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Automorphic realization of Langlands parameters for split groups over global function fields

I. Arithmetic foundations and the main theorems

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Abstract

Let G be a split connected reductive group over a global function field F. V. Lafforgue decomposed the space of cusp forms on G according to semisimple Langlands parameters. This is the first of three papers which prove a converse: every semisimple elliptic parameter is the parameter of a cusp form on G over F itself, which we call an automorphic realization of the parameter. This holds in every positive characteristic for classical groups, in characteristic greater than 3 for G_2 and F_4, greater than 5 for E_6, and greater than 7 for E_7 and E_8. A parameter that is not elliptic is the parameter of an automorphic form which is not cuspidal, under the bound for the smallest Levi subgroup through which the parameter factors. The form has the prescribed eigenvalue for every excursion operator, and we determine for which lattices in the centre it can be taken invariant.

The present paper contains the arithmetic part of the proof. It defines excursion operators on functions that are not assumed cuspidal, proves their compatibility with constant terms, shows that a function on which they act through an elliptic parameter is a cusp form, and passes from simply connected groups to isogenous groups and from Levi subgroups to the group. This reduces the theorems to the existence of a cusp form with a given elliptic parameter on a simply connected simple group. That is proved in the second paper by a geometric method, when the characteristic does not divide the order of the Weyl group, and in the third paper for classical groups, by descent from general linear groups.

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1 Introduction

Let F be the function field of a curve over a finite field and let G be a split reductive group over F. Vincent Lafforgue decomposed the space of cusp forms on G according to semisimple Langlands parameters $\Gamma_F \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$. His theorem attaches a parameter to each summand of the decomposition. It does not say which parameters occur.

This paper and its two sequels [II, III] address the converse. We show that every semisimple elliptic parameter is the parameter of a cusp form on G, for classical groups in every positive characteristic and for the exceptional groups once the characteristic avoids a short list of primes. Three features of the result should be stated at once. The cusp form lives over F itself: neither the field nor its field of constants is extended. The parameter may be ramified at finitely many places and may have disconnected, even finite, monodromy. And the parameter is recovered in the strongest available sense, through every excursion operator, so the form determines σ up to conjugacy and not only the conjugacy classes of its Frobenius elements.

1.1 Elliptic parameters

There are three theorems. Theorem A concerns elliptic parameters of semisimple groups, which are realized by cusp forms. Theorem B concerns all semisimple parameters of reductive groups. A parameter which is not elliptic is realized by an automorphic form which is not cuspidal, and the definitions which precede the theorem describe the space in which this form lives. Theorem C says which lattices in the centre are compatible with a given parameter. The first theorem is stated in this subsection, and Section 1.2 says how it is proved; a reader interested only in the elliptic case can stop there.

Let $X/\mathbb{F}_q$ be a smooth projective geometrically connected curve, let $F = \mathbb{F}_q(X)$, and put $p = \operatorname{char} F$. Fix a prime $\ell \neq p$, put $k = \overline{\mathbb{Q}}_\ell$, and write $\Gamma_F = \operatorname{Gal}(F^{\mathrm{sep}}/F)$. For a split connected reductive group G/F, the dual group $\widehat{G}$ is taken over k.

Definition 1.1. A *parameter* is a continuous homomorphism $\sigma : \Gamma_F \to \widehat{G}(k)$ unramified at all but finitely many places. It is *semisimple* if the identity component of the Zariski closure of its image is reductive. It is *elliptic* if

$$C_{\widehat{G}}(\operatorname{im}\sigma)/Z(\widehat{G}) \quad \text{is finite.}$$

The link between cusp forms and parameters is given by excursion operators. An excursion datum consists of a finite set I, a function $f \in k[\widehat{G}\backslash\widehat{G}^I/\widehat{G}]$, invariant under simultaneous left and right multiplication, and a tuple $\boldsymbol{\gamma} \in \Gamma_F^I$. Lafforgue's operator $S_{I,f,\boldsymbol{\gamma}}$ acts on cusp forms: it creates an invariant vector in the cohomology of shtukas with legs indexed by I, transports

it by $\boldsymbol{\gamma}$, and contracts with an invariant covector representing f [Laf18, §§0.3–0.5]. These operators commute, and a cusp form φ has parameter σ when

$$S^{\mathrm{fun}}_{I,f,\boldsymbol{\gamma}}\varphi = f\big((\sigma(\gamma_i))_{i\in I}\big)\varphi \qquad \text{for every excursion datum } (I,f,\boldsymbol{\gamma}). \tag{1.1}$$

We write $\nu_\sigma(I,f,\boldsymbol{\gamma})$ for the scalar on the right. A nonzero function φ which satisfies (1.1) is called an *automorphic realization* of σ, and we say that it *realizes* σ. The superscript in S^{fun} records that we need these operators on functions that are not yet known to be cuspidal; on cusp forms S^{fun} is Lafforgue's operator, and the extension is constructed in Section 5.

The characteristic enters through the following bound. For a simple root system put

Φ	A_n, B_n, C_n, D_n	G_2	F_4	E_6	E_7	E_8
$b(\Phi)$	0	3	3	5	7	7.

For a reductive group, $b(G)$ is the maximum of b over the simple factors of its derived group, and $b(G)=0$ for a torus.

Theorem A (Elliptic parameters)**.** *Suppose that G is split semisimple and $p > b(G)$. Let $\sigma : \Gamma_F \to \widehat{G}(k)$ be a semisimple elliptic parameter. Then there are an open compact subgroup $K \subset G(\mathbb{A}_F)$ and a nonzero cusp form φ_σ on $G(F)\backslash G(\mathbb{A}_F)/K$ satisfying (1.1).*

In the language of Lafforgue's decomposition, the theorem says that the excursion summand of the space of cusp forms with character ν_σ is nonzero at some finite level; a nonzero finite-dimensional excursion summand always contains a joint eigenvector, so the two formulations are equivalent. The theorem covers every group in the isogeny class of a semisimple group, not only the simply connected and adjoint ones.

In classical terms the theorem has the following consequence. We write Frob_v for a geometric Frobenius element at a place v.

Corollary 1.2 (Hecke eigenvalues)**.** *In the situation of Theorem A there are a finite set S of places of F and a cuspidal automorphic representation π of $G(\mathbb{A}_F)$ which is unramified outside S and whose Satake parameter at each place $v \notin S$ is the conjugacy class of the semisimple part of $\sigma(\mathrm{Frob}_v)$.*

The corollary is proved at the end of Section 13.1. It is weaker than the theorem: for some groups the conjugacy classes of the elements $\sigma(\mathrm{Frob}_v)$ do not determine σ up to conjugacy; the excursion operators do.

Two instances may help to fix ideas. If G is the split group of type E_8 and $p > 7$, every semisimple elliptic homomorphism $\Gamma_F \to E_8(k)$ is the parameter of a cusp form on G. If $G = \mathrm{Sp}_{2n}$, the same holds for parameters into $\mathrm{SO}_{2n+1}(k)$ in every characteristic, including $p = 2$. A parameter over $\mathbb{F}_5(t)$ with image $S_3 \subset \mathrm{PGL}_2(k)$ is followed through the first steps of the geometric construction in Sections II.6.4 and II.7.5.

Remark 1.3 (The characteristic restriction)**.** For the types G_2, F_4, E_6, E_7, E_8, the condition $p > b(G)$ is the same as $p \nmid |W_G|$. It is a hypothesis of our geometric method, in which it supplies nondegenerate invariant forms on the Lie algebra and its Levi subalgebras and keeps wild inertia inside tori when the parameter is lifted to characteristic zero. We do not expect it to reflect an obstruction to automorphy. For classical groups a second method removes it. For elliptic parameters of the exceptional types in the excluded characteristics, that is, for G_2 and F_4 in characteristics two and three, for E_6 in characteristics two, three and five, and for E_7 and E_8 in characteristics two, three, five and seven, the theorems make no assertion in either direction.

1.2 The two existence theorems

Theorem A, and with it Theorems B and C below, are deduced in this paper from two existence theorems for semisimple groups, which are proved in the two sequels.

Theorem 1.4 (Geometric construction [II, Theorem 12.2]). *Let G be split, simply connected and semisimple, and suppose that $p \nmid |W_G|$. Then every semisimple elliptic parameter $\sigma : \Gamma_F \to \widehat{G}(k)$ is the parameter of a cusp form on G: the conclusion of Theorem A holds.*

Theorem 1.5 (Classical groups [III, Theorem 13.1]). *Let G be split semisimple with simple factors of types A, B, C, D, in any positive characteristic. Then the conclusion of Theorem A holds for every semisimple elliptic parameter of G.*

type of the simple factors	construction	condition on p
A_n, B_n, C_n, D_n	descent and theta correspondence [III]	none
G_2, F_4, E_6, E_7, E_8	geometric [II]	$p \nmid \|W_G\|$

The proof of Theorem 1.4 starts from the geometric Langlands correspondence in characteristic zero. It provides a sheaf, and the function which the trace of Frobenius attaches to a nonzero sheaf can vanish identically. The main point is to arrange that the trace in question is a determinant $\det(1-\phi)$ with no zero factor, and that it stays nonzero under specialization to the given field; an overview is given in [II, Section 2]. The construction applies to every simply connected group with $p \nmid |W_G|$, but it is needed only for the five types in the second row of the table. The proof of Theorem 1.5 starts from L. Lafforgue's theorem for general linear groups and uses automorphic descent and theta correspondence; an overview is given in [III, Section 2].

The two sequels are independent of each other. Each uses Sections 4–10 of the present paper, and neither uses Theorems A–C or Sections 11–13.

1.3 Parameters which are not elliptic

A parameter that is not elliptic factors through a proper Levi subgroup of $\widehat{G}$, and an automorphic form with this parameter should come from the corresponding Levi subgroup of G by Eisenstein series. A minimal dual Levi $\widehat{M}$ containing $\operatorname{im}\sigma$ exists, and σ is elliptic as a parameter into $\widehat{M}$ (Proposition 11.1). The resulting forms are sought in a space larger than the space of cusp forms, which we now describe.

By an *automorphic form* we mean a smooth function on $G(F)\backslash G(\mathbb{A}_F)$, invariant under an open compact subgroup, whose orbit under the spherical Hecke algebra at one unramified place is finite-dimensional; such a function is automorphic in the usual sense (Proposition 6.11). Excursion operators with arbitrary Galois tuples are not known to act on all such functions. They do act on those that are invariant under a lattice in the centre and satisfy an integrality condition for the Hecke action; integrality is what makes the action of the Weil group extend continuously to the Galois group. The space $\mathcal{A}^{\mathrm{cont}}(G)$ consists of the automorphic forms satisfying these two conditions, together with their twists by characters of the constant field; it is defined in Definition 6.9, and Theorem 6.10 gives the action of all excursion operators on it.

Theorem B (Reductive groups). *Let G/F be split connected reductive and let $\sigma : \Gamma_F \to \widehat{G}(k)$ be a semisimple parameter. Let $\widehat{M}$ be a minimal dual Levi containing its image, and let M be the corresponding split Levi subgroup of G. If $p > b(M)$, there is a nonzero automorphic form $\varphi_\sigma \in \mathcal{A}^{\mathrm{cont}}(G)$ satisfying (1.1). If σ is elliptic, φ_σ can be chosen cuspidal; if σ is not elliptic, φ_σ can be chosen not cuspidal.*

When σ is not elliptic, the form φ_σ is not given as a value or a residue of an Eisenstein series induced from the cuspidal form on M. Its existence is deduced from the compatibility of the excursion operators with constant terms, by Nakayama's lemma applied to the functions with compact support modulo the cuspidal ones (Section 12). The form obtained in this way is orthogonal to the cuspidal functions with compact support, and this is why it is not cuspidal.

The twists by characters of the constant field cannot be omitted from the definition of $\mathcal{A}^{\mathrm{cont}}(G)$: when the centre of G has positive dimension, the central character of a continuous parameter need not have finite order.

The bound $p > b(M)$ depends on the parameter and can be weaker than $p > b(G)$. Every proper Levi subgroup of E_6, for example, has only classical simple factors, so a nonelliptic E_6-parameter is realized in every positive characteristic, while our construction for elliptic E_6-parameters needs $p > 5$. Proposition 11.4 gives the comparison for all root systems.

1.4 A prescribed quotient by the centre

One often works on the quotient of $G(F)\backslash G(\mathbb{A}_F)$ by a lattice in the centre, and it is natural to ask which lattices are compatible with a given parameter. Let $T_G = (Z_G^\circ)_{\mathrm{red}}$ and $C_{T_G} = T_G(F)\backslash T_G(\mathbb{A}_F)$. A *central lattice* is a free subgroup $\Xi \subset C_{T_G}$ on which the adelic degree map $\deg : C_{T_G} \to X_*(T_G)$ is injective with image of finite index. Dual to the inclusion $T_G \to G$ there is a map $\alpha_G : \widehat{G} \to \widehat{T}_G$, and class field theory attaches to σ the character

$$\omega_\sigma = \mathrm{CFT}_{T_G}(\alpha_G \sigma)$$

of C_{T_G}; our normalization sends a parameter χ of $\mathbb{G}_m$ to $t \mapsto \chi(\mathrm{Art}(t)^{-1})$, with Art the arithmetic Artin map (Section 3). Let $\mathcal{A}_\Xi(G) \subset \mathcal{A}^{\mathrm{cont}}(G)$ be the subspace of Ξ-invariant functions satisfying the integrality condition, without twists (Definition 6.9).

Theorem C (Central lattices)**.** *Under the hypotheses of Theorem B, let $\Xi \subset C_{T_G}$ be a central lattice.*

1. *If $\omega_\sigma|_\Xi = 1$, the form φ_σ of Theorem B can be chosen in $\mathcal{A}_\Xi(G)$.*
2. *If σ is elliptic, the excursion summand with character ν_σ in the space of cusp forms on $G(F)\backslash G(\mathbb{A}_F)/\Xi$ is nonzero if and only if $\omega_\sigma|_\Xi = 1$.*

Necessity in (2) is the statement that a cusp form with parameter σ has central character ω_σ. Sufficiency rests on the existence theorems. The three theorems are proved together in Section 13.

1.5 Relation to earlier work

General linear groups and excursion operators. Laurent Lafforgue proved the correspondence for irreducible GL_n-parameters with finite-order determinant [Laf02, Introduction, Theorem $(\mathrm{i})_n$–$(\mathrm{ii})_n$]. For a split reductive group, Vincent Lafforgue constructed the decomposition of cusp forms by parameters [Laf18, Theorem 0.1]. That theorem identifies the parameters of the summands that occur; that the summand of a prescribed parameter is nonzero is a separate assertion, and it is the subject of this paper. For a split semisimple group and a parameter which is everywhere unramified and has Zariski dense image, [BHKT19, Theorem 1.1] gives, after a finite Galois extension of the function field, a cuspidal representation whose unramified components have the Satake parameters prescribed by the parameter. Our conclusion keeps the field fixed.

Geometric Langlands. The geometric input is the restricted geometric Langlands equivalence of Gaitsgory and Raskin over an algebraically closed field of characteristic zero [GR25,

Theorem 1.3.9(ii)], together with its Whittaker functional. In positive characteristic their theorem gives an equivalence on a union of connected components of the spectral side, and whether this union is everything is their Conjecture 1.3.10. The subsequent preprint [Ope26c, Theorem 1.4], by OpenAI, states that it is, over $\overline{\mathbb{F}}_q$ under the four Lie-theoretic assumptions of the restricted theory, and over an arbitrary algebraically closed field when in addition p is very good and $p \nmid |W_G|$; its Corollary 8.2 gives the arithmetic formula for unramified parameters. We do not use that theorem. Our result concerns arithmetic parameters over F that are ramified at finitely many places and may have finite or disconnected monodromy, and it produces functions, not sheaves.

Our method does, however, owe several ideas to the three Langlands preprints [Ope26c, Ope26a, Ope26b] in OpenAI's public *math* repository. Testing a Hecke eigensheaf against a hypersurface transverse to the nilpotent cone, letting modification points collide, and working with framed bundles all appear there and are starting points for [II]. The statements of those preprints have different hypotheses from ours: the tame eigensheaf theorem [Ope26a, Theorem 1.1] assumes geometrically dense monodromy and one tame regular-unipotent puncture, and the Frobenius theorem [Ope26b, Theorem 1.1] treats SL_n with $p > n$ and allows a finite extension of constants. None of them is used as an input.

The steps of the geometric argument which are new, and the further works on which it draws, are described in [II, Section 1.2].

Descent and theta correspondence. The second construction starts from automorphy for general linear groups and uses automorphic descent [GRS99, GRS11], the theta correspondence [Ral84], and the theory of automorphic L-functions over function fields [Lom09, Lom19]. The work here is to prove nonvanishing over F, to carry out the necessary coordinate calculations in small characteristic, in particular in characteristic two, and to identify the parameter itself and not only its unramified Hecke eigenvalues.

What is not proved. The theorems assert that the excursion summand of σ is nonzero at some finite level; the form φ_σ is a nonzero element of it, and it is not unique. We do not determine multiplicities, the structure of packets, or compatibility with the local correspondence at the ramified places.

1.6 Organization

Section 2 lists the results we import, and Section 3 fixes conventions. The next seven sections contain the arithmetic used by both constructions: purity of elliptic parameters (Section 4); excursion operators on spaces of functions that are not assumed cuspidal (Sections 5 and 6); constant terms (Section 7); criteria for cuspidality (Section 8); irreducible constituents (Section 9); and transfer between the groups of an isogeny class (Section 10). Sections 11 and 12 treat Levi subgroups and Eisenstein series, and Section 13 proves Theorems A–C from Theorems 1.4 and 1.5. Appendix A contains the proof of a lemma on partial Frobenius maps which is used in Section 7. A list of notation is at the end of the paper.

The elliptic case is shorter than the general one. Theorem A uses from Section 7 only the elementary Lemma 7.7, and the elliptic case of Theorem B uses in addition Proposition 11.1. Theorem 7.1, Appendix A and Section 12 are needed only for parameters which are not elliptic.

A reader interested mainly in one of the two constructions needs only a few statements of this paper. For the geometric construction it is enough to know the definition of the operators S_e^{fun} (Definition 5.12) and the statements of Theorem 4.8 (elliptic parameters are pure) and of Proposition 8.10 (a function whose pairing with functions of compact support has the excursion eigenvalues of an elliptic parameter is a cusp form with that parameter). For the construction by descent one needs in addition the statements of Lemma 9.1 and of Theorem 10.9. Each sequel lists what it uses from this paper.

2 Results used from the literature

The proofs rest on a small number of deep theorems. We collect here, in the form in which they are used, those which enter the present paper, so that the reader can see what is imported and where. Precise references to the individual statements are repeated at each point of use. The results used only in the two constructions are listed in the sequels.

Input 1 (General linear groups)**.** *Let ϱ be a continuous irreducible n-dimensional $\overline{\mathbb{Q}}_\ell$-representation of Γ_F, unramified at almost all places, whose determinant has finite order.*

1. *There is a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ whose Satake parameters at the unramified places are the eigenvalues of geometric Frobenius elements in ϱ [Laf02, Introduction, Theorem $(\mathrm{i})_n$–$(\mathrm{ii})_n$].*
2. *The Frobenius eigenvalues of ϱ at the unramified places are algebraic numbers all of whose complex absolute values are one [Laf02, Theorem VII.6].*
3. *The representation π is tempered at every place [Laf02, Theorem VI.10(i)].*

In part (1) the Hecke operators act by right translation, as in Section 3. The statement does not depend on the conventions of [Laf02] for Hecke operators and for Frobenius elements: a change of either convention replaces ϱ by its dual and π by its contragredient, and the theorem applies to all irreducible representations.

Part (2) is the source of the purity theorem of Section 4. Parts (1) and (3) are not used in this paper; they are the starting point of [III].

Input 2 (Excursion operators on cusp forms)**.** *For a split reductive group G and a level N, the excursion operators $S_{I,f,\gamma}$ form a commutative algebra of endomorphisms of the space of cusp forms of level N on the quotient by a central lattice. They commute with the Hecke algebra, and at an unramified place v the operator attached to a representation V of $\widehat{G}$ and a Frobenius element at v is the Hecke operator corresponding to V under the Satake isomorphism. Every character of this algebra has the form ν_σ for a semisimple parameter σ, unique up to conjugacy [Laf18, Theorem 0.1, Definition–Proposition 11.3 and Proposition 11.7].*

Input 3 (Shtukas beyond the cuspidal part)**.** *We use the following results of Xue on the cohomology of stacks of shtukas with compact support, without passage to the cuspidal part.*

1. *With integral coefficients, and after dividing by a central lattice, the cohomology at the generic point of the space of legs is a finitely generated module over the spherical Hecke algebra at any one unramified place [Xue23, Theorem 5.0.4].*
2. *The specialization map of the sheaf, from its stalk at the geometric generic point of the diagonal to its stalk at the geometric generic point of the space of legs, is an isomorphism, and the action of partial Frobenius maps and the fundamental group factors through W_F^I [Xue23, Propositions 8.1.2 and 8.2.6]; after ℓ is inverted, the action is unramified outside the level [Xue20c, Proposition 5.0.4].*
3. *Excursion operators act on functions with compact support and satisfy the same relations as on cusp forms [Xue20b, §3.4].*
4. *The constant-term map along a parabolic subgroup $P = MU$ is realized by a morphism from the cohomology of shtukas for G to that for M, compatible with partial Frobenius maps and with creation and annihilation [Xue20a, Theorem 3.2.6 and Proposition 3.5.3], [Xue20b, Proposition 4.1.3].*

Input 4 (Automorphic forms over function fields)**.** *1. Cusp forms of fixed level on a group over a function field have compact support modulo the centre [Mor82, Chapter I, §5.3 and Proposition 5.3].*

2. Eisenstein series over function fields continue meromorphically, with the constant-term formulas of [BL24, (6.2), (6.4), (6.17)]. In the same work automorphic forms are characterized by finite systems of difference equations satisfied by the cuspidal projections of their constant terms [BL24, §9.1].

3 Conventions

We fix the normalizations of Frobenius, of trace functions, of the Satake equivalence and of class field theory used in the whole paper, and define the excursion summand attached to a parameter. The spaces of functions on which excursion operators act are constructed in Sections 5 and 6.

3.1 Frobenius and trace functions

Let $\deg_F : \Gamma_F \to \widehat{\mathbb{Z}}$ be the homomorphism given by the action on the constant field, normalized so that an arithmetic Frobenius element has degree 1. The Weil group W_F is the inverse image of $\mathbb{Z}$; it is dense in Γ_F. A geometric Frobenius element at a place v of degree d_v has degree $-d_v$. We write Fr_q for the Frobenius endomorphism of a scheme over $\mathbb{F}_q$, and Frob_q for the action of geometric Frobenius on cohomology.

For a bounded complex $\mathcal{K}$ with a Weil structure and a rational point x, the trace function is

$$t_{\mathcal{K}}(x) = \sum_i (-1)^i \operatorname{Tr}(\mathrm{Frob}_q; H^i(\mathcal{K}_x)),$$

and we write Str for an alternating trace of this kind. A shift by one changes the sign of the trace function. Tate twists and shifts are always kept when trace functions are compared. We fix one square root $q^{1/2} \in k$, once and for all. When q is the square of an integer we take the positive root; then in all cases there is an isomorphism of k with $\mathbb{C}$ which maps $q^{1/2}$ to the positive square root of q.

3.2 The Satake equivalence

The Satake sheaves, for G and for all its Levi subgroups, are those of [Laf18, Théorème 1.17], normalized to be pure of weight zero by the square root of q just fixed. The commutativity constraint is the modified one [Laf18, Remarque 1.19]. The sheaf of the irreducible representation of $\widehat{G}$ of highest weight ω is the intersection complex of the space of modifications $\phi : \mathcal{G} \to \mathcal{G}'$ at a point x in which the position of the source relative to the target is bounded by ω: for every dominant weight λ of G, with Weyl module V_λ,

$$\phi(\mathcal{G}_{V_\lambda}) \subset \mathcal{G}'_{V_\lambda}(\langle \lambda, \omega \rangle x) \tag{3.1}$$

[Laf18, Définition 1.2]. For a line bundle and $\omega = r$ the target is the source twisted by $\mathcal{O}(-rx)$. For GL_r and the standard representation the source contains the target with a quotient of length one, and for $I = \{1, 2\}$ and $\mathrm{Std} \boxtimes \mathrm{Std}^*$ one obtains Drinfeld's left shtukas, with the zero at the first leg [Laf18, Remarque 2.2 b)].

Some of the results we use are stated in terms of the loop group, and we record the translation. Trivialize the target on the formal disc at x, with a uniformizer t. A modification is then given by its source, a point $gG(\mathcal{O})$ of the affine Grassmannian; for GL_r it is the lattice $g\mathcal{O}^r$. By (3.1) the modifications bounded by ω form the closure of the orbit of $t^{-\omega}$. Let $\mathsf{Sat}_G^{\mathrm{lp}}$

be the Satake equivalence in which the representation of highest weight λ corresponds to the closure of the orbit of t^λ, as in [MV07] and [Zhu17, §5.3]. Fix a maximal torus of $\widehat{G}$ and an automorphism ϵ_G of $\widehat{G}$ which is $s \mapsto s^{-1}$ on this torus. Then, as tensor functors,

$$\mathsf{Sat}_G(V) \simeq \mathsf{Sat}_G^{\mathrm{lp}}(V \circ \epsilon_G). \tag{3.2}$$

Indeed, two tensor equivalences from the representations of $\widehat{G}$ to the same category differ by a tensor autoequivalence of the category of representations, which, k being algebraically closed, is composition with an automorphism of $\widehat{G}$, determined up to inner automorphisms. Here it sends the representation of highest weight ω to that of highest weight $-w_0\omega$, because $t^{-\omega}$ and $t^{-w_0\omega}$ lie in the same orbit. So does ϵ_G; and an automorphism which preserves the isomorphism class of every irreducible representation is inner, as one sees on the based root datum.

The automorphism ϵ_G preserves every Levi subgroup $\widehat{M}$ which contains the torus, and with $\epsilon_M = \epsilon_G|_{\widehat{M}}$ the formula (3.2) holds for M. Restriction of representations from $\widehat{G}$ to $\widehat{M}$ commutes with composition with ϵ_G and ϵ_M. Hence the restriction of Satake sheaves from G to M [Zhu17, Remark 5.3.16] corresponds to restriction of representations in our normalization too, and the summand on which the centre of $\widehat{M}$ acts by the weight μ lies on the component of $t^{-\mu}$ of the affine Grassmannian of M: it consists of modifications of M-bundles which change the degree from ν to $\nu - \mu$.

The normalization is determined by the Hecke operators, in the following sense.

Lemma 3.1 (Change of the Satake normalization)**.** *Let δ be an automorphism of $\widehat{G}$. If the Satake sheaf of every representation V of $\widehat{G}^I$ is replaced by that of $V \circ \delta^I$, the excursion operator $S_{I,f,\gamma}$ of Proposition 5.10 is replaced by $S_{I,f\circ\delta^I,\gamma}$. If the operators so obtained also have property* (3) *of that proposition, the compatibility with the Hecke operators at the unramified places, then δ is inner, and the excursion operators are unchanged.*

Proof. The operator $S_{I,f,\gamma}$ is constructed from a representation V of $\widehat{G}^I$, an invariant vector x and an invariant linear form ξ with $f(g) = \xi(gx)$. For $V \circ \delta^I$, with the same x and ξ, the function is $f \circ \delta^I$. This is the first assertion.

Let v be a place and V an irreducible representation of $\widehat{G}$, and let h_V be the spherical function at v which corresponds to V under the Satake isomorphism. By property (3) for the two families of operators, the Hecke operators of h_V and of $h_{V\circ\delta}$ coincide on $C_{N,E}$, for every level N which does not contain v. The spherical Hecke algebra at v acts faithfully on the union of these spaces. To see this let $h \neq 0$ be a spherical function and choose $r \in G(F_v)$ with $h(r^{-1}) \neq 0$. The set of $\gamma \in G(F)$ which lie in $r\,\mathrm{supp}(h)\prod_w G(\mathcal{O}_w)$ is finite, being the intersection of a discrete set and a compact set. For each $\gamma \neq 1$ in it there is a level N, prime to v, with $\gamma \notin r\,\mathrm{supp}(h)K_N$; take N which works for all of them, and let c be the characteristic function of the class of 1 in Y_N. The value at r of the Hecke operator of h applied to c is

$$\int_{G(F_v)} h(x)c(rx)\,dx = \int_{r^{-1}G(\mathcal{O}_v)} h(x)\,dx = h(r^{-1}) \neq 0.$$

Hence $h_V = h_{V\circ\delta}$, and by the Satake isomorphism $V \circ \delta$ is isomorphic to V. An automorphism with this property for every V is inner, and then $f \circ \delta^I = f$ for every function f which is invariant under the diagonal conjugation. □

By (3.2) and the lemma, the excursion operators constructed with $\mathsf{Sat}_G^{\mathrm{lp}}$ are those of the parameters $\epsilon_G^{-1} \circ \sigma$: a function with parameter σ in the sense of this paper has parameter $\epsilon_G^{-1} \circ \sigma$ in that normalization. The automorphism ϵ_G differs from the Chevalley involution ϑ_G of Section 3.5 by an inner automorphism, so this parameter is conjugate to $\vartheta_G \circ \sigma$.

3.3 The centre and class field theory

Let $T_G = (Z_G^\circ)_{\mathrm{red}}$ and

$$C_{T_G} = T_G(F)\backslash T_G(\mathbb{A}_F), \qquad \deg(t)(\chi) = \sum_v d_v\, v(\chi(t_v))$$

for a character χ of T_G. This defines a homomorphism $\deg : C_{T_G} \to X_*(T_G)$. The degree of the bundle attached to t is $-\deg(t)$.

Definition 3.2. A *central lattice* is a free subgroup $\Xi \subset C_{T_G}$ on which deg is injective and whose image under deg has finite index in $X_*(T_G)$. A function on $G(F)\backslash G(\mathbb{A}_F)$ is *invariant under* Ξ if it is invariant under translation by the elements of Ξ.

The quotient of C_{T_G} by a central lattice and an open compact subgroup is finite.

Class field theory is normalized by the arithmetic Artin map, $\mathrm{Art}_v(\varpi_v) = \mathrm{Frob}_v^{\mathrm{arith}}$, and a character χ of Γ_F corresponds to the character

$$\mathrm{CFT}_{\mathbb{G}_m}(\chi)(t) = \chi(\mathrm{Art}(t)^{-1})$$

of the idele class group; for a split torus one takes the product over a basis of characters. Right translation by a is denoted T_a, $(T_a h)(g) = h(ga)$. With these conventions the excursion operator of a character λ of $\widehat{T}_G$ and a geometric Frobenius element at v is $T_{\lambda(\varpi_v)}$ [Laf18, Proposition 6.2], and a parameter of the form $c^{\deg_F}$ corresponds to the character $t \mapsto c^{-\deg(t)}$. Let $\alpha_G : \widehat{G} \to \widehat{T}_G$ be the homomorphism dual to the inclusion $T_G \to G$. The *central character* of a parameter σ is

$$\omega_\sigma = \mathrm{CFT}_{T_G}(\alpha_G \sigma),$$

a character of C_{T_G}. This is the character which occurs in Theorem C.

3.4 Excursion summands

Definition 3.3. Let V be a finite-dimensional vector space on which the excursion operators S_e act, for all finite sets I, all functions f and all tuples in Γ_F^I; they generate a commutative algebra $\mathcal{B}$ of endomorphisms. Let $\mathfrak{m}_\sigma \subset \mathcal{B}$ be the ideal generated by the operators $S_e - \nu_\sigma(e)$. The *excursion summand* of σ in V is the generalized eigenspace

$$V_{\nu_\sigma} = \bigcup_{n \geq 1} \{v \in V : \mathfrak{m}_\sigma^n v = 0\}.$$

For a space which is a union of subspaces V of this kind, the excursion summand is the union of the V_{ν_σ}.

The definition involves the excursion operators for all tuples in the Galois group; the Hecke operators at the unramified places are among them. A nonzero excursion summand in a space of finite dimension contains a joint eigenvector of the excursion operators, with eigenvalues $\nu_\sigma(e)$: the commuting operators $S_e - \nu_\sigma(e)$ are nilpotent on it. So the existence of a nonzero summand and the existence of an eigenfunction are equivalent.

3.5 The Chevalley involution

Let ϑ_G be the Chevalley involution of $\widehat{G}$ which preserves the pinning. On the maximal torus of the pinning it is $t \mapsto w_0(t^{-1})$, where w_0 is the longest element of the Weyl group [Li19, Definition 3.1.1]. We put

$$\sigma^\dagger = \vartheta_G \circ \sigma.$$

The parameter $\sigma^\dagger$ appears because the pairing between functions with compact support and arbitrary functions is bilinear: transposition of an excursion operator replaces σ by $\sigma^\dagger$, see (5.6). The letter σ always denotes the parameter of the main theorems.

4 Purity of elliptic parameters

The geometric construction needs to know that the eigenvalues of Frobenius under the parameter, in every representation of the dual group, are algebraic numbers of absolute value one. The main theorems assume nothing of the kind. This section shows that for an elliptic parameter with values in a semisimple group the property is automatic (Theorem 4.8). By the theorem of L. Lafforgue, an irreducible representation of Γ_F has this property as soon as its determinant has finite order. A determinant of infinite order would be an algebraic character with infinite image of the Zariski closure of the image of the parameter, and such characters are excluded because the centre of the monodromy group is finite. The property is needed after every finite extension of F, and after such an extension the closure of the image becomes smaller. The proof therefore passes through the identity components of the closures, which do not change.

Throughout this section $k = \overline{\mathbb{Q}}_\ell$, with $\ell \neq p$. All closures of subgroups are Zariski closures. We use the degree map $\deg_F : \Gamma_F \to \widehat{\mathbb{Z}}$, for which the arithmetic Frobenius of the field of constants has degree one; its kernel is denoted Γ_F^{geom}.

