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SUBPLANES OF ORDER 3 AND MAXIMAL ARCS IN PROJECTIVE PLANES OF ORDER 12

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Abstract. Whether a projective plane of order 12 exists is open. A subplane of order 3 of such a plane is a case of equality in Bruck's bound $n \geq m^2 + m$; the 27 points on no line of the subplane then form a maximal arc of degree 3, and together with the subplane a maximal arc K of degree 4. We show that every line of a subplane of order 3 contained in K lies in at most two such subplanes, whereas in $\mathrm{PG}(3,3)$, which like K has 40 points, a line lies in four planes; the proof uses Egan and Wanless's classification of triples of mutually orthogonal Latin squares of order 9. We also show that $\mathrm{AG}(3,3)$ is not the design of any maximal arc of degree 3 in a plane of order 12, and that a permutation of the points that maps a subplane of order 3 onto itself and each line meeting it onto a line cannot have prime order other than 3. The proofs are computer-assisted, and their SAT refutations are certified.

1. Introduction

Projective planes of order n exist for all prime powers n, and no plane of any other order is known. The Bruck–Ryser theorem [BR49] excludes infinitely many orders, and a large computer search excluded order 10 [Lam91]. The smallest open order is 12. Most of the work on order 12 concerns collineations. A series of papers by Janko and van Trung, by Horvatić-Baldasar, Kramer and Matulić-Bedenić, by Suetake, and by Akiyama, Suetake and Tanaka shows that a collineation group of a plane of order 12 has order 1, 2 or 3 [AST23]. The introduction of [AST19] surveys the earlier steps. Hall and Wilkinson [HW84] studied the ternary code of such a plane.

This paper concerns subplanes. Bruck [Bru55] proved that a proper subplane of order m of a projective plane of order n satisfies $n = m^2$ or $n \geq m^2 + m$. For $n = 12$ this leaves $m \in \{2, 3\}$, and $m = 3$ is a case of equality: $12 = 3^2 + 3$. Janko and van Trung [JvT80] showed that a plane of order 12 has no collineation of order 13 that leaves a subplane of order 3 invariant. Their proof uses their classification of the Kirkman triple systems of order 27 with an automorphism of order 13 [JvT81], and they later excluded collineations of order 13 altogether [JvT82]. We do not assume any collineations.

Throughout, π is a projective plane of order 12 and π_0 is a subplane of order 3. A point outside π_0 is *free* if it lies on no line of π_0, and *non-free* otherwise. A line that meets π_0 in exactly one point is a *tangent*. We write F for the set of free points and $K = \pi_0 \cup F$.

A *maximal arc of degree d* in a projective plane of order n is a set of points that meets each line in 0 or d points. Such an arc has $(d-1)(n+1)+1$ points, and d divides n unless $d = n+1$. Denniston [Den69] constructed maximal arcs of all degrees d dividing q in $\mathrm{PG}(2,q)$ for q even, and Ball, Blokhuis and Mazzocca [BBM97] proved that $\mathrm{PG}(2,q)$ has none with $1 < d < q$ for q odd. Bruck's counts in the case of equality [Bru55, Lemma 3.1] imply that F has 27 points and is a maximal arc of degree 3, and that K is a maximal arc of degree 4 (Section 3). By the construction of Thas [Tha74] and Wallis [Wal73], K gives a partial geometry $\mathrm{pg}(8,9,6)$ and

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its dual $\mathrm{pg}(9, 8, 6)$. Brouwer's tables of strongly regular graphs list both with a question mark [Bro], so a plane of order 12 with a subplane of order 3 would settle these two entries.

Each tangent carries exactly three free points, and the free points with these triples form a Kirkman triple system of order 27; the 13 classes T_P of its resolution are formed by the triples on the tangents at the points P of π_0. The most symmetric candidate for this triple system is the affine space $\mathrm{AG}(3, 3)$, with the classes T_P as the classes of parallel lines. We call this the *affine case*. Then π_0 plays the role of the plane at infinity, and π extends the points and lines of $\mathrm{PG}(3, 3)$.

For a line L of π_0, let Γ_L be the graph on F in which two free points are adjacent if the line joining them meets L in a point of π_0. In the affine case each Γ_L is the disjoint union of three copies of K_9, formed by three parallel planes of $\mathrm{AG}(3, 3)$. Conversely, if this holds for every line L, the free points form a Hall triple system, hence $\mathrm{AG}(3, 3)$ by the theorem of Hall and Bruck [Hal60, Bru58], and we are in the affine case. A component of Γ_L with 9 points, together with the four points of π_0 on L, is a subplane of order 3 contained in K. So $\Gamma_L \cong 3K_9$ means that L lies in four subplanes of order 3 contained in K, just as a line of $\mathrm{PG}(3, 3)$ lies in four planes. Our main result shows that in a plane of order 12 the number is at most two.

Theorem A (Section 7)**.** *Let π be a projective plane of order* 12 *with a subplane π_0 of order* 3*, and let K be the union of π_0 and its free points. Then every line of a subplane of order* 3 *contained in K lies in at most two such subplanes, and K contains at most* 14 *subplanes of order* 3*.*

If $\Gamma_L \cong 3K_9$, the 81 tangents at the points of π_0 off L form a transversal design $\mathrm{TD}(5, 9)$, that is, three mutually orthogonal Latin squares of order 9. We go through the classification of these designs by Egan and Wanless [EW16] and show with a SAT solver that none of them extends to the configuration of free points and tangents that a plane would give. Together with a count showing that every component of Γ_L has 9, 18 or 27 points, this leaves two possibilities for Γ_L in a plane: it is connected, or it is the disjoint union of K_9 and a connected graph on 18 points.

In the affine case, the 117 points off K give a *covering set of spreads* of $\mathrm{PG}(3, 3)$, that is, a family of spreads such that every pair of skew lines lies in exactly one of them. Bruck [Bru69, Bru73] asked whether the points and lines of $\mathrm{PG}(3, q)$ extend to a projective plane of order $q(q + 1)$, and proposed to build such planes from covering sets of spreads. For $q \geq 4$ the existence of covering sets of spreads is open, with a partial result for $q = 4$ [Pri01]. For $q = 3$, a search of Hall and Roth [HR84], which omitted some cases, found no extension, and Prince [Pri99] gave a complete proof by showing that $\mathrm{PG}(3, 3)$ has no covering set of spreads; we recheck his theorem by a certified computation (Theorem 4.4). We strengthen the exclusion of the affine case in two ways.

Theorem B (Sections 4 and 5)**.**
(a) *In the affine case, already the incidences of the free points and of the non-free points on three lines of π_0 with the tangents are impossible, whether the three lines are concurrent or form a triangle.*
(b) *In no projective plane of order* 12 *is there a maximal arc of degree* 3 *whose points and secants form* $\mathrm{AG}(3, 3)$*.*

Since the free points in the affine case form an arc as in (b), each part gives a proof, independent of Prince's theorem, that the affine case does not occur.

A collineation of π that maps π_0 onto itself has order 1 or 3 (Proposition 3.4). We therefore consider a weaker kind of symmetry. The *core* of π relative to π_0 is the incidence structure formed by all points of π and the 130 lines that meet π_0; it omits the 27 lines that miss π_0. Similarly, the core of a maximal arc consists of all points and the secants of the arc. An automorphism of

a core is a permutation of the points that maps each line of the core onto a line of the core; for the core relative to π_0 we also require that it maps π_0 onto itself.

Theorem C (Section 6)**.** *Let π be a projective plane of order* 12*.*

(a) *The core of a maximal arc of degree* 3 *or* 4 *in π has no automorphism of order* 13*.*
(b) *If π_0 is a subplane of order* 3*, every automorphism of prime order of the core relative to π_0 has order* 3*.*

A core automorphism need not act on the lines outside the core, so the results on collineations cited above do not apply to it. Section 6 also excludes automorphisms of order 13 of the triple system that the secants cut out on a maximal arc of degree 3, and of the triple system of the free points with its labelled resolution; for the latter we recount the classification of [JvT81], whose list has four repetitions, so that there are 657 isomorphism classes rather than 661 (Remark 6.3). The same section shows that an automorphism of order 3 as in (b) induces on π_0 an elation or a collineation with exactly one fixed point, and only the latter if it fixes a free point.

Section 8 states open problems, among them the conjecture that no Steiner triple system of order 27 has a family of parallel classes covering every pair of disjoint triples exactly once; it would imply that no projective plane of order 12 contains a maximal arc of degree 3 or 4, and hence no subplane of order 3.

Several proofs depend on exhaustive computer searches [Knu00, BFF$^+$24b, BFF$^+$24a]. Their SAT refutations have LRAT certificates [CFHH$^+$17], checked by the verified checker of Lean 4 [dMU21]; Section 9 describes what else is checked, including the formalization of the results proved by hand other than Corollary 7.7 (the last step of Theorem A), and what is not formalized, notably the translation of the geometric statements into formulas and searches.

Section 2 collects definitions, and Section 3 derives the structure described above; the proofs of Theorems A–C are in the sections named in their headings.

2. Preliminaries

2.1. **Planes.** A projective plane of order n has n^2+n+1 points and as many lines. Each line has $n+1$ points, each point lies on $n+1$ lines, two points lie on exactly one line, and two lines meet in exactly one point. A subplane of order m is a set of points and lines that forms a projective plane of order m under the induced incidence. For $n=12$ there are 157 points and 157 lines, each line has 13 points, and each point lies on 13 lines. Since the projective plane of order 3 is unique, every subplane of order 3 is isomorphic to $\mathrm{PG}(2,3)$ [HP73].

The group $\mathrm{PGL}(3,3)$ has order $5616 = 2^4 \cdot 3^3 \cdot 13$. We use three facts about it.

- Every involution of $\mathrm{PGL}(3,3)$ is a homology: it fixes a line pointwise and a point off that line. Indeed, a matrix A with $A^2 = -I$ does not exist over $\mathbb{F}_3$ in dimension 3, since $\det(A)^2 = -1$ is not a square in $\mathbb{F}_3$. Hence $A^2 = I$, and A is diagonalizable with eigenvalues ± 1.
- An element of order 3 is either an elation, which fixes a line (its axis) pointwise and all lines through a point of the axis (its center), or it fixes exactly one point and one line, which are incident.
- The subgroups of order 13 are Sylow 13-subgroups and therefore conjugate. They are generated by Singer cycles, which act regularly on the points and on the lines.

