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Automorphic realization of Langlands parameters for split groups over global function fields

II. The geometric construction

Boyang Hu
Tsinghua University

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Abstract

Let G be a split simply connected semisimple group over a global function field F whose characteristic does not divide the order of the Weyl group of G. We prove that every semisimple elliptic Langlands parameter $\Gamma_F \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ is the parameter of a cusp form on G, in the sense of V. Lafforgue's excursion operators; we call such a form an automorphic realization of the parameter. The parameter may be ramified at finitely many places and may have finite monodromy, and neither F nor its field of constants is extended.

The starting point is the geometric Langlands correspondence in characteristic zero. It provides a sheaf, and the function which the trace of Frobenius attaches to a nonzero sheaf can vanish identically. The main point of the construction is to arrange that the trace in question is a determinant $\det(1-\phi)$ with no zero factor, and that it stays nonzero under specialization to the given field.

This is the second of three papers. Together with the first, which contains the arithmetic foundations, and the third, which treats classical groups in every characteristic, it gives automorphic realizations of semisimple parameters on all split reductive groups, with a restriction on the characteristic for the exceptional types only.

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1 Introduction

Let F be the function field of a curve over a finite field and let G be a split reductive group over F. Vincent Lafforgue decomposed the space of cusp forms on G according to semisimple Langlands parameters $\Gamma_F \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$. His theorem attaches a parameter to each summand of the decomposition. It does not say which parameters occur.

We prove that every semisimple elliptic parameter occurs when G is simply connected and the characteristic does not divide the order of the Weyl group. The proof is geometric: it starts from the geometric Langlands correspondence over a field of characteristic zero and ends with a function on the points of Bun_G over the given finite field. This is the second paper of a series of three. The first [I] contains the arithmetic foundations and deduces from the present theorem, and from its counterpart for classical groups [III], the existence of automorphic realizations of semisimple parameters on all split reductive groups.

1.1 The theorem

Let $X/\mathbb{F}_q$ be a smooth projective geometrically connected curve, let $F = \mathbb{F}_q(X)$, and put $p = \operatorname{char} F$. Fix a prime $\ell \neq p$, put $k = \overline{\mathbb{Q}}_\ell$, and write $\Gamma_F = \mathrm{Gal}(F^{\mathrm{sep}}/F)$. For a split connected reductive group G/F, the dual group $\widehat{G}$ is taken over k.

Definition 1.1. A *parameter* is a continuous homomorphism $\sigma : \Gamma_F \to \widehat{G}(k)$ unramified at all but finitely many places. It is *semisimple* if the identity component of the Zariski closure of its image is reductive. It is *elliptic* if

$$C_{\widehat{G}}(\operatorname{im}\sigma)/Z(\widehat{G}) \quad \text{is finite.}$$

The link between cusp forms and parameters is given by excursion operators. An excursion datum consists of a finite set I, a function $f \in k[\widehat{G}\backslash\widehat{G}^I/\widehat{G}]$, invariant under simultaneous left and right multiplication, and a tuple $\boldsymbol{\gamma} \in \Gamma_F^I$. Lafforgue's operator $S_{I,f,\boldsymbol{\gamma}}$ acts on cusp forms: it creates an invariant vector in the cohomology of shtukas with legs indexed by I, transports it by $\boldsymbol{\gamma}$, and contracts with an invariant covector representing f [Laf18, §§0.3–0.5]. These operators commute, and a cusp form φ has parameter σ when

$$S^{\mathrm{fun}}_{I,f,\boldsymbol{\gamma}}\varphi = f\big((\sigma(\gamma_i))_{i\in I}\big)\varphi \qquad \text{for every excursion datum } (I,f,\boldsymbol{\gamma}). \tag{1.1}$$

We write $\nu_\sigma(I,f,\boldsymbol{\gamma})$ for the scalar on the right. A nonzero cusp form φ which satisfies (1.1) is called an *automorphic realization* of σ, and we say that it *realizes* σ. The superscript in S^{fun} records that we need these operators on functions that are not yet known to be cuspidal; on cusp forms S^{fun} is Lafforgue's operator, and the extension is constructed in [I, Section 5].

Theorem 1.2. *Let G/F be split, simply connected and semisimple, and suppose that p does not divide the order of the Weyl group of G. Let $\sigma : \Gamma_F \to \widehat{G}(k)$ be a semisimple elliptic parameter. Then there are an open compact subgroup $K \subset G(\mathbb{A}_F)$ and a nonzero cusp form φ_σ on $G(F)\backslash G(\mathbb{A}_F)/K$ satisfying (1.1).*

This is Theorem 12.2. Three features of the result should be stated at once. The cusp form lives over F itself: neither the field nor its field of constants is extended. The parameter may be ramified at finitely many places and may have disconnected, even finite, monodromy. And the parameter is recovered through every excursion operator, so the form determines σ up to conjugacy and not only the conjugacy classes of its Frobenius elements.

For the types G_2 and F_4 the hypothesis is $p > 3$, for E_6 it is $p > 5$, and for E_7 and E_8 it is $p > 7$. For a group of classical type it is a condition which grows with the rank; these groups are treated in every characteristic in [III], by a different method, so the theorem is needed in [I] only for the five exceptional types. The hypothesis belongs to the method. It supplies nondegenerate invariant forms on the Lie algebra and its Levi subalgebras, and it keeps wild inertia inside tori when the parameter is lifted to characteristic zero. We do not expect it to reflect an obstruction to automorphy.

The function φ_σ is a nonzero element of the excursion summand of σ, and it is not unique. We do not determine multiplicities, the structure of packets, or compatibility with the local correspondence at the ramified places.

1.2 Relation to earlier work

General linear groups and excursion operators. Laurent Lafforgue proved the correspondence for irreducible GL_n-parameters with finite-order determinant [Laf02, Introduction, Theorem $(\mathrm{i})_n$–$(\mathrm{ii})_n$]. For a split reductive group, Vincent Lafforgue constructed the decomposition of cusp forms by parameters [Laf18, Theorem 0.1]. That theorem identifies the parameters of the summands that occur; that the summand of a prescribed parameter is nonzero is a separate assertion, and it is the subject of this series of papers. For a split semisimple group and a parameter which is everywhere unramified and has Zariski dense image, [BHKT19, Theorem 1.1] gives, after a finite Galois extension of the function field, a cuspidal representation whose unramified components have the Satake parameters prescribed by the parameter. Our conclusion keeps the field fixed.

Geometric Langlands. The geometric input is the restricted geometric Langlands equivalence of Gaitsgory and Raskin over an algebraically closed field of characteristic zero [GR25, Theorem 1.3.9(ii)], together with its Whittaker functional. In positive characteristic their theorem gives an equivalence on a union of connected components of the spectral side, and whether this union is everything is their Conjecture 1.3.10. The subsequent preprint [Ope26c, Theorem 1.4], by OpenAI, states that it is, over $\overline{\mathbb{F}}_q$ under the four Lie-theoretic assumptions of the restricted theory, and over an arbitrary algebraically closed field when in addition p is very good and $p \nmid |W_G|$; its Corollary 8.2 gives the arithmetic formula for unramified parameters. We do not use that theorem. Our result concerns arithmetic parameters over F that are ramified at finitely many places and may have finite or disconnected monodromy, and it produces functions, not sheaves.

Our method does, however, owe several ideas to the three Langlands preprints [Ope26c, Ope26a, Ope26b] in OpenAI's public *math* repository. Testing a Hecke eigensheaf against a hypersurface transverse to the nilpotent cone, letting modification points collide, and working with framed bundles all appear there and are starting points for the present paper. The statements of those preprints have different hypotheses from ours: the tame eigensheaf theorem [Ope26a, Theorem 1.1] assumes geometrically dense monodromy and one tame

regular-unipotent puncture, and the Frobenius theorem [Ope26b, Theorem 1.1] treats SL_n with $p > n$ and allows a finite extension of constants. None of them is used as an input.

What is new in the geometric argument is the combination of three steps: lifting the ramification of the parameter to a curve in characteristic zero on which it becomes unramified; constructing a coefficient whose Frobenius supertrace is a nonzero determinant; and comparing the whole category of coefficients, not one object, through specialization, which reduces the ambiguity in the Frobenius structure to a single scalar. A trace calculation with a homogeneous test function then yields a bundle defined over $\mathbb{F}_q$ at which the resulting function does not vanish. That calculation draws on [BBAMY26, Appendix A]; the Whittaker and spectral comparisons use [AGK+22, NY19, GKRV22, GR24, NT26].

1.3 Organization

Section 2 explains the construction without proofs and shows how the main results depend on one another. Section 3 lists the results we import, including those of [I], Section 4 fixes conventions, and Section 5 introduces the curves and the Hecke families which are used throughout. The construction itself occupies Sections 6–12. A parameter over $\mathbb{F}_5(t)$ with image $S_3 \subset \mathrm{PGL}_2(k)$ is followed through the first steps in Sections 6.4 and 7.5. A list of notation is at the end, and Appendix A lists the identities which were also checked by computer.

2 Overview of the construction

This section describes the proof of Theorem 1.2 without giving it. Its main difficulty is not visible from the statement, and the first subsection explains it. The last subsection lists the statements in which the steps are carried out and shows how they depend on one another.

2.1 The difficulty: a nonzero sheaf can have zero trace

Geometric Langlands produces sheaves, and we need a function. The function attached to a complex of sheaves $\mathcal{K}$ with an action of Frobenius is the alternating sum of the traces of Frobenius on the cohomology of its stalks. This function can vanish identically for a nonzero $\mathcal{K}$: on a point, the complex $k \oplus k[-1]$ with the identity as Frobenius already has trace zero. So a theorem that produces a nonzero Hecke eigensheaf does not, by itself, produce a nonzero automorphic function. To exclude this cancellation, over the given finite field and without enlarging it, is the central problem of the geometric construction.

The way out is to arrange that the alternating trace in question is a determinant. If ϕ is an endomorphism of a finite-dimensional vector space V, then the alternating trace of ϕ on the exterior algebra of V, with V placed in degree -1, is

$$\sum_i (-1)^i \operatorname{tr}(\phi; \Lambda^i V) = \det(1 - \phi; V).$$

A determinant is nonzero as soon as no eigenvalue of ϕ is equal to one, and this is a condition that purity can verify.

2.2 Where the exterior algebra comes from

The parameter σ lives on an open part of the curve X in characteristic p, and may be ramified. The first step of the construction (Section 7) replaces this situation by an unramified one in characteristic zero: there is a smooth projective curve C over an algebraically closed field of characteristic zero, with an everywhere unramified local system ρ. This curve degenerates,

through a nodal curve in characteristic zero, to a curve with two components in characteristic p, which is geometrically reduced and Gorenstein. One component is X, with the parameter σ; the other is auxiliary. On C the geometric Langlands equivalence of Gaitsgory and Raskin is available. It attaches to a coherent sheaf P on the space of local systems an automorphic sheaf $\mathcal{F}_P$ on the stack of G-bundles on C, and it identifies a certain functional of $\mathcal{F}_P$, its Whittaker coefficient, with the space of global sections of P.

It remains to choose P. The obvious choice is the structure sheaf of the residual gerbe of ρ, whose space of global sections is k. But its direct image on the stack of local systems need not be compact, and compactness is used twice below: for constructibility, and for the formula which expresses the Whittaker coefficient by ordinary global sections. Instead we remove a few points from the auxiliary component, fix the restriction of ρ to the punctured curve, and consider all ways of extending it across the punctures. As a set this space of extensions is small. As a derived space it also remembers the equations which say that the monodromy around each puncture is trivial, one equation for each direction in the Lie algebra, and these equations contribute exterior generators: the space is

$$\mathfrak{F}_\rho \simeq [\operatorname{Spec}\Lambda(\mathsf{B}_\rho[1])/Z_\rho],$$

where B_ρ is the sum of one copy $\widehat{\mathfrak{g}}_x^*$ of the dual of the Lie algebra of $\widehat{G}$ for each point x which is removed, and Z_ρ is the centralizer of the monodromy of ρ (Section 6). For an orbit of length d of Frobenius on the points which are removed, and a point x of the orbit, the d-th power of Frobenius acts on the summand $\widehat{\mathfrak{g}}_x^*$ by q^d times the coadjoint action of the image g_D of Frobenius under the parameter. The trace we need is therefore, up to the corrections discussed in the next paragraph,

$$\det(1 - q^d \operatorname{Ad}(g_D)^\vee; \widehat{\mathfrak{g}}_x^*),$$

and it is not zero: by the purity theorem of Section I.4 the eigenvalues of $\operatorname{Ad}(g_D)$ have absolute value one, so none of them is q^{-d}. In the example of Section 6.4, where $q = 5$, $d = 1$, $g_D = 1$ and the Lie algebra has dimension three, this determinant is $(1-5)^3 = -64$.

Two corrections are needed because of the quotient by Z_ρ. Global sections on the quotient are the invariants of Z_ρ, and taking invariants could reintroduce cancellation. The group of components Π_ρ of Z_ρ is finite; we tensor the structure sheaf with its regular representation,

$$P_{\mathrm{reg}} = \mathcal{O}_{\mathfrak{F}_\rho} \otimes k[\Pi_\rho],$$

because the invariants of a finite group in a representation tensored with the regular representation are the representation itself. The identity component of Z_ρ is a torus, since σ is elliptic, and Frobenius fixes no nontrivial character of it; so the parts of nonzero weight, which the invariants discard, would not have contributed to the trace in any case. The result is Proposition 6.10.

2.3 Why the comparison is made for all coefficients

We have computed a trace on the side of local systems, and we need it on the side of automorphic sheaves, in characteristic p. The equivalence of Gaitsgory and Raskin is an equivalence over an algebraically closed field. It identifies the two vector spaces, but it says nothing about the action of Frobenius on them, and an isomorphism of vector spaces that does not respect Frobenius is of no use for traces. If we transported only the object P_{reg}, the two actions of Frobenius could differ by an arbitrary automorphism.

For this reason we transport the functor and not one object. On the side of local systems the functor is

$$\Gamma_\rho : \operatorname{Perf}(\mathfrak{F}_\rho) \longrightarrow \operatorname{Vect}_k, \qquad P \longmapsto \Gamma(\mathfrak{F}_\rho, P),$$

and on the automorphic side it is $P \mapsto$ the Whittaker coefficient of $\mathcal{F}_P$. The two Frobenius actions then differ by an automorphism of the functor Γ_ρ, and such automorphisms are very few: by the Yoneda lemma they are given by the invertible functions of degree zero on $\mathfrak{F}_\rho$, which are the nonzero constants. So the two actions differ by one scalar $c \in k^\times$, the same for all P and in all degrees (Lemma 6.7). A scalar multiplies the alternating trace of P_{reg} by a nonzero number and cannot make it vanish.

2.4 From characteristic zero to characteristic p

The price is that the whole family $P \mapsto \mathcal{F}_P$, with all the maps between its members and all its Hecke identities, has to be carried from the smooth curve C in characteristic zero to the curve with two components in characteristic p. There are two degenerations, the formation of the nodes and the reduction modulo p, and we treat them together over a valuation ring of rank two (Section 8).

The family is extended by relative nearby cycles. To compare its Whittaker coefficient before and after specialization, we replace this coefficient by a local invariant, the vanishing cycles of $\mathcal{F}_P$ for a suitable function f on the stack of bundles at a suitable point, and we must show that this invariant is the same before and after specialization. It could change only at points where the differential of f lies in the singular support of the sheaves, in a suitable relative sense. For Hecke eigensheaves over a field the singular support is contained in the nilpotent cone; Theorem 9.2 is a form of this statement over a valuation ring: the restriction of the family to the hypersurface $f = 0$ still specializes well at every point where the differential of f is not nilpotent. The proof moves the point of a Hecke modification until the direction in question becomes visible there. The Hecke identity controls only the direct image of the sheaf along a proper map from the Hecke correspondence, and the theorem is a statement about one stalk before the direct image is taken. So one argument isolates the stalk inside a fibre of the correspondence (Lemma 9.8), and another removes the auxiliary variables introduced on the way (Lemmas 9.9 and 9.10). One nilpotent direction remains, the regular one. There f is chosen homogeneous of weight two for an action of the multiplicative group, and the part of the vanishing cycles on which -1 acts by -1 is shown to be the same at the three points of the base (Section 10).

2.5 Returning to the original curve and field

At this stage we know that a vanishing cycle of the sheaf in characteristic p has nonzero Frobenius trace. This is still not a value of the function. Because f is homogeneous, the trace on its vanishing cycles can be written as a sum of traces on stalks of the sheaf, at points defined over $\mathbb{F}_q$, some of them twisted by the action of -1. The twisted terms are ordinary traces at other rational points: if x satisfies $-\mathrm{Fr}_q(x) = x$ and $a^{q-1} = -1$, then ax is fixed by Fr_q, and the equivariance of the sheaf identifies the twisted trace at x with the ordinary trace at ax. So if the function vanished at every rational point, the trace on the vanishing cycles would vanish too. Hence the function is not zero at some bundle defined over $\mathbb{F}_q$; no extension of the finite field is made.

That bundle lives on the curve with two components. We restrict it to the auxiliary component, keep this restriction fixed, and let the bundle on X vary. The result is a nonzero function on bundles on X with a level structure along the points where the components meet (Section 12). The order of the two steps matters: the auxiliary bundle is the one found by the trace argument, and an arbitrary choice made beforehand would not be known to give a nonzero function.

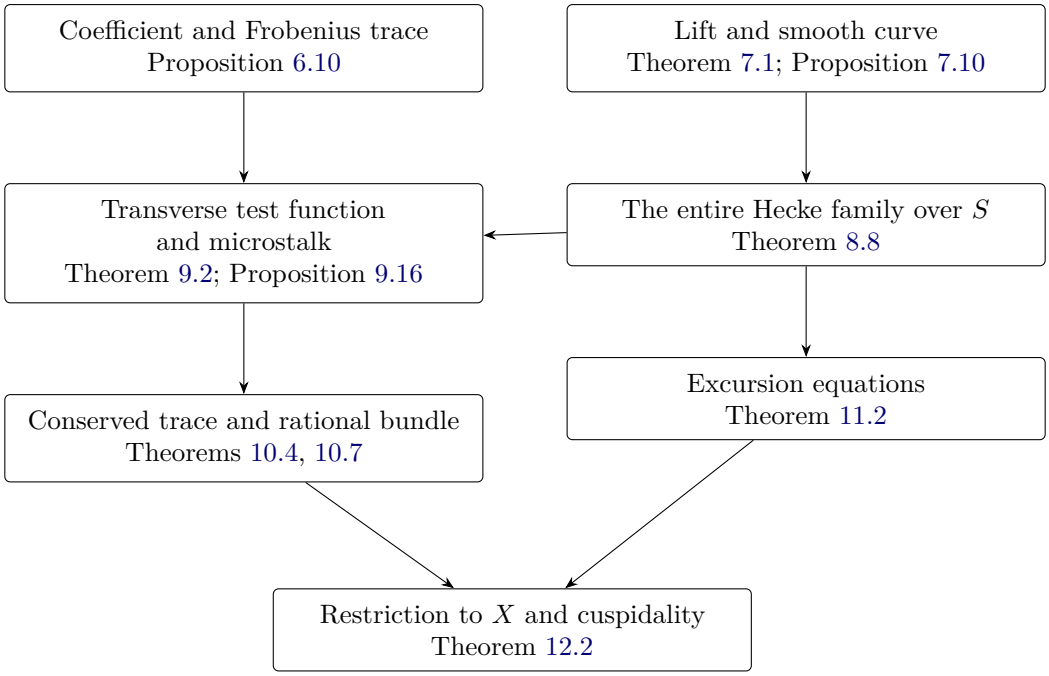


Figure 1: The two inputs to geometric realization: a nonzero rational trace and the excursion equations. The final restriction uses both.

2.6 The parameter

It remains to see that this function has parameter σ, in the sense of (1.1). Its Hecke eigenvalues at the unramified places are those of σ, because it comes from a Hecke family attached to σ. But for a general group the Hecke eigenvalues do not determine the parameter, and (1.1) asks for more: the eigenvalues of all excursion operators. An excursion operator is built from the cohomology of spaces of shtukas with several legs, and its three steps, creation, action of Galois elements, and annihilation, have counterparts for the family of sheaves: Hecke functors with several legs and their compatibilities when legs collide. Taking traces of Frobenius of these compatibilities gives the excursion equations (Theorem 11.2). They come out for the Chevalley dual of the parameter of the sheaf, so the construction is applied to the Chevalley dual of σ. This is why the Hecke identities of the family, with all legs and all collisions, were carried through every step above.

Two further points complete the argument. The trace argument gives the excursion equations for tuples of elements of the Weil group. The function is cuspidal. Otherwise one of its constant terms along a proper parabolic subgroup would have a parameter with values in a proper Levi subgroup of the dual group, and an elliptic parameter is not of this form (Section I.8). On the space of cusp forms the equations extend to tuples in the Galois group by continuity (Corollary I.6.14).

2.7 The statements and their dependencies

The nonzero number of the first subsections is followed through three statements.

1. Proposition 6.10: on the side of local systems, the Frobenius supertrace on the global sections of a certain perfect complex is a determinant, and it is nonzero by purity.

2. Proposition 10.9: the corresponding automorphic sheaf, specialized to characteristic p, has nonzero trace at some bundle defined over $\mathbb{F}_q$ on the curve with two components.

3. Proposition 12.1: the trace remains nonzero after the bundle on the auxiliary component is fixed, which gives a function on bundles on X.

Each passage must be compatible with Frobenius, and most of the work lies in the passage from (1) to (2). Figure 1 shows how these and the other main results depend on one another; an arrow means that the result at its tail is used in the result at its head. The calculation of the trace on the side of local systems and the construction of the curve can be read separately, up to the comparison under specialization. The proof of Theorem 12.2, in Section 12, assembles these results.

3 Results used

We collect the theorems on which the construction rests, in the form in which they are used. Precise references to the individual statements are repeated at each point of use.

3.1 From the literature

Input 1 (Restricted geometric Langlands in characteristic zero)**.** *Let C be a smooth proper curve over an algebraically closed field of characteristic zero. Gaitsgory and Raskin construct a functor $\mathbb{L}^r_G$ from the category of sheaves with nilpotent singular support on $\mathrm{Bun}_G(C)$ to the category of ind-coherent sheaves with nilpotent singular support on the stack $\mathrm{LS}^r_{\widehat{G}}(C)$ of local systems with restricted variation, and prove that in characteristic zero it is an equivalence [GR25, Theorem 1.3.9(i)–(ii)]. (Over a ground field of positive characteristic their theorem gives an equivalence onto the sheaves supported on a union of connected components; the hypotheses on the characteristic are those of [GR25, §0.1.9].) We use three of its properties: it intertwines the Hecke action with the spectral action of [AGK+22, Theorems 8.1.4 and 14.3.2]; on the image of quasi-coherent sheaves its inverse is given by the spectral action on the nilpotent projection of the Poincaré object [GR25, §§1.1.3–1.1.8]; and the Whittaker coefficient of the automorphic sheaf which corresponds to a quasi-coherent sheaf is the complex of $!$-global sections of the latter on the spectral side [GR25, Proposition 1.2.5].*

Input 2 (Universal local acyclicity over a valuation ring)**.** *Let S be the spectrum of an absolutely integrally closed valuation ring with generic point η, and let Y/S be separated and of finite presentation. Restriction to the generic fibre is an equivalence between complexes on Y that are universally locally acyclic over S and constructible complexes on Y_η, with inverse Rj_* [HS23, Theorem 4.1]. Universal local acyclicity can be tested on rank-one valuation rings [HS23, Corollary 3.10].*

Input 3 (Vanishing cycles over a two-dimensional base)**.** *We use Saito's results on nearby cycles for a bounded constructible complex and a morphism of finite type. If the morphism is locally acyclic relative to the complex outside a closed subscheme which is quasi-finite over the base, the nearby cycles and the vanishing cycles are constructible and commute with finite base change; if it is universally locally acyclic outside this subscheme, they commute with every base change [Sai15, Proposition 1.8 and Corollary 1.10].*

Input 4 (The Whittaker functional as a microstalk)**.** *For a smooth projective curve C over $\mathbb{C}$ and sheaves in the classical topology with nilpotent singular support on $\mathrm{Bun}_G(C)$, Nadler and Taylor define a Whittaker functional as the vanishing cycles, at the canonical bundle, of the sum of the simple root coordinates on the stack of twisted N-bundles, applied to the $!$-pullback of a sheaf. They prove that, after a shift, this functional computes the microstalk at the regular nilpotent Higgs field on the canonical bundle [NT26, (3.8) and Theorem 3.3.4]. Two further steps are needed here and are carried out in Proposition 9.16: the comparison of*

this functional with the Whittaker coefficient of Input 1, and the transfer to ℓ-adic sheaves over an arbitrary algebraically closed field of characteristic zero.

Some of these sources are recent preprints, among them [GR25], [NT26] and [BBAMY26]. The first two enter the proof of Theorem 12.2 exactly as stated in Inputs 1 and 4; from the third we take a method and no statement (Section 10). The results of L. Lafforgue, V. Lafforgue and Xue on which the arithmetic part rests are listed in [I, Section 2].

3.2 From the first paper

The arithmetic part of the argument is contained in [I]. We use the following definitions and statements; a reader who accepts them does not need the rest of that paper.

1. *Excursion operators on functions.* For a level N the excursion operators S_e, for excursion data $e = (I, f, \boldsymbol{\gamma})$ with $\boldsymbol{\gamma} \in W_F^I$, act on the functions with compact support on $Y_N = G(F)\backslash G(\mathbb{A}_F)/K_N$, and S_e^{fun} is defined on all functions on Y_N by transposition (Definition I.5.12). On cusp forms it is the operator of V. Lafforgue. The operators S_e are defined through the cohomology of the stacks of shtukas with integral coefficients (Section I.5), and the partial Frobenius maps on this cohomology have the properties stated in Lemma I.5.8.

2. *Purity.* A semisimple elliptic parameter is strongly pure (Theorem I.4.8). We also use Lemma I.4.1 on compact subgroups of $\mathrm{GL}_n(k)$, and Lemma I.4.4, by which the quotient of the arithmetic monodromy group of such a parameter by the geometric one is finite and cyclic.

3. *From a functional to a cusp form.* Let $\varphi \neq 0$ be a function on Y_N and σ_{coeff} a semisimple elliptic parameter unramified outside N. If the linear form $c \mapsto [c, \varphi]$ on functions with compact support has the excursion eigenvalues of σ_{coeff} for all tuples in the Weil group, then φ is a cusp form with parameter $\vartheta_G \sigma_{\mathrm{coeff}}$, for all tuples in Γ_F (Proposition I.8.10; see also Lemma I.8.11).

Those conventions of [I, Section 3] which are used in this paper are repeated in Section 4.

4 Conventions

We fix the normalizations of Frobenius and of trace functions used in the whole paper, define the excursion summand attached to a parameter, and recall the Chevalley involution. The spaces of functions on which excursion operators act are constructed in Sections I.5 and I.6. The conventions of [I, Section 3] on the centre and on class field theory are not needed here, because G is semisimple and simply connected.

4.1 Frobenius and trace functions

Let $\deg_F : \Gamma_F \to \widehat{\mathbb{Z}}$ be the homomorphism given by the action on the constant field, normalized so that an arithmetic Frobenius element has degree 1. The Weil group W_F is the inverse image of $\mathbb{Z}$; it is dense in Γ_F. A geometric Frobenius element at a place v of degree d_v has degree $-d_v$. We write Fr_q for the Frobenius endomorphism of a scheme over $\mathbb{F}_q$, and Frob_q for the action of geometric Frobenius on cohomology.

For a bounded complex $\mathcal{K}$ with a Weil structure and a rational point x, the trace function is

$$t_{\mathcal{K}}(x) = \sum_i (-1)^i \operatorname{Tr}(\mathrm{Frob}_q; H^i(\mathcal{K}_x)),$$

and we write Str for an alternating trace of this kind. A shift by one changes the sign of the trace function. Tate twists and shifts are always kept when trace functions are compared. The Satake equivalence, for G and for all its Levi subgroups, is normalized by one square root $q^{1/2} \in k$ chosen once and for all. When q is the square of an integer we take the positive root; then in all cases there is an isomorphism of k with $\mathbb{C}$ which maps $q^{1/2}$ to the positive square root of q.

4.2 Excursion summands

Definition 4.1. Let V be a finite-dimensional vector space on which the excursion operators S_e act, for all finite sets I, all functions f and all tuples in Γ_F^I; they generate a commutative algebra $\mathcal{B}$ of endomorphisms. Let $\mathfrak{m}_\sigma \subset \mathcal{B}$ be the ideal generated by the operators $S_e - \nu_\sigma(e)$. The *excursion summand* of σ in V is the generalized eigenspace

$$V_{\nu_\sigma} = \bigcup_{n \geq 1} \{v \in V : \mathfrak{m}_\sigma^n v = 0\}.$$

For a space which is a union of subspaces V of this kind, the excursion summand is the union of the V_{ν_σ}.

The definition involves the excursion operators for all tuples in the Galois group; the Hecke operators at the unramified places are among them. A nonzero excursion summand in a space of finite dimension contains a joint eigenvector of the excursion operators, with eigenvalues $\nu_\sigma(e)$: the commuting operators $S_e - \nu_\sigma(e)$ are nilpotent on it. So the existence of a nonzero summand and the existence of an eigenfunction are equivalent.

4.3 The Chevalley involution

Let ϑ_G be the Chevalley involution of $\widehat{G}$ which preserves the pinning. On the maximal torus of the pinning it is $t \mapsto w_0(t^{-1})$, where w_0 is the longest element of the Weyl group [Li19, Definition 3.1.1]. We put

$$\sigma^\dagger = \vartheta_G \circ \sigma.$$

The parameter $\sigma^\dagger$ appears because the pairing between functions with compact support and arbitrary functions is bilinear: transposition of an excursion operator replaces σ by $\sigma^\dagger$, see (I.5.6). The letter σ always denotes the parameter of the main theorems.

5 Geometric setup

This section fixes the geometric objects which are used from Section 6 on: the curves, the base over which they vary, the Hecke correspondences, and the notion of a Hecke family. It contains no proofs.

5.1 Curves and specializations

The curve of the closed fibre is the union $C_0 = X \cup_D C_2$ of the original curve and of an *auxiliary component* C_2. It is Gorenstein and geometrically reduced, but it is not nodal when D is not reduced. Its construction is expressed by the pushout square

$$\begin{array}{ccc} D & \hookrightarrow & C_2 \\ \downarrow & & \downarrow \\ X & \hookrightarrow & C_0. \end{array} \tag{5.1}$$

The original parameter and the Hecke legs remain on X. The component C_2 carries the extra puncture that produces the exterior algebra of Section 6. On the intermediate fibre, in characteristic zero, the two local systems have opposite inertia along D; this is the condition under which the glued local system extends to the smoothing. The divisor D may have multiplicities; its scheme structure records the level retained at the end of the construction.

Two specializations are involved: the smooth curve degenerates to a nodal one, and the nodal curve in characteristic zero reduces to characteristic p. Section 8.1 constructs a rank-two valuation base S with geometric points η, ξ, s in this order. The fibres have the following roles:

$$\begin{array}{ccccc} \mathcal{C}_\eta & \overset{\text{the nodes form}}{\rightsquigarrow} & \mathcal{C}_\xi & \overset{\text{reduction mod } p}{\rightsquigarrow} & C_0 \otimes \overline{\mathbb{F}}_q \\ \downarrow & & \downarrow & & \downarrow \\ \eta & \rightsquigarrow & \xi & \rightsquigarrow & s. \end{array} \tag{5.2}$$

The first fibre is smooth and proper in characteristic zero; the middle fibre has the two components of (5.1) in characteristic zero; the last is the curve C_0 in characteristic p, which is geometrically reduced and Gorenstein. In general only the middle fibre is nodal. The wavy arrows indicate specialization. All comparisons are made along this ordered flag.

For a finite set of legs I in the smooth open U, the Hecke functor is defined by the correspondence

$$\begin{array}{ccccc} \mathrm{Bun}_G(\mathcal{C}) \times U^I & \xleftarrow{h_0} & \mathrm{Hk}_{G,I} & \xrightarrow{h_1} & \mathrm{Bun}_G(\mathcal{C}) \times U^I \\ & \searrow & \downarrow & \swarrow & \\ & & U^I. & & \end{array} \tag{5.3}$$

After bounding the modifications by a Satake representation, the appropriate projection is proper.

5.2 The object which is transported: Hecke families

One kind of object is constructed in characteristic zero (Section 6), extended over the base S (Section 8), restricted to the curve X (Section 12), and finally used to produce a function (Section 11). We define it here once, in the generality needed for all four steps.

Let $\mathcal{Y}$ be a stack of G-bundles on a curve, with or without level structure, over a field or over S, and let U' be an open subset of the smooth locus of the curve, disjoint from the level. For a finite set I and a representation V of $\widehat{G}^I$, modifications of bundles at I points of U', with the Satake kernel of V, define the Hecke functor $\mathsf{H}_{I,V}$ from complexes on $\mathcal{Y}$ to complexes on $\mathcal{Y} \times U'^I$; when I has one element we write H_V. Let ρ be a $\widehat{G}$-local system on U', and Z a reductive group over k which acts on ρ by automorphisms. For V as above let V_ρ be the local system on U'^I attached to ρ and V; the group Z acts on it. For a local system $\mathcal{W}$ with an action of Z on a scheme B, and a functor $W \mapsto \mathcal{F}_W$ on representations of Z, put

$$\mathcal{F}(\mathcal{W}) = \bigoplus_W \mathcal{F}_W \boxtimes \underline{\mathrm{Hom}}_Z(W, \mathcal{W}) \qquad \text{on } \mathcal{Y} \times B,$$

the sum being over the irreducible representations of Z.

Definition 5.1 (Hecke families)**.** 1. A *Hecke family* on $\mathcal{Y}$, with legs in U' and eigenvalue ρ, is a k-linear functor $W \mapsto \mathcal{F}_W$ from the finite-dimensional representations of Z to complexes on $\mathcal{Y}$, together with isomorphisms

$$\mathsf{H}_{I,V}(\mathcal{F}_W) \simeq \mathcal{F}(W \otimes V_\rho) \qquad \text{on } \mathcal{Y} \times U'^I$$

for all I, V and W. They are required to be natural in W and in V, to be the identity for the trivial representation V, and to be compatible with the restriction to the diagonals of U'^I and with convolution, that is, with the composition of Hecke functors.

2. Let ϕ be an automorphism of the curve which preserves U' and the level structure, so that it acts on $\mathcal{Y}$, let α be an automorphism of Z, and let $\theta : \alpha^*\phi^*V_\rho \simeq V_\rho$ be isomorphisms commuting with Z and compatible with the operations on V. A *structure of compatibility with* ϕ on a Hecke family is a system of isomorphisms

$$c_W : \phi^*\mathcal{F}_W \simeq \mathcal{F}_{\alpha^*W},$$

natural in W, such that for all I, V and W the two isomorphisms from $\phi^*\mathsf{H}_{I,V}(\mathcal{F}_W)$ to $\mathcal{F}(\alpha^*W \otimes V_\rho)$ are equal: the Hecke isomorphism for W followed by c and θ, and $\mathsf{H}_{I,V}(c_W)$ followed by the Hecke isomorphism for α^*W.

3. When $\mathcal{Y}$ is defined over $\mathbb{F}_q$, the pullbacks by the Frobenius endomorphisms of $\mathcal{Y}$ and of U'^I are equivalences of the étale categories. A *Hecke family with Frobenius* is a Hecke family together with a comparison of the Hecke functors under these pullbacks, given by the Frobenius structure of the Satake kernels as in (F3) of Setup 11.1.

In this paper the word *coefficient*, used alone, refers to the argument of such a family: a representation W of Z, a local system $\mathcal{W}$ with an action of Z, or more generally a perfect complex P on the stack $\mathfrak{F}_\rho$ of Section 6. Whittaker coefficients, Fourier coefficients and fields of coefficients are always named in full.

The conditions (1) and (2), for $\mathcal{Y} = \mathrm{Bun}_{G,D}(X)$ and the Frobenius endomorphism, are written out with all normalizations as (F1)–(F3) in Setup 11.1. In these terms the steps of the construction are the following.

- Lemma 6.6: the functor $W \mapsto \mathcal{F}_{P_W}$ is a Hecke family on the stack of bundles of the smooth curve in characteristic zero, with legs outside D_{aux}, and it has a structure of compatibility with the semilinear Frobenius automorphism. (The lemma is stated for the larger family $P \mapsto \mathcal{F}_P$ indexed by perfect complexes; this is needed in Section 10.)
- Theorem 8.8: the family, with this structure, extends over S.
- Proposition 12.1: the restriction of the fibre over s to $\mathrm{Bun}_{G,D}(X)$ is a Hecke family with Frobenius.
- Theorem 11.2: the trace function of a Hecke family with Frobenius satisfies the excursion equations of its eigenvalue.

6 A perfect complex whose trace cannot cancel

As explained in Section 2, the first step of the geometric construction is to find a perfect complex whose Frobenius trace cannot vanish by cancellation. It is constructed in this section, in characteristic zero and before any sheaf is specialized. Its cohomology is an exterior algebra, so the alternating trace of Frobenius on it is a determinant, and purity shows that no factor of this determinant is zero.

The section is organized as follows. After fixing the data (Setup 6.1), we describe the stack on which the perfect complexes live, a derived fibre (Lemma 6.4), define the corresponding family of automorphic sheaves and compute its Whittaker coefficient (6.2), show that the family is compatible with Hecke functors and with Frobenius (Lemma 6.6), and finally compute the supertrace (Proposition 6.10).

Setup 6.1. Let C be a smooth proper curve over an algebraically closed field K_{geom} of characteristic zero, and let ρ be a semisimple $\widehat{G}$-local system on C with coefficients in $k = \overline{\mathbb{Q}}_\ell$. The curve and the local system carry a semilinear automorphism lifting the arithmetic q-Frobenius; let $g \in \widehat{G}(k)$ be its action on a marked fibre of ρ. Let Mon_ρ be the Zariski closure of the geometric monodromy. We assume that

1. Mon_ρ is reductive;
2. $\overline{\langle \mathsf{Mon}_\rho, g\rangle}/\mathsf{Mon}_\rho$ is finite;
3. $C_{\widehat{G}}(\mathsf{Mon}_\rho, g)$ is finite.

Such data are constructed from the given elliptic parameter in Section 7. The curve lives over K_{geom}, and k is the field of coefficients of the sheaves.

Put $Z_\rho = C_{\widehat{G}}(\mathsf{Mon}_\rho)$, $\alpha = \mathrm{Ad}(g)|_{Z_\rho}$ and $\Pi_\rho = \pi_0(Z_\rho)$. Thus Z_ρ is the centralizer of the geometric monodromy. Ellipticity makes the centralizer of the arithmetic monodromy finite, while Z_ρ can contain a torus.

Lemma 6.2 (The centralizer of the geometric monodromy)**.** *The group Z_ρ is reductive, Z_ρ° is a torus, and $1-\alpha : Z_\rho^\circ \to Z_\rho^\circ$ is an isogeny.*

Proof. The centralizer of a reductive subgroup of a reductive group in characteristic zero is reductive. By assumption (2) some power g^n lies in Mon_ρ; it commutes with Z_ρ, so $\alpha^n = 1$. The group of fixed points of α is $C_{\widehat{G}}(\mathsf{Mon}_\rho, g)$, which is finite by assumption (3).

Suppose that the derived group of Z_ρ° is not trivial. An automorphism of finite order of a connected reductive group stabilizes a Borel subgroup and a maximal torus of it [Ser07, §3.3, Proposition 3(b)–(c)]. Then α permutes the corresponding simple coroots, and the sum of the coroots in one orbit is a nonzero cocharacter fixed by α. Its image is a torus of dimension one fixed by α, which contradicts the finiteness of the group of fixed points. So Z_ρ° is a torus. The kernel of $1-\alpha$ on it is the group of fixed points, which is finite; so $1-\alpha$ is an isogeny. □

6.1 Filling punctures as a derived intersection

Choose a nonempty divisor D_{aux} on C, stable under the Frobenius automorphism and consisting of distinct points, and write $C^\circ = C - D_{\mathrm{aux}}$. For the stacks of local systems with restricted variation of [AGK+22, GR25], consider

$$Z_C = \mathrm{LS}^r_{\widehat{G}}(C), \qquad Z_{C^\circ} = \mathrm{LS}^r_{\widehat{G}}(C^\circ), \qquad \mathfrak{F}_\rho = BZ_\rho \times^R_{Z_{C^\circ}} Z_C.$$

Here $BZ_\rho \to Z_{C^\circ}$ is the residual gerbe of the point $\rho|_{C^\circ}$, and we write $j : \mathfrak{F}_\rho \to Z_C$ for the projection. The underlying classical stack of $\mathfrak{F}_\rho$ is BZ_ρ. The derived structure records the equations which express that the monodromy at the punctures is trivial; they are satisfied at ρ, but not transversally.

The stacks Z_C and Z_{C° are defined by means of lisse ℓ-adic sheaves. To compute with them we compare them with stacks of representations of topological fundamental groups. The comparison has three parts: a change of the base field, the passage from ℓ-adic sheaves to sheaves for the classical topology, and the passage from local systems with restricted variation to all local systems. We make them explicit, because the second and the third are used for the curve C°, which is not proper.

The base field. The curve C and the divisor D_{aux} are defined over an algebraically closed subfield $K_0 \subset K_{\mathrm{geom}}$ of finite transcendence degree over $\mathbb{Q}$, and such a field can be embedded in $\mathbb{C}$. For an extension of algebraically closed fields of characteristic zero and a scheme Y of finite type over the smaller field, pullback is an equivalence between the categories QLisse of Y and of its base change [AGK+22, §9.5.2, with its footnote, and Remark 9.5.3]. It is

compatible with tensor products and with restriction to open subschemes. The stack of local systems with restricted variation is defined in terms of the symmetric monoidal category QLisse with its t-structure. Therefore Z_C, Z_{C°, the map between them and the point ρ do not change when the pair (C, D_{aux}) is replaced by its model over K_0, and then by the base change of this model to $\mathbb{C}$. Lemmas 6.4 and 6.5 concern only these objects; the semilinear automorphism of Setup 6.1 does not occur in them. In their proofs we may therefore assume that $K_{\mathrm{geom}} = \mathbb{C}$.

Lemma 6.3 (Comparison with the Betti stacks)**.** *Suppose that $K_{\mathrm{geom}} = \mathbb{C}$. For $Y = C$ or $Y = C^\circ$ let $\mathrm{LS}^{\mathrm{B}}(Y)$ be the stack of $\widehat{G}$-local systems with coefficients in k on the topological space $Y(\mathbb{C})$, that is, the stack of maps from its homotopy type to $B\widehat{G}$ [AGK+22, §4.5.1].*

1. *There are monomorphisms of prestacks $\iota_Y : Z_Y \to \mathrm{LS}^{\mathrm{B}}(Y)$, compatible with restriction from C to C°.*
2. *Put*
$$T = BZ_\rho \times^R_{\mathrm{LS}^{\mathrm{B}}(C^\circ)} \mathrm{LS}^{\mathrm{B}}(C),$$
the fibre product being formed with the residual gerbe of $\rho|_{C^\circ}$ and the restriction map. The projection $T \to \mathrm{LS}^{\mathrm{B}}(C)$ factors through ι_C, and the resulting map $T \to \mathfrak{F}_\rho$ is an isomorphism, under which the projection becomes $\iota_C \circ j$.
3. *Let $\widehat{X}_\rho$ be the formal completion of $\mathrm{LS}^{\mathrm{B}}(C)$ along the fibre through ρ of the map to its coarse moduli space [AGK+22, §4.6.2]. The intersection $Z_{C,\rho}$ of Z_C and $\widehat{X}_\rho$ in $\mathrm{LS}^{\mathrm{B}}(C)$ is open and closed in each of them, and j factors through $Z_{C,\rho}$.*

Proof. Write $\mathrm{LS}^{\mathrm{B},r}(Y)$ for the stack of local systems with restricted variation in the classical topology [AGK+22, §4.7]; the results quoted from there apply to C and to C°, which are smooth connected varieties over $\mathbb{C}$.

Step 1: the monomorphisms. The comparison functor from ℓ-adic sheaves to ind-constructible sheaves of k-vector spaces for the classical topology is fully faithful and commutes with inverse images [GR25, §2.2.1]; on lisse sheaves it is symmetric monoidal [GR25, §2.2.4]. It preserves the usual t-structure, filtered colimits and lisse sheaves, so it maps the full subcategory $\mathrm{QLisse}(Y)$ of the first kind [AGK+22, Definition 1.2.6] to that of the second kind, and the functor so obtained is fully faithful and symmetric monoidal, being the restriction of a fully faithful functor to full subcategories. The functor from the second category QLisse to all locally constant sheaves is fully faithful as well [AGK+22, Proposition 4.7.2]. A fully faithful symmetric monoidal functor remains fully faithful after tensoring with $\mathrm{QCoh}(S)$, which is dualizable, and therefore induces a monomorphism between the prestacks of symmetric monoidal functors from $\mathrm{Rep}(\widehat{G})$; this is the argument of [AGK+22, Lemma 4.1.4], used in [AGK+22, §4.7.4] for the second functor. So we have monomorphisms

$$Z_Y \longrightarrow \mathrm{LS}^{\mathrm{B},r}(Y) \longrightarrow \mathrm{LS}^{\mathrm{B}}(Y),$$

and ι_Y is their composite. Both functors commute with restriction to C°. This proves (1).

Step 2: the projection factors through local systems with restricted variation. The monomorphism $\mathrm{LS}^{\mathrm{B},r}(C) \to \mathrm{LS}^{\mathrm{B}}(C)$ is a formal isomorphism [AGK+22, Proposition 4.7.8]. By the definition [AGK+22, §4.1.6] this means that a point of $\mathrm{LS}^{\mathrm{B}}(C)$ with values in an affine scheme S lies in $\mathrm{LS}^{\mathrm{B},r}(C)$ as soon as its restriction to the reduced classical scheme underlying S does. So let S be reduced and classical. On classical schemes the Betti stack classifies families of homomorphisms from the fundamental group to $\widehat{G}$, up to conjugation [AGK+22, §4.5]. A point of T with values in S is therefore given by a Z_ρ-torsor on S, a family of local systems on C parametrized by S, and an isomorphism between the restriction of this family to C°

and the twist by the torsor of the constant family with fibre $\rho|_{C^\circ}$. Since $\pi_1(C^\circ) \to \pi_1(C)$ is surjective, the family on C is the twist by the torsor of the constant family with fibre ρ. Locally on S for the étale topology the torsor is trivial, and the family is the image of the point ρ of $\mathrm{LS}^{\mathrm{B},r}(C)$ under $S \to \mathrm{pt}$. The prestack $\mathrm{LS}^{\mathrm{B},r}(C)$ is a stack for the étale topology [AGK+22, Main Theorem 1.4.5], so the point lies in it.

