Limit Cycles under Concave Per-Capita Growth

Zhenyu Sun

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Abstract

We study predator–prey systems with a fixed globally linear functional response and a strictly concave prey per-capita growth rate. For every $N \geq 1$ and $K \geq 2$, we construct a growth polynomial of degree $2N + 2\lceil K/2 \rceil + 2$ with exactly N positive periodic orbits, all hyperbolic. The construction retains a chosen center K-jet and the low-density growth rate and carrying capacity of its limiting profile, subject to an explicit relation between the center height and curvature. A limiting Lyapunov function and return-map estimates at three amplitude scales exclude additional cycles. A dissipative extension gives exact global counts with prescribed limiting center energies. We also prove that profiles with at least N hyperbolic cycles are C^2-dense in the admissible class, while preserving the entire germ at the maximum and both ecological endpoint values. The first nonzero canonical odd term determines the sharp differentiability threshold. Further results give parity restrictions for fixed focus jets, degree lower bounds, quantitative polynomial approximation under concavity and interpolation constraints, and amplitude and multiplicity interpolation by Abelian integrals. Finally, a single smooth strictly concave profile has infinitely many nested hyperbolic cycles converging to coexistence.

Keywords. Predator–prey system; exact cycle count; polynomial degree; concavity; finite jet; Abelian integral; differentiability.
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1 Introduction and main results

Consider

$$\dot{x} = x(\alpha(x) - \beta y), \qquad \dot{y} = y(k\beta x - m), \qquad k, m, \beta > 0. \tag{1}$$

Throughout the paper the three constants are fixed. The function α is the prey *per-capita* growth rate; the total prey growth in the absence of predation is $x\alpha(x)$. This distinction matters for shape assumptions. Put

$$a = \frac{m}{k\beta}.$$

If $\alpha(a) = c > 0$, the unique positive equilibrium is $E_c = (a, c/\beta)$.

Svirezhev's ICM report [6, p. 1682] asks whether (1), with a single-peaked growth rate, can possess a third, a fourth, or further limit cycles. In one regime, a subcritical Hopf bifurcation combined with the global phase portrait yields a two-cycle configuration. The question concerns a globally linear response βx. In some later ecological literature the name Holling I also denotes a linear response followed by saturation; we use the explicit expression βx throughout.

Multiple cycles have been constructed under several ecological shape constraints. Kuang [4] constructs a Gause-type system with a strictly concave prey isocline and at least three cycles. Wrzosek [7] obtains arbitrarily many cycles with a unimodal total prey-growth function and a concave functional response. Zegeling and Kooij [8] compare center, Hopf, and singular perturbation mechanisms in broader response families. These results motivate studying the

restrictions imposed by fixing the response while retaining strong shape conditions on the actual per-capita growth.

Our methods use established planar theory. The energy-coordinate reduction is the generalized Filippov transformation studied by Gasull and Guillamon [1]; their paired-branch criteria include the center and nonoscillation principles used below. The construction of multiple cycles through Abelian integrals has a direct predecessor in Gasull, Guillamon, Li, and Zhang [2]. The global zero bound in section 5 is an explicit application of Gavrilov and Iliev [3]. The results developed here concern simultaneous preservation of the fixed predator equation, concavity, local data, ecological endpoint values, and prescribed oscillations. Our explicit degree theorem uses only harmonic moments and a global Lyapunov argument. The Gavrilov–Iliev theorem enters the separate problem of interpolation throughout the regular annulus of an explicit center.

Definition 1.1. An admissible single-peaked profile at a is a function $\alpha \in C^\infty([0,\infty))$ satisfying

$$\alpha(0) > 0, \quad \alpha'(x) > 0 \ (0 < x < a), \quad \alpha'(a) = 0, \quad \alpha''(a) < 0, \quad \alpha'(x) < 0 \ (x > a),$$

and having a zero $x_1 > a$. It is *strictly concave* if $\alpha''(x) < 0$ for all $x \geq 0$.

For a strictly concave profile, the right-hand zero exists automatically: α' is negative and decreasing to the right of a, so $\alpha(x) \to -\infty$. Its positive zero is unique.

Our first result combines an exact global count with an explicit degree. For $\theta > 0$ define the analytic germ at a

$$C_\theta(x) = c - \theta m \left(\frac{x}{a} - 1 - \log \frac{x}{a} \right).$$

This is a center profile in the canonical coordinates used below. For functions defined near a, j_a^K denotes equality of derivatives through order K at a. Polynomial degrees in this paper refer to the growth function α; a growth polynomial of degree D gives a polynomial vector field of degree $D+1$ in (1).

Theorem 1.2 (Exact global count with a linear degree bound)**.** *Fix $k, m, \beta, c, \theta > 0$, integers $N \geq 1$, $K \geq 2$, and distinct numbers $0 < h_1 < \cdots < h_N$. Put*

$$D = 2N + 2\left\lceil \frac{K}{2} \right\rceil + 2 = \begin{cases} 2N + K + 2, & K \text{ even}, \\ 2N + K + 3, & K \text{ odd}, \end{cases} \tag{2}$$

and assume $c > \theta m(\mathsf{H}_D - 1)$, where $\mathsf{H}_D = \sum_{n=1}^{D} 1/n$. There exist $\alpha_0 > 0$, $b > a$, $\rho_0 > 0$, and an explicitly defined family of real polynomials $\widehat{P}_\rho$, $0 < \rho < \rho_0$, such that

$$\deg \widehat{P}_\rho = D, \qquad \widehat{P}_\rho''(x) < 0 \quad (x \geq 0), \qquad j_a^K \widehat{P}_\rho = j_a^K C_\theta,$$
$$\widehat{P}_\rho(0) = \alpha_0, \qquad \widehat{P}_\rho(b) = 0.$$

The system (1) with growth $\widehat{P}_\rho$ has exactly N positive periodic orbits, all hyperbolic. On the section $v = 0$, $s > 0$ of (3), their coordinates satisfy

$$s_i(\rho) = \rho\sqrt{2h_i} + O(\rho^2).$$

Their stabilities alternate, the outermost is attracting, and the coexistence equilibrium is a stable weak focus for even N and an unstable weak focus for odd N.

The numbers α_0, b are determined by the chosen limiting polynomial and then remain fixed throughout the family. For every fixed $c > 0$ and each (N, K), one can satisfy the displayed height condition by choosing $\theta > 0$ sufficiently small. For a preassigned center curvature, the same degree conclusion is subject to that condition. The proof, including the exact endpoint correction and exclusion of every additional periodic orbit, is given in section 4.

Two restrictions accompany this degree bound. First, when a prescribed jet fixes the stability of a focus, the total number of hyperbolic positive periodic orbits must be even for a stable focus and odd for an unstable focus (proposition 3.4). The center jet in theorem 1.2 leaves that stability free at a higher order. Second, degree bounds for arbitrary prescribed jets or for approximation of arbitrary baselines depend on the sizes of those data. Section 10 gives explicit lower bounds and a constructive data-dependent upper bound. This is a sufficient degree bound for the chosen center jet.

The proof separates the local creation of cycles from their global exclusion. In energy coordinates, the even part of the growth profile defines a reversible center, and the odd part determines the leading return displacement. Harmonic moments prescribe its simple zeros at a shrinking scale. A Lyapunov function for the limiting polynomial forces every possible cycle toward the equilibrium; return-map estimates then exclude cycles at both smaller and larger relative amplitudes.

The second result treats arbitrary local growth profiles.

Theorem 1.3 (Localized realization with preservation of the germ)**.** *Let α be an admissible single-peaked profile at a. For every $N \geq 1$ there are admissible profiles α_ρ, $\rho \downarrow 0$, such that:*

(i) $\alpha_\rho = \alpha$ on a neighborhood of a and outside an interval of length $O(\rho)$ centered at a;

(ii) $\|\alpha_\rho - \alpha\|_{C^2([0,\infty))} \to 0$;

(iii) (1) with α_ρ has at least N distinct hyperbolic positive limit cycles converging to E_c.

The neighborhood on which the germs agree may depend on ρ. If α is globally strictly concave, every α_ρ can be chosen globally strictly concave. The values $\alpha(0)$, $\alpha(a)$, the zero x_1, and the entire growth profile away from the shrinking interval are unchanged.

The C^2 conclusion can be refined. In the energy coordinate introduced in section 2, let the first nonzero odd term of the growth profile have degree d. Then $d \geq 3$ is odd, and the construction converges in C^{d-1}. A sufficiently small C^d perturbation that fixes the lower jet has no cycle in a fixed sufficiently small neighborhood of the equilibrium. This sharp threshold is proved in sections 7 and 8.

Polynomial profiles can retain additional ecological data.

Theorem 1.4 (Strictly concave polynomial realization)**.** *Let α be an admissible globally strictly concave profile at a, with positive zero x_1. Fix integers $N \geq 1$ and $K \geq 2$, a number $R > x_1$, and $\varepsilon > 0$. There is a real polynomial $\widehat{\alpha}$ such that*

$$\widehat{\alpha}''(x) < 0 \quad (x \geq 0), \qquad j_a^K \widehat{\alpha} = j_a^K \alpha, \qquad \widehat{\alpha}(0) = \alpha(0), \qquad \widehat{\alpha}(x_1) = 0,$$
$$\|\widehat{\alpha} - \alpha\|_{C^2([0,R])} < \varepsilon,$$

and (1) has at least N distinct hyperbolic positive limit cycles. Its unique maximum is a, and its unique positive zero is x_1.

The degree in this approximation theorem can increase with the jet, the approximation tolerance, and the chosen hyperbolicity margins. Section 10 quantifies this dependence. Its conclusion is an at-least-N statement for an arbitrary baseline. Exact global counts at specified

regular center energies are obtained by the dissipative extension theorem in section 6, which also admits polynomial approximation with its count preserved.

The local realization theorem has a smooth infinite counterpart.

Theorem 1.5 (One profile with infinitely many cycles)**.** *For every fixed $k, m, \beta > 0$ and $c > 0$, there is an admissible globally strictly concave C^∞ profile α, with maximum at a and $\alpha(a) = c$, for which (1) has infinitely many distinct nested hyperbolic limit cycles converging to E_c. All derivatives of α at a agree with those of*

$$\alpha_{\mathrm{cen}}(x) = c - m\left(\frac{x}{a} - 1 - \log\frac{x}{a}\right).$$

The explicit center yields an additional finite-dimensional theory. For odd perturbations of degrees $2r+3, \dots, 2r+2N+3$, the cyclicity of every compact regular subannulus is exactly N. The limiting energies, and even the actual section coordinates and multiplicities, can be prescribed. Finally, section 12 treats a fixed nonlinear response p. Under the simple break-even hypothesis, the condition $ap'(a) < 2p(a)$ permits strictly concave per-capita realizations; a response with elasticity at least two everywhere excludes cycles for every strictly concave positive low-density growth profile.

Organization. Sections 2 and 3 give the coordinate reduction, uniform confinement, and the global parity law. Section 4 proves theorem 1.2. Sections 5 and 6 treat prescribed regular amplitudes and exact global extension. Sections 7 to 10 develop local density, interpolation, and quantitative degree estimates. Sections 11 to 13 give smooth infinite realization, fixed nonlinear responses, and reproducible computations.

2 Energy coordinates and an explicit center

2.1 Basic geometry and periodic averages

The positive quadrant is invariant. For an admissible profile, every positive solution is bounded forward in time: $x(t)$ is bounded above by $\max\{x(0), x_1\}$, and $z = kx + y$ satisfies

$$\dot z = kx\alpha(x) - my = kx(\alpha(x) + m) - mz \le C - mz$$

for a constant C on this bounded prey interval. The Jacobian at $E_c = (a, c/\beta)$ has trace $a\alpha'(a)$ and determinant mc. In particular, at a growth maximum located at a, its eigenvalues are $\pm i\sqrt{mc}$.

Proposition 2.1 (Identities for periodic coexistence)**.** *For any positive nonconstant periodic orbit of period T, write $\langle u\rangle = T^{-1}\int_0^T u(t)\,dt$. Then*

$$\langle x\rangle = a, \qquad \beta\langle y\rangle = \langle\alpha(x)\rangle, \qquad \log\mu = \int_0^T x(t)\alpha'(x(t))\,dt,$$

where μ is its transverse Floquet multiplier. If α is strictly concave, then

$$0 < \beta\langle y\rangle < \alpha(a).$$

More precisely, if $0 < \kappa \le -\alpha'' \le K$ on the prey projection of the orbit, then

$$\frac{\kappa}{2}\langle (x-a)^2\rangle \le \alpha(a) - \beta\langle y\rangle \le \frac{K}{2}\langle (x-a)^2\rangle.$$

Proof. Integrate the equations for $\log y$ and $\log x$ over one period. In the divergence integral, these two identities cancel all terms except $x\alpha'(x)$, proving the multiplier formula. Strict Jensen inequality gives the strict average-growth bound: x cannot be constant on a nonconstant orbit, since the prey equation would then also force y to be constant. The two quantitative inequalities follow by Taylor's theorem at a and $\langle x - a \rangle = 0$. The prey projection contains a, by the first identity, so both curvature bounds apply throughout each Taylor segment. □

These identities constrain the means and stabilities of all the cycles constructed below, even when their number is arbitrarily large.

2.2 Canonical reduction

The following reduction is a specialization of the generalized Filippov transformation [1, §3]. We include the calculation to fix the time and energy conventions.

Lemma 2.2 (Exact energy-coordinate reduction)**.** *Suppose $\alpha(a) = c > 0$. Define*

$$s(x) = \operatorname{sgn}(x - a)\sqrt{2m\left(\frac{x}{a} - 1 - \log\frac{x}{a}\right)}, \qquad v = \log\frac{\beta y}{c}. \tag{3}$$

Then s is a smooth increasing diffeomorphism from $(0, \infty)$ onto $\mathbb{R}$, with $s(a) = 0$ and $s'(a) = \sqrt{m}/a$. The positive time change

$$\frac{d\tau}{dt} = xs'(x) = \frac{m(x/a - 1)}{s(x)}$$

transforms (1) into

$$s' = f(s) - c(e^v - 1), \qquad v' = s, \qquad f(s) = \alpha(x(s)) - c. \tag{4}$$

The time factor extends at $x = a$ to $\sqrt{m}$.

Proof. The expression inside the square root has a double zero at a with positive quadratic coefficient. The signed square root is therefore smooth there. Differentiation gives $sxs'(x) = m(x/a - 1)$. Both factors s and $x/a - 1$ have the same sign. Finally,

$$\dot{s} = xs'(x)(\alpha(x) - ce^v), \qquad \dot{v} = m(x/a - 1) = xs'(x)s,$$

which proves the transformed equations. Since $s(x) \to -\infty$ as $x \downarrow 0$ and $s(x) \to +\infty$ as $x \to \infty$, the transformation is global on the positive quadrant. □

At a nondegenerate maximum of α,

$$f(0) = f'(0) = 0, \qquad f''(0) = \frac{a^2}{m}\alpha''(a) < 0.$$

Write $f = E + O$ for its even and odd parts.

Lemma 2.3 (Even-part first integral)**.** *For every smooth even E with $E(0) = 0$, the field $s' = E(s) - c(e^v - 1)$, $v' = s$ has a nondegenerate center at the origin. There is a smooth local function H_E with a nondegenerate minimum such that, for the full field (4),*

$$\frac{dH_E}{d\tau} = K_w(s^2/2, v)\, sO(s), \qquad K_w > 0. \tag{5}$$

Proof. Put $w = s^2/2$ and write $E(s) = \mathcal{E}(w)$ for $w \geq 0$. The smooth function $\mathcal{E}$ has a smooth extension across zero. Consider the scalar nonautonomous equation

$$\frac{dw}{dv} = \mathcal{E}(w) - c(e^v - 1).$$

Let $K(w, v)$ be the value at $v = 0$ of the solution through (w, v). Smooth dependence gives $K_w > 0$, $K(w, 0) = w$, and

$$K_v + [\mathcal{E}(w) - c(e^v - 1)]K_w = 0.$$

Set $H_E(s, v) = K(s^2/2, v)$. Its quadratic part is $(s^2 + cv^2)/2$, so its sufficiently small positive levels are ovals. For the even field it is a first integral; the vector field has no other zero near the origin, hence these ovals are periodic. For the full field, substitution into the displayed equation gives (5). □

This is the local form of the paired-energy-branch center and nonoscillation criteria in [1, Theorems A and B]. In particular, if $O(s) = bs^d + o(s^d)$, where d is odd and $b \neq 0$, then $sO(s)$ has one strict sign away from $s = 0$ in a sufficiently small neighborhood. There are no periodic orbits in that neighborhood. For analytic f, the equilibrium is a center if and only if f is even.

For the explicit profile

$$\alpha_{\mathrm{cen}}(x) = c - \frac{1}{2}s(x)^2 = c - m\left(\frac{x}{a} - 1 - \log\frac{x}{a}\right), \tag{6}$$

one has

$$\alpha'_{\mathrm{cen}}(a) = 0, \qquad \alpha''_{\mathrm{cen}}(x) = -\frac{m}{x^2} < 0.$$

Its canonical system has the first integral

$$H_c(s, v) = \frac{1}{2}e^v s^2 + \frac{c}{2}(e^v - 1)^2. \tag{7}$$

Indeed, differentiating (7) along $s' = -c(e^v - 1) - s^2/2$, $v' = s$, gives zero. The profile (6) is used locally; global smooth strictly concave profiles agreeing with it near a are constructed in section 9.

For the explicit moment calculations we set $c = 1$. The general case reduces to this normalization by $s = \sqrt{c}\,u$ and the time change $d\sigma = \sqrt{c}\,d\tau$. All amplitude statements in section 5 use the normalized energy.

3 Global confinement, return maps, and parity

Throughout this section, $k, m, \beta > 0$ are fixed, $a = m/(k\beta)$, and α is an admissible profile with its maximum at a. Write $c = \alpha(a)$, $\alpha_0 = \alpha(0) > 0$, and let $x_1 > a$ be its positive zero. The results below use the single-peak assumptions; global strict concavity is not required.

