A Lean 4 formalization of the area law for three-dimensional $U(1)$ lattice gauge theory with the Wilson action

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October 2026

hexagon:2611.00393v1

Abstract

In three-dimensional $U(1)$ lattice gauge theory with the Wilson action, static charges are confined at every coupling: rectangular Wilson loops obey an area law, uniformly in the finite box. This was proved recently by the author in a separate paper; the proof runs to over three hundred pages with its supplement and adapts the Debye-screening analysis of Brydges and Federbush and the multiscale expansion of Göpfert and Mack. We describe a formalization of this proof in Lean 4 with Mathlib, which also covers the paper's corollaries and its appendix on comparing Wilson and Villain loops. The formal theorems take no hypotheses and use only the standard axioms of Lean. The proof became formalizable through explicit interfaces between its parts, written-out versions of the steps that the paper adapts from the literature or leaves out, and certified values for every numerical constant. The formalization also yielded 70 corrections and improvements to the paper, none of which changes its main results.

Contents

1 Introduction **2**

2 The Lean statements **4**

3 Architecture **7**

4 The parts of the formal proof, by section of [U1] **9**
4.1 Sections 2, 8 and 9: the comparison, the chain and the corollaries 9
4.2 Sections 3–4: the regularization 9
4.3 Section 5: the contour bound 10
4.4 Section 6: Stage 1 10
4.5 Section 7: Stage 2 10
4.6 What the interfaces had to make precise 11
4.7 The appendices and the supplement 12

5 Written-out proofs and their refereeing **12**

6 Certified numerics **12**
6.1 The threshold 13
6.2 Watson's integral 13

6.3 The symbolic constants . . . 13
6.4 The constants of the written-out proofs . . . 14

7 Appendix E of [U1] 14
7.1 What the appendix reuses . . . 14
7.2 Where the proofs take other routes . . . 14
7.3 The kernel decision for Theorem E.8 . . . 15

8 Audits 15
8.1 Statements . . . 15
8.2 Satisfiable hypotheses . . . 16
8.3 The library . . . 16

9 Findings about the paper 16

10 Reproduction 17

11 Conclusion 18

1 Introduction

In three-dimensional $U(1)$ lattice gauge theory, Polyakov's mechanism predicts that static charges are confined at every coupling: magnetic monopoles form a plasma that screens the magnetic field, and the electric flux between two opposite charges is squeezed into a tube of finite tension, so that their energy grows linearly with their distance. In the lattice model, the Wilson loop of a rectangle with sides R and T describes such a pair at distance R over a time T, and confinement means that it decays exponentially in the area RT. Göpfert and Mack [GM82] proved confinement at every coupling for the Villain action. For the Wilson action it remained open until the author's paper [U1]. This paper describes a formalization of that proof in Lean 4. We first recall the theorem. Let $\Lambda \subset \mathbb{Z}^3$ be a finite box, $\beta > 0$ the coupling, and let $\langle\cdot\rangle^{W}_{\Lambda,\beta}$ be the expectation in the $U(1)$ Wilson measure on Λ with free boundary conditions. For a rectangle C in a coordinate plane, with sides R and T, write θ_C for the sum of the link angles around C.

Theorem A ([U1, Theorem 1.2])**.** *There is a constant c such that for every $\beta > 0$ there are $\sigma(\beta) > 0$ and $R_1(\beta)$ with*

$$\langle \cos\theta_C \rangle^{W}_{\Lambda,\beta} \le \exp\bigl(-\sigma(\beta)RT + c(R+T)\bigr)$$

for every rectangle C in a coordinate plane with $\min(R,T) \ge R_1(\beta)$ and every finite box $\Lambda \supseteq C$. Moreover, for every $K > K_* = 377.38$ there is $\beta_0(K)$ such that for $\beta \ge \beta_0(K)$ one can take $\sigma(\beta) \ge 1.35 \cdot 10^{-4} c_g c_L (K\beta)^{-1/2} e^{-2.49436K\beta}$, with $c_g = \frac{1}{4}$ and $c_L = 1.6 \cdot 10^{-19}$, and for $\beta < \beta_0(K)$ the values of σ and R_1 chosen at $\beta_0(K)$.

From here on, numbers of sections, results and displays without further indication refer to [U1], and sections of this paper are cited with the sign §. [U1] also deduces an area law for compact gauge groups $G \subseteq U(n)$ that contain the scalars, with or without a Higgs field blind to the centre of $U(n)$ (Corollary 1.3), and for the Georgi–Glashow model at infinite Higgs coupling (Corollary 1.4). Appendix E of [U1] shows why the proof compares the Wilson loop with Villain

loops in which some couplings are infinite: on every finite box, an inequality between Wilson and Villain loops that holds for all charges, even with a perimeter factor, forces the Villain side to contain arbitrarily large couplings (Corollary E.4 and Proposition E.6).

The proof in [U1] is long, 155 pages with Parts S1 and S2 of its supplement and 185 more in Part S4, and its core carries over the Debye-screening analysis of Brydges and Federbush [BF80] and the multiscale expansion of Göpfert and Mack [GM82] to new domains, "with changes" and with all constants explicit. A proof of this kind is hard to check by reading, and the sources it adapts contain steps that need correction; [U1] itself notes one in a tree-graph count of [B78]. A formal proof checks every step and every constant, including those of the adapted arguments.

We formalize Theorem A, both corollaries, Fröhlich's comparison (Proposition 9.1) and all results of Appendix E in Lean 4 [Lean4] with Mathlib [Mathlib]. Seventeen Lean theorems (Theorem B and Table 1) state these results and the infinite-volume form of the area law (Remark 8.1).

The proof in [U1] has four steps, which Section 8 combines into a chain. A comparison (Section 2) bounds the Wilson loop by an average of Villain expectations at the coupling $\tau = K\beta$ over random sets of frozen plaquettes, whose Villain coupling is infinite. A deterministic regularization (Sections 3–4) turns each frozen set into a slit domain with good geometry, in which the frozen plaquettes are insulating walls. On each slit domain, an iterated Mayer expansion down to the Debye scale and a single-scale expansion, built on [BF80] and [GM82] (Sections 6–7), prove a deterministic area law in terms of the minimal cut area, the least number of unfrozen plaquettes in an integer 2-chain bounded by the loop. A Peierls estimate, which [U1] calls the min-cut lower tail and we call the contour bound (Section 5), shows that this area is close to RT outside a rare event. Section 9 deduces the corollaries. Appendix E stands apart from this chain; its proofs combine the dual formula for Gibbs expectations, Ginibre's inequality, Gaussian sums over lattices and Hölder's inequality with two computations on the boundary of a cube.

We organized the formal proof as a chain of interfaces, that is, Lean structures and propositions that state exactly what one part of the proof passes to the next, so that the parts could be formalized independently and the assembly checked separately (§3). Before formalizing the steps that [U1] adapts from the literature, we wrote each of them out as a self-contained document that states its inputs and outputs, and during the formalization we did the same for the steps that [U1] leaves out. Each document was checked line by line in a separate review, and the formal proof follows these nine documents (§5). Every numerical constant of the proof is certified in Lean by kernel-checked computations, and the kernel itself decides the two rational inequalities behind the bounds on Watson's integral (§6). Appendix E is formalized with the same definitions (§7). We also compiled a list of corrections and improvements to [U1], each with replacement text (§9).

