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THE PPT^2 CONJECTURE HOLDS FOR SPIN-FACTOR MAPS

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Abstract. The PPT^2 conjecture asserts that the composition of two completely positive and completely copositive maps is entanglement breaking. A counterexample in dimension ten has recently been announced, and for maps on M_4 the conjecture is open. Let $\mathcal{S} \subset \mathrm{M}_4$ be the real span of the identity γ_0 and five pairwise anticommuting Hermitian unitaries $\gamma_1, \dots, \gamma_5$. It is a spin factor whose positive cone is a Lorentz cone. We prove the conjecture for all compositions $\Psi \circ \Phi$ of completely positive maps $\Phi\colon \mathrm{M}_4 \to \mathrm{M}_k$ and $\Psi\colon \mathrm{M}_k \to \mathrm{M}_4$, with k arbitrary, such that the ranges of Ψ and of the adjoint Φ^* lie in invertible congruences of the complex span of $\mathcal{S}$. Such maps are automatically completely copositive, and the class contains maps whose Choi matrices are entangled. A theorem of Hildebrand on separability for pairs of Lorentz cones reduces the claim to one trace inequality between the coefficient matrices of the two Choi matrices. We derive this inequality from the operator inequality $\sum_{j=1}^{5} F_j F_0^{-1} F_j \preceq F_0$, valid whenever $\sum_{\mu=0}^{5} F_\mu \otimes \gamma_\mu \succeq 0$ and $F_0 \succ 0$, which in turn follows from the Kadison–Schwarz inequality. As a corollary, $\Psi \circ \Phi$ is entanglement breaking whenever Ψ is such a map and Φ is a completely positive map with Kraus operators of rank at most two.

1. Introduction

A linear map $\Phi\colon \mathrm{M}_m \to \mathrm{M}_n$ is completely positive if $\mathrm{id}_r \otimes \Phi$ is positive for all r, and completely copositive if Φ composed with the transpose is completely positive. We call Φ a PPT map if it is both. It is entanglement breaking if $(\mathrm{id}_r \otimes \Phi)(X)$ is separable for every positive semidefinite $X \in \mathrm{M}_r \otimes \mathrm{M}_m$, equivalently, if its Choi matrix $C_\Phi = \sum_{i,j} E_{ij} \otimes \Phi(E_{ij})$ is separable [14]. Christandl's PPT^2 conjecture [7] asserts that $\Psi \circ \Phi$ is entanglement breaking for all PPT maps $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ and $\Psi\colon \mathrm{M}_k \to \mathrm{M}_n$.

The conjecture holds for maps on M_3 [8, 4], and Henry proved it when the intermediate algebra is M_2 [11]. The iterates of a unital or trace-preserving PPT map approach the entanglement breaking maps [16], unital PPT channels become entanglement breaking after finitely many iterations [23], and so do all PPT maps [21]. The conjecture also holds for several families defined by symmetry or by randomness [9, 24, 17, 22, 20], among them the maps on M_4 that are covariant under the unitary symplectic group [22, 20]. A counterexample was announced recently [18]: a pair of PPT maps on M_{10} whose composition is not entanglement breaking, and a trace-preserving PPT map on M_{21} whose square is not entanglement breaking. For maps on M_4 the conjecture is open.

We prove it for a class of maps defined by an operator system of M_4 rather than by a symmetry. Let $\gamma_1, \dots, \gamma_5 \in \mathrm{M}_4$ be Hermitian matrices with

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij} I \qquad (1 \le i, j \le 5), \tag{1.1}$$

put $\gamma_0 = I$, and let $\mathcal{S} = \mathrm{span}_{\mathbb{R}}\{\gamma_0, \dots, \gamma_5\}$ and $\mathcal{S}_{\mathbb{C}} = \mathcal{S} + i\mathcal{S}$. For instance, with the Pauli matrices X, Y, Z one can take $(\gamma_1, \dots, \gamma_5) = (X \otimes I,\ Y \otimes I,\ Z \otimes X,\ Z \otimes Y,\ Z \otimes Z)$. The space $\mathcal{S}$ is a spin factor whose positive semidefinite elements form a Lorentz cone (Section 2).

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Definition 1.1. A completely positive map $\Psi\colon \mathrm{M}_k \to \mathrm{M}_4$ is a *spin-factor map* if $\Psi(\mathrm{M}_k) \subseteq T\mathcal{S}_{\mathbb{C}}T^\dagger$ for some $T \in \mathrm{GL}_4(\mathbb{C})$. A completely positive map $\Phi\colon \mathrm{M}_4 \to \mathrm{M}_k$ is a spin-factor map if its adjoint Φ^* is one.

Theorem 1.2. *Let $k \geq 1$, and let $\Phi\colon \mathrm{M}_4 \to \mathrm{M}_k$ and $\Psi\colon \mathrm{M}_k \to \mathrm{M}_4$ be spin-factor maps. Then Φ and Ψ are completely copositive, and $\Psi \circ \Phi$ is entanglement breaking.*

Here the intermediate dimension k is arbitrary, and for $k \leq 4$ the theorem is a special case of the PPT^2 conjecture on M_4; the two maps may use different matrices T. Since the class contains PPT maps that are not entanglement breaking (Example 6.1), Theorem 1.2 is not a consequence of the trivial case in which one factor is entanglement breaking. The map in Example 6.1 is, in addition, covariant under the unitary symplectic group, and its square is entanglement breaking by [22, 20]; what Theorem 1.2 adds for this map are its compositions with arbitrary spin-factor maps.

The proof has three steps. First, in the basis γ_μ the Choi matrix of the composition becomes, after a local congruence, $\sum_{\mu,\nu} X_{\mu\nu}\gamma_\mu \otimes \gamma_\nu$ for a real 6×6 matrix X. A theorem of Hildebrand [13] decides when X is a sum of products of vectors in the Lorentz cone. It turns separability into two semidefinite conditions (Proposition 2.2), one of which holds automatically. Second, by the symmetry of the Lorentz cone, the other condition reduces to the trace inequality

$$\sum_{i=1}^{5} \mathrm{Tr}(F_i G_i) \leq \mathrm{Tr}(F_0 G_0) \tag{1.2}$$

for all tuples (F_μ) and (G_μ) of Hermitian $k \times k$ matrices with $\sum_\mu F_\mu \otimes \gamma_\mu \succeq 0$ and $\sum_\mu G_\mu \otimes \gamma_\mu \succeq 0$ (Proposition 4.1). Third, we prove (1.2) (Proposition 3.3). The proof rests on the operator inequality

$$\sum_{j=1}^{5} F_j F_0^{-1} F_j \preceq F_0 \qquad (F_0 \succ 0), \tag{1.3}$$

which we derive from the Kadison–Schwarz inequality for a unital completely positive map that annihilates the products $\gamma_i\gamma_j$ with $i \neq j$ (Lemma 3.1). Remark 3.2 compares this with the three Pauli matrices on $\mathbb{C}^2$.

