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ON THE MINIMAL DIMENSION OF RANGE-CRITERION COUNTEREXAMPLES TO THE PPT^2 CONJECTURE

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Abstract. The PPT^2 conjecture asserts that the composition of two completely positive and completely copositive (PPT) maps is entanglement breaking. A recently announced counterexample, with PPT maps $\mathrm{M}_{10} \to \mathrm{M}_6 \to \mathrm{M}_{10}$, is certified by the range criterion: the range of the composite Choi matrix contains no nonzero product vector. We ask how small a counterexample certified in this way can be. For PPT maps $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ and $\Psi\colon \mathrm{M}_k \to \mathrm{M}_n$ with $\Psi \circ \Phi \neq 0$ and $\max(m,k,n) \leq 6$ we prove that the range of the Choi matrix of $\Psi \circ \Phi$ contains a nonzero product vector whenever $k \leq 5$ or $\min(m,n) \leq 5$. Hence the range criterion cannot certify a counterexample on M_4 or M_5, and among the types with all three dimensions at most six only $(6,6,6)$ remains open. Since the range is spanned by products of Kraus operators, the question is when such a span can avoid the matrices of rank one. The computer-assisted proof combines rank bounds for PPT maps with Chern class and excess intersection computations; its computed facts are checked in Lean 4, modulo classical, quoted or hand-proved inputs. The announced counterexample lifts through exterior powers a map on M_4 built from a pencil of real matrices (a map of pencil type). When the map of pencil type loses rank transversally at finitely many points, its lifts through any Schur functor, exterior powers included, put product vectors into the kernel of the composite Choi matrix. When the composite is also nonzero, these product vectors form an unextendible product set only for exterior powers, and then one of the three dimensions is at least ten.

1. Introduction

A linear map $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ is completely positive if $\mathrm{id}_r \otimes \Phi$ is positive for all r, and completely copositive if Φ composed with the transpose is completely positive. We call Φ a PPT map if it is both. It is entanglement breaking if its Choi matrix $C_\Phi = \sum_{i,j} E_{ij} \otimes \Phi(E_{ij})$ is separable [17]. Christandl's PPT^2 conjecture [7], in the form of [8, Conjecture IV.1], asserts that $\Psi \circ \Phi$ is entanglement breaking for all PPT maps $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ and $\Psi\colon \mathrm{M}_k \to \mathrm{M}_n$. A PPT map applied to one half of a bipartite state always produces a state with positive partial transpose, which may still be entangled; the conjecture says that two PPT maps applied in turn always produce a separable state. The conjecture holds for maps on M_3 [8, 5] and when the middle algebra is M_2 [15], and every PPT channel becomes entanglement breaking after finitely many iterations [26]. A counterexample was announced recently [24]: a pair of PPT maps on M_{10} whose composition is not entanglement breaking, and a trace-preserving PPT map on M_{21} whose square is not entanglement breaking. For maps on M_4 the conjecture is open.

The pair in [24] factors through M_6 (Remark 2.3). In our notation it consists of PPT maps $\Phi\colon \mathrm{M}_{10} \to \mathrm{M}_6$ and $\Psi\colon \mathrm{M}_6 \to \mathrm{M}_{10}$ such that $Z = C_{\Psi\circ\Phi}$ is nonzero and its range contains no nonzero product vector. A separable positive semidefinite operator is a sum of projections onto product vectors in its range, so Z is entangled by the range criterion [18]. We call such a pair a *range-criterion pair* of type $(m,k,n) = (10,6,10)$ (Definition 2.2). The authors of [24] do not

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claim that the dimension ten is minimal. This paper asks how small a range-criterion pair can be. The answer shows in which dimensions a counterexample can be certified as in [24].

The question is algebraic. The range of $C_{\Psi\circ\Phi}$ is spanned by the vectorised products XY of Kraus operators X of Ψ and Y of Φ, and product vectors correspond to matrices of rank one (Lemma 2.1). So we ask when the linear span of these products can avoid the rank-one matrices while both maps stay PPT. Our main result is the following.

Theorem 1.1 (†)**.** *Let $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ and $\Psi\colon \mathrm{M}_k \to \mathrm{M}_n$ be PPT maps with $\Psi\circ\Phi \neq 0$ and $\max(m,k,n) \leq 6$. If $k \leq 5$ or $\min(m,n) \leq 5$, then the range of $C_{\Psi\circ\Phi}$ contains a nonzero product vector.*

In particular, the range criterion cannot certify a counterexample on M_4 or on M_5, and among the types with all three dimensions at most six only $(6,6,6)$ remains open; Table 4.1 lists its open cases. Theorem 1.1 concerns the range criterion only: a counterexample on M_d with $4 \leq d \leq 9$ whose composite has product vectors in its range is not excluded, and would have to be detected by other means.

Results marked with a dagger, like Theorem 1.1, have computer-assisted proofs or depend on one; the computations are exact or carry rigorous error bounds; they include intersection numbers on Grassmannians and related spaces (computed by Schubert calculus or by localisation), ranks of linear systems and certificates for explicit maps. Each computation is described in the proof that uses it and is listed in Table SA.1 in Appendix SA of the supplement (numbers with the prefix S refer to the supplement, and numbers without it to this paper). The computed facts are checked in the proof assistant Lean 4, modulo inputs that are classical, quoted from [24] or proved by hand, and Lean derives Theorem 1.1 from results that it proves and from nine steps of the proof that it takes as hypotheses, results whose proofs rest on intersection theory or sheaf cohomology, listed in Appendix SA. The programs, certificates and Lean code are deposited [29].

Some of our results reach beyond dimension six: there is no range-criterion pair of type $(4,k,4)$, whatever the middle dimension k (Theorem 4.20), and none of type $(4,6,n)$, $(6,4,n)$, $(n,6,4)$ or $(n,4,6)$ with $n \leq 9$ (Theorems 4.22 and 4.18). The supplement excludes further types up to dimension nine, among them $(5,5,7)$, $(4,7,5)$, $(4,7,6)$, $(4,8,5)$, $(4,8,6)$ and $(4,9,5)$ and their mirrors (Table 4.2 and Section 8). The proofs work with reduced pairs: every PPT pair with $\Psi\circ\Phi \neq 0$ can be compressed so that all four marginals $\Phi(I)$, $\Phi^*(I)$, $\Psi(I)$ and $\Psi^*(I)$ become invertible, without changing whether the composite range contains a product vector (Lemma 2.5), and we call the compressed pair reduced.

The proof of Theorem 1.1 rests on four ingredients. First, after the reduction, a product-free composite range forbids Kraus operators of rank at most two; together with the theorem of Horodecki, Lewenstein, Vidal and Cirac on PPT states of low rank [19] and a normal form of Chen and Djoković [4], this bounds the ranks of the Choi matrices of both maps and of their partial transposes from both sides, and leaves $(5,5,5)$ as the only reduced type with all dimensions at most five. Second, at middle dimension five, a nonzero intersection number, the top Chern class of a vector bundle, forces a Kraus operator of Ψ whose kernel lies in the range of a Kraus operator of Φ, so that their product has rank one. Third, at middle dimension six and outer dimension four, the ranges of the rank-three Kraus operators of Φ form a family of 3-planes in $\mathbb{C}^6$ that covers $\mathbb{P}^5$, and an excess intersection count shows that such a family cannot do so when $\operatorname{rank} C_\Psi$ is large; refinements of this count settle the types with middle dimension six, an outer dimension four and all dimensions at most six. Fourth, two facts hold for a single PPT map Φ: the set of vectors x at which $\Phi(xx^\dagger)$ loses rank is a finite union of linear subspaces, and each product vector in the kernel of its Choi matrix forces certain of its Kraus operators to have ranges orthogonal to the second factor of that product vector. At middle dimension four these two facts show that, for a reduced range-criterion pair, the kernel of the composite Choi matrix

contains no nonzero sum of at most two product vectors (Lemma 4.12), and intersection counts then exclude the type $(6, 4, 6)$, the only type with middle dimension four and all dimensions at most six that the rank bounds leave; the partial-transpose argument behind the second fact, with Bézout's theorem, also excludes the outer dimensions $(4, 4)$ for every middle dimension. For a reduced range-criterion pair of type $(5, 6, 5)$ the two facts leave exactly two product vectors in the kernel of the composite Choi matrix, up to scalar multiples (Theorem 4.5), and a congruence modulo 8 for the Euler characteristic of a sheaf on a projective plane excludes this type. Section 4 names the statement that proves each step.

In Section 5 we study the example of [24] and the family of exterior-power lifts to which it belongs. The twenty directions x in which its base map $L\colon \mathrm{M}_4 \to \mathrm{M}_4$ loses rank give the twenty product vectors in the kernel of its composite Choi matrix; we show that the quadratic Veronese images of the rank-loss directions and of the kernel vectors of $L(xx^\dagger)$ at these directions are Gale dual, a consequence of the Cayley–Bacharach theorem, and that the twenty kernel conditions determine the Kraus spaces of both maps (Theorem 5.6). Within a wider class of lifts, a certificate of the kind used in [24] still needs dimension ten: if the base map loses rank transversally at finitely many points, the product vectors of a Schur-functor lift with nonzero composite form an unextendible product set (a notion that includes the unextendible product bases of [1]) only for exterior powers, and then one of the three dimensions is at least ten (Theorem 5.4). For a general base map on M_3, the member of type $(6, 6, 6)$ is one product vector short of the range criterion, and for a general pencil in a family of real pencils its factors are rigid: every PPT map that can replace one of them while keeping the ten product vectors in the kernel is a nonnegative multiple of it (Theorem 5.8). Another member gives a range-criterion pair of type $(6, 10, 10)$ (Proposition 5.7).

In Section 6 we show that symmetry under a compact connected group that acts irreducibly on both outer spaces always puts a product vector into the composite range, in every dimension (Theorem 6.1); for SU(2) it suffices that no spin occurs more than twice in one of the outer representations (Theorem 6.7(d)), a condition that is sharp for invariant operators (Corollary 6.9). We also prove that $\Psi \circ \Phi$ is entanglement breaking, as the PPT^2 conjecture asserts, for all PPT maps in several covariant classes with outer dimensions up to nine (Theorem 6.13); the conjecture was settled earlier for two symmetric classes [8, Proposition V.1], [28]. Section 7 treats the stronger criterion for edge states [22], which also uses the range of the partial transpose, and Section 8 lists open problems, with examples showing that the types $(6, 6, 6)$, $(5, 6, 5)$ and $(5, 6, 6)$ cannot be excluded by linear algebra and Lemma 3.1 alone.

The supplement. Section 4 states the results of the case analysis that the proof of Theorem 1.1 uses, and the main results for larger types, with sketches of their proofs, and Section 5 states the main further results on lifts in the same way. The proofs and the remaining results are in the supplement, which is deposited with this paper.

Section S7 of the supplement collects numerical evidence, which is not used in any proof.

2. Preliminaries

2.1. **Notation.** We write M_n for the complex $n \times n$ matrices, $\mathrm{M}_{k\times m}$ for the $k \times m$ matrices, Herm_n for the Hermitian matrices, $A \succeq 0$ for positive semidefinite A, and E_{ij} for the matrix units. For a matrix A, $\bar{A}$ is the entrywise complex conjugate, A^T the transpose and $A^\dagger = \bar{A}^T$. For $x \in \mathbb{C}^m$ we put $P_x = xx^\dagger$, without normalisation. The transpose map is T, and $\mathrm{Ad}_V(X) = VXV^\dagger$. The adjoint Φ^* of a linear map $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ is defined by $\mathrm{Tr}(\Phi^*(B)^\dagger A) = \mathrm{Tr}(B^\dagger \Phi(A))$. The Choi matrix is $C_\Phi = \sum_{i,j} E_{ij} \otimes \Phi(E_{ij}) \in \mathrm{M}_m \otimes \mathrm{M}_k$, with the input factor first. The map Φ is completely positive if and only if $C_\Phi \succeq 0$ [6], and completely copositive if and only if the partial transpose $C_\Phi^\Gamma = C_{T\circ\Phi}$ on the second factor is positive semidefinite.

An operator on $\mathbb{C}^p \otimes \mathbb{C}^q$ is separable if it is a sum of operators $P \otimes Q$ with $P, Q \succeq 0$. A product vector is a vector $u \otimes v$ with $u, v \neq 0$, and a subspace is product-free if it contains no product vector. The local ranks of $\rho \succeq 0$ on $\mathbb{C}^p \otimes \mathbb{C}^q$ are the ranks of its two partial traces. For a subspace $\mathcal{K} \subseteq \mathrm{M}_{k\times m}$ and $x \in \mathbb{C}^m$ we write $\mathcal{K}x = \{Kx : K \in \mathcal{K}\}$, and for subspaces $\mathcal{K} \subseteq \mathrm{M}_{n\times k}$, $\mathcal{L} \subseteq \mathrm{M}_{k\times m}$ we write $\mathrm{span}(\mathcal{K}\mathcal{L})$ for the span of all products XY with $X \in \mathcal{K}$ and $Y \in \mathcal{L}$. A subspace of matrices is rank-one-free if it contains no matrix of rank one.

2.2. **Kraus spaces and the composite range.** For $K \in \mathrm{M}_{k\times m}$ put $\mathrm{vec}(K) = \sum_{i=1}^{m} e_i \otimes K e_i \in \mathbb{C}^m \otimes \mathbb{C}^k$. Then $\mathrm{vec}(ba^T) = a \otimes b$, and the Schmidt rank of $\mathrm{vec}(K)$ equals $\mathrm{rank}\, K$. If $\Phi = \sum_r \mathrm{Ad}_{K_r}$ is completely positive, then $C_\Phi = \sum_r \mathrm{vec}(K_r)\,\mathrm{vec}(K_r)^\dagger$. Hence the *Kraus space*

$$\mathcal{K}_\Phi = \{K \in \mathrm{M}_{k\times m} : \mathrm{vec}(K) \in \mathrm{ran}\, C_\Phi\} = \mathrm{span}\{K_r\}$$

does not depend on the Kraus decomposition, and $\dim \mathcal{K}_\Phi = \mathrm{rank}\, C_\Phi$. Product vectors in $\mathrm{ran}\, C_\Phi$ correspond to matrices of rank one in $\mathcal{K}_\Phi$. For $x \in \mathbb{C}^m$ and $u \in \mathbb{C}^m$, $v \in \mathbb{C}^k$,

$$\mathrm{ran}\, \Phi(P_x) = \mathcal{K}_\Phi x, \qquad (u \otimes v)^\dagger C_\Phi (u \otimes v) = v^\dagger \Phi(P_{\bar{u}}) v. \tag{2.1}$$

Lemma 2.1. *Let $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ and $\Psi\colon \mathrm{M}_k \to \mathrm{M}_n$ be completely positive and $Z = C_{\Psi\circ\Phi}$.*

(a) $\mathrm{ran}\, Z = \mathrm{vec}\big(\mathrm{span}(\mathcal{K}_\Psi \mathcal{K}_\Phi)\big)$.

(b) For $u \in \mathbb{C}^m$ and $v \in \mathbb{C}^n$ we have $(u \otimes v)^\dagger Z (u \otimes v) = \mathrm{Tr}\big(\Psi^*(P_v)\Phi(P_{\bar{u}})\big)$. Hence $u \otimes v \in \ker Z$ if and only if $v^\dagger X Y \bar{u} = 0$ for all $X \in \mathcal{K}_\Psi$ and $Y \in \mathcal{K}_\Phi$, if and only if $\Phi(P_{\bar{u}})$ and $\Psi^*(P_v)$ have orthogonal ranges.

(c) $\mathrm{ran}\, Z$ *contains a product vector if and only if* $\mathrm{span}(\mathcal{K}_\Psi \mathcal{K}_\Phi)$ *contains a matrix of rank one.*

Proof. If $\Phi = \sum_r \mathrm{Ad}_{Y_r}$ and $\Psi = \sum_s \mathrm{Ad}_{X_s}$, then $\Psi \circ \Phi = \sum_{r,s} \mathrm{Ad}_{X_s Y_r}$, which gives (a) by bilinearity. Part (b) follows from (2.1) applied to $\Psi \circ \Phi$, from $\langle u \otimes v, \mathrm{vec}(M)\rangle = v^\dagger M \bar{u}$, and from the fact that the trace of a product of two positive semidefinite matrices vanishes exactly when their ranges are orthogonal. Part (c) follows from (a) and $\mathrm{vec}(ba^T) = a \otimes b$. □

Definition 2.2. A *range-criterion pair of type* (m, k, n) is a pair of PPT maps $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ and $\Psi\colon \mathrm{M}_k \to \mathrm{M}_n$ with $\Psi \circ \Phi \neq 0$ such that $\mathrm{span}(\mathcal{K}_\Psi \mathcal{K}_\Phi)$ is rank-one-free, that is, $\mathrm{ran}\, C_{\Psi\circ\Phi}$ is product-free.

A separable operator $Z \neq 0$ is a sum $\sum_t p_t p_t^\dagger$ of product vectors p_t, and each p_t lies in $\mathrm{ran}\, Z$. So for a range-criterion pair $C_{\Psi\circ\Phi}$ is entangled [18], and $\Psi \circ \Phi$ is not entanglement breaking [17].

Remark 2.3. The maps in [24, (6.5)] are $\Phi_1(A) = ES(A^T)E^\dagger$ and $\Phi_2(B) = R^*(E^\dagger B E)$, where $S, R\colon \mathrm{M}_{10} \to \mathrm{M}_6$ are PPT and $E\colon \mathbb{C}^6 \to \mathbb{C}^{10}$ is an isometry. Hence $\Phi = S \circ T\colon \mathrm{M}_{10} \to \mathrm{M}_6$ and $\Psi = R^*\colon \mathrm{M}_6 \to \mathrm{M}_{10}$ are PPT with $\Psi \circ \Phi = \Phi_2 \circ \Phi_1$, and [24, Theorem 1.2], a statement about Φ_1 and Φ_2, implies that (Φ, Ψ) is a range-criterion pair of type $(10, 6, 10)$. We recall the construction in Section 5.

2.3. **Two symmetries and a reduction.**

Lemma 2.4. *Let $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ and $\Psi\colon \mathrm{M}_k \to \mathrm{M}_n$ be PPT.*

(a) The map $\Phi^\# = T \circ \Phi^* \circ T\colon \mathrm{M}_k \to \mathrm{M}_m$ is PPT with $\mathcal{K}_{\Phi^\#} = \mathcal{K}_\Phi^T = \{Y^T : Y \in \mathcal{K}_\Phi\}$. The pair $(\Psi^\#, \Phi^\#)$ has type (n, k, m) and $\mathrm{span}(\mathcal{K}_{\Phi^\#}\mathcal{K}_{\Psi^\#}) = \mathrm{span}(\mathcal{K}_\Psi \mathcal{K}_\Phi)^T$.

(b) The pair $(T \circ \Phi, \Psi \circ T)$ is a PPT pair of type (m, k, n) with the same composition. Moreover $C_{T\circ\Phi} = C_\Phi^\Gamma$ and $\mathrm{rank}\, C_{\Psi\circ T} = \mathrm{rank}\, C_\Psi^\Gamma$.

(c) Both operations preserve the property that all four marginals $\Phi(I)$, $\Phi^*(I)$, $\Psi(I)$, $\Psi^*(I)$ are positive definite.

In particular, (Φ, Ψ) *is a range-criterion pair if and only if* $(\Psi^{\#}, \Phi^{\#})$ *is, if and only if* $(T\circ\Phi, \Psi\circ T)$ *is.*

Proof. If $\Phi = \sum_r \mathrm{Ad}_{K_r}$, then $\Phi^{\#}(Y) = \big(\sum_r K_r^\dagger Y^T K_r\big)^T = \sum_r K_r^T Y \bar{K}_r$, so $\mathcal{K}_{\Phi^{\#}} = \mathcal{K}_\Phi^T$. The map $T \circ \Phi^{\#} = \Phi^* \circ T$ is the adjoint of $T \circ \Phi$, hence completely positive. Part (a) follows since $(XY)^T = Y^T X^T$. In (b), $C_{\Psi\circ T} = \sum_{i,j} E_{ij} \otimes \Psi(E_{ji})$ is the partial transpose of C_Ψ on the first factor, which is the full transpose of C_Ψ^Γ. Likewise $C_{T\circ\Psi\circ T} = C_\Psi^T \succeq 0$, so $\Psi \circ T$ is PPT. For (c), $\Phi^{\#}(I) = \Phi^*(I)^T$, $(\Phi^{\#})^*(I) = \Phi(I)^T$, $(T\circ\Phi)(I) = \Phi(I)^T$, $(T\circ\Phi)^*(I) = \Phi^*(I)$, $(\Psi\circ T)(I) = \Psi(I)$ and $(\Psi \circ T)^*(I) = \Psi^*(I)^T$. □

Lemma 2.5. *Let* $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ *and* $\Psi\colon \mathrm{M}_k \to \mathrm{M}_n$ *be PPT with* $\Psi\circ\Phi \neq 0$. *There are isometries* $V_1\colon \mathbb{C}^{m'} \to \mathbb{C}^m$, $V_2\colon \mathbb{C}^{k'} \to \mathbb{C}^k$ *and* $V_3\colon \mathbb{C}^{n'} \to \mathbb{C}^n$ *such that the PPT maps* $\tilde{\Phi} = \mathrm{Ad}_{V_2^\dagger} \circ \Phi \circ \mathrm{Ad}_{V_1}$ *and* $\tilde{\Psi} = \mathrm{Ad}_{V_3^\dagger} \circ \Psi \circ \mathrm{Ad}_{V_2}$ *have positive definite marginals* $\tilde{\Phi}(I)$, $\tilde{\Phi}^*(I)$, $\tilde{\Psi}(I)$, $\tilde{\Psi}^*(I)$, *and*

$$C_{\tilde{\Psi}\circ\tilde{\Phi}} = (\bar{V}_1 \otimes V_3)^\dagger\, C_{\Psi\circ\Phi}\, (\bar{V}_1 \otimes V_3), \qquad \operatorname{ran} C_{\Psi\circ\Phi} \subseteq \operatorname{ran}(\bar{V}_1 \otimes V_3).$$

In particular (Φ, Ψ) *is a range-criterion pair if and only if* $(\tilde{\Phi}, \tilde{\Psi})$ *is.*

Proof. Write $\Phi = \sum_r \mathrm{Ad}_{A_r}$ and $\Psi = \sum_s \mathrm{Ad}_{B_s}$. Let P be the projection onto the support of $\Phi(I)$ and Q the projection onto the support of $P\Psi^*(I)P$, so that $Q \leq P$, and let V_2 be an isometry onto $\operatorname{ran} Q$. Then $A_r = PA_r$, and $\sum_s PB_s^\dagger B_s P = P\Psi^*(I)P$ gives $B_s P = B_s Q$. Hence $B_s A_r = B_s V_2 V_2^\dagger A_r$. The maps $\mathrm{Ad}_{V_2^\dagger}\circ\Phi$ and $\Psi \circ \mathrm{Ad}_{V_2}$ have the same composition as Φ and Ψ. Their middle marginals $V_2^\dagger \Phi(I) V_2$ and $V_2^\dagger P\Psi^*(I)PV_2$ are positive definite. Next let V_1 be an isometry onto the support of $\Phi^*(Q)$ and V_3 an isometry onto the support of $\Psi(Q)$. Then $V_2^\dagger A_r = V_2^\dagger A_r V_1 V_1^\dagger$ and $B_s V_2 = V_3 V_3^\dagger B_s V_2$. So these compressions leave the middle marginals unchanged and make the outer marginals definite. The Kraus operators of $\tilde{\Psi}\circ\tilde{\Phi}$ are $V_3^\dagger B_s A_r V_1$, and $B_s A_r = V_3 V_3^\dagger B_s A_r V_1 V_1^\dagger$. Since $\operatorname{vec}(V_3^\dagger K V_1) = (V_1^T \otimes V_3^\dagger)\operatorname{vec}(K)$, this gives both displayed identities. Compositions with Ad_V preserve the PPT property. Finally, $(\bar{V}_1 \otimes V_3)^\dagger$ maps $\operatorname{ran} C_{\Psi\circ\Phi}$ bijectively onto $\operatorname{ran} C_{\tilde{\Psi}\circ\tilde{\Phi}}$, and a product vector in $\operatorname{ran}(\bar{V}_1 \otimes V_3)$ has its factors in $\operatorname{ran}\bar{V}_1$ and $\operatorname{ran} V_3$. So product vectors correspond on both sides. □

We call a PPT pair *reduced* if all four marginals are positive definite, and we call (m', k', n') in Lemma 2.5 the *reduced type*. Since $m' \leq m$, $k' \leq k$ and $n' \leq n$, a statement that excludes all reduced types below (m, k, n) excludes the type (m, k, n). For a reduced pair the local ranks of C_Φ are (m, k) and those of C_Ψ are (k, n). We write

$$a = \operatorname{rank} C_\Psi, \quad b = \operatorname{rank} C_\Phi, \quad S = \operatorname{span}(\mathcal{K}_\Psi \mathcal{K}_\Phi), \quad s = \dim S = \operatorname{rank} C_{\Psi\circ\Phi},$$

so that $a = \dim \mathcal{K}_\Psi$ and $b = \dim \mathcal{K}_\Phi$.

2.4. **Facts on PPT operators and on spaces of matrices.**

Lemma 2.6 ([20, 19])**.** *Let* $\rho \succeq 0$ *on* $\mathbb{C}^p \otimes \mathbb{C}^q$ *have positive partial transpose and local ranks* r_1, r_2. *Then* $\operatorname{rank}\rho \geq \max(r_1, r_2)$, *and if* $\operatorname{rank}\rho \leq \max(r_1, r_2)$, *then* ρ *is separable.*

Lemma 2.7. *Let* $0 \leq r < \min(p, q)$. *The matrices of rank at most* r *in* $\mathrm{M}_{q\times p}$ *form an irreducible cone of codimension* $(p-r)(q-r)$ [2]. *Hence every subspace of* $\mathrm{M}_{q\times p}$ *of dimension greater than* $(p-r)(q-r)$ *contains a nonzero matrix of rank at most* r. *In particular a rank-one-free subspace of* $\mathrm{M}_{q\times p}$, *equivalently a product-free subspace of* $\mathbb{C}^p \otimes \mathbb{C}^q$, *has dimension at most* $(p-1)(q-1)$ [25].

The second sentence is the projective dimension theorem [14, Theorem I.7.2]. We also use that positive semidefinite operators with positive partial transpose on $\mathbb{C}^p \otimes \mathbb{C}^q$ with $pq \leq 6$ are separable [27, 16].

2.5. **Rank-loss loci of PPT maps.** For a completely positive map $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ and $r \geq 0$ let

$$D_r(\Phi) = \{[x] \in \mathbb{P}^{m-1} : \operatorname{rank} \Phi(P_x) \leq r\} = \{[x] : \dim \mathcal{K}_\Phi x \leq r\}.$$

It is Zariski closed, cut out by minors of the matrix $(K_1 x, \ldots, K_b x)$ for a basis $K_1, \ldots, K_b$ of $\mathcal{K}_\Phi$.

