Automorphic realization of Langlands parameters for split groups over global function fields

III. Classical groups

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Abstract

Let G be a split semisimple group whose simple factors are of classical type, over a global function field F of arbitrary positive characteristic. We prove that every semisimple elliptic Langlands parameter $\Gamma_F \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ is the parameter of a cusp form on G, in the sense of V. Lafforgue's excursion operators; we call such a form an automorphic realization of the parameter. Neither F nor its field of constants is extended.

We start from L. Lafforgue's theorem for general linear groups and use automorphic descent and the theta correspondence. The work is to prove that the descent does not vanish over F, to carry out the computations with root subgroups in small characteristic, in particular in characteristic two, and to identify the parameter itself and not only its unramified Hecke eigenvalues.

This is the third of three papers. Together with the first, which contains the arithmetic foundations, and the second, which treats the exceptional types by a geometric method, it gives automorphic realizations of semisimple parameters on all split reductive groups, with a restriction on the characteristic for the exceptional types only.

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Contents

1 Introduction **3**
1.1 The theorem . . . 3
1.2 Relation to earlier work . . . 4
1.3 Organization . . . 4

2 Overview of the construction **4**

3 Results used **6**
3.1 From the literature . . . 6
3.2 From the first paper . . . 7

4 Conventions **8**
4.1 Frobenius and the Satake equivalence . . . 8
4.2 The centre and class field theory . . . 8
4.3 Excursion summands . . . 8

5 Parameters and Fourier coefficients **9**
5.1 Recovering a parameter from its standard representation . . . 10
5.2 Poles of L-functions . . . 11
5.3 Recognizing the parameter of a function . . . 11
5.4 Type A and lifting along a central torus . . . 13
5.5 Two facts about Fourier coefficients . . . 14

6 An Eisenstein residue with a nonzero Fourier coefficient **15**
6.1 Whittaker functionals on general linear groups . . . 16
6.2 Regularized values of Eisenstein series . . . 17
6.3 The residue attached to self-dual blocks . . . 18

7 Orthogonal descent and its local parameter **21**
7.1 The descent datum . . . 21
7.2 The local Jacquet module . . . 23
7.3 The smallest case . . . 28
7.4 Two rearrangements of Fourier coefficients . . . 29
7.5 Nonvanishing for an odd orthogonal target . . . 31

8 Even orthogonal descent and its final Fourier coefficient **34**
8.1 The residual representation attached to a parameter . . . 34
8.2 A nonzero frequency . . . 35
8.3 The coefficients Λ_c . . . 36
8.4 Nonvanishing . . . 40

9 Even orthogonal descent in characteristic two **41**
9.1 Conventions in characteristic two . . . 42
9.2 The Cartier operator and characters of squares . . . 42
9.3 Moving the frequency along the rows . . . 43
9.4 The descent and its terminal plane . . . 45

10 Realization for orthogonal and spin groups **50**

11 Quadratic theta lifting and all spherical Hecke moments **52**
11.1 A model of the Weil representation in every characteristic . . . 53
11.2 The spherical vector . . . 55
11.3 The global lift . . . 56

12 Generic cuspidal support and symplectic realization **58**
12.1 Cuspidal support of a generic function . . . 58
12.2 Removing the two trivial lines . . . 63

13 The realization theorem for classical groups **64**

A Computer checks **64**

Notation **65**

Statement on the use of AI **66**

1 Introduction

Let F be the function field of a curve over a finite field and let G be a split reductive group over F. Vincent Lafforgue decomposed the space of cusp forms on G according to semisimple Langlands parameters $\Gamma_F \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$. His theorem attaches a parameter to each summand of the decomposition. It does not say which parameters occur.

For general linear groups every irreducible parameter whose determinant has finite order occurs, by the theorem of Laurent Lafforgue. We prove that every semisimple elliptic parameter occurs when the simple factors of G are of classical type, in every positive characteristic. The proof transports cusp forms from general linear groups to G by automorphic descent and by the theta correspondence. This is the third paper of a series of three. The first [I] contains the arithmetic foundations and deduces from the present theorem, and from its counterpart for the exceptional types [II], the existence of automorphic realizations of semisimple parameters on all split reductive groups, under the conditions on the characteristic stated there, which concern only the exceptional factors.

1.1 The theorem

Let $X/\mathbb{F}_q$ be a smooth projective geometrically connected curve, let $F = \mathbb{F}_q(X)$, and put $p = \operatorname{char} F$. Fix a prime $\ell \neq p$, put $k = \overline{\mathbb{Q}}_\ell$, and write $\Gamma_F = \operatorname{Gal}(F^{\mathrm{sep}}/F)$. For a split connected reductive group G/F, the dual group $\widehat{G}$ is taken over k.

Definition 1.1. A *parameter* is a continuous homomorphism $\sigma : \Gamma_F \to \widehat{G}(k)$ unramified at all but finitely many places. It is *semisimple* if the identity component of the Zariski closure of its image is reductive. It is *elliptic* if

$$C_{\widehat{G}}(\operatorname{im}\sigma)/Z(\widehat{G}) \quad \text{is finite.}$$

The link between cusp forms and parameters is given by excursion operators. An excursion datum consists of a finite set I, a function $f \in k[\widehat{G}\backslash\widehat{G}^I/\widehat{G}]$, invariant under simultaneous left and right multiplication, and a tuple $\boldsymbol{\gamma} \in \Gamma_F^I$. Lafforgue's operator $S_{I,f,\gamma}$ acts on cusp forms: it creates an invariant vector in the cohomology of shtukas with legs indexed by I, transports it by $\boldsymbol{\gamma}$, and contracts with an invariant covector representing f [Laf18, §§0.3–0.5]. These operators commute, and a cusp form φ has parameter σ when

$$S^{\mathrm{fun}}_{I,f,\boldsymbol{\gamma}}\varphi = f\big((\sigma(\gamma_i))_{i\in I}\big)\varphi \qquad \text{for every excursion datum } (I, f, \boldsymbol{\gamma}). \tag{1.1}$$

We write $\nu_\sigma(I, f, \boldsymbol{\gamma})$ for the scalar on the right. A nonzero cusp form φ which satisfies (1.1) is called an *automorphic realization* of σ, and we say that it *realizes* σ. The superscript in S^{fun} records that we need these operators on functions that are not yet known to be cuspidal; on cusp forms S^{fun} is Lafforgue's operator, and the extension is constructed in [I, Section 5].

Theorem 1.2. *Let G/F be split semisimple with simple factors of types A, B, C, D, in any positive characteristic. Let $\sigma : \Gamma_F \to \widehat{G}(k)$ be a semisimple elliptic parameter. Then there are an open compact subgroup $K \subset G(\mathbb{A}_F)$ and a nonzero cusp form φ_σ on $G(F)\backslash G(\mathbb{A}_F)/K$ satisfying* (1.1).

This is Theorem 13.1. The cusp form lives over F itself: neither the field nor its field of constants is extended. The theorem covers every group in an isogeny class, not only the simply connected and adjoint ones, and every characteristic, including $p = 2$. The parameter is recovered through every excursion operator. This is more than the equality of Hecke eigenvalues at the unramified places which the descent method gives directly, and for the

even orthogonal groups, where the standard representation determines a parameter only up to an outer automorphism, the difference matters.

The function φ_σ is a nonzero element of the excursion summand of σ, and it is not unique. We do not determine multiplicities, the structure of packets, or compatibility with the local correspondence at the ramified places.

1.2 Relation to earlier work

General linear groups and excursion operators. Laurent Lafforgue proved the correspondence for irreducible GL_n-parameters with finite-order determinant [Laf02, Introduction, Theorem $(\mathrm{i})_n$–$(\mathrm{ii})_n$]. For a split reductive group, Vincent Lafforgue constructed the decomposition of cusp forms by parameters [Laf18, Theorem 0.1]. That theorem identifies the parameters of the summands that occur; that the summand of a prescribed parameter is nonzero is a separate assertion, and it is the subject of this series of papers. For a split semisimple group and a parameter which is everywhere unramified and has Zariski dense image, [BHKT19, Theorem 1.1] gives, after a finite Galois extension of the function field, a cuspidal representation whose unramified components have the Satake parameters prescribed by the parameter. Our conclusion keeps the field fixed.

Descent and theta correspondence. The construction starts from automorphy for general linear groups and uses automorphic descent [GRS99, GRS11], the theta correspondence [Ral84], and the theory of automorphic L-functions over function fields [Lom09, Lom19]. The work here is to prove nonvanishing over F, to carry out the necessary coordinate calculations in small characteristic, in particular in characteristic two, and to identify the parameter itself and not only its unramified Hecke eigenvalues.

In characteristic two the orthogonal groups are defined by quadratic forms, and one step of the even orthogonal descent, which in odd characteristic divides a quadratic exponent by two, is replaced by an argument with the Cartier operator (Section 9).

1.3 Organization

Section 2 explains the construction without proofs, Section 3 lists the results we import, including those of [I], and Section 4 fixes conventions. Section 5 contains what is common to all cases: how the parameter is recovered from its standard representation, and the tools for comparing Fourier coefficients. Section 6 constructs the Eisenstein residue. Sections 7, 8 and 9 treat the orthogonal descents, and Section 10 deduces the theorem for orthogonal and spin groups. Sections 11 and 12 treat the symplectic groups by the theta correspondence, and Section 13 assembles the cases. A list of notation is at the end, and Appendix A lists the identities which were also checked by computer.

2 Overview of the construction

This section describes the proof of Theorem 1.2 without giving it. The construction starts from cusp forms on general linear groups, which exist by the theorem of L. Lafforgue, and transports them to G by integral operators. Functions are present from the beginning, and the difficulty is to show that the operators do not annihilate them.

In type A the parameter is lifted to a general linear group, and the correspondence of L. Lafforgue applies (Theorem 5.8). The rest of this outline is about the symplectic and orthogonal cases.

From the parameter to general linear groups. Let $\varrho = \mathrm{Std}\,\sigma$ be the composite of σ with the standard representation of $\widehat{G}$. Since σ is elliptic, ϱ is a sum of pairwise nonisomorphic irreducible representations ϱ_i, each of which preserves a form of the same type, symmetric or

alternating (Proposition 5.2). Each ϱ_i is the parameter of a cuspidal representation Π_i of a general linear group. The problem is to make a function on G out of the Π_i.

Orthogonal descent. As in the work of Ginzburg, Rallis and Soudry, we regard the product of these general linear groups as a Levi subgroup of a classical group S about twice as large as G, and form the Eisenstein series on S induced from the Π_i. That ϱ_i preserves a form of the given type is reflected in a pole of an L-function of Π_i, and this pole produces a pole of the Eisenstein series. Its residue is a space of functions on $S(F)\backslash S(\mathbb{A})$ which are square integrable modulo the centre (Section 6). The group S contains a unipotent subgroup U_{desc} with a character ψ_{desc} whose stabilizer contains G. The *descent* of a function E in the residue is the function

$$g \longmapsto \int_{U_{\mathrm{desc}}(F)\backslash U_{\mathrm{desc}}(\mathbb{A})} E(ug)\,\psi_{\mathrm{desc}}(u)^{-1}\,du, \qquad g \in G(\mathbb{A}).$$

In words: one integrates E over the directions of S which do not belong to G, against a character, and what remains is a function on $G(F)\backslash G(\mathbb{A})$.

Two separate questions. At an unramified place the local component of the residue is a subquotient of an explicit induced representation, and the corresponding local operation, the Jacquet module twisted by ψ_{desc}, is exact. The twisted Jacquet module of the induced representation can be computed, and its unramified constituents have the Satake parameter of σ (Section 7). In the smallest case, the descent from SO_5 to a torus of dimension one, it is the sum of two characters, inverse to each other, if the torus is split, and it is zero otherwise (Section 7.3). This gives the Hecke eigenvalues of the descent, provided the descent is not zero; it says nothing if it is zero. That the descent is not zero is a global statement. The residue has a nonzero Fourier coefficient of another kind, which comes from the Whittaker coefficients of the Π_i (Proposition 6.6). We show that the Whittaker coefficient of the descent is not zero by transforming one coefficient into the other. The tool is elementary (Lemma 5.9): when two unipotent subgroups X and Y are put in duality by the commutator and a character, an integral over $Y(F)\backslash Y(\mathbb{A})$ may be exchanged for an integral over $X(F)\backslash X(\mathbb{A})$ without affecting vanishing. A proof of nonvanishing is a sequence of such exchanges, sometimes a long one, and its whole difficulty is to keep track of the root subgroups involved (Sections 7 and 8). Figures 1–4 show these subgroups in small cases.

Small characteristic. The exchanges rest on the commutator relations between root subgroups, with their structure constants, and on Fourier analysis on $F\backslash\mathbb{A}$. For an odd orthogonal target the argument is the same in all characteristics, once the orthogonal group is defined by a quadratic form and not by a bilinear form. For an even orthogonal target one step moves a Fourier coefficient from the trivial character of a root subgroup to a nontrivial one, and in odd characteristic it does so by a computation in which a quadratic exponent is divided by two. In characteristic two we replace it by the following observation: the function $a \mapsto \psi(sa^2)$ is then an additive character of a, so it is of the form $\psi(C_\psi(s)a)$, and the map C_ψ, which is computed by the Cartier operator, is surjective (Section 9).

Symplectic groups. For a symplectic group we do not use a descent but the theta lift from an even orthogonal group, which on parameters adds a trivial line to the standard representation. Nonvanishing and the Hecke eigenvalues both follow from one computation, that of a Jacquet module of the Weil representation (Sections 11 and 12).

From Hecke eigenvalues to the parameter. In these cases the outcome is a nonzero function on G which has the Hecke eigenvalues of σ at almost all places and a nonzero Whittaker coefficient. This is less than what the theorem asserts, which concerns all excursion operators. The gap is closed by the arithmetic results of [I]: such a function is cuspidal because σ is elliptic, the cusp forms with these Hecke eigenvalues have parameters whose standard representations are isomorphic to ϱ, and for the groups in question this determines the

parameter up to conjugacy, or up to the outer automorphism in the even orthogonal case, which is then dealt with separately (Proposition 5.6).

3 Results used

We collect the theorems on which the construction rests, in the form in which they are used. Precise references to the individual statements are repeated at each point of use.

3.1 From the literature

Input 1 (General linear groups)**.** *Let ϱ be a continuous irreducible n-dimensional $\overline{\mathbb{Q}}_\ell$-representation of Γ_F, unramified at almost all places, whose determinant has finite order.*

1. *There is a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ whose Satake parameters at the unramified places are the eigenvalues of geometric Frobenius elements in ϱ [Laf02, Introduction, Theorem $(\mathrm{i})_n$–$(\mathrm{ii})_n$].*

2. *The Frobenius eigenvalues of ϱ at the unramified places are algebraic numbers all of whose complex absolute values are one [Laf02, Theorem VII.6].*

3. *The representation π is tempered at every place [Laf02, Theorem VI.10(i)].*

In part (1) the Hecke operators act by right translation, as in Section 4. The statement does not depend on the conventions of [Laf02] for Hecke operators and for Frobenius elements: a change of either convention replaces ϱ by its dual and π by its contragredient, and the theorem applies to all irreducible representations.

Parts (1) and (3) are the starting point of the construction; part (2) is the source of the purity theorem of [I, Section 4].

Input 2 (Eisenstein series, Whittaker functionals and the Weil representation)**.** *We follow the descent method of Ginzburg, Rallis and Soudry [GRS99, GRS11]: one forms the residue of an Eisenstein series on a larger classical group, induced from self-dual cuspidal representations of general linear groups, and takes a Fourier coefficient of it. No theorem is quoted from those works. The statements we do quote are the following.*

1. *Cusp forms of fixed level on a group over a function field have compact support modulo the centre [Mor82, Chapter I, §5.3 and Proposition 5.3].*

2. *A function of finite level agrees with its constant terms far in the cusps [Mor82, Chapter I, §5.9]; consequently an automorphic form on which the centre acts through a unitary character, and whose constant terms have exponents in the negative chamber, is square integrable modulo the centre (the estimate is written out in the proof of Lemma 6.4). For a maximal parabolic subgroup and unitary cuspidal data, at a fixed level, the intertwining operator and the Eisenstein series are rational functions on the complex torus of unramified characters by which the data are twisted. Along the real points of this torus, with parameter $s \in \mathbb{R}$ and $s = 0$ the trivial character, they are holomorphic at $s = 0$ and have at most simple poles for $s > 0$ [Mor82, Chapter III, Lemmas 3.2 and 3.5, Theorem 5.2].*

3. *Over a nonarchimedean local field of any characteristic, the intertwining operators for tempered inducing data are given by convergent integrals in the open positive chamber [Wal03, Proposition IV.2.1], and on the spherical vector of an unramified principal series they are given by the formula of Gindikin and Karpelevich [Cas80, Theorem 3.1].*

4. *The Whittaker functional on a principal series representation is given by an integral over the open cell [CS80, Theorems 1.4 and 1.6, Corollary 1.7], with the unramified value of [CS80, Theorem 5.4].*

5. *Eisenstein series over function fields continue meromorphically, with the constant-term formulas of [BL24, Theorem 2.3, (6.2), (6.4), (6.17)]; pseudo-Eisenstein series span the functions of compact support [BL24, §9.2, (9.3)]. In the same work automorphic forms are characterized by finite systems of difference equations satisfied by the cuspidal projections of their constant terms [BL24, §9.1].*

6. *The theta distribution of the Weil representation is invariant under rational points [Wei64, §§29–30 and 40–41]. The metaplectic cocycle in odd characteristic is described in [Rao93].*

What is proved here, and not taken from the literature, is the nonvanishing of the descent over F, the identification of its excursion parameter, and every argument in characteristic two.

3.2 From the first paper

The arithmetic part of the argument is contained in [I]. It is applied in Section 5, up to Proposition 5.6, and in the sections where the realization theorems are proved. We use the following definitions and statements; a reader who accepts them does not need the rest of that paper.

1. *Excursion operators on functions.* For a level N the excursion operators act on the functions with compact support on $Y_N = G(F)\backslash G(\mathbb{A}_F)/K_N$, through the cohomology of stacks of shtukas, and S_e^{fun} is defined on all functions on Y_N by transposition (Definition I.5.12). On cusp forms it is the operator of V. Lafforgue. The cohomology in question is finitely generated over the spherical Hecke algebra at one place (Proposition I.5.5).

2. *Purity and fields of coefficients.* A semisimple elliptic parameter with values in a semisimple group is strongly pure (Theorem I.4.8). We also use Lemma I.4.1 on compact subgroups of $\mathrm{GL}_n(k)$ and the decomposition of a parameter into a twist by the constant field and a parameter σ^{fin} (Lemma I.4.11).

3. *From Hecke eigenvalues to the parameter.* Proposition I.8.6, which shows that a function with the Hecke eigenvalues of an elliptic parameter at almost all places is cuspidal, and Lemma I.8.5, the form of this statement for a subalgebra of the Hecke algebra; Lemma I.8.2, by which the Hecke operators attached to one representation with finite kernel suffice; and Lemma I.8.8, which extracts an eigenfunction from a pairing.

4. *Irreducible constituents.* An admissible subspace of smooth vectors in a unitary representation is semisimple (Lemma I.9.1), and Proposition I.9.3 on images of restricted tensor products.

5. *Isogenies.* The pullback of cusp forms along a homomorphism with central kernel and normal image (Lemma I.10.1), the descent along the quotient by a central torus (Lemma I.10.4), and the reduction of the existence of a cusp form with a given elliptic parameter to simply connected simple groups (Corollary I.10.11).

Those conventions of [I, Section 3] which are used in this paper are repeated in Section 4.

4 Conventions

We fix the normalizations of Frobenius, of the Satake equivalence and of class field theory used in the whole paper, and define the excursion summand attached to a parameter. The spaces of functions on which excursion operators act are constructed in Sections I.5 and I.6.

4.1 Frobenius and the Satake equivalence

Let $\deg_F : \Gamma_F \to \widehat{\mathbb{Z}}$ be the homomorphism given by the action on the constant field, normalized so that an arithmetic Frobenius element has degree 1. A geometric Frobenius element at a place v of degree d_v has degree $-d_v$. The Satake equivalence, for G and for all its Levi subgroups, is normalized by one square root $q^{1/2} \in k$ chosen once and for all. When q is the square of an integer we take the positive root; then in all cases there is an isomorphism of k with $\mathbb{C}$ which maps $q^{1/2}$ to the positive square root of q.

4.2 The centre and class field theory

Let $T_G = (Z_G^\circ)_{\mathrm{red}}$ and

$$C_{T_G} = T_G(F)\backslash T_G(\mathbb{A}_F), \qquad \deg(t)(\chi) = \sum_v d_v\, v(\chi(t_v))$$

for a character χ of T_G. This defines a homomorphism $\deg : C_{T_G} \to X_*(T_G)$. The degree of the bundle attached to t is $-\deg(t)$.

Definition 4.1. A *central lattice* is a free subgroup $\Xi \subset C_{T_G}$ on which deg is injective and whose image under deg has finite index in $X_*(T_G)$. A function on $G(F)\backslash G(\mathbb{A}_F)$ is *invariant under* Ξ if it is invariant under translation by the elements of Ξ.

The quotient of C_{T_G} by a central lattice and an open compact subgroup is finite.

Class field theory is normalized by the arithmetic Artin map, $\mathrm{Art}_v(\varpi_v) = \mathrm{Frob}_v^{\mathrm{arith}}$, and a character χ of Γ_F corresponds to the character

$$\mathrm{CFT}_{\mathbb{G}_m}(\chi)(t) = \chi(\mathrm{Art}(t)^{-1})$$

of the idele class group; for a split torus one takes the product over a basis of characters. Right translation by a is denoted T_a, $(T_a h)(g) = h(ga)$. With these conventions the excursion operator of a character λ of $\widehat{T}_G$ and a geometric Frobenius element at v is $T_{\lambda(\varpi_v)}$ [Laf18, Proposition 6.2], and a parameter of the form $c^{\deg_F}$ corresponds to the character $t \mapsto c^{-\deg(t)}$. Let $\alpha_G : \widehat{G} \to \widehat{T}_G$ be the homomorphism dual to the inclusion $T_G \to G$. The *central character* of a parameter σ is

$$\omega_\sigma = \mathrm{CFT}_{T_G}(\alpha_G \sigma),$$

a character of C_{T_G}. This is the character which occurs in Theorem I.C.

4.3 Excursion summands

Definition 4.2. Let V be a finite-dimensional vector space on which the excursion operators S_e act, for all finite sets I, all functions f and all tuples in Γ_F^I; they generate a commutative algebra $\mathcal{B}$ of endomorphisms. Let $\mathfrak{m}_\sigma \subset \mathcal{B}$ be the ideal generated by the operators $S_e - \nu_\sigma(e)$. The *excursion summand* of σ in V is the generalized eigenspace

$$V_{\nu_\sigma} = \bigcup_{n \geq 1} \{v \in V : \mathfrak{m}_\sigma^n v = 0\}.$$

For a space which is a union of subspaces V of this kind, the excursion summand is the union of the V_{ν_σ}.

The definition involves the excursion operators for all tuples in the Galois group; the Hecke operators at the unramified places are among them. A nonzero excursion summand in a space of finite dimension contains a joint eigenvector of the excursion operators, with eigenvalues $\nu_\sigma(e)$: the commuting operators $S_e - \nu_\sigma(e)$ are nilpotent on it. So the existence of a nonzero summand and the existence of an eigenfunction are equivalent.

5 Parameters and Fourier coefficients

This paper proves Theorem 13.1: for the classical groups, every elliptic parameter is the parameter of a cusp form, in every characteristic. The starting point is L. Lafforgue's correspondence for general linear groups, applied to the standard representation $\varrho = \mathrm{Std}\,\sigma$ of the parameter. The form preserved by ϱ determines an Eisenstein series on a larger classical group; a Fourier coefficient of its residue is a function on the group we want, the *descent* of Ginzburg, Rallis and Soudry. For part of the symplectic case the descent is replaced by a theta lift.

There are two separate questions. A local computation of Jacquet modules identifies the Hecke eigenvalues of a descent *if the descent is nonzero*. That it is nonzero is a global statement, which we prove by comparing Fourier coefficients over F itself. The comparison uses explicit root subgroups, and this is where characteristic two needs separate calculations (Section 9). Once a nonzero function with the right Hecke eigenvalues is found, the results of [I] show that it is cuspidal and has the parameter σ, for the excursion operators of all tuples (Proposition 5.6).

The present section collects what is common to all cases: how σ is recovered from ϱ (Lemma 5.1, Proposition 5.2), the order of the poles of the relevant L-functions (Lemma 5.4), the recognition statement just mentioned, the case of type A (Theorem 5.8), and two general facts about Fourier coefficients (Lemmas 5.9 and 5.10).

Throughout the paper $k = \overline{\mathbb{Q}}_\ell$ with $\ell \neq p$. Integrals have complex values, compared with k through a fixed isomorphism $\iota : k \simeq \mathbb{C}$ which maps the chosen $q^{1/2}$ to the positive square root of q; so the Satake parameters defined with $q^{1/2}$ are those of normalized induction with the positive modulus character. All inductions are normalized. For a unipotent group U we put $[U] = U(F)\backslash U(\mathbb{A}_F)$, with the measure of volume one, and we fix a nontrivial character $\psi : F\backslash\mathbb{A}_F \to \mathbb{C}^\times$. The irreducible summands of ϱ are denoted ϱ_i. In computations with roots, E_{ab} is the matrix sending the basis vector b to a, and $x_\alpha(t)$ is the full matrix of the root subgroup, including its quadratic entries when the root is short. The table shows where each kind of input is established.

Input	Assertion needed	Location
Self-dual general-linear blocks	A residual representation with a nonzero Siegel–Whittaker coefficient	Proposition 6.6
Local descent	The parameter and the obstruction to a nonsplit endpoint	Proposition 7.1
Global root operations	A nonzero descent over F	Propositions 7.10, 8.5, 9.4
Quadratic kernels	Nonzero lifts and spherical Hecke equations	Corollary 11.6
Common arithmetic	Cuspidality and the excursion character	Proposition 5.6

5.1 Recovering a parameter from its standard representation

A parameter with values in a classical group is a representation together with a form. The first lemma says that the representation determines the form up to isometry. In even orthogonal dimension this leaves the outer automorphism: two parameters with the same standard representation are conjugate in the orthogonal group, and not necessarily in the special orthogonal group.

Lemma 5.1 (Forms and simultaneous conjugacy)**.** *Let r and r' be semisimple representations of a group Γ on a finite-dimensional k-vector space V which preserve, up to the same character χ of Γ, a nondegenerate symmetric or alternating form B. If $r \simeq r'$, then $r' = grg^{-1}$ for an isometry g of B. If B is symmetric and $\dim V$ is odd, g can be taken of determinant one.*

Proof. This is an instance of the following. Let T be a representation of $\mathrm{GL}(V)$ with an open orbit $\mathcal{O}$, let $b \in \mathcal{O}$, and suppose r and r' both multiply b by χ. Choose $a \in \mathrm{GL}(V)$ with $r' = ara^{-1}$. Then $b' = a^{-1}b$ is again multiplied by χ under r. If $b' = cb$ for some c in the centralizer C of r, then $g = ac$ stabilizes b and $grg^{-1} = r'$. So it suffices to show that C acts transitively on

$$\mathcal{N} = \mathcal{O} \cap E, \qquad E = \{t \in T : r(\gamma)t = \chi(\gamma)t \text{ for all } \gamma\}.$$

Let H be the Zariski closure of $r(\Gamma)$, a reductive group. The line kb is stable under $r(\Gamma)$, hence under H, and H acts on it through an algebraic character which extends χ; we denote this character by χ as well. At a point $t \in \mathcal{N}$ the map $\xi \mapsto \xi t$ from $\mathfrak{gl}(V)$ to T is surjective, because $\mathcal{O}$ is open, and it is H-equivariant when the action of H on T is twisted by χ^{-1}. Taking invariants under H is exact, so the Lie algebra $\mathfrak{gl}(V)^H$ of C maps onto E. Hence every orbit of C on $\mathcal{N}$ is open. But $\mathcal{N}$ is a nonempty open subset of the vector space E and therefore irreducible, so it cannot contain two disjoint nonempty open subsets. Thus $\mathcal{N}$ is a single orbit.

Take T to be the space of symmetric or alternating bilinear forms, in which the nondegenerate forms are an open orbit with stabilizer the isometry group. In odd dimension -1 is an isometry of determinant -1 which acts trivially by conjugation. □

Proposition 5.2 (The blocks of an elliptic parameter)**.** *Let r be a semisimple representation of Γ on V preserving a nondegenerate symmetric or alternating form B up to a character χ; in the symmetric case assume $\dim V \geq 3$. The following are equivalent.*

1. *The centralizer of r in the group of similitudes of B is finite modulo scalars.*
2. *V is an orthogonal direct sum of pairwise nonisomorphic irreducible subrepresentations V_i, and the restriction of B to each V_i is nondegenerate.*

If B is alternating and $\dim V_i = 2d_i$, then $\det V_i = \chi^{d_i}$.

Proof. Write $V = \bigoplus_W W \otimes M_W$, the sum being over the irreducible representations and M_W the multiplicity spaces. The form pairs the summand of W with that of $W^* = W^\vee \otimes \chi$ and with no other.

Assume (1). If some $W \not\simeq W^*$ occurs, then for every $t \in k^\times$ the transformation equal to t on $W \otimes M_W$, to t^{-1} on $W^* \otimes M_{W^*}$ and to 1 on the other summands is an isometry commuting with r. These form a torus which meets the scalars in a finite group, contrary to (1); in the symmetric case this uses $\dim V \geq 3$ when there is no other summand. So every W that occurs satisfies $W \simeq W^*$. By Schur's lemma the form on $W \otimes M_W$ is then $b_W \otimes b_M$, where b_W is the invariant form of W, unique up to a scalar, and b_M is a nondegenerate symmetric or alternating form on M_W. The isometry group of b_M commutes with r, and it

is infinite as soon as $\dim M_W \geq 2$. Hence $\dim M_W = 1$, the form b_M is symmetric, and b_W has the symmetry of B. This is (2).

Assume (2). A similitude commuting with r acts on each V_i by a scalar c_i, and c_i^2 is its multiplier, the same for all i. So it is a scalar times an element of a finite group of signs.

For the last assertion, a similitude of an alternating form in dimension $2d$ with multiplier c has determinant c^d. □

Remark 5.3. Ellipticity excludes multiplicities; it does not make the standard representation irreducible. In particular an orthogonal elliptic parameter has at most one line of invariants under Γ_F, which will determine the rank of the quadratic theta lift in Section 11.

5.2 Poles of L-functions

Lemma 5.4 (The order of the pole at $s = 1$)**.** *Let V be a semisimple lisse sheaf on a nonempty open $U \subset X$, pure of weight zero. Then the order of the pole of $L^U(s, V)$ at $s = 1$ is $\dim V^{\Gamma_F}$. The function has neither zeros nor poles at real $s > 1$, nor at $s = 1$ if $V^{\Gamma_F} = 0$.*

Proof. By the Grothendieck–Lefschetz formula,

$$L^U(s, V) = \prod_{j=0}^{2} \det\bigl(1 - q^{-s}\,\mathrm{Frob} \mid H_c^j(U_{\overline{\mathbb{F}}_q}, V)\bigr)^{(-1)^{j+1}}.$$

The eigenvalues of Frobenius on H_c^0 and H_c^1 have absolute values at most 1 and $q^{1/2}$ [Del80, Theorem 3.3.1 and §3.3.10], so these two factors have no zero for real $s \geq 1$. By Poincaré duality H_c^2 is the space of coinvariants of the geometric fundamental group on V, twisted by -1. Its eigenvalues have absolute value q, so the factor for $j = 2$ has no zero for $s > 1$, and its order of vanishing at $s = 1$ is the multiplicity of the eigenvalue 1 of Frobenius on the geometric coinvariants. Since V is semisimple and the geometric monodromy group is normal in the arithmetic one, the geometric invariants are a direct summand of V, isomorphic to the coinvariants, on which the quotient group acts. That quotient is reductive with a dense cyclic subgroup, hence diagonalizable, so Frobenius is semisimple there and the multiplicity of the eigenvalue 1 is $\dim V^{\Gamma_F}$. □

We shall apply the lemma to the blocks V_i of an elliptic parameter. By Schur's lemma the square L-function of the same sign as the form, $L(s, \Lambda^2 V_i \otimes \chi^{-1})$ or $L(s, \mathrm{Sym}^2 V_i \otimes \chi^{-1})$, has a simple pole at $s = 1$, and for $i \neq j$ the function $L(s, V_i \otimes V_j \otimes \chi^{-1})$ has neither a pole nor a zero there. This is what prevents cancellation in the Eisenstein coefficient of Section 6.