The same structure yields a statement in which only Ψ is restricted.

Corollary 1.3. *Let $\Psi\colon \mathrm{M}_k \to \mathrm{M}_4$ be a spin-factor map, and let $\Phi\colon \mathrm{M}_m \to \mathrm{M}_k$ be completely positive with Kraus operators of rank at most two. Then $\Psi \circ \Phi$ is entanglement breaking. In particular, $\Psi \circ \Phi$ is entanglement breaking for every completely positive $\Phi\colon \mathrm{M}_2 \to \mathrm{M}_k$, and for every PPT map $\Phi\colon \mathrm{M}_3 \to \mathrm{M}_3$.*

For the last statement we use that PPT states on $\mathbb{C}^3 \otimes \mathbb{C}^3$ have Schmidt number at most two, which is the dual form of the theorem of Yang, Leung and Tang that 2-positive maps $\mathrm{M}_3 \to \mathrm{M}_3$ are decomposable [26]. In the notation (m, k, n) for compositions $\mathrm{M}_m \to \mathrm{M}_k \to \mathrm{M}_n$, Corollary 1.3 covers the cases $(2, k, 4)$ and $(3, 3, 4)$, in which only Ψ needs to be a spin-factor map; passing to adjoints yields $(4, k, 2)$ and $(4, 3, 3)$, in which only Φ does. For $k \leq 3$, Theorem 1.2 can also be deduced from Corollary 1.3 (Remark 5.2). For comparison, An and Lee showed that for maps on M_3, $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are entanglement breaking for all completely positive Φ with Kraus operators of rank at most two exactly when the Choi matrix of Ψ is 1-undistillable [1].

Maps on matrix algebras that factor through Lorentz cones were studied by Aubrun and Müller-Hermes in connection with entanglement annihilation [2]; their examples include a Breuer–Hall map and the projections onto spin factors, such as the map Π in Example 6.1. In the language of matrix convex sets, (1.3) says that the free spectrahedron of a single irreducible

tuple of five anticommuting Hermitian unitaries lies in the matrix ball. Evert and Passer proved this inclusion for the universal tuple, where the defining inequality is two-sided and can be squared [10, Proposition 2.14]. For a single irreducible tuple of five matrices, however, the defining inequality is only one-sided, and the proof of Lemma 3.1 uses complete positivity instead of squaring.

2. Preliminaries

2.1. **Notation.** We write M_n for the complex $n \times n$ matrices, Herm_n for the Hermitian ones, $A \succeq 0$ for positive semidefinite A, and E_{ij} for the matrix units. The trace inner product is $\langle A, B\rangle = \mathrm{Tr}(A^\dagger B)$. The adjoint Φ^* of a linear map $\Phi\colon \mathrm{M}_m \to \mathrm{M}_n$ is defined by $\langle \Phi^*(B), A\rangle = \langle B, \Phi(A)\rangle$. A map is completely positive if and only if $C_\Phi \succeq 0$ [6]. Then Φ^* is completely positive as well. The map Φ is completely copositive if and only if the partial transpose $C_\Phi^\Gamma = \sum_{i,j} E_{ij} \otimes \Phi(E_{ij})^T$ is positive semidefinite. An operator on $\mathbb{C}^m \otimes \mathbb{C}^n$ is separable if it is a sum of operators $P \otimes Q$ with $P, Q \succeq 0$. More generally, the Schmidt number of a positive semidefinite operator W is the least r such that W is a sum of operators $|w\rangle\langle w|$ with vectors w of Schmidt rank at most r. Separability and the Schmidt number are preserved by local congruences $W \mapsto (A \otimes B)W(A \otimes B)^\dagger$ with invertible A and B.

For linear maps $\Phi\colon \mathrm{M}_4 \to \mathrm{M}_k$ and $\Psi\colon \mathrm{M}_k \to \mathrm{M}_4$, expanding $\Phi(E_{xy})$ in matrix units gives

$$C_{\Psi\circ\Phi} = \sum_{a,b=1}^{k} \overline{\Phi^*(E_{ab})} \otimes \Psi(E_{ab}), \tag{2.1}$$

where the bar denotes entrywise complex conjugation. We write $\rho = \sum_{a,b} E_{ab} \otimes \overline{\Phi^*(E_{ab})} = \overline{C_{\Phi^*}}$ and $\sigma = \sum_{a,b} E_{ab} \otimes \Psi(E_{ab}) = C_\Psi$. Both are operators on $\mathbb{C}^k \otimes \mathbb{C}^4$. They are positive semidefinite when Φ and Ψ are completely positive. We call the 4×4 blocks $\rho_{ab} = \overline{\Phi^*(E_{ab})}$ and $\sigma_{ab} = \Psi(E_{ab})$ the *outer blocks*. By (2.1), $C_{\Psi\circ\Phi} = \sum_{a,b} \rho_{ab} \otimes \sigma_{ab}$. Similarly, $C_\Phi = \sum_{a,b} \rho_{ab} \otimes E_{ab}$ is ρ with the tensor factors swapped.

2.2. **The spin factor.** Let $\gamma_1, \dots, \gamma_5$ satisfy (1.1). They generate a representation of the complex Clifford algebra on five generators on $\mathbb{C}^4$, which is irreducible because the irreducible representations of this algebra have dimension four. The element $c = \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5$ commutes with each γ_i and satisfies $c^2 = I$, so $c = \pm I$ by Schur's lemma. We use the following standard facts.

(S1) $\mathrm{Tr}(\gamma_\mu\gamma_\nu) = 4\delta_{\mu\nu}$ for $0 \le \mu, \nu \le 5$. A product of between one and four distinct generators is traceless: for one or two generators this follows from the anticommutation relations and the cyclicity of the trace. Since $c = \pm I$, a product of three or four distinct generators equals $\pm$ a product of the remaining two or one. In particular $\gamma_0, \dots, \gamma_5$ is a basis of $\mathcal{S}_\mathbb{C}$, and $\mathrm{Tr}(\gamma_\mu\gamma_i\gamma_j) = 0$ for all μ and all $i \ne j$.

