hexagon:2611.00396v1

Quadratic Strictly Positive Real Factors of Stable Transfer Functions

Jinlin Wang Zhenyu Sun

Abstract

We characterize factorizations $G = G_0 \circ H$ in which G and G_0 are stable, minimum-phase, biproper real-rational functions with simple poles and zeros and positive gains at infinity, and H is a quadratic strictly positive real function. Classical rational decomposition identifies the quadratic right-factor classes with real Möbius involutions fixing G. Within each class, we parameterize all normalized representatives and eliminate the scaling parameter and the frequency variable. The resulting sign conditions involve univariate polynomials of degree at most four. They determine all feasible denominator intervals and all factorizations in the class; for real-algebraic input, they give an exact decision procedure. We also give a scaling that maximizes the smaller of the inner positive-real margin and the outer spectral margin for a fixed denominator. For quadratic input, existence is equivalent to strict positive realness of G and to an explicit coefficient inequality. Examples exhibit a disconnected feasible parameter set, separate algebraic and analytic obstructions, and a quartic input with exactly three admissible classes.

Keywords. Rational decomposition; strictly positive real function; Möbius involution; Hurwitz polynomial; transfer function substitution.

1 Introduction

Positive-real functions arise in the analysis and synthesis of passive networks [3]. Substituting strictly positive real functions for the Laplace variable has been studied in stabilization, robustness, and composition problems; see [6, 8, 7]. State-space realizations of rational compositions and applications to electrical circuits and feedback networks are developed in [2]. The associated inverse problem asks whether a given transfer function is itself a composition with a positive-real inner function.

Fernández-Anaya and Martínez-García formulate this inverse question in [9, Problem 1.1, Problem 2]. Their formulation seeks an inner function in SPR_0, the strictly positive real class of relative degree zero, and asks for its largest possible denominator degree. We study quadratic inner factors and require the outer function to be stable, minimum-phase, and biproper. The input has the same spectral properties and has simple poles and zeros. We also require positive gains at infinity. These are the conditions imposed throughout this paper. An example in Section 7.3 shows that the condition on the outer factor changes the existence question.

At the algebraic level, rational decomposition is governed by intermediate fields and Möbius reparametrization. Algorithms and structural results are given in [11, 1, 4]. For a quadratic right factor, the corresponding function-field extension has a unique nontrivial automorphism. Thus algebraic quadratic factor classes correspond to involutions in the fixing group of the input rational function. We use this classical correspondence and the fixing-group viewpoint of [10] to separate the finite algebraic choice from the analytic existence question within each class.

For a fixed algebraic class, the analytic constraints reduce to a problem in one real parameter. If the corresponding involution is

$$\tau(s) = -\frac{bs + c}{as + b}, \qquad b^2 - ac \neq 0,$$

every quadratic representative normalized by $H(\infty) = 1$ has the form

$$H(s) = 1 - t\frac{as + b}{s^2 + d_1 s + d_0}, \qquad c = bd_1 - ad_0.$$

The denominator has one free real parameter. We eliminate t by comparing the real parts of $(as + b)/(s^2 + d_1 s + d_0)$ on the input spectrum and on the imaginary axis. We then eliminate the frequency variable by a quadratic positivity test. The final conditions involve univariate polynomials of degree at most four in the remaining denominator parameter.

The spectral separation criterion in Theorem 4.1 gives the full interval of admissible scalings for each denominator. Theorem 4.2 expresses denominator feasibility by polynomial signs, and Corollary 4.3 combines this criterion with the finite algebraic classification. There are at most $n - 1$ quadratic classes for an input of degree n. Proposition 4.5 selects an optimal scaling for each feasible denominator, and Theorem 6.1 gives an exact procedure for real-algebraic input. The quadratic-input criterion and examples in Sections 5 and 7 describe both existence and the distinct algebraic and analytic obstructions.

The algebraic classification uses the classical degree law and the intermediate-field correspondence. Our analysis determines which representatives in each class satisfy the simultaneous positive-real and outer spectral constraints, and reconstructs every such factorization.

2 The constrained problem and algebraic candidates

2.1 Transfer functions and normalization

Write

$$\mathbb{C}_+ = \{s \in \mathbb{C} : \operatorname{Re} s > 0\}, \qquad \mathbb{C}_- = \{s \in \mathbb{C} : \operatorname{Re} s < 0\}, \qquad K_+ = \overline{\mathbb{C}_+} \cup \{\infty\} \subset \mathbb{P}^1(\mathbb{C}).$$

A real polynomial is Hurwitz if all its roots lie in $\mathbb{C}_-$. The degree of a reduced rational function P/Q is $\max(\deg P, \deg Q)$. For a proper scalar transfer function this is its McMillan degree. A rational function is biproper if its numerator and denominator have equal degrees.

Definition 2.1. For $n \geq 1$, let $\mathcal{S}_n$ be the class of real-rational functions of degree n that are biproper, have all poles and zeros simple and in $\mathbb{C}_-$, and satisfy $G(\infty) > 0$. Numerators and denominators are always reduced.

A real-rational function H belongs to SPR_0 if it is biproper, its denominator is Hurwitz,

$$\operatorname{Re} H(i\omega) > 0 \quad (\omega \in \mathbb{R}), \qquad H(\infty) > 0.$$

The subclass of rational degree two is denoted by $\mathrm{SPR}_0^{(2)}$. The poles and zeros of an inner function in $\mathrm{SPR}_0^{(2)}$ may be repeated.

This is the rational frequency-domain convention for strict positive realness used here. The positive value at infinity supplies a uniform strict margin on the compactified imaginary axis.

Lemma 2.2 (Half-plane property)**.** *If $H \in \mathrm{SPR}_0$, then H is analytic on a neighborhood of K_+, and*

$$\min_{s \in K_+} \operatorname{Re} H(s) > 0.$$

In particular, every solution of $H(s) = \alpha$, with $\operatorname{Re}\alpha < 0$, lies in $\mathbb{C}_-$.

Proof. The denominator is Hurwitz and H is analytic at infinity. The continuous boundary values on $i\mathbb{R} \cup \{\infty\}$ have a positive minimum. The harmonic minimum principle, applied after a Cayley map from the half-plane to the disk, gives the same lower bound in the half-plane. Compactness then gives the assertion on K_+. □

Definition 2.3. An admissible quadratic factorization of $G \in \mathcal{S}_n$ is an identity

$$G = G_0 \circ H, \qquad H \in \mathrm{SPR}_0^{(2)}, \qquad G_0 \in \mathcal{S}_m$$

for some $m \geq 1$. Two such factorizations are equivalent if

$$\widetilde{H} = \phi \circ H, \qquad \widetilde{G}_0 = G_0 \circ \phi^{-1}$$

for a real Möbius transformation ϕ. Let $\mathcal{F}_2(G)$ denote the set of equivalence classes.

The standard degree law for rational composition [4, Lemma 1.1] gives $n = 2m$. If $r = H(\infty) > 0$, the replacement

$$H^\sharp = H/r, \qquad G_0^\sharp(w) = G_0(rw)$$

preserves all the conditions and yields $H^\sharp(\infty) = 1$. Hence it suffices to classify normalized representatives. A real Möbius transformation relating two normalized representatives fixes 1; normalization does not select a unique representative in a class.