2.2. **Maximal arcs.** Let K be a maximal arc of degree d in a plane of order n, with $1 < d \le n$. Lines meeting K are *secants*, the others are *external lines*. A point of K lies on $(|K|-1)/(d-1) = n+1$ secants. A point off K lies on $|K|/d$ secants. The *design* of K is the set K with the secants restricted to K as blocks. It is a Steiner system $S(2, d, |K|)$.

2.3. **Designs.** A Steiner system $S(2,k,v)$ is a set of v points with blocks of size k such that two points lie in exactly one block. A Steiner triple system $\mathrm{STS}(v)$ is an $S(2,3,v)$. A *parallel class* is a set of blocks that partitions the points, and a *resolution* is a partition of the blocks into parallel classes. A Kirkman triple system $\mathrm{KTS}(v)$ is an $\mathrm{STS}(v)$ with a resolution. An $\mathrm{STS}(27)$ has 117 triples, a parallel class consists of 9 triples, and a resolution has 13 classes.

The affine space $\mathrm{AG}(3,3)$ is the $\mathrm{STS}(27)$ on $\mathbb{F}_3^3$ whose triples are the lines $\{p, p+d, p+2d\}$. Three distinct points form a line if and only if their sum is 0. The lines with a given direction $\langle d\rangle$ form a parallel class, and the 13 directions form the plane at infinity $\mathrm{PG}(2,3)$. The projective space $\mathrm{PG}(3,3)$ has 40 points and 130 lines. A *spread* of $\mathrm{PG}(3,3)$ is a set of 10 lines that partitions the points.

A Steiner triple system is a *Hall triple system* if any three points not in a common triple lie in a subsystem isomorphic to $\mathrm{AG}(2,3)$, the $\mathrm{STS}(9)$. Hall [Hal60] showed that Hall triple systems correspond to commutative Moufang loops of exponent 3. Such a loop of order at most 27 is an elementary abelian group, since the smallest non-associative one has order 81 [Bru58]; see also [BL85]. Hence a Hall triple system on 27 points is isomorphic to $\mathrm{AG}(3,3)$.

A *partial geometry* $\mathrm{pg}(s,t,\alpha)$ is a set of points and lines such that each line has $s+1$ points, each point lies on $t+1$ lines, two points lie on at most one line, and for each point x and line $\ell \not\ni x$ exactly α points of ℓ are collinear with x. Its collinearity graph is strongly regular with parameters $v=(s+1)(st+\alpha)/\alpha$, $k=s(t+1)$, $\lambda=s-1+t(\alpha-1)$ and $\mu=\alpha(t+1)$. The dual structure is a $\mathrm{pg}(t,s,\alpha)$.

3. Structure

The structure described in this section follows from the counts in Bruck's equality case [Bru55, Lemma 3.1]; Dembowski [Dem58, pp. 76–77] gives the incidence numbers of the three classes of points and lines. We give the short proofs to fix notation. We fix π and π_0 and call the 13 lines of π_0 the *π_0-lines*. A tangent that meets π_0 in P is a *tangent at P*, and a line that misses π_0 is *external*.

Theorem 3.1. *The points of π are the 13 points of π_0, 117 non-free points (9 on each π_0-line L, forming the set N_L), and 27 free points. The lines of π are the 13 π_0-lines, 117 tangents (9 at each point of π_0), and 27 external lines. Moreover:*

(i) *a tangent at P carries P, one non-free point of each π_0-line not through P, and 3 free points;*
(ii) *an external line carries exactly one non-free point of each π_0-line;*
(iii) *a non-free point $X \in N_L$ lies on L, on one tangent at each point of π_0 off L, and on 3 external lines;*
(iv) *a free point lies on one tangent at each point of π_0, and on no other line.*

Proof. A point Q on no π_0-line is joined to the 13 points of π_0 by 13 distinct lines, because a line through Q with two points of π_0 would be a π_0-line. These are all the lines through Q, which proves (iv). A tangent at P meets each of the 9 π_0-lines not through P in a point outside π_0, which is non-free, and these 9 points are distinct, since two π_0-lines meet in a point of π_0. The remaining $13-1-9=3$ points lie on no π_0-line, which proves (i). A non-free point $X \in N_L$ lies on L and on the 9 lines XP with $P \notin L$. These lines are distinct tangents, since a line through X with two points of π_0 is a π_0-line and hence equal to L. The remaining 3 lines through X miss π_0, which proves (iii). Each π_0-line has $13-4=9$ non-free points, and a non-free point lies on only one π_0-line. Hence there are 117 non-free points and $157-13-117=27$ free points. By (iv) no external line contains a free point. An external line meets each π_0-line in a point outside π_0, and these 13 points are distinct, which proves (ii). The line counts follow by duality, since the dual of π_0 is a subplane of order 3 of the dual plane. □

Two free points lie on a unique line, which is a tangent at a unique point P of π_0. We call them *partners at* P if they lie on a common tangent at P. For $P \in \pi_0$, let T_P be the set of 9 triples of free points cut out by the tangents at P. For a π_0-line L, the graph Γ_L of the introduction joins two free points if they are partners at a point of L.

Corollary 3.2. (i) *The free points with the* 117 *triples of* $\bigcup_P T_P$ *form a Kirkman triple system* $\mathcal{K}$ *of order* 27 *with resolution* $\{T_P : P \in \pi_0\}$.

(ii) *For a* π_0*-line* L *and* $X \in N_L$*, let* C_X *be the set of free triples of the* 9 *tangents through* X*. Then* C_X *is a parallel class of* $\mathcal{K}$ *that contains one triple of each* T_P *with* $P \notin L$*, and the* 9 *classes* C_X *with* $X \in N_L$ *partition* $\bigcup_{P \notin L} T_P$.

(iii) *If* $X \in N_L$ *and* $X' \in N_{L'}$ *with* $L \neq L'$*, then* $|C_X \cap C_{X'}| \leq 1$.

(iv) *Each external line* e *gives a resolution* $\{C_{e \cap L} : L$ *a* π_0*-line*$\}$ *of* $\mathcal{K}$.

(v) *The graph* Γ_L *is* 8*-regular and is the union of the* 4 *triangle factors* T_P *with* $P \in L$.

Proof. (i) Two free points lie in exactly one triple, namely the free triple of their joining tangent. By Theorem 3.1(iv) each free point lies on exactly one tangent at P, so T_P is a parallel class.

(ii) The 9 tangents through X are tangents at the 9 points of π_0 off L, and they meet only in X, which is not free. Hence their free triples are pairwise disjoint and cover all 27 free points. A tangent at $P \notin L$ meets L in exactly one point of N_L, so each triple of T_P lies in exactly one class C_X with $X \in N_L$.

(iii) The points X and X' are distinct and lie on at most one common line. A common triple of C_X and $C_{X'}$ lies on a common tangent.

(iv) A triple $t \in T_P$ lies on a tangent at P, which meets e in a non-free point X. If L is the π_0-line through X, then $P \notin L$ and $t \in C_X$. If t lay in two of the classes, its tangent would contain two points of e and would equal e.

(v) Each free point has 2 partners at each of the 4 points of L. □

Definition 3.3. Let L be a π_0-line. A parallel class of $\mathcal{K}$ that contains one triple of T_P for each $P \notin L$ is a *transversal class* for L. A set of 9 transversal classes for L that partitions $\bigcup_{P \notin L} T_P$ is an *orthogonal resolution* for L. Orthogonal resolutions for distinct lines L and L' are *compatible* if a class of one and a class of the other share at most one triple.

An orthogonal resolution for L is a partial form of the usual notion of orthogonal resolutions of a design: its 9 classes are orthogonal to the 9 classes T_P with $P \notin L$. Our use of "compatible" differs from that of Tonchev [Ton17], where it refers to resolutions of the whole design. By Corollary 3.2, the classes C_X ($X \in N_L$) form an orthogonal resolution for L, and these resolutions are pairwise compatible. Throughout, the labels P of the classes T_P carry the incidence structure of π_0.

Proposition 3.4. *The stabilizer of* π_0 *in the collineation group of* π *has order* 1 *or* 3.

Proof. By [AST23] the collineation group of π has order 1, 2 or 3. An involution of a projective plane of order n is an elation, a homology or a Baer involution [HP73]. A Baer involution requires n to be a square, and a homology of order 2 requires n to be odd. So the involutions of π are elations. Suppose an involution σ maps π_0 onto itself. It does not fix π_0 pointwise, since the fixed points of an elation lie on its axis and π_0 contains a quadrangle. Hence σ induces an involution of $\mathrm{PGL}(3,3)$, which is a homology. It fixes a line and a point off that line, hence five points of π_0, not all collinear. But the fixed points of the elation σ lie on its axis, which meets π_0 in at most four points. □

Proposition 3.5. (i) *Every line meets* F *in* 0 *or* 3 *points, so* F *is a maximal arc of degree* 3*. Its secants are the* 117 *tangents.*

(ii) *Every line meets K in* 0 *or* 4 *points, so K is a maximal arc of degree* 4 *with* $(4-1)\cdot 13+1 = 40$ *points. Its secants are the* 130 *lines meeting π_0, and its external lines are the* 27 *external lines of π. These form a maximal arc of degree* 3 *in the dual plane.*

(iii) *The secants of K, restricted to K, form a Steiner system $\mathcal{D} = S(2, 4, 40)$ containing π_0 as a subdesign. For $X \notin K$, the* 10 *secants through X form a parallel class S_X of $\mathcal{D}$, and S_X contains exactly one block inside π_0. Two disjoint blocks of $\mathcal{D}$ lie in exactly one class S_X.*

(iv) *If $\pi_1 \subseteq K$ is a subplane of order* 3*, then K is the union of π_1 and its free points.*

(v) *The* 117 *points off K and the* 130 *secants form a partial geometry* $\mathrm{pg}(8, 9, 6)$*. Its dual is a* $\mathrm{pg}(9, 8, 6)$ *whose points are the blocks of $\mathcal{D}$ and whose lines are the classes S_X.*

Proof. (i) and (ii) follow from Theorem 3.1: a π_0-line meets K in $4 + 0$ points, a tangent in $1 + 3$, and an external line in none. A point off K is non-free and lies on 3 external lines, while a point of K lies on none. Thus the external lines form a dual maximal arc of degree 3.