Related work. Long proofs have been formalized before, among them the odd order theorem in Coq [GAA+13], the Kepler conjecture in HOL Light and Isabelle [HAB+17] and the main theorem of the Liquid Tensor Experiment in Lean [Sch22, LTE22]. As in the formalization of the sphere eversion theorem [vDMN23], large parts of our formal proof follow informal documents written for it; here they are the written-out proofs of §5, which also state the inputs and outputs of each part.

2 The Lean statements

The results of [U1] that we formalize are stated by the following theorems. We then give the main definitions in Lean syntax, with mathematical symbols for readability. All names are in the namespace `U1`.

Theorem B (formal)**.** *The seventeen theorems of Table 1 take no hypotheses, and for each of them* `#print axioms` *lists* `propext`, `Classical.choice` *and* `Quot.sound` *and nothing else. They formalize the two assertions of Theorem A, the infinite-volume form of the area law ([U1, Remark 8.1]), Corollaries 1.3 and 1.4 of [U1], each in a form relative to the area law of the* $U(1)$ *model and in a weaker form, Fröhlich's comparison (Proposition 9.1) and the eight results of [U1, Appendix E]; the seventeenth,* `setting_realized`*, checks the definition of the realized weights of Proposition E.6.*

[U1]	statement	theorem
Theorem 1.2, first assertion	`AreaLaw`	`Main.thm_1_2`
Theorem 1.2, second assertion	`AreaLawExplicit`	`Main.thm_1_2_explicit`
Remark 8.1		`Main.rem_8_1`
Corollary 1.3, relative form	`Cor13`	`Main.cor_1_3_rel`
Corollary 1.3, weaker form		`Main.cor_1_3`
Corollary 1.4, relative form	`Cor14`	`Main.cor_1_4_rel`
Corollary 1.4, weaker form		`Main.cor_1_4`
Proposition 9.1	`Prop91`	`Main.prop_9_1`
Lemma E.1	`LemmaE1`	`lem_E_1`
Theorem E.2	`TheoremE2`	`thm_E_2`
Lemma E.3	`LemmaE3`	`lem_E_3`
Corollary E.4	`CorollaryE4`	`cor_E_4`
Proposition E.6	`PropositionE6`	`prop_E_6`
Theorem E.8	`TheoremE8`	`thm_E_8`
Proposition E.10	`PropositionE10`	`prop_E_10`
Proposition E.12	`PropositionE12`	`prop_E_12`
realized weights of Proposition E.6	`RealizedWeightSpec`	`setting_realized`

Table 1: The seventeen theorems of Theorem B and the statements they prove. The names of the first eight rows are in the namespace `U1` (theorems in `U1/MainTheorem.lean`), those of the last nine in `U1.AppendixE` (statements in `U1/AppendixE/Statements.lean`, theorems in `U1/AppendixEMain.lean`).

Lattice and model. Sites are points of $\mathbb{Z}^3$. Bonds, plaquettes and cubes carry their orientations and incidence numbers, and the boundary maps satisfy $\partial\partial = 0$ (`bd1_bd2`). A box is a product of three integer intervals; `Box.complex` is the finite 2-complex of its bonds and plaquettes. The angle group is $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. On $\mathbb{T}^E$, with E the set of bonds of a finite complex, `FinComplex.wilsonExp` is the expectation for the probability measure proportional to $\prod_p e^{\beta\cos\theta_p}\,d\theta$, and the expectation of a constant is that constant (`wilsonExp_const`). The Villain measure at coupling t has the Villain weight $\tilde{v}_t(\theta_p)$ in place of $e^{\beta\cos\theta_p}$, where $\tilde{v}_t(\theta) = \sum_{n\in\mathbb{Z}} e^{-n^2/(2t)}e^{in\theta} = (2\pi t)^{1/2}\sum_{m\in\mathbb{Z}} e^{-t(\theta-2\pi m)^2/2}$. A rectangle `C : Rect` lies in a coordinate plane and has sides `C.R`, `C.T`; the loop that [U1] calls C, its boundary, is `C.chain`,

traversed first along the side of length `C.R` (`Rect.chain_eq_loop`). For a box `B : Box` (the box Λ of [U1]), the predicate `C.InBox B` says $\Lambda \supseteq C$. Then `B.wilsonLoop` β `C.chain` is $\langle \cos \theta_C \rangle^W_{\Lambda,\beta}$ with free boundary conditions.

In Lean, the first assertion of Theorem A reads

```
def AreaLawAt (β σ : ℝ) (R₁ : ℕ) (c : ℝ) : Prop :=
  ∀ C : Rect, R₁ ≤ C.R → R₁ ≤ C.T → ∀ B : Box, C.InBox B →
    B.wilsonLoop β C.chain ≤ Real.exp (-σ * C.R * C.T + c * (C.R + C.T))

def AreaLaw : Prop :=
  ∃ c : ℝ, ∀ β : ℝ, 0 < β → ∃ σ : ℝ, 0 < σ ∧ ∃ R₁ : ℕ, AreaLawAt β σ R₁ c
```

and the second reads

```
def AreaLawExplicit : Prop :=
  ∃ c : ℝ, ∀ K : ℝ, Kstar < K → ∃ β₀ : ℝ, 0 < β₀ ∧
    ∃ σ : ℝ → ℝ, ∃ R₁ : ℝ → ℕ,
    (∀ β, β₀ ≤ β →
      1.35e-4 * (1/4) * 1.6e-19 * (K * β) ^ (-(1:ℝ)/2)
        * Real.exp (-2.49436 * (K * β)) ≤ σ β) ∧
    ∀ β, 0 < β → AreaLawAt β (σ (max β β₀)) (R₁ (max β β₀)) c
```

with `Kstar` $= 377.38$. We compare these statements with [U1]. In the order of the quantifiers, the perimeter constant c comes before β, and before K in the second assertion, as in [U1]. For the first assertion alone this is no restriction, as [U1] notes after Theorem 1.2: a constant c that depends on β can be replaced by $c = 0$ if σ is halved and R_1 is enlarged. In the second assertion it is a restriction, since there halving σ would also halve the explicit lower bound on the string tension. The statements are also uniform in the box, since `AreaLawAt` quantifies over all rectangles with $\min(R, T) \geq R_1$ and all boxes that contain them, with free boundary conditions, so no infinite-volume limit and no other boundary condition enters the statement. In the explicit string tension, the constants are those of [U1, Theorem 1.2] with $c_g = \frac{1}{4}$ and $c_L = 1.6 \cdot 10^{-19}$ substituted, and for $\beta < \beta_0(K)$ the bound holds with the values chosen at $\beta_0(K)$, as in Theorem A. Finally, as in [U1], the second assertion implies the first with the same c; `AreaLawExplicit.toAreaLaw` proves this with $K = 378$.

