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Fixed points of the polar projection body operator and its square

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Abstract

Let $TK = (\Pi K)^\circ$ be the polar projection body of a convex body K in $\mathbb{R}^n$. In dimension $n \geq 3$, we prove that TK is a positive dilate of K exactly when K is an origin-centred ellipsoid, settling Petty's polar projection conjecture. The same classification holds under the weaker assumption that T^2K is a positive dilate of K. The proof uses a distributional Laplacian identity to compare two integrals of central section volumes. Equality supplies contractive hyperplane projections with every kernel direction, and James's theorem gives ellipsoidal rigidity. The argument applies to arbitrary convex bodies and yields the translative constant volume conjecture of Horváth and Lángi.

1 The polar projection problem

A convex body is a compact convex set with nonempty interior. Write $|\cdot|_d$ for d-dimensional volume, $|K| = |K|_n$, and $\kappa_d = |B_2^d|_d$. Let h_K be the support function of K and S_K its surface area measure, obtained by pushing forward boundary area under the almost everywhere defined outer unit normal ν_K. We use the normalization of Cauchy's projection formula

$$h_{\Pi K}(x) = \frac{1}{2}\int_{S^{n-1}} |\langle x, u\rangle|\, dS_K(u) = |x|\,|P_{x^\perp}K|_{n-1} \qquad (x \neq 0), \tag{1}$$

where P_H denotes orthogonal projection onto the hyperplane H. See [1, Lemma 2.1.2(ii)]. For a body C containing the origin in its interior, put

$$C^\circ = \{y : \langle x, y\rangle \leq 1 \text{ for all } x \in C\}, \qquad p_C(x) = \inf\{s > 0 : x \in sC\} = h_{C^\circ}(x).$$

Define $TK = (\Pi K)^\circ$. This is an origin-symmetric convex body, and $T(sK) = s^{1-n}TK$ for $s > 0$.

Petty's polar projection problem [8] asks whether $TK = aK$ for some $a > 0$ characterizes origin-centred ellipsoids in dimension $n \geq 3$ [5, Conjecture 2]. We prove the classification also for a return after two applications, with an arbitrary positive dilation.

Theorem 1.1 (Petty's conjecture and the second iterate)**.** *Let $n \geq 3$ and let $K \subset \mathbb{R}^n$ be a convex body. The following are equivalent.*

1. *K is an ellipsoid centred at the origin.*
2. *$TK = aK$ for some $a > 0$.*
3. *$T^2K = bK$ for some $b > 0$.*

When these conditions hold,

$$a = \frac{\kappa_n}{\kappa_{n-1}|K|}, \qquad b = a^{2-n}.$$

The fixed point and the return condition each imply $K = -K$ because T takes values in origin-symmetric bodies. The proof first compares the central sections of an origin-symmetric K and $L = TK$. Under $T^2K = bK$, the same comparison for L reverses the inequality after both sides are divided by b^{n-1}.

2 A flux identity for the polar projection body

For a body C containing the origin in its interior and a unit vector u, put $A_C(u) = |C \cap u^\perp|_{n-1}$. This is continuous in u by the radial formula for section volume.

Lemma 2.1. *Let $n \geq 2$, let K be a convex body, and set $L = TK$. Then*

$$\int_{S^{n-1}} \frac{1}{h_L(u)}\, dS_L(u) = \int_{S^{n-1}} A_L(u)\, dS_K(u). \tag{2}$$

Proof. By polarity and (1), $p_L = h_{\Pi K}$. In the sense of distributions, $\Delta|\langle x, u\rangle| = 2\mathcal{H}^{n-1} \lfloor u^\perp$. This follows by writing $x = z + tu$ with $z \in u^\perp$ and using $(|t|)'' = 2\delta_0$. Integration against compactly supported smooth test functions and Fubini's theorem therefore give the positive measure

$$M = \Delta p_L = \int_{S^{n-1}} \mathcal{H}^{n-1} \lfloor u^\perp\, dS_K(u). \tag{3}$$

The measures in the integral have uniformly bounded mass on each compact set. In particular $M(L) = \int A_L\, dS_K$. Also, $M(\partial L) = 0$. Indeed, each $u^\perp$ meets $\operatorname{int} L$, so $\partial L \cap u^\perp$ is the relative boundary of $L \cap u^\perp$ and has zero $(n-1)$-dimensional volume. To see this, if a point of $\partial L \cap u^\perp$ were interior to the section, a supporting normal there would have to be parallel to u. Its support value would then be zero, contrary to $0 \in \operatorname{int} L$.