Step 3: the projection factors through Z_C. The map $Z_C \to \mathrm{LS}^{\mathrm{B},r}(C)$ is an isomorphism onto a union of connected components [GR25, Lemma 2.2.5], that is, onto an open and closed substack. Its complement is open and closed as well. By step 2 every point of T with values in a field maps to the point ρ, after an extension of the field, and ρ is a point of Z_C. So the inverse image in T of the complement has no points and is empty, and the projection factors through Z_C.

Step 4: the isomorphism. If a map $A \to Y$ factors through a monomorphism $Y' \to Y$, then $A \times_Y Y' \simeq A$; and if two maps $A \to Y$ and $B \to Y$ factor through Y', then $A \times_{Y'} B \simeq A \times_Y B$. Both follow from $Y' \times_Y Y' \simeq Y'$. Applying the second to the monomorphism ι_{C° and the first to ι_C and the projection of T, which factors through it by step 3, we get

$$\mathfrak{F}_\rho = BZ_\rho \times_{Z_{C^\circ}} Z_C \simeq BZ_\rho \times_{\mathrm{LS}^{\mathrm{B}}(C^\circ)} Z_C \simeq T \times_{\mathrm{LS}^{\mathrm{B}}(C)} Z_C \simeq T,$$

all fibre products being derived. The composite is inverse to the map of (2), and it is compatible with the projections to $\mathrm{LS}^{\mathrm{B}}(C)$. This proves (2).

Step 5: the formal completion. The substack $\mathrm{LS}^{\mathrm{B},r}(C)$ of $\mathrm{LS}^{\mathrm{B}}(C)$ is the disjoint union of the formal completions of the fibres of the map to the coarse moduli space [AGK+22, Corollary 4.8.5]. So $\widehat{X}_\rho$ is open and closed in $\mathrm{LS}^{\mathrm{B},r}(C)$, and so is Z_C by step 3; hence their intersection is open and closed in each. The map j factors through Z_C by construction, and through $\widehat{X}_\rho$ because every point of $\mathfrak{F}_\rho$ with values in a reduced classical scheme maps into the fibre through ρ, by step 2. This proves (3). □

Lemma 6.4 (The derived fibre)**.** *There is an isomorphism*

$$\mathfrak{F}_\rho \simeq [\operatorname{Spec} \Lambda(\mathsf{B}_\rho[1])/Z_\rho], \qquad \mathsf{B}_\rho = \bigoplus_{x \in D_{\mathrm{aux}}} \widehat{\mathfrak{g}}_x^*, \tag{6.1}$$

where $\Lambda(\mathsf{B}_\rho[1])$ is the exterior algebra of B_ρ, placed in degrees ≤ 0 with zero differential, and Z_ρ acts through the coadjoint representation at each point. In particular $H^0\Gamma(\mathfrak{F}_\rho, \mathcal{O}) = k$. The objects $\mathcal{O} \otimes W$, for W a representation of finite dimension of Z_ρ, generate $\mathrm{Perf}(\mathfrak{F}_\rho)$ *under finite cones and retracts.*

Proof. We may assume that $K_{\mathrm{geom}} = \mathbb{C}$. By Lemma 6.3(2) the stack $\mathfrak{F}_\rho$ is then the fibre T of the restriction map between the Betti stacks, which we now compute.

Let h be the genus of C and $n = |D_{\mathrm{aux}}|$. The fundamental group of the punctured surface has generators $a_1, b_1, \ldots, a_h, b_h, c_1, \ldots, c_n$ and the relation $\prod_i [a_i, b_i] \prod_j c_j = 1$. We eliminate c_n: the punctured surface retracts onto a graph with $2h + n - 1$ loops, so its Betti stack is $[\widehat{G}^{2h+n-1}/\widehat{G}]$. The closed surface is obtained from it by attaching a disc along each boundary loop, and this is a homotopy pushout. The stack of maps to $B\widehat{G}$ turns it into a derived fibre product: the Betti stack of the closed surface is the fibre product of that of the punctured surface and of $B\widehat{G}$ over $[\widehat{G}^n/\widehat{G}]$, for the map which records the images of $c_1, \ldots, c_n$ and the inclusion of the identity. Since ρ has trivial monodromy at the punctures, the fibre over the residual gerbe of $\rho|_{C^\circ}$ is the quotient by Z_ρ of the spectrum of

$$k \otimes^L_{\mathcal{O}(\widehat{G}^n)} k,$$

both copies of k being the residue field at the identity. The identity is a smooth point with cotangent space B_ρ, and the Koszul resolution shows that the cohomology of this ring is the exterior algebra of B_ρ, with B_ρ in degree -1.

To obtain an isomorphism of algebras with an action of Z_ρ, and not only of their cohomology, we use formal coordinates at the identity which are compatible with the action. They exist: since Z_ρ is reductive, the surjections $\mathfrak{m}/\mathfrak{m}^{a+1} \to \mathfrak{m}/\mathfrak{m}^2$ of representations of Z_ρ are split, compatibly in a, and the image of a splitting is a space of coordinates. In such coordinates the Koszul complex is a resolution of k by modules with an action of Z_ρ, and after tensoring with k its differential is zero. This gives (6.1) for T, hence for $\mathfrak{F}_\rho$.

The identification of the cohomology in degree -1 with the cotangent space, and of the graded algebra of cohomology with its exterior algebra, is canonical. So an automorphism which preserves the fibre acts on the graded algebra of cohomology by the exterior powers of its action in degree -1. The splitting which was used to produce a model with zero differential is a choice, and it need not be compatible with Frobenius; it is not used for the action.

The functions of degree zero on $\mathfrak{F}_\rho$ are the invariants of Z_ρ in the part of degree zero of the exterior algebra, which is k. For a complex Q of modules over the exterior algebra with a compatible action of Z_ρ, the complex of maps from $\mathcal{O} \otimes W$ to Q is $(W^* \otimes Q)^{Z_\rho}$. This functor of Q commutes with filtered colimits, because taking invariants under a reductive group does. If Q has a nonzero cohomology class, the class lies in a finite-dimensional subrepresentation W of a cohomology group, and gives a nonzero map from a shift of $\mathcal{O} \otimes W$. So the objects $\mathcal{O} \otimes W$ are compact generators, and the perfect complexes, which are the compact objects, are obtained from them by finite cones and retracts. □

Lemma 6.5 (Compactness)**.** *For every $P \in \mathrm{Perf}(\mathfrak{F}_\rho)$, the object j_*P is compact in $\mathrm{QCoh}(Z_C)$.*

Proof. By Lemma 6.4 it is enough to consider $P = \mathcal{O} \otimes W$. We may assume that $K_{\mathrm{geom}} = \mathbb{C}$, and we use the notation of Lemma 6.3 and of the preceding proof. Write $X = \mathrm{LS}^{\mathrm{B}}(C)$ and $j^{\mathrm{B}} = \iota_C \circ j : \mathfrak{F}_\rho \to X$.

Step 1: the direct image on the Betti stack is perfect. The variety $\widehat{G}^{2h+n-1}$ is smooth, and the orbit of the semisimple representation $\rho|_{C^\circ}$ in it is closed and smooth; so the inclusion of this orbit is a regular immersion. The sheaf on $[\widehat{G}^{2h+n-1}/\widehat{G}] = \mathrm{LS}^{\mathrm{B}}(C^\circ)$ supported on the orbit and defined by the representation W of the stabilizer Z_ρ is therefore perfect. It is the direct image of W from the residual gerbe. The inclusion of the residual gerbe is a closed immersion, because the orbit is closed; so its base change j^{B} is an affine morphism, and direct images along these two maps commute with base change. By base change in the square which defines T, and by Lemma 6.3(2), the pullback of this sheaf to X is $j^{\mathrm{B}}_*(\mathcal{O} \otimes W)$. Hence $j^{\mathrm{B}}_*(\mathcal{O} \otimes W)$ is perfect. The stack X is the quotient of an affine derived scheme $\operatorname{Spec} R$ by $\widehat{G}$ [AGK+22, §§4.5–4.6.1]. Since $\widehat{G}$ is reductive, the functor of global sections on X commutes with colimits, and perfect complexes on X are compact in $\mathrm{QCoh}(X)$.

Step 2: sheaves on the formal completion. Let $\hat{\imath} : \widehat{X}_\rho \to X$ be the inclusion of the formal completion of Lemma 6.3(3). The coarse moduli space of X is the spectrum of the invariants $R^{\widehat{G}}$, and its underlying classical scheme is of finite type [AGK+22, §§4.6.1–4.6.2]; let $f_1, \dots, f_m \in H^0(R)^{\widehat{G}}$ generate the maximal ideal of the image of ρ in it. Then $\widehat{X}_\rho$ is the quotient by $\widehat{G}$ of the formal completion of $\operatorname{Spec} R$ along the locus $f_1 = \cdots = f_m = 0$, which is a formal affine scheme in the sense of [AGK+22, §7.1]. Let $\mathrm{QCoh}(X)_\rho \subset \mathrm{QCoh}(X)$ be the full subcategory of the objects whose restriction to the complement of the fibre is zero; it is closed under colimits. We claim that $\hat{\imath}^*$ restricts to an equivalence

$$\mathrm{QCoh}(X)_\rho \xrightarrow{\sim} \mathrm{QCoh}(\widehat{X}_\rho).$$

For the affine scheme $\operatorname{Spec} R$ in place of X this is [AGK+22, Proposition 7.1.5], and the same holds for the schemes $\widehat{G}^a \times \operatorname{Spec} R$ with the pullbacks of the functions f_i. These schemes form the Čech nerve of $\operatorname{Spec} R \to X$, and their formal completions form the Čech nerve of the corresponding cover of $\widehat{X}_\rho$. The equivalences are given by pullback, so they are compatible with the maps of the two nerves, and the condition on the support is compatible with these maps, because pullback commutes with restriction to the open complement of the fibre. The claim follows by descent for quasi-coherent sheaves.

Step 3: conclusion. The complex $j^{\mathrm{B}}_*(\mathcal{O} \otimes W)$ is supported at the point ρ, so it lies in $\operatorname{QCoh}(X)_\rho$. It is compact in $\operatorname{QCoh}(X)$ by step 1. Hence it is compact in the full subcategory $\operatorname{QCoh}(X)_\rho$: the inclusion commutes with colimits, so maps from the complex into a filtered colimit formed in the subcategory are computed in $\operatorname{QCoh}(X)$. By step 2 the complex $\hat{\imath}^* j^{\mathrm{B}}_*(\mathcal{O} \otimes W)$ is compact in $\operatorname{QCoh}(\widehat{X}_\rho)$. The map j^{B} factors through the monomorphism $\hat{\imath}$, so $\mathfrak{F}_\rho \times_X \widehat{X}_\rho = \mathfrak{F}_\rho$, and by base change $\hat{\imath}^* j^{\mathrm{B}}_*(\mathcal{O} \otimes W)$ is the direct image of $\mathcal{O} \otimes W$ under $\mathfrak{F}_\rho \to \widehat{X}_\rho$.

By Lemma 6.3(3) this map factors through $Z_{C,\rho}$, which is open and closed in $\widehat{X}_\rho$ and in Z_C. So $\operatorname{QCoh}(Z_{C,\rho})$ is a direct factor of $\operatorname{QCoh}(\widehat{X}_\rho)$ and of $\operatorname{QCoh}(Z_C)$, and the direct images of $\mathcal{O} \otimes W$ to $\widehat{X}_\rho$ and to Z_C both come from its direct image to $Z_{C,\rho}$. An object of a direct factor which is compact in the whole category is compact in the factor, and an object which is compact in a direct factor is compact in the whole category. Therefore $j_*(\mathcal{O} \otimes W)$ is compact in $\operatorname{QCoh}(Z_C)$. □

6.2 The automorphic family and its Whittaker coefficient

We use the equivalence of categories of the geometric Langlands correspondence with restricted variation in characteristic zero [GR25, Theorem 1.3.9(ii)], which we denote by $\mathbb{L}^r_G$. Let e be the fully faithful embedding of the category of quasi-coherent complexes into the spectral category with nilpotent singular support, normalized as in [GR25, §§1.3.1–1.3.4]; the normalization includes the graded line of the dualizing complex. Define

$$\mathcal{F}_P = (\mathbb{L}^r_G)^{-1} \mathsf{e}(j_* P), \qquad P \in \operatorname{Perf}(\mathfrak{F}_\rho).$$

The functor $P \mapsto \mathcal{F}_P$ is exact and k-linear. Since $j_* P$ is compact, [GR25, Theorem 1.1.7 and §1.3.4] shows that $\mathcal{F}_P$ is compact in the category of all automorphic sheaves; its restriction to every smooth chart of finite type is a bounded constructible complex.

The Whittaker coefficient

Let $\mathrm{Poinc}^{\mathrm{Vac}}_!$ be the Poincaré object defined by !-pushforward from the Whittaker stack, with the regular representation of the finite centre as coefficient (it is recalled in the proof of Lemma 6.6). The Whittaker coefficient is the functor Wh which it corepresents. The projection of $\mathrm{Poinc}^{\mathrm{Vac}}_!$ to the category of sheaves with nilpotent singular support corepresents the functor to $\operatorname{QCoh}(Z_C)$ followed by global sections. The two objects are different, but on our family the two functors agree:

$$\mathrm{Wh}(\mathcal{F}_P) \simeq \Gamma_!(Z_C, j_* P) \simeq \Gamma(\mathfrak{F}_\rho, P). \tag{6.2}$$

The first isomorphism is [GR25, Proposition 1.2.5]. For the second, $\Gamma_!$ agrees with Γ on compact objects [AGK+22, Remark 7.6.2], and $\Gamma(Z_C, j_* P) = \Gamma(\mathfrak{F}_\rho, P)$. In terms of the projected object, the adjunction for the functor $\mathbb{L}^r_{G,\mathrm{coarse}}$ to $\operatorname{QCoh}(Z_C)$ gives

$$\operatorname{RHom}(\mathrm{Poinc}^{\mathrm{Vac}}_{!,\mathrm{Nilp}}, \mathcal{F}_P) \simeq \Gamma(Z_C, j_* P), \tag{6.3}$$

since $\mathbb{L}^r_{G,\mathrm{coarse}} \mathcal{F}_P = j_* P$: the right adjoint of the fully faithful functor e maps $\mathsf{e}(j_* P)$ to $j_* P$ [GR25, Corollary 1.3.6(ii)].

Theta pairs

The curves in this paper carry a *theta pair*: a line bundle L with an isomorphism $L^2 \simeq \Omega_C$. Let λ_{pr} be the sum of the positive coroots of G, a cocharacter of the maximal torus T, and $\mathcal{P}_T = \lambda_{\mathrm{pr}}(L)$ the T-bundle induced from L. The theta pair enters the definition of the Whittaker stack through $\mathcal{P}_T$, and that of the canonical bundle in Section 9.6.

Pullback and transport. A semilinear automorphism u_C of the curve acts on the moduli stacks in two ways which are inverse to each other: an object can be pulled back by u_C, or transported by it. The direction of Frobenius in the rest of the paper depends on keeping the two apart, so we give them different names:

$$u_B(\mathcal{E}) = u_C^*\mathcal{E}, \qquad u_L(\tau) = u_C^*\tau, \qquad u_{\mathrm{aut}} = u_B^{-1}, \quad u_{\mathrm{spec}} = u_L^{-1}.$$

Thus u_{aut} and u_{spec} are transport by u_C, that is, pullback of bundles and of local systems by u_C^{-1}. When u_C is the arithmetic Frobenius acting on the coefficients, u_{aut} is the arithmetic Frobenius of the stack of bundles. The same convention must be used on the two stacks. Indeed the pullback of a modification at x is a modification at $u_C^{-1}x$, and for the universal local system V_{univ} evaluation gives a canonical isomorphism

$$(u_L \times u_C^{-1})^* V_{\mathrm{univ}} \simeq V_{\mathrm{univ}}.$$

The corresponding statements for the inverse maps u_{aut} and u_{spec} follow. The directions of all the Frobenius maps which occur in the paper are collected in the table of Remark 11.3.

Let $c : u_C^*\rho \xrightarrow{\sim} \rho$ be the chosen descent isomorphism. The point ρ is fixed by u_L, and on the residual gerbe BZ_ρ of this point u_L acts by the automorphism

$$\delta : Z_\rho \longrightarrow Z_\rho, \qquad z \longmapsto c\, u_C^*(z)\, c^{-1}.$$

So $u_{\mathrm{spec}} = u_L^{-1}$, and its lift $u_{\mathfrak{F}}$ to the derived fibre, act on the gerbe through δ^{-1}. The pullback of a vector bundle on BZ_ρ along the map induced by an automorphism γ of Z_ρ is the restriction of the representation along γ. Hence

$$u_{\mathfrak{F}}^* P_W \simeq P_{(\delta^{-1})^* W}. \tag{6.4}$$

We compute δ for the arithmetic Frobenius of Setup 6.1. The semilinear map u_C is the one with $u_C^{\#}(a) = a^q$ on the residue field. Let a be the chosen arithmetic lift of Frobenius in the fundamental group, used to base it, and g its image. Then u_C acts on the geometric fundamental group by $\gamma \mapsto a^{-1}\gamma a$, so that

$$(u_C^*\rho)(\gamma) = g^{-1}\rho(\gamma)g, \qquad c = g,$$

in the marked fibre, and $\delta = \mathrm{Ad}(g) = \alpha$. The orientation can be seen on a finite étale torsor: pulling back a section with coordinates b along u_C replaces them by b^q, so the canonical descent map on the fibre is the arithmetic Frobenius; the local system is a limit of such torsors. Therefore (6.4) reads

$$u_{\mathfrak{F}}^* P_W = P_{(\alpha^{-1})^* W}.$$

Lemma 6.6 (Hecke action and semilinear covariance)**.** *1. Let V be a representation of $\widehat{G}$, and let $V_\rho|_{C^\circ} = \bigoplus_a W_a \boxtimes L_a$ be a decomposition of the associated local system, with W_a a representation of Z_ρ and L_a a local system on C°. Then*

$$\mathsf{H}_V(\mathcal{F}_P) \simeq \bigoplus_a \mathcal{F}_{P\otimes W_a} \boxtimes L_a \tag{6.5}$$

over C°. The isomorphism does not depend on the decomposition. The same holds for every finite set I of legs and every representation of $\widehat{G}^I$, over $(C^\circ)^I$, and these isomorphisms are compatible with tensor products, composition, permutations of the legs and fusion. We refer to them as the Hecke identities *of the family; for $P = P_W = \mathcal{O}_{\mathfrak{F}_\rho} \otimes W$ they make $W \mapsto \mathcal{F}_{P_W}$ a Hecke family in the sense of Definition 5.1(1).*

2. *Let $u_C = u$ be a semilinear isomorphism of the curve C which preserves the pinning, the theta pair and the divisor D_{aux}, together with an isomorphism $u^*\rho \simeq \rho$. It induces an autoequivalence $u^*_{\mathfrak{F}}$ of $\mathrm{Perf}(\mathfrak{F}_\rho)$ and a natural isomorphism*

$$u^*_{\mathrm{aut}} \mathcal{F}_P \simeq \mathcal{F}_{u^*_{\mathfrak{F}} P},$$

compatible with the Hecke identities. For a composite of two semilinear isomorphisms the isomorphism is the composite of the two. In particular the chosen lift of Frobenius and all its powers act on the family compatibly.

Proof. The Hecke identities concern the spectral action. The covariance concerns the object to which the spectral action is applied, and it is this part which determines the Frobenius structure of the family.

Step 1: the Hecke identities. On $\mathfrak{F}_\rho \times C^\circ$ the universal local system is pulled back from $BZ_\rho \times C^\circ$, where the local system attached to V is $\bigoplus_a W_a \boxtimes L_a$ with W_a regarded as a vector bundle on BZ_ρ. The Hecke functor H_V corresponds under $\mathbb{L}^r_G$ to the tensor product with the local system attached to V, and by the projection formula for j

$$j_* P \otimes V_{\mathrm{univ}}|_{Z_C \times C^\circ} \simeq \bigoplus_a j_*(P \otimes W_a) \boxtimes L_a.$$

The left-hand side does not involve the decomposition. The compatibilities for all finite sets of legs are part of the equivalence between the spectral action and the action of Hecke functors on sheaves with nilpotent singular support [AGK+22, Theorems 8.1.4, 14.3.2 and §14.3.3].

Step 2: a formula for the family. Let $\mathcal{P}_C$ be the projection of $\mathrm{Poinc}^{\mathrm{Vac}}_!$ to the category of sheaves with nilpotent singular support. For $A \in \mathrm{QCoh}(Z_C)$ and an automorphic sheaf $\mathcal{K}$,

$$\mathrm{RHom}((\mathbb{L}^r_G)^{-1}\mathsf{e}A, \mathcal{K}) \simeq \mathrm{RHom}(A, \mathsf{e}^R \mathbb{L}^r_G \mathcal{K}) \simeq \mathrm{RHom}(A, \mathbb{L}^r_{G,\mathrm{coarse}} \mathcal{K}),$$

where e^R is the right adjoint of e. So $(\mathbb{L}^r_G)^{-1}\mathsf{e}$ is the left adjoint of $\mathbb{L}^r_{G,\mathrm{coarse}}$. By the construction of the latter [GR25, §§1.1.3–1.1.8] this left adjoint is $A \mapsto A \star \mathcal{P}_C$, where $\star$ denotes the spectral action. Hence

$$\mathcal{F}_P \simeq (j_* P) \star \mathcal{P}_C. \tag{6.6}$$

This formula does not involve the choice of an inverse of $\mathbb{L}^r_G$, and it fixes the normalization of e.

Step 3: covariance of the Poincaré object. Let $\mathscr{W}_C = \mathrm{Bun}_{N,\mathcal{P}_T}/T$ be the Whittaker stack and r the rank of G. The Poincaré object is defined by the diagram

$$\mathrm{Bun}_G(C) \xleftarrow{p_C} \mathscr{W}_C \xrightarrow{\chi_C} [\mathbb{A}^r/T] \xleftarrow{j_{\mathrm{reg}}} BZ(G)$$

as $p_{C,!}\chi^*_C$ of the sheaf $j_{\mathrm{reg},*}\,\mathrm{Reg}_{Z(G)}[-r]$, where $\mathrm{Reg}_{Z(G)} = (\mathrm{pt} \to BZ(G))_! k$, with the shifts of [GR24, §§3.3.3–3.3.7]. The component of χ_C attached to a simple root sends an N-torsor to the class of the extension it defines, an element of $H^1(C, \Omega_C)$, followed by the trace $H^1(C, \Omega_C) \to K_{\mathrm{geom}}$ of Serre duality.

With the convention of transport fixed above, u acts on the N-torsors, on their quotients attached to the simple roots and on the theta pair, and the trace of Serre duality commutes with semilinear pullback. So u carries the whole diagram to the diagram of the curve obtained by base change. Here χ_C is a morphism of stacks over K_{geom}, along which the sheaves, with coefficients in k, are pulled back. Pullback by an isomorphism commutes with $p_!$, χ^* and j_* and preserves $\mathrm{Reg}_{Z(G)}$. These base change isomorphisms give $u_{\mathrm{aut}}^* \operatorname{Poinc}_!^{\mathrm{Vac}} \simeq \operatorname{Poinc}_!^{\mathrm{Vac}}$.

Step 4: covariance of the projection. The projection to sheaves with nilpotent singular support is given by the universal Hecke projector of [AGK+22, §13.4.1, Corollary 13.4.6 and §15.4.7]. It is a colimit of Hecke functors, indexed by diagrams formed from finite sets of points of the curve, their collisions, representations of $\widehat{G}$ and the fibres of the universal local system. The pullbacks by u_{aut}, on the automorphic side, and by u_{spec}, on the spectral side, send each of these diagrams to the corresponding diagram for the curve obtained by base change, and they commute with tensor products, with colimits and with $\Gamma_!$ on the spectral side. So they commute with the projector. The comparison is canonical. By [GR25, Theorem 1.1.7] and [AGK+22, Proposition 17.2.3] the projector is the right adjoint of the inclusion of the sheaves with nilpotent singular support; the equivalence u_{aut}^* preserves this subcategory, and an equivalence which commutes with a fully faithful functor commutes with its right adjoint, compatibly with the counits. Applied to the Poincaré object this gives $u_{\mathrm{aut}}^* \mathcal{P}_C \simeq \mathcal{P}_C$.

Step 5: covariance of the family. For the same reason

$$u_{\mathrm{aut}}^*(A \star \mathcal{K}) \simeq (u_{\mathrm{spec}}^* A) \star (u_{\mathrm{aut}}^* \mathcal{K}).$$

The Cartesian square which defines $\mathfrak{F}_\rho$ is preserved by u, so $u_{\mathrm{spec}}^* j_* P \simeq j_* u_{\mathfrak{F}}^* P$. Substituting in (6.6) we obtain the asserted isomorphism, that is, a natural isomorphism which makes the square

$$\begin{array}{ccc} \mathrm{Perf}(\mathfrak{F}_\rho) & \xrightarrow{P \mapsto \mathcal{F}_P} & \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G(C)) \\ {\scriptstyle u_{\mathfrak{F}}^*}\downarrow & & \downarrow{\scriptstyle u_{\mathrm{aut}}^*} \\ \mathrm{Perf}(\mathfrak{F}_\rho) & \xrightarrow[P \mapsto \mathcal{F}_P]{} & \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G(C)) \end{array}$$

commute. For two semilinear isomorphisms u and v, the isomorphism attached to uv and the composite of those attached to u and v are both base change maps for the same diagrams, and base change maps are compatible with composition. The same argument applies to the maps of creation, fusion and annihilation in the Hecke action, so the isomorphisms are compatible with the Hecke identities. □

6.3 Why the whole family is needed

The comparison with a microstalk in Section 9.6 is at first a comparison over an algebraically closed field, without Frobenius. For a single complex this would give no control of the trace of Frobenius. The control comes from the following property of the functor of global sections on the whole category $\mathrm{Perf}(\mathfrak{F}_\rho)$.

Lemma 6.7 (Rigidity of the functor of global sections)**.** *Let $\Gamma_\rho(P) = \Gamma(\mathfrak{F}_\rho, P)$ on $\mathrm{Perf}(\mathfrak{F}_\rho)$. Every natural endomorphism of degree zero of the functor Γ_ρ acts on the cohomology groups $H^i\Gamma_\rho(P)$ by one scalar, the same for all P and all i. For this it suffices that the endomorphism be natural for the morphisms of the homotopy category and commute with the shift. Consequently, if a functor μ is isomorphic to $\Gamma_\rho \otimes L$ for a graded line L, and both functors are compatible with Frobenius, the two actions of Frobenius on cohomology differ, under the isomorphism, by a nonzero scalar which does not depend on P or on the degree.*

Proof. In the homotopy category of $\mathrm{Perf}(\mathfrak{F}_\rho)$ the functor $P \mapsto H^0\Gamma_\rho(P)$ is $\mathrm{Hom}(\mathcal{O}, P)$: it is corepresented by the structure sheaf, which is a perfect complex. By the Yoneda lemma for this ordinary category, its endomorphisms are the compositions with the elements of

$$\mathrm{End}(\mathcal{O}_{\mathfrak{F}_\rho}) = H^0\Gamma(\mathfrak{F}_\rho, \mathcal{O}) = k,$$

the last equality by Lemma 6.4. Concretely, a natural endomorphism is determined by its value on the identity of $\mathcal{O}$, an endomorphism of $\mathcal{O}$, and its value on a section $v : \mathcal{O} \to P$ is the composite of v with this endomorphism. So the given endomorphism acts on $H^0\Gamma_\rho(P)$ by a scalar c which does not depend on P. Since $H^i\Gamma_\rho(P) = H^0\Gamma_\rho(P[i])$ and the endomorphism commutes with the shift, it acts by c on $H^i\Gamma_\rho(P)$.

For the second assertion, transport the action of Frobenius on μ to $\Gamma_\rho \otimes L$ by the isomorphism, and compose it with the inverse of the action of Frobenius on $\Gamma_\rho \otimes L$. The result is a natural automorphism of the functor $\Gamma_\rho \otimes L$, of degree zero, and the first part applies to it after the choice of a generator of L; the scalar is not zero because the automorphism is invertible. □

Remark 6.8 (Which naturality is used)**.** Only the Yoneda lemma of an ordinary category enters. So the isomorphisms of functors to which the lemma is applied below, in particular the comparison with the microstalk of Proposition 9.16, are needed only as isomorphisms of functors on homotopy categories which commute with the shift. This is enough, because the functors take values in complexes of vector spaces over a field and all conclusions concern supertraces.

Remark 6.9. The lemma would be of no use for the single object P_{reg}. Its cohomology $\Gamma(\mathfrak{F}_\rho, P_{\mathrm{reg}})$ lives in several degrees, and an automorphism acting by different scalars in different degrees can turn a nonzero alternating trace into zero. Naturality on all of $\mathrm{Perf}(\mathfrak{F}_\rho)$ forces the same scalar in every degree.

Proposition 6.10 (The supertrace is a nonzero determinant)**.** *Suppose that every orbit of Frobenius in D_{aux} consists of the points which lift a closed point of degree d_{aux} of a fibre in characteristic p, and that the adjoint action of the Frobenius element $g_D \in \widehat{G}(k)$ of that point is pure of weight zero. Let*

$$P_{\mathrm{reg}} = \mathcal{O}_{\mathfrak{F}_\rho} \otimes k[\Pi_\rho]$$

with the action of Frobenius induced by α, which permutes the basis by $e_\gamma \mapsto e_{\alpha(\gamma)}$. Then Frobenius and its inverse have nonzero supertrace on $\Gamma(\mathfrak{F}_\rho, P_{\mathrm{reg}})$.

Proof. Write ϕ for the action of Frobenius. We compute in the graded algebra of cohomology $\Lambda = \Lambda(\mathsf{B}_\rho[1])$, on which Frobenius acts by the exterior powers of its action in degree -1. The group Z_ρ is linearly reductive, so $H^*\Gamma(\mathfrak{F}_\rho, P_{\mathrm{reg}}) = (k[\Pi_\rho] \otimes \Lambda)^{Z_\rho}$.

Step 1: removing the group of components. For a complex $\mathcal{K}$ of representations of Z_ρ the map

$$\mathcal{K}^{Z_\rho^\circ} \longrightarrow (k[\Pi_\rho] \otimes \mathcal{K})^{Z_\rho}, \qquad v \longmapsto \sum_{\gamma \in \Pi_\rho} e_\gamma \otimes \widetilde{\gamma} v,$$

where $\widetilde{\gamma} \in Z_\rho$ is a representative of γ, is an isomorphism: it does not depend on the representatives because v is invariant under Z_ρ°, and its inverse is the coefficient of e_1. For $\mathcal{K} = \Lambda$ it commutes with ϕ, since $\phi(\widetilde{\gamma} v) = \alpha(\widetilde{\gamma})\phi(v)$. So the supertrace to be computed is that of ϕ on $\Lambda^{Z_\rho^\circ}$.

Step 2: removing the torus. Decompose Λ according to the characters of the torus Z_ρ°. The map ϕ sends the part of weight χ to the part of weight $\chi \circ \alpha^{-1}$. If $\chi \circ \alpha^{-1} = \chi$, then χ is trivial on the image of $1 - \alpha$, which is Z_ρ° by Lemma 6.2; so $\chi = 0$. The parts of nonzero

weight are therefore permuted without fixed points and do not contribute to the trace, in any degree. Hence ϕ has the same supertrace on $\Lambda^{Z_\rho^\circ}$ and on Λ, and since $\Lambda^i \mathsf{B}_\rho$ is in degree $-i$,

$$\mathrm{Str}(\phi; \Gamma(\mathfrak{F}_\rho, P_{\mathrm{reg}})) = \sum_i (-1)^i \,\mathrm{tr}(\phi; \Lambda^i \mathsf{B}_\rho) = \det(1-\phi; \mathsf{B}_\rho). \tag{6.7}$$

Step 3: the determinant. Take an orbit of punctures; its length is $d = d_{\mathrm{aux}}$. At one of its points choose a local arithmetic lift a_D of Frobenius and a generator τ of the tame inertia. On the ℓ-primary tame quotient

$$a_D \tau a_D^{-1} = \tau^{q^d}.$$

Write $g_D = \rho(a_D)$, and let $\mathrm{Ad}(g_D)^\vee(\xi) = \xi \circ \mathrm{Ad}(g_D^{-1})$ be the coadjoint action on $\widehat{\mathfrak{g}}^*$. The arithmetic map on the spectral side is the transport u_{spec}. Let M be the monodromy at the puncture of a local system near ρ, a formal variable near the identity. After d steps, and after identifying the base local system by g_D^{-1}, the transport is

$$M \longmapsto g_D^{-1} M^{q^d} g_D,$$

whose linearization on the Lie algebra is $q^d \,\mathrm{Ad}(g_D^{-1})$. So the pullback of the equation $M = 1$ acts on its conormal space by $q^d \,\mathrm{Ad}(g_D)^\vee$. This is the action on H^{-1} of the Koszul complex of the derived intersection, and its exterior powers give the action on all the cohomology groups; the complex itself need not be given an action of Frobenius. Hence the arithmetic operator ϕ on B_ρ permutes the summands of the orbit cyclically, and

$$\phi^d|_{\widehat{\mathfrak{g}}_x^*} = q^d \,\mathrm{Ad}(g_D)^\vee.$$

The determinant of $1-\phi$ on the cyclic sum is $\det(1-\phi^d; \widehat{\mathfrak{g}}_x^*)$. Its factors are $1 - q^d\lambda$, with λ an eigenvalue of $\mathrm{Ad}(g_D)^\vee$, and $|\lambda| = 1$ for every complex embedding by purity. So no factor vanishes. The geometric operator is the inverse of the arithmetic one, on the conormal space and on the representation of the group of components together; its factors are $1 - q^{-d}\lambda^{-1}$, which do not vanish either. What is essential is purity: compactness of the image of the parameter gives only ℓ-adic information, and by itself would not exclude an eigenvalue of complex absolute value $q^{\pm d}$. □

Remark 6.11. The hypothesis on weights cannot be dropped: the determinant $\det(1-\phi; \mathsf{B}_\rho)$ vanishes as soon as ϕ has the eigenvalue 1. In the application the puncture is chosen over a closed point of the original curve (Lemma 8.3), so that Theorem I.4.8 applies to g_D.

6.4 An S_3-parameter and the determinant -64

In the following example the parameter, the group Z_ρ and the supertrace of Proposition 6.10 are computed by hand. The rest of the construction is not explicit: the automorphic function is obtained from these data by the results of Sections 7–12.

Take $F = \mathbb{F}_5(t)$, $G = \mathrm{SL}_2$, and $\widehat{G} = \mathrm{PGL}_2(k)$, with $\ell \neq 5$. The splitting field of $z^3 - z - t$ has geometric and arithmetic Galois group S_3. To see this directly, the map $t = x^3 - x$ of projective lines has separable degree three, so the cubic is geometrically irreducible. Its discriminant $4 - 27t^2 = 4 - 2t^2$ has two simple zeros and is not a square. The geometric Galois group is therefore S_3. The arithmetic group lies between it and S_3, so it is S_3 as well, and the splitting field has constant field $\mathbb{F}_5$.

Use the projective images of

$$r = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \qquad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

They satisfy $r^3 = s^2 = 1$ and $srs = r^{-1}$, and give the parameter $\sigma : \Gamma_F \twoheadrightarrow S_3 \hookrightarrow \mathrm{PGL}_2(k)$. Its ramification has order two at $t^2 = 2$ and order three at infinity, hence is tame at five. Its centralizer is

$$Z_\sigma = \{1, c\}, \qquad c = \left[\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}\right]. \tag{6.8}$$

Indeed, $c^2 = [-3I] = 1$; it commutes with r and commutes projectively with s. Conversely, diagonalizing r gives its torus centralizer, on which s acts by inversion. The fixed subgroup is μ_2. Thus σ is elliptic, although its centralizer is not trivial.

The point $t = 0$ has identity Frobenius, since the cubic splits there as $z(z-1)(z+1)$. In Section 7.5 the divisor D_{aux} is taken to be one rational point of the auxiliary component above $t = 0$. So $d_{\mathrm{aux}} = 1$, and the Frobenius element at the puncture is $g_D = 1$. It is also shown there that the geometric monodromy group of the local system ρ on the smooth curve is S_3; hence $Z_\rho = \Pi_\rho = \mu_2$, by (6.8). For any representation V of this group, the mutually inverse maps

$$V \longrightarrow (k[\mu_2] \otimes V)^{\mu_2}, \quad v \longmapsto e_1 \otimes v + e_c \otimes cv, \qquad e_1 \otimes v_1 + e_c \otimes v_c \longmapsto v_1$$

commute with Frobenius; this is step 1 of the proof of Proposition 6.10. Therefore the cohomology of $\Gamma(\mathfrak{F}_\rho, P_{\mathrm{reg}})$ is the exterior algebra Λ of that proof, with its grading. Here B_ρ is the conormal space $\mathfrak{pgl}_2^*$ at the one puncture, which has dimension three. The arithmetic operator of step 3 of that proof, which is defined by transport, acts by 5 id on B_ρ, in degree -1, because $q = 5$, $d_{\mathrm{aux}} = 1$ and $g_D = 1$. Its supertrace is

$$1 - 3 \cdot 5 + 3 \cdot 25 - 125 = (1-5)^3 = -64. \tag{6.9}$$

The geometric operator is its inverse, with supertrace $(1 - 5^{-1})^3 = 64/125$. Both numbers are nonzero in the coefficient field.

In the later sections this number is transported to the automorphic side. By Proposition 9.16 and Lemma 6.7, the supertrace of geometric Frobenius on the vanishing cycles studied in Section 10 is $64c/125$, where the scalar $c \in k^\times$ accounts for the comparison of the two Frobenius structures and for the graded line in that lemma. The scalar is not determined, and it need not be: the supertrace is not zero. Proposition 10.9 then gives a bundle on the special fibre of C_0, defined over $\mathbb{F}_5$, at which the Frobenius supertrace of the sheaf is not zero.

There is also an elementary check on the *parameter*. At $t = 0, 1, 2$ the cubic has respectively three, one and no roots in $\mathbb{F}_5$. Its Frobenius classes have orders one, two and three. The adjoint traces are respectively $3, -1, 0$. These are predicted Hecke eigenvalues of any realization of σ, not computed values of φ_σ.

For this parameter automorphy can also be proved directly. The matrices r and s define a linear representation of S_3 of dimension two, which is irreducible: r has two distinct eigenspaces, and s exchanges them. Its determinant has finite order, so by the theorem of L. Lafforgue it is the parameter of a cuspidal representation of GL_2. Take a cusp form in the corresponding summand and translate it on the right so that its value at the identity is not zero. Its restriction to $\mathrm{SL}_2(\mathbb{A}_F)$ is a nonzero cusp form, and it lies in the summand of σ by [Laf18, Proposition 12.5 and Remark 12.13]. This summand, at a finite level, contains a joint eigenvector of the excursion operators.

7 A smooth proper curve in characteristic zero carrying the parameter

This section constructs the data of Setup 6.1 from the given parameter σ: a smooth proper curve in characteristic zero with an unramified local system ρ, related to (X, σ) by specializa-

tion. The parameter σ lives on an open subset of X in characteristic p and may be wildly ramified there. The construction has four steps.

1. *Lifting* (Theorem 7.1). Lift X to a curve $\mathcal{X}$ over a ring of Witt vectors, and lift σ, with its whole compact image, to a representation of the fundamental group of the complement of a divisor $D \subset \mathcal{X}$. The divisor must not depend on the finite quotient of the image under consideration, and this is where $p \nmid |W_G|$ is used: it makes the images of wild inertia finite abelian with a tame action on them.

2. *A second component* (Lemma 7.6 and Proposition 7.7). Construct a cyclic cover $C_2 \to \mathcal{X}$ of degree d branched along D, and glue it to $\mathcal{X}$ along D. The degree is chosen so that the pullback of the local system to C_2 has inertia inverse to that of the original one, up to conjugation by elements commuting with Frobenius.

3. *Smoothing* (Proposition 7.7). The curve $C_0 = \mathcal{X} \cup_D C_2$, which has two components, is Gorenstein and is nodal on the fibre of characteristic zero, deforms to a smooth proper curve over $R_0[[\varepsilon]]$, with a square root of its dualizing line.

4. *Extending the local system* (Proposition 7.9 and Proposition 7.10). Because the inertia on the two branches at each node of the intermediate fibre, in characteristic zero, is inverse, the local system extends over the smoothing, and on the smooth fibre it is unramified. The ramification of σ has been traded for genus; the parameter itself is still present on the component $\mathcal{X}$ of the special fibre.

Throughout this section $p \nmid |W_G|$; recall that the Weyl groups of G and $\widehat{G}$ have the same order. We put $Q = \mathrm{im}(\sigma)$, a compact group, and choose a finite extension E of $\mathbb{Q}_\ell$ and a lattice stable under Q as in Lemma I.4.1.

7.1 Lifting the compact representation on one open

Theorem 7.1 (Lifting a parameter with its ramification)**.** *Let $X/\mathbb{F}_q$ be smooth, projective and geometrically connected, let $U = X - \Sigma$ for a finite set Σ, and let $\sigma : \pi_1(U) \to H(k)$ be continuous, where H is connected semisimple and $p \nmid |W_H|$. Fix a smooth projective lift $\mathcal{X}$ of X over $W(\mathbb{F}_q)$. After a finite totally ramified extension $R_0/W(\mathbb{F}_q)$ there is a divisor $D \subset \mathcal{X}_{R_0}$, finite and flat over R_0, whose special fibre has support Σ, and a continuous representation*

$$\sigma^{\mathrm{lift}} : \pi_1(\mathcal{X}_{R_0} - D) \longrightarrow Q$$

restricting to σ on the special fibre. Its image is Q, and the image of the geometric fundamental group of the generic fibre contains the image under σ of the geometric fundamental group of U. The same divisor D serves for all finite quotients of Q: the lift is constructed as a compatible system of torsors under these quotients on $\mathcal{X}_{R_0} - D$, with base points compatible with those of σ and with the action of Frobenius.

The point of the theorem is that D does not depend on the finite quotient. If each finite quotient were lifted separately, the branch loci could grow without bound. In the proof one first lifts the tame quotient of the representation, with ramification only along the lift of Σ, and then corrects the lift by a cocycle with values in a fixed finite group, which accounts for all further ramification.

Lemma 7.2 (Pro-p subgroups of the image)**.** *Every pro-p subgroup of Q is finite and abelian. If P is a normal pro-p subgroup of a closed subgroup Q_0 of Q, the image of the homomorphism $Q_0 \to \mathrm{Aut}(P)$ given by conjugation has order prime to p.*

Proof. The group Q has an open normal subgroup which is pro-ℓ, a congruence subgroup for the chosen lattice. A pro-p subgroup meets it trivially, so it embeds into the finite quotient and is a finite p-group. A finite nilpotent subgroup of $H(k)$ normalizes a maximal torus [Ser07, §3.3, Theorem 3″]. The image of our p-group in the Weyl group of this torus is a p-group, hence trivial since $p \nmid |W_H|$; so the group is contained in the torus and is abelian.

For the second assertion let S be a Sylow pro-p subgroup of Q_0. It contains the normal subgroup P, and it is abelian by the first part, so it acts trivially on P. The image of S in the finite group $\mathrm{im}(Q_0 \to \mathrm{Aut}(P))$ is a Sylow p-subgroup of this group, and it is trivial. □

The correction mentioned above is a cohomology class, and the next lemma will be used to lift it from the special fibre. For a complete discrete valuation ring R with uniformizer π and residue field κ_R, write

$$A_R = R[[z]], \qquad K_R = \mathrm{Frac}(A_R), \qquad K_R^{\mathrm{vert}} = \mathrm{Frac}\,\widehat{(A_R)_{(\pi)}}.$$

The field K_R^{vert} is the completion of K_R along the special fibre $\pi = 0$ of $\mathrm{Spec}\, A_R$. It is a complete discretely valued field with residue field $\kappa_R((t))$, where t is the image of z.

Lemma 7.3 (Approximation of cohomology classes)**.** *Let P be a finite abelian group killed by p^n, on which the Galois group of K_R acts through a finite quotient Δ of order prime to p. Let K'/K_R be the extension with group Δ. Suppose that $K' \otimes_{K_R} K_R^{\mathrm{vert}}$ is a field and that $\mu_{p^n} \subset K_R$. Then the restriction map*

$$H^1(K_R, P) \longrightarrow H^1(K_R^{\mathrm{vert}}, P)$$

is surjective.

Proof. Let $K'^{\mathrm{vert}} = K' \otimes_{K_R} K_R^{\mathrm{vert}}$. It is a complete discretely valued field in which K' is dense.

We first prove the surjectivity for K' in place of K_R. Over K' the module P is a sum of groups $\mathbb{Z}/p^h$ with trivial action, and $\mu_{p^h} \subset K'$. By Kummer theory it is enough to show that every $x \in (K'^{\mathrm{vert}})^\times$ is the product of an element of $K'^\times$ and a p^h-th power. Choose $y \in K'^\times$ with $v(x/y - 1) > 2v(p^h)$. By Hensel's lemma, applied to the polynomial $X^{p^h} - x/y$ and its approximate root 1, the element x/y is a p^h-th power.

Now let $\xi \in H^1(K_R^{\mathrm{vert}}, P)$, and let $\beta \in H^1(K', P)$ be a class whose restriction to K'^{vert} is the restriction of ξ. Put $m = [K' : K_R]$ and choose an integer m' with $mm' \equiv 1 \pmod{p^n}$. Since K'^{vert} is a field, restriction to K_R^{vert} commutes with corestriction from K', and the corestriction of the restriction of ξ is $m\xi$. So the class $m' \,\mathrm{Cor}_{K'/K_R}\, \beta$ restricts to $m'm\xi = \xi$. □

Lemma 7.4 (The local lift)**.** *Let κ be a finite field of characteristic p, let Q_0 be a closed subgroup of Q, and let $\sigma_0 : G_{\kappa((t))} \twoheadrightarrow Q_0$ be a continuous surjection. Let P be the image of the wild inertia group; it is finite abelian, and the conjugation action of Q_0 on it factors through a group of order prime to p, by Lemma 7.2. Let $p^n P = 0$ and $R = W(\kappa)[\zeta_{p^n}]$. Then there are a divisor $D^{\mathrm{hor}} \subset \mathrm{Spec}\, R[[z]]$, finite and flat over R and containing $z = 0$, and a continuous homomorphism*

$$\widetilde{\sigma}_0 : \pi_1(\mathrm{Spec}\, R[[z]] - D^{\mathrm{hor}}) \longrightarrow Q_0$$

whose composite with $G_{K_R^{\mathrm{vert}}} \to \pi_1(\mathrm{Spec}\, R[[z]] - D^{\mathrm{hor}})$ is the composite of σ_0 with $G_{K_R^{\mathrm{vert}}} \to G_{\kappa((t))}$. The equality is one of homomorphisms for compatible base points, and not only up to conjugation. The divisor D^{hor} serves for every finite quotient of Q_0.