For the corollaries, `U1.Cor13` states Corollary 1.3 relative to the area law: for every closed subgroup $G \subseteq U(n)$ that contains the scalars and every $\beta > 0$, the area law of the $U(1)$ model at coupling $2n\beta$ with constants (σ, R_1, c) implies $|\langle \mathrm{Tr}\, U_C \rangle| \leq 2n \exp(-\sigma RT + c(R + T))$ for the G model at β. The Higgs field φ may take values in any probability space, enters through a homomorphism ρ of G that is trivial on the scalars, and has any nonnegative integrable weight h with positive integral. Similarly, `U1.Cor14` states Corollary 1.4: in every finite box the $SU(2)$ model with a unit adjoint Higgs field converges, as the hopping coupling $\kappa \to \infty$, to the $U(1)$ Wilson model in the sense of expectations of continuous gauge-invariant functions (Proposition 9.3), and the area law of the $U(1)$ model at β with constants (σ, R_1, c) gives the same area law for the limits of the Wilson loops of the $SU(2)$ model (Corollary 9.4). With `thm_1_2`, `Cor13` gives Corollary 1.3 with the constants $\sigma(2n\beta)$, $R_1(2n\beta)$ and c, which depend on G only through n, and `Cor14` gives Corollary 1.4 with the constants $\sigma(\beta)$, $R_1(\beta)$ and c. `Cor13.of_areaLaw` and `Cor14.of_areaLaw` combine `Cor13` and `Cor14` with `AreaLaw` into the weaker forms `Main.cor_1_3`,

in which σ and R_1 are chosen after G, and `Main.cor_1_4`, in which the limit model has an area law with its own σ and R_1, not necessarily those of the $U(1)$ model.

The following formal statements are more general than those of [U1], but the paper's proofs cover them.

- `U1.Prop91` includes the Higgs weight of Corollary 9.2. It holds, as Proposition 9.1 does, on every finite 2-complex whose plaquettes are attached along closed edge paths and for every closed edge path ℓ, and it allows every Higgs weight h that `U1.Cor13` allows. With weight 1 it is Proposition 9.1, and on a box it is the first assertion of Corollary 9.2, which [U1] states on boxes only.
- In Corollaries 1.3 and 9.2 the map ρ is any group homomorphism trivial on the scalars, whereas the paper has a representation, and only the composite $(U, \varphi) \mapsto h((\rho(U_e))_e, \varphi)$ needs to be measurable and integrable. The paper's proof uses only $\rho(zU) = \rho(U)$ for scalar z.

Appendix E. Each of the eight results of [U1, Appendix E] is a closed proposition in `U1/AppendixE/Statements.lean`. The statements are typed on objects of Section 2 of [U1] that the main formalization defines: finite 2-complexes, the Villain weights $\tilde{v}_t$, the Wilson expectation, the frozen expectation and the Bessel functions I_ν. The setting of the appendix is defined on top of them: the Fourier coefficients $c_k(w)$ for normalized Haar measure on $\mathbb{T}$, plaquette weights, weights of Ginibre type, Villain mixtures with couplings at most τ, the Gibbs expectations $\langle\cdot\rangle_{X,w}$, the dual sums $\hat{Z}(n)$ of [U1, Lemma E.1], realized weights, and the boundary of the unit cube with its outward normals. The following conventions differ from the statements in [U1] without adding hypotheses, and those on τ and on the face p_0 make some statements slightly stronger. The coupling bound τ of a Villain mixture may be any real number. For $\tau < 0$ there is no Villain mixture, and for $\tau = 0$ the bounds $e^{-\kappa k^2/(2\tau)}$ of Theorem E.2(a), with $\kappa > 0$ determined by the complex and the loop, read $e^0 = 1$, since division by zero gives 0 in Lean. Theorem E.8 and Proposition E.12 hold at every face p_0 of the cube, whereas the paper fixes one. Theorem E.8 does not assume $t > 0$, since the condition $e^{-1/(2t)} = 15/16$ forces it. In Proposition E.6 the weights of Ginibre type are the normalized exponentials $e^f / \int e^f \, d\varphi/2\pi$ with $f = \sum_{j \le J} \lambda_j \cos j\varphi$, all $\lambda_j \ge 0$, and one cutoff J for all plaquettes; every finite family of plaquette weights of Ginibre type has this form (`exists_ginibreWeight_family`). §7 maps the statements to the paper.

The audit of the main results (§8) checks these statements against those of [U1]. Besides the comparison, the audit checks that no statement is true for a definitional reason: the expectations are normalized and take no junk values, such as the value 0 that Lean gives to a division by zero, under the hypotheses of the statements. As the partition function is positive (`integral_wilsonWeight_pos`), no expectation degenerates. The statements involve only the definitions of the models; in particular, no parameter of the proof, such as the constants λ_{PV}, μ and κ_S of §6 or the construction, enters a statement. Throughout the library, each Lean theorem that corresponds to a result of [U1] cites it by its number there, in its docstring. Some Lean names use the number of the result's LaTeX label instead, for instance `Thm761OfRegSpec` for Theorem 7.57 (label `thm:7.61`); the file `LABELS.tsv` of the archive lists the labels with the numbers the results carry in [U1]. The file `blueprint/BLUEPRINT.md` maps the definitions and results of [U1] that the statements, the chain and the interface `U1.R0` of §3 involve, among them those of Sections 2 and 9, to their Lean names, and `blueprint/APPENDIX_E.md` does the same for Appendix E.

3 Architecture

An interface is a Lean structure or proposition that collects the results passed from one part of the proof to another. The passage from one interface to the next is always a proved theorem, and hypotheses enter only as arguments of theorems, never as global assumptions.

The first interface is the structure `U1.R0`. For one deterministic construction, it packages the results of [U1] on which the chain of its Section 8 relies:

- the construction: for each rectangle C a box Λ° with $C \subset \Lambda^\circ \setminus \partial\Lambda^\circ$ (Lemma 3.3(e)) and, for each set D of frozen plaquettes in Λ° off its boundary, a hole set and a set of frozen plaquettes that contains D (property (R1) of Theorem 3.20);
- the screening mass $\tilde{m}$, with $m_D/2 \le \tilde{m} \le 2m_D$ for large τ, where m_D is the Debye mass;
- Theorem 7.57, the area law on every output of the construction, with its error term $\eta_S \to 0$;
- Theorem 5.1, the contour bound, for all large K, with its error term $\eta_T \to 0$.

For example, the field that carries Theorem 7.57 reads, with line breaks of our own,

```
thm_7_61 : ∀ τ, tau0 ≤ τ → ∀ C : Rect,
  loopScale r0 (mt τ) ≤ C.R → loopScale r0 (mt τ) ≤ C.T →
  ∀ D ⊆ (box τ C).intPlaqs,
    (box τ C).slitW (frozen τ C D) (holes τ C D) τ C.flat ≤
      Real.exp (-sigmaV r0 (mt τ) τ *
          (((box τ C).minCutArea (frozen τ C D) C.chain : ℝ)
            - etaS τ * C.R * C.T) + cN * (C.R + C.T))
```

Here `box`, `frozen` and `holes` are fields of the same structure that give the construction, `slitW` is the Villain loop on the slit domain, computed from the flat rectangle `C.flat` that the loop bounds, and `minCutArea` is the minimal cut area of the loop. Of the other names, `tau0` is the threshold in τ of Theorem 7.57, `r0` the fine side r_0 of a coarse cell, `mt` the screening mass $\tilde{m}$, `loopScale r0 (mt τ)` the lower bound R_1 on the sides, `intPlaqs` the set of interior plaquettes of the box, and `sigmaV`, `etaS` and `cN` are the constants σ_V, η_S and c_N of Theorem 7.57. The field thus says that, on the slit domain $G'(D)$ built from D, $W_{G'(D)}(C) \le \exp(-\sigma_V[A_{G'(D)}(C) - \eta_S RT] + c_N(R+T))$ for all $\tau \ge \tau_0$, all rectangles with $\min(R,T) \ge R_1$ and all D. The chain of Section 8 uses the field as it stands, without knowing how the construction produces its objects.

