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A thrackle on n vertices has fewer than $4n/3$ edges

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Abstract. A thrackle is a drawing of a graph in the plane in which every two edges meet exactly once, either at a common endpoint or at a proper crossing. Conway conjectured that a thrackle with n vertices has at most n edges. We prove that a thrackle with $n \geq 4$ vertices has at most $\frac{4}{3}(n-1)$ edges, improving the bound $\frac{18}{13}(n-1)$ obtained by Tóth and, independently, by Scoppetto. The proof combines Xu's discharging argument with two computer searches, by the method of Fulek and Pach, showing that two generalized dumbbells in the sense of Fulek and Pach are not thrackleable: $\mathrm{DB}(6, 8, -1)$ is a 6-cycle and an 8-cycle sharing one edge; $\mathrm{DB}(6, 10, -3)$ is a 6-cycle and a 10-cycle sharing a path of three edges. A formal proof in Lean 4 derives the bound from five properties of thrackles and checks the computational facts it uses; Section 6 states what this proof trusts.

1. Introduction

A *drawing* of a graph in the plane represents the vertices by distinct points and the edges by simple arcs between the points of their endpoints, so that no arc passes through a vertex other than its endpoints, two arcs have finitely many common points, and two arcs never touch without crossing. A *thrackle* is a drawing in which every two edges meet exactly once: two edges with a common endpoint meet only there, and two edges without a common endpoint cross exactly once, properly. A graph is *thrackleable* if it has a thrackle drawing. Let $t(n)$ be the largest number of edges of a thrackleable graph with n vertices. Conway's thrackle conjecture states that $t(n) \leq n$ for every n. It would be sharp, since every cycle of length at least five is thrackleable [13], [7, Section 1].

For integers $c', c'' \geq 3$ and $\ell \geq 0$, the *dumbbell* $\mathrm{DB}(c', c'', \ell)$ of Fulek and Pach [7] consists of a c'-cycle and a c''-cycle joined by a path of length ℓ; the two cycles are disjoint if $\ell > 0$ and share exactly one vertex if $\ell = 0$. Fulek and Pach extend the definition to negative ℓ: for $0 < k < \min(c', c'')$, $\mathrm{DB}(c', c'', -k)$ is the union of a c'-cycle and a c''-cycle whose intersection is a path with k edges. We call all these graphs *generalized dumbbells*; the ones in this paper have $\ell \leq 0$. A generalized dumbbell $\mathrm{DB}(c', c'', \ell)$ has $c' + c'' + \ell - 1$ vertices and one more edge than vertices, so a thrackleable one would disprove Conway's conjecture.

Lovász, Pach and Szegedy proved $t(n) \leq 2n - 3$ for $n \geq 2$ [10], and the constant was lowered several times since [3, 7, 9, 8, 14]. Fulek and Pach introduced a computational method for deciding whether a small graph is thrackleable [7], and Xu's discharging argument gives $t(n) \leq 1.393(n-1)$ for $n \geq 4$ [14], [11, Section 3.1]. The constant $\frac{18}{13}$ appeared in [8, Remark 2], conditional on a computer verification of the conjecture for $n \leq 11$ that had been stated without proof in [4]. In 2026 the bound $\frac{18}{13}(n-1)$ was obtained by Tóth in a bachelor thesis [12] and, independently and for $n \geq 4$, by Scoppetto [11, Theorem 1.1]. Scoppetto shows with an adapted reconstruction of the program of Fulek and Pach that the generalized dumbbell $\mathrm{DB}(6, 8, -2)$ is not thrackleable, deduces that every 8-face of the embedding used by Xu (Lemma 2.3) has at most four edges in common with 6-faces, and adapts Xu's discharging. We prove the following.

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Theorem 1.1. *Every thrackleable graph with $n \geq 4$ vertices has at most $\frac{4}{3}(n-1)$ edges.*

Since $t(1) = 0$, $t(2) = 1$ and $t(3) = 3$, Theorem 1.1 gives $t(n) < \frac{4}{3}n$ for every $n \geq 1$.

Scoppetto pointed out [11, Section 7.2] that if $\mathrm{DB}(6, 8, -1)$ is not thrackleable, and if this forces every 8-face to have no edge in common with a 6-face, then the discharging would give $\frac{15}{11}$. We carry this out, and we go further by bounding the number of edges that a face of size 10 has in common with 6-faces. The new computational ingredients are the following.

Theorem 1.2. *The generalized dumbbells* $\mathrm{DB}(6, 8, -1)$ *and* $\mathrm{DB}(6, 10, -3)$ *are not thrackleable.*

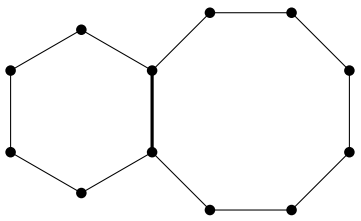


FIGURE 1. The generalized dumbbell $\mathrm{DB}(6, 8, -1)$: a 6-cycle and an 8-cycle sharing one edge (thick). It has 12 vertices and 13 edges.

We prove Theorem 1.2 with the method of Fulek and Pach [7]: a graph G is thrackleable if and only if, for some choice of the orders in which the edges cross each other, an associated graph G' is planar. Our search builds these orders edge by edge, tests the planarity of the partial graphs, and prunes with the complete lists of admissible orders for subgraphs, following an idea of Scoppetto [11]. Section 5 describes the search and the computations.