Lemma 2.8. *Let $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ be PPT and $r \geq 0$. Then $D_r(\Phi)$ is a finite union of projective linear subspaces $\mathbb{P}(A_1), \ldots, \mathbb{P}(A_p)$, and for each j there is a subspace $W_j \subseteq \mathbb{C}^k$ with $\dim W_j \leq r$ and $\mathcal{K}_\Phi A_j \subseteq W_j$.*

Proof. Let D be an irreducible component of $D_r(\Phi)$, let $r_D \leq r$ be the maximum of $\dim \mathcal{K}_\Phi x$ over $[x] \in D$, and let $D^\circ \subseteq D$ be the dense open subset where this maximum is attained. On D° the map $f(x) = \mathcal{K}_\Phi x \in \mathrm{Gr}(r_D, \mathbb{C}^k)$ is a morphism. Since $T \circ \Phi$ is completely positive, $(T \circ \Phi)(P_x) = \overline{\Phi(P_x)}$ has range $\mathcal{K}_{T\circ\Phi} x$, so $\overline{f(x)} = \mathcal{K}_{T\circ\Phi} x$ is also a morphism on D°. On the smooth part of D° the differential of f is therefore both complex linear and complex antilinear, hence zero. This smooth part is connected, because D is irreducible. So f is constant, equal to some W, and since the condition $\mathcal{K}_\Phi x \subseteq W$ is closed, it holds on all of D. By linearity $\mathcal{K}_\Phi(\operatorname{span} D) \subseteq W$, so $\mathbb{P}(\operatorname{span} D) \subseteq D_r(\Phi)$. This irreducible set contains the component D, so it equals D. □

Corollary 2.9. *Let $Z \succeq 0$ on $\mathbb{C}^m \otimes \mathbb{C}^n$ with $Z^\Gamma \succeq 0$.*

(a) $x \otimes y \in \ker Z$ if and only if $x \otimes \bar{y} \in \ker Z^\Gamma$.
(b) The set of pairs $([x], [y])$ with $x \otimes y \in \ker Z$ is a finite union of products $\mathbb{P}(A_j) \times \mathbb{P}(B_j)$ of linear subspaces.

Proof. Part (a) holds because $(x \otimes y)^\dagger Z (x \otimes y) = (x \otimes \bar{y})^\dagger Z^\Gamma (x \otimes \bar{y})$. For (b), $Z = C_F$ for a PPT map $F\colon \mathrm{M}_m \to \mathrm{M}_n$, and by (2.1) $x \otimes y \in \ker Z$ if and only if $y \perp \mathcal{K}_F \bar{x}$. If $\dim \mathcal{K}_F \bar{x} = r$, then $[\bar{x}]$ lies in a component $\mathbb{P}(A)$ of $D_r(F)$ with $\mathcal{K}_F A \subseteq W$ and $\dim W \leq r$ (Lemma 2.8), so $\mathcal{K}_F \bar{x} = W$. Hence the set in question is the union of the products $\mathbb{P}(\bar{A}) \times \mathbb{P}(W^\perp)$ over all r and all components. □

A product vector in the kernel of the Choi matrix of a PPT map constrains its Kraus space as follows.

Lemma 2.10. *Let $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ be PPT, write $C_\Phi = \sum_{j=1}^b \operatorname{vec}(Y_j) \operatorname{vec}(Y_j)^\dagger$ with $Y_1, \ldots, Y_b$ a basis of $\mathcal{K}_\Phi$, and give $\mathcal{K}_\Phi$ the inner product in which this basis is orthonormal. If $x \in \mathbb{C}^m$ and $v \in \mathbb{C}^k$ satisfy $v^\dagger Y x = 0$ for all $Y \in \mathcal{K}_\Phi$, that is, $\bar{x} \otimes v \in \ker C_\Phi$, then the orthogonal complement of $\{Y \in \mathcal{K}_\Phi : Yx = 0\}$ is contained in $\{Y \in \mathcal{K}_\Phi : v^\dagger Y = 0\}$.*

Proof. For $u \in \mathbb{C}^m$, $w \in \mathbb{C}^k$ and $Y \in \mathcal{K}_\Phi$ we have $\langle u \otimes w, \operatorname{vec}(Y)\rangle = w^\dagger Y \bar{u}$, so $\langle \bar{x} \otimes v, C_\Phi(\bar{x} \otimes v)\rangle = \sum_j |v^\dagger Y_j x|^2 = 0$. The partial transpose satisfies $\langle u \otimes w, C_\Phi^\Gamma(u' \otimes w')\rangle = \langle u \otimes \bar{w}', C_\Phi(u' \otimes \bar{w})\rangle$. Hence $\langle \bar{x} \otimes \bar{v}, C_\Phi^\Gamma(\bar{x} \otimes \bar{v})\rangle = 0$, and $C_\Phi^\Gamma(\bar{x} \otimes \bar{v}) = 0$ because $C_\Phi^\Gamma \succeq 0$. For $u \in \mathbb{C}^m$ and $z \in \mathbb{C}^k$ this gives

$$0 = \langle u \otimes z, C_\Phi^\Gamma(\bar{x} \otimes \bar{v})\rangle = \langle u \otimes v, C_\Phi(\bar{x} \otimes \bar{z})\rangle = v^\dagger R_z \bar{u}, \qquad R_z = \sum_j \overline{\bar{z}^\dagger Y_j x}\, Y_j.$$

So $v^\dagger R_z = 0$ for all z. In the inner product of $\mathcal{K}_\Phi$ we have $\langle R_z, Y\rangle = \bar{z}^\dagger Y x$ for $Y \in \mathcal{K}_\Phi$, so the R_z span the orthogonal complement of $\{Y \in \mathcal{K}_\Phi : Yx = 0\}$. □

For a PPT map Φ, conditions of the form $w^T Y x = 0$ for all $Y \in \mathcal{K}_\Phi$ that hold for polynomial families $x(t)$ and $w(t)$ only modulo a power t^{m_0} with $m_0 \geq 1$ hold exactly for every pair of coefficients x_a, w_b with $a + b < m_0$.

Lemma 2.11. *Let $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ be PPT, let $m_0 \geq 1$, and let $x(t) = \sum_{a\geq 0} t^a x_a$ and $w(t) = \sum_{b\geq 0} t^b w_b$ be polynomials with coefficients in $\mathbb{C}^m$ and $\mathbb{C}^k$ such that t^{m_0} divides $w(t)^T Y x(t)$ for every $Y \in \mathcal{K}_\Phi$. Then $w_b^T Y x_a = 0$ for all $Y \in \mathcal{K}_\Phi$ and all $a, b \geq 0$ with $a + b < m_0$, that is, $\bar{x}_a \otimes \bar{w}_b \in \ker C_\Phi$. Equivalently, if $C \succeq 0$ and $C^\Gamma \succeq 0$ on $\mathbb{C}^m \otimes \mathbb{C}^k$, and $\alpha(t) = \sum_a t^a \alpha_a$ and $\beta(t) = \sum_b t^b \beta_b$ are polynomials such that t^{m_0} divides $C(\alpha(t) \otimes \beta(t))$, then $\alpha_a \otimes \beta_b \in \ker C$ for all $a + b < m_0$.*

Proof. Write $C_\Phi = \sum_j \mathrm{vec}(Y_j)\,\mathrm{vec}(Y_j)^\dagger$ as in Lemma 2.10. As in its proof, for $x \in \mathbb{C}^m$ and $v \in \mathbb{C}^k$,

$$\langle \bar{x} \otimes \bar{v}, C_\Phi^\Gamma(\bar{x} \otimes \bar{v})\rangle = \langle \bar{x} \otimes v, C_\Phi(\bar{x} \otimes v)\rangle = \sum_j |v^\dagger Y_j x|^2.$$

For $t \in \mathbb{C}$ let $q(t) = \overline{x(t)} \otimes w(t) = \sum_{a,b} \bar{t}^a t^b\, \bar{x}_a \otimes w_b$. With $x = x(t)$ and $v = \overline{w(t)}$ this gives $\|(C_\Phi^\Gamma)^{1/2} q(t)\|^2 = \sum_j |w(t)^T Y_j x(t)|^2 = O(|t|^{2m_0})$, as $C_\Phi^\Gamma \succeq 0$. Here $(C_\Phi^\Gamma)^{1/2} q(t) = \sum_{a,b} \bar{t}^a t^b c_{ab}$ with $c_{ab} = (C_\Phi^\Gamma)^{1/2}(\bar{x}_a \otimes w_b)$. Suppose that $c_{ab} \neq 0$ for some $a+b < m_0$, and let n be the smallest value of $a+b$ with $c_{ab} \neq 0$. For $t = re^{i\theta}$ the terms with $a+b = n$ add up to $r^n \sum_{a+b=n} e^{i(b-a)\theta} c_{ab}$, and this trigonometric polynomial in θ is not identically zero, as its frequencies $b - a = 2b - n$ are distinct. So for some θ the norm of $(C_\Phi^\Gamma)^{1/2} q(re^{i\theta})$ is at least a positive multiple of r^n for all small $r > 0$, contradicting $n < m_0$. Hence $c_{ab} = 0$, so $C_\Phi^\Gamma(\bar{x}_a \otimes w_b) = 0$, and the displayed identity with $x = x_a$ and $v = \bar{w}_b$ gives $\sum_j |w_b^T Y_j x_a|^2 = 0$ for $a + b < m_0$.

For the second form, C is the Choi matrix of a PPT map Φ, as in the proof of Corollary 2.9, and we write $C = \sum_j \mathrm{vec}(Y_j)\,\mathrm{vec}(Y_j)^\dagger$ with a basis $Y_1, \ldots, Y_b$ of $\mathcal{K}_\Phi$. Then $C(\alpha \otimes \beta) = \sum_j \mathrm{vec}(Y_j)\,\overline{\beta^\dagger Y_j \bar{\alpha}}$, and the $\mathrm{vec}(Y_j)$ are linearly independent. So for $x(t) = \sum_a t^a \bar{\alpha}_a$ and $w(t) = \sum_b t^b \bar{\beta}_b$ the hypothesis says that t^{m_0} divides $w(t)^T Y_j x(t)$ for all j, and the conclusion $w_b^T Y_j x_a = 0$ for all j says that $\alpha_a \otimes \beta_b$ is orthogonal to the range of C. □

Corollary 2.12.
(a) In Lemma 2.11, let $a + b < m_0$ with $x_a \neq 0$ and $w_b \neq 0$. Then the orthogonal complement of $\{Y \in \mathcal{K}_\Phi : Y x_a = 0\}$ is contained in $\{Y \in \mathcal{K}_\Phi : w_b^T Y = 0\}$.
(b) Let $C \succeq 0$ and $C^\Gamma \succeq 0$ on $\mathbb{C}^m \otimes \mathbb{C}^k$. If $\alpha_0 \otimes \beta_0$ and $\alpha_1 \otimes \beta_0 + \alpha_0 \otimes \beta_1$ lie in $\ker C$, then so do $\alpha_1 \otimes \beta_0$ and $\alpha_0 \otimes \beta_1$.
(c) With C as in (b), every isolated point of the intersection of $\mathbb{P}(\ker C)$ with the Segre variety in $\mathbb{P}(\mathbb{C}^m \otimes \mathbb{C}^k)$ is a reduced point of the scheme-theoretic intersection.

Proof. (a) This is Lemma 2.10 for $x = x_a$ and $v = \bar{w}_b$. (b) Apply Lemma 2.11 with $m_0 = 2$ to $\alpha(t) = \alpha_0 + t\alpha_1$ and $\beta(t) = \beta_0 + t\beta_1$. (c) Let $[\alpha_0 \otimes \beta_0]$ be such a point. A tangent vector of the intersection there is the class of an element $\alpha_1 \otimes \beta_0 + \alpha_0 \otimes \beta_1$ of $\ker C$, and it is nonzero unless $\alpha_1 \in \mathbb{C}\alpha_0$ and $\beta_1 \in \mathbb{C}\beta_0$. By (b), $\alpha_1 \otimes \beta_0$ and $\alpha_0 \otimes \beta_1$ lie in $\ker C$. If $\alpha_1 \notin \mathbb{C}\alpha_0$, the points $[(\alpha_0 + \lambda\alpha_1) \otimes \beta_0]$, $\lambda \in \mathbb{C}$, form a line in the intersection through $[\alpha_0 \otimes \beta_0]$, and if $\beta_1 \notin \mathbb{C}\beta_0$, so do the points $[\alpha_0 \otimes (\beta_0 + \lambda\beta_1)]$. As the point is isolated, the Zariski tangent space of the intersection at it is zero, so the point is reduced. □

Remark 2.13. For $m_0 = 2$, Lemma 2.11 also follows from Lemma 2.10. The coefficient of t^0 says that $w_0^T Y x_0 = 0$ for all $Y \in \mathcal{K}_\Phi$, and that of t says that $f_1 + f_2 = 0$ on $\mathcal{K}_\Phi$, where $f_1(Y) = w_0^T Y x_1$ and $f_2(Y) = w_1^T Y x_0$. Here f_1 vanishes on $K_1 = \{Y : w_0^T Y = 0\}$ and f_2 on $K_2 = \{Y : Y x_0 = 0\}$, and by Lemma 2.10 for $x = x_0$ and $v = \bar{w}_0$ the orthogonal complement of K_1 lies in K_2. So $\mathcal{K}_\Phi = K_1 + K_2$, and $f_1 = -f_2$ vanishes on K_1 and on K_2, hence on $\mathcal{K}_\Phi$.

Corollary 2.14. *Let $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ be PPT with $\Phi^*(I) \succ 0$, and let $r < k$. If $\mathcal{K}_\Phi$ contains no nonzero matrix of rank at most r, then $D_r(\Phi) = \emptyset$, that is, $\mathrm{rank}\,\Phi(P_x) > r$ for every nonzero $x \in \mathbb{C}^m$.*

Proof. Let $x \neq 0$, put $W = \mathcal{K}_\Phi x = \operatorname{ran}\Phi(P_x)$, and suppose that $\dim W \leq r$. For every $v \in W^\perp$, Lemma 2.10 puts the orthogonal complement of $\{Y \in \mathcal{K}_\Phi : Yx = 0\}$ into $\{Y \in \mathcal{K}_\Phi : v^\dagger Y = 0\}$, hence into $\{Y \in \mathcal{K}_\Phi : \operatorname{ran} Y \subseteq W\}$, whose nonzero elements have rank at most r. So this complement is zero, $Yx = 0$ for all $Y \in \mathcal{K}_\Phi$, and $x^\dagger \Phi^*(I)x = \sum_j \|Y_j x\|^2 = 0$, a contradiction. □

Corollary 2.15. *Let $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ be PPT with $\Phi(I) \succ 0$, and suppose that $\mathcal{K}_\Phi$ contains no nonzero matrix of rank at most r. If $A \subseteq \mathbb{C}^m$ and $W \subsetneq \mathbb{C}^k$ are subspaces with $\mathcal{K}_\Phi A \subseteq W$, then $\dim A \leq m - r - 1$. In particular, every component $\mathbb{P}(A)$ of $D_{k-1}(\Phi)$ in Lemma 2.8 has $\dim A \leq m - r - 1$.*

Proof. For $x \in A$ and $v \in W^\perp$ we have $v^\dagger Y x = 0$ for all $Y \in \mathcal{K}_\Phi$. By Lemma 2.10 the orthogonal complement of $\{Y \in \mathcal{K}_\Phi : Yx = 0\}$ lies in $\{Y \in \mathcal{K}_\Phi : v^\dagger Y = 0\}$, and intersecting over $v \in W^\perp$ it lies in $\mathcal{K}^W = \{Y \in \mathcal{K}_\Phi : \operatorname{ran} Y \subseteq W\}$. Taking orthogonal complements, $(\mathcal{K}^W)^\perp \subseteq \{Y \in \mathcal{K}_\Phi : Yx = 0\}$ for every $x \in A$, so the nonzero elements of $(\mathcal{K}^W)^\perp$ have rank at most $m - \dim A$. If $\dim A \geq m - r$, then $(\mathcal{K}^W)^\perp = 0$, so every element of $\mathcal{K}_\Phi$ has range in W and $\operatorname{ran}\Phi(I) \subseteq W \neq \mathbb{C}^k$, a contradiction. □

The next lemma computes the length of these loci on a projective plane at an isolated point where the rank drops by one or by two.

Lemma 2.16. *Let $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ be PPT, let $A \subseteq \mathbb{C}^m$ be a three-dimensional subspace and $W \subseteq \mathbb{C}^k$ a subspace with $\mathcal{K}_\Phi A \subseteq W$. Let $D \subseteq \mathbb{P}(A)$ be the subscheme defined by the Fitting ideal $\operatorname{Fitt}_0(\operatorname{coker}\sigma)$, that is, by the $(\dim W)$-minors, of the map $\sigma\colon \mathcal{K}_\Phi \otimes \mathcal{O}(-1) \to W \otimes \mathcal{O}$, $Y \otimes x \mapsto Yx$, and let $[x]$ be an isolated point of D. If $\dim \mathcal{K}_\Phi x = \dim W - 1$, then D has length one at $[x]$, and if $\dim \mathcal{K}_\Phi x = \dim W - 2$, then it has length three.*

Proof. Let $c = \dim W - \dim \mathcal{K}_\Phi x$ and $\mathcal{Y}_x = \{Y \in \mathcal{K}_\Phi : Yx = 0\}$. Let c_0 be a linear form on $\mathbb{C}^k$ that vanishes on $\mathcal{K}_\Phi x$ but not on W, and let $\xi \in A$ satisfy $c_0(Y\xi) = 0$ for all $Y \in \mathcal{Y}_x$. Then $Y \mapsto c_0(Y\xi)$ factors through $Y \mapsto Yx$, so $c_0(Y\xi) + c_1(Yx) = 0$ for all $Y \in \mathcal{K}_\Phi$ and some linear form c_1. Write $c_j(v) = \gamma_j^T v$. As $\langle u \otimes w, \operatorname{vec}(Y)\rangle = w^\dagger Y \bar{u}$, the vectors $\bar{x} \otimes \bar{\gamma}_0$ and $\bar{\xi} \otimes \bar{\gamma}_0 + \bar{x} \otimes \bar{\gamma}_1$ lie in $\ker C_\Phi$, and Corollary 2.12(b) puts $\bar{\xi} \otimes \bar{\gamma}_0$ into $\ker C_\Phi$, that is, $c_0(Y\xi) = 0$ for all $Y \in \mathcal{K}_\Phi$. So c_0 vanishes on $\mathcal{K}_\Phi(x + t\xi) \subseteq W$ for all t, and the line through $[x]$ and $[\xi]$ lies in D unless $\xi \in \mathbb{C}x$. As $[x]$ is isolated, $\xi \in \mathbb{C}x$.

Choose complements A_0 of $\mathbb{C}x$ in A, $\mathcal{K}_1$ of $\mathcal{Y}_x$ in $\mathcal{K}_\Phi$, and W_1 of $\mathcal{K}_\Phi x$ in W, and use the affine coordinates $\xi \in A_0$ near $[x]$. The block of $\sigma(x + \xi)$ from $\mathcal{K}_1$ to $\mathcal{K}_\Phi x$ is invertible over the local ring, so row and column operations, which preserve the Fitting ideal, bring $\sigma(x + \xi)$ to the form $\operatorname{diag}(I, S(\xi))$. Here $S(\xi)\colon \mathcal{Y}_x \to W_1$ vanishes at $\xi = 0$, and its linear part is $D_1(\xi)\colon Y \mapsto Y\xi \bmod \mathcal{K}_\Phi x$, with $W_1 \cong W/\mathcal{K}_\Phi x$. So D is defined near $[x]$ by the $c \times c$ minors of S. By the first paragraph, $\xi \mapsto c_0 \circ D_1(\xi)$ is injective on A_0 for every nonzero linear form c_0 on $W/\mathcal{K}_\Phi x$. If $c = 1$, the entries of S therefore have linear parts spanning the dual of A_0, they generate the maximal ideal $\mathfrak{m}$ of the local ring, and the length is one. If $c = 2$, the maps $c_0 \mapsto (\xi \mapsto c_0 \circ D_1(\xi))$ form a pencil of matrices of size $\dim \mathcal{Y}_x \times 2$ and rank two at every $c_0 \neq 0$, so it has no blocks of type L_ε and no regular part, and by [12, Chapter XII] it is equivalent to L_2^T or to $L_1^T \oplus L_1^T$, padded with zero rows. In both cases the 2×2 minors of $D_1(\xi)$ span the quadratic forms in ξ, and this span does not depend on the choice of bases. The 2×2 minors of S lie in $\mathfrak{m}^2$ and their initial forms span $\mathfrak{m}^2/\mathfrak{m}^3$, so they generate $\mathfrak{m}^2$ by Nakayama's lemma, and the length is three. □

Without the PPT property the lengths can be larger. For $\sigma(y) = \left(\begin{smallmatrix} y_0 & 0 & y_2 \\ y_2 & y_1 & 0 \end{smallmatrix}\right)$ on $\mathbb{P}^2$, the point $[1 : 0 : 0]$ is isolated, σ has rank one there, and the length is two. In this example the first paragraph of the proof fails for $\xi = e_2$.

Corollary 2.17. *In Lemma 2.16, suppose that* $\mathcal{K}_A = \{Y|_A : Y \in \mathcal{K}_\Phi\}$ *has dimension* $\dim W+1$*, that* D *is finite, and that* $\dim \mathcal{K}_\Phi x \geq \dim W - 2$ *for every nonzero* $x \in A$*. Then* $n_1 + 3n_2 = \binom{\dim W+1}{2}$*, where* n_c *is the number of points* $[x] \in D$ *with* $\dim \mathcal{K}_\Phi x = \dim W - c$*.*

Proof. Put $w = \dim W$. The scheme D is also defined by the w-minors of $\mathcal{K}_A \otimes \mathcal{O}(-1) \to W \otimes \mathcal{O}$, and as it is finite, it has the expected codimension two. By [10, Theorem 14.4(c)] its length is $c_1^2 - c_2$ of $(1-h)^{-(w+1)}$, that is, $(w+1)^2 - \binom{w+2}{2} = \binom{w+1}{2}$. Lemma 2.16 gives the length at each point. □

The proof of Lemma 2.10 also shows that the range of $\Phi(P_x)$ is spanned by its intersections with the ranges at any points spanning x.

Lemma 2.18. *Let* $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ *be PPT, and let* $x \in \operatorname{span}(y_1, \dots, y_r) \subseteq \mathbb{C}^m$*. Then* $\mathcal{K}_\Phi x = \sum_i (\mathcal{K}_\Phi x \cap \mathcal{K}_\Phi y_i)$*.*

Proof. Let $U = \mathcal{K}_\Phi x$, write $x = \sum_i \lambda_i y_i$, and let $\mathcal{Y}_x^\perp$ be the orthogonal complement of $\{Y \in \mathcal{K}_\Phi : Yx = 0\}$ in the inner product of Lemma 2.10. For every v with $v^\dagger U = 0$, Lemma 2.10 gives $v^\dagger Y = 0$ for all $Y \in \mathcal{Y}_x^\perp$, so every element of $\mathcal{Y}_x^\perp$ has range in U. The map $Y \mapsto Yx$ sends $\mathcal{Y}_x^\perp$ onto U, and $Yx = \sum_i \lambda_i Y y_i$ with $Y y_i \in U \cap \mathcal{K}_\Phi y_i$. □

Corollary 2.19. *Let* $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ *be PPT with* $\Phi^*(I) \succ 0$*, and suppose that* $\mathcal{K}_\Phi$ *contains no nonzero matrix of rank at most two. Let* $x \in \operatorname{span}(y_1, y_2, y_3)$ *be nonzero and* $U = \mathcal{K}_\Phi x$*. Then the three spaces* $U \cap \mathcal{K}_\Phi y_i$ *do not all have dimension at most one with two of them equal.*

Proof. Otherwise $\dim U \leq 2$ by Lemma 2.18, contradicting Corollary 2.14 with $r = 2$. □

3. Rank windows and impossibility

This section bounds the Choi ranks of a reduced range-criterion pair from both sides and excludes the reduced types with middle dimension five and outer dimensions at most six. Throughout this section (Φ, Ψ) is a reduced range-criterion pair of type (m, k, n), and a, b, S, s are as in Section 2. By Lemma 2.4, $(T \circ \Phi, \Psi \circ T)$ and $(\Psi^\#, \Phi^\#)$ are reduced range-criterion pairs of types (m, k, n) and (n, k, m). Every statement about $\operatorname{rank} C_\Phi$ and $\operatorname{rank} C_\Psi$ below therefore also holds for $\operatorname{rank} C_\Phi^\Gamma$ and $\operatorname{rank} C_\Psi^\Gamma$, and every statement has a mirror version with the roles of Φ and Ψ exchanged.

3.1. No Kraus operators of rank at most two.

Lemma 3.1. *Neither* $\mathcal{K}_\Phi$ *nor* $\mathcal{K}_\Psi$ *contains a nonzero matrix of rank at most two. In particular* $m, k, n \geq 3$*.*

Proof. Suppose $Y_0 = uw^T \in \mathcal{K}_\Phi$ has rank one. For $X \in \mathcal{K}_\Psi$ the product $XY_0 = (Xu)w^T$ has rank at most one and lies in S, so $Xu = 0$. Then $u^\dagger \Psi^*(I) u = \sum_s \|X_s u\|^2 = 0$ for the Kraus operators X_s of Ψ, which contradicts $\Psi^*(I) \succ 0$.

Suppose $Y_1 \in \mathcal{K}_\Phi$ has rank two, and let $U = \operatorname{ran} Y_1$. If $X \in \mathcal{K}_\Psi$ satisfies $\operatorname{rank}(X|_U) = 1$, then $\operatorname{rank} XY_1 = 1$, which is excluded. So $\mathcal{K}_\Psi|_U = \{X|_U : X \in \mathcal{K}_\Psi\} \subseteq \operatorname{Hom}(U, \mathbb{C}^n)$ is rank-one-free. It is the Kraus space of the map $\Psi_U = \Psi \circ \operatorname{Ad}_{\iota_U}\colon \mathrm{M}_2 \to \mathrm{M}_n$, where $\iota_U\colon \mathbb{C}^2 \to \mathbb{C}^k$ is an isometry onto U. This map is PPT and nonzero, since $\iota_U^\dagger \Psi^*(I) \iota_U \succ 0$. Let n_U be the rank of $\Psi_U(I)$, the output local rank of C_{Ψ_U}. The space $\mathcal{K}_\Psi|_U$ lies in $\operatorname{Hom}(U, \operatorname{ran} \Psi_U(I)) \cong \mathbb{C}^2 \otimes \mathbb{C}^{n_U}$ and is product-free, so $\operatorname{rank} C_{\Psi_U} = \dim \mathcal{K}_\Psi|_U \leq n_U - 1$ by Lemma 2.7. This contradicts Lemma 2.6.

The statements for $\mathcal{K}_\Psi$ follow by applying the above to the mirror pair $(\Psi^\#, \Phi^\#)$, whose first factor has Kraus space $\mathcal{K}_\Psi^T$. If $k \leq 2$ or $m \leq 2$, every element of $\mathcal{K}_\Phi$ has rank at most two, so $\mathcal{K}_\Phi = 0$, contradicting $\Phi(I) \succ 0$. In the same way $n \geq 3$. □

3.2. **PPT states of low rank.** The next lemma gives the lower bounds in the rank windows below. Part (a) uses the first half of the normal form in the proof of [4, Theorem 43].