(S2) For $x \in \mathbb{R}^6$ the matrix $\sum_\mu x_\mu\gamma_\mu$ has the eigenvalues $x_0 \pm |\vec{x}|$, each twice, where $\vec{x} = (x_1, \dots, x_5)$. Hence $\sum_\mu x_\mu\gamma_\mu \succeq 0$ if and only if x lies in the Lorentz cone $L = \{x \in \mathbb{R}^6 : x_0 \ge |\vec{x}|\}$.

(S3) Two 5-tuples of Hermitian matrices on $\mathbb{C}^4$ satisfying (1.1) are unitarily equivalent if and only if their products $\gamma_1 \cdots \gamma_5$ coincide. Replacing a tuple by its negative changes the sign of the product, so any two spin factors of this form are unitarily equivalent. The tuple (γ_i^T) also satisfies (1.1). Its product is $\gamma_1^T \cdots \gamma_5^T = (\gamma_5 \cdots \gamma_1)^T = c^T = c$, since reversing the order of five anticommuting factors takes ten transpositions and $c = \pm I$. Consequently, there is a unitary V with $V\gamma_\mu V^\dagger = \gamma_\mu^T$ for all μ, so that $Y^T = VYV^\dagger$ for all $Y \in \mathcal{S}_\mathbb{C}$.

(S4) Let $\mathrm{SO}^+(1,5)$ be the identity component of the Lorentz group of the form $\eta = \mathrm{diag}(1,-1,-1,-1,-1,-1)$. For every $\Lambda \in \mathrm{SO}^+(1,5)$ there is $g \in \mathrm{GL}_4(\mathbb{C})$ with

$g\gamma_\mu g^\dagger = \sum_\nu \Lambda_{\nu\mu}\gamma_\nu$ for all μ. Indeed, $\exp(t\gamma_i/2)$ acts on $\mathcal{S}$ as the boost in the $(0,i)$-plane with rapidity t. The unitary $\exp(t\gamma_i\gamma_j/2)$ acts as a rotation by the angle t in the (i,j)-plane. These one-parameter groups generate $\mathrm{SO}^+(1,5)$.

For a real 6×6 matrix X put

$$\mathrm{op}(X) = \sum_{\mu,\nu=0}^{5} X_{\mu\nu}\,\gamma_\mu \otimes \gamma_\nu \in \mathrm{M}_4 \otimes \mathrm{M}_4.$$

This map is injective by (S1). If g and h realize $\Lambda_1, \Lambda_2 \in \mathrm{SO}^+(1,5)$ as in (S4), then

$$(g\otimes h)\,\mathrm{op}(X)\,(g\otimes h)^\dagger = \mathrm{op}(\Lambda_1 X \Lambda_2^T). \tag{2.2}$$

Lemma 2.1. *The matrices $\gamma_i \otimes \gamma_i$, $1 \le i \le 5$, commute pairwise. Their joint eigenspaces are one-dimensional and are indexed by the sixteen sign vectors $s \in \{\pm 1\}^5$ with $s_1 s_2 s_3 s_4 s_5 = 1$. Consequently, for $x \in \mathbb{R}^6$,*

(a) the eigenvalues of $\mathrm{op}(\mathrm{diag}(x))$ are $x_0 + \langle s, \vec{x}\rangle$, one for each such s;
(b) the eigenvalues of $\mathrm{op}(\mathrm{diag}(x)\eta)$ are $x_0 + \langle s, \vec{x}\rangle$, one for each $s \in \{\pm 1\}^5$ with $s_1 s_2 s_3 s_4 s_5 = -1$;
(c) if ξ_s is a unit vector in the joint eigenspace for s, then $\langle \xi_s | \gamma_\mu \otimes \gamma_\nu | \xi_s\rangle = \delta_{\mu\nu} s_\mu$, with $s_0 = 1$.

Proof. For $i \neq j$ the factors $\gamma_i\gamma_j = -\gamma_j\gamma_i$ in $(\gamma_i\otimes\gamma_i)(\gamma_j\otimes\gamma_j) = \gamma_i\gamma_j \otimes \gamma_i\gamma_j$ contribute two signs, so the matrices commute. Their product is $c \otimes c = I$, so each joint eigenvalue vector s satisfies $s_1 \cdots s_5 = 1$. For such s the projection onto the joint eigenspace is $P_s = 2^{-5}\sum_A s_A \gamma_A \otimes \gamma_A$, where A runs over the subsets of $\{1,\dots,5\}$, $s_A = \prod_{i\in A} s_i$, and γ_A is the ordered product of the γ_i with $i \in A$. By (S1) only $A = \emptyset$ and $A = \{1,\dots,5\}$ contribute to the trace, so $\mathrm{Tr}\, P_s = 2^{-5}(16+16) = 1$. This proves the first statement and (a). Part (b) follows from (a), since $\mathrm{diag}(x)\eta = \mathrm{diag}(x_0, -\vec{x})$ and $s \mapsto -s$ exchanges the two parities. For (c), let $\mu \neq \nu$. If one of the indices is 0 and the other is j, then $\gamma_\mu \otimes \gamma_\nu$ anticommutes with $\gamma_i \otimes \gamma_i$ for any $i \neq j$. If $\mu = i$ and $\nu = j$ are both nonzero, it anticommutes with $\gamma_i \otimes \gamma_i$. An operator that anticommutes with a Hermitian unitary has zero expectation in each of its eigenvectors. The diagonal entries are 1 for $\mu = \nu = 0$ and s_i for $\mu = \nu = i$. □

2.3. **Lorentz cones.** Let $\mathrm{Sep} \subset \mathbb{R}^{6\times 6}$ be the closed convex cone generated by the matrices ab^T with $a, b \in L$, and let Pos be the cone of matrices X with $a^T X b \ge 0$ for all $a, b \in L$, that is, the linear maps that take L into L. The group $\mathrm{SO}^+(1,5) \times \mathrm{SO}^+(1,5)$ acts on both cones by $X \mapsto \Lambda_1 X \Lambda_2^T$. We use three facts from Hildebrand's paper [13], citing its numbered results from the preprint version [12].

(H1) X lies in the interior of Pos if and only if Xb lies in the interior of L for every nonzero $b \in L$ [12, Lemma 3.5].
(H2) For every X in the interior of Pos there are $\Lambda_1, \Lambda_2 \in \mathrm{SO}^+(1,5)$ such that $\Lambda_1 X \Lambda_2^T$ is diagonal [12, Corollary 3.7].
(H3) A diagonal matrix D lies in Sep if and only if $D_{00} \ge \sum_{i=1}^5 |D_{ii}|$ [12, Lemma 3.12].