Smaller computations, collected in Section 3, show that in a triangle-free thrackleable graph no 6-cycle shares an edge with an 8-cycle and two distinct 6-cycles have no common vertex, and that $\mathrm{DB}(5, 5, -2)$ and $\mathrm{DB}(5, 6, -2)$ are not thrackleable. The second of these facts is used in [8, 11], although only the cases in which the two 6-cycles form a generalized dumbbell had been checked.

Section 4 then proves Theorem 1.1 along the lines of Xu [14] and Scoppetto [11]. A 2-connected triangle-free thrackleable graph has an embedding in the plane or the projective plane in which every face has even size, hence size at least six. In this embedding these facts show that the faces adjacent to a 6-face have size at least 10, and, with a further enumeration, that a face of size 10 has at most six edges in common with 6-faces; a discharging argument then gives the bound for triangle-free graphs. Conway's doubling reduces graphs with a triangle to bipartite ones, and disconnected graphs are handled separately. Section 6 describes a proof of Theorem 1.1 in Lean that also checks all the computational facts, including the two large searches. Section 7 explains why these facts give exactly the constant $\frac{4}{3}$.

2. Preliminaries

All graphs are finite and simple. For a graph G we write n for the number of vertices and m for the number of edges, and C_k for the cycle of length k. A subgraph of a thrackleable graph is thrackleable, since deleting edges and vertices from a thrackle leaves a thrackle. We use the following known facts.

Lemma 2.1 ([8, Lemma 4])**.** *The cycle C_4 is not thrackleable.*

In fact, C_4 cannot even be drawn so that every two edges without a common endpoint cross an odd number of times and adjacent edges meet only at their common endpoint [8, Lemma 4 and Section 5].

Let G be drawn as a thrackle, and let C be an odd cycle of G. Conway's doubling of C replaces every vertex v of C by two nearby vertices v_1, v_2, every edge vv' of C by the two edges v_1v_2' and v_2v_1', and attaches each other edge at v to v_1 or to v_2, according to the side of C on which it leaves v, so that every edge attached to v_1 crosses every edge attached to v_2 exactly once near v; all other edges are kept [7, Section 2]. The cycle C of length k becomes a cycle of length $2k$, so the number of vertices and the number of edges both grow by k.

Lemma 2.2 (Conway; [7, Lemma 5]; see also [3, Lemma 2])**.** *Let G be drawn as a thrackle, and let C be an odd cycle of G. Then the graph obtained by Conway's doubling of C is bipartite, and it can be drawn as a thrackle.*

An embedding of a graph in the projective plane is a *parity embedding* if every odd cycle is one-sided and every even cycle is two-sided. In a parity embedding every face has even size, where the size of a face is the length of its facial walk [8, Section 2].

Lemma 2.3 ([8, Corollary 1 and Proposition 1], based on [10, 4])**.** *Let G be a 2-connected triangle-free thrackleable graph. If G is bipartite, it has an embedding in the plane; otherwise it has a parity embedding in the projective plane. In either embedding every face has even size, and the facial walk of every face of size at most 8 is a cycle.*

Both embeddings are cellular: in the plane since G is connected, and in the projective plane since G contains an odd cycle C, which is one-sided, so that the complement of C is an open disc, and the faces of the connected graph $G \supseteq C$ in this disc are open discs. So every face has one facial walk, and every edge is traversed twice by the facial walks, once on each side. In the plane case the last statement of Lemma 2.3 holds for all faces, by the 2-connectivity of G.

Proposition 1 of [8] is stated for 2-connected thrackles, but its proof (in the arXiv version of [8]) uses that G is triangle-free; since that proof is brief, we include one for the projective case. There every odd cycle is one-sided and every even cycle two-sided, so a closed walk has odd length if and only if it is one-sided, that is, nonzero in the homology of the projective plane with coefficients in $\mathbb{Z}_2$ (decompose its edges, counted modulo 2, into cycles). Let W be the facial walk of a face D of size at most 8. Suppose first that W traverses an edge e twice. A simple closed curve γ that runs through D and meets G only once, crossing e, does not separate the projective plane, because e is not a bridge of G; so γ is one-sided, and $W = ePeQ$ traverses e twice in the same direction. The closed walks eP and eQ cross γ once, so they have odd length, which is at least 3 since G has no loops; as their lengths add up to $|W| \leq 8$, one of them has length 3 and is a triangle. If W traverses no edge twice but repeats a vertex, it splits into two closed trails of total length at most 8, one of which has length 3 or 4 and is a triangle or a 4-cycle. Both cases are excluded, so W is a cycle.

For an edge e of a graph G, let $N(e)$ be the set of edges of G that have no endpoint in common with e. Orient every edge of G. A *crossing system* $\Pi = (\pi_e)_{e \in E(G)}$ consists of a linear order π_e of $N(e)$ for every edge e; in a thrackle, π_e is the order in which the edges of $N(e)$ cross e when e is traversed in its direction. Given Π, the *augmented graph* $G'(G, \Pi)$ is obtained as follows [7, Section 4]. Each edge e becomes a path from its tail to its head through one new vertex X_{ef} for each $f \in N(e)$, in the order π_e, with one further new vertex between any two consecutive crossing vertices; the vertex $X_{ef} = X_{fe}$ lies on the paths of both e and f. Finally, for every crossing vertex $X = X_{ef}$, with neighbours a_1, a_2 on the path of e and b_1, b_2 on the path of f, the 4-cycle $a_1b_1a_2b_2$ is added.