2.2 Quadratic factors and involutions

Define

$$\Gamma_G = \{\gamma \in \mathrm{PGL}_2(\mathbb{R}) : G \circ \gamma = G\}, \qquad \mathcal{I}(G) = \{\tau \in \Gamma_G : \tau^2 = \mathrm{id},\ \tau \neq \mathrm{id}\}.$$

Proposition 2.4 (Classical quadratic correspondence)**.** *The equivalence classes of real rational decompositions $G = F \circ J$ with $\deg J = 2$, without analytic constraints on F, J, are in bijection with $\mathcal{I}(G)$. For $\tau \in \mathcal{I}(G)$, the corresponding field is*

$$\mathbb{R}(J) = \mathbb{R}(s)^{\langle \tau \rangle}.$$

Proof. The degree formula for function fields gives $[\mathbb{R}(s) : \mathbb{R}(J)] = 2$. In characteristic zero this quadratic extension has a unique nonidentity automorphism τ. Every $\mathbb{R}$-automorphism of $\mathbb{R}(s)$ is a real Möbius transformation, so $J \circ \tau = J$, and then $G \circ \tau = G$.

Conversely, an involution has a trace-zero matrix representative and can be written

$$\tau(s) = -\frac{bs + c}{as + b}, \qquad [a : b : c] \in \mathbb{RP}^2, \qquad \delta := b^2 - ac \neq 0. \tag{2.1}$$

Indeed its matrix square is a scalar matrix, and the Cayley–Hamilton identity forces trace zero for a nonidentity projective transformation. If $a \neq 0$, the invariant function

$$J_\tau(s) = s + \tau(s) = \frac{as^2 - c}{as + b}$$

has degree two: its numerator does not vanish at $-b/a$, since its value there is δ/a. If $a = 0$, then $b \neq 0$, and the invariant polynomial

$$J_\tau(s) = s^2 + (c/b)s$$

has degree two. In either case the quadratic extension $\mathbb{R}(s)/\mathbb{R}(J_\tau)$ has automorphism τ, and its fixed field is $\mathbb{R}(J_\tau)$. Thus $G = F \circ J_\tau$ for a real rational function F.

Two degree-two generators of the same field differ by a degree-one real rational function, hence by a real Möbius transformation. This proves the bijection. The general intermediate-field and fixing-group formulation is given in [4, 10]. □

Proposition 2.5 (Finiteness)**.** *For $G \in \mathcal{S}_n$, the group Γ_G acts freely on the pole set of G. Consequently*

$$|\Gamma_G| \mid n, \qquad |\mathcal{I}(G)| \le n-1, \qquad |\mathcal{F}_2(G)| \le n-1.$$

If n is odd, $\mathcal{I}(G)$ and $\mathcal{F}_2(G)$ are empty. Every involution in $\mathcal{I}(G)$ pairs the poles and the zeros of G without fixed points.

Proof. The identity $G \circ \gamma = G$ implies that γ permutes the poles. If γ fixes a simple pole p, the function $1/G$ has nonzero derivative at p, and is locally one-to-one. The identity $(1/G)\circ\gamma = 1/G$ forces γ to be the identity near p, and therefore globally. The map $\gamma \mapsto \gamma(p)$ is consequently injective. Thus the group is finite, and each pole orbit has $|\Gamma_G|$ elements. This proves divisibility and the bounds. An involution acts without fixed points on the pole set, so n is even. The same local argument applied to a simple zero proves the last statement. □

The finite classification above concerns factor classes. Each admissible class can contain a continuum of normalized representatives. The remaining sections determine this continuous family exactly.

2.3 Candidate extraction from poles, zeros, or coefficients

Let $P(G)$ and $Z(G)$ denote the pole and zero sets. The graph of (2.1), in affine coordinates, is

$$q_\tau(u,v) := auv + b(u+v) + c = 0. \tag{2.2}$$

It is nondegenerate because $\delta \neq 0$. When $a = 0$ this polynomial is affine-linear; its projective graph still has bidegree $(1,1)$.

Proposition 2.6 (A finite candidate list)**.** *Fix $p \in P(G)$ and $z \in Z(G)$. For each*

$$p' \in P(G) \setminus \{p\}, \qquad z' \in Z(G) \setminus \{z\},$$

solve

$$\begin{pmatrix} pp' & p+p' & 1 \\ zz' & z+z' & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{2.3}$$

over $\mathbb{R}$, interpreting complex equations through their real and imaginary parts. Each system produces at most one real projective candidate. After duplicates are removed, retain precisely those candidates for which τ pairs both $P(G)$ and $Z(G)$ without fixed points. The retained set is $\mathcal{I}(G)$. At most $(n-1)^2$ candidates are generated.

Proof. The two rows in (2.3) are linearly independent over $\mathbb{C}$. Otherwise their last coordinates imply equality of the rows. The two pairs would have equal sums and products, hence would be the same unordered pair, contrary to $P(G) \cap Z(G) = \varnothing$. The real kernel therefore has dimension at most one.

Any nonzero solution is nondegenerate. If $a \neq 0$ and $b^2 = ac$, then

$$q_\tau(u,v) = \frac{(au+b)(av+b)}{a}.$$

Both disjoint pairs in (2.3) would have to contain $-b/a$, which is impossible. If $a = 0$ and $b^2 = ac$, then $b = 0$, and the equations force $c = 0$.

Every element of $\mathcal{I}(G)$ occurs by taking $p' = \tau(p)$ and $z' = \tau(z)$. Conversely, if a candidate passes the pairing test, then $G \circ \tau$ and G have the same divisor on $\mathbb{P}^1$, so $G \circ \tau = \lambda G$ for a nonzero constant λ. A nonidentity involution has two distinct fixed points on $\mathbb{P}^1(\mathbb{C})$. Neither is a pole or zero of G, by the pairing test. Evaluation at either fixed point gives $\lambda = 1$; this is also valid at infinity because $G(\infty)$ is finite and nonzero. □

Proposition 2.7 (Coefficient recognition)**.** *Write $G = P/Q$ in reduced form and put*

$$B_G(u,v) = \frac{P(u)Q(v) - P(v)Q(u)}{u - v}. \tag{2.4}$$

The elements of $\mathcal{I}(G)$ correspond exactly to the factors of B_G of the form

$$auv + b(u+v) + c, \qquad a,b,c \in \mathbb{R}, \qquad b^2 - ac \neq 0,$$

taken up to nonzero real multiples.

Proof. Such a polynomial is irreducible over $\mathbb{C}$ and defines the graph $v = \tau(u)$. If $G \circ \tau = G$, the numerator in (2.4) vanishes on that graph. The graph polynomial divides it, and is coprime to $u - v$; hence it divides B_G. Conversely, a factor of the indicated form gives $G \circ \tau = G$ by substitution. The resulting rational identity extends to $\mathbb{P}^1$, including the points omitted by the affine formula. □

The numerator in (2.4) is the near-separated polynomial used in rational decomposition; see [1, 4]. Division by the diagonal factor isolates the nontrivial fiber relations. For exact rational input, real involutions may have real-algebraic coefficients. A coefficient-based implementation must therefore allow real-algebraic extensions. Factorization over $\mathbb{Q}$ alone need not expose all real quadratic classes.

