$$

For $j \le 1/s$, this is bounded by a constant times $s \sum_{j \le 1/s} A_{j+1} = o(1)$, since $A_j \to 0$. Split the remaining indices into blocks of width $1/s$. Potter bounds imply that the sum of their contributions is bounded by a term tending to zero at the first blocks plus $Ce^{sm}A_m$. Thus

$$M_{m_k}(s_k) - 1 \le o(1) + Ce^{s_k m_k} A_{m_k}. \tag{B3a}$$

The low-end term tends to zero because $s_k \sum_{j \le 1/s_k} A_{j+1} \to 0$, and the high-end term is $O(k^{\alpha\delta - \gamma}\ell(k)) = o(1)$. This proves the claim. In particular $M_{m_k}(s_k) < 2$ eventually and

$$\mathbb{E}[M_{m_k}(s_k)^{J_{m_k}}] \le \sum_{j \ge 0} (4\alpha_{m_k} M_{m_k}(s_k))^j \le 2.$$

Chernoff's bound gives $h_k^{(m_k)} \le 2k^{-\alpha} = o(A_k)$, proving (B3). □

The same estimate applies when the target length is asymptotic to $\ell T/32$ and the cutoff is δT, provided $32\delta < \ell$, where $\ell > 0$ is a fixed time ratio. This strict inequality is used in Appendix D below.

Appendix C. Recovery fronts, ray confinement and interval response

C.1. Boolean-row attachment

In this appendix a is an auxiliary catastrophe intensity with $0 < a \le \rho$, and a_+ denotes a fixed dominating intensity satisfying $a_+\chi_0 \le 1/32$. Write $S_1 = \sum_r r\nu_r$ and $b = a\chi_0$. The small-intensity phase estimates in Appendix A give

$$u_a = 1 - \beta_a > 0, \qquad b \le \beta_a \le \frac{1 - \sqrt{1 - 4b}}{2}. \tag{C1}$$

Throughout the proof, constants may depend on the fixed length law and fixed positive intensities. A superscript K denotes deletion of marks longer than K.

Lemma C.1 (mean attachment in one row). Let m_a be the mean number of sites attached to the left of a deterministic right ray by one fresh Boolean disaster row. Define

$$B_n = \sum_r \nu_r (r - n)_+, \qquad n \ge 0.$$

Then

$$m_a = \sum_{n \ge 0} (e^{aB_n} - 1), \qquad b \le m_a \le be^{2b/3}. \tag{C2}$$

Proof. Put $w = e^{-aS_1}$. Conditional on a site being uncovered, every interval affecting sites strictly to its right starts to its right. Revealing successive origin batches therefore makes uncovered sites a renewal process. If τ is its next uncovered-site gap, its renewal mass is we^{aB_n}, including $n = 0$. For $M(z) = \sum_{n \ge 0}(e^{aB_n} - 1)z^n$ and $F(z) = \mathbb{E}z^\tau$, the renewal equation gives

$$\frac{1}{1 - F(z)} = \frac{w}{1 - z} + wM(z).$$

Thus $Q(z) = w(1-F(z))/(1-z) = [1+(1-z)M(z)]^{-1}$. The coefficients satisfy $[z^n]Q(z) = w \Pr(\tau > n)$ and $\sum_n [z^n]Q(z) = 1$. A stationary covered block of length n can be partitioned by its last uncovered site on the left, giving its probability as $\sum_{j \ge n} w \Pr(\tau > j)$. Thus Q is also the probability generating function of the fixed-boundary attachment length, and its mean is $Q'(1-) = w\mathbb{E}[\tau(\tau - 1)/2] = M(1)$.

The derivative is justified by $\sum_n B_n = \chi_0 < \infty$ and monotonicity of B_n. Alternatively, truncate lengths, differentiate, and use monotone convergence. Finally,

$$aB_n \le e^{aB_n} - 1 \le ae^{aS_1} B_n, \qquad S_1 \le \frac{2}{3}\chi_0,$$

which proves (C2). □

C.2. Adaptive boundary drift

Let L_t be the left endpoint of the right-infinite hole component when the initial holes are $[1, \infty)$. Set $L_0 = 1$ and $J_t = L_t + 1 - L_{t+1} \ge 0$.

Lemma C.2 (drift at the actual boundary). For every boundary value x of positive probability,

$$m_a \le \mathbb{E}[J_t \mid L_t = x] \le m_a/u_a. \tag{C3}$$

Proof. Let H_t^0 be the full-open finite-history baseline and H_t^S the process whose initially open sources are $S = (-\infty, 0]$, coupled through the same rows. Fix a boundary value x and put $z = x - 1$. Write

$$O_z = \{z \notin H_t^S\}, \qquad A_z = \{z \notin H_t^0\}, \qquad R_x = \{[x, t] \subset H_t^S\}.$$

Here R_x is certain when $x = t + 1$. Since $[t + 1, \infty) \subset H_t^S$, the boundary event is exactly $\{L_t = x\} = O_z \cap R_x$.

For a hole set B, let $\widetilde{B}$ agree with B strictly left of z and force z, x to be open. Attach the hole ray $[x + 1, \infty)$ and add an independent fresh row D of intensity $a\nu$. Let $\mathcal{A}_x(\widetilde{B}, D)$ be its left attachment length, measured from x:

$$\{\mathcal{A}_x(\widetilde{B}, D) \ge n\} = \{[x - n + 1, x] \subset \widetilde{B} \cup D\}, \quad n \ge 1.$$

For $M < \infty$, define

$$f_M(B) = \mathbb{E}_D[\mathcal{A}_x(\widetilde{B}, D) \wedge M].$$

This is bounded and increasing in the hole coordinates strictly left of z. When $B = E(H_t^0)$, it depends only on the old coordinates strictly left of z. The new row is independent of these old coordinates.