Lemma 2.4 ([7, Section 4])**.** *A graph G is thrackleable if and only if $G'(G, \Pi)$ is planar for some crossing system Π.*

Sketch. A thrackle gives Π and a plane drawing of $G'(G, \Pi)$: the crossings become vertices, and the 4-cycles can be drawn along the arcs, close to the crossings. Conversely, in an embedding of

$G'(G,\Pi)$ in the plane the 4-cycle around X_{ef} together with the four edges at X_{ef} form a wheel, which is 3-connected and therefore has a unique embedding up to reflection; so the two edges of the path of e at X_{ef} are separated by those of f. Deleting the 4-cycles and turning the vertices X_{ef} into crossings gives a drawing in which every two edges without a common endpoint cross exactly once, at a proper crossing, and adjacent edges meet only at their common endpoint. □

3. The computational results

Besides Theorem 1.2, the proof of Theorem 1.1 uses the following computational facts.

Theorem 3.1. *The generalized dumbbells* $\mathrm{DB}(6,8,-2)$ *and* $\mathrm{DB}(6,8,-3)$ *are not thrackleable.*

For $\mathrm{DB}(6,8,-2)$ this is [11, Lemma 4.2]. For $\mathrm{DB}(6,8,-3)$, which is also $\mathrm{DB}(8,8,-5)$ and has 10 vertices, [11, Lemma 3.13] relies on the statement in [7] that Conway's conjecture holds for bipartite graphs with at most 10 vertices; since the computations in [7, Section 4] concern the generalized dumbbells $\mathrm{DB}(6,6,\ell)$, we checked $\mathrm{DB}(6,8,-3)$ directly. The proof of Theorem 3.1 is the computation described in Section 5.

Lemma 3.2. *Let C be a 6-cycle and F an 8-cycle of a triangle-free thrackleable graph, and suppose that C and F have at least one common edge. Then $C \cup F$ is* $\mathrm{DB}(6,8,-1)$.

Proof. Write $F = v_0v_1\dots v_7$ and $C = w_0w_1\dots w_5$, where $w_0w_1 = v_0v_1$ is a common edge; after reflecting F we may assume $w_0 = v_0$ and $w_1 = v_1$. Each of $w_2,\dots,w_5$ is either a vertex outside F or one of $v_2,\dots,v_7$, distinct ones for distinct w_i. We enumerated all these choices by computer and sorted the graphs $C \cup F$ into isomorphism classes. There are 343 classes. Of these, 266 contain a triangle; of the others, 60 contain a 4-cycle; of the remaining 17, four contain two distinct 6-cycles with a common vertex (among them $\mathrm{DB}(6,8,-4)$, which is also $\mathrm{DB}(6,6,-2)$), three are the generalized dumbbells $\mathrm{DB}(6,8,-k)$ for $k = 1,2,3$, and 10 are of none of these kinds. Graphs with a triangle are excluded by assumption, and graphs with a 4-cycle are excluded by Lemma 2.1. The generalized dumbbells $\mathrm{DB}(6,8,-2)$ and $\mathrm{DB}(6,8,-3)$ are not thrackleable by Theorem 3.1. For the 4+10 remaining classes the search of Section 5 found no planar augmented graph, so they are not thrackleable. Hence $C \cup F = \mathrm{DB}(6,8,-1)$. □

Corollary 3.3. *In a triangle-free thrackleable graph, no 6-cycle has an edge in common with an 8-cycle.*

Proof. By Lemma 3.2 the union of such cycles would be $\mathrm{DB}(6,8,-1)$, a subgraph of a thrackleable graph, contradicting Theorem 1.2. □

Fulek and Pach [8, Section 3] state that two distinct 6-cycles of a thrackle have no common vertex, citing [7], where Lemma 6 shows that the generalized dumbbells $\mathrm{DB}(6,6,-k)$, $0 \le k \le 3$, are not thrackleable (the case $k = 3$ is due to Lovász, Pach and Szegedy [10, Theorem 5.1]). Since two 6-cycles can also meet in more than one path or vertex, we checked these cases as in Lemma 3.2.

Lemma 3.4 (see [8, Section 3])**.** *Two distinct 6-cycles of a triangle-free thrackleable graph have no common vertex.*

Proof. Let $Q = v_0\dots v_5$ and $Q' = w_0\dots w_5$ be distinct 6-cycles with the common vertex $w_0 = v_0$. Each of $w_1,\dots,w_5$ is a vertex outside Q or one of $v_1,\dots,v_5$, distinct ones for distinct w_i. Up to isomorphism these choices give 84 graphs $Q \cup Q'$. Of these, 56 contain a triangle; of the others, 20 contain a 4-cycle; of the remaining eight, four are the generalized dumbbells $\mathrm{DB}(6,6,-k)$ for $0 \le k \le 3$, and four are of none of these kinds. The four generalized dumbbells are not thrackleable by [7, Lemma 6] and [10, Theorem 5.1], and our program confirms this. For the

remaining four graphs our program finds no planar augmented graph. With Lemma 2.1 this proves the claim. □

For the faces of size 10 we need, besides $\mathrm{DB}(6, 10, -3)$, two further generalized dumbbells. These three generalized dumbbells, $\mathrm{DB}(5, 5, -2)$, $\mathrm{DB}(5, 6, -2)$ and $\mathrm{DB}(6, 10, -3)$, are the graphs formed by three internally disjoint paths between two vertices, of lengths $2, 3, 3$, of lengths $2, 3, 4$ and of lengths $3, 3, 7$.