3.1 A common compact set for all periodic orbits

Lemma 3.1 (Uniform periodic-orbit bounds)**.** *Consider a family of admissible profiles satisfying*

$$\alpha(0) \geq d_0 > 0, \qquad \alpha(a) \leq c_1, \qquad x_1 \leq B.$$

Every positive nonconstant periodic orbit of every member of this family is contained in

$$Q = [\ell, B] \times [y_*, M], \qquad M = \frac{kB(c_1 + m)}{m}, \quad \ell = \frac{a}{2}\exp\left(-\frac{2\beta M}{m}\right), \quad y_* = \frac{d_0}{2\beta}\left(\frac{\ell}{a}\right)^{2m/d_0}. \tag{8}$$

In particular, no sequence of such periodic orbits can approach either coordinate axis or escape to infinity.

Proof. At a largest prey coordinate on a periodic orbit, $\alpha(x) = \beta y > 0$, whence $x < x_1 \le B$. For $z = kx + y$ the equations give

$$\dot{z} = kx(\alpha(x) + m) - mz \le kB(c_1 + m) - mz.$$

At a maximum of z on the orbit this implies $z \le M$, and hence $y \le M$.

The identity $\langle x \rangle = a$ from proposition 2.1 shows that a nonconstant periodic orbit has points on both sides of $x = a$. Every excursion into $x < a/2$ therefore begins at $x = a/2$. During this excursion,

$$\dot{y} \le -\frac{m}{2} y, \qquad \frac{d}{dt} \log x = \alpha(x) - \beta y \ge -\beta y.$$

The integral of y over any part of the excursion is at most $2M/m$. Thus $x \ge \ell$.

Put $y_0 = d_0/(2\beta)$. At a minimum of x, which is less than a, one has $y = \alpha(x)/\beta \ge d_0/\beta = 2y_0$. Consequently, if the orbit reaches $y < y_0$, that excursion starts with a downward crossing of $y = y_0$, at which $x < a$. As long as $y < y_0$ and $x < a$,

$$\dot{x} \ge \frac{d_0}{2} x.$$

The trajectory either exits this region or reaches $x = a$ within time at most $T_0 = 2\log(a/\ell)/d_0$. Since $\dot{y} \ge -my$, its predator coordinate during this interval is at least $y_0 e^{-mT_0} = y_*$. After an upward crossing of $x = a$, the prey coordinate cannot cross back downward while $y < y_0$: at $x = a$ the prey vector field points strictly to the right. Moreover, y increases while $x > a$. This proves the lower predator bound. □

The lemma needs only uniform bounds on three scalar quantities. In particular, it applies both to families retaining the ecological endpoint data exactly and to sufficiently small perturbations of those data. For a fixed profile, one may take $d_0 = \alpha_0$, $c_1 = c$, and $B = x_1$.

Lemma 3.2 (Confinement by a limiting Lyapunov function)**.** *Let $X_n \to X_*$ in C^1 on a neighborhood of a compact set Q, and let $\Gamma_n \subset Q$ be periodic orbits of X_n. Suppose that a smooth function H satisfies $X_* H \le 0$ there. Let $\mathcal{I}$ denote the largest compact invariant subset of $\{X_* H = 0\} \cap Q$.*

If $H \le h_*$ on $\mathcal{I}$, every Hausdorff limit of the Γ_n is contained in $\{H \le h_*\} \cap Q$. In particular, if $\mathcal{I} = \{E\}$ and H has its unique minimum on Q at E, then $\Gamma_n \to \{E\}$ in Hausdorff distance.

Proof. Take a subsequence converging in Hausdorff distance to a nonempty compact set $L \subset Q$. Continuous dependence of solutions, or equivalently compactness of the solutions through points of Γ_n, shows that L is invariant under X_* in both time directions. Choose $z \in L$ maximizing H. For $t \le 0$, monotonicity gives $H(\phi_t(z)) \ge H(z)$, while maximality gives the reverse inequality. The complete backward orbit through z therefore lies in $\{X_* H = 0\} \cap Q$, and H is constant on it. Its alpha-limit is a nonempty compact invariant subset of this zero-dissipation set, hence is contained in $\mathcal{I}$. It follows that $H(z) \le h_*$ and consequently $H \le h_*$ on L. In the stated special case, $H(z) = H(E)$, so uniqueness of the minimum forces $L = \{E\}$. The same argument applies to every convergent subsequence. □

We will also use the following neighborhood formulation. Suppose X_* has a closed center disc $\mathcal{D}$ such that $\mathcal{I} = \mathcal{D}$ and $H \le h_*$ on $\mathcal{D}$. Within an admissible class having a common compact set (8), every open neighborhood of $\{H \le h_*\} \cap Q$ contains all periodic orbits of every sufficiently C^1-close field. Otherwise a sequence of excluded periodic orbits would contradict lemma 3.2. This is the form needed when a local center is extended by a dissipative exterior.

3.2 Persistence and a global transverse section

Lemma 3.3 (Interior persistence and a global return map)**.** *Every positive solution is eventually contained in a compact subset of the positive quadrant. The half-line*

$$\Sigma = \{(a, y) : y > c/\beta\}$$

has a smooth first-return map $P : (c/\beta, \infty) \to (c/\beta, \infty)$ with $P' > 0$. Every positive periodic orbit meets Σ exactly once, and $P(y) < y$ for all sufficiently large y. If α is real analytic on $(0, \infty)$, then P is real analytic on its whole domain.

Proof. We first prove the persistence assertion directly. Eventually $x \le B_* := x_1 + a$: above this level the inequality $\dot{x} \le x\alpha(x)$ forces a downward crossing, and the level cannot be crossed upward. The preceding estimate for $z = kx + y$ then implies that eventually

$$y \le M_* := \frac{2kB_*(c + m)}{m}.$$

Set $\ell_* = (a/2)\exp(-2\beta M_*/m)$. A trajectory cannot remain in $x < a/2$ forever, since y would then decay exponentially, after which $\dot{x} \ge \alpha_0 x/2$ would force an exit. After a time at which $x \ge a/2$ and $y \le M_*$, every subsequent excursion into $x < a/2$ satisfies the same estimate as in lemma 3.1. Thus eventually $x \ge \ell_*$.

Put $y_0 = \alpha_0/(2\beta)$. A trajectory cannot remain in $y < y_0$ forever either. Choose $b > a$ so close to a that $\alpha(b) > 3\alpha_0/4$. While $x \le b$ and $y < y_0$, monotonicity of α on either side of its maximum gives $\dot{x} \ge \alpha_0 x/4$. The trajectory reaches $x = b$, and the vector field prevents a downward crossing of that level while $y < y_0$. Thereafter $\dot{y} \ge k\beta(b - a)y$, forcing an exit from $y < y_0$. Once $x \ge \ell_*$ is established and the trajectory has reached $y \ge y_0$, every subsequent downward excursion below y_0 satisfies the time-to-a estimate in the preceding lemma. Consequently, eventually

$$y \ge y_*^* := \frac{\alpha_0}{2\beta}\left(\frac{\ell_*}{a}\right)^{2m/\alpha_0} > 0.$$

This proves interior persistence without an assumption on the stability of the coexistence equilibrium.

Start now at a point of Σ. Since $\dot{x} < 0$ there, the trajectory enters $x < a$. If it never crossed $x = a$ again, then y would decrease to a positive limit. Its nonempty compact omega-limit set would lie on a horizontal line. Invariance forces $\dot{y} = 0$ on that set, hence $x = a$, and then forces the set to be the coexistence equilibrium $E_c = (a, c/\beta)$. This is impossible for a trajectory remaining in one half-plane. Indeed, in the canonical coordinates (4), with $p = s$ and $q = \sqrt{c}\, v$, the polar angle satisfies

$$\frac{d}{d\tau}\arg(p + iq) = \sqrt{c} + O((p^2 + q^2)^{1/2}).$$

It is bounded away from zero near E_c, so a trajectory tending to E_c cannot remain on one side of $x = a$.

The first subsequent crossing of $x = a$ is therefore finite and has $y < c/\beta$; equality would give the equilibrium, which a nonconstant trajectory cannot reach in finite time. The trajectory then enters $x > a$. The same argument, now with y increasing, gives a further finite crossing, necessarily on Σ. All these crossings are transverse. Smooth dependence of the flow and the implicit function theorem give a smooth first-return map. Its derivative is positive; explicitly, the standard variational formula expresses it as a positive exponential multiplied by the ratio

of the two negative transverse velocities. Analytic dependence gives the analytic assertion when α is analytic.

Every positive periodic orbit surrounds E_c, by the planar index formula, since there is no other equilibrium in the positive quadrant. Along Σ every crossing points to the left. The intersections of an oriented Jordan curve with a ray transverse to it alternate between entry and exit as one moves along the ray. The ray from E_c to infinity begins inside the curve and ends outside it. Since all crossing orientations here agree, there is exactly one intersection. Thus periodic orbits correspond precisely to fixed points of P.

Finally, a trajectory starting at (a, y) satisfies $x(t) < x_1$ for $t > 0$. With $M = kx_1(c+m)/m$, scalar comparison gives

$$z(t) \le M + (z(0) - M)e^{-mt} < z(0) \qquad \text{if } z(0) = ka + y > M.$$

At the return to Σ the prey coordinate is again a, so $P(y) < y$ for every such initial value. □

3.3 Parity and the analytic category

Proposition 3.4 (Parity of a finite hyperbolic cycle configuration)**.** *Suppose the coexistence equilibrium is a focus and all positive periodic orbits form a finite family of N hyperbolic cycles. When $N > 0$, the cycles are nested, the outermost cycle is attracting, and their stabilities alternate. Moreover,*

$$E_c \text{ stable} \implies N \text{ even}, \qquad E_c \text{ unstable} \implies N \text{ odd}.$$

Here $N = 0$ is allowed in the stable case. If the first nonzero canonical odd coefficient is $O(s) = bs^d + o(s^d)$, with d odd, then $b < 0$ gives the stable case and $b > 0$ the unstable case. A prescribed jet determining this coefficient therefore determines the parity of any such global configuration.

Proof. All cycles surround the unique equilibrium, so disjointness of trajectories makes them nested. By lemma 3.3, $P(y) - y$ has precisely N simple zeros and is negative for large y. Its sign alternates at the zeros. The outermost zero is therefore attracting, and the stabilities alternate inward. Near c/β the sign is negative for a stable focus and positive for an unstable focus, proving the parity assertion. The relation between b and stability follows from lemma 2.3: $\dot{H}_E$ has the sign of b away from $s = 0$, and the only invariant subset of the local zero-dissipation line is the equilibrium. □

The hypothesis concerns *all positive periodic orbits*. For a smooth growth rate, finiteness of the isolated limit cycles alone does not exclude a separate period annulus. Such an annulus would contribute an interval of fixed points to P and invalidate the simple-zero sign count. The constructions below exclude all additional periodic orbits, so this distinction causes no ambiguity.

Proposition 3.5 (No analytic center and finiteness of analytic cycles)**.** *If an admissible profile is real analytic on $(0, \infty)$, its coexistence equilibrium is a weak focus of finite order. It has only finitely many positive periodic orbits, each isolated. In particular, a globally strictly concave admissible polynomial cannot possess a period annulus or infinitely many positive limit cycles. If all its cycles are hyperbolic, their number satisfies proposition 3.4.*

Proof. The signed energy coordinate is a real analytic diffeomorphism from $(0, \infty)$ to $\mathbb{R}$, including across $x = a$. Hence $f(s) = \alpha(x(s)) - c$ is real analytic on $\mathbb{R}$. If the equilibrium were a center, the analytic center criterion following lemma 2.3 would make f even near zero. The

identity theorem would then make it even on all of $\mathbb{R}$. As $s \to -\infty$, one has $x(s) \to 0$, and therefore $f(s) \to \alpha_0 - c$. Evenness would imply $\alpha(x) \to \alpha_0 > 0$ as $x \to \infty$, contradicting strict decrease beyond the positive zero x_1.

Thus the analytic odd part has a first nonzero term of some finite odd degree $d \geq 3$. By lemma 2.3, the small oval levels of the even-part first integral are one-sided barriers. A periodic orbit can neither remain inside such an oval nor enter and leave it. Consequently a neighborhood of the equilibrium is met by no positive periodic orbit, and the equilibrium is a stable or unstable weak focus. The global return map in lemma 3.3 is analytic and is not the identity, since $P(y) < y$ for large y. Its fixed points lie in a compact subinterval of $(c/\beta, \infty)$: the upper bound follows from lemma 3.1, and the lower separation from c/β follows from the local absence of periodic orbits. An analytic function that is not identically zero has finitely many zeros in such a compact interval. Every fixed point is therefore isolated, and these are all the positive periodic orbits. □

3.4 Stability of an exact global cycle count

Proposition 3.6 (Exact-count stability under constrained perturbations)**.** *Let α_* be an admissible profile whose canonical odd part has a first nonzero term*

$$O_*(s) = b_* s^d + O(s^{d+2}), \qquad b_* \neq 0,$$

of finite odd degree $d \geq 3$. Suppose that its positive periodic orbits consist of exactly N hyperbolic cycles, where $N \geq 0$. For every $R > x_{1,*}$, there exists $\epsilon > 0$ such that every admissible profile $\widetilde{\alpha}$ satisfying

$$j_a^{d-1}\widetilde{\alpha} = j_a^{d-1}\alpha_*, \qquad \|\widetilde{\alpha} - \alpha_*\|_{C^d([0,R])} < \epsilon$$

also has exactly N positive periodic orbits, all hyperbolic. The endpoint values need not be retained exactly.

Proof. The fixed jet retains $c = \alpha_*(a)$ and hence the same coexistence equilibrium and section Σ. For sufficiently small ϵ, one has $\widetilde{\alpha}(0) \geq \alpha_*(0)/2$ and $\widetilde{\alpha}(R) < 0$. Admissibility then gives $\widetilde{x}_1 < R$. Thus lemma 3.1, with $d_0 = \alpha_*(0)/2$, $c_1 = c$, and $B = R$, provides a common compact rectangle for every periodic orbit under consideration.

Composition with the fixed analytic coordinate $x = x(s)$ preserves C^d closeness on a small interval. The canonical growth terms have identical jets through degree $d-1$, so Taylor's theorem gives

$$|\widetilde{O}(s) - O_*(s)| \leq \frac{\|\widetilde{O} - O_*\|_{C^d}}{d!}|s|^d.$$

Consequently, on a fixed sufficiently small interval, $s\widetilde{O}(s)$ has the sign of b_* whenever $s \neq 0$. By lemma 2.3, the corresponding even-part first integrals have oval levels on a common neighborhood, uniformly comparable to $(s^2 + cv^2)/2$. This uniformity follows from continuous dependence of the defining scalar equations and the uniform C^2 bounds on the even parts. Their derivatives along the full fields have the fixed sign of $s\widetilde{O}(s)$. These ovals are one-sided barriers: a periodic orbit can neither remain inside one nor enter and leave it. Hence every nearby profile has a common neighborhood of the equilibrium met by no positive periodic orbit.

The inner exclusion and the common compact rectangle put all periodic-orbit intersections with Σ in a fixed compact interval $I \subset (c/\beta, \infty)$. The target return map is defined on I by lemma 3.3; its return times are bounded and all crossings are transverse. The relevant trajectories stay below their positive prey zeros, hence in $x < R$. Smooth dependence therefore makes the nearby return maps C^1-close on I. The target displacement has precisely N simple

zeros there. Each persists uniquely in a small interval, while on the compact complement the displacement is bounded away from zero. If $N = 0$, this last observation applies to all of I. Thus the complete count remains N, and every surviving cycle remains hyperbolic. □

4 Exactly prescribed global counts at an explicit degree

This section gives a polynomial construction for which all positive periodic orbits can be counted. The construction combines a Taylor polynomial of even degree for a center profile with odd terms at a common shrinking scale. For the limiting field, a Lyapunov function decreases strictly off a line whose only invariant subset is the equilibrium. This limit and the two extreme amplitude regimes exclude every periodic orbit outside the prescribed scale.

4.1 A family of centers and a concave Taylor polynomial

Write $u = (x - a)/a$ and retain the energy coordinate s in (3). For $\theta > 0$, define

$$C_\theta(x) = c - \theta m(u - \log(1 + u)) = c - \frac{\theta}{2}s(x)^2, \qquad x > 0. \tag{9}$$

This local growth law is strictly concave, with $C''_\theta(x) = -\theta m/x^2$. The associated canonical field has the first integral

$$H_\theta(s, v) = \frac{1}{2}e^{\theta v}s^2 + c\left[\frac{e^{(\theta+1)v} - 1}{\theta + 1} - \frac{e^{\theta v} - 1}{\theta}\right]. \tag{10}$$

The second term has derivative $ce^{\theta v}(e^v - 1)$ and its unique minimum is zero at $v = 0$. Thus $H_\theta \geq 0$, with equality only at $(0, 0)$. For a general growth profile,

$$H'_\theta = e^{\theta v}s[\alpha(x(s)) - C_\theta(x(s))]. \tag{11}$$

These statements follow by direct differentiation. When $\theta = 1$, (10) agrees with (7).

Fix $N \geq 1$ and $K \geq 2$, and put

$$q = \lceil K/2 \rceil, \qquad d = 2q + 1, \qquad D = d + 2N + 1 = 2(N + q + 1). \tag{12}$$

In particular, D is even, $d > K$, and $D - 2 > K$. Let $\mathsf{H}_D = \sum_{n=1}^{D} 1/n$, and assume

$$c > \theta m(\mathsf{H}_D - 1). \tag{13}$$

The degree-D Taylor polynomial of C_θ at a is

$$A_D(x) = c + \theta m \sum_{n=2}^{D} \frac{(-1)^{n+1}}{n}u^n. \tag{14}$$

It satisfies $A_D(a) = c$, $A'_D(a) = 0$, and $A_D(0) = c - \theta m(\mathsf{H}_D - 1) > 0$.

Lemma 4.1 (Global concavity and a signed Taylor remainder)**.** *The polynomial A_D is strictly concave on the entire real line. For $u > -1$,*

$$A_D(x) - C_\theta(x) = -\theta m \int_0^u \frac{t^D}{1 + t}\, dt. \tag{15}$$

The difference in (15) has sign opposite to u whenever $u \neq 0$.