On O_z, the baseline and seeded old fields agree at every $y < z$. To see this, take a baseline path to y and a seeded path to z. Their pointwise minimum is an open path to y with an allowed source in S. Both old fields are open at z, so their eroded fields agree on $(-\infty, x]$ and have z, x open. On the full boundary event $O_z \cap R_x$, the eroded seeded ray begins at $x + 1$. Independence of D then gives

$$\mathbb{E}[J_t \wedge M \mid L_t = x] = \mathbb{E}[f_M(E(H_t^0)) \mid O_z, R_x].$$

Apply Lemma A.6 with checkpoint row 0, cap 0, and endpoint $z = x - 1$. On O_z, its part (i) identifies the seeded and full-source fields strictly left of z. The function $H \mapsto f_M(E(H))$ is increasing and reads only old hole coordinates strictly left of z: the eroder at such a site reads that site and its immediate left neighbour, and the two boundary coordinates have been clamped open. The event R_x is increasing in seeded holes strictly right of z. Parts (ii)–(iii), or directly (A6.2), therefore give

$$\mathbb{E}[f_M(E(H_t^0)) \mid O_z, R_x] \le \mathbb{E}[f_M(E(H_t^0)) \mid O_z] \le \mathbb{E}[f_M(E(H_t^0)) \mid A_z].$$

The fresh row is averaged in f_M before applying the comparison.

For the Palm bound, put $B = E(H_t^0)$ and $\Gamma_x = \{z, x \notin B\}$. Then $A_z \subset \Gamma_x$. The field B is stationary and ergodic in space, with open density $u_B > 0$. On Γ_x, the clamping used in f_M makes no change. For $n \ge 1$ let

$$N_n(B) = \#\{y \in [x - n + 1, x] : y \notin B\}, \qquad c_k = \Pr([1, k] \subset D).$$

Single-row compression gives

$$f_M(B) \le \sum_{n\ge 1} c_{N_n(B)} \qquad \text{on } \Gamma_x.$$

Under the open-site Palm law at x, enumerate open sites as

$$X_0 = x > X_1 > X_2 > \cdots, \qquad G_j = X_j - X_{j+1}.$$

The event Γ_x is $\{x \notin B, G_0 = 1\}$. Exactly G_j positive integers n have $N_n(B) = j+1$, namely

$$x - X_j + 1 \le n \le x - X_{j+1}.$$

Consequently the following is a pathwise identity under this Palm law:

$$\sum_{n\ge 1} c_{N_n(B)} = \sum_{j\ge 0} G_j c_{j+1}.$$

Point-shift invariance and Palm inversion give $\mathbb{E}[G_j \mid x \notin B] = 1/u_B$ for every j. Hence

$$\mathbb{E}[\mathbf{1}_{\Gamma_x} G_j] \le \mathbb{E}[\mathbf{1}_{\{x\notin B\}} G_j] = 1.$$

Tonelli's theorem and $\sum_{k\ge 1} c_k = m_a$ imply

$$\mathbb{E}[\mathbf{1}_{\Gamma_x} f_M(B)] \le m_a, \qquad \mathbb{E}[f_M(B) \mid A_z] \le \frac{m_a}{\Pr(A_z)} \le \frac{m_a}{u_a}.$$

Combining the two conditional comparisons and sending $M \to \infty$ proves the upper bound in (C3). For the lower bound, conditional on any old boundary value, fresh-row coverage of the n sites immediately left of the eroded ray forces $J_t \ge n$. The fresh row has its unconditional translated law, so summing its coverage probabilities gives $\mathbb{E}[J_t \mid L_t = x] \ge m_a$. □

C.3. The empty-exterior shape and its speed

Lemma C.3 (functional front limit). There exists v_a such that, for every fixed $C < \infty$,

$$\frac{1}{T}\mathbb{E} \max_{0\le t\le CT} |L_t - 1 - v_a t| \longrightarrow 0, \qquad 1 - m_a/u_a \le v_a \le 1 - m_a. \tag{C4}$$

Moreover, $v_a^K \downarrow v_a$ as $K \to \infty$.

Proof. At each fixed T, the entire hole field is contained in the original ray together with the union of the first T disaster rows. The latter union is a Boolean row of intensity Ta. Lemma C.1 implies finite mean attachment, so L_T is integrable. Put $d_T = \mathbb{E}(L_T - 1)$. Restarting only the contained ray at a block boundary deletes other holes and increases the future endpoint. Thus $d_{s+t} \le d_s + d_t$. Lemma C.2 bounds d_T/T between the two values in (C4); Fekete's lemma gives its finite limit v_a.

For a fixed block length n, perform the ray-only restart after every n rows. The successive restarted displacements are iid with mean d_n, because each new slab is independent of the preceding rows and spatial translation does not change its law. The restarted endpoint dominates L_{jn}. The strong law and the deterministic bound $L_{t+1} \le L_t + 1$ yield

$$\limsup_{t\to\infty} (L_t - 1)/t \le d_n/n \quad \text{a.s.}$$

Taking the countable infimum over n gives the upper bound v_a. Since $(L_t - 1)/t \le 1$, its positive deviation from v_a tends to zero in expectation. Convergence of its mean then gives the same for its negative deviation, proving L^1 convergence.