4.1 Three lemmas

Lemma 4.1 (A coefficient field for a compact group)**.** *A compact subgroup $\mathscr{K} \subset \mathrm{GL}_n(k)$ is contained in $\mathrm{GL}_n(E)$ for some finite extension $E/\mathbb{Q}_\ell$. It preserves an $\mathcal{O}_E$-lattice in E^n.*

Proof. The field k is the union of countably many finite extensions E_j of $\mathbb{Q}_\ell$, so $\mathscr{K}$ is the union of the countably many closed subgroups $\mathscr{K} \cap \mathrm{GL}_n(E_j)$. A compact group is a Baire space, so one of these subgroups has nonempty interior and is therefore open, hence of finite index. Adjoining to E_j the entries of representatives of the finitely many cosets gives E (see also [Con, Theorem 1]). For the second assertion, the $\mathcal{O}_E$-module spanned by the translates of a lattice under the compact group $\mathscr{K}$ is bounded, so it is a lattice, and it is stable under $\mathscr{K}$. □

Lemma 4.2 (Torus characters on geometric Galois)**.** *Let χ be a continuous homomorphism from Γ_F to the points of a torus over k, unramified at almost all places. Then $\chi(\Gamma_F^{\mathrm{geom}})$ is finite.*

Proof. In coordinates on the torus, χ is a finite family of characters, and it suffices to treat one character $\chi : \Gamma_F \to k^\times$. By Lemma 4.1 its values lie in $\mathcal{O}_E^\times$ for a finite extension E of $\mathbb{Q}_\ell$. By class field theory χ is a character of the idele class group. At a place v where it is ramified, the group of principal units of F_v is a pro-p group, while $\mathcal{O}_E^\times$ has an open subgroup which is pro-ℓ; so the image of the principal units is finite, and χ is trivial on the units congruent to one modulo a power of the maximal ideal. Hence χ is trivial on an open subgroup of the idele class group.

The group of idele classes of degree zero is compact [CF67, Chapter II, §16], so its image under χ is finite. The reciprocity map identifies the image of Γ_F^{geom} in the abelianization of Γ_F with the group of idele classes of degree zero [Zyw13, §1.2]. So $\chi(\Gamma_F^{\mathrm{geom}})$ is finite. □

Lemma 4.3 (Finite centre and algebraic characters)**.** *Let H be an algebraic group over a field of characteristic zero whose identity component is reductive. If the centre of H is finite, every algebraic character of H has finite image.*

Proof. Let $S = Z(H^\circ)^\circ$, a torus, on which the finite group H/H° acts by conjugation. The fixed points of this action form the group $S \cap Z(H)$, which is finite. So the lattice of characters of S invariant under H/H° is zero: its rank is the dimension of the fixed points. The restriction to S of a character χ of H is invariant, hence trivial. Since H° is generated

by S and by its derived group, on which every character is trivial, χ is trivial on H° and factors through the finite group H/H°. □

4.2 Monodromy groups

For a parameter σ with values in a group H, the *arithmetic monodromy group* H_{ar} is the closure of $\sigma(\Gamma_F)$, and the *geometric monodromy group* H_{geom} is the closure of $\sigma(\Gamma_F^{\mathrm{geom}})$. The second is a normal subgroup of the first.

Lemma 4.4 (Arithmetic monodromy)**.** *Suppose that H is semisimple and that $\sigma : \Gamma_F \to H(k)$ is continuous, finitely ramified, semisimple and elliptic. Then $H_{\mathrm{ar}}/H_{\mathrm{geom}}$ is a finite cyclic group.*

Proof. Let $A = H_{\mathrm{ar}}/H_{\mathrm{geom}}$. The image of an element of degree one generates a dense subgroup of A, because $\Gamma_F/\Gamma_F^{\mathrm{geom}} = \widehat{\mathbb{Z}}$. So A is commutative, and it suffices to prove that it is finite; a finite group with a dense cyclic subgroup is cyclic.

The identity component of H_{ar} is reductive because σ is semisimple, and the centre of H_{ar} is contained in the centralizer of σ, which is finite because σ is elliptic and H is semisimple. By Lemma 4.3 every character of H_{ar} has finite image. If A were infinite, its identity component would be a torus of positive dimension, since A is commutative with reductive identity component. A nontrivial character of that torus has a power which extends to a character of A, and this would be a character of H_{ar} with infinite image. □

Proposition 4.5 (Determinants after base extension)**.** *Under the hypotheses of Lemma 4.4, let F'/F be a finite extension and $r : H \to \mathrm{GL}(V)$ an algebraic representation. Every irreducible constituent of $r\sigma|_{\Gamma_{F'}}$ has a determinant of finite order.*

Proof. A purely inseparable extension does not change the Galois group, so we may assume F'/F separable. Then $\Gamma_{F'}$ has finite index in Γ_F, and $\Gamma_{F'}^{\mathrm{geom}} = \Gamma_{F'} \cap \Gamma_F^{\mathrm{geom}}$ has finite index in Γ_F^{geom}. Let H'_{ar} and H'_{geom} be the monodromy groups of the restriction of σ to $\Gamma_{F'}$.

The closure of a subgroup of finite index has the same identity component as the closure of the group: finitely many translates of the smaller closure cover the larger one. Together with Lemma 4.4 this gives

$$(H'_{\mathrm{ar}})^\circ = H_{\mathrm{ar}}^\circ = H_{\mathrm{geom}}^\circ = (H'_{\mathrm{geom}})^\circ,$$

so that $H'_{\mathrm{ar}}/H'_{\mathrm{geom}}$ is finite.

Now let χ be an algebraic character of H'_{ar}. The character $\chi \circ \sigma$ of $\Gamma_{F'}$ is continuous and finitely ramified, so by Lemma 4.2, applied to F', it has finite image on $\Gamma_{F'}^{\mathrm{geom}}$. By density $\chi(H'_{\mathrm{geom}})$ is finite, and since H'_{geom} has finite index in H'_{ar}, so is $\chi(H'_{\mathrm{ar}})$. The representation r of the reductive group H'_{ar} is semisimple, its irreducible constituents are those of $r\sigma|_{\Gamma_{F'}}$, and the determinant of each is a character χ as above. □

Remark 4.6. The proof shows that H_{ar}° is semisimple. Indeed, take F' so large that H'_{ar} is connected, that is, equal to H_{ar}°. Every character of this connected group has finite image, hence is trivial, and a connected reductive group without nontrivial characters is semisimple. Consequently the centre of H'_{ar} is finite for every finite extension F': it centralizes the semisimple group H_{ar}°, whose centralizer in H_{ar} is finite. For a general algebraic group with finite centre this can fail after passing to a subgroup of finite index: the group generated by a torus and the inversion has finite centre, and its identity component is the torus. Such a group is excluded here by class field theory, through Lemma 4.2.

4.3 Purity

Definition 4.7 (Strong purity)**.** A parameter with values in a semisimple group is *strongly pure of weight zero* if, for every finite extension F' of F and every algebraic representation of the group, the eigenvalues of the geometric Frobenius elements of $\Gamma_{F'}$ at the unramified places are algebraic numbers all of whose complex absolute values are one.

Theorem 4.8 (Purity from ellipticity)**.** *Every continuous, finitely ramified, semisimple, elliptic parameter with values in a semisimple group over k is strongly pure of weight zero.*

Proof. Let F'/F be a finite extension and r an algebraic representation. By Lemma 4.1 the representation $r\sigma|_{\Gamma_{F'}}$ and its irreducible constituents are defined over a finite extension of $\mathbb{Q}_\ell$. By Proposition 4.5 each constituent has a determinant of finite order. So Input 1(2) [Laf02, Theorem VII.6(i)–(ii), p. 198] applies to each constituent, on the open subset of the curve where it is unramified: its eigenvalues of Frobenius are algebraic numbers of absolute value one. The eigenvalues of $r\sigma|_{\Gamma_{F'}}$ are those of its constituents. □

Remark 4.9. Ellipticity cannot be omitted. The parameter $\gamma \mapsto \mathrm{diag}(q^{\deg_F(\gamma)}, q^{-\deg_F(\gamma)})$ with values in $\mathrm{SL}_2(k)$ is well defined and continuous, because q is an ℓ-adic unit; it is unramified and semisimple, and its eigenvalues of Frobenius do not have absolute value one. Its centralizer contains the diagonal torus.

4.4 Separating the central character

For a reductive group, the image of the parameter in the dual of the connected central torus need not be pure. Its restriction to Γ_F^{geom} has finite image, by Lemma 4.2. We remove a character of the Galois group of the field of constants, after which the whole image in the dual of the central torus is finite.

For a split connected reductive group G put $T_G = (Z_G^\circ)_{\mathrm{red}}$ and $Q_G = G/G_{\mathrm{der}}$. The restriction $r_G : T_G \to Q_G$ of the quotient map is an isogeny. On the dual side there are an exact sequence and a homomorphism

$$\begin{gathered} 1 \longrightarrow \widehat{Q}_G \xrightarrow{j_G} \widehat{G} \xrightarrow{\beta_G} \widehat{G_{\mathrm{der}}} \longrightarrow 1, \qquad \widehat{Q}_G = Z(\widehat{G})^\circ, \\ \alpha_G : \widehat{G} \longrightarrow \widehat{T}_G, \qquad \alpha_G j_G = \widehat{r}_G, \end{gathered} \tag{4.1}$$

so the restriction of α_G to the connected centre of $\widehat{G}$ is an isogeny. Recall from Section 3 the central character ω_σ of a parameter, and that with the normalization fixed there the character attached to $c^{\deg_F}$ is $t \mapsto c^{-\deg(t)}$.

Definition 4.10 (Twists by the field of constants)**.** A point c of a torus over k is a *unit point* if the values of all characters at c are ℓ-adic units. For a unit point c of $Z(\widehat{G})^\circ$ we denote by $c^{\deg_F}$ the continuous homomorphism $\Gamma_F \to Z(\widehat{G})^\circ(k)$ obtained by extending $n \mapsto c^n$ from $\mathbb{Z}$ to $\widehat{\mathbb{Z}}$ and composing with $\deg_F$.

The extension exists and is unique: in coordinates, c is a tuple of units of a finite extension E of $\mathbb{Q}_\ell$, and the group $\mathcal{O}_E^\times$ is profinite, so $n \mapsto c^n$ extends uniquely to a continuous homomorphism on $\widehat{\mathbb{Z}}$. The homomorphism $c^{\deg_F}$ is trivial on Γ_F^{geom}, in particular on every inertia group.

Lemma 4.11 (Central normalization)**.** *Every continuous, finitely ramified homomorphism $\sigma : \Gamma_F \to \widehat{G}(k)$ can be written as*

$$\sigma = \psi\, \sigma^{\mathrm{fin}}, \qquad \psi = c^{\deg_F},$$

with c a unit point of $Z(\widehat{G})^\circ$ and $\alpha_G\sigma^{\mathrm{fin}}$ of finite image. The parameters σ and σ^{fin} have the same image under β_G and the same restriction to every inertia group, and one is semisimple, or elliptic, if and only if the other is. If $\alpha_G\sigma$ has finite image one can take $\psi = 1$.

Proof. Choose $\gamma_0 \in \Gamma_F$ of degree one. By Lemma 4.2 the homomorphism $\alpha_G\sigma$ is finite on Γ_F^{geom}. Lift the point $\alpha_G\sigma(\gamma_0)$ of $\widehat{T}_G$ to a point c of $Z(\widehat{G})^\circ$ through the isogeny α_G. The point $\alpha_G\sigma(\gamma_0)$ is a unit point, because the image of $\alpha_G\sigma$ is compact, and then so is c: the valuations of the character values of a point and of its image under an isogeny determine each other. Put $\psi = c^{\deg_F}$ and $\sigma^{\mathrm{fin}} = \psi^{-1}\sigma$. The homomorphism $\alpha_G\sigma^{\mathrm{fin}}$ is trivial at γ_0 and finite on Γ_F^{geom}. Since the powers of γ_0 are dense in $\Gamma_F/\Gamma_F^{\mathrm{geom}}$, its image is finite.

The twist ψ has values in the kernel of β_G and is trivial on inertia, which gives the second assertion. Multiplying by a central element does not change centralizers, so ellipticity is preserved. For semisimplicity, let M be the closure of the image of (σ, ψ) in $\widehat{G} \times Z(\widehat{G})^\circ$. The unipotent radical of M° has trivial image in both factors, because the closures of the images of σ and of ψ have reductive identity components; so it is trivial. The closure of the image of σ^{fin} is the image of M under $(g, z) \mapsto z^{-1}g$, and its identity component is reductive. The same argument with ψ^{-1} gives the converse. □

5 Excursion operators on functions with compact support

The constructions of [II] and [III] produce functions on $G(F)\backslash G(\mathbb{A}_F)$, not automorphic representations, and at the moment they are produced these functions are not known to be cuspidal. Lafforgue's excursion operators are defined on cusp forms. To say that a function has parameter σ, and to prove afterwards that it is cuspidal, we need excursion operators on a larger space. This section and the next one construct them.

The starting point is Xue's extension of the theory of shtukas from the cuspidal part to cohomology with compact support (Input 3). It gives excursion operators on functions with compact support, for tuples of elements of the Weil group. Two things are missing for our purposes, and each occupies one section.

- Xue's results are stated after dividing by a lattice Ξ in the centre. Our functions live on $G(F)\backslash G(\mathbb{A}_F)$ itself, where the connected components are indexed by all degrees. In this section we remove the lattice (Proposition 5.10), prove that the operators are transposed by the natural pairing (Proposition 5.11), and use the transpose to let them act on arbitrary functions of finite level (Definition 5.12).
- The parameter σ is a representation of the Galois group, and its excursion character involves all Galois tuples. The passage from Weil tuples to Galois tuples needs integral structures and is the subject of Section 6.

In this section and the two following ones the space on which the operators act, and the tuples for which they are defined, change several times. The table records where one is at each point.

space	tuples in	statement
functions with compact support of level N	W_F^I	Proposition 5.10
all functions of level N, by transposition	W_F^I	Definition 5.12
finite Hecke quotients of the functions with compact support	Γ_F^I	Theorem 6.3
$\mathcal{A}^{\mathrm{cont}}(G)$, made of duals of such quotients	Γ_F^I	Theorem 6.10
constant terms of all functions of finite level	W_F^I	Theorem 7.1
constant terms of elements of duals of finite quotients	Γ_F^I	Corollary 7.6

5.1 Notation

Fix a split connected reductive group G/F. For a finite subscheme $N \subset X$, the *level*, write

$$K_N = \ker(G(\textstyle\prod_v \mathcal{O}_v) \to G(\mathcal{O}_N)), \qquad Y_N = G(F)\backslash G(\mathbb{A})/K_N.$$

Let $E/\mathbb{Q}_\ell$ be a finite extension containing the chosen square root $q^{1/2}$, with ring of integers $\mathcal{O} = \mathcal{O}_E$. The modules of functions with finite support on Y_N are denoted

$$C_{N,\mathcal{O}} = \mathcal{O}^{(Y_N)}, \qquad C_{N,E} = E^{(Y_N)};$$

we call their elements *functions with compact support*. For a place $u \notin N$, let $\mathscr{H}_{G,u,\mathcal{O}}$ be the spherical Hecke algebra with coefficients in $\mathcal{O}$, in which $G(\mathcal{O}_u)$ has volume one. The normalized Satake isomorphism identifies it with the representation ring of the split dual group over $\mathcal{O}$; the same square root $q^{1/2}$ is used for every Levi subgroup.

For a function c with compact support and an arbitrary function h on Y_N, put

$$[c,h]_N = \sum_{y \in Y_N} w_N(y)c(y)h(y), \qquad w_N([g]) = \frac{\mathrm{vol}(K_N)}{|G(F) \cap gK_Ng^{-1}|}.$$

The sum is finite. We omit the subscript when the level is fixed, and use the same notation on the quotient of Y_N by a central lattice, with the corresponding weights. Over E or over k this pairing identifies the space of all functions with the algebraic dual of the space of functions with compact support.

A *central lattice* is a subgroup $\Xi \subset C_{T_G}$ as in Definition 3.2, and the Ξ-*coinvariants* of a module of functions, or of a cohomology group, are its coinvariants under translation by Ξ; geometrically, they are the corresponding object for the quotient of the stack of shtukas by Ξ.

Let I be a finite set, V a representation of $\widehat{G}^I$ and $V_\mathcal{O}$ a stable $\mathcal{O}$-lattice in V. The cohomology of shtukas with legs indexed by I is

$$\mathcal{H}^j_{N,I,V_\mathcal{O}} = \bigoplus_{\delta \in \Lambda_G} \varinjlim_{\mu} R^j(p_{I,\mu,\delta})_! \mathcal{F}_{I,V_\mathcal{O}}, \qquad H^j_{N,I,V_\mathcal{O}} = (\mathcal{H}^j_{N,I,V_\mathcal{O}})_{\bar{\eta}_I}.$$

Here the degree of the first bundle of a shtuka is an element of $\Lambda_G = X_*(G/G_{\mathrm{der}})$; $p_{I,\mu,\delta}$ is the map to $(X-N)^I$ from the part of the stack of shtukas of level N where that degree is δ and the Harder–Narasimhan polygon is bounded by μ, the sheaf $\mathcal{F}_{I,V_\mathcal{O}}$ is the Satake sheaf in the normalization relative to X^I of [Xue23, §1.1.9], and $\bar{\eta}_I$ is a geometric generic point of X^I.

With coefficients in $\mathcal{O}$ the level N is assumed to be nonempty. The reason is the following property of the automorphism groups $A_y = G(F) \cap gK_Ng^{-1}$ of the points $y = [g]$ of Y_N.

Lemma 5.1 (Automorphism groups at nonempty level)**.** *If $N \neq \varnothing$, every group A_y is a finite p-group.*

Proof. The group A_y is discrete and compact, hence finite. Let $a \in A_y$, regarded as an automorphism of a bundle with a trivialization over N, and take a faithful representation of G of dimension n. The coefficients of the characteristic polynomial of a on the associated vector bundle are regular functions on the proper, geometrically connected curve X, hence constants. At a point of N the automorphism is the identity, so the polynomial is $(T-1)^n$. By the Cayley–Hamilton theorem $(a-1)^n = 0$, and $a^{p^k} = 1$ as soon as $p^k \geq n$. □

So $|A_y|$ is a unit of $\mathcal{O}$, and the cohomology with compact supports of the classifying groupoid of A_y with coefficients in $\mathcal{O}$ is $\mathcal{O}$ in degree zero and zero in the other degrees; if $|A_y|$ were not a unit, the torsion in the cohomology or the homology of A_y could enter. It follows that for $I = \varnothing$, $V = \mathbf{1}$ and $j = 0$ the module above is $C_{N,\mathcal{O}}$: the constant section on

the component of y corresponds to the characteristic function of y, and the factor $|A_y|^{-1}$ is put in the pairing by integration, through the weights w_N.

Functions of empty level are included in this setting. They are the functions of level N invariant under the finite group $G(\mathcal{O}_N)$, which commutes with the Hecke algebra at a place outside N and with the excursion operators. Since a submodule of a finitely generated module over a noetherian algebra is finitely generated, the finiteness statements below pass to these invariants with coefficients in $\mathcal{O}$. The projection onto the invariants by averaging divides by $|G(\mathcal{O}_N)|$; we use it only with coefficients in E.

5.2 Removing a central lattice

Xue fixes a discrete lattice in the centre [Xue23, Convention 0.0.6], and his finiteness theorem [Xue23, Theorem 5.0.4] concerns the quotient of the stack of shtukas by that lattice. We need the same statement for the direct sum over all degrees. The passage rests on two facts: translation by a central element identifies the components of different degrees, and such translations are invertible elements of a spherical Hecke algebra.

Lemma 5.2 (Central translations)**.** *At a fixed principal level, every translation by $T_G(\mathbb{A})$ is a product of invertible central elements of spherical Hecke algebras at places outside the level. Translation by $\lambda(\varpi_v)$ corresponds under the Satake isomorphism to the character $\alpha_G^*\lambda$ of $\widehat{G}$. The same holds for the action on the modules $H^j_{N,I,V_\mathcal{O}}$ and on their coinvariants under central lattices.*

Proof. Let $t \in T_G(\mathbb{A})$. By weak approximation in the split torus T_G there is $t_0 \in T_G(F)$ such that $t_0^{-1}t$ lies in K_N at the places of N. Translation by a rational central element is the identity on functions on $G(F)\backslash G(\mathbb{A})$, and on shtukas it is an isomorphism of all the data (bundles, level structures, modifications and Frobenius); so we may replace t by $t_0^{-1}t$. Translation by an element of $T_G(\prod_v \mathcal{O}_v) \cap K_N$ is trivial at level N. What remains is a finite product of elements $\lambda(\varpi_v)$ with $v \notin N$ and λ a cocharacter of T_G.

Translation by $\lambda(\varpi_v)$ is the Hecke operator of the double coset of $\lambda(\varpi_v)$, which is a single coset because the element is central. The corresponding Schubert variety is a point and the modulus character is trivial on it, so under the normalized Satake isomorphism the operator corresponds to the one-dimensional representation $\alpha_G^*\lambda$ of $\widehat{G}$. It is invertible, its inverse being the operator of $-\lambda$. On stacks of shtukas the Hecke correspondence of this double coset is the graph of the translation. The operator is the pullback along the translation by $\lambda(\varpi_v)$, so it moves the supports in the direction opposite to that in which the translation moves points. This gives the assertion for the modules $H^j_{N,I,V_\mathcal{O}}$ and for their coinvariants. □

Lemma 5.3 (An exact degree in a central quotient)**.** *Put $\Lambda_G = X_*(Q_G)$, where $Q_G = G/G_{\mathrm{der}}$, and grade the shtuka stack by the degree $\delta \in \Lambda_G$ of its first bundle. Fix a place $u \notin N$. The central ideles $\lambda(\varpi_u)$, for $\lambda \in X_*(T_G)$, generate a central lattice Ξ_u. Their degree shifts form*

$$\Lambda_u = -\deg(u)(r_G)_* X_*(T_G) \subset \Lambda_G, \qquad r_G : T_G \longrightarrow Q_G.$$

For every adjoint Harder–Narasimhan bound μ and every δ, the quotient map induces an equivalence

$$\mathrm{Cht}^{\leq\mu}_{G,N,I,V,\delta} \xrightarrow{\sim} \left(\mathrm{Cht}^{\leq\mu}_{G,N,I,V} / \Xi_u\right)_{\delta \bmod \Lambda_u}. \tag{5.1}$$

The equivalence is over the leg base, preserves the Satake coefficient, and is compatible with increasing μ. It also holds with $m\Xi_u$ and $m\Lambda_u$ for every $m \geq 1$.

Proof. Since r_G is an isogeny, $\lambda \mapsto -\deg(u)(r_G)_*\lambda$ is injective and Λ_u has finite index in Λ_G. Twisting a G-bundle by the T_G-bundle attached to $\lambda(\varpi_u)$ changes its degree in Λ_G

by $-\deg(u)(r_G)_*\lambda$ and does not change the associated bundle for the adjoint group, so it preserves the bound μ. Applying the same twist to all the bundles of a shtuka is compatible with the modifications, the level structure and the Frobenius isomorphism. Thus Ξ_u acts on the stack of shtukas, and $\lambda(\varpi_u)$ maps the part of degree δ isomorphically onto the part of degree $\delta - \deg(u)(r_G)_*\lambda$.

Consider the union of the parts whose degree lies in $\delta + \Lambda_u$. By the injectivity just noted, each of them is the translate of the part of degree δ by exactly one element of Ξ_u. So every object of the quotient of this union by Ξ_u is isomorphic to an object of degree δ, and a morphism in the quotient between two objects of degree δ involves the translation by an element of Ξ_u which does not change the degree, that is, by the identity. Hence the functor (5.1) is essentially surjective and fully faithful; in particular it preserves automorphism groups. The part of degree δ may have several connected components, indexed by a subset of $\pi_1(G)$; all of them are kept on both sides.

The Satake sheaf is defined in terms of the relative positions of the bundles of a shtuka at the legs, which are not changed by a simultaneous central twist; so it corresponds to itself under the equivalence. Everything commutes with restriction to open subsets of X^I and with enlarging μ, and the argument applies without change to the subgroup $m\Xi_u$. □

On cohomology this equivalence takes the following algebraic form. The lemma has two uses: part (2) transfers finite generation from the coinvariants to the whole module, and part (3) allows identities between operators to be checked on coinvariants.

Lemma 5.4 (Graded modules and central translations)**.** *Let $\mathscr{R}$ be an algebra graded by a free abelian group Γ, and let $\mathcal{V} = \bigoplus_{\delta\in\Gamma} \mathcal{V}_\delta$ be a graded $\mathscr{R}$-module. Suppose a finite-index sublattice $\Lambda \subset \Gamma$ acts by central homogeneous units $z_\lambda \in \mathscr{R}_\lambda$, with $z_{\lambda+\lambda'} = z_\lambda z_{\lambda'}$. Write $\mathcal{V}_\Lambda = \mathcal{V}/\sum_{\lambda\in\Lambda}(z_\lambda - 1)\mathcal{V}$. Then the following statements hold.*

1. *The map $\mathcal{V}_\delta \to (\mathcal{V}_\Lambda)_{\bar\delta}$ is an isomorphism.*
2. *If $\mathcal{V}_\Lambda$ is finite over $\mathscr{R}$, then $\mathcal{V}$ is finite over $\mathscr{R}$.*
3. *A vector of $\mathcal{V}$ that maps to zero in $\mathcal{V}_{m\Lambda}$ for every $m \geq 1$ is zero. Consequently compatible homomorphisms or operator identities can be tested on all these modules of coinvariants.*

Proof. (1) The quotient $\mathcal{V}_\Lambda$ is graded by Γ/Λ, and its part of degree $\bar\delta$ is a quotient of $\bigoplus_{\lambda\in\Lambda} \mathcal{V}_{\delta+\lambda}$. On this sum define

$$(v_{\delta+\lambda})_{\lambda\in\Lambda} \longmapsto \sum_{\lambda\in\Lambda} z_{-\lambda} v_{\delta+\lambda} \in \mathcal{V}_\delta. \tag{5.2}$$

The sum is finite. The map vanishes on the elements $(z_\lambda - 1)v$, so it factors through $(\mathcal{V}_\Lambda)_{\bar\delta}$, and it is inverse to the map of the statement in both directions.

(2) Choose finitely many generators of $\mathcal{V}_\Lambda$, homogeneous for the grading by Γ/Λ, and lift them to homogeneous elements $v_1, \dots, v_s$ of $\mathcal{V}$, of degrees δ_j. Let $v \in \mathcal{V}_\delta$, and write its image as $\sum_j a_j \bar v_j$ with $a_j \in \mathscr{R}$. Decomposing the a_j into homogeneous components $a_{j,\eta}$ and keeping only the terms of degree $\bar\delta$, we may assume $\eta + \delta_j - \delta \in \Lambda$ for all terms. The element

$$\sum_{j,\eta} z_{\delta-\eta-\delta_j} a_{j,\eta} v_j$$

of $\mathcal{V}$ then has degree exactly δ and the same image as v in $\mathcal{V}_\Lambda$, so it equals v by (1). Thus $v_1, \dots, v_s$ generate $\mathcal{V}$.

(3) A vector has components in finitely many degrees. For m large these degrees are distinct modulo $m\Lambda$, and by (1), applied to $m\Lambda$, each component maps injectively to $\mathcal{V}_{m\Lambda}$. So a vector whose image in every $\mathcal{V}_{m\Lambda}$ is zero is zero. For the last statement apply this to the value of the difference of two operators on a given vector. □

Proposition 5.5 (Finiteness over one spherical Hecke algebra)**.** *For every place $u \notin N$, every finite set I, every representation V with lattice $V_{\mathcal{O}}$ and every j, the module $H^j_{N,I,V_{\mathcal{O}}}$ is finitely generated over $\mathscr{H}_{G,u,\mathcal{O}}$. So are its coinvariants under any central lattice. In particular $C_{N,\mathcal{O}}$ is finitely generated over $\mathscr{H}_{G,u,\mathcal{O}}$.*

Proof. For the Ξ_u-coinvariants the finite generation is [Xue23, Theorem 5.0.4]; for $I = \varnothing$ the module is the space of functions with compact support, as in Example 2.3.13 of that paper. To pass to the whole module we apply Lemma 5.4. The algebra $\mathscr{H}_{G,u,\mathcal{O}}$ is graded by Λ_G, according to the change of the degree of a bundle, and $H^j_{N,I,V_{\mathcal{O}}}$ is a graded module over it. By Lemma 5.2 the translations by the elements of Ξ_u are central homogeneous units of the algebra, and by Lemma 5.3 the cohomology of the quotient by Ξ_u is the module of coinvariants. So part (2) of Lemma 5.4 gives the finite generation of $H^j_{N,I,V_{\mathcal{O}}}$.

For another central lattice Ξ, the same identification shows that the cohomology of the quotient by Ξ is the module of Ξ-coinvariants of $H^j_{N,I,V_{\mathcal{O}}}$, a quotient of a finitely generated module. All of this is done with coefficients in $\mathcal{O}$. □

Remark 5.6. An identity between operators cannot always be checked on the coinvariants under one lattice. For example, $z_\lambda - 1$ becomes zero after quotienting by Λ although it is nonzero before the quotient. Exact-degree maps can be compared on the coinvariants under one lattice; an identity involving several degree shifts is compared at all $m\Xi_u$ and then uses Lemma 5.4(3). This distinction will be used for the excursion identities.

5.3 The Weil action and its transpose

Recall that $W_F = \deg_F^{-1}(\mathbb{Z})$ is dense in Γ_F and contains every inertia group. In this section an excursion datum has its tuple in W_F^I. Its function is an invariant regular function $f \in E[\widehat{G}\backslash\widehat{G}^I/\widehat{G}]$. It can be expressed as $f((g_i)) = \xi((g_i)x)$, with x and ξ invariant under the diagonal group in a representation and its dual. The corresponding operator creates x, transports it along $(\gamma_i) \in W_F^I$, and contracts with ξ.

Definition 5.7 (Excursion data)**.** Write $e = (I, f, \boldsymbol{\gamma})$ for such a datum, and let ϑ_G be the pinned Chevalley involution of $\widehat{G}$. Define

$$e^\star = (I, f^\dagger, \boldsymbol{\gamma}^{-1}), \qquad f^\dagger((g_i)) = f((\vartheta_G(g_i^{-1}))). \tag{5.3}$$

Lemma 5.8 (Specialization and the Weil action for all degrees)**.** *The specialization map from the cohomology at the geometric generic point of the diagonal to $H^j_{N,I,V_{\mathcal{O}}}$ is an isomorphism, and it commutes with central translations. The action of partial Frobenius maps and of the fundamental group on $H^j_{N,I,V_{\mathcal{O}}}$ factors through W_F^I.*

Proof. Let H_Δ and H_η be the cohomology at the geometric generic point of the diagonal and at $\bar\eta_I$. The specialization map $H_\Delta \to H_\eta$ preserves the degree δ and commutes with central translations. For each δ it fits into the square

$$\begin{array}{ccc} H_{\Delta,\delta} & \xrightarrow{\ \mathrm{sp}^*\ } & H_{\eta,\delta} \\ \sim\downarrow & & \downarrow\sim \\ (H_\Delta/\Xi_u)_{\bar\delta} & \xrightarrow[\ \mathrm{sp}^*\ ]{} & (H_\eta/\Xi_u)_{\bar\delta}, \end{array}$$

in which H/Ξ_u is the cohomology of the quotient by Ξ_u, or equivalently the module of coinvariants, and the vertical maps are the isomorphisms of Lemma 5.3. The lower map is an isomorphism by [Xue23, Proposition 8.1.2], so the upper one is. Taking the sum over δ gives the first assertion.

The partial Frobenius morphisms of the stacks of shtukas act on the cohomology groups once the limit over μ has been taken. A weight of the centre of $\widehat{G}$ is an element of $X^*(Z(\widehat{G})) = \pi_1(G)$, and it has an image in Λ_G under the natural map $\pi_1(G) \to \Lambda_G$; this map is surjective with finite kernel, and it is not an isomorphism in general (for PGL_2 the source is $\mathbb{Z}/2$ and the target is zero). On the part of the Satake sheaf on which the centre of $\widehat{G}$ at a leg acts by a given weight, the partial Frobenius at that leg shifts the degree δ by the image of that weight. These maps commute with one another and their product is the total Frobenius, which acts by an isomorphism; so each of them is an isomorphism. Together with the action of the fundamental group of an open subset of X^I they define, as in [Laf18, Xue23], an action of a group which maps onto W_F^I. An element of the kernel has degree zero at every leg, so it preserves the degree δ, and it acts trivially on the coinvariants by [Xue23, Proposition 8.2.6]. By the injectivity of the vertical maps of the square it acts trivially on $H^j_{N,I,V_{\mathcal{O}}}$. □

Remark 5.9 (On the proof of the integral form of Drinfeld's lemma)**.** The proposition of Xue just quoted rests on [Xue23, Lemma 8.2.4]: if A is a finitely generated $\mathcal{O}$-algebra and M a finitely generated A-module with a continuous A-linear action of the Frobenius–Weil group, then the kernel of the map to W_F^I acts trivially on M. The proof given there reduces to the quotients $M/\mathfrak{m}^n M$, for the maximal ideals $\mathfrak{m}$ of A, and uses that $A/\mathfrak{m}^n$ is a torsion $\mathcal{O}$-module of finite type. That holds when the uniformizer ϖ lies in $\mathfrak{m}$, but not otherwise: for $A = \mathcal{O}[t, t^{-1}]$ and $\mathfrak{m} = (\varpi t - 1)$ the quotient $A/\mathfrak{m}$ is E. The lemma is nevertheless true, and the missing case is the following. If $\varpi \notin \mathfrak{m}$, then $A/\mathfrak{m}^n$ is an Artinian local algebra of finite type over E, hence of finite dimension over E, and so is $Q = M/\mathfrak{m}^n M$. By the definition of continuity [Xue23, Definition 8.2.3], M is a union of $\mathcal{O}$-submodules of finite type which are stable under the geometric part of the group and on which it acts continuously. Finitely many of them contain lifts of a basis of Q; the image of their sum is a lattice in Q on which the action is continuous. So the action on Q is a continuous action on a vector space of finite dimension over E, to which Drinfeld's lemma with coefficients in E applies [Xue20b, §3.2]. In both cases the kernel acts trivially on $M/\mathfrak{m}^n M$, and the end of Xue's proof, which uses Krull's intersection theorem at each maximal ideal, applies without change.