Proof. Differentiation gives

$$-A_D''(x) = \frac{\theta m}{a^2} S_D(u), \qquad S_D(u) = \sum_{j=0}^{D-2} (j+1)(-u)^j.$$

For $u < 0$ every summand is nonnegative and the constant term is positive. For $u \geq 0$, the finite geometric-sum identity gives

$$S_D(u) = \frac{1 + Du^{D-1} + (D-1)u^D}{(1+u)^2} > 0. \tag{16}$$

The derivative of $A_D - C_\theta$ with respect to u is $-\theta m u^D/(1+u)$. Its value at zero is zero, proving (15). Since D is even and $1 + t > 0$ on the integration segment, the integral has the sign of u. □

4.2 Moment interpolation and the polynomial formula

For an analytic germ G at a, denote its Taylor polynomial of order L by $T_{a,L}G$. Define

$$Q_j = T_{a,D-1}\big[s(x)^{d+2j}\big], \qquad 0 \leq j \leq N. \tag{17}$$

The polynomial Q_j has degree at most $D-1$ and vanishes at a to order $d+2j$. In particular,

$$Q_N = m^{(D-1)/2} u^{D-1}. \tag{18}$$

Let $0 < h_1 < \cdots < h_N$ be any prescribed scaled energies. For the harmonic oscillator $S' = -cV$, $V' = S$, direct parametrization gives

$$\oint_{(S^2+cV^2)/2=h} S^{2j+1}\, dV = C_j h^{j+1}, \qquad C_j = \frac{\pi}{\sqrt{c}\, 2^j} \binom{2j+2}{j+1}. \tag{19}$$

Choose $b_0, \ldots, b_N$ by the polynomial identity

$$M(h) := \sum_{j=0}^{N} b_j C_{q+j} h^{q+j+1} = -C_{q+N} h^{q+1} \prod_{i=1}^{N} (h - h_i). \tag{20}$$

This gives explicit coefficients in terms of the elementary symmetric functions of the h_i. The common factor $1/\sqrt{c}$ cancels, so the b_j do not depend on c. Moreover,

$$b_N = -1, \qquad \operatorname{sgn} b_0 = (-1)^{N+1}. \tag{21}$$

For a fixed small $\delta > 0$, set

$$P_\rho = A_D + \delta \sum_{j=0}^{N} b_j \rho^{2(N-j)} Q_j, \qquad 0 \leq \rho \leq 1. \tag{22}$$

Its coefficientwise limit is

$$P_0 = A_D - \delta Q_N. \tag{23}$$

To choose δ it suffices to compute the finite constants

$$L_0 = \sum_{j=0}^{N} |b_j|\, |Q_j(0)|, \qquad L_2 = \sum_{j=0}^{N} |b_j| \sup_{x \geq 0} \frac{|Q_j''(x)|}{-A_D''(x)},$$

and impose

$$\delta L_0 < \frac{c - \theta m(\mathsf{H}_D - 1)}{2}, \qquad \delta L_2 < \frac{1}{2}. \tag{24}$$

The suprema are finite: their denominators are positive, and their numerators have smaller degree. They can also be bounded by coefficient estimates or by the finite critical sets of rational functions. It follows that, uniformly for $0 \le \rho \le 1$,

$$P_\rho(0) > 0, \qquad -P_\rho'' \ge -\tfrac{1}{2}A_D'' > 0.$$

Every P_ρ has degree D, maximum c at a, and a unique positive zero to the right of a; its K-jet equals that of C_θ.

4.3 Fixing the endpoint data at the same degree

Let $b > a$ be the positive zero of P_0, put $z = (b-a)/a > 0$, and write $E_\rho = P_\rho - P_0$. All coefficients of E_ρ are $O(\rho^2)$. Define the corrected family

$$\widehat{P}_\rho = P_\rho + t_0(\rho)u^{D-2} + t_1(\rho)u^{D-1}, \tag{25}$$

where

$$\begin{pmatrix} 1 & -1 \\ z^{D-2} & z^{D-1} \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \end{pmatrix} = -\begin{pmatrix} E_\rho(0) \\ E_\rho(b) \end{pmatrix}. \tag{26}$$

The determinant is $z^{D-2}(1+z) > 0$, and the solution is

$$t_1 = \frac{E_\rho(0) - E_\rho(b)/z^{D-2}}{1+z}, \qquad t_0 = -\frac{zE_\rho(0) + E_\rho(b)/z^{D-2}}{1+z}. \tag{27}$$

Thus $t_0, t_1 = O(\rho^2)$, and, exactly,

$$\widehat{P}_\rho(0) = P_0(0), \qquad \widehat{P}_\rho(b) = 0. \tag{28}$$

The correction has degree at most $D-1$ and vanishes to order at least $D-2 > K$ at a. Hence it preserves both the degree and the required jet. Each change from P_0 has coefficients $O(\rho^2)$ and degree at most $D-1$. Its second derivative, divided by the strictly positive polynomial $-P_0''$ of degree $D-2$, is uniformly small on $[0,\infty)$. Consequently $\widehat{P}_\rho'' < 0$ globally for all sufficiently small ρ. The fixed values in (28) are the data of the chosen limiting polynomial; they need not be specified independently in advance.

4.4 A uniform return-map lemma

The following elementary form of first-order averaging makes the amplitude estimates precise. The even terms are retained in an exact center before the odd perturbation is varied.

Lemma 4.2 (Return maps near varying reversible centers)**.** *Consider fields*

$$S' = B_\nu(S) - g_\nu(V) + \epsilon\Phi_\nu(S), \qquad V' = S,$$

where B_ν is even, $B_\nu(0) = g_\nu(0) = 0$, $B_\nu \to 0$, $g_\nu \to cV$, and $\Phi_\nu \to \Phi$ in C^4 on a fixed neighborhood of the compact annulus in question. For ν sufficiently small, the even field has a period annulus containing any fixed compact harmonic annulus under consideration. On the section $V = 0$, $S > 0$, use $h = S^2/2$. Its return displacement obeys

$$\frac{\Delta(\nu, \epsilon, h)}{\epsilon} \longrightarrow \oint_{(S^2+cV^2)/2=h} \Phi(S)\, dV \tag{29}$$

in C^1 on compact positive energy intervals as $(\nu, \epsilon) \to 0$. The assertion also holds along any sequence of the indicated fields and amplitudes.

Proof. The scalar-flow construction in lemma 2.3, with $g_\nu(V)$ replacing $c(e^V-1)$, yields an exact first integral H_ν normalized by $H_\nu(S,0)=S^2/2$. On the compact sets in question it converges to $(S^2+cV^2)/2$. Alternatively, solutions close to the harmonic ovals cross the fixed line $S=0$ twice, and reversibility under $(S,V)\mapsto(-S,V)$ closes each orbit. Continuous dependence and transversality give uniform bounds on return times and their derivatives. The displacement is identically zero at $\epsilon=0$. Its first derivative there is obtained by integrating $K_w S\Phi_\nu(S)$ along an unperturbed oval, with $K_w\to 1$. This converges in C^1 to the right-hand side of (29). Taylor's theorem in the independent parameter ϵ gives a uniform $O(\epsilon^2)$ remainder. The same estimates apply to sequences; no smooth joint parametrization of competing amplitude ratios is required. □

4.5 Exclusion at all amplitude scales

Proof of theorem 1.2. The construction (25) already proves the degree, jet, concavity, and endpoint assertions. It remains to count all its positive periodic orbits.

First, (15) and (18) imply

$$s\big[P_0(x(s))-C_\theta(x(s))\big]<0 \qquad (s\neq 0).$$

Thus the derivative of (10) along the limiting field is strictly negative for $s\neq 0$. The only invariant subset of that line is the equilibrium: there $s'=-c(e^v-1)$ and $v'=0$. The endpoint values, maximum, and positive zero of $\widehat{P}_\rho$ are fixed, so lemma 3.1 confines every periodic orbit to one compact subset of the positive quadrant. Since $\widehat{P}_\rho\to P_0$ smoothly there, lemma 3.2 shows that all possible periodic orbits converge to E_c as $\rho\downarrow 0$.

Let O_ρ be the odd part of $\widehat{P}_\rho(x(s))-c$. On one fixed neighborhood of zero it has the uniform analytic factorization

$$\begin{aligned} O_\rho(s) = \delta\sum_{j=0}^{N} b_j\rho^{2(N-j)}s^{d+2j} \\ +\rho^2 s^{D-1}\mathcal{A}(s^2,\rho)+s^{D+1}\mathcal{B}(s^2,\rho), \end{aligned} \tag{30}$$

where $\mathcal{A},\mathcal{B}$ and all fixed finite derivatives are bounded for small s,ρ. Indeed, the difference between $Q_j(x(s))$ and s^{d+2j} starts at degree D. Since D is even, its odd part starts at degree $D+1$. The difference A_D-C_θ also starts at degree $D+1$. Finally, $u(s)=s/\sqrt{m}+O(s^2)$ and the correction coefficients are $O(\rho^2)$, so the odd part of the endpoint correction starts at $\rho^2 s^{D-1}$.

Every positive periodic orbit encloses the unique equilibrium and meets the ray $v=0$, $s>0$ exactly once: the Jordan-ray argument of lemma 3.3 applies since $v'=s>0$ on this ray. Let r be that intersection coordinate. The preceding confinement gives $r\to 0$. We exclude the two extreme ratios of r and ρ.

Large relative amplitudes. Suppose a sequence of cycles has $r/\rho\to\infty$. Set $s=rS$, $v=rV$; the initial point becomes $(1,0)$. Separate the even part of the scaled field and use the independent odd amplitude

$$\epsilon=\delta r^{D-2}.$$

By (30), its normalized odd force converges smoothly on compact sets to $b_N S^{D-1}$. The lower monomials have factors $(\rho/r)^{2(N-j)}$; the two remainder terms are bounded by constants times ρ^2/δ and r^2/δ, respectively. The even field converges to the harmonic oscillator. Applying lemma 4.2 at energy $1/2$, the displacement divided by ϵ tends to

$$b_N C_{q+N}(1/2)^{q+N+1}\neq 0.$$

This contradicts periodicity.

Small relative amplitudes. Suppose instead that $r/\rho \to 0$. Use the same spatial scaling but take

$$\epsilon = \delta \rho^{2N} r^{d-1}.$$

The normalized odd force now converges to $b_0 S^d$. The higher monomials carry factors $(r/\rho)^{2j}$, while the two errors are bounded by

$$C\frac{\rho^2}{\delta}(r/\rho)^{2N}, \qquad C\frac{r^2}{\delta}(r/\rho)^{2N}.$$

They tend to zero. The normalized displacement tends to $b_0 C_q (1/2)^{q+1} \neq 0$, again a contradiction.

Consequently there are constants $0 < A < B < \infty$ such that every cycle for every sufficiently small ρ satisfies

$$A\rho \leq r \leq B\rho. \tag{31}$$

For otherwise one could select a sequence contradicting one of the two exclusions. The constants may be enlarged to include all $\sqrt{2h_i}$ in their interior.

The prescribed scale. Scale $s = \rho S$, $v = \rho V$ and use $\epsilon = \delta \rho^{D-2}$. The even field converges smoothly to the harmonic oscillator, and the normalized odd force converges to $\sum b_j S^{d+2j}$. Its return displacement satisfies

$$\Delta(\rho, h) = \delta \rho^{D-2}[M(h) + O(\rho)] \quad \text{in } C^1([A^2/2, B^2/2]). \tag{32}$$

The $O(\rho)$ estimate follows from analytic dependence of the even field and odd remainders on ρ, with δ fixed. The polynomial M has exactly the N simple positive zeros h_i. In disjoint small intervals about them its derivative has fixed nonzero sign; on the complementary compact set its absolute value has a positive lower bound. Equation (32) therefore has exactly one simple zero in each of these intervals and none elsewhere. By (31), these exhaust all positive periodic orbits.

The zeros satisfy $h_i(\rho) = h_i + O(\rho)$, hence

$$\begin{aligned} s_i(\rho) &= \rho\sqrt{2h_i} + O(\rho^2), \\ \mu_i(\rho) &= 1 + \delta \rho^{D-2} M'(h_i) + O(\rho^{D-1}). \end{aligned} \tag{33}$$

Here μ_i is the nontrivial Floquet multiplier; coordinate and positive time changes preserve it. From (20),

$$\operatorname{sgn} M'(h_i) = (-1)^{N-i+1}.$$

Thus the outermost cycle is attracting and the stabilities alternate. The first nonzero odd coefficient at the equilibrium is $\delta b_0 \rho^{2N} s^d$. Its sign is $(-1)^{N+1}$, so the equilibrium is a stable weak focus for even N and an unstable weak focus for odd N, as also required by proposition 3.4. □

4.6 A degree-six example

Take $a = m = 1$, $c = 3$, $\theta = 1$, $N = 1$, $K = 2$, and $h_1 = 1$. Then $D = 6$, $d = 3$, $b_0 = 5/3$, and $b_1 = -1$. With $u = x - 1$,

$$\begin{aligned} A_6 &= 3 - \tfrac{1}{2}u^2 + \tfrac{1}{3}u^3 - \tfrac{1}{4}u^4 + \tfrac{1}{5}u^5 - \tfrac{1}{6}u^6, \\ Q_0 &= u^3 - u^4 + \tfrac{11}{12}u^5, \qquad Q_1 = u^5. \end{aligned}$$

Table 1: Degrees supplied by the explicit construction. Each row retains the indicated center jet and gives exactly N positive periodic orbits, subject to (13).

Number N	Jet order K	First free odd degree d	Degree D
1	2	3	6
2	2	3	8
3	2	3	10
4	2	3	12
3	4	5	12
3	5	7	14

Choose $\delta > 0$ by (24), and let $b = 1 + z > 1$ be the positive zero of $A_6 - \delta u^5$. In this case (27) reduces to

$$t_0 = \frac{5\delta\rho^2(2z-1)}{3z}, \qquad t_1 = -\frac{5\delta\rho^2(11z+12)}{36z}.$$

The corrected polynomial has the particularly simple form

$$\widehat{P}_\rho = A_6 - \delta u^5 + \frac{5\delta}{3}\rho^2 u^3(1+u)(1-u/z).$$

It is globally strictly concave for small ρ, has $\widehat{P}_\rho(0) = 31/20 + \delta$, positive zero b, and the prescribed quadratic center jet. Its system has exactly one positive periodic orbit, an attracting hyperbolic cycle with $s(\rho) = \sqrt{2}\rho + O(\rho^2)$.

For a fixed c, θ, condition (13) is a sufficient condition for this Taylor construction at the stated degree. For any fixed $c > 0$ and each pair (N, K), a sufficiently small $\theta > 0$ satisfies it. This latter choice changes the preserved center curvature. The degree estimate is an existence upper bound for this specified jet; section 10 quantifies the additional costs of other interpolation and approximation data.

5 Prescribed oscillations and constrained Abelian integrals

Consider the local family

$$s' = -(e^v - 1) - \tfrac{1}{2}s^2 + F(s), \qquad v' = s, \qquad F(s) = \sum_{j=0}^{N} \lambda_j s^{2r+2j+3}, \tag{34}$$

where $r, N \geq 0$ are integers. Put $q = e^v - 1$ and $Y = q + s^2/2$. Then the equations are exactly

$$s' = -Y + F(s), \qquad Y' = s - \tfrac{1}{2}s^3 + sF(s). \tag{35}$$

The unperturbed Hamiltonian is

$$K(s, Y) = \frac{Y^2}{2} + \frac{s^2}{2} - \frac{s^4}{8}.$$

Its central period annulus is $0 < K < 1/2$. This is the truncated-pendulum Hamiltonian. The original positive-predator condition is $q = Y - s^2/2 > -1$, which holds throughout this annulus. Indeed, on the oval $K = h$ one has $q \in [-\sqrt{2h}, \sqrt{2h}]$.

For $j \geq 0$ define

$$J_j(h) = \oint_{K=h} s^{2j+3}\, dY = 2\int_{-\sqrt{2h}}^{\sqrt{2h}} \left(\frac{2h - q^2}{1+q}\right)^{j+3/2} dq, \qquad 0 < h < \frac{1}{2}.$$

The oval is oriented by the unperturbed flow, so $J_j > 0$.

For completeness, the moments also admit the convergent representation

$$J_j(h) = \pi c_j h^{j+2} {}_2F_1\left(\frac{j}{2}+\frac{3}{4}, \frac{j}{2}+\frac{5}{4}; j+3; 2h\right), \qquad c_j = 2^{-j-1}\binom{2j+4}{j+2}. \tag{36}$$

Indeed, put $q = \sqrt{2h}\,t$ in the integral. Expanding $(1+\sqrt{2h}\,t)^{-j-3/2}$ and integrating term by term removes the odd terms; the beta integral gives the displayed hypergeometric series. Consequently $J_j(h) = \pi c_j h^{j+2}(1+O(h))$, with analytic remainders.

Along the perturbed field, $K' = s(1+q)F(s)$. The first return displacement is therefore $\oint F(s)\,dq = \oint F(s)\,dY$ to first order, since the difference is the integral of the exact form $sF(s)\,ds$. We use the usual Abelian-integral return-map principle, as in [2].

Theorem 5.1 (A constrained consequence of the Gavrilov–Iliev theorem)**.** *For every pair of integers $r, N \geq 0$, the ordered family $(J_r, J_{r+1}, \ldots, J_{r+N})$ is an extended complete Chebyshev system on $(0, 1/2)$. More strongly, a nonzero real linear combination of these functions has at most N zeros, counted with multiplicity, in $\mathbb{C} \setminus ([1/2, \infty) \cup \{0\})$.*

Proof. Set $s = 2x$, $Y = 2y$, and $e = h/2$. The rescaled Hamiltonian is $H(x,y) = y^2 + x^2 - x^4$. For its central ovals, define

$$B_k(e) = -\oint_{H=e} x^{2k} y\,dx,$$

using the counterclockwise orientation. Thus $B_k > 0$ on $(0, 1/4)$ and

$$J_j(h) = (2j+3)2^{2j+4} B_{j+1}(h/2). \tag{37}$$

Integration of $d(x^{2k+1}y^3)$ along an oval yields the recurrence

$$(2k+7)B_{k+2} = (2k+4)B_{k+1} - (2k+1)eB_k, \qquad k \geq 0. \tag{38}$$

Moreover, direct differentiation gives the Picard–Fuchs system

$$\begin{pmatrix} B_0 \\ B_1 \end{pmatrix} = A(e)\begin{pmatrix} B_0' \\ B_1' \end{pmatrix}, \qquad A(e) = \begin{pmatrix} 4e/3 & -2/3 \\ 2e/15 & 4e/5 - 4/15 \end{pmatrix}. \tag{39}$$

Here

$$\det A(e) = \frac{4e(4e-1)}{15}, \qquad \operatorname{tr} A(e) = (\det A(e))',$$

and the eigenvalues of A' are $4/3$ and $4/5$.