3 Normalized representatives of a quadratic class

Fix $\tau \in \mathcal{I}(G)$, and fix a representative (a,b,c) in (2.1). A set T_τ containing one point from each pole pair and one point from each zero pair has cardinality n. We first describe every normalized inner function in this class. Its values on T_τ will then determine the outer poles and zeros.

Lemma 3.1 (Normalized representatives)**.** *Every real degree-two biproper function H with $H(\infty) = 1$ and $H \circ \tau = H$ has a unique expression*

$$H_{D,t}(s) = 1 - tL_D(s), \qquad L_D(s) = \frac{as+b}{D(s)}, \qquad D(s) = s^2 + d_1 s + d_0, \tag{3.1}$$

where

$$t \neq 0, \qquad c = bd_1 - ad_0. \tag{3.2}$$

Conversely, (3.1)–(3.2) define such a function. Moreover,

$$\operatorname{Res}_s(D - t(as+b), D) = t^2(b^2 - ac) \neq 0. \tag{3.3}$$

Proof. Write $H = N/D$ as a reduced ratio of monic quadratics, $N = s^2 + n_1 s + n_0$. Direct expansion gives

$$N(u)D(v) - N(v)D(u) = (u-v)\big(Auv + B(u+v) + C\big),$$

where

$$A = d_1 - n_1, \quad B = d_0 - n_0, \quad C = n_1 d_0 - n_0 d_1 = Bd_1 - Ad_0.$$

The nontrivial fiber graph is the graph of τ, so $(A,B,C) = t(a,b,c)$ for a unique nonzero real t. This gives (3.1) and (3.2). Conversely, the same expansion gives the prescribed graph.

To compute the resultant, use $N - D = -t(as+b)$. If $a \neq 0$, the possible common root is $-b/a$, and

$$a^2 D(-b/a) = b^2 - abd_1 + a^2 d_0 = b^2 - ac.$$

The resultant is $t^2(b^2 - ac)$. If $a = 0$, then $b \neq 0$; $N - D = -tb$ gives the same formula. Hence the functions in the converse are reduced and have degree two. Uniqueness follows from their monic numerator and denominator. □

The involution fixes precisely the ramification points of H. At finite points they satisfy $as^2 + 2bs + c = 0$. By Proposition 2.5, none belongs to $P(G) \cup Z(G)$.

For a single real denominator parameter θ, use one of the following two projective normalizations:

Representative	$d_1(\theta)$	$d_0(\theta)$	Θ_τ
$a = 1$	θ	$b\theta - c$	$\{\theta : \theta > 0,\ b\theta - c > 0\}$
$a = 0,\ b = 1$	c	θ	$(0, \infty)$ if $c > 0$, $\varnothing$ if $c \le 0$.

(3.4)

Thus Θ_τ is exactly the set of Hurwitz denominators in the normalized class. Both rows are needed, including the affine involutions in the second row.

Lemma 3.2 (Quadratic SPR test)**.** *Let*

$$H(s) = \frac{s^2 + n_1 s + n_0}{s^2 + d_1 s + d_0}$$

be reduced, with $d_1, d_0 > 0$. Set

$$p = n_1 d_1 - n_0 - d_0, \qquad r = n_0 d_0.$$

Then $H \in \mathrm{SPR}_0^{(2)}$ if and only if

$$r > 0, \qquad (p \ge 0 \text{ or } p^2 < 4r). \tag{3.5}$$

Proof. For $x = \omega^2$,

$$\operatorname{Re} H(i\omega) = \frac{x^2 + px + r}{(d_0 - x)^2 + d_1^2 x}.$$

The denominator is positive for $x \ge 0$. The numerator is positive on $[0, \infty)$ precisely when $r > 0$ and either its vertex lies at a nonpositive point, $p \ge 0$, or its minimum $r - p^2/4$ is positive. □

Proposition 3.3 (Spectral reconstruction)**.** *Fix $\tau \in \mathcal{I}(G)$ and a representative $H_{D,t}$ from Lemma 3.1. Suppose $H_{D,t} \in \mathrm{SPR}_0^{(2)}$ and*

$$D(z) \neq 0, \qquad \operatorname{Re} H_{D,t}(z) < 0 \quad (z \in T_\tau). \tag{3.6}$$

Let $\alpha_1, \dots, \alpha_m$ be its values on the pole pairs and $\beta_1, \dots, \beta_m$ its values on the zero pairs, where $m = n/2$. Then the unique outer factor is

$$G_{0,D,t}(w) = \kappa_{D,t} \prod_{j=1}^{m} \frac{w - \beta_j}{w - \alpha_j}, \qquad \kappa_{D,t} = G(\infty) \prod_{j=1}^{m} \frac{1 - \alpha_j}{1 - \beta_j}. \tag{3.7}$$

It belongs to $\mathcal{S}_m$, and $G = G_{0,D,t} \circ H_{D,t}$. Conversely, every normalized admissible factorization in the class of τ satisfies (3.6) and is obtained in this way.

Proof. The identity $H_{D,t} \circ \tau = H_{D,t}$ shows that if a representative point is not a pole of $H_{D,t}$, neither is its partner. Each prescribed pair consists of two different points and is invariant under τ; it is therefore a fiber of the degree-two function $H_{D,t}$. Different pairs have different values, since a degree-two map cannot have four distinct points in one fiber. This also separates pole values from zero values. The two lists are invariant under conjugation, since $H_{D,t}$ and G have real coefficients. Their values lie in $\mathbb{C}_-$. Thus (3.7) is real-rational, has simple separated poles and zeros in $\mathbb{C}_-$, and has positive leading gain: each negative real value contributes a positive factor $1 - \alpha_j$ or $1 - \beta_j$, and each nonreal conjugate pair contributes a squared modulus.

The composition in (3.7) has exactly the prescribed pole and zero fibers, all simple. Poles of $H_{D,t}$ produce neither poles nor zeros of the composition because $G_{0,D,t}(\infty) = \kappa_{D,t} \neq 0, \infty$. At infinity the composition equals $G_{0,D,t}(1) = G(\infty)$. Equality of divisors and of the value at infinity proves the identity. Uniqueness follows because substitution by a nonconstant rational function is injective. The converse follows from the actual outer poles and zeros. □

4 A one-parameter analytic criterion

4.1 Eliminating the scaling parameter

For $H_{D,t} = 1 - tL_D$, strict positive realness requires $t \operatorname{Re} L_D(i\omega) < 1$, whereas the outer spectral constraints require $t \operatorname{Re} L_D(z) > 1$. Thus a common scaling exists exactly when the spectral values lie strictly on one side of the boundary range of $\operatorname{Re} L_D$.