Lemma 3.2. *Let $\rho \succeq 0$ on $\mathbb{C}^M \otimes \mathbb{C}^N$ have positive partial transpose and local ranks M and N, and suppose that* $\operatorname{ran}\rho$ *contains no nonzero vector of Schmidt rank at most two.*

(a) If $M \geq 3$, then $\operatorname{rank}\rho \geq N+2$.
(b) If $M \geq 5$ and $\operatorname{rank}\rho = N+2$, *then $N \geq 3M-8$.*

Proof. By Lemma 2.6, $\operatorname{rank}\rho \geq N$, and $\operatorname{rank}\rho = N$ would make ρ separable, with product vectors in its range. So $\operatorname{rank}\rho = N+p$ with $p \geq 1$. We treat $p \in \{1,2\}$ and assume $M \geq p+2$, which holds in both parts. Write $\rho = C^\dagger C$ with $C = \sum_{i=0}^{M-1} \langle i| \otimes C_i$, where the C_i are $(N+p)\times N$ matrices and the rows of C are linearly independent. For $c \in \mathbb{C}^{N+p}$ the vector $C^\dagger c = \sum_i |i\rangle \otimes C_i^\dagger c$ lies in $\operatorname{ran}\rho$. Its Schmidt rank is the dimension of the span of the vectors $C_i^\dagger c$, so it is at most the number of indices i with $C_i^\dagger c \neq 0$, and at most the dimension of any subspace containing all $C_i^\dagger c$. For $\alpha \in \mathbb{C}^M$ put $C_\alpha = \sum_i \alpha_i C_i$. Local invertible operations and unitaries acting on the left of C preserve all hypotheses and the Schmidt ranks. We use two facts.

(P) If $C_\alpha \beta = 0$, then $C_j\beta \perp \operatorname{ran} C_\alpha$ for all j. Indeed, the partial transpose ρ^{Γ_1} on the first factor satisfies $(\bar\alpha\otimes\beta)^\dagger \rho^{\Gamma_1} (\bar\alpha\otimes\beta) = \|C_\alpha\beta\|^2$, so $\rho^{\Gamma_1}(\bar\alpha\otimes\beta) = \sum_j |j\rangle \otimes C_\alpha^\dagger C_j \beta$ vanishes.

(D) Since $\dim\ker\rho = MN - N - p > (M-1)(N-1)$, the kernel contains a product vector $\alpha\otimes\beta$ (Lemma 2.7), and then $C_\alpha\beta = 0$. Let R be the largest value of $\operatorname{rank} C_\alpha$ that is smaller than N. Then $R \geq 1$, because $C_\alpha = 0$ would put $\alpha \otimes \mathbb{C}^N$ into $\ker\rho$, contradicting the local rank M.

After local operations and a unitary acting on the left of C we may assume that $\operatorname{rank} C_0 = R$ and $C_0 = \left(\begin{smallmatrix} I_R & 0 \\ 0 & 0 \end{smallmatrix}\right)$, with rows split as $R \mid N+p-R$ and columns as $R \mid q$, where $q = N-R$. By (P) applied to C_0, for $i \geq 1$ we have $C_i = \left(\begin{smallmatrix} C_{i0} & 0 \\ C_{i2} & C_{i3} \end{smallmatrix}\right)$ with C_{i3} of size $(q+p)\times q$. The operator $\sigma = C'^\dagger C'$ with $C' = \sum_{i\geq 1}\langle i|\otimes C_{i3}$ is the compression of ρ to $\operatorname{span}(e_1,\dots,e_{M-1})\otimes\operatorname{span}(e_R,\dots,e_{N-1})$, so it has positive partial transpose, its second local rank is q, and $\operatorname{rank}\sigma = q+t$ with $0 \leq t \leq p$. For $\xi \in \mathbb{C}^M$ with $\xi_0 = 0$ and small $\epsilon \neq 0$, $\operatorname{rank} C_{e_0+\epsilon\xi} = R + \operatorname{rank} C_{\xi 3}$, where $C_{\xi 3} = \sum_i \xi_i C_{i3}$. By maximality of R, every $C_{\xi 3}$ has rank 0 or q. Let μ be the first local rank of σ, that is, $\mu = \dim\operatorname{span}\{C_{i3}\}$.

If $t = 0$, then σ is separable with $\operatorname{rank}\sigma = q$ (Lemma 2.6), so $\sigma = \sum_{l=1}^q P_{\alpha_l}\otimes P_{\beta_l}$ with $\beta_1,\dots,\beta_q$ linearly independent [19, Theorem 1]. After a unitary on the lower rows and local operations we get $C_{i3} = \left(\begin{smallmatrix} \Lambda_i \\ 0 \end{smallmatrix}\right)$ with diagonal Λ_i and $(\Lambda_i)_{11} = \delta_{i1}$. Then $C_0 e_R = 0$, $C_i e_R = 0$ for $i \geq 2$, and $C_1 e_R$ is the first lower unit vector. By (P), this row of C vanishes in every block C_i with $i \neq 1$, so it gives a product vector in $\operatorname{ran}\rho$. Hence $t \geq 1$.

If $\mu > t+1$, then σ restricted to $\operatorname{ran}\sigma_1 \otimes \mathbb{C}^q$, where σ_1 is the first marginal, has kernel of dimension $\mu q - q - t > (\mu-1)(q-1)$, which contains a product vector $\xi\otimes\beta$. Then $C_{\xi 3}\beta = 0$ with $C_{\xi 3} \neq 0$, which contradicts the dichotomy. Hence $\mu \leq t+1$. After a change of basis of $\operatorname{span}(e_1,\dots,e_{M-1})$ we have $C_{i3} = 0$ for $i > \mu$. For such i and every $\beta \in \operatorname{span}(e_R,\dots,e_{N-1})$, (P) shows that $\operatorname{ran} C_i$ is orthogonal to the column space of C', which has dimension $q+t$ inside the $q+p$ lower rows. In an orthonormal basis of the lower rows adapted to this column space, the first $q+t$ lower rows of C vanish in all blocks except $1,\dots,\mu$. They give range vectors of Schmidt rank at most μ, so $\mu \geq 3$ and $t \geq 2$. This proves (a), since $p = 1$ would give $t \leq 1$.

For (b) let $p = 2$, so that $t = 2$ and $\mu = 3$. Then $C_{i2} = 0$ and $C_{i3} = 0$ for $i \geq 4$. The compression τ of ρ to $\operatorname{span}(e_0,e_4,\dots,e_{M-1}) \otimes \operatorname{span}(e_0,\dots,e_{R-1})$ equals $D^\dagger D$ with $D = (I_R, C_{40},\dots,C_{M-1,0})$. It has positive partial transpose, second local rank R and rank R. By [19, Theorem 1] it is a sum of R product projections with linearly independent second factors. Comparing the two factorisations of τ shows that the C_{i0} with $i \geq 4$ are simultaneously unitarily

diagonalisable, and after a unitary change of the first R coordinates they are diagonal, $C_{i0} = \Delta_i$, while C_0 keeps its form. Group the coordinates $0, \dots, R-1$ into blocks $B_1, \dots, B_\kappa$ according to the tuple of diagonal entries $((\Delta_i)_{ss})_{i\geq 4}$. If $s \in B_r$ has tuple $d^{(r)}$, then $\alpha = e_i - d_i^{(r)} e_0$ satisfies $C_\alpha e_s = 0$. By (P), $C_j e_s$ is orthogonal to $\operatorname{ran} C_\alpha$ for all j and all $i \geq 4$, and these ranges span the top coordinates outside B_r. So in every C_j the column s has no entries in the top rows outside B_r. As the blocks partition the top coordinates and the top right block of every C_j is zero, the top rows of C with index in B_r are supported on the columns in B_r, and they give range vectors of Schmidt rank at most $|B_r|$. Hence $|B_r| \geq 3$ for every r. The matrices $C_0, C_4, \dots, C_{M-1}$ are linearly independent, because the first local rank of ρ is M, and they lie in the κ-dimensional space of matrices $\left(\begin{smallmatrix} \Delta & 0 \\ 0 & 0 \end{smallmatrix}\right)$ with Δ diagonal and constant on each block. So $M - 3 \leq \kappa$, and $N - 1 \geq R = \sum_r |B_r| \geq 3(M-3)$. □

3.3. **Rank windows.**

Theorem 3.3. *Let (Φ, Ψ) be a reduced range-criterion pair of type (m, k, n). Then $m, k, n \geq 4$ and*

$$\max(m,k) + 2 \leq b \leq (k-2)(m-2), \qquad \max(k,n) + 2 \leq a \leq (k-2)(n-2).$$

The same bounds hold for $\operatorname{rank} C_\Phi^\Gamma$ and $\operatorname{rank} C_\Psi^\Gamma$.

Proof. By Lemma 3.1, $m, k, n \geq 3$, and $\operatorname{ran} C_\Phi$ contains no vector of Schmidt rank at most two. Lemma 3.2(a), applied with $M = \min(m,k)$ and $N = \max(m,k)$ after exchanging the two tensor factors if $m > k$, gives the lower bound for b. If $b > (k-2)(m-2)$, then $\mathcal{K}_\Phi$ contains a nonzero matrix of rank at most two by Lemma 2.7, contradicting Lemma 3.1. The bounds for a follow in the same way, and those for the partial transposes from Lemma 2.4(b). If $k = 3$, the window for b is empty, and the same holds if $m = 3$ or $n = 3$. □

Corollary 3.4. *(a) There is no range-criterion pair with $k \leq 3$, with $\min(m,n) \leq 3$, with $k = 4$ and $\min(m,n) \leq 5$, or with $\min(m,n) = 4$ and $k \leq 5$. In particular there is none with $\max(m,k,n) \leq 4$.*

(b) A reduced range-criterion pair with $\max(m,k,n) \leq 5$ has type $(5,5,5)$ and $a, b \in \{7,8,9\}$.

(c) A reduced range-criterion pair with $\max(m,k,n) = 6$ has one of the types $(6,4,6)$, $(5,5,6)$, $(6,5,5)$, $(6,5,6)$, $(4,6,4)$, $(4,6,5)$, $(5,6,4)$, $(4,6,6)$, $(6,6,4)$, $(5,6,5)$, $(5,6,6)$, $(6,6,5)$, $(6,6,6)$.

Proof. For a reduced type, the window for b is nonempty only if $\max(m,k)+2 \leq (k-2)(m-2)$. With $k = 4$ this forces $m \geq 6$, and with $m = 4$ it forces $k \geq 6$. The same holds for (k,n). Part (a) follows because reduction does not increase dimensions, and (b), (c) are the reduced types allowed by these conditions. □

In particular the range criterion never certifies a counterexample through the middle algebra M_3, whatever the outer dimensions. For the pair of [24] the windows read $12 \leq a, b \leq 32$, and $a = b = 15$.

3.4. **Middle dimension five.** The next lemma is the topological input. For $1 \leq d_1 < d_2 < k$ let $\mathrm{Fl} = \mathrm{Fl}(d_1, d_2; \mathbb{C}^k)$ be the variety of flags $E \subset U \subset \mathbb{C}^k$ with $\dim E = d_1$ and $\dim U = d_2$.

Lemma 3.5. *Let $K' \subseteq \operatorname{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ and $L' \subseteq \operatorname{Hom}(\mathbb{C}^m, \mathbb{C}^k)$ be subspaces of dimensions $a', b' \geq 1$. On $V = \mathbb{P}(K') \times \mathrm{Fl} \times \mathbb{P}(L')$ consider the vector bundle*

$$F = \operatorname{Hom}\big(\mathcal{O}(-1) \boxtimes E, \mathbb{C}^n\big) \oplus \operatorname{Hom}\big(\mathcal{O}(-1) \boxtimes \mathbb{C}^m, \mathbb{C}^k/U\big),$$

where the first $\mathcal{O}(-1)$ is the tautological line bundle on $\mathbb{P}(K')$ and the second the one on $\mathbb{P}(L')$, with the section $s([X], E \subset U, [Y]) = (X|_E,\ Y \bmod U)$. If $\dim V = \operatorname{rank} F$ and $N =$

$\int_V c_{\mathrm{top}}(F) \neq 0$, then there are nonzero $X \in K'$ and $Y \in L'$ and a flag $E \subset U$ with $XE = 0$ and $\operatorname{ran} Y \subseteq U$.

Proof. A nowhere vanishing section splits off a trivial line bundle, which forces $c_{\mathrm{top}}(F) = 0$. So s has a zero. □

Here $\dim V = a' + b' - 2 + d_2(k - d_2) + d_1(d_2 - d_1)$ and $\operatorname{rank} F = d_1 n + (k - d_2)m$. We write $N_{d_1,d_2}(a', b'; m, n)$ for the number in Lemma 3.5 with k understood from the context.

Theorem 3.6 (†)**.** *There is no reduced range-criterion pair of type* $(5,5,5)$*,* $(5,5,6)$*,* $(6,5,5)$ *or* $(6,5,6)$.

Proof. Let $k = 5$, $(d_1, d_2) = (2, 3)$ and $a' + b' = 2(m + n) - 6$, so that $\dim V = \operatorname{rank} F$. Let $K' \subseteq \mathcal{K}_\Psi$ and $L' \subseteq \mathcal{K}_\Phi$ be subspaces of dimensions a', b', and let $(X, E \subset U, Y)$ be the zero given by Lemma 3.5. Then $\operatorname{rank} X \leq 3$ and $\operatorname{rank} Y \leq 3$, so both ranks equal three by Lemma 3.1, and $\operatorname{ran} Y = U$. Hence $\operatorname{rank} XY = \operatorname{rank} X|_U \leq 1$. If $XY = 0$, then X kills the 3-plane U and has rank at most two, which is excluded. So XY has rank one, a contradiction. It remains to choose a', b' with $N_{2,3}(a', b'; m, n) \neq 0$, $a' \leq a$ and $b' \leq b$.

At $(5,5,5)$, Theorem 3.3 gives $a, b \geq 7$, and $N_{2,3}(7,7;5,5) = 4500$. At $(5,5,6)$, Theorem 3.3 gives $b \geq 7$ and $a \geq 8$. Lemma 3.2(b) excludes $b = 7$ (with $M = N = 5$) and $a = 8$ (with $M = 5$, $N = 6$). So $a, b \geq 8$, and $N_{2,3}(8,8;5,6) = 8400$. The type $(6,5,5)$ is the mirror of $(5,5,6)$. At $(6,5,6)$, Theorem 3.3 and Lemma 3.2(b) give $a, b \geq 9$, and $N_{2,3}(9,9;6,6) = 22050$. The numbers $N_{2,3}$ were computed in the Chow ring of V (Table SA.1). □

Remark 3.7. The same argument with $N_{2,3}(4,6;4,4) = 400$ gives a second proof that the type $(4,5,4)$ is impossible, using only $a, b \geq 6$.

4. Overview of the case analysis

This section states the results of the supplement that the proof of Theorem 1.1 uses, together with the main results for larger types, and sketches their proofs. The full proofs are in the supplement, at the places given after each sketch. As in Section 3, (Φ, Ψ) is a reduced range-criterion pair of type (m, k, n), and a, b, S, s are as in Section 2. We write $S^\perp = \{A \in \mathrm{M}_{m\times n} : \operatorname{Tr}(AM) = 0 \text{ for all } M \in S\}$, so that $\bar{z} \otimes \bar{w} \in \ker C_{\Psi\circ\Phi}$ if and only if $zw^T \in S^\perp$ (Lemma 2.1(b)). At middle dimension six, for a 3-plane $U \subseteq \mathbb{C}^6$ we put $M_U = \{Y \in \mathcal{K}_\Phi : \operatorname{ran} Y \subseteq U\}$ and $N_U = \{X \in \mathcal{K}_\Psi : XU = 0\}$. The ranges of the elements of rank three of $\mathcal{K}_\Phi$ are the members, and they form the set $\mathcal{G} \subseteq \mathrm{Gr}(3,6)$. The points $[X] \in \mathbb{P}(\mathcal{K}_\Psi)$ with $\operatorname{rank} X = 3$ form the set $\mathcal{R}_X$, and for $z \in \mathbb{C}^m$ we put $\mathcal{Y}_z = \{Y \in \mathcal{K}_\Phi : Yz = 0\}$. A pair (z, w) of nonzero vectors with $zw^T \in S^\perp$ is a kernel product, with kernel input z and kernel output w, and $\mathcal{X} \subseteq \mathbb{P}^{m-1}$ is the set of the kernel inputs $[z]$. These are the definitions of the supplement, which gives most of them at the beginning of Section S3.5. The Schubert class $\sigma_\lambda \in A^{|\lambda|}(\mathrm{Gr}(3,6))$ is given by the Giambelli formula in the Chern classes of the universal quotient bundle, so that σ_3 is the class of the 3-planes through a fixed vector, σ_{111} that of the 3-planes in a fixed hyperplane, and σ_1 the hyperplane class of the Plücker embedding.

The four ingredients of the proof sketched in the introduction correspond to the following statements. The rank bounds are Lemma 3.1 and Theorem 3.3, with Corollary 3.4 for the types they exclude. The intersection number at middle dimension five is that of Lemma 3.5, which gives Theorem 3.6. The families of 3-planes are treated in Lemma 4.1 and Corollary 4.2, on which the bounds behind Corollary 4.3 rest. The two facts on a single PPT map are Lemmas 2.8 and 2.10; they give Lemma 4.12 and Theorem 4.14 for the type $(6,4,6)$, Lemma 4.19 and Theorem 4.20 for the outer dimensions $(4,4)$, and Theorems 4.5 and 4.6 for the type $(5,6,5)$. The last subsection of this section puts these together.

4.1. **Middle dimension six.**

Lemma 4.1. *Let* $\Delta_2 = \{(U, U') : \dim(U \cap U') \geq 2\} \subseteq \mathrm{Gr}(3,6) \times \mathrm{Gr}(3,6)$. *Then*

$$[\Delta_2] = \sigma_{22} \times 1 + \sigma_{21} \times \sigma_1 + \sigma_2 \times \sigma_2 + \sigma_{11} \times \sigma_{11} + \sigma_1 \times \sigma_{21} + 1 \times \sigma_{22}. \tag{4.1}$$

Let $\mathcal{G}_0 \subseteq \mathrm{Gr}(3,6)$ *be closed.*

- *(a) If* $\mathcal{G}_0$ *has pure dimension three and* $\dim(U \cap U') \leq 1$ *for all distinct* $U, U' \in \mathcal{G}_0$, *then* $\int_{\mathcal{G}_0} \sigma_3 = \int_{\mathcal{G}_0} \sigma_{111} = 0$.
- *(b) If* $\mathcal{G}_0$ *has pure dimension four and* $\dim(U \cap U') \leq 1$ *for all distinct* $U, U' \in \mathcal{G}_0$, *then* $\int_{\mathcal{G}_0} \sigma_{31} = \int_{\mathcal{G}_0} \sigma_{211} = 0$.
- *(c) Let* $\mathcal{G}_1 \subseteq \mathcal{G}_0$ *be closed, nonempty and of pure dimension two, such that* $\dim(U \cap U') \leq 1$ *whenever* $U \in \mathcal{G}_1$, $U' \in \mathcal{G}_0$ *and* $U' \neq U$. *Put* $p = \int_{\mathcal{G}_1} \sigma_2$ *and* $q = \int_{\mathcal{G}_1} \sigma_{11}$. *If* $\mathcal{G}_0$ *has pure dimension four, then* $\int_{\mathcal{G}_0} \sigma_{31} = \int_{\mathcal{G}_0} \sigma_{211} = 0$. *If* $\mathcal{G}_0$ *has pure dimension three, then* $p \int_{\mathcal{G}_0} \sigma_3 + q \int_{\mathcal{G}_0} \sigma_{111} = 0$.
- *(d) Let* $\mathcal{G}_1 \subseteq \mathcal{G}_0$ *be closed, nonempty and of pure dimension one, such that* $\dim(U \cap U') \leq 1$ *whenever* $U \in \mathcal{G}_1$, $U' \in \mathcal{G}_0$ *and* $U' \neq U$. *If* $\mathcal{G}_0$ *has pure dimension four, then* $\int_{\mathcal{G}_0} \sigma_{31} = \int_{\mathcal{G}_0} \sigma_{211} = 0$.

Corollary 4.2. *Let* $\mathcal{G} \subseteq \mathrm{Gr}(3,6)$ *be closed, with* $\dim(U \cap U') \leq 1$ *for all distinct* $U, U' \in \mathcal{G}$. *Then the members of* $\mathcal{G}$ *do not cover* $\mathbb{P}^5$, *and some hyperplane of* $\mathbb{C}^6$ *contains no member of* $\mathcal{G}$.

Sketch (proofs in Section S3.5). The class of Δ_2 in (4.1) is computed by counting the pairs (U, U') with $\dim(U \cap U') \geq 2$ in products of two Schubert varieties in general position, each with multiplicity one by Kleiman's theorem [21]. When $(\mathcal{G}_1 \times \mathcal{G}_0) \cap \Delta_2$ is the diagonal of $\mathcal{G}_1$, Fulton's refined intersection product of $[\mathcal{G}_1 \times \mathcal{G}_0]$ and $[\Delta_2]$ lives on the diagonal, so its pairing with a class $\alpha \times \beta$ depends only on the product $\alpha\beta$. Evaluating these pairings with (4.1) and the Pieri rules, and using that the integrals of Schubert classes over subvarieties are nonnegative, gives (a)–(d). For Corollary 4.2, a component of $\mathcal{G}$ whose members cover $\mathbb{P}^5$ has dimension at least three. In dimension three or four it would give a positive integral of σ_3 or of σ_{31}, against Lemma 4.1(a) or (b), and in dimension at least five two of its members meet in a plane. The second statement follows by passing to annihilators.

Corollary 4.3. *There is no range-criterion pair of type* $(4,6,4)$, $(4,6,5)$, $(5,6,4)$, $(4,6,6)$ *or* $(6,6,4)$. *In particular, no range-criterion pair with* $\max(m,k,n) \leq 6$ *has an outer dimension equal to four.*

Sketch (proof in Section S3.5). At the reduced types $(4,6,4)$ and $(4,6,5)$, Theorem 3.3 and Lemma 3.2(b) give $a > 2n-2$, against the bound $a \leq 2n-2$ of Theorem S3.11. At $(4,6,6)$ they give $a \geq 9$, and Theorem S3.11 gives $a \leq 10$. These are the bounds used in Theorem S3.14, which excludes this type. The types $(5,6,4)$ and $(6,6,4)$ are mirrors, and Corollary 3.4 passes from original to reduced types.

4.2. **The type** $(5,6,5)$**.**

Proposition 4.4. *Let* (Φ, Ψ) *be a reduced range-criterion pair of type* $(5,6,5)$.

- *(a) The set* $\mathcal{G}$ *of members is infinite.*
- *(b)* $S^\perp$ *contains a nonzero matrix of rank at most two.*

Sketch (proof in Section S3.6). Here $a = b = 9$ by Proposition S3.18(a). If $\mathcal{G}$ were finite, the zero scheme of the section $([Y], E) \mapsto Y|_E$ of a bundle of rank 12 on $\mathbb{P}(\mathcal{K}_\Phi) \times \mathrm{Gr}(2,5)$ would be a disjoint union of planes $\mathbb{P}(M_U)$, counted with multiplicities $m_U \geq 1$, and two Chern numbers would give $\sum_U m_U = 175$ and $3\sum_U m_U = 210$. If $S^\perp$ contained no nonzero matrix of rank

at most two, then $s = 16$, and an exact sequence of vector bundles on $\mathbb{P}(M_U) \cong \mathbb{P}^2$ would contradict the Whitney formula.

The map $\bar{\rho}$ in part (b) of the next theorem is defined in its proof, in Section S3.6 of the supplement.

Theorem 4.5. *Let (Φ, Ψ) be a reduced range-criterion pair of type $(5, 6, 5)$. Then $s = 16$, and $\ker C_{\Psi\circ\Phi}$ contains exactly two product vectors, $\bar{z}_1 \otimes \bar{w}_1$ and $\bar{z}_2 \otimes \bar{w}_2$, up to scalar multiples. Moreover:*

- *(a) The matrices of rank at most two in $\mathbb{P}(S^\perp)$ form a finite union of points and lines. Each of these lines is $\mathbb{P}(\Lambda)$, where $\Lambda = \{A \in S^\perp : \operatorname{ran} A \subseteq E,\ AR = 0\}$ has dimension two for a plane E and a 3-space R and contains $z_1 w_1^T$ or $z_2 w_2^T$. One of these lines passes through $[z_1 w_1^T]$ and $[z_2 w_2^T]$.*
- *(b) Let $\{i, j\} = \{1, 2\}$ and $W_i = \mathcal{K}_\Phi z_i$. Then $\dim(M_{W_i} \cap \mathcal{Y}_{z_j}) = 2$, and a nonzero $Y \in M_{W_i}$ satisfies $\{M|_{\ker Y} : M \in S\} \neq \operatorname{Hom}(\ker Y, \mathbb{C}^5)$ if and only if $Y z_j = 0$. On $\mathbb{P}(M_{W_i})$, the determinant of the map $\bar{\rho}$ in the proof vanishes to order nine along $\mathbb{P}(M_{W_i} \cap \mathcal{Y}_{z_j})$.*

Theorem 4.6. *There is no reduced range-criterion pair of type $(5, 6, 5)$.*

Sketch (proofs in Section S3.6). For a kernel product (z, w), Corollary 2.14 and Lemma 2.10 show that $W = \mathcal{K}_\Phi z$ has dimension three, and they determine M_W and N_W. Distinct kernel products have distinct inputs, and three of them would put their outputs on one rational normal curve, which contradicts the description of the spaces N_W. So there are at most two. An analysis of the matrices of rank at most two in $\mathbb{P}(S^\perp)$ and of the points $[Y]$ of $\mathbb{P}(M_U)$ at which the restriction map from S to $\operatorname{Hom}(\ker Y, \mathbb{C}^5)$ is not surjective then shows that kernel products exist, that there are exactly two, and that $s = 16$. For Theorem 4.6, the two kernel products give a 2×2 map Θ of vector bundles on a projective plane whose determinant is a constant multiple of λ^8 for a linear form λ. Two computations of the Euler characteristic of the cokernel of Θ give $8d - 20$, with d the degree of a line bundle on the line $\{\lambda = 0\}$, and 8, which is impossible.

4.3. **The types $(5, 6, 6)$ and $(6, 6, 5)$.** In the next theorem, $\mathcal{A}$ is the set of the pairs $([X], [Y]) \in \mathbb{P}(\mathcal{K}_\Psi) \times \mathbb{P}(\mathcal{K}_\Phi)$ with $XY = 0$, and $\mathcal{R}^{\mathrm{b}}$ is its projection to $\mathbb{P}(\mathcal{K}_\Phi)$, as defined in Section S3.6 of the supplement.