The next proposition is the case of two six-dimensional Lorentz cones in Hildebrand's linear matrix inequality for Sep [12, Theorem 4.11]. We include the short derivation in our notation.

Proposition 2.2. *A matrix $X \in \mathbb{R}^{6\times 6}$ lies in* Sep *if and only if* $\mathrm{op}(X) \succeq 0$ *and* $\mathrm{op}(X\eta) \succeq 0$.

Proof. For $a, b \in L$ put $A = \sum_\mu a_\mu \gamma_\mu$ and $B = \sum_\nu b_\nu \gamma_\nu$. Then $\mathrm{op}(ab^T) = A \otimes B$ and $\mathrm{op}(ab^T\eta) = A \otimes \tilde{B}$ with $\tilde{B} = b_0 I - \sum_i b_i \gamma_i$. All three matrices are positive semidefinite by (S2), which proves the forward implication.

Conversely, suppose $\operatorname{op}(X) \succeq 0$ and $\operatorname{op}(X\eta) \succeq 0$. We first assume that $\operatorname{op}(X)$ is positive definite. For nonzero $a, b \in L$ we have $16\, a^T X b = \operatorname{Tr}\big(\operatorname{op}(X)(A \otimes B)\big) > 0$ by (S1), since $A \otimes B$ is nonzero and positive semidefinite. So X maps $L\setminus\{0\}$ into the interior of L, which places X in the interior of Pos by (H1). By (H2) there are $\Lambda_1, \Lambda_2 \in \mathrm{SO}^+(1,5)$ such that $D = \Lambda_1 X \Lambda_2^T$ is diagonal. By (2.2), $\operatorname{op}(D)$ is congruent to $\operatorname{op}(X)$. Since $\eta\Lambda_2\eta \in \mathrm{SO}^+(1,5)$ and $D\eta = \Lambda_1(X\eta)(\eta\Lambda_2\eta)^T$, also $\operatorname{op}(D\eta)$ is congruent to $\operatorname{op}(X\eta)$. Since both are positive semidefinite, Lemma 2.1 yields $D_{00} + \langle s, \vec{d}\rangle \geq 0$ for all $s \in \{\pm 1\}^5$, where $\vec{d} = (D_{11}, \ldots, D_{55})$. Choosing $s_i = -\operatorname{sign}(D_{ii})$ shows $D_{00} \geq \sum_i |D_{ii}|$. Then $D \in \mathrm{Sep}$ by (H3), and $X = \Lambda_1^{-1} D \Lambda_2^{-T} \in \mathrm{Sep}$. In general, $X + \varepsilon e_0 e_0^T$ satisfies both conditions with $\operatorname{op}(X + \varepsilon e_0 e_0^T) = \operatorname{op}(X) + \varepsilon I \succ 0$ and hence lies in Sep. Since Sep is closed, $X \in \mathrm{Sep}$. □

3. A matrix Lorentz inequality

Let $\mathcal{F}_k$ be the set of tuples $F = (F_0, \ldots, F_5)$ of Hermitian $k \times k$ matrices such that

$$\rho_F = \sum_{\mu=0}^{5} F_\mu \otimes \gamma_\mu \succeq 0 \quad \text{on } \mathbb{C}^k \otimes \mathbb{C}^4.$$

The outer blocks of ρ_F lie in $\mathcal{S}_{\mathbb{C}}$. Conversely, a positive semidefinite operator on $\mathbb{C}^k \otimes \mathbb{C}^4$ whose outer blocks lie in $\mathcal{S}_{\mathbb{C}}$ equals ρ_F for a unique $F \in \mathcal{F}_k$, by (S1). By (S3), $\rho_F^\Gamma = (I \otimes V)\rho_F(I \otimes V)^\dagger \succeq 0$, where Γ is the partial transpose on $\mathbb{C}^4$. This identity characterizes the operators with outer blocks in $\mathcal{S}_{\mathbb{C}}$. Indeed, if $\rho^\Gamma = (I \otimes V)\rho(I \otimes V)^\dagger$ for an operator ρ on $\mathbb{C}^k \otimes \mathbb{C}^4$, then its outer blocks are fixed by the linear map $\tau(Y) = V^\dagger Y^T V$ on M_4. This map fixes each γ_μ and maps $\gamma_i\gamma_j$ to $\gamma_j\gamma_i = -\gamma_i\gamma_j$. The matrices γ_μ and $\gamma_i\gamma_j$ with $i < j$ are pairwise orthogonal by (S1) and hence form a basis of M_4. Thus $\mathcal{S}_{\mathbb{C}}$ is the fixed space of τ. With V replaced by the identity, the identity $\rho^\Gamma = \rho$ defines the operators that are invariant under partial transposition. Chiribella, Davidson, Paulsen and Rahaman studied the maps with such Choi matrices and the corresponding analogue of the PPT² conjecture [5].

Lemma 3.1. *Let $F \in \mathcal{F}_k$ with $F_0 \succ 0$. Then*

$$\sum_{j=1}^{5} F_j F_0^{-1} F_j \preceq F_0.$$

Proof. Put $\hat{F}_\mu = F_0^{-1/2} F_\mu F_0^{-1/2}$ and $\hat{\rho} = \frac{1}{4}(F_0^{-1/2} \otimes I)\rho_F(F_0^{-1/2} \otimes I) = \frac{1}{4}\sum_\mu \hat{F}_\mu \otimes \gamma_\mu$. Then $\hat{\rho} \succeq 0$, $\hat{\rho}^\Gamma \succeq 0$ and $\hat{F}_0 = I$. Define $\Theta\colon \mathrm{M}_4 \to \mathrm{M}_k$ by

$$\Theta(W) = \operatorname{Tr}_2\big[\hat{\rho}\,(I \otimes W)\big] = \operatorname{Tr}_2\big[\hat{\rho}^\Gamma (I \otimes W^T)\big],$$

where Tr_2 is the partial trace over $\mathbb{C}^4$. If $\hat{\rho}^\Gamma = \sum_t |z_t\rangle\langle z_t|$ with $|z_t\rangle = \sum_{a,x}(M_t)_{ax}|a\rangle \otimes |x\rangle$, then $\Theta(W) = \sum_t M_t W M_t^\dagger$, so Θ is completely positive. By (S1), $\Theta(\gamma_\mu) = \hat{F}_\mu$ for all μ. In particular $\Theta(I) = I$, and

$$\Theta(\gamma_i\gamma_j) = \tfrac{1}{4}\sum_\mu \hat{F}_\mu \operatorname{Tr}(\gamma_\mu\gamma_i\gamma_j) = 0 \qquad (i \neq j).$$