Proof. In step 1 we choose a splitting over the tame quotient, and in step 2 we lift the homomorphism so obtained to characteristic zero, by means of the standard tame coverings. Its difference with σ_0 is a cocycle with values in the finite group P, which is lifted by Lemma 7.3. Step 3 is bookkeeping: it shows that the ramification of the result is contained in one divisor.

Step 1: splitting off the tame part. Let $\mathbb{T} = \widehat{\mathbb{Z}}^{(p')}(1) \rtimes G_\kappa$ be the tame quotient of $G_{\kappa((t))}$. Since P is the image of wild inertia, σ_0 induces $\bar\sigma_0 : \mathbb{T} \to Q_0/P$. Consider the pullback by $\bar\sigma_0$ of the extension

$$1 \to P \to Q_0 \to Q_0/P \to 1.$$

It is an extension of $\mathbb{T}$ by the finite abelian p-group P, and we claim that it splits. Extensions of profinite groups with finite kernel have continuous sections, so they are classified by continuous cohomology, and it suffices to show that $H^2(\mathbb{T}, P) = 0$. The tame inertia group $\widehat{\mathbb{Z}}^{(p')}(1)$ has order prime to p, so it has no cohomology in positive degrees with coefficients in P. By the Hochschild–Serre spectral sequence $H^2(\mathbb{T}, P)$ is then H^2 of G_κ with coefficients in a finite module, which is zero because $G_\kappa \simeq \widehat{\mathbb{Z}}$. A splitting is a continuous homomorphism

$$\eta : \mathbb{T} \longrightarrow Q_0$$

whose composite with $Q_0 \to Q_0/P$ is $\bar\sigma_0$.

Step 2: the correction. The unramified extensions of R and the coverings obtained by adjoining $z^{1/m}$, for m prime to p, define a homomorphism

$$\tau : \pi_1(\operatorname{Spec} R[[z]][1/z]) \longrightarrow \mathbb{T}.$$

The reductions of these coverings are the unramified and the tamely ramified coverings of $\kappa((t))$, so the restriction of τ to $G_{K_R^{\mathrm{vert}}}$ is the composite of $G_{K_R^{\mathrm{vert}}} \to G_{\kappa((t))}$ with the tame quotient map. We regard σ_0 as a homomorphism on $G_{K_R^{\mathrm{vert}}}$ in the same way. Then σ_0 and $\eta\tau$ have the same image in Q_0/P, and

$$\mathsf{c}_{\mathrm{vert}}(\gamma) = \sigma_0(\gamma)\,(\eta\tau(\gamma))^{-1} \in P$$

is a continuous 1-cocycle on $G_{K_R^{\mathrm{vert}}}$ for the action of this group on P by conjugation through $\eta\tau$. This action is defined on G_{K_R}, and it factors through a finite quotient Δ of order prime to p. The corresponding extension of K_R remains a field after tensoring with K_R^{vert}, because the restriction of τ to $G_{K_R^{\mathrm{vert}}}$ is still surjective. So Lemma 7.3 applies: the class of $\mathsf{c}_{\mathrm{vert}}$ is the restriction of a class in $H^1(K_R, P)$. Let c be a cocycle in this class. Its restriction differs from $\mathsf{c}_{\mathrm{vert}}$ by the coboundary of an element of P, and after changing c by the same coboundary the restriction of c is equal to $\mathsf{c}_{\mathrm{vert}}$. The cocycle relation says that

$$\widetilde{\sigma}_0(\gamma) = \mathsf{c}(\gamma)\,\eta\tau(\gamma)$$

is a homomorphism on G_{K_R}, and its restriction to $G_{K_R^{\mathrm{vert}}}$ is σ_0.

Step 3: the divisor. The homomorphism $\widetilde{\sigma}_0$ is determined by τ, which is unramified outside $z = 0$, and by the homomorphism

$$G_{K_R} \longrightarrow P \rtimes \Delta$$

with components c and the action on P, which has finite image. Let B be the normalization of $R[[z]]$ in the finite étale algebra over K_R defined by the latter. It is a finite $R[[z]]$-module since $R[[z]]$ is excellent, and it is free: a normal ring of dimension two is Cohen–Macaulay, and a finite Cohen–Macaulay module of maximal dimension over a regular local ring is free

by the formula of Auslander and Buchsbaum. Let δ be the discriminant of a basis of B. After completion along $\pi = 0$ the algebra is the one defined by $\mathsf{c}_{\mathrm{vert}}$ and the action, which come from $G_{\kappa((t))}$; so B is unramified at the prime (π) and $\delta \notin (\pi)$. By the Weierstrass preparation theorem $z\delta$ is the product of a unit and a distinguished polynomial in z. Hence

$$D^{\mathrm{hor}} = V(z\delta)$$

is finite and flat over R, and its special fibre is supported at $t = 0$.

The algebra B is étale outside $V(\delta)$, and the coverings which define τ are étale outside $z = 0$. So $\widetilde{\sigma}_0$ factors through $\pi_1(\operatorname{Spec} R[[z]] - D^{\mathrm{hor}})$. For a finite quotient of Q_0 we compose $\widetilde{\sigma}_0$ with the quotient map. No new discriminant and no new branch point occurs, and this is the uniformity asserted in the lemma. □

Proof of Theorem 7.1. Step 1 lifts the curve. Steps 2 to 4 glue the local lifts of Lemma 7.4 to the unique lift over the part of the curve where σ is unramified; this is an application of formal patching, and the only point to watch is that the divisor is constructed before a finite quotient of Q is chosen. Step 5 proves the statement on the geometric fundamental group, which is used in Proposition 7.10.

Step 1: the curve. The obstruction to lifting a smooth curve to the next infinitesimal thickening lies in $H^2(X, T_X)$, and the obstruction to lifting an ample line bundle lies in $H^2(X, \mathcal{O}_X)$; both groups vanish. So there is a formal lift with an ample line bundle, and it is algebraic by [Gro61, III$_1$, Theorem 5.4.5]. The locus where the lift is not smooth is closed and proper over the base and does not meet the special fibre, so it is empty.

Step 2: the divisor. Let p^n kill the images of the wild inertia groups at the points of Σ, and let $R_0 = W(\mathbb{F}_q)[\zeta_{p^n}]$. At each point of Σ the completed local ring of $\mathcal{X}_{R_0}$ is of the form $R[[z]]$ as in Lemma 7.4, and that lemma gives a divisor $V(z\delta)$ and a lift of the restriction of σ. On the formal completion of $\mathcal{X}_{R_0}$ along U, finite étale coverings of U lift uniquely. On the completed punctured discs around the points of Σ the two constructions agree, with their base points: finite étale coverings of such a punctured disc correspond to those of its special fibre $\operatorname{Spec} \kappa((t))$, and both constructions restrict to the given covering there.

The patching theorem for formal schemes [Pri00, Theorem 3.4] is an equivalence between coherent modules on $\mathcal{X}_{R_0}$ and compatible systems of modules on the formal completion along U and on the discs. We apply it first to the ideals $(z\delta)$ on the discs and the unit ideal on the formal completion along U. The result is an invertible ideal of $\mathcal{X}_{R_0}$ whose quotient is flat over R_0 and has finite support in the special fibre. The divisor D which it defines is proper and quasi-finite over R_0, hence finite.

Step 3: the torsors. Let Δ be a finite quotient of Q. Patching the normalizations B on the discs with the étale algebra on the formal completion along U we obtain a finite flat algebra $\mathcal{B}_\Delta$ on $\mathcal{X}_{R_0}$ with an action of Δ. We claim that it is a Δ-torsor outside D. Let $\mathcal{I}_D$ be the ideal of D. On each disc some power of $\mathcal{I}_D$ annihilates the module of relative differentials of $\mathcal{B}_\Delta$ and the kernel and cokernel of the map

$$\mathcal{B}_\Delta \otimes \mathcal{B}_\Delta \longrightarrow \prod_{a \in \Delta} \mathcal{B}_\Delta, \qquad b \otimes b' \longmapsto (b \cdot a(b'))_a,$$

since the algebra is a torsor outside $V(z\delta)$. On the formal completion along U these modules are zero. There are finitely many discs, so one power of $\mathcal{I}_D$ annihilates the three modules on all the pieces, and since patching is faithful the same holds on $\mathcal{X}_{R_0}$. So the modules vanish outside D, which is the claim.

Step 4: compatibility. The divisor D was constructed before Δ was chosen. For a surjection $\Delta' \to \Delta$ of finite quotients of Q, the maps between the algebras on the discs and on the

formal completion along U agree on the punctured discs, by the uniqueness of the lift of an étale covering with base point. Since patching is fully faithful they define a map $\mathcal{B}_\Delta \to \mathcal{B}_{\Delta'}$, and the maps for three quotients are compatible. The same argument applies to the actions of the groups, to the base points and to the descent data for Frobenius. The compatible system of torsors is a continuous homomorphism

$$\pi_1(\mathcal{X}_{R_0} - D) \longrightarrow \varprojlim_{\Delta} \Delta = Q.$$

Its restriction to the special fibre is σ, which is surjective; so its image is Q. If $\Sigma = \varnothing$ there are no discs, $D = \varnothing$, and the argument reduces to the lifting of étale coverings of a proper scheme.

Step 5: the geometric image. It suffices to prove the assertion in each finite quotient Δ. Let $\mathcal{Y}$ be the Δ-torsor over $\mathcal{X}_{R_0} - D$. It is connected: a connected component is finite étale over the connected base, hence surjective, so it meets the special fibre of $\mathcal{Y}$, which is connected because σ maps onto Δ. Let $L = \operatorname{Frac} R_0$ and let L' be the algebraic closure of L in the function field of $\mathcal{Y}$. The Galois group of L'/L is the quotient of Δ by the image of the geometric fundamental group of the generic fibre.

We claim that L'/L is unramified. A closed point of the special fibre of $\mathcal{Y}$ lifts, since $\mathcal{Y}$ is smooth over R_0, to a point with values in a finite unramified extension R_1 of R_0. The elements of L' integral over R_0 are regular functions on the normal scheme $\mathcal{Y}$, and evaluation at this point embeds L' into $\operatorname{Frac} R_1$.

Let R' be the ring of integers of L'. The corresponding quotient cover is $(\mathcal{X}_{R_0} - D) \times_{R_0} \operatorname{Spec} R'$: the evaluation argument gives the map to $\operatorname{Spec} R'$, and the two finite étale covers agree on the generic fibre. On the special fibre this is a constant-field cover, so the geometric fundamental group of U acts trivially on it. So the image of this group in Δ is contained in the image of the geometric fundamental group of the generic fibre. □

The divisor D is flat over R_0, so it is the closure of its generic fibre. In the sequel we replace D by the closure of the reduced generic fibre; this does not change the complement, and the generic fibre of D is now reduced. We are also free to enlarge D by divisors, finite and flat over R_0, along which σ^{lift} is unramified.

Fibres at tangential base points. Let x be a point of $D(\overline{L})$ and z a local coordinate of $\mathcal{X}_{\overline{L}}$ at x. A finite étale covering of $\mathcal{X}_{\overline{L}} - D$ whose inertia at x has order dividing n becomes unramified at x after the substitution $z = \zeta^n$: the normalization of its pullback is finite étale over the disc with coordinate ζ. Its fibre at $\zeta = 0$ does not depend on n, up to canonical isomorphism, and is called the fibre at the tangential base point defined by z. The group μ_n of the substitutions $\zeta \mapsto \theta\zeta$ acts on it, and this is the action of inertia. If $z' = za$ with $a(x) = 1$, the series $a^{1/n}$ with constant term one identify the fibres for z and for z', compatibly with n; so the fibre depends only on the tangent vector dual to z. If $a(x) = c \neq 1$ an identification requires a compatible system of roots of c.

Suppose that the tangent vector is defined over the residue field of x; this field is fixed by ϕ^r if the orbit of x under ϕ has length r. Then ϕ^r acts on the fibre at the tangential base point, and for a torsor under Q this action is given, in a frame of the fibre, by an element $g_x \in Q$. It depends on the tangent vector: replacing z by cz, with c in the residue field, multiplies g_x by the image under the inertia homomorphism of the Kummer class $(\phi^r(c^{1/n})/c^{1/n})_n$ of c.

Corollary 7.5 (Inertia and Frobenius at the boundary)**.** *Fix an algebraic closure $\overline{L}$ of L and a lift ϕ of the arithmetic Frobenius to $\overline{L}$. Let x be a point of $D(\overline{L})$ whose orbit under ϕ has length r. The image under σ^{lift} of the inertia group at x is quasi-unipotent: it is of the form*

$$\rho_x(\gamma) = s_x^{a(\gamma)} \exp(t_\ell(\gamma) N_x),$$

with s_x of finite order, N_x nilpotent and commuting with s_x, t_ℓ the ℓ-adic tame character and a a character with values in $\mathbb{Z}/\operatorname{ord}(s_x)$. Every element $g_x \in Q$ by which ϕ^r acts on the fibre at a tangential base point at x satisfies

$$g_x s_x g_x^{-1} = s_x^c, \quad (c, \operatorname{ord} s_x) = 1, \qquad \operatorname{Ad}(g_x) N_x = q^r N_x.$$

Proof. The inertia group at a point of a curve in characteristic zero is $\widehat{\mathbb{Z}}(1)$, and ϕ^r acts on it through the cyclotomic character. On the ℓ-part this character has the value q^r, because the roots of unity of order a power of ℓ are unramified over L.

An open subgroup of the inertia group is mapped into a pro-ℓ congruence subgroup of Q, hence factors through $\mathbb{Z}_\ell(1)$. Let A be the image of a generator. Then $g_x A g_x^{-1} = A^{q^r}$. So the map $\lambda \mapsto \lambda^{q^r}$ permutes the eigenvalues of A, in a faithful representation, and every eigenvalue satisfies $\lambda^{q^{rm}} = \lambda$ for some $m > 0$. The eigenvalues are therefore roots of unity, a power of A is unipotent, and the image of inertia is quasi-unipotent. The relation for N_x is the logarithm of $g_x A g_x^{-1} = A^{q^r}$ for this power, and the relation for s_x follows from the action of ϕ^r on the finite quotient of the inertia group, where the cyclotomic character has a value c prime to the order. All the coverings are defined over R_0, so these relations are compatible with the descent data for Frobenius. □

7.2 Reversing inertia without changing Frobenius

Two punctured curves with local systems can be glued at a node only if the inertia at the two branches is inverse to each other. The pullback to a cyclic covering of degree d branched at the point raises the inertia to the power d. The next lemma chooses d, together with the elements which will identify the fibres at the nodes.

Lemma 7.6 (Reversing inertia)**.** *Let ρ_x, s_x and N_x be as in Corollary 7.5, for the finitely many points x of $D(\overline{L})$, and let r_x be the length of the orbit of x under ϕ. There are an odd integer $d > 1$ prime to p, an integer $n \geq 1$ and elements $u_x \in \mathbb{Z}_\ell$, which depend only on Q, on the orders of the s_x and on the integers r_x, with the following property. For every x and every element $g_x \in Q$ by which ϕ^{r_x} acts on the fibre at a tangential base point at x, the element*

$$P_x = (g_x^n)^{u_x} \quad \text{if } N_x \neq 0, \qquad P_x = 1 \quad \text{if } N_x = 0,$$

of Q is defined and satisfies

$$P_x g_x = g_x P_x, \qquad P_x \rho_x(\gamma)^{-d} P_x^{-1} = \rho_x(\gamma).$$

If the tangential base points are permuted by ϕ on every orbit, so are the elements P_x.

The point of this formulation is that d is chosen before the tangent vectors: the curve of the next subsection depends on d, and the tangent vectors at its nodes are defined by its equation.

Proof. Let Q_1 be an open normal subgroup of Q contained in a congruence subgroup on which the ℓ-adic logarithm and exponential are defined and inverse to each other (level ℓ for odd ℓ, level 4 for $\ell = 2$). Choose $n \geq 1$ divisible by the index of Q_1 in Q and by the orders of the groups $(\mathbb{Z}/\operatorname{ord}(s_x))^\times$. Let g_x be as in the statement and write $r = r_x$. Then $g_x^n \in Q_1$, so $(g_x^n)^u$ is defined for $u \in \mathbb{Z}_\ell$; and g_x^n commutes with s_x, because $g_x s_x g_x^{-1} = s_x^c$ and $c^n \equiv 1$ modulo the order of s_x. If $N_x \neq 0$, the relation $\operatorname{Ad}(g_x^n) N_x = q^{rn} N_x$ shows that q^{rn} is congruent to one modulo the level, and $\log(q^{rn}) \neq 0$ because q is not a root of unity.

Choose $d > 1$ odd with $d \equiv -1$ modulo the orders of all the s_x, modulo p, and modulo ℓ^m, where $m \geq 2$ is at least the valuation of $\log(q^{r_x n})$ for every x with $N_x \neq 0$. Then d is prime to p, and

$$u_x = \frac{\log(-1/d)}{\log(q^{rn})} \in \mathbb{Z}_\ell.$$

These choices involve only Q, the orders of the s_x and the integers r_x. The element $P_x = (g_x^n)^{u_x}$ commutes with g_x and with s_x, and

$$\mathrm{Ad}(P_x)N_x = (q^{rn})^{u_x} N_x = -\tfrac{1}{d} N_x.$$

Therefore

$$P_x \rho_x(\gamma)^{-d} P_x^{-1} = s_x^{-da(\gamma)} \exp(-d\, t_\ell(\gamma) \cdot (-\tfrac{1}{d}) N_x) = \rho_x(\gamma),$$

since $d \equiv -1$ modulo the order of s_x. If $N_x = 0$ the same congruence gives $\rho_x^{-d} = \rho_x$. Finally, if the tangential base points are permuted by ϕ, the elements g_x at the points of an orbit are transported by ϕ, and so are the P_x. After r steps one returns to $g_x P_x g_x^{-1} = P_x$, so this is consistent. □

7.3 The curve and its square root of the dualizing line

Let $\Omega_{\mathcal{X}}$ be the relative dualizing sheaf of $\mathcal{X}$ over R_0, and let d be the integer of Lemma 7.6. Choose a sufficiently ample line bundle $\mathcal{A}$ on $\mathcal{X}$ and put

$$\mathcal{L} = \Omega_{\mathcal{X}} \otimes \mathcal{A}^2, \qquad \kappa_1 = \Omega_{\mathcal{X}}^{(d+1)/2} \otimes \mathcal{A}^d,$$

so that $\kappa_1^2 = \Omega_{\mathcal{X}} \otimes \mathcal{L}^d$. Let D_{ur} be a nonempty divisor, finite étale over R_0 and disjoint from the divisor of Theorem 7.1. Such a divisor exists: take a closed point of the special fibre outside that divisor; over the corresponding finite unramified extension of R_0 it lifts to a section, by smoothness and Hensel's lemma; the conjugates of this section have disjoint special fibres, hence are disjoint, and their union is defined over R_0. We choose a section $s \in H^0(\mathcal{X}, \mathcal{L}^d)$ whose divisor contains the divisor of the theorem and D_{ur}, has a simple zero along D_{ur} in every geometric fibre, and has reduced generic fibre. From now on D denotes the divisor of s; the local system σ^{lift} is unramified at the points which have been added.

Such a section exists if $\mathcal{A}$ is ample enough. Sections of $\mathcal{L}^d$ vanishing on the old divisor correspond to sections of a twist of $\mathcal{L}^d$, and by the vanishing theorem of Serre one can prescribe the value of the latter along D_{ur} and on the first infinitesimal neighbourhood of any point. The sections over R_0 with a prescribed reduction are Zariski dense in the space of sections over L, and those whose divisor on the generic fibre is not reduced form a proper closed subset, because the linear system separates tangent vectors. So a section with the required properties exists over R_0 itself.

Proposition 7.7 (The curve with two components and its smoothing)**.** *In the total space of $\mathcal{L}$, with tautological coordinate w, set*

$$C_2 : \ w^d = s, \qquad C_0 : \ w(w^d - s) = 0. \tag{7.1}$$

Let $\pi_0 : C_0 \to \mathcal{X}$ be the projection.

1. *The curve C_0 is the union of the zero section $\mathcal{X}$ and of C_2, glued along D. It is flat, projective and Gorenstein over R_0, with geometrically reduced and connected fibres, and its dualizing sheaf is $\pi_0^*\kappa_1^2$. So $\pi_0^*\kappa_1$ is a theta characteristic.*
2. *There is a flat projective family over $R_0[[\varepsilon]]$ with fibre C_0 at $\varepsilon = 0$, to which the theta characteristic extends, and whose fibre over the fraction field is smooth. Its dualizing sheaf has positive degree on every component of every fibre.*

Over the generic fibre of R_0 the divisor D is reduced and its points are ordinary double points of C_0. Over the closed fibre D may be nonreduced, and then that fibre of C_0 is not nodal: the local equation $w(w^3 - z^2) = 0$, in characteristic greater than three, defines a reduced Gorenstein curve whose singular point is not an ordinary double point. Nothing below uses that the closed fibre is nodal; the properties used are those of (1). The calculations in the proof are made for the whole family over R_0.

Proof. Step 1: the curve C_0. The two equations are monic in w, so C_2 and C_0 are finite and free over $\mathcal{X}$, of degrees d and $d+1$. They are Cartier divisors in the total space of $\mathcal{L}$, which is smooth over R_0; so they are Gorenstein. Let $h : C_2 \to \mathcal{X}$ be the projection. The curve C_2 meets the zero section in the divisor $D_2 = (w = 0)$ of C_2, which h maps isomorphically onto D, and $h^*D = dD_2$. The equality of ideals $(w) \cap (w^d - s) = (w(w^d - s))$ shows that C_0 is the pushout of $\mathcal{X}$ and C_2 along D, and this remains true after any base change. At a point of D_{ur} the section s is a uniformizer in every geometric fibre, so $w^d - s$ is an Eisenstein polynomial and the geometric fibres of C_2 are irreducible; they are reduced because d is invertible. Hence the fibres of C_0 are geometrically reduced and connected.

Step 2: the dualizing sheaf. The canonical bundle of the total space of $\mathcal{L}$ is the pullback of $\Omega_{\mathcal{X}} \otimes \mathcal{L}^{-1}$, and C_0 is the divisor of a section of the pullback of $\mathcal{L}^{d+1}$. The adjunction formula gives

$$\Omega_{C_0/R_0} \simeq \pi_0^*(\Omega_{\mathcal{X}} \otimes \mathcal{L}^d) = \pi_0^*\kappa_1^2. \tag{7.2}$$

On the component $\mathcal{X}$ this is $\Omega_{\mathcal{X}}(D)$. The derivative of $w(w^d - s)$ with respect to w is $-s$ on the component $w = 0$ and ds on the component $w^d = s$. Write θ_1 and θ_2 for the resulting local trivializations of the dualizing line, corresponding to the same local frame of κ_1^2 on the two components. Then

$$\theta_2 = (-1/d)\, c_h(h^*\theta_1), \tag{7.3}$$

where c_h is the isomorphism of the logarithmic Riemann–Hurwitz formula. This is the condition that the residues of the dualizing sheaf at the two branches of a node are opposite. If the degree of $\mathcal{A}$ is large, $\pi_0^*\kappa_1^2$ has positive degree on both components, and the generic fibre has at least three nodes.

Step 3: the smoothing. Since $\mathcal{L}^d \simeq \mathcal{O}(D)$ and $H^1(\mathcal{X}, \mathcal{L}) = 0$, the restriction $H^0(\mathcal{X}, \mathcal{L}^{d+1}) \to H^0(D, \mathcal{L}^{d+1}|_D)$ is surjective. The scheme D is finite over the local ring R_0, hence semilocal, so $\mathcal{L}|_D$ has a frame b_D. We choose a section t of $\mathcal{L}^{d+1}$ with

$$t|_D = -b_D^{d+1}. \tag{7.4}$$

In particular t is invertible on D. The normalization (7.4) is used in Section 7.4: it makes the tangent vectors of the two branches at a node correspond to each other. For an integer $e > 0$ let $\mathcal{C}$ be the curve over $R_0[[\varepsilon]]$ with equation

$$w(w^d - s) = \varepsilon^e t \tag{7.5}$$

in the total space of $\mathcal{L}$, and $\pi : \mathcal{C} \to \mathcal{X}$ the projection. As before it is finite and free over $\mathcal{X}$, projective and Gorenstein, with $\pi_*\mathcal{O}_{\mathcal{C}} = \bigoplus_{i=0}^{d} \mathcal{L}^{-i}$ and dualizing sheaf $(\pi^*\kappa_1)^2$. The first formula shows that the fibres are connected, and the second that the dualizing sheaf is positive on every component and that $\pi^*\kappa_1$ extends the theta characteristic. The locus of the base over which the fibre is geometrically reduced is open [The26, Tag 0C0E]; it contains the closed point of the local base, so it is the whole base.

It remains to see that the fibre over the fraction field of $R_0[[\varepsilon]]$ is smooth. Let $Z \subset \mathcal{C}$ be the closed subset where $\mathcal{C}$ is not smooth over $R_0[[\varepsilon]]$. Consider the fibre of $\mathcal{C}$ over the

generic point of the divisor $\varepsilon = 0$ of the base, which is $C_0 \otimes L$. Outside the nodes it is smooth. At a node, w and $w^d - s$ are part of a system of local coordinates of the total space of $\mathcal{L}$, because s has a simple zero there, and t is a unit; so, in a local frame of $\mathcal{L}$, with $x = w$ and $y = (w^d - s)/t$ the curve $\mathcal{C}$ is given near the node by $xy = \varepsilon^e$, and Z is contained in $\varepsilon = 0$ in a neighbourhood of the node. Now suppose that Z contains a point z of the fibre over the fraction field. The closure of z is proper over the base and maps onto it, so z specializes to a point of Z in $C_0 \otimes L$, necessarily a node. Every neighbourhood of this node then contains the point z of Z, at which $\varepsilon \neq 0$; this is a contradiction. □

Remark 7.8. The line bundle $\mathcal{A}$ is taken very positive for three reasons: there must be a section s with the prescribed divisor, there must be a direction of smoothing, and the dualizing line of C_0 must be positive on both components. The degree of $\mathcal{A}$ plays no role afterwards.

7.4 Extending the parameter across the smoothing

On the component $\mathcal{X}$ of C_0 we have the local system σ^{lift}, and on $C_2 - D_2$ we take its pullback by h. We work over $\overline{L}$ and use, near each point of D, a local frame b of $\mathcal{L}$ which extends the frame b_D of (7.4). In this frame w, s and t are functions, $t = -1$ on D, and

$$x = w, \qquad y = (w^d - s)/t$$

are the coordinates in which $\mathcal{C}$ is $xy = \varepsilon^e$ near the node. On the branch $\mathcal{X}$, where $w = 0$, the function $y = -s/t$ is a local coordinate; it is s times a function with value one at the node. On the branch C_2 the function $x = w$ is a local coordinate, and

$$h^*(y|_{\mathcal{X}}) = -w^d/t = x^d\, a(x), \qquad a(0) = 1. \tag{7.6}$$

Another frame extending b_D changes x and y by functions with value one at the node. So the tangent vectors defined by y on $\mathcal{X}$ and by x on C_2 are well defined, and they are permuted by ϕ, since b_D is defined over R_0. By (7.6) the fibre of $h^*\sigma^{\text{lift}}$ at the tangential base point x of C_2 is identified with the fibre of σ^{lift} at the tangential base point y of $\mathcal{X}$: one substitutes $y^{1/n} = x^{d/n} a^{1/n}$, with the root of a whose constant term is one. This identification commutes with the action of ϕ^r, which is therefore given on both fibres by the same element g_x of Q; and it transforms the action of $\theta \in \mu_n$ on the fibre for C_2 into that of θ^d on the fibre for $\mathcal{X}$. So the inertia on the branch in C_2 is ρ_x^d. With a section t whose value on D is not normalized, $a(0)$ would be a unit c of the residue field, and the two actions of ϕ^r would differ by the image under ρ_x of the Kummer class of c; this is the reason for (7.4).

From now on g_x denotes the action of ϕ^r at these tangential base points, and P_x the element of Lemma 7.6 for this g_x. At a node over a point x of the old divisor we identify the fibres on the two branches by P_x, and at a node over an added point by the identity. The two branches of a node have opposite orientations, so the condition for gluing is that ρ_x on the first branch corresponds to the inverse of ρ_x^d on the second, and this is the relation $P_x \rho_x^{-d} P_x^{-1} = \rho_x$. The relation $P_x g_x = g_x P_x$ says that the identifications are compatible with Frobenius.

Fix a compatible system of roots $\varepsilon^{1/m}$ of ε, put

$$R_{\text{P}} = \bigcup_m \overline{L}[[\varepsilon^{1/m}]],$$

and extend ϕ to R_{P} by its action on the coefficients, so that it fixes all the $\varepsilon^{1/m}$. The fraction field of R_{P} is algebraically closed.

Proposition 7.9 (Extension of the coverings to the smoothing)**.** *Let $\mathcal{C}$ be the family (7.5), considered over $\overline{L}[[\varepsilon]]$, and let Q be a profinite group with a countable basis of neighbourhoods of the identity. Suppose given*

- *a continuous homomorphism to Q of the fundamental group of $\mathcal{X}_{\overline{L}} - D$, and on $C_2 - D_2$ its pullback by h;*
- *for each node an element of Q identifying the fibres of the two branches at the tangential base points defined by y and by x, such that the inertia homomorphisms of the two branches become inverse to each other;*
- *an action of ϕ on these data, such that the identification at each node is carried by ϕ to the identification at its image.*

Then these data define a continuous homomorphism to Q of the fundamental group of the smooth locus of $\mathcal{C} \otimes R_{\mathrm{P}}$ over R_{P}, and in particular of its geometric generic fibre, which is smooth and proper. Its image is contained in the closed subgroup generated by the image of the given homomorphism and by the elements attached to the nodes, and it contains the image of the given homomorphism. The automorphism ϕ acts on the result, compatibly with all finite quotients of Q and with composition.

Proof. The construction is made for one finite quotient Δ of Q at a time, and the coverings are then shown to be compatible. The essential point is in step 2: the condition that the inertia on the two branches of a node is inverse is what is needed for the two coverings to glue to a finite étale covering of a stack whose coarse space is the nodal curve. Choose a decreasing sequence of open normal subgroups of Q which is a basis of neighbourhoods of the identity. (The group Q to which the proposition is applied has such a sequence because it is a compact subgroup of $\mathrm{GL}_n(E)$.)

Step 1: a stack of roots. Let n be an integer divisible by e and by the orders of the images in Δ of the inertia groups at the points of D. Put

$$\tau_n = \varepsilon^{e/n}, \qquad A_n = \overline{L}[[\tau_n]],$$

so that $\tau_n^n = \varepsilon^e$. On the open subset $t \neq 0$ of $\mathcal{C}_{A_n} = \mathcal{C} \otimes A_n$, which contains the nodes, w is a section of the pullback of $\mathcal{L}$, the quotient $y = (w^d - s)/t$ is a section of the pullback of $\mathcal{L}^{-1}$, and $wy = \tau_n^n$. Let $\mathscr{C}_n$ be the stack over this open subset whose objects are

$$(\mathcal{M},\ \mathcal{M}^{\otimes n} \simeq \pi^*\mathcal{L},\ u \in \Gamma(\mathcal{M}),\ v \in \Gamma(\mathcal{M}^{-1})), \qquad u^n = w, \quad v^n = y, \quad uv = \tau_n,$$

with $\mathcal{M}$ a line bundle. Where w is invertible, u trivializes $\mathcal{M}$, and the object is unique and has no automorphisms; the same holds where y is invertible. So $\mathscr{C}_n$ is isomorphic to $\mathcal{C}_{A_n}$ outside the nodes of the closed fibre, and we glue it with the complement of these nodes in $\mathcal{C}_{A_n}$ to obtain a stack over $\mathcal{C}_{A_n}$, still denoted $\mathscr{C}_n$. In a local frame of $\mathcal{L}$ near a node it is the quotient

$$[\{uv = \tau_n\}/\mu_n], \qquad x = u^n, \quad y = v^n, \quad \theta(u,v) = (\theta u, \theta^{-1}v). \tag{7.7}$$

Thus $\mathscr{C}_n$ is a Deligne–Mumford stack, proper over A_n, with coarse space $\mathcal{C}_{A_n}$. It is regular, since the chart $uv = \tau_n$ is and $\mathcal{C}_{A_n}$ is smooth over A_n outside the nodes. It coincides with $\mathcal{C}_{A_n}$ over the locus where $\mathcal{C}_{A_n}$ is smooth over A_n, in particular over the generic fibre. The closed fibre $\mathscr{C}_{n,0}$ has the chart $[\{uv = 0\}/\mu_n]$ at each node. Its normalization is the union of the stack of n-th roots of $\mathcal{X}_{\overline{L}}$ along D, with $v^n = y$, and of that of $C_{2,\overline{L}}$ along D_2, with $u^n = x$; and $\mathscr{C}_{n,0}$ is obtained by gluing them along the residual gerbes $B\mu_n$ of the nodes.

If $n' = mn$, then $\tau_n = \tau_{n'}^m$, and $(\mathcal{M}', u', v') \mapsto (\mathcal{M}'^{\otimes m}, u'^m, v'^m)$ defines a morphism $\mathscr{C}_{n'} \to \mathscr{C}_n \otimes_{A_n} A_{n'}$, which is the identity over the smooth locus. The automorphism ϕ acts on all these stacks and morphisms, because it fixes the τ_n and $\mathcal{L}$, s and t are defined over R_0.

Step 2: a covering of the closed fibre. The given torsor under Δ on $\mathcal{X}_{\overline{L}} - D$ extends uniquely to a finite étale torsor on the stack of n-th roots of $\mathcal{X}_{\overline{L}}$ along D: in the chart with coordinate v the normalization of its pullback is finite étale, because n is divisible by the order of inertia, and the group μ_n acts on it. Its restriction to the residual gerbe at a node is the fibre at the tangential base point defined by y, with the action of μ_n; by (7.7) an element θ acts by $\rho_x(\theta)^{-1}$. In the same way the pullback by h extends to the stack of roots of $C_{2,\overline{L}}$, and its restriction to the gerbe is the fibre at the tangential base point defined by x, which has been identified with the previous one; on it θ acts by $\rho_x(\theta)^d$. The element of Q attached to the node, taken in Δ, is an isomorphism between these two fibres, and the hypothesis that it makes the inertia homomorphisms inverse to each other says that it commutes with the two actions of μ_n. So it is an isomorphism between the restrictions of the two torsors to the gerbe $B\mu_n$.

Locally for the étale topology, a finite étale covering of the chart $[\{uv = 0\}/\mu_n]$ is the same as finite étale coverings of the two branches, with actions of μ_n, and an equivariant isomorphism between their fibres at the origin. Indeed the ring of $uv = 0$ is the fibre product of the rings of the two branches over the residue field of the origin; finite projective modules over a fibre product of rings along a surjection are obtained by patching ([Mil71, §2], [The26, Tag 0D2J]); and a finite locally free algebra over this ring is étale if it is étale over both branches. Hence the two torsors and the isomorphisms at the nodes define a finite étale torsor $\mathcal{Y}_{\Delta,0}$ under Δ on $\mathscr{C}_{n,0}$, whose restrictions to the two components are the given ones. The element attached to the node is part of the construction, so the torsor comes with its identification of fibres, and not only up to isomorphism.

Step 3: extension to the family. The stack $\mathscr{C}_n$ is proper over the complete local ring A_n. Therefore restriction to the closed fibre is an equivalence between the categories of finite étale coverings of $\mathscr{C}_n$ and of $\mathscr{C}_{n,0}$. Indeed, a finite étale covering extends uniquely over each infinitesimal neighbourhood of the closed fibre, with its morphisms. The resulting compatible system of coherent sheaves is algebraic by the existence theorem of Grothendieck for proper stacks [Ols05, Theorem 1.4]. The theorem gives an equivalence of categories of coherent sheaves which is compatible with tensor products, so the multiplication of the algebras, the action of the group and the morphisms between coverings are algebraic as well. The algebra obtained is étale over a neighbourhood of the closed fibre, and the closed subset where it is not étale has a closed image in $\operatorname{Spec} A_n$ which does not contain the closed point; so it is empty. Let $\mathcal{Y}_\Delta$ be the torsor on $\mathscr{C}_n$ which extends $\mathcal{Y}_{\Delta,0}$. Over the smooth locus of $\mathcal{C}_{A_n}$ it is a finite étale torsor on a scheme.

Step 4: the maps between the coverings. Let $\Delta' \to \Delta$ be two quotients of the sequence, with integers $n' = mn$. On the closed fibre of $\mathscr{C}_{n'}$ the torsor under Δ induced from $\mathcal{Y}_{\Delta',0}$ and the pullback of $\mathcal{Y}_{\Delta,0}$ are the same: they agree on the two components, the fibres at the tangential base points formed with n and with n' are identified canonically, and at each node both torsors are glued by the image of the same element of Q. Since restriction to the closed fibre is fully faithful, this isomorphism extends uniquely to $\mathscr{C}_{n'}$. For three quotients, the composite of two of these isomorphisms and the third have the same restriction to the closed fibre, so they are equal. On the smooth locus the morphisms $\mathscr{C}_{n'} \to \mathscr{C}_n$ are the identity. So the torsors $\mathcal{Y}_\Delta$, pulled back to R_{P}, form an inverse system of finite étale torsors on the smooth locus of $\mathcal{C} \otimes R_{\mathrm{P}}$. This is the continuous homomorphism to Q.

Step 5: the action of ϕ. On the closed fibre the action of ϕ on the data is a system of isomorphisms $\Phi_\Delta : \phi^* \mathcal{Y}_{\Delta,0} \simeq \mathcal{Y}_{\Delta,0}$ compatible with the maps of step 4. On the two components they are the given ones, and they are compatible with the gluing because the identification at each node is carried by ϕ to the identification at its image. By the equivalence of step 3, Φ_Δ and its inverse extend to $\mathscr{C}_n$, the extensions are inverse to each other and

compatible with the maps of step 4, and the extensions of the powers of ϕ are the powers of the extension. The action of ϕ itself is extended, and not only that of the power which fixes a given node. This is possible because ϕ fixes τ_n.

Step 6: the image. All the data have values in the closed subgroup of Q generated by the image of the given homomorphism and the elements attached to the nodes, so the homomorphism constructed has values in it.

For the lower bound, fix Δ and let $\mathcal{Y}'$ be a connected component of $\mathcal{Y}_\Delta$. Since $\mathcal{Y}'$ is proper over A_n, its closed fibre is connected: the idempotents of the closed fibre lift. The stack $\mathcal{Y}'$ is regular and connected, hence integral. Its closed fibre is a finite étale covering of the connected stack $\mathscr{C}_{n,0}$, so it has a point over the smooth locus of $\mathcal{C}_{A_n}$; as $\mathcal{Y}'$ is smooth over A_n at this point, the point lifts to a section of $\mathcal{Y}'$ over A_n. The ring of global functions of $\mathcal{Y}'$ is a domain, finite over A_n, and the section defines a homomorphism from it to A_n which is the identity on A_n. The kernel of this homomorphism is a prime ideal which meets A_n in zero, hence is zero. So the ring of global functions of $\mathcal{Y}'$ is A_n, and the generic fibre of $\mathcal{Y}'$ is geometrically connected. The same holds after replacing n by a multiple. So the connected components of the torsor on the geometric generic fibre correspond to those of $\mathcal{Y}_{\Delta,0}$.

Now the restriction of $\mathcal{Y}_{\Delta,0}$ to $\mathcal{X}_{\overline{L}} - D$ is the torsor of the given homomorphism, and a connected component of this restriction lies in one connected component of $\mathcal{Y}_{\Delta,0}$. So the image of the fundamental group of the geometric generic fibre in Δ contains the image of the given homomorphism. The image in Q is compact, hence closed, and a closed subgroup of Q which has this property in every quotient of the chosen sequence contains the image in Q of the given homomorphism.

The elements attached to the nodes themselves need not lie in the image. Only the products of these elements around the cycles of the dual graph, conjugated along paths, contribute to it. □

Proposition 7.10 (The unramified parameter on the smooth curve)**.** *Let* $\mathsf{Mon}_\sigma^{\mathrm{ar}}$ *and* $\mathsf{Mon}_\sigma^{\mathrm{geom}}$ *be the arithmetic and the geometric monodromy group of the given elliptic parameter* σ*, and* $g = \sigma(\mathrm{Frob})$ *the image of the chosen Frobenius element. The geometric generic fibre of the smoothing of Proposition 7.7 carries an unramified* $\widehat{G}$*-local system* ρ *whose geometric monodromy group* Mon_ρ *satisfies*

$$\mathsf{Mon}_\sigma^{\mathrm{geom}} \subset \mathsf{Mon}_\rho \subset \mathsf{Mon}_\sigma^{\mathrm{ar}}, \qquad \overline{\langle \mathsf{Mon}_\rho, g \rangle} = \mathsf{Mon}_\sigma^{\mathrm{ar}}.$$

The group Mon_ρ *is reductive and normalized by* g*. The local system* ρ *extends to the smooth locus of the family over* R_{P}*, and the restriction of the extension to the closed fibre is* σ^{lift} *on* $\mathcal{X}_{\overline{L}} - D$ *and the pullback of* σ^{lift} *by* h *on* $C_{2,\overline{L}} - D_2$*, compatibly with the action of Frobenius.*

Proof. Apply Proposition 7.9 to the local systems on the two components and to the elements P_x. All these data have values in Q, so the image of ρ is contained in Q and $\mathsf{Mon}_\rho \subset \mathsf{Mon}_\sigma^{\mathrm{ar}}$. Take the base point on the component which carries σ^{lift}. By the proposition the image contains the image of the geometric fundamental group of the generic fibre of $\mathcal{X}_{R_0} - D$ under σ^{lift}, which contains the image of the geometric fundamental group of U under σ by Theorem 7.1. This gives the first inclusion.

The quotient $\mathsf{Mon}_\sigma^{\mathrm{ar}}/\mathsf{Mon}_\sigma^{\mathrm{geom}}$ is finite and cyclic by Lemma I.4.4. So every group between the two is normal in $\mathsf{Mon}_\sigma^{\mathrm{ar}}$, in particular normalized by g, and has the same identity component, which is reductive. Finally $\mathsf{Mon}_\sigma^{\mathrm{geom}}$ and g generate a dense subgroup of $\mathsf{Mon}_\sigma^{\mathrm{ar}}$, which gives the last equality. □

7.5 The curves in the S_3 example

We carry out the construction of this section for the parameter of Section 6.4, in the notation of Lemma 7.6 and Proposition 7.7, and we find the rational point which was used there as

the auxiliary puncture. The numbers are chosen for simple arithmetic, not to minimize the genus.

Work over $R_0 = \mathbb{Z}_5$, with $\mathcal{X} = \mathbb{P}^1_{R_0}$ and coordinate t. The scheme of ordered roots of $z^3 - z - t$ is an étale S_3-torsor off $4 - 27t^2 = 0$ and infinity; it is the lift σ^{lift} of the parameter. Take $\mathcal{A} = \mathcal{O}(2)$, so that

$$\mathcal{L} = \Omega_{\mathcal{X}} \otimes \mathcal{A}^2 = \mathcal{O}(2), \qquad \kappa_1 = \Omega_{\mathcal{X}}^{(d+1)/2} \otimes \mathcal{A}^d = \mathcal{O}(d-1).$$

For the degree d of the cyclic cover we choose a positive integer satisfying

$$d \equiv -1 \pmod 6, \qquad d \equiv 1 \pmod 4, \qquad d \equiv 4 \pmod 5.$$

The first congruence gives the property of Lemma 7.6 with $P_x = 1$: the inertia groups of σ^{lift} have orders two and three, and $N_x = 0$ because the image is finite, so that $\rho_x(\gamma)^{-d} = \rho_x(\gamma)$. The second makes the d-th power map the identity on $\mathbb{F}_5^\times$, which makes a rational point easy to find. The third makes $2d - 3$ divisible by five, which gives the separable polynomial below. The least positive solution is $d = 29$.

Set $c_0 = 4/27$ and take the section s of $\mathcal{L}^{29} = \mathcal{O}(58)$ with affine polynomial

$$s(t) = (t^2 - c_0)(t^{55} + t + 1). \tag{7.8}$$

Its affine degree is 57, so its divisor D includes infinity with multiplicity one. Modulo five the derivative of $t^{55} + t + 1$ is 1; its remainder modulo $t^2 - 2$ is $4t + 1$, coprime to $t^2 - 2$. Thus D is finite étale of degree 58, contains all the branch points, and avoids $t = 0$. The factor $t^2 - c_0$ contains the branch points of the lift, not only their reductions; the zeros of the other factor form the divisor D_{ur}.

In the total space of $\mathcal{L}$, the curves (7.1) are

$$C_2 : \ w^{29} = s, \qquad C_0 : \ w(w^{29} - s) = 0.$$

The latter has components $\mathcal{X}$ and C_2, meeting along D. At each simple zero of s the cyclic equation is Eisenstein, so C_2 is geometrically irreducible; since 29 is a unit in R_0, it is smooth. The local system on it is the pullback of σ^{lift}. At a point of D the pullback raises the inertia to the power 29, and $\rho_x(\gamma)^{29} = \rho_x(\gamma)^{-1}$: the inertia on the two branches is inverse to each other, as is needed for gluing. Riemann–Hurwitz gives

$$2g(C_2) - 2 = 29(-2) + 58(28) = 1566, \qquad g(C_2) = 784.$$

There are 58 nodes, so the arithmetic genus of the nodal curve C_0 is $0 + 784 + 58 - 1 = 841$.

The theta characteristic is determined by the same degrees. By (7.2) the dualizing sheaf of C_0 is the pullback of $\Omega_{\mathcal{X}} \otimes \mathcal{L}^{29} = \mathcal{O}(-2) \otimes \mathcal{O}(2)^{29} = \mathcal{O}(56)$. Its square root $\pi_0^* \kappa_1$, the theta characteristic, is the pullback of $\mathcal{O}(28)$; no additional choice is involved.