Theorem 3.5. *The generalized dumbbells* $\mathrm{DB}(5, 5, -2)$ *and* $\mathrm{DB}(5, 6, -2)$ *are not thrackleable.*

The proof is again the computation of Section 5.

4. Proof of Theorem 1.1

Throughout this section $c = \frac{4}{3}$ and $r = 2c = \frac{8}{3}$. Lemma 4.4 uses the charges of [11, proof of Theorem 1.1, Case 1], which follow Xu [14], and Lemmas 4.5 and 4.6 and the proof of Theorem 1.1 follow Cases 1, 2 and 3 of the same proof. Let G be a 2-connected triangle-free thrackleable graph, embedded as in Lemma 2.3. A *k-face* is a face of size k. Every face has even size. A face of size 2 or 4 would have a cycle of length 2 or 4 as its facial walk, which is impossible since G is simple and C_4 is not thrackleable (Lemma 2.1). So every face has size at least 6, and the facial walks of the 6-faces and 8-faces are cycles. For a face D, a 6-*edge* of D is a traversal of an edge by the facial walk of D such that the other side of this edge belongs to a 6-face. An edge with a 6-face on one side is traversed only once by the facial walk of the face on its other side, so the 6-edges of D correspond to the edges that D has in common with 6-faces. We consider the traversals of a facial walk in their cyclic order, and call a maximal sequence of consecutive 6-edges a *run*.

Lemma 4.1. *Let $G \neq C_6$ be a 2-connected triangle-free thrackleable graph, embedded as in Lemma 2.3.*

(a) Distinct 6-faces have distinct boundary cycles.
(b) Every edge of a 6-face D has a face of size at least 10 on its other side.

Proof. (a) Suppose that distinct faces D, D' have the same cycle Q as facial walk. At every vertex v of Q both D and D' occupy an angle between the two edges of Q at v. If v had a third edge, at most one of the angles at v would be bounded by these two edges. So every vertex of Q has degree 2, and $G = Q$; for a 6-face this means $G = C_6$, which is excluded.

(b) The facial walk of D is a cycle Q and traverses each edge e of Q once, so the other side of e belongs to a face $D' \neq D$. If D' were a 6-face, its boundary cycle would, by (a), be a 6-cycle different from Q sharing the edge e with it, contradicting Lemma 3.4. If D' were an 8-face, its boundary would be an 8-cycle with the edge e in common with Q, contradicting Corollary 3.3. Since D' has even size at least 6, it has size at least 10. □

Lemma 4.2. *Let $G \neq C_6$ be a 2-connected triangle-free thrackleable graph, embedded as in Lemma 2.3. Then every 10-face has at most eight 6-edges.*

Proof. Let D be a 10-face, and consider the ten traversals of its facial walk in their cyclic order. Two consecutive traversals share a vertex, at which the walk passes from one to the next. If both are 6-edges, with 6-faces D_1 and D_2 on their other sides, then the boundary cycles of D_1 and D_2 have this vertex in common, so they coincide by Lemma 3.4, and $D_1 = D_2$ by Lemma 4.1(a). Hence the 6-edges of a run all have the same 6-face D^* on their other side. An edge traversed twice by the facial walk of D has D on both sides and gives no 6-edge. So the 6-edges of such a run are traversals of distinct edges of the boundary cycle of D^*, and there are at most six of them. In particular not all ten traversals are 6-edges. If the 6-edges form $b \geq 2$ runs, these are

cyclically separated by at least b other traversals, so there are at most $10 - b \leq 8$ of them; if $b \leq 1$, there are at most six. □

Lemma 4.3. *Let $G \neq C_6$ be a 2-connected triangle-free thrackleable graph, embedded as in Lemma 2.3. Then every run of 6-edges of a 10-face has at most two 6-edges, and every 10-face has at most six 6-edges.*