`U1.area_law_of_R0 : U1.R0 → U1.AreaLaw` formalizes Section 8. It covers the chain (1.4)–(1.8) at one coupling, the choice of constants $\sigma = (1 - \eta_T - \eta_S)\sigma_V$ and $c = c_N + 10$, with σ_V and c_N from Theorem 7.57, the bound $4|C| + 4 \le 10(R+T)$ and the case of small β in Remark 8.1, by monotonicity in β. It uses no numerical value of K_* or of $v_{\mathrm{Cb}}(0)$, which is half of Watson's integral, only $0 < v_{\mathrm{Cb}}(0) < \infty$.

The next interface, `U1.R1`, is `U1.R0` without the construction, which becomes data (`ConstructionMaps`) with a specification (`ConstructionSpec`). For the construction of Sections 3–4 the specification is proved (`U1.consMaps_spec`); `U1.R1Cons` is `U1.R1` for this construction.

In `U1.R2` only the screening mass is kept. Theorems 7.57 and 5.1 for the outputs of the construction become the propositions `U1.Thm761OfRegSpec K` r_0 $\tilde{m}$ and `U1.Thm51OfRegSpec K`, and the properties of the construction needed in Sections 5–7 become a third, `U1.RegSpec K`. Here `K` is the list of parameters of the construction (`Construction.ConsParams`), not the constant K of Theorem A. `U1.RegSpec K` says that every valid run of the construction satisfies

`Construction.RunSpec`, which collects Theorems 3.20 and 3.21 in the form the later sections require. A run is the construction applied to one rectangle C and one frozen set D at given r_0, $\tilde{m}$ and τ; it is valid (`Construction.Valid`) if its scales, defined in Definition 3.1, satisfy $L_c \geq 4$ for the block scale L_c and $\ell \geq 308 L_c$ for the removal scale ℓ, and its rectangle has sides at least the fine block scale $L = r_0 L_c$. The canonical point `U1.R2.canon` fixes $r_0 = 1$ and the screening mass `Gas.mtCanon lamPV`, with $\lambda_{\mathrm{PV}} = e^{-2213.5}$ (§6). The bounds on this mass are proved in `mtCanon_lamPV_bounds`. All three propositions are proved:

- `U1.RegSpec K` by `U1.consMaps_regSpec` (Sections 3–4);
- `U1.Thm51OfRegSpec K` by `U1.MinCut.thm_5_1_of_regSpec` (Section 5), and its form for every $K > K_*$ by `U1.MinCut.thm51Kstar_of_regSpec`;
- `U1.Thm761OfRegSpec` at the canonical point by the chain described next.

The part of the formalization in `U1/StageC/` covers Sections 6–7 and Appendices A–D of [U1] and the assembly described here. We call it Stage C, after its directory, which is named after the third stage of the work, not after a stage of the proof in [U1]. For the third proposition, a family of Stage C data (`U1.StageC.S77.StageCFamily`), the inputs of Sections 7.7–7.9 for the outputs of the construction, gives Theorem 7.57 on every output. The fields of these data are assembled from the results of Sections 6–7 and Appendix B into an explicit list: scalar constants, a threshold in τ, and for every valid run of the construction at large τ the per-run fields. In turn, all fields of the list are produced from one bound per run: the position bound of Proposition B.7 (Step 4 of its proof in Section S4.7 of [U1]), which bounds the sum over positions of the activity Q by a product of masses per cell times the exponential of the fractions of the localized energy F_1^{loc} returned by Lemma B.8. This bound is proved for every valid run at the canonical point and all large τ (`U1.StageC.Seed1.qMassRun_HK`). The theorem `U1.area_law_of_qmass` and its variants turn it into the main theorem and its corollaries, which `U1/MainTheorem.lean` obtains by applying them.

interface	content	proved by
`R0`	construction, screening mass, Thm 7.57, Thm 5.1	`R1.toR0`
`R1Cons`	`R1` at the construction of Sections 3–4	`consMaps_spec`, `R2.toR1Cons`
`R2`	screening mass; with the three propositions below it gives `R1Cons`	`R2.canon`, `mtCanon_lamPV_bounds`
`RegSpec K`	Theorems 3.20, 3.21 on every valid run	`consMaps_regSpec`
`Thm51OfRegSpec K`	Theorem 5.1 on the outputs	`MinCut.thm_5_1_of_regSpec`
`Thm761OfRegSpec`	Theorem 7.57 on the outputs	`StageC.Assembly.leaf_of_qmass` with `Seed1.qMassRun_HK`

Table 2: The interfaces of the formal proof and the theorems that discharge them.

Table 2 lists the interfaces. We use them because they make the dependency graph of the proof explicit and checkable. We formalize the chain of Section 8 and the corollaries against `U1.R0`, while the parts of the proof are formalized independently against fixed statements. However, if the hypotheses of an interface are unsatisfiable, all theorems derived from it are vacuous; the main theorems, which take no hypotheses, discharge every interface on their chain, as the audit of satisfiable hypotheses in §8 records.

4 The parts of the formal proof, by section of [U1]

We follow the paper's sections. For each part we give the Lean directory, the main results and the points where the formal proof differs from the paper's. The library has 887 files and 221,290 lines of Lean. More than half of it, 502 files and 122,175 lines in `U1/StageC/`, formalizes the two expansion stages of Sections 6–7 and their appendices.

4.1 Sections 2, 8 and 9: the comparison, the chain and the corollaries

The lattice, the models and the expectations are defined in `U1/Basic/` and `U1/Model/`, and the statements of Section 2 are in `U1/Reduction/`, which also proves Propositions 2.9 and 2.10; the directories named next hold the other proofs. Section 9 and the correlation inequalities of Section 2 (Theorem 2.1, Lemma 2.3) are both in `U1/Correlation/`; the inequalities are proved through positive kernels and the doubling map $(a, b) \mapsto (a+b, a-b)$ on compact abelian groups. The modified Bessel functions of real order and the mixture of Hartman and Watson [HW74] behind Lemma 2.2 are in `U1/Special/`, together with the lattice Coulomb and Yukawa kernels. Lemma 2.4 and Theorem 2.5 are in `U1/Annealed/`, the dual representation and the frozen limit (Lemmas 2.7, 2.8) in `U1/Dual/`, and Lemma 2.6 and the threshold K_* in `U1/Numerics/`. Cubical chains on boxes, used for Propositions 2.9 and 2.10 and in Section 5, are in `U1/Cubical/`.

The chain of Section 8 and its infinite-volume form (Remark 8.1) are formalized in `U1/Chain.lean`, and its variants for the other main results and the later interfaces in the other files `U1/Chain*.lean`. In Section 9, the comparison behind Proposition 9.1 says that, for any phases ψ_p, couplings $|J'_p| \le J_p$ and every α, the expectation of $\cos(\theta_\ell - \alpha)$ in the $U(1)$ model with weights $e^{J'_p \cos(\theta_p - \psi_p)}$ is at most that of $\cos\theta_\ell$ in the model with weights $e^{J_p \cos\theta_p}$. The formal proof checks the positivity behind it one plaquette at a time, so plaquettes with the same edge vector up to sign need not be combined, as they are in [U1]. The proof of Proposition 9.1 gives the constants n and $n\beta$, but the Lean statement uses the constants $2n$ and $2n\beta$ of [U1], which follow by monotonicity in β. Proposition 9.3 uses a unitary gauge chosen for each fixed Higgs configuration, so no measurable section is needed. The limit $\kappa \to \infty$ is a concentration argument on $SU(2)$ that uses only the invariance of Haar measure, its full support and Fubini's theorem.