Proposition 5.10 (Excursion operators on functions with compact support)**.** *For every excursion datum $e = (I, f, \boldsymbol{\gamma})$ with $\boldsymbol{\gamma} \in W_F^I$ there is an operator S_e on $C_{N,E}$ with the following properties.*

1. *The operators S_e commute with each other and satisfy the relations of [Xue20b, §3.4], which are those of [Laf18, Proposition 10.8]:*

   - *for fixed I and $\boldsymbol{\gamma}$, the map $f \mapsto S_{I,f,\boldsymbol{\gamma}}$ is a homomorphism of algebras;*
   - *for a map $\zeta : I \to J$ and $f^\zeta((g_j)_{j\in J}) = f((g_{\zeta(i)})_{i\in I})$, one has $S_{J,f^\zeta,(\gamma_j)_{j\in J}} = S_{I,f,(\gamma_{\zeta(i)})_{i\in I}}$;*
   - *for $\widetilde{f}((g_i),(g_i'),(g_i'')) = f((g_i g_i'^{-1} g_i''))$,*

   $$S_{I\sqcup I\sqcup I,\widetilde{f},(\gamma_i)\times(\gamma_i')\times(\gamma_i'')} = S_{I,f,(\gamma_i\gamma_i'^{-1}\gamma_i'')}.$$

   On the coinvariants under any central lattice they induce Xue's excursion operators.

2. *They commute with changes of level and with right translations.*

3. *For an unramified place v, a representation V of $\widehat{G}$ and a geometric Frobenius element at v, the operator attached to the character of V is the Hecke operator corresponding to V under the normalized Satake isomorphism.*

4. *Each S_e changes the degree in Λ_G by one of finitely many values.*

Proof. Write $f((g_i)) = \xi((g_i)x)$ as above. The operator S_e is the composite of four maps: the creation map defined by x, from $C_{N,E}$ to the cohomology at the generic point of the diagonal with coefficients V; the specialization isomorphism of Lemma 5.8; the action of $\boldsymbol{\gamma} \in W_F^I$ given by the same lemma; and the inverse specialization followed by the annihilation map defined by ξ. Creation and annihilation come from the isomorphisms of coalescence of legs.

Each of these maps commutes with central translations, and on the $m\Xi_u$-coinvariants the composite is Xue's excursion operator. That operator does not depend on the choice of (V, x, ξ), and the operators commute and satisfy the relations of coalescence [Xue20b, Lemma 3.4.2, Construction 3.4.6 and Lemmas 3.4.9, 3.4.10, 3.4.12 and 3.4.14]. To deduce the same statements on $C_{N,E}$, apply the difference of the two sides of an identity to a function with compact support. The result has components in finitely many degrees and vanishes in the $m\Xi_u$-coinvariants for every m, so it is zero by Lemma 5.4(3). This proves (1), and (2) follows in the same way from the corresponding maps between stacks of shtukas. Property (3) holds on all coinvariants by the compatibility of excursion operators with Hecke operators on cohomology with compact support [Xue23, Proposition 7.2.10], which extends [Laf18, Proposition 6.2], and is transferred in the same way.

For (4), decompose x according to the weights of the centre of $\widehat{G}^I$. A weight is a family of characters of $Z(\widehat{G})$, that is, of elements of $\pi_1(G)$; let $(\lambda_i)_{i\in I}$ be their images in Λ_G. Since x is invariant under the diagonal, $\sum_i \lambda_i = 0$. The degree is preserved by specialization and by the action of the fundamental group. The partial Frobenius morphism at the leg i maps shtukas of degree δ to shtukas of degree $\delta - \lambda_i$, so the pullback by it shifts the degree of a cohomology class by λ_i. Hence S_e shifts the degree by an element of the finite set

$$\Big\{ -\sum_i \deg_F(\gamma_i)\lambda_i : (\lambda_i) \text{ a weight of } x \Big\}.$$

For a geometric Frobenius element at a place v the supports are shifted by $+\deg(v)\lambda_i$. For a translation by a central element this is also the shift of the supports in Lemma 5.2. □

Proposition 5.11 (The transpose of an excursion operator)**.** *For functions c, h with compact support,*

$$[S_e c, h] = [c, S_{e^\star} h]. \tag{5.4}$$

In the basis of point masses, every S_e has finite rows and columns.

Proof. The proof rests on the pairing constructed by Li on the cohomology with compact support of the stacks of shtukas with bounded modification, before any passage to the cuspidal part [Li19, Remark 5.1.3 and (5-1)–(5-2)]. It pairs the cohomology with coefficients V with the cohomology whose coefficients are the dual of V twisted by the Chevalley involution, and it is invariant under the partial Frobenius morphisms [Li19, Proposition 5.2.3 and Corollary 5.2.4]. The transposition formula at the end of that paper is stated for cusp forms only; the argument for all functions with compact support is the following. The cup product which enters the pairing requires the diagonal of the stacks of shtukas with bounded modification, divided by a central lattice, to be proper. It is, because over the complement of a sufficiently large level these stacks are quotients of quasi-projective schemes by finite groups [Laf18, Remarque 4.3].

Write a_x for the creation map defined by x, b_ξ for the annihilation map defined by ξ, and sp for the specialization from the diagonal, so that

$$S_e = b_\xi \operatorname{sp}^{-1} \boldsymbol{\gamma} \operatorname{sp} a_x.$$

Li's pairing is defined on the cohomology with coefficients in E. This is enough, because the identity to be proved is between operators on $C_{N,E}$. Three properties of the pairing $\langle -, - \rangle_I$ in degree zero, between $H^0_{N,I,V_\mathcal{O}} \otimes_\mathcal{O} E$ and the same space for the dual coefficients twisted

by the Chevalley involution, are used. First, the annihilation map b_ξ is the transpose of the creation map a_ξ on the dual coefficients, and likewise for x; these are statements about maps of coefficients, so they are compatible with specialization. Secondly, the action of the fundamental group preserves the pairing, because the pairing is a morphism of sheaves. Thirdly, the partial Frobenius morphisms preserve it, by the result of Li quoted above. The last two properties say that $\boldsymbol{\gamma}$ preserves the pairing. Hence

$$\begin{aligned}[S_e c, h]_N &= \langle \boldsymbol{\gamma}\, \mathrm{sp}\, a_x c, \mathrm{sp}\, a_\xi h\rangle_I \\ &= \langle \mathrm{sp}\, a_x c, \boldsymbol{\gamma}^{-1}\mathrm{sp}\, a_\xi h\rangle_I \\ &= [c, b_x\, \mathrm{sp}^{-1}\boldsymbol{\gamma}^{-1}\mathrm{sp}\, a_\xi h]_N,\end{aligned}$$

where the inverse of the specialization map exists by Lemma 5.8. The operator in the last line is defined by the dual representation, the vectors ξ and x with their roles exchanged, and the tuple $\boldsymbol{\gamma}^{-1}$; its function is $f^\dagger$, so it is $S_{e^\star}$.

Two points remain to be justified, since Li's pairing is defined for a fixed bound μ and after division by a lattice in the centre. The vectors which occur in the calculation are finite in number, so they lie in the cohomology for some bound μ, and the pairings for different bounds are compatible. For the lattice, choose m so large that the finitely many degrees which occur are distinct modulo $m\Lambda_u$. By Lemma 5.3 the parts of the stacks of these degrees, with their Satake sheaves, are identified with the corresponding parts of the quotient by $m\Xi_u$, with the same automorphism groups. So the calculation may be made in the quotient, where the pairing is defined, and (5.4) follows.

For the last assertion write $S_e\delta_y = \sum_x a_{xy}\delta_x$. Each column of the matrix (a_{xy}) is finite because $S_e\delta_y$ has compact support. Let $(a^\star_{xy})$ be the matrix of $S_{e^\star}$ and $w_N(x) \neq 0$ the weight of the point x in the pairing. Then (5.4) says $w_N(x)a_{xy} = w_N(y)a^\star_{yx}$, so the row of (a_{xy}) at x is finite because the column of $(a^\star_{xy})$ at x is. □

By this proposition the operators extend from functions with compact support to all functions.

Definition 5.12 (Action on functions)**.** For any function h on Y_N, define $S_e^{\mathrm{fun}}h$ by

$$[c, S_e^{\mathrm{fun}}h] = [S_{e^\star}c, h]. \tag{5.5}$$

The operator $S_{e^\star}$ maps the characteristic function of a point to a function with compact support, so this formula determines the value of $S_e^{\mathrm{fun}}h$ at each point. On functions with compact support it agrees with S_e, by (5.4). If $L_h(c) = [c, h]$ and $\nu_\sigma(e) = f((\sigma(\gamma_i)))$, then

$$S_e^{\mathrm{fun}}h = \nu_\sigma(e)h \quad \text{for all } e \quad\Longleftrightarrow\quad L_hS_e = \nu_{\vartheta_G\sigma}(e)L_h \quad \text{for all } e. \tag{5.6}$$

Indeed $\nu_\sigma(e^\star) = \nu_{\vartheta_G\sigma}(e)$. The Chevalley involution appears because the pairing is bilinear and not sesquilinear. It is the reason why the constructions of [II] and [III] start from $\sigma^\dagger = \vartheta_G\sigma$ when the function they produce is tested through the pairing.

Proposition 5.13 (Central twisting)**.** *Let $\psi : \Gamma_F \to Z(\widehat{G})^\circ$ be a continuous homomorphism with finite image. Take a level on which the character attached to ψ by class field theory is trivial, and let T_ψ be the multiplication by this character. It satisfies*

$$S_eT_\psi = T_\psi S_{e^\psi}, \qquad f^\psi((g_i)) = f((g_i\psi(\gamma_i))). \tag{5.7}$$

The formula also holds for $\psi = c^{\deg_F}$ on W_F, with any $c \in Z(\widehat{G})^\circ(k)$.

Proof. Choose the modulus N so that the class-field character of ψ is trivial on the level subgroup of Q_G. By geometric class field theory with modulus there is a multiplicative Weil local system $\mathcal{L}_\psi$ of rank one on $\mathrm{Bun}_{Q_G,N}$: on the generalized Picard group of degree zero it is given by the Lang isogeny, and on the other components by the character of the degree. Its trace function is the character by which T_ψ multiplies. We describe what it induces on the cohomology of shtukas, before coinvariants are taken.

Consider the part of the stack of shtukas on which the weights at the legs are fixed, and let $\lambda_i \in X_*(Q_G)$ be the image of the weight at the leg i. On the underlying Q_G-bundles the modification at the leg i is the tensor product with the bundle attached to $-\lambda_i x_i$, with its canonical level structure away from x_i. Let $\mathcal{W}_i$ be the local system on $X - N$ obtained by pulling back $\mathcal{L}_\psi$ along the Abel–Jacobi map of $+\lambda_i x_i$. Its character is $(\lambda_i \circ \psi)^{-1}$. Indeed, at a place $v \notin N$ the image of the Abel–Jacobi map is represented by $\lambda_i(\varpi_v)^{-1}$, and with the normalization of class field theory of Section 3 this gives the value at the geometric Frobenius elements, which are dense.

Let $E_0, \dots, E_s$ be the successive Q_G-bundles of a shtuka. Since $\mathcal{L}_\psi$ is multiplicative, its fibre at the source of a modification is the tensor product of its fibre at the target with the fibres $\mathcal{W}_i(x_i)$ at the legs of the modification. Composing these isomorphisms along the chain, then the isomorphism $E_s \simeq {}^\tau E_0$ of the shtuka, then the Weil structure of $\mathcal{L}_\psi$ at E_0, we get

$$\mathcal{L}_\psi(E_0) \xrightarrow{\sim} \mathcal{L}_\psi(E_0) \otimes \bigotimes_{i \in I} \mathcal{W}_i(x_i),$$

that is, a section which trivializes the pullback of $\mathcal{W} = \boxtimes_i \mathcal{W}_i$ to the stack of shtukas.

This section is compatible with the partial Frobenius morphisms. Let b be the isomorphism of lines given by the first modification and d the composite of those given by the others. For the original shtuka the section is b, followed by d, followed by the Weil structure at E_0. For its image under the partial Frobenius it is d, followed by the pullback of b by Frobenius, followed by the Weil structure at E_1. The Weil structure is natural, so the last two maps can be replaced by the Weil structure at E_0 followed by b. The two composites differ by a cyclic permutation of three isomorphisms between lines, so they define the same section. Hence multiplication by the section is a map from the cohomology of shtukas to its tensor product with $\mathcal{W}$ which commutes with the fundamental group and with the partial Frobenius maps.

On the diagonal, creation and annihilation involve only families of weights with $\sum_i \lambda_i = 0$. For these the product of the Abel–Jacobi twists is canonically trivial, and the section is the function by which $\mathcal{L}_\psi$ multiplies on the stack without legs, that is, T_ψ. Transport the trivialization of $\mathcal{W}$ on the diagonal to its generic stalk by the chosen specialization map. In this basis, the equivariance just proved says that moving the multiplication by the section across the action of a tuple (γ_i) multiplies the result by $\prod_i \lambda_i(\psi(\gamma_i))$. Summing over the families of weights which occur in the matrix coefficient f gives (5.7).

For $\psi = c^{\deg_F}$, the operator T_ψ multiplies the functions supported in degree δ by a monomial in c. By Proposition 5.10(4), applied to a function supported in one degree, the two sides of (5.7) are Laurent polynomials in c. They agree when c has finite order, by the first part, and the points of finite order are Zariski dense in the torus; so they agree for all c. Finally, multiplication by a function is its own transpose for $[\ ,\ ]$, and $(e^\psi)^\star = (e^\star)^{\psi^{-1}}$, because the Chevalley involution is the inversion on the connected centre. So the transpose of the formula is the same formula for S_e^{fun}, with the same ψ. □

6 Galois tuples, finite Hecke quotients and the space $\mathcal{A}^{\mathrm{cont}}(G)$

Section 5 gave excursion operators for tuples in W_F^I. A parameter σ is defined on Γ_F, and its excursion character $\nu_\sigma(I, f, \boldsymbol{\gamma})$ makes sense for all $\boldsymbol{\gamma} \in \Gamma_F^I$. To state that a function has

parameter σ in this full sense, the operators must be defined for Galois tuples.

An action of W_F on a vector space over E does not extend to Γ_F in general: Frobenius could act on a line by ℓ, and the powers ℓ^n do not converge as n varies in $\widehat{\mathbb{Z}}$. It does extend, uniquely and continuously, when it preserves a lattice (Lemma 6.2). The plan of this section is therefore the following.

1. Divide the module of functions with compact support by an ideal of the integral spherical Hecke algebra at one place. By Proposition 5.5 the quotient is a finitely generated $\mathcal{O}$-module, and excursion operators for all Galois tuples act on it (Theorem 6.3).
2. Pass to duals. The dual of such a quotient is a space of functions annihilated by the ideal (Proposition 6.5). The space $\mathcal{A}^{\mathrm{cont}}(G)$ is spanned by the duals of the quotients with a central lattice and by their twists by characters of the degree (Definition 6.9); every finite set of its elements lies in the dual of one quotient without a central lattice (Theorem 6.10). It is smaller than the union of all such duals (Remark 6.13).
3. Show that the elements of $\mathcal{A}^{\mathrm{cont}}(G)$ are automorphic forms (Proposition 6.11), and give the criterion by which the functions constructed later are recognized as elements with parameter σ (Corollary 6.14).

6.1 From Weil tuples to Galois tuples

Definition 6.1 (Finite Hecke quotients)**.** An ideal $\mathfrak{a} \subset \mathscr{H}_{G,u,\mathcal{O}}$ is *cofinite* if the quotient is a finite $\mathcal{O}$-module. For $Y_{N,\Xi} = G(F)\backslash G(\mathbb{A})/(K_N\Xi)$, put

$$\mathcal{Q}_{N,\Xi,u,\mathfrak{a}} = \left(\mathcal{O}^{(Y_{N,\Xi})}/\mathfrak{a}\mathcal{O}^{(Y_{N,\Xi})}\right) \otimes_{\mathcal{O}} k.$$

For an integral character $\chi : \mathscr{H}_{G,u,\mathcal{O}} \to \mathcal{O}$ we also write

$$\mathcal{Q}_{N,\Xi,u,\chi,n} = \mathcal{Q}_{N,\Xi,u,(\ker\chi)^n}. \tag{6.1}$$

These spaces are finite-dimensional by Proposition 5.5. The powers in (6.1) are cofinite: the integral Satake algebra is Noetherian, its quotient by $\ker\chi$ is $\mathcal{O}$, and the successive quotients of the powers of this ideal are finite $\mathcal{O}$-modules.

Lemma 6.2 (Integral extension of a Weil action)**.** *Let $\mathcal{V}$ be a finitely generated $\mathcal{O}$-module with an $\mathcal{O}$-linear action of W_F whose restriction to Γ_F^{geom} is continuous. The action extends uniquely to a continuous action of Γ_F. Submodules stable under W_F are stable under Γ_F; homomorphisms which commute with W_F commute with Γ_F; and if several actions of W_F as above commute with one another, so do their extensions.*

Proof. The group $A = \mathrm{Aut}_{\mathcal{O}}(\mathcal{V})$ is profinite: since $\mathcal{V}$ is complete and separated for the ϖ-adic topology, it is the inverse limit of its images in the finite groups $\mathrm{Aut}(\mathcal{V}/\varpi^a\mathcal{V})$. Let $r : W_F \to A$ be the action, and choose $\gamma_{\mathrm{Frob}} \in W_F$ of degree one, so that

$$W_F = \Gamma_F^{\mathrm{geom}} \rtimes \gamma_{\mathrm{Frob}}^{\mathbb{Z}}, \qquad \Gamma_F = \Gamma_F^{\mathrm{geom}} \rtimes \gamma_{\mathrm{Frob}}^{\widehat{\mathbb{Z}}}.$$

In a profinite group the homomorphism $n \mapsto r(\gamma_{\mathrm{Frob}})^n$ extends to a continuous homomorphism $z \mapsto r(\gamma_{\mathrm{Frob}})^z$ of $\widehat{\mathbb{Z}}$. The relation

$$r(\gamma_{\mathrm{Frob}})^z\, r(g)\, r(\gamma_{\mathrm{Frob}})^{-z} = r(\gamma_{\mathrm{Frob}}^z g \gamma_{\mathrm{Frob}}^{-z}), \qquad g \in \Gamma_F^{\mathrm{geom}},$$

holds for $z \in \mathbb{Z}$, and both sides are continuous in z; so it holds for $z \in \widehat{\mathbb{Z}}$, and $g\gamma_{\mathrm{Frob}}^z \mapsto r(g)r(\gamma_{\mathrm{Frob}})^z$ is a continuous homomorphism on Γ_F which extends r. It is unique because W_F is dense in Γ_F.

The remaining assertions also follow from density. A submodule of $\mathcal{V}$ is closed, so its stabilizer in Γ_F is closed; the set of elements of Γ_F which commute with a given homomorphism, or with a given automorphism of $\mathcal{V}$, is closed as well. □

Theorem 6.3 (Continuous excursion actions on finite quotients)**.** *Let $\mathcal{Q}$ be one of the finite Hecke quotients of Definition 6.1, with or without a central lattice.*

1. *For every excursion datum $e = (I, f, \boldsymbol{\gamma})$ with $\boldsymbol{\gamma} \in \Gamma_F^I$ there is an operator S_e on $\mathcal{Q}$, continuous in $\boldsymbol{\gamma}$, which for $\boldsymbol{\gamma} \in W_F^I$ is induced by the operator of Proposition 5.10. These operators satisfy the same relations as on cusp forms.*
2. *The operators commute with the maps between such quotients induced by changes of level, of central lattice, of coefficient field and of the ideal.*
3. *There is an $\mathcal{O}$-lattice spanning $\mathcal{Q}$ that is stable under the integral spherical Hecke algebras at all places outside the level.*

Proof. The proof has five steps. Steps 1, 3 and 4 are formal: the maps which define the excursion operators pass to the quotient, the action of the Weil group extends by Lemma 6.2, and identities extend by density. The one point which needs an argument is step 2, the continuity of the action of the geometric fundamental group on a module which is defined as a quotient of an infinite limit. We treat the quotient without a central lattice; the argument with a central lattice is the same, and is in fact the case considered by Xue.

Step 1: finite modules at the generic point. Let I be a finite set and $V_{\mathcal{O}}$ a representation of $\widehat{G}^I$ on a free $\mathcal{O}$-module of finite rank. Besides the module $H_\eta = H^0_{N,I,V_{\mathcal{O}}}$ we consider the corresponding module H_Δ at the geometric generic point of the diagonal. The creation map $\mathcal{O}^{(Y_N)} \to H_\Delta$, the specialization isomorphism $H_\Delta \to H_\eta$, the action of W_F^I on H_η and the annihilation map $H_\Delta \to \mathcal{O}^{(Y_N)}$ all commute with $\mathscr{H}_{G,u,\mathcal{O}}$, because the Hecke correspondences at u act on the stacks of shtukas with legs outside u compatibly with all the structures. Here the specialization isomorphism and the action of W_F^I are those of [Xue23, Propositions 8.1.2 and 8.2.6], extended to the modules without a central lattice by Lemma 5.8. So all these maps pass to the quotients by $\mathfrak{a}$. By Proposition 5.5 the module H_η is finitely generated over $\mathscr{H}_{G,u,\mathcal{O}}$, and therefore

$$\mathcal{V} = H_\eta / \mathfrak{a} H_\eta$$

is a finitely generated $\mathcal{O}$-module.

Step 2: continuity. This is the point at which integral coefficients are needed. We show that the subgroup $(\Gamma_F^{\mathrm{geom}})^I$ of W_F^I acts continuously on $\mathcal{V}$. Fix $a \geq 1$, choose finitely many generators of $\mathcal{V}/\varpi^a\mathcal{V}$ and lift them to H_η. The module H_η is the inductive limit, over the bounds μ on the Harder–Narasimhan polygon and over finite sets of degrees, of the stalks at $\bar{\eta}_I$ of the cohomology sheaves of stacks of finite type over an open subset of X^I. So the chosen lifts come from one such stalk. It is the stalk of a constructible sheaf, on which the geometric fundamental group of the generic point of X^I acts continuously; an open subgroup of that group therefore fixes the lifts modulo ϖ^a. The geometric fundamental group maps onto $(\Gamma_F^{\mathrm{geom}})^I$: over the algebraically closed field of constants, a product of connected finite coverings of the geometric generic points of the factors is integral, which gives the surjectivity on every finite quotient. A continuous surjective homomorphism of profinite groups is open, and the action on $\mathcal{V}$ factors through W_F^I. Hence an open subgroup of $(\Gamma_F^{\mathrm{geom}})^I$ acts trivially on $\mathcal{V}/\varpi^a\mathcal{V}$. Since a is arbitrary, the action is continuous.

Step 3: extension to Galois tuples. For each $i \in I$, Lemma 6.2 extends the action of the i-th factor of W_F^I on $\mathcal{V}$ to an action of Γ_F. By the same lemma these extensions commute, and together they define a continuous action of Γ_F^I on $\mathcal{V}$. For $x \in V_{\mathcal{O}}$ and $\xi \in V_{\mathcal{O}}^*$ invariant

under the diagonal, and $\gamma \in \Gamma_F^I$, we let S_e be the composite of the creation map defined by x, the specialization isomorphism, the action of γ on $\mathcal{V}$, the inverse of specialization and the annihilation map defined by ξ. It is an endomorphism of $\mathcal{O}^{(Y_N)}/\mathfrak{a}\mathcal{O}^{(Y_N)}$, continuous in γ, and for a tuple in W_F^I it is induced by the operator of Proposition 5.10.

A function f on $\widehat{G}^I$ defined over E is of the form $f((g_i)) = \xi((g_i)x)$ for a representation V over E. Such a representation is a direct sum of irreducible representations, and each of them contains the Weyl module over $\mathcal{O}$ of the same highest weight; so V contains a lattice $V_{\mathcal{O}}$ which is a representation of the split group scheme $\widehat{G}^I$ over $\mathcal{O}$. Suitable nonzero multiples ax and $a'\xi$ lie in $V_{\mathcal{O}}$ and in its dual. We define S_e on $\mathcal{Q}$ as $(aa')^{-1}$ times the operator defined by ax and $a'\xi$.

Step 4: relations and compatibilities. For tuples in W_F^I the operators just defined are induced by those of Proposition 5.10. So for such tuples they do not depend on the choices made, they satisfy the relations of that proposition, and they commute with the maps induced by changes of level, central lattice, coefficient field and ideal. Each of these statements is an equality between two continuous functions of $\gamma \in \Gamma_F^I$ with values in a finite-dimensional space, or follows from such equalities, and W_F^I is dense in Γ_F^I. This proves (1) and (2).

Step 5: the lattice. For a place $v \notin N$ the algebra $\mathscr{H}_{G,v,\mathcal{O}}$ commutes with $\mathscr{H}_{G,u,\mathcal{O}}$, so it acts on $\mathcal{O}^{(Y_N)}/\mathfrak{a}\mathcal{O}^{(Y_N)}$. The image of this module in $\mathcal{Q}$ is a finitely generated $\mathcal{O}$-module which spans $\mathcal{Q}$ and is stable under all these algebras. □

We also record that the action of Γ_F^I on the space $\mathcal{V} \otimes_{\mathcal{O}} E$ of step 1 is unramified outside N:

$$\Gamma_F^I \longrightarrow \pi_1(X - N)^I \longrightarrow \mathrm{Aut}_E(\mathcal{V} \otimes E). \tag{6.2}$$

This is a statement with coefficients in E, and that is all that the construction of parameters uses. With a central lattice it is [Xue20c, Proposition 5.0.4], which is formulated with coefficients in E. Without a central lattice, let $v \notin N$ and consider an inertia group at v in one of the factors. It is contained in Γ_F^{geom}, so it preserves the degree on $H_\eta \otimes E$, and it acts trivially on the $m\Xi_u$-coinvariants of $H_\eta \otimes E$ for every m; by Lemma 5.4(3), applied after tensoring with E, it acts trivially on $H_\eta \otimes E$, hence on $\mathcal{V} \otimes E$. The kernel of the continuous action of Γ_F^I is a closed normal subgroup, and the quotient of Γ_F by the closed normal subgroup generated by the inertia groups at the places outside N is $\pi_1(X - N)$.

Remark 6.4. The construction works with the modules $\mathcal{Q}$ themselves and not with their semisimplifications under the Hecke algebra, so nilpotent parts of the operators and generalized eigenspaces are kept; this is used in Definition 3.3. Note also that the existence of a stable lattice implies that the Hecke eigenvalues are integral. At a Satake parameter, every character of the dual torus and its inverse are integral over the ring of invariants of the Weyl group, so their values are units.

Recall the involution $h \mapsto h^\star$, $h^\star(g) = h(g^{-1})$, of the spherical Hecke algebra. It is the transposition for the pairing $[\ ,\ ]$: for a Hecke operator T one has $[Tc, h] = [c, T^\star h]$. It preserves the integral algebra and maps cofinite ideals to cofinite ideals.

Proposition 6.5 (Dual quotients)**.** *The pairing* $[\ ,\ ]$ *identifies*

$$\mathcal{Q}^*_{N,\Xi,u,\mathfrak{a}} = \{h \in k^{Y_{N,\Xi}} : \mathfrak{a}^\star h = 0\}. \tag{6.3}$$

The same assertion holds without a central lattice. For tuples in W_F^I the transpose of the operator $S_{e^\star}$ on the quotient is the restriction of the operator S_e^{fun} of Definition 5.12.

Proof. The pairing identifies the dual of $k^{(Y_{N,\Xi})}$ with $k^{Y_{N,\Xi}}$. The functional $c \mapsto [c, h]$ factors through $\mathcal{Q}_{N,\Xi,u,\mathfrak{a}}$ if and only if $[Tc, h] = 0$ for all $T \in \mathfrak{a}$ and all c. Since $[Tc, h] = [c, T^\star h]$, this means $T^\star h = 0$ for all $T \in \mathfrak{a}$. The last assertion is the definition (5.5). □

By means of this proposition we transpose the operators of Theorem 6.3 for all Galois tuples: on the space (6.3) we define S_e^{fun} as the transpose of $S_{e^\star}$.

6.2 The spaces of functions

The quotients $\mathcal{Q}$ depend on a level, a place, an ideal and possibly a central lattice. We now assemble their duals into spaces of functions that do not depend on such choices, and in particular do not depend on the parameter one wishes to realize.

Definition 6.6. A finite collection of functions of level N has a *finite integral spherical orbit* at a place $u \notin N$ if, after a finite extension of E, it is contained in a finitely generated $\mathcal{O}_E$-module of functions that is stable under $\mathscr{H}_{G,u,\mathcal{O}_E}$.

Lemma 6.7 (Criterion for lying in the dual of a finite quotient)**.** *At fixed level and central lattice, a finite collection of functions lies in the dual of one finite Hecke quotient if and only if it has a finite integral spherical orbit at one place outside the level. It then has such an orbit at every place outside the level. The same holds for the quotients without a central lattice.*

Proof. Suppose that the functions lie in a finitely generated $\mathcal{O}_E$-module L of functions stable under $\mathscr{H}_{G,u,\mathcal{O}_E}$. The image of this algebra in the endomorphisms of L is a finitely generated $\mathcal{O}_E$-module, so the kernel $\mathfrak{b}$ of the action is a cofinite ideal. As $\mathfrak{b}L = 0$, formula (6.3) shows that L is contained in the dual of the quotient defined by the ideal $\mathfrak{b}^\star$.

Conversely, let the functions lie in the dual of a finite quotient $\mathcal{Q}$. By Theorem 6.3(3) there is a lattice in $\mathcal{Q}$ stable under the integral spherical algebras at all places outside the level. The functions in $\mathcal{Q}^*$ which have integral values on this lattice form a lattice in $\mathcal{Q}^*$, stable under the same algebras because they are stable under $\star$. After a finite extension of E, the given functions lie in a multiple of it. So they have a finite integral spherical orbit at every place outside the level.

It follows that finitely many functions which lie in the duals of finite quotients for different places and ideals lie in the dual of a single one: by what has been proved we may use the same place for all of them, and then the intersection of the ideals, which is cofinite because the quotient by it embeds into the product of the quotients. □

Lemma 6.8 (Integrality of parameter values)**.** *Let σ be a continuous semisimple parameter, unramified at a place v. The character of the spherical Hecke algebra at v defined by the Satake parameter $\sigma(\mathrm{Frob}_v)$ has integral values on the integral Hecke algebra, after a finite extension of E. The same holds for $\vartheta_G\sigma$.*

Proof. Under the normalized Satake isomorphism the integral Hecke algebra is spanned over $\mathcal{O}$ by the characters of the representations of $\widehat{G}$; recall that $q^{1/2}$ is a unit of $\mathcal{O}$. So it is enough to see that the trace of $\sigma(\mathrm{Frob}_v)$ in a representation V of $\widehat{G}$ is integral. The image of σ is compact and contained in $\widehat{G}(E')$ for a finite extension E' of E. A compact subgroup of $\mathrm{GL}(V)(E')$ stabilizes a lattice, so the traces of its elements are integral. The involution ϑ_G exchanges a representation and its dual, and the argument applies to $\vartheta_G\sigma$. □

Definition 6.9 (The spaces $\mathcal{A}_\Xi(G)$, $\mathcal{A}^{\mathrm{per}}(G)$ and $\mathcal{A}^{\mathrm{cont}}(G)$)**.** Define spaces of functions on $G(F)\backslash G(\mathbb{A}_F)$, as sums inside the space of all functions of finite level, by

$$\mathcal{A}_\Xi(G) = \bigcup_N \sum_{E,u,\mathfrak{a}} \mathcal{Q}^*_{N,\Xi,u,\mathfrak{a}}, \qquad \mathcal{A}^{\mathrm{per}}(G) = \sum_\Xi \mathcal{A}_\Xi(G), \tag{6.4}$$

and allow twists by characters of the degree with values in the centre by setting

$$\mathcal{A}^{\mathrm{cont}}(G) = \sum_{\psi = c^{\deg_F}} T_\psi \mathcal{A}^{\mathrm{per}}(G), \qquad c \text{ a unit point of } Z(\widehat{G})^\circ. \tag{6.5}$$

Here the dual of each quotient is regarded as a space of functions by (6.3), Ξ runs through the central lattices, and T_ψ is multiplication by the character of $G(\mathbb{A}_F)$ attached to ψ (Proposition 5.13). In words: $\mathcal{A}_\Xi(G)$ consists of the Ξ-invariant functions of finite level with a finite integral spherical orbit at one place, and $\mathcal{A}^{\mathrm{cont}}(G)$ is spanned by the twists of such functions, for varying Ξ, by characters of the degree.

Theorem 6.10 (Excursion operators on $\mathcal{A}^{\mathrm{cont}}(G)$)**.** *1. For every excursion datum e with tuple in Γ_F^I the operators of Theorem 6.3, on the duals of the finite quotients, define an operator S_e^{fun} on $\mathcal{A}^{\mathrm{cont}}(G)$ that preserves $\mathcal{A}_\Xi(G)$ and $\mathcal{A}^{\mathrm{per}}(G)$. It does not depend on the way a function is written as a sum of elements of such duals, and for tuples in W_F^I it is the operator of Definition 5.12.*

2. The three spaces are stable under right translations and Hecke operators, which commute with the operators S_e^{fun}.

3. They contain the cusp forms that are invariant under a central lattice, respectively the twists of such forms, and on cusp forms S_e^{fun} is Lafforgue's operator.

4. For a twist $\psi = c^{\deg_F}$ as in (6.5), and for a central parameter ψ of finite order after shrinking the lattice to the kernel of its class-field character,

$$S_e^{\mathrm{fun}}(T_\psi h) = T_\psi S_{e^\psi}^{\mathrm{fun}} h. \tag{6.6}$$

Proof. Step 1: one quotient for finitely many functions. We first show that every finite collection of elements of $\mathcal{A}^{\mathrm{cont}}(G)$ lies in the dual of one finite quotient without a central lattice. It is enough to consider functions of the form $T_\psi h'$ with h' in the dual of a quotient $\mathcal{Q}_{N,\Xi,u,\mathfrak{a}}$; we may take the same level for all of them, and by Lemma 6.7 the same place u outside that level. The function h' has a finite integral spherical orbit at u. Conjugation by T_ψ multiplies the Hecke operator of a double coset whose elements have valuation λ in $Q_G(F_u)$ by $c^{-\deg(u)\lambda}$, which is a unit. So it is an automorphism of the integral Hecke algebra, and $T_\psi h'$ has a finite integral spherical orbit at u as well. The assertion follows from Lemma 6.7, in its version without a central lattice.