5.3 Recognizing the parameter of a function

The constructions of this paper produce complex-valued functions. The next lemma moves them to the coefficient field k.

Lemma 5.5 (Transport of coefficients)**.** *Let φ be a complex cusp form of finite level, invariant under a central lattice. Then $\iota^{-1} \circ \varphi$ is a cusp form with values in a finite extension of $\mathbb{Q}_\ell$. It satisfies the same Hecke equations as φ, and its Whittaker coefficients are the images of those of φ.*

Proof. By Harder's theorem [Har74, Theorem 1.2.1 and Corollary 1.2.3] the support of φ at a given level is compact modulo the centre, and the lattice makes it finite modulo $G(F)$ and the level subgroup. So φ takes finitely many values, and their images under ι^{-1} lie in a finite extension of $\mathbb{Q}_\ell$. Hecke operators, constant terms and Whittaker coefficients at a given level are finite sums whose coefficients are rational numbers, powers of the chosen square root of q and roots of unity; ι^{-1} respects such sums. □

A smooth complex function that is not yet known to be cuspidal can also be composed with ι^{-1}: its constant terms, its Fourier coefficients and the Hecke operators are still finite sums. Only the argument given above for the finiteness of the field of values uses cuspidality.

Proposition 5.6 (Recognition by standard Hecke data)**.** *Suppose that the dual group of G is Sp_{2n}, SO_d with $d \geq 3$, GSp_{2n}, or GSO_{2n} with $n \geq 2$. Let $\sigma : \Gamma_F \to \widehat{G}(k)$ be continuous, finitely ramified and semisimple, with multiplier of finite order in the similitude cases. Let $h \neq 0$ be a smooth function of finite level on $G(F)\backslash G(\mathbb{A}_F)$, and let S be a finite set of places containing the places where h or σ is ramified. Suppose that at every place outside S the Hecke operators attached to the standard representation act on h through the coefficients of the characteristic polynomial of $\mathrm{Std}\,\sigma$ of Frobenius, possibly as generalized eigenvalues; in the similitude cases assume the same for the multiplier, and that h has the corresponding central character of finite order. (The set S may be larger than the set of ramified places: in the applications it also contains the places where an auxiliary additive character, a section or a level is ramified.)*

1. *If σ is elliptic, then h is cuspidal.*
2. *If h is cuspidal, the cuspidal excursion summand of σ is nonzero. If moreover a Whittaker coefficient of h is nonzero, this summand contains a function with a nonzero Whittaker coefficient.*

Proof. Step 1: parameters with the same standard data. Let σ' be a continuous semisimple parameter with the same characteristic polynomials of the standard representation at almost all places, and the same multiplier. By Chebotarev and Brauer–Nesbitt, $\mathrm{Std}\,\sigma' \simeq \mathrm{Std}\,\sigma$. Lemma 5.1 then shows that σ' is conjugate to σ in the stabilizer of the tensor, that is, in $\widehat{G}$ or, in even orthogonal type, in the full orthogonal or similitude group.

Step 2: cuspidality. Suppose that σ is elliptic and that h is not cuspidal. Let r be the standard representation of $\widehat{G}$, and for $u \notin S$ let $\mathscr{A}_u$ be the subalgebra of the spherical Hecke algebra at u generated by the operators which correspond to the coefficients of the characteristic polynomial of r, to the inverse of its determinant and, in the similitude cases, to the multiplier and its inverse. The spherical Hecke algebra is a finitely generated module over $\mathscr{A}_u$, by the proof of Lemma I.8.2(1). (The determinant of r is one in the isometry cases and a power of the multiplier in the similitude cases.) By hypothesis $\mathscr{A}_u$ acts on h through the generalized character χ_u given by the standard data of $\sigma(\mathrm{Frob}_u)$. A Satake parameter over χ_u has in the standard representation the eigenvalues of $\mathrm{Std}\,\sigma(\mathrm{Frob}_u)$, which are ℓ-adic units because σ is continuous; since the weights of the standard representation, together with the multiplier in the similitude cases, generate a subgroup of finite index in the character group of the dual torus, every character of the dual torus takes a unit value at such a parameter.

So Lemma I.8.5 applies. It gives a proper standard Levi subgroup M and a continuous semisimple parameter σ_M of M, unramified outside S, such that $i\sigma_M$ and σ have the same characteristic polynomials of the standard representation, and the same multiplier, at the Frobenius elements of all places outside S. By step 1, σ is conjugate to $i\sigma_M$ in $\widehat{G}$ or, in even orthogonal type, in the full orthogonal or similitude group. Conjugation by an element of that group preserves $\widehat{G}$ and maps $i(\widehat{M})$ onto a proper Levi subgroup of $\widehat{G}$. So the image of σ lies in a proper Levi subgroup of $\widehat{G}$, whose connected centre centralizes it and has positive dimension modulo $Z(\widehat{G})$. This contradicts ellipticity.

Step 3: the excursion summand. Assume h cuspidal. The space of cusp forms of its level and central character is finite dimensional and is the direct sum of its generalized excursion summands. Excursion operators commute with Hecke operators, so every component of h satisfies the same Hecke equations, and by step 1 the parameter of every nonzero component

is σ or its outer conjugate. Some component is nonzero, and if a Whittaker coefficient of h is nonzero then it is nonzero on some component.

Step 4: the outer automorphism. In even orthogonal type let s be the isometry, defined over F, which exchanges the two vectors of the last hyperbolic pair and fixes the other vectors of the basis. Conjugation by s is an automorphism of the special orthogonal group over F which induces the outer automorphism of the dual group. It lifts to GSpin. An element of the Clifford group which represents s is odd, so it does not lie in GSpin; but conjugation by it preserves the even Clifford group GSpin, and fixes the scalars. This holds in characteristic two as well, where s has Dickson invariant one. By Lemma I.10.1 this automorphism maps the summand of a parameter onto the summand of the outer conjugate of the parameter, and it maps a Whittaker coefficient to the Whittaker coefficient for the character which it transports. Hence the two summands are both nonzero. □

5.4 Type A and lifting along a central torus

Lemma 5.7 (Lifting along a central torus)**.** *Let $1 \to Z \to H' \to H \to 1$ be an exact sequence of algebraic groups over k with H semisimple, H' connected reductive and Z a central torus. Every continuous, finitely ramified parameter $\tau : \Gamma_F \to H(k)$ lifts to a continuous, finitely ramified parameter $\widetilde{\tau} : \Gamma_F \to H'(k)$ whose image in the abelianization of H' is finite. If τ is semisimple, or elliptic, or strongly pure, then so is $\widetilde{\tau}$, ellipticity being understood modulo the centre.*

Proof. A continuous lift, ramified at finitely many additional places, exists by [Con11, Lemma 5.2 and Proposition 5.3], and it takes values in a finite extension of $\mathbb{Q}_\ell$ by Lemma I.4.1. Since H is semisimple, Z is the connected centre of H' and maps onto the abelianization of H' with finite kernel. Multiplying the lift by a character with values in Z that factors through the degree, as in Lemma I.4.11, we make its image in the abelianization finite without changing its projection to H or its restriction to inertia.

If τ is semisimple, the unipotent radical of the Zariski closure of the image of $\widetilde{\tau}$ maps to a unipotent normal subgroup of the reductive closure of the image of τ, hence trivially; it lies in the torus Z and is trivial. The centralizer of $\widetilde{\tau}$ maps into that of τ, and the inverse image of the centre of H is the centre of H'; this gives ellipticity. For purity, let r be an irreducible representation of H'. The representation $\mathrm{End}(r)$ factors through H, so at an unramified place the ratios of the Frobenius eigenvalues of $r\widetilde{\tau}$ are algebraic of absolute value one under every embedding; and $\det r$ factors through the abelianization, so the product of the eigenvalues is a root of unity. Hence each eigenvalue is algebraic of absolute value one. □

Theorem 5.8 (Type A)**.** *Every continuous, finitely ramified, semisimple elliptic parameter with values in $\mathrm{PGL}_n(k)$ has a nonzero cuspidal excursion summand on $\mathrm{SL}_n(\mathbb{A}_F)$.*

Proof. By Lemma 5.7 the parameter lifts to a representation $\widetilde{\tau}$ of dimension n with determinant of finite order. It is irreducible: if it were a sum of two subrepresentations, scaling them independently would give a torus in the centralizer of the original parameter. By L. Lafforgue's theorem [Laf02, Introduction, assertion $(\mathrm{ii})_n$] there is a cuspidal representation of $\mathrm{GL}_n(\mathbb{A}_F)$ with parameter $\widetilde{\tau}$ and central character of finite order. Every excursion summand that meets it has, by the compatibility of excursion operators with Hecke operators and by Chebotarev, the parameter $\widetilde{\tau}$. Choose a function in such a summand, translate it so that it is nonzero at the identity, and restrict it to $\mathrm{SL}_n(\mathbb{A}_F)$. The restriction is cuspidal because the unipotent radicals of the parabolic subgroups of SL_n and GL_n coincide, and by Lemma I.10.1 it lies in the summand of the image of $\widetilde{\tau}$ in PGL_n. □

5.5 Two facts about Fourier coefficients

For a character ζ of $[U]$ write

$$\mathcal{F}_{U,\zeta} f(g) = \int_{[U]} f(ug)\zeta(u)^{-1}\, du.$$

Let $\mathcal{V}$ be a space of automorphic functions stable under right translation. We write $(U,\zeta) \sim (U',\zeta')$ if $\mathcal{F}_{U,\zeta}$ vanishes identically on $\mathcal{V}$ exactly when $\mathcal{F}_{U',\zeta'}$ does. This relates the two coefficients only as conditions on the whole space; it does not compare them on an individual function. The global arguments of this paper consist of chains of such equivalences, obtained from the following lemma and from conjugation by elements of $G(F)$.

Lemma 5.9 (Exchange of root subgroups)**.** *Let U be a unipotent group with a character ζ of $[U]$, and let X and Y be vector groups which normalize U, preserve ζ, meet U trivially and satisfy $[X,Y] \subset U$. Suppose that ζ extends to characters of UX and of UY, trivial on X and on Y, and that $(x,y) \mapsto \zeta([x,y])$ is a perfect pairing between X and Y defined over F. Then*

$$(UX,\zeta) \sim (U,\zeta) \sim (UY,\zeta).$$

Proof. Since $\mathcal{F}_{UX,\zeta}$ is the integral of $\mathcal{F}_{U,\zeta}$ over $[X]$, the vanishing of the latter implies that of the former. Conversely, for fixed g the function $x \mapsto \mathcal{F}_{U,\zeta} f(xg)$ on $[X]$ has a Fourier expansion, which is a finite sum because f is smooth. By the perfect pairing its characters are $x \mapsto \zeta([x,y])$ with $y \in Y(F)$. Write $[x,y] = xyx^{-1}y^{-1}$. The function $\mathcal{F}_{U,\zeta} f$ is equivariant on the left under $U(\mathbb{A}_F)$ and invariant on the left under $Y(F)$, so $\mathcal{F}_{U,\zeta} f(xy^{-1}g) = \zeta([x,y])^{-1}\mathcal{F}_{U,\zeta} f(xg)$. Hence the coefficient of this character is $\mathcal{F}_{UX,\zeta} f(y^{-1}g)$. Consequently, if $\mathcal{F}_{UX,\zeta}$ vanishes on $\mathcal{V}$, then all the coefficients vanish, and so does $\mathcal{F}_{U,\zeta}$. The statement for Y follows by symmetry. □

The second fact is local. It computes twisted Jacquet modules of induced representations and is the tool for Proposition 7.1.

Lemma 5.10 (Twisted Jacquet modules of induced representations)**.** *Let G be a reductive group over a nonarchimedean local field, $P = MU$ and P' parabolic subgroups, ζ a character of U, $H \subset M$ a subgroup preserving ζ, and χ a character of P'. For a representative w of a double coset in $P'\backslash G/P$, taken in the Weyl group of a maximal split torus contained in both Levi subgroups, write $P \cap w^{-1}P'w = M_w \ltimes U_w$ and*

$$Z_w = \{M_w\gamma \in M_w\backslash M : \zeta \text{ is trivial on } \gamma^{-1}U_w\gamma\}.$$

Assume that the conditions defining Z_w are closed conditions on the matrix entries of γ.

1. *The double coset of w contributes to the twisted Jacquet module $J_{U,\zeta}(i_{P'}^G\chi)$ only through the fibres over Z_w.*
2. *Suppose that Z_w is empty for all double cosets but one, and that for this one it is a finite disjoint union of closed orbits $M_w\gamma_j H \simeq B_j\backslash H$ admitting local sections. Put $U_j = U \cap (w\gamma_j)^{-1}P'w\gamma_j$. Then, as a representation of H,*

$$J_{U,\zeta}(i_{P'}^G\chi) \simeq \bigoplus_j \text{c-Ind}_{B_j}^H \left((\chi\delta_{P'}^{1/2}) \circ \mathrm{Ad}(w\gamma_j)\ \left|\det \mathrm{Ad}(\cdot)|_{\mathrm{Lie}\, U/\,\mathrm{Lie}\, U_j}\right|\right).$$

Proof. The proof has three parts: exactness of twisted coinvariants, the computation on a single fibre, and the passage from one fibre to a family of fibres, which is the only delicate point.

Step 1: exactness and a single fibre. A contracting cocharacter exhausts U by compact open subgroups U_n. Coinvariants for (U_n, ζ) are the image of the projector that averages against ζ^{-1}, which is exact in characteristic zero, and $J_{U,\zeta}$ is their direct limit; so $J_{U,\zeta}$ is exact. For a closed subgroup $L \subset U$, the twisted coinvariants of $\text{c-Ind}_L^U 1$ are zero if $\zeta|_L$ is nontrivial, and if $\zeta|_L$ is trivial they are one-dimensional, given by integration over $L\backslash U$ against ζ^{-1}. Indeed a function with compact support modulo L is supported in LK_0 for a compact open subgroup K_0, and averaging over a sufficiently large U_n proves both statements.

Step 2: families of fibres. By step 1 and the filtration of the induced representation by the double cosets, these can be treated one at a time. The piece attached to $P'wP$ is the space of sections with compact support of the inducing line bundle on $(M_w \ltimes U_w)\backslash P$. Over a chart of $M_w\backslash M$ choose a continuous section $b \mapsto \gamma(b)$ of $M \to M_w\backslash M$. Writing the points of the fibre over b as $wv\gamma(b)$ identifies this fibre with $U_w\backslash U$ and trivializes the line bundle on it. In these coordinates $u \in U$ acts on the fibre by right translation by $\gamma(b)u\gamma(b)^{-1}$, and the character becomes $\zeta_b(v) = \zeta(\gamma(b)^{-1}v\gamma(b))$.

Let a section have vanishing integrals over the fibres above Z_w, and let b be a point of the projection of its support. On a small compact open chart around b the supports in the fibres lie in the image of one compact open subgroup K' of U. If $b \notin Z_w$, enlarge K' so that it contains an element of U_w on which ζ_b is not trivial. In both cases the average of the section over K' against ζ_b vanishes at b, by step 1. It vanishes near b as well. Indeed, after shrinking the chart, the character $\zeta_{b'}$ on the compact set K' and the section, in the trivialization, do not depend on b'; and since the normalizer of a compact open subgroup of U is open in M, we may also assume that $\gamma(b')^{-1}K'\gamma(b') = \gamma(b)^{-1}K'\gamma(b)$ on the chart. So one compact open subgroup of U kills the section over the chart. The projection of the support is compact; cover it by finitely many such charts. A subgroup U_n which is large enough then kills the section.

Conversely, a function with compact support on Z_w extends to the base, by means of a finite partition into open and closed charts. On each chart choose K' so small that ζ_b is trivial on it for all b in the chart, and multiply the extension by the characteristic function of the image of K' in $U_w\backslash U$, divided by its volume. The integrals over the fibres of the section obtained are the given function on Z_w. This proves (1), and identifies the contribution of the double coset.

Step 3: the surviving layer. Under the hypothesis of (2) the map

$$f \longmapsto \left[h \longmapsto \int_{U_j\backslash U} f(w\gamma_j uh)\zeta(u)^{-1}\, du \right]$$

identifies the contribution of the jth orbit with a space of functions on $B_j\backslash H$. Substituting $u = bu'b^{-1}$ for $b \in B_j$ gives its transformation law: the factor $\chi\delta_{P'}^{1/2}$ of the induced representation evaluated at $w\gamma_j b(w\gamma_j)^{-1}$, times the modulus of the change of variables on $U_j\backslash U$. This is the compact induction of the statement. If B_j is a parabolic subgroup of a reductive group H, dividing the character by $\delta_{B_j}^{1/2}$ writes it as a normalized parabolic induction. □

6 An Eisenstein residue with a nonzero Fourier coefficient

Let σ be an elliptic symplectic or even orthogonal parameter whose multiplier χ has finite order. By Proposition 5.2 its standard representation is a sum of distinct blocks ϱ_i, each carrying a form of the same kind with multiplier χ. Since $(\det \varrho_i)^2 = \chi^{\dim \varrho_i}$, the determinants have finite order, and L. Lafforgue's theorem gives unitary cuspidal representations Π_i of $\mathrm{GL}_{m_i}(\mathbb{A}_F)$ with parameters ϱ_i. They satisfy $\Pi_i^\vee \otimes \omega_\sigma \simeq \Pi_i$, where ω_σ is the character corresponding to χ by class field theory.

The descent starts from an Eisenstein series on a group $S = \mathrm{GSpin}_{2m}$ or GSpin_{2m+1}, induced from $\boxtimes_i \Pi_i$ on a Levi subgroup contained in the Siegel Levi subgroup. This section proves what is needed about it (Proposition 6.6): its iterated residue is a nonzero, irreducible, square-integrable automorphic representation, and a specific Fourier coefficient, denoted $\mathcal{SW}$, does not vanish on it. This is the function-field form of the first step of the descent method of Ginzburg, Rallis and Soudry [GRS99, GRS11].

The proof has the usual shape. The constant terms of the Eisenstein series are sums over Weyl elements of intertwining operators, whose unramified parts are ratios of L-functions; the poles of these ratios at the point of interest come from Lemma 5.4. In the constant term along the Siegel parabolic subgroup exactly one term has a pole of maximal order, and its Whittaker coefficient along the Levi subgroup is an explicit nonzero product of L-values; this shows that the leading coefficient is not zero. Separately, every exponent of every cuspidal constant term of the leading coefficient is strictly negative; this shows that it is square integrable. At the ramified places nothing is computed: we exhibit test vectors for which the local coefficient is 1.

6.1 Whittaker functionals on general linear groups

Let $\psi_d(n) = \psi(\sum_{i<d} n_{i,i+1})$ on the group U_d of upper triangular unipotent matrices in GL_d, and $W_f(g) = \int_{[U_d]} f(ng)\psi_d(n)^{-1}\,dn$.

Lemma 6.1 (Whittaker functionals on general linear groups). *1. For a cusp form f on $\mathrm{GL}_d(\mathbb{A}_F)$,*

$$f(g) = \sum_{\gamma \in U_{d-1}(F)\backslash \mathrm{GL}_{d-1}(F)} W_f\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

In particular $W_f \neq 0$ if $f \neq 0$.

2. Over a nonarchimedean local field of any characteristic, the space of Whittaker functionals on an irreducible smooth representation of GL_d has dimension at most one.

3. The Whittaker functional $f \mapsto W_f(1)$ on a cuspidal representation $\otimes'_v \Pi_v$ is a product of local Whittaker functionals.

Proof. (1) Expand f along the last column of U_d, a vector group on which $\mathrm{GL}_{d-1}(F)$ acts. The term of the trivial character is a constant term along a proper parabolic subgroup and vanishes. The nontrivial characters form one orbit, and the stabilizer of a point is the mirabolic subgroup of GL_{d-1}. Repeating the argument inside the successive mirabolic subgroups gives the formula; at each stage the term of the trivial character is a constant term of f integrated over a unipotent group, and for fixed g all the sums are finite.

(2) By [BZ77, Theorem 2.5] an irreducible representation embeds in a representation induced from supercuspidal representations of a Levi subgroup. The space of Whittaker functionals of a representation of GL_d is the dual of its derivative of order d in the sense of Bernstein and Zelevinsky, the derivatives are exact functors [BZ77, Proposition 1.9(a)], and the derivative of order d of such an induced representation is one-dimensional [BZ77, Theorem 4.4, Lemma 4.5 and Corollary 4.6]. These results hold over any nonarchimedean local field.

(3) Choose a pure tensor on which the functional is nonzero. For a finite set of places, the restriction of the functional to the tensors that agree with the chosen one outside that set is, by (2), the product of the local functionals normalized to take the value 1 on the chosen components, times the value on the chosen tensor. These descriptions are compatible when the set grows. □

Lemma 6.2 (Whittaker functional of an induced representation)**.** *Let v be a place and consider the representation of $\mathrm{GL}_m(F_v)$ induced from $\boxtimes_i \Pi_{i,v}|\det|^{s_i}$, with the Whittaker functional given by the integral over the open cell of the Whittaker functionals of the $\Pi_{i,v}$.*

1. *If all the data are unramified, the value of this functional on the spherical vector, normalized by the spherical Whittaker vectors of the $\Pi_{i,v}$, is*
$$\prod_{i<j} L_v(1+s_i-s_j, \Pi_i \times \Pi_j^\vee)^{-1}.$$
2. *In general there is a section supported in the open cell, with compact support there, on which the functional is 1 for all (s_i).*

Proof. The functional is defined on the whole induced representation and is compatible with induction in stages [CS80, Theorems 1.4 and 1.6, Corollary 1.7]. For a principal series with Satake parameters z_a its value on the spherical vector is $\prod_{a<b}(1-q_v^{-1}z_a z_b^{-1})$ [CS80, Theorem 5.4]; the normalization is fixed by the case of rank one, where the integral over $\mathcal{O}_v$ is 1, the next shell contributes $-q_v^{-1}z_1z_2^{-1}$ and the others contribute zero. Writing each $\Pi_{i,v}$ as a subquotient of a principal series and dividing by the factors with a, b in the same block gives (1), first for distinct parameters and then in general because both sides are rational functions. For (2) take a section supported on a small compact open subset of the open cell, with values a vector on which the Whittaker functional of the inducing representation is nonzero, and with the dependence on the unipotent variable chosen to cancel the character. □

6.2 Regularized values of Eisenstein series

Let S be a split GSpin group and P_{S} its Siegel parabolic subgroup, with Levi subgroup $\mathrm{GL}_m \times \mathbb{G}_m$ and unipotent radical $U_{P_{\mathrm{S}}}$. The coefficient $\mathcal{SW}$ is the Whittaker coefficient, along the maximal unipotent subgroup $U_{\mathrm{GL},m}$ of GL_m, of the constant term along P_{S}:
$$\mathcal{SW}(f)(g) = \int_{[U_{\mathrm{GL},m}]} \int_{[U_{P_{\mathrm{S}}}]} f(ung)\psi_m(n)^{-1}\,du\,dn.$$

The Eisenstein series we need has a pole of order equal to the number of blocks, along several hyperplanes at once. The next lemma defines its leading term without choosing an order in which to take residues.

Lemma 6.3 (Regularized Eisenstein value)**.** *Let $E(f,x)$ be a family of Eisenstein series induced from cuspidal data with unitary central character, depending on a parameter x near 0 in a vector space, through unramified twists which are trivial on the centre of the ambient group; so the centre acts on all the members of the family through one unitary character. Suppose that every cuspidal constant term of $E(f,x)$ is a sum of terms of the form $u(x)/\prod_{j\in J} L_j(x)$, with u holomorphic and $J \subset \{1,\dots,N\}$, where $L_1,\dots,L_N$ are distinct linear forms. Put $\delta = \prod_{j=1}^N L_j$.*

1. *$\delta(x)E(f,x)$ is holomorphic at $x=0$.*
2. *Its value $\mathcal{R}(f)$ at 0 depends only on the value of the section f at $x=0$, and $\mathcal{R}$ is equivariant.*
3. *A term of a constant term contributes to the constant term of $\mathcal{R}(f)$ only if $J = \{1,\dots,N\}$.*

Proof. At a fixed level the Eisenstein series is meromorphic in x, and its values at all g have a common denominator locally [BL24, §9.5]; for a holomorphic family of sections, take the

product of the denominators for a basis of the inducing space of the given level, in its compact realization. Suppose that δE has a pole. Choose a smooth point of an irreducible component of its polar divisor which lies on no other component, and at which the leading coefficient of some value of δE is not zero. In a coordinate transverse to the divisor, this leading coefficient, as a function of g, is an automorphic function all of whose cuspidal constant terms vanish, by the hypothesis on the denominators. The centre acts on it through the same unitary character, and it is orthogonal to the cusp forms on which the centre acts through that character: in the region of convergence, the pairing of an Eisenstein series with a cusp form unfolds to an integral of a constant term of the cusp form, which is zero, and this persists by analytic continuation. An automorphic function with these two properties is zero [BL24, §9.1, equation (9.2), and §9.5]. This proves (1). Two sections with the same value at 0 differ by a sum $\sum_i x_i f_i(x)$, on which the regularized value vanishes; and the equivariance of E is an identity of meromorphic functions. This is (2). In (3), a term with $J \neq \{1, \dots, N\}$ becomes, after multiplication by δ, divisible by a linear form and vanishes at 0. □

Lemma 6.4 (Square integrability of the regularized value)**.** *In the situation of Lemma 6.3, where the centre acts through a unitary character, suppose that the exponents of all the terms contributing to the constant terms of $\mathcal{R}(f)$ are strictly negative on the relevant cones of reduction theory. Then the image of $\mathcal{R}$ consists of functions that are square integrable modulo the centre. If in addition the coefficient $\mathcal{SW}$ of $E(f,x)$ is $u(x)/\delta(x)$ with u holomorphic and $u(0) \neq 0$ for some f, then $\mathcal{R} \neq 0$, and $\mathcal{SW}(\mathcal{R}(f)) = u(0)$.*

Proof. The first statement is the criterion for square integrability in terms of constant terms. A function of finite level agrees with its constant terms far in the cusps [Mor82, Chapter I, §5.9], so a strictly negative exponent bounds the square norm on a Siegel set by a sum $C \sum_{d \in \mathbb{Z}^r_{\geq 0}} (1 + \|d\|)^a q^{-\epsilon \|d\|}$ with $\epsilon > 0$. The coefficient $\mathcal{SW}$ is an integral over a compact set, so it commutes with multiplication by δ and evaluation at 0. □

Corollary 6.5 (Irreducibility)**.** *Suppose moreover that at every place the local induced representation at $x = 0$ has an irreducible cosocle J_v, that is, a unique maximal proper subrepresentation with quotient J_v, and that at almost all places the spherical vector has nonzero image in it. Then the image of $\mathcal{R}$ is the irreducible representation $\bigotimes'_v J_v$.*

Proof. The image is a nonzero quotient of the restricted tensor product of the local induced representations, and it is unitary, being square integrable. Apply Proposition I.9.3. □

6.3 The residue attached to self-dual blocks

Proposition 6.6 (The residual representation)**.** *Let $m = \sum_i m_i$, and let Π_i be the distinct cuspidal representations described at the beginning of the section. Let $S = \mathrm{GSpin}_{2m}$ if the blocks are alternating and $S = \mathrm{GSpin}_{2m+1}$ if they are symmetric, and consider the Eisenstein series $E(f_t, t)$ on S induced from $\boxtimes_i \Pi_i |\det|^{t/2} \boxtimes \omega_\sigma$ on the parabolic subgroup contained in P_{S} with Levi subgroup $\prod_i \mathrm{GL}_{m_i} \times \mathbb{G}_m$. Let N be the number of blocks.*

1. *The limit $\mathcal{R}(f) = \lim_{t \to 1} (t-1)^N E(f_t, t)$ exists, and $\mathcal{R}$ is a nonzero equivariant map to functions that are square integrable modulo the centre.*
2. *The image of $\mathcal{R}$ is irreducible, and the connected centre acts on it through ω_σ.*
3. *The coefficient $\mathcal{SW}$ is not identically zero on the image of $\mathcal{R}$.*

Proof. Let S_{ram} be a finite set of places containing those where some Π_i or ψ is ramified, and put

$$L_i(z) = L^{S_{\mathrm{ram}}}(z, \Pi_i, r \otimes \omega_\sigma^{-1}), \qquad L_{ij}(z) = L^{S_{\mathrm{ram}}}(z, \Pi_i \times \Pi_j \otimes \omega_\sigma^{-1}),$$

where $r = \Lambda^2$ in the alternating case and $r = \mathrm{Sym}^2$ in the symmetric case. These are the L-functions of the representations of the dual Levi subgroup on the Lie algebra of the dual Siegel radical, whose weights are $z_a z_b/c$ with $a < b$, respectively $a \le b$.

The twists $|\det|^{s_i}$ which occur below are trivial on the centre of S, as Lemma 6.3 requires. Write an element of the maximal torus of the Levi subgroup $\mathrm{GL}_m \times \mathbb{G}_m$ as $(t_1, \dots, t_m; t_0)$. In type B_m the roots e_i vanish on the centre, which forces $t_i = 1$. In type D_m the roots $e_i \pm e_j$ force $t_1 = \cdots = t_m = \varepsilon$ with $\varepsilon^2 = 1$. In both cases $|\det|$ is one on the centre of each block, and the centre of S acts through the character $t \mapsto \omega_\sigma(t_0) \prod_i \omega_{\Pi_i}(\varepsilon)$, where ω_{Π_i} is the central character of Π_i and $\varepsilon = 1$ in type B. It does not depend on the twists, and it is unitary, because ω_σ has finite order and $\varepsilon^2 = 1$.

Step 1: the poles. We first let the exponents vary independently, inducing from $\boxtimes_i \Pi_i |\det|^{s_i}$. The constant terms are sums of intertwining operators, products of operators of rank one. The unramified parts of the latter are

$$\frac{L_{ij}(s_i + s_j)}{L_{ij}(1 + s_i + s_j)}, \qquad \frac{L_i(2s_i)}{L_i(1 + 2s_i)},$$

and similar ratios in the differences $s_i - s_j$. By Lemma 5.4 and the remark following it,

$$a_i = \operatorname*{Res}_{z=1} L_i(z) \neq 0, \qquad L_i(2) L_{ij}(1) L_{ij}(2) \neq 0 \quad (i \neq j).$$

We need the operators of rank one only near the real point at which all the s_i are equal to $1/2$; there the differences are 0 and the sums are 1. An operator in a difference is evaluated at the real point of the unitary axis, where the operators of rank one are holomorphic [Mor82, Chapter III, Theorem 5.2(i)]. Consider an operator in a sum $s_i + s_j$ with $i \neq j$, as a function of $z = s_i + s_j$. Its unramified part $L_{ij}(z)/L_{ij}(1 + z)$ is holomorphic at $z = 1$, by Lemma 5.4 and the remark which follows it; this is where $\Pi_j^\vee \otimes \omega_\sigma \simeq \Pi_j \not\simeq \Pi_i$ is used. At the places of S_{ram} the local operators are given by integrals which converge for $\Re z > 0$, because the representations $\Pi_{i,v}$ are tempered (Input 1(3)) [Wal03, Proposition IV.2.1]. So this operator is holomorphic at $z = 1$. The remaining operators, in $2s_i$, have at most simple poles [Mor82, Chapter III, Theorem 5.2(i)–(ii)]. So the hypothesis of Lemma 6.3 holds with the N linear forms $2s_i - 1$, and restricting to $s_i = t/2$ gives the existence of $\mathcal{R}(f) = \lim_{t \to 1} (t-1)^N E(f_t, t)$.