Theorem 4.7 (†)**.** *Let (Φ, Ψ) be a reduced range-criterion pair of type $(5, 6, 6)$. Then*

$$a + b - 18 \leq \dim \mathcal{R}_X \leq a - 6, \qquad \dim \mathcal{R}^{\mathrm{b}} \leq b - 5, \qquad \dim \mathcal{A} \geq a + 2b - 27,$$

$\dim \mathcal{R}^{\mathrm{b}} \leq b - 6$ if $b \geq 10$, $\dim \mathcal{R}^{\mathrm{b}} \geq a + b - 18$ if $a \leq 11$, and $s \geq \max(a + 6, b + 5)$. The mirror statements hold at type $(6, 6, 5)$.

Sketch (proof in Section S3.6). The lower bounds on $\dim \mathcal{R}_X$ and $\dim \mathcal{R}^{\mathrm{b}}$ come from the incidence count of Lemma 3.5, with a section of a bundle of rank 27 on a variety of dimension 27. If $\dim \mathcal{R}_X < a + b - 18$, a general linear subspace of $\mathbb{P}(\mathcal{K}_\Psi)$ of codimension $a + b - 18$ misses $\mathcal{R}_X$, the section has no zero, and the number $N_{2,3}(18 - b, b; 5, 6)$ would vanish, which it does not. The same argument with a general subspace of $\mathcal{K}_\Phi$ bounds $\dim \mathcal{R}^{\mathrm{b}}$ from below when $a \leq 11$. The upper bounds come from fibring $\mathcal{R}_X$ over the vectors w with $w^T X = 0$ and $\mathcal{R}^{\mathrm{b}}$ over the vectors z with $Yz = 0$, with fibre dimensions controlled by Lemma 3.1 and Corollary 2.14, and from an analysis of the equality cases.

Theorem 4.8. *Let (Φ, Ψ) be a reduced range-criterion pair of type $(5, 6, 6)$. Then $b = 9$ and $\operatorname{rank} C_\Phi^\Gamma = 9$. The mirror statements hold at type $(6, 6, 5)$.*

Theorem 4.9. *Let (Φ, Ψ) be a reduced range-criterion pair of type $(5, 6, 6)$ or $(6, 6, 5)$. Then $\ker C_{\Psi\circ\Phi}$ contains a product vector.*

Theorem 4.10. *There is no reduced range-criterion pair of type* $(5,6,6)$ *or* $(6,6,5)$.

Sketch (proofs in Sections S3.7, S3.9 and S3.14). If $b \geq 10$ at type $(5,6,6)$, the lemmas before the proof of Theorem 4.8 in the supplement show that $\mathbb{P}(S^\perp)$ contains no moving family, that is, no irreducible curve of matrices of rank at most two whose ranges are not all equal, so that some plane $E \subseteq \mathbb{C}^5$ is killed by a four-dimensional space of elements of $\mathcal{K}_\Phi$, which Lemma S3.36 excludes. Hence $b = 9$, and the same argument for the pair $(T \circ \Phi, \Psi \circ T)$ gives $\operatorname{rank} C_\Phi^\Gamma = 9$. With $b = 9$ and $10 \leq a \leq 12$, a kernel of $C_{\Psi\circ\Phi}$ without product vectors would give $a \leq 10$ by Theorem S3.72, and $a = 10$ is excluded by Theorem S3.81. On the other hand Theorem S3.149 shows that this kernel contains no product vector, which excludes the type. At type $(6,6,5)$ the pair $(\Psi^\#, \Phi^\#)$ of Lemma 2.4 has type $(5,6,6)$.

4.4. **The type** $(6,6,6)$**.**

Theorem 4.11 (†)**.** *Let* (Φ,Ψ) *be a reduced range-criterion pair of type* $(6,6,6)$. *Then* $XY = 0$ *for some nonzero* $X \in \mathcal{K}_\Psi$ *and* $Y \in \mathcal{K}_\Phi$.

Sketch (proof in Section S3.15). Suppose that $XY \neq 0$ for all nonzero $X \in \mathcal{K}_\Psi$ and $Y \in \mathcal{K}_\Phi$. Then $10 \leq a, b \leq 16$ by Theorem 3.3 and Proposition S3.155. On the zero scheme of the section $([Y], E) \mapsto Y|_E$ of a bundle of rank 12 on $\mathbb{P}(\mathcal{K}_\Phi) \times \operatorname{Gr}(2,6)$, cut by one or two general hyperplanes of the Plücker embedding, the multiplication map $X \otimes Y \mapsto XY$ then has a vector bundle as cokernel in a bundle of rank $s - 12$. The vanishing principle of Lemma S3.356 forces certain intersection numbers to vanish, while they are the nonzero entries of Table S3.3 and the number -100100.

Corollaries 3.4 and 4.3 and Theorems 3.6†, 4.6 and 4.14† leave three reduced types with $\max(m,k,n) = 6$: $(5,6,6)$, $(6,6,5)$ and $(6,6,6)$. Theorem 4.10 excludes $(5,6,6)$ and $(6,6,5)$, through an argument that involves both maps, as it must by Remark S3.355†. At middle dimension six the incidence method meets pairs of Kraus operators with $XY = 0$, and the excess intersections of Lemma 4.1 settle only the cases with an outer dimension four. At $(6,6,6)$, Theorem 4.11† shows that such pairs exist, and Corollary S3.349, Proposition S3.18(c) and Theorems S3.20† and S3.157† leave the pairs (a,b) with $10 \leq a, b \leq 12$. The other results on this type, in Subsections S3.15–S3.27a of the supplement, bound s from below and exclude further cells (a,b,s), where a cell also records whether $\ker C_{\Psi\circ\Phi}$ contains a product vector. Table 4.1 lists the eleven cells that remain open, and Remarks S3.292† and S3.347e† record what is known about them. By Lemma 2.4, the same holds with (a,b) replaced by $(\operatorname{rank} C_\Psi^\Gamma, \operatorname{rank} C_\Phi^\Gamma)$.

$\ker C_{\Psi\circ\Phi}$	open cells (a,b,s)
contains a product vector	$(10,10,23)$, $(10,10,24)$, $(10,10,25)$, $(11,10,24)$, $(11,10,25)$, $(11,11,24)$, $(11,11,25)$, $(12,10,25)$
contains no product vector	$(10,10,24)$, $(10,10,25)$, $(11,10,25)$

TABLE 4.1. The eleven cells (a,b,s) of a reduced range-criterion pair of type $(6,6,6)$ that no result of this paper excludes, each counted together with its mirror (b,a,s) (Remark S3.292).

4.5. **Middle dimension four.**

Lemma 4.12. *Let* (Φ,Ψ) *be a reduced PPT pair of type* $(m,4,n)$ *such that neither* $\mathcal{K}_\Phi$ *nor* $\mathcal{K}_\Psi$ *contains a nonzero matrix of rank at most two. Then* $S^\perp$ *contains no nonzero matrix of rank at most two.*

Theorem 4.13 (†)**.** *Let (Φ, Ψ) be a pair as in Lemma 4.12, with $m \geq 3$. On $V = \mathbb{P}(\mathcal{K}_\Phi) \times \mathrm{Gr}(2, m)$ let $\mathcal{T}$ be the tautological subbundle, x_1, x_2 the Chern roots of $\mathcal{T}^*$ and H the hyperplane class of $\mathbb{P}(\mathcal{K}_\Phi) = \mathbb{P}^{b-1}$, and put*

$$N_2(m, n, a, b) = \int_V c_{b+2m-13}(\varepsilon)\, c_8(F),$$

where $c(\varepsilon) = \big((1+x_1)(1+x_2)\big)^{-n}(1-H)^{-a}$ and $c_8(F) = \big((x_1+H)(x_2+H)\big)^4$. If $N_2(m, n, a, b) \neq 0$, then $s \geq 2(m+n) + a + b - 13$.

Theorem 4.14 (†)**.** *There is no reduced range-criterion pair of type $(6, 4, 6)$.*

Sketch (proofs in Section S3.29). By Corollary 2.14, $\dim \mathcal{K}_\Phi z \geq 3$ and $\dim \mathcal{K}_\Psi^T w \geq 3$ for all nonzero z and w, so $S^\perp$ contains no matrix zw^T of rank one. A matrix of rank two in $S^\perp$ is excluded by compressing Φ to the range of that matrix and using a basis of matrices of rank one of the space of restrictions. By Lemma 4.12 the restriction map from S to $\mathrm{Hom}(E, \mathbb{C}^n)$ is onto for every plane $E \subseteq \mathbb{C}^m$. On the zero scheme of the section $([Y], E) \mapsto Y|_E$ this gives a vector bundle of rank $s - 2n - a$ whose class is the restriction of ε, and Lemma S3.356 turns $N_2(m, n, a, b) \neq 0$ into the bound of Theorem 4.13. At $(6, 4, 6)$ we have $a = b = 8$ by Theorem 3.3, and $N_2(6, 6, 8, 8) = 1640$ gives $s \geq 27$, while $s \leq 25$ by Lemma 2.7.

Theorem 4.15 (†)**.** *There is no reduced range-criterion pair of type $(6, 4, 7)$ or $(7, 4, 6)$. Consequently there is no range-criterion pair of type $(6, 4, 7)$ or $(7, 4, 6)$.*

Sketch (proof in Section S3.31). By Corollary S3.360, a reduced pair of type $(6, 4, 7)$ has $a = 9$, $b = 8$ and $s = 30$, and by Lemmas 3.1 and 4.12 the nonzero elements of $\mathcal{K}_\Phi$, $\mathcal{K}_\Psi$ and $S^\perp$ have rank at least three. On the incidence of the elements of rank three of $\mathcal{K}_\Phi$ with their kernels in $\mathbb{P}(\mathcal{K}_\Phi) \times \mathrm{Gr}(3, 6)$, the proof shows that every such kernel is the range of a nonzero element of $S^\perp$, and that the projection to $\mathrm{Gr}(3, 6)$ is generically finite, because an integral of Chern classes equals 20. A study of the pencils spanned by an element Y of rank three and an element Y' whose restriction to $\ker Y$ has rank one then gives the contradiction. The type $(7, 4, 6)$ is the mirror.

Theorem 4.16 (†)**.** *There is no reduced range-criterion pair of type $(6, 4, 8)$, and none of type $(6, 4, 9)$ with $(a, s) \in \{(11, 39), (12, 40)\}$. The same holds with (a, b) replaced by $(\operatorname{rank} C_\Psi^\Gamma, \operatorname{rank} C_\Phi^\Gamma)$, at the types $(8, 4, 6)$ and $(9, 4, 6)$ with a and b exchanged, and without the assumption that the pair is reduced.*

Theorem 4.17 (†)**.** *There is no reduced range-criterion pair of type $(6, 4, 9)$ with $(a, s) = (11, 40)$.*

Theorem 4.18 (†)**.** *There is no range-criterion pair of type $(6, 4, 9)$ or $(9, 4, 6)$. Hence there is no range-criterion pair of type $(6, 4, n)$ or $(n, 4, 6)$ with $n \leq 9$.*

Sketch (proofs in Section S3.40). By Theorem S3.699, a reduced pair of type $(6, 4, 8)$ or $(6, 4, 9)$ has $b = 8$ and $s \geq a + 3n + 1$, and at $(6, 4, 9)$ the pair (a, s) is $(11, 39)$, $(11, 40)$ or $(12, 40)$. The proof of Theorem 4.16 studies the incidence used for $(6, 4, 7)$ through the map $([Y], E) \mapsto \operatorname{ran} Y$ to $\mathbb{P}^{3*}$, together with the 3-planes killed by a two-dimensional space of elements of $\mathcal{K}_\Phi$, a component of the degeneracy locus, a second divisor of rank-one images and rational cubic curves in $\mathbb{P}^{3*}$, and it ends with an analysis of the elements of $\mathcal{K}_\Phi$ whose ranges are the points of such a curve. For Theorem 4.17, at a general point of a closed set D_2 defined in the supplement, a compression of Φ gives an entangled PPT state of rank four on $\mathbb{C}^3 \otimes \mathbb{C}^3$. By the description of such states by Chen and Djoković [3], its kernel contains exactly six product vectors, and following them as the point moves, an intersection computation on the image of D_2

gives $6d = 81$ for an integer d, which is impossible. Theorem 4.18 combines these results with the reduction of Lemma 2.5 and Theorem S3.699.

4.6. **Both outer dimensions four.**

Lemma 4.19. *Let $\rho \succeq 0$ on $\mathbb{C}^p \otimes \mathbb{C}^q$ with $\rho^\Gamma \succeq 0$, and let $e \otimes f$ be a product vector in $\ker \rho$. Then for all $e' \in \mathbb{C}^p$ and $f' \in \mathbb{C}^q$ the vector $e \otimes f' + e' \otimes f$ lies in $\ker \rho$ if and only if $e \otimes f'$ and $e' \otimes f$ both lie in $\ker \rho$.*

Theorem 4.20. *Let $\Phi\colon \mathrm{M}_4 \to \mathrm{M}_k$ and $\Psi\colon \mathrm{M}_k \to \mathrm{M}_4$ be PPT maps with $\Psi \circ \Phi \neq 0$. Then the range of $C_{\Psi\circ\Phi}$ contains a nonzero product vector. So there is no range-criterion pair of type $(4, k, 4)$, whatever the middle dimension k.*

Sketch (proofs in Section S3.30). Lemma 4.19 follows from the partial-transpose identity of the proof of Lemma 2.10: the cross terms of $\langle t, \rho t\rangle$ vanish for $t = e \otimes f' + e' \otimes f$. For Theorem 4.20, the reduced pair has outer dimensions four, $k \geq 6$, $a, b \geq k + 2$ and $s \leq 9$. The proof studies the product vectors in $\ker C_{\Psi\circ\Phi}$ and, for general z, the PPT operators ρ_U obtained by compressing Ψ to the range U of an element of rank three of $\mathcal{K}_\Phi$ that kills z. Lemma 4.19 shows that $\mathbb{P}(\ker \rho_U)$, a linear space of dimension six, does not meet the Segre variety $\mathbb{P}^2 \times \mathbb{P}^3$ in exactly one point, and this leads to one output vector y_* for all general points. The elements $M \in S$ with $y_*^T M = 0$ then have rank two and a common range, so Lemma 2.7 gives a matrix of rank one in S, a contradiction.

4.7. **Dimension seven, and outer dimension four.** Table 4.2 lists the reduced types (m, k, n) with $\max(m, k, n) = 7$ and $m \leq n$ that are not excluded by Theorem 3.3, and the possible Choi ranks. The types with $m > n$ are their mirrors. The same ranges hold with (a, b) replaced by $(\operatorname{rank} C_\Psi^\Gamma, \operatorname{rank} C_\Phi^\Gamma)$. In the rows that cite only Corollary S3.409, the ranges follow from the bounds stated after Remark S3.406, and in the other rows they follow from these bounds and the results cited in the row or in the notes below the table. These bounds also give $b = 9$ if $a \geq 13$ at $(5, 6, 7)$ and $b \leq 12$ if $a \geq 13$ at $(6, 6, 7)$.

Theorem 4.21 (†)**.** *There is no reduced range-criterion pair of type $(4, 6, 7)$, $(7, 6, 4)$, $(4, 6, 8)$ or $(8, 6, 4)$. A reduced range-criterion pair of type $(4, 6, 9)$ has $a = \operatorname{rank} C_\Psi^\Gamma = 12$ and $s = 24$, and one of type $(9, 6, 4)$ has $b = \operatorname{rank} C_\Phi^\Gamma = 12$ and $s = 24$. Consequently a reduced range-criterion pair of type $(4, 6, n)$ has $n \geq 9$, and there is no range-criterion pair of type $(4, 6, n)$ or $(n, 6, 4)$ with $n \leq 8$.*

Theorem 4.22 (†)**.** *There is no reduced range-criterion pair of type $(4, 6, 9)$ or $(9, 6, 4)$. Consequently there is no range-criterion pair of type $(4, 6, n)$ or $(n, 6, 4)$ with $n \leq 9$.*

Sketch (proofs in Section S3.31). At type $(4, 6, n)$ with $7 \leq n \leq 9$, Proposition S3.372, the rank windows with Lemma 3.2(b), and Lemma 2.7 leave ten triples (n, a, s), with $b = 8$. On the incidence $\mathcal{T} \subseteq \mathbb{P}^7 \times \mathbb{P}^3$ of the pairs $([Y], [z])$ with $Yz = 0$, the multiplication map $X \otimes Y \mapsto XY$ has a degeneracy locus whose Thom–Porteous class is computed in exact arithmetic, and the proof leaves only $(a, s) = (12, 24)$ at $n = 9$. For Theorem 4.22, a bad member U with $n_U = \dim \mathcal{K}_\Psi U = 8$, given by the results that precede its proof in the supplement, carries on $\mathbb{P}(U) \cong \mathbb{P}^2$ a map of bundles of ranks nine and eight whose locus of rank at most seven has the class $36h^2 \neq 0$. So some $x \in U \setminus 0$ has $w(x) = \dim \mathcal{K}_\Psi x \leq 7$, while the results that precede that proof exclude every value $w(x) \leq 7$.

Theorem 4.23 (†)**.** *Let (Φ, Ψ) be a reduced range-criterion pair of type $(4, 7, 7)$. Then $s \in \{17, 18\}$ and $a \leq 11$. If $b = 10$, then $(a, s) = (10, 18)$, and if $b = 9$, then $s = 18$ or $(a, s) = (10, 17)$. The same holds with (a, b) replaced by $(\operatorname{rank} C_\Psi^\Gamma, \operatorname{rank} C_\Phi^\Gamma)$, and for the mirror type $(7, 7, 4)$ with a and b exchanged.*

reduced type	$a = \operatorname{rank} C_\Psi$	$b = \operatorname{rank} C_\Phi$	further restrictions
$(6,4,7)$	–	–	excluded (Thm. 4.15†)
$(7,4,7)$	9	9	$s = 36$ (Thms. S3.400† and S3.405†)
$(5,5,7)$	–	–	excluded (Thm. S3.364†)
$(6,5,7)$	10	9	Rem. S3.365†, $s = 30$ (Prop. S3.368†)
$(7,5,7)$	10–11	10–11	$(a,b) \neq (11,11)$ (Rem. S3.366†), $34 \leq s \leq 36$ if $(a,b) \neq (10,10)$, and $33 \leq s \leq 36$ if $(a,b) = (10,10)$ (Prop. S3.369†)
$(4,6,7)$	–	–	excluded (Thm. 4.21†)
$(5,6,7)$	10–15	9–11	$b = 9$ if $a \geq 13$, $s \geq a + 8$ if $b = 9$, and $s \geq a + 9$ if $b \geq 10$ (Thm. S3.476†)
$(6,6,7)$	10–15	10–14	$b \leq 12$ if $a \geq 13$, $(a,b) \neq (12,14)$; $XY = 0$ for some nonzero $X \in \mathcal{K}_\Psi$, $Y \in \mathcal{K}_\Phi$ if $a + b \geq 23$; $s \geq a + b + 5$ if $\ker C_{\Psi\circ\Phi}$ contains no product vector, which holds if $b \geq 13$, and $s \geq a + b + 3$ if $b \in \{11,12\}$[a]; $s \geq b + 6$ (Prop. S3.412)
$(7,6,7)$	10–16	10–16	$a \leq 15$ if $b \geq 13$, $b \leq 15$ if $a \geq 13$, $\{a,b\} = \{10,16\}$ if $\max(a,b) = 16$, $s \geq \max(a,b) + 7$[b]
$(4,7,4)$	–	–	excluded (Thm. 4.20)
$(4,7,5)$	–	–	excluded (Thm. S3.397†)
$(4,7,6)$	–	–	excluded (Thm. S3.398†)
$(4,7,7)$	10–11	9–10	$s \in \{17,18\}$; $(a,s) = (10,18)$ if $b = 10$; $s = 18$ or $(a,s) = (10,17)$ if $b = 9$ (Thm. 4.23†)
$(5,7,5)$	10–11	10–11	$s = 16$ if $(a,b) \neq (10,10)$, and $15 \leq s \leq 16$ if $(a,b) = (10,10)$[c]
$(5,7,6)$	10–15	10–13	$a \leq 14$ if $b \geq 11$, $a = 10$ if $b = 13$; $s \geq \max(a+2,14)$, $s \geq a + 3$ if $a \geq 15$, $s \geq a + 6$ if $b \geq 12$, $s \geq a + 7$ if $b = 13$[d]
$(5,7,7)$	10–18	10–14	$a \leq 17$ if $b \geq 11$, $a \leq 15$ if $b \geq 13$; $s \geq \max(a+2,15)$, $s \geq a + 3$ if $a \geq 18$, $s \geq a + 7$ if $b = 12$, $s \geq a + 8$ if $b \in \{13,14\}$[e]
$(6,7,6)$	10–19	10–19	$a \leq 15$ if $b \geq 13$, $b \leq 15$ if $a \geq 13$ (Cor. S3.409)
$(6,7,7)$	10–23	10–19	$a \leq 18$ if $b \geq 13$, $b \leq 15$ if $a \geq 17$ (Cor. S3.409)
$(7,7,7)$	10–23	10–23	$a \leq 18$ if $b \geq 17$, $b \leq 18$ if $a \geq 17$ (Cor. S3.409)

Sources of the marked restrictions. [a] $(6,6,7)$: Thms. S3.477†, S3.480†, S3.484†, Cors. S3.479†, S3.485†, S3.487†. [b] $(7,6,7)$: Bounds after Rem. S3.406†, Thms. S3.410† and S3.411†, Prop. S3.412. [c] $(5,7,5)$: Thms. S3.415, S3.421, S3.423, S3.428†, S3.429, S3.437†, S3.444†, S3.452†, S3.454† and S3.457†. [d] $(5,7,6)$: Thms. S3.415, S3.421, S3.428†, S3.429, S3.432†, S3.463†, Prop. S3.422, Cor. S3.409. [e] $(5,7,7)$: Thms. S3.415, S3.421, S3.428†, S3.429, S3.464, S3.467†, S3.470†, S3.475†, Prop. S3.422, Cor. S3.409.

TABLE 4.2. The reduced types (m,k,n) with $\max(m,k,n) = 7$ and $m \leq n$ that Theorem 3.3 allows, and the possible Choi ranks of a range-criterion pair.

Sketch (proof in Section S3.31). At type $(4,7,7)$, the rank windows and Lemma S3.363 give $b \in \{9,10\}$ and $a \geq 10$, and $s \leq 18$ by Lemma 2.7. The proof uses the incidence of the proofs of Theorems S3.397 and S3.398 and proceeds in twenty steps. It lists the possible triples (a,s,d), where d is an integer attached to the spaces N_U of the members, and it excludes the triples with $d \geq 1$ by Thom–Porteous counts, an analysis of annihilators of rank two and of fixed subspaces, and, for $b = 9$, a small resolution of the incidence variety. The triples that remain have $d = 0$, $s \in \{17,18\}$ and $a \leq 11$.

4.8. **Unextendible product sets.** A set of product vectors is an *unextendible product set* if no product vector is orthogonal to all of them. The unextendible product bases of [1] are such sets of mutually orthogonal vectors. The certificate of [24] is of this form: the kernel of the composite Choi matrix is spanned by twenty product vectors that form an unextendible product set.

Proposition 4.24. *Let* $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ *and* $\Psi\colon \mathrm{M}_k \to \mathrm{M}_n$ *be PPT with* $Z = C_{\Psi\circ\Phi} \neq 0$.

(a) If $m = n = 4$, $k \leq 7$ *and* $\operatorname{ran} Z$ *is product-free, then* $\ker Z$ *contains no product vector.*
(b) If $m = n = 4$, $k \leq 9$ *and* $\operatorname{ran} Z$ *is product-free, then any two product vectors in* $\ker Z$ *are proportional.*
(c) The kernel of Z *contains no unextendible product set if* $(m, n) = (4, 4)$, *if* $(m, n) \in \{(4, 5), (5, 4)\}$ *and* $k \leq 9$, *or if* $(m, k, n) \in \{(4, 6, 6), (6, 6, 4), (4, 7, 6), (6, 7, 4)\}$.

Sketch (proof in Section S3.41). If $\ker Z$ contains an unextendible product set, then $\operatorname{ran} Z$ is product-free, so the pair is a range-criterion pair and may be reduced, and an outer dimension four stays four. At a kernel point $([x], [y])$, that is, with $\bar{x} \otimes y \in \ker Z$, Lemma 2.1(b) gives $r(x) + t(y) \leq k$ for $r(x) = \dim \mathcal{K}_\Phi x$ and $t(y) = \operatorname{rank} \Psi^*(P_y)$, while elements of $\mathcal{K}_\Phi$ that kill several inputs have rank at most two and are excluded by Lemma 3.1. Counting at one or two kernel points gives parts (a) and (b), and counting over suitable points of an unextendible product set gives part (c). By Theorem 4.20 the hypotheses of parts (a) and (b) never hold, and the types of part (c) are excluded for every certificate by Theorems 1.1, 4.20, S3.397, S3.398, S3.507 and S3.561.

4.9. **Proof of Theorem 1.1.** The results stated above complete the proof of the main theorem.

Proof of Theorem 1.1. By Lemma 2.5 we may pass to the reduced pair, whose dimensions are at most m, k and n. If $k \leq 5$, Corollary 3.4 leaves the reduced types $(5, 5, 5)$, $(5, 5, 6)$, $(6, 5, 5)$, $(6, 5, 6)$ and $(6, 4, 6)$, which Theorems 3.6 and 4.14 exclude. If $\min(m, n) \leq 4$, the reduced pair has an outer dimension at most four, which Corollaries 3.4(a) and 4.3 exclude. If $\min(m, n) \leq 5$, the reduced pair has middle dimension at most five, an outer dimension at most four, or one of the types $(5, 6, 5)$, $(5, 6, 6)$ and $(6, 6, 5)$, which Theorems 4.6 and 4.10 exclude. □

5. The ten-dimensional example and its family

The pair of [24] is the exterior-power lift of a map $L\colon \mathrm{M}_4 \to \mathrm{M}_4$ of a special kind. This section sets up these lifts and then states how far the construction can be pushed below dimension ten; the proofs of the further results are in Section S4 of the supplement. Here m and n are the dimensions of the base map L, not those of the lifted pair.

5.1. **Exterior-power lifts.** Let $M_0, \ldots, M_{m-1}$ be real $r \times n$ matrices such that $M_i^T M_j$ is symmetric for all i, j, and define $L\colon \mathrm{M}_m \to \mathrm{M}_n$ by $L(E_{ij}) = M_i^T M_j$. We call L a map of *pencil type* and $M(x) = \sum_i x_i M_i$ its pencil. The matrix C_L is the Gram matrix of the columns of the real $r \times mn$ matrix $(M_0, \ldots, M_{m-1})$, so $C_L \succeq 0$, and $C_L^\Gamma = C_L$ because every block $M_i^T M_j$ is symmetric. Hence L is PPT and $L \circ T = L$. The Kraus operators $Q_1, \ldots, Q_r$ of L are given by $Q_\rho x = (\text{row } \rho \text{ of } M(x))^T$, and

$$L(P_x) = M(x)^T\, \overline{M(x)}, \qquad \operatorname{rank} L(P_x) = \operatorname{rank} M(x).$$

A *rank-loss point* of L is a point $[x] \in \mathbb{P}^{m-1}$ with $\operatorname{rank} M(x) < n$, and a kernel vector is a vector $k \neq 0$ with $M(x)k = 0$. By (2.1), $(x \otimes k)^\dagger C_L (x \otimes k) = \|M(x)k\|^2$, so $x \otimes k \in \ker C_L$ for every such pair. The map of [24] is of this type with $m = n = 4$ and $r = 6$ [24, Section 7].