Step 2: definition and independence. On the dual of a finite quotient without a central lattice, S_e^{fun} is the transpose of $S_{e^\star}$. Suppose that a function h lies in the duals of two finite quotients, with or without a central lattice. For tuples in W_F^I the two operators map h to the same function, namely the function $S_e^{\mathrm{fun}} h$ of Definition 5.12, which is defined without reference to a quotient (Proposition 6.5). For tuples in Γ_F^I the two images of h are continuous functions of the tuple with values in the sum of the two duals, a space of functions of finite dimension, and they agree on the dense subset W_F^I; so they agree. By step 1 this defines S_e^{fun} on $\mathcal{A}^{\mathrm{cont}}(G)$. Applied to a quotient by a central lattice Ξ, the same remark shows that $\mathcal{A}_\Xi(G)$ is preserved.

Step 3: twists. The twists by characters of the degree lie in duals of finite quotients, by step 1. For a central twist of finite order, enlarge the level so that it contains the places where the twist is ramified, and replace Ξ by the kernel of the class-field character of the twist on it. At a place u outside the level this character is constant on each double coset, so conjugation by the twist multiplies each Hecke operator of a double coset by a unit. Hence these twists lie in duals of finite quotients as well. For tuples in W_F^I, (6.6) is the transpose of (5.7). Both sides are continuous in $\boldsymbol{\gamma} \in \Gamma_F^I$: the datum e^ψ depends on $\boldsymbol{\gamma}$ through the elements $\psi(\gamma_i)$, which are continuous in γ_i because ψ is of finite order, or because c is a unit point. Indeed, the translate of f by an element of the central torus at each leg is the sum of the components of f of given central weights, each multiplied by the value of the corresponding

character at $\psi(\gamma_i)$. So the formula holds for all Galois tuples. This proves (4), and (1) follows since each $T_\psi \mathcal{A}^{\mathrm{per}}(G)$ is preserved.

Step 4: translations. Let $g \in G(\mathbb{A}_F)$ and let h be an element of one of the three spaces, of level N. Let N' be a level of the translate of h by g, and let $u \notin N \cup N'$ be a place with $g_u \in G(\mathcal{O}_u)$. By Lemma 6.7 the function h has a finite integral spherical orbit at u. Right translation by g commutes with the Hecke algebra at u and with the centre, so the translate has such an orbit as well and lies in the same space. For tuples in W_F^I the excursion operators commute with translations by Proposition 5.10(2), and for Galois tuples by continuity. A Hecke operator is a finite linear combination of right translations.

Step 5: cusp forms. By Harder's theorem the cusp forms of level N invariant under Ξ are supported on a finite subset of $Y_{N,\Xi}$. Those with values in $\mathcal{O}$ form a finitely generated module stable under the integral Hecke algebras, so cusp forms have a finite integral spherical orbit and belong to $\mathcal{A}_\Xi(G)$. On the cuspidal part of the cohomology the operators of Proposition 5.10 are those of [Laf18] by construction, and the latter are continuous in the Galois tuple; so the two actions agree on cusp forms. □

Proposition 6.11 (A finite spherical orbit at one place implies automorphy)**.** *A smooth function of finite level on $G(F)\backslash G(\mathbb{A}_F)$ whose orbit under the spherical Hecke algebra at one place outside the level is finite-dimensional is an automorphic form in the sense of [BL24, §9.1]. No growth condition is assumed.*

Proof. Let φ be the function, u the place, and $\mathfrak{b}$ an ideal of finite codimension in the spherical algebra $\mathscr{H}_{G,u}$ over k which annihilates φ.

Step 1: constant terms. Let $P = MU$ be a parabolic subgroup and φ_P the constant term of φ along P. Since φ has finite level, φ_P is determined by the finitely many functions

$$m \longmapsto \delta_P(m)^{-1/2} \varphi_P(mx)$$

on $M(F)\backslash M(\mathbb{A}_F)$, where x runs through a set of representatives of $G(\prod_v \mathcal{O}_v)/K_N$ with component one at u. The computation of the constant term of a convolution (Lemma 7.7) shows that each of these functions is annihilated by the image of $\mathfrak{b}$ under the normalized Satake homomorphism $\mathscr{H}_{G,u} \to \mathscr{H}_{M,u}$. In terms of a maximal split torus T of M this homomorphism is the inclusion

$$k[X_*(T)]^{W_G} \subset k[X_*(T)]^{W_M}.$$

The algebra $k[X_*(T)]$ is a finitely generated module over the first ring, which is Noetherian, and the second ring is a submodule of it. Hence $\mathscr{H}_{M,u}$ is a finitely generated module over $\mathscr{H}_{G,u}$, the ideal generated by the image of $\mathfrak{b}$ has finite codimension in $\mathscr{H}_{M,u}$, and each of the functions above has a finite-dimensional orbit under $\mathscr{H}_{M,u}$.

Step 2: the centre of the Levi subgroup. The criterion of [BL24, §9.1, Lemma, p. 227] is formulated in terms of a lattice A_P in the centre of $M(\mathbb{A}_F)$, which in the function field case consists of the elements $\lambda(\varpi_{v_0})$ for a place v_0 chosen once and for all and λ a cocharacter of the connected centre of M [BL24, §6.2]. We take $v_0 = u$. The translations by these elements belong to $\mathscr{H}_{M,u}$, so by step 1 the translates of φ_P by A_P span a space W of finite dimension. The elements of A_P act on W by commuting operators. Hence there are characters $\lambda_1, \dots, \lambda_n$ of A_P, with $n = \dim W$, such that

$$\prod_{i=1}^{n} (a_i - \lambda_i(a_i))\varphi_P = 0 \qquad \text{for all } a_1, \dots, a_n \in A_P,$$

where a_i stands for the translation by a_i: decompose W into generalized eigenspaces and use that a product of d commuting nilpotent endomorphisms of a space of dimension d is zero. This holds for every P, and it is the hypothesis of the criterion, which shows that φ is automorphic.

Step 3: coefficients. The criterion is stated for functions with complex values. Let $K \subset k$ be a finitely generated subfield which contains the chosen square root of q and the coefficients of a set of generators of $\mathfrak{b}$, and let $\mathfrak{b}_K$ be the ideal generated by these generators over K. By Proposition 5.5 the quotient of the space of functions with compact support on Y_N with values in K by $\mathfrak{b}_K^\star$ has finite dimension: this can be seen after extension of scalars to k. By transposition φ is a linear form on the extension of scalars of this quotient. So φ is a combination, with coefficients in k, of functions with values in K which are annihilated by $\mathfrak{b}_K$. The field K embeds into $\mathbb{C}$, and the first two steps apply to each of these functions. The space of automorphic forms with coefficients in k contains their span. □

Corollary 6.12. *Every element of $\mathcal{A}^{\mathrm{cont}}(G)$ is an automorphic form.*

Proof. By step 1 of the proof of Theorem 6.10 an element of this space lies in the dual of a finite quotient without a central lattice, so its orbit under the spherical Hecke algebra at one place has finite dimension. Proposition 6.11 applies. □

Remark 6.13. The dual of a finite quotient without a central lattice need not be contained in $\mathcal{A}^{\mathrm{cont}}(G)$. For example, take $G = \mathbb{G}_m$ over $F = \mathbb{F}_q(t)$, the trivial level and a Hecke place of degree one. The quotient of the functions with compact support on $\mathbb{Z}$ by the square of the augmentation ideal of the group algebra of the translations has a dual which contains the function $n \mapsto n$. On the span of this function and the constants, translation acts by a nontrivial Jordan block, whereas it acts semisimply on every element of $\mathcal{A}^{\mathrm{cont}}(G)$: on a function with a central lattice it has finite order, and a twist by a character of the degree multiplies it by a scalar. For this reason the subspaces V used below are intersections with $\mathcal{A}^{\mathrm{cont}}(G)$.

For a continuous semisimple parameter σ, let $\mathfrak{m}_\sigma$ be the ideal generated by the operators $S_e^{\mathrm{fun}} - \nu_\sigma(e)$ in the algebra of excursion operators of the space considered. Every finite collection of elements of $\mathcal{A}^{\mathrm{cont}}(G)$ lies in a subspace V of finite dimension stable under the excursion operators, namely the intersection of $\mathcal{A}^{\mathrm{cont}}(G)$ with the dual of a finite quotient: the intersection is stable because both spaces are (Theorem 6.10(1)). Applying Definition 3.3 to these subspaces gives the summand

$$\mathcal{A}^{\mathrm{cont}}(G)_\sigma = \bigcup_V \{h \in V : \mathfrak{m}_\sigma^a h = 0 \text{ for some } a \geq 1\} \tag{6.7}$$

of $\mathcal{A}^{\mathrm{cont}}(G)$; a joint eigenfunction is an element with $a = 1$.

Corollary 6.14 (Recognizing an eigenfunction for all Galois tuples)**.** *Let σ be a continuous, finitely ramified, semisimple parameter, and let h be a nonzero smooth function of finite level such that $S_e^{\mathrm{fun}} h = \nu_\sigma(e) h$ for every excursion datum e with tuple in W_F^I. Then:*

1. *h belongs to $\mathcal{A}^{\mathrm{cont}}(G)$ and is an automorphic form;*
2. *$S_e^{\mathrm{fun}} h = \nu_\sigma(e) h$ for every excursion datum with tuple in Γ_F^I;*
3. *$T_G(\mathbb{A})$ acts on h by the character ω_σ;*
4. *for every central lattice Ξ on which ω_σ is trivial, h lies in the dual of $\mathcal{Q}_{N,\Xi,u,\chi,1}$ for a suitable level N, place u and integral character χ.*

Proof. We may enlarge the level N so that σ is unramified outside N and ω_σ is trivial on $T_G(\mathbb{A}) \cap K_N$.

Step 1: the central character. By the proof of Lemma 5.2, the translation by an element of $T_G(\mathbb{A})$ is, on functions of level N, a product of translations by elements $\lambda(\varpi_v)$ with $v \notin N$, and such a translation is the Hecke operator of the representation $\alpha_G^*\lambda$ of $\widehat{G}$. By Proposition 5.10(3) a Hecke operator at v is an excursion operator with a Frobenius element at v, so h is an eigenvector of these translations. The eigenvalue is $\omega_\sigma(\lambda(\varpi_v))$. Indeed, by Section 3 the excursion operator S_e attached to $\alpha_G^*\lambda$ and to a geometric Frobenius element Frob_v is the translation $T_{\lambda(\varpi_v)}$ on functions with compact support; its transpose is $T_{\lambda(\varpi_v)^{-1}}$, so $S_e^{\mathrm{fun}} = T_{\lambda(\varpi_v)}$ on all functions. Its eigenvalue on h is

$$\nu_\sigma(e) = \lambda\big(\alpha_G\sigma(\mathrm{Frob}_v)\big) = \lambda\big(\alpha_G\sigma(\mathrm{Art}(\varpi_v)^{-1})\big) = \omega_\sigma(\lambda(\varpi_v)),$$

because $\mathrm{Art}(\varpi_v)$ is an arithmetic Frobenius element. Since ω_σ is trivial on $T_G(F)$ and on $T_G(\mathbb{A}) \cap K_N$, this proves (3).

Step 2: the case of a central character trivial on a lattice. Suppose that ω_σ is trivial on Ξ. Then h is invariant under Ξ. Let $u \notin N$. By (5.6) and Proposition 5.10(3), the linear form $c \mapsto [c, h]$ on functions with compact support is an eigenvector of the Hecke algebra at u, for the character χ defined by $\vartheta_G\sigma(\mathrm{Frob}_u)$. This character is integral by Lemma 6.8. The linear form vanishes on the image of $\ker\chi$, so h lies in the dual of $\mathcal{Q}_{N,\Xi,u,\chi,1}$, which is (4), and in particular $h \in \mathcal{A}_\Xi(G)$. On this space of finite dimension both $S_e^{\mathrm{fun}}h$ and $\nu_\sigma(e)h$ are continuous in the tuple, and they agree on the dense subset W_F^I of Γ_F^I; this is (2). The function is an automorphic form by Corollary 6.12.

Step 3: the general case. Write $\sigma = \psi\sigma^{\mathrm{fin}}$ as in Lemma 4.11. By Proposition 5.13 the function $T_\psi^{-1}h$ satisfies the hypothesis of the corollary for σ^{fin}, whose central character has finite order and is therefore trivial on some central lattice. By step 2, $T_\psi^{-1}h$ lies in $\mathcal{A}^{\mathrm{per}}(G)$ and satisfies (2) for σ^{fin}. Hence $h \in \mathcal{A}^{\mathrm{cont}}(G)$, and (2) for h and σ follows from (6.6). □

6.3 An example: the multiplicative group

For $G = \mathbb{G}_m$ the statements of this section are class field theory, and it may help to see them in this form. Fix a level N. The functions of level N are the functions on the abelian group

$$A_N = F^\times\backslash\mathbb{A}_F^\times/K_N,$$

which the degree maps onto $\mathbb{Z}$ with finite kernel. The dual group is $\mathbb{G}_m$, and a function on $\widehat{G}^I$ invariant under the diagonal translations is a combination of monomials $\prod_i z_i^{n_i}$ with $\sum_i n_i = 0$. For the monomial $z_1^n z_2^{-n}$ and the tuple $(\mathrm{Frob}_v, 1)$, with $v \notin N$, the excursion operator is the translation $T_{\varpi_v^n}$, where Frob_v is a geometric Frobenius element (Section 3).

Eigenfunctions. A character $\chi : A_N \to k^\times$ is an eigenfunction of all the translations, and $T_{\varpi_v^n}\chi = \chi(\varpi_v)^n\chi$. No character has compact support. This is the reason for which the excursion operators, which are defined first on functions with compact support, have to be transposed to spaces of functions without this property.

The space $\mathcal{A}^{\mathrm{cont}}(G)$. A function of level N which is invariant under a central lattice Ξ is a function on the finite group A_N/Ξ, and every such function has a finite integral orbit under the translations. So the part of $\mathcal{A}^{\mathrm{per}}(G)$ of level N is spanned by the characters of A_N of finite order. A twist by $c^{\deg_F}$ multiplies a function by $t \mapsto c^{-\deg(t)}$, and a character of A_N whose values are ℓ-adic units is the product of a character of finite order and of a character of this form, with c a unit. Hence the part of $\mathcal{A}^{\mathrm{cont}}(G)$ of level N is spanned by the characters of A_N with values in the ℓ-adic units. These are the characters $\mathrm{CFT}_{\mathbb{G}_m}(\sigma)$ attached to the

continuous characters σ of Γ_F: a character of the idele class group extends continuously to the profinite group Γ_F^{ab} if and only if its values are units.

Excursion summands. Let σ be a continuous character of Γ_F and $\chi = \mathrm{CFT}_{\mathbb{G}_m}(\sigma)$. By Corollary 6.14(3) a joint eigenfunction of the excursion operators with parameter σ is a multiple of χ. Conversely χ is such an eigenfunction. Indeed, write it as a twist of a character χ_0 of finite order. The excursion operators commute with the translations, for which the line of χ_0 is a joint eigenspace in the functions on A_N/Ξ; so χ_0 is a joint eigenfunction of the excursion operators, and its parameter is determined by the values $\chi_0(\varpi_v)$ at the Frobenius elements, by the theorem of Chebotarev. The twist is treated by (6.6). So the excursion summand of σ is the line of χ, and the decomposition of $\mathcal{A}^{\mathrm{cont}}(G)$ into excursion summands is its decomposition into characters.

What is left out. The character $t \mapsto c^{-\deg(t)}$, for c which is not a unit, is an eigenfunction of all the translations, but the corresponding character of the Weil group does not extend continuously to Γ_F, and the function is not in $\mathcal{A}^{\mathrm{cont}}(G)$. The function deg is killed by $(T_a - 1)^2$ for every a, without being a sum of eigenfunctions (Remark 6.13). For a group which is not a torus the eigenfunctions with a given parameter are no longer determined in this way; what Sections 7 and 8 add is a control of their constant terms.

7 Constant terms and the excursion character

An Eisenstein series must have the parameter of the function it is induced from, composed with the inclusion of dual Levi subgroups. To prove this in Section 12 we need the constant term to intertwine *all* excursion operators of G with those of the Levi subgroup, which is Theorem 7.1 below. Compatibility with Hecke operators at unramified places would not be enough there, since those determine a parameter only up to local conjugacy at each place.

The theorem is used in Section 12, that is, for the parameters which are not elliptic in Theorems B and C, and nowhere else. The criterion for cuspidality of Section 8, on which Theorem A and the two constructions [II, III] depend, uses only the elementary Lemma 7.7 at the end of this section.

Fix a split maximal torus and a Borel subgroup containing it, in the split model of G over the integers. Let $P = MU$ be a standard parabolic subgroup with its standard Levi subgroup, and let $i : \widehat{M} \hookrightarrow \widehat{G}$ be the inclusion of the dual Levi subgroup. For an excursion datum $e = (I, f, \boldsymbol{\gamma})$ of G put $i^*e = (I, f \circ i^{\times I}, \boldsymbol{\gamma})$, an excursion datum of M. Fix a Haar measure on $U(\mathbb{A})$, and let du be the measure which it induces on $U(F)\backslash U(\mathbb{A})$. Every element of $G(\mathbb{A})$ is of the form umg_0 with $u \in U(\mathbb{A})$, $m \in M(\mathbb{A})$ and $g_0 \in G(\prod_v \mathcal{O}_v)$; this is the Iwasawa decomposition. The normalized constant term of a function h is

$$\mathsf{CT}_P^{\mathrm{nor}} h(mg_0) = \delta_P(m)^{-1/2} \int_{U(F)\backslash U(\mathbb{A})} h(umg_0)\, du.$$

The integral exists for every smooth function of finite level, since $U(F)\backslash U(\mathbb{A})$ is compact. For fixed g_0 we regard $m \mapsto \mathsf{CT}_P^{\mathrm{nor}} h(mg_0)$ as a function on $M(F)\backslash M(\mathbb{A})$ and call it the *slice* of the constant term at g_0. A function of finite level has finitely many different slices.

Theorem 7.1 (Constant terms and excursion operators)**.** *Let the Satake isomorphisms of G and of M be normalized by the same square root of q and by geometric Frobenius elements. For every smooth function of finite level and every excursion datum e with $\boldsymbol{\gamma} \in W_F^I$,*

$$\mathsf{CT}_P^{\mathrm{nor}} S_e^{G,\mathrm{fun}} = S_{i^*e}^{M,\mathrm{fun}} \mathsf{CT}_P^{\mathrm{nor}}, \tag{7.1}$$

where the operator on the right is applied to each slice. If the function is invariant under a subgroup of $T_G(\mathbb{A})$, so are both sides.

On the level of cohomology of shtukas the comparison is due to Xue (Input 3(4)): there is a constant-term morphism $\mathsf{CT}_P^{\mathrm{geo}}$ from the cohomology for G to that for M, which for $I = \varnothing$ is the sum over the fibres of $\mathrm{Bun}_P \to \mathrm{Bun}_M$ with the weights of the stack [Xue20a, Theorem 3.2.6, Proposition 3.5.3, Example 3.5.15 and (3.44)], [Xue20b, Proposition 4.1.3]. What remains is to compare normalizations, and this is the content of the section. Three factors have to be accounted for.

1. The passage from the counting measure of the stack to Haar measure on $U(F)\backslash U(\mathbb{A})$ (Lemma 7.2).
2. The shift and Tate twist in the restriction of Satake sheaves from G to M (Lemma 7.3).
3. The degree of a Frobenius map on a torsor of jets that occurs when a partial Frobenius is compared for P and for M (Proposition 7.4). This factor depends on an auxiliary choice, the order of the jets, which must cancel. Its computation is independent of the rest of the section and is given in Appendix A.

The theorem is proved in Section 7.3.

7.1 Measure and Satake normalization

We fix a level N and put

$$R_U = K_N \cap U(\mathbb{A}), \qquad c_U = \mathrm{vol}(R_U), \qquad a_P = \det(\mathrm{Ad}_M\,|_{\mathrm{Lie}\,U}),$$

so that a_P is a character of M and $\delta_P(m) = |a_P(m)|$. For an M-bundle of degree $\nu \in \pi_1(M)$ write

$$\delta_P(\nu) = q^{\langle \nu, a_P\rangle},$$

the pairing being that of a_P with the image of ν in the cocharacter group of M/M_{der}. Two conventions are in force. The degree of the bundle attached to an adele is the negative of its valuation, so that $\delta_P(m) = \delta_P(\nu)$ if ν is the degree of the bundle attached to m. And a modification of relative position μ, from its source to its target, changes the degree from ν to $\nu - \mu$; for a line bundle and $\mu = r$ the target is the source twisted by $\mathcal{O}(-rx)$. This is the convention of Section 3.2, in which the position of the source relative to the target is bounded by the weights of the representation. So μ is also the weight by which the centre of $\widehat{M}$ acts on the corresponding summand of a representation.

Lemma 7.2 (The Haar factor)**.** *For $I = \varnothing$, the morphism $\mathsf{CT}_P^{\mathrm{geo}}$ and the normalized constant term of functions are related by*

$$\mathsf{CT}_P^{\mathrm{nor}} = c_U \delta_P^{1/2} \mathsf{CT}_P^{\mathrm{geo}}. \tag{7.2}$$

Proof. Let h be a function of level N and consider the point mg_0. Since g_0 normalizes K_N, the function $u \mapsto h(umg_0)$ on $U(F)\backslash U(\mathbb{A})$ is invariant under right translation by mR_Um^{-1}. The double cosets in $U(F)\backslash U(\mathbb{A})/mR_Um^{-1}$ are the points of the fibre of $\mathrm{Bun}_{P,N} \to \mathrm{Bun}_{M,N}$ over the point defined by m, and the automorphism group of the point defined by u, relative to $\mathrm{Bun}_{M,N}$, is

$$\Gamma_u = U(F) \cap u(mR_Um^{-1})u^{-1}.$$

The morphism $\mathsf{CT}_P^{\mathrm{geo}}$ gives this point the weight $|\Gamma_u|^{-1}$. In the integral, the same double coset has volume $\mathrm{vol}(mR_Um^{-1})|\Gamma_u|^{-1} = c_U\delta_P(m)|\Gamma_u|^{-1}$. So the integral is $c_U\delta_P(m)$ times the value of $\mathsf{CT}_P^{\mathrm{geo}}h$, and multiplication by $\delta_P(m)^{-1/2}$ gives (7.2). Dividing by a lattice in $T_G(\mathbb{A})$ changes neither the groups Γ_u nor the character a_P, so the formula holds with a central lattice as well. □

Lemma 7.3 (Restriction of Satake sheaves with their Frobenius)**.** *Let* $\mathsf{Sat}_G(V)$ *be the Satake sheaf of a representation* V *of* $\widehat{G}^I$*, normalized to be pure of weight zero. Let* i^0_μ *be the map which forgets the reduction to* P *on the space of local modifications of* P*-bundles of relative position* μ *in* $\pi_1(M)$*, and let* π^0_μ *be the map from this space to the modifications of* M*-bundles. We write* $n_\mu = \langle \mu, a_P \rangle$*; the pairing depends only on the image of* μ *in* $X_*(M/M_{\mathrm{der}})$*. There is an isomorphism*

$$\bigoplus_\mu (\pi^0_\mu)_! (i^0_\mu)^* \mathsf{Sat}_G(V)[-n_\mu](-n_\mu/2) \simeq \mathsf{Sat}_M(\operatorname{Res} V), \tag{7.3}$$

compatible with the actions of Frobenius and with fusion. Here $\operatorname{Res} V$ *is the restriction of* V *to* $\widehat{M}^I$*.*

Proof. Without the Frobenius structures the isomorphism, with its compatibility with tensor products, is [Xue20a, Theorem 3.2.6], and the compatibility with fusion is [Xue20b, Lemma 4.1.5]. The commutativity constraint of the Satake category is the one modified by the Koszul sign [Zhu17, (5.2.8) and Theorem 5.2.9], and the geometric Satake equivalence with integral coefficients is understood with the correction of [MV18].

We determine the sign of the shift with the conventions of Section 3.2. In the local model the target is trivialized, and a modification is given by the lattice of its source. A source of relative degree μ is the lattice $t^{-\mu}$, so the semi-infinite orbit $U((t))\, t^\lambda$ which contains it has $\lambda = -\mu$, and by (3.2) the sheaf is the one attached by $\mathsf{Sat}_G^{\mathrm{lp}}$ to $V \circ \epsilon_G$, in which the weight μ of V has become the weight λ. For restriction to a maximal torus, the cohomology with compact supports of a Satake sheaf on this orbit is concentrated in degree $\langle 2\rho_G, \lambda \rangle$ [MV07, Theorems 3.5 and 3.6]; for ℓ-adic sheaves in positive characteristic this is [Zhu17, Theorem 5.3.9(2)]. So the shift which brings it to degree zero is $[-\langle 2\rho_G, \mu \rangle]$. The restriction to the torus is the composite of the restriction from G to a Levi subgroup M and of the restriction from M to the torus [Zhu17, Remark 5.3.16]: the shift from G to M is the difference of the shifts from G and from M to the torus, that is, $[-n_\mu]$. For sheaves normalized to weight zero the Tate twist is half the shift. In [Xue20a, (3.17)] the shift and twist are written $[n](n/2)$; we use the isomorphism with the shift and twist just determined, and the following example is a check of the sign. Take $G = \mathrm{GL}_2$, with P the group of upper triangular matrices and V the standard representation. Then $a_P = (1, -1)$, and a reduction of a bundle to P is a line subbundle L_1 with quotient L_2. In the local model the target $\mathcal{G}_1$ of the modification is trivialized, with its reduction, and the source varies [Xue20a, Definition 1.1.11]. So the modifications are the lattices $\mathcal{G}_0 \supset \mathcal{G}_1$ with quotient of length one; they form a projective line, on which the Satake sheaf is $E[1](1/2)$. For $\mu = (1, 0)$, that is, when L_1 changes, the lattice $\mathcal{G}_0$ is the sum of $\mathcal{G}_1$ and the larger line, so the fibre of π^0_μ is a point and the restriction is $E[1](1/2)$; here $n_\mu = 1$. For $\mu = (0, 1)$ the fibre is an affine line and the restriction is $E[-1](-1/2)$; here $n_\mu = -1$. In both cases the shift and twist by $[-n_\mu](-n_\mu/2)$ gives the sheaf E in degree zero with trivial Frobenius, which is the Satake sheaf of the torus. (If the source were trivialized instead, the two fibres would be exchanged, and so would the roles of the unipotent groups of the source and of the target in the quotient by jets. The quotient stacks are the same. We use the local model as stated throughout, in particular in Lemmas A.1 and A.2.)

It remains to see that the isomorphism commutes with Frobenius. Both sides are additive in V, so we may assume that V is an exterior tensor product of representations of the factors. On the open subset where the legs are distinct both sides are then exterior products, and it suffices to consider one leg. Let T be a maximal torus of M. The restriction functor from M to T, normalized in the same way, is faithful, and its composite with the restriction from G to M is the restriction from G to T. For the intersection complexes normalized to weight zero, Frobenius acts trivially on every weight space of the restriction to T [Zhu17, Remark 5.5.15

and proof of Lemma 5.6.3]. Apply the restriction to T to the two sides of (7.3). The Frobenius endomorphisms of both become the identity, so the isomorphism commutes with them after restriction to T, and by faithfulness it commutes with them. The decomposition according to μ is the decomposition according to the weights of the centre of $\widehat{M}$, and is respected. By fusion the compatibility extends from the open subset to the diagonals. □

7.2 The partial Frobenius

The third factor is the subject of the following proposition.

Proposition 7.4 (Partial Frobenius and the constant term)**.** *Let μ_i be the relative position at the leg i and $n_i = \langle \mu_i, a_P \rangle$. Let F_i^G and F_i^M be the partial Frobenius maps on the cohomology of the shtukas of G and of M, for Satake sheaves normalized to weight zero, and $\mathsf{CT}_\nu^{\mathrm{geo}}$ the component of the constant-term morphism with values in the degree ν of M. Then*

$$\mathsf{CT}_\nu^{\mathrm{geo}} F_i^G = q^{-n_i/2} F_i^M \mathsf{CT}_{\nu-\mu_i}^{\mathrm{geo}},$$

and the morphism which is $c_U \delta_P(\nu)^{1/2} \mathsf{CT}_\nu^{\mathrm{geo}}$ in degree ν commutes with the partial Frobenius maps.

The proof is given in Appendix A. It compares the partial Frobenius morphisms of P and of M on a torsor under a group of jets. A Frobenius map of this torsor has degree q^{m_i}, where m_i depends on the order of the jets, which is an auxiliary choice (Lemmas A.1 and A.2). This factor cancels against the purity isomorphisms, and $q^{-n_i/2}$ remains. A compatibility of the constant-term morphism with the partial Frobenius morphisms is stated in [Xue20b, Lemma 4.1.7], without the normalization of the sheaves on M at each leg and the indexing of the components by the degree; for this reason we prove the proposition from the construction of the constant-term morphism. The appendix also explains the signs which occur when Satake sheaves are replaced by functions (Remark A.4), and checks the power of q by counting in the case of GL_2 (Remark A.5).

7.3 Completion of the comparison

We say that a linear operator between spaces of functions on discrete sets has *finite rows* if the value of the image at each point depends on the values of the function at finitely many points only.

Lemma 7.5 (Operators with finite rows)**.** *Two operators with finite rows which agree on the characteristic functions of points agree on all functions. A composite of operators with finite rows has finite rows.*

Proof. The value of the difference of the two operators at a point is a finite linear combination of values of the function, and the coefficients are the values of the difference on characteristic functions of points. For a composite, the value at a point depends on finitely many values of the intermediate function, each of which depends on finitely many values of the given one. □

Proof of Theorem 7.1. Step 1: functions with compact support. The cohomology of the shtukas of M appears in the constant-term morphism as a limit over truncations defined by the Harder–Narasimhan polygons of G. In a fixed degree of M these truncations and those defined by the polygons of M are cofinal in each other, since the inequalities for the roots differ by the central component of the degree, which is fixed [Xue20a, §1.5.17, (1.21)]. So the target of $\mathsf{CT}_P^{\mathrm{geo}}$ is the cohomology of the shtukas of M considered in Section 5, degree by

degree. Infinitely many degrees of M can occur, so the constant-term morphism takes its values in the product of the components of given degree.

By Lemma 7.2 and Proposition 7.4 the morphism $c_U \delta_P^{1/2} \mathsf{CT}_P^{\mathrm{geo}}$ is the normalized constant term for $I = \varnothing$ and commutes with the partial Frobenius maps and their inverses. It commutes with the action of the fundamental group and with specialization, as any morphism of sheaves, and with creation and annihilation by the compatibility with fusion in Lemma 7.3; the restriction to $\widehat{M}^I$ of the vectors x and ξ defines the function $f \circ i^{\times I}$. An excursion operator is a composite of finitely many of these maps. For a given degree at the end, only finitely many shifts of the degree occur in between, and every class which enters is represented at a finite bound of Harder–Narasimhan. Hence

$$\mathsf{CT}_P^{\mathrm{nor}} S_e^G c = S_{i^* e}^M \mathsf{CT}_P^{\mathrm{nor}} c$$

for a function c with compact support, where the right-hand side is understood as follows. The constant term of c need not have compact support, but its restriction to each degree of M has [Xue20a, Lemma 1.5.19]. By Proposition 5.10(4) the operator $S_{i^* e}^M$ shifts the degree by finitely many values. So its value at a point involves the constant term in finitely many degrees only, and it coincides with the value of $S_{i^* e}^{M,\mathrm{fun}}$.

Step 2: all functions. At a fixed level the normalized constant term has finite rows, because $U(F)\backslash U(\mathbb{A})$ is compact. The excursion operators have finite rows by Proposition 5.11. So both sides of (7.1) have finite rows, and by Lemma 7.5 the identity extends from functions with compact support to all functions of the given level. The slice at g_0 of the constant term of h is the slice at 1 of the constant term of a right translate of h, and excursion operators commute with right translations; this gives the identity on all slices. Finally the character δ_P is trivial on $T_G(\mathbb{A})$ and all the maps commute with translations by this group, which gives the last assertion. □

Corollary 7.6 (Constant terms and finite Hecke quotients)**.** *Let h lie in the dual of a finite Hecke quotient for G without a central lattice. Then every slice of its normalized constant term along P lies in the dual of a finite Hecke quotient for M without a central lattice, and* (7.1) *holds on h for all tuples $\boldsymbol{\gamma} \in \Gamma_F^I$.*

Proof. Let h be annihilated by $\mathfrak{a}^\star$, for a cofinite ideal $\mathfrak{a}$ of $\mathscr{H}_{G,u,\mathcal{O}}$. We use the finitely many slices at points g_0 whose component at u is one. Let $\mathrm{Sat}_P : \mathscr{H}_{G,u,\mathcal{O}} \to \mathscr{H}_{M,u,\mathcal{O}}$ be the normalized Satake homomorphism. As in the proof of Proposition 6.11, the second algebra is a finitely generated module over the first: the algebra $\mathcal{O}[X_*(T)]$ is finitely generated as a module over its Noetherian subring of invariants of W_G, and the ring of invariants of W_M is a submodule. Therefore the ideal

$$\mathfrak{a}_M = \mathrm{Sat}_P(\mathfrak{a}) \mathscr{H}_{M,u,\mathcal{O}}$$

is cofinite. The homomorphism Sat_P commutes with $\star$, since on both algebras $\star$ is induced by the inversion of $X_*(T)$. By Lemma 7.7, each slice is annihilated by $\mathfrak{a}_M^\star$, so it lies in the dual of the quotient for M defined by $\mathfrak{a}_M$. On these duals both sides of (7.1) are continuous in the tuple, by Theorem 6.3, and they agree on the dense subset W_F^I. □

7.4 Spherical Hecke operators: the classical formula

For Hecke operators at an unramified place the compatibility of Theorem 7.1 is elementary. We give the proof, because Section 8 uses only this case; nothing from the rest of the present section enters it.