Let $\Theta_5 = \mathrm{id}_5 \otimes \Theta$, which is unital and completely positive on $\mathrm{M}_5 \otimes \mathrm{M}_4$, and let $N = \sum_{j=1}^5 E_{1j} \otimes \gamma_j$. By Stinespring's theorem [25], $\Theta_5(W) = U^\dagger \pi(W) U$ for a $*$-representation π and an isometry U, so $\Theta_5(N)^\dagger\Theta_5(N) = U^\dagger\pi(N)^\dagger U U^\dagger \pi(N) U \preceq \Theta_5(N^\dagger N)$, which is the Kadison–Schwarz inequality [15]. Here $N^\dagger N = \sum_{i,j} E_{ij} \otimes \gamma_i\gamma_j$, so $\Theta_5(N^\dagger N) = I_5 \otimes I_k$. On the other side, $\Theta_5(N)^\dagger\Theta_5(N) = \sum_{i,j} E_{ij} \otimes \hat{F}_i\hat{F}_j = R^\dagger R$ for the row $R = (\hat{F}_1, \ldots, \hat{F}_5)$. Thus $\|R\| \leq 1$, so $RR^\dagger = \sum_j \hat{F}_j^2 \preceq I$. Conjugating by $F_0^{1/2}$ gives the claim. □

Remark 3.2. The proof of Lemma 3.1 uses two properties of $\gamma_1, \dots, \gamma_5$. By (S3), $\hat{\rho}^\Gamma \succeq 0$, so Θ is completely positive. The products $\gamma_i\gamma_j$ with $i \neq j$ are orthogonal to $\mathcal{S}$, so $\Theta_5(N^\dagger N) = I_5 \otimes I_k$. Both properties fail for the Pauli matrices X, Y, Z on $\mathbb{C}^2$: transposition acts on $\operatorname{span}_{\mathbb{R}}\{X, Y, Z\}$ as a reflection, which is not induced by a unitary, and $XY = iZ$. The analogue of the lemma fails as well: $I \otimes I - X \otimes X - Y \otimes Y - Z \otimes Z$ is four times the projection onto the antisymmetric subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2$, while $X^2 + Y^2 + Z^2 = 3I$. The partial transpose of this operator is not positive semidefinite, however, and with positive partial transpose the analogue holds, by a different argument. Let $\rho = I \otimes I + \sum_j F_j \otimes \sigma_j$ with $(\sigma_1, \sigma_2, \sigma_3) = (X, Y, Z)$ satisfy $\rho \succeq 0$ and $\rho^\Gamma \succeq 0$. Conjugation by $I \otimes Y$ maps ρ^Γ to $I \otimes I - \sum_j F_j \otimes \sigma_j$, so $-I \preceq \sum_j F_j \otimes \sigma_j \preceq I$. Squaring and applying $\frac{1}{2}\operatorname{Tr}_2$ gives $\sum_j F_j^2 \preceq I$, as in [10, Proposition 2.14]. For $\gamma_1, \dots, \gamma_5$ there is no such change of sign, since a unitary that anticommutes with all five would anticommute with $c = \pm I$.

Proposition 3.3. *For all $F, G \in \mathcal{F}_k$,*

$$\sum_{i=1}^{5} \operatorname{Tr}(F_i G_i) \leq \operatorname{Tr}(F_0 G_0).$$

Proof. Adding εI to F_0 and G_0 keeps both tuples in $\mathcal{F}_k$, so by continuity we may assume $F_0, G_0 \succ 0$. Put $A_i = F_0^{-1/2} F_i G_0^{1/2}$ and $B_i = G_0^{-1/2} G_i F_0^{1/2}$. Then $\operatorname{Tr}(F_i G_i) = \operatorname{Tr}(A_i B_i)$. By the Cauchy–Schwarz inequality,

$$\sum_i \operatorname{Tr}(F_i G_i) \leq \Big(\sum_i \operatorname{Tr}(A_i A_i^\dagger)\Big)^{1/2} \Big(\sum_i \operatorname{Tr}(B_i^\dagger B_i)\Big)^{1/2}.$$

By Lemma 3.1, $\sum_i \operatorname{Tr}(A_i A_i^\dagger) = \operatorname{Tr}\big(G_0 \sum_i F_i F_0^{-1} F_i\big) \leq \operatorname{Tr}(G_0 F_0)$. In the same way, $\sum_i \operatorname{Tr}(B_i^\dagger B_i) = \operatorname{Tr}\big(F_0 \sum_i G_i G_0^{-1} G_i\big) \leq \operatorname{Tr}(F_0 G_0)$. □

4. Proof of Theorem 1.2

4.1. **Normal form.** Let Φ and Ψ be as in Theorem 1.2, with $\Phi^*(\mathrm{M}_k) \subseteq T_1 \mathcal{S}_{\mathbb{C}} T_1^\dagger$ and $\Psi(\mathrm{M}_k) \subseteq T_2 \mathcal{S}_{\mathbb{C}} T_2^\dagger$. The outer blocks of $\rho = \overline{C_{\Phi^*}}$ lie in $\overline{T_1}\,\overline{\mathcal{S}_{\mathbb{C}}}\,\overline{T_1}^\dagger$. The tuple $(\overline{\gamma_i})$ also satisfies (1.1), so by (S3) there is a unitary V_1 with $\overline{\mathcal{S}_{\mathbb{C}}} = V_1 \mathcal{S}_{\mathbb{C}} V_1^\dagger$. With $A = (\overline{T_1} V_1)^{-1}$ and $B = T_2^{-1}$, the operators $\rho' = (I \otimes A)\rho(I \otimes A)^\dagger$ and $\sigma' = (I \otimes B)\sigma(I \otimes B)^\dagger$ are positive semidefinite with outer blocks in $\mathcal{S}_{\mathbb{C}}$. So $\rho' = \rho_F$ and $\sigma' = \rho_G$ for some $F, G \in \mathcal{F}_k$. Since $\rho_F^\Gamma \succeq 0$ and $\rho_G^\Gamma \succeq 0$, the operators ρ^Γ and σ^Γ are positive semidefinite, so Φ^* and Ψ, and hence Φ, are completely copositive. This proves the first claim of Theorem 1.2.