4.2 Sections 3–4: the regularization

In Lean, the construction of Section 3 is a function from frozen sets to slit domains (`U1/Regularization/`, and `U1/Geometry/` for the geometry). The grid of the construction is a product of one-dimensional partitions, and its properties (G1)–(G5) follow from one-dimensional facts in each direction. Both phases of the construction are iterations to a fixed point, with explicit bounds on the number of steps. For its submodular cut functionals, the smallest minimizers are intersections of all minimizers. We formalize the analysis of Section 4 for every grid that satisfies (G1)–(G5). The isoperimetric bounds of Theorem 4.5 and Lemma 4.11 use the discrete Loomis–Whitney inequality [LW49, Theorem 2], which we prove by slicing and the Cauchy–Schwarz inequality (`U1/Geometry/LoomisWhitney.lean`). In the cut rule of Definition 3.18, the selection of a fragmented pair is a parameter of the construction (the list of findings in [repo] has an entry on Definition 3.18: any order fixed before D works). Finally, `U1.consMaps_regSpec` proves Theorems 3.20 and 3.21 for every valid run of the construction.

4.3 Section 5: the contour bound

Theorem 5.1 is formalized in `U1/MinCut/`, with its cubical lemmas in `U1/Cubical/` and its counts of lattice animals in `U1/Trees/`. For the outputs of the construction it is proved by `U1.MinCut.thm_5_1_of_regSpec` in the form that the chain uses, the moment bound of (5.1). The proof of `U1.MinCut.thm_5_1` gives the tail bound of (5.1) with the sharper counting factor $2^{4|C|+2}$, but the theorem is stated with the factor $2^{4|C|+5}$ of [U1]. We formalize the separators, the Leaf Lemma, the charges and the classes of Subsections 5.4–5.6 for every selection of the cut rule.

4.4 Section 6: Stage 1

The first stage, a Mayer expansion to the Debye scale (`U1/StageC/S6/`), controls the Debye-screened Coulomb gas on the slit domain. The ingredients of this stage are the expansion with the forest formula and its coefficients, the screening mass, heat-kernel bounds on the slit graph, the depletion estimates, and the Yukawa and Coulomb kernels of `U1/Special/`. We state and prove the heat-kernel bound (HK2) of Theorem 6.8 for the tail mass at a fixed time, a form implied by the exit-time form of [U1]. Appendix D, the only place where (HK2) is applied, needs only this form.

4.5 Section 7: Stage 2

The second stage follows the paper's subsections.

- Subsections 7.0–7.5 and Appendix C (`U1/StageC/S70/`): the expansion of Göpfert and Mack [GM82, App. E] on the slit domain, walls and their tension, the collar, wall fields and classes, Proposition 7.22 and Theorem 7.27.
- Subsection 7.6 and Appendix B (`U1/StageC/S76/`): the bounds at a fixed configuration, with the per-cell form of [BF80, (9.824)] (Proposition B.7).
- Subsections 7.7–7.9 and Appendix A (`U1/StageC/S77/`): the tree form of the activities, the link sum, the Kirkwood–Salsburg system in the class algebra of [U1], a Beurling (weighted ℓ^1) algebra, the proof that frozen flux costs area, and the deterministic area law (Theorem 7.57).
- The assembly described in §3: `U1/StageC/Assembly/`, and the position bound in `U1/StageC/Seed1/`.
- Gaussian calculus (`U1/Gaussian/`); tree-graph bounds, Kruskal's identity, the Beurling algebra and the polymer gas (`U1/Trees/`, `U1/Cluster/`).

As an example of the analytic work, take Proposition 7.53, which shows that frozen flux costs area. The long wall fields are integer fields on the bonds that vanish inside the cells of the expansion, are constant on each interface between cells and have zero sum around every short closed walk; their classes $[N]$, the sums around closed walks, form a group Γ_{long}. For r in it, Z_r is the partition function at fixed class r, $Z = Z_0$, and $\mathrm{A}(r)$ is the least ℓ^1 norm of an integer field on the bonds whose sums around the cycles of the slit graph are those of r; the proposition states $Z_r \le e^{-\sigma' \mathrm{A}(r)} Z$, where σ' is the tension σ_V of Theorem 7.57 (§3). In [U1], each configuration of class $r \neq 0$ contains an irreducible family of wall polymers of class

r, one in which no nonempty subfamily has class 0. Put $\omega = e^{\sigma' \mathrm{A}}$ and $C_{\mathrm{FF}} = \sup_s \omega(s) Z_s / Z$. A union bound over these families and the re-expansion of each as a seed (Theorems 7.27(c) and 7.49) leave partition functions Z_s at shifted classes s, each at most $C_{\mathrm{FF}} Z / \omega(s)$. Since $\omega(r) \le \omega(r-s)\,\omega(s)$, the weight $\omega(r-s)$ of the shift is left over; Theorem 7.49(a) and the bound of Lemma 7.48 on the extensions of a seed absorb it, and with that of Proposition 7.52 on the sum over families this gives $\omega(r) Z_r / Z \le \frac{2}{31} C_{\mathrm{FF}}$ for $r \neq 0$. As $\omega(0) Z_0 / Z = 1$, it follows that $C_{\mathrm{FF}} \le \max(1, \frac{2}{31} C_{\mathrm{FF}})$, and since C_{FF} is finite (Proposition 7.47), $C_{\mathrm{FF}} = 1$, which is the proposition. The formal version, `prop_7_57` in `U1/StageC/S77/FrozenFlux.lean` (named after the label `prop:7.57`), is stated for any abelian group with a weight $\omega \ge 1$ such that $\omega(0) = 1$ and $\omega(s+t) \le \omega(s)\omega(t)$, under the hypotheses that $Z_s \ge 0$ (Lemma 7.42) and $\sum_s \omega(s) Z_s < \infty$ (Proposition 7.47) and, over abstract types of families and extensions, Theorem 7.49(a), the union bound with the re-expansion and the two bounds. The sums in these hypotheses take values in $[0, \infty]$, so they need no summability hypotheses of their own. The last step is a separate lemma (`le_of_sup_bound`). The theorem `frozen_flux` in `U1/StageC/S77/AreaLawInput.lean` applies `prop_7_57` to a structure whose fields are the objects of Sections 7.1–7.8 for one output of the construction and the properties of these objects that the proof uses, and the assembly in `U1/StageC/Assembly/` fills these fields.

4.6 What the interfaces had to make precise

The interfaces of Stage C had to state the inputs of Sections 6–7 in a form that the outputs of the construction actually satisfy. The paper states several of them more generally than its proof uses them, and a formal interface that demands the general form can be unsatisfiable. We call the restrictions that make such an interface satisfiable guards; the following examples show the kind of precision that was needed.