(iii) Two points of K lie on one line, which is a secant. A point $X \notin K$ lies on $13 - 3 = 10$ secants, which cover the 40 points of K, four on each, and therefore cover each point once. A block of $\mathcal{D}$ meets π_0 in 1 or 4 points. If S_X contains k blocks inside π_0, then $4k + (10 - k) = 13$, so $k = 1$. Two disjoint blocks meet in π in a point off K, which is the unique X with both blocks in S_X.

(iv) A π_1-line is a secant of K, and its four points in K are the points of π_1. Hence no point of $K \setminus \pi_1$ lies on a π_1-line, and $K \setminus \pi_1$ is contained in the set of free points of π_1. Both sets have 27 points.

(v) A secant has $13 - 4 = 9$ points off K, and a point off K lies on 10 secants, so $s = 8$ and $t = 9$. Let $X \notin K$ and let B be a secant not through X. Four of the secants through X meet B in a point of K, and the other 6 meet B in a point off K. Hence $\alpha = 6$. The parameters of the collinearity graphs are $(117, 80, 52, 60)$ and $(130, 81, 48, 54)$. □

By Proposition 3.5(ii), the lines that meet π_0 are the secants of K, so the core of π relative to π_0, defined in the introduction, is the core of the arc K. The core determines $\mathcal{K}$, the labels of its resolution, and the orthogonal resolutions of all π_0-lines.

Remark 3.6. Conversely, let $\mathcal{D}$ be an $S(2, 4, 40)$ with a family $\mathcal{S}$ of parallel classes such that any two disjoint blocks lie in exactly one class of $\mathcal{S}$. A block is disjoint from $130 - 1 - 4 \cdot 12 = 81$ blocks, so there are 5265 disjoint pairs. Since a class covers $\binom{10}{2} = 45$ pairs, $|\mathcal{S}| = 117$. Each block lies in $81/9 = 9$ classes, and two classes share at most one block. Adding one point for each class, incident with the blocks of the class, gives 157 points and 130 lines of size 13, in which two points lie on at most one line. This is the core of a putative plane. Of the $\binom{117}{2} = 6786$ pairs of classes, $130\binom{9}{2} = 4680$ share a block and $2106 = 27\binom{13}{2}$ do not. A projective plane of order 12 arises exactly when one can add 27 resolutions of $\mathcal{D}$ by classes of $\mathcal{S}$ such that two classes with no common block lie in exactly one of them. In the subplane case they correspond to the resolutions of Corollary 3.2(iv).

4. The affine case

In this section we assume the affine case: there is a bijection from F to $\mathbb{F}_3^3$ that maps each class T_P onto the set of lines of $\mathrm{AG}(3, 3)$ with one direction d_P. The 13 classes T_P partition the 117 lines of $\mathrm{AG}(3, 3)$, and $P \mapsto d_P$ is therefore a bijection from π_0 onto the set of directions. The first lemma shows that it is a collineation.

Lemma 4.1. *Let Δ be a set of four directions of* $\mathrm{AG}(3, 3)$ *that are not collinear in the plane at infinity. Then* $\mathrm{AG}(3, 3)$ *has no parallel class consisting of one line of each of the* 9 *directions outside Δ.*

Proof. Let φ be a linear form on $\mathbb{F}_3^3$. Over the 27 points, $\sum_x \varphi(x)^2 \equiv 0 \pmod 3$. Over a line $\{p + td : t \in \mathbb{F}_3\}$,

$$\sum_t \big(\varphi(p) + t\varphi(d)\big)^2 = 3\varphi(p)^2 + 6\varphi(p)\varphi(d) + 5\varphi(d)^2 \equiv 2\varphi(d)^2 \pmod 3,$$

and $\varphi(d)^2$ depends only on the direction $\langle d\rangle$. If the parallel class exists, summing over its lines gives $\sum_{d\notin\Delta} \varphi(d)^2 \equiv 0$ for every φ. If $\varphi \neq 0$, then φ vanishes on 4 directions, so the sum over all 13 directions is $9 \equiv 0$. Hence $\#\{d \in \Delta : \varphi(d) \neq 0\} \equiv 0 \pmod 3$ for every $\varphi \neq 0$. If Δ contains no three collinear directions, the line through two of them contains no other. If exactly three are collinear, the line through the fourth and one of the three contains no other. In both cases some $\varphi \neq 0$ vanishes on exactly two directions of Δ, and the count is 2. □

Corollary 4.2. *In the affine case, $P \mapsto \langle d_P\rangle$ is a collineation from π_0 onto the plane at infinity of* AG(3, 3).

Proof. Let L be a π_0-line and $X \in N_L$. By Corollary 3.2(ii), C_X is a parallel class with one line of each direction d_P, $P \notin L$. By Lemma 4.1 the four directions d_P, $P \in L$, are collinear. □

We identify π_0 with the plane at infinity accordingly. Then $K = F \cup \pi_0$ is the point set of PG(3, 3).

Lemma 4.3. *In the affine case, the design $\mathcal{D}$ of K is the design of points and lines of* PG(3, 3)*, and the classes S_X ($X \notin K$) form a covering set of spreads: a family of* 117 *spreads such that every pair of skew lines of* PG(3, 3) *lies in exactly one of them.*

Proof. The blocks of $\mathcal{D}$ inside π_0 are the lines at infinity, by Corollary 4.2. The other blocks have the form $\{P\} \cup t$ with $t \in T_P$, that is, an affine line together with its point at infinity. These are the 130 lines of PG(3, 3). A parallel class of $\mathcal{D}$ is a spread, and Proposition 3.5(iii) gives the covering property. □

Conversely, a covering set of spreads of PG(3, 3) gives the core of a putative plane of order 12 in the affine case, as in Remark 3.6.

Theorem 4.4 (Prince [Pri99])**.** *The lines of* PG(3, 3) *admit no covering set of spreads.*

Proof. Suppose $\mathcal{S}$ is a covering set of spreads. The following data come from exhaustive computation. PG(3, 3) has 8424 spreads and 5265 pairs of skew lines, and each pair of skew lines lies in exactly 72 spreads. Let $\ell_1 = \langle e_1, e_2\rangle$ and $\ell_2 = \langle e_3, e_4\rangle$. Exactly one member S of $\mathcal{S}$ contains both lines. The stabilizer of $\{\ell_1, \ell_2\}$ in PGL(4, 3) consists of the block-diagonal and block-antidiagonal matrices. It has order 2304 and three orbits on the 72 spreads through ℓ_1 and ℓ_2, of sizes 18, 6 and 48. It maps $\mathcal{S}$ to a family with the same property, so we may assume that S is one of three orbit representatives R_0, R_1, R_2. The other members of $\mathcal{S}$ share at most one line with S, and they cover the remaining 5220 skew pairs exactly once. For each i, a dancing-links search over the spreads that share at most one line with R_i finds no such exact cover. The same three problems, encoded as propositional formulas, are unsatisfiable. □

Under the Klein correspondence, lines of PG(3, 3) become points of the hyperbolic quadric $Q^+(5, 3)$, skew pairs become non-collinear pairs, and spreads become ovoids. Theorem 4.4 therefore says that the non-collinearity graph of $Q^+(5, 3)$ has no edge decomposition into ovoids.

Theorem 4.5 (Hall and Roth [HR84], Prince [Pri99], with Corollary 4.2)**.** *The affine case does not occur.*

Proof. By Lemma 4.3 the affine case gives a covering set of spreads of PG(3, 3), and by Theorem 4.4 there is none. □

The next proposition strengthens Theorem 4.5: it concerns only the free points and the classes C_X on three lines. Let $\mathcal{K} = \mathrm{AG}(3,3)$ with T_P the direction class of P, where π_0 is the plane at infinity.

Proposition 4.6. *For each line L at infinity there are exactly* 648 *transversal classes and* 63,666 *orthogonal resolutions. For two distinct lines there are exactly* 2304 *compatible pairs of orthogonal resolutions. No three lines admit pairwise compatible orthogonal resolutions, whether the three lines are concurrent or form a triangle.*

Proof. The counts come from an exhaustive enumeration. For three lines, the conditions form a propositional formula that encodes the classes T_P, an orthogonal resolution for each of the three lines and the compatibility condition. A SAT solver shows that the formula is unsatisfiable, both for a concurrent triple and for a triangle. Since $\mathrm{PGL}(3,3)$ acts transitively on each of these two kinds of triples, this covers all triples. □

In the language of Theorem 4.4: the members of a covering set of spreads that contain a given line at infinity give an orthogonal resolution for that line, so the spreads through three lines of one plane of $\mathrm{PG}(3,3)$ cannot be chosen consistently.

5. Maximal arcs whose design is AG(3, 3)

We now drop the subplane. Let F be a maximal arc of degree 3 in a projective plane of order 12. It has 27 points, and its design $\mathcal{T}$ is an STS(27).

Definition 5.1. A *class cover* of an STS(27) is a family of parallel classes such that any two disjoint triples lie in exactly one class of the family.

In an STS(27) each triple is disjoint from $117-1-3\cdot 12 = 80$ triples, so there are 4680 disjoint pairs. Since a parallel class covers $\binom{9}{2} = 36$ of them, a class cover has exactly 130 classes.

Lemma 5.2. *The* 130 *points off F give a class cover of $\mathcal{T}$: the class of X consists of the* 9 *secants through X.*

Proof. A point $X \notin F$ lies on $27/3 = 9$ secants, and they partition F. Two disjoint secants meet in exactly one point, which lies off F. Distinct points $X \neq X'$ give distinct classes, since at most one secant passes through both. □

Tonchev [Ton17, Theorem 2.1] characterizes the designs of maximal arcs by families of resolutions, and the class cover of Lemma 5.2 is the condition that appears in his proof. A class cover need not extend to a plane: the 40 external lines of F are still missing. The following theorem is therefore stronger than Theorem B(b).