Step 2: the exponents. We verify the hypothesis of Lemma 6.4. As explained at the end of this step, it suffices to consider the cuspidal components of the constant terms along the standard parabolic subgroups $P' = M'U'$. Let M be the inducing Levi subgroup. Since the inducing datum is cuspidal, the cuspidal component of the constant term along P' is a sum over the Weyl elements w with $wMw^{-1} = M'$ [BL24, (6.17) and (6.18)]; the terms induced from smaller Levi subgroups of M' are orthogonal to cusp forms of M'. By Lemma 6.3(3) a Weyl element w contributes to a remaining constant term only if it produces all N poles. Factor the intertwining operator of w along a reduced sequence of reflections in relative roots. The factors attached to two different blocks are holomorphic at the point, and the factor attached to the relative root β_i of the ith block which is fixed by the inducing datum has at most a simple pole; so w must make every β_i negative. The coroots which restrict to β_i on the centre of the inducing Levi subgroup are the $e_a + e_b$ with a, b in the block i, including $a = b$ when the block is of symmetric-square type. For such a coroot $\alpha^\vee$, the restriction of $w\alpha^\vee$ to the centre of $M' = wMw^{-1}$ is the relative root $w\beta_i$, which is negative. The coroots outside M' which are positive are those of the unipotent radical of P', and their restrictions to the centre of M' are positive relative roots; so $w\alpha^\vee$ is negative. Thus w makes negative all the coroots $e_a + e_b$ with a, b in the same block. Let

$$\gamma_i = \frac{2}{m_i} \sum_{a \text{ in block } i} e_a, \qquad \lambda = \frac{1}{2} \sum_{a=1}^{m} e_a = \sum_i \frac{m_i}{4} \gamma_i,$$

so that λ is the exponent at $t=1$ and the γ_i span the space of exponents fixed by the Weyl group of the inducing Levi subgroup, modulo the central direction of the ambient group. Each γ_i is a positive combination of these coroots:

$$\gamma_i = \frac{2}{m_i(m_i-1)} \sum_{a<b} (e_a + e_b) \qquad \text{or} \qquad \gamma_i = \frac{2}{m_i(m_i+1)} \sum_{a \le b} (e_a + e_b),$$

with a, b in the block i, according to whether the block is of alternating-square type, where $m_i \ge 2$, or of symmetric-square type; each coordinate occurs $m_i - 1$ times in the first sum and $m_i + 1$ times in the second. For a contributing w, each $w\gamma_i$ is therefore a positive combination of negative coroots, hence a combination with coefficients ≤ 0 of the simple coroots. Consider a simple root that does not belong to the Levi subgroup of the constant term. The coefficient of its coroot in $w\lambda$ is a sum of nonpositive numbers, and it is not zero: otherwise the coefficient would vanish in every $w\gamma_i$, whereas the $w\gamma_i$ span the space of exponents fixed by the Weyl group of M', modulo the central direction of the ambient group. For an inner product on the real space of coroots which is invariant under the Weyl group, this space is the orthogonal complement of the span of the coroots of M', and it maps isomorphically onto the quotient by that span. So the coefficient of a simple coroot outside M' does not vanish on the whole space. Hence $\langle w\lambda, d\rangle < 0$ for all $d \ne 0$ in the positive cone, which is the negativity to be proved; the half-sum of roots in the normalized induction cancels the modulus character in the integral of the square. It suffices to consider cuspidal constant terms: an exponent of an arbitrary constant term is the restriction of an exponent of a cuspidal constant term along a smaller parabolic subgroup, by the transitivity of constant terms, and the positive cone of the larger parabolic subgroup lies in the closed positive cone of the smaller. The regularized value is a Laurent coefficient, so its constant terms can contain polynomials in the degree multiplied by the same exponentials; a polynomial does not affect the convergence of the sum in the proof of Lemma 6.4, in which the factor $(1 + \|d\|)^a$ already appears.

Step 3: the coefficient $\mathcal{SW}$. This is the main point. In the constant term along P_{S}, only the terms with wMw^{-1} contained in the Siegel Levi subgroup remain, because the inducing datum is cuspidal. For such a term the roots which are differences of two indices of one block force all the indices of the block to have the same sign. If the sign of a block is positive, the roots which are sums of two of its indices remain positive, and the block contributes no pole. So a term which produces all the poles changes the sign of every block. Modulo the Weyl group of the Siegel Levi subgroup there is one such term, the one making the whole Siegel radical negative. Its term is a function on GL_m, induced from $\boxtimes_i \Pi_i$ with shifted exponents, and its Whittaker coefficient is computed on the Bruhat cells of GL_m: on a cell other than the open one the character ψ_m is nontrivial on a simple root subgroup contained in the stabilizer, so that cell contributes zero. On the open cell, at the places outside S_{ram} and for spherical data, the intertwining operator and Lemma 6.2(1) give the factor

$$\prod_i \frac{L_i(t)}{L_i(t+1)} \prod_{i<j} \frac{L_{ij}(t)}{L_{ij}(t+1)L_{ij}(1)},$$

and $(t-1)^N$ times this tends to $\prod_i a_i L_i(2)^{-1} \prod_{i<j} L_{ij}(2)^{-1} \ne 0$.

At a place of S_{ram} we choose the section so that the local factor is 1 for all values of the parameter. The cell in question is the product of the opposite Siegel radical, with coordinate z_P, and of the open cell of the Whittaker functional in GL_m, with coordinate z_A. The corresponding Weyl elements, whose inverses are the representatives which occur in the integral, satisfy

$$w_P = w_{0,S} w_{0,L_{\mathrm{S}}}, \qquad w_A = w_{0,M} w_{0,L_{\mathrm{S}}}, \qquad w_A w_P = -w_{0,M},$$

with additive lengths, where M is the inducing Levi subgroup and $w_{0,S} = -1$; in the even orthogonal case this holds because m is even. In the two successive integrations the product of the representatives is $w_P^{-1} u_P w_A^{-1} u_A = (w_A w_P)^{-1}(w_A u_P w_A^{-1}) u_A$. The root coordinates of the two unipotent factors are disjoint, and together they parametrize the open cell for the inducing parabolic subgroup and the Borel subgroup of the ambient group. Choose a small compact open neighbourhood Ω_v of the identity in these coordinates and a vector in the inducing representation whose Whittaker function is 1 at the identity, and let the section be $\mathrm{vol}(\Omega_v)^{-1}\psi_A(z_A)$ times this vector on Ω_v, extended by zero. On Ω_v the Levi components in the Iwasawa decomposition are units, so the exponents do not enter, and the local integral is 1.

Thus for suitable f the coefficient $\mathcal{SW}(E(f_t, t))$ is $u(t)/(t-1)^N$ with $u(1) \neq 0$, and Lemma 6.4 gives (1) and (3).

Step 4: irreducibility. Each $\Pi_{i,v}$ is tempered [Laf02, Theorem VI.10(i)]. The representation of $\mathrm{GL}_m(F_v)$ induced from $\boxtimes_i \Pi_{i,v}$ is irreducible and tempered: in terms of segments, those attached to tempered representations are centred on the unitary axis, and two such segments on the same cuspidal line are nested, hence not linked [Zel80, Theorem 9.7(a)]. By induction in stages the local induced representation at $t = 1$ is induced from the Siegel parabolic subgroup, from this tempered representation twisted by a positive exponent. By the Langlands classification it is a standard module, and a standard module has an irreducible cosocle [Sol25, Theorem A and §2]; at an unramified place the image of the spherical vector in it is nonzero. Corollary 6.5 gives (2); the central character is that of the inducing data. □

Remark 6.7. Purity of the blocks enters through the poles in step 1. Square integrability comes from the signs of the exponents in step 2. The test vectors at the ramified places in step 3 are needed because the Euler product over the unramified places says nothing about the remaining local factors of the coefficient.

7 Orthogonal descent and its local parameter

The residual representation of Proposition 6.6 lives on a special orthogonal group, or on the corresponding general spin group GSpin (an extension of the special orthogonal group by $\mathbb{G}_m$), about twice as large as the group we want. The descent is a Fourier coefficient of it along a unipotent subgroup U_{desc}, for a character ψ_{desc} whose stabilizer H in the Levi subgroup is the target group. This section does three things. It computes the twisted Jacquet module of the local components for $(U_{\mathrm{desc}}, \psi_{\mathrm{desc}})$ (Proposition 7.1), which gives the Hecke eigenvalues of the descent if it is nonzero. It proves two lemmas on rearranging Fourier coefficients (Lemmas 7.8 and 7.9). And it proves that the descent to an odd orthogonal group is nonzero, with a nonzero Whittaker coefficient, in every characteristic (Proposition 7.10). The descent to an even orthogonal group is treated in Sections 8 and 9.

Two features of the local computation are used later. In the even orthogonal case the Jacquet module has two summands, exchanged by the outer automorphism. And it vanishes when the stabilizer is not split, so that the stabilizer of a nonzero global descent is split at every unramified place.

7.1 The descent datum

Write a split even quadratic space as

$$V = \bigoplus_{i=1}^{m} (Fe_i \oplus Ff_i), \qquad Q\Big(\sum x_i e_i + y_i f_i\Big) = \sum x_i y_i,$$

with root vectors

$$E_{e_i-e_j} = E_{ij} - E_{f_jf_i}, \qquad E_{e_i+e_j} = E_{if_j} - E_{jf_i}, \qquad F_\alpha = E_\alpha^t.$$

In odd dimension there is an additional basis vector u, orthogonal to the others; its norm is 1 in characteristic two, and a convenient nonzero constant otherwise. Orthogonal groups are the smooth special orthogonal groups. In even dimension this is the kernel of the Dickson invariant on the orthogonal group of the quadratic form; outside characteristic two it is the kernel of the determinant. In odd dimension it is the kernel of the determinant $\mathrm{O}(Q) \to \mu_2$ in every characteristic; in characteristic two the bilinear form of an odd-dimensional space has the line of u as radical, and the quadratic form is not zero on it. The rank of $g-1$ modulo two, which is the Dickson invariant in even dimension, cannot be used in odd dimension: in characteristic two a short root element of the odd orthogonal group differs from the identity by a matrix of rank one. All unipotent groups below are the same for the orthogonal group and for GSpin.

In both configurations $r \geq 1$, and U_{desc} is the unipotent radical of the parabolic subgroup fixing the flag $\langle e_1\rangle \subset \cdots \subset \langle e_1, \ldots, e_r\rangle$, whose Levi subgroup is the product of a torus of rank r and the orthogonal group of the space spanned by the remaining basis vectors. The character ψ_{desc} is ψ of the sum of the coordinates on the simple roots $e_i - e_{i+1}$, $i < r$, plus a linear form λ of the last row, that is, of the component of U_{desc} in the root subgroups from e_r into the remaining space.

First configuration. Here $m = 2r+2$ and the ambient group has type D_m. The form λ has coefficients $(-1, 1)$ on $(e_r - e_m, e_r + e_m)$ and zero elsewhere; it is the pairing with $v = e_m - f_m$. Let H be the subgroup of the Levi subgroup whose torus component is trivial and whose orthogonal component fixes v. It stabilizes ψ_{desc}, and it is the special orthogonal group of the orthogonal complement of v in the remaining space, a split group of type B_{r+1}; in characteristic two this identification is explained below. The full stabilizer of ψ_{desc} in the Levi subgroup is slightly larger: the simple root coordinates force the r torus coordinates to have a common value a, the last row gives $gv = av$, and then $a^2 = 1$ because $Q(v) \neq 0$; so the stabilizer is $\mu_2 \times H$. Outside characteristic two H is its identity component. In characteristic two μ_2 is connected and not reduced, and H is the reduced identity component of the stabilizer.

Second configuration. Here $m = 2r$ and the ambient group has type B_m. We assume

$$\lambda = B(v, \cdot) \text{ with } Q(v) \neq 0 \quad (p \neq 2), \qquad \lambda(u) = 1 \quad (p = 2).$$

Then $V_\lambda = \ker \lambda$ is a nondegenerate even-dimensional quadratic space and $H = \mathrm{SO}(V_\lambda)$ has type D_r. It need not be split.

When the ambient group is GSpin, we write H also for the inverse image of this group.

In characteristic two the description of H in the first configuration requires an argument, because v lies in its own orthogonal complement. Put $v = e_m + f_m$ and $u_0 = f_m$, and write the remaining space as $Fv \oplus V_{\mathrm{mid}} \oplus Fu_0$, so that

$$Q(xv + w + yu_0) = x^2 + xy + Q_{\mathrm{mid}}(w).$$

An isometry of $v^\perp = Fv \oplus V_{\mathrm{mid}}$ fixing v has the form $w + xv \mapsto kw + (x + c(w))v$, with k symplectic for the bilinear form of V_{mid} and $c(w)^2 = Q_{\mathrm{mid}}(kw) + Q_{\mathrm{mid}}(w)$. Its extensions to the whole space send

$$u_0 \longmapsto u_0 + b + Av, \qquad B(kw, b) = c(w), \qquad A^2 + A = Q_{\mathrm{mid}}(b),$$

over every base ring R. The equation for A need not have a solution in R; it has one in the R-algebra $R' = R[A]/(A^2 + A - Q_{\mathrm{mid}}(b))$, which is finite étale and faithfully flat. Over R'

we thus have an extension. The reflection $x \mapsto x + B(x,v)v$ changes A into $A+1$ and has Dickson invariant one; composing with it on the open and closed part of $\operatorname{Spec} R'$ where the Dickson invariant of the extension is one, we get an extension of Dickson invariant zero. It is unique: the difference T of two values of A satisfies $T^2+T=0$, so it is an idempotent, and the two extensions differ by the reflection on the part where $T=1$. By uniqueness the two pullbacks of this extension to $R' \otimes_R R'$ coincide, and it descends to R. Thus restriction to $v^\perp$ identifies H with the special orthogonal group of the nondegenerate odd-dimensional quadratic space $v^\perp$, the split group SO_{2r+3}, as group schemes. On a remaining hyperbolic pair its short root subgroup is

$$h_j(s) = x_{e_j-e_m}(s)x_{e_j+e_m}(s), \qquad h_j(s)f_j = f_j + sv + s^2 e_j,$$

on which the torus acts through z_j.

7.2 The local Jacquet module

Let P' be the parabolic subgroup of the ambient group whose Levi subgroup consists of blocks GL_2 on consecutive pairs of basis vectors, n in number, where $n = r+1$ in the first configuration and $n=r$ in the second. In the first configuration there are two such parabolic subgroups P'_+ and P'_-, corresponding to the two families of maximal isotropic subspaces. For characters $\mu_1, \dots, \mu_n$ and ω_σ of the multiplicative group of the local field put

$$I_{P'}(\boldsymbol{\mu}; \omega_\sigma) = i_{P'}^G \left(\boxtimes_j \mu_j \circ \det \boxtimes \omega_\sigma \right),$$

where the last factor is a character of the central $\mathbb{G}_m$ of GSpin, which is a factor of the Levi subgroup, and is omitted for orthogonal groups. Let $z_1, \dots, z_n$ be the coordinates of the maximal torus of H and z_0 the coordinate of the central $\mathbb{G}_m$ of GSpin, that is, of the cocharacter e_0^*. In the dual coordinates the similitude character of GSpin is $2e_0 + \sum_i e_i$; the multiplier of the dual group GSp or GSO is the character e_0^* of the dual torus. Define characters of that torus by

$$\xi_\mu(z) = \omega_\sigma(z_0) \prod_{j=1}^n \mu_j(z_j), \qquad \xi_\mu^\iota(z) = \omega_\sigma(z_0) \prod_{j<n} \mu_j(z_j)(\omega_\sigma \mu_n^{-1})(z_n).$$

Proposition 7.1 (Orthogonal Jacquet module)**.** *Over any nonarchimedean local field and for arbitrary smooth characters μ_j and ω_σ,*

$$J_{U_{\mathrm{desc}}, \psi_{\mathrm{desc}}} I_{P'}(\boldsymbol{\mu}; \omega_\sigma) \simeq \begin{cases} i_{B_H}^H \xi_\mu, & \text{ambient type } D_{2r+2}, \\ i_{B_H}^H \xi_\mu \oplus i_{B_H}^H \xi_\mu^\iota, & \text{ambient type } B_{2r},\ V_\lambda \text{ split}, \\ 0, & \text{ambient type } B_{2r},\ V_\lambda \text{ nonsplit}. \end{cases}$$

In the first case the result is the same for P'_+ and for P'_-.

The proof occupies the rest of this subsection. It applies Lemma 5.10 with $P = MU_{\mathrm{desc}}$, the parabolic subgroup of which U_{desc} is the radical, and it has four parts. Only one double coset can contribute (Lemma 7.2). In this double coset the intersection of U_{desc} with the conjugate of P' is a product of root subgroups, which we list (Lemma 7.3). The locus on which the character is trivial on this intersection is a flag variety of H (Lemma 7.4); this is the main point. The characters given by Lemma 5.10 are ξ_μ and ξ_μ^ι (Lemma 7.5). In the first configuration we take $P' = P'_+$ in these four lemmas, and compare with P'_- afterwards (Lemma 7.6).

The character at a point of the Levi subgroup. The roots of U_{desc} are

$$e_i - e_j, \quad e_i + e_j \quad (1 \le i \le r,\ i < j \le m), \qquad e_i \quad (1 \le i \le r, \text{ ambient type } B). \tag{7.1}$$

Let W be the remaining space, spanned by the e_h, f_h with $h > r$ and by u in odd dimension, so that $M = \mathbb{G}_m^r \times G_W$ with G_W the special orthogonal group of W, or its inverse image in GSpin. The elements of the last row are the $n(y)$, $y \in W$, with $n(y)f_r \equiv f_r + y$ modulo e_r; on them $\psi_{\mathrm{desc}}(n(y)) = \psi(\lambda(y))$, and the root subgroups are

$$x_{e_r - e_h}(t) = n(-tf_h), \qquad x_{e_r + e_h}(t) = n(-te_h),$$

and in odd dimension the short root subgroup U_{e_r} consists of the $n(tu)$. Let γ be an element of M, with components $(a_1, \dots, a_r)$ and $g \in G_W$. Put $\psi^\gamma(n) = \psi_{\mathrm{desc}}(\gamma^{-1} n \gamma)$. Since $\gamma^{-1} n(y) \gamma = n(a_r^{-1} g^{-1} y)$, this is the character of U_{desc} of the same kind as ψ_{desc} in which the coefficients on the simple roots $e_i - e_{i+1}$, $i < r$, are multiplied by nonzero scalars and λ is replaced by $\lambda' = a_r^{-1} \lambda \circ g^{-1}$. It is trivial on every other root subgroup of (7.1). We shall use two properties of λ', valid for every γ.

(a) If $\lambda = B(v, \cdot)$, then $\lambda' = B(v', \cdot)$ with $v' = a_r^{-1} g v$, and $Q(v') = a_r^{-2} Q(v) \neq 0$. Writing $v' = \sum_{h>r} (x_h e_h + y_h f_h) + zu$, the coefficients of ψ^γ on $U_{e_r - e_h}$ and $U_{e_r + e_h}$ are $-x_h$ and $-y_h$, and $Q(v') = \sum_h x_h y_h + Q(u) z^2$.

(b) In the second configuration in characteristic two, $\lambda'(u) \neq 0$: the line of u is the radical of the bilinear form of W and $Q(u) \neq 0$, so g fixes u, and $\lambda'(u) = a_r^{-1} \lambda(u)$.

The condition of Lemma 5.10 at the point γ of the double coset of w is that ψ^γ be trivial on $U_w = U_{\mathrm{desc}} \cap w^{-1} P' w$. This group is the product of the root subgroups U_α, α in (7.1), for which $w\alpha$ is a root of P', that is, a positive root or a root $\pm(e_{2c-1} - e_{2c})$ of a block. So the condition is:

$$\text{if } w\alpha \text{ is a root of } P', \text{ the coefficient of } \psi^\gamma \text{ on } U_\alpha \text{ is zero.} \tag{7.2}$$

This is a condition on root subgroups and on the values of ψ at their points, and the arguments below are the same in every characteristic. The inducing character is trivial on all unipotent elements of P', including those of the blocks, so the μ_j do not enter the condition.

Lemma 7.2 (The contributing double coset)**.** *Let w be a signed permutation, and suppose that (7.2) holds at some point γ of M. Then w lies in the double coset of the element w_0 with*

$$w_0(e_i) = f_{2i-1}, \qquad w_0(e_{r+k}) = e_{2k} \quad (1 \le i, k \le r),$$

extended to the f_i by duality. In the first configuration w_0 fixes e_{m-1}, and it exchanges e_m and f_m exactly when r is odd; in the second configuration $w_0(u) = (-1)^r u$.

Proof. For $i \le r$ write

$$w(e_i) = e_{j_i} \text{ (sign +)} \quad \text{or} \quad w(e_i) = f_{j_i} \text{ (sign −)}, \qquad A_i = \lceil j_i/2 \rceil,$$

so that A_i is the block of P' containing the image.

Step 1: the sign of e_r is negative. Suppose $w(e_r) = e_j$. Let $h > r$. If $w(e_h) = e_l$, then $w(e_r + e_h) = e_j + e_l$ is a positive root, so $y_h = 0$. If $w(e_h) = f_l$, then $w(e_r - e_h) = e_j + e_l$, so $x_h = 0$. Hence $x_h y_h = 0$ for all h. In even dimension this gives $Q(v') = 0$, which contradicts (a). In odd dimension $w(e_r) = e_j$ is a positive short root, so $\lambda'(u) = 0$. In odd characteristic $\lambda'(u) = 2Q(u)z$, hence $z = 0$ and again $Q(v') = 0$. In characteristic two $\lambda'(u) \neq 0$ by (b), a contradiction.

Step 2: all signs are negative. If the signs of e_i and e_{i+1} are $(+,-)$, then $w(e_i - e_{i+1}) = e_{j_i} + e_{j_{i+1}}$ is a positive root, while the coefficient of ψ^γ on the simple root $e_i - e_{i+1}$ is not zero. So a positive sign cannot be followed by a negative one; since the last sign is negative, all are.

Step 3: the blocks. For $i < r$ we now have $w(e_i - e_{i+1}) = e_{j_{i+1}} - e_{j_i}$. If $A_{i+1} < A_i$, this is a positive root; if $A_{i+1} = A_i$, it is a root of a block. Both are excluded by (7.2), the coefficient on a simple root being nonzero. Hence

$$A_1 < A_2 < \cdots < A_r. \tag{7.3}$$

In the second configuration there are r blocks, so $A_i = i$. In the first there are $r+1$, and (7.3) leaves the possibility $A_r = r+1$. Suppose this is the case, and let $h > r$, $w(e_h) = e_l$ or f_l. Then $l \neq j_r$ and $\lceil l/2 \rceil \leq r+1 = A_r$, so $e_l - e_{j_r}$ is a root of P': a positive root if l lies in an earlier block than j_r, a root of the last block otherwise. It is the image of $e_r + e_h$ if $w(e_h) = e_l$ and of $e_r - e_h$ if $w(e_h) = f_l$. So again $x_h y_h = 0$ for every h and $Q(v') = 0$, a contradiction. Therefore $A_i = i$ for all i in both configurations.

Step 4: the representative. Multiplying w on the left by the Weyl group of the Levi subgroup of P', which exchanges the two indices of a block, we may assume $j_i = 2i-1$. The images of the e_h, $h > r$, are then, up to signs, the e_{2k} with $k \leq r$ and, in the first configuration, e_{m-1} and e_m. Multiplying on the right by the Weyl group of G_W, which consists of the signed permutations of the indices $h > r$, with an even number of sign changes when W has even dimension, we arrive at $w(e_i) = f_{2i-1}$ and $w(e_{r+k}) = e_{2k}$ for $1 \leq i, k \leq r$. In the first configuration we arrive in addition at $w(e_{m-1}) = e_{m-1}$, and then w exchanges e_m and f_m exactly when r is odd, since the number of sign changes of an element of the Weyl group of type D_m is even. In the second configuration $w(u) = (-1)^r u$, the determinant being one. This is w_0. □

Lemma 7.3 (The intersection)**.** *For $w = w_0$ the group $U_w = U_{\mathrm{desc}} \cap w^{-1} P' w$ is*

$$U_{\mathrm{int}} = \prod_{1 \leq k \leq i \leq r} U_{e_i + e_{r+k}},$$

a product of commuting root subgroups.

Proof. We apply w_0 to the list (7.1). In the following table $1 \leq i \leq r$, $i < j \leq r$ and $1 \leq k \leq r$.

root α of U_{desc}	$w_0\alpha$	root of P'
$e_i - e_j$	$e_{2j-1} - e_{2i-1}$	no
$e_i + e_j$	$-e_{2i-1} - e_{2j-1}$	no
$e_i - e_{r+k}$	$-e_{2i-1} - e_{2k}$	no
$e_i + e_{r+k}$, $k < i$	$e_{2k} - e_{2i-1}$	yes, a positive root
$e_i + e_{r+i}$	$e_{2i} - e_{2i-1}$	yes, a root of the ith block
$e_i + e_{r+k}$, $k > i$	$e_{2k} - e_{2i-1}$	no
$e_i \pm e_{m-1}$, $e_i \pm e_m$ (type D)	$-e_{2i-1} \pm e_{m-1}$, $-e_{2i-1} \pm e_m$	no
e_i (type B)	$-e_{2i-1}$	no

The roots marked "no" are negative, and none of them is a root $\pm(e_{2c-1} - e_{2c})$ of a block: in $e_{2j-1} - e_{2i-1}$ both indices are odd, in $e_{2k} - e_{2i-1}$ with $k > i$ the indices lie in different blocks, and in type D we have $2i - 1 < m - 1$. This gives U_{int}. The sum of two roots $e_i + e_{r+k}$ is not a root, so the root subgroups commute. □

Lemma 7.4 (The locus)**.** *Let $w = w_0$, and put $Z_k^0 = \langle e_{r+1}, \ldots, e_{r+k} \rangle$ for $k \leq r$.*

1. *The quotient $M_w\backslash M$ is the variety of isotropic flags $Z_1 \subset \cdots \subset Z_r$ in W with $\dim Z_k = k$, completed in the first configuration by a maximal isotropic subspace $Z_{r+2} \supset Z_r$ in a fixed family. The class of γ corresponds to the flag $(\gamma^{-1}Z_k^0)$.*
2. *The condition (7.2) at γ is $\lambda(Z_r) = 0$.*
3. *In the first configuration the locus defined by this condition is one orbit of H, isomorphic to $B_H\backslash H$ for a Borel subgroup B_H of H.*
4. *In the second configuration the locus is empty if V_λ is not split. If V_λ is split, it consists of two orbits of H, each isomorphic to $B_H\backslash H$ with the same Borel subgroup B_H.*

In all cases the locus is closed and the orbit maps have local sections. As base points we take $\gamma_1 = 1$, with the coordinates of W chosen so that $Z_r^0 \subset \ker\lambda$, and for the second orbit in (4) an element $\gamma_2 \in G_W$ which normalizes the torus, exchanges e_{2r} and f_{2r} and fixes the other basis vectors of V_λ.

Proof. Step 1: the flag variety. The group $M_w = M \cap w^{-1}P'w$ contains the torus $\mathbb{G}_m^r$, and its component in G_W is the parabolic subgroup whose roots β are those with $w\beta$ a root of P'. From $w(e_{r+k} - e_{r+k'}) = e_{2k} - e_{2k'}$ and $w(e_{r+k} + e_{r+k'}) = e_{2k} + e_{2k'}$, and the corresponding formulas for e_{m-1}, e_m and u, one finds that it is the stabilizer of the flag

$$Z_1^0 \subset \cdots \subset Z_r^0,$$

completed in the first configuration by the maximal isotropic subspace Z_{r+2}^0 spanned by Z_r^0, e_{m-1} and $w^{-1}e_m$. So $M_w\backslash M$ is the variety of isotropic flags $(Z_k) = \gamma^{-1}(Z_k^0)$ in W of dimensions $1, \dots, r$, followed in the first configuration by a maximal isotropic subspace Z_{r+2} in the family of Z_{r+2}^0. This is (1).

Step 2: the condition. By Lemma 7.3 the condition is that ψ^γ be trivial on the subgroups $U_{e_i+e_{r+k}}$ with $k \le i \le r$. The character ψ^γ is trivial on $U_{e_i+e_{r+k}}$ for $i < r$, and on $x_{e_r+e_{r+k}}(t) = n(-te_{r+k})$ it is $\psi(-t\lambda'(e_{r+k}))$. So the condition is $\lambda'(Z_r^0) = 0$, that is, $\lambda(Z_r) = 0$. This is (2).

Step 3: the first configuration. The condition says $Z_r \subset v^\perp$. Then $Y = Z_{r+2} \cap v^\perp$ has dimension $r+1$: it contains Z_r, and it is not Z_{r+2} because v, having nonzero norm, is not in $Z_{r+2}^\perp = Z_{r+2}$. So $(Z_1, \dots, Z_r, Y)$ is a complete isotropic flag of the space $v^\perp$ of dimension $2r+3$. Conversely, for such a flag the space $Y^\perp/Y$ is a split quadratic plane; its two isotropic lines give the two maximal isotropic subspaces of W containing Y, one in each family, and Z_{r+2} is the one in the prescribed family. So the locus is the variety of complete isotropic flags of $v^\perp$, on whose rational points H acts transitively with stabilizer a Borel subgroup: it is one orbit $B_H\backslash H$. These statements hold in characteristic two, where $v \in v^\perp$ and isotropic means that the quadratic form vanishes.

Step 4: the second configuration. The locus is the variety of complete isotropic flags of length r in W contained in $V_\lambda = \ker\lambda$, a space of dimension $2r$. It has no rational point if V_λ is not split. If V_λ is split, the stabilizer in H of $(Z_1, \dots, Z_{r-1})$ is a Borel subgroup B_H, and there are two choices of Z_r, one in each family of maximal isotropic subspaces. So the locus consists of two orbits of H, each isomorphic to $B_H\backslash H$ with the same B_H. In characteristic two the decomposition $W = Fu \oplus V_\lambda$, which exists because $\lambda(u) = 1$, gives the same description.

Step 5: local sections. The locus is defined by the closed condition $\lambda(Z_r) = 0$, and the orbit maps have local sections, given by the big cells. The elements γ_1 and γ_2 are points of the two orbits. □

Lemma 7.5 (The characters)**.** *Suppose that the locus of Lemma 7.4 is not empty, and let χ be the inducing character of $I_{P'}(\boldsymbol{\mu};\omega_\sigma)$. For the base point γ_j put $U_j = U_{\mathrm{desc}} \cap (w_0\gamma_j)^{-1}P'w_0\gamma_j$. Then, on the maximal torus of H,*

$$(\chi\delta_{P'}^{1/2}) \circ \mathrm{Ad}(w_0\gamma_j) \cdot \left|\det \mathrm{Ad}(\cdot)|_{\mathrm{Lie}\, U_{\mathrm{desc}}/\,\mathrm{Lie}\, U_j}\right| = \begin{cases} \xi_\mu\, \delta_{B_H}^{1/2}, & j = 1, \\ \xi_\mu^\iota\, \delta_{B_H}^{1/2}, & j = 2. \end{cases}$$

Proof. Step 1: the modulus characters. Let $\epsilon = 0$ in ambient type D and $\epsilon = 1$ in ambient type B. On the kth remaining pair (e_{r+k}, f_{r+k}) the exponents of $|z_k|$ in the half modulus of P', in the modulus of U_{int} and in the half modulus of B_H are

$$m - 2k + \frac{1+\epsilon}{2}, \qquad r - k + 1, \qquad m - r - k + \frac{\epsilon - 1}{2}.$$

The first is the coefficient $m - j + \epsilon/2$ of the half sum of positive roots of the ambient group at $j = 2k$, minus the contribution $-1/2$ of the block. The second counts the $r - k + 1$ roots $e_i + e_{r+k}$ in U_{int}; the modulus of the whole of U_{desc} is trivial on H because the weights z_k and z_k^{-1} occur in pairs. The first minus the second is the third. For the last coordinate in the first configuration the three exponents are $1/2, 0, 1/2$. So at $\gamma_1 = 1$, where $U_1 = U_{\mathrm{int}}$, the two modulus factors on the left combine to $\delta_{B_H}^{1/2}$.

Step 2: the inducing characters. Under w_0 the determinant of the kth block restricts to z_k on the torus of H. For GSpin, with coroots $e_i^* - e_j^*$ and $e_i^* + e_j^* - e_0^*$, the coordinates made negative by w_0 are the first r, and in the first configuration for odd r also the last one, since w_0 then exchanges e_m and f_m. All of them are trivial on H, so the central coordinate restricts to z_0. This gives ξ_μ at γ_1. At γ_2 the left side is evaluated at $\gamma_2 z \gamma_2^{-1}$, which is z with z_n inverted and, for GSpin, with z_0 multiplied by z_n. The half modulus of B_H does not involve z_n, the exponent $m - r - k + (\epsilon - 1)/2$ being zero for $k = r = n$; so step 1 applies without change, and the inducing character becomes ξ_μ^ι. □

Lemma 7.6 (The two parabolic subgroups)**.** *In the first configuration there is an automorphism of the ambient group which preserves U_{desc} and ψ_{desc}, is the identity on H, and carries P'_+ and its inducing character to P'_- and its inducing character.*

Proof. Consider the isometry which exchanges e_m and f_m and acts by -1 on the first r hyperbolic pairs. The exchange of e_m and f_m is the reflection in v. It fixes the vectors of $v^\perp$, so it commutes with H, and it changes the sign of λ. The sign -1 on the first r pairs commutes with H and restores the sign of λ without affecting the coordinates on the roots $e_i - e_{i+1}$, $i < r$. So the isometry preserves ψ_{desc} and commutes with H. The exchange of e_m and f_m also exchanges the two families of maximal isotropic subspaces, hence P'_+ and P'_-, and it carries the last block of one to the last block of the other compatibly with the determinants; the second factor lies in the maximal torus and changes neither the parabolic subgroups nor their characters. So conjugation by this isometry is the required automorphism of the orthogonal group.