Theorem 7.57 bounds the Villain loop $W_{G'}(C)$ on the slit domain G' in terms of its minimal cut area $A_{G'}(C)$. The paper proves it by a case split: either some bond of the slit graph crosses the rectangle, or none does. Only in the first case is the seed set X_1 of the expansion nonempty. An interface that asked for Stage C data on every output would be unsatisfiable, since data with $X_1 = \emptyset$ would force $W_{G'}(C) \le 0$, while $W_{G'}(C) > 0$. The formal interface therefore asks for the data only on crossing outputs, and on the others the bound is proved directly, from $W_{G'}(C) = 1$ and $A_{G'}(C) = 0$.

The bounds of Section 7 are stated for every mass in a band, but their inputs exist only at the actual screening mass of Section 6. The interface for Theorem 7.57 asks for them only at the one mass where the chain uses them. Similarly, the position bound of Proposition B.7 needs a per-cell constant that does not depend on the Villain coupling τ: its formal statement fixes the constants outside the quantifier over τ, and it is proved that each factor of the per-cell constant, measured in units of the screening mass, is bounded uniformly in τ at the canonical point.

The other guards are of the same kind. The per-run fields are required only for valid runs, for loops of at least the minimal length, for frozen sets inside the box and for crossing outputs; the seed bound with the loop present only for seeds that contain X_1; and some hypotheses of the lemmas leading to Proposition B.7 are bounded quantifiers, which in the unbounded form could not hold. Guards also appear in earlier sections: the rooted minimizers of the contour bound lie in the box and have the loop C as their boundary, as the paper's minimizers do, and on the output of the construction the map from sites to the $\mathcal{U}$-cells, the unions of cells from which the polymers of the expansion are built, is onto (`Construction.outWallGeom_ucell_surjective`). The audit of satisfiable hypotheses (§8) records these guards. Its probes, Lean files kept outside

the library and compiled against it, show for each guard that it is needed or that the objects of the proof satisfy it, except for the guards on the length of the loop and on the frozen sets, which [U1] itself imposes, and the one on minimizers, which holds by the definition of the rooted minimizers.

4.7 The appendices and the supplement

Appendices A, B and C, which carry [BF80, Section 9.1], [BF80, Section 9.8] and [GM82, Appendix E] over to slit domains, are formalized with the subsections of Section 7 listed above. Appendix D (heat kernel, coarse-to-fine transfer and covariance bounds on slit domains) is formalized with Section 6, and the constants of the supplement are in `U1/Numerics/` (§6).

5 Written-out proofs and their refereeing

The steps of [U1] that carry arguments of [GM82] and [BF80, Section 9] over to slit domains, with constants made explicit, cannot be formalized until the adapted arguments are written out. We therefore wrote out each of these parts as a self-contained document, which the formalization then followed. Six documents do this, for Stage 1 (Section 6, with Appendix D and the supplement), Subsections 7.0–7.5, Proposition 7.22 with Appendix C, Subsection 7.6, Appendix B, and Subsections 7.7–7.9 with Appendix A. Three more write out the steps that [U1] omits in Sections 3–4, the contour bound of Section 5 and the measure theory of Section 9. The nine documents have 233 pages in all.

All the documents have the same form. Each states its inputs (the results it takes from other parts) precisely and in the form in which it uses them, and its outputs (the results it supplies to other parts). Every step is marked as taken unchanged from a source, proved in full, or an input; a step taken from a source is cited by equation and page, and equations quoted from the sources carry page references. If a step could not be completed, it is marked as a gap, with the missing piece stated. Results of [U1] are cited by their numbers there.

Each document has a referee report, written with generative AI tools separately from the writing of the document it checks. Each report re-derives all proofs line by line, recomputes all constants, checks that each input is proved elsewhere or assigned to another part, and lists its findings by severity. The reports check the quoted equations against the sources or against notes that transcribe them, apart from a few that they take as cited. We archive each document with the corrections made after its review, together with its referee report, in the record [repo]. The current versions, with the corrections of the findings list applied, form Part S4 of [U1]. The findings that concern [U1] itself are in the list of findings.

In Lean, the modules of Sections 3–7 follow the documents statement by statement. The interfaces of these modules are the documents' inputs and outputs, and their docstrings cite the document's statement and the paper result. The documents also go beyond the paper in places. For example, they state the seeded sums of Lemma 7.48 for a class of wall fields defined by local primitives. The fixes made in the documents are among the findings.

6 Certified numerics

The proof uses numerical constants in three ways: the threshold $K_* = 377.38$ and the explicit string tension of the second assertion; the value $v_{\mathrm{Cb}}(0) = (-\Delta_{\mathbb{Z}^3})^{-1}(0,0)$ of the lattice Green's

function at the origin, half of Watson's integral; and the many inequalities between constants in the supplement of [U1] and in the written-out proofs. We certify all of them in Lean, in `U1/Numerics/` or, for some claims of the written-out proofs, where they are used; §7 uses the same method for the sign conditions of Theorem E.8.

6.1 The threshold

Lemma 2.6 bounds the density $\delta = e^{-c_\delta(K)\beta}$ of frozen plaquettes through a saddle-point exponent $c_\delta^{\mathrm{sp}}(K)$, and the Peierls estimate of Section 5 needs $c_\delta(K)/(2K) > 6\pi^2 v_{\mathrm{Cb}}(0) = 14.96613\ldots$; the number 14.9662 below is this value rounded up. We certify the values of $c_\delta^{\mathrm{sp}}(K)/(2K)$ quoted in Lemma 2.6 through a parametrization $K = e^t/(2t)$, rational brackets for the saddle point and Taylor bounds on exp checked by `norm_num`. In particular $c_\delta^{\mathrm{sp}}(K)/(2K) \geq 14.9663$ for $K \geq 377.38$, and the root of $c_\delta^{\mathrm{sp}}(K)/(2K) = 14.9662$ is unique and lies in $[377.36345, 377.36355]$. In the proof of the qualitative assertion, K_* enters only through the contour bound, which the library proves for every $K > K_*$.

6.2 Watson's integral

The bounds

$$0.2527310098586 \leq v_{\mathrm{Cb}}(0) \leq 0.2527310098631$$

reduce to two inequalities between rational numbers, one for each bound, each built from a head of the series expansion of $v_{\mathrm{Cb}}(0)$ and a bound on its tail. Both are decided by the kernel (`decide +kernel`), which adds no axiom; the resulting theorem is `U1.Numerics.vCb0_bounds`, and it gives $2.494355087 \leq \pi^2 v_{\mathrm{Cb}}(0) \leq 2.494355088$ (`U1.Numerics.pi_sq_vCb0_bounds`). The explicit string tension and the statements for large β use these bounds.

6.3 The symbolic constants

The paper leaves some constants symbolic, among them the parameters μ and κ_S of Sections 6 and 7 (item 12 of [U1, S1.1.1]) and the Pauli–Villars parameter λ_{PV} (item 13), and states the conditions they must satisfy. In the formal proof we fix them, together with the parameter C of Theorem 6.5 (not the loop C):

$$C = \tfrac{1}{4}, \qquad \mu = 4401, \qquad \kappa_S = 7, \qquad \lambda_{\mathrm{PV}} = \lambda_\star(\kappa_S, \mu) := e^{-(\kappa_S + \mu/2 + 6)} = e^{-2213.5},$$

and take $\theta_0 = \frac{1}{100}$, $\eta_H = 0.62$, $p_4 = 15$ and $c_\lambda = \frac{1}{24}$ from the paper. We choose the constants in an order in which each condition concerns only constants already chosen: the absolute constants and the heat-kernel constants; then $c_{10} = \frac{7}{9} r_C$, with $r_C = \kappa_0/2 \approx 2.8 \cdot 10^{-183}$, and $c_{10}'' = 2200$, which is certified admissible (the least admissible value is about 2137.3); then $\mu = 1 + 2c_{10}''$; then κ_S; then λ_{PV}; and finally β large. This order is not circular, because no condition bounds μ or κ_S from above, and c_{10}'' involves none of μ, κ_S, λ_{PV} and β. Each condition has a certificate in `U1/Numerics/`, most of them in `U1/Numerics/StageC/`; for example, `c10pp_certificate` certifies the condition of Lemma A.6 on c_{10}'', which with $\mu = 1 + 2c_{10}''$ gives that of Proposition B.7, and `T730_global` certifies condition (7.30), which is the binding condition on κ_S.