We apply Theorem 1 and Proposition 4 of Gavrilov and Iliev [3]. Their parameters are $\lambda = 3/4$, $\mu = 5/4$, and $\lambda^* = 3/4$. For $\sigma \geq 3/4$, let

$$\mathcal{V}_\sigma = \{I(e) = P(e)B_0(e) + Q(e)B_1(e) :\ P, Q \in \mathbb{R}[e], \\ I(e) = O(|e|^\sigma) \text{ sectorially at infinity}\}.$$

Their theorem gives at most $\dim \mathcal{V}_\sigma$ zeros, counted with multiplicity, in $D = \mathbb{C} \setminus [1/4, \infty)$ for every nonzero element of this space. Their dimension formula is

$$\dim \mathcal{V}_\sigma = \lfloor \sigma - 3/4 \rfloor + \lfloor \sigma - 5/4 \rfloor + 2.$$

All hypotheses are displayed in (39): there are two distinct finite singular values $0, 1/4$, and the oval integrals are analytic at 0.

For $L \geq 0$, put $\sigma_L = (2L+3)/4$. The Fuchsian exponents at infinity give $B_0 = O(|e|^{3/4})$ and $B_1 = O(|e|^{5/4})$. Recurrence (38) therefore implies $B_k = O(|e|^{(2k+3)/4})$. Consequently

$$\operatorname{span}(B_0, \dots, B_L) \subset \mathcal{V}_{\sigma_L}.$$

Since $B_k(e) = b_k e^{k+1}(1 + O(e))$ with $b_k > 0$, these $L+1$ functions are linearly independent. The displayed dimension formula gives $\dim \mathcal{V}_{\sigma_L} = L+1$, so the inclusion is equality.

Now take $L = r + N + 1$. A linear combination of $B_{r+1}, \dots, B_{r+N+1}$ is an element of $\mathcal{V}_{\sigma_L}$ and vanishes to order at least $r+2$ at $e = 0$. The complex zero bound is $L + 1 = r + N + 2$, leaving at most N zeros away from the origin. Rescaling by (37) proves the stronger assertion. Applying the same argument to every prefix proves the ECT property. □

Elementary local alternative. The global theorem uses the cited complex nonoscillation result. A fully elementary local statement follows from

$$J_j(h) = \pi c_j h^{j+2}(1 + O(h)), \qquad c_j = 2^{-j-1}\binom{2j+4}{j+2}.$$

For $0 \leq \ell \leq N$,

$$W(J_r, \dots, J_{r+\ell})(h) = \left(\prod_{j=0}^{\ell} \pi c_{r+j}\right)\left(\prod_{j=1}^{\ell} j!\right) h^{(\ell+1)(r+2)}(1 + O(h)).$$

Thus all Wronskians are positive on a sufficiently short positive interval. This alternative is useful when only local analyticity of a more general restoring force is available.

Corollary 5.2 (Interpolation of oscillation amplitudes)**.** *Fix $r, N \geq 0$ and distinct energies $0 < h_1 < \cdots < h_N < 1/2$. There is a unique, up to a nonzero common factor, polynomial $P(z) = \sum_{j=0}^{N} b_j z^j$ such that*

$$M(h) = \sum_{j=0}^{N} b_j J_{r+j}(h)$$

vanishes at all the h_i. These are exactly its zeros on $(0, 1/2)$, and all are simple. One may replace the N simple-zero conditions by Hermite conditions at finitely many distinct energies with total multiplicity N; the conclusion then holds with the prescribed multiplicities.

Proof. The homogeneous interpolation system has a nonzero solution. The ECT property bounds its total number of zeros by N, which proves simplicity in the first case and exact multiplicity in the second. The interpolation matrix has rank N: otherwise the first N basis functions would admit a nonzero combination satisfying N zero conditions, contradicting their ECT property. Hence its kernel is one-dimensional. The last coefficient b_N is nonzero for the same reason. □

Theorem 5.3 (Sharp compact-annulus cyclicity)**.** *For each $r, N \geq 0$, the $(N+1)$-parameter family (34) has cyclicity exactly N on every nondegenerate compact subannulus $h_- \leq K \leq h_+$ with $0 < h_- < h_+ < 1/2$. Here cyclicity counts cycles bifurcating from the regular ovals of that compact subannulus as $\lambda \to 0$. The N limiting energies may be prescribed arbitrarily in (h_-, h_+). The resulting hyperbolic cycles have alternating stability, with either choice of the first stability type.*

Proof. The return displacement on a fixed transversal is analytic in energy and parameters and has the uniform expansion

$$\Delta(h,\lambda)=\sum_{j=0}^{N}\lambda_j J_{r+j}(h)+O(\|\lambda\|^2)$$

in every finite differentiability norm on a slightly larger compact energy interval. If an upper bound N failed for parameters tending to zero, normalize by $\|\lambda\|$ and take a convergent subsequence of directions on the unit sphere. The normalized displacements converge to a nonzero combination of the J_{r+j}. Generalized Rolle's theorem (or the local analytic zero theorem) shows that $N+1$ converging zeros, with coalescing zeros counted with their limiting multiplicities, would give at least $N+1$ zeros for this limiting combination. Theorem 5.1 excludes this. For the lower bound, use Corollary 5.2 and take $\lambda=\varepsilon b$. Each simple Melnikov zero persists as a hyperbolic fixed point. Derivatives of a real function at successive simple zeros alternate in sign; replacing b by $-b$ reverses all stabilities. □

Exact transversal placement. The actual, perturbed section coordinates can also be fixed exactly. Normalize the interpolating vector by $b_N=1$ and put $\lambda_N=\varepsilon$, $\lambda_j=\varepsilon b_j(\varepsilon)$ for $j<N$. The equations $\Delta(h_i,\lambda)/\varepsilon=0$ have at $\varepsilon=0$ the invertible Jacobian $(J_{r+j}(h_i))_{i=1,\dots,N;,j=0,\dots,N-1}$. The implicit function theorem gives analytic coefficients $b_j(\varepsilon)$ for which each h_i is an exact fixed point of the return map. The same argument with the Hermite evaluation matrix fixes prescribed multiple cycles and their exact multiplicities: the first derivative not constrained to vanish tends, after division by ε, to the corresponding nonzero Melnikov derivative. These claims concern transversal coordinates; the perturbed closed curves themselves are naturally deformations of the prescribed unperturbed ovals.

Finite-jet preservation. In the original prey coordinate, the perturbation is

$$\sum_{j=0}^{N}\lambda_j s(x)^{2r+2j+3}.$$

Since $s(a)=0$ and $s'(a)>0$, all derivatives of the prey growth function through order $2r+2$ at a agree exactly with those of the fixed center profile $1-s(x)^2/2$. In particular its value and negative second derivative are fixed independently of the number of cycles. Taking r large preserves any prescribed finite jet of this center profile while realizing arbitrarily many cycles at arbitrary energies in the period annulus. The prescribed jet in this statement belongs to the fixed center profile. Section 7 treats arbitrary nondegenerate maximum profiles by localized smooth perturbations.

Remark 5.4. The cyclicity bound concerns regular ovals on a compact subannulus in the specified finite-dimensional family. The boundary two-saddle polycycle and the dynamics outside the globally extended local model are separate bifurcation problems. Prescribed positions throughout the whole annulus are a statement about the local canonical family. A simultaneous globally concave realization with $\alpha(0)>0$ uses sufficiently small ovals. For example, the left tangent intercept of the center profile at $\ell<a$ is $\alpha_{\mathrm{cen}}(\ell)-\ell\alpha'_{\mathrm{cen}}(\ell)=c+m\log(\ell/a)$; a concave extension positive at zero cannot retain a center interval whose left endpoint makes this intercept nonpositive.

6 Exact global realization at prescribed amplitudes

The moment interpolation of section 5 prescribes the number and locations of zeros on a center annulus. To make this a global count, we construct a strictly dissipative exterior, then control both the equilibrium and the last center oval. This construction also provides exact global polynomial realizations through constrained approximation, without a restriction on the prescribed center height $c > 0$.

6.1 A concave extension with a dissipative exterior

Keep $k, m, \beta, c > 0$ fixed and put

$$u(x) = \frac{s(x)}{\sqrt{c}} = \operatorname{sgn}(x-a)\sqrt{\frac{2m}{c}\left(\frac{x}{a} - 1 - \log\frac{x}{a}\right)}.$$

After the positive time change $d\sigma = \sqrt{c}\, d\tau$, the center profile (6) becomes

$$u' = -(e^v - 1) - \frac{1}{2}u^2, \qquad v' = u.$$

Its normalized first integral is

$$H(u,v) = \frac{H_c(\sqrt{c}\,u, v)}{c} = \frac{1}{2}e^v u^2 + \frac{1}{2}(e^v - 1)^2. \tag{40}$$

Thus the energies below are exactly the normalized energies used in section 5.

Lemma 6.1 (A strictly dissipative concave extension)**.** *For every sufficiently small $b > 0$ there exists a globally strictly concave admissible profile α_b such that*

$$\alpha_b(x) = \alpha_{\mathrm{cen}}(x) \quad \text{when } |u(x)| \le b,$$

and

$$\alpha_b(x) > \alpha_{\mathrm{cen}}(x) \quad (u(x) < -b), \qquad \alpha_b(x) < \alpha_{\mathrm{cen}}(x) \quad (u(x) > b). \tag{41}$$

It has its maximum c at a and satisfies $\alpha_b(0) > 0$. If $0 < b < \sqrt{2}$, its positive periodic orbits are precisely the center ovals

$$0 < H \le h_b, \qquad h_b = \frac{b^2}{2} - \frac{b^4}{8}.$$

The largest compact invariant subset of its zero-dissipation set for H is the closed center disc $\mathcal{D}_b = \{H \le h_b\}$.

Proof. Let $L < a < R$ be defined by $u(L) = -b$ and $u(R) = b$. Choose a smooth function $w : [0,\infty) \to (0,\infty)$ with

$$w(x) = \frac{m}{x^2} \quad (L \le x \le R), \qquad 0 < w(x) < \frac{m}{x^2} \quad (0 < x < L), \qquad w(x) > \frac{m}{x^2} \quad (x > R).$$

It can be chosen small and constant near zero, and positive and constant for all sufficiently large x. Smooth flat transitions at L and R give the stated strict inequalities off the matching interval. On the left, first take b small enough that $\int_L^a m/x\, dx < c$, and then make the transition interval and the remaining curvature sufficiently small that

$$\int_0^a x w(x)\, dx < c.$$

Define

$$\alpha_b(x) = c - \int_a^x (x-t)w(t)\,dt. \tag{42}$$

Then $\alpha_b'' = -w < 0$, $\alpha_b'(a) = 0$, and $\alpha_b(a) = c$. The last integral inequality gives $\alpha_b(0) > 0$. Its positive right-hand zero exists and is unique because the profile is strictly concave. Twice integrating the curvature comparisons from a proves both exact matching on $[L, R]$ and (41).

In the normalized coordinates, differentiation of (40) gives

$$H' = e^v u\,\frac{\alpha_b(x(u)) - \alpha_{\mathrm{cen}}(x(u))}{c} \le 0, \tag{43}$$

with strict inequality whenever $|u| > b$. Every periodic orbit must therefore remain in the strip $|u| \le b$, where the field agrees exactly with the center.

To identify the compact invariant part of that strip, put $q = e^v - 1$ and $p = e^{v/2}u$. Then

$$H = \frac{p^2+q^2}{2}, \qquad q > -1, \qquad (p', q') = \sqrt{1+q}\,(-q, p).$$

The center ovals $0 < H < 1/2$ are complete circles in these coordinates. Levels $H \ge 1/2$ meet the excluded line $q = -1$ in their closure, and no nonconstant trajectory on such a level can remain in a compact subset of the (u, v)-plane in both time directions. On the oval of energy $h < 1/2$,

$$\max u^2 = 2\big(1 - \sqrt{1-2h}\big).$$

The oval remains in $|u| \le b$ exactly when $h \le h_b$. The equilibrium and these ovals consequently form the entire compact invariant part of the zero-dissipation strip. □

The ecological values $\alpha_b(0)$ and $x_{1,b}$ produced by this extension are fixed once the curvature w is chosen. The next theorem preserves both values throughout its perturbation family; it does not assert that two arbitrary external endpoint values can be imposed independently of the chosen extension.

6.2 Counting every positive periodic orbit

Theorem 6.2 (Exact global realization by a dissipative extension)**.** *Fix $N \ge 1$, an odd integer $d = 2r + 3 \ge 3$, and an extension α_b from lemma 6.1. For any distinct energies*

$$0 < h_1 < \cdots < h_N < h_b$$

there is a smooth one-parameter family α_ε, $0 < \varepsilon < \varepsilon_0$, of globally strictly concave admissible profiles whose positive periodic orbits consist of exactly N hyperbolic cycles. As $\varepsilon \to 0^+$, these cycles converge to the specified center ovals $H = h_i$. Moreover,

$$j_a^{d-1}\alpha_\varepsilon = j_a^{d-1}\alpha_{\mathrm{cen}}, \qquad \alpha_\varepsilon(0) = \alpha_b(0), \qquad \alpha_\varepsilon(x_{1,b}) = 0.$$

In particular, the maximum remains c at a, and the positive zero remains exactly $x_{1,b}$.

Proof. By corollary 5.2, choose

$$F(u) = u^d P(u^2), \qquad P(z) = \sum_{j=0}^{N} b_j z^j,$$

so that the first energy displacement of the center is

$$M(h) = \sum_{j=0}^{N} b_j J_{r+j}(h)$$

and has precisely the simple zeros $h_1, \dots, h_N$ on $(0, 1/2)$. Normalize the common sign by $M(h_b) < 0$. Both b_0 and b_N are nonzero. If either vanished, the remaining N consecutive moment functions would have a nonzero linear combination with N zeros, contrary to theorem 5.1.

Choose an even smooth cutoff $\chi(u)$ that equals one on a neighborhood of $[-b, b]$ and whose compact support, expressed in the prey coordinate, is contained in $(0, x_{1,b})$. Define

$$\alpha_\varepsilon(x) = \alpha_b(x) + c\varepsilon\chi(u(x))F(u(x)). \tag{44}$$

Such a support is possible because the center profile is positive at both endpoints of the matching interval when $b < \sqrt{2}$. The perturbation is compactly supported away from zero and $x_{1,b}$. Its C^2 norm is $O(\varepsilon)$, so the strictly negative curvature on its support remains negative for small ε. Outside that support the profile is unchanged. The order d of F gives the required jet, and the support gives the two exact endpoint values. Strict concavity then preserves the unique maximum and positive zero.

We now exclude periodic orbits in four regions. This is necessary because simple zeros of M alone count only the nearby cycles.

The equilibrium region. Choose a small fixed $h_0 > 0$ such that $2h_0 < h_1$ and $uF(u)$ has one nonzero sign for $0 < |u|$ on the disc $H \le 2h_0$. This is possible because $b_0 \neq 0$. There the baseline is the exact center and $\chi = 1$, so

$$H' = \varepsilon e^v uF(u).$$

Its sign is fixed, with equality only on $u = 0$. An orbit cannot remain in that line unless it is the equilibrium. The levels of H therefore provide one-sided barriers: a periodic orbit contained in this disc is impossible, and one meeting $H \le h_0$ could not both enter and leave the slightly larger disc. Hence no periodic orbit meets $H \le h_0$.

The compact center annulus. Use the section $u = 0, v > 0$, with energy coordinate $h = (e^v - 1)^2/2$. On $h_0 \le h \le h_b$, the unperturbed first return is the identity, and smooth dependence gives

$$\Delta(h, \varepsilon) := P_\varepsilon(h) - h = \varepsilon M(h) + O(\varepsilon^2) \quad \text{in } C^1. \tag{45}$$

The variation formula is the one used in section 5; the cutoff is identically one on a neighborhood of every unperturbed oval under consideration. Around each simple zero h_i, the implicit function theorem gives one and only one zero of Δ for small positive ε. On the compact complement of these neighborhoods, M is bounded away from zero, so (45) gives no other zeros. There are exactly N simple fixed points on this annulus.

A collar of the last center oval. The orbit $H = h_b$ is regular. Its return map extends smoothly to a fixed collar $h_b \le h \le h_b + \zeta$, with $h_b + \zeta < 1/2$, and to small positive and negative ε. For the baseline, the Lyapunov inequality (43) gives $\Delta(h, 0) \le 0$ throughout this collar. At its inner boundary,

$$\partial_\varepsilon \Delta(h_b, 0) = M(h_b) < 0.$$

By continuity, after reducing ζ and the allowed amplitude, $\partial_\varepsilon \Delta < 0$ on the whole collar. Thus $\Delta(h, \varepsilon) < 0$ for every sufficiently small positive ε and every h in the collar. There are no periodic orbits with section coordinate there.

The remaining exterior. The profiles converge smoothly to α_b and retain $\alpha_b(0)$, c, and $x_{1,b}$, so lemma 3.1 gives a common compact set for all their periodic orbits. By lemmas 3.2 and 6.1, every such periodic orbit lies in $\{H < h_b + \zeta\}$ for sufficiently small ε. Otherwise a sequence of exterior periodic orbits would have a Hausdorff limit whose maximal energy exceeds the energy of the last center oval, contradicting the confinement lemma. This excludes the remaining exterior uniformly.

Finally, lemma 3.3 shows that every positive periodic orbit meets the upper section exactly once. The four regions therefore exhaust all periodic orbits, and the N simple return zeros already found are the complete list. Simplicity makes every cycle hyperbolic, and their convergence to $H = h_i$ follows from (45). □

The sign normalization in this construction also fixes the local stability. Since $M(h_b) < 0$ and its N zeros are simple, its sign near zero is $(-1)^{N+1}$. The same is the sign of b_0, because J_r is the lowest-order positive moment. Thus the equilibrium is stable for even N and unstable for odd N, precisely as required by proposition 3.4. The outermost cycle is always attracting.

6.3 Polynomial transfer with no additional cycles

Corollary 6.3 (Exact global polynomial realization at fixed center data)**.** *Fix one sufficiently small nonzero ε in theorem 6.2, and let d be the first nonzero canonical odd degree of α_ε. For every $R > x_{1,b}$ and integers $J \geq d-1$, $L \geq d$, there are globally strictly concave polynomials converging to α_ε in $C^L([0,R])$ and satisfying exactly*

$$j_a^J \alpha_n = j_a^J \alpha_\varepsilon, \qquad \alpha_n(0) = \alpha_b(0), \qquad \alpha_n(x_{1,b}) = 0,$$

such that, for all sufficiently large n, their positive periodic orbits consist of exactly N hyperbolic cycles. Consequently, for every finite jet of the specified center profile (6), with arbitrary fixed $c > 0$, an exact global polynomial realization exists. This statement asserts finiteness of the required degree; the explicit linear degree estimate is furnished separately by theorem 1.2 under its stated height condition.