Let u be a place. Haar measures on $G(F_u)$, $M(F_u)$ and $U(F_u)$ are normalized so that the groups of points with values in $\mathcal{O}_u$ have volume one. The spherical Hecke algebra $\mathscr{H}_{G,u}$ consists of the functions with compact support on $G(F_u)$ which are invariant on both sides under $G(\mathcal{O}_u)$, and $f \in \mathscr{H}_{G,u}$ acts on functions invariant on the right under $G(\mathcal{O}_u)$ by

$$(T_f h)(g) = \int_{G(F_u)} h(gx) f(x)\, dx.$$

Put

$$\mathrm{Sat}_P(f)(m) = \delta_P(m)^{1/2} \int_{U(F_u)} f(mn)\, dn \qquad (m \in M(F_u)).$$

Let $B = TN$ be a Borel subgroup contained in P, with $T \subset M$. The Satake transform of G is $f \mapsto (t \mapsto \delta_B(t)^{1/2} \int_{N(F_u)} f(tn)\, dn)$ [Car79, §4.2]. Since $N = (N \cap M)U$ and $\delta_B = \delta_{B\cap M}\delta_P$ on T, it is the composite of Sat_P with the Satake transform of M. So Sat_P is a homomorphism of algebras $\mathscr{H}_{G,u} \to \mathscr{H}_{M,u}$, and the Satake isomorphisms of G and of M, normalized by the same square root of q, identify it with the inclusion $k[X_*(T)]^{W_G} \subset k[X_*(T)]^{W_M}$. On the dual side it is the restriction of representations from $\widehat{G}$ to $\widehat{M}$.

Lemma 7.7 (Constant terms and spherical Hecke operators)**.** *Let h be a smooth function of finite level on $G(F)\backslash G(\mathbb{A})$, invariant on the right under $G(\mathcal{O}_u)$, and let $g_0 \in G(\prod_v \mathcal{O}_v)$ have component one at u. For $f \in \mathscr{H}_{G,u}$ the slice at g_0 of $\mathsf{CT}_P^{\mathrm{nor}}(T_f h)$ is the image under $T_{\mathrm{Sat}_P(f)}$ of the slice at g_0 of $\mathsf{CT}_P^{\mathrm{nor}} h$.*

Proof. By the Iwasawa decomposition every $x \in G(F_u)$ is of the form $x = m'n'k$ with $m' \in M(F_u)$, $n' \in U(F_u)$ and $k \in G(\mathcal{O}_u)$, and $dx = dm'\, dn'\, dk$, because $dm'\, dn'$ is a left Haar measure on $P(F_u)$. The elements of $G(F_u)$ commute with g_0, and h is invariant under k. So for $n \in U(\mathbb{A})$ and $m \in M(\mathbb{A})$

$$(T_f h)(nmg_0) = \int_{M(F_u)} \int_{U(F_u)} h(nmm'n'g_0)\, f(m'n')\, dn'\, dm'.$$

We integrate over $n \in U(F)\backslash U(\mathbb{A})$. Since $nmm'n' = n \cdot mm'n'(mm')^{-1} \cdot mm'$ and the measure on $U(F)\backslash U(\mathbb{A})$ is invariant under translations, the integral of $h(nmm'n'g_0)$ is $\delta_P(mm')^{1/2}\, \mathsf{CT}_P^{\mathrm{nor}} h(mm'g_0)$, whatever n' is. Multiplying by $\delta_P(m)^{-1/2}$ we get

$$\mathsf{CT}_P^{\mathrm{nor}}(T_f h)(mg_0) = \int_{M(F_u)} \mathsf{CT}_P^{\mathrm{nor}} h(mm'g_0)\, \delta_P(m')^{1/2} \int_{U(F_u)} f(m'n')\, dn'\, dm',$$

which is the assertion. □

If the Hecke algebra is made to act through $f(x^{-1})$ instead of $f(x)$, the lemma holds in the same form, because Sat_P commutes with the involutions $f \mapsto f(x^{-1})$ of the two algebras: on both, the involution is induced by the inversion of $X_*(T)$.

8 Criteria for cuspidality

The main result of this section is Proposition 8.6: a function of finite level whose excursion character is an elliptic parameter is cuspidal. The idea is that a nonzero constant term along a proper parabolic subgroup $P = MU$ would produce a parameter σ_M of the Levi subgroup M such that σ and the composite $i\sigma_M$ of σ_M with $\widehat{M} \to \widehat{G}$ have conjugate images of Frobenius at almost all places. For a general reductive group this does not make the two parameters conjugate. It does make their composites with the adjoint representation of $\widehat{G}$ isomorphic, and the spaces of invariants of these two representations have different dimensions when σ is

elliptic and M is proper. The argument uses only the Hecke operators at unramified places and the classical formula for the constant term of a convolution (Lemma 7.7).

Around this result there are three auxiliary statements, used by the constructions of [II] and [III], which deliver their functions with different amounts of information.

- Lemma 8.2: a function known to satisfy only the Hecke equations of one faithful representation lies in the spaces of Section 6.2.
- Lemma 8.8: from a function whose pairing with one test function has the right Hecke eigenvalues, one can extract a function with the right generalized Hecke character and nonzero pairing.
- Proposition 8.10: the form in which the geometric construction delivers its function, as a functional on functions with compact support.

8.1 Functions with the Hecke equations of one representation

Let u be a place and W a finite-dimensional space of functions stable under the spherical Hecke algebra at u. The joint eigencharacters of the algebra on W correspond, by the Satake isomorphism, to finitely many points of the dual torus up to the Weyl group; we call them the *Satake parameters* of W.

Definition 8.1 (Unit-valued Satake parameters)**.** The space W has *unit-valued Satake parameters* if every character of the dual torus takes an ℓ-adic unit value at each of its Satake parameters.

The condition does not depend on the representative modulo the Weyl group. It does not say that every eigenvalue of a Hecke operator is a unit: such an eigenvalue is a sum of values of characters, and can be zero.

Lemma 8.2 (From one representation to the whole Hecke algebra)**.** *Let $r : \widehat{G} \to \mathrm{GL}(V)$ be a representation with finite kernel, and u a place outside the level of a smooth function h of finite level. Let $\mathscr{C}_r$ be the subalgebra of the spherical Hecke algebra at u generated by the operators which correspond to the coefficients of the characteristic polynomial of r and to the inverse of its determinant.*

1. *If $\mathscr{C}_r h$ has finite dimension, the orbit of h under the whole spherical Hecke algebra at u has finite dimension.*
2. *If moreover, at every joint eigencharacter of $\mathscr{C}_r$ on $\mathscr{C}_r h$, the roots of the characteristic polynomial are ℓ-adic units, then this orbit has unit-valued Satake parameters, and it is spanned by a lattice stable under the integral spherical Hecke algebra.*
3. *If in addition h is invariant under a central lattice Ξ, it belongs to $\mathcal{A}_\Xi(G)$.*

Proof. (1) Let $\widehat{T}$ be the dual torus and $\mathscr{L} = k[X^*(\widehat{T})]$, so that the spherical Hecke algebra is $\mathscr{H} = \mathscr{L}^{W_G}$. We show that $\mathscr{L}$ is integral over $\mathscr{C}_r$. For a weight μ of r the monomial t^μ is a root of the characteristic polynomial of r, a monic polynomial with coefficients in $\mathscr{C}_r$; and $t^{-\mu}$ is a root of the reciprocal polynomial, which becomes monic after multiplication by the inverse of the determinant. Since the kernel of r is finite, the weights of r generate a subgroup of finite index in $X^*(\widehat{T})$. So for every λ some power of t^λ is a product of monomials $t^{\pm\mu}$, and t^λ is integral over $\mathscr{C}_r$.

The algebra $\mathscr{L}$ is finitely generated and integral over $\mathscr{C}_r$, hence a finitely generated module over it. The ring $\mathscr{C}_r$ is Noetherian, so the submodule $\mathscr{H}$ is finitely generated as well. If $a_1, \dots, a_s$ are generators, then $\mathscr{H}h = \sum_j a_j \mathscr{C}_r h$ has finite dimension.

(2) At a Satake parameter of $\mathscr{H}h$ the values of the weights of r are roots of the characteristic polynomial at a joint eigencharacter of $\mathscr{C}_r$, hence units. Since the weights generate a subgroup of finite index, a power of the value of any character is a unit, and so is the value itself.

For the lattice, choose finitely many generators $T_1, \dots, T_m$ of the integral spherical Hecke algebra as an algebra over $\mathcal{O}$. Their eigenvalues on $\mathscr{H}h$ are sums of values of characters at the Satake parameters, hence integral. So the characteristic polynomial of each T_j on $\mathscr{H}h$ has integral coefficients, and by the theorem of Cayley and Hamilton the algebra generated over the ring of integers by the commuting operators T_j is spanned by finitely many monomials in them. The image under this algebra of a finite set which spans $\mathscr{H}h$ is the required lattice.

(3) This follows from (2) and Lemma 6.7. □

For the functions constructed later, the roots in (2) are eigenvalues of the image of a Frobenius element under a continuous parameter, and they are units because the image of the parameter is compact (Lemma 6.8). This is where the continuity of the parameter is used.

8.2 Ellipticity implies cuspidality

The first lemma keeps track of all excursion operators on a constant term. It depends on Theorem 7.1 and is used only in Section 12.

Lemma 8.3 (A cuspidal constant term inherits all relations)**.** *Let $h \neq 0$ be a smooth function of finite level on $G(F)\backslash G(\mathbb{A})$ whose orbit under the spherical Hecke algebra at one place outside the level has finite dimension and unit-valued Satake parameters. There are a standard Levi subgroup $M \subset G$ and a nonzero cuspidal function on M which is a joint eigenfunction of the excursion operators of M, for tuples in the Weil group, with a parameter σ_M which is continuous, semisimple and ramified at finitely many places, such that*

$$Bh = 0 \quad \Longrightarrow \quad \nu_{i\sigma_M}(B) = 0$$

for every polynomial B in excursion operators of G with tuples in the Weil group. Here $i : \widehat{M} \hookrightarrow \widehat{G}$ is the inclusion, and $\nu_{i\sigma_M}(B)$ is the value of the polynomial when each operator S_e is replaced by $\nu_{i\sigma_M}(e)$. If h is not cuspidal, M can be chosen proper.

Proof. Step 1: a cuspidal constant term. Among the standard parabolic subgroups, G included, along which the constant term of h is not zero, choose one whose Levi subgroup M has minimal semisimple rank, and let h_M be a nonzero slice of the normalized constant term. The constant term of h_M along a proper parabolic subgroup of M is a slice of the constant term of h along a smaller parabolic subgroup of G, which vanishes by the choice of M. So h_M is cuspidal. If h is not cuspidal, some proper parabolic subgroup has a nonzero constant term, and M is proper.

Step 2: a finite spherical orbit. Let u be the place of the hypothesis. Take the slice at a point of $G(\prod_v \mathcal{O}_v)$ whose component at u is one; this is possible because h is invariant under $G(\mathcal{O}_u)$. If B is a polynomial in Weil excursion operators of G with $Bh = 0$, then $(i^*B)h_M = 0$ by Theorem 7.1. Let J be the annihilator in $\mathscr{H}_{G,u}$ of the orbit of h. By Lemma 7.7 the ideal $\mathrm{Sat}_P(J)\mathscr{H}_{M,u}$ annihilates h_M. Since $\mathscr{H}_{G,u}/J$ has finite dimension and $\mathscr{H}_{M,u}$ is a finitely generated module over $\mathscr{H}_{G,u}$, the orbit $\mathscr{H}_{M,u}h_M$ has finite dimension. Its elements are cusp forms, and they are annihilated by every i^*B as above, because spherical Hecke operators commute with Weil excursion operators. Let v be a joint eigenvector of $\mathscr{H}_{M,u}$ in this orbit. Its Satake parameter, taken modulo the Weyl group of G, is one of the Satake parameters of the orbit of h. So all characters of the dual torus take unit values at it.

Step 3: a central lattice. Let Ξ be the group generated by the images of the elements $\lambda(\varpi_u)$, $\lambda \in X_*(T_M)$. It is a central lattice of M. The translations by its elements are spherical

Hecke operators, so Ξ acts on v through a character with unit values. The degree maps Ξ isomorphically onto a subgroup of finite index of $X_*(Q_M)$, and the group of units of the ring of integers of k is divisible. So there is a point c of $Z(\widehat{M})^\circ(k)$, at which all characters take unit values, such that the character of $M(\mathbb{A})$ attached to $\psi = c^{\deg_F}$ is the inverse of this character on Ξ. The point c is defined over a finite extension of $\mathbb{Q}_\ell$, and $n \mapsto c^n$ extends continuously to $\widehat{\mathbb{Z}}$. The function $v' = T_\psi v$ is cuspidal and invariant under Ξ. For a polynomial R in Weil excursion data of M let $R^{\psi^{-1}}$ be its image under the substitution $e \mapsto e^{\psi^{-1}}$ of (5.7). That formula gives $R^{\psi^{-1}} v' = T_\psi R v$, which is zero for $R = i^*B$ with $Bh = 0$.

Step 4: a parameter for M. By [Laf18, Theorem 0.1] the cusp form v', which is invariant under Ξ, is a sum of components indexed by parameters of M. Let τ be the parameter of a nonzero component. The projections onto the components commute with the excursion operators, so the operators $R^{\psi^{-1}}$ of step 3 annihilate this component. On the summand of τ the only eigenvalue of $R^{\psi^{-1}}$ is $\nu_\tau(R^{\psi^{-1}})$, which is therefore zero. Let w be a joint eigenvector of the excursion operators in the summand, and put $\sigma_M = \tau\psi^{-1}$. This parameter is continuous, finitely ramified and semisimple, because ψ is a continuous unramified character with values in the centre. The cusp form $T_{\psi^{-1}} w$ is not zero, and the Weil excursion operators act on it through σ_M. For B with $Bh = 0$,

$$\nu_{i\sigma_M}(B) = \nu_{\sigma_M}(i^*B) = \nu_\tau\big((i^*B)^{\psi^{-1}}\big) = 0.$$

□

Definition 8.4 (Generalized excursion character)**.** A function h has *generalized excursion character σ for Weil tuples* if for every excursion datum e with tuple in W_F^I there is an integer $n_e \geq 1$ with

$$(S_e^{\mathrm{fun}} - \nu_\sigma(e))^{n_e} h = 0.$$

The exponent may depend on e.

Lemma 8.5 (A function which is not cuspidal has a parameter in a Levi subgroup)**.** *Let h be a smooth function of finite level and S a finite set of places containing the level of h. For each place $u \notin S$ let $\mathscr{A}_u$ be a finitely generated subalgebra of the spherical Hecke algebra $\mathscr{H}_{G,u}$ over which $\mathscr{H}_{G,u}$ is a finitely generated module, and χ_u a character of $\mathscr{A}_u$ such that for every $T \in \mathscr{A}_u$ some power of $T - \chi_u(T)$ annihilates h. Suppose that h is not cuspidal, and that at one place $u_0 \notin S$ every character of $\mathscr{H}_{G,u_0}$ which extends χ_{u_0} is given by a Satake parameter at which all characters of the dual torus take ℓ-adic unit values. Then there are a proper standard Levi subgroup M of G and a continuous semisimple parameter $\sigma_M : \Gamma_F \to \widehat{M}(k)$, unramified outside S, such that for every $u \notin S$ the character of $\mathscr{H}_{G,u}$ given by the Satake parameter $i\sigma_M(\mathrm{Frob}_u)$ restricts to χ_u on $\mathscr{A}_u$.*

Proof. The argument is the classical one for automorphic forms: a nonzero constant term along a parabolic subgroup which is minimal for this property is cuspidal (step 1), it satisfies the Hecke equations obtained from those of h by the Satake transform (step 2), and a cuspidal function has a parameter by the theorem of V. Lafforgue (step 5). Steps 3 and 4 are needed because h is not assumed invariant under a lattice in the centre: they show that the constant term becomes invariant under such a lattice after a twist.

Step 1: a cuspidal constant term. As in step 1 of the proof of Lemma 8.3, choose a proper standard parabolic subgroup $P = MU$ along which the constant term of h is not zero, with M of minimal semisimple rank, and a nonzero slice h_M of the normalized constant term, at a point $g_0 \in G(\prod_v \mathcal{O}_v)$. Since h is invariant under $G(\mathcal{O}_u)$ for $u \notin S$, the slice does not depend on the components of g_0 at these places, and we take them equal to one. The slice

is a cuspidal function on $M(F)\backslash M(\mathbb{A})$. If K_N is the level of h, then h_M is invariant under $M(\mathbb{A}) \cap g_0 K_N g_0^{-1}$, which contains $M(\mathcal{O}_u)$ for every $u \notin S$.

Step 2: its Hecke equations. Let $u \notin S$, $T \in \mathscr{A}_u$ and $n \geq 1$. By Lemma 7.7 the slice at g_0 of the normalized constant term of $(T - \chi_u(T))^n h$ is $(\mathrm{Sat}_P(T) - \chi_u(T))^n h_M$. Hence

$$\big(\mathrm{Sat}_P(T) - \chi_u(T)\big)^n h_M = 0 \qquad (T \in \mathscr{A}_u,\ n \text{ large}). \tag{8.1}$$

Step 3: a finite-dimensional orbit. Take $u = u_0$. If $T_1, \dots, T_s$ generate $\mathscr{A}_{u_0}$ and $(\mathrm{Sat}_P(T_j) - \chi_{u_0}(T_j))^{n_j} h_M = 0$, the orbit of h_M under $\mathrm{Sat}_P(\mathscr{A}_{u_0})$ is spanned by the functions $\prod_j (\mathrm{Sat}_P(T_j) - \chi_{u_0}(T_j))^{a_j} h_M$ with $0 \leq a_j < n_j$. The algebra $\mathscr{H}_{M,u_0}$ is a finitely generated module over $\mathrm{Sat}_P(\mathscr{H}_{G,u_0})$: in terms of a maximal torus T these are the invariants of W_M and of W_G in the algebra $k[X_*(T)]$, which is a finitely generated module over its Noetherian subring of invariants of W_G. By hypothesis $\mathscr{H}_{G,u_0}$ is a finitely generated module over $\mathscr{A}_{u_0}$. So the space

$$V = \mathscr{H}_{M,u_0} h_M$$

has finite dimension. Its elements are cuspidal functions on M of the same level as h_M, and they satisfy (8.1) at every $u \notin S$, since the Hecke operators at u_0 commute with those at u. Let $v \in V$ be a joint eigenvector of $\mathscr{H}_{M,u_0}$, and $t \in \widehat{T}$ its Satake parameter, which is defined up to W_M. By (8.1) the character of $\mathscr{H}_{G,u_0}$ defined by t extends χ_{u_0}, so by hypothesis all characters of $\widehat{T}$ take unit values at t.

Step 4: a central lattice and a twist. Let $T_M = (Z_M^\circ)_{\mathrm{red}}$, and let Ξ be the image in C_{T_M} of the group of the elements $\lambda(\varpi_{u_0})$, $\lambda \in X_*(T_M)$, where ϖ_{u_0} is a uniformizer at u_0. The degree maps Ξ isomorphically onto $d_{u_0} X_*(T_M)$, so Ξ is a central lattice of M. Translation by $\lambda(\varpi_{u_0})$ is the Hecke operator in $\mathscr{H}_{M,u_0}$ of the characteristic function of a coset of $M(\mathcal{O}_{u_0})$ by a central element, and its Satake transform is the monomial attached to λ. Hence Ξ acts on v by a character χ_Ξ, whose values $\lambda(t)$ are units of the ring of integers $\mathcal{O}_k$ of k.

Put $Q_M = M/M_{\mathrm{der}}$, a split torus with dual torus $Z(\widehat{M})^\circ$, and let $\deg_M : M(\mathbb{A}) \to X_*(Q_M)$ be the degree, defined as for T_G in Section 3. Since $T_M \to Q_M$ is an isogeny, $\deg_M$ maps Ξ isomorphically onto a subgroup of finite index. The group $\mathcal{O}_k^\times$ is divisible, so χ_Ξ extends to a homomorphism $c : X_*(Q_M) \to \mathcal{O}_k^\times$, that is, to a point c of $Z(\widehat{M})^\circ$ at which all characters take unit values. It is defined over a finite extension of $\mathbb{Q}_\ell$. Let $\chi_c = c \circ \deg_M$. It is a character of $M(F)\backslash M(\mathbb{A})$, trivial on $M(\mathcal{O}_u)$ for all u, and $v' = \chi_c^{-1} v$ is a nonzero cuspidal function of finite level, invariant under Ξ. By Harder's theorem [Mor82, Chapter I, §5.3, p. 104] its support is compact modulo Ξ.

Step 5: a parameter of M. By [Laf18, Theorem 0.1], applied to M and Ξ, the space of cuspidal functions of the level of v' which are invariant under Ξ is the direct sum of subspaces $\mathfrak{H}_\tau$ indexed by continuous semisimple parameters τ of M, unramified outside S. The projections onto the summands are polynomials in excursion operators, which commute with the Hecke operators. For $u \notin S$ every $T \in \mathscr{H}_{M,u}$ has on $\mathfrak{H}_\tau$ the single eigenvalue $\widehat{T}(\tau(\mathrm{Frob}_u))$, where $\widehat{T}$ denotes the Satake transform of T, a function on $\widehat{T}/W_M$. Let $w \neq 0$ be the component of v' in one of the spaces $\mathfrak{H}_\tau$.

For $f \in \mathscr{H}_{M,u}$ and a function ϕ on $M(\mathbb{A})$ the definition of T_f gives $T_f(\chi_c^{-1}\phi) = \chi_c^{-1} T_{\chi_c^{-1} f}\phi$. So (8.1), which holds for v, gives

$$\big(T_{\chi_c \mathrm{Sat}_P(T)} - \chi_u(T)\big)^n v' = 0 \qquad (u \notin S,\ T \in \mathscr{A}_u,\ n \text{ large}),$$

and the same holds for w. Comparing with the eigenvalue on $\mathfrak{H}_\tau$ we find

$$\chi_u(T) = \chi_c \widehat{\mathrm{Sat}_P(T)}(\tau(\mathrm{Frob}_u)) \qquad (u \notin S,\ T \in \mathscr{A}_u). \tag{8.2}$$

Multiplication by χ_c on $\mathscr{H}_{M,u}$ corresponds under the Satake isomorphism to a translation of the dual torus. Indeed, the Satake transform of a function supported on the double coset of $\lambda(\varpi_u)$ is a combination of monomials, the exponents of which all have the image $\bar{\lambda}$ of λ in $X_*(Q_M)$; and on this double coset χ_c is equal to $c(d_u\bar{\lambda}) = \bar{\lambda}(c^{d_u})$. So the Satake transform of $\chi_c f$ is the translate by c^{d_u} of that of f. Moreover the Satake transform of $\mathrm{Sat}_P(T)$ is that of T, regarded as a function on $\widehat{T}/W_M$. So the right-hand side of (8.2) is the value of the Satake transform of T at $c^{d_u}\tau(\mathrm{Frob}_u)$.

Recall that a geometric Frobenius element at u has degree $\deg_F(\mathrm{Frob}_u) = -d_u$. Since c lies in a compact subgroup of $Z(\widehat{M})^\circ(k)$, the homomorphism $\gamma \mapsto c^{-\deg_F(\gamma)}$ is defined and continuous, and

$$\sigma_M(\gamma) = \tau(\gamma)\, c^{-\deg_F(\gamma)}$$

is a continuous parameter of M, unramified outside S. It is semisimple: the Zariski closures of the images of τ and of σ_M have the same image in $\widehat{M}/Z(\widehat{M})^\circ$, and a closed subgroup of $\widehat{M}$ whose image there has reductive identity component has reductive identity component itself. Finally $\sigma_M(\mathrm{Frob}_u) = c^{d_u}\tau(\mathrm{Frob}_u)$, so (8.2) is the assertion. □

Proposition 8.6 (Ellipticity implies cuspidality)**.** *Let $\sigma : \Gamma_F \to \widehat{G}(k)$ be a continuous semisimple elliptic parameter, and let h be a smooth function of finite level with generalized excursion character σ for Weil tuples. Then h is cuspidal. If h is invariant under a central lattice, its support is compact modulo this lattice.*

More generally, let S be a finite set of places containing the level of h and the places where σ is ramified, and for $u \notin S$ let χ_u be the character of $\mathscr{H}_{G,u}$ given by the Satake parameter $\sigma(\mathrm{Frob}_u)$. If for every $u \notin S$ and every $T \in \mathscr{H}_{G,u}$ some power of $T - \chi_u(T)$ annihilates h, then h is cuspidal.

Proof. We prove the second assertion first. The Hecke operators give only the conjugacy classes of the Frobenius elements, and the point is that this suffices: it determines the composite of a parameter with the adjoint representation, and the invariants of this representation detect ellipticity.

Suppose that h is not cuspidal. At every $u \notin S$ the Satake parameter $\sigma(\mathrm{Frob}_u)$ is unit-valued, because σ is continuous (Lemma 6.8). So Lemma 8.5 applies, with $\mathscr{A}_u = \mathscr{H}_{G,u}$, and gives M and σ_M. For $u \notin S$ the semisimple parts of $\sigma(\mathrm{Frob}_u)$ and of $i\sigma_M(\mathrm{Frob}_u)$ define the same character of $\mathscr{H}_{G,u}$; so they are conjugate in $\widehat{G}$, and have the same trace in the adjoint representation of $\widehat{G}$ on its Lie algebra $\widehat{\mathfrak{g}}$. The two representations

$$\mathrm{Ad}\circ\sigma, \qquad \mathrm{Ad}\circ i\sigma_M$$

of Γ_F on $\widehat{\mathfrak{g}}$ are continuous and unramified outside S, so by the theorem of Chebotarev their characters are equal. Both are semisimple, because the identity components of the Zariski closures of the images of σ and of $i\sigma_M$ are reductive and k has characteristic zero. So they are isomorphic, and their spaces of invariants have the same dimension. The invariants of the first form the Lie algebra of the centralizer of the image of σ, whose dimension is $\dim Z(\widehat{G})$ because σ is elliptic. The invariants of the second contain the Lie algebra of $Z(\widehat{M})^\circ$, and $\dim Z(\widehat{M}) > \dim Z(\widehat{G})$ because M is proper. This is a contradiction. So h is cuspidal.

Let now h have generalized excursion character σ for Weil tuples, and let S be the set of places where h or σ is ramified. For $u \notin S$ every element T of $\mathscr{H}_{G,u}$ is a linear combination $\sum_j a_j S_{e_j}$ of excursion operators with tuples in the Weil group, and $\sum_j a_j \nu_\sigma(e_j) = \chi_u(T)$ (Proposition 5.10(3)). These operators commute, and $(S_{e_j} - \nu_\sigma(e_j))^{n_j} h = 0$; expanding the power of $T - \chi_u(T) = \sum_j a_j(S_{e_j} - \nu_\sigma(e_j))$ with exponent $\sum_j n_j$ shows that it annihilates h. So the hypothesis of the second assertion holds, and h is cuspidal. The assertion on the support is the theorem of Harder quoted above. □

Remark 8.7. Equality of Satake parameters at the unramified places gives only local conjugacy of σ and $i\sigma_M$, which for a general reductive group is weaker than conjugacy. The proof does not need more: local conjugacy already determines the composite with every representation of $\widehat{G}$ up to isomorphism, and the adjoint representation detects ellipticity. Identifying the parameter of a cuspidal function is a different matter, and uses the excursion operators of all tuples: in [II] through Proposition 8.10, and in [III] through Proposition III.5.6. Lemma 8.5 is stated for a subalgebra $\mathscr{A}_u$ because the construction of [III] controls only the Hecke operators of the standard representation.

8.3 Extracting a function from one pairing

Lemma 8.8 (Extraction from a pairing)**.** *Fix a level K and a central lattice Ξ, and let $\mathscr{H}^{\mathrm{good}}$ be the algebra generated by the spherical Hecke algebras at the places outside the level. Let v be a function on $G(F)\backslash G(\mathbb{A})/K\Xi$, let c be a function with compact support on the same set, and let $\eta : \mathscr{H}^{\mathrm{good}} \to k$ be a character, such that*

$$[c, Tv] = \eta(T)[c, v] \quad (T \in \mathscr{H}^{\mathrm{good}}), \qquad [c, v] \neq 0.$$

Then there is a function h on the same set with $[c, h] \neq 0$ which is annihilated by a power of a maximal ideal of the spherical Hecke algebra at one place and by a power of $T - \eta(T)$ for every $T \in \mathscr{H}^{\mathrm{good}}$. It is an automorphic form. If the Satake parameter of η at that place is unit-valued, h belongs to $\mathcal{A}_\Xi(G)$.

Proof. Let C be the space of functions with compact support on $G(F)\backslash G(\mathbb{A})/K\Xi$ with values in k, let u be a place outside the level and $\mathscr{H} = \mathscr{H}_{G,u,k}$.

Step 1: a finite algebra. By Proposition 5.5, C is a finitely generated module over $\mathscr{H}$. Let A be the image of $\mathscr{H}^{\mathrm{good}}$ in $\mathrm{End}_{\mathscr{H}}(C)$. Evaluation on a finite set of generators embeds $\mathrm{End}_{\mathscr{H}}(C)$ into a finite sum of copies of C; so A is a finitely generated module over the Noetherian ring $\mathscr{H}$.

Step 2: a maximal ideal. Write $L_v(d) = [d, v]$ for $d \in C$ and $\eta^\star(T) = \eta(T^\star)$. Transposing the hypothesis we get

$$L_v(Tc) = \eta^\star(T) L_v(c), \qquad T \in \mathscr{H}^{\mathrm{good}}.$$

Since $L_v(c) \neq 0$, the character $\eta^\star$ vanishes on the operators which act by zero on C. So it is a character of A; let $\mathfrak{m}$ be its kernel. The image of c in the localization $C_\mathfrak{m}$ is not zero: otherwise $sc = 0$ for some $s \notin \mathfrak{m}$, while $L_v(sc) = \eta^\star(s) L_v(c) \neq 0$.

Step 3: a finite quotient. The ideal $\mathfrak{j} = \mathfrak{m} \cap \mathscr{H}$ is maximal, because A is integral over the image of $\mathscr{H}$. By the intersection theorem of Krull, applied to the finitely generated module $C_\mathfrak{m}$ over the Noetherian local ring $A_\mathfrak{m}$, there is an n such that the image of c is not in $\mathfrak{j}^n C_\mathfrak{m}$. The space $C/\mathfrak{j}^n C$ has finite dimension and is a module over the Artinian ring $A/\mathfrak{j}^n A$, which is a product of local rings; the factor attached to $\mathfrak{m}$ cuts out the direct summand $C_\mathfrak{m}/\mathfrak{j}^n C_\mathfrak{m}$. Choose a linear form on this summand which does not vanish on the image of c, and extend it by zero on the other summands. On the summand every $T - \eta^\star(T)$ is nilpotent, so the linear form is annihilated by a power of the transpose of $T - \eta^\star(T)$.

Step 4: the function. By the pairing, the linear form is $d \mapsto [d, h]$ for a function h. Then $[c, h] \neq 0$; the function h is annihilated by the ideal $(\mathfrak{j}^n)^\star$ of $\mathscr{H}$; and it is annihilated by a power of $T - \eta(T)$ for every $T \in \mathscr{H}^{\mathrm{good}}$. Its orbit under $\mathscr{H}$ has finite dimension, so it is an automorphic form by Proposition 6.11. The only Satake parameter of this orbit is that of η at u; if it is unit-valued, the orbit is spanned by a stable lattice as in the proof of Lemma 8.2(2), and $h \in \mathcal{A}_\Xi(G)$ by Lemma 6.7. □

Remark 8.9. The power $\mathfrak{j}^n$ cannot be replaced by $\mathfrak{j}$. For $C = k[t]$ and $c = t$, the functional with value 1 on t and 0 on the other monomials satisfies the hypothesis for the character $t \mapsto 0$, but c vanishes in C/tC; it survives in C/t^2C.

8.4 The form used by the geometric construction

The geometric construction of [II] produces a function φ and proves an eigenvalue equation for the functional $c \mapsto [c, \varphi]$ on functions with compact support. Transposing this equation into one for φ itself introduces the Chevalley involution, by (5.6). The following proposition records the outcome.

Proposition 8.10 (From a functional to a cusp form)**.** *Let G/F be split semisimple, let N be a level, and let $\varphi \neq 0$ be a function on $\mathrm{Bun}_{G,N}(\mathbb{F}_q)$. Let $\sigma_{\mathrm{coeff}} : \pi_1(X - \operatorname{supp} N) \to \widehat{G}(k)$ be continuous, semisimple and elliptic. Suppose that the linear form $L_\varphi = [\ , \varphi]$ on functions with compact support satisfies*

$$L_\varphi S_e = \nu_{\sigma_{\mathrm{coeff}}}(e) L_\varphi \tag{8.3}$$

for every excursion datum e with tuple in the Weil group. Then φ is cuspidal with compact support, and

$$S^{\mathrm{fun}}_{I,f,\boldsymbol{\gamma}} \varphi = f\big(((\vartheta_G \sigma_{\mathrm{coeff}})(\gamma_i))_{i \in I}\big)\varphi \qquad \text{for all } \boldsymbol{\gamma} \in \Gamma_F^I.$$

Proof. For a function c with compact support and a tuple in the Weil group, (5.5) and (5.6) give

$$[c, S^{\mathrm{fun}}_e \varphi] = [S_{e^\star} c, \varphi] = \nu_{\sigma_{\mathrm{coeff}}}(e^\star)[c, \varphi] = \nu_{\vartheta_G \sigma_{\mathrm{coeff}}}(e)[c, \varphi].$$

Taking for c the characteristic functions of points, whose weights in the pairing are not zero, we get $S^{\mathrm{fun}}_e \varphi = \nu_{\vartheta_G \sigma_{\mathrm{coeff}}}(e)\varphi$. The parameter $\vartheta_G \sigma_{\mathrm{coeff}}$ is continuous, finitely ramified, semisimple and elliptic, like σ_{coeff}. So φ is cuspidal by Proposition 8.6, with compact support because G is semisimple, and the equations hold for all Galois tuples by Corollary 6.14. The level is N throughout. □

8.5 The Chevalley involution and excursion summands

The parameters σ and $\vartheta_G \sigma$ play symmetric roles for the existence of cusp forms. On the spaces of cusp forms invariant under a central lattice this is the transposition formula of W.-W. Li [Li19, Lemma 5.3.3 and Proposition 5.3.4].