Fix a Hurwitz denominator D satisfying (3.2). On the compactified imaginary axis put

$$m_D = \min_{\omega \in \mathbb{R} \cup \{\infty\}} \operatorname{Re} L_D(i\omega), \qquad M_D = \max_{\omega \in \mathbb{R} \cup \{\infty\}} \operatorname{Re} L_D(i\omega). \tag{4.1}$$

The values are finite and attained, and $m_D \le 0 \le M_D$, since $L_D(\infty) = 0$. When $D(z) \neq 0$ for $z \in T_\tau$, put

$$r_z(D) = \operatorname{Re} L_D(z), \qquad r_-(D) = \min_{z \in T_\tau} r_z(D), \qquad r_+(D) = \max_{z \in T_\tau} r_z(D). \tag{4.2}$$

Theorem 4.1 (Spectral separation and all scaling intervals)**.** *For the fixed denominator D, an admissible normalized factorization exists if and only if $D(z) \neq 0$ on T_τ and one of the following strict separation conditions holds:*

$$r_-(D) > M_D, \tag{4.3}$$

$$r_+(D) < m_D. \tag{4.4}$$

The two conditions are mutually exclusive. All corresponding scaling parameters are, respectively,

$$\mathcal{J}_D^+ = \left(\frac{1}{r_-(D)}, \frac{1}{M_D} \right), \qquad \text{under (4.3)}, \tag{4.5}$$

$$\mathcal{J}_D^- = \left(\frac{1}{m_D}, \frac{1}{r_+(D)} \right), \qquad \text{under (4.4)}. \tag{4.6}$$

In these formulas $1/M_D = +\infty$ when $M_D = 0$, and $1/m_D = -\infty$ when $m_D = 0$.

Proof. The frequency inequalities for $H_{D,t} = 1 - tL_D$ give the open interval

$$\frac{1}{m_D} < t < \frac{1}{M_D},$$

with the stated conventions. Its point $t = 0$ represents a constant function and will be excluded by the spectral inequalities. Those inequalities are exactly

$$t\, r_z(D) > 1 \qquad (z \in T_\tau).$$

If $t > 0$, they require $r_-(D) > 0$ and $t > 1/r_-(D)$. Intersection with the SPR interval is nonempty precisely when $r_-(D) > M_D$, and gives (4.5). If $t < 0$, they require $r_+(D) < 0$ and $t < 1/r_+(D)$, giving (4.4) and (4.6). Since $m_D \le 0 \le M_D$, both separation conditions cannot hold. Proposition 3.3 completes the proof. □

The extrema in this theorem require only a quadratic equation. Set

$$e = ad_1 - b, \qquad f = bd_0, \qquad q = d_1^2 - 2d_0, \qquad r = d_0^2. \tag{4.7}$$

For $x = \omega^2$,

$$\operatorname{Re} L_D(i\omega) = R_D(x) := \frac{ex + f}{x^2 + qx + r}. \tag{4.8}$$

The denominator equals $(d_0 - x)^2 + d_1^2 x > 0$. Consequently m_D, M_D are obtained by evaluating R_D at 0, at infinity with value zero, and at the nonnegative real roots of

$$ex^2 + 2fx + fq - er = 0. \tag{4.9}$$

This follows by differentiating (4.8). When $e = 0$, the equation is linear; $e = f = 0$ is excluded by $d_0, d_1 > 0$ and $(a, b) \neq (0, 0)$.

4.2 Eliminating the frequency variable

The difference between a spectral value and a boundary value has a quadratic numerator in $x = \omega^2$. Testing the strict sign of this difference on the compactified interval $[0, \infty]$ eliminates x, leaving only the denominator parameter.

Use one of the normalized charts in (3.4), and regard $D = D_\theta$ as a function of its single parameter. For $z \in T_\tau$, define the real polynomials

$$U_z(\theta) = \mathrm{Re}\big((az+b)\overline{D_\theta(z)}\big), \qquad V_z(\theta) = |D_\theta(z)|^2, \tag{4.10}$$

$$B_z(\theta) = qU_z - eV_z, \qquad C_z(\theta) = rU_z - fV_z, \tag{4.11}$$

$$E_z(\theta) = 4U_zC_z - B_z^2. \tag{4.12}$$

Here e, f, q, r are evaluated at $d_0(\theta), d_1(\theta)$.

Theorem 4.2 (Univariate polynomial criterion)**.** *Fix $\tau \in \mathcal{I}(G)$. A parameter $\theta \in \Theta_\tau$ admits an admissible normalized factorization in its class if and only if either*

$$\left.\begin{array}{ll} U_z > 0, & C_z > 0, \\ B_z \geq 0 \quad or & E_z > 0 \end{array}\right\} \quad \text{for every } z \in T_\tau, \tag{4.13}$$

or

$$\left.\begin{array}{ll} U_z < 0, & C_z < 0, \\ B_z \leq 0 \quad or & E_z > 0 \end{array}\right\} \quad \text{for every } z \in T_\tau. \tag{4.14}$$

These are respectively the positive and negative branches of Theorem 4.1. In either chart,

$$\deg_\theta U_z \leq 1, \quad \deg_\theta V_z \leq 2, \quad \deg_\theta B_z \leq 2, \quad \deg_\theta C_z \leq 2, \quad \deg_\theta E_z \leq 4. \tag{4.15}$$

The strict signs of U_z automatically exclude $D_\theta(z) = 0$.

Proof. If $D(z) = 0$, then $U_z = V_z = 0$, so neither set of conditions holds. Otherwise $V_z > 0$, and

$$r_z(D) - R_D(x) = \frac{U_z x^2 + B_z x + C_z}{V_z(x^2 + qx + r)}.$$

The condition $r_z(D) > M_D$ includes strict separation from $R_D(\infty) = 0$. It is therefore equivalent to $U_z > 0$ and positivity of the numerator polynomial for every $x \geq 0$. For $U > 0$,

$$Ux^2 + Bx + C > 0 \quad (x \geq 0) \quad \Longleftrightarrow \quad C > 0, \quad B \geq 0 \text{ or } B^2 < 4UC.$$

The difference is continuous on $[0, \infty]$ and has positive limit U_z/V_z at infinity, so this positivity gives a uniform positive gap. This gives (4.13). Applying the same test to $-U_zx^2 - B_zx - C_z$ gives (4.14).

We verify the degree cancellations explicitly. In the chart $a = 1$, write

$$u_1 = |z+b|^2, \qquad u_0 = \mathrm{Re}\big((z+b)\overline{z^2-c}\big), \qquad v_0 = |z^2-c|^2.$$

Then

$$U_z = u_1\theta + u_0, \qquad V_z = u_1\theta^2 + 2u_0\theta + v_0.$$

Substitution gives

$$B_z = -(u_0 + bu_1)\theta^2 + (2cu_1 - v_0)\theta + 2cu_0 + bv_0, \tag{4.16}$$

$$C_z = -(b\theta - c)\big((bu_0 + cu_1)\theta + cu_0 + bv_0\big). \tag{4.17}$$

In the chart $a = 0, b = 1$, set

$$\eta = \mathrm{Re}(z^2 + cz), \qquad \nu = |z^2 + cz|^2.$$

Then

$$U_z = \theta + \eta, \qquad V_z = \theta^2 + 2\eta\theta + \nu,$$
$$B_z = -\theta^2 + c^2\theta + c^2\eta + \nu, \qquad C_z = -\theta(\eta\theta + \nu).$$

All the stated degree bounds follow. □

Corollary 4.3 (Complete characterization)**.** *Let $G \in \mathcal{S}_n$. Its admissible quadratic factorization classes are in bijection with those $\tau \in \mathcal{I}(G)$ for which the univariate conditions (4.13) or (4.14) have a solution in Θ_τ. For each such class, (3.4), (4.5)–(4.6), and (3.7) give all normalized admissible factorizations.*

Proof. Every admissible factorization can be normalized, determines an involution by Proposition 2.4, and has the form in Lemma 3.1. Theorems 4.1 and 4.2 give the necessary and sufficient analytic conditions. Proposition 3.3 supplies the unique outer function. Two normalized inner functions are equivalent exactly when their involutions agree, by Proposition 2.4. □

4.3 Feasible intervals and their endpoints

Let $\mathcal{E}_\tau \subset \Theta_\tau$ be the denominator parameters satisfying Theorem 4.2.