Proof. Apply proposition 9.2 with the stated jet and endpoint constraints. The resulting polynomials are globally strictly concave, have their maximum c at a, and have the same positive zero $x_{1,b}$. They converge in C^d because $L \geq d$, and retain the jet through $d-1$ because $J \geq d-1$. The target profile has exactly N hyperbolic periodic orbits and a nonzero canonical odd coefficient of degree d by theorem 6.2. All hypotheses of proposition 3.6 therefore hold, proving that every sufficiently accurate polynomial approximant has exactly N positive periodic orbits, all hyperbolic.

To retain the jet of (6) through any chosen finite order K, take an odd $d > K$ in theorem 6.2 and then choose $J \geq d-1$. The construction works for every $c > 0$ because the matching interval of the smooth extension can be chosen sufficiently small. □

The differentiability requirement in this transfer has a specific purpose. Mere C^1 closeness preserves the already hyperbolic cycles, but does not exclude new cycles approaching a degenerate focus. The C^d estimate together with the fixed lower jet preserves the leading odd sign and supplies the missing local barrier. The approximation retains the specified finite jet.

7 Localized creation of cycles with preservation of the germ

Consider the canonical system

$$\dot{s} = f(s) - c(e^v - 1), \qquad \dot{v} = s, \qquad c > 0, \tag{46}$$

where f is smooth near zero and

$$f(0) = f'(0) = 0, \qquad f''(0) < 0.$$

Write $f = E + O$, where E is even and O is odd. Suppose first that

$$O(s) = bs^d + O(s^{d+2}), \qquad b \neq 0, \qquad d = 2r+1 \geq 3. \tag{47}$$

Theorem 7.1 (Localized cycle creation)**.** *Let $N \geq 1$ and let $0 < h_1 < \cdots < h_N$ be prescribed. There is an odd function $R \in C_c^\infty(\mathbb{R})$, vanishing on a neighborhood of zero, such that, for all sufficiently small $\rho > 0$, the function*

$$f_\rho(s) = f(s) + \rho^d R(s/\rho)$$

has the following properties.

1. *$f_\rho = f$ both on a neighborhood of zero and outside an interval $|s| < B\rho$, where B is independent of ρ.*
2. *$\|f_\rho - f\|_{C^{d-1}} = O(\rho)$, and $f_\rho'(s)$ has the sign of $-s$ throughout a fixed sufficiently small neighborhood of zero.*
3. *The system obtained by replacing f with f_ρ in (46) has N distinct hyperbolic limit cycles. On the section $v = 0$, $s > 0$, their intersection coordinates satisfy*

$$s_i(\rho) = \rho\sqrt{2h_i} + O(\rho^2).$$

4. *Their multipliers have the form*

$$\mu_i(\rho) = 1 + \rho^{d-1}(M_0'(h_i) + O(\rho)),$$

where $M_0'(h_i) \neq 0$. The signs of these derivatives alternate; the first has sign $-\operatorname{sgn}(b)$.

Proof. We first construct R. For the harmonic oscillator

$$\dot S = -cV, \qquad \dot V = S,$$

let γ_h denote the oval $(S^2 + cV^2)/2 = h$. Its odd moments are

$$\oint_{\gamma_h} S^{2j+1}\, dV = K_j h^{j+1}, \qquad K_j = \frac{\pi}{\sqrt{c}\, 2^j}\binom{2j+2}{j+1} > 0. \tag{48}$$

Choose a compact interval $I \Subset (0, \infty)$ containing all h_i in its interior. There are unique numbers $a_1^0, \ldots, a_N^0$ such that

$$bK_r h^{r+1} + \sum_{\ell=1}^{N} a_\ell^0 K_{r+\ell} h^{r+\ell+1} = bK_r h^{r+1} \prod_{i=1}^{N}(1 - h/h_i). \tag{49}$$

Take a smooth even cutoff χ that equals one on an interval containing the S-projections of all γ_h, $h \in I$, and has compact support. Take another smooth even cutoff θ_κ that vanishes for $|S| \leq \kappa$ and equals one for $|S| \geq 2\kappa$. The matrix

$$A_{i\ell}(\kappa) = \oint_{\gamma_{h_i}} \theta_\kappa(S) S^{2(r+\ell)+1}\, dV$$

converges as $\kappa \downarrow 0$ to $K_{r+\ell} h_i^{r+\ell+1}$. The latter matrix is invertible: after factoring positive row and column factors, it is a Vandermonde matrix. Consequently, for all sufficiently small fixed κ, there are unique $a_\ell(\kappa)$, converging to a_ℓ^0, for which

$$M_0(h_i) = 0, \qquad M_0(h) := \oint_{\gamma_h} \left[bS^d + \theta_\kappa(S) \sum_{\ell=1}^{N} a_\ell(\kappa) S^{2(r+\ell)+1} \right] dV.$$

Moreover, M_0 and its first derivative converge uniformly on I to the right-hand side of (49). This follows directly from the parametrization

$$S = \sqrt{2h}\cos\vartheta, \qquad V = \sqrt{2h/c}\sin\vartheta.$$

The cutoff error is supported where $|S| \le 2\kappa$; differentiation in h remains harmless because h is bounded away from zero. In particular, the prescribed zeros remain simple and retain the derivative signs in (49). Fix such a κ and put

$$R(S) = \chi(S)\theta_\kappa(S)\sum_{\ell=1}^{N} a_\ell(\kappa)S^{2(r+\ell)+1}.$$

Now rescale $s = \rho S$, $v = \rho V$. The modified system becomes

$$\dot S = B_\rho(S,V) + \rho^{d-1}D_\rho(S), \qquad \dot V = S,$$

where

$$B_\rho(S,V) = \frac{E(\rho S)}{\rho} - c\frac{e^{\rho V}-1}{\rho}, \qquad D_\rho(S) = \frac{O(\rho S)}{\rho^d} + R(S).$$

Both expressions extend smoothly to $\rho = 0$, with

$$B_0(S,V) = -cV, \qquad D_0(S) = bS^d + R(S).$$

For every sufficiently small ρ, the unperturbed field (B_ρ, S) is reversible under $(S,V) \mapsto (-S,V)$ and has an elliptic linearization at the origin. Its local orbits are periodic. One elementary proof uses the two successive intersections of a near-harmonic orbit with the fixed line $S = 0$; reversibility closes the orbit. Equivalently, the scalar-flow construction of K in lemma 2.3 gives a smooth first integral H_ρ normalized by $H_\rho(S,0) = S^2/2$; as $\rho \to 0$ it converges smoothly on the required compact set to $(S^2 + cV^2)/2$. Consequently, on any fixed compact annulus containing the ovals indexed by I, the periodic-orbit construction and the transversality of the section are uniform in ρ.

Introduce an independent perturbation parameter ϵ and write

$$\dot S = B_\rho(S,V) + \epsilon D_\rho(S), \qquad \dot V = S.$$

On the section $V = 0$, $S > 0$, use $h = S^2/2$. Its displacement is smooth in (ρ, ϵ, h) and vanishes identically when $\epsilon = 0$. Uniformly in $C^1(I)$,

$$\Delta(\rho,\epsilon,h) = \epsilon\big(M_\rho(h) + O(\epsilon)\big), \qquad M_\rho = M_0 + O(\rho).$$

At $\rho = 0$, the formula for M_0 follows by integrating the derivative of $(S^2 + cV^2)/2$ along γ_h. Substituting $\epsilon = \rho^{d-1}$ gives

$$\rho^{1-d}\Delta(\rho,\rho^{d-1},h) = M_0(h) + O(\rho) \quad \text{in } C^1(I).$$

The implicit function theorem yields one hyperbolic cycle near each h_i, with $h_i(\rho) = h_i + O(\rho)$ and the stated multipliers.

Finally, for each $0 \le j \le d-1$,

$$\left\| \frac{d^j}{ds^j}(f_\rho - f) \right\|_\infty \le \rho^{d-j}\|R^{(j)}\|_\infty.$$

This proves the norm estimate. Because R vanishes near zero, $R'(S)/S$ is bounded, and

$$\frac{f_\rho'(s) - f'(s)}{s} = \rho^{d-2}\frac{R'(s/\rho)}{s/\rho} \longrightarrow 0$$

uniformly on the support. Since $f'(s)/s \to f''(0) < 0$, the derivative sign is preserved. All germ and support assertions are immediate. □

Corollary 7.2 (Strictly concave prey growth)**.** *Fix $k, m, \beta > 0$ and set $a = m/(k\beta)$. Let α be any admissible smooth prey growth function with maximum at a and $\alpha(a) > 0$. If its canonical odd part has a first nonzero term of degree d, then for every N there are growth functions α_ρ with at least N hyperbolic positive limit cycles, equal to α near a and outside a shrinking neighborhood of a, and converging to α in C^{d-1}. In particular, the carrying capacity, the value at zero, the tail, and the entire germ at the maximum are unchanged.*

If $\alpha'' < 0$ globally, the functions α_ρ may also be chosen globally strictly concave. Hence profiles arbitrarily close in C^2 to an admissible quadratic profile $\alpha(x) = c - b(x-a)^2$, $c > ba^2$, can have arbitrarily many hyperbolic cycles.

Proof. Use the smooth local coordinate $s(x) = \operatorname{sgn}(x-a)\sqrt{2m(x/a - 1 - \log(x/a))}$ and put $f(s) = \alpha(x(s)) - \alpha(a)$. Add $\rho^d R(s(x)/\rho)$ to α. The coordinate change preserves the germ, support, and norm conclusions. Since $d \geq 3$, the perturbation is C^2-small. Strict concavity is preserved on its shrinking support, where the original second derivative is bounded above by a negative constant; outside the support the growth function is unchanged. For the quadratic profile the first canonical odd term has degree three:

$$f(s) = -\frac{ba^2}{m}s^2 - \frac{2ba^2}{3m^{3/2}}s^3 + O(s^4).$$

□

8 The sharp differentiability threshold

Proposition 8.1. *Under (47), there are $s_0 > 0$, a neighborhood U of the origin, and $\epsilon_0 > 0$ such that every smooth $\widetilde{f}$ satisfying*

$$j_0^{d-1}\widetilde{f} = j_0^{d-1}f, \qquad \|\widetilde{f} - f\|_{C^d([-s_0,s_0])} < \epsilon_0$$

has no periodic orbit contained in U for its canonical system. Thus the C^{d-1} approximation assertion cannot generally be upgraded to C^d while preserving the indicated jet.

Proof. Shrink the s-interval so that $O(s)/s^d$ has the sign of b and absolute value at least $|b|/2$. Taylor's theorem and the jet condition give

$$|(\widetilde{f} - f)_{\mathrm{odd}}(s)| \leq \frac{\|\widetilde{f} - f\|_{C^d}}{d!}|s|^d.$$

For a sufficiently small norm bound, the odd part $\widetilde{O}$ therefore has the sign of bs for $s \neq 0$.

Apply lemma 2.3 to the even part $\widetilde{E}$. Its local function $H_{\widetilde{E}}$ satisfies

$$\dot{H}_{\widetilde{E}} = K_w\, s\widetilde{O}(s), \qquad K_w > 0.$$

This has the fixed sign of b away from $s = 0$, and its integral over a nonconstant periodic orbit cannot vanish. The scalar flow defining K exists on a common neighborhood under the stated small C^d bound, by continuous dependence and uniform C^2 bounds on the even part. □

8.1 The flat odd-part case

If O is flat at zero, then for every finite k and every N one can still create at least N hyperbolic cycles while preserving the entire germ and making $\|f_\rho - f\|_{C^k} \to 0$. Choose an odd integer

$d = 2r + 1 > \max\{k, 2\}$ and use the same outer and inner cutoffs. This time the leading monomial must also be inserted:

$$R_\kappa(S) = \chi(S)\theta_\kappa(S)\sum_{\ell=0}^{N} a_\ell(\kappa)S^{d+2\ell}, \qquad a_0(\kappa) = 1.$$

Solve the N equations $\oint_{\gamma_{h_i}} R_\kappa(S)\, dV = 0$ for $a_1(\kappa), \ldots, a_N(\kappa)$. At $\kappa = 0$ their coefficient matrix is the invertible Vandermonde-type matrix used above, and the target displacement is $K_r h^{r+1} \prod_i (1 - h/h_i)$. For small fixed κ the coefficients converge, the prescribed zeros remain simple, and their derivative signs are unchanged.

Now $O(\rho S)/\rho^d \to 0$ smoothly on compact sets, including smooth extension in ρ at zero by flatness. The limiting displacement is $\oint R_\kappa(S)\, dV$. The same two-parameter return-map proof gives the cycles, and the norm estimate is $O(\rho^{d-k})$. Taking k successively larger and choosing the corresponding scales sufficiently small gives a sequence converging in the smooth Fréchet topology, with any prescribed sequence of finite lower bounds on cycle counts.

Proof of theorem 1.3. If the canonical odd part has a first nonzero Taylor coefficient, theorem 7.1 and corollary 7.2 apply, and $d \geq 3$ gives C^2 convergence. If all its Taylor coefficients vanish, apply the flat case with $k = 2$. The perturbations vanish near the maximum and outside their shrinking support; hence the low-density value, positive zero, and the germ are exactly preserved. On the support, strict concavity follows from C^2 convergence and the negative curvature of the original profile near its maximum. Outside the support the profile is unchanged. □

Remark 8.2. Germ preservation means that, for each member of the approximating family, there is a neighborhood on which the functions agree. The size of that neighborhood shrinks with the perturbation scale. In the finite-order odd case, the family converges in C^{d-1} and need not converge in C^d. These are the topology and locality features used by the realization theorem.

9 Global concavity and polynomial realization

Local oscillation constructions can be extended without changing the vector field on the selected compact cycles. The following elementary extension lemma will also be used for nonlinear responses.

Lemma 9.1 (Concave extension)**.** *Let a smooth local function A be given near $a > 0$, with $A(a) > 0$, $A'' < 0$ locally, and*

$$L := A(a) - aA'(a) > 0.$$

After restricting its local domain if necessary, it extends to a smooth function on $[0, \infty)$ that is strictly concave, positive at zero, has a unique nondegenerate maximum at a positive point, and has a unique positive zero to the right of that maximum.

Proof. Write $A_0 = A(a)$, $D = A'(a)$, and prescribe a global smooth weight $w > 0$ agreeing with $-A''$ on a small interval around a. Define

$$\widetilde{A}(x) = A_0 + D(x - a) - \int_a^x (x - t)w(t)\, dt.$$

Then $\widetilde{A} = A$ locally and $\widetilde{A}'' = -w < 0$. The two conditions needed on the left extension are

$$\int_0^a tw(t)\, dt < L, \qquad D + \int_0^a w(t)\, dt > 0.$$

The first follows by restricting the prescribed interval about a and taking a sufficiently small positive background elsewhere. If $D < 0$, the second can be obtained by adding a positive bump of mass larger than $-D$ close to zero. Its contribution to the first, weighted integral can be made arbitrarily small by placing its support sufficiently close to zero. Smooth positive interpolation at the ends of the prescribed interval does not affect these strict inequalities. On the right choose w eventually to be a positive constant. Thus $\widetilde{A}(0) > 0$, $\widetilde{A}'(0) > 0$, $\widetilde{A}'(x) \to -\infty$, and $\widetilde{A}(x) \to -\infty$. Strict concavity gives the unique nondegenerate maximum and the claimed zero. □

Taking $A = \alpha_{\mathrm{cen}}$ from (6) gives $A(a) - aA'(a) = c > 0$. Thus there are admissible globally strictly concave profiles that agree exactly with this integrable model on a sufficiently small neighborhood of a.

9.1 Polynomial approximation with exact constraints

Proposition 9.2. *Let $\alpha \in C^\infty([0,\infty))$ be strictly concave, with $\alpha(a) = c > 0$, $\alpha'(a) = 0$, $\alpha(0) > 0$, and $\alpha(x_1) = 0$ for $x_1 > a$. Fix $R > x_1$, a finite jet order $K \geq 2$, and a finite approximation order L. Then α can be approximated on $[0,R]$ in C^L by polynomials α_n satisfying, exactly,*

$$\alpha_n''(x) < 0 \quad (x \geq 0), \qquad j_a^K \alpha_n = j_a^K \alpha, \qquad \alpha_n(0) = \alpha(0), \qquad \alpha_n(x_1) = 0.$$

In particular, any finite collection of hyperbolic positive limit cycles is preserved by a sufficiently accurate such approximation.

Proof. Put $w = -\alpha'' > 0$ on $[0,R]$, choose $0 < d < \min_{[0,R]} w$, and define $q = \sqrt{w-d}$. Polynomial approximation with Hermite interpolation gives polynomials $p_n \to q$ in any prescribed finite differentiability norm, with $j_a^{K-2} p_n = j_a^{K-2} q$. One elementary construction approximates a sufficiently high derivative of q uniformly and integrates repeatedly, using the exact lower derivatives of q at a as integration constants.

Choose $2M > K - 2$. Introduce

$$p_{n,t}(x) = p_n(x)\big[1 + t_0(x-a)^{2M} + t_1(x-a)^{2M+1}\big], \qquad w_{n,t} = p_{n,t}^2 + d.$$

These weights are globally positive polynomials and retain the required jet at a. The two equations to be imposed are

$$\int_0^a x w_{n,t}(x)\,dx = c - \alpha(0), \qquad \int_a^{x_1} (x_1 - x) w_{n,t}(x)\,dx = c.$$

At $t = 0$, the Jacobian of the two left-hand sides has rows (A_-, B_-) and (A_+, B_+), where

$$A_- > 0, \quad B_- < 0, \qquad A_+ > 0, \quad B_+ > 0.$$

Indeed, apart from positive weights, the entries are integrals of $2p_n^2(x-a)^{2M}$ and $2p_n^2(x-a)^{2M+1}$ over the left and right intervals, respectively. Thus the determinant is strictly positive. View the two moments as a smooth map of (p,t), with p in a neighborhood of q in $C([0,R])$. At $(q,0)$ the target moments hold and the displayed Jacobian is invertible. The implicit function theorem with p as a Banach-space parameter gives a continuous solution $t(p)$ near q, with $t(q) = 0$. Hence $t_n = t(p_n) \to 0$ solves both moment equations exactly.