Theorem 5.3. $\mathrm{AG}(3,3)$ *has no class cover.*

Proof. By exhaustive search, $\mathrm{AG}(3,3)$ has 17,641 parallel classes, in agreement with [CMM92, p. 444]. Its 4680 disjoint pairs of lines are 468 parallel and 4212 skew pairs, and each pair lies in between 79 and 142 classes. Suppose $\mathcal{C}$ is a class cover. It contains exactly one class through the parallel lines $\ell_0 = \{(0,0,t)\}$ and $\ell_1 = \{(1,0,t)\}$, which is one of the 79 classes containing both lines. The stabilizer of $\{\ell_0, \ell_1\}$ in $\mathrm{AGL}(3,3)$ maps class covers to class covers. It has order 648 and 6 orbits on these 79 classes, of sizes 9, 1, 6, 3, 6 and 54. So we may assume that the class of $\mathcal{C}$ through ℓ_0 and ℓ_1 is one of six orbit representatives. This gives six cases, numbered 0 to 5 in this order, in each of which one class is fixed. In case 1 it is the class of all lines parallel to ℓ_0, which the whole stabilizer fixes.

We split further by the same argument. Suppose classes $C_1, \ldots, C_m$ are fixed. Choose a disjoint pair p covered by none of them, and let H be a group of affine maps that fix each C_i as

a set of lines. In a class cover containing $C_1, \dots, C_m$, the member covering p contains both lines of p and shares no disjoint pair with any C_i. We call such classes *admissible*. The stabilizer H_p of p in H maps the cover to another cover containing $C_1, \dots, C_m$, so we may assume that the member covering p is a representative of an H_p-orbit of admissible classes. Case 1 is split in four levels, as in Table 1. This gives 23 leaves: cases 0, 2, 3, 4, 5 and the 18 leaves of Table 1.

For each leaf, a SAT solver shows that no class cover contains its fixed classes. The formula has a variable for each of the 17,641 classes, a unit clause for each fixed class, and for each disjoint pair a clause requiring at least one of its classes and a cardinality constraint allowing at most one. □

TABLE 1. The split of case 1 in the proof of Theorem 5.3. At each node, H is the group used for the split, H_p the stabilizer of the chosen pair, and the children are the representatives of the H_p-orbits of admissible classes. At the first split H also fixes $\{\ell_0, \ell_1\}$.

node	admissible classes	$\lvert H\rvert$	$\lvert H_p\rvert$	children
case 1	36	648	72	b0–b5
b2	13	1944	108	c0–c3
c1	10	972	108	d0, d1
d1	33	972	12	e0–e8
leaves: b0, b1, b3, b4, b5, c0, c2, c3, d0, e0–e8 (18 in all)				

Proof of Theorem B. Part (a) is Proposition 4.6. For (b), if F were such an arc, Lemma 5.2 would give a class cover of $\mathrm{AG}(3,3)$, contradicting Theorem 5.3. □

6. Symmetric cores

6.1. **Definitions.** An *automorphism of the core relative to* π_0 is a permutation σ of the points of π such that $\sigma(\pi_0) = \pi_0$ and σ maps each line meeting π_0 onto a line meeting π_0. It maps π_0-lines to π_0-lines and tangents to tangents, so it permutes the free points and the non-free points. It induces a collineation g of π_0, maps T_P to T_{gP}, and maps C_X to $C_{\sigma X}$. An *automorphism of the core of a maximal arc* K is a permutation of the points of π that maps each secant of K onto a secant. It maps K onto itself, because the points of K lie on 13 secants and the other points on fewer. It therefore induces an automorphism of the design of K. By Proposition 3.5(ii), an automorphism of the core relative to π_0 is an automorphism of the core of $K = \pi_0 \cup F$.

For collineations, Roth [Rot64, Theorem 7] showed that a collineation of prime order whose fixed structure is a subplane of order m with $m^2 \neq n$ forces $n \geq m^2 + m + 2$. His argument uses a line that misses the subplane, so it does not apply to automorphisms of the core. In Lemma 6.1, σ is an automorphism of the core relative to π_0. Throughout this section, g denotes the collineation of π_0 induced by such an automorphism σ.

Lemma 6.1. (i) *The fixed free points of σ form a subsystem of $\mathcal{K}$. Each triple inside it lies in a class T_P with $gP = P$.*

(ii) *If q is a fixed free point and $gP = P$, the tangent at P through q is fixed, and σ fixes or swaps the two partners of q at P.*

(iii) *If g is the identity and σ fixes a free point, then σ is the identity.*

Proof. (i) Two fixed free points lie on a unique tangent, which is fixed. Its point $P \in \pi_0$ is fixed, and its third free point is fixed. (ii) is clear. (iii) Let q be the fixed free point. The 13 tangents

through q are fixed. For a π_0-line L, the 9 tangents through q at the points off L meet L in the 9 points of N_L, so all non-free points are fixed. Then all tangents are fixed, since each of them joins a point of π_0 to a non-free point. If a free point x were moved, the 13 tangents through x would be fixed, so all of them would contain σx as well, which is impossible. □

6.2. **Automorphisms of order 13.** Janko and van Trung [JvT81, Theorems A and B] classified the Kirkman triple systems of order 27 whose 13 parallel classes are permuted by an automorphism ρ of order 13. They showed that none of them has 9 pairwise disjoint triples, in distinct ρ-orbits and in distinct parallel classes, such that the 13 images of the set of the 4 omitted parallel classes form the lines of a projective plane of order 3. This excludes a collineation of order 13 that leaves π_0 invariant [JvT80]. Such a collineation fixes one of the 27 external lines, since $27 \equiv 1 \pmod{13}$. Let X be a non-free point on that line. The images of C_X are the classes of the 13 non-free points on the line, and they partition the triples, so C_X meets every ρ-orbit of triples once. Its omitted classes are those at the points of a π_0-line. This argument uses an external line, which the core does not contain, and a plane of order 12 has no collineation of order 13 at all [JvT82]. So the next theorem concerns exactly the case that these papers leave open.

Theorem 6.2. *If π_0 is a subplane of order 3 of a projective plane of order 12, the core relative to π_0 has no automorphism of order 13.*

Proof. Let σ be such an automorphism. If g were the identity, σ would permute the 9 tangents at each point of π_0 in orbits of size 1 or 13, so it would fix all core lines, and all points would be fixed. Hence g has order 13. All subgroups of order 13 of $\mathrm{PGL}(3,3)$ are conjugate. So after replacing σ by a suitable power and relabelling π_0, we may assume that g is a fixed Singer cycle. Since $27 \equiv 1 \pmod{13}$, σ fixes a free point q^*. A second fixed free point would lie on a tangent through q^* at a fixed point of π_0, and g fixes no point. The other 26 free points therefore form two orbits of size 13. If one orbit contained both partners of q^* at a point P, say x and $\sigma^k x$, then σ^k would map the tangent q^*x onto itself and would fix P, so $13 \mid k$ and $\sigma^k x = x$, a contradiction. Hence we may label the partners of q^* at P as a_P and b_P with $\sigma(a_P) = a_{gP}$ and $\sigma(b_P) = b_{gP}$. A non-free point $X \in N_L$ lies on exactly one tangent through q^*. We label X by L and by the point $P \notin L$ of this tangent. Then σ acts on these labels by $(L, P) \mapsto (gL, gP)$.

With these labels, the core gives a satisfying assignment of a propositional formula. Its variables describe the classes T_P, that is, which free points are partners at P, and, for each π_0-line L, which non-free point of L lies on the tangent at P through a free point q. Its clauses say that these data form a Kirkman triple system with pairwise compatible orthogonal resolutions for all 13 lines, that they respect the labels above, and that they are invariant under σ. A SAT solver shows that this formula is unsatisfiable. □

Remark 6.3. We recounted the classification of [JvT81] with an independent program: an exact cover over the possible first parallel classes, and orbits under the normalizer of $\langle\rho\rangle$, which has order 4056. There are 657 isomorphism classes, 629 of affine type and 28 of antiaffine type in the sense of [JvT81]. Exactly three are diagonal and none is projective, as in [JvT81, Theorem B]. Each of the 661 entries of the list in [JvT81] belongs to one of the 657 classes, and each class occurs. The entries (74) and (77) are the images of (72), and the entries (75) and (76) the images of (73), under the multiplier μ of order 3 of [JvT81, Section 1]. So the list contains four repetitions, and the count in [JvT81, Theorem A] should read 657, of which 629 are of affine type. The count 661 is also quoted in later tables of Kirkman triple systems of order 27 [MR90, CMM92]. Four entries, (230), (339), (395) and (412), repeat a point as printed; each has exactly one single-symbol correction that gives a valid first parallel class, and we used it. None of this affects the conclusions of [JvT80, JvT82].

The classification also handles the case in which only the triple system of the free points, with its labelled resolution, is symmetric.