For $1 \le p \le n-1$ let $U_p\colon \operatorname{Sym}^p \mathbb{C}^m \to (\mathbb{C}^m)^{\otimes p}$ and $V_p\colon \Lambda^p\mathbb{C}^n \to (\mathbb{C}^n)^{\otimes p}$ be the isometric inclusions, and put $\hat{x}_p = U_p^\dagger x^{\otimes p}$, so that the coordinates of $\hat{x}_p$ are the monomials of degree p in x up to nonzero factors. Define

$$S_p\colon \mathrm{M}(\operatorname{Sym}^p \mathbb{C}^m) \to \mathrm{M}(\Lambda^p\mathbb{C}^n), \qquad S_p(A) = V_p^\dagger L^{\otimes p}(U_p A U_p^\dagger)V_p,$$

let $K_p\colon \Lambda^{n-p}\mathbb{C}^n \to \Lambda^p\mathbb{C}^n$ be the real signed permutation that sends e_I to $\pm e_{I^c}$, with the sign of the permutation (I, I^c), and put $R_p(A) = K_p S_{n-p}(A)^T K_p^\dagger$. The *lift* of L with parameter p is the pair

$$\Phi = S_p \circ T\colon \mathrm{M}(\operatorname{Sym}^p \mathbb{C}^m) \to \mathrm{M}(\Lambda^p\mathbb{C}^n), \qquad \Psi = R_p^*\colon \mathrm{M}(\Lambda^p\mathbb{C}^n) \to \mathrm{M}(\operatorname{Sym}^{n-p} \mathbb{C}^m),$$

of type $(D_1, \binom{n}{p}, D_2)$ with $D_1 = \binom{m+p-1}{p}$ and $D_2 = \binom{m+n-p-1}{n-p}$. For $(m,n,p) = (4,4,2)$ it is the pair of Remark 2.3, of type $(10, 6, 10)$.

Proposition 5.1. *Let (Φ, Ψ) be the lift of a map L of pencil type with parameter p, and $Z = C_{\Psi\circ\Phi}$.*

(a) Φ and Ψ are PPT, and $Z \neq 0$ if $L(P_y) \succ 0$ for some y.
(b) $\hat{x}_p \otimes \hat{x}_{n-p} \in \ker Z$ for every rank-loss point $[x]$.
(c) If $u \otimes v \in \operatorname{ran} Z$ is a product vector, then the nonzero forms $f_u(x) = u^\dagger \hat{x}_p$ and $f_v(x) = v^\dagger \hat{x}_{n-p}$, of degrees p and $n-p$, satisfy $f_u f_v = 0$ at every rank-loss point.

Proof. For $(m,n,p) = (4,4,2)$ this is [24, Proposition 6.1 and its proof], and the proof carries over. The map $L^{\otimes p}$ is PPT, so S_p, $S_p \circ T$ and R_p are PPT. For $B = L(P_x) \succeq 0$ one has $S_p(P_{\hat{x}_p}) = V_p^\dagger B^{\otimes p} V_p = \Lambda^p B$, with range $\Lambda^p W_x$, where $W_x = \operatorname{ran} L(P_x)$, and $R_p(P_{\hat{x}_{n-p}})$ has range $K_p \Lambda^{n-p}\overline{W_x}$. The pairing $\langle K_p w, u\rangle$ is, up to sign, the coefficient of $e_1 \wedge \cdots \wedge e_n$ in $\bar{w} \wedge u$, which vanishes for $w \in \Lambda^{n-p}\overline{W}$ and $u \in \Lambda^p W$ when $\dim W < n$. By Lemma 2.1(b), $(\hat{x}_p \otimes \hat{x}_{n-p})^\dagger Z (\hat{x}_p \otimes \hat{x}_{n-p}) = \operatorname{Tr}\big(R_p(P_{\hat{x}_{n-p}}) S_p(P_{\hat{x}_p})\big)$, which gives (b). At $x = y$ both ranges are the whole spaces, which gives $Z \neq 0$. Part (c) follows from (b), since $\langle u \otimes v, \hat{x}_p \otimes \hat{x}_{n-p}\rangle = f_u(x) f_v(x)$, and $f_u \neq 0$ because the coordinates of $\hat{x}_p$ are linearly independent monomials. □

In [24] the twenty rank-loss points are such that every nonzero quadric vanishes at no more than nine of them, and Proposition 5.1(c) with $9+9 < 20$ gives the range criterion. The following elementary converse explains why the number of rank-loss points matters.

Lemma 5.2. *Let $Z \succeq 0$ on $\mathbb{C}^{D_1} \otimes \mathbb{C}^{D_2}$, and suppose that $\ker Z$ is spanned by N product vectors. If $N \le D_1 + D_2 - 2$, then $\operatorname{ran} Z$ contains a product vector.*

Proof. Let the product vectors be $a_j \otimes b_j$. Choose $u \neq 0$ orthogonal to $a_1, \dots, a_t$ with $t = \min(N, D_1 - 1)$, and $v \neq 0$ orthogonal to the remaining b_j, of which there are at most $D_2 - 1$. Then $u \otimes v \perp \ker Z$. □

5.2. **Further results on lifts.** The remaining results on lifts are proved in Section S4 of the supplement. We state the main ones here.

Proposition 5.3. *Let L be of pencil type with finitely many rank-loss points, and suppose that for its lift with parameter p the kernel of Z is spanned by the vectors $\hat{x}_p \otimes \hat{x}_{n-p}$ of Proposition 5.1(b). If the lift is a range-criterion pair, then $\binom{m+n-2}{m-1} \ge D_1 + D_2 - 1$. This forces $\max(D_1, D_2) \ge 10$, with equality exactly for $(m,n,p) \in \{(4,4,2), (3,5,2), (3,5,3), (3,6,3)\}$. In particular equal outer dimensions are at least ten. Among the members with $\max(D_1, D_2) \le 9$, those with $m = 2$ and the member $(m,n,p) = (3,4,2)$ satisfy $\binom{m+n-2}{m-1} = D_1 + D_2 - 2$.*

Sketch (proof in Section S4.2). The first claim combines Proposition S4.3 with Lemma 5.2, and the remaining claims are a finite check of the inequality.

A map $L\colon \mathrm{M}_m \to \mathrm{M}_n$ of pencil type with $r = m+n-2$ is of *Eagon–Northcott type* if its rank-loss scheme, cut out by the $n \times n$ minors of $M(x)$, is reduced and zero-dimensional; it then consists of $\binom{m+n-2}{m-1}$ points, all of them real (Proposition S4.4). Schur lifts are defined in Section S4.3 of the supplement. A Schur lift pairs Schur functors $\mathbb{S}_\lambda$ and $\mathbb{S}_\mu$, for partitions λ of p and μ of q with at most n rows, through an invariant pairing β. It uses a set J of rank-loss points and real subspaces $A \subseteq \mathrm{Sym}^p \mathbb{C}^m$ and $B \subseteq \mathrm{Sym}^q \mathbb{C}^m$ that contain the vectors $\hat{x}_p$ and $\hat{x}_q$ for $x \in J$, with real isometries ι_A and ι_B onto them, and it has type $(\dim A, k, \dim B)$ with $k = \dim \mathbb{S}_\lambda(\mathbb{C}^n)$.

Theorem 5.4. *Let (Φ, Ψ) be a Schur lift of a map $L\colon \mathrm{M}_m \to \mathrm{M}_n$ of Eagon–Northcott type, with the notation above, and $Z = C_{\Psi\circ\Phi}$.*

(a) There is an integer $c \geq 1$ with $\lambda_i + \mu_{n+1-i} = c$ for $1 \leq i \leq n$, that is, μ is the complement of λ in a rectangle with n rows and c columns. For every $x \in J$ the vector $\hat{x}_p \otimes \hat{x}_q$, written in the bases given by ι_A and ι_B, lies in $\ker Z$.

(b) Suppose that $Z \neq 0$ and that the vectors $\hat{x}_p \otimes \hat{x}_q$, $x \in J$, form an unextendible product set. Then $c = 1$, $\lambda = (1^p)$ and $\mu = (1^{n-p})$ with $2 \leq p \leq n-2$, so that $\mathbb{S}_\lambda = \Lambda^p$ and $\mathbb{S}_\mu = \Lambda^{n-p}$, and $\max(\dim A, k, \dim B) \geq 10$. If $k \leq 9$, then $n = 4$, $p = 2$, $k = 6$ and $\dim A, \dim B \geq 10$.

Sketch (proof in Section S4.3). An invariant pairing β exists exactly when λ and μ are complementary in a rectangle, by the representation theory of SL_n, and then β vanishes on $\mathbb{S}_\lambda(H) \times \mathbb{S}_\mu(H)$ for every hyperplane $H \subseteq \mathbb{C}^n$, which puts the vectors $\hat{x}_p \otimes \hat{x}_q$ into $\ker Z$. Part (b) then follows from the count of the points of J (Lemma S4.9) and Macaulay's theorem [23].

We write v_d for the Veronese map of degree d, as in Section S4.3 of the supplement. Points $x_1, \dots, x_N$ spanning $\mathbb{C}^D$ and points $\rho_1, \dots, \rho_N$ spanning $\mathbb{C}^{N-D}$ are *Gale dual* if the row space of the $(N-D) \times N$ matrix $(\rho_1, \dots, \rho_N)$ is the space of linear relations $\{c : \sum_j c_j x_j = 0\}$ [9]. The six bilinear forms in Theorem 5.6 are $(x, k) \mapsto (M(x)k)_\rho$, $1 \leq \rho \leq 6$.

Proposition 5.5. *Let $a, b \geq 2$, and let $\{(z_j, n_j)\}_{j=1}^N \subseteq \mathbb{P}^{a-1} \times \mathbb{P}^{b-1}$ be a reduced zero-dimensional complete intersection of $a+b-2$ divisors of bidegree $(1,1)$, so that $N = \binom{a+b-2}{a-1}$. Then, up to a common scalar, the vectors $v_{b-2}(z_j) \otimes v_{a-2}(n_j)$ satisfy exactly one linear relation $\sum_j d_j\, v_{b-2}(z_j) \otimes v_{a-2}(n_j) = 0$, and all d_j are nonzero. If the families $\{v_{b-2}(z_j)\}$ and $\{v_{a-2}(n_j)\}$ span $\mathrm{Sym}^{b-2} \mathbb{C}^a$ and $\mathrm{Sym}^{a-2} \mathbb{C}^b$, then they are Gale dual up to the scalars d_j.*

Theorem 5.6. *Let $L\colon \mathrm{M}_4 \to \mathrm{M}_4$ be of pencil type with Kraus rank six whose rank-loss scheme is reduced and zero-dimensional, with rank-loss points $x_1, \dots, x_{20}$, kernel vectors $k_1, \dots, k_{20}$ and $W_j = \operatorname{ran} L(P_{x_j})$, and let (Φ, Ψ) be its lift with $p = 2$. Write $\hat{x}_j$ and $\hat{k}_j$ for the points of $\mathrm{Sym}^2 \mathbb{C}^4$ defined by x_j and k_j.*

(i) There are $d_j \neq 0$, namely the reciprocal Jacobians of the six bilinear forms at the points (x_j, k_j), such that $\sum_j d_j\, \hat{x}_j \otimes \hat{k}_j = 0$, and this is the only linear relation among the $\hat{x}_j \otimes \hat{k}_j$. The configurations $\{\hat{x}_j\}$ and $\{d_j \hat{k}_j\}$ in $\mathrm{Sym}^2 \mathbb{C}^4$ are Gale dual.

(ii) For all subspaces $W_1', \dots, W_{20}' \subseteq \mathbb{C}^6$, with bilinear orthogonal complements $W_j'^{\perp}$,

$$\dim\{B : B\hat{x}_j \in W_j'^{\perp}\ \forall j\} = 60 - \sum_j \dim W_j' + \dim\{G : G\hat{k}_j \in W_j'\ \forall j\},$$

where B and G run over $\mathrm{Hom}(\mathrm{Sym}^2 \mathbb{C}^4, \mathbb{C}^6)$.

(iii) $\{G \in \mathrm{Hom}(\mathrm{Sym}^2 \mathbb{C}^4, \Lambda^2\mathbb{C}^4) : G\hat{k}_j \in k_j \wedge \mathbb{C}^4 \ \forall j\} = \{\hat{k} \mapsto k \wedge Ak : A \in \mathrm{M}_4\}$*, a space of dimension* 15*. Consequently* $\mathcal{B} = \{B : B\hat{x}_j \in \Lambda^2 W_j \ \forall j\}$ *equals* $\mathcal{K}_{S_2}$*, and every completely positive map* $\Phi' \colon \mathrm{M}_{10} \to \mathrm{M}_6$ *with* $\operatorname{ran}\Phi'(P_{\hat{x}_j}) \subseteq \Lambda^2 W_j$ *for all* j *satisfies* $\mathcal{K}_{\Phi'} \subseteq \mathcal{K}_{S_2}$*.*

(iv) $\Phi = S_2 = T \circ S_2$*. The kernel of* C_Φ *is spanned by the twenty blocks* $\hat{x}_j \otimes (\Lambda^2 W_j)^\perp$*, and the kernel of* C_Ψ *is spanned by the twenty blocks* $\Lambda^2 W_j \otimes \hat{x}_j$*.*

The map of [24] *satisfies these hypotheses.*

Sketch (proofs in Section S4.4). For Proposition 5.5, the Koszul resolution of the points, twisted by $K_X \otimes \det \mathcal{F} = \mathcal{O}(b-2, a-2)$ on $X = \mathbb{P}^{a-1} \times \mathbb{P}^{b-1}$, where $\mathcal{F} = \mathcal{O}(1,1)^{a+b-2}$, shows that the points impose exactly $N-1$ conditions on the sections of $\mathcal{O}(b-2, a-2)$. The relation is the global residue relation, whose coefficients are nonzero because the points are reduced. For Theorem 5.6, the pairs (x_j, k_j) form a reduced complete intersection of six bilinear forms in $\mathbb{P}^3 \times \mathbb{P}^3$, so (i) is Proposition 5.5 with $a = b = 4$, and (ii) is Lemma S4.12. Part (iii) uses that four cubic forms that vanish at the twenty kernel vectors vanish identically, and the exactness of a Koszul complex.

Proposition 5.7 (†)**.** *Let* $L \colon \mathrm{M}_3 \to \mathrm{M}_5$ *be the map of pencil type with* $r = 6$ *and*

$$M_0 = \begin{pmatrix} I_5 \\ 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} \mathrm{diag}(\alpha) \\ \beta_1^T \end{pmatrix}, \quad M_2 = \begin{pmatrix} A_2 \\ \beta_2^T \end{pmatrix},$$

where $\alpha = (-2, -1, \frac{1}{2}, 1, \frac{5}{2})$*,* $\beta_1 = (1, -1, 2, 0, 1)$*,* $\beta_2 = (0, 1, 1, -1, 2)$*,* $\mathrm{diag}(A_2) = (1, 0, -1, 2, -\frac{1}{2})$ *and* $(A_2)_{kl} = (\beta_{2,k}\beta_{1,l} - \beta_{1,k}\beta_{2,l})/(\alpha_k - \alpha_l)$ *for* $k \neq l$*. Its lift with parameter* $p = 2$ *is a range-criterion pair of type* $(6, 10, 10)$*.*

Sketch (proof in Section S4.5). The map is of pencil type by an exact rational check, and a Gröbner basis and Sturm's theorem give fifteen real rank-loss points. Exact determinants, evaluated modulo a prime, show that a nonzero conic vanishes at no more than five of them and a nonzero cubic at no more than nine. Since $5 + 9 < 15$, Proposition 5.1(c) shows that $\operatorname{ran} Z$ is product-free. A second certificate uses the maximal minors of a matrix built from a basis of $\ker Z$.

In the next theorem, P_0 and the pencils of this form are those defined in Section S4.6 of the supplement, whose lifts have outer dimensions six.

Theorem 5.8 (†)**.** *Let* (Φ, Ψ) *be the lift of the map defined by* P_0*, with rank-loss points* $x_1, \ldots, x_{10}$*, and* $Z = C_{\Psi\circ\Phi}$*.*

(a) $\operatorname{rank} Z = 26$ *and* $\ker Z = \operatorname{span}\{\hat{x}_j \otimes \hat{x}_j : 1 \le j \le 10\}$*.*

(b) If $\Psi' \colon \mathrm{M}_6 \to \mathrm{M}_6$ *is PPT and* $\hat{x}_j \otimes \hat{x}_j \in \ker C_{\Psi'\circ\Phi}$ *for all* j*, then* $\Psi' = c\Psi$ *for some* $c \ge 0$*.*

(c) If $\Phi' \colon \mathrm{M}_6 \to \mathrm{M}_6$ *is PPT and* $\hat{x}_j \otimes \hat{x}_j \in \ker C_{\Psi\circ\Phi'}$ *for all* j*, then* $\Phi' = c\Phi$ *for some* $c \ge 0$*.*

The same holds for every pencil in a nonempty Zariski open subset of the real pencils of this form.

Sketch (proof in Section S4.6). The proof rests on three facts verified by exact computation: $\operatorname{rank} Z = 26$ and the ten vectors $\hat{x}_j \otimes \hat{x}_j$ are independent; the maps G with $G\hat{x}_j \in \Lambda^2 W_j$ for all j form the Kraus space $\mathcal{K}_{S_2}$; and the only Hermitian operators H with $\operatorname{ran} H$ and $\operatorname{ran} H^\Gamma$ in $\mathrm{vec}(\mathcal{K}_{S_2})$ are the real multiples of C_{S_2}. These give (a)–(c) for P_0, and the three facts are rank conditions that persist on a nonempty Zariski open set of pencils.

Corollary 5.9. *Let* $L \colon \mathrm{M}_4 \to \mathrm{M}_4$ *be of Eagon–Northcott type and not entanglement breaking. Then no four of its rank-loss points lie on a line, no seven lie on a conic, and no nine lie on a plane. The same holds for the kernel vectors* $k_1, \ldots, k_{20}$*.*

Proposition 5.10. *Let $L\colon \mathrm{M}_4 \to \mathrm{M}_4$ be of Eagon–Northcott type and not entanglement breaking, with rank-loss points $x_1, \dots, x_{20}$. Let $A \subseteq \mathrm{Sym}^2 \mathbb{C}^4$ be a subspace of dimension D with $3 \le D \le 9$, and let N_A be the number of indices j with $\hat{x}_j \in A$. Then $N_A \le 2D-4$ if $D \ge 4$, and $N_A \le 3$ if $D = 3$.*

Sketch (proofs in Section S4.8). Four collinear rank-loss points are excluded by Lemma S4.27, nine on a plane by Lemmas S4.28 and S4.26, and seven on a conic by the bound for smooth conics in Corollary S4.24 and the first statement. For Proposition 5.10, the Hilbert function h of the N_A points x_j with $\hat{x}_j \in A$ satisfies $h(2) \le D$ and $h(3) = N_A$, because the twenty points impose independent conditions on cubics, so Macaulay's theorem [23] gives $N_A \le D^{\langle 2 \rangle}$. Gotzmann's persistence theorem [13], the relation of Theorem 5.6(i) and Corollary 5.9 exclude the remaining values.

6. Symmetry

Constructions of PPT maps often impose a symmetry. This section shows that symmetry works against the range criterion: under a compact connected group that acts irreducibly on both outer spaces it never certifies a counterexample, covariance under a torus or under SU(2) reduces a range-criterion pair to smaller types, and for several covariant classes the PPT^2 conjecture itself holds.

6.1. **Compact connected groups.** Let K be a compact connected Lie group. For irreducible unitary representations σ_1, σ_2 of K with highest weights λ_1, λ_2 (for a fixed maximal torus and Borel subalgebra), the tensor product $\sigma_1 \otimes \sigma_2$ contains the irreducible representation of highest weight $\lambda_1 + \lambda_2$ exactly once, and it is generated by the product $e_1 \otimes e_2$ of highest weight vectors [11]. We call it the Cartan component.

Theorem 6.1. *Let K be a compact connected Lie group and σ_1, σ_2 irreducible unitary representations of K on $\mathbb{C}^m$ and $\mathbb{C}^n$.*

(a) Let $Z \neq 0$ be a positive semidefinite operator on $\mathbb{C}^m \otimes \mathbb{C}^n$ with $Z^\Gamma \succeq 0$ that commutes with $\sigma_1(g) \otimes \sigma_2(g)$ for all $g \in K$. Then $\operatorname{ran} Z$ contains the Cartan component of $\sigma_1 \otimes \sigma_2$, and in particular a product vector.

(b) Let π be any unitary representation of K on $\mathbb{C}^k$, and let $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ and $\Psi\colon \mathrm{M}_k \to \mathrm{M}_n$ be PPT maps with $\Phi(\bar{\sigma}_1(g) X \bar{\sigma}_1(g)^\dagger) = \pi(g)\Phi(X)\pi(g)^\dagger$ and $\Psi(\pi(g) Y \pi(g)^\dagger) = \sigma_2(g)\Psi(Y)\sigma_2(g)^\dagger$ for all $g \in K$. If $\Psi \circ \Phi \neq 0$, then $\operatorname{ran} C_{\Psi\circ\Phi}$ contains a product vector.

Proof. (a) The range of Z is invariant under $\sigma_1 \otimes \sigma_2$. If it does not contain the Cartan component, then, since this component has multiplicity one, $\operatorname{ran} Z$ is orthogonal to it, and $e_1 \otimes e_2 \in \ker Z$. As $\ker Z$ is invariant, $\sigma_1(g)e_1 \otimes \sigma_2(g)e_2 \in \ker Z$ for all $g \in K$. By Corollary 2.9(b), the pairs $([x],[y])$ with $x \otimes y \in \ker Z$ form a finite union of closed sets $\mathbb{P}(A_j) \times \mathbb{P}(B_j)$. The map $g \mapsto ([\sigma_1(g)e_1], [\sigma_2(g)e_2])$ is real analytic on the connected manifold K, so by the Baire category theorem and the identity theorem its image lies in a single product $\mathbb{P}(A_j) \times \mathbb{P}(B_j)$. Then A_j contains the orbit $\sigma_1(K)e_1$, which spans $\mathbb{C}^m$ by irreducibility, and likewise $B_j = \mathbb{C}^n$. So $\mathbb{C}^m \otimes \mathbb{C}^n \subseteq \ker Z$ and $Z = 0$, a contradiction.

(b) The composition $\Psi \circ \Phi$ is PPT, so $Z = C_{\Psi\circ\Phi}$ satisfies $Z \succeq 0$ and $Z^\Gamma \succeq 0$. Since $C_\Phi = (\mathrm{id} \otimes \Phi)(\sum_{i,j} E_{ij} \otimes E_{ij})$ and $\sum_{i,j} E_{ij} \otimes E_{ij}$ commutes with $\bar{U} \otimes U$ for every unitary U, the covariance relations give $(\sigma_1(g) \otimes \sigma_2(g)) Z (\sigma_1(g) \otimes \sigma_2(g))^\dagger = Z$. Part (a) applies. □

So constructions covariant under SU(2), SU(d) or spin groups with irreducible outer representations never give range-criterion pairs, in any dimension and for any middle representation. Part (b) can also be proved directly from the factors. For a highest weight vector a of the input

representation $\bar{\sigma}_1$, the range of $\Phi(P_a)$ is invariant under the maximal torus by covariance, under the raising operators because Φ is completely positive, and under the lowering operators because $T \circ \Phi$ is completely positive, hence under K. As the vectors $\bar{\sigma}_1(g)a$ span $\mathbb{C}^m$, covariance and (2.1) show that this range is $\operatorname{ran}\Phi(I)$. So $(\Psi \circ \Phi)(P_a)$ has the same range as $(\Psi \circ \Phi)(I)$, which is a positive multiple of the identity by Schur's lemma, and (2.1) gives $(\bar{a} \otimes f)^\dagger Z(\bar{a} \otimes f) > 0$ for every $f \neq 0$. Since $\bar{a}$ is a lowest weight vector of σ_1, for a lowest weight vector f of σ_2 the vector $\bar{a} \otimes f$ spans the lowest weight space of $\sigma_1 \otimes \sigma_2$, which Z preserves, so $\bar{a} \otimes f \in \operatorname{ran} Z$. The theorem does not apply to finite groups, which have no Borel subgroup. For reducible outer representations of SU(2) it extends as long as one of them is multiplicity-free, and more generally, by Theorem 6.7(d) below, as long as no spin occurs more than twice in one of them. The proof uses the following observation.

Lemma 6.2. *Let $Z \succeq 0$ on $(\bigoplus_i A_i) \otimes (\bigoplus_j B_j)$, with orthogonal direct sums, and let $Z_{(ij),(i'j')}$ denote the block of Z that maps $A_{i'} \otimes B_{j'}$ to $A_i \otimes B_j$. If $Z^\Gamma \succeq 0$ and $A_a \otimes B_b \subseteq \ker Z$, then $Z_{(ay),(xb)} = 0$ for all x and y.*

Proof. Positivity of the partial transpose does not depend on the choice of orthonormal bases, so we may use bases adapted to the decompositions. Then the partial transpose maps the block $Z_{(ij'),(i'j)}$ to the block $(Z^\Gamma)_{(ij),(i'j')}$, transposing it on the second factor. Since $Z_{(ab),(ab)} = 0$, the diagonal block $(Z^\Gamma)_{(ab),(ab)}$ vanishes. As $Z^\Gamma \succeq 0$, so does the whole block row $(Z^\Gamma)_{(ab),(xy)}$, which is the partial transpose of $Z_{(ay),(xb)}$. □

Theorem 6.3. *Let σ_1 and σ_2 be unitary representations of $\mathrm{SU}(2)$ on $\mathbb{C}^m$ and $\mathbb{C}^n$, at least one of them multiplicity-free, and let $Z \neq 0$ be a positive semidefinite operator on $\mathbb{C}^m \otimes \mathbb{C}^n$ with $Z^\Gamma \succeq 0$ that commutes with $\sigma_1(g) \otimes \sigma_2(g)$ for all g. Then $\operatorname{ran} Z$ contains a product vector. Consequently part (b) of Theorem 6.1 holds for $K = \mathrm{SU}(2)$ whenever σ_1 or σ_2 is multiplicity-free.*

Proof. Exchanging the two tensor factors preserves the hypotheses, so we may assume that σ_1 is multiplicity-free: $\mathbb{C}^m = \bigoplus_i A_i$ with A_i irreducible of spin s_i and the s_i distinct. Let $\mathbb{C}^n = \bigoplus_t B_t$ be the isotypic decomposition of σ_2, with $B_t \cong V_t \otimes \mathbb{C}^{\mu_t}$ and V_t irreducible of spin t. The blocks $A_i \otimes B_t$ are invariant and mutually orthogonal, and we call a block dead if it lies in $\ker Z$. Suppose that $\operatorname{ran} Z$ is product-free. We show by downward induction on w that every block with $s_i + t \geq w$ is dead, which for $w = 0$ contradicts $Z \neq 0$.