Proof. Let D be a 10-face with facial walk W. By the proof of Lemma 4.2, the 6-edges of a run all have the same 6-face on their other side; let Q be its boundary cycle. They are traversals of distinct edges of Q, so there are $s \leq 6$ of them. A facial walk returns along the edge it has just traversed only at a vertex of degree 1; since G is 2-connected, W never does, so the run traverses s consecutive edges of Q (all of Q if $s = 6$). The other side of each edge of Q is the 6-face, not D, so every traversal of an edge of Q by W is a 6-edge. Hence W traverses each edge of Q at most once, and by maximality the edges of W just before and just after the run are not edges of Q. Label $Q = 0\,1\,2\,3\,4\,5$ and $W = w_0 w_1 \dots w_9$ so that $w_i = i$ for $0 \leq i \leq s$ (with $w_6 = 0$ if $s = 6$). We enumerated by computer all closed walks W in which each of $w_{s+1}, \dots, w_9$ is a vertex of Q, an earlier vertex of W or a new vertex (new vertices numbered in order of first appearance), such that: W never returns along the edge it has just traversed, also at w_0; $w_s w_{s+1}$ and $w_9 w_0$ are not edges of Q; W traverses every edge of Q at most once and every other edge at most twice; and the union of Q and the edges of W has neither triangles nor 4-cycles. With this labelling, there is no such walk for $s = 6$. For $s = 5$ there are three, and in each of them the union contains two distinct 6-cycles with a common vertex, which Lemma 3.4 excludes. For $s = 4$ there are four, and in each of them the union contains a 6-cycle and an 8-cycle with a common edge, which Corollary 3.3 excludes. For $s = 3$ there are 19. In five of them the union contains two distinct 6-cycles with a common vertex, and in five others a 6-cycle and an 8-cycle with a common edge. The remaining nine unions fall into five isomorphism classes: one is $\mathrm{DB}(5, 5, -2)$, two contain it, one contains $\mathrm{DB}(5, 6, -2)$, and one is $\mathrm{DB}(6, 10, -3)$. Explicitly, in four of the nine walks W contains a path of length 2 from 3 to 0 whose middle vertex is not on Q, which forms $\mathrm{DB}(5, 5, -2)$ with Q; in four others W contains a path of length 3, with inner vertices not on Q, between two vertices at distance 2 on Q, which forms $\mathrm{DB}(5, 6, -2)$ with Q; in the last, W is the run followed by a path of length 7 from 3 to 0 through new vertices. By Theorems 1.2 and 3.5 none of the nine unions is thrackleable. So every run has at most two 6-edges. If the 6-edges of D form b runs, there are at most $2b$ of them and, as in the proof of Lemma 4.2, at most $10 - b$ when $b \geq 2$; so there are at most six. □

Lemma 4.4. *Let G be a 2-connected triangle-free thrackleable graph other than C_6. Then $m \leq c(n-1)$, and $m \leq c(n-2)$ if G is bipartite.*

Proof. Embed G as in Lemma 2.3, and let f be the number of faces. Put $k = 2$ if G is bipartite and $k = 1$ otherwise. Since the embedding is cellular, $n - m + f = k$ by Euler's formula.

Give every vertex v the charge $d(v) - r$, where $d(v)$ is its degree, and every face D of size $|D|$ the charge $\frac{r-2}{2}|D| - r$, which is $\frac{|D|-8}{3}$. The sum of the vertex degrees and the sum of the face sizes are both $2m$, so the total charge is

$$2m - rn + (r-2)m - rf = -r(n - m + f) = -kr.$$

Now let every face of size at least 10 send $\frac{1}{9}$ across each of its 6-edges. By Lemma 4.1(b), every 6-face receives $\frac{1}{9}$ across each of its six edges and ends with charge $-\frac{2}{3} + \frac{6}{9} = 0$. An 8-face sends nothing and keeps 0. A 10-face sends at most $\frac{6}{9}$, by Lemma 4.3, and keeps at least $\frac{2}{3} - \frac{6}{9} = 0$. A face of size $d \geq 12$ sends at most $\frac{d}{9}$ and keeps at least $\frac{d-8}{3} - \frac{d}{9} = \frac{2d-24}{9} \geq 0$. The total charge

does not change, so

$$2m - rn = \sum_v \big(d(v) - r\big) \le -kr,$$

that is, $m \le \frac{r}{2}(n-k) = c(n-k)$. □

Lemma 4.5. *Every triangle-free thrackleable graph with $n \ge 1$ vertices has at most $c(n-1)$ edges.*

Proof. We use induction on n. If G is disconnected, apply the induction hypothesis to its components $G_1, \dots, G_j$: then $m \le c(n-j) \le c(n-1)$. If G is connected and has a cut vertex, then G is the union of two connected graphs G_1, G_2 with exactly one common vertex, both smaller than G, and $m = m_1 + m_2 \le c(n_1 - 1) + c(n_2 - 1) = c(n-1)$. The graphs with one or two vertices satisfy the bound. If G is 2-connected, the bound follows from Lemma 4.4, or from $6 \le 5c$ if $G = C_6$. □

Lemma 4.6. *Every connected thrackleable graph with $n \ge 4$ vertices that contains a triangle has at most $\frac{4}{3}(n-1)$ edges.*

Proof. Draw G as a thrackle, and let H be the graph obtained by Conway's doubling of a triangle. By Lemma 2.2, H is bipartite and thrackleable, and it is connected since G is connected. It has $N = n+3 \ge 7$ vertices and $M = m+3$ edges, so $H \ne C_6$, and it contains the 6-cycle that replaces the triangle. For a block B of H with v_B vertices and e_B edges, put $g(B) = c(v_B - 1) - e_B$. Since H is connected, $\sum_B (v_B - 1) = N - 1$, so

$$c(N-2) - M = \sum_B g(B) - c.$$

A block is a single edge, with $g(B) = c - 1 = \frac{1}{3}$, or it is 2-connected and bipartite. If it is C_6, then $g(B) = 5c - 6 = \frac{2}{3}$. Otherwise $e_B \le c(v_B - 2)$ by Lemma 4.4, and $g(B) \ge c$. If some block is 2-connected and not C_6, then $c(N-2) - M \ge 0$. Otherwise some block is C_6, since H contains a cycle, and there is at least one further block, since $H \ne C_6$; so $c(N-2) - M \ge \frac{2}{3} + \frac{1}{3} - c = -\frac{1}{3}$. In both cases $M \le c(N-2) + \frac{1}{3}$, that is, $m + 3 \le \frac{4}{3}(n+1) + \frac{1}{3}$, which is $m \le \frac{4}{3}(n-1)$. □