Proposition 6.4. *Let π_0 be a subplane of order 3 of a projective plane of order 12. Suppose that a permutation σ of order 13 of the free points maps each class T_P onto a class T_{gP}, where g is a collineation of π_0. Then no π_0-line has an orthogonal resolution. Such a plane therefore does not exist.*

Proof. Since $27 \equiv 1 \pmod{13}$, σ fixes a free point x. If g were the identity, σ would fix the triple of each class through x, hence each of its two other points, since σ has odd order; then σ would fix all 27 points. So g has order 13, it is a Singer cycle, and $\mathcal{K}$ with the resolution $\{T_P\}$ is one of the systems of Remark 6.3. Fix P_0. Then $T_{g^jP_0} = \sigma^j(T_{P_0})$ for all j. All subgroups of order 13 of $\mathrm{PGL}(3,3)$ are conjugate, and the collineations of π_0 that normalize $\langle g\rangle$ induce the multipliers 1, 3, 9 on it. So, up to isomorphism and collineations of π_0, each of the 657 systems carries four labellings, one for each coset of $\{1,3,9\}$ in $\mathbb{Z}_{13}^*$. By the action of g, all π_0-lines behave alike. An exhaustive search over the $657 \cdot 4$ labelled systems finds that 2594 of them have no transversal class for a line, 33 have between 1 and 24 and no orthogonal resolution, and the remaining one, $\mathrm{AG}(3,3)$ with the natural labelling, has 648. A plane would give an orthogonal resolution for every line by Corollary 3.2(ii), so $\mathcal{K}$ would be $\mathrm{AG}(3,3)$ in the affine case, which does not occur by Theorem 4.5. □

An automorphism of order 13 of the core induces such a permutation of the free points, so Proposition 6.4 contains Theorem 6.2. So does the next theorem, which assumes no subplane, in the special case $K = \pi_0 \cup F$. We keep all three, because their proofs rest on different computations (a SAT formula, the classification of Remark 6.3 with a search, and an enumeration of $\mathbb{Z}_{13}$-invariant designs with a search) and so check one another.

Theorem 6.5. *Let K be a maximal arc of degree 4 or 3 in a projective plane of order 12. The core of K has no automorphism of order 13.*

Proof. We treat degree 4 first. Let σ be an automorphism of order 13 of the core, and let $\mathcal{D}$ be the design of K, an $S(2,4,40)$. If σ fixed all points of K, it would fix all secants and then all points. The fixed points of σ in K form a closed set: the block through two fixed points is fixed, and its other two points are fixed because σ has order 13. A closed set with $w \geq 2$ points is a block or an $S(2,4,w)$, so $w \equiv 1$ or 4 (mod 12). The number of fixed points is $\equiv 40 \equiv 1 \pmod{13}$, so it is 1, 14, 27 or 40. The values 14 and 27 are not $\equiv 1$ or 4 (mod 12), and 40 is excluded by the first observation. Hence it is 1. Call the fixed point ∞ and write the other points as (o,i) with $o \in \{0,1,2\}$, $i \in \mathbb{Z}_{13}$ and $\sigma(o,i) = (o,i+1)$.

No block is fixed, since a fixed block would consist of fixed points. Hence the 13 blocks through ∞ form one orbit. Its base block meets each point orbit once, since otherwise two blocks through ∞ would share a second point. Translating each point orbit separately commutes with σ, so we may take the base block $B_\infty = \{\infty, (0,0), (1,0), (2,0)\}$. The pairs of points fall into 60 orbits under σ: 6 inside each point orbit, 13 between any two point orbits, and 3 through ∞. The orbit of B_∞ covers 6 of them. The other 117 blocks form 9 orbits, whose base blocks cover each of the remaining 54 pair orbits exactly once. An exact-cover search over the 3762 candidate base blocks gives 3,291,704 labelled designs.

Now consider the parallel classes S_X for $X \notin K$. A fixed X would give a fixed class, whose block through ∞ would be fixed. Hence the 117 points off K form 9 orbits of size 13. Within an orbit, the 13 classes contain the 13 blocks through ∞, one each, so exactly one class of each orbit contains B_∞. Call these classes $S_0, \ldots, S_8$. They partition the 81 blocks disjoint from B_∞, and since two classes share at most one block,

$$|S_k \cap \sigma^d S_l| \leq 1 \qquad \text{for all } k, l \text{ and } d \neq 0. \tag{6.1}$$

Conversely, if parallel classes $S_0, \dots, S_8$ through B_∞ partition the blocks disjoint from B_∞ and satisfy (6.1), then the 117 classes $\sigma^d S_k$ cover every pair of disjoint blocks exactly once. Indeed, each point of a block B not through ∞ lies in B_∞ after exactly one translation, and these 4 translations are distinct, since a translate with two points of B_∞ would equal B_∞, which contains ∞. So B has exactly 4 translates that meet B_∞, and it lies in 9 of the classes, which pairwise share only this block and so cover its 81 disjoint blocks once. A dancing-links search over all 3,291,704 designs, with condition (6.1) checked during the search, finds no design that admits such classes.

For degree 3 the argument is the same, with an STS(27). The number of fixed points is 1, 14 or 27, a closed set with $w \geq 2$ points has $w \equiv 1$ or $3 \pmod 6$, and 27 is excluded as before, so it is 1. There are two point orbits, the base triple $T_\infty = \{\infty, (0,0), (1,0)\}$ and $130 = 10 \cdot 13$ points off K. There are 164 candidate base triples and 38,288 labelled STS(27), and none admits 10 classes through T_∞ with the analogous properties. □

For degree 3 a symmetry of the design alone suffices.

Theorem 6.6. *In no projective plane of order* 12 *is there a maximal arc of degree* 3 *whose design has an automorphism of order* 13*.*

Proof. Let F be such an arc and $\mathcal{T}$ its design. The degree-3 version of the first two paragraphs of the proof of Theorem 6.5 uses only the design and its automorphism of order 13. Hence $\mathcal{T}$ is isomorphic to one of the 38,288 labelled systems. By Lemma 5.2, $\mathcal{T}$ has a class cover. Four of the systems are $\mathrm{AG}(3,3)$, which has none by Theorem 5.3. For each of the other 38,284 systems we list all parallel classes and search for a class cover by dancing links, with no symmetry assumed. No system has one, and each search ends after at most 5 nodes. In 35,540 of these systems some disjoint pair lies in no parallel class at all. □

6.3. **Automorphisms of order 2 and 3.**

Lemma 6.7. *No automorphism of order* 3 *of the core relative to* π_0 *fixes a free point and induces an elation on* π_0*.*

Proof. Let σ be such an automorphism, q a fixed free point, and g the induced elation with axis a and center c. The fixed points of g are the four points of a.

The fixed free points. For $P \in a$, σ fixes the two partners of q at P by Lemma 6.1(ii), since an element of order 3 cannot swap two points. A fixed free point $x \neq q$ lies on a fixed tangent with q, at a fixed point of π_0, that is, at a point of a. Thus the fixed free points are q and its 8 partners at the points of a. By Lemma 6.1(i) they form a subsystem $C_0 \cong \mathrm{AG}(2,3)$. For $P \in a$, the tangent at P through a fixed free point is fixed, so its other two free points are fixed, again because σ has order 3. Hence the classes T_P, $P \in a$, restrict to the four parallel classes of C_0.

A fixed class C_X. Let $\ell \neq a$ be a π_0-line through c, so $g\ell = \ell$, and let $P \in a \setminus \{c\}$, so $P \notin \ell$. The tangent at P through q is fixed and meets ℓ in a fixed point $X \in N_\ell$. Hence C_X is invariant under σ.

Contradiction. Let $t \in C_X$ lie in a class $T_{P'}$ with $gP' \neq P'$. Then $\sigma t \in C_X$ lies in a different class, so $\sigma t \neq t$ and $t \cap \sigma t = \emptyset$, and t contains no fixed point. The fixed points of g off ℓ are the three points of $a \setminus \{c\}$. So C_0 is covered by the three triples of C_X in the classes T_P with $P \in a \setminus \{c\}$, and each of these triples lies in C_0. They are pairwise disjoint lines of the affine plane C_0 from three different parallel classes. But lines from different parallel classes meet. □

Proposition 6.8. *The core relative to* π_0 *has no automorphism of order* 2*.*

Proof. Let σ be an involution of the core. Since 27 is odd, σ fixes a free point, which we call 0. The induced collineation g is the identity or a homology. Let s be the set of fixed points of g

at which σ swaps the two partners of 0. By Lemma 6.1, the fixed free points other than 0 are the partners of 0 at the fixed points of g outside s, and they form, with 0, a subsystem of size $1 + 2(|\mathrm{Fix}\, g| - |s|)$. A proper subsystem of an STS(27) has 1, 3, 7, 9 or 13 points.

If g is the identity, σ is the identity by Lemma 6.1(iii). Hence g is a homology with axis a and center $c \notin a$, and $|\mathrm{Fix}\, g| = 5$. The subsystem size forces $|s| \in \{1, 2, 4, 5\}$. Up to the centralizer of g there are seven cases: $s = \{c\}$ or $s = \{P_1\}$ with $P_1 \in a$; two cases with $|s| = 2$; two cases with $|s| = 4$; and $s = a \cup \{c\}$.

$|s| = 2$. The 7 fixed free points form a Fano plane. By Lemma 6.1(i) its 7 lines lie in the 5 classes T_P with $gP = P$. Two triples of one class are disjoint, while two Fano lines meet, so each class contains at most one of them, which is impossible for 7 lines and 5 classes.

$|s| = 5$. The only fixed free point is 0. For a free point $x \neq 0$, the tangent through x and σx is fixed, so its point in π_0 is fixed, and its third free point is fixed, so it is 0. Then x is a partner of 0 at a fixed point of g. There are only 10 such partners, but 26 free points $x \neq 0$.

$|s| = 1$. The 9 fixed free points form a subsystem $C_0 \cong \mathrm{AG}(2, 3)$. As in the proof of Lemma 6.7, a triple of a class C_X with $\sigma X = X$ that lies in a class $T_{P'}$ with $gP' \neq P'$ contains no fixed point.

- Let $s = \{c\}$. For $P \in a$, the tangent at P through 0 is a line of C_0 in T_P. These four lines pass through 0, so they have four different directions in C_0, and the lines of C_0 in T_P are pairwise disjoint, hence all of the direction δ_P of the first one. Let ℓ be a π_0-line through c, let $P_0 = \ell \cap a$, and let X be the point where the tangent at some $P \in a \setminus \{P_0\}$ through 0 meets ℓ. Then X is fixed. The classes T_c and T_{P_0} do not occur in C_X, since $c, P_0 \in \ell$. So C_0 is covered by the three triples of C_X in the classes T_P with $P \in a \setminus \{P_0\}$, and each of them lies in C_0. These are three pairwise disjoint lines of C_0 with three different directions, which is impossible.
- Let $s = \{P_1\}$ with $P_1 \in a$. Let ℓ be the π_0-line through c and a point $P_0 \in a \setminus \{P_1\}$, and let X be the point where the tangent at P_1 through 0 meets ℓ. Then X is fixed, and the fixed points of g off ℓ are the three points of $a \setminus \{P_0\}$. The triple of C_X in T_{P_1} consists of 0 and its two partners at P_1, which are swapped, so it contains one fixed point. The two other triples of C_X in fixed classes contain at most 3 fixed points each. Hence C_X covers at most $1 + 3 + 3 = 7$ of the 9 fixed free points, although it is a parallel class.