By (2.1), $(A \otimes B) C_{\Psi \circ \Phi} (A \otimes B)^\dagger = \sum_{a,b} \rho'_{ab} \otimes \sigma'_{ab}$. Expanding the blocks in the basis γ_μ, we obtain

$$(A \otimes B) C_{\Psi \circ \Phi} (A \otimes B)^\dagger = \operatorname{op}(X(F, G)), \qquad X(F, G)_{\mu\nu} = \operatorname{Tr}\big(F_\mu G_\nu^T\big). \tag{4.1}$$

It therefore suffices to show that $\operatorname{op}(X(F, G))$ is separable for all $F, G \in \mathcal{F}_k$. If $X(F, G) = \sum_t a_t b_t^T$ with $a_t, b_t \in L$, then $\operatorname{op}(X(F, G)) = \sum_t A_t \otimes B_t$ with $A_t = \sum_\mu (a_t)_\mu \gamma_\mu \succeq 0$ and $B_t = \sum_\nu (b_t)_\nu \gamma_\nu \succeq 0$. So it suffices to show $X(F, G) \in \mathrm{Sep}$. The operator $\operatorname{op}(X(F, G))$ is congruent to the Choi matrix of the completely positive map $\Psi \circ \Phi$, hence positive semidefinite. By Proposition 2.2, Theorem 1.2 follows from the next proposition.

Proposition 4.1. *For all $F, G \in \mathcal{F}_k$, $\operatorname{op}(X(F, G)\eta) \succeq 0$.*

4.2. **The dual cone.** Let $\mathcal{P} = \{Y \in \mathbb{R}^{6\times 6} : \mathrm{op}(Y) \succeq 0\}$ and let $\mathcal{P}^* = \{Z : \mathrm{Tr}(Y^T Z) \geq 0 \text{ for all } Y \in \mathcal{P}\}$ be its dual cone. Since $\mathcal{P}$ is closed and convex, $Y \in \mathcal{P}$ if and only if $\mathrm{Tr}(Y^T Z) \geq 0$ for all $Z \in \mathcal{P}^*$.

Lemma 4.2. *(a) $\mathcal{P}$ and $\mathcal{P}^*$ are invariant under $Y \mapsto \Lambda_1 Y \Lambda_2^T$ for $\Lambda_1, \Lambda_2 \in \mathrm{SO}^+(1,5)$.*

(b) A diagonal matrix lies in $\mathcal{P}^*$ if and only if it is a nonnegative combination of the matrices $\mathrm{diag}(1,s)$ with $s \in \{\pm 1\}^5$ and $s_1 s_2 s_3 s_4 s_5 = 1$.

(c) Every Z in the interior of $\mathcal{P}^*$ equals $\Lambda_1 D \Lambda_2^T$ for some $\Lambda_1, \Lambda_2 \in \mathrm{SO}^+(1,5)$ and some diagonal $D \in \mathcal{P}^*$.

Proof. (a) The invariance of $\mathcal{P}$ follows from (2.2). Since the transpose of a matrix in $\mathrm{SO}^+(1,5)$ is again in $\mathrm{SO}^+(1,5)$, the identity $\mathrm{Tr}(Y^T \Lambda_1 Z \Lambda_2^T) = \mathrm{Tr}((\Lambda_1^T Y \Lambda_2)^T Z)$ implies the invariance of $\mathcal{P}^*$.

(b) For every Y, $\mathrm{Tr}(Y^T Z_\xi) = \langle \xi | \mathrm{op}(Y) | \xi \rangle$, where $(Z_\xi)_{\mu\nu} = \langle \xi | \gamma_\mu \otimes \gamma_\nu | \xi \rangle$. So $Z_\xi \in \mathcal{P}^*$ for all vectors ξ. In particular $\mathrm{diag}(1,s) = Z_{\xi_s} \in \mathcal{P}^*$ for even s by Lemma 2.1(c). Conversely, let $\mathrm{diag}(d) \in \mathcal{P}^*$. By Lemma 2.1(a), $\mathrm{diag}(x) \in \mathcal{P}$ whenever $x_0 + \langle s, \vec{x} \rangle \geq 0$ for all even s. For such x we get $\langle x, d \rangle \geq 0$. By Farkas' lemma, d is a nonnegative combination of the vectors $(1,s)$ with s even.

(c) If $a, b \in L$, then $ab^T \in \mathcal{P}$ by the proof of Proposition 2.2, so every $Z \in \mathcal{P}^*$ satisfies $a^T Z b \geq 0$. Hence $\mathcal{P}^* \subseteq \mathrm{Pos}$, and interior points of $\mathcal{P}^*$ are interior points of Pos. By (H2) there are $\Lambda_1, \Lambda_2 \in \mathrm{SO}^+(1,5)$ with $D = \Lambda_1^{-1} Z \Lambda_2^{-T}$ diagonal. By (a), $D \in \mathcal{P}^*$. □

4.3. **Proof of Proposition 4.1.** Fix $F, G \in \mathcal{F}_k$ and write $X = X(F,G)$. Since op is injective, the cone $\mathcal{P}$ contains no line. Therefore $\mathcal{P}^*$ has nonempty interior and is the closure of its interior. It suffices to show $\mathrm{Tr}(\eta X^T Z) \geq 0$, that is, $\mathrm{Tr}((X\eta)^T Z) \geq 0$, for every Z in the interior of $\mathcal{P}^*$. By Lemma 4.2, $Z = \sum_s c_s \Lambda_1 \mathrm{diag}(1,s) \Lambda_2^T$ with $c_s \geq 0$, the sum running over even s. For each term,

$$\mathrm{Tr}\big((X\eta)^T \Lambda_1 \mathrm{diag}(1,s) \Lambda_2^T\big) = \mathrm{Tr}\big((\Lambda_1^T X \Lambda_3 \eta)^T \mathrm{diag}(1,s)\big), \qquad \Lambda_3 = \eta \Lambda_2 \eta \in \mathrm{SO}^+(1,5),$$

since $\eta \Lambda_2 = \Lambda_3 \eta$. Let $g, h \in \mathrm{GL}_4(\mathbb{C})$ realize Λ_1^T and Λ_3^T as in (S4). The operators $(I \otimes g)\rho_F (I \otimes g)^\dagger$ and $(I \otimes h)\rho_G (I \otimes h)^\dagger$ are positive semidefinite with outer blocks in $\mathcal{S}_\mathbb{C}$, so they equal $\rho_{F'}$ and $\rho_{G'}$ with $F' = \Lambda_1^T F$ and $G' = \Lambda_3^T G$. Here $(\Lambda F)_\nu = \sum_\mu \Lambda_{\nu\mu} F_\mu$, and $X(F',G') = \Lambda_1^T X \Lambda_3$. The term above therefore equals