Lemma 8.11 (Duality of excursion summands)**.** *Let N be a level, Ξ a central lattice, and C^{cusp} the space of cusp forms of level N invariant under Ξ, with values in k.*

1. *The pairing $[\ ,\ ]$ of $Y_{N,\Xi} = G(F)\backslash G(\mathbb{A})/K_N\Xi$ is nondegenerate on C^{cusp}.*
2. *For all excursion data e with tuples in Γ_F^I and all $c, h \in C^{\mathrm{cusp}}$, $[S_e c, h] = [c, S_{e^\star} h]$.*
3. *The pairing identifies the excursion summand of $\vartheta_G \sigma$ in C^{cusp} with the dual of the excursion summand of σ. In particular one is zero if and only if the other is.*

Proof. (1) By Harder's theorem the cusp forms of level N invariant under Ξ are supported on a finite subset Y_0 of $Y_{N,\Xi}$. For a function of level N the constant term at a point is a finite sum of values of the function with positive rational coefficients, if the Haar measures are chosen so that open compact subgroups have rational volumes. So C^{cusp} is the space of solutions in k^{Y_0} of a system of linear equations with rational coefficients, and it is spanned by the space $C^{\mathrm{cusp}}_{\mathbb{Q}}$ of solutions with rational values. The weights $w_N(y)$ of the pairing are positive rational numbers, so the pairing is positive definite on $C^{\mathrm{cusp}}_{\mathbb{Q}}$. Hence it is nondegenerate on $C^{\mathrm{cusp}}_{\mathbb{Q}}$ and on $C^{\mathrm{cusp}} = C^{\mathrm{cusp}}_{\mathbb{Q}} \otimes k$.

(2) The pairing with the weights $1/|\operatorname{Aut}(y)|$ is the pairing by integration for the measure which gives K_N the volume one, as in [Li19, Remark 2.2.3]. Dividing by the lattice does not change the stabilizers: the lattice acts freely on the adelic double quotient, because the degree, with values in $X_*(Q_G)$, is injective on it. The identity for all Galois tuples is then [Li19, Lemma 5.3.3], with $e^\star = (I, f^\dagger, \boldsymbol{\gamma}^{-1})$.

(3) Let V_ν and $V_{\nu'}$ be the generalized eigenspaces in C^{cusp} of two characters of the algebra generated by the excursion operators. Suppose that $\nu(e) \neq \nu'(e^\star)$ for some e. Then $S_{e^\star} - \nu(e)$ is invertible on $V_{\nu'}$, while a power of $S_e - \nu(e)$ annihilates V_ν; by (2), $[V_\nu, V_{\nu'}] = 0$. Since C^{cusp} is the direct sum of the generalized eigenspaces, (1) shows that the pairing between V_ν and $V_{\nu'}$ is perfect when $\nu'(e^\star) = \nu(e)$ for all e. For $\nu = \nu_\sigma$ this character ν' is $\nu_{\vartheta_G \sigma}$, because $\nu_\sigma(e^\star) = \nu_{\vartheta_G\sigma}(e)$ by (5.6). □

9 Irreducible constituents and nonzero Whittaker functionals

The constructions of [III] give spaces of cusp forms stable under right translation, together with a Whittaker functional that does not vanish on them. To state results about generic representations we must pass to an irreducible subrepresentation on which the functional is still nonzero. This short section provides the tool: such spaces are semisimple, because they are admissible and carry an invariant inner product after the coefficients are transported to $\mathbb{C}$.

Fix an isomorphism of fields $\iota : k \xrightarrow{\sim} \mathbb{C}$ which maps $q^{1/2}$ to the positive square root of q. We use it to transport functions and representations; algebraic identities are preserved. Let Ξ be a central lattice. A cusp form of a given level which is invariant under Ξ has compact support modulo Ξ: its support is compact modulo the centre [Mor82, Chapter I, Proposition 5.3], and the quotient of the adelic points of the centre by its rational points and by Ξ is compact. So the cusp forms invariant under Ξ are square integrable on $G(F)\backslash G(\mathbb{A})/\Xi$, and right translation, which commutes with Ξ, is unitary on them. At a fixed level they form a space of finite dimension [Laf18, Theorem 0.1]. So every space of cusp forms invariant under Ξ which is stable under right translation is an admissible subspace of the smooth vectors of a unitary representation. No character of the centre is fixed here: when the centre of G is not connected, a space of this kind can contain several of them.

The lemma below is a statement about smooth representations. After it has been applied, an irreducible subrepresentation and a functional which does not vanish on it are transported back to k. The excursion operators commute with this transport. A given vector lies in a finite-dimensional space of functions of fixed level, so it is defined over a finite extension of the field of coefficients already in use.

Lemma 9.1 (Admissible subspaces of unitary representations)**.** *Let $\mathcal{H}$ be a unitary representation of a totally disconnected locally compact group on a Hilbert space, and V a subspace of smooth vectors which is stable under the group and admissible. Then V is a semisimple representation. Every vector of V lies in a finite sum of irreducible subrepresentations, and a linear form which does not vanish on the vector does not vanish on one of them.*

Proof. Let $U \subset V$ be a subrepresentation and $\overline{U}$ its closure in $\mathcal{H}$. For a compact open subgroup K, the average over K is a continuous projection of $\mathcal{H}$ onto the vectors fixed by K. It maps a sequence in U converging to a vector of $(\overline{U})^K$ to a sequence in U^K with the same limit. Hence

$$(\overline{U})^K = \overline{U^K} = U^K,$$

the last equality because U^K has finite dimension, by admissibility, and is therefore closed.

Since the representation is unitary, $\overline{U}$ and its orthogonal complement are stable under the group, and the orthogonal projection onto $\overline{U}$ commutes with the group. It maps V^K

into $(\overline{U})^K = U^K$. Every vector of V is fixed by some K, so the projection maps V onto U. Thus every subrepresentation of V is a direct summand, and V is semisimple. The subrepresentation generated by one vector is semisimple and finitely generated, hence a finite sum of irreducible subrepresentations. A linear form which vanishes on all of them vanishes on the vector. □

Remark 9.2. Admissibility is used in the equality $(\overline{U})^K = U^K$. Without it, orthogonal projection would only map V to the closure of U, and the algebraic statement would not follow.

Proposition 9.3 (Images of restricted tensor products)**.** *Let G/F be split connected reductive. For every place v let V_v be a smooth admissible complex representation of $G(F_v)$ of finite length whose largest semisimple quotient W_v is irreducible, and let $\mathcal{T} = \bigotimes'_v V_v$ be the restricted tensor product with respect to vectors which, for almost all v, are fixed by $G(\mathcal{O}_v)$ and have nonzero image in W_v. Let*

$$T : \mathcal{T} \longrightarrow \mathcal{H}^\infty$$

be a nonzero $G(\mathbb{A})$-equivariant map into the smooth vectors of a unitary representation $\mathcal{H}$ of $G(\mathbb{A})$. Then the image of T is irreducible and isomorphic to $\bigotimes'_v W_v$.

Proof. Fix a place v and a pure tensor w in the restricted tensor product of the $V_{v'}$ with $v' \neq v$. The map $x \mapsto T(x \otimes w)$ from V_v to $\mathcal{H}^\infty$ commutes with $G(F_v)$. Its image is admissible: taking the vectors fixed by a compact open subgroup is an exact functor on smooth complex representations, by averaging over the subgroup. The image is a subrepresentation of the restriction of $\mathcal{H}$ to $G(F_v)$. By Lemma 9.1 the image is semisimple, so the map factors through W_v. This holds for all v and w. Therefore T vanishes on the sum of the subspaces obtained by replacing one factor V_v by the kernel of $V_v \to W_v$. The quotient of $\mathcal{T}$ by this sum is $\bigotimes'_v W_v$: this is clear for finite tensor products, and the restricted tensor product is their union.

The representation $\bigotimes'_v W_v$ is irreducible, as a restricted tensor product of irreducible admissible representations. So the nonzero map which T induces on it is injective. □

10 Passing to isogenous groups while preserving the parameter

The papers [II] and [III] realize parameters for simply connected groups. This section transfers the result to the other groups with the same root system: to a quotient $G = H_0/C$ of a semisimple group by a finite central subgroup (Theorem 10.9), and from the derived group of a reductive group to the group itself (Lemma 10.7). As a consequence Theorem A reduces to simply connected simple groups (Corollary 10.11).

The difficulty is the following. Let $\mathsf{a} : \widehat{G} \to \widehat{H_0}$ be the isogeny dual to $H_0 \to G$. A parameter σ of G determines the parameter $\mathsf{a}\sigma$ of H_0, but several parameters of G can have the same image, and a cusp form on H_0 with parameter $\mathsf{a}\sigma$ does not single one of them out. The proof proceeds in three steps: embed H_0 in a group G^{env} which maps onto G with kernel a central torus; extend the cusp form to G^{env} and twist it by a character so that its parameter is the prescribed one; and descend to G. That the descended form has parameter σ, and not only a parameter with the same image under a, rests on the surjectivity of a restriction map between rings of invariant functions (Lemma 10.3).

All groups in this section are split. For a parameter σ we write $\mathcal{A}_{\mathrm{cusp}}(G)_\sigma$ for the excursion summand of σ in the space of cusp forms on $G(F)\backslash G(\mathbb{A})$; functions on this set of a given level are identified with functions on the points of the stack of bundles with level structure as in [Laf18, Remark 8.21]. The cusp forms of a given level which are invariant under a given central lattice form a space of finite dimension [Laf18, Theorems 0.1 and 12.3]. We use the notation T_G, Q_G, α_G, β_G of (4.1), and write $\pi_a : G \to Q_G$ for the quotient map. For

a homomorphism $\psi : \Gamma_F \to Z(\widehat{G})^\circ = \widehat{Q}_G$ which has finite image or is of the form $c^{\deg_F}$, let $\chi_\psi = \mathrm{CFT}_{Q_G}(\psi)$, normalized as in Section 3. As in Proposition 5.13, T_ψ is the multiplication by this character:

$$(T_\psi h)(g) = \chi_\psi(\pi_a(g))h(g). \tag{10.1}$$

10.1 Functoriality and descent of all invariant functions

Lemma 10.1 (Pullback along a central map)**.** *Let $v : G_1 \to G_2$ be a homomorphism whose image is normal and whose kernel is central, as a group scheme, and let $\widehat{v} : \widehat{G}_2 \to \widehat{G}_1$ be the dual homomorphism. For compatible levels and central lattices,*

$$v^* \mathcal{A}_{\mathrm{cusp}}(G_2)_\sigma \subseteq \mathcal{A}_{\mathrm{cusp}}(G_1)_{\widehat{v}\sigma}.$$

The kernel of v is not assumed to be smooth. The map v^* commutes with right translations.

Proof. Step 1: cuspidality. Let $P_1 = P_{G_1}(\lambda)$ be the parabolic subgroup of G_1 defined by a cocharacter λ, and $P_2 = P_{G_2}(v\lambda)$. Since the kernel of v is central, v maps each root subgroup of G_1 isomorphically onto a root subgroup of G_2. Since the image of v is normal, the other roots of G_2 belong to simple factors which commute with the image, and they are orthogonal to $v\lambda$. So v maps the unipotent radical of P_1 isomorphically onto that of P_2, and P_2 is proper if P_1 is. Consequently the constant term of v^*h along P_1 is the pullback of the constant term of h along P_2, and v^* preserves cusp forms.

Step 2: excursion operators. For every function f on $\widehat{G}_1^I$ invariant under the diagonal translations,

$$S^{G_1}_{I,f,\gamma} v^* = v^* S^{G_2}_{I,f\circ\widehat{v}^I,\gamma}$$

by [Laf18, Proposition 12.5], where the corresponding morphism of stacks of shtukas is shown to be compatible with creation, with the action of the Galois groups and with annihilation. By the comparison of roots in step 1, the Schubert varieties which correspond have the same dimension, so the Satake sheaves have the same normalization. For a central isogeny v and a coweight λ of G_1, the map from the reduced Schubert variety of λ to that of $v\lambda$ is a finite universal homeomorphism. To see this, work over an algebraically closed extension of the residue field. The two positive loop groups have the same root subgroups, and each is generated by its root subgroups and by the integral points of the split torus. The torus fixes the point of the coweight and normalizes the root subgroups, so the map is surjective on the open orbits. It is injective on geometric points: v is finite, so a loop of G_1 whose image in G is integral is integral. On the closures of the orbits the map is proper with dense image, hence surjective; being proper and injective on geometric points, it is finite and radicial. The étale site is invariant under such a map, so the intersection complexes correspond, with the same dimensions and twists. For the simply connected cover this map is discussed in [HLR24, Introduction, (1.1)].

It follows that v^* transforms a polynomial in the operators $S^{G_2}_e - \nu_\sigma(e)$ into the corresponding polynomial in the operators $S^{G_1}_{e'} - \nu_{\widehat{v}\sigma}(e')$, which gives the inclusion. The compatibility with right translations is [Laf18, Remark 12.4]. □

Proposition 10.2 (Products)**.** *If the parameters σ_j have nonzero cuspidal excursion summands on the groups G_j, the parameter $\prod_j \sigma_j$ has a nonzero cuspidal excursion summand on $\prod_j G_j$.*

Proof. A product of nonzero cusp forms on the factors is a nonzero cusp form on the product. For a given set of legs, the stack of shtukas of the product is the fibre product of the stacks of shtukas of the factors over the space of the legs, and for an exterior tensor product of

representations the complex of coefficients is the exterior product of those of the factors. By the Künneth formula for direct images with compact supports, the creation maps, the action of the Galois groups and the annihilation maps are, on the tensor product of the parts of degree zero, the tensor products of the corresponding maps for the factors. Only these parts enter an excursion operator: creation starts in degree zero on each factor, and the three operations preserve the degree on each factor. A function on $\prod_j \widehat{G}_j^I$ invariant under the diagonal translations is a sum of products of such functions on the factors, so every excursion operator of the product is a sum of tensor products of excursion operators of the factors. On the tensor product of the excursion summands of the σ_j it is therefore the sum of the scalar $\nu_{\prod \sigma_j}(e)$ and of a nilpotent operator. No condition on the images of the σ_j is needed. □

Lemma 10.3 (Invariant functions restrict surjectively)**.** *Let $H \hookrightarrow H^{\mathrm{env}}$ be a closed immersion of reductive groups over k such that $H^{\mathrm{env}} = Z(H^{\mathrm{env}})^\circ H$. For every $m \geq 0$ the restriction map*

$$k[(H^{\mathrm{env}})^m]^{H^{\mathrm{env}}} \longrightarrow k[H^m]^H$$

between the rings of functions invariant under simultaneous conjugation is surjective.

Proof. The restriction $k[(H^{\mathrm{env}})^m] \to k[H^m]$ is surjective, and it commutes with conjugation by H^{env}, which normalizes H. A reductive group in characteristic zero is linearly reductive, so the map remains surjective on invariants of H^{env}. The centre of H^{env} acts trivially on H^m, so the invariants of H^{env} in $k[H^m]$ are the invariants of H. □

Lemma 10.4 (Descent along a quotient by a central torus)**.** *Let $\pi_c : G^{\mathrm{env}} \to G$ be the quotient by a split central torus T, and $\widehat{\pi_c} : \widehat{G} \to \widehat{G^{\mathrm{env}}}$ the dual homomorphism. Every cusp form on G^{env} which is invariant under $\mathsf{T}(\mathbb{A})$ and under a central lattice, and lies in the excursion summand of $\widehat{\pi_c}\sigma$, is the pullback of a cusp form on G, invariant under a central lattice, in the excursion summand of σ. Joint eigenfunctions correspond to joint eigenfunctions, the values of Whittaker functionals are the same, and irreducible cuspidal subrepresentations correspond to each other.*

Proof. By Hilbert's Theorem 90 the map π_c is surjective on points with values in F and in the fields F_v. At all but finitely many places the exact sequence extends to the split models over $\mathcal{O}_v$, and it is surjective on points with values in $\mathcal{O}_v$ because $\mathrm{Pic}(\mathcal{O}_v) = 0$. Hence $G^{\mathrm{env}}(\mathbb{A}) \to G(\mathbb{A})$ is surjective with kernel $\mathsf{T}(\mathbb{A})$. So π_c^* is a bijection from functions on $G(F)\backslash G(\mathbb{A})$ to functions on $G^{\mathrm{env}}(F)\backslash G^{\mathrm{env}}(\mathbb{A})$ invariant under $\mathsf{T}(\mathbb{A})$. The unipotent radicals of corresponding parabolic subgroups are identified by π_c, so constant terms and Whittaker coefficients correspond, and the action of $G^{\mathrm{env}}(\mathbb{A})$ on the invariant functions is the action of $G(\mathbb{A})$.

Next we choose compatible central lattices. Let $\pi_c^* h$ be invariant under $\mathsf{T}(\mathbb{A})$ and under a central lattice Ξ' of G^{env}. The centre of G is the quotient of the centre of G^{env} by T, so π_c maps $T_{G^{\mathrm{env}}}$ onto T_G with kernel T, and h is invariant under the image of Ξ'. Choose elements of Ξ' whose degrees in $X_*(T_G)$ are a basis of a subgroup of finite index. Their images generate a central lattice Ξ_G of G, under which h is invariant; and these elements, together with a central lattice of T, generate a central lattice of G^{env} which maps onto Ξ_G and under which $\pi_c^* h$ is invariant.

It remains to compare excursion operators. By Lemma 10.1, applied with these lattices, for a function f^{env} on $(\widehat{G^{\mathrm{env}}})^I$,

$$S^{G^{\mathrm{env}}}_{I,f^{\mathrm{env}},\gamma}\, \pi_c^* = \pi_c^*\, S^G_{I,f,\gamma}, \qquad f = f^{\mathrm{env}} \circ \widehat{\pi_c}^{\,I}.$$

We claim that every function f on $\widehat{G}^I$ invariant under the diagonal translations is of this form. For $I = \varnothing$ there are only constants. Otherwise choose $i_0 \in I$. The map $(g_i) \mapsto (g_i g_{i_0}^{-1})_{i \neq i_0}$

identifies the functions invariant under the diagonal translations with the functions on $\widehat{G}^{I-\{i_0\}}$ invariant under simultaneous conjugation, and likewise for $\widehat{G^{\mathrm{env}}}$. The homomorphism $\widehat{\pi_c}$ is a closed immersion, and

$$\widehat{G^{\mathrm{env}}} = Z(\widehat{G^{\mathrm{env}}})^\circ\, \widehat{\pi_c}(\widehat{G}).$$

So the claim follows from Lemma 10.3.

Now let $\pi_c^* h$ lie in the excursion summand of $\widehat{\pi_c}\sigma$. For an excursion datum $e = (I, f, \boldsymbol{\gamma})$ of G choose f^{env} as in the claim and let e^{env} be the corresponding datum. Then $\nu_{\widehat{\pi_c}\sigma}(e^{\mathrm{env}}) = \nu_\sigma(e)$, and

$$\pi_c^* \prod_e (S_e^G - \nu_\sigma(e))h = \prod_e (S_{e^{\mathrm{env}}}^{G^{\mathrm{env}}} - \nu_{\widehat{\pi_c}\sigma}(e^{\mathrm{env}}))\pi_c^* h$$

for every finite product. Since π_c^* is injective, the same power of the ideal $\mathfrak{m}_\sigma$ annihilates h; for the first power this is the statement about eigenfunctions. □

10.2 The central character of a cuspidal summand

Proposition 10.5 (The central character of a cuspidal summand)**.** *Let Ξ be a central lattice for G. On the cusp forms invariant under Ξ which lie in the excursion summand of σ, the group $T_G(\mathbb{A})$ acts by the character*

$$\omega_\sigma = \mathrm{CFT}_{T_G}(\alpha_G \sigma). \tag{10.2}$$

If there is a nonzero cusp form of this kind, $\alpha_G\sigma$ has finite image.

Proof. Fix a level N such that σ is unramified outside N and ω_σ is trivial on $T_G(\mathbb{A}) \cap K_N$. On functions of level N invariant under Ξ the group $T_G(\mathbb{A})$ acts through a finite quotient, and the action commutes with the excursion operators. So the summand is the direct sum of its eigenspaces for $T_G(\mathbb{A})$, and it suffices to show that a character ω which occurs is ω_σ.

Let $v \notin N$ and let λ be a cocharacter of T_G. The translation by $\lambda(\varpi_v)$ is the Hecke operator at v of the character $\alpha_G^*\lambda$ of $\widehat{G}$ (Lemma 5.2), hence an excursion operator, and on the excursion summand of σ its only eigenvalue is $\omega_\sigma(\lambda(\varpi_v))$, by the normalization of Section 3. So ω and ω_σ agree on these elements. Both characters are trivial on $T_G(F)$ and on $T_G(\mathbb{A}) \cap K_N$, and by the proof of Lemma 5.2 the elements $\lambda(\varpi_v)$ with $v \notin N$ generate $T_G(\mathbb{A})$ modulo these two subgroups. Hence $\omega = \omega_\sigma$.

In particular ω_σ factors through a finite group, and by class field theory $\alpha_G\sigma$ has finite image. □

10.3 Extension from the derived group

Let G be split connected reductive. Recall from (4.1) that $\ker \beta_G = \widehat{Q}_G = Z(\widehat{G})^\circ$ and that the restriction of α_G to $\widehat{Q}_G$ is an isogeny.

Lemma 10.6 (Extension by zero from the derived group)**.** *Let φ_0 be a nonzero cusp form on G_{der}. There are a central lattice Ξ of G and a cusp form on G, invariant under Ξ, whose restriction to $G_{\mathrm{der}}(\mathbb{A})$ is φ_0.*

Proof. Step 1: extension by zero. Let K_{der} be an open compact subgroup under which φ_0 is invariant. Since G_{der} is semisimple, the support of φ_0 is compact [Mor82, Chapter I, Proposition 5.3]. The group $Q_G(F)$ is discrete in $Q_G(\mathbb{A})$, so there is an open compact subgroup $K_G \subset G(\mathbb{A})$ with

$$\pi_a(K_G) \cap Q_G(F) = \{1\}, \qquad K_G \cap G_{\mathrm{der}}(\mathbb{A}) \subset K_{\mathrm{der}}.$$

We claim that the map

$$G_{\mathrm{der}}(F)\backslash G_{\mathrm{der}}(\mathbb{A})/(K_G \cap G_{\mathrm{der}}(\mathbb{A})) \longrightarrow G(F)\backslash G(\mathbb{A})/K_G$$

is injective. Indeed, let $h_2 = \gamma h_1 \kappa$ with $h_1, h_2 \in G_{\mathrm{der}}(\mathbb{A})$, $\gamma \in G(F)$ and $\kappa \in K_G$. Applying π_a we get $\pi_a(\gamma)\pi_a(\kappa) = 1$, hence $\pi_a(\gamma) = \pi_a(\kappa) = 1$ by the choice of K_G, and $\gamma \in G_{\mathrm{der}}(F)$, $\kappa \in G_{\mathrm{der}}(\mathbb{A})$. Let φ^c be the function on $G(F)\backslash G(\mathbb{A})/K_G$ which is φ_0 on the image of this map and zero elsewhere. It has compact support, contained in the image of $\mathcal{S} = G(F)G_{\mathrm{der}}(\mathbb{A})K_G$.

Step 2: cuspidality. The unipotent radical U of a parabolic subgroup of G is contained in G_{der}, and $\mathcal{S}$ is stable under left multiplication by $G_{\mathrm{der}}(\mathbb{A})$, which is normal. So the constant term of φ^c vanishes at points outside $\mathcal{S}$. At a point $\gamma h \kappa$ of $\mathcal{S}$ we have

$$\int_{U(F)\backslash U(\mathbb{A})} \varphi^c(u\gamma h\kappa)\, du = \int_{U'(F)\backslash U'(\mathbb{A})} \varphi_0(u'h)\, du', \qquad U' = \gamma^{-1}U\gamma,$$

and U' is the unipotent radical of a proper parabolic subgroup of G_{der}. The integral is zero because φ_0 is cuspidal.

Step 3: the lattice. Since $r_G : T_G \to Q_G$ is an isogeny, there is a central lattice Ξ of G on which the degree of the image in Q_G is injective. Put $\varphi(g) = \sum_{\xi\in\Xi} \varphi^c(g\xi)$. The image in Q_G of a point of $\mathcal{S}$ has degree zero, so at most one term of the sum is not zero, and for $g \in G_{\mathrm{der}}(\mathbb{A})$ it is the term $\xi = 1$. Thus φ is a cusp form invariant under Ξ which restricts to φ_0. □

Lemma 10.7 (From the derived group to the group)**.** *Let $\sigma : \Gamma_F \to \widehat{G}(k)$ be continuous, finitely ramified and semisimple, and suppose that $\beta_G\sigma$ has a nonzero cuspidal excursion summand on G_{der}.*

1. *There is a nonzero cuspidal joint eigenfunction of the excursion operators in $\mathcal{A}^{\mathrm{cont}}(G)$ with parameter σ.*
2. *If $\alpha_G\sigma$ has finite image, there is a full central lattice Ξ such that the excursion summand of σ in the cusp forms invariant under Ξ is not zero.*

Proof. The function is obtained in three moves: extend a cusp form from G_{der} to G by zero, choose in the extension a component with a parameter σ_{in} which has the right image in $\widehat{G_{\mathrm{der}}}$, and twist by the character of G/G_{der} which corresponds to the quotient of σ by σ_{in}. The point which makes this work for the parameter itself, and not only for its conjugacy class at each place, is in step 1: the two parameters are made equal after projection, as homomorphisms, so that their quotient is a homomorphism to the centre.

Step 1: a parameter with the right image in the derived group. Let $\varphi_0 \neq 0$ be in the excursion summand of $\beta_G\sigma$, and let φ be an extension as in Lemma 10.6. The cusp forms on G of the level of φ invariant under Ξ are the direct sum of their excursion summands, and each summand belongs to a parameter; write $\varphi = \sum_\tau \varphi_\tau$ accordingly. By Lemma 10.1, applied to the inclusion of G_{der}, the restriction of φ_τ to G_{der} lies in the excursion summand of $\beta_G\tau$. The excursion summands on G_{der} of parameters which are not conjugate are independent, so φ_0 is the sum of the restrictions of the φ_τ with $\beta_G\tau$ conjugate to $\beta_G\sigma$. Choose such a τ with nonzero restriction and call it σ_{in}. After conjugating σ_{in} by an element of $\widehat{G}$ which lifts the conjugating element, we have $\beta_G\sigma_{\mathrm{in}} = \beta_G\sigma$.

Step 2: the difference of the two parameters. Put

$$z(\gamma) = \sigma(\gamma)\sigma_{\mathrm{in}}(\gamma)^{-1} \in \ker \beta_G = \widehat{Q}_G.$$

Since z has values in the centre, it is a homomorphism: in $\sigma(\gamma\delta)\sigma_{\mathrm{in}}(\gamma\delta)^{-1}$ write $\sigma(\delta) = z(\delta)\sigma_{\mathrm{in}}(\delta)$ and move $z(\delta)$ to the left. It is continuous and finitely ramified. By Lemma 4.11, applied to the torus $\widehat{Q}_G$, we can write $z = z_0\psi$ with z_0 of finite image and $\psi = c^{\deg_F}$.

Step 3: twisting. By (6.6) the operator T_{z_0} maps the excursion summand of σ_{in} into that of $\sigma_{\mathrm{in}}z_0$, after Ξ has been replaced by its subgroup of finite index $\Xi \cap \ker \chi_{z_0}$, which is again a central lattice. (This can also be seen from Lemma 10.1 and Proposition 10.2: $T_{z_0}h$ is the pullback of the function $h \otimes \chi_{z_0}$ on $G \times Q_G$ by $g \mapsto (g, \pi_a(g))$, and the dual of this homomorphism is the multiplication $\widehat{G} \times \widehat{Q}_G \to \widehat{G}$.) In the same way T_ψ maps the excursion summand of $\sigma_{\mathrm{in}}z_0$ to a space of functions in $\mathcal{A}^{\mathrm{cont}}(G)$ of finite dimension on which the operators $S_e^{\mathrm{fun}} - \nu_\sigma(e)$ are nilpotent, since $\sigma_{\mathrm{in}}z_0\psi = \sigma$. Both twists preserve cusp forms, because the characters are trivial on unipotent radicals. The space obtained is not zero, so it contains a joint eigenvector. This proves (1).

Step 4: the case of a finite central character. By Proposition 10.5, $\alpha_G\sigma_{\mathrm{in}}$ has finite image. If $\alpha_G\sigma$ has finite image, so has $\alpha_G z$. The restriction of α_G to $\widehat{Q}_G$ has finite kernel, so z has finite image. Then we may take $\psi = 1$, and the twist by $z_0 = z$ gives cusp forms invariant under a central lattice. This proves (2). □

Corollary 10.8 (Genericity is preserved)**.** *In Lemma 10.7(2), suppose that the Whittaker functional does not vanish on the excursion summand of $\beta_G\sigma$ on G_{der}. Then the excursion summand of σ on G contains a globally generic irreducible cuspidal representation.*

Proof. Choose φ_0 with nonzero Whittaker coefficient. The maximal unipotent subgroups of G and of G_{der} are the same, so the extension φ has the same Whittaker coefficient at the identity. In step 1 of the preceding proof the sum of the φ_τ with $\beta_G\tau$ conjugate to $\beta_G\sigma$ restricts to φ_0, so one of them has a nonzero Whittaker coefficient, and we take it for σ_{in}. The twist by z multiplies the Whittaker coefficient at the identity by a nonzero scalar. The resulting cusp form lies in the excursion summand of σ and is invariant under a central lattice. The representation which it generates under right translation consists of cusp forms invariant under this lattice, in the same summand. By Section 9 and Lemma 9.1 it contains an irreducible subrepresentation on which the Whittaker functional is not zero. □

10.4 Finite central quotients and the prescribed lift

Theorem 10.9 (Transfer to a central quotient)**.** *Let H_0 be split semisimple, let $C \subset Z_{H_0}$ be a finite diagonalizable group scheme, and put $G = H_0/C$. Let $\mathsf{a} : \widehat{G} \to \widehat{H_0}$ be the dual isogeny. If $\mathsf{a}\sigma$ has a nonzero cuspidal excursion summand on H_0, then σ has a nonzero cuspidal excursion summand on G. If the first summand contains a globally generic irreducible cuspidal representation, so does the second.*

Proof. Embed C in a split torus T and put

$$G^{\mathrm{env}} = (H_0 \times \mathsf{T})/\{(c, c^{-1}) : c \in C\}, \qquad \mathsf{Q} = \mathsf{T}/C. \tag{10.3}$$

The derived group of G^{env} is H_0, and T is its maximal central torus. The centre of G^{env} is $(Z_{H_0} \times \mathsf{T})/C$, and the quotient of this centre by T is Z_{H_0}/C. So the centre is not connected in general, and in characteristic p it can be nonreduced; we use only the torus T. There are exact sequences

$$1 \to \mathsf{T} \to G^{\mathrm{env}} \xrightarrow{\pi_c} G \to 1, \qquad 1 \to H_0 \to G^{\mathrm{env}} \to \mathsf{Q} \to 1.$$

The dual sequences are

$$\begin{aligned} &1 \to \widehat{G} \xrightarrow{\widehat{\pi_c}} \widehat{G^{\mathrm{env}}} \xrightarrow{\alpha} \widehat{\mathsf{T}} \to 1, \\ &1 \to \widehat{\mathsf{Q}} \to \widehat{G^{\mathrm{env}}} \xrightarrow{\beta} \widehat{H_0} \to 1, \qquad \beta\,\widehat{\pi_c} = \mathsf{a}, \end{aligned} \tag{10.4}$$

where α and β are the maps $\alpha_{G^{\mathrm{env}}}$ and $\beta_{G^{\mathrm{env}}}$ of (4.1).

Consider the parameter $\widehat{\pi_c}\sigma$ of G^{env}. Its image under β is $\mathsf{a}\sigma$, which has a nonzero cuspidal excursion summand on H_0, and its image under α is trivial. By Lemma 10.7(2) there are nonzero cusp forms on G^{env}, invariant under a central lattice, in the excursion summand of $\widehat{\pi_c}\sigma$. By Proposition 10.5 the group $\mathsf{T}(\mathbb{A})$ acts on them by the character $\omega_{\widehat{\pi_c}\sigma}$, which is trivial. By Lemma 10.4 they are pullbacks of cusp forms on G in the excursion summand of σ. The statement on generic representations follows from Corollary 10.8 and the last assertion of Lemma 10.4. □

Remark 10.10. For $\mathrm{Sp}_6 \to \mathrm{PGSp}_6$ one can take $G^{\mathrm{env}} = \mathrm{GSp}_6$, and the dual groups are $\mathrm{Spin}_7 \subset \mathrm{GSpin}_7 \to \mathrm{SO}_7$. Any two lifts of the same SO_7-valued homomorphism to GSpin_7 differ by a central character. The proof does not stop there: it conjugates the parameter found on GSp_6 so that the two images in SO_7 are equal as homomorphisms, and then twists by the quotient z of the two parameters. So it produces a cusp form with the prescribed parameter $\widehat{\pi_c}\sigma$, and not only one with the same representation of dimension seven. In the descent to PGSp_6 it is the surjectivity of Lemma 10.3 which shows that no excursion operator is lost.