For $s \in [j,k]$, the one-sided slope bound gives

$$(L_k - 1 - v_a k) - (1 - v_a)(k - s) \le L_s - 1 - v_a s \le (L_j - 1 - v_a j) + (1 - v_a)(s - j).$$

A finite time grid proves the maximum version in (C4). Under common thinning, $L_T^K \downarrow L_T$, with integrable lower bound L_T and upper bound $T+1$. Hence $d_T^K \to d_T$. Monotonicity gives $v_a^K \geq v_a$, while $\limsup_K v_a^K \leq d_T/T$ for each T. Taking the infimum over T proves the speed continuity. □

Corollary C.4 (quantitative weak-intensity speed). For $b = a\chi_0 \leq 1/32$,

$$1 - 2b - 4b^2 \leq \kappa_a = 2v_a - 1 \leq 1 - 2b, \qquad |\kappa_a^{-1} - (1+2b)| \leq 10b^2. \tag{C5}$$

Proof. The small root in (C1) implies $\beta_a \leq b + (5/2)b^2$ and also $\beta_a \leq 14b/13$ throughout this budget. Thus

$$\frac{e^{2b/3}}{u_a} \leq \frac{1}{(1-2b/3)(1-14b/13)} \leq 1 + 2b.$$

Combine this with (C2)–(C4). The resulting lower bound for κ_a is at least $239/256 > 9/10$; division of $1 - \kappa_a(1+2b)$ by κ_a gives the stated inverse bound. In particular $u_a, \kappa_a \to 1$ as $a \downarrow 0$. □

C.4. Transfer to a predictable nonstationary exterior

Lemma C.5 (front in a dominated exterior). Fix a dominating intensity a_+ with $a_+\chi_0 \leq 1/32$. Let the initial baseline holes B_T be coupled below a stationary a_+ hole field, and let an insertion location q_T be measurable with respect to the past, with $|q_T| \leq KT$ with probability tending to one. All future rows are independent of that past. If their intensity is $a \leq a_+$ and their length cutoff $m_T \to \infty$, the left boundary of the inserted right ray satisfies

$$\sup_{0 \leq t \leq CT} |L_t - q_T - v_a t|/T \longrightarrow 0 \quad \text{in probability.} \tag{C6}$$

The right boundary of a left ray has velocity $1 - v_a$.

Proof. Fix $\varepsilon > 0$ and an integer N. The stationary dominating field is a spatially ergodic factor of independent origin histories, with positive open density. Divide a deterministic interval containing all permitted q_T into finitely many cells of length proportional to εT. With probability tending to one, every cell contains at least N open sites. Hence the interval immediately left of any permitted q_T contains N baseline-open sites, chosen by a rule that reads only the past.

Start from a single open site x and use the same future damage for all initial sets. At any age when its descendant set is nonempty, its rightmost descendant equals the rightmost descendant from the completely occupied half-line $(-\infty, x]$, and its leftmost descendant equals the leftmost descendant from $[x, \infty)$. Indeed, a half-line path ending beyond a surviving single-source path must cross it; exchanging prefixes makes that endpoint reachable from x. The same crossing argument identifies the full-source baseline between the two single-source extremes. These pathwise statements hold for increments $0, 1$ and for their reflection $0, -1$.

For the extinction comparison, use reflected propagation $B_-C = C \cup (C-1)$, killing $K_D C = C \setminus D$, ordinary propagation-then-kill $F_D = K_D B_-$, and actual backward update $G_D = B_- K_D$. Row by row, $B_- F_D = G_D B_-$. Thus

$$G_{D_t} \cdots G_{D_1}(B_- C) = B_- F_{D_t} \cdots F_{D_1} C.$$

Since $B_- A$ is empty exactly when A is empty, ordinary extinction from C equals actual backward extinction from $B_- C$. Finite-history duality and its increasing limit identify the latter probability with $p_a(B_- C)$. Compression and monotone coupling therefore give, for $|C| = N$,

$$\Pr_C^a(F \text{ eventually dies}) = p_a(B_- C) \leq h_a(|B_- C|) \leq h_{a_+}(N).$$

The last bound tends to zero as $N \to \infty$ and remains an upper bound under every deletion of future marks. Reflection gives the same statement for forward increments $0, 1$. For each selected source, the relevant half-line endpoints in the truncated future are sandwiched between those for the full a law and any fixed length cutoff $K_0 \leq m_T$. Lemma C.3, followed by $K_0 \to \infty$, gives their common velocity. The sources are selected before the future rows, so a union bound over the fixed

N sources is valid. One surviving source supplies a lower bracket for the ray's rightmost open point; the completely open left half-line supplies the upper bracket. Their initial positions differ by at most εT. Let $T \to \infty$, then $N \to \infty$, then $\varepsilon \downarrow 0$.

For fixed N and ε, condition on the entire past after selecting the sources. Each selected source has the same translated future half-line law. The probability that any of the N endpoint bounds fails is therefore at most Ne_T, where $e_T \to 0$ by Lemma C.3 and the cutoff sandwich. The probability that every selected family dies is at most $h_{a_+}(N)$. The past event that the required open sites exist has probability tending to one. These three bounds justify the order $T \to \infty$, then $N \to \infty$, then $\varepsilon \downarrow 0$, even for a past-dependent insertion location. The reflected boundary statement follows under $x \mapsto t - x$, which preserves allowed increments and the Poisson interval law. The finite-grid argument gives the supremum over bounded scaled times in (C6).

In particular, this conclusion holds for every deterministic sequence of older history lengths $k_T \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, with the corresponding all-open-history or upper stationary exterior, whenever these exteriors have the common stationary domination assumed above. It holds for the full future law and for every deterministic cutoff sequence tending to infinity. For the family consisting of the full law and cutoff $\lfloor \delta T \rfloor$, a failure of uniformity in the history length or in this cutoff choice would give deterministic near-maximizers along a subsequence, contradicting (C6) for that sequence. The same argument treats insertion locations in a fixed compact scaled range. These choices are made before the future is revealed; predictable locations retain the stated past-measurability condition. □

C.5. Exact two-ray confinement and the compact response

Insert $I = [a, b]$ into any baseline hole field H_0. In the common future, let p_s be the rightmost occupied site descended from the baseline occupied sources strictly left of a, and q_s the leftmost occupied site descended from baseline sources strictly right of b. Their infinite source sets exist in the stationary case by positive density. Then the inserted process is all holes on $[p_s + 1, q_s - 1]$ when $p_s < q_s - 1$, and equals the baseline outside that interval. Once $p_s \geq q_s - 1$, both fields agree permanently. Indeed, at a point left of p_s, exchange prefixes of a baseline path and a left-source path to p_s if the baseline source was in I or to its right. This yields a left-source path to the point. The right side is symmetric. If $p_s \geq q_s - 1$, the two regions cover every integer point. Conversely, any inserted-process occupied source is on one of the two exterior sides, so a point strictly between the two extremes cannot be occupied in that process. Both extremes are defined by actual paths in the common forcing. Influence is also in the deterministic cone $[a, b + s]$.