6.4 The constants of the written-out proofs

In the six written-out proofs of Sections 6 and 7 and their appendices, every numerical claim is inventoried with its location, its kind (arithmetic, parametric over a stated range, definitional or analytic) and whether it holds as written. The arithmetic and parametric claims, except the two figures of a numerical experiment described below, are also checked independently in 50-digit arithmetic. Lean lemmas in `U1/Numerics/StageC/` certify 563 claims, among them 96 of the 98 arithmetic and 465 of the 516 parametric ones; the other parametric and analytic claims are proved where they are used, and the two remaining arithmetic claims, figures of a numerical experiment reported in a remark, are checked against its recorded run. Of the 1677 claims, nine fail as written: seven roundings in the last digits, one wrong digit and one equality that holds only as an inequality. Six more cannot be checked from the documents alone. All fifteen are corrected or resolved in the documents, without changing any conclusion. The inventory, with the status of each claim and the lemma that certifies it where there is one, is the file `U1/Numerics/StageC/CLAIMS.md` of the archive.

7 Appendix E of [U1]

Appendix E of [U1] is formalized in `U1/AppendixE/` and `U1/AppendixEMain.lean`, in the namespace `U1.AppendixE`. These files are in the import closure of the root module, so the audits and the build cover them. The formalization follows the proofs of the appendix. Table 1 gives each result with its closed statement in `U1/AppendixE/Statements.lean` and the theorem that proves it. The remarks E.5, E.7, E.9 and E.11 are not results, and the numerical values quoted after Theorem E.8 are not formalized.

7.1 What the appendix reuses

The appendix uses the definitions of the main formalization. The Gibbs expectation of the appendix is the expectation of Section 2 (`gibbsExp_eq_gexp`), and with the Wilson weight on every plaquette it is the Wilson expectation of Theorem A (`gibbsExp_wilsonW`). The dual formula of Lemma E.1 is proved for general plaquette weights with the character integrals and Fourier series of the dual representation (`U1/Dual/`), by the steps that give the Villain case. Theorem E.2(b) and Proposition E.6(a) use the Ginibre framework of Section 2 (`U1/Correlation/`): the admissibility of $e^{\lambda\cos\varphi}$ and Ginibre's order, composed with $\varphi \mapsto j\varphi$ and with products, give the monotonicity in every coefficient $\lambda_{p,j}$. Corollary E.4, Proposition E.6 and the certificate below use the modified Bessel functions I_ν and their series (`U1/Special/`).

7.2 Where the proofs take other routes

At three steps the formal proofs differ from those in [U1].

- Lemma E.3 is proved without Poisson summation on the lattice. With $g(y) = e^{-\|y\|^2}$ and its periodization $G(y) = \sum_{m\in\Gamma} g(m+y)$, the Gaussian convolution identity and the unfolding of the integral over a fundamental domain F of Γ give $c\sum_{m\in\Gamma} e^{-\|m+x\|^2/2} = \int_F G(z)G(x-z)\,dz$ with $c > 0$. Since G is even, the Cauchy–Schwarz inequality bounds the right side by its value at $x = 0$. The lemma is also proved for discrete subgroups that are not of full rank, with the factor $e^{-\|x-Px\|^2/2}$ for an orthogonal projection P onto a subspace containing Γ.

- In Theorem E.2(a), the plaquettes with coupling $t_p = 0$ are handled directly instead of by the limit $s \downarrow 0$. The cycles that contribute form a coset of the sublattice of cycles vanishing on these plaquettes, and the form of Lemma E.3 with a projection factor applies to it.

- In Proposition E.12, Poisson summation for the one-dimensional theta functions comes from Mathlib's summation formula for quadratic exponentials. For the ratio R_d of theta functions that [U1] uses, with d frozen faces, it gives $(1 - 3q)/(1 + q) \le R_d \le 1$ with $q = e^{-2\pi^2\tau/(6-d)} \le e^{-\pi^2\tau/3}$, in place of the bound $(1 - 2\varrho)/(1 + 2\varrho)$ of [U1], and the conclusion for $\tau \ge 3$ is the same.

7.3 The kernel decision for Theorem E.8

On the boundary of the cube the cycles are the multiples of the sum of the faces (`mem_cycles2_cube_iff`); the cell counts, the incidences and $\partial\partial = 0$ of the cube are decided by the kernel. Lemma E.1 then gives the closed form $\mathcal{W}(x) = (A_C + xB_C)/(A_0 + xB_0)$ of [U1] for the loop around p_0 as a function of the coefficient $x = c_1$ of the weight on p_0, so $\mathcal{W}'$ has the sign of $S = A_0B_C - A_CB_0$, and the derivative with every face perturbed has the sign of a second number U. Both are series in the coefficients c_k of the weight, homogeneous of degree 11. A soundness lemma (`cube_signs_of_bounds`) turns rational enclosures of c_k for $k \le H + 2$, together with a geometric bound on the remaining coefficients, into rational upper bounds for S and U; here $H = 4$.

- For the Villain weight, $c_k = (15/16)^{k^2}$ is rational, and $(h + 6)^2 \ge 36 + 12h$ gives the geometric bound $c_{6+h} \le c_6(15/16)^{12h}$ for $h \ge 0$.

- For the Wilson weight w_8^W, homogeneity allows $c_k = I_k(8)$. Sixteen terms of the series $I_k(8) = \sum_{m\ge0} 4^{2m+k}/(m!\,(m+k)!)$ and a geometric remainder enclose $I_k(8)$, and $I_{k+1}(8) \le 4I_k(8)/(k + 1)$ gives the geometric bound.

As for Watson's integral, the kernel decides (`decide +kernel`) that the resulting rational upper bounds are negative. The bounds give $S \le -0.277$ and $U \le -0.325$ for the Villain weight and $S \le -0.2529\, I_0(8)^{11}$ and $U \le -0.2958\, I_0(8)^{11}$ for the Wilson weight. The margins are large, so the low truncation suffices.

8 Audits

The proof checker does not notice if a statement says less than the paper, if a hypothesis is unsatisfiable, or if a build depends on files that are not in the repository. The audits check for each of these. Like the referee reports on the written-out proofs, they were made separately from the writing of the formal proofs. The audit reports are in the directory `audit/` of the Lean archive and, for the release checks on the audited commit `e8e86d3`, in a separate file of [repo] (§10).

8.1 Statements

The audit of the main results compares each Lean statement of a paper result with its statement in [U1], clause by clause: quantifiers and their order, ranges of parameters, boundary conditions, normalizations and the objects quantified over. Deliberate generalizations are recorded as such.