For GSpin we lift it. In the Clifford algebra, normalized by $e_i^2 = f_i^2 = 0$ and $e_i f_i + f_i e_i = 1$, put

$$\tilde{t}_r = \prod_{i=1}^{r} (e_i + f_i)(e_i - f_i).$$

This is an even element of the Clifford group whose image in the orthogonal group is -1 on the first r hyperbolic pairs and the identity on their complement, and conjugation by the vector v induces conjugation by the reflection in v. So the automorphism of GSpin which lifts

our isometry is conjugation by $\tilde{t}_r v$; it preserves the even Clifford group although v is odd. It is trivial on H: the element $\tilde{t}_r$ lies in the even Clifford algebra of the first r hyperbolic pairs and commutes with the even Clifford algebra of the remaining space, and v anticommutes with the vectors of $v^\perp$, hence commutes with their even products. In characteristic two each factor $(e_i + f_i)(e_i - f_i)$ is $(e_i + f_i)^2 = 1$, the automorphism is conjugation by v, and the isometry has Dickson invariant one. □

Proof of Proposition 7.1. Take $P' = P'_+$ in the first configuration. The condition (7.2) is closed in γ. By Lemma 5.10(1) a double coset contributes to the Jacquet module only through the points at which this condition holds, and by Lemma 7.2 there are such points only in the double coset of w_0. If V_λ is not split, there are none in this double coset either (Lemma 7.4), and the Jacquet module is zero. Otherwise the locus is a finite disjoint union of closed orbits $B_H\backslash H$ with local sections, and Lemma 5.10(2) gives the Jacquet module as the sum over these orbits of the compact inductions from B_H of the characters of Lemma 7.5. Since $B_H\backslash H$ is compact, these are the normalized inductions $i^H_{B_H}\xi_\mu$ and $i^H_{B_H}\xi^\iota_\mu$.

In the first configuration, composing $I_{P'_+}(\boldsymbol{\mu};\omega_\sigma)$ with the automorphism of Lemma 7.6 gives $I_{P'_-}(\boldsymbol{\mu};\omega_\sigma)$, and it does not change the twisted Jacquet module as a representation of H. □

Let $a_j = \mu_j(\varpi)$ and $c = \omega_\sigma(\varpi)$ for unramified characters. Since Jacquet modules and invariants under a compact open subgroup are exact, the proposition shows that every spherical subquotient of the twisted Jacquet module of a subquotient of $I_{P'}(\boldsymbol{\mu};\omega_\sigma)$ has multiplier c and standard eigenvalues

$$a_1, \dots, a_n, c/a_n, \dots, c/a_1.$$

Remark 7.7. The vanishing in the nonsplit case holds because the flag variety of Lemma 7.4 has no rational point. It allows us to construct a global descent without controlling the discriminant of V_λ in advance, and to conclude afterwards that V_λ is split. In characteristic two the discriminant is replaced by an Artin–Schreier class, the Arf invariant.

7.3 The smallest case

It may help to see the proof of Proposition 7.1 in the smallest case, the second configuration with $r = 1$. The ambient group is SO_5, with basis e_1, e_2, f_2, f_1 and u, and $W = \langle e_2, f_2, u\rangle$. The group U_{desc} is the unipotent radical of the stabilizer of the line of e_1. It is abelian, $y \mapsto n(y)$ identifies it with W, and its root subgroups are $U_{e_1-e_2}$, $U_{e_1+e_2}$ and U_{e_1}. There is no simple root $e_i - e_{i+1}$ with $i < r$, so $\psi_{\mathrm{desc}} = \psi \circ \lambda$ for a linear form λ on W whose kernel V_λ is a nondegenerate plane, and $H = \mathrm{SO}(V_\lambda)$ is a torus of dimension one, split if and only if V_λ is a hyperbolic plane. The parabolic subgroup P' is the stabilizer of the isotropic plane $\langle e_1, e_2\rangle$, its Levi subgroup is GL_2, and $I_{P'}(\mu)$ is induced from $\mu \circ \det$.

There are two double cosets, according to the sign of $w(e_1)$ (Lemma 7.2). If the sign is positive we may take $w = 1$. All three roots of U_{desc} are then roots of P', so (7.2) asks that $\lambda' = 0$, which never happens: this double coset does not contribute. If the sign is negative, $w(e_1) = f_1$, $w(e_2) = e_2$ and $w(u) = -u$. The images of $e_1 - e_2$, $e_1 + e_2$ and e_1 are $-e_1 - e_2$, $e_2 - e_1$ and $-e_1$, of which only the second is a root of P', a root of the block. So $U_{\mathrm{int}} = U_{e_1+e_2}$ (Lemma 7.3), and the condition at the point $\gamma = (a_1, g)$ is $\lambda(g^{-1}e_2) = 0$.

The quotient $M_w\backslash M$ is the variety of isotropic lines $Z_1 = g^{-1}\langle e_2\rangle$ in the quadratic space W of dimension three, a conic, and the condition says that Z_1 lies in the plane V_λ (Lemma 7.4). So the locus is the intersection of a conic with a line. It consists of two rational points if V_λ is split and has no rational point otherwise; in the second case the Jacquet module is zero. In the split case choose the coordinates so that $V_\lambda = \langle e_2, f_2\rangle$; the two points are the lines

of e_2 and of f_2, both fixed by H, whose element z acts by z on e_2 and by z^{-1} on f_2. Here $B_H = H$.

For the characters (Lemma 7.5): at the first point the element z is sent by w to the element of the block with determinant z, so the inducing character gives $\mu(z)$. The half modulus of P' contributes $|z|$, because the half sum of the positive roots of SO_5 is $(3/2, 1/2)$ and that of the block is $(1/2, -1/2)$. The roots of U_{desc} have weights z^{-1}, z and 1 on H, and removing U_{int}, of weight z, leaves $|z|^{-1}$. The two cancel. At the second point z is replaced by z^{-1}. Hence

$$J_{U_{\mathrm{desc}},\psi_{\mathrm{desc}}} I_{P'}(\mu) \simeq \begin{cases} \mu \oplus \mu^{-1}, & V_\lambda \text{ split}, \\ 0, & V_\lambda \text{ not split}, \end{cases}$$

as a representation of H, which in the split case is the multiplicative group of the local field. The two summands are the two points in which the line meets the conic, and in general the two families of maximal isotropic subspaces of V_λ play the same role.

7.4 Two rearrangements of Fourier coefficients

The global argument uses the following two operations on root subgroups. All the commutator pairings in them have coefficient ± 1, so they are valid in characteristic two.

Lemma 7.8 (Separating a row). *In the odd orthogonal group, let U be the standard maximal unipotent subgroup, $U^0_{\mathrm{last}} = \{g \in U : g_{u,f_m} = 0\}$, and $L_i(g) = g_{i,m} + b g_{i,-m}$ for a constant b, where the indices m and $-m$ refer to e_m and f_m. Let $\nu : U^0_{\mathrm{last}} \to F$ be an additive homomorphism, so that $\psi \circ \nu$ is a character, which is trivial on $U_{e_r - e_j}$ and on $U_{e_j - e_m}$ for $r < j < m$. Then $(U_{\mathrm{tail}}, \psi(\nu + L_r)) \sim (U_{\mathrm{row}}, \psi(\nu + L_r))$ for the subgroups of U^0_{last}*

$$U_{\mathrm{tail}} = \{L_i = 0 : r < i < m\}, \qquad U_{\mathrm{row}} = \{g_{r,i} = 0 : r < i < m\}.$$

Proof. Write the form as $Q = a_0 z^2 + \sum x_i y_i$. On U^0_{last} the entry $g_{m,-m} = -a_0 g^2_{u,f_m}$ vanishes. Hence in the entry $(i, -m)$ of a product gg' the terms passing through u and e_m vanish, and those passing through the f_j vanish because the matrices are triangular. Adding b times this entry to the entry (i, m) gives

$$L_i(gg') = L_i(g) + L_i(g') + \sum_{i<j<m} g_{ij} L_j(g').$$

For $j = r+1, \dots, m-1$ in turn, let the common subgroup be the one defined by $g_{ri} = 0$ for $r < i \le j$ and $L_i = 0$ for $j \le i < m$, and exchange $U_{e_r - e_j}$ for $U_{e_j - e_m}$ by Lemma 5.9. Before the exchange the entry g_{rj} is free and $L_j = 0$; after it $g_{rj} = 0$ and L_j is free. So the first step starts from U_{tail} and the last ends at U_{row}. The displayed formula shows that the common subgroup is a group and that its character is invariant. The commutator of the two root subgroups is $U_{e_r - e_m}$, on which the character is nontrivial, and the pairing is $\pm st$. In odd characteristic the condition defining U^0_{last} is equivalent to $g_{m,u} = 0$. □

Lemma 7.9 (Interleaving two chains). *Let $A_1, \dots, A_r$ and $B_1, \dots, B_b$ be two sequences of weights, with $b = r - d \ge 1$ and $d \in \{-1, 0, 1\}$. Consider the unipotent group generated by the root subgroups of the positive roots $A_i - A_{i'}$ and $B_j - B_{j'}$ within each sequence and of the cross roots $A_i - B_j$ with $i < r$, with a character which is ψ of the sum of the coordinates on the adjacent roots $A_i - A_{i+1}$ and $B_j - B_{j+1}$, and possibly of further coordinates on an auxiliary subgroup which is normalized by all these root subgroups and by $U_{B_j - A_{i+1}}$ for $j \le i - d$, the character being invariant. Then the coefficient is equivalent, in the sense of $\sim$, to the one in which the cross roots are replaced by*

$$A_i - B_j \ (j > i - d), \qquad B_j - A_i \ (i \ge 2, \ j < i - d).$$

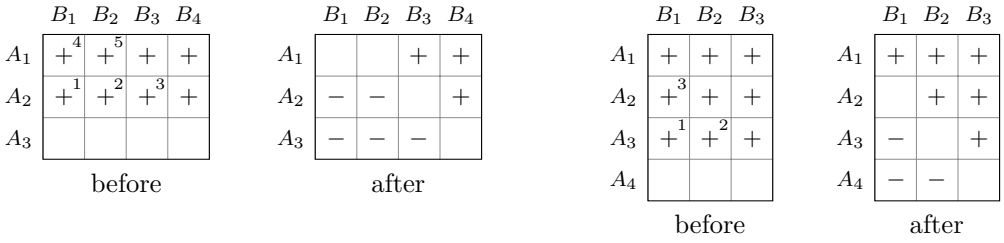


Figure 1: The cross roots of Lemma 7.9. A cell in the row of A_i and the column of B_j contains the sign + if the root subgroup of $A_i - B_j$ belongs to the group, and the sign − if that of $B_j - A_i$ does. Left: $r = 3$, $b = 4$, $d = -1$, the case of step 2 of the proof of Proposition 7.10 for $n = 4$. Right: $r = 4$, $b = 3$, $d = 1$, the case of step 2 of the proof of Lemma 8.3 for $n = 4$. The small numbers give the order of the exchanges; each replaces the sign + of a cell by the sign − in the cell below it. At the end the signs are those of the degrees of the roots $A_i - B_j$ for a grading which decreases by 2 along each chain and in which A_i and B_{i-d} have the same degree, except that for $d = -1$ the root $B_1 - A_1$ is absent.

Proof. The roots involved are those of the general linear group on basis vectors indexed by $A_1, \dots, A_r, B_1, \dots, B_b$, where $[1 + sE_{ab}, 1 + tE_{bc}] = 1 + stE_{ac}$ for distinct a, b, c. Put $X_{ij} = U_{A_i - B_j}$ and $Y_{ij} = U_{B_j - A_{i+1}}$. Their commutator is $U_{A_i - A_{i+1}}$, on which the character is ψ, so the pairing between X_{ij} and Y_{ij} is $\psi(st)$. We exchange X_{ij} for Y_{ij} by Lemma 5.9, for $i = r-1, r-2, \dots, 1$ in this order and, for each i, for $j = 1, \dots, i-d$ in this order; here $i - d \le b - 1$. At the end the cross roots $A_i - B_j$ that remain are those with $j > i - d$, and the root $B_j - A_h$ has been added, at the step $(h-1, j)$, exactly when $h \ge 2$ and $j < h - d$. This is the statement. It remains to check the hypotheses of Lemma 5.9 at the step (i, j).

The common subgroup. Remove X_{ij}. The root $A_i - B_j$ could be a sum of two remaining roots only as $(A_i - A_h) + (A_h - B_j)$ with $h > i$ or as $(A_i - B_h) + (B_h - B_j)$ with $h < j$. But $A_h - B_j$ with $h > i$ was removed at an earlier value of i, and $A_i - B_h$ with $h < j$ at an earlier value of j. So the remaining roots form a group.

Conjugation by Y_{ij}. The root $B_j - A_{i+1}$ can be added to a remaining root in four ways:

remaining root	sum
$A_h - B_j$	$A_h - A_{i+1}$
$A_{i+1} - A_h$	$B_j - A_h$
$B_h - B_j$	$B_h - A_{i+1}$
$A_{i+1} - B_h$	$B_j - B_h$.

In the first row $h < i$, since the root with $h = i$ has just been removed and those with $h > i$ earlier; the sum is a nonadjacent root of the first chain. In the second row $h > i + 1$, and the sum was added at the step $(h-1, j)$. In the third row $h < j$, and the sum was added at the step (i, h). In the fourth row $A_{i+1} - B_h$ remains only if $h > i + 1 - d \ge j + 1$, and the sum is a nonadjacent root of the second chain. In all cases the sum belongs to the common subgroup and the character is trivial on it.

Conjugation by X_{ij}. The possible sums are $A_i - B_h$ with $h > j$, $A_h - B_j$ with $h < i$, and $A_i - A_h$ with $h \ge i + 2$. The first two have not been removed yet and the third is not adjacent. An adjacent root of the second chain could only arise from a root $B_h - A_i$, and such roots are added only at the later value $i - 1$. So X_{ij} normalizes the common subgroup and preserves its character. The auxiliary subgroup is covered by the hypothesis. □

Figure 1 shows the cross roots before and after the exchanges, in the two cases in which the lemma is applied below.

7.5 Nonvanishing for an odd orthogonal target

The next proposition shows that the descent in the first configuration is nonzero. The Siegel–Whittaker coefficient is the coefficient $\mathcal{SW}$ of Section 6.

Proposition 7.10 (Nonvanishing of the odd orthogonal descent)**.** *Let $\mathcal{V}$ be a space of automorphic functions on SO_{4n} or GSpin_{4n}, $n \geq 2$, stable under right translation. If the Siegel–Whittaker coefficient is not identically zero on $\mathcal{V}$, then the descent of $\mathcal{V}$ to the split group of type B_n has a nonzero principal Whittaker coefficient.*

Proof. The Whittaker coefficient of the descent is a Fourier coefficient of $\mathcal{V}$ along a unipotent subgroup of the ambient group. The coefficients to be compared are taken along subgroups generated by the root subgroups of positive weight for a cocharacter D. In the gradings used below the smallest positive weight is 2. The quotient of such a subgroup by the subgroup of the weights greater than 2 is a vector group, with the root subgroups of weight 2 as coordinates, and an element f of weight -2 defines a character of it, by the pairing over the integers explained in step 3. We transform the pair for the descent, by exchanges of root subgroups and by one conjugation over F, into the pair $(D_{\mathrm{SW}}, f_{\mathrm{SW}})$ of a principal nilpotent element of the Siegel Levi subgroup, and then relate the latter to the Siegel–Whittaker coefficient. The conjugation in step 3 is the essential point.

Step 1: combining the two coefficients. Put $m = 2n$ and $r = n - 1$. The composite of the descent and of the Whittaker coefficient of H is the coefficient along the product of U_{desc} and of the maximal unipotent subgroup of H. Let N be the unipotent radical of the parabolic subgroup with Levi subgroup $\mathrm{GL}_{m-1} \times \mathrm{SO}_2$, and $N_c \subset N$ the subgroup of the roots $e_i + e_j$ with $i < j < m$. The quotient N/N_c has coordinates x_i and y_i, for the roots $e_i - e_m$ and $e_i + e_m$, and the upper triangular unipotent subgroup of GL_{m-1} acts on the two columns x and y in the usual way. We write a point, modulo N_c, as (A, x, y), with A in that unipotent subgroup. In the initial group $x_j = y_j$ for $n \leq j < m$, because its short root subgroup in the row j is $x_{e_j - e_m}(s) x_{e_j + e_m}(s)$. The character is the sum of the adjacent coordinates of the two chains of A, together with $L_r + y_{m-1}$, where $L_i = -x_i + y_i$.

For $j = n, \dots, m-1$ in turn we exchange $U_{e_r - e_j}$ for $U_{e_j - e_m}$. The common subgroup of the exchange is defined by $A_{ri} = 0$ for $n \leq i \leq j$ and $L_i = 0$ for $j \leq i < m$; it is a subgroup, by the rule for the multiplication of the columns. The root subgroup of $e_r - e_j$ preserves L_r because $L_j = 0$, and that of $e_j - e_m$ preserves it because $A_{rj} = 0$. Both normalize the common subgroup and preserve the other coordinates of the character, and the commutator of two of their elements changes L_r by $\pm st$. The subgroup N_c is central in N and is normalized by the upper triangular unipotent subgroup of GL_{m-1}, hence by all the groups in question; and it is contained in the common subgroup and in the kernel of the character. So it suffices to check all this in the quotient. After the exchanges the conditions $L_j = 0$ have been replaced by $A_{rj} = 0$. The resulting character is the pairing with

$$f = \sum_{i<r} F_{e_i - e_{i+1}} - F_{e_r - e_m} + F_{e_r + e_m} + \sum_{j=n}^{m-2} F_{e_j - e_{j+1}} + F_{e_{m-1} + e_m}.$$

Step 2: interleaving. Apply Lemma 7.9 with $A_i = e_i$, $B_j = e_{n+j-1}$ and $d = -1$; the auxiliary subgroup is N. The root elements of GL_{m-1} which are exchanged act on the columns only in rows other than r and $m-1$: the root $A_i - B_j$ has row $i < r$, and the root $B_j - A_{i+1}$ has row $n + j - 1 \leq m - 2$, since $j \leq i + 1 \leq r$. So they preserve the auxiliary character $L_r + y_{m-1}$. The group obtained is the subgroup $\{\delta = 0\}$ of the group of positive degree for the grading

$$D(e_i) = 2(n - i) \ (i < n), \qquad D(e_j) = 2(m - j) \ (j \geq n), \qquad D(f_i) = -D(e_i),$$

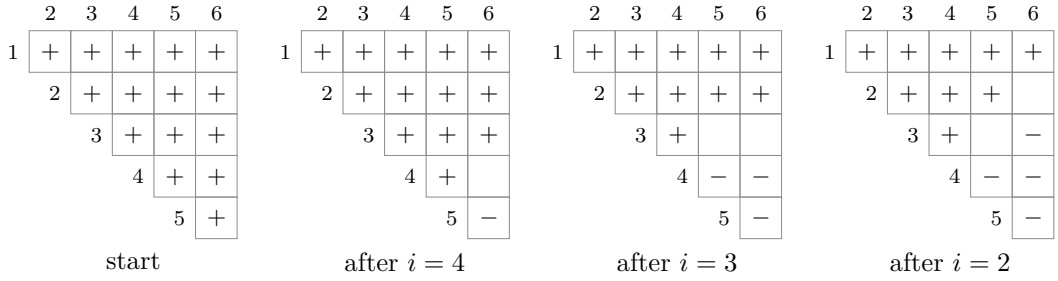


Figure 2: Step 4 of the proof of Proposition 7.10 for $m = 6$. The cell in row k and column l, $k < l$, contains the sign + if the root subgroup of $e_k + e_l$ belongs to the group and the sign − if that of $-e_k - e_l$ does. At the start all the signs are +, as for the Siegel–Whittaker coefficient. For $i = 4$ and $i = 3$ the root $e_{i+1} + e_{i+2}$ is removed first; each exchange then replaces the sign + of a cell (i, j) by the sign − in the cell $(i + 1, j)$ below it. At the end the sign is + for $k + l < 8$ and − for $k + l > 8$, and the cells with $k + l = 8$ are empty: these are the roots of positive degree for D_{SW}.

where δ is the coordinate on the root $e_n - e_1$, of degree 2; all other positive degrees are at least 2 as well. The characters of the whole group of positive degree that extend ours are the pairings with

$$f_t = f + tE_{e_1 - e_n}, \qquad t \in F.$$

Step 3: conjugation to a principal nilpotent element of the Siegel Levi subgroup. For $t \in \{0, -1\}$ define vectors

$$\begin{aligned}
v_1 &= e_n, \qquad & w_1 &= f_n, \\
v_k &= te_{k-1} + e_{n+k-1}, \qquad & w_k &= (2t+1)f_{k-1} + (t+1)f_{n+k-1} \quad (2 \le k \le n), \\
v_{n+1} &= -te_m + (t+1)f_m, \qquad & w_{n+1} &= (t+1)e_m - tf_m, \\
v_{n+2+h} &= (-1)^h\big((2t+1)f_{r-h} + tf_{m-1-h}\big), & & \\
w_{n+2+h} &= (-1)^h\big((t+1)e_{r-h} - e_{m-1-h}\big) \quad (0 \le h \le n-2). & &
\end{aligned}$$

For these two values of t they form a hyperbolic basis: $Q(v_i) = Q(w_i) = 0$, $B(v_i, w_j) = \delta_{ij}$ and $B(v_i, v_j) = B(w_i, w_j) = 0$. So $P_t : e_i \mapsto v_i$, $f_i \mapsto w_i$ is an isometry, and

$$P_t^{-1} f_t P_t = \sum_{i<m} F_{e_i - e_{i+1}} =: f_{\mathrm{SW}}, \qquad P_t^{-1} D P_t = D_{\mathrm{SW}}, \quad D_{\mathrm{SW}}(e_i) = m + 2 - 2i.$$

These are identities between matrices with integer entries; the defect of each for general t is divisible by $t(t+1)$, for instance $f_t^m = (-1)^{n-1} 2t(t+1) E_{f_n, e_n}$ whereas $f_{\mathrm{SW}}^m = 0$. The span of the v_i meets that of the e_i in dimension n for $t = 0$ and $n + 1$ for $t = -1$. Hence P_t lies in the special orthogonal group, of Dickson invariant zero, for exactly one of the two values:

$$t_* = 0 \ \text{ if } n \text{ is even}, \qquad t_* = -1 \ \text{ if } n \text{ is odd}.$$

In characteristic two the pairing between the root vectors F_α and E_α that defines the characters is the reduction of the integral pairing $\mathrm{tr}(AB)/2$ on the Lie algebra of the even orthogonal group over $\mathbb{Z}$, and the identities above hold over $\mathbb{Z}$.

Step 4: from $(D_{\mathrm{SW}}, f_{\mathrm{SW}})$ to the Siegel–Whittaker coefficient. The group of positive degree for D_{SW} contains the roots $e_i + e_j$ with $i + j < m + 2$ and $-e_i - e_j$ with $i + j > m + 2$, whereas the Siegel–Whittaker coefficient is along the group of all positive roots. We pass from

the second to the first; Figure 2 shows the case $m = 6$. For $i = m-2, \dots, 2$ in decreasing order: if $i \geq n$, first remove the root $e_{i+1} + e_{i+2}$, on which the character is trivial; then, for $j = m, \dots, \max(i+2, m+2-i)$ in decreasing order, exchange

$$U_{e_i+e_j} \quad \text{for} \quad U_{-e_{i+1}-e_j}.$$

At the step (i, j) the positive roots $e_k + e_l$ present are those with $k < i$, or $k = i$ and $l \leq j$, or $k > i$ and $k + l < m + 2$, and the negative roots $-e_k - e_l$ present are those with $k \geq i + 2$ and $k + l > m + 2$, or $k = i + 1$ and $l > j$; in both lists $k < l$. Call these sets P and N. Together with the roots $e_k - e_l$, $k < l$, of the upper triangular group of GL_m they form a closed set of roots which contains no pair of opposite roots, and therefore the product of their root subgroups is a unipotent group. We check this directly. Recall that $j \geq \max(i+2, m+2-i)$.

- P *is stable under lowering an index.* If $e_k + e_l \in \mathsf{P}$ and one of the two indices is replaced by a smaller one, the result is in P. The only case which is not immediate is that of $k, l > i$ with $k + l < m + 2$ when the new index is i; then the other index is at most $m - i < j$. This gives the sums of a root $e_a - e_b$ with a root of P.

- N *is stable under raising an index.* If $-e_k - e_l \in \mathsf{N}$ and an index is replaced by a larger one, the result is in N: in the case $k = i + 1$, $l > j$, raising k gives two indices which are at least $i + 2$, with sum larger than $m + 2$ because $l > m + 2 - i$. This gives the sums of a root $e_a - e_b$ with a root of N.

- *A root of* P *plus a root of* N. If $e_a + e_b \in \mathsf{P}$ and $-e_b - e_c \in \mathsf{N}$ with $a \neq c$, the sum is $e_a - e_c$, and we must show $a < c$. Both b and c are at least $i + 1$. If $a \leq i$ there is nothing to prove. If $a > i$, then $a + b < m + 2$. When $b, c \geq i + 2$ and $b + c > m + 2$ we get $c > m + 2 - b > a$. When $b = i + 1$ and $c > j$ we get $a < m + 1 - i \leq j - 1 < c$. The case $c = i + 1$, $b > j$ does not occur, since it would give $a < m + 2 - b < i$.

- *No opposite roots.* A pair $\{k, l\}$ cannot give a root of P and one of N: for $k \geq i + 2$ the conditions $k + l < m + 2$ and $k + l > m + 2$ exclude each other, and for $k = i + 1$ the conditions $l < m + 1 - i$ and $l > j$ do.

The same verification applies with the root $e_i + e_j$ removed, so the common subgroup of the two groups of an exchange is a group. The only commutator on which the character is nontrivial is the root $e_i - e_{i+1}$, with coefficient ± 1: it is the commutator of the two exchanged root subgroups. The other sums of an exchanged root with a present root which could be simple roots do not occur, because $e_{i+1} + e_{j-1} \notin \mathsf{P}$, as $i + j \geq m + 2$, and because no root of N involves the index i. Consider the root removed at the beginning of the step for $i \geq n$. Among the positive roots which are sums and are still present, $e_{i+1} + e_{i+2}$ is the only one with both indices greater than i; lowering one of its indices gives a root which is still present, and the removed root is not obtained by lowering an index of a root which is kept. Among the negative roots which are sums, none involves the index $i + 1$, and with a root $-e_{i+2} - e_l$ one obtains $e_{i+1} - e_l$ with $l \geq i + 3$, on which the character is trivial. So the root subgroup which is removed normalizes the subgroup which is kept, and its character. The coefficient before a removal is an integral of the coefficient after it, so nonvanishing is preserved, and each exchange is an equivalence by Lemma 5.9. At the end the roots are those of positive degree for D_{SW}.

Step 5: conclusion. If the Siegel–Whittaker coefficient is not identically zero on $\mathcal{V}$, then by step 4 neither is the coefficient for $(D_{\mathrm{SW}}, f_{\mathrm{SW}})$, and conjugating by $P_{t_*} \in G(F)$ neither is the coefficient for (D, f_{t_*}). The latter is a Fourier coefficient, along the root subgroup of $e_n - e_1$, of the coefficient on $\{\delta = 0\}$ of step 2, which is therefore not identically zero. By steps 2 and 1 the Whittaker coefficient of the descent is not identically zero. □

Remark 7.11. Of the two values $t = 0, -1$ exactly one gives an isometry in the identity component of the orthogonal group; which one depends on the parity of n. The argument works in characteristic two because it uses the quadratic form and not only its bilinear form: the latter does not see the component or the action of the short root subgroup. The identities of step 3, the determinant of P_t and the dimensions of the intersections have been verified by computer for $n \leq 5$. The sequences of exchanges in steps 1, 2 and 4 and in Lemma 7.9, and the statements of Lemmas 7.2 and 7.3, have been verified in small rank by a program. At each exchange it checks the conditions of Lemma 5.9 on the sets of roots and on the supports of the characters, in the form in which the program encodes them. The same was done for Lemmas 8.3 and 8.4.

8 Even orthogonal descent and its final Fourier coefficient

For an even orthogonal target the descent depends on a vector v in the last row of the unipotent radical, and it is the stabilizer of a nondegenerate v that has the right type. The Siegel–Whittaker coefficient of the residue only gives a coefficient in which the corresponding frequency is zero. The problem of this section is to pass from frequency zero to a nonzero frequency $c \in F^\times$. In odd characteristic this is done by a Fourier–Jacobi argument (Lemma 8.1); the argument in characteristic two is different and is given in Section 9. In both cases the vector is found over F without control of its norm, and the vanishing of the local Jacquet module in the nonsplit case (Proposition 7.1) then shows that the stabilizer is split. The result is Proposition 8.5.

The argument has the following shape. A family of Fourier coefficients Λ_c, $c \in F$, is defined on a unipotent subgroup U_{corr} of the source group. For $c \neq 0$ it is equivalent to the Whittaker coefficient of the descent with endpoint of norm $c/2$ (Lemma 8.3); for $c = 0$ it is implied by the Siegel–Whittaker coefficient (Lemma 8.4). The group U_{rad} of the parabolic subgroup with Levi subgroup $\mathrm{GL}_2^{n-1} \times \mathrm{SO}_5$ is normalized by a Jacobi group which preserves λ_{rad}. After integration over $(U_{\mathrm{rad}}, \psi\lambda_{\mathrm{rad}})$ one has a function on the Jacobi group, and the Λ_c are its Fourier coefficients. A function on the Jacobi group with a nonzero coefficient at $c = 0$ has a nonzero coefficient at some $c \neq 0$.

8.1 The residual representation attached to a parameter

Let σ take values in $\mathrm{GSp}_{2n}(k)$ or $\mathrm{GSO}_{2n}(k)$, with $n \geq 2$, and suppose it is continuous, finitely ramified, semisimple, elliptic modulo scalars and strongly pure, with multiplier χ of finite order and $\omega_\sigma = \mathrm{CFT}(\chi)$. The descent goes

$$\mathrm{GSpin}_{4n} \longrightarrow \mathrm{GSpin}_{2n+1}, \qquad \mathrm{GSpin}_{4n+1} \longrightarrow \mathrm{GSpin}_{2n}$$

in the two cases. By Proposition 5.2, $\mathrm{Std}\,\sigma = \bigoplus_i \varrho_i$ with distinct irreducible ϱ_i of dimensions m_i, $\sum m_i = 2n$, each carrying a form of the same kind with multiplier χ. In the alternating case the m_i are even and $\det \varrho_i = \chi^{m_i/2}$; in the symmetric case odd dimensions, including one, are allowed, and $(\det \varrho_i)^2 = \chi^{m_i}$. By L. Lafforgue's theorem, and class field theory in dimension one, there are unitary cuspidal representations Π_i with parameters ϱ_i; they satisfy $\Pi_i^\vee \otimes \omega_\sigma \simeq \Pi_i$ and are tempered at every place [Laf02, Theorem VI.10(i)]. The hypotheses of Proposition 6.6 hold, by the remark after Lemma 5.4, and we obtain an irreducible residual representation π of the source group, on which the connected centre acts through ω_σ, with nonzero Siegel–Whittaker coefficient.

At a place where everything is unramified, let the standard eigenvalues of σ be a_j and c/a_j, $1 \leq j \leq n$, with $c = \chi(\mathrm{Frob}_v)$. The Satake parameter of π_v on the Siegel Levi subgroup GL_{2n} consists of all these numbers shifted by $q_v^{-1/2}$. Replacing c/a_j by its partner under

the Weyl group of the source, we can write it as the pairs $(a_j q_v^{-1/2}, a_j q_v^{1/2})$, so that π_v is a spherical subquotient of the representation $I_{P'}(\boldsymbol{\mu}; \omega_\sigma)$ of Proposition 7.1 with $\mu_j(\varpi) = a_j$. In type B_{2n} the Weyl group contains all the sign changes needed. In type D_{2n} it contains only an even number of them; one uses P'_+ if n is even and P'_- if n is odd, which accounts for the remaining sign. Since only spherical subquotients are involved, this holds also when eigenvalues coincide.