For the section of $\mathcal{L}^{30} = \mathcal{O}(60)$ in (7.5), which is called t there, we take $1 + t^3 s$, where t is the coordinate. It is nonzero everywhere on D, including infinity, where its leading coefficient is 1. It satisfies the normalization (7.4): in the standard frames of $\mathcal{L}$ along D its restriction to D is 1, and $1 = -b^{30}$ for a primitive fourth root of unity $b \in \mathbb{Z}_5$. With $e = 6$ the family (7.5) is

$$w(w^{29} - s) = \varepsilon^6 (1 + t^3 s), \tag{7.9}$$

and it smooths every node. The exponent 6 is a common multiple of the two inertia orders; locally the equation is $xy = \varepsilon^6$ up to a unit. Propositions 7.9 and 7.10 give the unramified local system ρ on the smooth proper generic fibre, and Proposition 7.7 extends the theta characteristic. The arithmetic and the geometric monodromy group of σ are both S_3, so Proposition 7.10 gives $\mathsf{Mon}_\rho = S_3$.

At $t = 0$ the equation $w^{29} = s(0) = -4/27$ has the solution $w = 3$ modulo five, and Hensel's lemma gives its unique lift w_0 in $\mathbb{Z}_5$. The derivative of the left side of (7.9) with respect to w at this point is $29w_0^{29}$, a unit, so the point extends to a section of the smoothing; this is the divisor D_{aux} of Lemma 8.3, with $d_{\mathrm{aux}} = 1$. It lies on the auxiliary component, away from D. The torsor of ordered roots is split at $t = 0$, so the Frobenius element at this puncture is $g_D = 1$, which gives the determinant in (6.9). The legs of the Hecke modifications on the original component still range over $\mathcal{X} - D$.

The genus 841 is the cost of satisfying the three elementary congruences with the same cyclic cover. The example makes the inverse inertia, the theta characteristic and the rational puncture explicit; only their existence enters the realization theorem.

8 Specializing the whole Hecke family

Section 6 constructed a family of automorphic sheaves $P \mapsto \mathcal{F}_P$ on the smooth curve in characteristic zero, and Section 7 showed that this curve degenerates, first to the nodal curve $\mathcal{X} \cup_D C_2$ in characteristic zero and then to characteristic p. The goal of this section is Theorem 8.8: the family extends over a base S that contains both degenerations, as a family of complexes which are universally locally acyclic (ULA) over S, together with its Hecke identities and its compatibility with Frobenius. The whole family is extended, with all maps between its members, because the argument of Lemma 6.7 needs naturality in P.

The tool is the theorem of Hansen and Scholze (Input 2): over an absolutely integrally closed valuation ring, ULA complexes are the same as constructible complexes on the generic fibre. So objects extend automatically and uniquely, and maps between ULA complexes extend uniquely. What has to be proved is that everything appearing in a Hecke identity is ULA. For the sheaf this holds by construction, for the local systems on the curve it requires that they extend over S (Lemma 8.2), and for the Satake kernels it is a theorem of Gaitsgory and Raskin over a Witt ring, which we transfer to S (Lemma 8.5).

8.1 The base and the total local system

Let $\overline{L}$ be the algebraic closure of $\operatorname{Frac} R_0$ chosen above, and R_{m} its valuation ring. Introduce

$$R_{\mathrm{P}} = \bigcup_{n \geq 1} \overline{L}[[\varepsilon^{1/n}]], \qquad R = R_{\mathrm{P}} \times_{\overline{L}} R_{\mathrm{m}}, \qquad S = \operatorname{Spec} R,$$

so that R consists of the series in R_{P} whose constant term lies in R_{m}. It is a valuation ring of rank two. Its spectrum S has three points η, ξ, s, with residue fields the field of Puiseux series over $\overline{L}$, the field $\overline{L}$, and $\overline{\mathbb{F}}_q$. Over ξ the curve $\mathcal{C}_S$ is the nodal curve C_0 in characteristic zero, and over s it is its reduction. The semilinear automorphism ϕ acts on the coefficients of the series and fixes the chosen roots of ε.

Lemma 8.1 (The valuation ring)**.** *The field $\bigcup_n \overline{L}((\varepsilon^{1/n}))$ is algebraically closed. The ring R is a valuation ring of this field, and it is absolutely integrally closed.*

Proof. A polynomial over the union has its coefficients in a field $\overline{L}((u))$ with $u = \varepsilon^{1/n}$. A finite extension of this complete discretely valued field has residue field $\overline{L}$, and it is of the form $\overline{L}((v))$ for a uniformizer v. Write $u = v^e a(v)$ with a a unit. Since the residue characteristic is zero, a has an e-th root, and after replacing v by $va(v)^{1/e}$ we have $u = v^e$. So the extension is contained in the union.

Let f be a nonzero element of the field. If its leading exponent is positive, it lies in R_{P} with constant term zero, hence in R; if the exponent is negative, f^{-1} lies in R. If the exponent is zero, f is a unit of R_{P} whose constant term is a nonzero element of $\overline{L}$, and either

this constant term or its inverse lies in R_{m}. So R is a valuation ring. A valuation ring with algebraically closed fraction field is absolutely integrally closed. □

Let $\mathcal{C}_S$ be the base change to S of the family of Proposition 7.7, and $U = \mathcal{C}_S^{\mathrm{sm}}$ its smooth locus over S. Over R_{m} the family is the Gorenstein curve C_0 with its two components, nodal on its generic fibre, whose smooth locus carries σ^{lift} and its pullback by h; over R_{P} it carries the homomorphism of Proposition 7.9.

Lemma 8.2 (The local system over the base)**.** *These homomorphisms glue to a continuous homomorphism from the fundamental group of U to Q, and so define, for every representation of $\widehat{G}$, a local system on U, compatibly with tensor products. We denote this system by $\mathcal{E}$. The gluing is compatible with base points, with the finite quotients of Q and with the action of ϕ.*

Proof. By construction the two homomorphisms agree on the fibre over $\overline{L}$, which is the smooth locus of $C_0 \otimes \overline{L}$. The equality $R = R_{\mathrm{P}} \times_{\overline{L}} R_{\mathrm{m}}$ says that S is the pushout of $\operatorname{Spec} R_{\mathrm{P}}$ and $\operatorname{Spec} R_{\mathrm{m}}$ along $\operatorname{Spec} \overline{L}$, which is closed in the first. Such pushouts have the expected universal property for maps to algebraic stacks with suitable diagonal, and they are compatible with flat base change [AHHLR24, Theorem 4.2 and Lemma 4.3(2)]; so U is the pushout of its restrictions to the two rings.

A torsor under a finite quotient Q/K of Q is a map to the classifying stack $B(Q/K)$, whose diagonal is finite. By the universal property the torsors on the two pieces, with their identification over $\overline{L}$, define a torsor on U, and maps of torsors glue in the same way. Since a map out of the pushout is determined by its restrictions, the glued maps satisfy the same identities; this applies to the transition maps, to the action of the groups and to the action of ϕ. The inverse limit over K is the required continuous homomorphism. □

Lemma 8.3 (The auxiliary puncture)**.** *Let y be a closed point, of degree d_{aux}, of the special fibre of C_2 over $\mathbb{F}_q$ which does not lie on D_2. There is a divisor D_{aux} of the family $\mathcal{C}$, finite étale over $R_0[[\varepsilon]]$, whose closed fibre is y. On the generic fibre ϕ permutes its points cyclically, and $\phi^{d_{\mathrm{aux}}}$ acts on the fibre of $\mathcal{E}$ at such a point as the Frobenius element of y acts on the fibre of $h^*\sigma$. In particular its adjoint action is pure of weight zero.*

Proof. The family $\mathcal{C}$ is smooth over $R_0[[\varepsilon]]$ at y. Let R_1/R_0 be the unramified extension of degree d_{aux}. By Hensel's lemma y lifts to a section over $R_1[[\varepsilon]]$. Its d_{aux} conjugates are pairwise disjoint: the locus where two of them coincide is a closed subscheme of the local scheme $\operatorname{Spec} R_1[[\varepsilon]]$ which does not contain the closed point. Their union descends to a divisor D_{aux} finite étale over $R_0[[\varepsilon]]$.

After base change to S each of the sections is a section over S. An absolutely integrally closed valuation ring is strictly henselian, so the restriction of $\mathcal{E}$ to a section is constant, and its fibres over η and over s are identified. The identification commutes with $\phi^{d_{\mathrm{aux}}}$, which maps the section to itself. Over s this automorphism is the Frobenius element of y. The last assertion follows from Theorem I.4.8, applied to the image of y in X. □

Put $U^\circ = U - D_{\mathrm{aux}}$; this is where the legs of Hecke modifications are allowed. It is dense in every component of every geometric fibre. Since the puncture D_{aux} lies on the auxiliary component, U° contains all of $X - \operatorname{supp} D$ on the original component.

8.2 ULA extension and relative Satake

For a map $Y \to S$, a complex is universally locally acyclic (ULA) if every local specialization map is an isomorphism after every base change. By Input 2, for Y/S separated and of finite presentation, restriction to the generic fibre is an equivalence

$$D^{\mathrm{ULA}}(Y/S, k) \xrightarrow{\sim} D_c(Y_\eta, k), \qquad \text{with inverse } Rj_*. \tag{8.1}$$

This includes rational and algebraic ℓ-adic coefficients [HS23, Theorem 4.1]. It commutes with external products [HS23, Corollary 4.2(iv)], and with smooth pullback, because smooth pullback preserves universal local acyclicity. We use it in the following form: *a map between the generic fibres of two ULA complexes extends uniquely*, and an identity between such maps holds over S if it holds over η.

On a stack with a smooth atlas, a complex is called *locally ULA* if its pullbacks to the schemes of a smooth atlas are ULA over S.

Lemma 8.4 (ULA extension on the stack of bundles)**.** *Let $\mathcal{B} = \mathrm{Bun}_G(\mathcal{C}_S/S)$. Restriction to the generic fibre is an equivalence between the locally ULA complexes on $\mathcal{B}$ which are bounded and constructible on every quasi-compact open substack, and the complexes on $\mathcal{B}_\eta$ which are bounded and constructible on every quasi-compact open substack.*

Proof. The stack $\mathcal{B}$ is locally of finite presentation over S with affine diagonal. Choose a smooth atlas by affine schemes of finite presentation over S; the fibre products of its members over $\mathcal{B}$ are then separated and of finite presentation, and (8.1) applies to all of them. A complex on $\mathcal{B}$, or on $\mathcal{B}_\eta$, is a compatible system of complexes on the schemes Y_n of the Čech nerve of the atlas Y_0. Let a complex on $\mathcal{B}_\eta$ be given. Let E_0 be the ULA extension to Y_0 of its restriction to the generic fibre of the atlas, and let E_n be the pullback of E_0 to Y_n by the first projection. Smooth pullback preserves ULA complexes and commutes with restriction to the generic fibre, so E_n is the ULA extension of the n-th member of the given system. Let $f : Y_n \to Y_m$ be a map of the nerve. The composite of f with the first projection of Y_m is one of the projections of Y_n to Y_0, which are smooth. So f^*E_m is ULA; this includes the degeneracy maps, which are not smooth. By the full faithfulness in (8.1) the isomorphisms $f^*E_m \simeq E_n$ of the given system extend, together with the identities and the higher compatibilities between them. The E_n therefore form a complex on $\mathcal{B}$ which extends the given one, and the same argument applies to maps between complexes. On a quasi-compact open substack one can take a finite atlas, which gives the boundedness; constructibility on each member of the atlas is part of the theorem of Hansen and Scholze. □

To extend a Hecke identity, both of its sides must be ULA. For the side that is a Hecke transform this requires a Satake kernel over S that is ULA over the space of legs, including the loci where legs collide. The results available in the literature are over other bases. Zhu's fusion theorem is over a field [Zhu17, §5.4, Proposition 5.4.2, Remark 5.4.12, Theorem 5.5.17 and Corollary 5.5.19]. Gaitsgory and Raskin construct the kernel in mixed characteristic over a ring of Witt vectors and prove that it is ULA for both projections [GR25, Theorem 4.2.2, §4.2.3 and (4.14)]. The following lemma transfers these kernels to S.

Lemma 8.5 (Satake kernels over the base S)**.** *Let V be a representation of $\widehat{G}^I$. On the Hecke correspondence over S with legs in U°, bounded by the weights of V, there is a complex $\mathcal{S}^{\mathrm{val}}_{I,V}$ with the following properties.*

1. *Its restrictions to the fibres over η, ξ and s are the Satake kernels of V over those fields.*
2. *It is ULA over the product of the stack of bundles and the space of legs, for the projection to either end.*
3. *The isomorphisms expressing compatibility of Satake kernels with maps of representations, with convolution, with changes of local coordinate and with fusion along the diagonals extend from the generic fibre.*

Proof. Steps 1 and 2 define the kernel, by extending the Satake sheaf of the generic fibre. The content of the lemma is in step 3: over a larger base, which contains a ring of Witt vectors, the kernel so defined coincides with the one constructed by Gaitsgory and Raskin,

whose properties are known. Steps 4 and 5 descend these properties to S and extend the compatibilities.

Step 1: local models. The question is local on the space of legs. Fix a geometric point of $(U^\circ)^I$, and divide the legs into groups: two legs are in the same group if they coincide at this point. For each group choose an étale coordinate on the smooth curve U° near the common image of its legs. We work on the locus where legs of different groups do not meet. By completion along the graphs of the legs, the space of modifications at the legs of one group is identified with the Beilinson–Drinfeld Grassmannian of the affine line, pulled back by the coordinates; for several groups one obtains the product. On the part bounded by the weights of V, the action of the group of arcs and the changes of coordinates factor through jets of finite order. Adding a trivialization of the jets of that order is a smooth covering, because the group of jets is smooth. In this way the construction is reduced to the bounded Schubert varieties in the Grassmannians, which are separated and of finite presentation over S, with the diagonals included.

Step 2: definition of the kernel. On each of these Schubert varieties we apply (8.1) to the Satake sheaf of the generic fibre, and call the extension $\mathcal{S}^{\mathrm{val}}_{I,V}$. It is ULA over S by construction. Properties (1) and (2) do not follow from this, and are proved in the next two steps by comparison with the kernel of Gaitsgory and Raskin.

Step 3: comparison over a larger base. Let C_{mix} be the completion of $\overline{L}$ and $\mathcal{O}_{\mathrm{mix}}$ its valuation ring, which contains $W(\overline{\mathbb{F}}_q)$. The field C_{mix} is algebraically closed [BGR84, §3.4.1, Proposition 3]. Define

$$R^{\mathrm{W}} = \left(\bigcup_n C_{\mathrm{mix}}[[\varepsilon^{1/n}]]\right) \times_{C_{\mathrm{mix}}} \mathcal{O}_{\mathrm{mix}}, \qquad S^{\mathrm{W}} = \operatorname{Spec} R^{\mathrm{W}}.$$

By the proof of Lemma 8.1 this is again an absolutely integrally closed valuation ring. The inclusion $R \to R^{\mathrm{W}}$ is local and injective, so R^{W} is torsion free over the valuation ring R, hence flat, and faithfully flat [The26, Tags 0539 and 00HQ].

We take the kernels from the projective line. By [GR25, Theorem 4.2.2 and §4.2.3], applied to $\mathbb{P}^1$ over $W(\overline{\mathbb{F}}_q)$, with the legs restricted to $\mathbb{A}^1$ and after pullback along the section of either end given by the trivial bundle, there is a kernel on the Beilinson–Drinfeld Grassmannian of the affine line which is ULA over the space of legs and restricts to the Satake kernels over the fields. Pull it back to R^{W}. With the trivializations of jets of step 1, the kernel for either end is the pullback of this one; it does not depend on the trivializations, because the kernel on the model is equivariant for the changes of trivialization. The ULA property is preserved by base change, so the pullback is ULA over the same product for our local model. It is therefore also ULA over S^{W}: the space of legs is smooth over S^{W}, and so are the charts of the stack of bundles, because the obstruction to deforming a bundle on a curve lies in the H^2 of the adjoint bundle, which is zero.

We now have two complexes on the local model over S^{W}: the pullback of $\mathcal{S}^{\mathrm{val}}_{I,V}$ and the pullback of the kernel of Gaitsgory and Raskin. Both are ULA over S^{W}, and on the generic fibre both are the Satake sheaf of V, with the same shifts and twists. By (8.1) over S^{W} they are isomorphic, by a unique isomorphism extending the identification on the generic fibre. Hence the pullback of $\mathcal{S}^{\mathrm{val}}_{I,V}$ has properties (1) and (2).

Step 4: descent. The map $S^{\mathrm{W}} \to S$ is faithfully flat and quasi-compact, hence a covering for the topology in which the ULA property descends [The26, Tag 0ETC], [HS23, Proposition 3.7]; the same holds after base change to the charts of the product of the stack of bundles and the space of legs. So $\mathcal{S}^{\mathrm{val}}_{I,V}$ has property (2) over S. Property (1) over S follows from step 3: each of the three algebraically closed fields over which S has a fibre embeds in the corresponding

field for S^{W}, and groups of morphisms between constructible complexes do not change under an extension of algebraically closed fields. The isomorphisms which express the invariance of the kernel under changes of coordinates and under the action of jets are maps between complexes which are ULA over S, and are given on the generic fibre; so they extend uniquely, and the extensions agree on overlaps of charts. This defines $\mathcal{S}^{\mathrm{val}}_{I,V}$ on the Hecke correspondence of the curve $\mathcal{C}_S$.

Step 5: compatibilities. A map of representations induces a map of kernels on the generic fibre, which extends uniquely. For convolution the complex to be pushed forward is ULA, by the argument with local products of Lemma 8.6, and the proper direct image from the bounded convolution space preserves this property. For a diagonal, the restriction of the kernel is again ULA over S by (2); so the fusion isomorphism of the generic fibre extends uniquely, and by the comparison of step 3 the extension is the fusion isomorphism on the special fibre. Associativity and the other identities between these isomorphisms hold because they hold on the generic fibre. Finally, the construction involves only the formal neighbourhoods of the legs, which lie in the smooth locus; the nodes of the fibres play no role. □

Lemma 8.6 (Hecke transforms are ULA over the legs)**.** *Let $\mathcal{K}$ be a locally ULA complex on $\mathcal{B}$, I a finite set and V a representation of $\widehat{G}^I$. The Hecke transform of $\mathcal{K}$ with kernel $\mathcal{S}^{\mathrm{val}}_{I,V}$ is locally ULA over $(U^\circ)^I_S$, the diagonals included. The same holds for a composite of such transforms.*

Proof. Let a be large enough for the bound defined by V. We trivialize the bundles along the divisor $a\sum_i \Gamma_i$, where Γ_i is the graph of the i-th leg. In an étale coordinate t this divisor has the equation $\prod_{i\in I}(t-t_i)^a = 0$, which is monic of degree $a|I|$; so the divisor is finite and locally free over the space of legs, also where legs coincide. By the infinitesimal lifting criterion, the Weil restriction of G along it is smooth. Let $T = (U^\circ)^I_S$. When a trivialization at the first end is added to the data, we get a smooth covering $B' \to \mathcal{B}\times_S T$, over which the bounded Hecke correspondence becomes the product $B'\times_T \mathrm{Gr}_{I,\le V}$ with the local model $\mathrm{Gr}_{I,\le V}$ of the Grassmannian over T. On this product the complex to be pushed forward is the exterior product over T of the pullback of $\mathcal{K}$ to B' and of the Satake complex of the local model. The first factor is ULA over T, by base change and smooth pullback; the second is, by Lemma 8.5(2); and an exterior product over the common base of two ULA complexes is ULA [HS23, Corollary 4.2(iv) and Corollary 3.10]. The property descends along the smooth covering and is preserved by the proper direct image. For a composite of several transforms, give the legs of the successive transforms different labels, apply the statement for one transform repeatedly, each time over the space of the legs already introduced, and finally pull back along the map which identifies the labels that are to coincide. On a diagonal, the fusion isomorphism of Lemma 8.5(3) and proper base change identify the restriction of the transform with the transform for the representation obtained by fusion. □

The next statement concerns Hecke families in the sense of Definition 5.1, on $\mathcal{B}_\eta$ and with legs in U°_η. It applies in the same way when the family is indexed by a small stable category, as the family $P\mapsto \mathcal{F}_P$ of Lemma 6.6 is, and not only by the representations of Z: the proof uses only that the Hecke identities are isomorphisms between complexes which are ULA.

Proposition 8.7 (Extension of a Hecke family)**.** *Consider a Hecke family on $\mathcal{B}_\eta$ whose members are bounded and constructible on every quasi-compact open substack, and suppose that the local systems in its Hecke identities extend to local systems on powers of U°. Then the family has a unique locally ULA extension to $\mathcal{B}$, and the Hecke identities extend to it. If the family carries an action of a semilinear automorphism, or of a smooth group, compatible with the Hecke identities, the action extends as well.*

Proof. The members of the family extend by Lemma 8.4. In a Hecke identity the source is a Hecke transform of an extended member, which is ULA by Lemma 8.6, and the target is the exterior product of an extended member with a local system, which is ULA. Both restrict on the generic fibre to the two sides of the given identity, by Lemma 8.5(1). So the isomorphism and its inverse extend uniquely, and every identity between such isomorphisms holds for the extensions.

The same reasoning applies to all the data at once. A family with its Hecke identities and their compatibilities is an object of a limit of categories, indexed by the finite sets of legs, the maps of representations and the diagrams of convolution and fusion; an action of a group, or of an automorphism, adds the terms of its nerve to the index. Restriction to the generic fibre is an equivalence on each term of the limit, hence on the limit. Each finite part of these data is defined over a finite extension of $\mathbb{Q}_\ell$, so the passage to coefficients in k causes no difficulty. The extension is functorial in the member of the family, so it applies to an exact functor on a small stable category. □

Theorem 8.8 (The family over the valuation base)**.** *The family $P \mapsto \mathcal{F}_P$ of Section 6 extends uniquely to a family of locally ULA complexes on $\mathrm{Bun}_G(\mathcal{C}_S)$, bounded and constructible on every quasi-compact open substack, together with the maps induced by maps in $\mathrm{Perf}(\mathfrak{F}_\rho)$, the Hecke identities for all finite sets of legs in U°, and the action of Frobenius.*

Proof. We apply Proposition 8.7 to the functor $P \mapsto \mathcal{F}_P$ on $\mathrm{Perf}(\mathfrak{F}_\rho)$. Its Hecke identities (6.5) involve the local systems $L_a = \mathrm{Hom}_{Z_\rho}(W_a, \mathcal{E}(V)_\eta)$, and we have to show that they extend to local systems on U°. The local system $\mathcal{E}(V)$ is defined on U by Lemma 8.2. The action of Z_ρ on $\mathcal{E}(V)_\eta$ has its matrix coefficients in a subcoalgebra $C_V \subset \mathcal{O}(Z_\rho)$ of finite dimension, and is given by a coaction

$$\mathcal{E}(V)_\eta \longrightarrow \mathcal{E}(V)_\eta \otimes C_V.$$

Both sides are restrictions of local systems on U, which are ULA. So the coaction extends, with the identities which express that it is counital and coassociative and its compatibility with maps of representations. Since Z_ρ is reductive, the extension of L_a is then defined as a direct summand of a local system, and is a local system. The action of Frobenius on the family (Lemma 6.6(2)) extends by the proposition. □

Remark 8.9. The theorem gives a family on the special fibre, but it gives no control over its trace function. Restriction from S to the closed fibre is not conservative, and nothing proved so far excludes that the trace function of $\mathcal{F}_{P,s}$ vanishes identically. Sections 9 and 10 address this.

9 Hecke symmetry and transverse test functions

Theorem 8.8 extended the family $P \mapsto \mathcal{F}_P$ over the valuation base S as a family of universally locally acyclic (ULA) complexes. To reach a trace we shall take vanishing cycles of these complexes along a function f on a chart of Bun_G, and for that we must know how they behave on the hypersurfaces $f = t$. Restriction to a hypersurface does not preserve the ULA property in general. The main result of this section, Theorem 9.2, says that for the members of a Hecke family it does, at every point where df lies outside the global nilpotent cone. This is a relative form, over a valuation ring and in mixed characteristic, of the theorem that Hecke eigensheaves have nilpotent singular support [AGK+22, Main Theorems 14.4.3 and 14.4.4]; this theorem is the converse of the theorem of Nadler and Yun [NY19] that the Hecke transforms of a sheaf with nilpotent singular support are lisse along the curve. Its proof follows the same mechanism: a non-nilpotent covector can be moved by a Hecke

modification into the direction of the moving point, where the local systems attached to the family are lisse.

The rest of the section applies the theorem. We show that near the canonical bundle the nilpotent cone has a single smooth branch (Proposition 9.12), construct a function f, homogeneous of weight two and defined over $\mathbb{F}_q$, whose differential meets the nilpotent cone only at that point and transversally there (Lemma 9.14), and identify the resulting vanishing cycles on the generic fibre with the Whittaker coefficient (Proposition 9.16).

A *test function* is a regular function f on a smooth chart of a stack of bundles. Its differential is the covector against which the singular support of a complex is tested.

9.1 The comparison theorem

Setup 9.1. Let S_1 be the spectrum of an absolutely integrally closed valuation ring of rank one, with generic point η and closed point s. Let $\mathcal{C}/S_1$ be a projective Gorenstein curve with geometrically reduced fibres, and let $U \subset \mathcal{C}$ be an open subscheme, smooth over S_1, that is dense in every component of every fibre. Let G be split semisimple, and assume that the residue characteristic of S_1 is zero or does not divide $|W_G|$. Put $\mathcal{B} = \mathrm{Bun}_G(\mathcal{C}/S_1)$.

The cotangent space of $\mathcal{B}$ at a bundle E on a fibre C is the space $H^0(C, \mathrm{ad}(E)^* \otimes \Omega_C)$ of Higgs fields, where Ω_C is the dualizing line bundle. The *global nilpotent cone* Nilp consists of the pairs (E, θ) with θ nilpotent at every point. For a smooth chart $\pi : Y \to \mathcal{B}$ we write $\pi^\circ \,\mathrm{Nilp} \subset T^*Y$ for the set of covectors $d\pi^*\theta$ with θ nilpotent; these covectors vanish on the tangent spaces of the fibres of π.

Let $\mathcal{K}$ be a complex on a separated space Z of finite presentation over S_1, and let $i_Z : Z_s \to Z$ and $j_Z : Z_\eta \to Z$ be the inclusions of the fibres. The *specialization cone*, which is computed on an étale chart when Z is not a scheme, is

$$\Delta_Z(\mathcal{K}) = \mathrm{Cone}\big(i_Z^*\mathcal{K} \longrightarrow i_Z^* R j_{Z,*} j_Z^* \mathcal{K}\big),$$

a complex on Z_s. By Input 2, a complex $\mathcal{K}$ with constructible generic fibre is ULA over S_1 if and only if $\mathcal{K} \simeq R j_{Z,*} j_Z^* \mathcal{K}$, that is, if and only if $\Delta_Z(\mathcal{K}) = 0$. On an open subscheme of Z, the complex is ULA over S_1 if and only if all the stalks of $\Delta_Z(\mathcal{K})$ vanish there. Three properties follow from the base-change theorems and [HS23, Theorem 4.1]: the formation of Δ commutes with proper pushforward and with smooth pullback, and $\Delta_Z(\mathcal{K} \otimes \mathcal{L}) = \Delta_Z(\mathcal{K}) \otimes \mathcal{L}|_{Z_s}$ when $\mathcal{L}$ is lisse on Z.

For a dominant coweight λ let V_λ be the irreducible representation of $\widehat{G}$ of highest weight λ, and let H_{V_λ} be the corresponding Hecke functor with one leg in U, that is, the functor $\mathsf{H}_{I,V}$ of Section 5 for a set I with one element; it sends complexes on $\mathcal{B}$ to complexes on $\mathcal{B} \times_{S_1} U$.

Theorem 9.2 (Transverse Hecke comparison)**.** *In Setup 9.1, let $\mathcal{K}$ be a locally bounded constructible complex on $\mathcal{B}$ that is ULA over S_1. Put $\lambda_j = |W_G|\varpi_j^\vee$, where the $\varpi_j^\vee$ are the fundamental coweights, and suppose that for every j there is an isomorphism*

$$\mathsf{H}_{V_{\lambda_j}}(\mathcal{K}) \simeq \bigoplus_a \mathcal{K}_{j,a} \boxtimes \mathcal{E}_{j,a}, \tag{9.1}$$

in which each $\mathcal{K}_{j,a}$ is a complex on $\mathcal{B}$ that is ULA over S_1 and each $\mathcal{E}_{j,a}$ is lisse on U.

Let $\pi : Y \to \mathcal{B}$ be a smooth chart, separated and of finite presentation over S_1, let $f \in \Gamma(Y, \mathcal{O}_Y)$, and let y be a point of $V(f)_s$ such that $df_y \notin \pi^\circ \,\mathrm{Nilp}$. Then

$$\Delta_{V(f)}(\pi^*\mathcal{K}|_{V(f)})_y = 0.$$

Remark 9.3 (Naturality)**.** The conclusion says that the canonical map $i^*\mathcal{K}' \to i^*Rj_*j^*\mathcal{K}'$, for $\mathcal{K}' = \pi^*\mathcal{K}|_{V(f)}$, is an isomorphism at y. Since this map is canonical, it commutes with every isomorphism induced by an automorphism of the data $(\mathcal{C}, Y, f, \mathcal{K})$. In particular, when these data are defined over a finite field and $\mathcal{K}$ carries a Weil structure, the isomorphism at y commutes with Frobenius.

Outline of the proof. If df_y does not vanish on the fibre of π through y, then $V(f)$ is smooth over $\mathcal{B}$ near y and there is nothing to prove. Otherwise $df_y = d\pi^*\theta$ for a Higgs field θ that is not nilpotent, and the argument has four steps, carried out in Sections 9.2–9.5.

1. *Linear algebra* (Lemma 9.4). At a point $u \in U$ where θ is not nilpotent, θ can be conjugated into a maximal Levi subalgebra $\mathfrak{m}_j$ in such a way that it acts invertibly on $\mathfrak{g}/\mathfrak{m}_j$ and pairs nontrivially with λ_j.

2. *Geometry of the Hecke correspondence* (Lemma 9.6). Modify the bundle at u by λ_j inside this Levi subgroup. Then θ extends to the modified bundle. Near this modification the conormal variety of the Hecke correspondence is étale over the cotangent bundles of both ends, and its covector in the direction of the moving point u is not zero.

3. *Isolation* (Lemma 9.8). The Hecke transform of $\mathcal{K}$ is ULA in the direction of u by (9.1). Since the Hecke transform is a proper pushforward, this gives the vanishing of the cohomology of a specialization cone over a whole fibre of the correspondence. Step (2) shows that near the chosen modification this cone is supported at one point, so its stalk there is a direct summand of zero.

4. *Stabilization* (Lemmas 9.9 and 9.10). The hypersurface on which the stalk of step (3) lives is, in suitable coordinates, the hypersurface $f + \sum r_i s_i = 0$. Adding the nondegenerate quadratic form $\sum r_i s_i$ in new variables does not change the specialization cone, up to a shift and twist.

The steps are combined at the end of Section 9.5.

A vanishing theorem for a proper pushforward does not by itself give information about the complex being pushed forward, because a nonzero complex on a proper variety can have zero cohomology. Step (3) is where this is addressed, and it is the reason the proof needs the geometric information of step (2) and not only the identity (9.1).

9.2 Placing a non-nilpotent Higgs field in a Levi subalgebra

Let κ be the normalized invariant bilinear form on $\mathfrak{g}$, defined over $\mathbb{Z}$ on the Chevalley form of the Lie algebra.

Lemma 9.4 (Detecting non-nilpotence)**.** *Work over an algebraically closed ground field $\Bbbk$, of characteristic zero or of characteristic $p \nmid |W_G|$.*

1. *Each $\lambda_j = |W_G|\varpi_j^\vee$ is a dominant cocharacter in the coroot lattice.*

2. *The form κ is nondegenerate on $\mathfrak{g}$, on every Levi subalgebra, and on the centre of every Levi subalgebra.*

3. *Let $a(t)\,dt \in \mathfrak{g}[[t]]\,dt$ with $a(0)$ not nilpotent. After conjugation by an element of $G[[t]]$ there is an index j such that $a(t) \in \mathfrak{m}_j[[t]]$, where M_j is the standard maximal Levi subgroup obtained by omitting the simple root α_j, and*

$$\operatorname{ad}(a(0)) \text{ is invertible on } \mathfrak{g}/\mathfrak{m}_j, \qquad \kappa(a(0), d\lambda_j) \neq 0. \tag{9.2}$$

Proof. (1) For a coweight ν and $w \in W_G$ the difference $\nu - w\nu$ lies in the coroot lattice $Q^\vee$, and $\sum_w w\nu = 0$ because G is semisimple. Hence $|W_G|\nu = \sum_w (\nu - w\nu) \in Q^\vee$.

(2) In a basis of simple coroots, the matrix of κ on the Cartan subalgebra is the symmetrized Cartan matrix of G. Its determinant is a product of determinants of Cartan matrices of the simple factors of G and of ratios of squared lengths of roots. For a standard Levi subgroup, the matrix of κ on the span of its simple coroots is obtained in the same way from the Cartan matrix of its subdiagram, with the lengths of the roots inherited from G. All the primes which divide these determinants divide $|W_G|$. Away from these primes the pairing between two opposite root spaces is perfect as well. A Levi subalgebra is the sum of the whole Cartan subalgebra and of pairs of opposite root spaces, so κ is nondegenerate on it. The same argument, with the span of its coroots in place of the Cartan subalgebra, shows that κ is nondegenerate on its derived subalgebra. The orthogonal complement of the derived subalgebra in the Levi subalgebra is the centre: an element orthogonal to all brackets commutes with the Levi subalgebra, by invariance and nondegeneracy. So κ is nondegenerate on the centre. A change of the lattice of cocharacters within the isogeny class changes the indices by divisors of the Cartan determinants, which are invertible in the characteristic considered.

(3) We first treat the constant term. Conjugate $a(0)$ into a Borel subalgebra and write it as $h + n'$ with h in the Cartan subalgebra $\mathfrak{t}$ and n' in the nilpotent radical. Conjugating by root subgroups in order of increasing height removes the components of n' in the root spaces $\mathfrak{g}_\beta$ with $\beta(h) \neq 0$. We obtain $a(0) = h + n$ with n nilpotent and $[h, n] = 0$, and $h \neq 0$ because $a(0)$ is not nilpotent.

Next we show that, after conjugation by an element of the Weyl group, the roots vanishing on h are exactly those of a standard Levi subgroup M_I. This is automatic in characteristic zero; in characteristic p we argue as follows. Let $W_h \subset W_G$ be the stabilizer of h. Its order is prime to p, so averaging over W_h is a projector on $X_*(T) \otimes \mathbb{Z}_{(p)}$. Let Λ_h be its image, the lattice of W_h-invariants; then h, being W_h-invariant, lies in $\Lambda_h \otimes \Bbbk \subset \mathfrak{t}$. If $\beta(h) = 0$, the reflection s_β fixes h, so it belongs to W_h and β vanishes on Λ_h. If $\beta(h) \neq 0$, then β does not vanish on Λ_h, since otherwise it would vanish on $\Lambda_h \otimes \Bbbk$. Choose a rational vector in $\Lambda_h \otimes \mathbb{Q}$ outside the hyperplanes of the roots of the second kind, and move it into the closed dominant chamber by an element of W_G. The roots vanishing on it are those of a standard Levi subgroup M_I, and after the corresponding conjugation they are the roots vanishing on h.

Now $h + n \in \mathfrak{m}_I$, and h lies in the centre of $\mathfrak{m}_I$. The vectors $d\lambda_j$ with $j \notin I$ form a basis of that centre, because $|W_G|$ is invertible. By (2) there is $j \notin I$ with $\kappa(h, d\lambda_j) \neq 0$, and $\kappa(n, d\lambda_j) = 0$ because root spaces are orthogonal to $\mathfrak{t}$. Since $j \notin I$ we have $\mathfrak{m}_I \subset \mathfrak{m}_j$, and every root β outside $\mathfrak{m}_j$ satisfies $\beta(h) \neq 0$. Thus $\mathrm{ad}(h)$ is invertible on $\mathfrak{g}/\mathfrak{m}_j$, and so is $\mathrm{ad}(h + n)$, because $\mathrm{ad}(n)$ is nilpotent and commutes with $\mathrm{ad}(h)$. This proves (9.2) for the constant term.

Finally we treat the whole series. The conjugation map $G \times^{M_j} \mathfrak{m}_j \to \mathfrak{g}$ has differential $(v, m) \mapsto [v, a(0)] + m$ at the point $(1, a(0))$, which is an isomorphism from $\mathfrak{g}/\mathfrak{m}_j \oplus \mathfrak{m}_j$ to $\mathfrak{g}$ by the invertibility just proved. The map is therefore étale there, and Hensel's lemma lifts $a(t)$ uniquely to a $\Bbbk[[t]]$-point of $G \times^{M_j} \mathfrak{m}_j$ with the given reduction. Its first coordinate is a $\Bbbk[[t]]$-point of G/M_j; since M_j is smooth it lifts to $G[[t]]$, and conjugating by this element puts $a(t)$ in $\mathfrak{m}_j[[t]]$. □

9.3 The Hecke correspondence near the chosen modification

Fix a geometric point of U in a fibre C of $\mathcal{C}$, with local coordinate t, a G-bundle E_0 on C with a Higgs field θ, and a trivialization of E_0 near the point in which $\theta = a(t)\, dt$ satisfies (9.2) for some j. Let E_1 be the modification of E_0 at the point defined by t^{λ_j} in this trivialization.

Since λ_j is central in M_j and $a(t) \in \mathfrak{m}_j[[t]]$, conjugation by t^{λ_j} fixes $a(t)$, so θ is also a Higgs field θ' on E_1. We call $h_* = (E_0, E_1)$, with its leg, the *chosen modification*. It lies in the open stratum of the Hecke correspondence bounded by λ_j, whose fibres over either end have dimension $a = \langle 2\rho_G, \lambda_j \rangle$, where $2\rho_G$ is the sum of the positive roots.

The Hecke correspondence is a correspondence between stacks. We describe it near h_* by schemes, in the following form.

Definition 9.5 (Hecke charts)**.** A *Hecke chart* at the chosen modification consists of

- two schemes Y_0 and Y_1 with smooth maps to $\mathcal{B}$, both smooth of relative dimension n over S_1;
- an étale map $M \to Y_0 \times Y_1 \times U$ and a closed subscheme $\Gamma \subset M$, with projections h_0, h_1 and leg to Y_0, Y_1 and U;
- an étale map over $Y_1 \times U$ from Γ to the open stratum of the pullback to $Y_1 \times U$ of the Hecke correspondence bounded by λ_j, under which the first end of the modification attached to a point h of Γ is the bundle of $h_0(h)$;
- a point of Γ over the chosen modification, again denoted by h_*.

The Higgs fields θ and θ' define covectors on Y_0 at $h_0(h_*)$ and on Y_1 at $h_1(h_*)$, by pullback from $\mathcal{B}$. The following properties are required.

1. The scheme Γ is smooth over S_1, of codimension $c = n - a$ in M, and its projections to $Y_0 \times U$ and to $Y_1 \times U$ are smooth of relative dimension a.
2. There is a covector ξ at the leg such that $(\theta, -\theta', -\xi)$ is conormal to Γ at h_*. The two projections from the conormal bundle $N^*\Gamma$ to $T^*Y_0 \times U$ and to $T^*Y_1 \times U$, which forget the covector at the other end and the covector at the leg, are étale at this point.
3. The covector ξ is not zero.

Lemma 9.6 (Hecke charts of framed bundles)**.** *There are a divisor Q on $\mathcal{C}$, disjoint from the leg, and a Hecke chart (Y_0, Y_1, M, Γ) at the chosen modification in which Y_0 and Y_1 are étale charts of the stack of G-bundles with a trivialization along Q. Its covector at the leg is*

$$\xi = \pm\kappa(a(0), d\lambda_j)\, dx, \tag{9.3}$$

where x is the coordinate of the leg. The charts and the divisor Q can be chosen so that properties (1)–(3) hold in a neighbourhood of h_* in the family over S_1.

Proof. Steps 1 and 2 construct the charts and describe their tangent spaces at the chosen modification. The content is in step 3: the condition that the Higgs field remains regular when the modification moves is expressed by the pairing (9.4), and this pairing is nondegenerate because of (9.2). Steps 4 and 5 deduce the properties (2) and (3) of a Hecke chart.

Step 1: charts. A bound on the relative position bounds the poles of a modification, in a faithful representation V, by a multiple mx of the leg. Choose Q disjoint from the leg and so large that

$$H^0\big(\mathrm{Hom}(V(E_0), V(E_1))(mx - Q)\big) = 0, \qquad H^0\big(\mathrm{End}(V(E_i))(-Q)\big) = 0 \quad (i = 0, 1).$$

The second vanishing says that a bundle with a trivialization along Q has no nontrivial automorphisms, so the framed bundles near E_i form an algebraic space, of which we take étale charts Y_i. The modifications considered identify the two framings along Q, so each framing determines the other. When the two framed ends and the leg are fixed, an infinitesimal

variation of the modification is, in the faithful representation, a section of the first vector bundle of the display. This bundle has no sections, so the map from the Hecke correspondence to $Y_0 \times Y_1 \times U$ is unramified at h_*. An unramified morphism becomes a closed immersion after an étale base change of its target near the point [The26, Tag 0G1D]: there are an étale map $M \to Y_0 \times Y_1 \times U$ and a closed immersion into M of a neighbourhood Γ of h_* which lifts the given map. By semicontinuity the two vanishings hold in a neighbourhood. The projections of the open stratum to either end and the leg are smooth with fibres the open orbit of dimension a in the affine Grassmannian. So Γ is smooth of dimension $n+1+a$ and of codimension $(2n+1)-(n+1+a) = n-a$. Finally, a framing of the second end of a modification determines a framing of the first. So the pullback to $Y_1 \times U$ of the Hecke correspondence is the pullback of the correspondence between framed bundles, and the scheme of which Γ is an étale neighbourhood is its base change along the étale chart Y_0. Hence Γ is étale over this pullback, with first end the bundle of h_0. These are the data of a Hecke chart, with property (1).

Step 2: the local model. All tangent spaces are taken over the residue field of the leg. Put $\mathfrak{g}_{\mathrm{in}} = \mathfrak{g}[[t]]$, $\mathfrak{g}_{\mathrm{out}} = \mathrm{Ad}(t^{\lambda_j})\mathfrak{g}_{\mathrm{in}}$ and $\mathfrak{i} = \mathfrak{g}_{\mathrm{in}} \cap \mathfrak{g}_{\mathrm{out}}$, and let $\mathfrak{g}_b$ be the subspace of $\mathfrak{g}$ on which λ_j has weight b. Then

$$\mathfrak{g}_{\mathrm{in}}/\mathfrak{i} = \bigoplus_{b>0} \mathfrak{g}_b[t]/t^b\mathfrak{g}_b[t], \qquad \mathfrak{g}_{\mathrm{out}}/\mathfrak{i} = \bigoplus_{b<0} t^b\mathfrak{g}_b[t]/\mathfrak{g}_b[t].$$

The first space is the tangent space at h_* to the modifications of E_0 at the fixed leg, and the second is the tangent space to the modifications of E_1. Both have dimension a, and the pairing $(v_+, v_-) \mapsto \operatorname{Res} \kappa(v_+, v_-)\, dt$ between them is perfect.

Step 3: the Hessian. Fix the end E_0, its Higgs field θ and the leg, and move the modification by v_+. To first order the modified bundle is changed by conjugation by $1+v_+$, and θ remains regular on it if and only if $[v_+, a(t)]\, dt$ has no polar part with respect to $\mathfrak{g}_{\mathrm{out}}$. By the duality of step 2 this is the vanishing, for all v_-, of

$$(v_+, v_-) \longmapsto \operatorname{Res} \kappa(a(t), [v_+, v_-])\, dt = \operatorname{Res} \kappa(\mathrm{ad}(a(t))v_+, v_-)\, dt. \tag{9.4}$$

We claim that this pairing is nondegenerate. Since $a(t) \in \mathfrak{m}_j[[t]]$ has weight zero, $\mathrm{ad}(a(t))$ preserves each $\mathfrak{g}_b[t]/t^b$. It is triangular for the filtration by powers of t, with diagonal blocks $\mathrm{ad}(a(0))|_{\mathfrak{g}_b}$, and these are invertible for $b \neq 0$ by (9.2). Hence $\mathrm{ad}(a(t))$ is an automorphism of $\mathfrak{g}_{\mathrm{in}}/\mathfrak{i}$, and the claim follows from the perfect pairing of step 2. The same holds with the two ends exchanged.

To relate this to the global geometry, let $\mathcal{A}$ be the sheaf of infinitesimal automorphisms of the modification h_*; it is the adjoint bundle away from the leg and has stalk $\mathfrak{i}$ there. At h_*, the framed Hecke correspondence with fixed leg has tangent complex $R\Gamma(\mathcal{C}, \mathcal{A}(-Q))[1]$, and under Serre duality on the Gorenstein curve the maps induced on cotangent spaces by the two projections are given by the local spaces of step 2.

Step 4: the projections are étale. Consider the projection $N^*\Gamma \to T^*Y_0 \times U$. Its source and target are smooth of the same dimension $2n+1$, so it is enough to show that its tangent map at the chosen point is injective. A tangent vector in the kernel fixes E_0, the covector θ and the leg. What remains are the a tangent directions of the modification and the c directions in the fibre of the conormal bundle. Because the projection $\Gamma \to Y_1 \times U$ is smooth, the differentials in the directions of Y_0 of the c equations of Γ are linearly independent; equivalently, the map from a fibre of the conormal bundle to the covectors at the first end is injective. Now let a tangent vector in the kernel move the modification by v_+. Along it the covector θ at E_0 stays in the image of the conormal bundle, which means that θ continues to extend to the

moved second end, to first order. By step 3 this forces $v_+ = 0$. The vector then lies in a fibre of the conormal bundle and has zero image in the covectors at the first end, so it is zero by the injectivity just noted. The argument for the other projection is the same with the ends exchanged. This proves (2), given that $(\theta, -\theta')$ extends to a conormal vector, which is the statement that θ and θ' agree away from the leg.