Proof of Theorem 1.1. Let G be a thrackleable graph with $n \ge 4$ vertices and components $G_1, \dots, G_j$, where G_i has n_i vertices and m_i edges. If G_i is triangle-free, then $m_i \le c(n_i - 1)$ by Lemma 4.5. If G_i contains a triangle and $n_i \ge 4$, then $m_i \le c(n_i - 1)$ by Lemma 4.6. If G_i contains a triangle and $n_i \le 3$, then $G_i = C_3$ and $m_i = 3 = c(n_i - 1) + \frac{1}{3}$. Let t be the number of components equal to C_3. If $j = 1$, then $t = 0$ since $n \ge 4$, and $m \le c(n-1)$. If $j \ge 2$, then

$$m \le c(n-j) + \frac{t}{3} \le c(n-j) + \frac{j}{3} \le c(n-j) + c(j-1) = c(n-1),$$

since $j \le 4(j-1)$ for $j \ge 2$. □

5. The search

5.1. **The search tree.** Let G be a graph with oriented edges $e_0, e_1, \dots, e_{m-1}$, listed in a fixed order. For $0 \le j \le m$ let G_j be the subgraph formed by $e_0, \dots, e_{j-1}$, and for a crossing system Π of G let $\Pi|_{G_j}$ be the crossing system of G_j obtained by deleting all edges outside G_j from the orders. If $G'(G, \Pi)$ is planar, then so is $G'(G_j, \Pi|_{G_j})$, which is obtained from it by deleting edges and vertices and contracting edges.

The program builds crossing systems edge by edge, following the backtracking scheme of [7, Appendix A]. When the edge e_j is added, it is traced from its tail: each step chooses the next crossing of e_j, that is, an edge f of G_j in $N(e_j)$ that e_j has not crossed yet, and a position in the current order π_f at which e_j is inserted. After each step the program tests the planarity of

the partial augmented graph, in which the edges of G_j are complete, e_j is drawn up to its last crossing, and the 4-cycle at that crossing is present only on existing arms. This graph is obtained from the augmented graph of every completion by deleting edges and vertices and contracting edges, so if it is not planar, no completion is, and the branch is pruned. When e_j has crossed all edges of $N(e_j) \cap E(G_j)$, its last segment is added and the complete augmented graph $G'(G_{j+1}, \cdot)$ is tested. Every crossing system Π of G determines, for each j, the restriction $\Pi|_{G_{j+1}}$ and the order in which e_j crosses the edges of G_j, hence exactly one branch of the search tree, and this branch is pruned only at a nonplanar graph obtained from $G'(G, \Pi)$ by deleting and contracting. So G is thrackleable if and only if the search reaches a planar complete augmented graph of G. The filters of Section 5.2 never prune the branch of a system Π with $G'(G, \Pi)$ planar.

Planarity is tested by the left-right algorithm [5, 2].

5.2. **Filters.** If Π is a crossing system of G with $G'(G, \Pi)$ planar and S is a subgraph of G, then $G'(S, \Pi|_S)$ is planar. So if the list of all crossing systems Σ of S with $G'(S, \Sigma)$ planar is known, the program can prune every partial system whose restriction to S, as far as it is determined, extends to no system in the list. We call such lists *filters*. They are computed by the same program, which can list all complete planar systems of a graph instead of stopping at the first. During the search the program keeps, for each filter, the set of listed systems that are still compatible with the partial system, as a bit set, and updates it with precomputed bit sets for relations of the form "on the edge e, the crossing with x comes before the crossing with y". A branch is pruned when one of these sets becomes empty. Scoppetto used filters of this kind, also stored as bit sets, for C_6, C_8 and C_8 with one further edge of $\mathrm{DB}(6, 8, -2)$ [11, Section 4].

5.3. **The computations.** For $\mathrm{DB}(6, 8, -1)$ we label the vertices $0, \dots, 11$ so that the common edge is $e_0 = 01$, the 6-cycle consists of e_0 and the path $1, 2, 3, 4, 5, 0$ (edges $e_1, \dots, e_5$, with e_5 oriented from 0 to 5), and the 8-cycle consists of e_0 and the path $1, 6, 7, \dots, 11, 0$ (edges $e_6, \dots, e_{12}$). The filters are the 8-cycle $e_0, e_6, \dots, e_{12}$ (2,544 systems) and the two graphs obtained from it by adding the pendant edge e_1 or e_5 (60,196 systems each). We first listed the complete planar crossing systems of the subgraph formed by $e_0, \dots, e_9$ that pass the filters; there are 1,215,372 of them, and every complete planar system of $\mathrm{DB}(6, 8, -1)$ restricts to one of them. The permutation $\sigma = (0\,1)(2\,5)(3\,4)(6\,11)(7\,10)(8\,9)$ of the vertices is an automorphism of $\mathrm{DB}(6, 8, -1)$. For a crossing system Π let $\sigma\Pi$ be the system with $\pi'_{\sigma(e)} = \sigma(\pi_e)$, reversed when σ reverses the orientation of e; then $G'(G, \sigma\Pi) \cong G'(G, \Pi)$. The set $R = \{e_0, \dots, e_5, e_9\}$ is mapped to itself by σ (the set $\{e_0, \dots, e_9\}$ is not), so $(\sigma\Pi)|_R = \sigma(\Pi|_R)$. We kept a system only if its restriction to R, written as the sequence of the orders on $e \in R$ in increasing order of e, is lexicographically at most its image under σ. Since σ is an involution and the order is total, for every complete planar system Π of $\mathrm{DB}(6, 8, -1)$, Π or $\sigma\Pi$ restricts to a kept system. This leaves 694,211 systems, and the program continued the search from each of them; the parts were run in parallel. All parts ended without a planar complete augmented graph.