Corollary 10.11 (Reduction to simply connected simple groups)**.** *Let G be split semisimple, with simply connected cover $\prod_a G_a$, the G_a simple, and let σ be an elliptic parameter of G. The images σ_a of σ in the groups $\widehat{G}_a$ are elliptic, and if every σ_a has a nonzero cuspidal excursion summand on G_a, then σ has one on G. Thus Theorem A follows from the same statement for simply connected simple groups, with the bounds on the characteristic attached to their types. The central subgroup by which the cover is divided need not be a product of subgroups of the factors.*

Proof. The dual of the covering map is a central isogeny $\widehat{G} \to \prod_a \widehat{G}_a$. It induces an isomorphism of Lie algebras compatible with the adjoint actions, so the Lie algebra of the centralizer of σ is the direct sum of the Lie algebras of the centralizers of the σ_a. Hence each σ_a is elliptic. If each σ_a has a nonzero cuspidal excursion summand, so has the product parameter on $\prod_a G_a$ by Proposition 10.2, and Theorem 10.9, applied once to the product and the central subgroup, gives the summand of σ on G. □

11 The smallest Levi containing a parameter

The papers [II] and [III] treat elliptic parameters. A semisimple parameter σ that is not elliptic factors through the dual of a Levi subgroup M of G, and is elliptic as a parameter of M if M is chosen minimal (Corollary 11.2). The cusp form on M given by the elliptic case is then transferred to G by Eisenstein series, in Section 12. The Levi subgroup M is determined by the centralizer of σ, and the restriction on the characteristic in Theorem B is the one needed for M, not for G (Proposition 11.4).

Proposition 11.1 (Minimal Levi subgroups containing a reductive subgroup)**.** *Let $\widehat{G}$ be a connected reductive group over k and $H \subset \widehat{G}$ a closed subgroup with H° reductive. Let $C = C_{\widehat{G}}(H)$.*

1. *The group C° is reductive.*
2. *If S is a maximal torus of C°, then*

$$\widehat{M} = C_{\widehat{G}}(S) \tag{11.1}$$

is a Levi subgroup of $\widehat{G}$ containing H and minimal with this property, $Z(\widehat{M})^\circ = S$, and

$$C_{\widehat{M}}(H)^\circ = Z(\widehat{M})^\circ. \tag{11.2}$$

3. Every minimal Levi subgroup containing H is conjugate to $\widehat{M}$ by an element of C°.

Proof. (1) In characteristic zero a group whose identity component is reductive is linearly reductive: an invariant complement under H° can be averaged over the finite group H/H°. So H is completely reducible in $\widehat{G}$, and the identity component of its centralizer is reductive [BMR05, Lemma 2.6 and Proposition 3.12].

(2) The centralizer of a torus is a Levi subgroup, and $\widehat{M}$ contains H because $S \subset C$. We have $S \subset Z(\widehat{M})^\circ \subset C^\circ$, the second inclusion because $H \subset \widehat{M}$; as $Z(\widehat{M})^\circ$ is a torus containing the maximal torus S, they are equal. Next,

$$C_{\widehat{M}}(H)^\circ = C_C(S)^\circ = C_{C^\circ}(S) = S,$$

since the centralizer of a maximal torus of the connected reductive group C° is that torus. Finally, if $\widehat{L} \subset \widehat{M}$ is a Levi subgroup containing H, then $Z(\widehat{L})^\circ$ is a torus in C° containing S, hence equal to S, and $\widehat{L} = C_{\widehat{G}}(Z(\widehat{L})^\circ) = \widehat{M}$.

(3) Let $\widehat{L}$ be a minimal Levi subgroup containing H. The torus $Z(\widehat{L})^\circ$ lies in C°; let S' be a maximal torus of C° containing it. Then $C_{\widehat{G}}(S') \subset C_{\widehat{G}}(Z(\widehat{L})^\circ) = \widehat{L}$ is a Levi subgroup containing H, so it equals $\widehat{L}$ by minimality. The maximal tori S and S' are conjugate in C°. □

Corollary 11.2 (Factorization through a minimal Levi subgroup)**.** *Let G be split connected reductive and $\sigma : \Gamma_F \to \widehat{G}(k)$ a continuous, finitely ramified, semisimple parameter. After conjugation,*

$$\sigma = i \circ \sigma_M, \qquad i : \widehat{M} \hookrightarrow \widehat{G}, \tag{11.3}$$

where M is a standard Levi subgroup of G and $\sigma_M : \Gamma_F \to \widehat{M}(k)$ is semisimple and elliptic. It is ramified at the same places as σ and takes values in a finite extension of $\mathbb{Q}_\ell$.

Proof. Apply Proposition 11.1 to the Zariski closure H of the image of σ, and conjugate the resulting Levi subgroup to the dual of a standard Levi subgroup of G. The closure of the image is still H, whose identity component is reductive, and (11.2) says that σ_M is elliptic. Continuity and ramification are not affected, since σ_M is the same map with a smaller target. For the field of values, enlarge that of Lemma 4.1 to contain the entries of the conjugating element. □

Example 11.3. For $\widehat{G} = \mathrm{GL}_n$, write a semisimple parameter as $\bigoplus_a \rho_a \otimes k^{m_a}$ with pairwise nonisomorphic irreducible ρ_a. Its centralizer is $\prod_a \mathrm{GL}_{m_a}$, and a maximal torus of it, the diagonal torus in the multiplicity spaces, gives

$$\widehat{M} = \prod_a \prod_{j=1}^{m_a} \mathrm{GL}_{\dim \rho_a}.$$

For instance $\chi \oplus \chi \oplus \rho$, with ρ irreducible of dimension two, has minimal Levi subgroup $\mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_2$. The Levi subgroup $\mathrm{GL}_2 \times \mathrm{GL}_2$ also contains the image, but $\chi \oplus \chi$ is not elliptic in GL_2.

11.1 The characteristic bound belongs to this Levi

Recall the bounds of Theorem B:

$$\begin{gathered} b(A_r) = b(B_r) = b(C_r) = b(D_r) = 0, \qquad b(G_2) = b(F_4) = 3, \\ b(E_6) = 5, \qquad b(E_7) = b(E_8) = 7, \qquad b(\varnothing) = 0, \end{gathered} \tag{11.4}$$

and, for a reductive group, b is the maximum over the simple factors of its derived group. Let also

$$b^{\text{proper}}(G) = \max_{M \subsetneq G \text{ Levi}} b(M), \tag{11.5}$$

which is 0 if G is a torus.

Proposition 11.4 (Bounds for Levi subgroups)**.** *For every Levi subgroup $M \subset G$,*

$$b(M) \leq b(G). \tag{11.6}$$

If the root system of G is irreducible, then

type of G	$A, B, C, D, G_2, F_4, E_6$	E_7	E_8
$b^{\text{proper}}(G)$	0	5	7.

If the root system of G has at least two irreducible components, then

$$b^{\text{proper}}(G) = b(G). \tag{11.7}$$

Proof. The Dynkin diagram of a Levi subgroup is obtained from that of G by removing vertices. A proper connected subdiagram of the diagram of F_4 or E_6 is of classical type. A proper connected subdiagram of E_7 has at most six vertices and is of classical type or of type E_6, which occurs. The diagram of E_8 contains that of E_7, and $b \leq 7$ for all its proper subdiagrams. Subdiagrams of classical diagrams and of G_2 are classical. This gives the first two assertions. If there are at least two components, remove a vertex from a component other than one on which b is maximal. □

Remark 11.5. A central torus does not count as a component in (11.7). For a given parameter the condition that matters is $p > b(M)$ for its minimal Levi subgroup M. For example every parameter of a group of type F_4 or E_6 which is not elliptic factors through a Levi subgroup all of whose factors are classical, and is covered in every characteristic.

12 Eisenstein transfer of the excursion character

Let φ_M be an eigenfunction of the excursion operators on a Levi subgroup M, with parameter σ_M. We want an eigenfunction on G with parameter $i\sigma_M$ (Theorem 12.1). The Eisenstein series induced from φ_M would be one, but it may have a pole at the point in question. We avoid choosing a value or a residue. The Eisenstein series are used, in the region of convergence and for a family of unramified twists of φ_M, only to prove that every relation among the excursion operators on functions with compact support holds for the values given by $i\sigma_M$ (Lemma 12.2): for the twists this follows from the constant term, and a relation is a Laurent polynomial in the twist, so it holds at the untwisted point as well. When $M \neq G$ the same is true of the relations which hold only modulo the cuspidal functions with compact support, because a convergent Eisenstein series is orthogonal to these functions (Lemma 12.5). Hence $i\sigma_M$ defines a character of the algebra generated by the excursion operators on the quotient of the space of functions with compact support by the cuspidal ones. This quotient is a finitely generated faithful module over that algebra, so the character occurs in a quotient of it (Lemma 12.6). The function obtained is orthogonal to the cuspidal functions with compact support, and a cuspidal function with this property is zero (Lemma 12.4).

Theorem 12.1 (Transfer from a Levi subgroup)**.** *Let $P = MU$ be a parabolic subgroup of a split connected reductive group G, with dual inclusion $i : \widehat{M} \hookrightarrow \widehat{G}$. Let $\sigma_M : \Gamma_F \to \widehat{M}(k)$ be continuous, finitely ramified and semisimple, and suppose there is a nonzero smooth function*

φ_M of finite level on M on which the Weil excursion operators act through $\sigma_M|_{W_F}$. Then there is a nonzero automorphic form $\varphi \in \mathcal{A}^{\mathrm{cont}}(G)$ with connected central character $\omega_{i\sigma_M}$ such that

$$S^{G,\mathrm{fun}}_{I,f,\boldsymbol{\gamma}}\varphi = f((i\sigma_M(\gamma_j))_{j\in I})\varphi \qquad (\boldsymbol{\gamma} \in \Gamma_F^I). \tag{12.1}$$

For every central lattice Ξ on which $\omega_{i\sigma_M}$ is trivial, φ lies in the dual of a finite quotient defining $\mathcal{A}_\Xi(G)$, with the first power of the spherical ideal. If $M \neq G$, the form φ can be chosen not cuspidal.

In the assertion on the lattices the ideal is the spherical ideal $(\ker \chi_{i\sigma_M,u})^\star$ at a place u outside the level, over a sufficiently large finite coefficient field. Here φ is an eigenfunction, and not only a generalized one. The function φ_M may be a twist of a cuspidal eigenfunction by a continuous central character.

12.1 Relations among excursion operators

For a parameter τ of G and a polynomial r in excursion data, $\nu_\tau(r)$ denotes the value of r when each datum $(I, f, \boldsymbol{\gamma})$ is replaced by $f((\tau(\gamma_j))_j)$.

Lemma 12.2 (Relations among excursion operators)**.** *In Theorem 12.1 assume that φ_M is cuspidal. There is a level K of G such that, with $\mathcal{C} = C_c(G(F)\backslash G(\mathbb{A})/K, k)$, every polynomial r in Weil excursion data whose operator on $\mathcal{C}$ is zero satisfies $\nu_{i\sigma_M}(r) = 0$.*

Proof. Step 1: the level and the section. Let $K_0 = G(\prod_v \mathcal{O}_v)$, and let $K \subset K_0$ be a principal level such that $M(\mathbb{A}) \cap K$ fixes φ_M. By the Iwasawa decomposition there are representatives g_a of $P(\mathbb{A})\backslash G(\mathbb{A})/K$ in K_0, and we take 1 as one of them. Define Φ by $\Phi(umk) = \delta_P(m)^{1/2}\varphi_M(m)$ on $P(\mathbb{A})K$ and by $\Phi = 0$ on the other double cosets. This is well defined: the image of $P(\mathbb{A}) \cap K$ in $M(\mathbb{A})$ is $M(\mathbb{A}) \cap K$, which fixes φ_M and on which δ_P is trivial. Its restrictions

$$\varphi_{M,a}(m) = \delta_P(m)^{-1/2}\Phi(mg_a)$$

are equal to φ_M for $g_a = 1$ and to zero for the other representatives, and the excursion operators of M act on them through σ_M. If $M = G$ there is nothing more to do: an operator which is zero on $\mathcal{C}$ is zero on all functions of level K by Lemma 7.5, in particular on φ_M. Assume $M \neq G$.

Step 2: the family of twists. For c in the torus $Z(\widehat{M})^\circ$ let χ_c be the unramified character of $M(\mathbb{A})$ attached to the homomorphism $c^{\deg_F}$ of W_F, and let Φ_c be the section obtained by multiplying the $\varphi_{M,a}$ by χ_c. By Proposition 5.13 the excursion operators act on the twisted functions through the Weil parameter $\sigma_M c^{\deg_F}$. This parameter need not extend continuously to Γ_F, which is why Weil tuples are used. For a fixed r,

$$c \longmapsto \nu_{i(\sigma_M c^{\deg_F})}(r)$$

is a Laurent polynomial in c: the degrees of the Weil elements in r are integers and the functions f are regular.

Step 3: Eisenstein series. By Corollary 6.14 the function φ_M takes values in a finite extension of $\mathbb{Q}_\ell$ and has central character ω_{σ_M}. We transport it to complex coefficients by ι; the identities between values of functions that express the action of the excursion operators are finite sums and are preserved. For c in a nonempty open subset of the region where the twist is sufficiently positive, the Eisenstein series $E_P(\Phi_c)$ converges absolutely.

Step 4: the constant term. Let r be zero on $\mathcal{C}$, hence on all functions of level K. The excursion operators are compatible with the normalized constant term along P, so

$$(i^*r)\,\mathsf{CT}_P^{\mathrm{nor}} E_P(\Phi_c) = 0,$$

where i^*r is the corresponding polynomial in excursion data of M. Since φ_M is cuspidal, the constant term is a sum over $W(M,M) = N_G(M)/M$,

$$\mathsf{CT}_{P,g_a}^{\mathrm{nor}} E_P(\Phi_c) = v_U \varphi_{M,c,a} + \sum_{\substack{w \in W(M,M) \\ w \neq 1}} \varphi_{M,w,c,a}, \qquad v_U = \mathrm{vol}(U(F)\backslash U(\mathbb{A})) \neq 0.$$

The first term comes from the identity double coset; the others are images of intertwining operators, on which the connected centre of M acts through $w(\omega_{\sigma_M}\chi_c)$ [BL24, (6.2), (6.4), (6.17), and §2.2]. The double cosets whose intersection with the Levi subgroup is a proper parabolic subgroup contribute constant terms of φ_M, which vanish.

Step 5: specialization. The group $W(M,M)$ acts faithfully on T_M, whose centralizer is M. The isogeny $T_M \to Q_M$ commutes with the action of $W(M,M)$. For each $w \neq 1$ there is therefore a cocharacter of T_M whose image in $X_*(Q_M)$ is moved by w; let $a_w \in T_M(\mathbb{A})$ be its value at a uniformizer at some place. Then the degrees of a_w and of $w^{-1}a_w$ in $X_*(Q_M)$ are different, so for $\chi_w = w(\omega_{\sigma_M}\chi_c)$ and $\chi_1 = \omega_{\sigma_M}\chi_c$ the equality $\chi_1(a_w) = \chi_w(a_w)$ is a monomial equation in c which is not an identity. We remove from the torus the finitely many proper closed subsets defined by these equations. Let T_{a_w} be the translation by a_w. The operator

$$e_c = \prod_{w \neq 1} \frac{T_{a_w} - \chi_w(a_w)}{\chi_1(a_w) - \chi_w(a_w)}$$

is the identity on the first term of the constant term and kills the others, and it commutes with excursion operators. Applying it to the equation of step 4 gives

$$\nu_{i(\sigma_M c^{\deg_F})}(r)\, v_U \varphi_{M,c,a} = 0.$$

For the index a of step 1 the function is not zero, so the scalar vanishes for all c in a Zariski dense subset of the torus. Being a Laurent polynomial it vanishes identically, and at $c = 1$ we get $\nu_{i\sigma_M}(r) = 0$. □

Remark 12.3. The specialization at $c = 1$ is that of a Laurent polynomial, so it is valid even if the Eisenstein series has a pole there.

For a level K and $\mathcal{C} = C_c(G(F)\backslash G(\mathbb{A})/K, k)$, let $\mathcal{C}_{\mathrm{cusp}} \subset \mathcal{C}$ be the subspace of cuspidal functions.

Lemma 12.4 (Cuspidal functions with compact support)**.** *1. The subspace $\mathcal{C}_{\mathrm{cusp}}$ is stable under the operators S_e, for excursion data with tuples in the Weil group.*

2. Let h be a cuspidal function of level K. If $[c,h] = 0$ for every $c \in \mathcal{C}_{\mathrm{cusp}}$, then $h = 0$.

Proof. (1) On $\mathcal{C}$ the operator S_e is S_e^{fun}, which commutes with the normalized constant terms along all parabolic subgroups, by Theorem 7.1. So it maps cuspidal functions to cuspidal functions.

(2) Let $Q_G = G/G_{\mathrm{der}}$, and let $\deg_G : G(\mathbb{A}) \to X_*(Q_G)$ be the degree. For $\nu \in X_*(Q_G)$ let h_ν be the product of h and of the characteristic function of the elements of degree ν. The unipotent radicals of the parabolic subgroups lie in G_{der}, on whose adelic points the degree is zero; so h_ν is cuspidal. Its support is compact. Indeed, let Ξ be a central lattice. The

translates of h_ν by the elements of Ξ are supported in different degrees, because the degree is injective on Ξ and $T_G \to Q_G$ is an isogeny. So their sum is a cuspidal function of level K which is invariant under Ξ. By Harder's theorem [Mor82, Chapter I, §5.3, p. 104] the support of this sum is compact modulo Ξ, and its part in degree ν, which is the support of h_ν, is compact.

Suppose that $h_\nu \neq 0$, and let A be its support, a finite subset of $G(F)\backslash G(\mathbb{A})/K$. A function supported in A is cuspidal if and only if its values satisfy a system of linear equations, whose coefficients are volumes of subsets of the spaces $U(F)\backslash U(\mathbb{A})$ which are invariant under compact open subgroups; with a suitable normalization of the measures they are rational numbers. So the space of cuspidal functions supported in A is $D \otimes_{\mathbb{Q}} k$, where D is the space of those with rational values. The weights of the points in the pairing are positive rational numbers. Hence the pairing is positive definite on D, and its extension to $D \otimes_{\mathbb{Q}} k$ is nondegenerate, the determinant of its matrix in a basis of D being a nonzero rational number. As h_ν belongs to $D \otimes_{\mathbb{Q}} k$, there is c in this space with $[c, h_\nu] \neq 0$. Then $c \in \mathcal{C}_{\mathrm{cusp}}$, and $[c, h] = [c, h_\nu]$ because c is supported in degree ν. This contradicts the hypothesis. So $h_\nu = 0$ for every ν, and $h = 0$. □

Lemma 12.5 (Relations modulo cuspidal functions)**.** *In Lemma 12.2 suppose that $M \neq G$, and let K be the level of its proof. Every polynomial r in Weil excursion data whose operator maps $\mathcal{C}$ into $\mathcal{C}_{\mathrm{cusp}}$ satisfies $\nu_{i\sigma_M}(r^\star) = 0$, where $r^\star$ is the polynomial obtained from r by replacing each excursion datum e by $e^\star$.*

Proof. We use the sections Φ_c and the Eisenstein series of steps 2 and 3 of that proof, with complex coefficients as there. Let c be in the region of absolute convergence. For $d \in \mathcal{C}_{\mathrm{cusp}}$,

$$[d, E_P(\Phi_c)] = 0.$$

Indeed, the pairing is a finite sum of values of the Eisenstein series, each of which is an absolutely convergent sum over $P(F)\backslash G(F)$. Unfolding, it is the integral over $P(F)\backslash G(\mathbb{A})$ of $d\,\Phi_c$. The section Φ_c is invariant on the left under $U(\mathbb{A})$, so integrating first over $U(F)\backslash U(\mathbb{A})$ replaces d by its constant term along P, which is zero.

Now let r map $\mathcal{C}$ into $\mathcal{C}_{\mathrm{cusp}}$. Since $(e^\star)^\star = e$, the definition (5.5) gives, for every $b \in \mathcal{C}$,

$$[b, r^\star E_P(\Phi_c)] = [rb, E_P(\Phi_c)] = 0,$$

where $r^\star$ acts through the operators S_e^{fun}. Taking for b the characteristic functions of points, we get $r^\star E_P(\Phi_c) = 0$. This is the only property of the polynomial which is used in steps 4 and 5 of the proof of Lemma 12.2. Applied to $r^\star$ they give $\nu_{i(\sigma_M c^{\deg_F})}(r^\star) = 0$ for all c in a Zariski dense subset of the torus, hence identically, and at $c = 1$ we get $\nu_{i\sigma_M}(r^\star) = 0$. □

12.2 A nonzero character quotient

Lemma 12.6 (A character of a faithful finite module occurs in a quotient)**.** *Let $\mathcal{B}$ be a commutative k-algebra and $\mathcal{C}$ a finitely generated faithful $\mathcal{B}$-module. For every character $\eta : \mathcal{B} \to k$, the vector space $\mathcal{C}/(\ker\eta)\mathcal{C}$ is nonzero and finite dimensional.*

Proof. If $(\ker\eta)\mathcal{C} = \mathcal{C}$, then by Nakayama's lemma [The26, Tag 00DV(1)] some element of $1 + \ker\eta$ annihilates $\mathcal{C}$; it is nonzero, contradicting faithfulness. The quotient is spanned over $\mathcal{B}/\ker\eta = k$ by the images of finitely many generators. □

Proof of Theorem 12.1. Step 1: reduction to cuspidal φ_M. The Hecke operators at an unramified place act on φ_M through the Satake parameter of σ_M at that place. Since σ_M is continuous, the eigenvalues of this parameter in every representation of $\widehat{M}$ are ℓ-adic units;

so the eigenvalues of the Hecke operators, which are traces, are integral over $\mathcal{O}$. (They need not be units: a sum of units can vanish.) By Lemma 8.3, applied in M, there are a Levi subgroup $M_0 \subset M$ and a cuspidal function on M_0 which is an eigenfunction of the excursion operators with a parameter σ_{M_0}, and which satisfies every relation satisfied by φ_M. With $j : \widehat{M}_0 \hookrightarrow \widehat{M}$, the parameters $j\sigma_{M_0}$ and σ_M then have the same values on all excursion data, and are conjugate because they are continuous and semisimple. So we may replace M by M_0 and assume φ_M cuspidal. If $M = G$ the theorem follows from Corollary 6.14.

Step 2: a character of the excursion algebra. Now $M \neq G$. Take K and $\mathcal{C}$ as in Lemma 12.2, and put $\overline{\mathcal{C}} = \mathcal{C}/\mathcal{C}_{\mathrm{cusp}}$. By Lemma 12.4(1) the Weil excursion operators act on $\overline{\mathcal{C}}$; let $\mathcal{B}$ be the image of the algebra of polynomials in Weil excursion data in the endomorphisms of $\overline{\mathcal{C}}$. We claim that $\eta = \nu_{\vartheta_G i\sigma_M}$ is a character of $\mathcal{B}$, that is, vanishes on every r whose operator on $\overline{\mathcal{C}}$ is zero. Such an r maps $\mathcal{C}$ into $\mathcal{C}_{\mathrm{cusp}}$, and by Lemma 12.5

$$\eta(r) = \nu_{\vartheta_G i\sigma_M}(r) = \nu_{i\sigma_M}(r^\star) = 0.$$

In particular $\overline{\mathcal{C}} \neq 0$, since $\eta(1) = 1$.

Step 3: a function which is not cuspidal. At a place u outside the level the spherical Hecke algebra acts on $\overline{\mathcal{C}}$ through $\mathcal{B}$, and $\overline{\mathcal{C}}$ is finitely generated over it, because $\mathcal{C}$ is (Proposition 5.5); hence it is finitely generated over $\mathcal{B}$, and it is a faithful $\mathcal{B}$-module by definition. By Lemma 12.6 there is a nonzero linear form on $\overline{\mathcal{C}}/(\ker\eta)\overline{\mathcal{C}}$. We regard it as a linear form on $\mathcal{C}$ which vanishes on $\mathcal{C}_{\mathrm{cusp}}$. Through the pairing $[\,\cdot\,,\cdot\,]$ it is given by a function $\varphi \neq 0$ of level K, and for every Weil excursion datum e and every $c \in \mathcal{C}$,

$$[c, S_e^{\mathrm{fun}}\varphi] = [S_{e^\star}c, \varphi] = \eta(e^\star)[c,\varphi] = \nu_{i\sigma_M}(e)[c,\varphi].$$

Taking for c the characteristic functions of points gives $S_e^{\mathrm{fun}}\varphi = \nu_{i\sigma_M}(e)\varphi$. Moreover $[c,\varphi] = 0$ for every $c \in \mathcal{C}_{\mathrm{cusp}}$. If φ were cuspidal it would be zero, by Lemma 12.4(2). So φ is not cuspidal.

Step 4: from Weil to Galois tuples. Corollary 6.14 applies to φ and the continuous parameter $i\sigma_M$. It shows that φ is an automorphic form in $\mathcal{A}^{\mathrm{cont}}(G)$, that (12.1) holds for all Galois tuples, that the connected centre acts through $\omega_{i\sigma_M}$, and that for each lattice Ξ as in the statement φ lies in the dual of the corresponding finite quotient, with the first power of the ideal. □

13 Proofs of the realization theorems

We now prove the three theorems of the introduction. The geometric construction (Theorem 1.4) and the classical one (Theorem 1.5) both give, for an elliptic parameter of a simply connected simple group, a nonzero cuspidal excursion summand. Between them they cover every type under the hypothesis on the characteristic. Theorem A follows by taking products and passing to an isogenous group (Section 10); Theorem B by passing to a minimal Levi subgroup and applying Theorem 12.1; Theorem C from the central character.

13.1 Semisimple groups

The classical types are covered in every positive characteristic by Theorem 1.5. For the exceptional types we use the geometric construction, which requires that p does not divide

the order of the Weyl group:

type	$\lvert W\rvert$	prime divisors
G_2	$2^2\,3$	$2, 3$
F_4	$2^7\,3^2$	$2, 3$
E_6	$2^7\,3^4\,5$	$2, 3, 5$
E_7	$2^{10}\,3^4\,5\,7$	$2, 3, 5, 7$
E_8	$2^{14}\,3^5\,5^2\,7$	$2, 3, 5, 7.$

For these types the condition $p \nmid |W_G|$ is the condition $p > b(\Phi)$ of (11.4).

Proof of Theorem A. If G is trivial, the constant function one is the required form. Otherwise let $G^{\mathrm{sc}} = \prod_{a\in A} G_a$ be the simply connected cover of G, with the G_a simple, and let

$$\mathsf{a} : \widehat{G} \longrightarrow \widehat{G^{\mathrm{sc}}} = \prod_{a\in A} \widehat{G}_a, \qquad \sigma_a = \mathrm{pr}_a\, \mathsf{a}\sigma.$$

Step 1: the parameters of the factors. The map a is a central isogeny of groups over a field of characteristic zero, so it induces an isomorphism of Lie algebras compatible with the adjoint actions, and

$$\mathrm{Lie}\, C_{\widehat{G}}(\sigma) \simeq \bigoplus_{a\in A} \mathrm{Lie}\, C_{\widehat{G}_a}(\sigma_a).$$

The left-hand side is zero because σ is elliptic. So each σ_a is elliptic. It is semisimple because the image of a reductive group under a homomorphism is reductive, and it is continuous and finitely ramified.

Step 2: the factors. For each a the cuspidal excursion summand of σ_a on G_a is nonzero at some finite level: for a factor of type G_2, F_4, E_6, E_7 or E_8 by Theorem 1.4, whose hypothesis holds since $p > b(G) \geq b(G_a)$, and for the other factors by Theorem 1.5. The summand has finite dimension, so it contains a joint eigenvector v_a of the excursion operators (see the remark after Definition 3.3).

Step 3: the product. Let $v = \bigotimes_a v_a$, a function on $G^{\mathrm{sc}}(F)\backslash G^{\mathrm{sc}}(\mathbb{A}_F)$. It is not zero: evaluate it at a point whose components are points where the v_a do not vanish. It is cuspidal: a proper parabolic subgroup of a product is proper in at least one factor, and the constant term of v along it has as a factor a constant term of the corresponding cusp form v_a. It is an eigenvector of the excursion operators with character $\nu_{\mathsf{a}\sigma}$: for every finite set I of legs, the ring of functions on $\widehat{G^{\mathrm{sc}}}^I$ invariant under left and right diagonal translation is the tensor product of the corresponding rings of the factors, since each factor acts on its own coordinates only; so every excursion operator of G^{sc} is a sum of products of excursion operators of the factors, and it acts on v by its value at $\mathsf{a}\sigma$.

Step 4: the isogenous group G. By Theorem 10.9, applied to $G = G^{\mathrm{sc}}/C$ and to the summand of $\mathsf{a}\sigma$ which contains v, the cuspidal excursion summand of σ on G is nonzero. The theorem gives the parameter σ itself, and not only its image $\mathsf{a}\sigma$. A joint eigenvector in this summand is the form φ_σ. □

Proof of Corollary 1.2. Let S contain the places at which σ is ramified and those at which the component of K is not $G(\mathcal{O}_v)$. For $v \notin S$ and a representation V of $\widehat{G}$, the excursion operator attached to V and Frob_v is the Hecke operator $h_{V,v}$ which corresponds to V under the Satake isomorphism (Input 2). So

$$h_{V,v}\varphi_\sigma = \mathrm{Tr}(\sigma(\mathrm{Frob}_v); V)\, \varphi_\sigma.$$

The representation of $G(\mathbb{A}_F)$ generated by φ_σ is contained in the space of cusp forms, so it is semisimple. Let π be an irreducible constituent on which φ_σ has a nonzero projection. The projection commutes with $G(\mathbb{A}_F)$, hence with the Hecke operators. Its value on φ_σ is therefore, for every $v \notin S$, a nonzero vector of π fixed by $G(\mathcal{O}_v)$ on which the spherical Hecke algebra at v acts through the same character. □

13.2 Reductive groups and the central character

Proof of Theorem B. By Corollary 11.2 we may write $\sigma = i\sigma_M$, where $i : \widehat{M} \hookrightarrow \widehat{G}$ is the inclusion of a minimal Levi subgroup containing the image and σ_M is elliptic as a parameter of M.

Step 1: the derived group of M. Let $\beta_M : \widehat{M} \to \widehat{M_{\mathrm{der}}}$ be the projection dual to the inclusion of the derived group; its kernel is the connected centre of $\widehat{M}$. The parameter $\beta_M\sigma_M$ is elliptic. Indeed, if it were not, it would factor through a proper Levi subgroup of $\widehat{M_{\mathrm{der}}}$ by Proposition 11.1; the inverse image of that subgroup in $\widehat{M}$ is a proper Levi subgroup of $\widehat{M}$, defined by the same set of simple roots, and it contains the image of σ_M, which contradicts the minimality of $\widehat{M}$.

Step 2: a cusp form on M. Since $p > b(M)$, Theorem A gives a nonzero cuspidal excursion summand of $\beta_M\sigma_M$ on M_{der}; if M_{der} is trivial, the constant function one. By Lemma 10.7 there is a nonzero cuspidal joint eigenfunction φ_M on M with parameter σ_M and connected central character ω_{σ_M}. That lemma allows a central character of infinite order, which it treats by a twist by a character of the degree; so no condition on the central character is needed here.

Step 3: transfer to G. If $M = G$, then σ is elliptic and φ_M is the required cusp form. Otherwise Theorem 12.1 gives a function $\varphi_\sigma \in \mathcal{A}^{\mathrm{cont}}(G)$ which is not cuspidal, in particular not zero, and which satisfies (1.1) for all tuples of elements of Γ_F and has connected central character ω_σ. It is an automorphic form by Corollary 6.12. □

Proof of Theorem C. (1) The form φ_σ constructed above has connected central character ω_σ. If this character is trivial on Ξ, then φ_σ is invariant under Ξ, and by Corollary 6.14 and the last assertion of Theorem 12.1 it lies in $\mathcal{A}_\Xi(G)$.

(2) If $\omega_\sigma|_\Xi = 1$, the cusp form of (1) is a nonzero element of the summand. Conversely, suppose that the summand is not zero. By Proposition 10.5 the connected centre acts on it through ω_σ. On the other hand the translation by an element ξ of Ξ is the identity on functions on $G(F)\backslash G(\mathbb{A}_F)/\Xi$. Since the summand is not zero, $\omega_\sigma(\xi) = 1$. □

The characteristic of F enters the proof of Theorem B only through the realization on M_{der}, which requires $p > b(M)$. Proposition 11.4 gives $b(M) \le b(G)$ and the sharper bounds for proper Levi subgroups.

A The partial Frobenius and the constant term

This appendix proves Proposition 7.4. The proposition enters the proof of Theorem 7.1, and through it the realization of the parameters which are not elliptic; it is not used for elliptic parameters. The proof is a computation of normalizations: it determines a power of q and shows that an auxiliary choice cancels. It is independent of the rest of the paper, which is why it is placed here. The notation is that of Section 7.

A.1 The Frobenius of the torsor of jets

Let $Y \subset X^I$ be an open subset on which the legs, and the finitely many translates of the legs by powers of Frobenius which occur in the composite of maps under consideration, are pairwise distinct. Choose an integer d such that, for all the bounds on modifications which occur, the action of the groups of jets on the bounded affine Grassmannians factors through the jets of order d. For $H \in \{G, P, M\}$ let $H_{i,d}$ be the group of jets of order d at the leg i, and put

$$U_{i,d} = \ker(P_{i,d} \to M_{i,d}), \qquad m_i = \dim U_{i,d}, \qquad m = \sum_{i \in I} m_i.$$

Fix a leg i. For $a = 1, 2$ let Cht^a_H be the stack of shtukas of H whose modifications are taken in the order given by the partition $(\{i\}, I \setminus \{i\})$ for $a = 1$ and $(I \setminus \{i\}, \{i\})$ for $a = 2$. The partial Frobenius morphism at i maps Cht^1_H to Cht^2_H. We write LocMod^a_H for the corresponding stacks of local modifications, the quotients of the bounded affine Grassmannians by the groups of jets.

The constant-term morphism of Xue is defined by means of the stack

$$\mathrm{Cht}^{a,\mathrm{corr}}_M = \mathrm{Cht}^a_M \times_{\mathrm{LocMod}^a_M} \mathrm{LocMod}^a_P,$$

which adds to a shtuka of M the local data of a modification of P-bundles. The map $\pi_{a,d} : \mathrm{Cht}^a_P \to \mathrm{Cht}^{a,\mathrm{corr}}_M$ is smooth of relative dimension m [Xue20a, Lemma 3.1.8 and (3.9)]. The partial Frobenius morphisms of P and of M are compatible with these maps, and define

$$c : \mathrm{Cht}^1_P \longrightarrow \mathcal{B}'_i = \mathrm{Cht}^{1,\mathrm{corr}}_M \times_{\mathrm{Fr}^{\mathrm{corr}}_{M,i}, \mathrm{Cht}^{2,\mathrm{corr}}_M} \mathrm{Cht}^2_P\,.$$

The map c sends a shtuka of P to its image in the corrected stack for M together with its image under the partial Frobenius.