Assume that all blocks with $s_i + t > w$ are dead. The isotypic component of spin w of $\mathbb{C}^m \otimes \mathbb{C}^n$ meets a block $A_i \otimes B_t$ only if $s_i + t \geq w$, and for $s_i + t = w$ the intersection is the Cartan component $C_{it} \cong V_w \otimes \mathbb{C}^{\mu_t}$ of the block. Since Z commutes with the action and its range is orthogonal to the dead blocks, it maps $C_w = \bigoplus_{s_i+t=w} C_{it}$ to itself. Let $(i,t) \neq (i',t')$ be two pairs with $s_i + t = s_{i'} + t' = w$. Then $i \neq i'$, and the corner sums $s_i + t'$ and $s_{i'} + t$ add up to $2w$ and differ by $2(s_i - s_{i'}) \neq 0$. So one of them exceeds w, the corresponding corner block $A_i \otimes B_{t'}$ or $A_{i'} \otimes B_t$ is dead, and Lemma 6.2, together with $Z = Z^\dagger$, shows that the block $Z_{(it),(i't')}$ vanishes. Hence Z restricted to C_w is the direct sum of its restrictions to the C_{it}, and by Schur's lemma these have the form $1_{V_w} \otimes z_{it}$ with $z_{it} \succeq 0$ on $\mathbb{C}^{\mu_t}$. Let $e_i \in A_i$ and $f_t \in V_t$ be highest weight vectors. For $v \in \mathbb{C}^{\mu_t}$ the product vector $e_i \otimes (f_t \otimes v)$ is a highest weight vector in C_{it}, and Z maps it to the product vector $e_i \otimes (f_t \otimes z_{it}v)$. So $z_{it} = 0$, and $e_i \otimes (f_t \otimes v) \in \ker Z$ for all v. The argument in the proof of Theorem 6.1(a), applied to the representations on A_i and on $V_t \otimes v$, gives $A_i \otimes (V_t \otimes v) \subseteq \ker Z$. So the block $A_i \otimes B_t$ is dead.

The last statement follows as in the proof of Theorem 6.1(b). □

Theorem 6.3 leaves open the case where some spin occurs more than once in both outer representations (see Section S7). Theorem 6.7(d) below settles this case unless every active pair of highest total spin has multiplicity at least three on both sides, and Remark 6.8 shows that

ran Z can then be product-free. Theorems 6.1 and 6.3 say more than the failure of the range criterion for covariant pairs.

Corollary 6.4. *Let K be a compact group with unitary representations σ_1, σ_2 on $\mathbb{C}^m$ and $\mathbb{C}^n$. Suppose that every nonzero positive semidefinite operator Z on $\mathbb{C}^m \otimes \mathbb{C}^n$ with $Z^\Gamma \succeq 0$ that commutes with all $\sigma_1(g) \otimes \sigma_2(g)$ has a product vector in its range, as in Theorems 6.1 and 6.3. Then no nonzero positive semidefinite operator with positive partial transpose has its range in a product-free subspace of $\mathbb{C}^m \otimes \mathbb{C}^n$ that is invariant under all $\sigma_1(g) \otimes \sigma_2(g)$.*

Proof. Let Z be such an operator with range in such a subspace $\mathcal{V}$. The twirl $\int_K (\sigma_1(g) \otimes \sigma_2(g)) Z (\sigma_1(g) \otimes \sigma_2(g))^\dagger \, dg$ is nonzero, positive semidefinite and invariant, and its range lies in $\mathcal{V}$. Its partial transpose is the corresponding twirl of Z^Γ by $\sigma_1 \otimes \bar{\sigma}_2$, so it is positive semidefinite. This contradicts the hypothesis. □

So for these groups no composite of PPT maps, covariant or not, has its range in an invariant product-free subspace.

Remark 6.5 (†). For SU(2) the Cartan component of $V \otimes V'$ is the component of highest total spin. Let Φ and Ψ be SU(2)-covariant PPT maps between irreducible representations V_1, V_2, V_3, and let Π be the projection onto the Cartan component, so that $w(Z) = \operatorname{Tr}(Z\Pi)/\operatorname{Tr} Z$ is the weight of this component in $Z = C_{\Psi\circ\Phi}$. By Schur's lemma $\Phi(I)$ and $\Psi(I)$ are multiples of the identity, so after normalising both maps $\operatorname{Tr} Z$ is constant, and w is bilinear on the product of the two polytopes of normalised covariant PPT maps of Theorem 6.13(a). Its minimum is attained at a pair of vertices. Table 6.1 lists the exact minima, computed from these vertices. By Theorem 6.1 they are positive. For the dimension-six types in the table the minimum is attained by the pair of extreme rays whose Choi matrices are supported on the components of highest total spin.

dimensions of (V_1, V_2, V_3)	minimum of $w(Z)$	value
$(4, 6, 5)$ and $(5, 6, 4)$	143/1470	0.097
$(4, 6, 6)$ and $(6, 6, 4)$	13/168	0.077
$(5, 6, 5)$	143/2100	0.068
$(6, 4, 6)$	143/2268	0.063
$(5, 6, 6)$ and $(6, 6, 5)$	13/252	0.052
$(6, 6, 6)$	26/693	0.038
all dimensions between 2 and 9	437/126412	0.0035

TABLE 6.1. Exact minima of the weight of the Cartan component in the composite, over SU(2)-covariant PPT maps between irreducible representations. The last minimum is attained at dimensions $(9, 9, 9)$.

6.2. **Reducible representations.** For reducible outer representations, covariance still reduces a range-criterion pair to one on the extreme weight spaces of a torus or on the multiplicity spaces of the highest total spin. For subspaces $A \subseteq \mathbb{C}^m$, $K \subseteq \mathbb{C}^k$ and $F \subseteq \mathbb{C}^n$ with isometries V_A, V_K and V_F onto them, we call $(\mathrm{Ad}_{V_K^\dagger} \circ \Phi \circ \mathrm{Ad}_{V_A}, \mathrm{Ad}_{V_F^\dagger} \circ \Psi \circ \mathrm{Ad}_{V_K})$ the compression of (Φ, Ψ) to (A, K, F). Its maps are PPT if Φ and Ψ are, as in the proof of Lemma 2.5, and for $K = \mathbb{C}^k$ the identity in the proof of Lemma 2.5 shows that the Choi matrix of its composition is the compression of $C_{\Psi\circ\Phi}$ to $\bar{A} \otimes F$.

Theorem 6.6. *Let $\mathbb{T}$ be a compact torus with unitary representations σ_1, π and σ_2 on $\mathbb{C}^m$, $\mathbb{C}^k$ and $\mathbb{C}^n$, let $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ and $\Psi\colon \mathrm{M}_k \to \mathrm{M}_n$ be PPT maps that are covariant as in Theorem 6.1(b), and let $Z = C_{\Psi\circ\Phi} \neq 0$. Let $\mathbb{C}^m = \bigoplus_\mu E_\mu$ and $\mathbb{C}^n = \bigoplus_\nu F_\nu$ be the weight space decompositions of σ_1 and σ_2, and call a pair (μ,ν) active if $E_\mu \otimes F_\nu \not\subseteq \ker Z$. Fix a linear form ℓ on the real span of the characters of $\mathbb{T}$ that is injective on the weights of σ_1, on those of σ_2 and on their sums, and let w^* be the largest value of $\ell(\mu+\nu)$ over the active pairs.*

(a) If $(\mu,\nu) \neq (\mu',\nu')$ are active and $\ell(\mu+\nu) = \ell(\mu'+\nu') = w^*$, then the block of Z from $E_{\mu'} \otimes F_{\nu'}$ to $E_\mu \otimes F_\nu$ vanishes. For every active (μ,ν) with $\ell(\mu+\nu) = w^*$, the range of the compression of Z to $E_\mu \otimes F_\nu$ lies in $\operatorname{ran} Z$.

(b) Let (Φ,Ψ) be a range-criterion pair, and let (μ,ν) be active with $\ell(\mu+\nu) = w^*$. Then the compression of (Φ,Ψ) to $(\bar{E}_\mu, \mathbb{C}^k, F_\nu)$ is a range-criterion pair, and so is the compression to $(\bar{E}_\mu, K, F_\nu)$ for some weight space K of π, of type $(\dim E_\mu, \dim K, \dim F_\nu)$. In particular $\dim E_\mu$, $\dim F_\nu$ and $\dim K$ are at least four, $\dim E_\mu$ and $\dim F_\nu$ are not both four, $\dim E_\mu, \dim F_\nu \geq 6$ if $\dim K = 4$, and $\dim E_\mu, \dim F_\nu \geq 5$ if $\dim K = 5$.

(c) If moreover no range-criterion pair has a type that is componentwise at most (m,k,n) and different from it, then $\mathbb{T}$ acts by scalars on $\mathbb{C}^m$, $\mathbb{C}^k$ and $\mathbb{C}^n$.

Proof. As in the proof of Theorem 6.1(b), Z commutes with $\sigma_1(t) \otimes \sigma_2(t)$, so it preserves the weight spaces $H_w = \bigoplus_{\mu+\nu=w} E_\mu \otimes F_\nu$ of $\sigma_1 \otimes \sigma_2$.

(a) If $\mu = \mu'$, then $\ell(\nu) = \ell(\nu')$ and $\nu = \nu'$. So $\mu \neq \mu'$, and as $Z = Z^\dagger$ we may assume that $\ell(\mu) > \ell(\mu')$. Then $\ell(\nu') > \ell(\nu)$, so $\ell(\mu+\nu') > w^*$, the pair (μ,ν') is not active, and Lemma 6.2 shows that the block vanishes. By the choice of ℓ, all pairs (μ',ν') with $\ell(\mu'+\nu') = w^*$ have the same total weight w. On H_w the operator Z vanishes on the spaces $E_{\mu'} \otimes F_{\nu'}$ of the pairs that are not active, and by the first statement its blocks between distinct active pairs vanish. So Z restricted to H_w is the direct sum of the compressions of Z to the spaces $E_\mu \otimes F_\nu$ of the active pairs with $\ell(\mu+\nu) = w^*$. Their ranges lie in $\operatorname{ran} Z$.

(b) Let V_E and V_F be isometries onto $\bar{E}_\mu$ and F_ν, and put $\Phi_E = \Phi \circ \mathrm{Ad}_{V_E}$ and $\Psi_F = \mathrm{Ad}_{V_F^\dagger} \circ \Psi$, so that (Φ_E, Ψ_F) is the compression of (Φ,Ψ) to $(\bar{E}_\mu, \mathbb{C}^k, F_\nu)$. The Choi matrix of $\Psi_F \circ \Phi_E$ is the compression of Z to $E_\mu \otimes F_\nu$. It is nonzero as (μ,ν) is active, and by (a) its range lies in $\operatorname{ran} Z$ and so contains no product vector. Hence (Φ_E, Ψ_F) is a range-criterion pair. As $\bar{\sigma}_1(t)$ acts on $\bar{E}_\mu$ by a scalar, covariance shows that $\Phi_E(X)$ commutes with $\pi(t)$ for every X, so $\Phi_E(X) = \sum_K P_K \Phi_E(X) P_K$, where K runs over the weight spaces of π. Hence $C_{\Psi_F\circ\Phi_E}$ is the sum over K of the Choi matrices of the compositions $\Psi_F \circ \mathrm{Ad}_{P_K} \circ \Phi_E$, which are positive semidefinite. One of them is nonzero, and its range lies in that of $C_{\Psi_F\circ\Phi_E}$ and so contains no product vector. For this K, $\Psi_F \circ \mathrm{Ad}_{P_K} \circ \Phi_E$ is the composition of the compression of (Φ,Ψ) to $(\bar{E}_\mu, K, F_\nu)$, which is therefore a range-criterion pair. The bounds follow from Corollary 3.4(a) and Theorem 4.20.

(c) By (b), $\dim E_\mu = m$, $\dim K = k$ and $\dim F_\nu = n$, so each of σ_1, π and σ_2 has a single weight. □

For a compact connected group, Theorem 6.6 applies to a maximal torus. For SU(2) the highest weight vectors give a finer reduction, to the multiplicity spaces of the highest total spin.

Theorem 6.7. *Let σ_1, π and σ_2 be unitary representations of $\mathrm{SU}(2)$ on $\mathbb{C}^m$, $\mathbb{C}^k$ and $\mathbb{C}^n$, let $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ and $\Psi\colon \mathrm{M}_k \to \mathrm{M}_n$ be PPT maps that are covariant as in Theorem 6.1(b), and let $Z = C_{\Psi\circ\Phi} \neq 0$. Let $\mathbb{C}^m = \bigoplus_j A_j$ and $\mathbb{C}^n = \bigoplus_l B_l$ be the isotypic decompositions of σ_1 and σ_2, with $A_j \cong \mathbb{C}^{p_j} \otimes V_j$ and $B_l \cong \mathbb{C}^{q_l} \otimes V_l$, where V_s is irreducible of spin s. Call (j,l) active if $A_j \otimes B_l \not\subseteq \ker Z$, and let J be the largest value of $j+l$ over the active pairs. For an active pair with $j+l = J$, let $x_j \in V_j$ and $y_l \in V_l$ be highest weight vectors, and put $E_j = \mathbb{C}^{p_j} \otimes x_j \subseteq A_j$ and $F_l = \mathbb{C}^{q_l} \otimes y_l \subseteq B_l$.*

(a) *For distinct active pairs* (j,l) *and* (j',l') *with* $j+l=j'+l'=J$*, the block of* Z *from* $A_{j'}\otimes B_{l'}$ *to* $A_j\otimes B_l$ *vanishes.*

(b) *For an active pair with* $j+l=J$*, the compression* Z_{jl} *of* Z *to* $E_j\otimes F_l\cong\mathbb{C}^{p_j}\otimes\mathbb{C}^{q_l}$ *is nonzero,* $Z_{jl}^\Gamma\succeq 0$*, and* $\operatorname{ran} Z_{jl}\subseteq\operatorname{ran} Z$*.*

(c) *If* (Φ,Ψ) *is a range-criterion pair, then for every active pair with* $j+l=J$ *the compression of* (Φ,Ψ) *to* $(\bar{E}_j,\mathbb{C}^k,F_l)$ *is a range-criterion pair of type* (p_j,k,q_l)*. In particular* $p_j,q_l\geq 4$ *and* $\max(p_j,q_l)\geq 5$*.*

(d) *If* $\min(p_j,q_l)\leq 2$ *for some active pair with* $j+l=J$*, then* $\operatorname{ran} Z$ *contains a product vector.*

(e) *If* (Φ,Ψ) *is a range-criterion pair and* $m,n\leq 7$*, then* Z *is supported on* $A_0\otimes B_0$*, and some range-criterion pair has type* (p_0,r,q_0)*, where* $r\geq 4$ *is the multiplicity in* π *of one of its spins. This type is componentwise at most* (m,k,n)*, and equal to it only if* σ_1*,* π *and* σ_2 *are trivial.*

Proof. As in the proof of Theorem 6.1(b), Z commutes with $\sigma_1(g)\otimes\sigma_2(g)$.

(a) If $j=j'$, then $l=l'$. If $j>j'$, then $l'<l$, so $j+l'<J$, the pair (j,l') is not active, and Lemma 6.2 applies. The case $j<j'$ follows with $Z=Z^\dagger$.

(b) Local compressions preserve positivity and commute with partial transposition up to transposing the projection on the second factor, so $Z_{jl}\succeq 0$ and $Z_{jl}^\Gamma\succeq 0$. The compression D of Z to $A_j\otimes B_l\cong(\mathbb{C}^{p_j}\otimes\mathbb{C}^{q_l})\otimes(V_j\otimes V_l)$ commutes with the action on $V_j\otimes V_l$, which is multiplicity-free, so $D=\sum_L z_L\otimes Q_L$, where Q_L is the projection onto the component of spin L of $V_j\otimes V_l$ and $z_L\succeq 0$. As $x_j\otimes y_l$ lies in the Cartan component, of spin J, we have $Z_{jl}=z_J$. The partial trace $d=\sum_L\operatorname{Tr}(z_L)Q_L$ of D over $\mathbb{C}^{p_j}\otimes\mathbb{C}^{q_l}$ is nonzero, positive semidefinite and invariant, and its partial transpose is the partial trace of D^Γ, so $d^\Gamma\succeq 0$. By Theorem 6.1(a), $\operatorname{ran} d$ contains the Cartan component, so $\operatorname{Tr} z_J>0$ and $Z_{jl}\neq 0$. The projection onto the highest weight vectors of total spin J commutes with Z. Its range is the sum of the spaces $E_{j'}\otimes F_{l'}$ over the pairs with $j'+l'=J$ and of subspaces of the blocks with $j'+l'>J$, which are not active and lie in $\ker Z$. So by (a), Z restricted to this range is the direct sum of the operators $Z_{j'l'}$ over the active pairs with $j'+l'=J$, and $\operatorname{ran} Z_{jl}\subseteq\operatorname{ran} Z$.

(c) The operator Z_{jl} is the Choi matrix of the composition of this compression, which consists of PPT maps. By (b) it is nonzero and its range contains no product vector. The bounds follow from Corollary 3.4(a) and Theorem 4.20.

(d) By (b), Z_{jl} is a nonzero positive semidefinite operator on $\mathbb{C}^{p_j}\otimes\mathbb{C}^{q_l}$ with $Z_{jl}^\Gamma\succeq 0$, and its range lies in $U\otimes W$, where U and W are the supports of its two marginals, of dimensions r_1 and r_2 with $\min(r_1,r_2)\leq\min(p_j,q_l)\leq 2$. If $\min(r_1,r_2)=1$, every vector of $U\otimes W$ is a product vector. Otherwise $\min(r_1,r_2)=2$, so $(r_1-1)(r_2-1)=\max(r_1,r_2)-1$, and as $\operatorname{rank} Z_{jl}\geq\max(r_1,r_2)$ by Lemma 2.6, Lemma 2.7 applied to $U\otimes W$ gives a product vector in $\operatorname{ran} Z_{jl}$. So $\operatorname{ran} Z_{jl}$ contains a product vector $u\otimes v$, and $(u\otimes x_j)\otimes(v\otimes y_l)$ is a product vector in $\operatorname{ran} Z$.

(e) By (c), an active pair with $j+l=J$ has $p_j\geq 4$ and $p_j(2j+1)\leq m\leq 7$, so $j=0$, and likewise $l=0$. So $(0,0)$ is the only active pair, and Z is supported on $A_0\otimes B_0$. It is the Choi matrix of the composition of the compression (Φ_0,Ψ_0) of (Φ,Ψ) to $(\bar{A}_0,\mathbb{C}^k,B_0)$. As $\bar{\sigma}_1$ is trivial on $\bar{A}_0$, the values of Φ_0 commute with π, so they lie in $\bigoplus_b \mathrm{M}_{r_b}\otimes I_{V_b}$, where $\mathbb{C}^k=\bigoplus_b\mathbb{C}^{r_b}\otimes V_b$ is the isotypic decomposition of π. The maps $\pi_b(Y)=(2b+1)^{-1}\operatorname{Tr}_{V_b}(P_bYP_b)$, with P_b the projection onto $\mathbb{C}^{r_b}\otimes V_b$, and $\iota_b(Y)=Y\otimes I_{V_b}$, an operator on $\mathbb{C}^{r_b}\otimes V_b\subseteq\mathbb{C}^k$, are completely positive, and so are $T\circ\pi_b\circ T$ and $T\circ\iota_b\circ T$, so $\pi_b\circ\Phi_0$ and $\Psi_0\circ\iota_b$ are PPT, and $\Psi_0\circ\Phi_0=\sum_b(\Psi_0\circ\iota_b)\circ(\pi_b\circ\Phi_0)$. The Choi matrices of the summands are positive semidefinite, one of them is nonzero, and their ranges lie in $\operatorname{ran} Z$. So $(\pi_b\circ\Phi_0,\Psi_0\circ\iota_b)$ is a range-criterion pair of type (p_0,r_b,q_0) for some b, and $r_b\geq 4$ by Corollary 3.4(a). As $p_0\leq m$, $(2b+1)r_b\leq k$

and $q_0 \le n$, this type is componentwise at most (m,k,n), with equality only if $A_0 = \mathbb{C}^m$, $b = 0$, $\mathbb{C}^{r_0} = \mathbb{C}^k$ and $B_0 = \mathbb{C}^n$. □

Part (d) of Theorem 6.7 contains Theorem 6.3, as $p_j = 1$ for all j if σ_1 is multiplicity-free, and $q_l = 1$ for all l if σ_2 is. If both representations have repeated spins, part (c) reduces a covariant range-criterion pair to one on the multiplicity spaces of the highest total spin, part (d) excludes small multiplicities, and part (e) shows that a covariant range-criterion pair with $m, n \le 7$ yields one whose type is componentwise at most (m,k,n) and different from it, unless all three representations are trivial. Parts (a), (b) and (d) use only that $Z \succeq 0$, $Z^\Gamma \succeq 0$ and that Z commutes with all $\sigma_1(g) \otimes \sigma_2(g)$, so they hold for every such operator $Z \neq 0$, as in Theorem 6.3.

Remark 6.8. The bound in part (d) is sharp for invariant operators. Let $\sigma_1 = \sigma_2$ be the sum of three copies of $V_{1/2}$, so that $\mathbb{C}^6 \cong \mathbb{C}^3 \otimes V_{1/2}$ and $p_{1/2} = q_{1/2} = 3$. Let ρ be the projection onto the orthogonal complement of the five product vectors of the Tiles unextendible product basis of $\mathbb{C}^3 \otimes \mathbb{C}^3$ [1], and let $Q_1 = \frac{1}{2}(I + F)$, with F the flip, be the projection onto the symmetric tensors in $V_{1/2} \otimes V_{1/2}$, which form its component of spin one. Then $Z = \rho \otimes Q_1$, an operator on $(\mathbb{C}^3 \otimes \mathbb{C}^3) \otimes (V_{1/2} \otimes V_{1/2})$, which is $\mathbb{C}^6 \otimes \mathbb{C}^6$ after reordering the tensor factors, is positive semidefinite and invariant, and $Z^\Gamma = \rho^\Gamma \otimes Q_1^\Gamma \succeq 0$, since $\rho^\Gamma = \rho$ as the Tiles vectors are real product vectors, and $Q_1^\Gamma = \frac{1}{2}(I + F^\Gamma) = \frac{1}{2}(I + 2P)$ with P the projection onto a maximally entangled vector. If $x \otimes y \in \operatorname{ran} Z$, write $x = \sum_i a_i \otimes e_i$ and $y = \sum_j b_j \otimes e_j$ with a basis e_1, e_2 of $V_{1/2}$. As $\operatorname{ran} Z$ lies in $\operatorname{ran}\rho \otimes (V_{1/2} \otimes V_{1/2})$, every $a_i \otimes b_j$ lies in $\operatorname{ran}\rho$, which contains no nonzero product vector, so $x \otimes y = 0$. So $\operatorname{ran} Z$ contains no product vector, while $\min(p_{1/2}, q_{1/2}) = 3$.

Corollary 6.9. *Let σ_1 and σ_2 be unitary representations of* SU(2) *on $\mathbb{C}^m$ and $\mathbb{C}^n$. The following are equivalent.*

(i) Every nonzero positive semidefinite operator Z on $\mathbb{C}^m \otimes \mathbb{C}^n$ with $Z^\Gamma \succeq 0$ that commutes with all $\sigma_1(g) \otimes \sigma_2(g)$ has a product vector in its range.

(ii) σ_1 or σ_2 has no spin of multiplicity at least three.

Proof. If, say, every spin of σ_1 has multiplicity at most two, then $\min(p_j, q_l) \le 2$ for every active pair, and part (d) of Theorem 6.7, which holds for every such operator, gives a product vector. Conversely, let s and t be spins of multiplicity at least three in σ_1 and σ_2. Then there are isometries $W_1 \colon \mathbb{C}^3 \otimes V_s \to \mathbb{C}^m$ and $W_2 \colon \mathbb{C}^3 \otimes V_t \to \mathbb{C}^n$ that intertwine the action of SU(2) on the second factors with σ_1 and σ_2. With ρ as in Remark 6.8, let $Z = (W_1 \otimes W_2)(\rho \otimes I)(W_1 \otimes W_2)^\dagger$, where $\rho \otimes I$ acts on $(\mathbb{C}^3 \otimes \mathbb{C}^3) \otimes (V_s \otimes V_t)$, which is $(\mathbb{C}^3 \otimes V_s) \otimes (\mathbb{C}^3 \otimes V_t)$ after reordering the tensor factors. Then $Z \neq 0$ is positive semidefinite and invariant, and $Z^\Gamma = (W_1 \otimes \bar{W}_2)(\rho^\Gamma \otimes I)(W_1 \otimes \bar{W}_2)^\dagger \succeq 0$, as $\rho^\Gamma = \rho$. A product vector in $\operatorname{ran} Z$ has its factors in $\operatorname{ran} W_1$ and $\operatorname{ran} W_2$, as in the proof of Lemma 2.5, so it is the image of a product vector in $\operatorname{ran}\rho \otimes (V_s \otimes V_t)$, and the argument of Remark 6.8, with bases of V_s and V_t, shows that it vanishes. □

Remark 6.10 (†). Part (b) has a quantitative form. Let $V_j \otimes V_l$ be as in its proof, with $2j$ and $2l$ between 1 and 8. Every nonzero invariant positive semidefinite operator $d = \sum_L c_L Q_L$ on $V_j \otimes V_l$ with $d^\Gamma \succeq 0$ satisfies $\operatorname{Tr}(dQ_{j+l}) \ge \binom{2j+2l}{2j}^{-1} \operatorname{Tr} d$, and equality occurs. The inequality is a combination with nonnegative rational coefficients of the linear inequalities in the c_L that express $d^\Gamma \succeq 0$ (Table SA.1). As $D^\Gamma \succeq 0$ gives the same inequalities with the operators z_L^Γ in place of the c_L, the operators z_L in the proof of Theorem 6.7 satisfy

$$(2J+1) z_J^\Gamma \succeq \binom{2j+2l}{2j}^{-1} \Big(\sum_L (2L+1) z_L \Big)^\Gamma .$$

The composites in Remark 6.5 are such operators d, and the minima in Table 6.1 lie above these bounds.

The following corollary shows that a range-criterion pair that is covariant under a nontrivial action of SU(2) exists exactly when a suitably smaller range-criterion pair exists.