Step 5: the covector at the leg. Moving the leg from 0 to x replaces t^{λ_j} by $(t-x)^{\lambda_j}$, whose logarithmic derivative in x at $x = 0$ is $-d\lambda_j/t$. Pairing with $\theta = a(t)\,dt$ gives $\mp \operatorname{Res} \kappa(a(t), d\lambda_j)\,dt/t = \mp\kappa(a(0), d\lambda_j)$, which is not zero by (9.2). This proves (3) and (9.3). □

Theorem 9.2 concerns an arbitrary smooth chart of $\mathcal{B}$. After pullback to the stack of framed bundles such a chart is not étale but smooth, of some relative dimension d. A smooth morphism need not be a product in any étale neighbourhood, but it admits, locally, an étale map to the product of its target with $\mathbb{A}^d$. The next lemma shows that this is enough: a Hecke chart can carry d additional parameters, which a modification leaves unchanged.

Lemma 9.7 (Hecke charts with parameters)**.** *Let (Y_0, Y_1, M, Γ) be a Hecke chart at the chosen modification, and let $(\varpi, w) : Y_0^+ \to Y_0 \times \mathbb{A}^d$ be an étale map with a point y_0^+ over $h_0(h_*)$. Put $Y_1^+ = Y_1 \times \mathbb{A}^d$, let M^+ be the pullback of M to $Y_0^+ \times Y_1^+ \times U$, and let $\Gamma^+ \subset M^+$ be the part of the inverse image of Γ on which the coordinate w' of the factor $\mathbb{A}^d$ of Y_1^+ is equal to $w(y)$, where y is the first end. Then $(Y_0^+, Y_1^+, M^+, \Gamma^+)$, with the point $h_*^+ = (h_*, y_0^+)$, is a Hecke chart at the chosen modification, of relative dimension $n + d$ and with the same covector at the leg.*

Proof. The maps $Y_0^+ \to Y_0 \to \mathcal{B}$ and $Y_1^+ \to Y_1 \to \mathcal{B}$ are smooth. The map $M^+ \to M \times \mathbb{A}^d \times \mathbb{A}^d$ defined by w and w' is étale, because (ϖ, w) is, and Γ^+ is the pullback along it of the product of Γ with the diagonal of $\mathbb{A}^d$. Hence Γ^+ is closed in M^+ and étale over $\Gamma \times \mathbb{A}^d$, which is étale over the pullback to $Y_1^+ \times U$ of the Hecke correspondence; the first end of a modification is the bundle of $\varpi(y)$, as required. The points of Γ^+ are the pairs of a point h of Γ and a point of Y_0^+ over $h_0(h)$, and h_*^+ is one of them.

Consequently Γ^+ is smooth of codimension $c + d = (n + d) - a$ in M^+. Its projection to $Y_0^+ \times U$ is a base change of the projection of Γ to $Y_0 \times U$, and its projection to $Y_1^+ \times U$ is the composite of the étale map to $\Gamma \times \mathbb{A}^d$ with the product of the projection of Γ to $Y_1 \times U$ and the identity of $\mathbb{A}^d$. Both are smooth of relative dimension a. This is (1).

The conormal bundle of Γ^+ is the pullback of the product of $N^*\Gamma$ with the conormal bundle of the diagonal, and the latter maps isomorphically to the cotangent bundle of either factor $\mathbb{A}^d$. The pullbacks of θ and θ' to Y_0^+ and Y_1^+ vanish on the directions of $\mathbb{A}^d$, so $(\theta, -\theta', -\xi)$ is conormal to Γ^+ at h_*^+, with the same ξ. The two projections from $N^*\Gamma^+$ to $T^*Y_0^+ \times U$ and to $T^*Y_1^+ \times U$ are étale at every point lying over a point at which the projections from $N^*\Gamma$ are étale. This gives (2) and (3). □

9.4 Isolating one stalk in a Hecke fibre

The hypothesis (9.1) is about the Hecke transform of $\mathcal{K}$, that is, about an integral of $\mathcal{K}$ over all modifications with a given second end. The theorem is about $\mathcal{K}$ itself, near one point and in the codirection of one test function f. This subsection passes from the integral to one of its terms. We introduce a function F of the two ends of a modification and of c auxiliary variables. On Γ it is f of the first end, and at the base point its differential in the direction of the second end and the leg is (θ', ξ). Because $\xi \neq 0$, the integral over the modifications which end at the base point has zero specialization cone. Among these modifications the chosen one is isolated in the locus where the integrand can have a nonzero cone, because the

projection from the conormal bundle of Γ to $T^*Y_1 \times U$ is étale there. So the term of the chosen modification has zero cone by itself (Lemma 9.8). The next subsection identifies this term with the specialization cone of $\mathcal{K}$ on the hypersurface $f = 0$.

In this subsection and the next, (Y_0, Y_1, M, Γ) is a Hecke chart at the chosen modification, of relative dimension n, and $c = n - a$. Let f be a function on Y_0, and let $y_0 = h_0(h_*)$, a point of the special fibre of Y_0 lying over E_0. Assume that

$$f(y_0) = 0, \qquad df(y_0) = \theta,$$

where θ is the covector on Y_0 defined by the Higgs field. Write $B = Y_1 \times U$, and let $x : M \to Y_0$ and $b : M \to B$ be the two components of the étale map $M \to Y_0 \times B$. Choose equations $g_1, \dots, g_c$ for Γ in M near h_*. On $M \times \mathbb{A}^c$ put

$$F(m, \zeta) = f(x(m)) - \sum_{j=1}^{c} \zeta_j g_j(m).$$

Here $x(m)$ is the image of m in Y_0. It is the first end of a modification ending at $b(m)$ exactly when m lies in Γ. Since M is étale over $Y_0 \times B$, a tangent vector at m is a pair of tangent vectors at $x(m)$ and at $b(m)$. By Definition 9.5(2) the covector $(\theta, -\theta', -\xi)$ is conormal to Γ at h_*, so there is a unique ζ_* with $\sum_j \zeta_{*,j}\, dg_j = (\theta, -\theta', -\xi)$ at h_*. We write m_* for the image of h_* in M, b_* for $b(m_*)$ and b' for (m_*, ζ_*). Then $F(b') = 0$ and

$$d_x F = 0, \qquad d_\zeta F = 0, \qquad d_b F = (\theta', \xi) \qquad \text{at } b'. \tag{9.5}$$

Let $q : \overline{\mathsf{H}} \to B$ be the pullback to B of the Hecke correspondence bounded by λ_j, with its proper projection to the second end and the leg: a point of $\overline{\mathsf{H}}$ over (y_1, u) is a modification at u, bounded by λ_j, whose second end is the bundle of y_1. Let $\bar{h}_0 : \overline{\mathsf{H}} \to \mathcal{B}$ be the first end, and let A be the Satake kernel of V_{λ_j} on $\overline{\mathsf{H}}$. The first end of a modification is a point of the stack $\mathcal{B}$ and not of the chart Y_0; but Γ is an étale neighbourhood of h_* in $\overline{\mathsf{H}}$, and on it $\bar{h}_0$ is the composite of $h_0 : \Gamma \to Y_0$ with the chart $Y_0 \to \mathcal{B}$. Pulling back along $V(F) \to B$ gives a proper map

$$q' : \overline{\mathsf{H}} \times_B V(F) \longrightarrow V(F),$$

and we call the pullback of $\bar{h}_0^* \mathcal{K} \otimes A$ to its source the *integrand*. The pushforward of the integrand along q' is the pullback of $\mathsf{H}_{V_{\lambda_j}}(\mathcal{K})$ to $V(F)$.

Lemma 9.8 (The isolated contribution)**.** *Assume* (9.1)*, with $\mathcal{K}$ and all $\mathcal{K}_{j,a}$ ULA over S_1. Then the specialization cone of the integrand has zero stalk at the point (h_*, b').*

Proof. Let $\mathcal{D}$ be the specialization cone of the integrand.

Step 1: the cohomology of $\mathcal{D}$ on the fibre of q' over b' vanishes. By (9.5) and $\xi \neq 0$, the differential of F at b' is not zero in the direction of the leg. Hence the projection $V(F) \to Y_1$ is smooth near b'. The pullback of $\mathcal{K}_{j,a}$ along a smooth map is ULA over S_1, and it stays so after tensoring with the pullback of the lisse sheaf $\mathcal{E}_{j,a}$. By (9.1), the pullback of the Hecke transform to $V(F)$ therefore has zero specialization cone at b'. Since Δ commutes with the proper pushforward q'_*,

$$R\Gamma(q'^{-1}(b'), \mathcal{D}) = 0. \tag{9.6}$$

This statement concerns the whole fibre, which is the closed Schubert variety of modifications ending at b_*; it has no analogue for an open part of the fibre.

Step 2: near h_*, the restriction of $\mathcal{D}$ to the fibre is supported at h_*. Since Δ commutes with étale pullback, we may compute near h_* on the étale neighbourhood Γ of h_* in $\overline{\mathsf{H}}$. The kernel A is lisse of rank one there. Up to this factor the integrand is the pullback of $\mathcal{K}$ along

$$\Gamma \times_B V(F) \longrightarrow Y_0, \qquad (h, m, \zeta) \longmapsto h_0(h).$$

Since the pullback of $\mathcal{K}$ to the smooth chart Y_0 is ULA, the cone $\mathcal{D}$ vanishes at every point where this map is smooth. Its source is the hypersurface $F(m,\zeta)=0$ in $(\Gamma\times_B M)\times\mathbb{A}^c$, a scheme which is étale over $\Gamma\times Y_0\times\mathbb{A}^c$; so the map is smooth at a point unless dF vanishes there on all tangent vectors with $dh_0=0$. At a point (h,b') of the fibre over b', the derivatives of F in x and ζ are those of (9.5) and vanish. The remaining tangent vectors are those of Γ at h with $dh_0=0$, on which dF is the covector (θ',ξ). Hence, if the cone is not zero at (h,b'), then (θ',ξ) vanishes on $\ker(dh_0)\subset T_h\Gamma$; in other words, $(\theta_0,-\theta',-\xi)$ is conormal to Γ at h for some covector θ_0 at $h_0(h)$.

Let $\mathcal{Z}$ be the set of these points h; it is closed in the fibre. Since the projection of Γ to $Y_0\times U$ is smooth, dh_0 is surjective, so θ_0 is determined by h, and it depends continuously on $h\in\mathcal{Z}$. Thus $h\mapsto(h;\theta_0,-\theta',-\xi)$ is a continuous section of $N^*\Gamma$ over $\mathcal{Z}$. Its values lie in the fibre of the projection $N^*\Gamma\to T^*Y_1\times U$ over the fixed point defined by (b_*,θ') and the leg, and its value at h_* is the point $(h_*;\theta,-\theta',-\xi)$. By Definition 9.5(2) this projection is étale at that point, which is therefore isolated in its fibre. Hence h_* is isolated in $\mathcal{Z}$. As $\Gamma\to\overline{\mathsf{H}}$ is étale, hence open, there is a neighbourhood $\mathcal{N}$ of h_* in $q'^{-1}(b')$ such that $\mathcal{D}|_{\mathcal{N}}$ is supported at h_*.

Step 3: conclusion. Put $F_*=q'^{-1}(b')$. By step 2 the complex $\mathcal{D}|_{F_*}$ vanishes on $\mathcal{N}-\{h_*\}$. So it is the direct image of its restriction to the union of the two disjoint closed subsets $\{h_*\}$ and $F_*-\mathcal{N}$, and it is the direct sum of a complex supported at h_*, with stalk $\mathcal{D}_{(h_*,b')}$, and of a complex supported away from h_*. Thus $\mathcal{D}_{(h_*,b')}$ is a direct summand of (9.6) and vanishes. □

9.5 Removing the auxiliary variables

Lemma 9.8 is a statement about the hypersurface $F=0$ in $(\Gamma\times_B M)\times\mathbb{A}^c$. We now show that this hypersurface differs from $V(f)$ by a nondegenerate quadratic form in additional variables, and that such a form does not affect the specialization cone.

Lemma 9.9 (Hyperbolic coordinates)**.** *Near (h_*,b') there are functions $r_1,\dots,r_n$ and $s_1,\dots,s_n$ on $(\Gamma\times_B M)\times\mathbb{A}^c$, vanishing at (h_*,b'), such that the map*

$$(\Gamma\times_B M)\times\mathbb{A}^c\longrightarrow(Y_0\times U)\times\mathbb{A}^{2n},\qquad(h,m,\zeta)\longmapsto\big(h_0(h),\mathrm{leg}(h),r,s\big),$$

is étale at (h_*,b') and

$$F=f(h_0(h))+\sum_{i=1}^{n}r_is_i. \tag{9.7}$$

Proof. Choose étale coordinates $x_1,\dots,x_n$ on Y_0 near y_0 and put $s_i=x_i(x(m))-x_i(h_0(h))$. Because the x_i are étale coordinates, the locus $s=0$ coincides near the point with the locus $x(m)=h_0(h)$, which inside $\Gamma\times_B M$ is $\Gamma\times_{Y_0\times B}M$. This scheme is étale over Γ, and the closed immersion $\iota:\Gamma\to M$ defines a section of it, which is an open immersion. So near (h_*,b') the functions s_i generate the ideal of the graph $m=\iota(h)$, the variable ζ being free. A function that vanishes on the graph is therefore a combination of the s_i: there are functions F_i and G_{ji} with

$$f(x(m))-f(h_0(h))=\sum_i s_iF_i,\qquad g_j(m)=\sum_i s_iG_{ji};$$

the second function vanishes on the graph because $g_j(\iota(h))=0$ for $h\in\Gamma$. With $r_i=F_i-\sum_j\zeta_jG_{ji}$, substitution into the definition of F gives (9.7) as an identity of functions. We differentiate (9.7) with respect to the étale coordinate x_i, with h fixed: on the locus $s=0$ the function r_i is the partial derivative of F in x_i; in particular r vanishes at (h_*,b') by (9.5).

It remains to show that the map is étale. Its source and target are smooth over S_1, of relative dimensions

$$\dim\big((\Gamma \times_B M) \times \mathbb{A}^c\big) = (n+1+a) + n + (n-a) \qquad \text{and} \qquad (n+1) + 2n,$$

which are equal, so it suffices to show that a tangent vector at (h_*, b') killed by the differential of the map is zero. Such a vector has $dh_0 = 0$ and $d\,\mathrm{leg} = 0$, and then $ds = 0$ gives $dx = 0$. What remains are the a directions of Γ in the fibre of (h_0, leg) and the c directions of ζ. On the locus $s = 0$, to which the vector is tangent, r_i is the partial derivative of F in x_i, that is, the i-th component of the covector $df - \sum_j \zeta_j d_x g_j$ at the point $h_0(h)$ of Y_0. The first term does not change along the remaining directions, since they fix $h_0(h)$. The derivative of r along them is therefore, up to sign, the tangent map of the projection from $N^*\Gamma$ to the covector at the first end, with the first end and the leg fixed. That map is injective by Definition 9.5(2), so the vector is zero. □

Lemma 9.10 (Quadratic stabilization)**.** *Let Z be a scheme smooth over S_1, let f be a function on Z, and let $\mathcal{K}$ be a complex on Z that is ULA over S_1. Let $\mathcal{K}_m$ be the pullback of $\mathcal{K}$ to the hypersurface*

$$\mathcal{Q}_m = \Big\{ f + \sum_{i=1}^{m} r_i s_i = 0 \Big\} \subset Z \times \mathbb{A}^{2m},$$

and identify $V(f)$ with the locus $r = s = 0$ in $\mathcal{Q}_m$. Then

$$\Delta_{Q_m}(\mathcal{K}_m)|_{V(f)_s} \simeq \Delta_{V(f)}(\mathcal{K}|_{V(f)})(-m)[-2m].$$

In particular, if $\Delta_{Q_m}(\mathcal{K}_m)$ has zero stalk at a point $z \in V(f)_s$, then so does $\Delta_{V(f)}(\mathcal{K}|_{V(f)})$.

Proof. By induction it is enough to treat $m = 1$: the function $f + \sum_{i<m} r_i s_i$ on $Z \times \mathbb{A}^{2m-2}$ and the pullback of $\mathcal{K}$, which is again ULA, reduce the case of m to that of $m-1$.

For $m = 1$, compactify the fibres of $\mathcal{Q}_1 \to Z$ to the family of conics

$$\bar{q} : \overline{\mathcal{Q}} = \{rs + fw^2 = 0\} \subset \mathbb{P}^2_Z \longrightarrow Z,$$

so that $\mathcal{Q}_1$ is the open subscheme $w = 1$. This family is the pullback along $f : Z \to \mathbb{A}^1$ of the universal family $\{rs + tw^2 = 0\} \subset \mathbb{P}^2 \times \mathbb{A}^1$. The total space of the universal family is smooth, and the map $[r : s : w] \mapsto [r : w] = [-tw : s]$ identifies it with the blowup of $\mathbb{P}^1 \times \mathbb{A}^1$ at the point $([0:1], 0)$. The formulas for the cohomology of a blowup and of a projective bundle [Rio14, Theorem 1.2 and Proposition 2.2.2.1], proper base change along f and the projection formula give

$$R\bar{q}_* \bar{q}^* \mathcal{K} \simeq \mathcal{K} \oplus \mathcal{K}(-1)[-2] \oplus i_*(\mathcal{K}|_{V(f)})(-1)[-2],$$

where $i : V(f) \to Z$ is the inclusion. The decomposition is constructed on the universal family, so it requires no regularity of $V(f)$.

Apply Δ_Z, which commutes with $R\bar{q}_*$ and with i_*. On the right the first two summands vanish because $\mathcal{K}$ is ULA, and the third becomes $i_*\Delta_{V(f)}(\mathcal{K}|_{V(f)})(-1)[-2]$. On the left we get $R\bar{q}_*\Delta_{\overline{\mathcal{Q}}}(\bar{q}^*\mathcal{K})$. The map $\bar{q}$ is smooth outside the locus $r = s = 0$ over $V(f)$, which lies in the chart $w = 1$ and is the copy of $V(f)$ in the statement. So $\Delta_{\overline{\mathcal{Q}}}(\bar{q}^*\mathcal{K})$ is supported on that locus, on which $\bar{q}$ restricts to the isomorphism onto $V(f)$. Comparing the two sides gives the assertion. □

Proof of Theorem 9.2. Step 1: the smooth case. If df_y does not vanish on the tangent space of the fibre of π through y, then $V(f) \to \mathcal{B}$ is smooth near y, and $\pi^*\mathcal{K}|_{V(f)}$ is ULA near y as a smooth pullback of $\mathcal{K}$.

Step 2: reduction to a chart with parameters. Otherwise $df_y = d\pi^*\theta$ for a Higgs field θ on the bundle $E_0 = \pi(y)$, and θ is not nilpotent by hypothesis. Some invariant polynomial of θ is then a nonzero section of a line bundle on the reduced curve, so it is nonzero at a point u of the dense open U. Apply Lemma 9.4(3) to the germ of θ at u; this gives j and the chosen modification h_*, and Lemma 9.6 gives Q and a Hecke chart $(Y_0', Y_1', M', \Gamma')$ of framed bundles.

Let $\mathcal{B}_Q \to \mathcal{B}$ be the stack of bundles with a trivialization along Q; it is smooth over $\mathcal{B}$, and $Y_0' \to \mathcal{B}_Q$ is étale. The fibre product $Y \times_{\mathcal{B}} Y_0'$ is therefore smooth over Y, and it is smooth over Y_0', of some relative dimension d, because π is smooth. Choose a point z of it over y and over the first end of the chosen point of Γ'. By the local structure of smooth morphisms, a neighbourhood Y_0 of z admits an étale map to $Y_0' \times \mathbb{A}^d$ over Y_0'. By Lemma 9.7, Y_0 is the first scheme of a Hecke chart (Y_0, Y_1, M, Γ) at the chosen modification, in which $h_0(h_*) = z$. Since Δ commutes with smooth pullback, it suffices to prove the theorem for the chart Y_0, the pullback of f and the point $y_0 = z$. At that point the pullback of f vanishes, and its differential is the pullback of θ from $\mathcal{B}$, because df_y is. These are the hypotheses of Section 9.4 for this Hecke chart.

Step 3: conclusion. By Lemma 9.8, the specialization cone of the integrand vanishes at (h_*, b'). Near that point the integrand is, up to a lisse factor of rank one, the pullback of $\mathcal{K}$ along h_0 to the hypersurface $F = 0$ in $(\Gamma \times_B M) \times \mathbb{A}^c$. By Lemma 9.9 this hypersurface is étale, near (h_*, b'), over the hypersurface $\{f + \sum r_i s_i = 0\} \subset (Y_0 \times U) \times \mathbb{A}^{2n}$, and on the latter the complex is pulled back along the projection to Y_0. The point (h_*, b') maps to the point with $r = s = 0$ over (y_0, u), where u is the leg of h_*. Since Δ commutes with étale pullback, Lemma 9.10, applied with $Z = Y_0 \times U$ and the pullback of $\mathcal{K}$, which is ULA, shows that the specialization cone of the pullback of $\mathcal{K}$ to $V(f) \times U$ vanishes at (y_0, u). As $V(f) \times U \to V(f)$ is smooth, the same holds on $V(f)$ at y_0. □

Corollary 9.11 (Local acyclicity along a transverse test function)**.** *For the family of Theorem 8.8 over the base S, let $Y \to \mathrm{Bun}_G(\mathcal{C}_S)$ be a smooth chart and let $f : Y \to \mathbb{A}^1_S$ be a smooth function whose differential lies outside the pullback of the nilpotent cone except along a section x_*. Then, for every perfect coefficient P, the pullback of $\mathcal{F}_P$ to Y is ULA over $\mathbb{A}^1_S$ away from x_*.*

Proof. By the valuative criterion [HS23, Corollary 3.10] it suffices to test universal local acyclicity after base change along maps $\operatorname{Spec} V \to \mathbb{A}^1_S$ from absolutely integrally closed valuation rings of rank one. If the coordinate of $\mathbb{A}^1$ pulls back to $t \in V$, the base change of Y is the hypersurface $f - t = 0$ in Y_V. The Hecke identities (9.1) hold for $\mathcal{F}_P$ over S with the ULA complexes $\mathcal{F}_{P \otimes W_a}$ and lisse factors on U° (Lemma 6.6 and Theorem 8.8), and they survive base change to V, as does the density of the open set of legs in every component. Theorem 9.2, applied to the function $f - t$ on Y_V, gives the vanishing of the specialization cone at every point outside x_*. The maps $\operatorname{Spec} V \to \mathbb{A}^1_S$ include those that change t and the characteristic at once, and those that lie in a single fibre of S. □

9.6 The nilpotent cone at the canonical bundle

Let C be a fibre of our family of curves, with dualizing line bundle Ω_C, and let $(L, L^2 \simeq \Omega_C)$ be the theta pair that is part of the data of the family (Section 6). Fix a pinning of G and put

$$E_* = \lambda_{\mathrm{pr}}(L) \times^T G,$$

where λ_{pr} is the sum of the positive coroots. A root of height h contributes the line bundle Ω_C^h to $\operatorname{ad} E_*$, so the pinned generators of the negative simple root spaces define sections of $\operatorname{ad} E_* \otimes \Omega_C$ that vanish nowhere. Their sum is a Higgs field θ_* on E_* that is regular nilpotent at every point. We call (E_*, θ_*) the *canonical point* of the nilpotent cone.

Proposition 9.12 (The nilpotent cone at the canonical point)**.** *Suppose that Ω_C has positive degree on every component of C. Near (E_*, θ_*) the global nilpotent cone is smooth and Lagrangian, relative to the base. If $m_1, \dots, m_r$ are the exponents of G, its tangent space at (E_*, θ_*) is*

$$\bigoplus_i H^1(C, \Omega_C^{-m_i}), \qquad \text{of rank} \quad \sum_i \operatorname{rk} H^1(C, \Omega_C^{-m_i}) = \dim G\,(g(C) - 1).$$

The pair (E_*, θ_*) has no infinitesimal automorphisms, and these statements commute with base change.

Proof. Step 1: deformations. The centralizer of a regular nilpotent element is a smooth commutative group scheme [Ric17, Proposition 4.4.1], and the principal grading acts on its Lie algebra with weights $-2m_i$. In characteristic zero the second assertion is classical: for the element f of a principal $\mathfrak{sl}_2$-triple, the Lie algebra is a sum of irreducible representations with highest weights $2m_i$, and the centralizer of f is spanned by their lowest weight vectors. In characteristic p we deduce it as follows. Let $A = \mathbb{Z}[1/|W_G|]$; the primes which are not very good divide $|W_G|$, so the results of [Ric17, §4] hold over A. Let f be the sum of the root vectors of the pinning for the negative simple roots. Its centralizer is smooth over A, by [Ric17, Proposition 4.4.1] applied with the opposite Borel subgroup, so the Lie algebra of the centralizer is a finite locally free A-module. The principal cocharacter acts on f by t^{-2}, hence normalizes the centralizer and acts on this module. The weight spaces are direct summands, and their ranks are constant because $\operatorname{Spec} A$ is connected. So the weights are the same in characteristic p as in characteristic zero. If the group is semisimple but not simply connected, these results are applied to its simply connected cover: under our hypothesis on the characteristic the cover is a central étale isogeny, it induces an isomorphism of Lie algebras, and the inverse image of the centralizer of an element of the Lie algebra is the centralizer of the same element. Twisting by E_* gives a smooth commutative group scheme J over C with $\operatorname{Lie} J = \bigoplus_i \Omega_C^{-m_i}$. Because the orbit map of the regular nilpotent orbit is smooth, every regular nilpotent Higgs field near θ_* is locally conjugate to θ_*, and the pairs (E, θ) with θ regular nilpotent near (E_*, θ_*) are the J-torsors near the trivial one. Their automorphisms, deformations and obstructions are H^0, H^1 and H^2 of $\operatorname{Lie} J$. Here $H^0 = 0$ by the positivity of Ω_C and $H^2 = 0$ because C is a curve. Hence H^1 is locally free and commutes with base change, and the space of such pairs is smooth with the stated tangent space. Since being regular is an open condition and C is proper, every nilpotent Higgs field near θ_* is regular nilpotent, so this space is a neighbourhood of (E_*, θ_*) in the nilpotent cone. The rank is computed by Riemann–Roch and $\sum_i (2m_i + 1) = \dim G$.

Step 2: the Lagrangian property. Let (E, θ) be a point of the germ. By step 1 it is locally isomorphic to (E_*, θ_*), and a tangent vector to the germ at this point is represented by a cocycle γ with values in the centralizer of θ, which is locally $\operatorname{Lie} J$. The Liouville form, evaluated on this tangent vector, is the Serre duality pairing of γ with θ, that is, the class of the cocycle $\kappa(\theta, \gamma)$ with values in Ω_C. Locally θ has principal weight -2 and γ has weights $-2m_i < 0$, while κ pairs only opposite weights; so $\kappa(\theta, \gamma) = 0$. Thus the Liouville form vanishes at every point of the germ, and therefore so does the symplectic form.

The rank computed in step 1 is $\dim G\,(g-1)$, the dimension of the stack of bundles. Now let $h : Y \to \operatorname{Bun}_G$ be a smooth chart of relative dimension r, for instance a chart of framed bundles. The pullback of the germ to T^*Y consists of the covectors $dh^*\theta$ at the points of Y over the germ. It is smooth of dimension $\dim G\,(g-1) + r = \dim Y$, half the dimension of T^*Y, the r directions of the fibres of h being included. The Liouville form of T^*Y vanishes on it: evaluated on a tangent vector it is the pairing of θ with the image of that vector in the tangent space of Bun_G, which is the Liouville form of $T^* \operatorname{Bun}_G$ on a vector tangent to the

germ, and that is zero by what was just shown. So the pullback of the germ to every smooth chart is Lagrangian; this is the form in which the statement is used. □

9.7 A homogeneous test function defined over $\mathbb{F}_q$

We now construct a chart at the canonical point and a test function on it. Two requirements shape the construction. The function must be homogeneous for a $\mathbb{G}_m$-action, because Section 10 uses the action to compare vanishing cycles at different points of S. And the chart, its distinguished point and the function must be defined over $\mathbb{F}_q$, because the final trace is taken over that field.

Work first over the complete local base $S_0 = \operatorname{Spec} R_0[[\varepsilon]]$, whose closed point s_0 has residue field $\mathbb{F}_q$; the objects over S are obtained by pullback. Choose a divisor Q in the smooth locus of the curve over S_0, so large that $H^0(C, \operatorname{End}(V(E_*))(-Q)) = 0$ for a faithful representation V; by the theorem on cohomology and base change this holds on every fibre. Bundles near E_* with a trivialization along Q then have no nontrivial automorphisms, and since deformations of bundles on a curve are unobstructed they form a smooth algebraic space $\mathscr{X}_{\mathrm{fr}}$ over S_0, which is quasi-separated with affine diagonal. The divisor Q is finite over the local base S_0, so L is trivial along Q; a trivialization of L along Q gives one of E_*, through its torus reduction, and hence a section $c : S_0 \to \mathscr{X}_{\mathrm{fr}}$.

Changing the trivialization of a bundle along Q by the constant element $\lambda_{\mathrm{pr}}(t)$ defines an action of $\mathbb{G}_m$ on $\mathscr{X}_{\mathrm{fr}}$. It does not change the underlying bundle, and it fixes c, because $\lambda_{\mathrm{pr}}(t)$ is an automorphism of E_*. This automorphism multiplies θ_* by t^{-2}. So θ_*, as a covector at c, is an eigenvector of the action, of weight 2 or -2 according to the conventions for induced actions. We fix the direction of the action, replacing t by t^{-1} if necessary, so that the weight is two in the following sense: on a chart with an action, the differential at a fixed point of a function of weight two has the same weight as θ_*. Finally, let $\mathbb{G}_m$ act on the cotangent bundle by the induced action composed with that scaling of the fibres, by t^2 or t^{-2}, which fixes θ_*. This action preserves the graph of the differential of every function of weight two. It preserves the pullback of the nilpotent cone, which is stable under the induced action, since that action does not change the underlying bundle, and under scaling. The symplectic form is homogeneous for it, and we normalize the grading it induces on tangent spaces at fixed points so that the symplectic form has weight two.

Lemma 9.13 (An equivariant chart through the canonical point)**.** *There are a $\mathbb{G}_m$-equivariant étale morphism $u : Y \to \mathscr{X}_{\mathrm{fr}}$ with Y affine over S_0, and a $\mathbb{G}_m$-fixed section $x_* : S_0 \to Y$ with $u \circ x_* = c$, such that $x_*(s_0)$ is an $\mathbb{F}_q$-rational point. After replacing Y by a $\mathbb{G}_m$-stable affine open subscheme containing x_*, there are homogeneous étale coordinates on Y that vanish along x_*.*

Proof. Step 1: the chart. By [AHR26, Corollary 10.2], applied to the fixed point $c(s_0)$ of the action of the group $\mathbb{G}_m$ of multiplicative type on $\mathscr{X}_{\mathrm{fr}}$, which is locally of finite presentation, there is a $\mathbb{G}_m$-equivariant étale map $u : (Y, y_0) \to (\mathscr{X}_{\mathrm{fr}}, c(s_0))$ with Y affine, inducing an isomorphism of residue fields and of stabilizers at y_0. So y_0 is an $\mathbb{F}_q$-rational fixed point.

Step 2: the section. Form the fibre product

$$\begin{array}{ccc} V = S_0 \times_{\mathscr{X}_{\mathrm{fr}}} Y & \xrightarrow{v} & Y \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle u} \\ S_0 & \xrightarrow[c]{} & \mathscr{X}_{\mathrm{fr}}. \end{array}$$

The map p is étale, V is affine because the diagonal of $\mathscr{X}_{\mathrm{fr}}$ is affine, and y_0 defines a rational point of the closed fibre of p. Since S_0 is henselian, there is a unique section h of p through this point [The26, Tag 04GR]. Put $x_* = v \circ h$.

Step 3: the section is fixed. Let $\mathbb{G}_m$ act trivially on S_0. Then V inherits an action over S_0. The geometric fibres of p are discrete and $\mathbb{G}_m$ is connected, so the action is trivial on every geometric fibre. The locus in $\mathbb{G}_m \times_{S_0} V$ where the action map agrees with the projection is open, because V/S_0 is étale, and it contains every geometric point. So the action on V is trivial, and h, hence x_*, is fixed.

Step 4: homogeneous coordinates. Let $\mathcal{I}$ be the ideal of $x_*(S_0)$ in Y. The conormal module $\mathcal{I}/\mathcal{I}^2$ is free of finite rank over the local ring of S_0 and is graded by the weights of $\mathbb{G}_m$. Lift a homogeneous basis to homogeneous elements of $\mathcal{I}$; this is possible because taking a homogeneous component commutes with the surjection $\mathcal{I} \to \mathcal{I}/\mathcal{I}^2$. These functions define an equivariant map from Y to an affine space with a linear action, which is étale along x_*. The locus where it is not étale is closed and $\mathbb{G}_m$-stable. Take a function vanishing on that locus and nonzero at y_0; its component of weight zero has the same value at y_0 and still vanishes on the locus, since the ideal of a stable closed subscheme is graded. Inverting it gives a stable affine open subscheme containing x_* on which the map is étale. □

The symplectic vector space in the next proof is the tangent space E of the cotangent bundle of Y at (x_*, θ_*), with its grading by the weights of $\mathbb{G}_m$. It contains two graded Lagrangian subspaces: the tangent space Λ_{vert} of the cotangent fibre, and the tangent space Λ_{nilp} of the nilpotent cone, which is Lagrangian by Proposition 9.12. The graph of the differential of a function f with $df(x_*) = \theta_*$ is a Lagrangian complement of Λ_{vert}, and it is graded if f is homogeneous of weight two. We need it to be a complement of Λ_{nilp} as well.

Lemma 9.14 (A transverse homogeneous test function)**.** *After replacing Y by its product with a graded vector space and by a $\mathbb{G}_m$-stable affine open neighbourhood of x_*, there is a function f on Y, homogeneous of weight two and smooth over the base, such that the graph of df meets the pullback of the nilpotent cone exactly along x_*, and transversally:*

$$\operatorname{graph}(df) \cap \mathrm{Nilp}_Y = x_*(S_0). \tag{9.8}$$

Proof. Step 1: linear algebra. Put $I = \Lambda_{\mathrm{vert}} \cap \Lambda_{\mathrm{nilp}}$ and choose a graded decomposition

$$\mathsf{E} = (I \oplus I^\vee(2)) \oplus (U_+ \oplus U_-), \qquad \Lambda_{\mathrm{nilp}} = I \oplus U_+, \qquad \Lambda_{\mathrm{vert}} = I \oplus U_-,$$

in which the two summands in parentheses are orthogonal, U_+ and U_- are Lagrangian in the second, and $I^\vee(2)$ is an isotropic graded complement of I in the first. Such a decomposition exists: graded complements of I in Λ_{nilp} and Λ_{vert} pair perfectly, and a graded complement of I in the orthogonal of $U_+ \oplus U_-$ is made isotropic by subtracting half of its alternating self-pairing.

A common graded Lagrangian complement of U_+ and U_- need not exist; see Remark 9.15. We therefore enlarge the chart. Replace Y by $Y \times U_-^{\mathrm{new}}$, where U_-^{new} is a copy of U_- as a graded vector space. This adds to E the symplectic space $U_-^{\mathrm{new}} \oplus U_+^{\mathrm{new}}$, where U_+^{new} consists of the new cotangent directions; we identify it with a copy of U_+ in such a way that the new symplectic form is the negative of the old one under the two identifications. The vertical subspace becomes $\Lambda_{\mathrm{vert}} \oplus U_+^{\mathrm{new}}$, and the tangent space of the pulled-back nilpotent cone becomes $\Lambda_{\mathrm{nilp}} \oplus U_-^{\mathrm{new}}$. Consider

$$\Lambda = I^\vee(2) \oplus \{u_+ + u_+^{\mathrm{new}}\} \oplus \{u_- + u_-^{\mathrm{new}}\},$$

where $u_\pm$ runs through $U_\pm$ and $u_\pm^{\mathrm{new}}$ is its copy. It is graded, and it is isotropic because the two symplectic forms cancel on the last two summands; it has half the dimension, so it is Lagrangian. Comparing components shows that it meets both $\Lambda_{\mathrm{nilp}} \oplus U_-^{\mathrm{new}}$ and $\Lambda_{\mathrm{vert}} \oplus U_+^{\mathrm{new}}$ in zero.

Step 2: the function. As a Lagrangian complement of the vertical subspace, Λ is the graph of a symmetric bilinear form on the tangent space of the enlarged chart at x_*, homogeneous of weight two. This holds on the closed fibre; lift the form to a homogeneous symmetric form in the homogeneous coordinates of Lemma 9.13 over S_0, and let q be half the associated quadratic form. Let e_0 be the linear function of weight two in these coordinates with $de_0 = \theta_*$ along x_*, and put $f = e_0 + q$. Then $df = \theta_*$ along x_* and the tangent space of $\mathrm{graph}(df)$ at $x_*(s_0)$ is Λ. The determinant expressing transversality to the nilpotent cone is a unit at s_0, hence a unit over the local base S_0.

Step 3: shrinking. Let $Z_f \subset Y$ be the closed subscheme where $\mathrm{graph}(df)$ meets Nilp_Y. By transversality it is étale over S_0 along x_*, and f is smooth there because $\theta_* \neq 0$. We shrink Y twice. First we remove the closed subset where Z_f is not étale over S_0 or f is not smooth; it is stable under the action and does not meet $x_*(S_0)$. After this Z_f is étale and separated over S_0, so the section x_* is open and closed in it, and we remove $Z_f - x_*(S_0)$, which is again closed and stable. Each time the result is an affine open subset, stable under the action, which contains the whole section: if J is the homogeneous ideal of the subset to be removed and $\mathcal{I}$ the ideal of the section, then $J + \mathcal{I} = \mathcal{O}_Y$, so J contains an element equal to 1 along the section; its component of degree zero has the same property, and we invert it. □

Remark 9.15 (Why the chart is enlarged)**.** In a symplectic plane with weights 0 and 2, the two weight lines are the only graded Lagrangian subspaces, so they have no common graded complement. After one coordinate is added, the quadratic form xy of weight two has Hessian determinant -1 and supplies one. The enlargement in Lemma 9.14 does this for all weights at once. It depends only on the chart, so the same function f serves for every coefficient P.

Pulling back along $S \to S_0$ gives the chart, the section x_* and the test function f used in Section 10. By construction their special fibres and the point $x_*(s_0)$ are defined over $\mathbb{F}_q$.

9.8 The microstalk on the generic fibre

On the generic fibre the curve is smooth and proper over a field of characteristic zero, and the vanishing cycles of f at x_* can be computed.

Proposition 9.16 (The microstalk is the Whittaker coefficient)**.** *On the geometric generic fibre there is an isomorphism of functors on* $\mathrm{Perf}(\mathfrak{F}_\rho)$*,*

$$(R\Phi_f \mathcal{F}_P)_{x_*}[-1] \simeq \Gamma(\mathfrak{F}_\rho, P) \otimes L_{\mathrm{shift}}, \tag{9.9}$$

where L_{shift} *is a graded line that does not depend on* P*.*

The proof is given at the end of the section. It has four ingredients, of different kinds, which we state as separate lemmas: a passage from étale sheaves to sheaves in the classical topology (Lemma 9.17); the theorem of Nadler and Taylor (Input 4); a computation of vanishing cycles of a homogeneous function (Lemmas 9.19, 9.20 and 9.21); and the comparison of two kernels by a calculation with D-modules (Lemma 9.22).

Both sides of (9.9) are composites of $P \mapsto \mathcal{F}_P$ with a functor on constructible sheaves with nilpotent singular support on $\mathrm{Bun}_G(C)$, where C is the generic fibre, a smooth proper curve over the algebraically closed field K_{geom} of characteristic zero: on the left the functor $\mu = (R\Phi_f(-))_{x_*}[-1]$, on the right, by (6.2), the functor $\mathrm{Wh} = \mathrm{RHom}(\mathrm{Poinc}_!^{\mathrm{Vac}}, -)$. So

the proposition says that $\mu \simeq \mathrm{Wh} \otimes L_{\mathrm{shift}}$. Vanishing cycles are normalized by the triangle $i^* \to R\Psi \to R\Phi$.

The functors μ and Wh make sense in three settings: for étale sheaves over K_{geom}, for étale sheaves over $\mathbb{C}$, and for sheaves of k-vector spaces in the classical topology. We write μ^{cl} and $\mathrm{Wh}^{\mathrm{cl}}$ for the functors in the third setting.

Lemma 9.17 (Reduction to the classical topology)**.** *Choose an embedding of K_{geom} in $\mathbb{C}$. If there is an isomorphism $\mu^{\mathrm{cl}} \simeq \mathrm{Wh}^{\mathrm{cl}} \otimes L_{\mathrm{shift}}$ of functors on sheaves of k-vector spaces with nilpotent singular support, in the classical topology, on the stack of bundles of the complex curve $C \otimes_{K_{\mathrm{geom}}} \mathbb{C}$, then there is an isomorphism $\mu \simeq \mathrm{Wh} \otimes L_{\mathrm{shift}}$ of functors on étale sheaves with nilpotent singular support on $\mathrm{Bun}_G(C)$.*

Proof. An embedding exists, because K_{geom} has the cardinality of the continuum.

The form of the two functors. Let $\pi : Y \to \mathrm{Bun}_G(C)$ be the chart, a smooth map from a scheme of finite type. Then $\mu(\mathcal{F})$ is, up to a shift, the stalk at the rational point x_* of the vanishing cycles of $\pi^*\mathcal{F}$ for the function f on Y. The object $\mathrm{Poinc}_!^{\mathrm{Vac}}$ is the direct image with compact supports, along a map p_{W} to $\mathrm{Bun}_G(C)$ from a stack of finite type, of a complex $\mathcal{K}_{\mathrm{W}}$ obtained by the six operations from constant sheaves [GR25, §1.1.1][GR24, §3.3]; so $\mathrm{Wh}(\mathcal{F}) = \mathrm{RHom}(\mathcal{K}_{\mathrm{W}}, p_{\mathrm{W}}^! \mathcal{F})$. Thus each of the two functors is a pullback to a stack of finite type, followed by operations on that stack: the functor RHom out of a fixed complex in one case, vanishing cycles at a point of a scheme in the other.

From K_{geom} to $\mathbb{C}$. For stacks of finite type over the field K_{geom}, the six operations on constructible complexes commute with the base change to $\mathbb{C}$: for direct images this is the generic base change theorem over the base $\mathrm{Spec}\, K_{\mathrm{geom}}$ [Del77, Th. finitude, Théorème 1.9]; inverse images, direct images with compact supports and tensor products commute with every base change; and the extraordinary inverse image and the internal Hom are reduced, by a decomposition into strata on which the complexes are lisse, to direct images along immersions and to smooth pullbacks. Moreover the pullback functor is fully faithful on constructible complexes, the cohomology of such a complex being unchanged by an extension of algebraically closed fields. Vanishing cycles commute with the change of trait from the henselization of $\mathbb{A}^1_{K_{\mathrm{geom}}}$ at the origin to that of $\mathbb{A}^1_{\mathbb{C}}$ [Del77, Th. finitude, Proposition 3.7]. Singular supports are compatible with extension of the base field: for torsion coefficients this is [Bei16, Theorem 1.4(iii)], and for coefficients in an algebraic extension of $\mathbb{Q}_\ell$ it is [Bar24, Theorem 1.5(xi)]. We apply this on smooth charts of finite type; the nilpotent cone is also compatible with the extension of the base field, so nilpotence is preserved.

From étale sheaves to the classical topology. Over $\mathbb{C}$ there is a fully faithful functor from constructible étale k-sheaves to constructible sheaves of k-vector spaces in the classical topology, which commutes with the direct and inverse images of both kinds [GR25, §2.2.1][BBD82, §6.1.2] and with vanishing cycles [DK73, Exposé XIV, Théorème 2.8], and which preserves nilpotence of the singular support on $\mathrm{Bun}_G(C)$ [GR25, §2.2.2]. Being fully faithful, it preserves RHom. It sends the complex $\mathcal{K}_{\mathrm{W}}$ to the complex defined by the same recipe.

Conclusion. Let an denote the composite of the two functors of comparison, from nilpotent étale sheaves on $\mathrm{Bun}_G(C)$ to nilpotent sheaves in the classical topology on the stack of bundles of the complex curve. By the two paragraphs above there are isomorphisms of functors $\mu \simeq \mu^{\mathrm{cl}} \circ \mathrm{an}$ and $\mathrm{Wh} \simeq \mathrm{Wh}^{\mathrm{cl}} \circ \mathrm{an}$. They are isomorphisms of functors on the homotopy categories which commute with the shift, and this is all we need (Remark 6.8). Composing the given isomorphism with an gives the assertion. □

The comparison with the classical topology is used only to produce this isomorphism of functors. The action of Frobenius is brought in afterwards, on the étale side (Proposition 10.9, step 3).

From here to the proof of Proposition 9.16, C is a smooth proper complex curve, and all sheaves are sheaves of k-vector spaces in the classical topology. Let $i : \mathrm{Bun}_N^\omega \to \mathrm{Bun}_G$ be the stack of N-bundles twisted by the torus bundle of E_*, and let $f_N : \mathrm{Bun}_N^\omega \to \mathbb{A}^1$ be the sum of the coordinates on the simple root spaces. The one-parameter group λ_{pr} acts on Bun_N^ω, fixing E_* and contracting to it, and f_N has weight two. So f_N descends to

$$\bar{f} : \mathrm{Bun}_N^\omega /\mathbb{G}_m \longrightarrow [\mathbb{A}^1/\mathbb{G}_m],$$

where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ with weight two. Let $i_{\mathbb{G}_m} : \mathrm{Bun}_N^\omega /\mathbb{G}_m \to \mathrm{Bun}_G$ be the map induced by i, let $j : B\mu_2 \to [\mathbb{A}^1/\mathbb{G}_m]$ be the open point, and let $k[\mu_2]$ be the regular representation.

Two steps below identify the cohomology of a space which an action contracts to a point with the stalk at that point. We state this once, in the form in which it is used.

Lemma 9.18 (Contraction by an action with positive weights)**.** *Let $\mathbb{R}_{>0}$ act on $E = \mathbb{C}^n$ by*

$$r \cdot (z_1, \dots, z_n) = (r^{w_1} z_1, \dots, r^{w_n} z_n)$$

with integers $w_i > 0$, and let $\mathcal{F}$ be a bounded constructible complex on E whose cohomology sheaves are locally constant along the orbits. Then the restriction map

$$R\Gamma(E, \mathcal{F}) \longrightarrow \mathcal{F}_0$$

is an isomorphism. More generally, let $f : E \to \mathbb{C}$ be a polynomial which is homogeneous of weight $d > 0$ for this action, let $\epsilon > 0$, and let $U_\epsilon \subset E$ be the open subset where $|f| < \epsilon$. Then the restriction map $R\Gamma(U_\epsilon, \mathcal{F}) \to \mathcal{F}_0$ is an isomorphism.