Table 1 summarizes the runs. For $\mathrm{DB}(6, 8, -2)$ we used the analogous filters (C_8 and C_8 with the pendant edges e_2 or e_5 in the labelling of the program) and split the search at the subgraph of the first nine edges, without symmetry reduction. For $\mathrm{DB}(6, 8, -3)$, $\mathrm{DB}(5, 5, -2)$, $\mathrm{DB}(5, 6, -2)$, the 14 classes of Lemma 3.2 and the four remaining graphs of Lemma 3.4 we used as filters the complete planar systems of all cycles of length 5 to 8 in the graph. The generalized dumbbells $\mathrm{DB}(6, 6, -k)$ were run without filters ($k = 3$) or with filters formed by C_6, C_8, and C_6 with pendant edges or paths.

For $\mathrm{DB}(6, 10, -3)$ we label the vertices $0, \dots, 11$ so that the three paths from 0 to 3 are $0, 1, 2, 3$ and $0, 4, 5, 3$, which form the 6-cycle (edges $e_0, \dots, e_5$), and $0, 6, 7, \dots, 11, 3$ (edges $e_6, \dots, e_{12}$), and every edge is oriented along its path from 0 to 3. The filters are the two 10-cycles formed by

$e_6, \dots, e_{12}$ and one of the other two paths. Listed by the path from 0 to 3 followed by $e_6, \dots, e_{12}$, the two cycles are the same oriented graph with the same edge order, so both filters use one list, of the 6,059,712 complete planar systems of this oriented 10-cycle; for comparison, C_9 has 72,916. A second computation of the list, for C_{10} with every edge oriented around the cycle and split differently, gave a list that the isomorphism between the two cycles maps onto the first, the orders on the seven edges with opposite orientations being reversed. We then listed the 32,626 complete planar systems of the subgraph formed by $e_0, \dots, e_8$ (a 6-cycle with a pendant path of three edges) that pass the filters, and their 1,211,878 extensions by e_9; every complete planar system of $\mathrm{DB}(6, 10, -3)$ restricts to one of them. Without symmetry reduction, the program continued the search from each of them; the parts were run in parallel. All parts ended without a planar complete augmented graph.

graph	split states	planarity tests	time
$\mathrm{DB}(6, 8, -1)$	694,211	207,380,593,775	7,765,559 s
$\mathrm{DB}(6, 8, -2)$	26,550	3,520,550,932	128,254 s
$\mathrm{DB}(6, 8, -3)$	–	6,024,873	34 s
$\mathrm{DB}(6, 10, -3)$	1,211,878	83,401,645,992	2,439,459 s

TABLE 1. The runs for Theorems 1.2 and 3.1. The times are thread-seconds and leave out the listing of the split states and the filter list of $\mathrm{DB}(6, 10, -3)$. All runs report "not thrackleable".

5.4. **Checks.** We checked the computations in four ways.

- The left-right planarity test agrees with our implementation of that of Demoucron, Malgrange and Pertuiset [6] on random graphs and on augmented graphs sampled during the search for $\mathrm{DB}(6, 8, -1)$.
- The program finds 1, 0, 1, 8, 92 and 2,544 complete planar systems for the cycles $C_3, \dots, C_8$, and 60,196 for C_8 with a pendant edge; the numbers for C_6, C_8 and C_8 with a pendant edge are those found by Scoppetto with his program [11, Section 4]. It finds that $\mathrm{DB}(6, 6, -k)$ is not thrackleable for $0 \le k \le 3$, in agreement with [10, Theorem 5.1] and [7, Lemma 6], and it reproduces Scoppetto's result for $\mathrm{DB}(6, 8, -2)$, which was obtained with a different program.
- A second implementation of the search, written independently of the first, uses the edge addition planarity test of Boyer and Myrvold [1], no symmetry reduction, its own splittings and, except for $\mathrm{DB}(6, 10, -3)$, its own edge orders. It repeated the checks above and the computations of Theorems 1.2, 3.1 and 3.5 and Lemmas 3.2 and 3.4 with the same results.
- A third program, also written independently, repeated the enumeration of the walks of Lemma 4.3 with the same results.

6. A proof in Lean

Theorem 1.1 is also proved in Lean 4 with Mathlib, from five properties of thrackles that are quoted or proved in Sections 2 and 4. The formal statement concerns an arbitrary property T of finite simple graphs and assumes the following of it.

(P1) T passes to subgraphs.
(P2) C_4 does not have T.

(P3) Every 2-connected triangle-free graph with T, with n vertices and m edges, has a list of f closed walks that together traverse every edge twice, have even length at least 4, are cycles if their length is at most 8 and never return along the edge just traversed, such that $m+1 \le n+f$, $m+2 \le n+f$ if the graph is bipartite, and two of the walks are cycles with the same edges only if the graph is that cycle.