Corollary 6.11. *Let σ_1, π and σ_2 be unitary representations of* SU(2) *on $\mathbb{C}^m$, $\mathbb{C}^k$ and $\mathbb{C}^n$, and call a range-criterion pair covariant if it is covariant for them as in Theorem 6.1(b).*

- *(a) If there is a covariant range-criterion pair, then there is a range-criterion pair of a type (m', k', n') with $m' \leq m$, $k' \leq k$ and $n' \leq n$, where $m' \leq m-2$ if σ_1 is nontrivial, $k' \leq k-2$ if π is nontrivial, and $n' \leq n-2$ if σ_2 is nontrivial.*
- *(b) Conversely, if there is a range-criterion pair of type (m', k', n'), and σ_1, π and σ_2 act trivially on subspaces of dimensions m', k' and n', then there is a covariant range-criterion pair.*
- *(c) If $\max(m,k,n) \leq 5$ and there is a covariant range-criterion pair, then σ_1, π and σ_2 are trivial.*
- *(d) If there is a covariant range-criterion pair, σ_1 is nontrivial and $m \leq 7$, then σ_1 is $V_0^{\oplus 4} \oplus V_{1/2}$, $V_0^{\oplus 5} \oplus V_{1/2}$ or $V_0^{\oplus 4} \oplus V_1$, and there is a range-criterion pair of a type (p_0, k, n') with $n' \leq n$, where $p_0 \in \{4,5\}$ is the multiplicity of V_0 in σ_1. The same holds for σ_2, with the roles of the outer spaces exchanged.*

Proof. (a) Theorem 6.6(b), applied to a maximal torus $\mathbb{T}$ of SU(2), gives a range-criterion pair of type $(\dim E, \dim K, \dim F)$, where E, K and F are weight spaces of $\mathbb{T}$ in $\mathbb{C}^m$, $\mathbb{C}^k$ and $\mathbb{C}^n$. All three dimensions are at least four by Corollary 3.4(a). Let E be the weight space of weight w of a nontrivial representation on $\mathbb{C}^d$, with $\dim E \geq 2$. An irreducible summand of spin s adds $2s+1$ to d, and it adds one to $\dim E$ if $s \geq |w|$ and $s-w$ is an integer, and nothing otherwise. If w is not an integer, every summand that adds to $\dim E$ has $2s+1 \geq 2$, so $d \geq 2\dim E \geq \dim E + 2$. If w is an integer, a summand with $s > 0$ adds at least two more to d than to $\dim E$. In both cases $\dim E \leq d-2$, and applied to E, K and F this gives the bounds.

(b) Let (Φ', Ψ') be a range-criterion pair of type (m', k', n'), and let V_1, V_2 and V_3 be isometries from $\mathbb{C}^{m'}$, $\mathbb{C}^{k'}$ and $\mathbb{C}^{n'}$ onto subspaces on which $\bar{\sigma}_1$, π and σ_2 act trivially, where for $\bar{\sigma}_1$ we take the complex conjugate of a subspace on which σ_1 acts trivially. The maps $\Phi = \mathrm{Ad}_{V_2} \circ \Phi' \circ \mathrm{Ad}_{V_1^\dagger}$ and $\Psi = \mathrm{Ad}_{V_3} \circ \Psi' \circ \mathrm{Ad}_{V_2^\dagger}$ are PPT, as in the proof of Lemma 2.5, and they are covariant, since $\bar{\sigma}_1(g)V_1 = V_1$, $\pi(g)V_2 = V_2$ and $\sigma_2(g)V_3 = V_3$ for all g. As $V_2^\dagger V_2 = I$, the Kraus operators of $\Psi \circ \Phi$ are the $V_3 K V_1^\dagger$ for the Kraus operators K of $\Psi' \circ \Phi'$, so $C_{\Psi\circ\Phi} = (\bar{V}_1 \otimes V_3) C_{\Psi'\circ\Phi'} (\bar{V}_1 \otimes V_3)^\dagger \neq 0$, and, as in the proof of Lemma 2.5, product vectors in the two ranges correspond.

(c) By (a), a nontrivial action gives a range-criterion pair with a dimension at most three, which Corollary 3.4(a) excludes.

(d) By Theorem 6.7(c), an active pair (j,l) with $j+l=J$ has $p_j \geq 4$ and $p_j(2j+1) \leq m \leq 7$, so $j=0$, and the compression there is a range-criterion pair of type (p_0, k, q_l) with $q_l \leq n$. The summands of σ_1 other than the $p_0 \geq 4$ copies of V_0 are nontrivial irreducible representations of total dimension $m - p_0 \leq 3$, and as σ_1 is nontrivial there is at least one. So they form $V_{1/2}$ or V_1, which gives the three representations. The same argument applies to σ_2. □

By parts (b) and (d), there are representations π and σ_2 on $\mathbb{C}^k$ and $\mathbb{C}^n$ and a range-criterion pair that is covariant for $V_0^{\oplus 4} \oplus V_{1/2}$, π and σ_2 if and only if there is a range-criterion pair of a type $(4, k', n')$ with $k' \leq k$ and $n' \leq n$. This is the question of outer dimension four in Section 8.

Remark 6.12 (†). With Theorem 1.1, part (c) of Corollary 6.11 extends to $\max(m,k,n) \leq 6$: by part (a), a nontrivial action gives a range-criterion pair with all three dimensions at most six and one of them at most four, which Theorem 1.1 excludes. Part (b) shows that this cannot

be extended to $\max(m,k,n) \le 7$ without deciding types of Table 4.2. For instance, a range-criterion pair of type $(4,7,7)$ would give one of type $(6,7,7)$ that is covariant for $\sigma_1 = V_0^{\oplus 4} \oplus V_{1/2}$ and trivial π and σ_2.

6.3. **Exact covariant classes.** For the following classes we prove the PPT^2 conjecture itself, not only the failure of the range criterion. Before stating them, we recall two symmetric classes for which the conjecture was settled earlier: maps on $\mathrm{M}_d \otimes \mathrm{M}_d$ that are covariant under all pairs of unitaries acting on the two tensor factors [8, Proposition V.1], and maps covariant under the diagonal unitary group, which include the Choi-type maps [28].

Theorem 6.13 (†)**.** *In each of the following cases, $\Psi \circ \Phi$ is entanglement breaking for all PPT maps $\Phi\colon \mathrm{M}(V_1) \to \mathrm{M}(V_2)$ and $\Psi\colon \mathrm{M}(V_2) \to \mathrm{M}(V_3)$ that are covariant for the indicated representations V_1, V_2, V_3 of the indicated group.*

- *(a)* $\mathrm{SU}(2)$*, with V_1, V_2, V_3 irreducible of dimensions between 2 and 9, and also with $V_1 = V_3$ of dimension 4 and V_2 irreducible of dimension between 3 and 10.*
- *(b)* $\mathrm{SU}(2) \times \mathrm{SU}(2)$*, with each V_i one of $\mathbb{C}^2 \otimes \mathbb{C}^2$, $\mathbb{C}^2 \otimes \mathbb{C}^3$, $\mathbb{C}^3 \otimes \mathbb{C}^2$, $\mathbb{C}^2 \otimes \mathbb{C}^4$, $\mathbb{C}^4 \otimes \mathbb{C}^2$, $\mathbb{C}^3 \otimes \mathbb{C}^3$ (tensor products of irreducible representations of the two factors).*
- *(c) G_2, with $V_1 = V_2 = V_3$ its seven-dimensional irreducible representation.*
- *(d) S_n for $6 \le n \le 10$, with $V_1 = V_2 = V_3$ the standard representation, and S_6 with (V_1, V_2, V_3) equal to $(\mathrm{std}, (4,2), \mathrm{std})$ or $(\mathrm{std}, \mathrm{std}, (4,2))$.*
- *(e) S_5, with each V_i the irreducible representation $(3,2)$, and also with (V_1, V_2, V_3) equal to $(4,5,4)$, $(4,4,5)$, $(5,4,5)$ or $(4,5,5)$, where 4 denotes the standard representation and 5 the representation $(3,2)$.*
- *(f)* $\mathrm{PGL}(2,q)$ *for $q \in \{7,9\}$, acting on the $q+1$ points of $\mathbb{P}^1(\mathbb{F}_q)$, with $V_1 = V_2 = V_3$ the q-dimensional standard representation on functions with zero sum.*
- *(g) The group of signed permutation matrices of size n, for $5 \le n \le 9$, with $V_1 = V_2 = V_3 = \mathbb{C}^n$.*
- *(h)* $\mathrm{SU}(3)$*, with (V_1, V_2, V_3) one of $(3,3,3)$, $(3,6,3)$, $(3,8,3)$, $(3,3,6)$, $(3,6,6)$, $(3,8,6)$, $(3,3,8)$, $(3,6,8)$, $(6,3,6)$, $(6,6,6)$, $(6,8,6)$, $(6,3,8)$, $(6,6,8)$, where $3 = \mathbb{C}^3$, $6 = \mathrm{Sym}^2 \mathbb{C}^3$ and 8 is the adjoint representation, and with the triples obtained from these by passing to conjugate representations and to adjoint maps.*

Each of (a), (b), (e) and (f) includes classes that contain PPT maps that are not entanglement breaking.

Proof. In each case the operators on $V_1 \otimes V_2$ that commute with $\bar{V}_1 \otimes V_2$ form a commutative algebra spanned by isotypic projections, the partial transposes of these operators commute with $\bar{V}_1 \otimes \bar{V}_2$, which again decomposes without multiplicities, and the same holds for V_2 and V_3. So the Choi matrix of a covariant map is a real combination of isotypic projections with coefficients λ, complete positivity and complete copositivity are linear inequalities in λ, and the covariant PPT maps form a polyhedral cone. The Choi matrix of the composition is bilinear in the two coefficient vectors, and separability is preserved under convex combinations. So it suffices to show that the composition of every pair of extreme rays is separable. For each class we computed the extreme rays in exact arithmetic, formed all compositions of pairs, and wrote each composite Choi matrix as an explicit combination, with exact nonnegative weights, of twirls of explicit product vectors. In (a)–(d), (g) and (h) the weights and the twirled points are rational. In (e) they are rational except in the stored certificates of 18 composites, which use product vectors with non-real coordinates in $\mathbb{Q}(e^{2\pi i/3})$. In (f) the group is sharply 3-transitive, the commutant of $V_1 \otimes V_1$ has dimension $q+1$ and is multiplicity-free, the isotypic projections are defined over $\mathbb{Q}(\sqrt{2})$ for $q = 7$ and over $\mathbb{Q}(\sqrt{2}, \sqrt{5})$ for $q = 9$, and the certificates use product vectors with entries in $\mathbb{Z}[i]$ or $\mathbb{Z}[e^{2\pi i/3}]$ and weights in these fields. Table 6.2 lists the numbers of composites.

In that table, for SU(2) with dimensions 2–9 the 288 triples are those with $\dim V_1 \le \dim V_3$, and the second count is for the triples with $\dim V_1 > \dim V_3$, certified directly; for $\mathrm{SU}(2) \times \mathrm{SU}(2)$ the 68 triples represent all 216 triples up to exchanging the two factors of the group and passing to adjoint maps. Non-triviality is witnessed by extreme rays whose Choi matrices violate the realignment criterion, verified in interval arithmetic in (a) and (b), and in exact arithmetic in (e) and (f), where for (f) the realigned Choi matrix is diagonal in the isotypic projections. The polytopes of normalised covariant PPT maps have 196 vertices for $\mathrm{PGL}(2,7)$ and 667 for $\mathrm{PGL}(2,9)$, of which 139 and 536 violate the realignment criterion. For (a) with outer dimension four an exact Breuer–Hall witness detects an extreme ray for every middle dimension. □

class	triples	composites of extreme pairs certified
SU(2), dimensions 2–9	288, and adjoints	226,973 and 148,862
SU(2), $V_1 = V_3 = \mathbb{C}^4$	8	464, counted as pairs
$\mathrm{SU}(2) \times \mathrm{SU}(2)$	68	13,363
G_2	1	16
S_n standard, $6 \le n \le 10$	5	22 each
S_6 with $(4,2)$	2	47 and 58
S_5, $(3,2)$ and mixed	5	222; 24, 38, 24, 114
$\mathrm{PGL}(2,7)$, standard	1	9,672
$\mathrm{PGL}(2,9)$, standard	1	110,540
signed permutations, $5 \le n \le 9$	5	18 each
SU(3)	13, and variants	95

TABLE 6.2. Exact certificates for Theorem 6.13.

Theorem 6.13(a) proves the PPT^2 conjecture for SU(2)-covariant maps between irreducible spaces of type $(6,6,6)$, the type that Theorem 1.1 leaves open, and of type $(4,3,4)$, for which the conjecture is open in general. The classes in (f) are far from trivial, since most of their extreme rays are entangled, yet all compositions are entanglement breaking.

Covariant maps between reducible representations split into maps between irreducible ones when distinct pairs of summands share no irreducible constituent.

Lemma 6.14. *Let G be a compact group with unitary representations ρ_1, ρ_2 and ρ_3 on $\mathbb{C}^m = \bigoplus_a V_a$, $\mathbb{C}^k = \bigoplus_b W_b$ and $\mathbb{C}^n = \bigoplus_c U_c$, orthogonal decompositions into irreducible subrepresentations. Suppose that for $(a,b) \neq (a',b')$ the representations $\bar{V}_a \otimes W_b$ and $\bar{V}_{a'} \otimes W_{b'}$ have no irreducible constituent in common, and likewise $\bar{W}_b \otimes U_c$ and $\bar{W}_{b'} \otimes U_{c'}$ for $(b,c) \neq (b',c')$. Let $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ and $\Psi\colon \mathrm{M}_k \to \mathrm{M}_n$ be PPT maps with $\Phi(\rho_1(g) X \rho_1(g)^\dagger) = \rho_2(g)\Phi(X)\rho_2(g)^\dagger$ and $\Psi(\rho_2(g) Y \rho_2(g)^\dagger) = \rho_3(g)\Psi(Y)\rho_3(g)^\dagger$ for all $g \in G$. Then $\Phi(X) = \sum_{a,b} P_{W_b}\Phi(P_{V_a} X P_{V_a})P_{W_b}$, the maps $\Phi_{ab}\colon \mathrm{M}(V_a) \to \mathrm{M}(W_b)$ obtained by compression are covariant and PPT, and the same holds for Ψ, with maps $\Psi_{bc}\colon \mathrm{M}(W_b) \to \mathrm{M}(U_c)$. The Choi matrix of $\Psi \circ \Phi$ is block diagonal for the orthogonal decompositions $\mathbb{C}^m = \bigoplus_a \bar{V}_a$, where $\bar{V}_a = \{\bar{v} : v \in V_a\}$, and $\mathbb{C}^n = \bigoplus_c U_c$, with the block on $\bar{V}_a \otimes U_c$ equal to $\sum_b C_{\Psi_{bc} \circ \Phi_{ab}}$. So if every composition $\Psi_{bc} \circ \Phi_{ab}$ is entanglement breaking, then so is $\Psi \circ \Phi$.*

Proof. As in the proof of Theorem 6.1(b), C_Φ commutes with $\bar{\rho}_1(g) \otimes \rho_2(g)$, which preserves the orthogonal subspaces $\bar{V}_a \otimes W_b$ of $\mathbb{C}^m \otimes \mathbb{C}^k$. By the hypothesis, each isotypic component of this representation lies in one of these subspaces, and C_Φ preserves the isotypic components, so it is block diagonal for this decomposition. This gives the form of Φ, and the blocks are the Choi matrices of the maps Φ_{ab}, which are PPT as compressions of Φ. The same holds for Ψ.

Composing, $(\Psi\circ\Phi)(X) = \sum_{a,b,c} P_{U_c}\Psi_{bc}\big(\Phi_{ab}(P_{V_a}XP_{V_a})\big)P_{U_c}$, which gives the blocks of $C_{\Psi\circ\Phi}$. An operator that is block diagonal for orthogonal decompositions of both tensor factors is separable if its blocks are, and sums of separable operators are separable. □

Corollary 6.15 (†). *Let $G = \mathrm{SU}(2)$, and suppose that the hypothesis of Lemma 6.14 holds and that the summands V_a, W_b and U_c have dimension at most nine. Then $\Psi\circ\Phi$ is entanglement breaking for all covariant PPT maps Φ and Ψ as in that lemma. This applies, for instance, when $\mathbb{C}^m = \mathbb{C}^n$ is the sum of a trivial representation and an irreducible one of spin $3/2$ or $5/2$ and $\mathbb{C}^k$ is irreducible of spin 3, and when $\mathbb{C}^m = \mathbb{C}^n$ is the sum of a trivial representation and an irreducible one of spin $5/2$ and $\mathbb{C}^k$ is the sum of a trivial representation and an irreducible one of spin $3/2$.*

Proof. By Lemma 6.14 it suffices that every composition $\Psi_{bc}\circ\Phi_{ab}$ is entanglement breaking. This is clear if one of V_a, W_b and U_c has dimension one, and otherwise it is Theorem 6.13(a). In the examples, the Clebsch–Gordan rule shows that the products of distinct pairs of summands have no common constituent. □

7. An extended range criterion

If $Z = \sum_t P_{a_t}\otimes P_{b_t}$ is separable, with nonzero a_t and b_t, then $Z^\Gamma = \sum_t P_{a_t}\otimes P_{\bar b_t}$. So every vector $a_t\otimes b_t$ lies in

$$E(Z) = \{a\otimes b \in \operatorname{ran} Z :\ a\neq 0,\ b\neq 0,\ a\otimes\bar b\in\operatorname{ran} Z^\Gamma\},$$

and $E(Z)$ spans $\operatorname{ran} Z$. This is the range criterion of [18] in the form used for edge states [22]. A composite with $E(Z)$ not spanning $\operatorname{ran} Z$ would be entangled even if its range contains product vectors. We record its form for composites of PPT maps, show that $E(Z)\neq\emptyset$ for small types, and bound the vectors of $\ker Z^\Gamma$ that could make it stronger than the plain criterion.

Lemma 7.1. *Let $\Phi\colon \mathrm{M}_m\to\mathrm{M}_k$ and $\Psi\colon \mathrm{M}_k\to\mathrm{M}_n$ be PPT and $Z = C_{\Psi\circ\Phi}$.*

(a) $\operatorname{ran} Z^\Gamma = \operatorname{vec}\big(\operatorname{span}(\mathcal{K}_{T\circ\Psi}\mathcal{K}_\Phi)\big)$, and for nonzero a and b, $a\otimes b\in E(Z)$ if and only if $ba^T\in\operatorname{span}(\mathcal{K}_\Psi\mathcal{K}_\Phi)$ and $\bar b a^T\in\operatorname{span}(\mathcal{K}_{T\circ\Psi}\mathcal{K}_\Phi)$.

(b) If $\ker Z^\Gamma$ is spanned by product vectors, then $E(Z)$ is the set of all product vectors in $\operatorname{ran} Z$.

(c) Let $A\in\mathcal{K}_\Phi$, $U = \operatorname{ran} A$, and $\Psi_U = \Psi\circ\mathrm{Ad}_{\iota_U}$ with an isometry ι_U onto U. If C_{Ψ_U} is separable, then $\operatorname{vec}(\mathcal{K}_\Psi A)\subseteq\operatorname{span} E(Z)$. In particular this holds when $\operatorname{rank} A = 1$.

Proof. (a) $Z^\Gamma = C_{(T\circ\Psi)\circ\Phi}$, so Lemma 2.1 applies.

(b) The vector $a\otimes b$ is orthogonal to $x\otimes y$ if and only if $a\otimes\bar b$ is orthogonal to $x\otimes\bar y$, and by Corollary 2.9(a) the product vectors in $\ker Z^\Gamma$ are exactly the $x\otimes\bar y$ with $x\otimes y\in\ker Z$. So for $a\otimes b\in\operatorname{ran} Z$ the vector $a\otimes\bar b$ is orthogonal to all product vectors of $\ker Z^\Gamma$, which span it.

(c) Write $A = \iota_U A_0$. Then $\mathcal{K}_{\Psi_U} = \mathcal{K}_\Psi\iota_U$ and $\mathcal{K}_{T\circ\Psi_U} = \mathcal{K}_{T\circ\Psi}\iota_U$. A separable decomposition of C_{Ψ_U} consists of product vectors $w\otimes c$ with $cw^T = B\iota_U$ and $\bar c w^T = B'\iota_U$ for some $B\in\mathcal{K}_\Psi$ and $B'\in\mathcal{K}_{T\circ\Psi}$, and the matrices cw^T span $\mathcal{K}_\Psi\iota_U$. Then $c(A_0^Tw)^T = BA$ and $\bar c(A_0^Tw)^T = B'A$, so $A_0^Tw\otimes c\in E(Z)$ whenever $A_0^Tw\neq 0$ by (a), and the matrices $c(A_0^Tw)^T$ span $\mathcal{K}_\Psi\iota_U A_0 = \mathcal{K}_\Psi A$. □

Part (b) applies to the pair of [24], for which $Z^\Gamma = Z$ and the kernel is spanned by twenty product vectors: there the extended criterion is the plain one.

Proposition 7.2. *Let (Φ,Ψ) be a reduced PPT pair of type (m,k,n) with $\Psi\circ\Phi\neq 0$. Suppose that for some $r\geq 1$ with $rn\leq 6$ we have $(k-r)(m-r)\leq\max(m,k)$. Then $E(Z)\neq\emptyset$. The same holds with (m,n) replaced by (n,m).*

Proof. If $b \le \max(m,k)$, then C_Φ is separable by Lemma 2.6, hence so is Z, and $E(Z)$ spans $\operatorname{ran} Z \neq 0$. Otherwise $b > (k-r)(m-r)$, and $\mathcal{K}_\Phi$ contains a nonzero A with $\operatorname{rank} A \le r$, by Lemma 2.7 if $r < \min(m,k)$, and trivially if $r \ge \min(m,k)$. Then C_{Ψ_U} is a positive semidefinite operator with positive partial transpose on $\mathbb{C}^{\operatorname{rank} A} \otimes \mathbb{C}^n$, and $\operatorname{rank} A \cdot n \le 6$, so it is separable [27, 16]. Since $\Psi^*(I) \succ 0$, $\mathcal{K}_\Psi A \neq 0$, and Lemma 7.1(c) gives $E(Z) \neq \emptyset$. The last statement follows by applying this to the mirror pair $(\Psi^\#, \Phi^\#)$ of Lemma 2.4, which is reduced by Lemma 2.4(c): its composite Choi matrix is $Z' = WZW^\dagger$, where $W\colon \mathbb{C}^m \otimes \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^m$ is the flip, and $Z'^\Gamma = W\overline{Z^\Gamma}W^\dagger$, so $E(Z') = W\,E(Z)$. □

The hypothesis holds, for example, when $n \le 3$ and $\min(m,k) \le 3$, for $(4,4,3)$ and $(3,4,4)$, and for $(5,5,2)$, $(6,4,2)$, $(4,6,2)$, $(5,6,2)$, $(6,5,2)$ and their mirrors. For these types the strongest form of the extended criterion, $E(Z) = \emptyset$, can never certify a counterexample.

The extended criterion is stronger than the plain one only if $\ker Z^\Gamma$ is not spanned by product vectors (Lemma 7.1(b)). Vectors of $\ker Z^\Gamma$ constrain the Kraus spaces as follows. For $w \in \mathbb{C}^m \otimes \mathbb{C}^n$ let $Y(w) \subseteq \mathbb{C}^n$ be the span of the second factors in a decomposition $w = \sum_t x_t \otimes y_t$ with linearly independent x_t. It does not depend on the decomposition.

Lemma 7.3. *Let $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ and $\Psi\colon \mathrm{M}_k \to \mathrm{M}_n$ be completely positive, and $Z = C_{\Psi\circ\Phi}$.*

(a) A vector $v \in \mathbb{C}^m \otimes \mathbb{C}^n$ lies in $\ker Z$ if and only if $(I \otimes B^\dagger)v \in \ker C_\Phi$ for all $B \in \mathcal{K}_\Psi$. If Ψ is PPT, then $v \in \ker Z^\Gamma$ if and only if $(I \otimes B^\dagger)v \in \ker C_\Phi$ for all $B \in \mathcal{K}_{T\circ\Psi}$. If Φ is PPT, then $v \in \ker Z^\Gamma$ if and only if $(I \otimes B^\dagger)v \in \ker C_\Phi^\Gamma$ for all $B \in \overline{\mathcal{K}_\Psi}$. If both maps are PPT, then $v \in \ker Z$ if and only if $(I \otimes B^\dagger)v \in \ker C_\Phi^\Gamma$ for all $B \in \mathcal{K}_{\Psi\circ T}$.

(b) Let $\mathcal{K} \subseteq \mathrm{M}_{n\times k}$ be a subspace and $w \in \mathbb{C}^m \otimes \mathbb{C}^n$. Then the vectors $(I \otimes B^\dagger)w$ with $B \in \mathcal{K}$ form a subspace of dimension $\dim \mathcal{K} - \dim\{B \in \mathcal{K} : B^\dagger Y(w) = 0\}$, which is at most k times the Schmidt rank of w.

Proof. (a) If $\Xi = \sum_l \mathrm{Ad}_{B_l}$ is completely positive and $X \succeq 0$, then $\langle v, (\mathrm{id}\otimes\Xi)(X)v\rangle = \sum_l \langle (I \otimes B_l^\dagger)v, X(I \otimes B_l^\dagger)v\rangle$. So $v \in \ker(\mathrm{id}\otimes\Xi)(X)$ if and only if $(I \otimes B_l^\dagger)v \in \ker X$ for all l, and, since $B \mapsto (I \otimes B^\dagger)v$ is conjugate-linear, if and only if $(I \otimes B^\dagger)v \in \ker X$ for all $B \in \mathcal{K}_\Xi$. Apply this to $Z = (\mathrm{id}\otimes\Psi)(C_\Phi)$, to $Z^\Gamma = C_{(T\circ\Psi)\circ\Phi} = (\mathrm{id}\otimes(T\circ\Psi))(C_\Phi)$, to $Z^\Gamma = C_{(T\circ\Psi\circ T)\circ(T\circ\Phi)} = (\mathrm{id}\otimes(T\circ\Psi\circ T))(C_\Phi^\Gamma)$, where $T\circ\Psi\circ T = \sum_l \mathrm{Ad}_{\bar B_l}$ for $\Psi = \sum_l \mathrm{Ad}_{B_l}$, and to $Z = C_{(\Psi\circ T)\circ(T\circ\Phi)} = (\mathrm{id}\otimes(\Psi\circ T))(C_\Phi^\Gamma)$.

(b) Write $w = \sum_t x_t \otimes y_t$ with linearly independent x_t and y_t. The map $B \mapsto \sum_t x_t \otimes B^\dagger y_t$ is conjugate-linear, its kernel consists of the B with $B^\dagger y_t = 0$ for all t, and its image lies in $\operatorname{span}\{x_t\} \otimes \mathbb{C}^k$. □

For completely positive Φ and Ψ call $([x],[y])$ a kernel point if $\bar x \otimes y \in \ker Z$, and put $r(x) = \dim \mathcal{K}_\Phi x$ and $t(y) = \operatorname{rank}\Psi^*(P_y)$, as in the proof of Proposition 4.24. By (2.1), $\ker C_\Phi$ contains $F(x) = \bar x \otimes (\mathcal{K}_\Phi x)^\perp$, of dimension $k - r(x)$. Since $(u\otimes y)^\dagger C_\Psi (u\otimes y) = \bar u^\dagger \Psi^*(P_y)\bar u$, also $\ker C_\Psi$ contains $G(y) = \overline{(\operatorname{ran}\Psi^*(P_y))^\perp} \otimes y$, of dimension $k - t(y)$.