8.2 A nonzero frequency

For the rest of this section assume $p \neq 2$. Let H be the Heisenberg group with coordinates (x, y, z) and multiplication

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + (xy' - yx')/2),$$

on which SL_2 acts; its upper unipotent element $n(b)$ acts by $(x, y, z) \mapsto (x, y - bx, z)$. We write X, Y, Z for the three coordinate subgroups. The subgroup generated by Y, Z and the $n(b)$ is abelian, and we consider its characters $\psi(z + cb)$, $c \in F$.

Lemma 8.1 (A nonzero frequency)**.** *Let $\mathcal{V}$ be a space of automorphic functions on the Jacobi group $\mathsf{H} \rtimes \mathrm{SL}_2$, stable under right translation. If the Fourier coefficient for the character $\psi(z)$, that is $c = 0$, is not identically zero on $\mathcal{V}$, then the coefficient for $\psi(z + cb)$ is not identically zero for some $c \in F^\times$.*

Proof. We pair with a theta series. This turns the coefficients into the Fourier coefficients, along the upper unipotent subgroup, of a genuine function on the metaplectic group, and a genuine function cannot be invariant under both unipotent subgroups of SL_2.

Step 1: unfolding. Let ω_ψ be the Schrödinger representation,

$$\omega_\psi(x, y, z)\phi(t) = \psi(z + yt + xy/2)\phi(t + x), \qquad \omega_\psi(n(b))\phi(t) = \psi(bt^2/2)\phi(t).$$

The functional $\phi \mapsto \sum_{\xi \in F} \phi(\xi)$ is invariant under the rational points of the Jacobi group in its metaplectic cover [Wei64, §§29,40–41, Proposition 5 and Theorem 6]. For $f \in \mathcal{V}$ and a Schwartz function ϕ, integrate f against the theta series of $\omega_{\psi^{-1}}$ over $[\mathsf{H}]$; the result is a genuine automorphic function Φ on the metaplectic group. Let $\mathcal{F}_c f$ be the coefficient of f for $\psi(z + cb)$. We claim that the Fourier coefficient of Φ for the character $\psi(cb)$ of the upper unipotent subgroup is

$$\int_{\mathbb{A}} \mathcal{F}_c f(X(x)g)(\omega_{\psi^{-1}}(g)\phi)(x)\, dx.$$

Indeed, write the Heisenberg variable in the order YZX. The term of the theta series with index ξ has the phase $\psi(-z - y\xi - b(\xi + x)^2/2)$; translating the variable by the rational element $X(\xi)$ removes ξ and combines the sum with the integral over $[X]$ into an integral over $X(\mathbb{A})$, and the relation

$$X(x)n(b) = Y(bx)Z(bx^2/2)n(b)X(x)$$

cancels the quadratic phase. All the quotients are compact, so the manipulations are justified. Since the functions $\omega_{\psi^{-1}}(g)\phi$ are arbitrary Schwartz functions of x, the hypothesis $\mathcal{F}_0 \neq 0$ implies that Φ is not identically zero for suitable f and ϕ, and it suffices to show that some Fourier coefficient of some such Φ with $c \neq 0$ is nonzero.

Step 2: a genuine function has a nonzero frequency. Suppose all the coefficients with $c \neq 0$ vanished, for all f and ϕ. Then every Φ would be invariant under the upper unipotent subgroup, and, conjugating by the rational Weyl element, under the lower one. But the

canonical lifts of these two subgroups to the metaplectic group generate the kernel of the covering at a place v: with $w(t) = n_+(t)n_-(-t^{-1})n_+(t)$ and $h(t) = w(t)w(-1)$ one has $h(s)h(t) = (s,t)_v h(st)$ [Rao93, Theorem 5.3, Corollary 5.5(2), Lemma 5.1(iii), Theorem A.4], and the Hilbert symbol of a uniformizer and a nonsquare unit is -1, because norms from the unramified quadratic extension have even valuation. So Φ would be invariant under the nontrivial element of the kernel, on which a genuine function changes sign, and Φ would vanish. □

The Jacobi group occurs in the source group as follows. Consider a quadratic space of dimension five with basis (e_1, e_2, u, f_2, f_1), where $B(e_i, f_j) = \delta_{ij}$ and $B(u,u) = 1$, and let

$$x_{e_i}(t) = 1 + tE_{i,0} - tE_{0,-i} - \tfrac{1}{2}t^2 E_{i,-i}$$

be its short root subgroups, the index 0 referring to u and $-i$ to f_i.

Lemma 8.2 (The Jacobi group in SO_5)**.** *The assignments*

$$X(x) \mapsto x_{e_2}(-2x), \quad Y(y) \mapsto x_{e_1}(y), \quad Z(z) \mapsto x_{e_1+e_2}(-2z), \quad n(b) \mapsto x_{e_1-e_2}(b/2)$$

extend to an embedding of the Jacobi group in SO_5*, in which* SL_2 *is the subgroup* $\mathrm{SL}(\langle e_1, e_2\rangle)$ *of the Levi subgroup* GL_2*. It lifts to* GSpin_5*.*

Proof. One computes

$$X(x)Y(y)Z(z - xy/2) = \begin{pmatrix} 1 & 0 & y & xy - 2z & -y^2/2 \\ 0 & 1 & -2x & -2x^2 & xy + 2z \\ 0 & 0 & 1 & 2x & -y \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In the product of two such matrices the entry $(1,4)$ is $xy - 2z + x'y' - 2z' + 2yx'$, which is the entry attached to $(x + x', y + y', z + z' + (xy' - yx')/2)$; the other entries are checked in the same way. Conjugation by $1 + (b/2)(E_{1,2} - E_{-2,-1})$ replaces y by $y - bx$, and conjugation by $1 + 2a(E_{2,1} - E_{-1,-2})$ replaces x by $x - ay$; both fix z. The group SL_2 and the unipotent elements lift canonically to the spin group. □

Now let $m = 2n$ and consider the source group of type B_m, with the notation of Section 7. Let U_{rad} be the unipotent radical of the parabolic subgroup with Levi subgroup $\mathrm{GL}_2^{n-1} \times \mathrm{SO}_5$, the last factor acting on the span of $e_{m-1}, e_m, u, f_m, f_{m-1}$, with the character

$$\lambda_{\mathrm{rad}}(g) = \sum_{i=1}^{m-2} g_{i,i+2},$$

the sum of the traces of the $n-1$ blocks of size two that describe U_{rad} modulo its commutator subgroup. An element $(g, \dots, g, h)$ of the Levi subgroup preserves this character if $hj = jg$, where j is the inclusion of $\langle e_{m-1}, e_m\rangle$ in the five-dimensional space. So the Jacobi group of Lemma 8.2, with SL_2 embedded diagonally in all the blocks, normalizes U_{rad} and preserves λ_{rad}, and acts on the coefficients of automorphic functions along $(U_{\mathrm{rad}}, \psi\lambda_{\mathrm{rad}})$.

8.3 The coefficients Λ_c

Let U be the standard maximal unipotent subgroup of the source group and $U^0_{\mathrm{last}} = \{g \in U : g_{m,0} = 0\}$. Let $U_{\mathrm{corr}} \subset U^0_{\mathrm{last}}$ be the subgroup on which the entries $g_{1,2}, g_{3,4}, \dots, g_{m-1,m}$ have a common value, denoted a. In terms of the preceding paragraph,

$$U_{\mathrm{corr}} = U_{\mathrm{rad}} \rtimes (YZn), \qquad a = b/2,$$

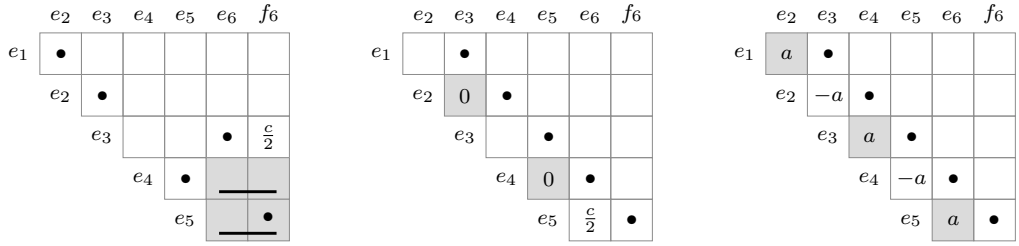


Figure 3: Three of the subgroups of U^0_{last} of Table 1, with their characters, for $n = 3$ and $m = 6$. The cells are the entries $g_{i,j}$ in the rows of $e_1, \dots, e_5$ and the columns of $e_2, \dots, e_6$ and f_6; the remaining root subgroups of U^0_{last} belong to the three groups, and the characters are trivial on them. A shaded cell is an entry subject to a condition: it vanishes, or it has the common value a, or it is one of two entries joined by a bar, which satisfy $L_i = 0$. The symbol in a cell is the coefficient of that entry in the character, a dot standing for 1; the character Λ_c contains in addition the term $\frac{c}{2}a$. In $U_{\mathrm{Wh},c}$ the character follows the two chains e_1, e_2, e_3 and e_4, e_5. Row separation, interleaving and the isometry of step 2 of the proof of Lemma 8.3, which sends e_1, e_2, e_3 to e_1, e_3, e_5 and e_4, e_5 to e_2, e_4 and exchanges e_6 and f_6, carry it to the second diagonal of U_{alt}.

where YZn is the abelian subgroup of the Jacobi group. For $c \in F$ define on U_{corr}

$$\Lambda_c(g) = \sum_{i=1}^{m-2} g_{i,i+2} - a \sum_{j=1}^{n-1} g_{2j,2j+1} + g_{m-1,-m} + \frac{c}{2}a. \tag{8.1}$$

The first sum is not λ_{rad} of the component of g in U_{rad}: a block C_i of g is the corresponding block of that component times $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$, and the trace of $C_i \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)^{-1}$ is the trace of C_i minus $a g_{2i,2i+1}$. This is the second sum. The entry $g_{m-1,-m}$ is $-2z$. Thus

$$\Lambda_c = \lambda_{\mathrm{rad}} - 2z + cb/4,$$

and $\psi\Lambda_c$ is the character λ_{rad} on U_{rad} times the character of YZn which, for the additive character $\psi(-2\,\cdot)$, is the one with frequency $-c/8$ in Lemma 8.1.

The rest of this section passes between the coefficients along five families of subgroups of U^0_{last}. Table 1 lists these subgroups with their characters, and then the order in which the coefficients are reached in the proof of Proposition 8.5. The groups $U_{\mathrm{Wh},c}$ and U_{alt} are defined in Lemma 8.3 and its proof, the groups $U_{\mathrm{chain},r}$ in the proof of Lemma 8.4. Figure 3 shows three of them for $n = 3$.

Lemma 8.3 (The coefficient Λ_c and the descent)**.** *For $c \in F$ put $L_i = g_{i,m} + (c/2)g_{i,-m}$, let $U_{\mathrm{Wh},c} \subset U^0_{\mathrm{last}}$ be defined by $L_i = 0$ for $n < i < m$, and let*

$$\lambda_{\mathrm{Wh},c} = \sum_{i<n} g_{i,i+1} + L_n + \sum_{i=n+1}^{m-2} g_{i,i+1} + g_{m-1,-m}$$

on $U_{\mathrm{Wh},c}$. Then $(U_{\mathrm{Wh},c}, \psi\lambda_{\mathrm{Wh},c}) \sim (U_{\mathrm{corr}}, \psi\Lambda_c)$. For $c \neq 0$ the coefficient along $(U_{\mathrm{Wh},c}, \psi\lambda_{\mathrm{Wh},c})$ is the descent of Section 7, in the second configuration with $r = n$ and endpoint $-v_c$, where $v_c = e_m + (c/2)f_m$, followed by a Whittaker coefficient of its stabilizer.

subgroup	defined in U^0_{last} by	character, composed with ψ
$U_{\mathrm{chain},m-1}$	no condition	$\lambda_{m-1} = \sum_{i<m} g_{i,i+1}$
$U_{\mathrm{chain},r}$, $r \le m-2$	$g_{i,m} = 0$ for $r < i < m$	$\lambda_r = \sum_{i<r} g_{i,i+1} + g_{r,m} + \sum_{i=r+1}^{m-2} g_{i,i+1} + g_{m-1,-m}$
$U_{\mathrm{Wh},c}$	$L_i = 0$ for $n < i < m$	$\lambda_{\mathrm{Wh},c} = \sum_{i<n} g_{i,i+1} + L_n + \sum_{i=n+1}^{m-2} g_{i,i+1} + g_{m-1,-m}$
U_{alt}	$g_{2j,2j+1} = 0$ for $j < n$	$\lambda_{\mathrm{alt},c} = \sum_{i=1}^{m-2} g_{i,i+2} + g_{m-1,-m} + \frac{c}{2} g_{m-1,m}$
U_{corr}	$g_{2j-1,2j} = a$ for $j \le n$	$\Lambda_c = \sum_{i=1}^{m-2} g_{i,i+2} - a \sum_{j<n} g_{2j,2j+1} + g_{m-1,-m} + \frac{c}{2} a$

	from		to	by
1	Siegel–Whittaker coefficient	$\Rightarrow$	$(U_{\mathrm{chain},m-1}, \lambda_{m-1})$	the first is the integral of the second over U_{e_m} (Lemma 8.4, step 1)
2	$(U_{\mathrm{chain},r}, \lambda_r)$	$\Rightarrow$	$(U_{\mathrm{chain},r-1}, \lambda_{r-1})$	for $r = m-1, \dots, n+1$: row separation, removal of $U_{e_r+e_m}$, exchange of U_{e_r} for U_{-e_m}, conjugation by w_r (Lemma 8.4, steps 2 and 3)
3	$(U_{\mathrm{chain},n}, \lambda_n)$	$=$	$(U_{\mathrm{Wh},0}, \lambda_{\mathrm{Wh},0})$	the definitions agree
4	$(U_{\mathrm{Wh},0}, \lambda_{\mathrm{Wh},0})$	$\sim$	$(U_{\mathrm{alt}}, \lambda_{\mathrm{alt},0})$	row separation, interleaving of the chains $e_1, \dots, e_n$ and $e_{n+1}, \dots, e_{m-1}$, an isometry (Lemma 8.3, step 2)
5	$(U_{\mathrm{alt}}, \lambda_{\mathrm{alt},0})$	$\sim$	$(U_{\mathrm{corr}}, \Lambda_0)$	exchange of $\prod_{j<n} U_{e_{2j-1}-e_{2j}}$ for $\prod_{j<n} U_{e_{2j}-e_{2j+1}}$ (Lemma 8.3, step 1)
6	$(U_{\mathrm{corr}}, \Lambda_0)$	$\Rightarrow$	$(U_{\mathrm{corr}}, \Lambda_c)$, some $c \neq 0$	Fourier expansion on the Jacobi group, after integration over $(U_{\mathrm{rad}}, \psi\lambda_{\mathrm{rad}})$ (Lemma 8.1; Proposition 8.5, step 1)
7	$(U_{\mathrm{corr}}, \Lambda_c)$	$\sim$	$(U_{\mathrm{Wh},c}, \lambda_{\mathrm{Wh},c})$	lines 5 and 4 for this c, through $(U_{\mathrm{alt}}, \lambda_{\mathrm{alt},c})$
8	$(U_{\mathrm{Wh},c}, \lambda_{\mathrm{Wh},c})$	$=$	descent, then Whittaker	$U_{\mathrm{Wh},c} = U_{\mathrm{desc},n} \rtimes U_{H_c}$: the descent with $r = n$ and endpoint $-v_c$, then a Whittaker coefficient of H_c (Lemma 8.3, step 3)

Table 1: The coefficients of Section 8, for $m = 2n$ and $p \neq 2$. Above: the subgroups and their characters; here $n \le r \le m-1$ and $c \in F$, $L_i = g_{i,m} + (c/2) g_{i,-m}$, and a denotes the common value of the entries $g_{1,2}, g_{3,4}, \dots, g_{m-1,m}$. Below: the route from the Siegel–Whittaker coefficient to the Whittaker coefficient of the descent. The sign $\Rightarrow$ means that if the coefficient on the left is not identically zero on the space of functions considered, neither is the one on the right; $\sim$ is the relation of Section 5.5. The frequency c changes only in line 6. That c is then a square, so that H_c is split, is step 2 of the proof of Proposition 8.5.

Proof. Both coefficients are compared with a third. Let $U_{\mathrm{alt}} \subset U^0_{\mathrm{last}}$ be defined by the vanishing of the entries $g_{2j,2j+1}$, with the character $\psi\lambda_{\mathrm{alt},c}$, where

$$\lambda_{\mathrm{alt},c}(g) = \sum_{i=1}^{m-2} g_{i,i+2} + g_{m-1,-m} + (c/2)g_{m-1,m}.$$

Step 1: $U_{\mathrm{corr}} \sim U_{\mathrm{alt}}$. The two groups contain $U_{\mathrm{corr}} \cap U_{\mathrm{alt}}$, and are obtained from it by adding $\prod_{j<n} U_{e_{2j}-e_{2j+1}}$, respectively $\prod_{j<n} U_{e_{2j-1}-e_{2j}}$. Conjugation by $U_{e_{2j}-e_{2j+1}}$ changes two neighbouring entries on the second superdiagonal by opposite multiples of a, which cancel in (8.1); conjugation by a root subgroup $U_{e_{2j-1}-e_{2j}}$ preserves the character of U_{alt} because the entries $g_{2j,2j+1}$ vanish there. The commutator pairing, in coordinates s_j on the first group and t_j on the second, with $t_n = 0$, is $\sum_{j<n}(t_j - t_{j+1})s_j$, a perfect pairing. So Lemma 5.9 applies.

Step 2: $U_{\mathrm{Wh},c} \sim U_{\mathrm{alt}}$. By Lemma 7.8 with $r = n$ and $b = c/2$ the conditions $L_i = 0$ can be replaced by $g_{n,j} = 0$ for $n < j < m$. Then apply Lemma 7.9 to the chains $A_i = e_i$, $i \le n$, and $B_j = e_{n+j}$, $j < n$, with $d = 1$. The auxiliary subgroup is the unipotent radical of the parabolic subgroup with Levi subgroup $\mathrm{GL}_{m-1} \times \mathrm{SO}_3$; the character on it only involves the rows of A_n and B_{n-1}, paired with $e_m + (c/2)f_m$ and with f_m, and the commutators that occur in the exchanges lie in other rows. The isometry over F

$$A_i \mapsto e_{2i-1}, \quad B_j \mapsto e_{2j}, \quad e_m \leftrightarrow f_m, \quad u \mapsto -u,$$

of determinant one, carries the resulting group and character to U_{alt} and $\psi\lambda_{\mathrm{alt},c}$.

Step 3: the case $c \neq 0$. The equations $L_i = 0$ say that the rows $n < i < m$ are orthogonal to v_c, so that $U_{\mathrm{Wh},c} = U_{\mathrm{desc},n} \rtimes U_{H_c}$, where $U_{\mathrm{desc},n}$ is the descent radical with $r = n$ and U_{H_c} is a maximal unipotent subgroup of $H_c = \mathrm{GSpin}(v_c^\perp)$, the orthogonal complement being taken in the space spanned by the last n hyperbolic pairs and u. In the convention $n(y)f_i \equiv f_i + y \bmod Fe_i$ of (7.1) and the following formulas, $g_{i,m} = -B(y, e_m)$ and $g_{i,-m} = -B(y, f_m)$. Thus L_n is the pairing with $-v_c$, and $\psi\lambda_{\mathrm{Wh},c}$ restricts to ψ_{desc} with endpoint $-v_c$. Its restriction to U_{H_c} has coefficient one on the simple roots $e_i - e_{i+1}$, $n < i < m-1$. The last term $g_{m-1,-m}$ pairs the last row with $-f_m$, whose component in $v_c^\perp$ is $z_c = -f_m + c^{-1}v_c$. This vector is orthogonal to the other basis vectors of $v_c^\perp$ and $B(z_c, z_c) = -c^{-1}$; over a field where H_c splits, it has nonzero components along both of the last two simple root vectors. So the character is a nondegenerate character of the maximal unipotent subgroup of the quasi-split group H_c. □

Lemma 8.4 (From the Siegel–Whittaker coefficient to frequency zero)**.** *Let $\mathcal{V}$ be a space of automorphic functions on GSpin_{4n+1}, stable under right translation. If the Siegel–Whittaker coefficient is not identically zero on $\mathcal{V}$, neither is the coefficient along $(U_{\mathrm{corr}}, \psi\Lambda_0)$.*

Proof. By Lemma 8.3 with $c = 0$ it suffices to treat the coefficient along $(U_{\mathrm{Wh},0}, \psi\lambda_{\mathrm{Wh},0})$. We reach it through a chain of coefficients indexed by $r = m-1, m-2, \dots, n$, each implied by the preceding one.

Step 1: the chain. For $n \le r \le m-2$ let $U_{\mathrm{chain},r} \subset U^0_{\mathrm{last}}$ be defined by $g_{i,m} = 0$ for $r < i < m$, with the character

$$\lambda_r = \sum_{i<r} g_{i,i+1} + g_{r,m} + \sum_{i=r+1}^{m-2} g_{i,i+1} + g_{m-1,-m}.$$

For $r = m-1$ let $U_{\mathrm{chain},m-1} = U^0_{\mathrm{last}}$ and $\lambda_{m-1} = \sum_{i<m} g_{i,i+1}$. For $r = n$ this is $(U_{\mathrm{Wh},0}, \lambda_{\mathrm{Wh},0})$. The Siegel–Whittaker coefficient is the integral of the coefficient for $r = m-1$ over the root subgroup U_{e_m}, so the latter is not identically zero.

Step 2: from r to $r-1$: an exchange. Let $n < r \le m-1$. By Lemma 7.8 with $b = 0$ we may replace the conditions $g_{i,m} = 0$ by $g_{r,j} = 0$ for $r < j < m$. Remove the root subgroup $U_{e_r+e_m}$, on which the character is trivial; this preserves nonvanishing. What remains is $U_{\mathrm{com},r} \rtimes Z_r$ with $Z_r = U_{e_r}$, where $U_{\mathrm{com},r}$ is the group with roots

$$\Phi^+ \setminus \{e_m,\ e_r,\ e_r + e_m,\ e_r - e_j\ (r < j < m)\}.$$

Let $Y_r = U_{-e_m}$, with

$$x_{-e_m}(s) = 1 + s(E_{f_m,u} - E_{u,e_m}) - \tfrac{1}{2}s^2 E_{f_m,e_m}.$$

We exchange Z_r for Y_r by Lemma 5.9. Both normalize $U_{\mathrm{com},r}$ and preserve its character: conjugation by Y_r produces from $e_i + e_m$ the roots e_i and $e_i - e_m$, and from e_i the root $e_i - e_m$, with $i \neq r$; conjugation by Z_r produces from $e_i - e_r$ the roots e_i and $e_i + e_r$, and from e_i the root $e_i + e_r$. All of these belong to $U_{\mathrm{com},r}$ and the character is trivial on them; the root $e_{m-1} + e_m$, on which it is not trivial, does not occur because $r < m$. Finally

$$[x_{-e_m}(s), x_{e_r}(t)] = x_{e_r-e_m}(st),$$

as one checks in the five-dimensional space with basis (e_r, e_m, u, f_m, f_r), and the character on $U_{e_r-e_m}$ is ψ. So the pairing is perfect.

Step 3: from r to $r-1$: a conjugation. Let w_r be the isometry which fixes the first $r-1$ hyperbolic pairs, sends (e_i, f_i) to $(-e_{i+1}, -f_{i+1})$ for $r \le i \le m-2$, sends (e_{m-1}, f_{m-1}) to $(-f_m, -e_m)$ and (e_m, f_m) to (e_r, f_r), and sends u to $-u$. It has determinant one, the exchange of e and f in one pair being compensated by the sign on u. For $i \le m-2$ put $i' = i$ if $i < r$ and $i' = i+1$ otherwise. The roots supported on the indices at most $m-2$ are mapped to all the positive roots which involve neither r nor m. The roots $e_i \pm e_{m-1}$ are mapped to $e_{i'} \mp e_m$, and e_{m-1} to $-e_m$. The sums $e_i + e_m$ with $i \le m-2$ are mapped to the sums which involve e_r and not e_m, and $e_{m-1} + e_m$ to $e_r - e_m$. The differences $e_i - e_m$ which are allowed, those with $i < r$, are mapped to $e_i - e_r$. Altogether these are the roots of $U_{\mathrm{com},r}$ and $-e_m$, so

$$w_r U_{\mathrm{chain},r-1} w_r^{-1} = U_{\mathrm{com},r} Y_r,$$

and on the roots where λ_{r-1} is nontrivial it acts as follows, for $r \le m-2$:

root	image
$e_i - e_{i+1}\ (i < r-1)$	$e_i - e_{i+1}$
$e_{r-1} - e_m$	$e_{r-1} - e_r$
$e_i - e_{i+1}\ (r \le i \le m-3)$	$e_{i+1} - e_{i+2}$
$e_{m-2} - e_{m-1}$	$e_{m-1} + e_m$
$e_{m-1} + e_m$	$e_r - e_m$.

The coefficient is $+1$ in each row: in a root $e_i - e_j$ the two signs cancel, and in the last two rows the exchange of e_m and f_m turns the sign of $E_{e_i+e_j}$ into that of $E_{e_i-e_j}$. For $r = m-1$ the last two rows are replaced by $e_{m-2} - e_{m-1}$ and $e_{m-1} - e_m$. So conjugation by $w_r \in G(F)$ carries the coefficient for $r-1$ to the coefficient obtained in step 2. □

8.4 Nonvanishing

Proposition 8.5 (Nonvanishing of the even orthogonal descent, $p \neq 2$)**.** *Assume $p \neq 2$. The descent of the residual representation of Section 8.1 on* GSpin_{4n+1} *to the split group* GSpin_{2n} *is nonzero and has a nonzero Whittaker coefficient.*

Proof. Step 1: a nonzero descent for some c. The residual representation has a nonzero Siegel–Whittaker coefficient. By Lemma 8.4 the coefficient Λ_0 is not identically zero on it. The coefficients Λ_c are the Fourier coefficients, for the characters of Lemma 8.1, of the functions on the Jacobi group obtained by integrating over $(U_{\mathrm{rad}}, \psi\lambda_{\mathrm{rad}})$. So Λ_c is not identically zero for some $c \in F^\times$, and by Lemma 8.3 the descent with endpoint $-v_c$ has a nonzero Whittaker coefficient on H_c. The conjugations by isometries used in these lemmas lift to elements of $\mathrm{GSpin}(F)$, because $H^1(F, \mathbb{G}_m) = 0$, and two lifts differ by a rational point of the centre, which acts trivially on automorphic functions; so all the coefficients are coefficients of π.

Step 2: c is a square. By Witt cancellation the space $v_c^\perp$ is split if and only if $Q(v_c)/Q(u)$ is a square; since $Q(v_c) = c/2$ and $Q(u) = 1/2$, this means that c is a square. If it were not, by Chebotarev there would be a place v where π and ψ are unramified and H_c is not split. There π_v is a subquotient of the representation $I_{P'}$ of Section 8.1, and its twisted Jacquet module vanishes by Proposition 7.1 and exactness. On the other hand a nonzero descent gives a nonzero functional on this Jacquet module. Translate a vector with nonzero descent on the right so that the integral which defines its descent is not zero at the identity; write it as a sum of pure tensors, one of which has a nonzero integral; fix the components of this tensor at the places other than v, and let the component at v vary. By a change of variable in the adelic integral the functional obtained is equivariant for $U_{\mathrm{desc},n}(F_v)$ and $\psi_{\mathrm{desc},v}$, so it factors through the Jacquet module, which is zero. This contradiction shows that H_c is split. By Witt's theorem and a suitable element of the torus the endpoint $-v_c$ and the character are conjugate over F to the split datum. □

Remark 8.6. The Fourier–Jacobi argument only shows that some frequency $c \neq 0$ occurs. That the stabilizer is split follows from the unramified local components together with the global nonvanishing. Because these two steps are separate, the first can be replaced by another argument in characteristic two without changing the second. The identities of Lemma 8.2 and the commutator in step 2 of the proof of Lemma 8.4 have been verified by computer.

9 Even orthogonal descent in characteristic two

Section 8 produced, in odd characteristic, a nonzero generic descent from GSpin_{4n+1} to split GSpin_{2n}. The step that fails in characteristic two is the Fourier–Jacobi argument: it moves a Fourier coefficient from frequency zero to a nonzero frequency on the unipotent subgroup of the SL_2 in the Jacobi group, a long root group, and it uses a quadratic exponent divided by two. This section gives a replacement. Its result is Proposition 9.4 together with Corollary 9.5; the realization theorems for orthogonal groups, in all characteristics, are then deduced in Section 10.

Throughout this section $p = 2$. The argument has three parts.

1. In characteristic two the short root element contains a square, and the commutator that should move the frequency has character $\psi(sa^2)$ instead of $\psi(sa)$. The map $a \mapsto a^2$ is not surjective on F, so this looks weaker. But $a \mapsto \psi(sa^2)$ is an additive character of a, and the Cartier operator computes it: it equals $\psi(C_\psi(s)a)$ for an explicit additive map $C_\psi : F \to F$ that is surjective. This allows us to prescribe the frequency (Lemmas 9.1 and 9.2).

2. Iterating this along the rows of the unipotent radical moves the Siegel–Whittaker coefficient of the residual representation to a coefficient with a prescribed short-root frequency (Lemma 9.3).

3. A second Fourier expansion then produces a descent whose target is the special orthogonal group of a quadratic space containing a plane $x^2 + xy + c_0^2 y^2$. Here the

frequency c_0 cannot be prescribed, and the plane may a priori be anisotropic. The local Jacquet module of Proposition 7.1 vanishes for an anisotropic plane, which forces the Arf invariant of the plane to be trivial (Proposition 9.4 and Corollary 9.5).

The matrix identities of this section and its statements about sets of roots were also checked by computer, in small rank. These checks are not part of the proofs; Appendix A lists them, with the programs and the ranges covered.

9.1 Conventions in characteristic two

We use the quadratic space with basis $e_1, \dots, e_m, u, f_m, \dots, f_1$ and form

$$Q\Big(zu + \sum_i (x_i e_i + y_i f_i)\Big) = z^2 + \sum_i x_i y_i.$$

Its polar form B has radical spanned by u. As before, $E_{a,b}$ denotes the matrix sending the basis vector b to a and the other basis vectors to zero. The short root elements and their coordinates are

$$x_{e_i}(a) = 1 + aE_{u,f_i} + a^2 E_{e_i,f_i}, \qquad x_{-e_i}(a) = 1 + aE_{u,e_i} + a^2 E_{f_i,e_i}, \qquad c_i(g) = g_{u,f_i}.$$

The root e_i has coroot $2e_i^*$ in the special orthogonal group and $2e_i^* - e_0^*$ in GSpin; the quadratic entry a^2 does not affect this. The map $a \mapsto x_{e_i}(a)$ is a homomorphism, and the short root groups of two different hyperbolic pairs commute, because $B(u,u) = 0$.

Fix a separating element $t \in F$, so that dt is a basis of $\Omega^1_{F/\mathbb{F}_q}$, and write the nontrivial additive character ψ of $F\backslash\mathbb{A}_F$ as $\psi = \psi_{\eta\, dt}$ with $\eta \in F^\times$:

$$\psi(a) = \prod_v \psi_v(a_v), \qquad \psi_v(a) = (-1)^{\mathrm{Tr}_{\kappa(v)/\mathbb{F}_2} \mathrm{Res}_v(a\eta\, dt)}.$$

For a unipotent F-group N write $[N] = N(F)\backslash N(\mathbb{A}_F)$, with Haar measure of volume one. For a character ζ of $N(\mathbb{A}_F)$ trivial on $N(F)$, the Fourier coefficient of an automorphic function ϕ is

$$\mathcal{F}_{N,\zeta}\phi(g) = \int_{[N]} \phi(ng)\zeta(n)^{-1}\, dn.$$

We say that the coefficient (N,ζ) is *nonzero on* a space of automorphic functions stable under right translation if $\mathcal{F}_{N,\zeta}\phi(g) \neq 0$ for some ϕ in the space and some g.

9.2 The Cartier operator and characters of squares

Lemma 9.1 (The character of a square)**.** *Every element of F can be written uniquely as $A^2 + tB^2$ with $A, B \in F$. Put*

$$C_t(A^2 + tB^2) = B, \qquad C_\psi(s) = \eta^{-1} C_t(\eta s).$$

Then for $s \in F$ and $a \in \mathbb{A}_F$,

$$\psi(sa^2) = \psi(C_\psi(s)a), \tag{9.1}$$

and $C_\psi(t\eta\beta^2) = \beta$ for every $\beta \in F$. In particular $C_\psi : F \to F$ is surjective.