The boundary of L is Lipschitz and has a unique outer unit normal at almost every point [1, Lemma 1.5.6]. At such a point y, write $u = \nu_L(y)$. Since $p_L = h_{L^\circ}$, its subgradients at y are exactly the $g \in L^\circ$ with $\langle g, y\rangle = 1$. They are the supporting normals scaled to take value one at y. Thus the subdifferential is a singleton and

$$\nabla p_L(y) = \frac{u}{h_L(u)}. \tag{4}$$

To integrate (3), let φ_ε be a nonnegative smooth mollifier of integral one supported in εB_2^n, and put $p_\varepsilon = p_L * \varphi_\varepsilon$. The integral of $M * \varphi_\varepsilon$ over L tends to $M(L)$: indeed,

$$\int_L \Delta p_\varepsilon(x)\, dx = \int_{\mathbb{R}^n} \left(\int_L \varphi_\varepsilon(x - z)\, dx \right) dM(z),$$

and the inner integral is bounded by one and tends to $\mathbf{1}_L(z)$ off ∂L. Its support lies in the fixed compact set $L + B_2^n$ when $\varepsilon \leq 1$. This proves convergence since $M(\partial L) = 0$.

Choose $r > 0$ with $rB_2^n \subset L$. All subgradients of p_L have norm at most $1/r$. At a point y of differentiability, any limit of subgradients at points approaching y belongs to $\partial p_L(y)$ by the subgradient inequality. Since this set is a singleton, nearby gradients converge to $\nabla p_L(y)$ wherever they exist. It follows that $\nabla p_\varepsilon(y) \to \nabla p_L(y)$ at the boundary points used in (4), and $|\nabla p_\varepsilon| \leq 1/r$. The divergence theorem for the smooth field ∇p_ε on $\operatorname{int} L$ [1, Theorem 2.1.4] now gives

$$M(L) = \lim_{\varepsilon \downarrow 0} \int_{\partial L} \langle \nabla p_\varepsilon(y), \nu_L(y)\rangle\, d\mathcal{H}^{n-1}(y) = \int_{S^{n-1}} \frac{1}{h_L(u)}\, dS_L(u).$$

The last limit uses dominated convergence and the definition of S_L. Comparing with the value of $M(L)$ from (3) proves (2). □

3 The section comparison and its equality case

Proposition 3.1. *Let $n \geq 2$, let $K = -K$ be a convex body, and put $L = TK$. Then*

$$\int_{S^{n-1}} A_K(u)\, dS_L(u) \;\leq\; \int_{S^{n-1}} A_L(u)\, dS_K(u). \tag{5}$$

If $n \geq 3$ and equality holds, then K is an origin-centred ellipsoid.

Proof. For a unit vector u, choose a support point $y \in \partial L$ with $\langle y, u\rangle = h_L(u) > 0$. The map

$$Q_{y,u} z = z - \frac{\langle z, u\rangle}{h_L(u)} y$$

projects onto $u^\perp$ along $\mathbb{R}y$. Its restriction from $y^\perp$ to $u^\perp$ is inverse to the restriction of orthogonal projection onto $y^\perp$ from $u^\perp$. The latter map has absolute Jacobian $\langle y/|y|, u\rangle = h_L(u)/|y|$, the cosine of the angle between the normals. This follows by using the common $(n-2)$-dimensional subspace of the hyperplanes, with the parallel case giving Jacobian one. As $Q_{y,u}$ kills y, it acts on K by first orthogonally projecting onto $y^\perp$ and then applying this inverse. Hence

$$|Q_{y,u}K|_{n-1} = \frac{|y|\,|P_{y^\perp}K|_{n-1}}{h_L(u)} = \frac{h_{\Pi K}(y)}{h_L(u)} = \frac{1}{h_L(u)}. \tag{6}$$

The last equality uses $y \in \partial L$ and $p_L = h_{\Pi K}$.

Since $Q_{y,u}$ fixes $u^\perp$, the image $Q_{y,u}K$ contains $K \cap u^\perp$. This gives $A_K(u) \leq 1/h_L(u)$. Integrating this inequality with respect to S_L and using Lemma 2.1 proves (5).

Suppose $n \geq 3$ and equality holds. Then $A_K(u) = 1/h_L(u)$ for S_L-almost every u. For every support point y of L in such a direction, the two convex bodies $Q_{y,u}K$ and $K \cap u^\perp$ have equal volume in $u^\perp$. The inclusion gives $Q_{y,u}K = K \cap u^\perp$, so $Q_{y,u}K \subset K$. By the definition of S_L, this conclusion holds with $u = \nu_L(y)$ on a set $G \subset \partial L$ of full boundary area. Every nonempty relatively open patch of ∂L has positive boundary area, as seen by orthogonally projecting a local Lipschitz graph onto its base. Thus G is dense in ∂L even when S_L is supported on a proper subset of the sphere.

Fix $v \neq 0$ and set $y = v/p_L(v) \in \partial L$. Take $y_j \in G$ with $y_j \to y$, and write $u_j = \nu_L(y_j)$ and $g_j = u_j/h_L(u_j) \in L^\circ$. By compactness of L° a subsequence of g_j converges to g, with $\langle g, y\rangle = 1$. The maps $z \mapsto z - \langle g_j, z\rangle y_j$ converge to $Qz = z - \langle g, z\rangle y$. Closedness of K gives $QK \subset K$. The identity $\langle g, y\rangle = 1$ makes Q a projection with range $\ker g$ and kernel $\mathbb{R}v$. Thus for every $z \in \ker g$ and $t \in \mathbb{R}$,

$$p_K(z) = p_K(Q(z + tv)) \leq p_K(z + tv).$$

In Birkhoff's terminology, the hyperplane $\ker g$ is orthogonal to v: each element of the hyperplane is in the first argument of the orthogonality relation. James's characterization of real inner product spaces [6, Theorem 5, p. 565] requires just such a hyperplane for each $v \neq 0$ in dimension at least three. Therefore p_K is an inner product norm and its unit ball K is an origin-centred ellipsoid. □

The pointwise comparison $A_K(u) \leq 1/h_{TK}(u)$ is the classical inclusion $IK \subseteq \Pi K$, where the intersection body IK has radial function A_K [7]. Formula (6) identifies equality in that comparison with a contractive projection.