Proposition 4.4 (Interval structure)**.** *The set $\mathcal{E}_\tau$ is an open semialgebraic subset of $\mathbb{R}$. It is a finite union of open intervals and has at most $9n+3$ connected components. Each finite endpoint is a root of a nonzero polynomial among*

$$d_0,\ d_1, \quad U_z,\ B_z,\ C_z,\ E_z \quad (z \in T_\tau). \tag{4.18}$$

If G has real-algebraic coefficients, the set $\mathcal{E}_\tau$ and all its finite endpoints can be determined exactly by univariate real-root isolation and sign evaluation.

Proof. The polynomial conditions give semialgebraicity. For openness, take a feasible parameter. In a sufficiently small neighborhood the denominator remains Hurwitz and is nonzero at all $z \in T_\tau$. The functions L_{D_θ} depend continuously on θ, uniformly on the compactified imaginary axis. This follows either from continuity on that compact set or from a uniform expansion at infinity. Hence m_D, M_D, r_-, r_+ are continuous there. The strict conditions in Theorem 4.1 persist.

Delete identically zero polynomials from the list (4.18), while recording their zero sign in the logical conditions. The remaining polynomials have at most

$$2 + n(1 + 2 + 2 + 4) = 9n + 2$$

distinct real roots in total. On every complementary open interval all their signs are constant. Every component of the open set $\mathcal{E}_\tau$ contains at least one such interval, which proves the component bound and the endpoint claim.

For real-algebraic input, the poles and zeros are algebraic. The linear systems in Proposition 2.6 give algebraic representatives of the candidates. Thus all coefficients in (4.18) lie in an explicit real algebraic number field. Exact root isolation, ordering, and sign determination over this field give the intervals; these are standard univariate operations in real algebraic geometry [5]. □

In applying this proposition, the roots used to partition the line must themselves be evaluated. For example, $E_z = 0$ with $B_z > 0$, $U_z > 0$, and $C_z > 0$ can satisfy the positive branch. Such a point joins adjacent feasible cells. The appearance of a weak inequality $B_z \geq 0$ causes no failure of openness: when $B_z = 0$ and $U_z, C_z > 0$, one has $E_z = 4U_zC_z > 0$.

The degree bound four concerns the parameter polynomials over the field generated by the spectral data and the involution coefficients. The degree of an endpoint over $\mathbb{Q}$ can be larger.

4.4 Explicit scaling and simultaneous half-plane margins

For a feasible denominator D, define

$$\gamma_D(t) = \min\left\{ \min_{\omega\in\mathbb{R}\cup\{\infty\}} \operatorname{Re} H_{D,t}(i\omega), \min_{z\in T_\tau}(-\operatorname{Re} H_{D,t}(z)) \right\}. \tag{4.19}$$

The first term is a strict positive-real margin. The second is the minimum distance, measured by real part, of the reconstructed outer poles and zeros from the imaginary axis. Normalization gives $H_{D,t}(\infty) = 1$, so $\gamma_D(t) \le 1$.

Proposition 4.5 (Optimal scaling for a fixed denominator)**.** *Under* (4.3)*, the choice*

$$t_* = \frac{2}{r_-(D) + M_D} \tag{4.20}$$

is admissible and maximizes $\gamma_D(t)$ over all admissible scalings. The maximum is

$$\gamma_D(t_*) = \frac{r_-(D) - M_D}{r_-(D) + M_D} > 0.$$

Under (4.4)*, the corresponding choice and value are*

$$t_* = \frac{2}{r_+(D) + m_D}, \qquad \gamma_D(t_*) = \frac{m_D - r_+(D)}{-m_D - r_+(D)} > 0. \tag{4.21}$$

If $M_D > 0$, respectively $m_D < 0$, the optimizer is unique. If $M_D = 0$, respectively $m_D = 0$, the maximum is 1 and the displayed choice is one of the optimizers.

Proof. On the positive branch,

$$\gamma_D(t) = \min\{1 - tM_D,\ tr_-(D) - 1\}.$$

The decreasing and increasing affine functions have equal values at (4.20). The strict separation condition places that point inside $\mathcal{J}_D^+$. If $M_D > 0$, their intersection is the unique maximum. If $M_D = 0$, the first function is constant and the maximum 1 is attained for every $t \ge 2/r_-(D)$. On the negative branch,

$$\gamma_D(t) = \min\{1 - tm_D,\ tr_+(D) - 1\},$$

and the same argument gives (4.21). □

The optimization in Proposition 4.5 fixes both the involution and the denominator. It supplies a definite representative for every feasible denominator; the remaining parameter may subsequently be selected to meet an additional design condition.

Corollary 4.6 (Rational reconstruction)**.** *Suppose $G \in \mathbb{Q}(s)$ and an involution in $\mathcal{I}(G)$ has a rational representative (a, b, c). If its feasible set is nonempty, the corresponding class contains an admissible factorization with $H, G_0 \in \mathbb{Q}(s)$.*

Proof. Choose a rational θ in an open feasible denominator interval, and then a rational t in its open scaling interval. The normalized inner function has rational coefficients. For fixed rational G, H, the coefficients of the unique outer factor are obtained from the rational coefficient identity $G = G_0 \circ H$. After a normalization of its numerator and denominator, this is a consistent linear system over $\mathbb{Q}$, and thus has a rational solution. Uniqueness of the outer rational function identifies that solution with (3.7). □

5 Complete classification for quadratic input

For an input of degree two, the outer function necessarily has degree one. The constrained existence question then has a closed coefficient answer.

Theorem 5.1 (Quadratic input)**.** *Let*

$$G(s) = k\,\frac{s^2 + u_1 s + u_0}{s^2 + v_1 s + v_0} \in \mathcal{S}_2, \qquad k > 0.$$

The following conditions are equivalent:

(i) G *admits an admissible quadratic factorization;*

(ii) $G \in \mathrm{SPR}_0^{(2)}$*;*

(iii)

$$u_1 v_1 > (\sqrt{u_0} - \sqrt{v_0})^2. \tag{5.1}$$

When these conditions hold, $\mathcal{F}_2(G)$ *has exactly one element. Otherwise it is empty.*

A factorization can be constructed as follows. Put

$$\delta_G = \min_{s \in K_+} \operatorname{Re} G(s) > 0, \qquad M_G = \max_{s \in K_+} |G(s)|, \qquad \ell = \delta_G/2,$$

choose $R > 2M_G^2/\delta_G$*, and set*

$$H(s) = \frac{G(s) - \ell}{R - G(s)}, \qquad G_0(w) = \frac{Rw + \ell}{w + 1}. \tag{5.2}$$

Proof. An admissible degree-one outer function has the form

$$G_0(w) = \kappa \frac{w + b}{w + a}, \qquad a, b, \kappa > 0, \quad a \neq b.$$

For $\operatorname{Re} w \geq 0$,

$$\operatorname{Re} G_0(w) = \kappa \frac{|w|^2 + (a + b)\operatorname{Re} w + ab}{|w + a|^2} > 0.$$

The half-plane property of the inner factor therefore implies (ii).