$|s| = 4$. All homologies of $\mathrm{PGL}(3, 3)$ are conjugate, so we may fix g. Along each 2-cycle of g we label the partners of 0 so that σ maps the first partner at P to the first partner at gP. At the fixed points of g, σ fixes the two partners outside s and swaps them on s. With these labels, and with the non-free points labelled through 0 as in the proof of Theorem 6.2, the action of σ is determined by s. In each of the two cases, a SAT solver shows that the core encoding of Theorem 6.2, with all 13 lines and with the clauses for invariance under σ instead of those for the Singer cycle, is unsatisfiable. □

Corollary 6.9. *Let σ be an automorphism of prime order p of the core relative to π_0. Then $p = 3$. Moreover σ either fixes no free point and induces an elation or a collineation with exactly one fixed point on π_0, or it fixes a free point and induces a collineation with exactly one fixed point.*

Proof. If $p \geq 5$ and $p \neq 13$, then p does not divide $|\mathrm{PGL}(3, 3)| = 2^4 \cdot 3^3 \cdot 13$, so g is the identity. Since $p \nmid 27$, σ fixes a free point, and σ is the identity by Lemma 6.1(iii). The cases $p = 13$ and $p = 2$ are Theorem 6.2 and Proposition 6.8. Let $p = 3$. If g is the identity, then σ fixes no free point by Lemma 6.1(iii). Then for a free point q, the tangent through q and σq at some point P is mapped to the tangent through σq and $\sigma^2 q$ at P. These two tangents share σq, so they coincide, and each orbit of σ on F is a triple of $\mathcal{K}$ whose tangent is fixed. Such a fixed tangent at P meets each π_0-line $L \not\ni P$ in a fixed point X. The 9 tangents through a fixed $X \in N_L$

are fixed, so C_X consists of 9 orbits of σ, one in each class $T_{P'}$ with $P' \notin L$. As there are only 9 orbits, the points P of the fixed tangents are exactly the 9 points off L. This holds for each π_0-line L that carries a fixed non-free point, and there are at least two such lines. But only 6 points of π_0 lie off two distinct lines. Therefore g is not the identity. It is an elation or has exactly one fixed point. Lemma 6.7 excludes an elation together with a fixed free point. □

The cases left open by Corollary 6.9 have a rigid fixed structure.

Lemma 6.10. *Let σ be an automorphism of order 3 of the core relative to π_0 that fixes no free point, and let g be the induced collineation of π_0. For a point P of π_0 with $gP = P$, let k_P be the number of triples of T_P fixed by σ.*

(i) A triple fixed by σ lies in a class T_P with $gP = P$, and it is an orbit of σ. Each k_P is 0, 3, 6 or 9, and the sum of the k_P is at most 9.

(ii) If $gP = P$ and $g\ell = \ell$ for a π_0-line ℓ not through P, then k_P is the number of fixed points of σ in N_ℓ.

(iii) If g is an elation with axis a and center c, then k_P has the same value $k \in \{0, 3\}$ for the three points $P \in a \setminus \{c\}$, and $k = 3$ implies $k_c = 0$.

Proof. (i) Since σ maps T_P onto T_{gP}, a fixed triple lies in a class with $gP = P$. It contains no fixed point, so it is a union of orbits of size 3, that is, one orbit. The automorphism permutes the 9 triples of T_P in orbits of size 1 or 3, so $k_P \equiv 9 \equiv 0 \pmod 3$. An orbit is a single triple, so it lies in one class, and there are only 9 orbits.

(ii) The classes C_X, $X \in N_\ell$, partition the triples of the classes $T_{P'}$ with $P' \notin \ell$. Hence $X \mapsto$ (the triple of C_X in T_P) is a bijection from N_ℓ onto T_P. It commutes with σ, because $\sigma C_X = C_{\sigma X}$ and $gP = P$. So it maps the fixed points onto the fixed triples.

(iii) The elation fixes every line through c. For $P \in a \setminus \{c\}$ and a π_0-line $\ell \neq a$ through c we have $P \notin \ell$, so by (ii) k_P is the number of fixed points in N_ℓ, which does not depend on P. By (i), $k_c + 3k \leq 9$, so $k \in \{0, 3\}$, and $k = 3$ forces $k_c = 0$. □

By part (i), the elation case without fixed free points splits into 35 cases, one for each choice of the numbers k_P with $P \in a$. Part (iii) leaves 5 of them. A SAT solver refutes each of the other 30 cases. The 5 remaining cases, the 4 cases (by k_P at the unique fixed point) of the other type without fixed free points, and the type with a fixed free point remain open.

Proof of Theorem C. Part (a) is Theorem 6.5, and part (b) is Corollary 6.9. □

7. A reduction of the general case

In this section $\mathcal{K}$ is a Kirkman triple system of order 27 whose resolution $\{T_P\}$ is labelled by the points P of $\mathrm{PG}(2, 3)$. For a line L of $\mathrm{PG}(2, 3)$, the graph Γ_L joins two points if their triple lies in T_P for some $P \in L$. In the situation of Corollary 3.2 this is the graph Γ_L of Section 3.

Theorem 7.1. *Suppose that Γ_L is a disjoint union of three copies of K_9 for every line L. Then there is an isomorphism from $\mathcal{K}$ to $\mathrm{AG}(3, 3)$ that maps each class T_P onto the set of lines of one direction d_P, and $P \mapsto d_P$ is a collineation onto the plane at infinity.*

Proof. Step 1: the classes of a triangle are collinear. Let q_1, q_2, q_3 be points not in a common triple, and let P_{ij} be the class of the triple through q_i and q_j. These three classes are distinct, since each point lies in only one triple of each class. Let L be the line through P_{12} and P_{13}. In Γ_L we have $q_1 \sim q_2$ and $q_1 \sim q_3$. Since Γ_L is a union of cliques, $q_2 \sim q_3$, so $P_{23} \in L$.

Step 2: $\mathcal{K}$ is a Hall triple system. With L as in Step 1, let C be the component of Γ_L containing q_1, q_2 and q_3. If two points of a triple of T_P with $P \in L$ lie in C, so does the third. Hence the triples of the classes T_P, $P \in L$, inside C cover each pair of points of C exactly once,

and C is a closed subsystem $\mathrm{STS}(9) \cong \mathrm{AG}(2,3)$. Thus any three points not in a common triple lie in a subsystem $\mathrm{AG}(2,3)$.

Step 3: $\mathcal{K} \cong \mathrm{AG}(3,3)$. This follows from Step 2 and the theorem of Hall and Bruck (Section 2).

Step 4: the labelling. A closed set of 9 points of $\mathrm{AG}(3,3)$ is a plane. By the argument of Step 2, every component of Γ_L is closed. So the three components of Γ_L are pairwise disjoint planes, and they are the cosets of a 2-dimensional subspace W_L. For $P \in L$, each triple of T_P lies in a component of Γ_L, so its direction lies in W_L. For distinct lines $L \neq L'$ we have $W_L \neq W_{L'}$. Otherwise the 7 classes T_P with $P \in L \cup L'$ would contribute 63 triples inside the three cosets, which contain only $3 \cdot 12 = 36$ triples. Now let P be a point and $L \neq L'$ two lines through P. Every triple of T_P has its direction in the 1-dimensional space $W_L \cap W_{L'}$. Hence T_P is the direction class of $d_P = W_L \cap W_{L'}$. Distinct classes have distinct directions, since a direction class consists of 9 lines. Finally $d_P \subseteq W_L$ for all $P \in L$, so the injective map $P \mapsto d_P$ maps lines to lines. □

The components of Γ_L have a geometric meaning. Let $K = \pi_0 \cup F$ be the maximal arc of degree 4 of Proposition 3.5.

Lemma 7.2. *In a plane of order* 12 *with a subplane* π_0 *of order* 3*, let* L *be a* π_0*-line and* C *a component of* Γ_L *with* 9 *points. Then* C *and the four points of* π_0 *on* L *are the points of a subplane* π_1 *of order* 3*, and* $\pi_0 \cap \pi_1$ *consists of the points of* L*. Conversely, every subplane of order* 3 *contained in* K *that meets* π_0 *in the points of a line* L *arises in this way.*

Proof. Since Γ_L is 8-regular, C is a 9-clique. Two points of C lie on a tangent at a point of L, whose third free point is adjacent to both, so it lies in C. Take as lines L and the 12 tangents at points of L that carry a triple inside C. Two points of C lie on one of these tangents, and two points of L lie on L. A point of C and a point $P \in L$ lie on the tangent at P through that point, which carries a triple inside C because C is a component. Each of the 13 lines contains 4 of the 13 points, so these points and lines form a projective plane of order 3. Conversely, let $\pi_1 \subseteq K$ meet π_0 in the points of L. Its other 9 points are free, and its lines other than L meet π_0 in one point of L, so they are tangents at points of L. Hence these 9 points are pairwise adjacent in Γ_L and form a component. □

Lemma 7.3. *In a plane of order* 12 *with a subplane* π_0 *of order* 3*, two distinct subplanes of order* 3 *contained in* K *meet in the four points of a line of both.*

Proof. Let $\pi' \subseteq K$ be a subplane of order 3. A line of π' meets K only in its 4 points of π', since K is an arc of degree 4. So no point of $K \setminus \pi'$ lies on a line of π', and since both sets have 27 points, $K \setminus \pi'$ is the set of free points of π'. By Proposition 3.5 applied to π', every line meets $K \setminus \pi'$ in 0 or 3 points. A line that meets K meets it in 4 points, so it meets π' in 1 or 4 points. Now let $\pi'' \subseteq K$ be another subplane of order 3. Each line of π' meets K, hence meets π'', in a point of π'. So $\pi' \cap \pi''$ meets every line of π'. It is also closed: the line through two common points is a line of both, and its 4 points in K lie in both. A closed set of points of $\mathrm{PG}(2,3)$ that meets every line is a line or the whole plane, and $\pi' \neq \pi''$. □

So Γ_L is a disjoint union of three copies of K_9 if and only if L lies in four subplanes of order 3 contained in K: three components give three subplanes besides π_0, and conversely every subplane through L other than π_0 meets π_0 in L by Lemma 7.3, so it comes from a component. By Corollary 3.2, Theorem 7.1 applies to the free points of a plane, so Theorems 7.1 and 4.5 exclude planes in which every line of π_0 lies in four subplanes of order 3 contained in K; Theorem 7.6 below excludes a single such line. By Lemma 7.3 the subplanes of order 3 in K meet pairwise in lines.