4 Fixed points and two applications of the operator

Proof of Theorem 1.1. Assume $T^2K = bK$ and write $L = TK$. Both K and L are origin symmetric. Proposition 3.1 gives (5). Applied to L and $TL = bK$, it gives

$$\begin{aligned} b^{n-1} \int_{S^{n-1}} A_L \, dS_K &= \int_{S^{n-1}} A_L \, dS_{bK} \\ &\le \int_{S^{n-1}} A_{bK} \, dS_L = b^{n-1} \int_{S^{n-1}} A_K \, dS_L. \end{aligned}$$

Here $S_{bK} = b^{n-1} S_K$ and $A_{bK} = b^{n-1} A_K$. Dividing by b^{n-1} gives the reverse of (5). Its equality case shows that K is an ellipsoid. Also, if $TK = aK$, the homogeneity of T gives $T^2K = a^{2-n}K$, so the fixed point condition is covered by the same argument.

For the converse and the constants, recall the segment formula

$$|C + [0, v]| = |C| + h_{\Pi C}(v) \tag{7}$$

for every convex body C and $v \in \mathbb{R}^n$ [5, Lemma 1]. For $v \neq 0$, slicing along the lines parallel to v adds $|v|$ to the length of each nonempty fibre, which gives (7) by (1). If $K = UB_2^n$ for $U \in GL(n)$, a change of variables by U in (7) yields

$$h_{\Pi K}(v) = |\det U| \, h_{\Pi B_2^n}(U^{-1}v) = \kappa_{n-1} |\det U| \, |U^{-1}v|.$$

Thus $TK = (\kappa_{n-1} |\det U|)^{-1} K$. Since $|K| = \kappa_n |\det U|$, this gives a, and homogeneity gives b. □

Remark 4.1 (Dimension two). In the plane, for $K = -K$ and a rotation J through $\pi/2$, brightness is perpendicular width, so $\Pi K = 2JK$. Then $TK = \frac{1}{2} JK^\circ$ and $T^2K = K$ for every such K. Nonellipsoidal one-application fixed points up to dilation also exist in the plane. For instance a regular hexagon of circumradius one satisfies $K^\circ = (2/\sqrt{3})JK$ and $TK = K/\sqrt{3}$. See [5, Theorem 2] for the connection with Radon curves.

5 The translative constant volume conjecture

Corollary 5.1 (Horváth–Lángi)**.** *Let $n \ge 3$ and let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body. Suppose the volume of the convex hull of any two touching translates of K is independent of the pair of translates. Then K is an ellipsoid. Conversely, ellipsoids have this property.*

Proof. Translate K so that $K = -K$. Two translates touch when they intersect and their interiors are disjoint. By translation invariance it suffices to use K and $K + v$, which touch exactly for $v \in \partial(K - K) = \partial(2K)$. Moreover, convexity and (7) give

$$|\operatorname{conv}(K \cup (K + v))| = |K + [0, v]| = |K| + h_{\Pi K}(v).$$

Thus $h_{\Pi K}$ is constant, say $d > 0$, on $\partial(2K)$. On that boundary $p_K(v) = 2$, so positive homogeneity gives $h_{\Pi K} = (d/2) p_K$ everywhere. Since $p_{TK} = h_{\Pi K}$, this is $TK = (2/d)K$. Theorem 1.1 gives the result. For an ellipsoid, the same identities and $TK = aK$ imply constancy. □

Horváth and Lángi posed the conjecture in [4, Section 4, Conjecture 2] and established its equivalence with Petty's polar projection problem in [5, Theorem 3].

A Earlier approaches

For $TK = aK$, the second moment of the boundary functionals under cone measure gives $\beta E \subset K \subset \sqrt{\beta}E$ for an ellipsoid E, with $\beta = 2/(na|K|)$. This yields a Banach–Mazur bound of order $n^{1/4}$. The associated embedding into L^1 has relative projection constant $n\beta$. The absolute bound in [3] allows β above the ellipsoidal value, and general subspaces of L^1 can exceed the Euclidean projection constant [2]. The sign kernel calculation gives a squared residual $1 - n\beta^2$, positive even for ellipsoids.

For two rotational blocks, estimates at the axes and an averaging identity from affine shears forced an ellipsoidal profile. Fractional moments of the normals reduced two arbitrary block norms to the rotational case. For homothetic ellipsoidal parallel sections, a contraction of the section centres forced their collinearity.

Use of AI

OpenAI's ChatGPT was used to explore proof strategies, develop and check mathematical arguments, and draft and revise the exposition.

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