Lemma C.5 applies to stationary exteriors and to any number of finite baseline-history rows, and to length cutoffs tending to infinity. For $|I|/n \to a_0 > 0$, the inner boundary positions, divided by n, converge uniformly on compact time intervals to $l + vt$ and $r + (1 - v)t$. Therefore the discrepancy between the perturbed and baseline indicators differs from

$$1_{\{x \in O_s\}} 1_{\{(x/n, s/n) \in \Gamma_{[l,r]}\}} \tag{C7}$$

by $o(n^2)$ in mean on any compact time horizon. To verify the integrated error, the endpoint bounds put the symmetric difference in strips of width at most εn except on an event of probability $o(1)$. The deterministic cone has $O(n^2)$ vertices. Send $n \to \infty$ then $\varepsilon \downarrow 0$. The same reasoning treats the interval's disappearance, because an empty fluid width and positive endpoint margins imply crossing.

The baseline stationary field is spatially ergodic, being a factor of independent origin columns. At any deterministic row, its average on an interval of length tending to infinity converges in L^1 to u. The distribution is the same at all rows. For deterministic triangle sections, subdivide the compact scaled time range by a finite grid, or integrate their rowwise L^1 error and use bounded convergence; sections of width $o(n)$ contribute arbitrarily small normalized mass. It follows that for every continuous compactly supported f,

$$\frac{1}{n^2} \sum_{s,x} f(x/n, s/n) \Delta_I(x, s) \longrightarrow u \int_{\Gamma_{[l,r]}} f(y, t)\, dy\, dt \quad \text{in } L^1. \tag{C8}$$

Rowwise spatial ergodicity supplies the compact mass limit. The endpoint part of this argument is also valid for a finite-history background dominated by the stationary field, a fact used for the insertion estimates in Appendix E.

Appendix D. Uniform history error and finite-source extinction tails

D.1. The actual backward update and spatial compression

For a finite target C, define $B_-C = C \cup (C-1)$, $K_DC = C \setminus D$, and the actual backward update $G_D = B_-K_D$. Let $U_t(C)$ be its iteration. Let $p_t(C)$ be the probability that C is all holes after t rows from all-open sites, and $p_*(C)$ its stationary value, with $p_*(\varnothing) = 1$. Finite-history path reversal gives

$$p_t(C) = \Pr_C(U_t = \varnothing). \tag{D1}$$

Appendix B gives $p_*(C) \le h_{|C|} \le CA_{|C|}$ for nonempty C. Stationary harmonicity makes $p_*(U_t)$ a bounded martingale. On survival $|U_t| \to \infty$ almost surely. To see this, for every fixed cardinality bound M, a supported mark length supplies a uniform positive one-row probability to kill all at most M sites: choose at most one mark per site and use independence of their distinct coordinates. Repeated visits to this cardinality range therefore give geometric chances of extinction. A surviving trajectory can visit it only finitely often. Compression then makes $p_*(U_t) \to 0$ on survival, whereas it equals one after extinction. Taking expectations proves

$$p_*(C) = \Pr_C(\tau < \infty), \qquad D_C(t) := p_*(C) - p_t(C) = \Pr_C(t < \tau < \infty). \tag{D2}$$

For $C = \varnothing$, set $D_C(t) = 0$. The Markov property gives $D_C(s+t) = \mathbb{E}_C D_{U_s}(t)$. For a length truncation retaining no positive-rate mark, $p_*^{(m)}(C) = p_t^{(m)}(C) = 0$ for nonempty C, so its history defect is identically zero. This degenerate case satisfies the later uniform bounds directly; the supported-mark argument above is used only for truncations retaining a positive-rate mark.

D.2. Uniform restart from arbitrary finite sets

There is a common $\eta_B \to 0$, valid for the original law and all its length truncations, such that

$$\sup_{0<|C|<\infty} \Pr_C(0 < |U_{8B}| < \lceil B/2 \rceil) \le \eta_B. \tag{D3}$$

Here is a proof that includes dispersed initial targets. First consider ordinary propagation-then-kill with reflected increments $0, -1$, from one source x. If its descendants survive, their two extremes equal the corresponding half-line extremes: an exterior half-line path ending beyond a surviving source path can exchange prefixes with it. Between the extremes its occupied sites equal the full-source baseline by the same planar exchange. At age $8B$, the two ports $[x-5B, x-4B-1]$ and $[x-3B, x-2B-1]$ lie with a fixed proportional margin inside the limiting envelope, because Lemmas C.3–C.4 give $v > 0.96$. The baseline dominates its stationary occupied field, whose density is $u > 15/16$. Spatial ergodicity makes each port contain at least $\lceil B/2 \rceil$ occupied sites with probability tending to one. The endpoint good event can be sandwiched between the original law and a fixed length truncation; baseline density and the relevant endpoint bounds improve under removing damage. Thus a single error probability $\delta_B \to 0$ controls survival without dense output for every length truncation.