These are restriction maps. So they are natural in $\mathcal{F}$, and compatible with the restriction from U_ϵ to $U_{\epsilon'}$ for $\epsilon' < \epsilon$.

Proof. Write $z_i = x_i + \sqrt{-1}\, y_i$, and let $v = \sum_i w_i(x_i\partial_{x_i} + y_i\partial_{y_i})$ be the vector field of the action. Away from the origin the action is free, and locally E is the product of an interval of an orbit and a transversal. The cohomology sheaves of $\mathcal{F}$ are locally constant along the orbits, so over $E \setminus \{0\}$ the microsupport of $\mathcal{F}$ consists of covectors which vanish on v [KS85, Proposition 4.1.2(ii)]. At the origin $v = 0$.

Choose an integer $D > 0$ divisible by all the w_i, and put

$$\rho(z) = \sum_i |z_i|^{2D/w_i}.$$

This is a polynomial in the x_i and y_i. It is proper, it vanishes only at the origin, and $d\rho(v) = 2D\rho$. So away from the origin $d\rho$ does not vanish on v, and the conormal directions of the level sets of ρ do not meet the microsupport of $\mathcal{F}$. The open sets $E_s = \{\rho < s\}$, $s > 0$, are relatively compact, increase with s and exhaust E. By the theorem on noncharacteristic deformations [KS85, Theorem 4.4.1(1)] the restriction maps

$$R\Gamma(E, \mathcal{F}) \longrightarrow R\Gamma(E_s, \mathcal{F})$$

are isomorphisms. The sets E_s form a basis of neighbourhoods of the origin, so the colimit of the right-hand side as $s \to 0$ is $\mathcal{F}_0$. This proves the first assertion.

For the second, replace ρ by the function

$$\rho_\epsilon = \rho + \frac{|f|^2}{\epsilon^2 - |f|^2}$$

on U_ϵ. It is smooth and vanishes only at the origin. The set where $\rho_\epsilon \leq s$ is compact: on it $\rho \leq s$ and $|f|^2 \leq s\epsilon^2/(1+s) < \epsilon^2$. Since $|f|^2$ is homogeneous of weight $2d$,

$$d\rho_\epsilon(v) = 2D\rho + \frac{2d\,\epsilon^2|f|^2}{(\epsilon^2 - |f|^2)^2},$$

which is positive away from the origin. The set U_ϵ is not stable under the whole action, but the cohomology sheaves of $\mathcal{F}$ are locally constant along the parts of the orbits which lie in it, and the statement about the microsupport uses nothing more. The argument then continues as before, with the sets $\{\rho_\epsilon < s\}$. □

Lemma 9.19 (Vanishing cycles on the line with the action of weight two)**.** *For a constructible complex $\mathcal{G}$ on $[\mathbb{A}^1/\mathbb{G}_m]$, with pullback $\mathcal{G}'$ to $\mathbb{A}^1$, there is an isomorphism*

$$(R\Phi_t\mathcal{G}')_0[1] \simeq \mathrm{RHom}(j_*k[\mu_2][-1], \mathcal{G}), \tag{9.10}$$

natural in $\mathcal{G}$.

Proof. Complexes on this stack split according to the two characters of the central subgroup μ_2.

For the trivial character we are on the quotient $\mathcal{Q}_1$ of $\mathbb{A}^1$ by the action of weight one, with open point $j_1 : \mathrm{pt} \to \mathcal{Q}_1$, closed point $i_1 : B\mathbb{G}_m \to \mathcal{Q}_1$ and atlas $a : \mathbb{A}^1 \to \mathcal{Q}_1$, and we must show that $(R\Phi_t\mathcal{G}')_0 \simeq \mathrm{RHom}(j_{1,*}k, \mathcal{G})$. With our normalization, the stalk of the vanishing cycles is the cone of the specialization map from the stalk $\mathcal{G}'_0$ to the stalk $\mathcal{G}'_1$ at the point 1; the latter is the stalk of the nearby cycles, because $\mathcal{G}'$ is equivariant. Both stalks are corepresentable functors of $\mathcal{G}$. Indeed $\mathcal{G}'_1 = \mathrm{RHom}(j_{1,!}k, \mathcal{G})$; and $\mathcal{G}'_0 = R\Gamma(\mathbb{A}^1, \mathcal{G}')$ by Lemma 9.18, and this is $\mathrm{RHom}(a_!k[2], \mathcal{G})$ because a is smooth of relative dimension one. By the Yoneda lemma the specialization map is induced by a morphism $\mathrm{sp} : j_{1,!}k \to a_!k[2]$, and the stalk of the vanishing cycles is $\mathrm{RHom}(\mathcal{T}, \mathcal{G})$, where $\mathcal{T}$ is the fibre of sp. It remains to see that $\mathcal{T} \simeq j_{1,*}k$. For $\mathcal{G} = j_{1,*}k$ the vanishing cycles are a line in degree zero, and $\mathrm{RHom}(\mathcal{T}, j_{1,*}k) = \mathrm{RHom}(j_1^*\mathcal{T}, k)$; so $j_1^*\mathcal{T} \simeq k$. The restriction $i_1^*\mathcal{T}$ is $i_1^*a_!k[1] = p_!k[1]$, where $p : \mathrm{pt} \to B\mathbb{G}_m$, and $i_1^*j_{1,*}k = p_*k$. These two objects are isomorphic: for a torsor under $\mathbb{G}_m$, both are the cone of the Euler class $k[-2] \to k$. The endomorphisms of degree zero of p_*k are the scalars, since $p^*p_*k = k \oplus k[-1]$. So the map $i_1^*\mathcal{T} \to i_1^*j_{1,*}j_1^*\mathcal{T}$ that glues $\mathcal{T}$ from its two restrictions is either an isomorphism, in which case $\mathcal{T} \simeq j_{1,*}k$, or zero, in which case $\mathcal{T} \simeq j_{1,!}k \oplus i_{1,*}i_1^*\mathcal{T}$. The second case is excluded by taking $\mathcal{G} = j_{1,!}k$: its vanishing cycles are a line, whereas RHom from the direct sum to $j_{1,!}k$ has total dimension three.

For the nontrivial character, the stalk and the costalk at the origin vanish, because the central μ_2 acts trivially on every complex on $B\mathbb{G}_m$; so $\mathcal{G}$ is the direct image of its restriction V to the open point, and both sides of (9.10) are $V[1]$. □

Lemma 9.20 (A linear chart of the stack of twisted unipotent bundles)**.** *There are an affine space with a linear action of $\mathbb{G}_m$, all of whose weights are positive, and a map from it to Bun_N^ω, equivariant for the action of λ_{pr}, which is a torsor under a unipotent group and maps the origin to E_*.*

Proof. Let N^ω be the group scheme over the curve obtained by twisting N by the fixed torus bundle, so that Bun_N^ω classifies its torsors. Filter N by the height of the roots. The successive quotients of N^ω are central, and they are the direct sums of the line bundles $\mathcal{L}_\alpha$ attached to the roots α of a given height. Choose an effective divisor Q so large that $H^0(\mathcal{L}_\alpha(-Q)) = 0$ for all positive roots α, and let $\mathrm{Bun}_{N,Q}^\omega$ classify the torsors with a trivialization along Q. Lifting a torsor with its trivialization from one quotient of the filtration to the next meets no obstruction, since H^2 vanishes on a curve; the lifts have no automorphisms; and they form a

torsor under the vector group $\bigoplus_\alpha H^1(\mathcal{L}_\alpha(-Q))$, the sum being over the roots of the given height. So $\mathrm{Bun}^\omega_{N,Q}$ is obtained from a point by successive torsors under vector groups. The group λ_{pr} acts on all these data, and on the vector group of the roots of height h with weight $2h$. A torsor under a vector group over an affine scheme is trivial, and it has a trivialization invariant under λ_{pr}: its sections form an affine space under a rational representation of $\mathbb{G}_m$, and $\mathbb{G}_m$ is linearly reductive. By induction $\mathrm{Bun}^\omega_{N,Q}$ is an affine space with a linear action of positive weights, whose origin is the trivial torsor. The map $\mathrm{Bun}^\omega_{N,Q} \to \mathrm{Bun}^\omega_N$ which forgets the trivialization is a torsor under the group of sections of N^ω over Q, a unipotent group. It commutes with the action of λ_{pr}, and it maps the origin to the point of Bun^ω_N defined by the trivial torsor, whose image in Bun_G is E_*. □

Lemma 9.21 (Vanishing cycles of the homogeneous function)**.** *For a constructible complex $\mathcal{F}$ on Bun_G there is an isomorphism*

$$(R\Phi_{f_N} i^! \mathcal{F})_{E_*}[-1] \simeq \mathrm{RHom}(\mathcal{P}_1, \mathcal{F}), \qquad \mathcal{P}_1 = i_{\mathbb{G}_m,!}\, \bar{f}^*(j_* k[\mu_2])[-1],$$

natural in $\mathcal{F}$.

Proof. Put $\mathcal{J} = i^!_{\mathbb{G}_m} \mathcal{F}$, a complex on the quotient. Its pullback $\mathcal{J}'$ to Bun^ω_N is $i^!\mathcal{F}[-2]$, because the quotient map is smooth of relative dimension one. By adjunction and Lemma 9.19,

$$\mathrm{RHom}(\bar{f}^* j_* k[\mu_2][-1], \mathcal{J}) \simeq \big(R\Phi_t(f_{N,*}\mathcal{J}')\big)_0[1],$$

the stalk at the origin of the vanishing cycles of the direct image of $\mathcal{J}'$ under f_N; here smooth base change identifies the pullback of $\bar{f}_*\mathcal{J}$ with $f_{N,*}\mathcal{J}'$.

The map f_N is not proper, and vanishing cycles do not commute with a direct image which is not proper. What makes the comparison possible here is the contraction: we claim that

$$\big(R\Phi_t(f_{N,*}\mathcal{J}')\big)_0 \simeq (R\Phi_{f_N}\mathcal{J}')_{E_*},$$

naturally in $\mathcal{J}'$. Both sides are cones of specialization maps. On the left the stalk of $f_{N,*}\mathcal{J}'$ at 0 is the cohomology of the inverse image of a small disc, which the action contracts to E_*; so it is $\mathcal{J}'_{E_*}$, by Lemma 9.18 applied on the chart of Lemma 9.20. The nearby stalk is the cohomology of the whole fibre $f_N^{-1}(1)$, because over the punctured line f_N is a product, after a covering of degree two, by homogeneity and equivariance. On the right the stalk is again $\mathcal{J}'_{E_*}$, and the nearby stalk is the cohomology of a Milnor fibre, which the action identifies with the part of $f_N^{-1}(1)$ where a norm ν is less than a radius that tends to infinity as the value of f_N tends to 0. So the claim amounts to this: for R large the restriction

$$R\Gamma(f_N^{-1}(1), \mathcal{J}') \longrightarrow R\Gamma(f_N^{-1}(1) \cap \{\nu < R\}, \mathcal{J}')$$

is an isomorphism.

We prove this on the chart of Lemma 9.20. The map from the chart to Bun^ω_N is smooth, so nearby cycles commute with pullback along it, and its fibres are contractible, so for a complex pulled back from Bun^ω_N it preserves the cohomology of the fibre of f_N and of the inverse image of the disc. On the affine space, ν is the maximum of the functions $|x_i|^{1/w_i}$, where $w_i > 0$ is the weight of the coordinate x_i. It is a proper semialgebraic function, and $\mathcal{J}'$ is constructible for a finite algebraic stratification. We claim that the complex $R\nu_*\mathcal{J}'$ on the half-line is locally constant on an interval (R_0, ∞). Put $u = \nu/(1+\nu)$, a function with values in $[0,1)$, and let Γ be the graph of u on the fibre $f_N^{-1}(1)$, as a subset of $\mathbb{P} \times \mathbb{R}$, where $\mathbb{P}$ is the projective space which contains the affine space. It is a semialgebraic set with compact closure. So the extension by zero of $\mathcal{J}'$ from Γ to $\mathbb{P} \times \mathbb{R}$ is an $\mathbb{R}$-constructible complex with compact support, and its direct image on $\mathbb{R}$ is $\mathbb{R}$-constructible [KS85, Proposition 8.3.1]. In

particular this direct image is locally constant on an interval $(1-\delta, 1)$. Since ν is proper, the points of the closure of Γ which do not belong to Γ lie over $u = 1$; so over $[0, 1)$ the direct image is $Ru_*\mathcal{J}'$, and the claim follows. Covering the half-line by $[0, R)$ and (R_0, ∞) now shows that the restriction to $\nu < R$ is an isomorphism for $R > R_0$.

The isomorphism so obtained is natural in $\mathcal{J}'$. For each R the restriction map displayed above is a natural transformation between functors of $\mathcal{J}'$, and these transformations are compatible when R increases. The action identifies the nearby stalk with the right-hand side for all large R, compatibly with the restriction maps between different radii. So there is one natural transformation from the cohomology of the fibre $f_N^{-1}(1)$ to the nearby stalk; what depends on $\mathcal{J}'$ is only the radius beyond which the individual restriction maps are isomorphisms. Together, and with $\mathcal{J}' = i^!\mathcal{F}[-2]$, this gives the statement. It says that the Whittaker functional is corepresented by the Whittaker sheaf [NT26, Remark 1.1.1]. □

Lemma 9.22 (The kernel of the Poincaré object)**.** *The object $\mathcal{P}_1$ of Lemma 9.21 is isomorphic to* $\mathrm{Poinc}_!^{\mathrm{Vac}}$.

Proof. The object $\mathrm{Poinc}_!^{\mathrm{Vac}}$ is defined in the same way as $\mathcal{P}_1$, with the torus T in place of λ_{pr}, the map $\chi : \mathrm{Bun}_N^{\omega} \to \mathbb{A}^r$ to the simple root spaces in place of $f_N = \mathrm{sum} \circ \chi$, and the kernel $\mathcal{K}_T = j_{\mathrm{reg},*}R_{Z(G)}[-r]$ on $[\mathbb{A}^r/T]$, where r is the rank of G, $j_{\mathrm{reg}} : BZ(G) \to [\mathbb{A}^r/T]$ is the open point and $R_{Z(G)}$ is the direct image with compact supports of k under $\mathrm{pt} \to BZ(G)$ [GR24, §§3.3.3–3.3.7]. By base change along χ it suffices to prove that

$$q_!\,\mathrm{sum}^*(j_*k[\mu_2][-1]) \simeq \mathcal{K}_T, \qquad q : [\mathbb{A}^r/\mathbb{G}_m] \to [\mathbb{A}^r/T]. \tag{9.11}$$

Both sides are obtained by the six operations from constant sheaves, so they are defined with coefficients in $\mathbb{Q}$, on the same analytic stack. For two bounded constructible complexes with coefficients in $\mathbb{Q}$ on $[\mathbb{A}^r/T]$, which has finitely many orbits, the space Hom^0 has finite dimension and commutes with the extension of the coefficients to $\mathbb{C}$. The isomorphisms form a Zariski open subset of this space, defined over $\mathbb{Q}$, because a morphism is an isomorphism if it is one on the stalks at a point of each orbit; and such an open subset has a rational point if it has a complex point. So we may take $k = \mathbb{C}$ and use D-modules. Under the Riemann–Hilbert correspondence $\mathcal{K}_T$ corresponds to the direct image with compact supports, under $\mathbb{A}^r \to [\mathbb{A}^r/T]$, of the pullback of the exponential D-module under the sum [GR24, Remark 3.3.4]. By base change and transitivity of direct images, (9.11) follows once we know that $j_*k[\mu_2][-1]$ corresponds to the direct image $\mathcal{E}_2$ with compact supports of the exponential D-module under $\mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$, for the action of weight two. This direct image is regular holonomic: on the open orbit its pullback by the orbit map $z \mapsto z^2$ is a trivial connection, so it is a sum of regular Kummer connections, and the only other stratum is the origin. For the trivial character of μ_2 we are in the case of weight one, treated in [GR24, Remark 3.3.2]. The part of $\mathcal{E}_2$ with nontrivial character is the direct image of its restriction to the open point, as in the proof of Lemma 9.19. That restriction is the cohomology with compact supports of $\mathbb{G}_m$ with coefficients in the product of the exponential D-module and the local system of order two. It is dual to the cohomology of the complex $h \mapsto h' - (1 + 1/(2x))h$ on Laurent polynomials. The kernel of this operator is zero, as one sees from the term of highest degree. Its cokernel is one-dimensional: the relations $[x^n] = (n - \frac{1}{2})[x^{n-1}]$, for all integers n, show that it is spanned by the class of 1, and that class is not zero, because the linear form $x^n \mapsto c_n$ with $c_0 = 1$ and $c_n = (n - \frac{1}{2})c_{n-1}$ vanishes on the image. So the restriction is the nontrivial character, in the same degree as for the trivial character. This proves (9.11). □

Proof of Proposition 9.16. By Lemma 9.17 it suffices to construct an isomorphism $\mu^{\mathrm{cl}} \simeq \mathrm{Wh}^{\mathrm{cl}} \otimes L_{\mathrm{shift}}$ for a complex curve; we drop the superscripts.

Step 1: the theorem of Nadler and Taylor. By Lemma 9.14 the graph of df meets the nilpotent cone transversally at the smooth point (E_*, θ_*). For such a function the functor of vanishing cycles factors through the category of sheaves localized at this point of the cotangent bundle, which is equivalent to the category of complexes of vector spaces [NT26, Proposition 2.3.1 and its proof]; so μ is the microstalk at (E_*, θ_*), up to a shift. We need the comparison of the functors attached to two transverse functions as an isomorphism of functors, natural in the sheaf, and not only for each sheaf separately. It is. Choose once a contact transformation near the point, with its kernel. The two functors defined by the kernel are inverse to each other [KS85, Theorems 6.3.4 and 6.3.5], and the kernel is determined up to tensor product with an invertible complex of coefficients [KS85, Remark 6.3.6], which over the field k is a graded line; the coefficients in that section are a ring of finite weak global dimension, so k is allowed. After the transformation the functor attached to a transverse function is the tensor product with a graded line which depends only on the function [KS85, Lemmas 7.2.3 and 7.2.4]. The comparison of two transverse functions is then the choice, made once, of an isomorphism between two graded lines. By Input 4, which is [NT26, Theorem 3.3.4], the microstalk at the same point is also computed, up to a shift, by $\mathcal{F} \mapsto (R\Phi_{f_N} i^! \mathcal{F})_{E_*}$.

Step 2: conclusion. By Lemma 9.21 the last functor is, up to a shift, $\mathrm{RHom}(\mathcal{P}_1, -)$, and by Lemma 9.22 this is Wh. So $\mu \simeq \mathrm{Wh} \otimes L_{\mathrm{shift}}$, where the graded line L_{shift} collects the shifts. It does not depend on the sheaf, and the isomorphism is one of functors: the isomorphism of step 1 is one of functors, that of Lemma 9.21 is natural, and Lemma 9.22 is an isomorphism between two objects which corepresent functors, so it induces a natural isomorphism between them whatever isomorphism is chosen. The remaining ambiguity is a scalar, by Lemma 6.7. □

10 From a microstalk to a rational trace

We now have a chart Y at the canonical point, with a fixed section x_* and a test function f of weight two (Section 9.7); we know that the automorphic family is ULA over S and, away from x_*, along f (Corollary 9.11); and we know the vanishing cycles of f at x_* on the generic fibre (Proposition 9.16). This section turns that information into the statement we need, Proposition 10.9: in characteristic p, the member of the family with coefficient P_{reg} has nonzero Frobenius trace at some bundle defined over $\mathbb{F}_q$. There are three steps.

1. *Comparing the two ends of S* (Theorem 10.4). Vanishing cycles along f and specialization over S need not commute. We show that the discrepancy carries the trivial action of the scaling group, so that it disappears on the part of the vanishing cycles on which $-1 \in \mathbb{G}_m$ acts by -1.

2. *Frobenius* (proof of Proposition 10.9). The comparison of step (1) is an isomorphism of functors of the coefficient P. By Lemma 6.7 the two Frobenius structures then differ by one scalar, and the nonzero supertrace of Proposition 6.10 survives.

3. *From vanishing cycles to a stalk* (Theorem 10.7). A nonzero trace on vanishing cycles at x_* is not yet a nonzero value of the trace function. A trace formula for homogeneous functions shows that it forces a nonzero stalk trace at some $\mathbb{F}_q$-rational point of the chart.

The nonzero number which is being followed sits on a different object at each stage:

the trace of Frobenius is known to be nonzero on	by
$\Gamma(\mathfrak{F}_\rho, P_{\mathrm{reg}})$, on the spectral side	Proposition 6.10
the vanishing cycles of f at x_* on the generic fibre, up to one scalar	Proposition 9.16 and step (2)
their minus part on the special fibre	Theorem 10.4
the stalk at a rational point of the chart	Theorem 10.7
the stalk at a bundle defined over $\mathbb{F}_q$	Proposition 10.9

10.1 Scaling actions and the minus part

Let $a : \mathbb{G}_m \times Y \to Y$ be an action. A complex $\mathcal{K}$ on Y is *strongly equivariant* if it is equipped with an isomorphism $a^*\mathcal{K} \simeq \mathrm{pr}_Y^*\mathcal{K}$ that restricts to the identity along the unit section and satisfies the cocycle condition on $\mathbb{G}_m \times \mathbb{G}_m \times Y$. At a fixed point of the action, the group $\mathbb{G}_m$ then acts on the stalk of $\mathcal{K}$.

Setup 10.1. Let S be the spectrum of an absolutely integrally closed valuation ring in which 2 and ℓ are invertible. Let Y/S be separated and of finite presentation, with an action of $\mathbb{G}_m$, a fixed section $x_* : S \to Y$, and a function $f : Y \to \mathbb{A}^1_S$ of weight two that vanishes along x_*. Put $Z = f^{-1}(0)$. Let P be a strongly equivariant complex on Y.

Since f has weight two, the element $\epsilon = -1$ of $\mathbb{G}_m$ preserves f, and it therefore acts on the vanishing cycles of f at the fixed point. For a geometric point b of S define

$$\mu_b(P) = (R\Phi_{f_b} P_b)_{x_{*,b}}[-1], \qquad \mu_b^-(P) = \frac{1-\epsilon}{2}\,\mu_b(P).$$

The coefficients are rational, so the projector makes sense also when $\ell = 2$. The action of ϵ comes from the geometry of Y. The monodromy of the vanishing cycles, which comes from the inertia group at $0 \in \mathbb{A}^1$, is a different action.

10.2 Vanishing cycles at the two ends of the base

The comparison rests on a result of Saito on nearby cycles over a base of dimension two (Input 3), which is stated for noetherian schemes and torsion coefficients. Our base is a valuation ring and our coefficients are rational, so two reductions are needed. The first lemma provides the integral structure, and the second carries out both reductions.

Lemma 10.2 (A lattice for both acyclicity conditions)**.** *Let R be an absolutely integrally closed valuation ring in which ℓ is invertible, and $S = \operatorname{Spec} R$. Let Y be separated and of finite presentation over S, let $x_* : S \to Y$ be a closed section, and let $f : Y \to \mathbb{A}^1_S$. Put $U = Y - x_*(S)$. Let P be a bounded constructible complex with rational coefficients that is ULA over S and such that $P|_U$ is ULA over $\mathbb{A}^1_S$. Then, for a suitable finite extension $E/\mathbb{Q}_\ell$, there is a constructible complex P^+ of $\mathcal{O}_E$-modules with $P^+[1/\ell] \simeq P$ that has the same two properties.*

Proof. Step 1: a lattice on U. The ring R is integrally closed, hence so are $R[t]$ and its localizations. Thus $\mathbb{A}^1_S$ is normal and irreducible, in particular geometrically unibranch. By [HS23, Proposition 6.11], $P|_U$ admits a lattice $\mathcal{L}_U$ that is ULA over $\mathbb{A}^1_S$. Since $\mathbb{A}^1_S \to S$ is smooth, $\mathcal{L}_U$ is also ULA over S [HS23, §3, discussion following Proposition 3.3].

Step 2: extension over the generic fibre. The generic fibre Y_η is a noetherian scheme of finite type over a field. Write $j_\eta : U_\eta \to Y_\eta$ and $i_\eta : x_{*,\eta} \to Y_\eta$ for the inclusions. The complex P_η is determined by $j_\eta^* P_\eta$, by $i_\eta^* P_\eta$, and by the gluing morphism

$$i_{\eta,*} i_\eta^* P_\eta \longrightarrow j_{\eta,!} j_\eta^* P_\eta[1].$$

On the open part take the lattice $j_\eta^*\mathcal{L}_U$ of step 1, and on the closed part take any lattice of $i_\eta^*P_\eta$, a bounded complex of finite free modules. By [HS23, Corollary 2.4], morphisms between the rational complexes are the morphisms between the lattices with ℓ inverted. So after multiplying the lattice on the closed part by a large power of a uniformizer, the gluing morphism is defined integrally. Its fibre is a lattice $\mathcal{L}_\eta$ in P_η whose restriction to U_η is $j_\eta^*\mathcal{L}_U$.

Step 3: extension over S. By [HS23, Theorem 4.1] with integral coefficients, $\mathcal{L}_\eta$ has a unique extension P^+ to Y that is ULA over S. On U, both $P^+|_U$ and $\mathcal{L}_U$ are ULA over S with the same generic fibre, so the full faithfulness in that theorem gives $P^+|_U \simeq \mathcal{L}_U$. Hence $P^+|_U$ is ULA over $\mathbb{A}^1_S$. By the same uniqueness, with rational coefficients, $P^+[1/\ell] \simeq P$. The reductions of P^+ modulo powers of the uniformizer of $\mathcal{O}_E$ retain both properties, by the criterion for change of coefficients in the proof of [HS23, Proposition 3.8]. □

Lemma 10.3 (Successive specializations)**.** *In the situation of Lemma 10.2, assume $f \circ x_* = 0$ and put $Z = f^{-1}(0)$. Choose a specialization $s \leftarrow \eta$ of geometric points of S, and compatible geometric generic points of the strict localizations of $\mathbb{A}^1$ at 0 over s and over η. Let*

$$\mathcal{D}_P = \mathrm{Cone}\left(P_{x_{*,s}} \longrightarrow (R\Psi_{Z/S}(P|_{Z_\eta}))_{x_{*,s}}\right).$$

Then there is a distinguished triangle

$$\mathcal{D}_P[-1] \longrightarrow \mu_s(P) \longrightarrow \mu_\eta(P) \longrightarrow \mathcal{D}_P. \tag{10.1}$$

It is natural in P: the three terms are exact functors on the stable ∞-category of complexes satisfying the hypotheses, and the maps are natural transformations between them.

Proof. Step 1: torsion coefficients on a noetherian model. In the strict localization of $\mathbb{A}^1_S$ at $(0, s)$ consider the four points

$$\begin{array}{ccc} (0,s) & \longleftarrow & (0,\eta) \\ \uparrow & & \uparrow \\ (\bar{t},s) & \longleftarrow & (\bar{t},\eta), \end{array}$$

where $\bar{t}$ denotes the chosen geometric generic points of the line. The locus where P is not ULA over $\mathbb{A}^1_S$ is contained in $x_*(S)$, which is finite over its image, the zero section of $\mathbb{A}^1_S$. On a noetherian model and with torsion coefficients, [Sai15, Proposition 1.8(2), (1.17)] applied along the top and right edges of the square gives a distinguished triangle

$$A_0 \longrightarrow A_1 \longrightarrow \mu_\eta(P)[1], \qquad A_0 = (R\Psi_{Z/S}P|_{Z_\eta})_{x_{*,s}}.$$

Along the left and bottom edges, the generic point of the line does not meet x_*, and the same proposition identifies A_1 with $(R\Psi_{f_s}P_s)_{x_{*,s}}$. Both A_0 and A_1 receive the canonical map from $P_{x_{*,s}}$, compatibly, and the cones of these maps are $\mathcal{D}_P$ and $\mu_s(P)[1]$. The octahedral axiom for the composite $P_{x_{*,s}} \to A_0 \to A_1$, shifted by -1, gives (10.1).

Step 2: descent to a noetherian model. Take the lattice P^+ of Lemma 10.2 and put $P_m = P^+ \otimes^L_{\mathcal{O}_E} \mathcal{O}_E/\varpi^m$. The schemes, the section and f are of finite presentation and descend to a noetherian model; a finite affine cover with its presentations gives a bound on the relative dimension that persists under every further base change. For each fixed m, the complex P_m descends after refining the model, and so do the two ULA conditions, by [HS23, Proposition 3.7] applied to Y/S and to $U/\mathbb{A}^1_S$. This is done for one m at a time, on a model which depends on m.

Step 3: return to S and passage to the limit. Apply step 1 on the model. By the compatibility of Saito's proposition with base change, the resulting terms, pulled back to R, are the nearby

and vanishing cycles of P_m over R. Let d be a common bound for the relative dimensions of the models of $Y \to \mathbb{A}^1_S$ and of $Z \to S$. By base change the stalks of the nearby cycles in the triangle are the cohomology complexes of Milnor fibres. If P^+ has Tor-amplitude in $[a, b]$, these stalks have Tor-amplitude in $[a, b + d]$ and commute with change of coefficients [Sai15, Lemma 1.6(1) and Corollary 1.10(1)]. The cones and shifts in the triangle change the bounds by integers which do not depend on m. The same formula identifies the reduction of each term for $m + 1$ with the term for m, compatibly with the maps, which are the canonical adjunction, specialization and cone maps. Different models give the same maps, by passing to a common refinement. We can therefore take the derived limit over m on the original base and invert ℓ. By the uniform bound the terms of the limit are the nearby cycles of P^+ and, after inverting ℓ, of P; compare [Sai15, Corollary 1.10(1)]. This gives the triangle for P.

Step 4: naturality. The maps in (10.1) are composites of the canonical transformations of the six operations, and these are defined for rational coefficients without reference to a lattice. The lattice was used only to prove that the cones are as stated. The naturality in P follows; in particular, no automorphism of P is required to preserve a lattice. □

Theorem 10.4 (The minus part is the same at both ends)**.** *In Setup 10.1, suppose that P is ULA over S and that $P|_{Y - x_*(S)}$ is ULA over $\mathbb{A}^1_S$. Then for a specialization $s \leftarrow \eta$ of geometric points of S, and choices as in Lemma 10.3, the map of (10.1) restricts to an isomorphism*

$$\mu_s^-(P) \simeq \mu_\eta^-(P),$$

natural in P in the sense of that lemma.

Proof. The triangle (10.1) is compatible with the action of ϵ on its three terms, since ϵ is an automorphism of the data (Y, f, x_*, P). It therefore suffices to show that ϵ acts trivially on the cohomology of $\mathcal{D}_P$: the image of the projector $(1 - \epsilon)/2$ on $\mathcal{D}_P$ is then a direct summand with zero cohomology, hence zero, and applying the projector to the triangle gives the isomorphism.

The subscheme $Z = f^{-1}(0)$ is stable under $\mathbb{G}_m$, and $P|_{Z_\eta}$ is strongly equivariant. Its extension $Rj_{Z,*}(P|_{Z_\eta})$ is ULA over S by [HS23, Theorem 4.1], and the restriction of this extension to Z_s is $R\Psi_{Z/S}(P|_{Z_\eta})$. The action and projection maps $\mathbb{G}_m \times Z \to Z$ are smooth, so the pullbacks of the extension along them are ULA over S; they are identified on the generic fibre by the equivariant structure, and the full faithfulness of [HS23, Theorem 4.1] extends the identification, with its unit and cocycle conditions, over S. Hence $R\Psi_{Z/S}(P|_{Z_\eta})$ is strongly equivariant, compatibly with the map from $P|_{Z_s}$, and $\mathbb{G}_m$ acts on $\mathcal{D}_P$.

For a strongly equivariant complex and a fixed point, the action of $\mathbb{G}_m$ on each cohomology group of the stalk is locally constant on $\mathbb{G}_m$ and is the identity at the unit. As $\mathbb{G}_m$ is connected, every element acts trivially; in particular ϵ does. □

When the data carry a Weil structure that commutes with the scaling action, Frobenius commutes with ϵ and preserves $\mu_s^-(P)$.

The comparison is useful only if the generic microstalk is not killed by the projector. The next lemma arranges this.

Lemma 10.5 (Changing the sign)**.** *Suppose that the generic fibre has characteristic zero and that ϵ acts on the functor μ_η by the scalar $+1$. Replace*

$$(Y, f, P) \quad \text{by} \quad (Y \times \mathbb{A}^1_y,\ f + y^2,\ \mathrm{pr}_Y^* P[1]),$$

where $\mathbb{G}_m$ acts on y with weight one. Then the two ULA conditions of Theorem 10.4 still hold, μ_η is tensored with a fixed graded line, and ϵ acts on it by -1. If f is smooth, so is $f + y^2$.

Proof. The pullback of P is ULA over S as a smooth pullback. For the condition along the new function, cover the complement of $(x_*, 0)$ by two open sets. Where $y \neq 0$, the map $(w, y) \mapsto (w, f(w) + y^2)$ to $Y \times \mathbb{A}^1$ is étale, as 2 is invertible, and the pullback of P to $Y \times \mathbb{A}^1$ is ULA over the second factor. Where $w \neq x_*$, the pullback of P is ULA over $\mathbb{A}^2$ with respect to (f, y) by hypothesis, and the map $(t, y) \mapsto t + y^2$ is smooth.

On the generic fibre, the Thom–Sebastiani theorem [Ill16, Theorem 4.5 and Remark 4.11(a)] identifies the vanishing cycles of $f + y^2$, with the normalization given by the shift, with the local convolution of those of f and those of y^2. The generic fibre lies over an algebraically closed field of characteristic zero. There the monodromy is tame, and local convolution is isomorphic to the tensor product, as a functor of the two complexes, once a compatible system of trivializations of the convolution kernels has been chosen [Ill16, Corollaries 5.25 and 5.26, Remark 5.27]. We choose one. Both functors are defined by kernels, and the isomorphism is an isomorphism between the kernels [Ill16, Proposition 5.18 and Corollary 5.25]; so it applies to complexes as well as to sheaves. The resulting isomorphism is natural in P, which is what the next steps use; its compatibility with Frobenius is not needed, because Frobenius is brought in afterwards (Proposition 10.9, step 3). The nearby fibre of y^2 consists of two points, its reduced cohomology is a line, and $y \mapsto -y$ exchanges the points and acts on the line by -1. This changes the sign of the action of ϵ. Finally df is a component of $d(f + y^2)$, so smoothness is preserved. □

10.3 From vanishing cycles to a rational point

We now work over $\mathbb{F}_q$, and write Fr_q for the q-power map on points and Frob_q for geometric Frobenius on cohomology. The theorem of this subsection must produce a point over $\mathbb{F}_q$ itself; a point over an extension would give an automorphic function over the wrong field.

The proof uses hyperbolic restriction, and at one point exchanges it with the cohomology of an inertia group. We isolate that exchange. It is also the source of the factor $1 - q$ in the trace computation.

Lemma 10.6 (Inertia and hyperbolic restriction)**.** *Let I_{in} be the inertia group of a trait of residue characteristic $p \neq \ell$, and let $\mathcal{A}$ be a bounded constructible complex with a continuous action of I_{in} on a scheme of finite type over $\overline{\mathbb{F}}_q$. Suppose the scheme carries an action of $\mathbb{G}_m$ commuting with that of I_{in} on $\mathcal{A}$, let x be a rational fixed point, and let $\operatorname{Hyp}\mathcal{A}$ be the cohomology with compact supports of the attractor of x, the set of points y with $\lim_{t\to 0} ty = x$, with coefficients in $\mathcal{A}$. Then*

$$\operatorname{Hyp} R\Gamma(I_{\mathrm{in}}, \mathcal{A}) \simeq R\Gamma(I_{\mathrm{in}}, \operatorname{Hyp}\mathcal{A}),$$

compatibly with geometric Frobenius and with any further symmetry of finite order. If I_{in} acts trivially on the cohomology of $\operatorname{Hyp}\mathcal{A}$, then

$$\operatorname{Str}\big(\mathrm{Frob}_q; R\Gamma(I_{\mathrm{in}}, \operatorname{Hyp}\mathcal{A})\big) = (1 - q)\operatorname{Str}\big(\mathrm{Frob}_q; \operatorname{Hyp}\mathcal{A}\big). \tag{10.2}$$

Both statements hold for the eigenspaces of such a symmetry.

Proof. Step 1: the exchange. Work first with torsion coefficients. The action of wild inertia factors, on a finite stratification, through a finite p-group. Since p is invertible in the coefficients, its invariants are the image of the averaging idempotent, and there is no higher cohomology. The part of tame inertia of order prime to ℓ is treated in the same way. These idempotents commute with pullback and with pushforward with compact supports. The remaining quotient of I_{in} is $\mathbb{Z}_\ell(1)$. For a topological generator τ, its continuous cohomology is computed by the complex

$$[\,\mathcal{B} \xrightarrow{\tau - 1} \mathcal{B}\,]$$

in degrees 0 and 1, where $\mathcal{B}$ is the invariant part just described. Hyperbolic restriction is a pullback followed by a pushforward with compact supports, so it commutes with this cone. This gives the comparison for torsion coefficients. For integral and rational coefficients, use the compatible system of torsion complexes: the two geometric operations have finite cohomological dimension and the complex above has length one, so the bounds are uniform and the derived limits give the comparison. A different choice of τ changes the complex but not the functor it computes, so the isomorphism is compatible with the action of Frobenius on I_{in} by conjugation and with commuting symmetries.

Step 2: the trace. There is a spectral sequence, equivariant for Frobenius,

$$H^a(I_{\mathrm{in}}, H^b(\operatorname{Hyp}\mathcal{A})) \Longrightarrow H^{a+b}(R\Gamma(I_{\mathrm{in}}, \operatorname{Hyp}\mathcal{A})).$$

If I_{in} acts trivially on $H^b(\operatorname{Hyp}\mathcal{A})$, its only nonzero columns are $H^b(\operatorname{Hyp}\mathcal{A})$ for $a = 0$ and $H^b(\operatorname{Hyp}\mathcal{A})(-1)$ for $a = 1$. Geometric Frobenius acts on $k(-1)$ by q, and the alternating trace of the abutment equals that of the E_2 page, which is $(1 - q)$ times the alternating trace on $\operatorname{Hyp}\mathcal{A}$. Passing to an eigenspace of a commuting symmetry is another idempotent, to which the same argument applies. □

Theorem 10.7 (A nonzero microstalk trace gives a rational point)**.** *Let $p \neq 2$. Let $Y/\mathbb{F}_q$ be smooth and affine with an action of $\mathbb{G}_m$, let $f : Y \to \mathbb{A}^1$ be smooth of weight two, and let $x_* \in Y(\mathbb{F}_q)$ be a fixed point. Let P be a bounded constructible Weil complex on Y, strongly equivariant compatibly with its Weil structure, such that the vanishing cycles of f on $f^{-1}(0)$ are supported at x_*. If*

$$\operatorname{Str}(\operatorname{Frob}_q; \mu^-(P)) \neq 0,$$

then there is $x \in Y(\mathbb{F}_q)$ at which the stalk of P has nonzero Frobenius supertrace.

The action of Frobenius on vanishing cycles depends on a lift of Frobenius to the Galois group of the generic point of the henselization of $\mathbb{A}^1$ at the origin, and two lifts differ by an element of the inertia group. In the theorem, and wherever Frob_q acts on μ^-, the lift is taken in the subgroup of index two attached to the covering $t = u^2$. Step 3 of the proof shows that the inertia group of that covering acts trivially on the cohomology of $\mu^-(P)$; so the supertrace does not depend on the lift within this subgroup.

The role of the scaling action can be seen on points. In the argument below there appear points x of Y that are fixed not by Fr_q but by $\epsilon \operatorname{Fr}_q$. Choose $a \in \overline{\mathbb{F}}_q^{\times}$ with $a^{q-1} = -1$. Then

$$\operatorname{Fr}_q(ax) = a^q \operatorname{Fr}_q(x) = (-a)(\epsilon x) = ax,$$

so ax is a rational point, and the equivariant structure of P, being compatible with the Weil structure, identifies the trace of $\epsilon \operatorname{Frob}_q$ on the stalk at x with the trace of Frob_q on the stalk at ax. Twisted fixed points thus contribute values of the trace function at rational points, and no extension of $\mathbb{F}_q$ is needed.

Proof. Steps 1 to 4 compute one hyperbolic restriction in two ways. One gives the vanishing cycles at x_*, the other the cohomology with compact supports of the part of a fibre of f which the action contracts to x_*, and the outcome is the equality of supertraces (10.4). This is the content of the proof. Step 5 is then the trace formula for the fibre.

Step 1: a function of weight one. Make the base change $t = u^2$: put $Y' = Y \times_{\mathbb{A}^1_t} \mathbb{A}^1_u$ and $P' = \operatorname{pr}_Y^* P$. For the action $c(x, u) = (cx, cu)$, the function u on Y' has weight one. The involution $\epsilon'(x, u) = (\epsilon x, u)$ preserves u. Since f is smooth, Y' is smooth. The nearby cycles of u for P', as a complex with Weil structure, are those of f for P; only the inertia group is replaced by its subgroup of index two [Ill06, §1.1(b)].

Step 2: hyperbolic restriction. Let $Z = (u = 0)$, with inclusion $i : Z \to Y'$ and complement $j : Y' - Z \to Y'$, and let F be the fixed locus of $\mathbb{G}_m$ in Y'. It lies in Z, because u has weight one. Hyperbolic restriction takes values in complexes on F; the point x_* need not be isolated in F, and stalks at x_* are taken at the end. For $X = Y'$ or $X = Z$, let X^+ and X^- be the attractor and the repeller, with their inclusions $p^\pm : X^\pm \to X$ and the limit maps $q^\pm : X^\pm \to \mathsf{F}$, and put

$$\mathrm{Hyp}_X^+ = q_!^+ (p^+)^*, \qquad \mathrm{Hyp}_X^- = q_*^- (p^-)^!.$$

Braden's theorem gives an isomorphism $\mathrm{Hyp}_X^+ K \simeq \mathrm{Hyp}_X^- K$ of complexes on F for every equivariant complex K on X [Bra03, Theorem 1][Ric19, Theorem 2.6]. Because u has positive weight, the repeller of Y' lies in Z, so that $Z^- = (Y')^-$ and $p^-_{Y'} = i \circ p^-_Z$. For an equivariant complex Q on $Y' - Z$ we claim

$$\mathrm{Hyp}_Z^+ \, i^* R j_* Q \simeq \mathrm{Hyp}_{Y'}^+ \, j_! Q[1] \tag{10.3}$$

as complexes on F. Indeed

$$\mathrm{Hyp}_Z^+ \, i^* R j_* Q \simeq \mathrm{Hyp}_Z^- \, i^! j_! Q[1] = \mathrm{Hyp}_{Y'}^- \, j_! Q[1] \simeq \mathrm{Hyp}_{Y'}^+ \, j_! Q[1].$$

The first isomorphism is Braden's theorem on Z for the equivariant complex $i^* R j_* Q \simeq i^! j_! Q[1]$, the equality holds because $(p^-_{Y'})^! = (p^-_Z)^! i^!$, and the last isomorphism is Braden's theorem on Y' for $j_! Q$. Both complexes are equivariant.

We now take the stalk at x_* of both sides of (10.3). By base change for the direct image with compact supports, the stalk at x_* of $\mathrm{Hyp}_X^+ K$ is the cohomology with compact supports, with coefficients in K, of the fibre of q^+ over x_*, which is the attractor of x_* in X. We write Hyp_X for this functor; it is the one of Lemma 10.6. All maps commute with Frobenius and with ϵ'. We apply the resulting isomorphism $\mathrm{Hyp}_Z \, i^* R j_* Q \simeq \mathrm{Hyp}_{Y'} \, j_! Q[1]$ to $Q = P'|_{Y'-Z}$, take the part on which ϵ' acts by -1, and compute the Frobenius supertrace of both sides.

Step 3: the left side. We have $i^* R j_* Q = R\Gamma(I_{\mathrm{in}}, R\Psi_u P')$, where I_{in} is the inertia group of $\mathbb{A}^1_u$ at 0. By Lemma 10.6 the left side of (10.3) is $R\Gamma(I_{\mathrm{in}}, \mathrm{Hyp}_Z \, R\Psi_u P')$.

We first identify the minus part of $\mathrm{Hyp}_Z \, R\Psi_u P'$. On Z the involution ϵ' is the action of $-1 \in \mathbb{G}_m$; hence it is trivial on the cohomology of $\mathrm{Hyp}_Z \, i^* P'$. Indeed, the action by scaling is a family of automorphisms of the attractor, parametrized by $\mathbb{G}_m$, together with an isomorphism of the complex of coefficients. The direct image with compact supports along the projection to $\mathbb{G}_m$ is a constant complex with an endomorphism, which is the identity at the unit element, and therefore on the cohomology at every point of the connected group $\mathbb{G}_m$. The triangle relating $i^* P'$, nearby cycles and vanishing cycles therefore gives

$$(\mathrm{Hyp}_Z \, R\Psi_u P')^- \simeq (\mathrm{Hyp}_Z \, R\Phi_u P')^- = \mu^-(P)[1];$$

the last equality holds because the vanishing cycles are supported at x_*, and the attractor of x_* meets this support in the point x_* alone; so here hyperbolic restriction is the stalk.

Next, I_{in} acts trivially on the cohomology of $(R\Psi_u P')_{x_*}$. To see this, consider on $Y' \times \mathbb{G}_m$ the function λu and the pullback P'^{λ} of P' from Y'. Verdier [Ver83, §7, Proposition 7.1(a)] proves that the nearby cycles $R\Psi_{\lambda u}(P'^{\lambda})$ form a monodromic complex in the direction of $\mathbb{G}_m$, that wild inertia acts trivially on them, and that tame inertia acts by the inverse of the canonical monodromy of this complex. On the other hand the automorphism $\varphi(y, \lambda) = (\lambda y, \lambda)$ of $Y' \times \mathbb{G}_m$ carries the function u, pulled back from Y', to λu, and by strong equivariance it carries the pullback of P' to P'^{λ}. Hence

$$R\Psi_{\lambda u}(P'^{\lambda}) \simeq a^* R\Psi_u(P'),$$

compatibly with the action of I_{in}, where a is the action map. In particular wild inertia acts trivially on $R\Psi_u(P')$. Along $\{x_*\}\times\mathbb{G}_m$ the right side is constant, because x_* is a fixed point, so its canonical monodromy is trivial there, and tame inertia acts trivially on the cohomology of the stalk at x_*. The isomorphisms of the preceding paragraph are compatible with I_{in}, and on the minus part the stalk of the vanishing cycles at x_* is identified with that of the nearby cycles. So I_{in} acts trivially on the cohomology of $(\mathrm{Hyp}_Z\, R\Psi_u P')^-$.