$$X(F',G')_{00} - \sum_{i=1}^5 s_i \, X(F',G')_{ii}.$$

Since s is even, the set of indices i with $s_i = -1$ splits into pairs $\{i,j\}$. For each pair, conjugation by the unitary $\gamma_i \gamma_j$ changes the signs of γ_i and γ_j and fixes the other γ_μ. Let u be the product of these unitaries. Then $(I \otimes u)\rho_{G'}(I \otimes u)^\dagger = \rho_{G''}$ with $G''_0 = G'_0$ and $G''_i = s_i G'_i$. The term equals $X(F',G'')_{00} - \sum_i X(F',G'')_{ii}$. Finally, the full transpose of $\rho^\Gamma_{G''} \succeq 0$ is $\sum_\mu G''^T_\mu \otimes \gamma_\mu$, so the tuple $(G''^T_\mu)_\mu$ lies in $\mathcal{F}_k$. Proposition 3.3 applied to F' and this tuple gives

$$X(F',G'')_{00} - \sum_{i=1}^5 X(F',G'')_{ii} = \mathrm{Tr}\big(F'_0 G''^T_0\big) - \sum_{i=1}^5 \mathrm{Tr}\big(F'_i G''^T_i\big) \geq 0.$$

Hence every term is nonnegative, which proves Proposition 4.1 and Theorem 1.2. □

5. Kraus operators of rank two

Proposition 5.1. *Let $T \in \mathrm{GL}_4(\mathbb{C})$. Every positive semidefinite operator on $\mathbb{C}^2 \otimes \mathbb{C}^4$ whose outer blocks lie in $T\mathcal{S}_\mathbb{C}T^\dagger$ is separable.*

Proof. After the congruence by $I \otimes T^{-1}$ the operator is $W = \sum_{a=0}^{3}\sum_{\mu=0}^{5} X_{a\mu}\sigma_a \otimes \gamma_\mu$ with $X \in \mathbb{R}^{4\times 6}$, where $\sigma_0 = I$ and $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices. Let $L_4 \subset \mathbb{R}^4$ be the Lorentz cone. For $a \in L_4$ and $b \in L$ we have $8\, a^T X b = \operatorname{Tr}\big(W(\sum_\alpha a_\alpha\sigma_\alpha \otimes \sum_\mu b_\mu\gamma_\mu)\big) \geq 0$, so X maps L into L_4. Replacing X by $X + \varepsilon e_0 e_0^T$, which replaces W by $W + \varepsilon I$, we may assume $W \succ 0$, as in the proof of Proposition 2.2. Then the trace above is positive for nonzero a and b. So Xb lies in the interior of L_4 for all nonzero $b \in L$, and X is an interior point of the cone of maps from L to L_4 [12, Lemma 3.5]. Hildebrand's diagonalization [12, Corollary 3.7] provides $\Lambda \in \mathrm{SO}^+(1,3)$ and $\Lambda' \in \mathrm{SO}^+(1,5)$ such that $D = \Lambda X \Lambda'^T$ is diagonal, $D = \sum_{a=0}^{3} d_a e_a e_a^T$. The group $\mathrm{SL}_2(\mathbb{C})$ acts on $\operatorname{span}_{\mathbb{R}}\{\sigma_a\}$ by congruence through $\mathrm{SO}^+(1,3)$, so $\sum_a d_a\sigma_a \otimes \gamma_a$ is congruent to W and positive semidefinite. The three matrices $\sigma_i \otimes \gamma_i$, $i = 1,2,3$, are commuting Hermitian unitaries. Each product of a nonempty subset of them is traceless. For subsets of size one or two the Pauli factor is traceless, while $\sigma_1\sigma_2\sigma_3 \otimes \gamma_1\gamma_2\gamma_3 = iI \otimes \gamma_1\gamma_2\gamma_3$ is traceless by (S1). As in Lemma 2.1, all sign vectors in $\{\pm 1\}^3$ occur as joint eigenvalues, so $d_0 + \sum_{i=1}^{3} s_i d_i \geq 0$ for all s, that is, $d_0 \geq \sum_i |d_i|$. By [12, Lemma 3.12], D is a sum of matrices ab^T with $a \in L_4$ and $b \in L$. Hence so is X, and W is separable. Closedness of the separable cone handles the limit $\varepsilon \to 0$. □

Proof of Corollary 1.3. Write $\Phi = \sum_t K_t(\cdot)K_t^\dagger$ with $\operatorname{rank} K_t \leq 2$, so that $\Psi \circ \Phi = \sum_t \Psi \circ \mathrm{Ad}_{K_t}$, and fix one term $K = \sum_{j=1}^{2} |u_j\rangle\langle v_j|$ with $u_j \in \mathbb{C}^k$ and $v_j \in \mathbb{C}^m$. A direct computation gives

$$C_{\Psi\circ\mathrm{Ad}_K} = \sum_{j,l=1}^{2} |\overline{v_j}\rangle\langle\overline{v_l}| \otimes \Psi\big(|u_j\rangle\langle u_l|\big).$$

This operator is positive semidefinite, it is supported on $\operatorname{span}\{\overline{v_1}, \overline{v_2}\} \otimes \mathbb{C}^4$, and its outer blocks with respect to an orthonormal basis of this span lie in $\Psi(\mathrm{M}_k) \subseteq T\mathcal{S}_{\mathbb{C}}T^\dagger$. By Proposition 5.1 it is separable, and so is the sum over t.

Every completely positive map on M_2 has Kraus operators of rank at most two. If $\Phi\colon \mathrm{M}_3 \to \mathrm{M}_3$ is PPT, then C_Φ is a PPT state on $\mathbb{C}^3 \otimes \mathbb{C}^3$, which has Schmidt number at most two [26]. Then $C_\Phi = \sum_t |w_t\rangle\langle w_t|$ with vectors w_t of Schmidt rank at most two, which correspond to Kraus operators of rank at most two. □

Remark 5.2. For $k \leq 3$, Theorem 1.2 also follows from Corollary 1.3. Let $F \in \mathcal{F}_k$, and let P be a projection of rank two on $\mathbb{C}^k$. The compression $(P \otimes I)\rho_F(P \otimes I)$, viewed as an operator on $P\mathbb{C}^k \otimes \mathbb{C}^4$, is positive semidefinite with outer blocks in $\mathcal{S}_{\mathbb{C}}$, hence separable by Proposition 5.1. For $k \leq 2$ this shows that ρ_F is separable. For $k = 3$, a lemma of Chen, Yang and Tang [3, Lemma 15(i)] shows that ρ_F has Schmidt number at most two. In the setting of Theorem 1.2, C_Φ is ρ with the tensor factors swapped, and ρ is a local congruence of some ρ_F (Section 4). So Φ has Kraus operators of rank at most two, and Corollary 1.3 applies. We do not know whether ρ_F has Schmidt number at most two for all k. If it does, Theorem 1.2 is a special case of Corollary 1.3.