Define

$$\alpha_n(x) = c - \int_a^x (x-u) w_{n,t_n}(u)\,du.$$

Then $\alpha_n'' = -w_{n,t_n} < 0$ globally, $\alpha_n(a) = c$, and $\alpha_n'(a) = 0$. The weight jet gives the full requested growth-rate jet, and the two moment equations give the two endpoint values. The differentiable approximation follows from the construction. Every positive periodic orbit of the target system lies in $0 < x < x_1$, since at a maximum of x along such an orbit one has $\alpha(x) > 0$. Thus C^1 approximation on this interval suffices for the persistence of its finitely many hyperbolic cycles. □

Proof of theorem 1.4. First apply theorem 1.3 to the prescribed profile, taking the perturbation small enough that the resulting strictly concave profile α_* is within $\varepsilon/2$ of α in $C^2([0, R])$. It has N hyperbolic cycles, agrees with α near a, and has the same values at 0 and x_1. Apply proposition 9.2 to α_*, retaining the jet through order K and the two endpoint values. Choose the approximation within $\varepsilon/2$ in C^2 and within the C^1 persistence neighborhood of all N cycles. The resulting polynomial has all the asserted properties. □

Corollary 9.3 (Fixed quadratic data)**.** *Fix $b, c > 0$ with $c > ba^2$, and let $\alpha_0(x) = c - b(x-a)^2$. For every $N \geq 1$ and $K \geq 2$ there are globally strictly concave polynomials with at least N hyperbolic cycles such that*

$$\alpha(0) = c - ba^2, \quad \alpha(a) = c, \quad \alpha''(a) = -2b, \quad \alpha^{(j)}(a) = 0 \ (3 \leq j \leq K), \quad x_1 = a + \sqrt{c/b}.$$

They can be chosen arbitrarily close to α_0 in C^2 on any prescribed compact interval containing $[0, x_1]$.

Remark 9.4. Hyperbolicity also permits a small change in the predator mortality or the linear predation coefficient after the construction. The cycles persist while the break-even density moves away from the growth-rate maximum. Thus the resulting oscillations persist in systems with a hyperbolic coexistence equilibrium as well.

10 Cycle number, retained jets, and polynomial degree

The degree in theorem 1.2 is explicit and linear in the number of cycles and the retained jet order. Quantitative approximation of a prescribed growth profile involves additional data. This section separates these two questions, gives necessary degree bounds, and makes the approximation in proposition 9.2 effective. Throughout this section,

$$\|g\|_{C^j(I)} = \max_{0 \leq i \leq j} \sup_{x \in I} |g^{(i)}(x)|.$$

10.1 The degree forced by a center jet

Proposition 10.1. *Fix $\theta > 0$ and $K \geq 2$. Any polynomial P satisfying $P'' < 0$ on $[0, \infty)$ and $j_a^K P = j_a^K C_\theta$, where C_θ is the center profile in (9), has degree at least*

$$D_{\text{jet}} = 2 \left\lceil \frac{K}{2} \right\rceil. \tag{50}$$

If $c > \theta m(\mathsf{H}_{D_{\text{jet}}} - 1)$, this lower bound is attained by an admissible globally strictly concave polynomial. Here $\mathsf{H}_j = \sum_{i=1}^{j} 1/i$.

Proof. For $u = (x - a)/a$, the center germ is

$$C_\theta(x) = c + \theta m \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n} u^n.$$

Its derivative of every order $n \geq 2$ at a is nonzero. Hence $\deg P \geq K$. If K is odd and $\deg P = K$, the prescribed jet determines P completely, and its leading coefficient is positive. Then P'' is positive for all sufficiently large x, contrary to the hypothesis. This proves (50).

For an even integer d, the Taylor polynomial

$$T_d(x) = c + \theta m \sum_{n=2}^{d} \frac{(-1)^{n+1}}{n} u^n$$

satisfies

$$-T_d''(x) = \frac{\theta m}{a^2} \sum_{j=0}^{d-2} (j+1)(-u)^j > 0.$$

For $u < 0$ every summand is nonnegative and the constant term is positive. For $u \geq 0$, the sum is

$$\frac{1 + du^{d-1} + (d-1)u^d}{(1+u)^2} > 0.$$

Moreover, $T_d(a) = c$, $T_d'(a) = 0$, and $T_d(0) = c - \theta m(\mathsf{H}_d - 1)$. Taking $d = D_{\text{jet}}$ proves the attainment assertion under the stated height condition. □

Consequently, the explicit degree (2) is $D_{\text{jet}} + 2N + 2$. This is the degree cost of the construction in theorem 1.2; it does not assert that this degree is optimal among all profiles with N cycles. The height condition also has a precise role. For fixed $c > 0$ one may choose θ sufficiently small for each N, K, whereas fixing θ fixes the curvature of the center jet and retains the condition $c > \theta m(\mathsf{H}_D - 1)$ in that construction. Its two fixed endpoint values are the values determined by the limiting polynomial in theorem 1.2. Approximation with arbitrary prescribed admissible endpoint data is covered below by a data-dependent bound.

10.2 A sharp lower bound from the left endpoint

Proposition 10.2 (The curvature budget at the left endpoint)**.** *Let P have degree $D \geq 2$, and suppose*

$$P''(x) < 0 \quad (x \geq 0), \qquad P(a) = c, \quad P'(a) = 0, \quad P''(a) = -\kappa < 0, \quad P(0) = \alpha_0 > 0.$$

Then

$$\frac{c - \alpha_0}{\kappa a^2} \geq b_D, \qquad b_D = \begin{cases} \dfrac{4}{D(D+2)}, & D \text{ even}, \\ \dfrac{4}{(D-1)(D+3)}, & D \text{ odd}. \end{cases} \tag{51}$$

In particular,

$$D \geq \sqrt{1 + \frac{4\kappa a^2}{c - \alpha_0}} - 1. \tag{52}$$

The constants b_D are sharp infima under the stated shape, value, slope, and curvature conditions, with the endpoint value allowed to vary. For the center curvature $\kappa = \theta m/a^2$, the denominator on the left of (51) is θm.

Proof. Set $w = -P''$ and $q(t) = w(at)/\kappa$. Then $q > 0$ on $[0, \infty)$, $q(1) = 1$, and $\deg q = d = D - 2$. Integrating the curvature twice gives

$$\frac{c - \alpha_0}{\kappa a^2} = \int_0^1 tq(t)\, dt. \tag{53}$$

We first establish two elementary polynomial facts.

Every real polynomial nonnegative on $[0,\infty)$ can be written as

$$q(t) = A(t)^2 + tB(t)^2, \qquad \deg A \le \lfloor d/2 \rfloor, \quad \deg B \le \lfloor (d-1)/2 \rfloor. \tag{54}$$

To see this, factor q over the reals. Its positive real roots have even multiplicity. Apart from squares and a positive constant, the factors are powers of t, factors $t+\lambda$ with $\lambda > 0$, and irreducible quadratics $(t-u)^2+v^2$ with $v \neq 0$. The first two types have the required form. For the last, put $\rho = \sqrt{u^2+v^2}$ and use

$$(t-u)^2 + v^2 = (t-\rho)^2 + 2(\rho - u)t.$$

The class is closed under multiplication because

$$(A^2+tB^2)(C^2+tE^2) = (AC - tBE)^2 + t(AE+BC)^2.$$

The degree bounds in (54) follow because the leading coefficients of the two nonzero summands are positive.

For $r = 1, 2$ and any polynomial p of degree at most n, one has

$$\int_0^1 t^r p(t)^2 \, dt \ge \frac{p(1)^2}{(n+1)(n+r+1)}. \tag{55}$$

Indeed, the Rodrigues polynomials

$$L_j^{(r)}(t) = \frac{1}{j! t^r} \frac{d^j}{dt^j} [t^{j+r}(t-1)^j], \qquad 0 \le j \le n,$$

have degree j, value one at $t = 1$, and are orthogonal for the weight t^r on $[0,1]$. Integration by parts j times, followed by the beta integral, gives

$$\int_0^1 t^r \left(L_j^{(r)}(t)\right)^2 dt = \frac{1}{2j+r+1}.$$

Expanding $p = \sum_{j=0}^n a_j L_j^{(r)}$ and applying Cauchy–Schwarz therefore yields

$$p(1)^2 \le \left(\sum_{j=0}^n \frac{a_j^2}{2j+r+1} \right) \left(\sum_{j=0}^n (2j+r+1) \right) = (n+1)(n+r+1) \int_0^1 t^r p(t)^2 \, dt.$$

If $d = 2n$ with $n \ge 1$, (54) and (55) give

$$\begin{aligned} \int_0^1 tq(t) \, dt &\ge \frac{A(1)^2}{(n+1)(n+2)} + \frac{B(1)^2}{n(n+2)} \\ &\ge \frac{A(1)^2 + B(1)^2}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)}. \end{aligned}$$

For $n = 0$, $q = 1$ and the same conclusion follows directly. If $d = 2n+1$, both A and B have degree at most n, so

$$\int_0^1 tq(t) \, dt \ge \frac{A(1)^2}{(n+1)(n+2)} + \frac{B(1)^2}{(n+1)(n+3)} \ge \frac{1}{(n+1)(n+3)}.$$

Substitution of $d = D-2$ proves (51). Since both parity-dependent bounds are at least $4/[D(D+2)]$, (52) follows.

For sharpness, define the endpoint reproducing kernel

$$K_n^{(r)}(t,1) = \sum_{j=0}^{n} (2j+r+1) L_j^{(r)}(t).$$

Take, respectively, for even and odd d,

$$q(t) = \left[\frac{K_n^{(1)}(t,1)}{K_n^{(1)}(1,1)}\right]^2, \qquad q(t) = t\left[\frac{K_n^{(2)}(t,1)}{K_n^{(2)}(1,1)}\right]^2.$$

These polynomials are nonnegative on the entire half-line, have degree exactly d, satisfy $q(1) = 1$, and attain the corresponding bound for nonnegative curvature. The polynomials

$$q_\varepsilon(t) = \frac{q(t)+\varepsilon}{1+\varepsilon}, \qquad \varepsilon > 0,$$

are strictly positive on $[0,\infty)$, retain their degree and the normalization at one, and approach the same bound. Integrating $P_\varepsilon''(x) = -\kappa q_\varepsilon(x/a)$ with $P_\varepsilon(a) = c$, $P_\varepsilon'(a) = 0$ gives admissible profiles for all sufficiently small ε whenever $c > \kappa a^2 b_D$. This proves the claimed sharpness as an infimum in the strictly concave class. □

The budget in (53) explains why degree estimates retaining independently prescribed ecological values must depend on dimensionless ratios such as $(c-\alpha_0)/(\kappa a^2)$. Additional derivatives in a prescribed jet can impose further restrictions.

10.3 The size of an arbitrary prescribed jet

Proposition 10.3. *Let P have degree $D \geq 2$, with $P'' < 0$ on $[0,\infty)$, $P(a) = c > 0$, $P'(a) = 0$, and $P(0) > 0$. Put $n = D-2$. Then, for every $j \geq 2$,*

$$|P^{(j)}(a)| \leq \frac{4^{j-1}c}{a^j} n^{2j-4}(n+1)^2. \tag{56}$$

For $j = 2$, the factor n^0 is interpreted as one. In particular,

$$-P''(a) \leq \frac{4c(n+1)^2}{a^2}, \qquad |P'''(a)| \leq \frac{16cn^2(n+1)^2}{a^3}.$$

Proof. For $w = -P'' > 0$,

$$\int_0^a x w(x)\,dx = c - P(0) < c.$$

Hence $\int_I w \leq 2c/a$ on $I = [a/2, a]$. The reproducing kernel for polynomials of degree at most n on an interval of length $|I|$ has absolute value at most $(n+1)^2/|I|$. To verify this, express the kernel in shifted Legendre polynomials and use $|P_j(t)| \leq 1$ for $-1 \leq t \leq 1$, together with $\sum_{j=0}^{n}(2j+1) = (n+1)^2$. Reproduction and positivity of w give

$$\|w\|_{C^0(I)} \leq \frac{(n+1)^2}{|I|}\int_I w \leq \frac{4c(n+1)^2}{a^2}.$$

Markov's polynomial inequality, scaled to I [5, eq. (4)], gives $\|w'\|_{C^0(I)} \leq (4n^2/a)\,\|w\|_{C^0(I)}$. Iterating it $j-2$ times and bounding the intermediate degrees by n proves (56). Derivatives above the degree of P vanish, as required. □

Corollary 10.4. *There is no degree bound depending only on N, K, a, c for polynomial realization of every admissible prescribed jet, already for $K = 3$. This remains true if the peak curvature is fixed and no approximation condition is imposed.*

Proof. Choose $\kappa > 0$ with $c > \kappa a^2/2$, a smooth compactly supported function ψ with $\psi(0) = 0$, $\psi'(0) \neq 0$, and $\epsilon > 0$ small enough that $\kappa + \epsilon\psi > 0$. For large L, set

$$w_L(x) = \kappa + \epsilon\psi(L(x-a)), \qquad A_L(x) = c - \int_a^x (x-z)w_L(z)\,dz.$$

These profiles are globally strictly concave, have their maximum c at a, and have the same curvature $A_L''(a) = -\kappa$. Since

$$\int_0^a x w_L(x)\,dx = \kappa a^2/2 + O(L^{-1}),$$

their values at zero are positive for all sufficiently large L. Their curvature is constant outside a bounded interval, so each has a unique positive zero to the right of a. However, $|A_L'''(a)| = \epsilon L|\psi'(0)| \to \infty$. Any polynomial with the same three-jet must therefore have degree tending to infinity by proposition 10.3, irrespective of any requested cycles. □

10.4 The degree cost of approximation near a fixed focus

Proposition 10.5. *Let A be a fixed admissible polynomial of degree D_0, with maximum at a, and suppose that the first nonzero odd term of $A(x(s)) - A(a)$ has degree $d \geq 3$. Let I be a compact prey interval whose interior contains a. There are a neighborhood U of the coexistence equilibrium and a constant $C > 0$ such that every polynomial P of degree D satisfying*

$$j_a^{d-1}P = j_a^{d-1}A, \qquad \|P - A\|_{C^2(I)} \leq \varepsilon$$

and having a periodic orbit contained in U obeys

$$\varepsilon \max\{D, D_0, 1\}^{d-2} \geq C. \tag{57}$$

Consequently, for sufficiently small ε, $D \geq C_1\varepsilon^{-1/(d-2)}$. For an admissible quadratic $A(x) = c - b(x-a)^2$, this gives $D \geq C_1/\varepsilon$ when its two-jet is retained and a cycle is required in the fixed neighborhood.

Proof. Reduce I, if necessary, to $[a-\ell, a+\ell] \Subset (0, \infty)$. Put $r = d-2$ and $Q = P'' - A''$, whose degree is at most $n = \max\{D, D_0, 2\} - 2$. The interior Bernstein inequality [5, eq. (1)] reads

$$|Q'(x)| \leq \frac{n\,\|Q\|_{C^0(I)}}{\sqrt{\ell^2 - (x-a)^2}}.$$

Apply it successively on the intervals $I_j = [a - \ell_j, a + \ell_j]$, where

$$\ell_j = \ell\left(1 - \frac{j}{2r}\right), \qquad 0 \leq j \leq r.$$

Since $\ell_{j-1}^2 - \ell_j^2 \geq \ell^2/(2r)$, each step costs at most $\sqrt{2r}\,n/\ell$. Thus

$$\left\|Q^{(j)}\right\|_{C^0(I_r)} \leq \left(\frac{\sqrt{2r}\,n}{\ell}\right)^j \|Q\|_{C^0(I)}, \qquad 0 \leq j \leq r.$$

The case $n = 0$ is immediate, since all derivatives of Q vanish. Combining these estimates with the assumed bounds for orders zero and one gives

$$\|P - A\|_{C^d(I_r)} \le B_r \varepsilon \max\{D, D_0, 1\}^{d-2}, \qquad B_r = \max\{1, \sqrt{2r}/\ell\}^r. \tag{58}$$

Choose a fixed s-interval whose image under $x = x(s)$ lies in I_r. Composition with this analytic coordinate multiplies the C^d bound by a finite constant depending only on the coordinate and the interval; it also preserves equality of the lower jets. The no-cycle neighborhood in proposition 8.1 therefore applies whenever the right-hand side of (58) is sufficiently small. Its contrapositive proves (57). For the quadratic profile,

$$A(x(s)) - c = -\frac{ba^2}{m}s^2 - \frac{2ba^2}{3m^{3/2}}s^3 + O(s^4),$$

so $d = 3$, giving the final assertion. □

The polynomial-baseline hypothesis is essential to this proof: $P'' - A''$ must be a polynomial before Bernstein's inequality can be applied. For an arbitrary smooth baseline, quantitative approximation also requires control of its derivatives or its modulus of continuity.

Proposition 10.6. *Even with the entire germ at a, the values at 0 and a, and the positive zero fixed, there is a family of admissible smooth strictly concave profiles for which polynomial C^2 approximation to a fixed positive tolerance requires unbounded degree.*

Proof. Start with $A(x) = c - \kappa(x-a)^2/2$, where $c > \kappa a^2/2$, and write $b = a + \sqrt{2c/\kappa}$. Choose disjoint compact intervals in (a, b), a smooth cutoff $0 \le \chi \le 1$ supported in the first and equal to one on a nondegenerate interval J, and a nonzero nonnegative smooth function ψ supported in the second. Set

$$\gamma_L = \frac{\int_a^b (b-x)\chi(x)\sin(Lx)\,dx}{\int_a^b (b-x)\psi(x)\,dx}, \qquad w_L = \kappa + \frac{\kappa}{4}[\chi(x)\sin(Lx) - \gamma_L\psi(x)].$$

Integration by parts gives $\gamma_L = O(L^{-1})$. Thus $w_L > 0$ for large L. Define $A_L(x) = c - \int_a^x (x-z)w_L(z)\,dz$. The supports lie away from a, so every A_L has the same germ at a and the same value at zero as A. The definition of γ_L gives $A_L(b) = 0$. Strict concavity makes b its unique positive zero.