Formula (10.2), with the shift by one, now shows that the supertrace of Frobenius on the minus part of the left side of (10.3) is

$$-(1-q)\,\mathrm{Str}(\mathrm{Frob}_q;\mu^-(P)).$$

Step 4: the right side. The attractor of x_* in $Y'-Z$ is, by homogeneity, the product of $\mathbb{G}_m$ and its fibre over $u=1$. Let C_1 be the cohomology with compact supports of that fibre with coefficients in P'. The involution ϵ' acts trivially on the factor $\mathbb{G}_m$, whose cohomology with compact supports has Frobenius supertrace $q-1$. With the shift, the supertrace on the minus part of the right side is $-(q-1)\,\mathrm{Str}(\mathrm{Frob}_q;C_1^-)$. Comparing with step 3 and dividing by $1-q$,

$$\mathrm{Str}(\mathrm{Frob}_q;\mu^-(P)) = -\,\mathrm{Str}(\mathrm{Frob}_q;C_1^-). \tag{10.4}$$

Step 5: conclusion. Suppose that the stalk of P has zero supertrace at every point of $Y(\mathbb{F}_q)$. By the trace formula for cohomology with compact supports, $\mathrm{Str}(\mathrm{Frob}_q;C_1)$ is a sum of such stalk traces and vanishes. The same formula applied to $\epsilon'\,\mathrm{Frob}_q$, that is, to the quadratic twist of the fibre over $u=1$, expresses $\mathrm{Str}(\epsilon'\,\mathrm{Frob}_q;C_1)$ as a sum over points fixed by $\epsilon\,\mathrm{Fr}_q$, and by the remark preceding the proof each term is a stalk trace at a rational point. So this vanishes too. Half the difference of the two is the supertrace on C_1^-, which contradicts (10.4). □

Remark 10.8 (The two factors)**.** The factors $1-q$ in step 3 and $q-1$ in step 4 have different origins: the first comes from the cohomology of inertia, the second from the cohomology with compact supports of a scaling orbit. Their quotient is the sign in (10.4). The argument needs the part on which ϵ acts by -1: on the invariant part the contribution of i^*P' in step 3 does not drop out.

10.4 Nonvanishing for the automorphic family

Proposition 10.9 (A rational bundle with nonzero trace)**.** *Let $P\mapsto\mathcal{F}_P$ be the family over S of Theorem 8.8. The complex $\mathcal{F}_{P_{\mathrm{reg}},s}$ on the special fibre has nonzero Frobenius supertrace at some bundle on the curve $C_0=X\cup_D C_2$ defined over $\mathbb{F}_q$.*

Proof. If G is trivial, the stack of bundles of each fibre is a point, and the complex attached to P_{reg} is a graded line, on which Frobenius is invertible. Otherwise $p\nmid|W_G|$ implies $p\neq 2$.

Step 1: the hypotheses of the comparison. Pull the family back to the chart Y of Lemma 9.14, with the shift by the relative dimension. The pullbacks are strongly equivariant for the scaling action, which is induced by a change of trivialization and does not change the underlying bundle. They are ULA over S, and by Corollary 9.11 they are ULA over $\mathbb{A}^1_S$ away from x_*. The chart and the test function depend only on the nilpotent cone, so this holds for every perfect coefficient P on the same chart.

Step 2: the sign. By Proposition 9.16, the functor μ_η on $\mathrm{Perf}(\mathfrak{F}_\rho)$ is $\Gamma(\mathfrak{F}_\rho,-)\otimes L_{\mathrm{shift}}$. The involution ϵ is a natural automorphism of this functor, hence a scalar by Lemma 6.7, and the

scalar is 1 or -1. If it is 1, apply Lemma 10.5 to the whole family. In either case $\mu_\eta = \mu_\eta^-$, and Theorem 10.4, applied along the flag of S, gives an isomorphism of functors

$$\mu_s^- \simeq \mu_\eta \simeq \Gamma(\mathfrak{F}_\rho, -) \otimes L_{\mathrm{shift}}. \tag{10.5}$$

Step 3: Frobenius. Both sides of (10.5) carry a Frobenius covariance with respect to the same autoequivalence of $\mathrm{Perf}(\mathfrak{F}_\rho)$: the left side from the Weil structure of the family on the special fibre, with Frobenius acting on vanishing cycles through a lift chosen as explained after Theorem 10.7, the right side from the action of Frobenius on $\mathfrak{F}_\rho$ (Lemma 6.6). By Lemma 6.7 they differ by one nonzero scalar. Evaluated at P_{reg}, this identifies geometric Frobenius on $\mu_s^-(\mathcal{F}_{P_{\mathrm{reg}}})$, up to that scalar and the action on the line L_{shift}, with the inverse of arithmetic Frobenius on $\Gamma(\mathfrak{F}_\rho, P_{\mathrm{reg}})$, including the inverse of the permutation of the basis of $k[\Pi_\rho]$. Its supertrace is not zero, by Proposition 6.10, which covers the inverse of Frobenius.

Step 4: a rational point. The finitely many objects and maps involving P_{reg} are defined over a finite extension of $\mathbb{Q}_\ell$. On the special fibre of the chart, which is defined over $\mathbb{F}_q$, the vanishing cycles of f are supported on x_* by step 1. So Theorem 10.7 applies and gives an $\mathbb{F}_q$-rational point of the chart at which the stalk of the pullback has nonzero supertrace. The pullback from the stack of bundles to the chart is smooth, and it changes stalk traces by a sign and a power of q. The image of the point is the required bundle. □

11 The arithmetic parameter of the trace

Section 10 produced a nonzero trace function φ. This section shows that it has the right parameter: Theorem 11.2 says that the functional $h \mapsto [h, \varphi]$ on functions with compact support is an eigenvector of every excursion operator, with eigenvalue given by ρ. By Proposition I.8.10 this makes φ a cusp form with parameter $\vartheta_G \rho$ (Corollary 11.13). Here ϑ_G is the Chevalley involution of $\widehat{G}$ (Section 4); it appears because the pairing with functions of compact support exchanges a parameter with its Chevalley dual. Accordingly, in Section 12 the construction is applied to $\vartheta_G \sigma$ in place of σ.

For a single Hecke eigensheaf the argument is due to Braverman and Varshavsky and to V. Lafforgue: the eigensheaf defines, for every finite set of legs, a map from the cohomology of shtukas to the fibre of its eigenvalue, and these maps are compatible with creation, annihilation and the action of the Weil groups; an excursion operator is then computed on the eigenvalue. Two things have to be added here. First, the compatibility of these maps with partial Frobenius morphisms was left unwritten in [Laf18, Proposition 15.1]; we prove it in Proposition 11.10. Secondly, we have a family $\mathcal{F}_W$ and not a single eigensheaf. Its Hecke identities involve only the geometric local system, and a geometric local system has several arithmetic structures: the image g of Frobenius can be multiplied by any z in the centralizer Z of the geometric monodromy. The maps just described therefore take values in functions of z, and an excursion operator acts on them as multiplication by the function $z \mapsto \nu_{\rho_z}(e)$, where ρ_z is the parameter with Frobenius image modified by z (Proposition 11.12). The function of z attached to the coefficient $R = k[\pi_0(Z)]$ is supported on the twisted conjugacy class of $z = 1$, on which ρ_z is conjugate to ρ (Lemma 11.6). This gives the theorem.

11.1 Setting and statement

Setup 11.1. Let G be split, simply connected and semisimple, let $Y = \mathrm{Bun}_{G,D}$ for the original curve X and a nonempty effective divisor D, which may be nonreduced, and let $U = X - \operatorname{supp} D$. The coefficient field k is an algebraic closure of $\mathbb{Q}_\ell$, with a fixed square root of q. Sheaves are considered on the base change of Y, U and of the stacks below to $\overline{\mathbb{F}}_q$,

and Fr denotes the q-Frobenius endomorphism of such a base change; it is linear over $\overline{\mathbb{F}}_q$. For a finite set I and $J \subset I$, the partial Frobenius Frob_J of U^I is Fr on the coordinates in J and the identity on the others.

The local system. Let ρ be a $\widehat{G}$-local system on U. For a representation V of $\widehat{G}^I$ let V_ρ be the associated lisse sheaf on U^I; on $\overline{\mathbb{F}}_q$ it comes with isomorphisms $\theta_J : \mathrm{Frob}_J^* V_\rho \simeq V_\rho$. Let $Z \subset \widehat{G}$ be a reductive subgroup centralizing the image of the geometric fundamental group. The group Z^I then acts on V_ρ over $\overline{\mathbb{F}}_q$; for $z \in Z$ we write z_J for its action through the legs in J, and $z = z_I$. Let ϕ be a geometric Frobenius element of the Weil group of U and put $g = \rho(\phi)^{-1}$, the image of an arithmetic Frobenius element. We assume that g normalizes Z and put $\alpha = \mathrm{Ad}(g)|_Z$. Since θ induces on a fibre the action of ϕ,

$$\theta_J \circ \mathrm{Frob}_J^*(z_J) = \alpha^{-1}(z)_J \circ \theta_J. \tag{11.1}$$

For $z \in Z$ let ρ_z be the parameter of the Weil group with the same restriction to the geometric fundamental group as ρ and with $\rho_z(\phi) = zg^{-1}$; equivalently, V_{ρ_z} is V_ρ with θ_J replaced by $z_J\theta_J$. This is a homomorphism because Z centralizes the geometric image. For an excursion datum $e = (I, f, \boldsymbol{\gamma})$ with $\boldsymbol{\gamma} = (\gamma_i) \in W_F^I$, the value $\nu_{\rho_z}(e) = f((\rho_z(\gamma_i))_{i\in I})$ is a regular function of z.

The family. We are given a k-linear functor $W \mapsto \mathcal{F}_W$ from finite-dimensional representations of Z to locally bounded constructible complexes on Y. For a lisse sheaf $\mathcal{W}$ on a scheme B with an action of Z we put $\mathcal{F}(\mathcal{W}) = \bigoplus_W \mathcal{F}_W \boxtimes \underline{\mathrm{Hom}}_Z(W, \mathcal{W})$ on $Y \times B$, the sum being over the irreducible representations. Let Hecke_I be the stack over U^I of modifications of G-bundles with level structure along D, with its maps a, b to Y given by the source and the target and its map ℓ to U^I, and let $\mathcal{S}_{I,V}$ be the Satake kernel on it, in the normalization of [Laf18, Definition 4.5 and (4.8)]. The Hecke functor is $\mathsf{H}_{I,V}(K) = (b, \ell)_!(\mathcal{S}_{I,V} \otimes a^*K)$. The family is a Hecke family with Frobenius in the sense of Definition 5.1. With the normalizations just fixed, this means the following.

(F1) *Hecke isomorphisms.* For all I, V and W there are isomorphisms $\mathsf{H}_{I,V}(\mathcal{F}_W) \simeq \mathcal{F}(W \otimes V_\rho)$ on $Y \times U^I$, where Z acts diagonally on $W \otimes V_\rho$. They are natural in W and in V, are the identity for the trivial representation, and are compatible with restriction to diagonals of U^I and with convolution: for $I = J \sqcup J'$ and $V = V_1 \boxtimes V_2$ the isomorphism for (I, V) is the composite of those for (J, V_1) and (J', V_2) under the convolution isomorphism of Satake kernels. Here naturality in V, restriction to diagonals and convolution refer to the corresponding structures of the kernels $\mathcal{S}_{I,V}$ [Laf18, Notation 4.9, Propositions 4.10 and 4.12].

(F2) *Frobenius covariance.* There are isomorphisms $c_W : \mathrm{Fr}^* \mathcal{F}_W \simeq \mathcal{F}_{\alpha^*W}$, natural in W, where $r_W : Z \to \mathrm{GL}(W)$ denotes the action of Z on a representation W, and α^*W is W with the action $z \mapsto r_W(\alpha(z))$.

(F3) *Compatibility.* The Satake kernel has its Frobenius structure $\Phi_{\mathcal{S}} : \mathrm{Fr}^* \mathcal{S}_{I,V} \simeq \mathcal{S}_{I,V}$ [Laf18, (3.5)], which gives $\mathrm{Fr}^* \mathsf{H}_{I,V}(K) \simeq \mathsf{H}_{I,V}(\mathrm{Fr}^* K)$, where on the left Fr is the Frobenius of $Y \times U^I$. Under this identification the two isomorphisms

$$\mathrm{Fr}^* \mathsf{H}_{I,V}(\mathcal{F}_W) \longrightarrow \mathcal{F}(\alpha^*W \otimes V_\rho),$$

namely the Hecke isomorphism for W followed by c and θ_I, and $\mathsf{H}_{I,V}(c_W)$ followed by the Hecke isomorphism for α^*W, are equal.

In (F3) we used that $\theta_I : \alpha^* \mathrm{Fr}^* V_\rho \to V_\rho$ commutes with the action of Z, by (11.1). The family of Theorem 8.8, restricted as in Proposition 12.1, has this structure. For simply connected G the Schubert dimensions are even, so no sign occurs in the commutativity constraint [Laf18, Remarks 1.19 and 1.21].

The trace function. A map $\lambda : \alpha^* W \to W$ of representations of Z makes $\mathcal{F}_W$ a Weil complex, with structure map $\mathcal{F}_\lambda \circ c_W : \mathrm{Fr}^* \mathcal{F}_W \to \mathcal{F}_W$. We call such a map λ a *closing map.* Let

$$\Pi = \pi_0(Z), \qquad R = k[\Pi], \qquad \lambda_R : \alpha^* R \to R, \quad e_\gamma \mapsto e_{\alpha^{-1}(\gamma)},$$

where Z acts on R by left translation, and let φ be the trace function of the Weil complex $(\mathcal{F}_R, \lambda_R)$.

Theorem 11.2 (Excursion equations for the trace)**.** *In Setup 11.1, assume that Z° is a torus and that $1 - \alpha : Z^\circ \to Z^\circ$ is an isogeny. For every excursion datum $e = (I, f, \boldsymbol{\gamma})$ with $\boldsymbol{\gamma} \in W_F^I$, the pairing*

$$L_\varphi(h) = \sum_{[y] \in Y(\mathbb{F}_q)} \frac{h(y)\varphi(y)}{|\operatorname{Aut}(y)(\mathbb{F}_q)|} \qquad (h \in C_c(Y(\mathbb{F}_q), k)) \tag{11.2}$$

satisfies

$$L_\varphi S_e = \nu_\rho(e) L_\varphi, \qquad \nu_\rho(e) = f((\rho(\gamma_i))_{i \in I}).$$

Remark 11.3 (Conventions)**.** Let u_C be the arithmetic Frobenius of the curve, acting on the coefficients. With the conventions which precede Lemma 6.6, the automorphisms of the stack of bundles and of the spectral stack are the transports $u_{\mathrm{aut}}(\mathcal{E}) = (u_C^{-1})^* \mathcal{E}$ and $u_{\mathrm{spec}}(\tau) = (u_C^{-1})^* \tau$. The descent isomorphism $c : u_C^* \rho \simeq \rho$ has matrix g in the marked fibre, so u_{spec} acts on the centralizer through α^{-1}, where $\alpha = \mathrm{Ad}(g)|_Z$, and the covariance of Lemma 6.6 is

$$u_{\mathrm{aut}}^* \mathcal{F}_W \simeq \mathcal{F}_{(\alpha^{-1})^* W}.$$

On the special fibre $u_{\mathrm{aut}} \circ \mathrm{Fr}_Y$ is the absolute Frobenius, whose pullback is canonically the identity on étale sheaves. Inverting the covariance gives

$$\mathrm{Fr}_Y^* \mathcal{F}_W \simeq \mathcal{F}_{\alpha^* W},$$

which is (F2). On the legs the corresponding map is u_C^{-1}, whose pullback is identified with Fr_U^* in the same way, and the isomorphism of local systems is

$$\theta = (u_C^{-1})^*(c^{-1}) : (u_C^{-1})^* \rho \longrightarrow \rho.$$

Its matrix in the marked fibre is g^{-1}, and $\theta \circ \mathrm{Fr}^*(z) = \alpha^{-1}(z) \circ \theta$; equivalently $\theta : \alpha^* \mathrm{Fr}^* V_\rho \to V_\rho$ is equivariant under Z. By the identity for the universal local system and the Hecke identities, the inverse of the arithmetic covariance is the square (F3). So (F3) is a property of the family which has been constructed, and the direction of α is determined by the construction.

For the regular representation R of the group of components the map which plays the role of a closing map for the arithmetic covariance is $(\alpha^{-1})^* R \to R$, $e_\gamma \mapsto e_{\alpha(\gamma)}$; we call it the arithmetic closing map. The closing map of the setup, the geometric one, is its inverse $\lambda_R : \alpha^* R \to R$, $e_\gamma \mapsto e_{\alpha^{-1}(\gamma)}$. The arithmetic and the geometric operator on $\Gamma(\mathfrak{F}_\rho, P_{\mathrm{reg}})$ are inverse to each other, on the exterior algebra Λ and on R together, and Proposition 6.10 shows that the supertraces of both are nonzero. The trace function of a Weil complex is the trace of the geometric operator.

The following table collects the two directions. The left column is the arithmetic one, defined by transport by u_C; the right column is the geometric one, defined by pullback by Frobenius, and the maps on the right are the inverses of those on the left.

	arithmetic	geometric
map of the stack of bundles	$u_{\mathrm{aut}} : \mathcal{E} \mapsto (u_C^{-1})^*\mathcal{E}$	Fr_Y; the composite $u_{\mathrm{aut}} \circ \mathrm{Fr}_Y$ is the absolute Frobenius
covariance of the family	$u_{\mathrm{aut}}^* \mathcal{F}_W \simeq \mathcal{F}_{(\alpha^{-1})^* W}$	$\mathrm{Fr}_Y^* \mathcal{F}_W \simeq \mathcal{F}_{\alpha^* W}$, which is (F2)
local system on the legs	$c : u_C^* \rho \to \rho$, with matrix g	$\theta : (u_C^{-1})^* \rho \to \rho$, with matrix g^{-1}
closing map on $R = k[\Pi]$	$e_\gamma \mapsto e_{\alpha(\gamma)}$	$\lambda_R : e_\gamma \mapsto e_{\alpha^{-1}(\gamma)}$
operator on $\widehat{\mathfrak{g}}_x^*$, for an orbit of length d	$q^d \operatorname{Ad}(g_D)^\vee$	$q^{-d}(\operatorname{Ad}(g_D)^\vee)^{-1}$
factor of the supertrace, for an eigenvalue λ of $\operatorname{Ad}(g_D)^\vee$	$1 - q^d\lambda$	$1 - q^{-d}\lambda^{-1}$
the example of Section 6.4	-64	$64/125$

11.2 Functions on the centralizer

Twisted conjugation is the action $h \star z = \alpha(h) z h^{-1}$ of Z on itself. If z is replaced by $h \star z$ then zg^{-1} is replaced by its conjugate under $\alpha(h)$, so that $\rho_{h\star z} = \operatorname{Ad}(\alpha(h)) \circ \rho_z$.

Definition 11.4 (Trace data and matrix coefficients)**.** Let V be a finite-dimensional representation of Z. We write $A(V)$ for the space of regular maps $a : Z \to V^*$ such that

$$a(h \star z) = a(z) \circ r_V(\alpha(h))^{-1} \qquad (h, z \in Z).$$

A *trace datum* for V is a pair (W, λ) of a finite-dimensional representation W of Z and a map $\lambda : \alpha^*(W \otimes V) \to W$ of representations. Its *matrix coefficient* is

$$a_{W,\lambda}(z)(v) = \operatorname{Tr}\big(r_W(z) \circ \lambda(- \otimes v) \mid W\big).$$

For the trivial representation, $A(\mathbf{1})$ is the space $\mathcal{O}(Z)^\star$ of regular functions invariant under twisted conjugation, and $a_{W,\lambda}(z) = \operatorname{Tr}(r_W(z)\lambda)$.

Lemma 11.5 (Matrix coefficients)**.** *1. The function $a_{W,\lambda}$ belongs to $A(V)$, and $\lambda \mapsto a_{W,\lambda}$ induces an isomorphism*

$$\bigoplus_W \operatorname{Hom}_Z\big(\alpha^*(W \otimes V), W\big) \xrightarrow{\sim} A(V),$$

the sum being over the irreducible representations of Z.

2. If $W = \bigoplus_i W_i \otimes M_i$ with W_i irreducible and pairwise nonisomorphic, and λ_i denotes the partial trace over M_i of the component of λ from $\alpha^*(W_i \otimes M_i \otimes V)$ to $W_i \otimes M_i$, then $a_{W,\lambda} = \sum_i a_{W_i,\lambda_i}$.

3. Let $V = V_1 \otimes V_2$, let V_1' be another representation and $\vartheta : V_1' \to V_1$ a linear map with $\vartheta \circ r_{V_1'}(z) = r_{V_1}(\alpha^{-1}(z)) \circ \vartheta$. Put $W' = W \otimes V_1$ and

$$\lambda' : \alpha^*(W \otimes V_1 \otimes V_1' \otimes V_2) \longrightarrow W \otimes V_1, \qquad w \otimes v_1 \otimes v_1' \otimes v_2 \longmapsto \lambda(w \otimes v_1 \otimes v_2) \otimes \vartheta(v_1').$$

Then (W', λ') is a trace datum for $V_1' \otimes V_2$, and

$$a_{W',\lambda'}(z) = a_{W,\lambda}(z) \circ \big(r_{V_1}(z)\vartheta \otimes \mathrm{id}_{V_2}\big).$$

Proof. (1) Write $\lambda_v = \lambda(- \otimes v) \in \operatorname{End}(W)$. That λ is a map of representations means $\lambda_{\alpha(h)v}\, r_W(\alpha(h)) = r_W(h)\lambda_v$. Hence

$$a(z)(v) = \operatorname{Tr}\big(r_W(z) r_W(h)^{-1} \lambda_{\alpha(h)v} r_W(\alpha(h))\big) = a(h \star z)(\alpha(h)v),$$

which is the condition defining $A(V)$. Representations of the reductive group Z are semisimple, and by the algebraic Peter–Weyl theorem the maps $\mathrm{End}(W) \to \mathcal{O}(Z)$, $T \mapsto \mathrm{Tr}(r_W(z)T)$, for W irreducible, are injective and $\mathcal{O}(Z)$ is the direct sum of their images. Twisted conjugation acts on the image of $\mathrm{End}(W)$ through $T \mapsto r_W(h)Tr_W(\alpha(h))^{-1}$, that is, through the natural action on $\mathrm{Hom}(\alpha^*W, W)$. Tensoring with V^* and taking invariants gives the isomorphism.

(2) Only the components of λ from the ith summand to itself contribute to the trace, and the trace over $W_i \otimes M_i$ is the trace over W_i of the partial trace.

(3) That λ' is a map of representations follows from the property of ϑ. For fixed v_1' and v_2 the operator $r_{W'}(z)\lambda'(-\otimes v_1' \otimes v_2)$ on $W \otimes V_1$ sends $w \otimes v_1$ to $r_W(z)\lambda(w \otimes v_1 \otimes v_2) \otimes r_{V_1}(z)\vartheta(v_1')$. The trace of an operator of the form $w \otimes v_1 \mapsto T(w \otimes v_1) \otimes c$, with $T : W \otimes V_1 \to W$ and $c \in V_1$, is the trace of $w \mapsto T(w \otimes c)$. This gives the formula. □

Lemma 11.6 (The regular coefficient)**.** *Assume that Z° is a torus and that $1 - \alpha$ is an isogeny of Z°. Then*

$$a_{R,\lambda_R} = |\Pi^\alpha|\, 1_{\mathscr{O}}, \qquad \mathscr{O} = \{\alpha(h)h^{-1} : h \in Z\}. \tag{11.3}$$

The set $\mathscr{O}$ is a union of connected components of Z, and for every excursion datum e,

$$\nu_{\rho_z}(e)\, a_{R,\lambda_R}(z) = \nu_\rho(e)\, a_{R,\lambda_R}(z) \qquad (z \in Z). \tag{11.4}$$

Proof. The operator $r_R(z)\lambda_R$ sends e_γ to $e_{\bar{z}\alpha^{-1}(\gamma)}$, where $\bar{z}$ is the image of z in Π. Its trace is the number of γ with $\bar{z} = \gamma\alpha^{-1}(\gamma)^{-1}$. Writing $\gamma = \alpha(\delta)$, these are the solutions of $\bar{z} = \alpha(\delta)\delta^{-1}$; if there is one, the others are its products on the right with elements of Π^α.

Suppose $\bar{z} = \alpha(\delta)\delta^{-1}$ and let h_0 be a lift of δ. Then $z = \alpha(h_0)\, t\, h_0^{-1}$ with $t \in Z^\circ$. The map $s \mapsto \alpha(s)s^{-1}$ is an endomorphism of the torus Z°, and it is surjective because $1 - \alpha$ is an isogeny. Writing $t = \alpha(s)s^{-1}$ gives $z = \alpha(h_0 s)(h_0 s)^{-1} \in \mathscr{O}$. Conversely the image in Π of an element of $\mathscr{O}$ has the required form. This proves (11.3) and shows that $\mathscr{O}$ is the inverse image of a subset of Π.

For $z = \alpha(h)h^{-1} = h \star 1$ we have $\rho_z = \mathrm{Ad}(\alpha(h)) \circ \rho$, and excursion functions are invariant under simultaneous conjugation. This proves (11.4). □

Remark 11.7. With the trivial coefficient in place of R, all the parameters ρ_z would contribute. The regular representation of $\pi_0(Z)$ restricts the contributions to certain components of Z, and the condition that $1 - \alpha$ is an isogeny of Z° makes those components a single twisted conjugacy class. Both parts of Lemma 6.2 are used here.

11.3 Traces of correspondences

We recall the trace of a cohomological correspondence relative to a base, in the form in which it will be used. Let B be a scheme of finite type over $\overline{\mathbb{F}}_q$ and M a scheme separated and of finite type over B. Complexes on M are assumed bounded, constructible and universally locally acyclic over B; this holds for $K \boxtimes N$ when $M = Y_0 \times B$, K is a complex on Y_0 and N is lisse on B.

Definition 11.8 (Correspondences with kernel and their traces)**.** Let K be a complex on M and K' a complex on a second scheme M' of the same kind; in most of what follows $M' = M$, but in step 5 of the proof of Proposition 11.10 the two are different. A *correspondence with kernel* from K to K' consists of a scheme H separated and of finite type over B with a map $(a, b) : H \to M \times_B M'$, a complex A on H, the kernel, and a map

$$\eta : A \otimes a^*K \longrightarrow b^!K'.$$

The composite with a correspondence (H', A', η') from K' to a complex K'' on a third such scheme is supported on $H \times_{b,M',a'} H'$, has kernel $A \boxtimes_{M'} A'$, and the map of the composite is deduced from η and η' by the base change map, as in [Laf18, §4.1]. A map $f : K \to K'$ is a correspondence supported on M with kernel k. For an endomorphism F of M over B and a map $\phi : F^*K \to K'$, we write Ψ_ϕ for the correspondence from K to K' supported on M, with $(a, b) = (F, \mathrm{id})$ and kernel k.

Let N be a lisse sheaf on B and $\epsilon = (H, A, \eta)$ a correspondence with kernel from K to $K \otimes N$. Let C be its fixed locus, the inverse image of the diagonal of $M \times_B M$, with maps $i : C \to H$, $\pi : C \to M$ and $p : C \to B$. Let $p_H : H \to B$ be the structure map. The map η is the same as a map $A \to (a, b)^!(D_{M/B}K \boxtimes_B K) \otimes p_H^* N$, by the duality

$$D_{M/B}K \boxtimes_B K' \simeq R\mathcal{H}om(\mathrm{pr}_1^* K, \mathrm{pr}_2^! K') \tag{11.5}$$

of [LZ22, Proposition 2.5]. The *trace* of ϵ is the composite

$$\mathrm{tr}(\epsilon) : \ i^*A \longrightarrow i^*(a, b)^!(D_{M/B}K \boxtimes_B K) \otimes p^*N \longrightarrow \pi^!(D_{M/B}K \otimes K) \otimes p^*N \longrightarrow \omega_{C/B} \otimes p^*N, \tag{11.6}$$

in which the middle arrow is the base change map $i^*(a, b)^! \to \pi^!\Delta^*$ and the last is evaluation. The *pairing* of ϵ is the map

$$\Pi(\epsilon) : \ p_! i^* A \xrightarrow{p_! \,\mathrm{tr}(\epsilon)} p_! p^! N \longrightarrow N.$$

In the language of [LZ22, Construction 2.6], a correspondence with kernel from K to K' is a morphism $(H, A) \otimes (M, K) \to (M, K')$ in the symmetric monoidal 2-category of cohomological correspondences over B. The object (M, K) is dualizable because K is universally locally acyclic [LZ22, Theorem 2.16], and $\mathrm{tr}(\epsilon)$ is the trace of an endomorphism of a dualizable object [LZ22, Construction 1.6], formed with the kernel (H, A) as a parameter and with values in (B, N). No condition on the kernel is needed.

The statements of [LZ22] are for torsion coefficients, and we pass to the coefficients k in the usual way. The complexes used below are finite in number and defined over the ring of integers $\mathcal{O}_E$ of a finite extension E of $\mathbb{Q}_\ell$. Their reductions modulo the powers of a uniformizer ϖ are universally locally acyclic where the complexes are, so the trace and its properties are available modulo ϖ^m for every m, compatibly with reduction. To pass to the limit one needs the cohomology with compact supports of the fixed loci, with coefficients modulo ϖ^m, to be bounded independently of m. The fixed loci which occur are stacks of shtukas of level D with bounded modifications, truncated by the Harder–Narasimhan polygon (step 1 of the proof of Theorem 11.2). Let $D' \geq D$ be a divisor with the same support and large multiplicities. The stacks of level D', with the same bounds, are quasi-projective schemes, and those of level D are their quotients by the finite group $\ker(G(\mathcal{O}_{D'}) \to G(\mathcal{O}_D))$ [Laf18, Remarque 4.3]; here we use that D is not empty and that G is semisimple. This group has a filtration with quotients which are vector spaces over $\mathbb{F}_q$, so it is a p-group and its order is invertible in the coefficients. Hence the cohomology of the quotient is the invariant part of the cohomology of the scheme, and the bounds for schemes of finite type apply. Finally, the maps between the complexes are defined after ℓ is inverted, and they need not preserve the integral structures. Since the trace and the identities proved below are multilinear in the maps, they can be checked after the maps have been multiplied by a power of the uniformizer. In the application D contains the divisor D_{ur} of Section 7, which is not empty.

Lemma 11.9 (Properties of the trace)**.** *(a)* Sums and products. *The trace is additive for decompositions of K into direct sums: the trace of an endomorphism of $K_1 \oplus K_2$ is the sum of the traces of its two diagonal components. If $K = K_1 \boxtimes \mathcal{M}$ with $\mathcal{M}$ lisse on B and $\epsilon = \epsilon_1 \otimes m$ with $m : \mathcal{M} \to \mathcal{M} \otimes N'$, then $\mathrm{tr}(\epsilon) = \mathrm{tr}(\epsilon_1) \otimes \mathrm{Tr}(m)$.*

(b) Base change. *The trace and the pairing are compatible with base change along a map* $B' \to B$.

(c) Proper maps of kernels. *Let* $\epsilon = (H, A, \eta)$ *and* $\epsilon' = (H', A', \eta')$ *be correspondences from* K *to* $K \otimes N$*, let* $m : H \to H'$ *be a proper map compatible with the maps to* $M \times_B M$*, and let* $\kappa : m^* A' \to A$*. Suppose that* η' *is the pushforward of* η*, that is, the composite*

$$A' \otimes a'^* K \to m_*(m^* A' \otimes a^* K) \xrightarrow{\kappa} m_*(A \otimes a^* K) \xrightarrow{\eta} m_* m^! b'^! (K \otimes N) \to b'^! (K \otimes N).$$

Then $\Pi(\epsilon')$ *is the composite of* $\Pi(\epsilon)$ *with the map* $p'_! i'^* A' \to p_! i^* A$ *given by pullback along the map of fixed loci* $C \to C'$ *and by* κ*.*

(d) Cyclic symmetry. *Let* K_0 *and* K_1 *be complexes on schemes* M_0 *and* M_1 *as above, possibly different, let* ϵ_1 *be a correspondence from* K_0 *to* K_1 *and* ϵ_2 *one from* K_1 *to* $K_0 \otimes N$*. The fixed loci of* $\epsilon_2 \circ \epsilon_1$*, an endomorphism of* K_0*, and of* $(\epsilon_1 \otimes N) \circ \epsilon_2$*, an endomorphism of* K_1*, are the same, and* $\operatorname{tr}(\epsilon_2 \circ \epsilon_1) = \operatorname{tr}((\epsilon_1 \otimes N) \circ \epsilon_2)$.

(e) Frobenius. *Let* $B = \operatorname{Spec} \overline{\mathbb{F}}_q$*, let* M *be defined over* $\mathbb{F}_q$ *with Frobenius* Fr*, and let* $\phi : \mathrm{Fr}^* K \to K$*. The fixed locus of* Ψ_ϕ *is the finite set* $M(\mathbb{F}_q)$*, and* $\operatorname{tr}(\Psi_\phi)$ *is the function* $y \mapsto \operatorname{Tr}(\phi_y \mid K_y)$.

(f) Open subschemes. *Let* $M^\circ \subset M$ *be an open subscheme, and let* ϵ° *be the restriction of a correspondence* $\epsilon = (H, A, \eta)$ *from* K *to* $K \otimes N$ *to the open subset* $H^\circ = a^{-1}(M^\circ) \cap b^{-1}(M^\circ)$ *of* H*. Its fixed locus is the open subset* $C^\circ = \pi^{-1}(M^\circ)$ *of* C*, the trace of* ϵ° *is the restriction of the trace of* ϵ *to* C°*, and* $\Pi(\epsilon^\circ)$ *is the composite of* $\Pi(\epsilon)$ *with the map* $p^\circ_! i^{\circ *} A \to p_! i^* A$ *of extension by zero.*

Proof. Statements (a), (b), (c) and (e) are formal or are in the literature; (d) requires an argument, because it is stated there without parameters, and (f) is a direct verification.

(a), (b) The symmetric monoidal 2-category of [LZ22, Construction 2.6] is additive in the sheaf, which gives the first part of (a); the second holds because $(B, \mathcal{M})$ is dualizable and the trace of a tensor product of endomorphisms is the product of the traces. Base change is a symmetric monoidal functor [LZ22, Proposition 2.26], and such functors preserve traces; this is (b).

(c) The pair (m, κ) is a morphism $\mu : (H', A') \to (H, A)$ of that category, supported on H, and the hypothesis says that m is a 2-morphism from $\epsilon \circ (\mu \otimes \mathrm{id})$ to ϵ'. Composition with the coevaluation and the evaluation of (M, K) is functorial for 2-morphisms, so m induces a 2-morphism between the traces, which is the assertion; this is the mechanism of the relative Lefschetz–Verdier formula [LZ22, Theorem 2.21], here in the case where M does not change.

(d) This is the symmetry of the pairing $\langle u, v \rangle \simeq \langle v, u \rangle$ of [LZ22, Construction 1.6], in a form with parameters. Let X and Y be dualizable objects of a symmetric monoidal 2-category, with evaluations $\mathrm{ev}_X, \mathrm{ev}_Y$ and coevaluations $\mathrm{coev}_X, \mathrm{coev}_Y$, and let

$$u : P_1 \otimes X \longrightarrow Y, \qquad v : P_2 \otimes Y \longrightarrow X \otimes T$$

be morphisms, where P_1, P_2 and T are arbitrary objects. Consider the morphism $\langle u, v \rangle : P_1 \otimes P_2 \to T$ obtained as follows: apply $\mathrm{coev}_X \otimes \mathrm{coev}_Y$ to create $X \otimes X^\vee \otimes Y \otimes Y^\vee$; apply u to $P_1 \otimes X$ and v to $P_2 \otimes Y$; and apply ev_Y to the result of u with $Y^\vee$, and ev_X to the factor X of the result of v with $X^\vee$. The trace of $v \circ (P_2 \otimes u)$, an endomorphism of X with parameters, is by definition obtained from coev_X, the composite, and ev_X. Inserting between u and v the identity of Y, written as the composite $(Y \otimes \mathrm{ev}_Y) \circ (\mathrm{coev}_Y \otimes Y)$, we obtain $\langle u, v \rangle$. In the same way the trace of $(u \otimes T) \circ (P_1 \otimes v)$, an endomorphism of Y with parameters, becomes $\langle u, v \rangle$ after insertion of the identity of X between v and u. So the two traces are isomorphic,

by an isomorphism built from the triangle identities of X and Y and the symmetry; the objects P_1, P_2 and T are only carried along. We apply this to $X = (M_0, K_0)$, $Y = (M_1, K_1)$, $T = (B, N)$ and the kernels of ϵ_1 and ϵ_2 as P_1 and P_2; the supports of the two composites are identified by exchanging the two factors of the fibre product.

(e), (f) Statement (e) says that the local terms of the Frobenius correspondence are the traces on the fibres [GV25, Corollary 0.9]. For (f), each map in (11.6) is compatible with restriction to open subsets, since for an open immersion the two pullbacks coincide and commute with all the functors involved; and $p^\circ = p \circ e$ for the open immersion $e : C^\circ \to C$, so that the integration map of p° is that of p composed with the counit $e_! e^* \to \mathrm{id}$. Property (f) concerns open immersions and extension by zero, and property (c) concerns proper maps m: an open immersion is not an admissible 2-morphism in (c). □

11.4 The trace pairing

For a finite set I and a representation V of $\widehat{G}^I$ let $\mathcal{H}_{I,V}$ be the sheaf $\mathcal{H}^0_{D,I,V} \otimes k$ of Section I.5, on U^I over $\overline{\mathbb{F}}_q$: it is the direct limit over μ of $R^0 p_!$ of the Satake sheaf $\mathcal{F}_{I,V}$ on the stacks $\mathrm{Cht}_I^{\leq \mu}$ of shtukas with level D. The stack Cht_I is the locus $b = \mathrm{Fr} \circ a$ in Hecke_I, and $\mathcal{F}_{I,V}$ is the restriction of $\mathcal{S}_{I,V}$, where the condition $b = \mathrm{Fr} \circ a$ is understood as a 2-fibre product: an identification of the two bundles, with their level structures, is part of the datum. The partial Frobenius morphisms of [Laf18, §3] induce maps $F_J : \mathrm{Frob}_J^* \mathcal{H}_{I,V} \to \mathcal{H}_{I,V}$ [Laf18, §4.3]. For $I = \varnothing$ the sheaf $\mathcal{H}_{\varnothing,\mathbf{1}}$ is the space $C_c(Y(\mathbb{F}_q), k)$.

Let $\mathcal{A}_{I,V}$ be the sheaf on U^I over $\overline{\mathbb{F}}_q$ whose sections are the regular maps a from Z to sections of V_ρ^* satisfying the condition of Definition 11.4 for the diagonal action of Z. By Lemma 11.5(1) it is the direct sum, over the irreducible representations W, of the lisse sheaves

$$\mathcal{Q}_W = \underline{\mathrm{Hom}}_Z(\alpha^*(W \otimes V_\rho), W).$$

It follows from (11.1) that

$$\Theta_J : \mathcal{A}_{I,V} \longrightarrow \mathrm{Frob}_J^* \mathcal{A}_{I,V}, \qquad (\Theta_J a)(z) = a(z) \circ z_J \theta_J,$$

is well defined; it is an isomorphism. Evaluation at a point z is a map from $\mathcal{A}_{I,V}$ to the dual of V_{ρ_z} which carries Θ_J^{-1} to the partial Frobenius structure of that dual.

A *trace datum over a base* $\beta : B \to U^I$ is a triple $(\mathcal{W}, N, \lambda)$ of a lisse sheaf $\mathcal{W}$ on B with an action of Z, a lisse sheaf N on B and a map $\lambda : \alpha^*(\mathcal{W} \otimes \beta^* V_\rho) \to \mathcal{W} \otimes N$ commuting with Z. Its matrix coefficient $a_{\mathcal{W},\lambda}$, defined fibrewise, is a section of $\beta^* \mathcal{A}_{I,V} \otimes N$.

Proposition 11.10 (The trace pairing)**.** *In Setup 11.1, for every finite set I and every representation V of $\widehat{G}^I$ there is a map of sheaves on U^I over $\overline{\mathbb{F}}_q$,*

$$\Pi_{I,V} : \mathcal{A}_{I,V} \otimes \mathcal{H}_{I,V} \longrightarrow k,$$

with the following properties.

1. Functoriality. *For a map $u : V \to V'$ of representations, $\Pi_{I,V'}(a', \mathcal{H}(u)h) = \Pi_{I,V}(a' \circ u, h)$. For a map of finite sets $\zeta : I \to I'$, the restriction of $\Pi_{I,V}$ to the corresponding diagonal is Π_{I',V^ζ} under the coalescence isomorphism.*

2. No legs. *For a trace datum (W, λ) for the trivial representation, with trace function $\varphi_{W,\lambda}$ of $(\mathcal{F}_W, \lambda)$,*

$$\Pi_\varnothing(a_{W,\lambda}, h) = \sum_{[y] \in Y(\mathbb{F}_q)} \frac{h(y)\varphi_{W,\lambda}(y)}{|\operatorname{Aut}(y)(\mathbb{F}_q)|}.$$

3. Partial Frobenius. *For $J \subset I$, sections a of $\mathcal{A}_{I,V}$ and h of* $\mathrm{Frob}_J^* \mathcal{H}_{I,V}$,

$$\Pi_{I,V}(a, F_J(h)) = \mathrm{Frob}_J^* \Pi_{I,V}(\Theta_J(a), h). \tag{11.7}$$

The proof occupies the rest of this subsection. We first define a pairing for every trace datum.

Step 1: construction. Let $(\mathcal{W}, N, \lambda)$ be a trace datum over $\beta : B \to U^I$. Fix μ and let $Y_0 \subset Y$ be a quasi-compact open substack defined over $\mathbb{F}_q$ which contains the bundles that occur in $\mathrm{Cht}_I^{\leq \mu}$, for instance one of the open substacks of the Harder–Narasimhan filtration. Put $M = Y_0 \times B$ and $F = \mathrm{Fr} \times \mathrm{id}_B$. By adjunction the Hecke isomorphism (F1) is a correspondence with kernel $\epsilon_{\mathcal{W}}$ from $\mathcal{F}(\mathcal{W})$ to $\mathcal{F}(\mathcal{W} \otimes \beta^* V_\rho)$, supported on $\mathrm{Hecke}_I \times_{U^I} B$ with kernel $\mathcal{S}_{I,V}$. Let

$$\phi_\lambda : \ F^* \mathcal{F}(\mathcal{W} \otimes \beta^* V_\rho) \xrightarrow{c} \mathcal{F}(\alpha^*(\mathcal{W} \otimes \beta^* V_\rho)) \xrightarrow{\mathcal{F}(\lambda)} \mathcal{F}(\mathcal{W}) \otimes N.$$

The fixed locus of $\Psi_{\phi_\lambda} \circ \epsilon_{\mathcal{W}}$ is $\mathrm{Cht}_I \times_{U^I} B$, and its kernel there is $\mathcal{F}_{I,V}$. We define

$$\Pi^{\mathcal{W},\lambda} = \Pi(\Psi_{\phi_\lambda} \circ \epsilon_{\mathcal{W}}) : \ \beta^* \mathcal{H}_{I,V} \longrightarrow N,$$

after restriction to the truncation by μ and passage to the cohomology sheaf of degree zero. By Lemma 11.9(f) the result does not depend on Y_0 and is compatible with the transition maps in μ, which are extensions by zero.

The relative trace is used here for Artin stacks; the construction and the relative formula of Lefschetz and Verdier hold in that generality, with proper morphisms replaced by proper representable ones [LZ22, Remark 2.24]. The hypotheses are satisfied. On each truncation the stacks which occur in (11.6) are of finite type over B. The complexes on the stacks of bundles at the source and at the target are exterior products with lisse complexes on B, so they are ULA over B. The maps of supports along which we push forward are bounded convolution maps and relative Frobenius maps of bounded Hecke projections, which are representable and proper. The trace, its cyclicity and its restriction to open substacks are then formed with the same duality and base change maps as for schemes. This gives the pairing at level D over the whole of U^I, the diagonals included.

For an irreducible representation W, the tautological map $\alpha^*(W \otimes V_\rho) \to W \otimes \mathcal{Q}_W^\vee$ is a trace datum over U^I and gives $\Pi^W : \mathcal{Q}_W \otimes \mathcal{H}_{I,V} \to k$. We define $\Pi_{I,V}$ as the sum of the maps Π^W.

Step 2: general trace data. We claim that for every trace datum over a base,

$$\Pi^{\mathcal{W},\lambda} = (\beta^* \Pi_{I,V} \otimes N)(a_{\mathcal{W},\lambda} \otimes -). \tag{11.8}$$

Decompose $\mathcal{W} = \bigoplus_W W \otimes \mathcal{M}_W$ with $\mathcal{M}_W = \underline{\mathrm{Hom}}_Z(W, \mathcal{W})$, so that $\mathcal{F}(\mathcal{W}) = \bigoplus_W \mathcal{F}_W \boxtimes \mathcal{M}_W$. The Hecke isomorphisms and c are natural in the coefficient, hence respect this decomposition and act trivially on the sheaves $\mathcal{M}_W$. By Lemma 11.9(a) the trace of $\Psi_{\phi_\lambda} \circ \epsilon_{\mathcal{W}}$ is the sum over W of the traces obtained from the partial trace over $\mathcal{M}_W$ of the diagonal component of λ. The same decomposition holds for $a_{\mathcal{W},\lambda}$, by Lemma 11.5(2).

Step 3: functoriality and the case without legs. The Hecke isomorphisms are natural in V and compatible with restriction to diagonals, and traces are compatible with base change (Lemma 11.9(b)). This gives (1). For $I = \varnothing$ the correspondence ϵ_W is the identity and Ψ_{ϕ_λ} is the Frobenius correspondence of the Weil complex $(\mathcal{F}_W, \lambda)$. The stack of its fixed points is the finite groupoid of the bundles defined over $\mathbb{F}_q$ in the chosen truncation, and the local term on the component of a bundle y is the trace on the stalk at y [GV25, Theorem 0.8(a)

and Corollary 0.9]. The integral over this component carries the weight $|\operatorname{Aut}(y)(\mathbb{F}_q)|^{-1}$: the point is an atlas of $B\operatorname{Aut}(y)(\mathbb{F}_q)$, finite étale of degree $|\operatorname{Aut}(y)(\mathbb{F}_q)|$, and the trace map of the atlas followed by the integral over the component is the integral over the point. This gives (2).