(P4) Conway's doubling of a triangle turns a connected graph with T, a triangle and at least four vertices into a connected bipartite graph with T and three more vertices and three more edges.

(P5) Every graph with T and at least one edge has, for every orientation and order of its edges, a crossing system whose augmented graph is planar.

Thrackleability has these properties: (P1) holds since a subgraph of a thrackle is a thrackle; (P3) holds for the facial walks of an embedding as in Lemma 2.3, by the remarks after that lemma, Euler's formula, the proof of Lemma 4.1(a) and the remark on facial walks in the proof of Lemma 4.3; and (P2), (P4) and (P5) are Lemmas 2.1, 2.2 (with the proof of Lemma 4.6) and 2.4. In Lean, a graph counts as planar if it has neither a K_5 nor a $K_{3,3}$ minor; a graph drawn in the plane without crossings has no such minor, so Lemma 2.4 gives (P5).

The formal proof has two parts. The first derives the bound from (P1)–(P5) and 23 computational facts, namely that 23 given graphs have no crossing system with a planar augmented graph; these graphs include the six generalized dumbbells of Theorems 1.2, 3.1 and 3.5. This part makes its own enumerations for Lemmas 3.2, 3.4 and 4.3 and checks them in the kernel; its trusted base is Lean's kernel, including the kernel's built-in arithmetic on natural numbers, and it uses no axioms beyond the three standard ones (`propext`, `Classical.choice` and `Quot.sound`). The second part proves the 23 facts by a checker whose soundness theorems are proved in Lean with the three standard axioms; the checker accepts that an augmented graph is not planar only together with a K_5 or $K_{3,3}$ minor of it, which it verifies. Its checks run as compiled code (with `native_decide`), each entering the proof as one auxiliary axiom, so the bound follows from (P1)–(P5) with the three standard axioms and 3,480 auxiliary axioms and relies on Lean's compiler as well as its kernel; a second build from the same sources, made separately on other machines, ran all these checks again with the same results. The Lean files are in the directory `lean/` of the repository given in the data availability statement, and its README gives the theorem names and the details of the checks and of both builds.

7. Remarks

In the discharging of Lemma 4.4, a face of size d with h six-edges needs

$$\frac{r-2}{2}\,d - r - h\Big(1-\frac{r}{3}\Big) \ge 0,$$

where $1-\frac{r}{3}$ is the amount sent across each 6-edge, that is, $c=\frac{r}{2} \ge \frac{3(d+h)}{3d-6+2h}$. Table 2 lists this requirement for the faces that matter. The bounds $h \le 5$ of [14] and $h \le 4$ of [11] for 8-faces give their constants $\frac{39}{28} = 1.3928\ldots$ and $\frac{18}{13}$. The value $\frac{15}{11}$ is the requirement both of 8-faces with $h \le 2$ and of a 10-face with ten 6-edges, and at $\frac{15}{11}$ the 8-faces no longer bind: with $h \le 1$ they require only $\frac{27}{20}$. So for $\frac{15}{11}$ Corollary 3.3 gives more than is needed. For $\frac{4}{3}$, which is less than $\frac{27}{20}$, it is needed in full, since an 8-face with one 6-edge would end with charge $-\frac{1}{9}$. For the 10-faces, Lemma 4.2 alone would give $\frac{27}{20}$, which is also the requirement of an 8-face with one 6-edge; at most seven 6-edges would give $\frac{51}{38}$, and Lemma 4.3 gives $\frac{4}{3}$.

The charges cannot give less than $\frac{4}{3}$, which is the requirement of an 8-face without 6-edges: for $c < \frac{4}{3}$ an 8-face has negative charge $3r-8$ before any charge is moved. A smaller constant would therefore need a discharging in which the 8-faces receive charge.

d	h	required c
8	5	39/28 (the case of [14])
8	4	18/13 (the case of [11])
8	3	11/8
8	2	15/11
8	1	27/20
8	0	4/3
10	10	15/11
10	8	27/20
10	7	51/38
10	6	4/3

TABLE 2. The constant c required by a face of size d with h six-edges in the discharging of Lemma 4.4.

Data availability

The programs, the input graphs, the filter lists, the split states (for $\mathrm{DB}(6, 8, -1)$ those after the symmetry reduction), the data of the comparison of the planarity tests and the logs of all runs and enumerations are archived at doi:10.5281/zenodo.24065371 and in the repository `https://github.com/tejasvi/thrackle-bound/tree/v5`, together with the second implementation of Section 5.4 (directory `independent/`) and the Lean files of Section 6 (directory `lean/`); the certificate data that the Lean checks read are two further archives of the record. The search program is written in C and the enumerations in Python 3. The large runs for $\mathrm{DB}(6, 8, -1)$ and $\mathrm{DB}(6, 8, -2)$ used static builds of earlier revisions of the search program, which differ from the archived version only in command-line options and in a faster implementation of the filter step, and the runs for $\mathrm{DB}(6, 10, -3)$ used a static build of the archived version; these binaries, their sources and the differences are archived, and each binary is reproduced bit for bit by its recorded build command.

Use of AI tools

Generative AI tools were used to find and prove the results, to write the programs, and to draft the text. The author reviewed the arguments and the programs and takes responsibility for the content.

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