On J, $w_L = \kappa + \kappa\sin(Lx)/4$. If $\|P - A_L\|_{C^2([0,b])} < \kappa/8$, then $-P'' - \kappa$ has alternating signs at the successive extrema of $\sin(Lx)$ in J. There are $L|J|/\pi + O(1)$ such extrema. The intermediate value theorem forces at least that number minus one of distinct zeros of $-P'' - \kappa$. Hence $\deg P \ge L|J|/\pi - O(1)$, proving the assertion. □

10.5 A quantitative Hermite–Bernstein construction

We now give explicit sufficient degree bounds for a specified smooth target and a specified approximation tolerance. Both the finite jet and the two endpoint values are imposed exactly, while strict concavity holds on the entire half-line.

Theorem 10.7. *Let α be admissible and globally strictly concave, with $\alpha(a) = c$, $\alpha'(a) = 0$, and positive zero $x_1 > a$. Fix $R > x_1$, integers $K, L \ge 2$, and $\varepsilon > 0$. The constants defined below give an integer $n \ge 1$ and a polynomial $\widehat{\alpha}$ such that*

$$\widehat{\alpha}''(x) < 0 \quad (x \ge 0), \qquad j_a^K\widehat{\alpha} = j_a^K\alpha, \qquad \widehat{\alpha}(0) = \alpha(0), \qquad \widehat{\alpha}(x_1) = 0, \tag{59}$$

$$\|\widehat{\alpha} - \alpha\|_{C^L([0,R])} \le \varepsilon, \qquad \deg\widehat{\alpha} \le 2n + 4\max\{K, L\}. \tag{60}$$

All constants involve only finite derivatives, integrals, and norms of the prescribed target on $[0, R]$.

Here are the constants and the construction. Put

$$w = -\alpha'', \qquad 0 < \zeta < \min_{[0,R]} w, \qquad q = \sqrt{w - \zeta}, \qquad H = \max\{K, L\} - 2.$$

Choose the smallest even integer $J > H$, and define $g_0 = (x-a)^J$, $g_1 = (x-a)^{J+1}$. For a continuous function p and $t = (t_0, t_1) \in \mathbb{R}^2$, set

$$p_t = p(1 + t_0 g_0 + t_1 g_1), \qquad w_{p,t} = p_t^2 + \zeta,$$

$$\Phi(p,t) = \begin{pmatrix} \displaystyle\int_0^a x w_{p,t}(x)\,dx - [c - \alpha(0)] \\ \displaystyle\int_a^{x_1} (x_1 - x) w_{p,t}(x)\,dx - c \end{pmatrix}.$$

The matrix $J_0 = D_t \Phi(q, 0)$ is invertible: its first row has signs $(+,-)$ and its second has signs $(+,+)$. Let

$$Q = \|q\|_{C^0([0,R])}, \qquad P_* = Q + 1, \qquad B = \sup_{[0,R]} (|g_0| + |g_1|),$$
$$W = \tfrac{1}{2}\max\{a^2, (x_1 - a)^2\}, \qquad \Lambda = \left\|J_0^{-1}\right\|_{\infty\to\infty}, \qquad A = W(2Q+1),$$

$$\begin{gathered} F = 2AB, \qquad G = 2WP_*^2 B^2, \qquad \tau = \min\{1, (4\Lambda G)^{-1}\}, \\ \eta_0 = \min\{1, (4\Lambda F)^{-1}, \tau/(2\Lambda A)\}, \qquad C_t = 2\Lambda A. \end{gathered} \tag{61}$$

For the differentiable estimate, define

$$\begin{aligned} A_* &= \left\|q^{(H+1)}\right\|_{C^0([0,R])} \max_{0\le j\le H} \frac{R^{H-j+1}}{(H-j)!}, \\ Q_H &= \|q\|_{C^H([0,R])}, \\ B_H &= \max_{0\le j\le H} \sum_{\ell=0}^{j} \binom{j}{\ell} \left(\left\|g_0^{(j-\ell)}\right\|_{C^0([0,R])} + \left\|g_1^{(j-\ell)}\right\|_{C^0([0,R])} \right), \\ C_{q,H} &= 1 + (Q_H + 1) C_t B_H, \\ C_H &= \max\{1, R, R^2/2\}\, 2^H C_{q,H}(2Q_H + C_{q,H}). \end{aligned}$$

Finally choose

$$\eta = \min\{\eta_0, \varepsilon/C_H\}, \qquad n = \max\left\{1, \left\lceil (A_*/\eta)^2 \right\rceil\right\}. \tag{62}$$

The extra lower bound $n \ge 1$ includes the case $A_* = 0$.

Proof of theorem 10.7. Let $B_n g$ be the degree-n Bernstein polynomial of $g = q^{(H)}$ on $[0, R]$. If Z has the binomial distribution with parameters n and x/R, then $B_n g(x) = \mathbb{E}[g(RZ/n)]$. The Lipschitz bound and Cauchy–Schwarz give

$$|B_n g(x) - g(x)| \le \left\|q^{(H+1)}\right\|_{C^0} \sqrt{\frac{x(R-x)}{n}} \le \frac{R}{2\sqrt{n}} \left\|q^{(H+1)}\right\|_{C^0}.$$

Subtract the error at a, and integrate the resulting polynomial H times, using the exact derivatives of q at a as the integration constants. The resulting polynomial p_n has degree at most $n + H$, retains $j_a^H q$, and satisfies

$$\left\|p_n^{(j)} - q^{(j)}\right\|_{C^0([0,R])} \le \frac{R^{H-j+1}}{(H-j)!\sqrt{n}} \left\|q^{(H+1)}\right\|_{C^0([0,R])}, \qquad 0 \le j \le H. \tag{63}$$

For $H = 0$, this just means the Bernstein polynomial corrected by a constant. Thus (62) ensures $\|p_n - q\|_{C^H} \le \eta$.

We next solve the two moment equations with quantitative control. Write $p = p_n$. Since $\eta \le 1$, one has $\|p\|_{C^0} \le P_*$. The moment residual satisfies

$$\|\Phi(p, 0)\|_\infty \le W(2Q + 1)\eta = A\eta.$$

For $\|t\|_\infty \le \tau$, differentiating the moments gives

$$\|D_t\Phi(p, t) - J_0\|_{\infty\to\infty} \le 2W(2Q + 1)B\eta + 2WP_*^2B^2\tau = F\eta + G\tau.$$

Consequently, the map $\mathcal{T}(t) = t - J_0^{-1}\Phi(p, t)$ has Lipschitz constant at most $1/2$ on the closed τ-ball. Also $\|\mathcal{T}(0)\|_\infty \le \Lambda A\eta \le \tau/2$, so the ball is invariant. The contraction theorem gives a fixed point with

$$\Phi(p, t) = 0, \qquad \|t\|_\infty \le 2\Lambda A\eta = C_t\eta. \tag{64}$$

The product rule, the definition of B_H, and (64) yield

$$\|p_t - q\|_{C^H} \le \eta + (Q_H + 1)C_tB_H\eta = C_{q,H}\eta.$$

Using $\|uv\|_{C^H} \le 2^H \|u\|_{C^H} \|v\|_{C^H}$, we obtain

$$\left\|p_t^2 - q^2\right\|_{C^H} \le 2^H C_{q,H}(2Q_H + C_{q,H})\eta.$$

Define the growth polynomial by

$$\widehat{\alpha}(x) = c - \int_a^x (x - z)\left[p_t(z)^2 + \zeta\right] dz.$$

Its second derivative is negative on the whole half-line. Integrating the preceding error estimate twice, with matching value and slope at a, gives

$$\|\widehat{\alpha} - \alpha\|_{C^{H+2}([0,R])} \le C_H\eta \le \varepsilon.$$

Because $J > H$, multiplication by $1 + t_0g_0 + t_1g_1$ retains the jet of p_n through order H. Squaring then retains the curvature jet, and integration retains the growth jet through order $H + 2 \ge K$. The two zero moments give the values at 0 and x_1 exactly. This proves (59) and the approximation assertion.

Finally,

$$\deg p_t \le n + H + J + 1, \qquad \deg \widehat{\alpha} \le 2n + 2H + 2J + 4.$$

Since the smallest even integer $J > H$ satisfies $J \le H + 2$, the last expression is at most $2n + 4H + 8 = 2n + 4\max\{K, L\}$, proving (60). □

For a specified target with a finite family of hyperbolic cycles, one may take $L = 2$ and choose ε below a persistence tolerance for those cycles. An exact-count argument additionally specifies the required displacement margins outside the cycle neighborhoods and the finite-jet condition near the equilibrium; the higher-L form of theorem 10.7 supplies the needed approximation without losing the endpoint or concavity constraints. The degree depends explicitly on these tolerances and on the derivatives of the target. Corollary 10.4 and propositions 10.5 and 10.6 show why such dependence cannot generally be removed.

11 Infinitely many cycles in one smooth profile

The finite realization theorem can be iterated. The central issue is persistence: later perturbations supported at smaller values of $|s|$ still affect the outer cycles, because those cycles also cross the strip where s is small. We choose the later perturbations so that their combined C^1 norm remains within the persistence neighborhood of every previously selected cycle.

Proof of theorem 1.5. Choose a globally strictly concave extension α_b of (6) by lemma 9.1, with $\alpha_b(0) > 0$. Its positive zero is denoted by x_1. Restrict the coordinate patch to a compact interval $I \Subset (0, x_1)$ containing a, on which $\alpha_b = \alpha_{\mathrm{cen}}$. The normalization at the end of section 2 permits us to work with $c = 1$. For general c, a perturbation $R(u)$ in the normalized coordinate $u = s/\sqrt{c}$ corresponds to the growth perturbation $cR(s(x)/\sqrt{c})$; flatness, smallness, and the following persistence argument are unchanged.

In the Duffing coordinates of section 5, write

$$s' = -Y + R(s), \qquad Y' = s - \frac{s^3}{2} + sR(s), \qquad K(s,Y) = \frac{Y^2}{2} + U(s), \quad U(s) = \frac{s^2}{2} - \frac{s^4}{8}.$$

Choose $r_1 > 0$ so small that $|s| \le r_1$ lies inside the coordinate patch and U is strictly increasing on $[0, r_1]$.

One insertion. Suppose that the current perturbation R_* vanishes on $|s| \le r$. Choose a regular oval $K = h_*$ whose maximal $|s|$ is less than $r/4$, and a compact interval E about h_* whose ovals all have maximal $|s| < r/2$. The ratio J_1/J_0 has positive derivative near zero, since

$$J_0(h) = 3\pi h^2(1 + O(h)), \qquad J_1(h) = 5\pi h^3(1 + O(h)).$$

For sufficiently small h_*, the odd polynomial

$$P(s) = s^3\left(s^2 - \frac{J_1(h_*)}{J_0(h_*)}\right)$$

has a Melnikov function $M(h) = \oint_{K=h} P(s)\,dY$ with a simple zero at h_*. Shorten E so that M' is positive on E and M has opposite signs at its endpoints.

Let χ_o be a smooth even cutoff taking values in $[0,1]$, equal to one on $|s| \le r/2$, and supported in $|s| < 3r/4$. Let χ_δ be another such cutoff, zero on $|s| \le \delta$, and one on $|s| \ge 2\delta$. The function $G_\delta = \chi_o\chi_\delta P$ has a Melnikov function converging to M in $C^1(E)$ as $\delta \downarrow 0$. Here is an explicit estimate. If $A(h) > 0$ is defined by $U(A(h)) = h$, then for odd G,

$$\oint_{K=h} G(s)\,dY = 4\int_0^{A(h)} \frac{G(s)U'(s)}{\sqrt{2(h - U(s))}}\,ds. \tag{65}$$

The cutoff changes this integral only on $[0, 2\delta]$. For $h \in E$ the denominator stays away from zero there, whereas $P(s)U'(s) = O(s^4)$. The integral difference, and its derivative in h, are therefore $O(\delta^5)$. Thus, for small fixed δ, the cutoff moment still has a simple zero inside E.

For every sufficiently small nonzero ε, the perturbation $R_* + \varepsilon G_\delta$ consequently creates a hyperbolic cycle near that oval. This follows from the C^1 return-map expansion

$$\Delta(h, \varepsilon) = \varepsilon \oint_{K=h} G_\delta(s)\,dY + O(\varepsilon^2).$$

The new perturbation remains zero on $|s| \le \delta$, and its amplitude can be made arbitrarily small in any prescribed finite collection of derivative norms.

Diagonal construction. Let $H_n = \varepsilon_n G_n$ and $R_n = \sum_{j=1}^{n} H_j$. Choose the insertion bands successively so that

$$\operatorname{supp} H_n \subset \{\delta_n \le |s| < 3r_n/4\}, \qquad r_{n+1} < \min(\delta_n/2, r_n/4).$$

The next insertion is then always made in an exact center region. Choose the associated energy intervals, and compact annular neighborhoods V_n of the inserted cycles, pairwise disjoint and converging to the origin.

After stage n, choose $\eta_n > 0$ such that any change of R_n of $C^1([-r_1, r_1])$ norm less than η_n preserves a hyperbolic cycle in V_n, with the same stability. This is possible by hyperbolic periodic-orbit persistence. At every later insertion choose the nonzero amplitude so small that

$$\|H_n\|_{C^{n+2}([-r_1,r_1])} \le 2^{-n-2} \min\{1, \kappa, \eta_1, \dots, \eta_{n-1}\}, \tag{66}$$

where the absent persistence radii at the first stage are omitted. Cycle creation places no lower positive bound on the amplitude, so these constraints are compatible with every insertion.

For each fixed derivative order, the tail of (66) is summable. Hence $R = \sum_{n\ge1} H_n$ converges in C^∞. Each summand vanishes near zero, and therefore $R^{(j)}(0) = 0$ for every j. For every fixed n,

$$\|R - R_n\|_{C^1} \le \sum_{j>n} \|H_j\|_{C^1} < \eta_n.$$

Thus the limiting system retains a hyperbolic cycle in every V_n. They are distinct, surround the same equilibrium, and converge to it.

Preservation of concavity. Composition with the fixed energy coordinate is bounded in C^2 on I. Choose κ sufficiently small that the total growth-rate perturbation has second derivative of absolute value less than $\frac{1}{2}\min_I(-\alpha_b'')$. Add this perturbation to α_b on the patch, extending it by zero outside. It is supported strictly inside the patch. The resulting global α is smooth and strictly concave, and has exactly the full jet of α_{cen} at a. Its value at zero and its positive zero x_1 are unchanged. The coordinate changes and positive changes of time preserve the hyperbolicity and nesting of the cycles. □

Corollary 11.1 (Nonanalyticity and limiting periods)**.** *The profile in theorem 1.5 is nonanalytic at a. The original-time periods T_n and nontrivial Floquet multipliers μ_n of its shrinking cycles satisfy*

$$T_n \longrightarrow \frac{2\pi}{\sqrt{mc}}, \qquad \mu_n \longrightarrow 1.$$

Proof. An analytic profile with the specified full jet would equal α_{cen} near a and have a center neighborhood, contradicting the presence of isolated cycles approaching the equilibrium. The linearization in the original time has eigenvalues $\pm i\sqrt{mc}$. On shrinking circles the angular flow converges uniformly to that of the linear oscillator, which gives the period limit. The multiplier limit follows from the divergence formula, since the divergence tends to its equilibrium value zero and the periods stay bounded. □

Individual cycles are hyperbolic, but the family is not uniformly hyperbolic near its accumulation point. The polynomial approximation of section 9 preserves any finite selected subfamily; it does not retain this entire infinite family in one polynomial model.

12 Fixed nonlinear responses and an elasticity threshold

The localization argument allows the functional response to be fixed in a considerably larger class. Consider

$$\dot{x} = x\alpha(x) - p(x)y = p(x)(h(x) - y), \qquad \dot{y} = yg(x), \qquad h(x) = \frac{x\alpha(x)}{p(x)}. \tag{67}$$

Here p, g are fixed smooth functions on the positive prey axis, $p(x) > 0$ there, and $a > 0$ is a simple predator break-even point: $g(a) = 0$, $g'(a) > 0$. Smooth extension to the boundary may be imposed when required for the ecological model; all the cycles below stay in the interior. The usual conversion relation is $g = kp - m$.

The classical generalized Filippov transformation [1, §3] is

$$s(x) = \operatorname{sgn}(x - a)\sqrt{2\int_a^x \frac{g(u)}{p(u)}\,du}, \qquad v = \log(y/c), \qquad c = h(a) > 0. \tag{68}$$

Near a, s is smooth and increasing, with $s'(a) = \sqrt{g'(a)/p(a)}$. Since $ss' = g/p$, the time factor $d\tau/dt = p(x)s'(x) = g(x)/s(x)$ is positive and gives

$$s' = h(x(s)) - ce^v, \qquad v' = s.$$

Thus the previous canonical constructions prescribe the prey isocline h. To control the actual per-capita rate $\alpha = (p/x)h$, an additional condition is needed.

Define the response elasticity by

$$\mathcal{E}_p(x) = \frac{xp'(x)}{p(x)}.$$

Theorem 12.1 (Strictly concave realization below elasticity two)**.** *If $\mathcal{E}_p(a) < 2$, then for every $N \geq 1$ there is a smooth globally strictly concave per-capita growth rate α, positive at zero, with a unique positive maximum and a positive carrying capacity, for which (67) has at least N hyperbolic positive limit cycles near the coexistence equilibrium at $x = a$.*

Conversely, a globally strictly concave α with $\alpha(0) > 0$ can have a nonhyperbolic coexistence equilibrium at this simple break-even point only if $\mathcal{E}_p(a) < 2$.

Proof. Put $r(x) = p(x)/x$. For sufficiency, first prescribe locally

$$\alpha_{\mathrm{loc}}(x) = r(x)\left[c - \frac{1}{2}s(x)^2\right].$$

At a this function satisfies

$$\alpha''_{\mathrm{loc}}(a) = cr''(a) - r(a)\frac{g'(a)}{p(a)} = cr''(a) - \frac{g'(a)}{a},$$
$$\alpha_{\mathrm{loc}}(a) - a\alpha'_{\mathrm{loc}}(a) = c\left[\frac{2p(a)}{a} - p'(a)\right] > 0.$$

Choose $c > 0$ sufficiently small that the second derivative is negative, and restrict to an interval on which strict concavity holds. The canonical isocline is the explicit center $h = c - s^2/2$. The normalization $s = \sqrt{c}\,u$ reduces it to section 5. Choose N sufficiently small target energies so that all corresponding ovals project strictly inside this interval. The moment interpolation theorem gives N simple Melnikov zeros. A sufficiently small nonzero amplitude creates N hyperbolic cycles, while preserving the two strict inequalities above and local strict concavity.