The sizes of the components are restricted by counting. Here $\mathcal{K}$ is again a Kirkman triple system with a labelled resolution, as at the start of this section.

Lemma 7.4. *Let L be a line of $\mathrm{PG}(2,3)$ and C a union of components of Γ_L with c points. Then the number of triples of the classes T_P with $P \notin L$ that lie in C or in its complement is $(c-9)(c-18)/2$. Hence every component of Γ_L has 9, 18 or 27 points, and if C is a component with 9 points, every triple of a class T_P with $P \notin L$ has exactly one point in C.*

Proof. A point lies in one triple of each class, so it has 8 neighbours in Γ_L, and two points that are not adjacent lie in a triple of a class T_P with $P \notin L$. Since C is a union of components, C contains $\binom{c}{2} - 4c = c(c-9)/2$ pairs of points that are not adjacent, and its complement contains $(27-c)(18-c)/2$. A triple of a class off L with k points in C contains $\binom{k}{2} + \binom{3-k}{2}$ of these pairs, that is, 3 if $k \in \{0,3\}$ and 1 otherwise. Let N be the number of these 81 triples with $k \in \{0,3\}$. Summing over them gives $81 + 2N = c(c-9)/2 + (27-c)(18-c)/2 = c^2 - 27c + 243$, so $2N = (c-9)(c-18)$. Since $N \geq 0$, no union of components has between 10 and 17 points. A component has at least 9 points, so its complement, if not empty, has at least 9 as well. Hence every component has 9, 18 or 27 points. For a component with 9 points, $N = 0$. □

So Γ_L is $3K_9$, or the disjoint union of K_9 and a connected graph on 18 points, or connected. In planes the first case does not occur.

Lemma 7.5. *In a plane of order 12 with a subplane π_0 of order 3, let L be a π_0-line such that Γ_L has three components A_1, A_2, A_3 with 9 points. Let A_0 be the set of points of π_0 off L, and N_L the set of points of L off π_0. Then each tangent at a point $Q \in A_0$ meets L in a point of N_L and each A_i in one point, and the 81 sets of five points obtained in this way are the blocks of a transversal design $\mathrm{TD}(5,9)$ with groups A_0, A_1, A_2, A_3 and N_L.*

Proof. A tangent at Q meets π_0 only in $Q \notin L$, so it meets L in a point of N_L. Its three free points are pairwise non-adjacent in Γ_L, so they lie in different components. Two points from different groups lie on exactly one of these tangents. A point $Q \in A_0$ and a free point or a point of N_L are joined by a line that is not a line of π_0, since a line of π_0 through Q has no free point and meets L in a point of π_0; so it is a tangent at Q. Two free points in different components are joined by a tangent at a point of π_0 that is not on L, by the definition of Γ_L. A free point and a point $X \in N_L$ are joined by a tangent at a point of π_0, which is not on L because the line is not L. □

By Lemma 7.2, each A_i with the points of π_0 on L is a subplane of order 3. Taking it in place of π_0 gives the same 81 lines, with A_i in the role of A_0. Conversely, the design determines the classes T_P for $P \notin L$: two free points in different components lie in a triple of T_P, where P is the point of A_0 on their block.

Theorem 7.6. *In a projective plane of order 12 with a subplane π_0 of order 3, no graph Γ_L is a disjoint union of three copies of K_9. Equivalently, no line of π_0 lies in four subplanes of order 3 contained in K.*

Proof. Suppose that Γ_L is, and let $A_0, \ldots, A_3$, N_L be as in Lemma 7.5. A $\mathrm{TD}(5,9)$ is the same as an orthogonal array with 5 columns and 81 rows over 9 symbols, or three mutually orthogonal Latin squares of order 9. Egan and Wanless [EW16] classified these up to permutations of the columns and of the symbols in each column: there are 371 classes, 232 of them maximal. So the design of Lemma 7.5 is isomorphic to one of their representatives, by an isomorphism that maps N_L to one of the five columns. By the remark after Lemma 7.5, any of the other four columns may then play the role of A_0. For each representative and each choice of the column for N_L, we take the other four columns, in their order, as A_0, A_1, A_2, A_3, and write a formula

for the following: a bijection from the symbols of A_0 to the points of π_0 off L, an affine plane on each of A_1, A_2, A_3 whose parallel classes are labelled by the points of L, and pairwise compatible orthogonal resolutions for all 13 lines of π_0, in which the classes T_P with $P \notin L$ and the orthogonal resolution for L are those given by the design. The stabilizer of L in $\mathrm{PGL}(3,3)$ acts on the points off L as $\mathrm{AGL}(2,3)$, so we may fix the images of two symbols of A_0 and choose the image of a third among two points. All $371 \cdot 5 = 1855$ formulas are unsatisfiable. □

Corollary 7.7. *In a projective plane of order* 12 *with a subplane of order* 3*, every line of a subplane of order* 3 *contained in* K *lies in at most two such subplanes, and* K *contains at most* 14 *subplanes of order* 3*.*

Proof. Every subplane π' of order 3 contained in K can play the role of π_0, with the same arc K (proof of Lemma 7.3). If a line ℓ of π' lay in three subplanes in K, the graph Γ_ℓ relative to π' would have two components with 9 points by Lemmas 7.2 and 7.3, hence three by Lemma 7.4, against Theorem 7.6. By Lemma 7.3, every other subplane in K meets π' in a line, and distinct subplanes meet π' in distinct lines, so there are at most 13 of them. □

Proof of Theorem A. This is Corollary 7.7. □

The hypothesis of Theorem 7.1 is not a consequence of a single orthogonal resolution.

Proposition 7.8. *There is a Kirkman triple system of order* 27 *with a resolution labelled by the points of* $\mathrm{PG}(2,3)$ *and an orthogonal resolution for one line* L*, in which* Γ_L *is connected.*

Proof. In a plane, a free point q determines the map from the points off L to N_L that sends P to the point where the tangent at P through q meets L. We describe the example by such maps. Identify the 9 points of $\mathrm{PG}(2,3)$ off L, and a set N of 9 further symbols, with $\mathbb{Z}_3^2$, where (x,y) has index $3x+y$. Let f_1, f_2, f_3 be the maps on $\{0,\dots,8\}$ whose lists of values are

$$(0,2,6,8,5,3,1,4,7), \quad (0,6,7,2,8,4,3,5,1), \quad (5,1,3,0,7,8,4,6,2),$$

and for $t \in \mathbb{Z}_3^2$ let $f_{i,t}(P) = f_i(P-t)+t$. The points of the system are the 27 maps $f_{i,t}$. These maps are permutations, each pair (P,X) satisfies $f(P)=X$ for exactly three of them, and two of them agree in at most one argument. For $P \notin L$, the class T_P consists of the triples $\{f : f(P)=X\}$ with $X \in N$. The graph Γ that joins two maps agreeing nowhere is 8-regular and connected, and it has 81 triangles, whereas $3K_9$ has 252. A search finds a decomposition of Γ into four triangle factors, and these are the classes T_P with $P \in L$. Then two maps lie in exactly one triple, $\Gamma_L = \Gamma$, and the classes $C_X = \{\{f : f(P)=X\} : P \notin L\}$, $X \in N$, form an orthogonal resolution for L. □

8. Open problems

We keep the setting of Section 7: $\mathcal{K}$ is a Kirkman triple system of order 27 with a resolution labelled by the points of $\mathrm{PG}(2,3)$. In a plane of order 12 with a subplane π_0 of order 3, the free points form such a system, and all 13 lines of π_0 have pairwise compatible orthogonal resolutions (Corollary 3.2). Hence a positive answer to Problem 8.1, together with Theorem 7.6, or a negative answer to Problem 8.2 would show that no projective plane of order 12 contains a subplane of order 3. So would a proof of Conjecture 8.4 or of Conjecture 8.5 below.

Problem 8.1 (two-line problem)**.** *If two distinct lines* L *and* L' *admit compatible orthogonal resolutions, is* Γ_L *a disjoint union of three copies of* K_9*?*

Problem 8.2 (three-line problem)**.** *Is there a* $\mathcal{K}$ *with pairwise compatible orthogonal resolutions for three concurrent lines?*

By Proposition 4.6, AG(3, 3) with the natural labelling is not an example. Without the word "compatible", Problem 8.1 has a negative answer, even if every line admits an orthogonal resolution.

Example 8.3. Let H be the Heisenberg group of order 27: its elements are the triples $(a, b, c) \in \mathbb{F}_3^3$, with product $(a, b, c)(a', b', c') = (a + a', b + b', c + c' + ab')$. Each element $(a, b, c) \neq (0, 0, 0)$ has order 3, and we label the subgroup it generates by the point $[a : b : c+ab]$ of PG(2, 3), which does not depend on the choice of the generator. Let T_P be the set of left cosets of the subgroup labelled P. Two elements $x \neq y$ lie in a common left coset only of the subgroup generated by $x^{-1}y$, so these 13 classes form a resolution of a Steiner triple system on H, labelled by the points of PG(2, 3). The bijection $(a, b, c) \mapsto (a, b, c + ab)$ from H to $\mathbb{F}_3^3$ maps every triple onto a line of AG(3, 3), so the system is AG(3, 3), but only $T_{[0:0:1]}$ is mapped onto a direction class; as the automorphisms of AG(3, 3) are affine, the same holds for every isomorphism. A computer search shows that every line of PG(2, 3) admits an orthogonal resolution, with 135 transversal classes for each line, but that Γ_L is a disjoint union of three copies of K_9 only for the four lines through $[0 : 0 : 1]$, the label of the centre of H; for the other nine lines it is connected.