For the kill-then-expand ancestor update use the exact identity

$$G_{D_T} \cdots G_{D_1} C = B_- F_{D_T} \cdots F_{D_2}(C \setminus D_1), \qquad F_D = K_D B_-. \tag{D4}$$

The preceding single-source argument uses $T-1$ ordinary rows and its final expansion only adds sites; changing $8B$ by one does not change the positive port margins.

For a predictable nonempty finite C, select at most N sites before the next slab is revealed. If $|C| < N$, select all of them. On a nonempty terminal output, at least one selected family is still present, and small output forces that family to have a bad dense-output event. If $|C| \geq N$, select exactly N sites by any fixed rule. Either at least one selected family is still present and has a bad dense-output event, or all selected families have disappeared by age $8B$. Only in this latter branch do we use compression to bound their joint disappearance probability by h_N, for either update. A union bound therefore gives the two separate estimates

$$\Pr_C(0 < |U_{8B}| < \lceil B/2 \rceil) \leq \begin{cases} |C|\delta_B, & 0 < |C| < N, \\ h_N + N\delta_B, & |C| \geq N, \end{cases} \qquad \leq h_N + N\delta_B. \tag{D4a}$$

Choose $N = N(B) \to \infty$ slowly enough that $N\delta_B \to 0$. Selection is made before future rows, so the bound holds conditionally for every finite input and can be restarted. This proves (D3).

D.3. Weighted restart closes the late tail

Put $F_m(t) = \sup_{C \neq \varnothing} D_C^{(m)}(t)$ for the law truncated at length m, with $m = \infty$ allowed. Splitting U_{8B} into empty, large and small nonempty sets, and retaining the remaining-time defect on the small branch, yields

$$F_m(t + 8B) \leq h^{(m)}_{\lceil B/2 \rceil} + \eta_B F_m(t). \tag{D5}$$

For a large L, take $B = \lfloor L/16 \rfloor$ and $t = L - 8B$, so $t/L \to 1/2$ and $\lceil B/2 \rceil / L \to 1/32$. Uniformly in m, $h_b^{(m)} \leq h_b \leq C_0 A_L$, and regular variation gives $A_t \leq K A_L$. Increase the fixed threshold until $K\eta_B \leq 1/2$. Strong induction on L, with a constant enlarged over the finitely many initial ages, proves

$$F_m(L) \leq C A_L \qquad (m \geq 2 \text{ or } m = \infty). \tag{D6}$$

For $L = \lfloor \lambda n \rfloor$, $m = \lfloor \delta n \rfloor$ and $\lambda > 32\delta$, the first term in (D5) is $o(A_n)$ by Appendix B.2, since its target length exceeds the cutoff by a fixed proportion. The second is $\eta_B O(A_n) = o(A_n)$. Hence

$$F_{\lfloor \delta n \rfloor}(\lfloor \lambda n \rfloor) = o(A_n), \qquad \lambda > 32\delta. \tag{D7}$$

These are uniform finite-target history bounds. They also control ordinary future extinction from any finite seed. Indeed $B_- F_D = G_D B_-$, and $B_- C$ is empty exactly when C is empty, so ordinary extinction through age L from C equals backward extinction through the same age from $B_- C$. Reflection gives the forward increments $0, 1$. In particular

$$\sup_{C \neq \varnothing} \Pr_C(L < \tau^{\text{ordinary}} < \infty) \leq C A_L, \quad \beta(C) = p_*(C \cup (C+1)) \leq h_2 < 1. \tag{D8}$$

Both identities use exactly the stated number of rows.

D.4. A spatial covariance bound

For two sites separated by d, truncate the past to $L = \lfloor d/4 \rfloor$ rows, costing at most $C A_L$ in the mean of each indicator by (D6). Delete lengths above $m = L$, at cost bounded by

$$\rho L \sum_{r > m} \nu_r(r + L) = O(A_d)$$

in each cone. The two remaining finite origin ranges are disjoint for large d (their widths are at most $L + m + O(1)$). Their observations are independent. A bounded-variable covariance comparison therefore gives

$$0 \leq \operatorname{Cov}(1_{\{0 \in H_0^*\}}, 1_{\{d \in H_0^*\}}) \leq C A_d. \tag{D9}$$

The lower sign follows from association of independent damage coordinates; the finite-coordinate induction proof is in Appendix A.1. This estimate is sufficient for the two-history argument below.

Appendix E. The single-gap amplitude and its complete age law

The dependency order is A–B, C, D, candidate classification, then the one- and two-history amplitudes.

E.1. Exact two-history representation

Put $G_g = [0, g-1]$. Let $p_g(L)$ denote extinction of its actual stationary initial sources by age L, and let π_g denote eventual extinction.

Let $H^{(1)}$ have the original stationary law, independent of future rows, and $C_0 = G_g \setminus H^{(1)}$. Ordinary propagation-then-kill extinction from C_0 by age L has probability $p_L(B_+C_0)$, by (D8) and reflection, where $B_+C = C \cup (C+1)$. At a fixed terminal age, the shear-reflection $x \mapsto t - x$ sends a path increment 0 to 1 and 1 to 0. It sends every length-r interval in a row to a length-r interval, and its origin map is a bijection. Spatial homogeneity and row independence therefore preserve the Poisson mark law. Introduce an independent same-oriented reflected hole field $H_L^{(2)}$ made from exactly L rows. Conditional on C_0, the event $B_+C_0 \subset H_L^{(2)}$ is equivalent to $C_0 \subset H_L^{(2)} \cap (H_L^{(2)} - 1)$. Therefore

$$p_g(L) = \Pr(Q_{g,L}), \quad Q_{g,L} = \{G_g \subset H^{(1)} \cup [H_L^{(2)} \cap (H_L^{(2)} - 1)]\}. \tag{E1}$$

Couple the second history by adding older rows up to stationarity. The limiting event $Q_{g,\infty}$ has probability π_g. This auxiliary coupling computes the probability for each fixed L. The actual forward trajectory supplies the observation filtration.