Step 4: Frobenius covariance as an identity of correspondences. Let J be a finite set, V_1 a representation of $\widehat{G}^J$, $\beta : B \to U^J$ a base and $\mathcal{W}$ a lisse sheaf on B with an action of Z. In the two fibre products which follow, the Hecke stack is the bounded one which supports the kernel for V_1. Let $H = \mathrm{Hecke}_J \times_{U^J,\beta} B$ and $H^\circ = \mathrm{Hecke}_J \times_{U^J,\mathrm{Fr}\,\beta} B$, the Hecke correspondences over B with legs at β and at $\mathrm{Fr}\circ\beta$, and let ϵ and ϵ° be the correspondences with kernel given by (F1) on them. Consider the two composites

$$\begin{aligned}\epsilon_{\alpha^*\mathcal{W}} \circ \Psi_c :&\ \mathcal{F}(\mathcal{W}) \rightsquigarrow \mathcal{F}(\alpha^*\mathcal{W}) \rightsquigarrow \mathcal{F}(\alpha^*\mathcal{W} \otimes \beta^* V_{1,\rho}),\\ \Psi_{c,\theta} \circ \epsilon^\circ_{\mathcal{W}} :&\ \mathcal{F}(\mathcal{W}) \rightsquigarrow \mathcal{F}(\mathcal{W} \otimes \beta^* \operatorname{Fr}^* V_{1,\rho}) \rightsquigarrow \mathcal{F}(\alpha^*\mathcal{W} \otimes \beta^* V_{1,\rho}),\end{aligned}$$

where Ψ_c is the Frobenius correspondence of the covariance c, and $\Psi_{c,\theta}$ that of c followed by $\mathcal{F}(1 \otimes \theta_J)$. The first is supported on H, with source map $F \circ a$ and target map b. The second is supported on $T = H^\circ \times_{b,M,F} M$. The relative Frobenius

$$r : H \longrightarrow T, \qquad u \longmapsto \big(\mathrm{Fr}(u), b(u)\big),$$

is compatible with the source and target maps. It is a base change of the relative Frobenius of the map $(b, \ell) : \mathrm{Hecke}_J \to Y \times U^J$, which is representable and of finite type on bounded parts; so r is a finite universal homeomorphism. The kernel on T is the pullback of $\mathcal{S}_{J,V_1}$, and its pullback by r is $\mathrm{Fr}^*\,\mathcal{S}_{J,V_1}$, which $\Phi_{\mathcal{S}}$ identifies with the kernel on H.

Claim. The correspondence $\Psi_{c,\theta} \circ \epsilon^\circ_{\mathcal{W}}$ is the pushforward of $\epsilon_{\alpha^*\mathcal{W}} \circ \Psi_c$ along $(r, \Phi_{\mathcal{S}})$, in the sense of Lemma 11.9(c).

Indeed, by adjunction a correspondence with kernel A_T supported on T, with source and target maps s and t, is the same as a map $t_!(A_T \otimes s^*K) \to K'$, and its being the pushforward of a correspondence supported on H means that this map is the composite of the corresponding map for H with the isomorphism

$$t_!(A_T \otimes s^*K) \xrightarrow{\sim} t_! r_* r^*(A_T \otimes s^*K) \xrightarrow{\Phi_{\mathcal{S}}} (tr)_!(A_H \otimes (sr)^*K),$$

in which the first arrow is an isomorphism because r is a universal homeomorphism. For $\Psi_{c,\theta} \circ \epsilon^\circ_{\mathcal{W}}$ we have, by base change, $t_!(A_T \otimes s^*K) = \mathrm{Fr}^*\,\mathsf{H}_{J,V_1}(\mathcal{F}(\mathcal{W}))$, pulled back to M, and the displayed isomorphism is the identification $\mathrm{Fr}^*\,\mathsf{H}_{J,V_1}(K) \simeq \mathsf{H}_{J,V_1}(\mathrm{Fr}^*\,K)$ of (F3). The two maps to be compared are therefore the two isomorphisms of (F3), for the coefficient $\mathcal{W}$ in place of W; they are equal by (F3) and additivity.

Step 5: partial Frobenius. By (1) and additivity in V we may assume $V = V_1 \boxtimes V_2$, with V_1 a representation of $\widehat{G}^J$ and V_2 one of $\widehat{G}^{J'}$, $J' = I - J$. By (11.8) it suffices to consider $a = a_{W,\lambda}$ for a trace datum (W, N, λ) over $B = U^I$ with W a representation.

The open substacks. The correspondences below relate bundles which differ by bounded modifications, and an open substack of Y of finite type is not stable under such modifications. If the middle bundle of an iterated modification were required to lie in the same open substack as its ends, the convolution maps would in general not be proper, and Lemma 11.9(c) could not be applied. We therefore use three open substacks. Fix μ. Bounded modifications change the Harder–Narasimhan polygon of a bundle by a bounded amount, so there are quasi-compact open substacks, defined over $\mathbb{F}_q$,

$$Y_{\mathrm{a}} \subset Y_{\mathrm{b}} \subset Y_{\mathrm{c}} \subset Y$$

with the following properties: Y_{a} contains the first bundle of every shtuka in the truncation by μ of $\mathrm{Cht}_I^{(J',J)}$; a bundle obtained from a bundle of Y_{a} by a modification with legs in J' bounded by V_2 lies in Y_{b}; and a bundle obtained from a bundle of Y_{b} by a modification with legs in J bounded by V_1 lies in Y_{c}. Being defined over $\mathbb{F}_q$, they are stable under Fr and under its inverse on points.

Let ϵ_1 be the Hecke correspondence with legs in J from $\mathcal{F}_W$ on $Y_{\mathrm{b}} \times B$ to $\mathcal{F}(W \otimes V_{1,\rho})$ on $Y_{\mathrm{c}} \times B$, and ϵ_2 the one with legs in J' from $\mathcal{F}(W \otimes V_{1,\rho})$ on $Y_{\mathrm{c}} \times B$ to $\mathcal{F}(W \otimes V_\rho)$ on $Y_{\mathrm{b}} \times B$. They are composed below in two orders, which are shown in Figure 2.

$$\begin{array}{ccccc} \mathcal{E}_0 & \xrightarrow[J]{\epsilon_1} & \mathcal{E}_1 & \xrightarrow[J']{\epsilon_2} & \mathrm{Fr}\,\mathcal{E}_0 \\ Y_{\mathrm{b}} & & Y_{\mathrm{c}} & & Y_{\mathrm{b}} \end{array}$$

$$\begin{array}{ccccc} \mathcal{E}_1 & \xrightarrow[J']{\epsilon_2} & \mathrm{Fr}\,\mathcal{E}_0 & \xrightarrow[J]{\epsilon_1} & \mathrm{Fr}\,\mathcal{E}_1 \\ Y_{\mathrm{a}} & & Y_{\mathrm{b}} & & Y_{\mathrm{c}} \end{array}$$

Figure 2: The two orders of the modifications in step 5. Above: a fixed point of $\Psi_{\phi_\lambda} \circ \epsilon_2 \circ \epsilon_1$ is a shtuka in which the modification with legs in J comes first; its ends lie in Y_{b}, and the middle bundle then lies in Y_{c}. Below: after the cyclic move the modification with legs in J' comes first; the first bundle is taken in Y_{a}, and then the middle bundle lies in Y_{b} and the last one in Y_{c}. Under each bundle is the open substack which contains it.

The composite $\epsilon_2 \circ \epsilon_1$ is supported on the part of the iterated Hecke stack $\mathrm{Hecke}^{(J,J')}$ whose two ends lie in Y_{b} and whose middle bundle lies in Y_{c}. By the choice of Y_{c} the condition on the middle bundle is automatic. So this is the whole iterated Hecke stack over the ends in Y_{b}, its map m to Hecke_I is proper, and by (F1) the correspondence ϵ_W, taken with $Y_0 = Y_{\mathrm{b}}$ in step 1, is its pushforward along m. The map of kernels required in Lemma 11.9(c) is the counit: the convolution kernel is the proper direct image of the iterated kernel, and the counit maps the pullback of this direct image to the iterated kernel. The fixed locus of $\Psi_{\phi_\lambda} \circ \epsilon_2 \circ \epsilon_1$ is the stack $\mathrm{Cht}_I^{(J,J')}$ of shtukas with two modifications, and, by proper base change of this direct image to the fixed locus, the pullback on cohomology is the isomorphism through which $\mathcal{H}_{I,V}$ does not depend on the partition [Laf18, Définition 4.1]. By Lemma 11.9(c) and then (d),

$$\Pi^{W,\lambda} = \Pi\big(\Psi_{\phi_\lambda} \circ \epsilon_2 \circ \epsilon_1\big) = \Pi\big((\epsilon_1 \otimes N) \circ \Psi_{\phi_\lambda} \circ \epsilon_2\big),$$

where the last correspondence is an endomorphism of $\mathcal{F}(W \otimes V_{1,\rho})$ on $Y_{\mathrm{c}} \times B$ with values in N; here (d) is applied to complexes on the two different spaces $Y_{\mathrm{b}} \times B$ and $Y_{\mathrm{c}} \times B$. In this last correspondence the middle bundle, which is $\mathrm{Fr}\,\mathcal{E}_0$, is constrained to lie in Y_{b}, so its support is in general not the whole of the iterated Hecke stack $\mathrm{Hecke}^{(J',J)}$ over ends in Y_{c}, and its convolution map need not be proper. We therefore restrict it to Y_{a}. By Lemma 11.9(f) the pairing of the restriction is the pairing itself on the classes supported over Y_{a}, extended by zero, and these include the classes of the truncation by μ. Over Y_{a} the constraint is automatic: the middle bundle is obtained from a bundle of Y_{a} by a modification with legs in J', so it lies in Y_{b}. Hence the restricted correspondence is supported on the whole iterated Hecke stack over ends in Y_{a}, and its convolution map is proper. Conversely, for a shtuka $(\mathcal{E}_1 \to \mathrm{Fr}\,\mathcal{E}_0 \to \mathrm{Fr}\,\mathcal{E}_1)$ with $\mathcal{E}_1 \in Y_{\mathrm{a}}$, the bundle $\mathcal{E}_0$ lies in Y_{b}; so no part of a class supported over Y_{a} is lost on the other side.

Now $\Psi_{\phi_\lambda} = \mathcal{F}(\lambda) \circ \Psi_c$. By naturality of the Hecke isomorphisms in the coefficient, $(\epsilon_1 \otimes N) \circ \mathcal{F}(\lambda) = \mathcal{F}(\lambda \otimes 1) \circ \epsilon_1$, where on the right ϵ_1 is taken with the coefficient $\alpha^*(W \otimes V_\rho)$. By the claim of step 4, applied to $\mathcal{W} = W \otimes V_\rho$, the correspondence $\epsilon_1 \circ \Psi_c$ has pushforward

$\Psi_{c,\theta}\circ\epsilon_1^\circ$ along r. Composition with the other correspondences preserves this relation. Moreover

$$\mathcal{F}(\lambda\otimes 1)\circ\Psi_{c,\theta}=\Psi_{\phi_{\lambda'}},\qquad \lambda'=(\lambda\otimes 1)\circ(1\otimes\theta_J):\alpha^*(\mathcal{W}'\otimes \mathrm{Frob}_J^*\,V_\rho)\to\mathcal{W}'\otimes N,$$

with $\mathcal{W}'=W\otimes V_{1,\rho}$. Thus $(\mathcal{W}',N,\lambda')$ is a trace datum over the base $\mathrm{Frob}_J:U^I\to U^I$, and $\Psi_{\phi_{\lambda'}}\circ\epsilon_1^\circ\circ\epsilon_2$, restricted to Y_{a}, is the correspondence defining its pairing, for the partition (J',J) and with $Y_0=Y_{\mathrm{a}}$ in step 1; its convolution map is proper, as we have just seen, so Lemma 11.9(c) applies to it. On fixed loci r induces the map

$$\mathrm{Cht}_I^{(J,J')}\longrightarrow \mathrm{Cht}_I^{(J',J)}\times_{U^I,\mathrm{Frob}_J}U^I,$$

which sends a shtuka $(\mathcal{E}_0\to\mathcal{E}_1\to\mathrm{Fr}\,\mathcal{E}_0)$ to $(\mathcal{E}_1\to\mathrm{Fr}\,\mathcal{E}_0\to\mathrm{Fr}\,\mathcal{E}_1)$. This is the partial Frobenius morphism of [Laf18, §3], and the identification of kernels, which is $\Phi_{\mathcal{S}}$ on the factor with legs in J and the identity on the other, is the isomorphism of [Laf18, Propositions 3.3, 3.4 and 4.10]. Hence the pullback on cohomology is the map F_J of [Laf18, §4.3]; truncations are treated as there, using [Laf18, Lemme 3.1]. Lemma 11.9(c), applied to the finite universal homeomorphism r, now gives

$$\Pi^{W,\lambda}\circ F_J=\Pi^{\mathcal{W}',\lambda'}\qquad\text{on }\ \mathrm{Frob}_J^*\,\mathcal{H}_{I,V},$$

first on the classes which come from the truncation by μ of the target of the partial Frobenius morphism. For the pullback of such a class the first bundle lies in Y_{b}: it is the inverse image under Frobenius of the result of a bounded modification at the legs J' of a bundle in Y_{a}. So the two maps of the equality compute the same trace on these classes. Every class comes from such a truncation, and both maps are compatible with extension by zero; the equality follows on the direct limit. By (11.8) the right side is $\mathrm{Frob}_J^*\,\Pi_{I,V}(a_{\mathcal{W}',\lambda'},-)$, and by Lemma 11.5(3), applied fibrewise with $\vartheta=\theta_J$ and using (11.1), we have $a_{\mathcal{W}',\lambda'}=\Theta_J(a_{W,\lambda})$. This proves (11.7) and completes the proof of Proposition 11.10. □

Remark 11.11 (Relation to a statement of V. Lafforgue)**.** In step 5 the closing map λ stays in the path when the first modification is moved past Frobenius, and the coefficient changes from W to $W\otimes V_{1,\rho}$. For a single eigensheaf, when Z is trivial, the matrix coefficients form the sheaf V_ρ^*, and (11.7) says that the adjoint maps $\mathcal{H}_{I,V}\to V_\rho$ commute with the partial Frobenius maps. This is the statement left unwritten in the proof of [Laf18, Proposition 15.1]; compare [AGK+21, §4.5.5] for sheaves with nilpotent singular support in the absence of level structure. The only input specific to the situation is (F3).

11.5 Evaluating an excursion

Let $H_{I,V}$ be the fibre of $\mathcal{H}_{I,V}$ at the geometric generic point $\bar\eta_I$ of U^I. By Lemma I.5.8 the partial Frobenius maps and the fundamental group act on it through W_F^I. The fibre of $\mathcal{A}_{I,V}$ at $\bar\eta_I$ is the space $A(V)$ for the fibre of V_ρ.

Proposition 11.12 (Excursion operators on matrix coefficients)**.** *For every excursion datum e with tuple in W_F^I, every $a\in\mathcal{O}(Z)^\star$ and every $h\in C_c(Y(\mathbb{F}_q),k)$,*

$$\Pi_\varnothing(a,S_eh)=\Pi_\varnothing(\nu_{\rho_\bullet}(e)\,a,h),$$

where $\nu_{\rho_\bullet}(e)\,a$ is the function $z\mapsto\nu_{\rho_z}(e)a(z)$.

Proof. The map $\Pi_{I,V}$ is a map of sheaves, so its fibre at $\bar\eta_I$ is invariant under the geometric fundamental group of U^I, and by Proposition 11.10(3) it is invariant under the partial Frobenius maps, acting on $\mathcal{A}_{I,V}$ by Θ_J^{-1}. The Frobenius–Weil group of [Laf18, Remarque 8.18] is generated by the geometric fundamental group and lifts of the partial Frobenius morphisms.

Evaluation at z carries its action on $A(V)$ to the action on the dual of the fibre of V_{ρ_z}, which factors through the product of the Weil groups. The evaluations being jointly injective, the action on $A(V)$ factors through W_F^I and is

$$(\gamma \cdot a)(z) = a(z) \circ \rho_z(\gamma)^{-1}.$$

Thus $\Pi_{I,V}(a, \gamma h) = \Pi_{I,V}(\gamma^{-1} \cdot a, h)$ for $\gamma \in W_F^I$, $a \in A(V)$ and $h \in H_{I,V}$.

Write $f((g_i)) = \langle \xi, (g_i)x \rangle$ with $x \in V$ and $\xi \in V^*$ invariant under the diagonal group. The operator S_e is the composite of the creation operator of x, the action of $\gamma = (\gamma_i)$ on $H_{I,V}$ and the annihilation operator of ξ. Creation is the map $\mathcal{H}(x) : \mathcal{H}_{\{0\},\mathbf{1}} \to \mathcal{H}_{\{0\},V^\triangle}$ followed by the coalescence isomorphism and by specialization to $\bar{\eta}_I$; annihilation is defined in the same way from ξ. The sheaf $\mathcal{A}_{I,V}$ is a sum of lisse sheaves and $\Pi_{I,V}$ is a map of sheaves, so it commutes with specialization. By Proposition 11.10(1),

$$\Pi_{I,V}(a', \mathcal{C}_x^\sharp h) = \Pi_\varnothing(a'(\cdot)(x), h), \qquad \Pi_\varnothing(a, \mathcal{C}_\xi^\flat h') = \Pi_{I,V}(a\,\xi, h'),$$

where $a\,\xi \in A(V)$ is the function $z \mapsto a(z)\xi$; it satisfies the condition of Definition 11.4 because ξ is invariant under Z. Combining the three steps,

$$\Pi_\varnothing(a, S_e h) = \Pi_{I,V}(\gamma^{-1} \cdot (a\,\xi), \mathcal{C}_x^\sharp h) = \Pi_\varnothing\big(z \mapsto a(z)\langle \xi, \rho_z(\gamma)x \rangle, h\big),$$

and $\langle \xi, \rho_z(\gamma)x \rangle = \nu_{\rho_z}(e)$. □

Proof of Theorem 11.2. By Proposition 11.10(2) and (11.8), $L_\varphi = \Pi_\varnothing(a_{R,\lambda_R}, -)$. By Proposition 11.12 and (11.4),

$$L_\varphi(S_e h) = \Pi_\varnothing\big(\nu_{\rho_\bullet}(e)\, a_{R,\lambda_R}, h\big) = \nu_\rho(e)\, \Pi_\varnothing(a_{R,\lambda_R}, h) = \nu_\rho(e) L_\varphi(h).$$ □

Corollary 11.13 (The parameter of a nonzero trace function)**.** *If ρ is semisimple and elliptic and $\varphi \neq 0$, then φ is a cusp form with compact support and a joint eigenfunction of the excursion operators for all Galois tuples. Its parameter is $\vartheta_G \rho$, where ϑ_G is the Chevalley involution.*

Proof. By Theorem 11.2 the linear form L_φ satisfies (I.8.3) with $\sigma_{\text{coeff}} = \rho$, for all tuples in the Weil group. Proposition I.8.10 then gives both assertions; the Chevalley involution comes from the transposition in that proposition. □

12 Returning to the original curve

At this point we have a family of Weil complexes on $\mathrm{Bun}_G(C_0)$, for the curve $C_0 = X \cup_D C_2$ over $\mathbb{F}_q$, and a G-bundle $\mathcal{G}_0$ on C_0, defined over $\mathbb{F}_q$, at which the member with coefficient P_{reg} has nonzero trace (Proposition 10.9). We want a function on bundles on X. A bundle on C_0 is a pair of bundles on X and on C_2 with an identification along D, so we fix the bundle on C_2 and let the one on X vary. The bundle on C_2 must be the restriction of $\mathcal{G}_0$: then $\mathcal{G}_0$ itself lies in the resulting family, and the function obtained is not zero. Theorem 12.2 is then assembled from the results of Sections 6–11.

12.1 Fixing the bundle on the auxiliary component

Recall from Definition 5.1 the notion of a Hecke family with Frobenius; for $\mathcal{Y} = \mathrm{Bun}_{G,D}(X)$ its conditions are (F1)–(F3) of Setup 11.1.

Proposition 12.1 (Restriction with a trivialization on the boundary)**.** *Let $C_0 = X \cup_D C_2$ be the special fibre of the curve of Proposition 7.7; the divisor D, identified with $D_2 \subset C_2$, may be nonreduced. Let $\mathrm{Bun}_{G,D}(X)$ be the stack of G-bundles on X with a trivialization along D. Suppose given a Hecke family with Frobenius on $\mathrm{Bun}_G(C_0)$ whose legs lie in an open subset containing $X - \operatorname{supp} D$, and a point $\mathcal{G}_0 \in \mathrm{Bun}_G(C_0)(\mathbb{F}_q)$ at which one member of the family has nonzero trace. Then there is a morphism*

$$\mathsf{gl} : \mathrm{Bun}_{G,D}(X) \longrightarrow \mathrm{Bun}_G(C_0)$$

over $\mathbb{F}_q$ such that the pullback of the family by gl is a family with the structure (F1)–(F3) for legs in $X - \operatorname{supp} D$, and the pullback of that member has nonzero trace at a point of $\mathrm{Bun}_{G,D}(X)(\mathbb{F}_q)$.

Proof. Step 1: the morphism. Let $\mathcal{G}_2$ be the restriction of $\mathcal{G}_0$ to C_2. Its restriction to D_2 is trivial: D_2 is a finite scheme over $\mathbb{F}_q$, a G-torsor over a finite field is trivial by the theorem of Lang, and a section over the reduced subscheme lifts to D_2 because the torsor is smooth. Choose a trivialization of $\mathcal{G}_2$ along D_2. A bundle on X with a trivialization along D can then be glued to $\mathcal{G}_2$ along $D \simeq D_2$, and this defines gl.

Since D may be nonreduced, and its scheme structure determines the level of the function we are going to construct, we give the details of the gluing. Locally the ring of C_0 is the fibre product of the rings of X and of C_2 over the ring of D, by Proposition 7.7, and this remains true after base change to any test scheme. Finite projective modules over such a fibre product are equivalent to pairs of finite projective modules over the two rings with an isomorphism over the ring of D, compatibly with tensor products [The26, Tag 0D2J]. To pass to G-bundles choose a faithful representation $G \to \mathrm{GL}_n$; the quotient GL_n / G is affine, because G is geometrically reductive. A G-bundle is a vector bundle of rank n with a section of the associated bundle with fibre GL_n / G. The vector bundles glue by the equivalence just quoted, and a section of a scheme affine over C_0 is the same as a pair of compatible sections over the two components. The same applies to isomorphisms. Hence, if $\mathcal{B}_1$ and $\mathcal{B}_2$ are the stacks of bundles on X and on C_2 with a trivialization along D,

$$\mathrm{Bun}_G(C_0) \simeq [\mathcal{B}_1 \times \mathcal{B}_2 / G_D], \qquad G_D = \mathrm{Res}_{D/\mathbb{F}_q}(G \times D),$$

and the choice of $\mathcal{G}_2$ with its trivialization gives

$$\mathrm{Bun}_{G,D}(X) \simeq \mathrm{Bun}_G(C_0) \times_{\mathrm{Bun}_G(C_2)} \{\mathcal{G}_2\}. \tag{12.1}$$

Step 2: the trace. The bundle $\mathcal{G}_0$ defines a point $\mathcal{G}_1^{\mathrm{fr}}$ of the fibre (12.1) over $\mathbb{F}_q$, which gl maps to $\mathcal{G}_0$. For the member $\mathcal{K}$ of the family with nonzero trace at $\mathcal{G}_0$, the stalk of $\mathsf{gl}^*\mathcal{K}$ at $\mathcal{G}_1^{\mathrm{fr}}$ is the stalk of $\mathcal{K}$ at $\mathcal{G}_0$, with the same action of Frobenius. So

$$\mathrm{Str}\big(\mathrm{Frob}_q, (\mathsf{gl}^*\mathcal{K})_{\mathcal{G}_1^{\mathrm{fr}}}\big) = \mathrm{Str}\big(\mathrm{Frob}_q, \mathcal{K}_{\mathcal{G}_0}\big) \neq 0.$$

Step 3: the Hecke structure. A modification of a bundle on C_0 at points of $X - \operatorname{supp} D$ does not change its restriction to C_2. So the Hecke correspondence of $\mathrm{Bun}_{G,D}(X)$ with legs in $X - \operatorname{supp} D$ is the base change by gl of the Hecke correspondence of $\mathrm{Bun}_G(C_0)$, for the projection to the source and for the projection to the target. The formal neighbourhoods of the legs are the same for X and for C_0, so the two correspondences have the same Satake kernels; this includes the loci where legs coincide, where one uses the sum of the graphs of the legs. Let h_0 and h_1 be the projections of a bounded part of the correspondence of C_0

and A_H the kernel, and let h_0^X, h_1^X and A_H^X be the corresponding objects for X. Since h_1 is proper on a bounded part, proper base change gives

$$(\mathsf{gl} \times \mathrm{id}_{U^I})^* h_{1,!}(h_0^* \mathcal{K} \otimes A_H) \simeq h_{1,!}^X((h_0^X)^* \mathsf{gl}^* \mathcal{K} \otimes A_H^X).$$

Base change isomorphisms are compatible with maps of representations, with convolution and with restriction to diagonals. So the isomorphisms (F1) of the family on $\mathrm{Bun}_G(C_0)$ give isomorphisms (F1) for its pullback, with the same compatibilities. Everything is defined over $\mathbb{F}_q$, so the isomorphisms (F2) pull back as well and (F3) is preserved. □

12.2 Geometric realization when p does not divide the order of the Weyl group

Theorem 12.2 (Geometric realization)**.** *Let G/F be split, simply connected and semisimple, and suppose that $p \nmid |W_G|$. For every semisimple elliptic parameter $\sigma : \Gamma_F \to \widehat{G}(k)$ there are an open compact subgroup $K \subset G(\mathbb{A}_F)$ and a nonzero cusp form φ_σ on $G(F)\backslash G(\mathbb{A}_F)/K$ such that*

$$S^{\mathrm{fun}}_{I,f,\boldsymbol{\gamma}} \varphi_\sigma = f((\sigma(\gamma_i))_{i \in I}) \varphi_\sigma$$

for every excursion datum $(I, f, \boldsymbol{\gamma})$ with $\boldsymbol{\gamma} \in \Gamma_F^I$. In particular the excursion summand of σ in the space of cusp forms of level K is not zero.

The proof occupies the rest of this subsection. It assembles the results of the preceding sections in three stages, which produce a family of sheaves on the special fibre and a rational point at which one of them has nonzero trace. The step which is new here is Proposition 12.3: the family restricted to bundles on X has the structure required by Theorem 11.2.

If G is trivial the constant function has the required properties. Otherwise the Weyl group contains a reflection, so $p \neq 2$. We apply the construction to the parameter $\sigma^\dagger = \vartheta_G \sigma$, which is again continuous, finitely ramified, semisimple and elliptic. The reason is that the function obtained at the end has the parameter $\vartheta_G \sigma^\dagger$ (Corollary 11.13), which is conjugate to σ.

The smooth curve and its local system. By Lemmas I.4.1 and I.4.4 the parameter $\sigma^\dagger$ has values in a finite extension of $\mathbb{Q}_\ell$, its arithmetic and geometric monodromy groups $\mathsf{Mon}^{\mathrm{ar}}_{\sigma^\dagger}$ and $\mathsf{Mon}^{\mathrm{geom}}_{\sigma^\dagger}$ are reductive, and the quotient of the first by the second is finite and cyclic. Theorem 7.1 lifts $\sigma^\dagger$ to characteristic zero. Proposition 7.7 gives a proper family $\mathcal{C}$ of curves with a theta characteristic, whose special fibre is $C_0 = X \cup_D C_2$ and whose generic fibre is smooth of characteristic zero, and Proposition 7.10 gives a local system ρ on the geometric generic fibre such that

$$\mathsf{Mon}^{\mathrm{geom}}_{\sigma^\dagger} \subset \mathsf{Mon}_\rho \subset \mathsf{Mon}^{\mathrm{ar}}_{\sigma^\dagger}, \qquad \overline{\langle \mathsf{Mon}_\rho, g \rangle} = \mathsf{Mon}^{\mathrm{ar}}_{\sigma^\dagger},$$

where g is the image under $\sigma^\dagger$ of the chosen arithmetic Frobenius element. These data satisfy the conditions of Setup 6.1: Mon_ρ is reductive, the quotient in (2) is finite, and the centralizer in (3) is the centralizer of $\mathsf{Mon}^{\mathrm{ar}}_{\sigma^\dagger}$, which is finite because $\sigma^\dagger$ is elliptic and G is semisimple. By Lemma 6.2 the group

$$Z_\rho = C_{\widehat{G}}(\mathsf{Mon}_\rho), \qquad \alpha = \mathrm{Ad}(g)|_{Z_\rho},$$

is reductive, its identity component is a torus, and $1 - \alpha$ is an isogeny of this torus. Since $\mathsf{Mon}^{\mathrm{geom}}_{\sigma^\dagger} \subset \mathsf{Mon}_\rho$, the group Z_ρ centralizes the image under $\sigma^\dagger$ of the geometric fundamental group. It will be the group Z of Setup 11.1.

The family and the trace of its coefficient. Choose the divisor D_{aux} on the second component as in Lemma 8.3. For a finite-dimensional representation W of Z_ρ let

$$P_W = \mathcal{O}_{\mathfrak{F}_\rho} \otimes W, \qquad P_{\mathrm{reg}} = P_{k[\pi_0(Z_\rho)]}.$$

Section 6 attaches to these objects the sheaves $\mathcal{F}_{P_W}$ on the stack of bundles of the generic fibre, with their Hecke identities and their compatibility with Frobenius (Lemma 6.6). By Proposition 6.10, whose hypothesis on weights holds by Lemma 8.3, Frobenius and its inverse have nonzero supertrace on $\Gamma(\mathfrak{F}_\rho, P_{\mathrm{reg}})$.

A rational bundle with nonzero trace. By Theorem 8.8 the whole family extends over the base S. The functor $P \mapsto \mathcal{F}_P$ is extended, and not only the object $\mathcal{F}_{P_{\mathrm{reg}}}$; so the comparison with a microstalk is natural in P, and by Lemma 6.7 the two actions of Frobenius differ by one scalar. These arguments are carried out in Sections 9 and 10, and their conclusion is Proposition 10.9: there is a bundle $\mathcal{G}_0 \in \mathrm{Bun}_G(C_0)(\mathbb{F}_q)$ at which $\mathcal{F}_{P_{\mathrm{reg}},s}$ has nonzero trace of geometric Frobenius. (Geometric Frobenius is the inverse of the Frobenius automorphism used in Lemma 6.6, see Remark 11.3; this is why both Frobenius and its inverse were considered in the preceding paragraph.)

Proposition 12.3 (The family on bundles on X)**.** *With the data just constructed, let*

$$\mathcal{F}^X_W = \mathsf{gl}^* \mathcal{F}_{P_W,s} \qquad \text{on } \mathrm{Bun}_{G,D}(X),$$

with gl *as in Proposition 12.1 for the bundle* $\mathcal{G}_0$*. Then* $W \mapsto \mathcal{F}^X_W$ *is a family with the structure (F1)–(F3) of Setup 11.1, for the group* $Z = Z_\rho$ *with its automorphism* α *and for the local system on* $X - \operatorname{supp} D$ *which is the restriction of the local system* $\mathcal{E}$ *on the base* S*, namely* $\sigma^\dagger$ *(Proposition 7.10). The Frobenius structure on the Satake kernels which enters (F3) is that of [Laf18, (3.5) and Propositions 4.10–4.14].*

Proof. By Proposition 12.1 it suffices to see that the functor $W \mapsto \mathcal{F}_{P_W,s}$ on $\mathrm{Bun}_G(C_0)$ has this structure, with legs in U°_s, and this is obtained from Lemma 6.6 as follows.

Step 1: the Hecke identities (F1). For $P = P_W$ and a decomposition $V_\rho = \bigoplus_a W_a \boxtimes L_a$ into isotypic parts, with W_a irreducible and $L_a = \underline{\mathrm{Hom}}_{Z_\rho}(W_a, V_\rho)$, the Hecke identity (6.5) reads

$$\mathsf{H}_V(\mathcal{F}_{P_W}) \simeq \bigoplus_a \mathcal{F}_{P_{W \otimes W_a}} \boxtimes L_a.$$

The functor $W \mapsto \mathcal{F}_{P_W}$ is additive and Z_ρ is reductive, so $\mathcal{F}_{P_{W\otimes W_a}} = \bigoplus_{W'} \mathcal{F}_{P_{W'}} \otimes \mathrm{Hom}_{Z_\rho}(W', W \otimes W_a)$, the sum being over the irreducible representations. Hence the right-hand side is

$$\bigoplus_{W'} \mathcal{F}_{P_{W'}} \boxtimes \underline{\mathrm{Hom}}_{Z_\rho}(W', W \otimes V_\rho) = \mathcal{F}(W \otimes V_\rho)$$

in the notation of Setup 11.1. The same computation applies to every finite set of legs. The isomorphisms are natural in W because the Hecke identities are natural in P, and their compatibility with maps of representations V, with restriction to diagonals and with convolution is the compatibility asserted in Lemma 6.6(1). By Theorem 8.8 these isomorphisms, with their compatibilities, extend over S, for the Hecke functors defined by the kernels $\mathcal{S}^{\mathrm{val}}_{I,V}$. The restriction of these Hecke functors to the fibre over s is the Hecke functor of that fibre, by proper base change and Lemma 8.5(1) and (3). So the restriction to s has the isomorphisms (F1).

Step 2: the Frobenius covariance (F2). Let u_C be the arithmetic Frobenius of the curve and $c : u_C^* \rho \simeq \rho$ the descent isomorphism, with matrix g in the marked fibre. Pullback of local

systems by u_C acts on BZ_ρ through $z \mapsto c\, u_C^*(z)\, c^{-1} = \alpha(z)$. The spectral map of Lemma 6.6 is the transport, the inverse of this pullback, so

$$u_{\mathfrak{F}}^* P_W = P_{(\alpha^{-1})^* W}, \qquad u_{\mathrm{aut}}^* \mathcal{F}_{P_W} \simeq \mathcal{F}_{P_{(\alpha^{-1})^* W}}.$$

This covariance extends over S. On the special fibre u_{aut}^* is inverse to Fr^*, and the inverse of the covariance is $\mathrm{Fr}^* \mathcal{F}_W \simeq \mathcal{F}_{\alpha^* W}$.

Step 3: the compatibility (F3). The inverse covariance acts on the legs by u_C^{-1}, and the corresponding isomorphism of local systems is $\theta = (u_C^{-1})^*(c^{-1})$, with matrix g^{-1} in the marked fibre. It is an equivariant map $\alpha^* \mathrm{Fr}^* V_\rho \to V_\rho$ under Z_ρ. The compatibility of the covariance with the Hecke identities, in Lemma 6.6(2), read in the inverse direction, is (F3). It is preserved by the extension over S and by the restriction to s, like all the maps which enter it.

Step 4: the Frobenius structure of the Satake kernels. The Frobenius structure obtained in this way must be the one used in (F3). The semilinear pullback of the Satake kernels on the generic fibre, and the comparison maps, are compatible with convolution and with fusion, and so are their extensions, by Lemma 8.5(3) and the uniqueness of ULA extensions. So on the special fibre we have two Frobenius structures on the Satake kernels compatible with the tensor structure: this one, and that of [Laf18, (3.5) and Propositions 4.10–4.14]. The kernels are taken with their Tate twists. On the open stratum of the Schubert variety of a coweight λ the kernel is the constant sheaf with the shift and twist $[d_\lambda](d_\lambda/2)$, where $d_\lambda = \langle 2\rho_G, \lambda\rangle$ is even because G is simply connected. Both structures restrict there to the canonical structure of the constant sheaf and of the Tate twist, for which the semilinear comparison was the canonical one. An isomorphism between intersection complexes is determined by its restriction to the open stratum, so the two structures agree on the kernel of every irreducible representation, for one leg, and by naturality on all the kernels. The tensor structure gives the same conclusion: for one leg the ratio of the two structures is an automorphism of the Satake functor compatible with tensor products. By the Satake equivalence it is given by an element of $\widehat{G}$ which commutes with $\widehat{G}$, that is, by an element of the centre of $\widehat{G}$, and this centre is trivial because G is semisimple and simply connected. For several legs the two structures agree by factorization and fusion. □

Proof of Theorem 12.2. Step 1: the parameter. Let φ_σ be the trace function of the member of the family of Proposition 12.3 for the regular representation of $\pi_0(Z_\rho)$. The sheaf $\mathcal{F}_{P_{\mathrm{reg}},s}$ has nonzero trace at $\mathcal{G}_0$, so φ_σ is not zero (Proposition 12.1). On this representation the arithmetic closing map sends e_γ to $e_{\alpha(\gamma)}$, and its inverse is the map λ_R of Setup 11.1. By Theorem 11.2 the linear form defined by φ_σ on functions with compact support is an eigenvector of the excursion operators for tuples in the Weil group, with the eigenvalues of $\sigma^\dagger$. By Corollary 11.13, φ_σ is a cusp form with compact support and an eigenfunction of the excursion operators for all Galois tuples, with parameter $\vartheta_G \sigma^\dagger$, that is, σ up to conjugation. (See Remark 12.4 for the dependence of this step on conventions.)

Step 2: the level. Write $D = \sum_v n_v v$ and

$$K_D = \prod_{v \notin \mathrm{supp}\, D} G(\mathcal{O}_v) \times \prod_{v \in \mathrm{supp}\, D} \ker\left(G(\mathcal{O}_v) \longrightarrow G(\mathcal{O}_v/\mathfrak{m}_v^{n_v})\right).$$

Since G is smooth, $G(\mathcal{O}_v)$ maps onto $G(\mathcal{O}_v/\mathfrak{m}_v^{n_v})$, so a trivialization along D is the same as a level structure for K_D, also when D is not reduced. For split simply connected G the points of $\mathrm{Bun}_{G,D}(X)$ over $\mathbb{F}_q$ are $G(F)\backslash G(\mathbb{A}_F)/K_D$ [Laf18, Remark 8.21], and φ_σ is a function on this set. The bundle $\mathcal{G}_0$, its restriction to C_2 and the trivialization are defined over $\mathbb{F}_q$, and

the values of φ_σ lie in a finite extension of $\mathbb{Q}_\ell$. So the conclusion holds for the given field F and the given prime ℓ. □

Remark 12.4 (Independence of conventions)**.** Which of σ and $\vartheta_G\sigma$ is the parameter of φ_σ depends on conventions of several of the works we use: whether the Hecke functor attached to V in [AGK+22, GR25] corresponds to the functor $\mathsf{H}_{I,V}$ of [Laf18] or to the one attached to the dual representation, and the transposition in Proposition I.8.10. The two normalizations of the Satake equivalence which occur, and the effect of passing from one to the other, are described in [I, Section 3.2]. The existence statement of Theorem 12.2 does not depend on them: by Lemma I.8.11 the cuspidal excursion summand of σ at a given level is not zero if and only if that of $\vartheta_G\sigma$ is not zero. For the same reason it does not matter whether ϑ_G is an involution or only has inner square.

Remark 12.5. The condition $p \nmid |W_G|$ is used in two places. In Theorem 7.1 it makes the images of wild inertia lie in tori, so that the parameter lifts to characteristic zero with a divisor of ramification independent of the finite quotient. In Lemma 9.4 it makes the invariant form nondegenerate on all Levi subalgebras. There is no hypothesis on the ramification of σ or on the component group of the centralizer of its geometric monodromy; these affect the curves C_2 and C and the level of φ_σ only.

12.3 The example of Section 6.4, concluded

For the parameter $\sigma : \Gamma_F \to S_3 \subset \mathrm{PGL}_2(k)$ over $F = \mathbb{F}_5(t)$ of Sections 6.4 and 7.5, the theorem gives the following. The group is $G = \mathrm{SL}_2$, and $p = 5$ does not divide $|W_G| = 2$. The genus of the smooth curve is 841, and its special fibre is $C_0 = \mathbb{P}^1 \cup_D C_2$, with D a divisor of degree 58 on $\mathbb{P}^1$ which contains the points where σ is ramified. The group Z_ρ has two elements, the perfect complex P_{reg} is $\mathcal{O} \otimes k[\mu_2]$. On its global sections the arithmetic supertrace is $(1-5)^3 = -64$, and the geometric one is $(1-5^{-1})^3 = 64/125$. Since these numbers are not zero, Proposition 10.9 provides a bundle on C_0 over $\mathbb{F}_5$ at which the trace function of $\mathcal{F}_{P_{\mathrm{reg}},s}$ does not vanish, and the theorem provides a nonzero cusp form on $\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A}_F)/K_D$ with parameter σ. What the construction does not give is a formula: neither the bundle nor any value of the cusp form is determined explicitly, and the level K_D is far from minimal. (For this group the result also follows from [III]; the example is meant to show the size of the objects.)

A Computer checks

Some of the computations in this paper were also carried out by computer. No proof depends on such a computation: in each case the argument is given in the text, and the program is a check on it. The programs are written in Python, with the library SymPy, and are in the directory `anc` of the source of this paper. They compute with matrices, for subspaces of Lie algebras.

Statement	What is checked	Program and range
Lemma 9.4(2)	the primes dividing the discriminant of the invariant form on the span of the simple coroots of each standard Levi divide $\lvert W_G\rvert$	`check_kappa.py`; A_n, B_n, C_n, D_n for $n \le 6$, and all exceptional types

Statement	What is checked	Program and range
Lemma 9.6, step 3; Lemma 9.10; Lemma 9.14, step 1; Lemma 9.22	the rank of the pairing (9.4), with degenerate cases as controls; the identities of the chart of the universal conic; the graded Lagrangian complement; the recurrence and the rank for a truncation in Laurent polynomials	`check_transverse_chart.py`; five blocks and orders of pole in $\mathfrak{gl}_2, \mathfrak{gl}_3, \mathfrak{gl}_4$, 66 graded cases with initial dimension at most 10, and the chart identities symbolically

Notation

The table lists notation used in more than one section, with the place where it is introduced.

Notation	Meaning and location
X, F, Γ_F, W_F	The original curve, its function field, its absolute Galois group and its Weil group (Section 4).
p, q, ℓ, k	The geometric characteristic, size of the constant field, coefficient prime, and $k = \overline{\mathbb{Q}}_\ell$.
K_{geom}	The characteristic-zero geometric field of the smooth curve C, distinct from k (Section 6).
$\deg_F, d_v$	Arithmetic Galois degree and the degree of a closed point v; geometric Frobenius at v has degree $-d_v$.
$\mathrm{Fr}_q, \mathrm{Frob}_q$	The q-power map of a scheme and geometric Frobenius acting on sheaf cohomology, respectively.
$G, \widehat{G}, M, \widehat{M}$	The split group, its dual, a Levi subgroup and the corresponding dual Levi.
σ, ν_σ	The prescribed parameter and its character on excursion data, defined by (1.1).
$\vartheta_G, \sigma^\dagger$	The Chevalley involution preserving the pinning, and $\vartheta_G\sigma$ (Section 4).
$e = (I, f, \boldsymbol{\gamma})$, S_e, S_e^{fun}	An excursion datum, its excursion operator on functions with compact support, and the transposed operator on all functions ([I, Definitions 5.7 and 5.12]).
$C_0 = X \cup_D C_2$	The curve, geometrically reduced and Gorenstein, obtained by gluing X and the auxiliary component C_2 along the divisor D, which determines the level of φ_σ (Section 7).
S, η, ξ, s	The spectrum of a valuation ring of rank two and its three points (Section 8.1).
D_{aux}, ρ	The additional divisor on the auxiliary component, and the unramified local system on the smooth proper curve in characteristic zero (Setup 6.1).
$\mathcal{E}$	The local system on the smooth locus of the family of curves over S (Lemma 8.2).
e	The embedding of quasi-coherent complexes into the spectral category (Section 6).
gl	The gluing morphism $\mathrm{Bun}_{G,D}(X) \to \mathrm{Bun}_G(C_0)$ (Proposition 12.1).
f	In an excursion datum $(I, f, \boldsymbol{\gamma})$, a function on $\widehat{G}^I$; in Sections 9 and 10, a test function on a chart.
I, I_{in}	A finite set of legs; an inertia group (Section 10).
Z_ρ, Π_ρ	Centralizer of the geometric monodromy and its component group (Section 6).

Notation	Meaning and location
$\mathfrak{F}_\rho, P_W$	The derived fibre of the restriction to the punctured curve, and the perfect complex $\mathcal{O}_{\mathfrak{F}_\rho} \otimes W$ on it.
$\mathcal{F}_P$	The automorphic sheaf attached to a perfect complex P on $\mathfrak{F}_\rho$.
$\Delta_Y(\mathcal{K})$	The specialization cone of $\mathcal{K}$ on Y; it vanishes exactly when $\mathcal{K}$ is universally locally acyclic (Section 9).
$\mathsf{H}_{I,V}$, H_V	The Hecke functor attached to a representation V of $\widehat{G}^I$, with legs in the smooth locus of the curve; H_V when I has one element (Section 5).
Nilp	The global nilpotent cone in the cotangent stack of Bun_G.
μ^-	The part of the microstalk at the regular nilpotent point on which $-1 \in \mathbb{G}_m$ acts by -1 (Section 10).

The following terms are introduced in this paper, or are used in a sense fixed here.

Term	Where it is defined
parameter; semisimple; elliptic	Definition 1.1
excursion summand	Definition 4.1
automorphic realization; to realize a parameter	Section 1, after (1.1)
Hecke family; Hecke family with Frobenius	Definition 5.1
test function; specialization cone	Section 9
chosen modification	Section 9.3
Hecke chart	Definition 9.5
trace datum; matrix coefficient	Definition 11.4
correspondence with kernel; its trace and pairing	Definition 11.8

Statement on the use of AI

The mathematical arguments of the three papers of this series were developed with GPT-6 Astra Pro. The papers were then organized and rewritten with Claude Fable 5.2, and in the course of this several arguments were replaced or expanded. The three papers were then reviewed in detail, line by line, by GPT and Claude models, in sessions which had not taken part in writing them, and the errors found in this way were corrected.

Boyang Hu organized the work and is responsible for the manuscript.

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