Lemma A.1 (Rotation and its relative Frobenius)**.** *The map c is a universal homeomorphism, finite and flat of degree q^{m_i}.*

Proof. All fibre products in this proof are taken with their scheme structures, and the degree of a relative Frobenius morphism is computed on its fibres as schemes. The étale site, and with it the sheaves, are the same for a scheme and for its reduction.

The proof has two steps. The first recalls, for each of the groups G, P and M, the description of a partial Frobenius morphism as a base change of the relative Frobenius of a torsor under a group of jets, due to V. Lafforgue. The second compares these descriptions for P and for M, and this is where the degree q^{m_i} appears: it is the order of the kernel of Frobenius on a form of a group of jets of the unipotent radical.

Step 1: the rotation square. Let H be one of the three groups. The objects introduced in this step concern the leg i only: they live over the curve $X - N$ in which this leg varies, and not over Y. Let Z_H be the bounded space of local modifications at one point, with its action of $H_{i,d}$, and

$$\mathrm{LocMod}_{H,i} = [H_{i,d} \backslash Z_H], \qquad \mathcal{B}_{H,i} = \mathrm{Bun}_{H,N} \times \mathrm{LocMod}_{H,i}\,.$$

A point of $\mathrm{LocMod}_{H,i}$ is a point x of $X - N$, a torsor $\mathcal{P}$ under $H_{i,d}$ and an equivariant map $\mathcal{P} \to Z_H$. So $\mathcal{B}_{H,i}$ carries two torsors under $H_{i,d}$: the torsor $\mathcal{P}$, and the torsor $\mathcal{T}$ of trivializations of the restriction of the universal bundle to the divisor $d\Gamma_x$. Let $\mathrm{Mod}_{H,i}$ be the stack of modifications $c_1 : E_0 \dashrightarrow E_1$ at one point of $X - N$, with the given bound, where E_1 has a structure of level N. Sending c_1 to the pair formed by E_1 and the local modification of c_1 defines a map

$$\mathrm{loc}_H : \mathrm{Mod}_{H,i} \longrightarrow \mathcal{B}_{H,i},$$

and a modification with a given image is the same as an isomorphism of torsors $\mathcal{P} \simeq \mathcal{T}$. So loc_H is a torsor under the group scheme $\mathcal{A}_H = \underline{\mathrm{Aut}}(\mathcal{P})$ over $\mathcal{B}_{H,i}$, an inner form of $H_{i,d}$ [Laf18, Construction 1.8 and Remark 1.11]. This group scheme is smooth of relative dimension $\dim H_{i,d}$. Let

$$F_{\mathrm{Mod}/\mathcal{B}} : \mathrm{Mod}_{H,i} \longrightarrow \mathrm{Mod}^{(q)}_{H,i} = \mathcal{B}_{H,i} \times_{F,\mathcal{B}_{H,i}} \mathrm{Mod}_{H,i}$$

be the Frobenius of $\mathrm{Mod}_{H,i}$ relative to $\mathcal{B}_{H,i}$, where F is the Frobenius endomorphism of $\mathcal{B}_{H,i}$. It is equivariant with respect to the relative Frobenius of $\mathcal{A}_H$, a faithfully flat homomorphism whose kernel $\mathcal{A}_H[F]$ is finite, locally free of rank $q^{\dim H_{i,d}}$ and infinitesimal. So $F_{\mathrm{Mod}/\mathcal{B}}$ is a torsor under $\mathcal{A}_H[F]$.

For a bundle E over $X \times S$, with S a scheme over $\mathbb{F}_q$, write ${}^\tau E = (\mathrm{id}_X \times \mathrm{Frob}_{q,S})^* E$. A point of Cht^1_H consists of legs $(x_j)_{j\in I}$ in Y and a pair of modifications

$$c_1 : E_0 \dashrightarrow E_1, \qquad c_2 : E_1 \dashrightarrow {}^\tau E_0,$$

the first at the leg x_i and the second at the other legs. Its image under the partial Frobenius is the pair $(c_2, {}^\tau c_1)$; the leg i has become $\mathrm{Frob}(x_i)$ and the other legs are unchanged. Accordingly Cht^2_H is considered over the open subset of X^I which is the image of Y under the Frobenius of the factor i. Let

$$\mathcal{T}'_H = \mathrm{Cht}^2_H \times_{\mathrm{LocMod}_{H,i},F} \mathrm{LocMod}_{H,i},$$

where the map from Cht^2_H is the local modification of the second modification, the one at the leg i, and F is the Frobenius endomorphism of $\mathrm{LocMod}_{H,i}$. So a point of $\mathcal{T}'_H$ is a point of Cht^2_H, whose second modification we call c_{mov}, a point z of $\mathrm{LocMod}_{H,i}$, and an isomorphism between the local modification of c_{mov} and $F(z)$. Only the leg i is involved in z; the other legs belong to the point of Cht^2_H and are not touched. If E_1 is the first bundle of the point of Cht^2_H, then $b_0 = (E_1, z)$ is a point of $\mathcal{B}_{H,i}$, the target of c_{mov} is ${}^\tau E_1$, and the data identify $\mathrm{loc}_H(c_{\mathrm{mov}})$ with $F(b_0) = ({}^\tau E_1, F(z))$. This defines a map from $\mathcal{T}'_H$ to $\mathrm{Mod}^{(q)}_{H,i}$. The square

$$\begin{array}{ccc} \mathrm{Cht}^1_H & \xrightarrow{\mathrm{rot}_i} & \mathcal{T}'_H \\ \downarrow & & \downarrow \\ \mathrm{Mod}_{H,i} & \xrightarrow[F_{\mathrm{Mod}/\mathcal{B}}]{} & \mathrm{Mod}^{(q)}_{H,i} \end{array} \tag{A.1}$$

in which the left vertical map is $(c_1, c_2) \mapsto c_1$, the upper map is the partial Frobenius with z the local modification of c_1, and the lower map is the relative Frobenius, is 2-Cartesian. Indeed, a point of the fibre product consists of c_1, of a point (c_2, c_{mov}) of Cht^2_H and of an isomorphism $c_{\mathrm{mov}} \simeq {}^\tau c_1$ which induces the given identification of local modifications. The target of c_1 is the first bundle E_1 of the second datum, and $c_2 : E_1 \dashrightarrow {}^\tau E_0$ is then the second modification of a shtuka in Cht^1_H, whose legs lie in Y because their image under the Frobenius of the factor i lies in the image of Y. This construction is inverse to the map from Cht^1_H, on objects and on isomorphisms. The statement is the variant, with trivializations at the target, of [Laf18, Proposition 3.4 and Remark 3.5]; the argument does not use that the group is reductive.

It follows that $\mathrm{rot}_i : \mathrm{Cht}^1_H \to \mathcal{T}'_H$ is a base change of $F_{\mathrm{Mod}/\mathcal{B}}$. So it is representable, finite and locally free of degree $q^{\dim H_{i,d}}$, and it is a torsor under the pullback of $\mathcal{A}_H[F]$.

Step 2: the map c. A point of $\mathcal{B}'_i$ consists of a shtuka (c_1^M, c_2^M) of M in Cht^1_M, of local modifications of P-bundles at the legs which lift those of this shtuka, of a shtuka s' of P in Cht^2_P, and of an identification of the images of these data in $\mathrm{Cht}^{2,\mathrm{corr}}_M$. Let z_P be the local modification of P-bundles at the leg i, and z_M its image. Then (s', z_P) is a point of $\mathcal{T}'_P$, and

by step 1 for $H = P$ the shtukas of P with image (s', z_P) under rot_i form a torsor $\mathcal{R}_P$ under $\mathcal{A}_P[F]$, the fibre of $F_{\mathrm{Mod}/\mathcal{B}}$ for P. In the same way the image of (s', z_P) in $\mathcal{T}'_M$ defines a torsor $\mathcal{R}_M$ under $\mathcal{A}_M[F]$, of which the shtuka (c_1^M, c_2^M) is a point. The fibre of c over the given point of $\mathcal{B}'_i$ is the fibre over this point of the map $\mathcal{R}_P \to \mathcal{R}_M$ induced by $P \to M$: a shtuka of P in Cht^1_P lies over the given point of $\mathcal{B}'_i$ if and only if its image under rot_i is (s', z_P) and its image in Cht^1_M is (c_1^M, c_2^M).

The map $\mathcal{R}_P \to \mathcal{R}_M$ is equivariant with respect to the homomorphism $\mathcal{A}_P[F] \to \mathcal{A}_M[F]$. Since $P_{i,d}$ is the semidirect product of $U_{i,d}$ and $M_{i,d}$, compatibly with Frobenius, the homomorphism $\mathcal{A}_P \to \mathcal{A}_M$ is smooth and surjective, its kernel $\mathcal{U}$ is a form of $U_{i,d}$, and

$$1 \longrightarrow \mathcal{U}[F] \longrightarrow \mathcal{A}_P[F] \longrightarrow \mathcal{A}_M[F] \longrightarrow 1$$

is an exact sequence of finite locally free group schemes: locally for the flat topology it is the sequence of the Frobenius kernels of $U_{i,d}$, $P_{i,d}$ and $M_{i,d}$. So the fibres of $\mathcal{R}_P \to \mathcal{R}_M$ are torsors under $\mathcal{U}[F]$, which is finite, locally free of rank $q^{\dim U_{i,d}} = q^{m_i}$, and infinitesimal. This holds over every scheme mapping to $\mathcal{B}'_i$. Therefore c is representable, finite and locally free of degree q^{m_i}, and a universal homeomorphism.

As a check, the composite of c with the map $\mathcal{B}'_i \to \mathcal{T}'_P$ which sends a point as above to (s', z_P) is rot_i for P, of degree $q^{\dim P_{i,d}}$; and $\mathcal{B}'_i \to \mathcal{T}'_P$ is a base change of rot_i for M, of degree $q^{\dim M_{i,d}}$. The quotient of the two degrees is q^{m_i}. □

Lemma A.2 (The trace of the map c)**.** *Let $\pi_0 = \pi_{1,d}$, and let $\pi_1 : \mathcal{B}'_i \to \mathrm{Cht}^{1,\mathrm{corr}}_M$ be the base change of $\pi_{2,d}$, so that $\pi_0 = \pi_1 c$. Under the purity isomorphisms for the smooth maps π_0 and π_1, the trace map of π_0 composed with the pullback by c is q^{m_i} times the trace map of π_1. The same holds after any base change.*

Proof. Both π_0 and π_1 are smooth of relative dimension m. The constant-term morphism uses the trace map

$$(\pi_0)_!(\pi_0)^*[2m](m) \simeq (\pi_0)_!(\pi_0)^! \xrightarrow{\epsilon_{\pi_0}} \mathrm{id},$$

as in [Xue20a, (3.22)–(3.23)]. Write $\pi_0 = \pi_1 c$. For a sheaf $\mathcal{K}$ pulled back from $\mathrm{Cht}^{1,\mathrm{corr}}_M$ to $\mathcal{B}'_i$, the purity isomorphisms of π_0 and π_1 identify the comparison between the two trace maps with

$$c_! c^* \mathcal{K} \xrightarrow{c_! \,\mathrm{cl}_c} c_! c^! \mathcal{K} \xrightarrow{\epsilon_c} \mathcal{K}, \tag{A.2}$$

where cl_c is the fundamental class of the finite flat map c; this follows from the rule $\epsilon_{\pi_0} = \epsilon_{\pi_1} \circ (\pi_1)_! \epsilon_c$ for the composition of trace maps. For a finite flat map, the adjunction map $\mathcal{K} \to c_* c^* \mathcal{K}$ followed by (A.2) is multiplication by the degree [Lau76, Theorem 0.2.1 and (0.3.2)], which is q^{m_i} by Lemma A.1. For the constant sheaf this is an identity between maps of sheaves in degree zero, because c is a finite universal homeomorphism. The projection formula gives it for every $\mathcal{K}$, complexes included. Therefore

$$\mathrm{tr}_{\pi_0} \circ c^* = q^{m_i} \,\mathrm{tr}_{\pi_1}\,.$$

Only the leg i contributes, since the maps on the trivializations at the other legs are the same on the two sides. Base change preserves the purity isomorphisms and the trace maps, and so does an isomorphism of the trivializations. □

A.2 Proof of the proposition

Proof of Proposition 7.4. Step 1 fixes the notation. The computation is in step 2. The square formed by the constant-term maps and the partial Frobenius morphisms is not Cartesian, and the factor $q^{-n_i/2}$ of (A.6) is the product of the factor q^{m_i} of Lemma A.2 and of the twist $q^{-n_i/2-m_i}$ of the sheaves. Step 3 passes to cohomology.

Step 1: notation, and the map of coefficients. We work over Y and over its image under the Frobenius of the factor i. For $H \in \{G, P, M\}$ let LocMod_H be the stack of local modifications at all the legs, which over these open subsets is the product over $j \in I$ of the stacks $\mathrm{LocMod}_{H,j}$ of the proof of Lemma A.1. Let

$$\epsilon_H^a : \mathrm{Cht}_H^a \to \mathrm{LocMod}_H, \qquad \iota : \mathrm{LocMod}_P \to \mathrm{LocMod}_G, \qquad \rho : \mathrm{LocMod}_P \to \mathrm{LocMod}_M$$

be the maps which take local modifications and which are induced by $P \subset G$ and $P \to M$, and let Φ_H be the endomorphism of LocMod_H which is the Frobenius endomorphism of the factor i and the identity on the other factors. The maps ι and ρ commute with Φ, and

$$\epsilon_H^2 \circ \mathrm{Fr}_{H,i} = \Phi_H \circ \epsilon_H^1, \tag{A.3}$$

because the local modification of ${}^\tau c_1$ is the image under Frobenius of that of c_1.

Let $\mathcal{S}_G$ be the Satake sheaf of V on LocMod_G, an exterior product over the legs, and $\mathcal{S}_M = \bigoplus_\mu \mathcal{S}_{M,\mu}$ that of $\mathrm{Res}\, V$, decomposed according to the weights $\mu = (\mu_j)_{j\in I}$ of the centre of $\widehat{M}^I$. Every sheaf $\mathcal{K}$ on a stack over $\mathbb{F}_q$ has a canonical isomorphism with its pullback by the Frobenius endomorphism of the stack, and these isomorphisms are compatible with pullbacks, with direct images with proper supports and with base change. Applied to the factor i they give isomorphisms

$$\phi_G : \Phi_G^* \mathcal{S}_G \simeq \mathcal{S}_G, \qquad \phi_M : \Phi_M^* \mathcal{S}_M \simeq \mathcal{S}_M.$$

By (A.3) they induce isomorphisms $\mathrm{Fr}_{H,i}^* \mathcal{F}_H^2 \simeq \mathcal{F}_H^1$ for the sheaves $\mathcal{F}_H^a = (\epsilon_H^a)^* \mathcal{S}_H$ on the stacks of shtukas, and the partial Frobenius map F_i^H on cohomology is the pullback by $\mathrm{Fr}_{H,i}$ followed by this isomorphism [Laf18, §§3 and 4.3]. For the constant sheaf $E(a)$ on a stack over $\mathbb{F}_q$ the canonical isomorphism is the action of the geometric Frobenius, that is, multiplication by q^{-a} when $E(a)$ is identified with its pullback as a sheaf coming from $\mathrm{Spec}\, \mathbb{F}_q$.

Lemma 7.3, together with the fact that the cohomology with compact supports of the classifying stack of $U_{j,d}$ is $E[2m_j](m_j)$ [Xue20a, (3.18)], gives an isomorphism of sheaves on LocMod_M

$$\lambda : \rho_! \iota^* \mathcal{S}_G \simeq \bigoplus_\mu \mathcal{S}_{M,\mu} \otimes \mathcal{L}_\mu, \qquad \mathcal{L}_\mu = \bigotimes_{j\in I} E[n_{\mu_j} + 2m_j](\tfrac{n_{\mu_j}}{2} + m_j), \tag{A.4}$$

which is an exterior product over the legs of isomorphisms defined over $\mathbb{F}_q$. Let $\widetilde{\rho}_a : \mathrm{Cht}_M^{a,\mathrm{corr}} \to \mathrm{Cht}_M^a$ and $\widetilde{\epsilon}_a : \mathrm{Cht}_M^{a,\mathrm{corr}} \to \mathrm{LocMod}_P$ be the two projections, so that the map $\pi^a : \mathrm{Cht}_P^a \to \mathrm{Cht}_M^a$ is $\widetilde{\rho}_a \pi_{a,d}$ and $\epsilon_P^a = \widetilde{\epsilon}_a \pi_{a,d}$. The map of coefficients which defines the constant-term morphism [Xue20a, (3.18)–(3.23)] is the composite

$$\begin{aligned} \kappa_a : \ (\pi^a)_! (\epsilon_P^a)^* \iota^* \mathcal{S}_G &= (\widetilde{\rho}_a)_! (\pi_{a,d})_! \pi_{a,d}^* \widetilde{\epsilon}_a^* \iota^* \mathcal{S}_G \\ &\xrightarrow{\mathrm{tr}} (\widetilde{\rho}_a)_! \widetilde{\epsilon}_a^* \iota^* \mathcal{S}_G[-2m](-m) \\ &\xrightarrow{\sim} (\epsilon_M^a)^* \rho_! \iota^* \mathcal{S}_G[-2m](-m) \\ &\xrightarrow{\lambda} \bigoplus_\mu \mathcal{F}_{M,\mu}^a[n_\mu](\tfrac{n_\mu}{2}), \end{aligned} \tag{A.5}$$

where the first map is the trace map of the smooth map $\pi_{a,d}$, the second is the base change isomorphism of the Cartesian square which defines $\mathrm{Cht}_M^{a,\mathrm{corr}}$, and in the third the twist (m) of $\mathcal{L}_\mu$ is cancelled against $(-m)$; here $n_\mu = \sum_j n_{\mu_j}$. The component of $\mathsf{CT}^{\mathrm{geo}}$ on the summand μ is obtained by identifying $E(n_\mu/2)$ with E; these identifications are chosen once and for all, compatibly with tensor products.

Step 2: the identity of maps of sheaves. We claim that, on the summand of weight μ,

$$\kappa_1 \circ (\pi^1)_!(\epsilon_P^1)^*\iota^*(\phi_G) \circ \mathrm{BC} = q^{-n_i/2}\, \phi_M \circ \mathrm{Fr}_{M,i}^*(\kappa_2), \qquad n_i = n_{\mu_i}, \tag{A.6}$$

as maps from $\mathrm{Fr}_{M,i}^*(\pi^2)_!(\epsilon_P^2)^*\iota^*\mathcal{S}_G$ to $\mathcal{F}_{M,\mu}^1[n_\mu]$. Here BC is the base change map of the commutative square formed by π^1, π^2 and the partial Frobenius morphisms of P and of M: the pullback of sheaves along the map from Cht_P^1 to the fibre product. The square is not Cartesian, and this is the point of the computation. We compare the three maps of (A.5) for $a = 1$ and $a = 2$ in turn.

(a) *The trace map.* The square formed by $\pi_{1,d}$, $\pi_{2,d}$ and the partial Frobenius morphisms factors through the Cartesian square with corner $\mathcal{B}_i'$ and the map c. The trace map of $\pi_{2,d}$ is compatible with the base change to $\mathcal{B}_i'$, and by Lemma A.2 the trace map of $\pi_{1,d} = \pi_1 c$ composed with the pullback by c is q^{m_i} times the trace map of π_1. Hence, for every sheaf $\mathcal{K}$ on $\mathrm{Cht}_M^{2,\mathrm{corr}}$,

$$\mathrm{tr}_{\pi_{1,d}} \circ \mathrm{BC}_d = q^{m_i}\, (\mathrm{Fr}_{M,i}^{\mathrm{corr}})^*(\mathrm{tr}_{\pi_{2,d}})$$

as maps from $(\mathrm{Fr}_{M,i}^{\mathrm{corr}})^*(\pi_{2,d})_!\pi_{2,d}^*\mathcal{K}$ to $(\mathrm{Fr}_{M,i}^{\mathrm{corr}})^*\mathcal{K}[-2m](-m)$, where BC_d is the base change map of the square. In this identity the sheaf $E(-m)$ is identified with its pullback as a sheaf coming from $\operatorname{Spec}\mathbb{F}_q$; the partial Frobenius morphisms are morphisms of stacks over $\mathbb{F}_q$.

(b) *The base change isomorphism.* The Cartesian squares which define $\mathrm{Cht}_M^{1,\mathrm{corr}}$ and $\mathrm{Cht}_M^{2,\mathrm{corr}}$ are the two ends of a commutative cube whose other edges are $\mathrm{Fr}_{M,i}^{\mathrm{corr}}$, $\mathrm{Fr}_{M,i}$, Φ_P and Φ_M. Base change maps for composite squares are composites of base change maps. So the second map of (A.5) transforms the base change map of $\widetilde{\rho}$ with respect to the partial Frobenius morphisms into the pullback by ϵ_M^1 of the base change map of ρ with respect to Φ_P and Φ_M.

(c) *The local isomorphism.* Since the canonical isomorphisms with Frobenius pullbacks are compatible with ι^*, with $\rho_!$ and with base change, the isomorphism $\rho_!\iota^*(\phi_G)$, composed with the base change map of ρ with respect to Φ, is the canonical isomorphism of the sheaf $\rho_!\iota^*\mathcal{S}_G$ for the factor i. The isomorphism λ of (A.4) is an exterior product of isomorphisms defined over $\mathbb{F}_q$, so it transforms this canonical isomorphism into that of the right-hand side, which on the summand μ is the tensor product of ϕ_M and of the canonical isomorphism of the factor i of $\mathcal{L}_\mu$. By step 1 the latter is multiplication by

$$q^{-n_i/2-m_i}.$$

Combining (a), (b) and (c): the composite on the left of (A.6) is $\phi_M \circ \mathrm{Fr}_{M,i}^*(\kappa_2)$ multiplied by

$$q^{m_i} \cdot q^{-n_i/2-m_i} = q^{-n_i/2},$$

in which the auxiliary dimension m_i has cancelled, in agreement with the fact that the constant-term morphism does not depend on d. The shifts do not enter an identity between maps of sheaves; their effect on trace functions is discussed in Remark A.4.

It remains to pass to cohomology with compact supports. Fix the degree of the M-bundle at the target and use the bounded correspondences of [Xue20a, §3.5]. On the dense open subset of Proposition 3.5.3 of that paper the pullback from G to P is the pullback along a proper map of the correspondence. Without a central lattice the same proper maps are obtained on the open and closed components of given degree of the correspondence with a central lattice. For the partial Frobenius maps take the inverse image of the bounded stack at the target, as in [Laf18, §4.3], and extend by zero to a larger bound which contains it. The comparison of trace maps and base change maps above is compatible with these restrictions and with the extensions by zero when the bound increases [Xue20a, §3.5.9]. Hence (A.6) passes to the limit. This is an identity on the generic stalk, which is where the partial

Frobenius maps enter the construction of the action of W_F^I. Over the diagonals no Frobenius is involved: there the constant-term morphism is compatible with fusion by Lemma 7.3 and [Xue20b, Lemma 4.1.5], and on the diagonal the summand of weight μ lies in the summand whose weight is the sum of the μ_j, with n additive.

Step 3: normalization. The constant-term morphism on cohomology is the pullback from the shtukas of G to those of P, which commutes with the partial Frobenius maps, followed by the direct image of κ_a. The partial Frobenius morphism of M maps the shtukas of degree ν whose modification at the leg i has weight μ_i to shtukas of degree $\nu - \mu_i$. So taking cohomology in (A.6) gives the first formula of the proposition. With the factor of Lemma 7.2,

$$\begin{aligned} c_U \delta_P(\nu)^{1/2} \mathsf{CT}_\nu^{\mathrm{geo}} F_i^G &= F_i^M c_U \delta_P(\nu)^{1/2} q^{-n_i/2} \mathsf{CT}_{\nu-\mu_i}^{\mathrm{geo}} \\ &= F_i^M c_U \delta_P(\nu-\mu_i)^{1/2} \mathsf{CT}_{\nu-\mu_i}^{\mathrm{geo}}, \end{aligned}$$

because $\delta_P(\nu - \mu_i) = q^{-n_i}\delta_P(\nu)$. This is the second assertion. Note finally that for a shtuka the sum of the μ_i has image zero in the cocharacter group of M/M_{der}, because the pullback by Frobenius does not change the degree of a bundle. So the sum of the shifts $[-n_i]$ is zero, and the constant-term morphism preserves the cohomological degree. □

Remark A.3. The map c of Lemma A.1 is a universal homeomorphism, so it induces an equivalence of étale sites. Its trace map is nevertheless multiplication by its degree q^{m_i} and not the identity, and this is the factor that cancels the dependence on the order of the jets.

Remark A.4 (Signs of trace functions)**.** Proposition 7.4 is an identity between maps of cohomology groups, and no sign occurs in it. Signs occur when Satake sheaves are replaced by functions. The shift by n_i multiplies the trace function of Section 3 by $(-1)^{n_i}$, and a negative half Tate twist cannot absorb this sign at all places, since at a place of degree d it would contribute $(-1)^d$. The function attached to a Satake sheaf $\mathcal{S}$ supported on a connected component c of the affine Grassmannian is therefore taken with the sign of [Laf18, before Remark 1.21]:

$$h_{\mathcal{S}} = (-1)^{p_G(c)} t_{\mathcal{S}}, \qquad p_G(c) = \langle 2\rho_G, \lambda \rangle \bmod 2 \quad \text{for } \lambda \in c.$$

This sign is multiplicative for convolution, and with it the function of the sheaf attached to V is the element of the spherical Hecke algebra whose Satake transform is the character of V. For example, for PGL_2 and the minuscule coweight the intersection complex has trace $-q_v^{-1/2}$ on its support, and the function is $q_v^{-1/2}$ there; here q_v is the cardinality of the residue field of the place at which the trace is taken. In the restriction at the leg i, the component μ_i of the modifications of M lies over a component c of the modifications of G, and $n_i \equiv p_G(c) - p_M(\mu_i)$ (mod 2) by the definition of n_i. So the three signs $(-1)^{p_G(c)}$, $(-1)^{n_i}$ and $(-1)^{p_M(\mu_i)}$ cancel, and the comparison of functions has the factor $q_v^{-n_i/2}$ and no sign. For a semisimple simply connected group, in particular in [II], all coweights lie in the coroot lattice and $p_G = 0$.

Remark A.5 (A check on the normalization)**.** In Proposition 7.4 the scalar $q^{-n_i/2}$ appears when the Satake sheaves of M are normalized to weight zero at each leg separately, and the degree changes from ν to $\nu - \mu_i$ because the partial Frobenius replaces the first bundle of a shtuka by the second. Both can be checked on Hecke operators. For an excursion operator which is a Hecke operator at an unramified place, Theorem 7.1 says that $\mathsf{CT}_P^{\mathrm{nor}}$ intertwines a spherical Hecke operator of G with its image under the homomorphism of Hecke algebras which corresponds to the restriction of representations from $\widehat{G}$ to $\widehat{M}$. That homomorphism is the Satake transform with the factor $\delta_P^{1/2}$, and the statement is the classical one: if φ lies in a representation induced from $\pi \otimes \delta_P^{1/2}$, then $m \mapsto \delta_P(m)^{-1/2}\varphi(mg)$ transforms according to π. With another power of q in Proposition 7.4 the exponent of δ_P would be different.

Here is the check in the simplest case, by counting. Let $G = \mathrm{GL}_2$ and P the upper triangular subgroup, so that $\mathsf{CT}^{\mathrm{geo}} h(L_1, L_2)$ is the sum of $h(E)$ over the extensions E of L_2 by L_1, divided by $|\operatorname{Hom}(L_2, L_1)|$, and $\delta_P(L_1, L_2) = q^{\deg L_1 - \deg L_2}$. Let v be a place of degree one, and T_v the Hecke operator of the double coset which enlarges lattices; it is the transpose of the one which shrinks them. It sums a function over the $q+1$ bundles $E' \supset E$ with E'/E of length one at v. For an extension E, one of these bundles is the pushout of E along $L_1 \to L_1(v)$, and the other q are extensions of $L_2(v)$ by L_1. The pushout map $\operatorname{Ext}^1(L_2, L_1) \to \operatorname{Ext}^1(L_2, L_1(v))$ is surjective with kernel of order $q\,|\operatorname{Hom}(L_2, L_1)|/|\operatorname{Hom}(L_2, L_1(v))|$, by the cohomology sequence of $0 \to L_2^{-1}L_1 \to L_2^{-1}L_1(v) \to k_v \to 0$. And the pairs $E \subset E'$ of the second kind correspond, by pullback along $L_2 \to L_2(v)$, to the extensions E' of $L_2(v)$ by L_1, with the same automorphisms. Hence

$$\mathsf{CT}^{\mathrm{geo}}(T_v h)(L_1, L_2) = q\,\mathsf{CT}^{\mathrm{geo}} h(L_1(v), L_2) + \mathsf{CT}^{\mathrm{geo}} h(L_1, L_2(v)).$$

Since $\delta_P(L_1(v), L_2) = q\,\delta_P(L_1, L_2)$ and $\delta_P(L_1, L_2(v)) = q^{-1}\delta_P(L_1, L_2)$, the two coefficients are $q^{1/2}(\delta_P(m')/\delta_P(m))^{1/2}$, where m' is the pair on the right. So $\delta_P^{1/2}\mathsf{CT}^{\mathrm{geo}}$ intertwines $q^{-1/2}T_v$ with the sum of the two translations, as the proposition says: the coefficient of the term in which the degree of the M-bundle is changed by $-\mu$ is $q^{-n_\mu/2}$ times the symmetric one.

Notation

The table lists notation used in more than one section, with the place where it is introduced.

Notation	Meaning and location
X, F, Γ_F, W_F	The original curve, its function field, its absolute Galois group and its Weil group (Sections 1 and 3).
p, q, ℓ, k	The geometric characteristic, size of the constant field, coefficient prime, and $k = \overline{\mathbb{Q}}_\ell$.
$E, \mathcal{O}_E$	A finite extension of $\mathbb{Q}_\ell$ and its ring of integers (Section 5).
$\deg_F, d_v$	Arithmetic Galois degree and the degree of a closed point v; geometric Frobenius at v has degree $-d_v$.
$\mathrm{Fr}_q, \mathrm{Frob}_q$	The q-power map of a scheme and geometric Frobenius acting on sheaf cohomology, respectively.
$G, \widehat{G}, M, \widehat{M}$	The split group, its dual, a Levi subgroup and the corresponding dual Levi.
$K_N, \mathscr{H}_{G,u,\mathcal{O}_E}$	The open compact subgroup of level N and the integral spherical Hecke algebra at $u \notin N$ (Section 5).
σ, ν_σ	The prescribed parameter and its character on excursion data, defined by (1.1).
$\vartheta_G, \sigma^\dagger$	The Chevalley involution preserving the pinning, and $\vartheta_G\sigma$ (Section 3).
T_G, Ξ, ω_σ	The connected centre with its reduced structure, a central lattice, and the central character of σ (Definition 3.2).
Q_G, α_G, β_G	The quotient G/G_{der} and the homomorphisms $\widehat{G} \to \widehat{T}_G$, $\widehat{G} \to \widehat{G_{\mathrm{der}}}$ (4.1).
a	The isogeny of dual groups dual to a central isogeny $H_0 \to G$ (Section 10).
$\mathcal{A}_\Xi(G), \mathcal{A}^{\mathrm{cont}}(G)$	The functions invariant under Ξ with a finite integral spherical orbit, and the span of the twists of such functions by characters of the degree (Definition 6.9).
$Y_N, C_{N,\mathcal{O}}$	The double coset space of level N and the functions with compact support on it (Section 5).

Notation	Meaning and location
$e = (I, f, \boldsymbol{\gamma}),\ e^\star$	An excursion datum and its transpose (Definition 5.7).
S_e, S_e^{fun}	The excursion operator on functions with compact support, and its extension to all functions of finite level (Definition 5.12).
$\mathcal{Q}_{N,\Xi,u,\mathfrak{a}}$	A finite Hecke quotient (Definition 6.1).
T_ψ	Multiplication by the class-field character of a central parameter ψ (Proposition 5.13).
$c^{\deg_F}$	The twist by a unit point c of $Z(\widehat{G})^\circ$ (Definition 4.10).
$\mathsf{CT}_P^{\mathrm{nor}}$	The normalized constant term (Section 7).
ι	A fixed isomorphism $k \simeq \mathbb{C}$ which maps $q^{1/2}$ to the positive square root of q (Section 3).

The following terms are introduced in this paper, or are used in a sense fixed here.

Term	Where it is defined
parameter; semisimple; elliptic	Definition 1.1
excursion datum; excursion character ν_σ	Section 1; Definition 5.7
automorphic form	Section 1, after Theorem A
central lattice; invariant under Ξ	Definition 3.2
excursion summand	Definition 3.3
automorphic realization; to realize a parameter	Section 1, after (1.1)
arithmetic monodromy group; strongly pure of weight zero	Section 4; Definition 4.7
unit point; twist by the field of constants	Definition 4.10
central lattice; periodization	Section 5
functions with compact support	Section 5
finite Hecke quotient; cofinite ideal	Definition 6.1
finite integral spherical orbit	Definition 6.6
unit-valued Satake parameters	Definition 8.1
generalized excursion character; for Weil tuples	Definition 8.4
operator with finite rows	Section 7

Statement on the use of AI

The mathematical arguments of the three papers of this series were developed with GPT-6 Astra Pro. The papers were then organized and rewritten with Claude Fable 5.2, and in the course of this several arguments were replaced or expanded. The three papers were then reviewed in detail, line by line, by GPT and Claude models, in sessions which had not taken part in writing them, and the errors found in this way were corrected.

Boyang Hu organized the work and is responsible for the manuscript.

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