6. An example

Example 6.1. Take $(\gamma_1, \ldots, \gamma_5) = (X \otimes I,\ Y \otimes I,\ Z \otimes X,\ Z \otimes Y,\ Z \otimes Z)$ and

$$\rho_0 = \tfrac{1}{48}\operatorname{op}\big(\operatorname{diag}(3,1,1,1,1,1)\big) = \tfrac{1}{48}\Big(3\, I \otimes I + \sum_{i=1}^{5} \gamma_i \otimes \gamma_i\Big).$$

By Lemma 2.1(a) the eigenvalues of $48\rho_0$ are $3 + \langle s, (1,1,1,1,1)\rangle$ over the even s, that is, 8 once, 4 ten times and 0 five times. So ρ_0 is a state. Let $\Lambda_0\colon \mathrm{M}_4 \to \mathrm{M}_4$ be the completely positive map with $C_{\Lambda_0} = 4\rho_0$. Its values $\Lambda_0(E_{ij})$ are the outer blocks of $4\rho_0$, so $\Lambda_0(\mathrm{M}_4) \subseteq \mathcal{S}_{\mathbb{C}}$.

Let $V = X \otimes Y$. Then V is a Hermitian unitary with $V^T = -V$ and $V\gamma_\mu V = \gamma_\mu^T$ for all μ. Let $\Delta(A) = \frac{1}{4}\operatorname{Tr}(A)I$, let $\top$ be the transpose map, and let $\mathrm{Ad}_V(A) = VAV$. We claim that

$$\Lambda_0 = \tfrac{2}{3}\Delta + \tfrac{1}{6}\big(\top + \mathrm{Ad}_V\big). \tag{6.1}$$

The Choi matrices of Δ, $\top$ and Ad_V are $\frac{1}{4}I \otimes I$, the swap operator and $(I \otimes V)\big(\sum_{i,j} E_{ij} \otimes E_{ij}\big)(I \otimes V)$. The matrices I, γ_i and $i\gamma_i\gamma_j$ with $i < j$ form an orthogonal basis $\mathcal{B}$ of M_4 consisting of Hermitian unitaries. In this basis the last two Choi matrices are $\frac{1}{4}\sum_{B\in\mathcal{B}} B \otimes B$ and $\frac{1}{4}\sum_{B\in\mathcal{B}} B \otimes VB^TV$. Here $VB^TV = B$ for $B \in \{I, \gamma_1, \dots, \gamma_5\}$ and $VB^TV = -B$ for the other elements of $\mathcal{B}$. So the Choi matrix of the right side of (6.1) is $\frac{1}{6}I\otimes I + \frac{1}{12}\big(I\otimes I + \sum_i \gamma_i \otimes \gamma_i\big) = 4\rho_0$. Since Δ, $\top$ and Ad_V are self-adjoint, $\Lambda_0^* = \Lambda_0$. Thus Λ_0 is a spin-factor map in both senses of Definition 1.1, and it is PPT by Theorem 1.2.

If U is a unitary with $U^TVU = V$, then $\Lambda_0(UAU^\dagger) = \overline{U}\Lambda_0(A)U^T$. So Λ_0 is conjugate symplectic covariant in the sense of [20, Remark 2.11]. By [20, Theorem 5.3], $\Lambda_0 \circ \Lambda_0$ is entanglement breaking (see also [22]). The state ρ_0 is the Pál–Vértesi state for $d = 4$ [19], in the form of [20, Example 2.14] with V in place of Ω_4.

The state ρ_0 is entangled [19]. We include the short proof in our notation. The projection $\Pi(A) = \frac{1}{4}\sum_\mu \operatorname{Tr}(A\gamma_\mu)\gamma_\mu$ onto $\mathcal{S}_\mathbb{C}$ is a positive map: if $A \succeq 0$ and $e \in \mathbb{R}^5$ is a unit vector, then $\operatorname{Tr}(A\sum_i e_i\gamma_i) \le \operatorname{Tr} A$, since $\sum_i e_i\gamma_i$ has eigenvalues ± 1. Thus $(\operatorname{Tr} A\gamma_\mu)_\mu \in L$. If $\rho_0 = \sum_t P_t \otimes Q_t$ with $P_t, Q_t \succeq 0$, then applying $\Pi \otimes \Pi$, which fixes ρ_0, writes $\operatorname{diag}(3,1,1,1,1,1)$ as an element of Sep. This contradicts (H3), since $3 < 5$. Hence Λ_0 is not entanglement breaking. By Theorem 1.2, $\Psi \circ \Lambda_0$ and $\Lambda_0 \circ \Phi$ are entanglement breaking for all completely positive maps $\Psi, \Phi\colon \mathrm{M}_4 \to \mathrm{M}_4$ with $\Psi(\mathrm{M}_4) \subseteq T_1\mathcal{S}_\mathbb{C}T_1^\dagger$ and $\Phi^*(\mathrm{M}_4) \subseteq T_2\mathcal{S}_\mathbb{C}T_2^\dagger$, where $T_1, T_2 \in \mathrm{GL}_4(\mathbb{C})$.

Remark 6.2. In the terminology of [8], Proposition 5.1 says that each spin-factor map $\Psi\colon \mathrm{M}_k \to \mathrm{M}_4$ is 2-entanglement breaking: $(\mathrm{id}_2 \otimes \Psi)(Y)$ is positive semidefinite with outer blocks in $T\mathcal{S}_\mathbb{C}T^\dagger$, hence separable, for all positive semidefinite $Y \in \mathrm{M}_2 \otimes \mathrm{M}_k$. For maps on M_3 the analogous property holds for all PPT maps, since PPT operators on $\mathbb{C}^2 \otimes \mathbb{C}^3$ are separable; this is where the proof for M_3 in [8] starts. On M_4, by contrast, it fails for general PPT maps, because PPT operators on $\mathbb{C}^2 \otimes \mathbb{C}^4$ can be entangled.

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