Definition 1.1 (the Wilson and Villain measures, the Wilson loop and the area law), both assertions of Theorem 1.2 (with $K_* = 377.38$), the infinite-volume form of the area law (Remark 8.1) and Proposition 9.1 (more general than in [U1]) are faithful, and Corollaries 1.3 and 1.4 in the paper's form are `U1.Main.cor_1_3_rel` and `U1.Main.cor_1_4_rel` together with `thm_1_2`. The statements of Section 2 match the paper, with two restrictions that no statement of a main theorem involves: Lemma 2.2 is stated without the uniqueness of the mixture ω_β, and Lemma 2.3(d) for couplings on D that are all equal to s, as $s \to \infty$. The interfaces restate the results of [U1] that they cite, field by field. A separate audit finds that the nine closed statements of Appendix E, those of its eight results and of the realized weights, match the text of [U1] without extra hypotheses, and Lean probes show that each of their hypotheses is satisfied by the paper's own objects: the Wilson and Villain weights are plaquette weights, Villain weights are Villain mixtures, the Wilson weight is of Ginibre type, a one-plaquette complex realizes its own weight, and $\partial\delta_{p_0} \neq 0$ on the cube. Both audits were made against version 3 of [U1]; version 4 states Definition 1.1, Theorem 1.2, its corollaries and the results of Sections 2, 8 and 9 and of Appendix E with the same mathematical content.

8.2 Satisfiable hypotheses

Every hypothesis structure on the chain from the construction to Theorem 1.2 is inhabited by closed terms of the library, since the proofs of the main theorems build each interface at the canonical parameters. These structures are satisfiable because of the guards, and the probes in `audit/probes/` check the guards as described in §4.6; all 38 probe files compile against the audited commit.

8.3 The library

The release checks build the library from a fresh export of the audited commit and run `#print axioms` on every theorem of Theorem B, with the result stated there. The source contains no `sorry`, no `axiom` declaration and no `native_decide`, and no constant of the library uses `sorryAx`. Every constant stored in the compiled library is replayed through the kernel, which accepts all of them. The theorems of Theorem B with all their dependencies, and separately every named constant of the library with its dependencies, are exported with `lean4export`; both exports contain only the same three axioms, and two independent kernel implementations, `nanoda` and `lean4lean`, accept both. The report of the release checks (§10) gives the details.

9 Findings about the paper

We recorded 70 entries about version 1 of [U1]: 5 errors, 11 gaps and 11 unstated hypotheses, besides 43 points of presentation, notation or simplification. They include every place where the formalization, the written-out proofs, their referee reports or the audits found a statement or proof wrong, incomplete or relying on something unstated. For each, the list gives replacement text for [U1] and, where it applies, how the formal proof and the written-out proofs treat it. No entry changes Theorem A or its corollaries, and where an entry changes a statement of [U1], the paper's proofs already use the new statement. The list is the file `changes.md` of the record [repo], and `PROVENANCE.md` there says for each entry where it was found and which documents are involved, among them the findings ledger `FINDINGS.md` of the Lean archive. Versions 2 and later of [U1] apply every entry that is not marked optional.

The five errors are statements or proof steps that are false as written. Display (A.3) takes the supremum over the interpolation parameters inside the sum over elementary terms, while Proposition A.7 combines all terms with one link datum and bounds that combination uniformly in the parameters, which needs the supremum outside the sum; the corrected form has the same proof and the same constants. Section D.3(c) bounds the number of points at graph distance k in a slit graph by $4k^2 + 2$, the size of an ℓ^1-sphere of $\mathbb{Z}^3$, which fails on slit graphs: in a three-dimensional H-tree the sphere of radius 189 about the root has $2^{18} = 262144$ points, while $4 \cdot 189^2 + 2 = 142886$. What the proof uses is the bound $\sum_y e^{-a\,d(x,y)} \le 64a^{-3}$ for $0 < a \le 2$, which follows directly from $d \ge \|\cdot\|_1$. Proposition 7.22 is stated for a label on a region, which does not determine the objects of the expansion; it is proved and used for wall fields, and the fix states it for them. In Lemma 7.29(f), the claim that the ratios $\mathcal{Z}_p(X)/Z$ are unchanged holds only for $X = \emptyset$; the error is harmless, since from Lemma 7.29 on the paper builds every activity and every $\mathcal{Z}_p(X)$ with the modified objects. Finally, the growth constant K_g of Appendix B, and the same count after display (A.9), omit the rounding to lattice cells; only the independence of K_g from τ is used later, so nothing else changes.

The eleven gaps are steps that are missing or only asserted. Among them, Lemma 3.5 is stated without proof; the reference density of Section 6 is taken in infinite volume, where the paper neither defines it as a limit over finite volumes nor proves the bound it uses; Lemma C.1 and Proposition 7.22 use charge moments that the paper does not state; absolute convergence in Theorem 7.27 is proved only for the objects built with the modified covariance, which are the only ones used later; and display (A.9) puts a distance factor on links of the first type, which have no covariance line, so that Proposition B.7 cannot pay for it. The fix drops the factor for these links, and since the count of link data (Lemma A.6) never uses it, all constants are unchanged. Among the simplifications is the sharper counting factor $2^{4|C|+2}$ in the tail bound of Theorem 5.1, noted in §4.3.

Findings about the sources. The paper already records its corrections to [B78] and [BF80] (in the proof of Lemma A.1, in Remark A.5 and in Proposition A.7), and the formal proof uses them. The referee report on the written-out proof of Appendix A found one more, which [U1] has recorded in Remark A.5 since version 2: in [BF80, pp. 213–217] the sums are listed in the reverse of the order in which they are performed, so the constraint of the step 3Σ, that each set contain one of the cubes "mentioned in 10Σ", refers to the variable of an inner sum. The count of Proposition A.4 instead uses a valid order, in which each constraint links a variable only to variables outer to it.

10 Reproduction

The Lean archive of [repo] is a git archive of the release files of a commit that differs from the audited commit `e8e86d3` only in its README. The archive pins Lean `v4.35.0-rc3` and Mathlib at commit `c55e6e786f49`. With `elan` installed, the three commands of its README, `lake exe cache get`, `lake build U1` and `lake env lean PrintAxioms.lean`, fetch the Mathlib build for the pinned commit, build the library and print the statements and axioms of the theorems of Theorem B. A clean build of the library took 382 seconds of wall-clock time on a 60-core machine, with a peak of 3.7 GiB in its largest process. The release checks were run on `e8e86d3`, and their report `release-checks-e8e86d3.md` in [repo] gives the figures of the checks not given here. The audit reports in `audit/` describe earlier commits, most recently `254a4d3` and `1873b51`,

whose library, with comments removed, is that of `e8e86d3`.

11 Conclusion

We draw two lessons that may carry over to formalizations of other long analytic proofs. First, arguments that a paper borrows "with changes" are best written out in full, inputs and outputs included, before anyone formalizes them. Second, when the parts of a proof are joined by interfaces, each interface needs its own check that the objects the proof actually produces satisfy it (§4.6).

Acknowledgements

Generative AI tools were used to write code, run analyses and draft text; the author directed the work and takes full responsibility for its content.

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[repo] The Zenodo record of this paper (all versions), `https://doi.org/10.5281/zenodo.23480809`, whose current version also holds the formalization and its working records: the Lean sources of the audited commit `e8e86d3` (§10), with the report of their release checks; the written-out proofs and their referee reports; and the list of findings.