E.2. Weighted candidate classification

Lemma E.1 (weighted candidate classification). The following estimate applies to the event $Q_{g,L}$ defined in Section E.1 and to its stationary limit; its one-history version concerns full closure of G_g. It uses only Appendices C and D. Fix δ, M, H and a deterministic origin box $\mathcal{Z}_g \subset [-Kg, Kg]$ containing the relevant cones. For color c, a candidate $e = (r, z, t)$ has $\delta g < r \le Mg$, $z \in \mathcal{Z}_g$ and $0 \le t \le Hg$, restricted to $t < L$ in the finite second history. Write Q for $Q_{g,L}$ or $Q_{g,\infty}$, as applicable, and $Q^{c,e}$ for Q after inserting this atom into its own, otherwise unconditioned background. The two backgrounds remain independent before this insertion. Put

$$I_e^{\mathrm{fl}} = [z + vt, z + r - 1 + (1-v)t], \quad J_1 = G_g, \quad J_2 = G_g \cup (G_g + 1), \qquad \iota_c(e) = 1_{\{J_c \subset I_e^{\mathrm{fl}}\}}, \tag{E2}$$

and define the candidate misclassification probability

$$e_g^c(r, z, t) = \mathbb{E}\big|1_{Q^{c,e}} - \iota_c(e)\big|. \tag{E3}$$

For a single history replace Q by full closure of G_g and use J_1. The one-site enlargement in J_2 enforces both factors of $H^{(2)} \cap (H^{(2)} - 1)$.

Fix $\varepsilon > 0$ small. We partition candidates into four classes. Throughout, constants may depend on the fixed caps and on ε, and the estimates are uniform in the number of older baseline-history rows, including stationarity.

(i) **Near-zero age.** The candidates with $t \le \varepsilon g$ have total mean at most $C\varepsilon A_g + o(A_g)$, since $|\mathcal{Z}_g| = O(g)$ and $\sum_{\delta g < r \le Mg} \nu_r = O_{\delta,M}(A_g/g^2)$. Their error is bounded by their mean.

(ii) **Parameter boundary.** For each (r, t), at most $C(\varepsilon g + 1)$ origins put either endpoint of I_e^{fl} within $\varepsilon g + 2$ of the corresponding target endpoint. Their total mean is again $C\varepsilon A_g + o(A_g)$. This includes changes of the containing indicator caused by a lattice endpoint or the extra site in J_2.

(iii) **Strict interior containment.** If $t > \varepsilon g$ and both endpoints contain J_c by more than $\varepsilon g + 2$, the exact two-ray confinement in C.5 makes every target site a hole after insertion unless an endpoint differs from its fluid position by at least εg. Lemma C.5 bounds that probability by $a_g(\varepsilon) \to 0$, uniformly in r, z, t in the fixed box and in the older history length. One way to check the uniformity is to take any sequence violating it, pass to a convergent subsequence of the compact scaled parameters, and apply the endpoint statement there. Uniform domination by the same stationary field controls the initial-source selection for every history length. Since the whole candidate box has mean $O(A_g)$, this class costs at most $Ca_g(\varepsilon)A_g = o(A_g)$.

(iv) **A missed deterministic grid segment.** In every remaining candidate outside (i)–(iii) for which $\iota_c(e) = 0$, the fluid interval misses a target segment of length at least εg. Choose a subsegment of length at least $\varepsilon g/4$ from a deterministic grid of at most C_ε such segments. It lies a positive distance from the fluid interval. In the two backgrounds before insertion, the number of sites open in both on this segment is at least $b_\varepsilon g$, except with probability $b_g(\varepsilon) \to 0$. Indeed each finite-history or cutoff background is dominated by an independent original stationary hole field; their common-open sites therefore dominate those of two independent stationary comparisons. Equation (D9), Chebyshev's inequality and a union bound over the fixed grid prove the assertion. For the single-history version use its positive open density instead of u^2.

To bound the changes on this missed segment, make the background family explicit. For each sufficiently large g, set $\mathfrak{M}_g = \{\infty, \lfloor \delta g \rfloor\}$, where $m = \infty$ denotes the original length law and finite m retains precisely the lengths at most m. For $m \in \mathfrak{M}_g$ and $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, let the background at the insertion row be formed by k homogeneous rows of this law from all-open sites; $k = \infty$ denotes its upper stationary limit. Continue with independent future rows of the same law and the same intensity ρ. Couple every such initial background below the original stationary hole field. Let $\Delta^{(m)}_{r,k}(x,t)$ be the discrepancy at age t caused by inserting $[0, r-1]$ into this background. Define

$$d_g(r,t) = \sup_{m \in \mathfrak{M}_g} \sup_{k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}} \mathbb{E} \sum_{x \notin [vt, r-1+(1-v)t]} \Delta^{(m)}_{r,k}(x,t). \qquad \sup_{\delta g < r \leq Mg} \sum_{t \leq Hg} d_g(r,t) = o(g^2). \tag{E4}$$

For a fixed $\eta > 0$, let $\epsilon_g(\eta)$ be the supremum, over $m \in \mathfrak{M}_g$, $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $\delta g < r \leq Mg$, of the probability that either inner ray boundary differs from its fluid position by more than ηg at some $t \leq Hg$. The sequence formulation following (C6) gives $\epsilon_g(\eta) \to 0$: any failed supremum supplies deterministic near-maximizing history lengths, cutoff choices and interval lengths, to which that statement applies. Each parameter choice is deterministic and precedes the future realization.