Proof. Since F is a function field in one variable over a perfect field of characteristic two, $[F : F^2] = 2$, and $t \notin F^2$ because $dt \neq 0$. So $1, t$ is a basis of F over F^2. The same holds for each completion: $[F_v : F_v^2] = 2$ by the expansion of Laurent series into even and odd parts, and t is not a square in F_v, since otherwise $F = F^2(t)$ would be contained in F_v^2, although F contains an element of valuation one at v.

Write $\eta s = A^2 + tB^2$ with $A, B \in F$. Let $\mathcal{C}$ be the Cartier operator on differentials of F_v; it is additive and satisfies $\mathcal{C}(g^2\,dt) = 0$ and $\mathcal{C}(g^2 t\,dt) = g\,dt$. For $a \in F_v$ this gives

$$\mathcal{C}(sa^2\eta\,dt) = \mathcal{C}((Aa)^2\,dt + (Ba)^2 t\,dt) = aB\,dt = C_\psi(s)a\,\eta\,dt.$$

In a local uniformizer z the Cartier operator sends $\sum_j a_j z^j\,dz$ to $\sum_j a_{2j+1}^{1/2} z^j\,dz$, so the residue of $\mathcal{C}\omega$ is the square root of the residue of ω. The trace from $\kappa(v)$ to $\mathbb{F}_2$ is invariant under the square root. Hence $\psi_v(sa^2) = \psi_v(C_\psi(s)a)$ at every place v, including those where $\eta\,dt$ has a zero or a pole, and taking the product gives (9.1). Finally $\eta \cdot t\eta\beta^2 = t(\eta\beta)^2$, so $C_\psi(t\eta\beta^2) = \eta^{-1}\eta\beta = \beta$. □

Lemma 9.2 (Prescribing a frequency)**.** *Let N be a unipotent group with a character ζ trivial on $N(F)$, and let $X = x(\mathbb{G}_a)$ and $Y = y(\mathbb{G}_a)$ be one-parameter groups that normalize N and preserve ζ. Assume that $X \cap N = Y \cap N = 1$, that $[X, Y] \subset N$, and that*

$$\zeta([x(s), y(a)]) = \psi(sa^2).$$

If the coefficient (N, ζ) is nonzero on a space $\mathcal{V}$ of automorphic functions stable under right translation, then for every $\beta \in F$ the coefficient on NY of the character extending ζ by $y(a) \mapsto \psi(\beta a)$ is nonzero on $\mathcal{V}$.

Proof. For $\phi \in \mathcal{V}$ put $h = \mathcal{F}_{N,\zeta}\phi$ and, for $\beta \in F$,

$$h_\beta(g) = \int_{F\backslash\mathbb{A}_F} h(y(a)g)\psi(\beta a)\,da.$$

By the hypotheses on Y this is the coefficient of ϕ in the statement. For fixed g the function $a \mapsto h(y(a)g)$ on $F\backslash\mathbb{A}_F$ is locally constant, so it is a finite sum of characters $\psi(\alpha a)$ with $\alpha \in F$. If $h \neq 0$, choose ϕ, g and α with $h_\alpha(g) \neq 0$.

Let $s \in F$. Since ϕ is left invariant under $x(s)$ and $x(s)$ normalizes N and preserves ζ, we have $h(x(s)g') = h(g')$ for all g'. Moving $x(s)$ past $y(a)$ produces the commutator, an element of N on which ζ takes the value $\psi(sa^2)$; signs play no role, since ψ takes the values ± 1. With Lemma 9.1 we obtain

$$h_\beta(x(s)g) = \int_{F\backslash\mathbb{A}_F} h(y(a)g)\psi((\beta + C_\psi(s))a)\,da = h_{\beta + C_\psi(s)}(g). \tag{9.2}$$

Given β, take $s = t\eta(\alpha + \beta)^2$. Then $C_\psi(s) = \alpha + \beta$ and $h_\beta(x(s)g) = h_\alpha(g) \neq 0$. The translation is by an element of $G(F)$, so the nonzero coefficient is a coefficient of a function in $\mathcal{V}$ over F. □

9.3 Moving the frequency along the rows

Let G be SO_{2m+1} or GSpin_{2m+1}, let U be its upper triangular maximal unipotent subgroup in the basis above, and put $U^0_{\mathrm{last}} = \{g \in U : c_m(g) = 0\}$, the subgroup generated by the positive root groups other than U_{e_m}. Matrix entries are indexed by the basis; we write $g_{i,j}$ for the entry at (e_i, e_j) and $g_{i,-j}$ for the entry at (e_i, f_j). For $1 \leq r < m-1$ define

$$\begin{aligned} U_{\mathrm{chain},r} &= \{g \in U^0_{\mathrm{last}} : g_{i,m} = 0 \ (r < i < m)\},\\ U_{\mathrm{row},r} &= \{g \in U^0_{\mathrm{last}} : g_{r,j} = 0 \ (r < j < m)\},\\ \lambda_r &= \sum_{i<r} g_{i,i+1} + g_{r,m} + \sum_{i=r+1}^{m-2} g_{i,i+1} + g_{m-1,-m}. \end{aligned}$$

In terms of roots, λ_r is the sum of the coordinates on the chain $e_1 - e_2, \dots, e_{r-1} - e_r$, on $e_r - e_m$, on the chain $e_{r+1} - e_{r+2}, \dots, e_{m-2} - e_{m-1}$, and on $e_{m-1} + e_m$. For $r = m-1$ both groups are defined to be U^0_{last}, and $\lambda_{m-1} = \sum_{i<m} g_{i,i+1}$. We call $(U^0_{\mathrm{last}}, \psi\lambda_{m-1})$ the *endpoint coefficient*. The Siegel–Whittaker coefficient $\mathcal{SW}$ of Section 6 is its integral over the remaining root group U_{e_m}, so if $\mathcal{SW}$ is nonzero on a space of functions, the endpoint coefficient is too.

Lemma 9.3 (Row iteration)**.** *Let $\mathcal{V}$ be a space of automorphic functions on G, stable under right translation, on which the Siegel–Whittaker coefficient is nonzero. Then for every r with $1 \le r \le m-1$ the coefficient $(U_{\mathrm{chain},r}, \psi\lambda_r)$ is nonzero on $\mathcal{V}$. Moreover, for every r with $1 \le r < m-1$ and every $\beta \in F$, the coefficient $(U_{\mathrm{chain},r}, \psi(\lambda_r + \beta c_{m-1}))$ is nonzero on $\mathcal{V}$.*

Proof. The case $r = m-1$ is the endpoint coefficient. We show, for $2 \le r \le m-1$ and $\beta \in F$, that if $(U_{\mathrm{chain},r}, \psi\lambda_r)$ is nonzero on $\mathcal{V}$ then so is $(U_{\mathrm{chain},r-1}, \psi(\lambda_{r-1} + \beta c_{m-1}))$. Taking $\beta = 0$ and descending induction on r gives the first assertion, and a general β in the last step gives the second.

The step which is special to characteristic two is the second. The commutator of the two root groups is quadratic in one of the variables, and Lemma 9.2 turns the character $\psi(sa^2)$ which results into a character which is linear in a. Steps 3 and 5 are verifications on roots.

Step 1: from the chain group to the row group. By Lemma 7.8 with $b = 0$, the coefficients of $\psi\lambda_r$ on $U_{\mathrm{chain},r}$ and on $U_{\mathrm{row},r}$ vanish identically together. Put

$$Z_r = U_{e_r}, \qquad X_r = U_{e_r+e_m}, \qquad Y = U_{-e_m},$$

and let N_r be generated by the positive root groups other than those of

$$e_m, \quad e_r, \quad e_r + e_m, \quad e_r - e_j \ \ (r < j < m).$$

Then $U_{\mathrm{row},r} = N_r Z_r X_r$. The character $\psi\lambda_r$ is trivial on Z_r and on X_r; for $r = m-1$ this uses that λ_{m-1} has no term on a root $e_i + e_m$. Integrating over fewer variables, we see that the coefficient $(N_r Z_r, \psi\lambda_r)$ is nonzero on $\mathcal{V}$.

Step 2: the commutator. We apply Lemma 9.2 to $N = N_r Z_r$, $X = X_r$ and Y. In the ordered basis (e_r, e_m, u, f_m, f_r) of the relevant five-dimensional space,

$$x_{e_r+e_m}(s) = 1 + s(E_{e_r,f_m} + E_{e_m,f_r}), \qquad x_{-e_m}(a) = 1 + aE_{u,e_m} + a^2 E_{f_m,e_m},$$

and a direct computation gives

$$[x_{e_r+e_m}(s), x_{-e_m}(a)] = x_{e_r}(sa)\, x_{e_r-e_m}(sa^2)\,: \tag{9.3}$$

both sides fix e_r, u, f_m and send

$$e_m \longmapsto e_m + sa^2 e_r, \qquad f_r \longmapsto f_r + sa\,u + sa^2 f_m + s^2 a^2 e_r.$$

The first factor on the right lies in Z_r, where the character is trivial, and the second has character $\psi(sa^2)$, because λ_r contains the coordinate on $e_r - e_m$ with coefficient one. So $[X_r, Y] \subset N_r Z_r$ with the commutator character required by Lemma 9.2.

Step 3: normalization. We check that X_r and Y normalize $N_r Z_r$ and preserve its character.

Conjugation by $Y = U_{-e_m}$ moves a root group $U_{e_i+e_m}$ into $U_{e_i} U_{e_i-e_m}$. For $i \ne r$ both of these lie in N_r and carry the trivial character, and $U_{e_r+e_m}$ is not in $N_r Z_r$. The remaining positive root groups in $N_r Z_r$ commute with Y: the only positive root that could interact with $-e_m$ otherwise is e_m, which is excluded, and the short root groups of the other hyperbolic pairs commute with Y in characteristic two.

Conjugation by $X_r = U_{e_r+e_m}$ can only produce root groups $U_{e_i+e_m}$, from $e_i - e_r$, and $U_{e_i+e_r}$, from $e_i - e_m$. These lie in N_r and carry the trivial character. The one root of this form on which λ_r is not trivial, $e_{m-1} + e_m$, would have to come from $e_{m-1} - e_r$, which is not a positive root since $r \le m-1$.

Step 4: prescribing the frequency on Y. Lemma 9.2 now shows that the coefficient on $N_r Z_r Y$ of the character $\psi\lambda_r$ extended by $y(a) \mapsto \psi(\beta a)$ is nonzero on $\mathcal{V}$, for every $\beta \in F$. Since Z_r commutes with Y and carries the trivial character, the same holds for the group $N_r Y$.

Step 5: identification with the previous row. Let w_r be the element of the Weyl group, represented over F, that fixes e_i, f_i for $i < r$ and u, and satisfies

$$w_r(e_i) = e_{i+1} \quad (r \le i \le m-2), \qquad w_r(e_{m-1}) = f_m, \qquad w_r(e_m) = e_r,$$

with the corresponding action on the dual vectors. We claim that

$$w_r U_{\mathrm{chain},r-1} w_r^{-1} = N_r Y$$

and that conjugation carries $\psi(\lambda_{r-1} + \beta c_{m-1})$ to the character of step 4. For $r \le m-2$ the roots on which $\lambda_{r-1} + \beta c_{m-1}$ is nontrivial are transformed as follows.

root of $U_{\mathrm{chain},r-1}$	image under w_r
$e_i - e_{i+1}$ $(i < r-1)$	$e_i - e_{i+1}$
$e_{r-1} - e_m$	$e_{r-1} - e_r$
$e_i - e_{i+1}$ $(r \le i \le m-3)$	$e_{i+1} - e_{i+2}$
$e_{m-2} - e_{m-1}$	$e_{m-1} + e_m$
$e_{m-1} + e_m$	$e_r - e_m$
e_{m-1}	$-e_m$

The images in the first five rows are exactly the roots of λ_r, and the last row carries the coordinate c_{m-1} to the coordinate of Y. For $r = m-1$ the last two roots of λ_{m-2} are carried to $e_{m-2} - e_{m-1}$ and $e_{m-1} - e_m$, the last two roots of the endpoint character.

For the groups we compare the sets of roots. On the indices, w_r is the map $i \mapsto i$ for $i < r$, $i \mapsto i+1$ for $r \le i \le m-2$, $m-1 \mapsto -m$ and $m \mapsto r$. It maps the indices $1, \dots, m-2$ onto the indices other than r and m, preserving their order; so the roots of $U_{\mathrm{chain},r-1}$ supported on the first $m-2$ indices give all the positive roots which involve neither r nor m. The roots $e_i \pm e_{m-1}$ with $i \le m-2$ give the roots $e_j \mp e_m$ with $j \notin \{r, m\}$, and e_{m-1} gives $-e_m$. The sums $e_i + e_m$ with $i \le m-2$ give the sums which involve e_r and not e_m, the sum $e_{m-1} + e_m$ gives $e_r - e_m$, and the differences $e_i - e_m$ with $i < r$, the only ones in $U_{\mathrm{chain},r-1}$, give $e_i - e_r$. These are the roots of $N_r Y$. The matrix of w_r permutes the vectors of the basis, so it maps each root subgroup of the table to its image with the same parameter, also in characteristic two. The two characters are homomorphisms which agree on these root subgroups, hence they agree.

Since the functions in $\mathcal{V}$ are left invariant under the rational element w_r, the coefficient $(U_{\mathrm{chain},r-1}, \psi(\lambda_{r-1} + \beta c_{m-1}))$ of ϕ at g equals the coefficient of step 4 at $w_r g$. It is therefore nonzero on $\mathcal{V}$. □

9.4 The descent and its terminal plane

Now take $m = 2n$ with $n \ge 2$, so that G is SO_{4n+1} or GSpin_{4n+1}. Let V_{tail} be the span of the basis vectors other than $e_1, \dots, e_n, f_1, \dots, f_n$. A linear form λ on V_{tail} with $\lambda(u) = 1$ defines the descent character

$$\psi_\lambda(g) = \psi\Big(\sum_{i<n} g_{i,i+1} + \lambda(\mathrm{pr}_{V_{\mathrm{tail}}}(g f_n))\Big)$$

on the unipotent radical of the parabolic subgroup stabilizing the flag spanned by $e_1, \dots, e_n$. Its connected stabilizer in the Levi subgroup is $\mathrm{GSpin}(\ker \lambda)$, and the *descent* with terminal covector λ is the Fourier coefficient with respect to ψ_λ, viewed as a function on that group. The descent is *generic* if it has a nonzero Whittaker coefficient.

The construction below involves two linear forms on the last three-dimensional space $V_{\mathrm{term}} = \langle e, u, f \rangle$, where $e = e_m$ and $f = f_m$. One, called λ_{c_0}, defines the group to which we descend; the other, called κ_{W}, defines a Whittaker character on that group.

Proposition 9.4 (A generic descent in characteristic two)**.** *Let $\mathcal{V}$ be a space of automorphic functions on GSpin_{4n+1}, stable under right translation, on which the Siegel–Whittaker coefficient is nonzero. Then there is $c_0 \in F$ such that $\mathcal{V}$ has a nonzero generic descent of length n whose terminal covector is*

$$\lambda_{c_0}(zu + xe + yf) = z + x + c_0 y$$

on V_{term}, extended by zero on the other hyperbolic pairs of V_{tail}. The restriction of Q to $\ker \lambda_{c_0} \cap V_{\mathrm{term}}$ is the binary form $x^2 + xy + c_0^2 y^2$, whose Arf invariant is the class of c_0 in $F/\{s^2 + s : s \in F\}$. If the connected centre acts on $\mathcal{V}$ through a character, it acts on the descent through the same character.

Proof. The proof transforms the coefficient provided by row iteration into a Whittaker coefficient of a descent, through a sequence of Fourier exchanges (Lemmas 5.9 and 7.9) and one Fourier expansion. Each exchange preserves the property of being nonzero on $\mathcal{V}$. The expansion, in step 3, is the one step which is not an equivalence: it produces a frequency c_0 which cannot be prescribed, and this is why Corollary 9.5 is needed afterwards.

Table 2 records the group and the character after each step. They have the same shape throughout. The weights of a basis of $\langle e_1, \dots, e_{m-1} \rangle$, the standard basis in steps 1 to 3 and a new basis from step 4 on, are divided into two chains. Write N for the unipotent radical of the parabolic subgroup with Levi subgroup $\mathrm{GL}_{m-1} \times \mathrm{SO}(V_{\mathrm{term}})$; it is generated by the root groups of the sums of two weights of the chains and by the terminal rows $r_i(V_{\mathrm{term}})$ of step 5. The group is generated by N, by the root groups of the positive differences within each chain, and by the root groups of certain differences between the two chains. The character is ψ of the sum of the coordinates on the adjacent roots of each chain and of one linear form on the terminal row of the last weight of each chain, written in the coordinates $v = zu + xe + yf$ of V_{term}. In this description the roots $A_{n-1} - \epsilon_{\mathrm{last}}$, $B_n + \epsilon_{\mathrm{last}}$ and B_n of step 1 are the coordinates y on the row of A_{n-1}, and x and z on the row of B_n.

Comparing the first line with the last: the chain of length n carries the form $z + x$ at the start and $z + x + c_0 y$ at the end, the chain of length $n - 1$ carries the form y throughout, and the differences between the chains, which at the start are of the form (short chain) minus (long chain), are at the end all the differences (long chain) minus (short chain). The last configuration is the one of a descent along the long chain followed by a Whittaker coefficient along the short chain. Figure 4 shows the differences between the chains at each step for $n = 4$.

Step 1: the starting coefficient. By Lemma 9.3 with $r = n - 1$ and $\beta = 1$, the coefficient $(U_{\mathrm{chain},n-1}, \psi(\lambda_{n-1} + c_{m-1}))$ is nonzero on $\mathcal{V}$. Rename the coordinate weights

$$A_i = e_i \ (i < n), \qquad B_j = e_{n-1+j} \ (j \le n), \qquad \epsilon_{\mathrm{last}} = e_m,$$

so that $B_n = e_{m-1}$. After the row separation of Lemma 7.8, the character has coefficient one on the adjacent roots of the two chains $A_1, \dots, A_{n-1}$ and $B_1, \dots, B_n$, and on $A_{n-1} - \epsilon_{\mathrm{last}}$, $B_n + \epsilon_{\mathrm{last}}$ and B_n. The group is generated by the positive root groups other than those of ϵ_{last} and of $A_{n-1} - B_j$.

step	the two chains	differences between the chains	forms on the last rows	obtained by
1	$A_1, \dots, A_{n-1}$ and $B_1, \dots, B_n$	$A_i - B_j$ with $i < n-1$	y on A_{n-1}, $z + x$ on B_n	row iteration with $r = n-1$ and $\beta = 1$; row separation
2	the same	$A_i - B_j$ with $j > i+1$, $B_j - A_i$ with $i \geq 2$, $j \leq i$	the same	interleaving with $d = -1$
3	the same	those of step 2 and $B_1 - A_1$	the same; in addition $c_0\delta$ on $B_1 - A_1$	Fourier expansion along δ; this produces c_0
4	$A'_1, \dots, A'_n$ and $B'_1, \dots, B'_{n-1}$	$A'_i - B'_j$ with $j > i-1$, $B'_j - A'_i$ with $j < i-1$	$z + x + c_0 y$ on A'_n, y on B'_{n-1}	the same group in the basis changed by g_{pair}
4	the same	$A'_i - B'_j$ with $i < n$	the same	interleaving with $d = 1$, in the opposite direction
6	the same, renamed $A_1, \dots, A_n$ and $B_1, \dots, B_{n-1}$	all $A_i - B_j$	the same	exchange of U_{B_j} for $U_{A_n - B_j}$, $j < n$

Table 2: The group and the character after each step of the proof of Proposition 9.4; step 4 has two stages. In steps 1 to 3, $A_i = e_i$ and $B_j = e_{n-1+j}$. In step 4 the new weights are those of the vectors $b_1, c_0 a_1 + b_2, \dots, c_0 a_{n-1} + b_n$ for the chain A' and $a_1, \dots, a_{n-1}$ for the chain B'; the term $c_0\delta$ of step 3 has become part of the coordinate on the adjacent root $A'_1 - A'_2$. The forms are $\lambda_{c_0} = z + x + c_0 y$ and $\kappa_{\text{W}} = y$. In the last line the terminal rows of the weights B_j are restricted to $L_{c_0} = \ker \lambda_{c_0}$, so that the group is no longer generated by N and root groups; it is the group R of step 6.

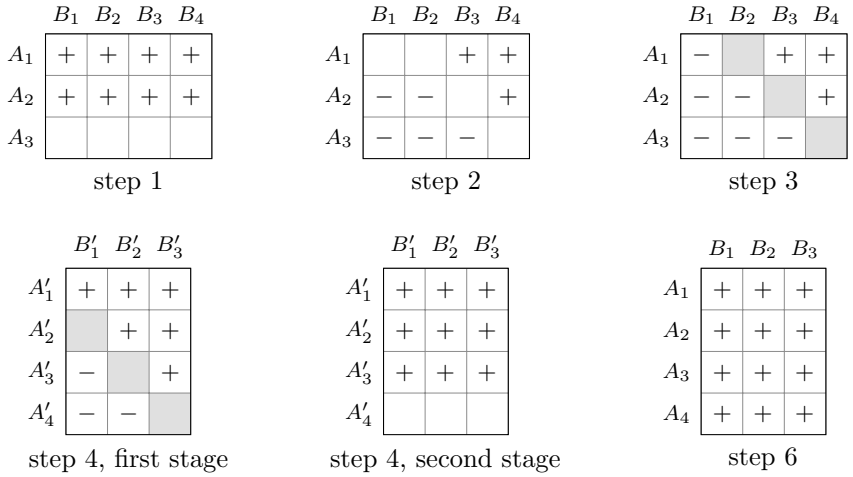


Figure 4: The differences between the two chains in Table 2, for $n = 4$, with the conventions of Figure 1: the sign + stands for a root (row) minus (column) and the sign − for its opposite. In step 3 and in the first stage of step 4 the signs are those of the weights for the grading of step 3. A shaded cell marks two members of the chains which have the same weight for that grading: these are the blocks (A_i, B_{i+1}) in which g_{pair} acts in step 4, and after the change of basis the pairs (A'_{i+1}, B'_i). In step 6 the chains have been renamed, and the terminal rows of the B_j are restricted to L_{c_0}.

Step 2: interleaving the two chains. The chains have lengths $r = n-1$ and $b = n = r+1$, so Lemma 7.9 applies with $d = -1$. The exchanges it performs are

$$U_{A_i - B_j} \longleftrightarrow U_{B_j - A_{i+1}}, \qquad i = n-2, \dots, 1, \quad j = 1, \dots, i+1.$$

Its auxiliary group is here the unipotent radical of the parabolic subgroup with Levi $\mathrm{GL}_{m-1} \times \mathrm{SO}(V_{\mathrm{term}})$. We verify the hypothesis on this group. Conjugation by a root group $U_{A_i - B_j}$ in the list changes only the row of A_i, with $i < n-1$, and conjugation by its partner $U_{B_j - A_{i+1}}$ changes only the row of B_j, with $j \le i+1 < n$. Neither is one of the two rows, those of A_{n-1} and B_n, on which the character of the auxiliary group is nontrivial, so that character is preserved. After the exchanges, the cross roots in the group are

$$A_i - B_j \quad (j > i+1), \qquad B_j - A_i \quad (i \ge 2,\ j \le i).$$

For $n = 2$ there are no exchanges and this description holds from the start.

Step 3: a Fourier expansion and the frequency c_0. Introduce a grading: B_1 has weight $2n$, both A_i and B_{i+1} have weight $2(n-i)$, ϵ_{last} has weight zero, and dual vectors have the opposite weights. Let $U_{>0}$ be generated by the root groups of positive weight. The positive cross weights are those of $A_i - B_j$ with $j > i+1$ and of $B_j - A_i$ with $j \le i$. Comparing with step 2, the group obtained there is the kernel in $U_{>0}$ of the coordinate δ on the single root $B_1 - A_1$. Every positive weight is at least two, so δ, which has weight two, is an additive character of $U_{>0}$; the squares of the coordinates on short roots have weight at least four, so they do not affect it. Expanding along δ, we find $c_0 \in F$ such that the coefficient on $U_{>0}$ of the character extended by $\psi(c_0\delta)$ is nonzero on $\mathcal{V}$. The expansion gives no control over c_0.

Step 4: change of basis in the Levi subgroup. The Levi subgroup of the grading has blocks B_1, the pairs (A_i, B_{i+1}), and V_{term}. On the part of weight two, the character is given by the vector $(c_0, 1)$ in the first row, by the trace on each intermediate 2×2 block, and on the last block by the matrix

$$C_{\mathrm{term}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

whose columns correspond to the roots with $-\epsilon_{\mathrm{last}}$, with $+\epsilon_{\mathrm{last}}$, and to the short root. Apply the matrix $g_{\mathrm{pair}} = \left(\begin{smallmatrix} c_0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, of determinant one, in each block (A_i, B_{i+1}). Then

$$g_{\mathrm{pair}}^{-1} \begin{pmatrix} c_0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad g_{\mathrm{pair}}^{t} C_{\mathrm{term}} = \begin{pmatrix} c_0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

and the trace characters on the intermediate blocks are unchanged. In the new basis there is a long chain b_1, $c_0 a_1 + b_2, \dots, c_0 a_{n-1} + b_n$ and a short chain $a_1, \dots, a_{n-1}$, with the dual basis transformed by the inverse transpose. Call the vectors of the long chain A'_i, $1 \le i \le n$, and those of the short chain B'_j, $1 \le j \le n-1$. Their weights are $2(n-i+1)$ and $2(n-j)$. So the root $A'_i - B'_j$ has weight $2(j-i+1)$: the positive roots between the two chains are the $A'_i - B'_j$ with $j > i-1$ and the $B'_j - A'_i$ with $j < i-1$. This is the situation at the end of Lemma 7.9 with $d = 1$. Each exchange of that lemma involves two roots $A'_i - B'_j$ and $B'_j - A'_{i+1}$ with $i < n$ and $j \le i-1$. Their root subgroups act on the auxiliary radical only in the rows of A'_i and of B'_j, which are not the last rows A'_n and B'_{n-1} of the two chains; so they preserve the character on these last rows. We apply the lemma in the opposite direction, and get a nonzero coefficient in which the long chain ends with the linear form λ_{c_0} on V_{term} and the short chain ends with

$$\kappa_{\mathrm{W}}(zu + xe + yf) = y.$$

The Levi subgroup acts linearly on the part of weight two, including the short root coordinate, so these are identities between characters and not only between their differentials.

Step 5: vector groups in the terminal space. To compare this coefficient with a descent followed by a Whittaker coefficient we need the following subgroups. For a hyperbolic index i and $v \in V_{\mathrm{term}}$ define $r_i(v)$ by

$$r_i(v)f_i = f_i + v + Q(v)e_i, \qquad r_i(v)w = w + B(w,v)e_i \quad (w \in V_{\mathrm{term}}), \qquad r_i(v)e_i = e_i,$$

and by the identity on the other basis vectors. Since $Q(v+w) = Q(v) + Q(w) + B(v,w)$, the map $v \mapsto r_i(v)$ is a homomorphism, and one checks on the basis that

$$[r_i(v), r_j(w)] = x_{e_i+e_j}(B(v,w)), \tag{9.4}$$

$$x_{e_i-e_j}(s)\, r_j(v)\, x_{e_i-e_j}(s)^{-1} = r_j(v)\, r_i(sv)\, x_{e_i+e_j}(sQ(v)). \tag{9.5}$$

The plane $L_{c_0} = \ker \lambda_{c_0} \subset V_{\mathrm{term}}$ has equation $z = x + c_0 y$, so Q restricts to $x^2 + xy + c_0^2 y^2$ on it. The vector $e + u$ lies in L_{c_0}, has $Q(e+u) = 1$, and satisfies $B(e+u, v) = y$ for $v = zu + xe + yf$; that is, it represents $\kappa_{\mathrm{W}}|_{L_{c_0}}$. Consequently the adjacent roots of the short chain, together with κ_{W} on its last row, define a generic character of the maximal unipotent subgroup of the quasi-split group $\mathrm{GSpin}(\ker \lambda_{c_0})$, whether or not L_{c_0} is isotropic.

Step 6: descent followed by a Whittaker coefficient. Rename the long and short chains $A_1, \dots, A_n$ and $B_1, \dots, B_{n-1}$. Taking the descent with terminal covector λ_{c_0} along the long chain and then the Whittaker coefficient along the short chain is a Fourier coefficient on the group R generated by all root groups $U_{e_i+e_j}$ among the chain indices, the positive differences within each chain, all $U_{A_i-B_j}$, and the terminal rows $r_{A_i}(V_{\mathrm{term}})$ and $r_{B_j}(L_{c_0})$. Its character is nontrivial on terminal rows only for A_n and B_{n-1}, where it is λ_{c_0} and κ_{W}. We pass from R to the group of step 4 by the exchanges, for $j = 1, \dots, n-1$,

$$X_j = U_{A_n - B_j}, \qquad Y_j = r_{B_j}(Fu) = U_{B_j},$$

and verify the hypotheses of Lemma 5.9.

When X_j is removed, the remaining generators still generate a group with these root subgroups. Among the differences, $A_n - B_j$ could reappear as a commutator only from $A_n - B_i$ and $B_i - B_j$ with $i < j$, and $A_n - B_i$ has been removed at an earlier step. Commutators with the terminal rows give terminal rows and sums, never a difference. The group X_j normalizes what remains, and preserves the character. Indeed, conjugating by X_j a difference $A_i - A_n$ or $B_j - B_k$ which is present produces the root $A_i - B_j$ with $i < n$, or $A_n - B_k$ with $k > j$; both are present and carry the trivial character. Conjugating a sum produces sums. And by (9.5) the action of X_j on the terminal rows adds to the row of A_n a vector of the row of B_j; this row is still L_{c_0}, since it is enlarged only after the exchange, and λ_{c_0} vanishes on L_{c_0}.

The group Y_j commutes with all the terminal rows, by (9.4) and because $B(u, \cdot) = 0$. Its commutators with differences come from $B_i - B_j$ with $i < j$ and from $A_i - B_j$ with $i < n$. They lie in the rows of the B_i with $i < j$, which have already been enlarged, and in the rows of the A_i with $i < n$, on all of which the character is trivial; the sums which appear carry the trivial character as well. Moreover Y_j meets the common subgroup trivially, because $Fu \cap L_{c_0} = 0$. Finally, by (9.5) with $v = au$,

$$[X_j(s), Y_j(a)] = r_{A_n}(sa\,u)\, x_{A_n+B_j}(sa^2),$$

on which the character is $\psi(\lambda_{c_0}(sa\,u)) = \psi(sa)$. This is a perfect pairing, so the exchange applies. It replaces the row $r_{B_j}(L_{c_0})$ by $r_{B_j}(V_{\mathrm{term}})$ and removes X_j. Since $\kappa_{\mathrm{W}}(u) = 0$, the character on the last row of the short chain is unchanged.

After all $n-1$ exchanges we have the group and character of step 4. Hence the coefficient on R is nonzero on $\mathcal{V}$. Since all the groups involved have compact quotients, Fubini's theorem identifies this coefficient with the Whittaker coefficient of the descent of a function in $\mathcal{V}$, which is therefore nonzero.

Step 7: central character and Arf invariant. Unipotent elements lift uniquely to GSpin. The rational elements of Levi subgroups and the Weyl elements used above lift because $H^1(F, \mathbb{G}_m) = 0$, and two lifts differ by a rational point of the centre, which acts trivially on automorphic functions. So the construction preserves the character through which the connected centre acts. Finally the Arf invariant of $x^2 + xy + c_0^2 y^2$ is the class of c_0^2, and $c_0^2 = c_0 + (c_0^2 + c_0)$ has the same class as c_0. □

Corollary 9.5 (The descent is to the split group)**.** *In Proposition 9.4, suppose that $\mathcal{V}$ is an irreducible automorphic representation whose local components at almost all places are subquotients of the representation induced from GL_2^n considered in Proposition 7.1. Then $c_0 = s^2 + s$ for some $s \in F$, and the generic descent is a nonzero generic automorphic function on split GSpin_{2n} over F.*

Proof. Suppose that c_0 is not of the form $s^2 + s$. Then $z^2 + z = c_0$ defines a separable quadratic extension of F, and by Chebotarev's theorem there is a place v, among the places where the local component is as in the statement, that does not split in it. At such a place the plane L_{c_0} is anisotropic, and the twisted Jacquet module of Proposition 7.1 is zero. But the nonzero global descent coefficient gives a nonzero functional on that Jacquet module. Indeed, translate a vector with nonzero descent on the right so that the integral which defines the descent is not zero at the identity, write it as a sum of pure tensors, one of which has a nonzero integral, and fix the components of this tensor at the places other than v. A change of variable in the integral shows that the functional obtained on the component at v has the required equivariance. This contradiction shows that $c_0 = s^2 + s$.