Conversely, suppose (ii). Lemma 2.2 gives $\delta_G > 0$, and compactness gives $M_G < \infty$. Since $M_G \geq \delta_G$, the chosen R exceeds M_G. For (5.2),

$$\operatorname{Re} H(s) = \frac{R(\operatorname{Re} G(s) - \ell) + \ell \operatorname{Re} G(s) - |G(s)|^2}{|R - G(s)|^2} \geq \frac{R\delta_G/2 + \delta_G^2/2 - M_G^2}{|R - G(s)|^2} > 0$$

on K_+. There is no denominator zero on K_+. If $G = P/Q$ is reduced, the polynomials $P - \ell Q$ and $RQ - P$ have degree two because $\ell < G(\infty) < R$. They are coprime: a common root would imply $(R - \ell)Q = 0$, hence $P = Q = 0$. Thus H has degree two, is biproper, has a Hurwitz denominator, and has positive value at infinity. The outer function in (5.2) has pole -1, zero $-\ell/R$, and positive gain R, so it belongs to $\mathcal{S}_1$. The composition identity is direct.

Finally, the Hurwitz conditions give $u_0, u_1, v_0, v_1 > 0$. After removing the positive scalar k, the frequency numerator is

$$x^2 + (u_1 v_1 - u_0 - v_0)x + u_0 v_0, \qquad x \geq 0.$$

By Lemma 3.2, its strict positivity is equivalent to

$$u_1 v_1 - u_0 - v_0 > -2\sqrt{u_0 v_0},$$

which is (5.1). The quadratic extension $\mathbb{R}(s)/\mathbb{R}(G)$ has exactly one nontrivial automorphism. Proposition 2.4 therefore gives exactly one algebraic quadratic class, and hence the asserted admissible class count. □

Corollary 5.2 (A complete boundary family)**.** *For $\rho > 1$, set*

$$G_\rho(s) = \frac{s^2 + s + 1}{s^2 + s + \rho}.$$

Then $G_\rho \in \mathcal{S}_2$, and an admissible quadratic factorization exists exactly when $1 < \rho < 4$. At $\rho = 4$,

$$\operatorname{Re} G_4(i\omega) = \frac{(\omega^2 - 2)^2}{(4 - \omega^2)^2 + \omega^2},$$

so strict positive realness fails at $\omega = \pm\sqrt{2}$. For $\rho > 4$, the real part is negative at some frequency.

Proof. The quadratic input criterion reduces to $1 > (1 - \sqrt{\rho})^2$, which is $\rho < 4$ for $\rho > 1$. The boundary formula follows by direct substitution. If $\rho > 4$, the frequency numerator $x^2 - \rho x + \rho$ takes the negative value $\rho - \rho^2/4$ at $x = \rho/2$. □

6 Exact identification and reconstruction

The characterization gives an exact procedure for input with real-algebraic coefficients. A complex pole or zero is represented by its real and imaginary parts as real algebraic numbers. All sign decisions are exact. Arbitrary real coefficients retain the mathematical characterization of Corollary 4.3; the computational statement below uses the specified algebraic input model.

Theorem 6.1 (Exact decision and complete output)**.** *For $G \in \mathcal{S}_n$ with real-algebraic coefficients, the following procedure terminates and returns all admissible quadratic factor classes, their complete normalized parameter families, and an explicit representative of every nonempty class.*

1. *If n is odd, return the empty set. Otherwise compute the exact simple poles and zeros.*
2. *Generate and filter the candidates by Proposition 2.6. Alternatively, use Proposition 2.7 over the real algebraic numbers.*
3. *For each retained τ, choose a chart in (3.4) and a representative set T_τ. Form the polynomials (4.10)–(4.12).*
4. *Isolate and order all real roots of the nonzero polynomials (4.18). On each complementary open interval and at every root, check both Hurwitz inequalities $d_0(\theta) > 0, d_1(\theta) > 0$ and the logical conditions of Theorem 4.2. Merge the feasible cells into open intervals.*
5. *Discard the candidate if no interval remains. Otherwise record its intervals, the positive or negative branch on each interval, and the scaling intervals (4.5)–(4.6).*
6. *Choose a parameter in any retained interval. Compute the boundary extrema using (4.9), choose t_* from Proposition 4.5, and reconstruct G_0 by (3.7).*

The candidate-generation stage uses at most $(n-1)^2$ projective candidates. For each retained candidate, the analytic stage uses at most $4n + 2$ parameter polynomials, all of degree at most four.

Proof. The completeness of the algebraic list is Proposition 2.6. Each listed coefficient is algebraic. The parameter conditions are exactly those in Theorem 4.2; root isolation and sign evaluation produce all their solutions by Proposition 4.4. There are finitely many such operations. A nonempty interval contains a rational sample point, and the extrema at that point are algebraic by (4.9). The displayed choice of t_* is therefore algebraic and belongs to the required interval. Reconstruction is exact by Proposition 3.3. Conversely every admissible factorization lies in one retained class and is described by its recorded parameter and scaling intervals. The operation counts follow from (2.3) and (4.18). □

The bounds in Theorem 6.1 count candidates and polynomial degrees. Root-field construction and arithmetic heights enter a bit-complexity analysis separately. The procedure returns an algebraic reason for exclusion: failure of the fixing-group test, an empty Hurwitz denominator domain, or an empty set of feasible sign cells. No frequency sampling is needed for any of these decisions.

7 Inverse identification and obstructions

The following examples identify all quadratic classes directly from the input rational function and determine which classes are admissible.

7.1 A quartic input with one class and two feasible intervals

Consider the coefficient input

$$G(s) = \frac{60s^4 + 453s^3 + 1428s^2 + 2163s + 1368}{60s^4 + 430s^3 + 1320s^2 + 1970s + 1260}.$$

Its numerator and denominator factor as

$$P(s) = 3(4s^2 + 15s + 19)(5s^2 + 19s + 24), \qquad Q(s) = 10(2s^2 + 7s + 9)(3s^2 + 11s + 14).$$

The four quadratic factors are Hurwitz, have negative discriminants, and have no common roots. Thus the input is stable, minimum-phase, biproper, and has simple poles and zeros, with $G(\infty) = 1$.