On the complementary event, exact two-ray confinement bounds the number of changed sites outside the fluid interval by $2\eta g + O(1)$ at every age, including near the fluid interval's disappearance. On the exceptional event, the deterministic influence cone gives the uniform bound $C_{M,H} g$. Consequently

$$\sup_{\delta g < r \leq Mg} \sum_{0 \leq t \leq Hg} d_g(r,t) \leq C_{M,H} g^2 [\eta + \epsilon_g(\eta) + g^{-1}]. \tag{E5}$$

The suprema in m, k are controlled before the age sum. First let $g \to \infty$ and then $\eta \downarrow 0$ to prove (E4). The two allowed future laws have the same limiting speed v because their finite cutoff tends to infinity and C.3 gives speed continuity.

For $Q^{c,e}$ to hold, all common-open sites in the chosen missed segment must be changed by the inserted color. Markov's inequality therefore gives an error at most $b_g(\varepsilon) + d_g(r,t)/(b_\varepsilon g)$. The same bound applies in color two: a site open in both backgrounds can be covered by its second-color paired holes only if that site's own second-color state changes. Summing over origins and ages yields the explicit Palm-weighted estimate

$$\begin{aligned}
&\rho \sum_{\delta g<r\le Mg} \nu_r \sum_{z\in\mathcal{Z}_g} \sum_{0\le t\le Hg} e_g^c(r,z,t) \\
&\quad \le C\varepsilon A_g + o(A_g) + C\big[a_g(\varepsilon)+b_g(\varepsilon)\big]A_g + \frac{C_\varepsilon \rho}{g}|\mathcal{Z}_g| \sum_{\delta g<r\le Mg} \nu_r \sum_{t\le Hg} d_g(r,t) \qquad \text{(E6)} \\
&\quad = O_{\delta,M,H}(\varepsilon A_g) + o_{\delta,M,H,\varepsilon}(A_g).
\end{aligned}$$

For the last equality use $|\mathcal{Z}_g| = O(g)$, (E4), and $\sum_{\delta g<r\le Mg}\nu_r = O(A_g/g^2)$. Restricting the second-color ages to $t < L$ only decreases this error bound. The estimate holds also when that color's older history has length $L-t-1$ depending on the candidate. Thus first letting $g\to\infty$ and then $\varepsilon\downarrow 0$ gives the required $o(A_g)$ coefficient error on each fixed candidate box. The length and history caps are released afterwards in the order specified in (E12). This proves the classification estimate. □

E.3. One history: finite and infinite ages

Put $G_g = [0, g-1]$. Let $h_{g,L}$ be the probability that this length-g block is all holes after exactly L rows from all-open sites, and let $h_g = \lim_L h_{g,L}$. For $L/g\to\theta\in(0,\infty)$ we prove

$$\frac{h_{g,L}}{A_g} \longrightarrow \frac{\rho}{\kappa}[1-(1+\kappa\theta)^{-\gamma}], \qquad \frac{h_g}{A_g} \longrightarrow \frac{\rho}{\kappa}. \qquad \text{(E7)}$$

Fix $0<\delta<1$, a large length cap M, and a finite scaled history cap $H>\theta+1$. Delete marks longer than Mg in the finite backward cone. Their total mean, divided by A_g, is bounded by

$$CH[M^{2-p} + (1+H)M^{1-p}], \qquad \text{(E8)}$$

which tends to zero at fixed H as $M\to\infty$. Among the remaining marks designate as candidates the lengths $\lfloor\delta g\rfloor < r \le Mg$, birth ages $0\le t<L$, and all origins meeting the cone. Their Poisson mean is $O_{\delta,M,H}(A_g)$; two or more candidates have probability $O(A_g^2)$. The target event depends only on the finite backward cone, so deleting all candidate coordinates outside that cone leaves the event unchanged. Under the zero- or one-candidate condition it can therefore be evaluated using the full-plane cutoff background with that specified candidate; the exterior used for its ray comparison introduces no change to the target law. With zero candidates, full block closure has probability at most $h_g^{(\lfloor\delta g\rfloor)} = o(A_g)$ by Appendix B.2.

With one candidate of scaled length s, origin ξ, and age z, Lemma C.5 puts its two ray boundaries at $\xi+vz$ and $\xi+s+(1-v)z$, uniformly over the underlying finite-history baseline and away from vanishing age strips. If this interval contains $[0,1]$ with a fixed positive margin, the block is all holes with probability $1-o(1)$. If it misses a segment of positive scaled length, take one of finitely many deterministic grid subsegments inside the missed part. The cutoff baseline is coupled below an original stationary hole field independent of the distinguished candidate, which has a positive density of open sites there. The ray confinement of C.5 prevents the inserted candidate from closing that subsegment with probability tending to one. A strip of width ε around the parameter boundaries has intensity $O(\varepsilon A_g)$ on the fixed candidate box. The same cone bound treats ages near zero.

The classification lemma in Section E.2 makes this summation precise for the one-history case as well. It applies uniformly to the cutoff baseline under the zero/one-candidate expansion: complete that background outside the target cone by the homogeneous cutoff law, which leaves the target unchanged. The endpoint and outside-fluid estimates are uniform over the number of older baseline rows. Summing against the candidate intensities therefore costs $o(A_g)$ after removing the parameter boundary strips.

The exact zero/one Poisson expansion and regular variation now give

$$\frac{h_{g,L}}{\rho A_g} \longrightarrow \frac{1}{C_A}\int_1^\infty s^{-p}\int_0^\theta (s-1-\kappa z)_+\,dz\,ds = \frac{1-(1+\kappa\theta)^{-\gamma}}{\kappa}. \qquad \text{(E9)}$$

To obtain the integral, release the length cap after the fixed finite history; the cutoff $\delta < 1$ excludes no fluid-containing root. The derivative in θ of the integral divided by C_A is $\gamma(1+\kappa\theta)^{-\gamma-1}$. For stationarity use $0 \le h_g - h_{g,\lfloor Hg \rfloor} \le CA_{\lfloor Hg \rfloor}$ from (D6), divide by A_g, and send $H \to \infty$ after $g \to \infty$. This proves (E7). Monotonicity covers ages with $L/g \to 0$ or infinity when needed.