For such c_0, the vectors

$$e_{\mathrm{spl}} = su + s^2 e + f, \qquad f_{\mathrm{spl}} = (s+1)u + (s^2+1)e + f$$

lie in L_{c_0}, satisfy $Q(e_{\mathrm{spl}}) = Q(f_{\mathrm{spl}}) = 0$ and $B(e_{\mathrm{spl}}, f_{\mathrm{spl}}) = 1$, and κ_{W} takes the value one on each. So L_{c_0} is a hyperbolic plane over F. The isometry that fixes u and carries this pair to the standard hyperbolic pair identifies the target of the descent with the split group, and the generic character with a generic character of its standard maximal unipotent subgroup. □

Remark 9.6 (Two different uses of frequencies)**.** The proof uses Fourier expansion in two ways. In row iteration the frequency on a short root group can be prescribed, by Lemma 9.2. In step 3 of Proposition 9.4 the expansion only provides some frequency c_0, and no rational conjugation is available to change its class modulo $\{s^2 + s\}$: such a conjugation would amount to the surjectivity of $s \mapsto s^2 + s$ on F. That the class of c_0 is trivial is proved by the local argument of Corollary 9.5.

10 Realization for orthogonal and spin groups

We return to arbitrary positive characteristic and collect the results of Sections 6–9. The Eisenstein residue and the global Fourier coefficients have produced a nonzero generic descent. Two things remain: the descent must be cuspidal, and its excursion parameter, not only its unramified Hecke eigenvalues, must be σ. Both follow from ellipticity through Proposition 5.6.

Recall the situation of Section 8.1. The parameter σ takes values in $\mathrm{GSp}_{2n}(k)$ or $\mathrm{GSO}_{2n}(k)$; it is continuous, finitely ramified, semisimple, elliptic modulo scalars and strongly pure, and

its multiplier χ has finite order. The residual representation attached to σ lives on GSpin_{4n} or GSpin_{4n+1}, and its descent is a space of functions on split GSpin_{2n+1} or GSpin_{2n}, respectively.

Lemma 10.1 (The descent is cuspidal)**.** *Every function in the descent of the residual representation of Section 8.1 to the split group, and every right translate of such a function, is cuspidal.*

Proof. Let h be a function in the descent; it is complex valued, and we consider $\iota^{-1}h$. Let u_v be an element of the descent radical at a place v. Right translation by u_v multiplies the descent integral by $\psi_{\mathrm{desc},v}(u_v)$: at a point of the target group one conjugates u_v by this point and changes the variable of integration. So, when the components at the places other than v are fixed, the descent factors through the twisted Jacquet module at v, and the resulting map is equivariant for the target group at v. At a place where h is spherical and the residual representation is unramified, the local component of the latter is a subquotient of the induced representation of Proposition 7.1. The twisted Jacquet functor and the functor of invariants under a hyperspecial subgroup are exact. It follows that $\iota^{-1}h$ satisfies the Hecke equations of σ for the standard representation and for the multiplier, possibly in generalized form. This holds also when some of the inducing characters coincide, and for both summands in the second case of the proposition. Since σ is elliptic, Proposition 5.6 shows that $\iota^{-1}h$ is cuspidal. A constant term at a fixed argument is a finite sum, so h is cuspidal as well. A right translate of h is spherical outside a larger finite set of places, and the same argument applies to it. □

Theorem 10.2 (Orthogonal realization with multiplier)**.** *Let G be split GSpin_{2n+1} with $n \geq 1$, or split GSpin_{2n} with $n \geq 2$, over a global function field F of any characteristic. Let σ be a continuous, finitely ramified, semisimple parameter with values in the dual group $\mathrm{GSp}_{2n}(k)$ or $\mathrm{GSO}_{2n}(k)$, elliptic modulo scalars and strongly pure, whose multiplier χ has finite order. Then the excursion summand of ν_σ in the space of cusp forms on G contains a generic irreducible cuspidal automorphic representation. Its connected central character is $\mathrm{CFT}(\chi)$.*

Proof. Step 1: a nonzero generic descent. For $\mathrm{GSpin}_3 = \mathrm{GL}_2$ the theorem is Input 1. Otherwise take the residual representation of Section 8.1, which has a nonzero Siegel–Whittaker coefficient by Proposition 6.6. For the odd target, Proposition 7.10 gives a nonzero generic descent of length $n-1$. For the even target, a nonzero generic descent of length n to the split group over F is given by Proposition 8.5 in odd characteristic, and by Proposition 9.4 and Corollary 9.5 in characteristic two. In all cases the connected centre acts through $\omega_\sigma = \mathrm{CFT}(\chi)$.

Step 2: cuspidality and the parameter. By Lemma 10.1 the descent is cuspidal. Take a function in it with nonzero Whittaker coefficient and transport it to k by ι^{-1}; Lemma 5.5 shows that it is defined over a finite extension of $\mathbb{Q}_\ell$, at a fixed level and for a fixed central lattice. It satisfies the standard and multiplier Hecke equations of σ, so Proposition 5.6 provides a function in the excursion summand of ν_σ whose Whittaker coefficient is still nonzero. In the even case that proposition may first require replacing σ by its conjugate under the outer automorphism; applying the corresponding rational outer automorphism of G to the function removes this ambiguity.

Step 3: an irreducible representation. Transport the function back to $\mathbb{C}$. By Lemma I.9.1 the representation it generates is a finite sum of irreducible cuspidal representations, and the Whittaker functional is nonzero on one of them. Excursion operators are given by finite sums at each fixed level, so they commute with the transport of coefficients; and right translations preserve excursion summands. This representation therefore lies in the excursion summand of ν_σ. □

Theorem 10.3 (Spin groups)**.** *Let G be split Spin_{2n+1} with $n \geq 1$ or split Spin_{2n} with $n \geq 2$, in any characteristic, and let σ be a continuous, finitely ramified, semisimple, elliptic and strongly pure parameter with values in $\mathrm{PSp}_{2n}(k)$ or $\mathrm{PSO}_{2n}(k)$, respectively. Then the excursion summand of ν_σ in the space of cusp forms on G is nonzero.*

Proof. By Lemma 5.7, σ lifts to a parameter with values in $\mathrm{GSp}_{2n}(k)$ or $\mathrm{GSO}_{2n}(k)$ with multiplier of finite order, and the lift is again semisimple, elliptic modulo scalars and strongly pure. Theorem 10.2 realizes the lift on GSpin. Take a nonzero function in that realization, translate it on the right so that its value at the identity is not zero, and restrict it to Spin. The restriction is cuspidal, because the two groups have the same unipotent radicals of parabolic subgroups. By Lemma I.10.1 and the invariance of excursion operators under right translation [Laf18, Remark 12.4], the restriction lies in the excursion summand of the image of the lift in the dual group of Spin, which is σ. The cases D_2 and D_3 are included. □

Corollary 10.4 (Special orthogonal groups)**.** *Let H be split SO_{2n+1} with $n \geq 1$ or split SO_{2n} with $n \geq 2$, in any characteristic, and let σ be a continuous, finitely ramified, semisimple elliptic parameter with values in $\mathrm{Sp}_{2n}(k)$ or $\mathrm{SO}_{2n}(k)$, respectively, whose standard representation is pure of weight zero for every embedding of the coefficients into $\mathbb{C}$. Then the excursion summand of ν_σ in the space of cusp forms on H contains a generic irreducible cuspidal automorphic representation.*

Proof. Regard σ as a parameter with values in $\mathrm{GSp}_{2n}(k)$ or $\mathrm{GSO}_{2n}(k)$ with trivial multiplier. The irreducible summands of its standard representation have determinants of finite order, and the purity theorem of Section I.4 gives the hypotheses of the residual construction. Theorem 10.2 gives a generic cuspidal representation of GSpin with trivial connected central character. Its functions are invariant under the central torus $\mathbb{G}_m$, which is smooth and split in every characteristic, including two, and so they descend to the quotient H. Lemma I.10.4 identifies the excursion summand containing the descended representation as that of ν_σ, and the Whittaker coefficient is unchanged. □

11 Quadratic theta lifting and all spherical Hecke moments

To pass from the even orthogonal group SO_{2m} to the symplectic group Sp_{2m} we use the theta lift for the dual pair of equal rank. On parameters it adds a trivial line to the standard representation. Two things have to be proved about the lift of a generic cusp form: that it is nonzero, and that its Whittaker coefficient Λ satisfies the expected Hecke equations at all unramified places, that is, $\Lambda(T\Theta) = \eta(T)\Lambda(\Theta)$ for the spherical Hecke operators T. We obtain both from one computation, that of the twisted Jacquet module of the Weil representation along the maximal unipotent subgroup of the symplectic group (Lemma 11.1): it is the space of compactly supported Whittaker functions on the orthogonal group. Globally this gives a formula for the Whittaker coefficient of the lift as an integral of the Whittaker function of the cusp form against a test function at our disposal (Proposition 11.5), from which nonvanishing is read off. At unramified places the test function attached to a Hecke operator applied to the spherical vector is computed explicitly (Proposition 11.3); it is governed by the branching rule from SO_{2m+1} to SO_{2m}, and this gives these equations. A generalized Hecke eigenfunction with a nonzero Whittaker coefficient is then extracted by Lemma I.8.8 (Corollary 11.6). The Weil representation is written in a model that uses the quadratic form and not only its bilinear form, so that the argument applies in characteristic two.

11.1 A model of the Weil representation in every characteristic

Let $V = V_+ \oplus V_+^\vee$ with $\dim V_+ = m \geq 2$,

$$Q(x,\xi) = \xi(x), \qquad B((x,\xi),(x',\xi')) = \xi(x') + \xi'(x),$$

and put $G_0 = \mathrm{SO}(V)$ and $G_1 = \mathrm{Sp}_{2m}$. Over a local field, G_1 acts by right multiplication on $\mathrm{Mat}_{m,2m} = V_+ \otimes F^{2m}$, hence on its Schwartz space. Writing a matrix as $(\mathbf{x},\mathbf{y})$ with two blocks of size $m \times m$, we transport this action by the partial Fourier transform

$$(\mathcal{F}\Phi)(\mathbf{x},\boldsymbol{\xi}) = \int \Phi(\mathbf{x},\mathbf{y})\psi(-\operatorname{tr}(\mathbf{y}^t\boldsymbol{\xi}))\,d\mathbf{y}$$

to a representation $\mathscr{W}_\psi$ of G_1 on $\mathcal{S}(V^m)$, where $z = (z_1,\dots,z_m)$ and $z_i = (x_i,\xi_i) \in V$. Let $m(A) = \mathrm{diag}(A, A^{-t})$, $n(C) = \left(\begin{smallmatrix} 1 & C \\ 0 & 1 \end{smallmatrix}\right)$ with $C = C^t$, and $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then

$$\mathscr{W}_\psi(m(A))\phi(z) = |\det A|^m \phi(zA), \tag{11.1}$$

$$\mathscr{W}_\psi(n(C))\phi(z) = \psi\left(\sum_i C_{ii}Q(z_i) + \sum_{i<j} C_{ij}B(z_i,z_j)\right)\phi(z), \tag{11.2}$$

$$\mathscr{W}_\psi(w)\phi(z) = \int \phi(z')\psi\left(\sum_i B(z_i,z_i')\right)dz'. \tag{11.3}$$

The first follows from the substitution $\mathbf{y} \mapsto \mathbf{y}A^{-t}$ in the Fourier transform. For the second, substitute $\mathbf{y} + \mathbf{x}C$ for $\mathbf{y}$; the phase that remains is $\psi(\operatorname{tr}(C^t\mathbf{x}^t\boldsymbol{\xi}))$, and its diagonal and off-diagonal terms are the two sums displayed. The third is the Fourier transform in all variables. Since these operators come from an action of G_1 by pullback, they define a representation of G_1 itself; no metaplectic cover occurs, and no quadratic phase is divided by two. The group $O(V)$ acts by $r(h)\phi(z) = \phi(h^{-1}z)$, and the two actions commute.

Whittaker data. Fix a hyperbolic basis e_i, f_i of V with $e_i \in V_+$. Let U_1 be the maximal unipotent subgroup of G_1 consisting of the $m(u)n(C)$ with u upper unipotent, and for $c \in F^\times$ let

$$\chi_{1,c}(m(u)n(C)) = \psi\left(\sum_{i<m} u_{i,i+1} + cC_{mm}\right).$$

On the orthogonal side let U_0 be the maximal unipotent subgroup of G_0 fixing the flag of the e_i, and

$$\mathbf{v}_c = (e_1,\dots,e_{m-1},e_m + cf_m) \in V^m.$$

The span of $\mathbf{v}_c$ is stable under U_0, which gives a homomorphism frame_c from U_0 onto the group U_{GL_m} of upper unipotent matrices, defined by $n\mathbf{v}_c = \mathbf{v}_c\,\mathrm{frame}_c(n)$. Let $\psi_{\mathrm{GL}}(u) = \psi(\sum_i u_{i,i+1})$ and $\chi_{0,c} = \psi_{\mathrm{GL}} \circ \mathrm{frame}_c$. This is a nondegenerate character of U_0, with coefficients $1,\dots,1,c$ on the simple roots, and every nondegenerate character of U_0 is conjugate under the torus to one of these: the first $m-1$ coefficients can be normalized to 1, and the ratio of the last two is then c.

Lemma 11.1 (The Whittaker quotient of the Weil representation)**.** *Over a nonarchimedean local field of any characteristic, the map $\phi \mapsto \mathcal{J}_\phi$,*

$$\mathcal{J}_\phi(h) = \int_{U_{\mathrm{GL}_m}} \phi(h^{-1}\mathbf{v}_c u)\psi_{\mathrm{GL}}(u)^{-1}\,du, \tag{11.4}$$

induces an isomorphism of representations of G_0

$$J_{U_1,\chi_{1,c}}(\mathscr{W}_\psi) \simeq \mathfrak{c}\mathrm{Ind}_{U_0}^{G_0}\chi_{0,c}^{-1}.$$

In particular every smooth function on G_0 that transforms by $\chi_{0,c}^{-1}$ on the left under U_0 and has compact support modulo U_0 is of the form $\mathcal{J}_\phi$.

Proof. We take coinvariants in two stages, first along the subgroup of the $n(C)$ and then along that of the $m(u)$. By (11.2) the first stage restricts functions to a fibre of the map which sends z to the values $Q(z_i)$ and $B(z_i, z_j)$; the second, by (11.1), is an integral over U_{GL_m} acting on the right. The points of the fibre where z has rank less than m do not contribute, and the others form one orbit of G_0.

Step 1: the fibre. The coinvariants for the characters $C \mapsto \psi(cC_{mm})$ of the group of symmetric matrices are the restrictions of Schwartz functions to the closed set

$$\mathcal{Q}_c = \{z : Q(z_i) = 0 \ (i < m),\ Q(z_m) = c,\ B(z_i, z_j) = 0 \ (i < j)\};$$

a function vanishing on $\mathcal{Q}_c$ is killed by averaging against the character, and restriction is surjective. In characteristic two the conditions on the $Q(z_i)$ are independent of those on the $B(z_i, z_j)$. At a point where z has rank m the differentials of the equations are independent: in terms of $\beta_{ij} = B(z_i, dz_j)$, which are independent linear forms, they are β_{ii} and $\beta_{ij} + \beta_{ji}$. So this part of $\mathcal{Q}_c$ is smooth in every characteristic.

Step 2: frames of rank less than m. Stratify the rest of $\mathcal{Q}_c$ by the first index j such that z_j lies in the span of $z_1, \dots, z_{j-1}$; here $j < m$, because $Q(z_m) = c \neq 0$ while the span of the others is isotropic. Acting by an element of U_{GL_m}, which preserves ψ_{GL} up to conjugation within the stratum, we may assume $z_j = 0$. Then the root subgroup $u_{j,j+1}(a)$ fixes z while ψ_{GL} is nontrivial on it, so the fibre of the coinvariants at z is zero. This is the statement at one point; it extends to the stratum as follows. Fix a compact subset of the stratum. On a neighbourhood of one of its points where a given minor of $(z_1, \dots, z_{j-1})$ is invertible, the coefficients which express z_j in terms of $z_1, \dots, z_{j-1}$ are continuous functions, and so is the element of U_{GL_m} which they define. Conjugating by it a fixed element of $u_{j,j+1}$ on which the character is not trivial, we get an element of the stabilizer of the frame which depends continuously on the frame and on which the character is not trivial. Cover the compact subset by finitely many such neighbourhoods with compact closure. All the elements obtained lie in one compact open subgroup of U_{GL_m}, and the average over this subgroup against the character kills every function supported in the compact subset: each orbit of the subgroup which meets the support has in its stabilizer an element of the subgroup on which the character is not trivial. Since coinvariants are exact, the strata can be removed one after the other, in the order given by the closed conditions that the first j columns are dependent.

Step 3: frames of rank m. These form one orbit of G_0. Indeed the first $m-1$ vectors span an isotropic subspace and can be moved to $e_1, \dots, e_{m-1}$; then z_m is, modulo their span, of the form $xe_m + yf_m$ with $xy = c$, the torus makes $x = 1$, and the unipotent radical of the stabilizer of the span removes the other coordinates. This uses the full orthogonal group; but the reflection in $e_m - cf_m$ fixes $\mathbf{v}_c$ and has determinant -1, or Dickson invariant one in characteristic two, so G_0 is transitive as well. The stabilizer of $\mathbf{v}_c$ is $S_c = \ker \mathrm{frame}_c$. Writing an element of U_0 as

$$n = \mathrm{diag}(u, u^{-t}) \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}, \qquad T^t = -T, \quad T_{ii} = 0,$$

one finds $\mathrm{frame}_c(n) = u(1 + c(Te_m)e_m^t)$, so that S_c is defined by $u = 1 - c(Te_m)e_m^t$; it is isomorphic to an affine space, in every characteristic. Since the orbit and the stabilizer are smooth, the orbit is $S_c \backslash G_0$.

Step 4: the integral. On functions on $S_c \backslash G_0$ the remaining coinvariants, for $U_{\mathrm{GL}_m} = U_0/S_c$ and the character ψ_{GL}, are given by the integral (11.4), which converges because $u \mapsto h^{-1}\mathbf{v}_c u$ is a closed embedding. Its image consists of all functions with compact support modulo U_0, and its kernel is spanned by the differences $\phi(\cdot u) - \psi_{\mathrm{GL}}(u)\phi$, as one sees with local sections

of $G_0 \to U_0\backslash G_0$. A Schwartz function whose support meets the strata of step 2 is, by that step, congruent modulo the relations to one with compact support in the orbit of frames of rank m, and the integral vanishes on the relations. Finally $\mathcal{J}_{r(h_0)\phi}(h) = \mathcal{J}_\phi(hh_0)$. □

11.2 The spherical vector

Let v be a place where the residue characteristic is arbitrary, ψ has conductor $\mathcal{O}_v$ and $c \in \mathcal{O}_v^\times$. Let K_0 and K_1 be the hyperspecial subgroups of G_0 and G_1 defined by the lattice spanned by the basis, and $\phi^0 = 1_{V(\mathcal{O}_v)^m}$. By (11.1)–(11.3) it is fixed by K_1, and it is fixed by K_0. For $J = G_0, G_1$ and a dominant weight γ of the dual group let T_γ^J be the Hecke operator corresponding under the normalized Satake isomorphism to the character of the irreducible representation of highest weight γ, and t_γ the corresponding element of the torus:

$$t_\lambda = \mathrm{diag}(\varpi^{\lambda_i}, \varpi^{-\lambda_i}) \in G_0, \qquad t_\mu = m(\mathrm{diag}(\varpi^{\mu_i})) \in G_1.$$

Lemma 11.2 (Whittaker functionals and Hecke operators)**.** *Let J be G_0 or G_1, let Λ be a Whittaker functional on a smooth representation of J, for a character of conductor $\mathcal{O}_v$ on each simple root, and let v_0 be a vector fixed by the hyperspecial subgroup. Then*

$$\Lambda(T_\gamma^J v_0) = \delta_{B_J}(t_\gamma)^{-1/2}\Lambda(t_\gamma v_0). \tag{11.5}$$

Proof. No irreducibility is assumed, so this is a statement about the module of compactly supported Whittaker functions invariant under the hyperspecial subgroup, of which the function $g \mapsto \Lambda(gv_0)$ is a linear functional. Precisely, pair compactly supported functions which transform by ψ^{-1} under the maximal unipotent subgroup U with Whittaker functions, which transform by ψ, by integration over $U\backslash J$; this pairing is bilinear, with no complex conjugation. The compact module has a basis b_γ indexed by the dominant γ: b_γ is supported on $Ut_\gamma K$ and normalized by $b_\gamma(t_\gamma) = \delta_{B_J}(t_\gamma)$, so that $\langle b_\gamma, W\rangle = W(t_\gamma)$ for every Whittaker function W invariant under K. For a Hecke operator T let $T^*(g) = T(g^{-1})$; then $\langle T^*b, W\rangle = \langle b, TW\rangle$. With $W(g) = \Lambda(gv_0)$ the two sides of (11.5) are $\langle b_0, T_\gamma^J W\rangle = \langle (T_\gamma^J)^* b_0, W\rangle$ and $\delta_{B_J}(t_\gamma)^{-1/2}\langle b_\gamma, W\rangle$. So the lemma follows from the identity

$$(T_\gamma^J)^* b_0 = \delta_{B_J}(t_\gamma)^{-1/2} b_\gamma$$

in the compact module; the transpose matters when $-w_0\gamma \neq \gamma$. To verify the identity, pair both sides with the Whittaker functions W_t of the unramified principal series, normalized by $W_t(1) = 1$, for t in a Zariski-dense set of unramified parameters. By the formula of Casselman and Shalika [CS80, Theorem 5.4], $W_t(t_\gamma) = \delta_{B_J}(t_\gamma)^{1/2}\chi_\gamma(t)$, where χ_γ is the character of highest weight γ; and $T_\gamma^J W_t = \chi_\gamma(t) W_t$. So both sides pair with W_t to $\chi_\gamma(t)$. An element of the compact module is a finite combination of the b_λ, and its pairing with W_t is the corresponding combination of the functions $\delta_{B_J}(t_\lambda)^{1/2}\chi_\lambda(t)$ of t. These are regular functions which are linearly independent, also on a Zariski-dense set; so an element which pairs to zero with all these W_t is zero. □

A weight $\lambda = (\lambda_1, \dots, \lambda_m)$ of SO_{2m} is said to interlace a dominant weight μ of SO_{2m+1}, written $\lambda \prec \mu$, if

$$\mu_i \geq \lambda_i \geq \mu_{i+1}\ (i < m), \qquad \mu_m \geq |\lambda_m|. \tag{11.6}$$

Such λ are dominant. Put $d_0(\lambda) = \sum_i (m-i)\lambda_i$ and $d_1(\mu) = \sum_i (m-i+1)\mu_i$, so that $\delta_{B_0}(t_\lambda)^{1/2} = q_v^{-d_0(\lambda)}$ and $\delta_{B_1}(t_\mu)^{1/2} = q_v^{-d_1(\mu)}$.

Proposition 11.3 (Hecke operators on the spherical vector)**.** *For dominant μ and λ,*

$$\mathcal{J}_{\mathscr{W}_\psi(t_\mu)\phi^0}(t_\lambda) = \begin{cases} q_v^{-d_0(\lambda)-d_1(\mu)}, & \lambda \prec \mu, \\ 0, & \text{otherwise.} \end{cases} \tag{11.7}$$

Consequently, if W_λ^0 denotes the function in $\mathrm{cInd}_{U_0}^{G_0} \chi_{0,c}^{-1}$ supported on $U_0 t_\lambda K_0$ with value $q_v^{-d_0(\lambda)}$ at t_λ, then

$$\mathcal{J}_{\mathscr{W}_\psi(T_\mu^{G_1})\phi^0} = \sum_{\lambda \prec \mu} W_\lambda^0,$$

and, if Λ is a Whittaker functional for $\chi_{0,c}$ on a smooth representation of G_0 and v_0 is a vector fixed by K_0,

$$\int_{U_0\backslash G_0} \Lambda(hv_0)\, \mathcal{J}_{\mathscr{W}_\psi(T_\mu^{G_1})\phi^0}(h)\, dh = \sum_{\lambda \prec \mu} \Lambda(T_\lambda^{G_0} v_0).$$

The operator $\sum_{\lambda\prec\mu} T_\lambda^{G_0}$ corresponds under the Satake isomorphism to the restriction of the character of highest weight μ of SO_{2m+1} along $s \mapsto \mathrm{diag}(s,1)$.

Proof. By (11.1), $\mathscr{W}_\psi(t_\mu)\phi^0$ is $q_v^{-m\sum\mu_j}$ times the characteristic function of the z with $z_j \in \varpi^{-\mu_j} V(\mathcal{O}_v)$. In the integral (11.4) at $h = t_\lambda$ the variable is the upper unipotent matrix u, and the jth column of $t_\lambda^{-1}\mathbf{v}_c u$ has coordinates $\varpi^{-\lambda_i} u_{ij}$ on e_i, $i \le j$, with in addition the coordinate $c\varpi^{\lambda_m}$ on f_m for $j = m$. The integrand is therefore nonzero only if $\lambda_j \le \mu_j$ for all j, from the diagonal entries $u_{jj} = 1$, and $\lambda_m \ge -\mu_m$, from the coordinate on f_m; and then the domain of integration is $u_{ij} \in \varpi^{\lambda_i-\mu_j}\mathcal{O}_v$ for $i < j$. The integral over the variable $u_{i,i+1}$ against ψ is the volume $q_v^{\mu_{i+1}-\lambda_i}$ of its domain if this domain is contained in $\mathcal{O}_v$, that is if $\lambda_i \ge \mu_{i+1}$, and zero otherwise; the other variables contribute the volumes $q_v^{\mu_j-\lambda_i}$. This gives the conditions (11.6), and the product of the volumes and of the factor $q_v^{-m\sum\mu_j}$ is $q_v^{-d_0(\lambda)-d_1(\mu)}$.

The function $\mathcal{J}_{\mathscr{W}_\psi(t_\mu)\phi^0}$ is invariant under K_0 on the right, so by the Iwasawa decomposition it is determined by its values at the t_λ, which vanish unless λ is dominant because of the character $\chi_{0,c}$. Applying Lemma 11.2 for G_1 to the functionals $\phi \mapsto \mathcal{J}_\phi(h)$, which are Whittaker functionals on $\mathscr{W}_\psi$ by Lemma 11.1, we get $\mathcal{J}_{\mathscr{W}_\psi(T_\mu^{G_1})\phi^0} = q_v^{d_1(\mu)}\mathcal{J}_{\mathscr{W}_\psi(t_\mu)\phi^0}$, which is the sum of the W_λ^0. The integral of $\Lambda(hv_0)$ against W_λ^0 is, by the Iwasawa decomposition, $\delta_{B_0}(t_\lambda)^{-1/2}\Lambda(t_\lambda v_0)$, and this is $\Lambda(T_\lambda^{G_0}v_0)$ by Lemma 11.2 for G_0. The last assertion is the branching rule

$$\mathrm{tr}_{B_m,\mu}(\mathrm{diag}(s,1)) = \sum_{\lambda\prec\mu} \mathrm{tr}_{D_m,\lambda}(s) \quad [\text{Mol06, §4.1, (4.4)}],$$

in which each λ occurs with multiplicity one. □

Remark 11.4. For $m = 2$ and $\mu = (1,0)$ the interlacing weights are $\lambda = (0,0)$ and $(1,0)$, with values q_v^{-2} and q_v^{-3} in (11.7); multiplied by $q_v^{d_1(\mu)} = q_v^2$ they are 1 and q_v^{-1}, the values of W_λ^0. The branching rule is $\mathrm{Std}_{\mathrm{SO}_5}|_{\mathrm{SO}_4} = \mathrm{Std}_{\mathrm{SO}_4} \oplus 1$: the lift adds a trivial line to the standard parameter.

11.3 The global lift

Let f be a cusp form on $G_0(\mathbb{A})$ in a generic cuspidal representation, with Satake parameters s_v^0 at the unramified places, and choose $c \in F^\times$ such that its Whittaker coefficient $\mathrm{Wh}_{0,c} f$

for $\chi_{0,c}$ is not zero. For a Schwartz function ϕ on $V(\mathbb{A})^m$ define

$$\theta(h,g;\phi)=\sum_{z\in V(F)^m}(r(h)\mathscr{W}_\psi(g)\phi)(z),\qquad \Theta(f,\phi)(g)=\int_{[G_0]}f(h)\theta(h,g;\phi)\,dh.$$

The theta function is invariant under $G_0(F)\times G_1(F)$: for the three kinds of generators of $G_1(F)$ this follows from the product formula, from $\psi(F)=1$, and from the Poisson summation formula [Wei64, §§29–30,40–41]. The support of a cusp form of given level is compact modulo the centre [Mor82, Chapter I, Proposition 5.3], and here the centre is finite, so that its group of adelic points is compact. Choose a compact subset of $G_0(\mathbb{A})$ which maps onto this support. For h in this subset and g in a compact set, the supports of the functions $r(h)\mathscr{W}_\psi(g)\phi$ lie in one compact subset of $V(\mathbb{A})^m$, which contains only finitely many points of $V(F)^m$. So the integral defining the lift, its integrals over compact unipotent quotients and the unfolding below are finite sums. With Schwartz functions taking algebraic values they make sense over k as well, after Lemma 5.5.

Proposition 11.5 (Whittaker coefficient of the theta lift)**.** *The Whittaker coefficient of the lift for $\chi_{1,c}$ is*

$$\mathrm{Wh}_{1,c}\,\Theta(f,\phi)(g)=\int_{U_0(\mathbb{A})\backslash G_0(\mathbb{A})}\mathrm{Wh}_{0,c}\,f(h)\,\mathcal{J}_{\mathscr{W}_\psi(g)\phi}(h)\,dh. \tag{11.8}$$

Moreover there is ϕ, equal to ϕ_v^0 outside finitely many places, such that $v=\Theta(f,\phi)$ and the functional $\Lambda=\mathrm{Wh}_{1,c}(-)(1)$ satisfy

$$\Lambda(Tv)=\eta(T)\Lambda(v),\qquad \Lambda(v)\neq 0$$

for every T in the tensor product of the spherical Hecke algebras at the places outside a finite set. This set contains the places v where f is not spherical, where the conductor of ψ is not $\mathcal{O}_v$, or where c is not a unit; and η is the character whose standard parameter at v is $\mathrm{Std}(s_v^0)\oplus 1$.

Proof. Step 1: unfolding. The global computation is parallel to the proof of Lemma 11.1. Integrating against the character of the $n(C)$ restricts the sum over $V(F)^m$ to $\mathcal{Q}_c(F)$. In an orbit of $U_{\mathrm{GL}_m}(F)$ consisting of frames of rank less than m, choose the frame in which the first dependent column is zero, $z_j=0$, as in Lemma 11.1. Unfolding the sum over this orbit leaves an integral over the quotient by the stabilizer of the frame. The stabilizer contains $u_{j,j+1}$, on which the character is not trivial, so the integral over $u_{j,j+1}(F)\backslash u_{j,j+1}(\mathbb{A})$ vanishes. The points of rank m form the orbit $S_c(F)\backslash G_0(F)$, and unfolding the sum against the integral over $[G_0]$, then combining the integral over $[S_c]$ with the integral over $[U_{\mathrm{GL}_m}]$ into an integral of f over $[U_0]$, gives (11.8).

Step 2: Hecke operators. Let v be a place which is not in the finite set of the statement, and let $\phi=\phi_v^0\otimes\phi^v$. We apply Proposition 11.3 to the representation of $G_0(F_v)$ generated by f and to its Whittaker functional $f'\mapsto\mathrm{Wh}_{0,c}\,f'(h^v)$. With (11.8) we get

$$\Lambda(T_\mu^{G_1}\Theta(f,\phi))=\sum_{\lambda\prec\mu}\Lambda(\Theta(T_\lambda^{G_0}f,\phi))=\eta(T_\mu^{G_1})\Lambda(\Theta(f,\phi)),$$

since f is an eigenvector of the operators $T_\lambda^{G_0}$ and the sum of the eigenvalues is the character of highest weight μ at $\mathrm{diag}(s_v^0,1)$. The components at the other places were arbitrary, so the argument can be repeated at other unramified places, and products of operators at one place are linear combinations of the $T_\mu^{G_1}$.