Direct coefficient arithmetic gives

$$\frac{P(u)Q(v) - P(v)Q(u)}{u - v} = -60(uv + u + v - 1)R(u, v),$$

where

$$\begin{aligned} R(u, v) = {} & 23u^2v^2 + 85u^2v + 108u^2 + 85uv^2 + 314uv \\ & + 399u + 108v^2 + 399v + 507. \end{aligned}$$

As a quadratic polynomial in u, its discriminant is

$$\operatorname{disc}_u R = -2711v^4 - 20048v^3 - 62534v^2 - 94176v - 59823.$$

This quartic is squarefree: its discriminant is $63880506703872 \neq 0$. Consequently it is not a square in $\mathbb{C}(v)$, so R has no factor linear in u over $\mathbb{C}(v)$. In particular, it has no additional nondegenerate symmetric factor of bidegree $(1, 1)$. Hence the unique quadratic right-factor class is determined by

$$[a : b : c] = [1 : 1 : -1], \qquad \tau(s) = -\frac{s - 1}{s + 1}.$$

All normalized representatives of this class with Hurwitz denominator have

$$D_d(s) = s^2 + ds + d + 1, \qquad d > 0, \qquad H_{d,t}(s) = 1 - t\frac{s + 1}{D_d(s)}, \qquad t \neq 0.$$

Write $L_d = (s + 1)/D_d$. Since

$$L_d = \frac{L_4}{1 + (d - 4)L_4},$$

and the four spectral fibers satisfy $L_4(z) = k + 1$, $k = 1, 2, 3, 4$, their values are

$$L_d(z) = \frac{1}{d - 4 + 1/(k + 1)}.$$

The four denominators must be nonzero. They have one common sign precisely when $d < 7/2$ or $d > 19/5$; for $7/2 < d < 19/5$ at least two have opposite signs, except at excluded zero denominators. Thus the spectral inequalities $tL_d(z) > 1$ are equivalent to

$$\begin{cases} t < d - 19/5, & 0 < d < 7/2, \\ t > d - 7/2, & d > 19/5. \end{cases}$$

These spectral inequalities must also be intersected with the SPR interval in t.

For any fixed d, frequency-domain positivity of $1 - tL_d$ is an open interval of t containing zero. This elementary convexity permits an exact threshold test. In the positive case, an admissible $t > d - 7/2$ exists if and only if the threshold $t_+ = d - 7/2$ lies strictly inside the SPR interval. At this threshold,

$$H_{d,t_+}(s) = \frac{s^2 + (7/2)s + 9/2}{D_d(s)}, \qquad p = \frac{5d - 11}{2} > 0, \quad r = \frac{9}{2}(d+1) > 0$$

for every $d > 19/5$. Hence all such d are feasible.

In the negative case, the corresponding threshold is $t_- = d - 19/5$. At that value,

$$H_{d,t_-}(s) = \frac{s^2 + (19/5)s + 24/5}{D_d(s)}, \qquad p = \frac{14d - 29}{5}, \quad r = \frac{24}{5}(d+1).$$

For $d \geq 29/14$, the SPR inequality is immediate. For $0 < d < 29/14$, it is

$$196d^2 - 1292d + 361 < 0.$$

The smaller root of this quadratic is

$$d_* = \frac{323 - 76\sqrt{15}}{98},$$

and the larger root exceeds $29/14$. Therefore the complete set of feasible denominator parameters is

$$(d_*, 7/2) \ \cup \ (19/5, \infty).$$

At $d = d_*$, the necessary spectral threshold is an endpoint of the SPR interval, so strict feasibility fails. The endpoints $7/2$ and $19/5$ make one of the spectral evaluations infinite.

For the particular denominator $d = 4$, one has

$$\operatorname{Re} L_4(i\omega) = \frac{3x + 5}{x^2 + 6x + 25}, \qquad x = \omega^2 \geq 0.$$

Its minimum on the compactified half-line is zero and its maximum is

$$M_4 = \frac{\sqrt{10} - 1}{8}, \qquad x_{\max} = \frac{-5 + 4\sqrt{10}}{3}.$$

Consequently the full scaling interval for this denominator is

$$\frac{1}{2} < t < \frac{8(1 + \sqrt{10})}{9}.$$

For every t in this interval, reconstruction gives

$$G_{0,t}(w) = \frac{3}{10} \frac{(w - 1 + 4t)(w - 1 + 5t)}{(w - 1 + 2t)(w - 1 + 3t)}.$$

All of its poles and zeros are negative, simple, and distinct, and the identity $G = G_{0,t} \circ H_{4,t}$ holds exactly. Choosing $t = 1$ yields

$$H_{4,1}(s) = \frac{s^2 + 3s + 4}{s^2 + 4s + 5}, \qquad G_{0,1}(w) = \frac{3}{10}\frac{(w+3)(w+4)}{(w+1)(w+2)}.$$

For the denominator $d = 4$, Proposition 4.5 selects

$$t_* = \frac{16}{15 + \sqrt{10}}, \qquad \gamma_* = \frac{17 - \sqrt{10}}{15 + \sqrt{10}}.$$

Indeed $r_- = 2$ and $\gamma_4(t) = \min\{1 - t(\sqrt{10} - 1)/8,\ 2t - 1\}$, whose two affine terms agree at t_*. This gives an explicit admissible representative with maximal simultaneous margin for this denominator.

7.2 An even-degree algebraic obstruction

Consider

$$G_{\mathrm{alg}}(s) = \frac{(s+5)(s+6)(s+7)(s+8)}{(s+1)(s+2)(s+3)(s+4)}.$$

All poles and zeros are simple negative integers. A quadratic right factor would give an involution that pairs the four poles without fixed points. There are exactly three possible pole pairings. Their corresponding involutions fail to preserve the zero set, as the following exact check shows:

Pole pairs	$[a : b : c]$	$\tau(-5)$
$\{-1,-2\},\{-3,-4\}$	$[2 : 5 : 11]$	$-14/5$
$\{-1,-3\},\{-2,-4\}$	$[2 : 5 : 14]$	$-11/5$
$\{-1,-4\},\{-2,-3\}$	$[0 : 1 : 5]$	0

None of the values in the last column belongs to $\{-5,-6,-7,-8\}$. Hence G_{alg} has no real quadratic right factor, before any SPR constraint is imposed.

7.3 An analytic obstruction despite an SPR right factor

Let

$$G_*(s) = \frac{s^2 + s + 1}{s^2 + s + 9}.$$

The poles and zeros are simple and have real part $-1/2$. Its quadratic fiber involution is $\tau(s) = -s - 1$, but

$$\operatorname{Re} G_*(i\sqrt{2}) = -\frac{5}{51} < 0.$$

Theorem 5.1 therefore excludes every factorization satisfying the prescribed stable outer-factor conditions. Nevertheless,

$$G_* = \widehat{G}_0 \circ \widehat{H}, \qquad \widehat{H}(s) = \frac{s^2 + s + 1}{s^2 + s + 2}, \qquad \widehat{G}_0(w) = \frac{w}{8 - 7w}.$$

Here $\widehat{H}$ is SPR because its frequency numerator is $(\omega^2 - 1)^2 + 1$. The pole of $\widehat{G}_0$ is $8/7$, so it is unstable. This also distinguishes the constrained problem from a version in which only the inner factor is required to be SPR.