E.4. Excluding escape from a finite candidate box

We prove that $Q_{g,\infty}$ occurs, up to $o(A_g)$ after releasing the caps, only if one of the two histories has a candidate macroscopic atom in a bounded past cone. This proof uses only the original budget $c \le 1/32$.

Choose $0 < b < \min(\gamma, 1)$, an integer Q with $Qb > \gamma$, and then a fixed $\delta < 1/(132Q)$. Put $m = \lfloor \delta g \rfloor$, $w = \lfloor \delta g \rfloor$, $l = \lfloor 64\delta g \rfloor$. In two independent cutoff histories of length l, choose Q blocks of width w with starting points separated by $w + l + m + 3$. They fit inside G_g and their origin cones are disjoint. Thus their common-open counts are independent.

For two independent original stationary fields, the common-open indicator has mean u^2 and covariance at separation d equal to $2u^2C(d) + C(d)^2$, where $C(d)$ is the hole covariance. Equation (D9) and Potter bounds give a block-average variance at most $K_\delta g^{-b}$. Each short cutoff history has at least the common-open sites of its independent original stationary comparison. For a sufficiently small fixed $\varepsilon > 0$, Chebyshev therefore bounds the probability that a block's short-history count is at most $2\varepsilon g$ by $K_\delta g^{-b}$. Multiplication over the independent short cones gives $O(g^{-Qb}) = o(A_g)$.

For any longer cutoff history, including stationarity, the expected loss of common-open sites relative to these l-row histories is at most $2gF_m(l) = o(gA_g)$ by (D7), since $64\delta > 32\delta$. Markov's inequality transfers the preceding bound, proving

$$\Pr\{\text{common-open count in } G_g \text{ for two long cutoff histories} \le \varepsilon g\} = o(A_g). \tag{E10}$$

Now retain the last $\lfloor Hg \rfloor$ rows of both full histories. Let $Y_{g,H}$ count sites open in both finite-history fields. If $Q_{g,\infty}$ holds and $Y_{g,H} \ge \varepsilon g$, at least εg of these sites must be closed by stationary-minus-finite-history discrepancies in one of the two histories. The mean number of those discrepancies is at most $2gCA_{\lfloor Hg \rfloor}$, so

$$\Pr(Q_{g,\infty}, Y_{g,H} \ge \varepsilon g) \le 2CA_{\lfloor Hg \rfloor}/\varepsilon. \tag{E11}$$

In these finite cones count both histories' candidates with lengths $m < r \le Mg$ and origins in a box of radius $(M + H + 2)g$. If the candidate count N is zero and no length above Mg hits either cone, the finite histories equal their cutoff versions. Equations (E8), (E10)–(E11) imply

$$\limsup_{g\to\infty} \frac{\Pr(Q_{g,\infty}, N = 0)}{A_g} \le KH[M^{2-p} + (1+H)M^{1-p}] + K_\varepsilon H^{-\gamma}. \tag{E12}$$

First send $M \to \infty$ at fixed H, then $H \to \infty$. The escape coefficient vanishes. The argument uses the original intensity and its budget throughout.

E.5. The two-color coefficient

For the finite candidate family, the Poisson add-one formula integrates the probability of Q after adding a distinguished atom into its own background. It follows either by summing the independent Poisson probabilities, or by the identity $\mathbb{E}[N_e F(N)] = \lambda_e \mathbb{E}F(N + 1_e)$ and summing the finite family. Since its total mean is $O(A_g)$, replacing $N1_Q$ by $1_{\{N\ge 1\}}1_Q$ costs $O(A_g^2)$.

Each color consequently contributes the integral in (E9) over its permitted root ages. Combining (E12) with all ages available in both stationary histories proves

$$\pi_g \sim \frac{2\rho}{\kappa} A_g. \tag{E13}$$

For the finite-L representation (E1), suppose $L/g \to \theta \in (0, \infty)$ and take $H > \theta + 1$. In the same finite candidate box split $N = N_L + N_{\text{old}}$, where N_{old} consists only of second-color roots at ages

at least L. These coordinates are independent of $Q_{g,L}$, because that event uses exactly the last L second-history rows. The now established bound $\Pr(Q_{g,L}) \le \pi_g = O(A_g)$ gives

$$\Pr(Q_{g,L}, N_{\text{old}} \ge 1) = O(A_g^2). \tag{E14}$$

Thus $\Pr(Q_{g,L}, N_L = 0)$ is bounded by the stationary escape term plus $O(A_g^2)$. The local Palm argument applies with the first color stationary and the second color's inserted root having a possibly short finite baseline history. Lemma C.5 was proved uniformly for exactly these histories. The first color has all ages and the second only $0 \le t < L$. Their coefficients are respectively ρ/κ and $\rho[1 - (1 + \kappa\theta)^{-\gamma}]/\kappa$. Hence

$$\frac{p_g(L)}{A_g} \longrightarrow \frac{\rho}{\kappa}[2 - (1 + \kappa\theta)^{-\gamma}]. \tag{E15}$$

At $\theta = 0$, use $p_g(0) = h_g$ and squeeze by $p_g(\lceil \varepsilon g \rceil)$, then let $\varepsilon \downarrow 0$. At infinity use monotonicity and (E13), or the independent sufficient events consisting of initial full closure and finite-age extinction from the full gap. These give (6.3) on the compactified half-line. A failed uniform-in-age approximation has a subsequence L/g converging in $[0, \infty]$, so (6.3)'s deterministic-sequence statement implies the uniform version used in Section 6.

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