For the extension, choose in advance an interval so short that the weighted curvature integral on the left in the proof of lemma 9.1 is less than half the displayed positive quantity. Then choose the ovals inside a smaller interval and the perturbation sufficiently small. The same extension construction retains the entire prey projection of these cycles. It produces a global strictly concave function with $\alpha(0) > 0$, a positive maximum, and a positive right zero, without changing those cycles.

For necessity, the Jacobian at $(a, h(a))$ has trace $p(a)h'(a)$ and determinant $p(a)h(a)g'(a) > 0$. Nonhyperbolicity therefore implies $h'(a) = 0$. Strict concavity gives

$$x\alpha'(x) < \alpha(x) - \alpha(0) \qquad (x > 0).$$

On the other hand,

$$h'(x) = \frac{x\alpha'(x) + (1 - \mathcal{E}_p(x))\alpha(x)}{p(x)}. \tag{69}$$

At a, the condition $h'(a) = 0$ consequently yields

$$(2 - \mathcal{E}_p(a))\alpha(a) > \alpha(0) > 0,$$

which proves $\mathcal{E}_p(a) < 2$. □

The maximum of α need not occur at a for a nonlinear response. In the local center construction, $\alpha'(a) = c(p(x)/x)'|_{x=a}$. Thus the fixed-response statement distinguishes the maximum of per-capita growth from the maximum of the prey isocline.

Proposition 12.2 (A global exclusion condition)**.** *Suppose α is concave, $\alpha(0) > 0$, and has a positive zero x_1. If $\mathcal{E}_p(x) \geq 2$ for every $0 < x < x_1$, then (67) has no positive periodic orbit, for any smooth function g.*

Proof. Concavity and $\alpha(0) > 0$ imply $\alpha > 0$ on $(0, x_1)$ and $\alpha < 0$ to the right of x_1. At the maximum prey coordinate of a positive periodic orbit, $x\alpha(x) = p(x)y > 0$; thus every such orbit is contained in $0 < x < x_1$. By concavity and (69), throughout this strip

$$h'(x) \leq \frac{(2 - \mathcal{E}_p(x))\alpha(x) - \alpha(0)}{p(x)} < 0.$$

For the Dulac multiplier $B(x, y) = 1/[p(x)y]$, the divergence of the multiplied field is $h'(x)/y < 0$. The strip is simply connected, so the Bendixson–Dulac criterion excludes positive periodic orbits. □

The local condition at a and the global exclusion condition have different scopes. The single inequality $\mathcal{E}_p(a) \geq 2$ precludes a nonhyperbolic coexistence equilibrium under the stated concavity assumptions; it does not by itself exclude distant cycles.

Fixed response $p(x)$	Elasticity $\mathcal{E}_p(x)$	Local criterion
βx	1	Always satisfied
$Mx/(K + x)$	$K/(K + x) < 1$	Always satisfied
$Mx^2/(K^2 + x^2)$	$2K^2/(K^2 + x^2) < 2$	Always satisfied

Here $M, K > 0$. Each row admits the realization theorem at any positive simple break-even point. In particular, varying the response function is unnecessary for these local many-cycle constructions: it can be fixed in advance.

13 Algebraic verification and numerical illustrations

The constructions above are analytic existence and counting proofs. The separate computational supplement contains exact symbolic checks of selected instances of the degree construction and numerical illustrations of the Abelian-integral interpolation.

13.1 Exact algebra for the degree construction

The script `quantitative_computations/verify_quantitative_algebra.py` constructs A_D, Q_j, and b_j with exact symbolic arithmetic for the six pairs (N, K) in table 1. It verifies the degree and jet identities, the lowest nonzero order of every Q_j, the harmonic-moment identity (20), the curvature formula (16), and the signed Taylor remainder. The same-degree correction is checked against its two endpoint equations and its vanishing order at a. The degree-six example is recorded separately with rational coefficients.

The outputs contain every polynomial and coefficient, a degree table, and the individual verification results. These checks verify algebraic identities for the six selected pairs (N, K). The existence of a sufficiently small parameter interval, and the exclusion of all additional periodic orbits on that interval, are supplied by the analytic proof of theorem 1.2.

13.2 Moment interpolation for three and four cycles

The following computations illustrate the regular-annulus moment interpolation and locate selected cycles by direct integration. They use floating-point arithmetic and do not supply a certified global cycle count at the displayed parameter value.

Set $k = m = \beta = c = 1$ and use the local profile

$$\alpha(x) = 1 - \frac{s(x)^2}{2} + \delta s(x)^3 P_N(s(x)^2), \qquad s(x) = \operatorname{sgn}(x-1)\sqrt{2(x - 1 - \log x)}. \tag{70}$$

Write $P_N(z) = \sum_{j=0}^{N} b_j z^j$ and normalize $b_N = -1$. For $N = 3$, prescribe Melnikov zeros at $0.05, 0.10, 0.15$; for $N = 4$, prescribe them at $0.04, 0.08, 0.12, 0.16$. Solving $\sum b_j J_j(h_i) = 0$ with 80 decimal digits gives approximately

$$\begin{aligned} P_3(z) &= 0.00462584243209378 - 0.0975516389901764z + 0.575793634905719z^2 - z^3, \\ P_4(z) &= -0.000734820723917012 + 0.0220639437260988z - 0.209685998923226z^2 \\ &\quad + 0.783964939740499z^3 - z^4. \end{aligned}$$

The complete coefficients are included in the accompanying CSV and JSON files. Independent quadrature of the defining moment integral agrees with (36) with a computed relative discrepancy below 3.2×10^{-81} at the check energy $h = 0.117$. Figure 1 shows the prescribed zeros.

13.3 Direct return maps and multipliers

To integrate the system efficiently, put $q = y - 1$ and $p = s\sqrt{y}$. Here p denotes an oscillator coordinate. With $d\sigma/d\tau = \sqrt{1+q}$, the exact equations are

$$\frac{dq}{d\sigma} = p, \qquad \frac{dp}{d\sigma} = -q + \delta \frac{p^3}{(1+q)^{3/2}} P_N\left(\frac{p^2}{1+q}\right). \tag{71}$$

Starting at $q = 0$, $p = \sqrt{2h} > 0$, use the next positive crossing of $q = 0$ to define the return map in the coordinate $h = p^2/2$. Its fixed points are found by bracketed scalar root searches.

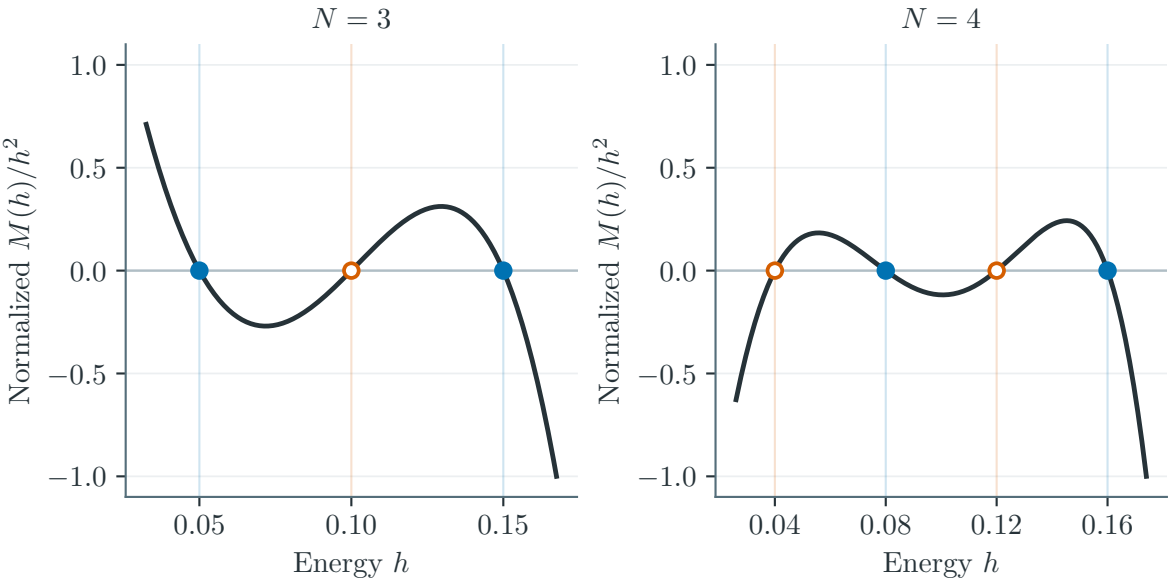


Figure 1: The interpolating Melnikov functions for three and four selected cycles. Each is divided by h^2 and then by its maximum absolute value on the displayed interval. These positive normalizations preserve zeros and their crossing signs. Filled blue markers indicate the attracting branches for positive δ; open orange markers indicate repelling branches.

The transverse multiplier is computed by integrating the divergence of (71) over the periodic orbit. An independent calculation in the original variables uses

$$\log \mu = \int_\Gamma x\alpha'(x)\, dt.$$

This formula follows by integrating the full divergence and using $\int(\alpha(x)-y)dt = \int(x-1)dt = 0$ on a periodic orbit.

For $\delta = 0.1$, the results are in table 2. These examples keep the interpolating coefficients fixed as δ changes; their actual section positions therefore differ slightly from the prescribed first-order energies. The exact placement construction of section 5 would additionally adjust those coefficients.

N	Target h_i	Actual section energy	Multiplier μ	Stability
3	0.05	0.0499999109	0.999918785905	Attracting
3	0.10	0.1000002816	1.000183252618	Repelling
3	0.15	0.1499988200	0.999059900849	Attracting
4	0.04	0.0400000066	1.000007729473	Repelling
4	0.08	0.0799999857	0.999988415660	Attracting
4	0.12	0.1200000385	1.000029541335	Repelling
4	0.16	0.1599998569	0.999819716489	Attracting

Table 2: Numerically located fixed points of the exact return map for $\delta = 0.1$. Energies are rounded to ten decimal places. The computed multipliers indicate alternating stability. The calculations use floating-point arithmetic without interval bounds.

The DOP853 integrations were repeated with two controls: $(\mathrm{rtol}, \mathrm{atol}, \mathrm{max_step}) = (2 \cdot 10^{-12}, 2 \cdot 10^{-14}, 0.075)$ and $(5 \cdot 10^{-14}, 2 \cdot 10^{-15}, 0.02)$. The largest change in a section energy was

below 10^{-11}; the largest multiplier change was below 10^{-13}. The two divergence calculations of $\log \mu$ agreed to within 2×10^{-15}.

To recover the original prey coordinate, the code uses

$$x = -W_0(-e^{-1-s^2/2}) \quad (s < 0), \qquad x = -W_{-1}(-e^{-1-s^2/2}) \quad (s > 0),$$

with a local inverse series near the branch point. The original predator coordinate is $y = 1 + q$. The three selected cycles in these coordinates are shown in figure 2.

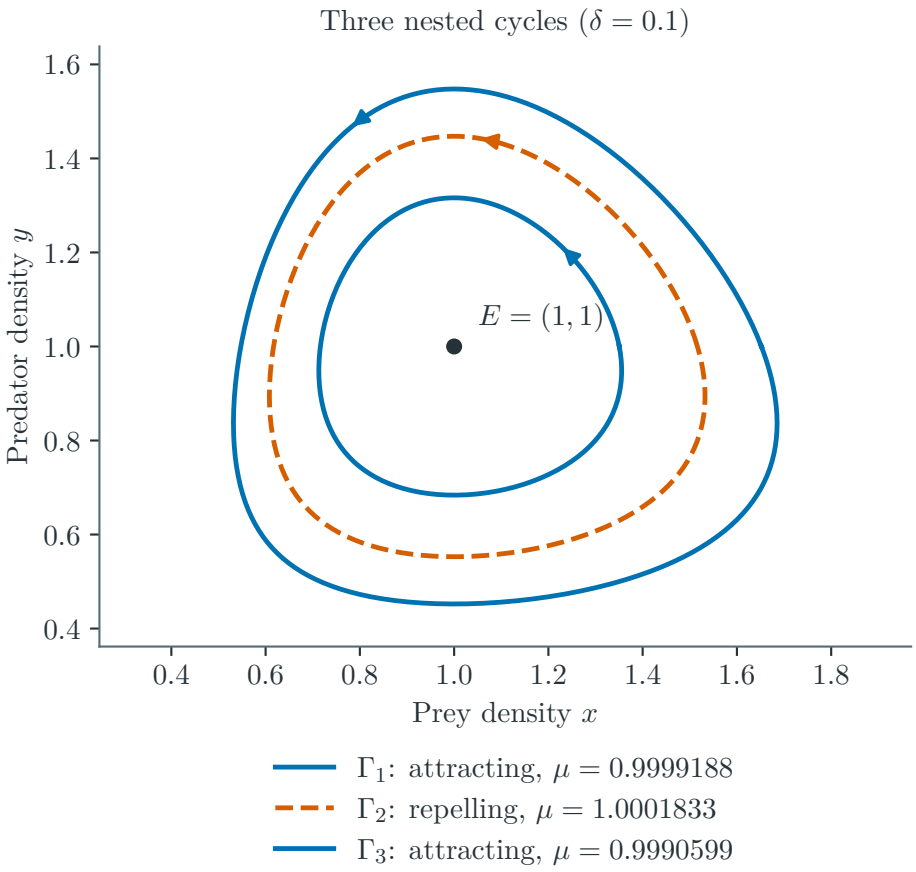


Figure 2: Three numerically located limit cycles in the original positive quadrant for the local profile (70). Solid blue curves are attracting, the dashed orange curve is repelling, and arrows show the flow direction. All surround $E = (1, 1)$.

13.4 Shape checks and reproduction

The analytic expression for α'' was evaluated on intervals slightly larger than the cycles' prey projections. A grid of 50,001 points, together with the peak point and local extremum refinement, gave the estimates

N	checked prey interval	estimated maximum of α''
3	[0.5086701260, 1.7083293776]	−0.3636065093
4	[0.4945100775, 1.7384221971]	−0.3383078145.

These are numerical checks of local strict concavity. The global smooth and polynomial shape conclusions rest on the extension and approximation proofs, not on this grid. In particular,

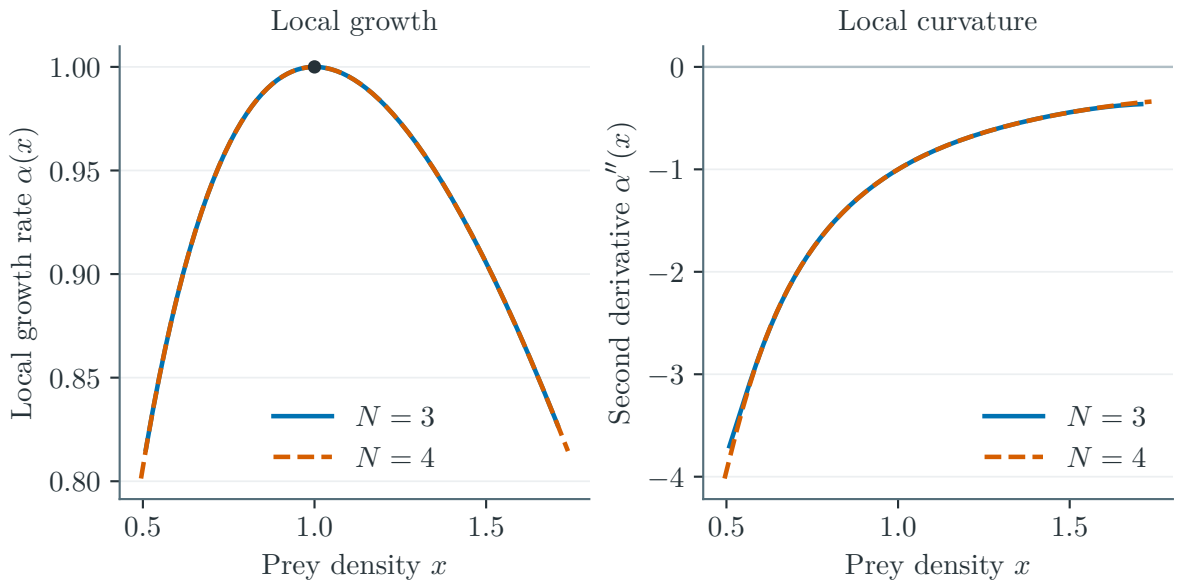


Figure 3: The local growth profiles in the three-cycle and four-cycle examples, together with their analytically evaluated second derivatives on the checked intervals. The two growth curves are visually close. This figure illustrates strict concavity locally; the global realization is supplied by the analytic construction.

(70) contains $\log x$ and is not itself claimed to be the final growth law at $x = 0$ or for arbitrarily large x. The local profiles and their second derivatives appear in figure 3.

The separate computational supplement contains `computations/reproduce_cycles.py`, instructions, all coefficients and cycle samples, and editable figure sources. Running the script with Python 3.10 or later, NumPy, SciPy, mpmath, Matplotlib, and a LaTeX installation recreates the numerical outputs and figures. The supplied results record the software versions, solver settings, and all sensitivity comparisons. No numerical assertion is used to prove the existence, density, cyclicity, or infinite-cycle theorems.

14 Further questions

The construction gives the sufficient degree $D = 2N + 2\lceil K/2 \rceil + 2$ for its chosen center jet. The jet-only lower bound is $2\lceil K/2 \rceil$, and the endpoint budget and the sizes of higher derivatives give additional independent obstructions. Determining the smallest degree compatible with all these data, or a sharp global cycle bound at a fixed polynomial degree, requires further information beyond the three-scale construction.

A separate problem is to impose independently specified low-density growth, carrying capacity, and focus jets while requiring a global exact count. The parity law is necessary when the focus stability is fixed. The arbitrary-data approximation theorem realizes any finite lower bound on the number of cycles, whereas the exact-count theorems use center jets and compatible global reference profiles. A classification of exact realization for arbitrary prescribed data would connect these two forms of flexibility.

The explicit coefficients and concavity inequalities also permit quantitative work on the small-parameter range. The present proof gives $\rho_0 > 0$ by compactness and uniform return-map estimates. Certified bounds on ρ_0, return-map remainders, and multiplier separation would turn the symbolic families into finite numerical certificates at selected parameter values. The high-order focus makes these estimates more demanding as N or K increases.

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