7.4 A quartic input with three admissible classes

Set $x = s + 4$ and consider

$$G_3(s) = \frac{x^4 + 3x^2 + 1}{x^4 + 4x^2 + 1}.$$

Each polynomial $y^2 + ky + 1$, for $k = 3, 4$, has distinct negative roots. Thus all poles and zeros of G_3 are simple, have real part -4, and are mutually distinct. In the x coordinate, the coefficient calculation is

$$\frac{P(u)Q(v) - P(v)Q(u)}{u - v} = (u + v)(uv - 1)(uv + 1).$$

The three factors give precisely the involutions

$$x \longmapsto -x, \qquad x \longmapsto 1/x, \qquad x \longmapsto -1/x.$$

All three classes satisfy the full analytic constraints. Explicit representatives and outer factors are

$$H_1(s) = \frac{x^2 + 4}{x^2 + 1/4}, \qquad G_{0,1}(w) = 5\frac{w^2 + 28w + 16}{w^2 + 208w + 16},$$
$$H_2(s) = \frac{(x-1)^2}{(x+1)^2} = \frac{(s+3)^2}{(s+5)^2}, \qquad G_{0,2}(w) = \frac{5w^2 + 6w + 5}{6w^2 + 4w + 6},$$
$$H_3(s) = \frac{x^2 - 3x - 1}{x^2 + 3x - 1} = \frac{s^2 + 5s + 3}{s^2 + 11s + 27}, \qquad G_{0,3}(w) = \frac{14w^2 + 8w + 14}{15w^2 + 6w + 15}.$$

In all three cases $G_3 = G_{0,j} \circ H_j$. The SPR frequency parameters (p, r) are respectively

$$(111/4, 325), \qquad (26, 225), \qquad (25, 81).$$

They are strictly positive. Every outer numerator and denominator is a Hurwitz quadratic with nonzero discriminant, and each numerator is coprime to its denominator. Thus this example has exactly three admissible Möbius classes, attaining the general bound $n - 1$ for $n = 4$. The repeated numerator and denominator roots of H_2 are allowed: the simple-spectrum hypothesis concerns G_3 and the outer factors, and the chosen spectral fibers of H_2 are unramified.

8 Dimension and perturbations of admissible inputs

For even n, the set of admissible inputs has codimension $n - 2$ in the coefficient space of degree-n biproper functions. This quantifies the restriction imposed by exact quadratic decomposition, even though all spectral and positivity conditions on a given factorization are open.

Proposition 8.1 (Dimension of the admissible input set)**.** *Let $n = 2m \geq 2$. In the $(2n + 1)$-dimensional real coefficient space of reduced biproper degree-n functions, the set*

$$\mathcal{D}_n = \{G \in \mathcal{S}_n : \mathcal{F}_2(G) \neq \varnothing\}$$

is a nonempty semialgebraic set of dimension $n + 3$. Its codimension is $n - 2$. In particular, for $n \geq 4$ it has empty interior in $\mathcal{S}_n$.

Proof. A reduced biproper degree-k function is described by a monic denominator, an arbitrary degree-k numerator, and a nonzero leading numerator coefficient, giving $2k + 1$ real parameters. Consider pairs (G_0, H) of degrees $m, 2$ satisfying the required outer and inner conditions and such that $G_0 \circ H$ has simple poles and zeros. They form an open semialgebraic subset $\mathcal{U}$ of a coefficient space of dimension

$$(2m + 1) + 5 = n + 6.$$

Openness of the SPR condition follows from uniform strict positivity on the compactified imaginary axis. Simplicity of the composite spectrum excludes finitely many critical-value incidences, an open condition. Semialgebraicity follows from the quadratic SPR test, Hurwitz inequalities, and nonvanishing discriminants and resultants.

The set $\mathcal{U}$ is nonempty. Choose $H = (s^2 + 3s + 4)/(s^2 + 4s + 5)$, and choose m distinct negative real outer poles and m distinct negative real outer zeros, with all $2m$ values different and avoiding the finitely many critical values of H. Lemma 2.2 places their preimages in $\mathbb{C}_-$, and avoidance of the critical values makes them simple.

The composition map $\mathcal{U} \to \mathcal{D}_n$ is semialgebraic and surjective, so its image $\mathcal{D}_n$ is semialgebraic. For fixed G, its fiber is a finite union of Möbius reparametrization orbits intersected with $\mathcal{U}$, by Propositions 2.4 and 2.5. The action

$$(G_0, H) \longmapsto (G_0 \circ \phi^{-1}, \phi \circ H)$$

is free, since $\phi \circ H = H$ with H nonconstant implies $\phi = \mathrm{id}$. Each orbit has dimension three. At an admissible pair all the defining conditions are strict and the degrees persist, so its intersection with $\mathcal{U}$ contains a three-dimensional relatively open neighborhood. Every nonempty fiber therefore has dimension three. Hardt's triviality theorem and additivity of dimension on products [5, Theorem 5.45 and Proposition 5.28] give

$$\dim \mathcal{D}_n = (n + 6) - 3 = n + 3.$$

The final assertion follows by comparing with $2n + 1$. □

For a fixed involution and a feasible denominator, the strict separation conditions persist under small changes of the denominator parameter. If t also varies within its admissible interval, the inner and outer half-plane margins stay positive. More generally, small changes of an admissible pair (G_0, H), preserving degrees and the strict spectral conditions, remain admissible and yield nearby inputs. Proposition 8.1 specifies why unrestricted changes of the input coefficients can remove an exact decomposition when $n \geq 4$.

Within each admissible quadratic class, the denominator remains free to vary over the intervals determined by Theorem 4.2. Proposition 4.5 gives the optimal scaling for each such choice.

References

[1] César Alonso, Jaime Gutiérrez, and Tomás Recio. A rational function decomposition algorithm by near-separated polynomials. *Journal of Symbolic Computation*, 19(6):527–544, 1995.

[2] Daniel Alpay and Izchak Lewkowicz. Composition of rational functions: State-space realization and applications. *Linear Algebra and its Applications*, 580:359–383, 2019.

[3] Brian D. O. Anderson and Sumeth Vongpanitlerd. *Network Analysis and Synthesis: A Modern Systems Theory Approach*. Prentice-Hall, Englewood Cliffs, NJ, 1973.

[4] Mohamed Ayad and Peter Fleischmann. On the decomposition of rational functions. *Journal of Symbolic Computation*, 43(4):259–274, 2008.

[5] Saugata Basu, Richard Pollack, and Marie-Françoise Roy. *Algorithms in Real Algebraic Geometry*, volume 10 of *Algorithms and Computation in Mathematics*. Springer, Berlin, second edition, 2006.

[6] Guillermo Fernández-Anaya. Preservation of SPR functions and stabilization by substitutions in SISO plants. *IEEE Transactions on Automatic Control*, 44(11):2171–2174, 1999.

[7] Guillermo Fernández-Anaya, J. J. Flores-Godoy, and A. Rodríguez-Palacios. Properties of strictly positive real functions: Products and compositions. *International Journal of Systems Science*, 41(4):457–466, 2010.

[8] Guillermo Fernández-Anaya, J. C. Martínez-García, and Vladimír Kučera. H_∞-robustness properties preservation in SISO systems when applying SPR substitutions. *International Journal of Control*, 76(7):728–740, 2003.

[9] Guillermo Fernández-Anaya and Juan Carlos Martínez-García. Stability and composition of transfer functions. In Vincent D. Blondel and Alexandre Megretski, editors, *Unsolved Problems in Mathematical Systems and Control Theory*, pages 3–7. Princeton University Press, Princeton, NJ, 2004. Problem 1.1.

[10] Jaime Gutiérrez, Rosario Rubio, and David Sevilla. Computing the fixing group of a rational function. arXiv:0805.2331, 2008.

[11] Richard Zippel. Rational function decomposition. In *Proceedings of the 1991 International Symposium on Symbolic and Algebraic Computation*, pages 1–6, New York, 1991. ACM.