Step 3: nonvanishing. Translate f so that $\mathrm{Wh}_{0,c} f(1) \neq 0$, and let $K = \prod_v K_v$ be a compact open subgroup of $G_0(\mathbb{A})$ fixing f, hyperspecial at almost all places, such that $\chi_{0,c}$ is trivial on $U_0(\mathbb{A}) \cap K$. By Lemma 11.1 at the remaining places, and with ϕ_v^0 at the others, there is ϕ such that $\mathcal{J}_\phi$ is the function supported on $U_0(\mathbb{A})K$ which is $\chi_{0,c}^{-1}$ on $U_0(\mathbb{A})$. Then the right side of (11.8) at $g = 1$ is $\mathrm{Wh}_{0,c} f(1)$ times the volume of $(U_0(\mathbb{A}) \cap K)\backslash K$. □

The proposition gives the values of one functional Λ on the vectors Tv. A generalized eigenfunction is obtained from it as follows. The functional Λ at the fixed level is a pairing with a function of compact support. The identities $\Lambda(Tv) = \eta(T)\Lambda(v)$ with $\Lambda(v) \neq 0$ show that this pairing does not vanish after localization at the maximal ideal of the transposed character: an operator T that is invertible there cannot kill it, because $\Lambda(Tv) = \eta(T)\Lambda(v) \neq 0$. Lemma I.8.8 then gives a finite quotient of the Hecke module, killed by a power of that maximal ideal, in which the pairing is nonzero, and a functional on that quotient which is nonzero on it is the required function.

Corollary 11.6 (A generic function with the lifted Hecke character)**.** *There is an automorphic function of finite level on $G_1(\mathbb{A})$ on which the spherical Hecke algebras at the unramified places act through η, as a generalized character, and whose principal Whittaker coefficient is nonzero.*

12 Generic cuspidal support and symplectic realization

This section proves the realization theorem for symplectic groups (Theorem 12.5). Let σ be an elliptic parameter with values in SO_{2n+1} and $\varrho = \mathrm{Std}\,\sigma$. By Proposition 5.2 the space of invariants of ϱ has dimension zero or one. If it has dimension one, $\varrho \ominus 1$ is an elliptic even orthogonal parameter of dimension $2n$, and the theta lift of Section 11 of a generic cusp form with that parameter has standard parameter ϱ on Sp_{2n}. If it has dimension zero, we must start from $\varrho \oplus 1$ on SO_{2n+2}, and the lift, on Sp_{2n+2}, has standard parameter $\varrho \oplus 1^{\oplus 2}$. The lift is then not cuspidal, and its cuspidal support is a representation of $\mathrm{GL}_1 \times \mathrm{Sp}_{2n}$ whose second factor has parameter ϱ. To make this precise we need a version of cuspidal support which applies to a generalized Hecke eigenfunction and which keeps track of the Whittaker coefficient (Proposition 12.2); this lemma is the main work of the section.

12.1 Cuspidal support of a generic function

Lemma 12.1 (No doubled line in a cuspidal representation)**.** *Let π be an irreducible cuspidal automorphic representation of split Sp_{2s}, $s \geq 1$, and let τ be a continuous semisimple representation of Γ_F of dimension $2s+1$, unramified at almost all places. Suppose that the eigenvalues of Frobenius on τ at the unramified places are algebraic numbers all of whose complex absolute values are one, and that at almost all places the characteristic polynomial of Frobenius on τ is that of the standard Satake parameter of π. Then the space of invariants of Γ_F in τ has dimension at most one.*

Proof. We use one Eisenstein series of rank one. Let $H = \mathrm{Sp}_{2s+2}$, and let $P = MU$ be the stabilizer of an isotropic line, with $M = \mathbb{G}_m \times \mathrm{Sp}_{2s}$. The roots of U are $e_1 \pm e_j$, for $2 \leq j \leq s+1$, and $2e_1$. Let w_0 represent the element of the Weyl group which changes the sign of e_1; it normalizes M and maps U to the opposite unipotent radical. For $z \in \mathbb{C}$ let $I(z)$ be the space induced from $|\cdot|^z \otimes \pi$, with the normalization by $\delta_P^{1/2}$, and

$$M(z) : I(z) \longrightarrow I(-z), \qquad (M(z)\Phi)(g) = \int_{U(\mathbb{A})} \Phi(w_0^{-1} u g)\, du,$$

the intertwining operator. The integral converges when $\Re z$ is large, and on the sections of a given level $M(z)$ is a rational function of q^{-z}.

Step 1: the poles are simple. The parabolic subgroup P is maximal, and the inducing representation is cuspidal and unitary: the centre of Sp_{2s} is finite, so its cusp forms are square integrable. So a pole of $M(z)$ at a real point $z > 0$ is simple [Mor82, Chapter III, Theorem 5.2(i)].

Step 2: the unramified places. Write $\pi = \bigotimes'_v \pi_v$, and let S be a finite set of places outside of which π and τ are unramified and the characteristic polynomials agree. For $\Re z$ large and a pure tensor $\Phi = \bigotimes_v \Phi_v$, the operator $M(z)$ is the product of the local operators $A_v(z)$ defined by the same integral over $U(F_v)$. Let $v \notin S$, and let t_v be the Satake parameter of π_v, an element of SO_{2s+1}. The representation π_v is a subrepresentation of an unramified principal series of Sp_{2s}, so $I_v(z)$ is a subrepresentation of an unramified principal series of H, on which $A_v(z)$ is the intertwining operator of the Weyl element w_0. The coroots of the roots of U are $e_1 \pm e_j$ and e_1: on them the character of the torus which defines this principal series takes the values $q_v^{-z}a$, where a runs over the eigenvalues of t_v in the standard representation. So the formula of Gindikin and Karpelevich [Cas80, Theorem 3.1] gives, for the spherical vector Φ_v^0 of $I_v(z)$ whose value at the identity is the spherical vector ξ_v^0 of π_v,

$$A_v(z)\Phi_v^0 = \frac{\det(1 - q_v^{-1-z}t_v)}{\det(1 - q_v^{-z}t_v)}\,\Phi_{v,-}^0 = \frac{L_v(z,\tau)}{L_v(1+z,\tau)}\,\Phi_{v,-}^0,$$

where $\Phi_{v,-}^0$ is the spherical vector of $I_v(-z)$ with the same value at the identity. The formula holds for regular characters in the region where the integral converges absolutely; both sides are holomorphic in the character there, so it holds for the characters above, whose entries along the Levi subgroup may coincide, and then for all z by meromorphic continuation.

Step 3: the other places. Let $v \in S$. Choose a nonzero vector ξ_v of π_v and a compact open subgroup U_v of $U(F_v)$, and let $\Phi_{v,z} \in I_v(z)$ be the section which is supported in the open subset $P(F_v)w_0^{-1}U_v$ and is given there by

$$\Phi_{v,z}(p\,w_0^{-1}u) = \delta_P^{1/2}(p)\,|a|^z\,\pi_v(m)\xi_v \qquad (u \in U_v),$$

where (a, m) is the image of p in $M(F_v)$. The map $P \times U \to Pw_0^{-1}U$ is bijective, so this is well defined; it is a smooth vector, and its restriction to a maximal compact subgroup depends holomorphically on z. For every z,

$$(A_v(z)\Phi_{v,z})(1) = \int_{U(F_v)} \Phi_{v,z}(w_0^{-1}u)\,du = \mathrm{vol}(U_v)\,\xi_v,$$

an integral over a compact set.

Step 4: conclusion. Take Φ_z with the components of step 3 at the places of S and the spherical vectors elsewhere. It is a holomorphic family of sections of a fixed level, and by steps 2 and 3, first for $\Re z$ large and then for all z by continuation,

$$(M(z)\Phi_z)(1) = \frac{L^S(z,\tau)}{L^S(1+z,\tau)} \prod_{v \in S} \mathrm{vol}(U_v) \cdot \xi,$$

where ξ is the nonzero vector of π with components ξ_v at the places of S and ξ_v^0 elsewhere. The representation τ defines a semisimple lisse sheaf on the complement of S, pure of weight zero. By Lemma 5.4 the function $L^S(z,\tau)$ has a pole at $z = 1$ of order $\dim \tau^{\Gamma_F}$, and $L^S(2,\tau)$ is finite and not zero. So $M(z)$ has a pole at $z = 1$ of order at least $\dim \tau^{\Gamma_F}$, and by step 1 this dimension is at most one. □

For generic π the functorial transfer to general linear groups [Lom19, Theorem 7.2], which is proved under Kim's Assumption A [dCL24, §4], gives more: τ is multiplicity free.

The function produced by Corollary 11.6 is a generalized Hecke eigenfunction with a nonzero Whittaker coefficient, and it need not be cuspidal. For a Levi subgroup M, write D_M for the image of the adelic degree map in $\mathrm{Hom}(X_F^*(M), \mathbb{Z})$.

Proposition 12.2 (Generic cuspidal support)**.** *Let J be split connected reductive, and let h be a function of finite level on $J(F)\backslash J(\mathbb{A})$ on which the spherical Hecke algebras at the unramified places act through a generalized character η, and whose principal Whittaker coefficient is nonzero.*

1. *There are a standard Levi subgroup M, possibly equal to J, a generic irreducible cuspidal representation π of $M(\mathbb{A})$ whose central character has finite order on the connected centre, and a character $\chi : \mathsf{D}_M \to k^\times$, such that the Hecke character of $\pi \otimes (\chi \circ \deg)$ corresponds to η under the normalized Satake restriction from J to M.*
2. *If at one unramified place the eigenvalues given by η in a faithful representation of the dual group are ℓ-adic units, then χ takes values in the units of a finite extension of $\mathbb{Q}_\ell$, and $\pi \otimes (\chi \circ \deg)$ has a continuous parameter.*

Proof. The space of functions with compact support is spanned by pseudo-Eisenstein series built from cuspidal data. We separate those built from generic cuspidal representations from the others, show by pairing with a Poincaré series that η occurs in the first part, and then specialize the degree variables. The step which needs care is the third. The image of the Hecke algebra in the endomorphisms of the functions with compact support does not act on the module from which the generic part is induced, so the specialization is made with the abstract Hecke algebra, and with translations by a central lattice which do not belong to it and only commute with it.

Step 1: pseudo-Eisenstein series. Fix the level K and an unramified place u, and let $\mathcal{C} = C_c(J(F)\backslash J(\mathbb{A})/K, k)$. It is a finitely generated module over the spherical Hecke algebra $\mathscr{H}_u$ (Proposition I.5.5). By [BL24, §9.2, (9.3)], $\mathcal{C}$ is the sum, over the standard Levi subgroups M, of the images of the pseudo-Eisenstein maps, which send a compactly supported cuspidal function on M, placed on finitely many pieces of the Iwasawa decomposition, to the sum of its translates under $P(F)\backslash J(F)$. With the normalization by the half modulus these maps are compatible with the Satake restriction. The statement holds over k by transport under ι, because at a fixed level the sums and the integrals which define constant terms are finite. The level on the Levi subgroup depends on the piece of the Iwasawa decomposition: we choose a representative $g_a \in J(\mathbb{A})$ of each double coset in $P(\mathbb{A})\backslash J(\mathbb{A})/K$, and let $K_{M,a}$ be the image in $M(\mathbb{A})$ of $P(\mathbb{A}) \cap g_a K g_a^{-1}$. The representatives can be chosen with component one at the unramified places, and there the local Satake restriction acts on each piece.

A compactly supported cuspidal function on M of given level is a finite sum of functions supported in one degree $d \in \mathsf{D}_M$. Such a function is the restriction to degree d of a cusp form which is periodic under a central lattice Ξ whose degrees are large enough for the translates of the support to be disjoint. Periodic cusp forms of fixed level form a finite-dimensional space [BL24, §9.2, proof after (9.3)], and the representation which it generates is a finite sum of irreducible cuspidal representations by Lemma I.9.1; their central characters have finite order on the connected centre, because the connected centre modulo Ξ and the level subgroup is finite. So the cuspidal functions with compact support are spanned by the functions $1_{\deg=d}\varphi$, with φ in an irreducible cuspidal representation π of this kind. Let $\mathcal{C}^{\mathrm{gen}}$, respectively $\mathcal{C}^{\mathrm{nong}}$, be the span of the pseudo-Eisenstein series of those for which π is generic, respectively not generic. Then

$$\mathcal{C} = \mathcal{C}^{\mathrm{gen}} + \mathcal{C}^{\mathrm{nong}},$$

and both are stable under the Hecke operators at all unramified places: a Hecke operator of M of a given degree shifts d, and the pseudo-Eisenstein maps are compatible with Satake restriction.

Step 2: the character η occurs in the generic part. Choose g_0 at which the Whittaker coefficient of h, for the character ψ_B of the maximal unipotent subgroup U_B, is nonzero. Since h is invariant under K, the character is trivial on $U_B(\mathbb{A}) \cap g_0 K g_0^{-1}$, and $\phi(ng_0k) = \psi_B(n)^{-1}$ is a well defined function on $U_B(\mathbb{A})g_0K$. Its Poincaré series $P_0 = \sum_{\gamma \in U_B(F)\backslash J(F)} \phi(\gamma\,\cdot)$ has compact support on $J(F)\backslash J(\mathbb{A})$, because the support of ϕ is compact modulo $U_B(\mathbb{A})$ and $[U_B]$ is compact. Unfolding gives

$$\langle h, P_0\rangle \neq 0, \qquad \langle E, P_0\rangle = 0 \quad \text{for } E \in \mathcal{C}^{\mathrm{nong}}.$$

For the second, unfold along the Bruhat decomposition, with representatives w of minimal length in their classes modulo the Weyl group of M. If the cell of w is not the open one, there is a simple root α such that $w\alpha$ is a root of the unipotent radical N of the parabolic subgroup. The section is invariant on the left under $N(\mathbb{A})$, so its translate by w is invariant under the root group of α, on which ψ_B is not trivial; the contribution of the cell is zero. The open cell contributes the Whittaker coefficient on M of the function from which the section is induced. This function is a function in the nongeneric representation, restricted to one degree; the restriction does not affect integrals over unipotent groups, so the coefficient is zero.

Let A be the image of the Hecke algebra of all unramified places in $\mathrm{End}_{\mathscr{H}_u}(\mathcal{C})$. It is a finitely generated module over the noetherian ring $\mathscr{H}_u$, being a submodule of a finite sum of copies of $\mathcal{C}$. An operator which vanishes on $\mathcal{C}$ vanishes on all functions of level K, so η is a character of A. We claim that

$$\mathrm{Ann}_A(\mathcal{C}^{\mathrm{gen}}) \subset \ker \eta.$$

Let T annihilate $\mathcal{C}^{\mathrm{gen}}$. Its transpose $\mathsf{T}^\star$ satisfies $\langle E, \mathsf{T}^\star P_0\rangle = \langle \mathsf{T}E, P_0\rangle = 0$ for E in $\mathcal{C}^{\mathrm{gen}}$ and also for E in $\mathcal{C}^{\mathrm{nong}}$, which is stable under T. Hence $\mathsf{T}^\star P_0 = 0$ and $\langle \mathsf{T}h', P_0\rangle = 0$ for every function h' of level K. If $\eta(\mathsf{T}) \neq 0$, then T is invertible on the generalized eigenspace of h: with $(\mathsf{T} - \eta(\mathsf{T}))^a h = 0$ and

$$h_0 = \eta(\mathsf{T})^{-1} \sum_{j=0}^{a-1} (-\eta(\mathsf{T})^{-1}(\mathsf{T} - \eta(\mathsf{T})))^j h$$

we have $\mathsf{T}h_0 = h$, and $\langle h, P_0\rangle = 0$, a contradiction. This proves the claim. Since $\mathcal{C}^{\mathrm{gen}}$ is a finitely generated A-module whose annihilator is contained in the maximal ideal $\ker\eta$, Nakayama's lemma gives $\mathcal{C}^{\mathrm{gen}} \otimes_A k_\eta \neq 0$.

Step 3: specializing the degree. Finitely many generic π suffice to generate $\mathcal{C}^{\mathrm{gen}}$ over $\mathscr{H}_u$. For each of them choose a central lattice Ξ, generated by elements at the place u, which acts trivially on π [BL24, §6.2], and let $\Lambda = \deg \Xi \subset \mathsf{D}_M$, a sublattice of finite index, and $R = k[\Lambda]$. On each of the finitely many pieces of the Iwasawa decomposition consider the functions $1_{\deg = d}\varphi$ with $\varphi \in \pi^{K_{M,a}}$. Together they span

$$\mathcal{D}_\pi = \bigoplus_{\bar{d} \in \mathsf{D}_M/\Lambda} R \otimes_k V_{\bar{d}}^{\mathrm{cut}},$$

where $V_{\bar{d}}^{\mathrm{cut}}$ is the direct sum, over these pieces, of the spaces of restrictions to one degree of the class $\bar{d}$. The lattice Λ acts through the translations by Ξ. So $\mathcal{D}_\pi$ is a free R-module of finite rank.

Two algebras act on $\mathcal{D}_\pi$. Let $\mathscr{H}^{\mathrm{ur}}$ be the Hecke algebra of J at all unramified places, as an abstract algebra. It acts on $\mathcal{C}$, through its image A, and on $\mathcal{D}_\pi$, through the Satake restriction to M; $\mathcal{D}_\pi$ is stable because a Hecke operator of M is a sum of operators which shift the degree and map each $\pi^{K_{M,a}}$ into itself. The pseudo-Eisenstein map $\mathcal{D}_\pi \to \mathcal{C}^{\mathrm{gen}}$ is equivariant for $\mathscr{H}^{\mathrm{ur}}$. The image A itself need not act on $\mathcal{D}_\pi$, since this map need not be injective; so we work with $\mathscr{H}^{\mathrm{ur}}$ and with the ideal $\mathfrak{m}_\eta \subset \mathscr{H}^{\mathrm{ur}}$, the kernel of the character defined by η. The translations by Ξ are not elements of $\mathscr{H}^{\mathrm{ur}}$: already for a group of semisimple rank one, the Satake algebra of the torus is $k[z, z^{-1}]$, that of the group is $k[z + z^{-1}]$, and translation is multiplication by z. But they commute with the action of $\mathscr{H}^{\mathrm{ur}}$ on $\mathcal{D}_\pi$, because Ξ is central in M. Commutation is all that is needed.

By step 2, $\mathcal{C}^{\mathrm{gen}}/\mathfrak{m}_\eta\mathcal{C}^{\mathrm{gen}} \neq 0$. The sum of the maps $\mathcal{D}_\pi \to \mathcal{C}^{\mathrm{gen}}$ over the finitely many chosen π is surjective and equivariant, so by right exactness $Q = \mathcal{D}_\pi/\mathfrak{m}_\eta\mathcal{D}_\pi \neq 0$ for one of them. The submodule $\mathfrak{m}_\eta\mathcal{D}_\pi$ is stable under R, because the two actions commute. So Q is a nonzero finitely generated R-module, and by Nakayama's lemma there is a maximal ideal of R, that is, a character $\zeta : \Lambda \to k^\times$, such that the fibre of Q at ζ is not zero. Extend ζ to $\chi : \mathsf{D}_M \to k^\times$.

The fibre of $\mathcal{D}_\pi$ at ζ is described by the identity

$$\sum_{\nu\in\Lambda} \zeta(\nu)^{-1} T_\nu(1_{\deg=d}\varphi) = \chi(d)^{-1} 1_{\deg\equiv d \bmod \Lambda}\,(\varphi \otimes \chi),$$

where a_ν is an element of Ξ of degree ν and $T_\nu f(x) = f(xa_\nu)$. At each point only one term of the sum is not zero. So the fibre is identified with its image: restricted to one chosen degree in each class, passing to the quotient by Ξ gives back the corresponding space of restrictions. Since

$$1_{\deg\equiv d \bmod \Lambda} = \frac{1}{|\mathsf{D}_M/\Lambda|} \sum_{\varepsilon\in(\mathsf{D}_M/\Lambda)^\vee} \varepsilon(\deg - d),$$

the fibre is contained in the sum of the cuspidal representations $\pi \otimes \chi\varepsilon$. They are generic, since twisting by a character of the degree does not affect unipotent integrals, and $\mathscr{H}^{\mathrm{ur}}$ acts on their spherical vectors by characters. As the fibre of Q at ζ is a nonzero quotient of this sum on which $\mathscr{H}^{\mathrm{ur}}$ acts through η, one of these characters is η. Replacing χ by $\chi\varepsilon$ gives (1).

Step 4: continuity. Assume the hypothesis of (2). All the data are defined over a finite extension of $\mathbb{Q}_\ell$. The cuspidal representation π has a continuous parameter, namely that of a nonzero excursion summand, and its eigenvalues at u in a faithful representation r of the dual group of J are units. On the weight space of a weight w_i of the connected centre of the dual group of M in r, the twist by χ multiplies the eigenvalues by the value of w_i on the element of that centre corresponding to χ, with coordinates z_j. By hypothesis the products are units, so

$$\sum_j (w_i)_j \,\mathrm{val}_\ell(z_j) = 0 \quad \text{for every } i.$$

Since r is faithful the w_i span the rational character space of the centre, and all the z_j are units; this is not affected by the extraction of a root needed to pass from the degree of u to degree one. A unit of a finite extension of $\mathbb{Q}_\ell$ generates a subgroup with compact closure, so the character of $\mathbb{Z}$ it defines extends continuously to $\widehat{\mathbb{Z}}$. This gives (2). □

Remark 12.3. The usual cuspidal support would produce a Levi subgroup and a cuspidal representation but could lose genericity. Here the Poincaré series P_0 of step 2 forces η to occur in the generic part before the degree variables are specialized. The hypothesis on units is used only in step 4, to make the character of the degree continuous.

12.2 Removing the two trivial lines

Proposition 12.4 (Removing the two trivial lines)**.** *Let $\sigma : \Gamma_F \to \mathrm{SO}_{2r+1}(k)$ be continuous, finitely ramified, semisimple, elliptic and strongly pure, and assume that $\varrho = \mathrm{Std}\,\sigma$ has no nonzero invariant vector, that is, $\varrho^{\Gamma_F} = 0$. (A line on which Γ_F acts by a nontrivial character is allowed.) Suppose there is a function of finite level on Sp_{2r+2} with a nonzero principal Whittaker coefficient on which the spherical Hecke algebras act, as a generalized character, with standard parameter $\varrho \oplus 1^{\oplus 2}$. Then the excursion summand of σ on Sp_{2r} contains a generic irreducible cuspidal representation.*

Proof. Proposition 12.2 applies: for a symplectic group the coefficients of the standard characteristic polynomial generate the spherical Hecke algebra, and the Satake parameters are unit-valued because σ is continuous. It gives a Levi subgroup $M = \prod_i \mathrm{GL}_{d_i} \times \mathrm{Sp}_{2s}$ and a generic cuspidal representation of it with continuous parameters τ_i of the factors GL_{d_i} and τ_s of Sp_{2s}, such that

$$\bigoplus_i (\tau_i \oplus \tau_i^\vee) \oplus \mathrm{Std}\,\tau_s \simeq \varrho \oplus 1^{\oplus 2}$$

by Chebotarev. The right side is pure of weight zero, because σ is strongly pure, and it contains the trivial representation twice. So by Lemma 12.1 it is not $\mathrm{Std}\,\tau_s$ alone: the Levi subgroup is proper. By Proposition 5.2 the irreducible constituents of ϱ are nontrivial, self-dual and of multiplicity one. A nontrivial constituent of some τ_i would be a constituent of ϱ, hence self-dual, and would occur also in $\tau_i^\vee$, with multiplicity at least two on the left. So each τ_i is a sum of trivial representations; being the parameter of a cuspidal representation of GL_{d_i} it is irreducible, so $d_i = 1$ and $\tau_i = 1$. Each such factor accounts for two trivial lines, and there are exactly two. Hence there is one factor GL_1, $s = r$, and $\mathrm{Std}\,\tau_s \simeq \varrho$.

The component on Sp_{2r} is a generic cuspidal representation whose Hecke eigenvalues are those of ϱ. Every excursion summand which meets it has a parameter with standard representation ϱ, by Chebotarev, and such a parameter is conjugate to σ by Lemma 5.1 in odd dimension. □

Theorem 12.5 (Symplectic groups)**.** *Every continuous, finitely ramified, semisimple, elliptic and strongly pure parameter $\sigma : \Gamma_F \to \mathrm{SO}_{2n+1}(k)$ has a nonzero cuspidal excursion summand on split Sp_{2n}, in every positive characteristic. For $n \geq 3$, if its standard representation has no nonzero invariant vector, this summand contains a globally generic irreducible cuspidal representation.*

Proof. For $n = 1$ this is the case of type A_1 (Theorem 5.8), and for $n = 2$ it follows from $\mathrm{Sp}_4 \simeq \mathrm{Spin}_5$ and Theorem 10.3. Let $n \geq 3$, and let $n_0 \in \{0, 1\}$ be the dimension of the space of invariants of $\varrho = \mathrm{Std}\,\sigma$. Put

$$\varrho_0 = \begin{cases} \varrho \ominus 1, & m = n, \quad \text{if } n_0 = 1, \\ \varrho \oplus 1, & m = n + 1, \quad \text{if } n_0 = 0. \end{cases}$$

This is an orthogonal representation of dimension $2m$ and determinant one whose irreducible constituents are orthogonal and distinct, hence an elliptic parameter for SO_{2m} by Proposition 5.2. By Corollary 10.4 there is a generic cuspidal representation of split SO_{2m} with this parameter, and by Corollary 11.6 a function on Sp_{2m} with nonzero Whittaker coefficient which is a generalized Hecke eigenfunction with standard parameter $\varrho_0 \oplus 1$.

If $n_0 = 1$ this parameter is ϱ, and Proposition 5.6 shows that the function is cuspidal and that the summand of σ is nonzero. If $n_0 = 0$ it is $\varrho \oplus 1^{\oplus 2}$ on Sp_{2n+2}, and Proposition 12.4 gives a generic cuspidal representation in the summand of σ on Sp_{2n}. □

13 The realization theorem for classical groups

This section assembles the results of the preceding sections into the realization theorem for all split semisimple groups with factors of classical type.

Theorem 13.1 (Classical groups)**.** *Let G/F be split connected semisimple with simple factors of types A, B, C, D, in any positive characteristic, and let $\sigma : \Gamma_F \to \widehat{G}(k)$ be a semisimple elliptic parameter in the sense of Definition 1.1. Then the cuspidal excursion summand of σ is nonzero at some finite level. This holds for every group in each isogeny class.*

This is the conclusion of Theorem I.A for these groups, and it is Theorem 1.2: a nonzero excursion summand at a finite level is a finite-dimensional space on which the excursion operators commute and act through the generalized character of σ, so it contains a cusp form satisfying (1.1). In the statements of this paper the words continuous and finitely ramified repeat what Definition 1.1 requires of a parameter.

Proof. By Corollary I.10.11 it suffices to treat the simply connected simple groups, the passage to products and to an arbitrary central quotient being made there. The parameters of the simple factors are again continuous, finitely ramified, semisimple and elliptic, hence strongly pure by Theorem I.4.8. For SL_n the assertion is Theorem 5.8; for Spin_{2n+1} and Spin_{2n} it is Theorem 10.3; for Sp_{2n} it is Theorem 12.5. The isomorphisms in low rank ($B_1 = C_1 = A_1$, $D_2 = A_1 \times A_1$, $D_3 = A_3$, $B_2 = C_2$) are included in these statements.

One point in the reduction should be kept in mind. The central subgroup by which the product of the simply connected factors is divided need not be a product of subgroups of the factors, and it need not be reduced when the characteristic divides its order. The transfer of Corollary I.10.11 is therefore applied once, to the product and the given subgroup, and not factor by factor. □

A Computer checks

Some of the computations in this paper were also carried out by computer. No proof depends on such a computation: in each case the argument is given in the text, and the program is a check on it. Statements about a classical group of arbitrary rank can be checked by a program only in the ranks indicated, and for the other ranks the argument in the text is the only proof.

The programs are written in Python, with the libraries SymPy and NumPy, and are in the directory `anc` of the source of this paper. The program `exchange_checker.py` is a common library. It keeps track of a set of roots and of the roots on which a character is not trivial, and at each exchange it tests that the sets are closed and normalized under addition of roots, that the character has the required support, and that the sum of the two exchanged roots lies in this support. These are the conditions on roots in Lemma 5.9; the constants of the commutators and the Fourier pairing over the adeles are treated in the text. The other programs compute with matrices, for subspaces of Lie algebras or for the groups themselves.

Statement	What is checked	Program and range
Lemmas 7.2 and 7.3	exactly one double coset of Weyl elements contributes, and the intersection of unipotent groups is as stated	`check_orthogonal_jacquet.py`; B_2, B_4, B_6 (also with the criterion of characteristic two), D_4, D_6

Statement	What is checked	Program and range
Lemma 7.9	for the two chains, the conditions on the sets of roots and on the supports of the characters	`check_interleave.py`, $2 \le r \le 7$
Proposition 7.10, steps 1, 2 and 4	the sequences of exchanges and the final subgroups and characters	`check_odd_descent_steps12.py`, $n \le 4$; `check_sw_exchange.py`, $m \le 12$
Proposition 7.10, step 3	the hyperbolic basis, the determinant of P_t and the conjugation formula	`check_odd_descent.py`; $n \le 5$
Lemma 8.2	the embedding of the Jacobi group and the commutator used in Lemma 8.4	`check_jacobi_embedding.py`; symbolic
Lemma 8.3, step 2	the equivalence of the two Fourier coefficients, in the Lie algebra, and the additivity of Λ_c on the group	`check_even_descent_wh.py`, $n = 2, 3, 4$, with several rational values of c; `check_lambda_c.py`, $n = 2$
Lemma 8.4, steps 2 and 3	the chain of exchanges and the signs in the final table	`check_even_descent_chain.py`; $n \le 4$
(9.3), (9.4), (9.5), Corollary 9.5	matrix identities over polynomial rings in characteristic two	`check_char2.py`; symbolic
Lemma 9.3, step 5; Proposition 9.4, steps 1–3	the sets of roots	`check_roots.py`; $m \le 9$, $n \le 7$

Notation

The table lists notation used in more than one section, with the place where it is introduced.

Notation	Meaning and location
X, F, Γ_F	The original curve, its function field and its absolute Galois group.
p, q, ℓ, k	The geometric characteristic, size of the constant field, coefficient prime, and $k = \overline{\mathbb{Q}}_\ell$.
$\deg_F, d_v$	Arithmetic Galois degree and the degree of a closed point v; geometric Frobenius at v has degree $-d_v$.
$G, \widehat{G}, M, \widehat{M}$	The split group, its dual, a Levi subgroup and the corresponding dual Levi.
σ, ν_σ	The prescribed parameter and its character on excursion data, defined by (1.1).
$e = (I, f, \boldsymbol{\gamma})$, S_e, S_e^{fun}	An excursion datum, its excursion operator on functions with compact support, and the transposed operator on all functions ([I, Definitions 5.7 and 5.12]).
T_G, Ξ, ω_σ	The connected centre with its reduced structure, a central lattice, and the central character of σ (Definition 4.1).
ι	The fixed isomorphism $k \simeq \mathbb{C}$ (Section 5).
$\psi, [N], \mathcal{F}_{N,\zeta}$	The additive character, the quotient $N(F)\backslash N(\mathbb{A}_F)$, and a Fourier coefficient (Sections 5 and 9).
$\mathcal{SW}$	The Siegel–Whittaker coefficient (Section 6).
$x_\alpha(t), E_{a,b}$	A root group element and an elementary matrix (Section 5).

Notation	Meaning and location
$r_i(v)$	The unipotent element attached to a vector v of the terminal space (Section 9).
$\varrho, \varrho_i, \Pi_i$	The standard representation $\mathrm{Std}\,\sigma$ of a classical parameter, an irreducible summand of it, and the corresponding cuspidal representation of a general linear group (Section 5).

The following terms are introduced in this paper, or are used in a sense fixed here.

Term	Where it is defined
parameter; semisimple; elliptic	Definition 1.1
central lattice; invariant under Ξ	Definition 4.1
excursion summand	Definition 4.2
automorphic realization; to realize a parameter	Section 1, after (1.1)
descent; descent datum	Section 7

Statement on the use of AI

The mathematical arguments of the three papers of this series were developed with GPT-6 Astra Pro. The papers were then organized and rewritten with Claude Fable 5.2, and in the course of this several arguments were replaced or expanded. The three papers were then reviewed in detail, line by line, by GPT and Claude models, in sessions which had not taken part in writing them, and the errors found in this way were corrected.

Boyang Hu organized the work and is responsible for the manuscript.

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