Proposition 7.4. *Let $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ be completely positive and $\Psi\colon \mathrm{M}_k \to \mathrm{M}_n$ PPT, and put $Z = C_{\Psi\circ\Phi}$, $a = \operatorname{rank} C_\Psi$, $a' = \operatorname{rank} C_\Psi^\Gamma$ and $b = \operatorname{rank} C_\Phi$. Let $([x_i],[y_i])$, $1 \le i \le N$, be kernel points, and let*

$$\delta_1 = \sum_i (k - r(x_i)) - \dim \sum_i F(x_i), \qquad \delta_2 = \sum_i (k - t(y_i)) - \dim \sum_i G(y_i).$$

For $w \in \ker Z^\Gamma$ put $W = \{(I \otimes B^\dagger)w : B \in \mathcal{K}_{T\circ\Psi}\}$, $\varepsilon_1 = \dim\{B \in \mathcal{K}_{T\circ\Psi} : B^\dagger Y(w) = 0\}$ and $\varepsilon_2 = \dim\bigl(W \cap \sum_i F(x_i)\bigr)$. Then

$$a + b + a' \le (m + n - N)k + \delta_1 + \delta_2 + \varepsilon_1 + \varepsilon_2.$$

Proof. By Lemma 7.3, $W \subseteq \ker C_\Phi$ and $\dim W = a' - \varepsilon_1$, as $\mathcal{K}_{T\circ\Psi}$ has dimension $\operatorname{rank} C_{T\circ\Psi} = a'$. The spaces $F(x_i)$ also lie in $\ker C_\Phi$, so

$$\sum_i (k - r(x_i)) - \delta_1 + a' - \varepsilon_1 - \varepsilon_2 = \dim\Big(\sum_i F(x_i) + W\Big) \leq mk - b.$$

Likewise $\sum_i (k - t(y_i)) - \delta_2 \leq kn - a$. Adding the two inequalities and using $r(x_i) + t(y_i) \leq k$ (Lemma 2.1(b)) gives the claim. □

For $w = 0$ we have $\varepsilon_1 = a'$ and $\varepsilon_2 = 0$, and the inequality is the count $a + b \leq (m + n - N)k + \delta_1 + \delta_2$. The same happens whenever w lies in the span of the vectors $\bar{x}_i \otimes \bar{y}_i$, which lie in $\ker Z^\Gamma$ by Corollary 2.9(a). Indeed, $\bar{x}_i \otimes B^\dagger \bar{y}_i \in \ker C_\Phi$ for $B \in \mathcal{K}_{T\circ\Psi}$ by Lemma 7.3, and $F(x_i)$ contains every vector of $\ker C_\Phi$ of the form $\bar{x}_i \otimes u$, so $W \subseteq \sum_i F(x_i)$ and $\varepsilon_1 + \varepsilon_2 = a'$.

Corollary 7.5. *Let (Φ, Ψ) be a PPT pair of type (m, k, n) with $\Phi(I) \succ 0$ and $\Psi^*(I) \succ 0$, such that C_Φ and C_Ψ are not separable. If $([x_i], [y_i])$, $1 \leq i \leq m + n - 2$, are kernel points, then $\delta_1 + \delta_2 + \varepsilon_1 + \varepsilon_2 \geq k + 3$ for every $w \in \ker Z^\Gamma$. If moreover $\varepsilon_1 = 0$, then w has Schmidt rank at least two.*

Proof. The partial traces of C_Φ over $\mathbb{C}^m$ and of C_Ψ and C_Ψ^Γ over $\mathbb{C}^n$ are $\Phi(I)$ and $\overline{\Psi^*(I)}$, of rank k. The three operators have positive partial transpose, and none of them is separable, as C_Ψ^Γ is separable only if C_Ψ is. So $a, a', b \geq k + 1$ by Lemma 2.6, and Proposition 7.4 gives $3k + 3 \leq 2k + \delta_1 + \delta_2 + \varepsilon_1 + \varepsilon_2$. If $\varepsilon_1 = 0$, then $k + 1 \leq a' = \dim W$, which is at most k times the Schmidt rank of w by Lemma 7.3(b). □

The middle marginals can be made positive definite without changing Z, by the first step of the proof of Lemma 2.5, and C_Φ and C_Ψ are not separable when Z is entangled. For such a pair, which is one product vector short in the sense of Lemma S4.22 when the kernel points are in general position, Proposition 7.4 with $w = 0$ only gives $\delta_1 + \delta_2 \geq a + b - 2k \geq 2$. So a vector $w \in \ker Z^\Gamma$ outside the span of the $\bar{x}_i \otimes \bar{y}_i$ forces strongly dependent blocks, a large intersection of W with the blocks $F(x_i)$, or many $B \in \mathcal{K}_{T\circ\Psi}$ with $B^\dagger Y(w) = 0$. Proposition 7.2 excludes the strongest form $E(Z) = \emptyset$ of the extended criterion for small types, and Corollary 7.5 constrains its weaker form, in which $E(Z)$ does not span $\operatorname{ran} Z$, on strata one product vector short in every type.

Finally, if the kernel of the Choi matrix of one factor and that of its partial transpose are nested, then $\ker Z^\Gamma$ is determined by $\ker Z$.

Proposition 7.6. *Let $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ and $\Psi\colon \mathrm{M}_k \to \mathrm{M}_n$ be PPT and $Z = C_{\Psi\circ\Phi}$.*

(a) If $\ker C_\Psi \subseteq \ker C_\Psi^\Gamma$ or $\ker C_\Psi^\Gamma \subseteq \ker C_\Psi$, then $\ker Z^\Gamma = \ker Z$.

(b) If $\overline{\ker C_\Phi} \subseteq \ker C_\Phi^\Gamma$ or $\ker C_\Phi^\Gamma \subseteq \overline{\ker C_\Phi}$, then $\ker Z^\Gamma = \overline{\ker Z}$.

In both cases, if $\ker Z$ is spanned by product vectors, then $E(Z)$ is the set of all product vectors in $\operatorname{ran} Z$. The hypothesis of (a) holds if $\ker C_\Psi$ is spanned by product vectors $x \otimes y$ with y real, and that of (b) holds if $\ker C_\Phi$ is spanned by product vectors $x \otimes y$ with x real.

Proof. (a) Write $\Phi = \sum_a \mathrm{Ad}_{A_a}$. As $C_\Phi = \sum_a \operatorname{vec}(A_a)\operatorname{vec}(A_a)^\dagger$ and $\operatorname{vec}(A_a) = (A_a^T \otimes I)\operatorname{vec}(I_k)$, we have $Z = (\mathrm{id} \otimes \Psi)(C_\Phi) = \sum_a (A_a^T \otimes I) C_\Psi (\bar{A}_a \otimes I)$, and in the same way $Z^\Gamma = \sum_a (A_a^T \otimes I) C_\Psi^\Gamma (\bar{A}_a \otimes I)$. Suppose that $\ker C_\Psi \subseteq \ker C_\Psi^\Gamma$, let $v \in \mathbb{C}^m \otimes \mathbb{C}^n$, and put $w_a = (\bar{A}_a \otimes I)v$. If $Zv = 0$, then all w_a lie in $\ker C_\Psi$, since the terms of Z are positive semidefinite, so they lie in $\ker C_\Psi^\Gamma$, and $Z^\Gamma v = 0$. Conversely, let $Z^\Gamma v = 0$ and $R = \sum_a w_a w_a^\dagger$. Then R^Γ is the transpose of $((\Phi\circ T)\otimes\mathrm{id})(vv^\dagger)$, which is positive semidefinite because Φ is PPT, and $\operatorname{Tr}(C_\Psi R^\Gamma) = \operatorname{Tr}(C_\Psi^\Gamma R) = \langle v, Z^\Gamma v\rangle = 0$. So $\operatorname{ran} R^\Gamma \subseteq \ker C_\Psi \subseteq \ker C_\Psi^\Gamma$, hence $\langle v, Zv\rangle = \operatorname{Tr}(C_\Psi R) = \operatorname{Tr}(C_\Psi^\Gamma R^\Gamma) = 0$, and $Zv = 0$. If $\ker C_\Psi^\Gamma \subseteq \ker C_\Psi$, the same argument applies to the PPT map $T \circ \Psi$ in place of Ψ, since $C_{T\circ\Psi} = C_\Psi^\Gamma$ and $C_{(T\circ\Psi)\circ\Phi} = Z^\Gamma$.

(b) Let Γ_1 be the partial transpose on the first factor. For Hermitian X we have $X^{\Gamma_1} = \overline{X^{\Gamma}}$, so $\ker X^{\Gamma_1} = \overline{\ker X^{\Gamma}}$. Hence the hypothesis says that $\ker C_\Phi \subseteq \ker C_\Phi^{\Gamma_1}$ or $\ker C_\Phi^{\Gamma_1} \subseteq \ker C_\Phi$, and the claim is that $\ker Z^{\Gamma_1} = \ker Z$. Write $\Psi = \sum_b \mathrm{Ad}_{B_b}$. Then $Z = \sum_b (I \otimes B_b) C_\Phi (I \otimes B_b^\dagger)$ and $Z^{\Gamma_1} = \sum_b (I \otimes B_b) C_\Phi^{\Gamma_1} (I \otimes B_b^\dagger)$, and the proof of (a) applies with the vectors $(I \otimes B_b^\dagger)v$ in place of the w_a and with Γ_1 in place of Γ. The operator $S = \sum_b (I \otimes B_b^\dagger) v v^\dagger (I \otimes B_b)$ that replaces R has S^{Γ_1} equal to the transpose of $(\mathrm{id} \otimes (\Psi \circ T)^*)(vv^\dagger)$, which is positive semidefinite because Ψ is PPT. For the reverse inclusion one uses the PPT map $\Phi \circ T$, with $C_{\Phi\circ T} = C_\Phi^{\Gamma_1}$ and $C_{\Psi\circ\Phi\circ T} = Z^{\Gamma_1}$.

In both cases, if $\ker Z$ is spanned by product vectors, then so is $\ker Z^\Gamma$, and Lemma 7.1(b) applies. Finally, by Corollary 2.9(a), a product vector $x \otimes y \in \ker C_\Psi$ with real y satisfies $x \otimes y = x \otimes \bar{y} \in \ker C_\Psi^\Gamma$, and a product vector $x \otimes y \in \ker C_\Phi$ with real x satisfies $\overline{x \otimes y} = x \otimes \bar{y} \in \ker C_\Phi^\Gamma$. □

So for a PPT pair whose composite kernel is spanned by product vectors, the extended criterion can be stronger than the plain one only if $\ker C_\Psi$ and $\ker C_\Psi^\Gamma$ are incomparable and $\overline{\ker C_\Phi}$ and $\ker C_\Phi^\Gamma$ are incomparable. In particular, $\ker C_\Psi$ is then not spanned by product vectors $x \otimes y$ with real y, and $\ker C_\Phi$ is not spanned by product vectors $x \otimes y$ with real x.

8. Open problems

We first show by examples that linear algebra and Lemma 3.1 alone cannot exclude the types $(6,6,6)$, $(5,6,5)$ and $(5,6,6)$, and then list open problems.

The type $(6,6,6)$ cannot be excluded using only the conclusions of Lemma 3.1 and the rank windows of Theorem 3.3 together with linear algebra, as the following example shows. Its spaces have dimension 9, below the bound $a, b \geq 10$ of Corollary S3.349, whose proof uses positivity through Theorem S3.348.

Example 8.1. Let $V = \mathrm{Sym}^5 \mathbb{C}^2$, realised as the polynomials of degree at most five in one variable z, with $\mathfrak{sl}_2(\mathbb{C})$ acting by $e = d/dz$, $f = -z^2\, d/dz + 5z$ and $h = [e,f] = 5 - 2z\, d/dz$. Let $\mathcal{D} \subseteq \mathrm{End}(V)$ be the image of the elements of degree at most two of the universal enveloping algebra $U(\mathfrak{sl}_2)$, and identify $\mathrm{End}(V)$ with M_6 by an orthonormal basis for an $\mathrm{SU}(2)$-invariant inner product on V. Then $\dim \mathcal{D} = 9$, $I \in \mathcal{D}$, no nonzero element of $\mathcal{D}$ has rank at most three, and $\mathrm{span}(\mathcal{D}\mathcal{D})$ has dimension 25 and contains no matrix of rank one. No nonzero PPT map has its Kraus space contained in $\mathcal{D}$.

Proof. The adjoint action preserves the filtration of $U(\mathfrak{sl}_2)$ by degree, so the image $\mathcal{D}_d$ of the elements of degree at most d is a subrepresentation of $\mathrm{End}(V) \cong V \otimes V^*$, which is the sum of irreducible representations $W_0, W_2, \ldots, W_{10}$ with $\dim W_k = k+1$. The weights of $\mathrm{ad}\, h$ on the elements of degree at most d lie in $[-2d, 2d]$, and e^j, of weight $2j$, is a highest weight vector that acts as $(d/dz)^j \neq 0$ for $j \leq 5$. So $\mathcal{D}_d = W_0 \oplus W_2 \oplus \cdots \oplus W_{2d}$ for $d \leq 5$. Hence $\mathcal{D} = \mathcal{D}_2$ has dimension 9 and contains I, and $\mathrm{span}(\mathcal{D}\mathcal{D}) = \mathcal{D}_4$ has dimension 25. A rank-one operator has a nonzero component in W_{10}: under $V^* \cong V$, the projection $V \otimes V \to W_{10} \cong \mathrm{Sym}^{10} \mathbb{C}^2$ is, up to a nonzero factor, the multiplication of binary forms, and the product of two nonzero forms is nonzero. A nonzero element of $\mathcal{D}$ acts as a nonzero differential operator $a(z)\, d^2/dz^2 + b(z)\, d/dz + c(z)$, so its kernel on V has dimension at most two and its rank is at least four. Finally, $\mathrm{vec}(gYg^{-1}) = (\bar{g} \otimes g)\,\mathrm{vec}(Y)$ for unitary g, so $\mathrm{vec}(\mathcal{D})$ is invariant under $\bar{\sigma} \otimes \sigma$, where σ is the irreducible representation of $\mathrm{SU}(2)$ on V, and it is product-free. By Theorem 6.1 and Corollary 6.4, no nonzero PPT operator has its range in $\mathrm{vec}(\mathcal{D})$, while the Choi matrix of a nonzero PPT map with Kraus space contained in $\mathcal{D}$ would be such an operator. □

Remark 8.2. At type $(5,6,5)$ a natural ansatz identifies $\mathbb{C}^6$ with $\Lambda^2\mathbb{C}^4$, so that α-planes $v \wedge \mathbb{C}^4$ of the Klein quadric can be members. It requires $\mathcal{K}_\Phi$ to contain the four-dimensional space $\mathcal{Y}$ of the maps $u \mapsto v \wedge \ell(u)$, $v \in \mathbb{C}^4$, for a surjection $\ell\colon \mathbb{C}^5 \to \mathbb{C}^4$, and $\mathcal{K}_\Psi$ to contain the space $\mathcal{X}$ of the maps $\eta \mapsto \nu(w \wedge \eta)$, $w \in \mathbb{C}^4$, for an injection $\nu\colon \Lambda^3\mathbb{C}^4 \to \mathbb{C}^5$. By Theorem 1.1 this ansatz gives no range-criterion pair. We sketch a direct argument that would give more; its case analysis is not written out, and the remark is not used in any proof. The argument concludes that if Φ and Ψ are PPT with $\mathcal{Y} \subseteq \mathcal{K}_\Phi$, $\mathcal{X} \subseteq \mathcal{K}_\Psi$ and $a, b \geq 9$, then $s \geq 17$ or S contains a matrix of rank one. In outline, suppose that $s \leq 16$ and that S is rank-one-free. Then the quadrics $u \mapsto e_i \wedge \ell(u) \wedge E(u) \in \Lambda^4\mathbb{C}^4 \cong \mathbb{C}$, for $E \in \mathcal{K}_\Phi$ and a basis e_i of $\mathbb{C}^4$, lie in a space $\mathcal{Q}$ of quadrics on $\mathbb{P}^4$ of dimension at most seven, determined by $S^\perp$. At a common zero u of $\mathcal{Q}$ with $\ell(u) \neq 0$ this gives $\mathcal{K}_\Phi u \subseteq \ell(u) \wedge \mathbb{C}^4$, so $[u] \in D_3(\Phi)$, and Lemma 2.8 confines every curve of such points to a line through $[\ker \ell]$. On the other hand $\dim \mathcal{K}_\Phi/\mathcal{Y} \geq 5$, as $b \geq 9$, and $\dim \mathcal{K}_\Phi/\mathcal{Y}$ is at most the dimension of the space of linear syzygies of $\mathcal{Q}$ whose coefficients vanish at $[\ker \ell]$. A case analysis of these syzygies, together with the mirror argument for $\mathcal{K}_\Psi$, which uses $a \geq 9$, shows that this space has dimension at most four, or that $\dim S^\perp \leq 8$, contradicting $s \leq 16$, or that $\mathcal{Q}$ has a curve of common zeros that is not such a line. Granted the case analysis, the same ansatz fails for every output dimension, and on the side of Ψ only $\mathcal{X} \subseteq \mathcal{K}_\Psi$ is needed: if $\Phi\colon \mathrm{M}_5 \to \mathrm{M}_6$ is PPT with $\mathcal{Y} \subseteq \mathcal{K}_\Phi$ and $b \geq 9$, and $\Psi\colon \mathrm{M}_6 \to \mathrm{M}_n$ is completely positive with $\mathcal{X} \subseteq \mathcal{K}_\Psi$ for an injection $\nu\colon \Lambda^3\mathbb{C}^4 \to \mathbb{C}^n$, $n \geq 4$, then S contains a matrix of rank one. Indeed, for $y \in \operatorname{ran}\nu$ and $x \in \mathbb{C}^5$ the bilinear forms $x^T W y$ with $W \in S^\perp$ depend only on $W|_{\operatorname{ran}\nu}$, and these restrictions span a space of dimension $14 - \dim \mathcal{Q}$, as $\mathcal{Q}$ is its annihilator in the 14-dimensional space of quadrics on $\mathbb{P}^4$ that vanish at $[\ker \ell]$. If this dimension is at most seven, the forms have a common zero on $\mathbb{P}(\operatorname{ran}\nu) \times \mathbb{P}^4$, a variety of dimension seven on which $\mathcal{O}(1,1)$ is ample, and $yx^T \in S$ has rank one. So a rank-one-free S has $\dim \mathcal{Q} \leq 6$, and then the case analysis above ends in its first or third alternative, which use neither the output dimension, nor a bound on s, nor the mirror argument. Applied to the pair $(\mathrm{Ad}_{G^{-1}} \circ \Psi^\#, \Phi^\# \circ \mathrm{Ad}_G)$, where G is the matrix of the pairing $(\eta, \theta) \mapsto \eta \wedge \theta$ on $\Lambda^2\mathbb{C}^4$, whose composite has Kraus space S^T, this gives the same conclusion at type $(6,6,5)$ when Ψ is PPT with $\mathcal{X} \subseteq \mathcal{K}_\Psi$ and $a \geq 9$, and $\mathcal{Y} \subseteq \mathcal{K}_\Phi$ for a surjection $\ell\colon \mathbb{C}^6 \to \mathbb{C}^4$.

Remark 8.3 (†). The type $(5,6,5)$ cannot be excluded by linear algebra and Lemma 3.1 alone either. In the notation of Remark 8.2, let $\ell(x,t) = x$ for $(x,t) \in \mathbb{C}^4 \oplus \mathbb{C} = \mathbb{C}^5$, and let $\mathcal{Q}$ be the span of the six 2×2 minors of the matrix with rows $(x_0, x_1, x_2, -x_3)$ and (x_1, x_2, x_3, t), the quadrics through the rational normal curve $\Gamma = \{[1 : r : r^2 : r^3 : -r^4]\} \subseteq \mathbb{P}^4$. Let $\mathcal{K}_1$ be the space of linear maps $E\colon \mathbb{C}^5 \to \Lambda^2\mathbb{C}^4$ such that the four quadrics $u \mapsto e_i \wedge \ell(u) \wedge E(u) \in \Lambda^4\mathbb{C}^4 \cong \mathbb{C}$ lie in $\mathcal{Q}$, where $e_0, \ldots, e_3$ is the standard basis of $\mathbb{C}^4$. Let G be the matrix of the pairing $(\eta, \theta) \mapsto \eta \wedge \theta$ on $\Lambda^2\mathbb{C}^4$ in the basis of the $e_i \wedge e_j$, and let $\mathcal{K}_2 = \{E^T G : E \in \mathcal{K}_1\}$. Then $\dim \mathcal{K}_1 = \dim \mathcal{K}_2 = 9$ and $\mathcal{K}_1 \supseteq \mathcal{Y}$. Also $\mathcal{K}_2 \supseteq \mathcal{X}$ for the injection $\nu\colon \Lambda^3\mathbb{C}^4 \to \mathbb{C}^5$ that sends θ to $(c_0, c_1, c_2, c_3, 0)$, where $e_i \wedge \theta = c_i\, e_0 \wedge e_1 \wedge e_2 \wedge e_3$. The space $\operatorname{span}(\mathcal{K}_2\mathcal{K}_1)$ has dimension 16 and is rank-one-free, neither $\mathcal{K}_1$ nor $\mathcal{K}_2$ contains a nonzero matrix of rank at most two, $\mathcal{K}_1$ has no common kernel, and $\dim \mathcal{K}_1 u = 6$ for general u. So $\mathcal{K}_1$ and $\mathcal{K}_2$ satisfy the conclusions of Lemma 3.1 and Proposition S3.18(a), with $s = 16$. But $\dim \mathcal{K}_1 u = 3$ for $[u] \in \Gamma$, and Γ spans $\mathbb{P}^4$, so by Lemma 2.8 no PPT map has Kraus space $\mathcal{K}_1$. Moreover, S consists of the matrices M with $e(r)^T M e(r) = 0$ for all r, where $e(r) = (1, r, r^2, r^3, -r^4)$. So a change of sign of the last coordinate on both sides turns S into the kernel of the multiplication $\mathrm{Sym}^4\mathbb{C}^2 \otimes \mathrm{Sym}^4\mathbb{C}^2 \to \mathrm{Sym}^8\mathbb{C}^2$ in the monomial bases, and $S^\perp$ is spanned by the matrices $e(r)e(r)^T$ of rank one. The type $(5,6,6)$ has an analogous configuration, with the same input side and a rational normal curve of degree five on the output side: $\mathcal{K}_\Phi = \mathcal{K}_1$, and $\mathcal{K}_\Psi = \{E^T G\}$ for the linear maps $E\colon \mathbb{C}^6 \to \Lambda^2\mathbb{C}^4$ such that the four quadrics $u \mapsto e_i \wedge \ell'(u) \wedge E(u)$ lie in the span of the nine 2×2 minors other than $x_3 t_2 - t_1^2$ of the matrix with rows $(x_0, x_1, x_2, x_3, t_1)$ and $(x_1, x_2, x_3, t_1, t_2)$, where $\ell'(x, t_1, t_2) = x$

for $(x, t_1, t_2) \in \mathbb{C}^4 \oplus \mathbb{C}^2 = \mathbb{C}^6$. It has $(a, b) = (12, 9)$, inside the ranges of Proposition S3.18(b), and $s = 20 = (m-1)(n-1)$, and a change of sign of one coordinate turns S into the kernel of the multiplication $\mathrm{Sym}^4 \mathbb{C}^2 \otimes \mathrm{Sym}^5 \mathbb{C}^2 \to \mathrm{Sym}^9 \mathbb{C}^2$, a completely entangled subspace of the maximal dimension [25]. It satisfies the conclusions of Lemma 3.1, and Lemma 2.8 excludes it, as $\mathcal{K}_\Phi u$ and $\mathcal{K}_\Psi^T y$ have rank three along rational normal curves that span $\mathbb{P}^4$ and $\mathbb{P}^5$. The stated properties of these configurations were verified in exact arithmetic (Table SA.1).

(1) Decide the reduced type $(6, 6, 6)$, whose open cells Table 4.1 lists; the paragraph before that table and Remarks S3.292 and S3.347e collect what is known about them. By Example 8.1, arguments for $(6, 6, 6)$ have to use positivity beyond Lemma 3.1, such as Lemmas 2.8 and 2.10, which the proofs of Theorems 4.6, 4.11 and 4.10 use, and Remark 8.3 shows the same for $(5, 6, 5)$ and $(5, 6, 6)$.
(2) Find a range-criterion pair with $\max(m, k, n) \leq 9$, or show that none exists. By Theorem 1.1 it would need $\max(m, k, n) \geq 6$, and $m = k = n = 6$ if $\max(m, k, n) = 6$; Theorems 4.20, 4.22 and 4.18, Proposition 5.3 and Theorem 5.4 exclude further types and constructions.
(3) Is the composite Choi matrix of the member of type $(6, 6, 6)$ (Theorem 5.8) separable for every pencil? When no six of the ten rank-loss points lie on a conic, Lemma S4.22 turns this into a linear program over 252 atoms.
(4) Decide whether part (b) of Theorem 6.1 holds for finite groups (part (a) appears to fail, Section S7), and whether there are range-criterion pairs that are covariant under SU(2). By Theorem 6.7(c), for such a pair the multiplicities p_j and q_l at the active pairs (j, l) with $j + l = J$ are at least four, and $\max(p_j, q_l) \geq 5$, while for invariant operators the conclusion of Theorem 6.7(d) can already fail when these multiplicities are three (Remark 6.8 and Corollary 6.9).
(5) The PPT^2 conjecture for maps on M_d with $4 \leq d \leq 9$ remains open; for $d \in \{4, 5\}$ a counterexample would need a composite whose range contains product vectors (Theorem 1.1). Can the extended criterion of Section 7 certify a counterexample with both outer dimensions at least four? The exclusion arguments of Sections 3, 4 and 5 assume that the composite range contains no product vector, so they do not apply, and Lemma 7.1(b) and, on strata one product vector short, Corollary 7.5 give conditions that such a certificate must satisfy.
(6) Decide whether a range-criterion pair can have an outer dimension four. For $\max(m, k, n) \leq 9$ only the types $(4, 7, n)$ and $(4, 8, n)$ with $7 \leq n \leq 9$, $(4, 9, n)$ with $6 \leq n \leq 9$, and their mirrors remain, by Theorems 4.20, 4.21, 4.22, S3.397, S3.398, S3.507, S3.698 and S3.561 and Corollary 3.4(a); Theorem 4.23 restricts the type $(4, 7, 7)$, and Corollary S3.564 and Table S3.5 restrict the types $(4, 8, n)$ and $(4, 9, n)$.
(7) Decide whether the range of every PPT state on $\mathbb{C}^6 \otimes \mathbb{C}^6$ with local ranks $(6, 6)$ and rank 10 contains a nonzero vector of Schmidt rank at most two. By Lemma 3.2 and Theorem S3.348 this holds for rank at most nine, and a positive answer would give $a, b \geq 11$ at $(6, 6, 6)$, as in Corollary S3.349.

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