Among the other systems we examined, orthogonal resolutions are rare: twelve single-line structures as in Proposition 7.8, all invariant under $\mathbb{Z}_3^2$, admit no orthogonal resolution for any second line. If $\mathcal{K}$ has an orthogonal resolution for every line, and a permutation of order 13 of its points maps T_P to T_{gP} for a Singer cycle g and the resolution of L to that of gL, for all P and L, then a SAT computation shows that the component of its fixed point in each graph Γ_L is a K_9.

The question for all 13 lines at once, whether pairwise compatible orthogonal resolutions for all lines force each Γ_L to be a disjoint union of three copies of K_9, is weaker and has the same consequence. A symmetry reduction turns it into four explicit SAT instances, which general-purpose solvers have not decided; nor have they decided the cases of order 3 left open by Corollary 6.9. In the proof of Theorem 7.6, by contrast, the transversal design fixes most of the triple system; for the two cases that Theorem 7.6 and Lemma 7.4 leave, we know no classification that would play this role.

Class covers. The next conjecture concerns maximal arcs directly. By Lemma 5.2 it would rule out maximal arcs of degree 3 in planes of order 12, and by duality also those of degree 4: the external lines of a maximal arc of degree 3 form a maximal arc of degree 4 in the dual plane, and conversely, as in the proof of Proposition 3.5(ii). It is stronger than the nonexistence of these arcs, since a class cover need not come from a plane.

Conjecture 8.4. *No Steiner triple system of order* 27 *has a class cover.*

The lines of a partial geometry whose collinearity graph is the disjointness graph of the triples of an STS(27) are sets of 9 pairwise disjoint triples, that is, parallel classes. So Conjecture 8.4 says that this graph, which is strongly regular with parameters $(117, 80, 52, 60)$, is never the collinearity graph of a pg$(8, 9, 6)$.

AG(3, 3) has no class cover by Theorem 5.3, which needs a deep search, while every other system we tested fails after at most 5 search nodes: the 38,284 non-affine systems with an automorphism of order 13 (Theorem 6.6) and several hundred tripled, cyclic, random and nearly affine systems.

A weaker property is *(DP)*: any two disjoint triples lie in a common parallel class. A system with a class cover satisfies (DP), but (DP) does not characterize AG(3, 3). The *tripling* construction gives an STS(27) from a Latin square Q of order 9: take three copies AG$(2, 3) \times \{i\}$, $i \in \mathbb{Z}_3$, with their 36 lines, and the 81 triples $\{(x, 0), (y, 1), (Q(x, y), 2)\}$. Some of these systems satisfy (DP) although they have far fewer parallel classes than AG(3, 3).

A system is *residually resolvable* at a triple t if the 80 triples disjoint from t split into 10 parallel classes of the 24 points off t. A class cover gives such a splitting at every triple t, by the members of the cover that contain t; for the design of a maximal arc of degree 3, these are the classes of the 10 points off the arc on the secant t. By Theorem 5.3, the following conjecture therefore implies Conjecture 8.4.

Conjecture 8.5. *An* STS(27) *that is residually resolvable at every triple is* AG$(3,3)$.

AG$(3,3)$ is residually resolvable at all 117 triples, while each tripled system with (DP) that we found fails at most of its triples.

9. Verification

All searches are exhaustive. Exact-cover problems are solved by Algorithm X with dancing links and the minimum-column rule [Knu00]. In the SAT encodings, a constraint "exactly one of $x_1, \dots, x_k$" is the clause $x_1 \vee \cdots \vee x_k$ together with an encoding of "at most one", by pairwise clauses if $k \leq 6$ and by a sequential counter otherwise; constraints "exactly 2" and "exactly 3" are encoded by sequential counters. Symmetry enters only through orbit representatives under explicit groups, as described in the proofs. The SAT solvers are kissat [BFF$^+$24b] and CaDiCaL [BFF$^+$24a]. For every formula used in a proof, CaDiCaL wrote a certificate in the LRAT format [CFHH$^+$17], which `lrat-check` [WHH14] and then the verified LRAT checker of the standard library of Lean 4 [dMU21] checked.

For all formulas except the 1855 of Theorem 7.6, the check by Lean is a theorem, proved without Mathlib and evaluated by `native_decide`, which trusts the Lean compiler and runtime as well as the kernel. Since the checker is verified, neither the SAT solver nor the program that trims the certificates is trusted. For the 1855 formulas, the same checker ran as a compiled Lean program.

For Theorem 7.6, the 371 designs are the representatives published with [EW16] (`https://users.monash.edu.au/~iwanless/data/MOLS/`), and our program checks that each consists of three pairwise orthogonal Latin squares. As a control, with a design obtained from AG$(3,3)$ with the natural labelling, the formulas are satisfiable when only L and one further line are required. A second implementation of the encoding, written from the mathematical description alone and with different variables, refutes 24 of the formulas, among them the four slowest, and also finds the control satisfiable.

In each geometric Lean theorem, the passage from a geometric statement to a formula enters as a hypothesis, that is, a proposition defined in the source and passed as an argument, not an axiom; for Theorem 6.2, for example, it says that a plane of order 12 with a subplane of order 3 whose core has an automorphism of order 13 yields a model of the formula. These hypotheses comprise the soundness of the encodings, the uniqueness of PG$(2,3)$, the conjugacy arguments that fix the action on π_0, and the symmetry splits. Lean checks the computational parts of some of them: the lists of spreads and parallel classes, the formulas derived from these lists, and the orbit decompositions of the first symmetry step.

The searches in the proofs of Proposition 6.4 and Theorems 6.5 and 6.6 are run in Lean as well. A search written in Lean enumerates the objects in question (the first parallel classes through $\{\infty, 0, 13\}$, and the $\mathbb{Z}_{13}$-invariant $S(2,4,40)$ and STS(27)), a theorem proves that every such object is among its results, and a second procedure, proved sound, rejects every object except the affine ones; `native_decide` evaluates both. In this way Lean proves that for every first parallel class and each of the four labellings, an orthogonal resolution for a line forces exactly 648 transversal classes for this line; that no $\mathbb{Z}_{13}$-invariant $S(2,4,40)$ or STS(27) admits classes as in the proof of Theorem 6.5; and that every $\mathbb{Z}_{13}$-invariant STS(27) with a class cover is a translate of one of four systems, which the kernel shows to be copies of AG$(3,3)$. These

theorems need no hypothesis, since they are stated for the objects themselves. The rest of the proofs of Proposition 6.4 and Theorems 6.5 and 6.6 lies outside them: the reductions from a plane to these objects, the identification of the system with 648 transversal classes as $\mathrm{AG}(3,3)$, and the use of Theorems 4.5 and 5.3.

The results proved by hand that are marked "kernel" in Table 2 are formalized in Lean 4 with Mathlib [The20] and checked by the kernel, with only the standard axioms; the statements are made for an arbitrary projective plane of order 12, with a subplane of order 3 where needed, or, in Lemma 4.1, Theorem 7.1 and Lemma 7.4, for the abstract objects of those statements. Table 2, in the appendix, lists for each result that uses a computation or is formalized how it is established. The evidence in Section 8 other than Example 8.3 rests on searches without certificates, except the Singer-symmetric formula, whose certificate `lrat-check` verified. Independent programs reproduce most counts and conclusions. The programs, the Lean sources and the formulas other than the 1855 of Theorem 7.6 are available on Zenodo at `https://doi.org/10.5281/zenodo.23687637`. The certificates are too large to store; the README there explains how to regenerate them and the remaining formulas, lists for each result the programs, the logs, the formulas and the Lean theorems, names the programs of the independent checks, and records the versions of the solvers and of Lean.

Appendix A. Verification status of the results

Table 2: How the results are established. Lean theorems about formulas, and searches proved complete, are evaluated by `native_decide`; the theorems about formulas have the hypotheses described in Section 9.

Result	Method	Certificate	Lean
Thms. 4.4, 4.5	dancing links and SAT, three cases	LRAT	theorems
Prop. 4.6	enumeration; SAT, two formulas	LRAT	theorems for the SAT part
Thm. 5.3	SAT, 23 leaves	LRAT	theorems
Thm. 6.2	SAT, one formula	LRAT	theorem
Rem. 6.3	exact cover and orbit count	—	the 14,520 classes recounted
Prop. 6.4	search over $657 \cdot 4$ labelled systems	—	search proved complete
Thm. 6.5	dancing links over 3,291,704 designs and 38,288 systems	—	searches proved complete
Thm. 6.6	dancing links over 38,284 systems	—	search proved complete
Lem. 6.7	SAT form of the lemma, one formula	LRAT	theorem
Prop. 6.8	hand proof and SAT; every case also by SAT	LRAT	theorems
Cor. 6.9	hand proof from Thm. 6.2 and Prop. 6.8	—	kernel, with these two results as hypotheses
Lem. 6.10	SAT for the 30 cases excluded by hand	LRAT, `lrat-check` only	—
Thm. 7.6	SAT, 1855 formulas	LRAT	compiled checker (Section 9)
Cor. 7.7	hand proof from Thm. 7.6 and Lems. 7.2–7.4	—	—
Prop. 7.8	explicit example	—	kernel
Ex. 8.3	explicit example; search	—	kernel: the isomorphism, the resolutions and the lines with $\Gamma_L \cong 3K_9$

Result	Method	Certificate	Lean
Thms. 3.1, 7.1; Cors. 3.2, 4.2; Props. 3.4, 3.5; Lems. 4.1, 4.3, 5.2, 6.1, 6.7, 6.10, 7.2–7.5	hand proofs	—	kernel; hypotheses: the theorem of Hall and Bruck for Thm. 7.1, [AST23] and the classification of involutions [HP73] for Prop. 3.4

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