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SMALE'S SIXTH PROBLEM FOR SIX BODIES IN THE PLANE: FINITENESS OF CENTRAL CONFIGURATIONS FOR GENERIC MASSES, WITH A PROOF FORMALIZED IN LEAN

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Abstract. Smale's sixth problem asks whether, for every choice of positive masses, the number of central configurations up to similarity is finite. We prove that it is finite for six bodies in the plane and all positive masses outside a proper algebraic subset of the mass space. This subset, which we describe only in part, contains every mass vector with three equal masses or with one mass equal to the sum of two others. Finiteness for all positive masses remains open. Infinitely many central configurations would give a curve of complex solutions; following Albouy and Kaloshin, we exclude such a curve using the valuations of the mutual distances at its points at infinity. The proof is computer-assisted and uses linear programs, Gröbner bases and interval arithmetic. The theorem is proved in the Lean proof assistant, whose kernel checks the computations as well.

1. Introduction

A configuration $q = (q_1, \dots, q_n)$ of n distinct points in the plane, carrying positive masses $m_1, \dots, m_n$, is a *central configuration* if there are $\lambda \in \mathbb{R}$ and $c \in \mathbb{R}^2$ such that

$$\lambda\, m_i(q_i - c) = \sum_{j \neq i} \frac{m_i m_j\,(q_i - q_j)}{|q_i - q_j|^3} \qquad (i = 1, \dots, n). \tag{1.1}$$

Central configurations are the shapes of the homographic solutions of the n-body problem, and in particular of its relative equilibria [23]. A configuration similar to a central configuration is again one. Smale's sixth problem [26] (Problem 9 of [1]) asks whether, for every choice of positive masses, the number of central configurations up to similarity is finite.

For $n = 3$ there are five, the two equilateral triangles of Lagrange and the three collinear configurations of Euler. Moulton [24] counted the collinear central configurations for all n. Hampton and Moeckel [13] proved finiteness for four bodies and all positive masses, and Albouy and Kaloshin [2] for five bodies in the plane when the masses lie outside a closed algebraic subset of codimension two of the mass space. Jensen and Leykin [16] gave a tropical method for generic masses and completed its computation for $n \leq 5$. Moczurad and Zgliczyński [22] counted the central configurations at several mass vectors of the exceptional set of [2]. For five bodies, finiteness for all positive masses was proved in [27]. Chen and Tong [8] state the same theorem and give a different proof.

For six bodies, Moczurad and Zgliczyński counted the planar central configurations of equal masses with a computer-assisted proof [21]. The prevariety of Jensen and Leykin, computed for one choice of mass valuations, had recession cones that were not all pointed [16, §4.3]. Chang and Chen [5, 6, 7] mechanized the method of [2] for six bodies. Their first algorithm reduces the problem to 117 diagrams, their second excludes 31 of them, and their third excludes one more at all positive masses and 61 others outside explicit hypersurfaces of masses. They left 24 diagrams

2020 *Mathematics Subject Classification.* 70F10, 70F15, 14Q20, 65G40, 68V20.

Key words and phrases. Central configurations, relative equilibria, six-body problem, Smale's sixth problem, valuations, Gröbner bases, interval arithmetic, Lean.

unresolved, together with the question of a dominant polynomial bounded on all of them [5, §11, pp. 54–55]. Aquino-Michaels [3] then excluded four of the 24 diagrams (numbers 52, 63, 65 and 67) for masses outside proper algebraic subsets, excluded diagram 6 on components that come from real configurations, and reduced the remaining question to 11 diagrams. Our main result is the following.

Theorem 1.1. *There is a nonzero polynomial $F \in \mathbb{Z}[m_1, \dots, m_6]$ with the following property. For every $m \in (\mathbb{R}_{>0})^6$ with $F(m) \neq 0$, the planar six-body problem with masses m has finitely many central configurations up to similarity.*

The proof produces F as a product of finitely many nonzero polynomials, most of which are known only to exist, so we do not compute F and describe its zero set only in part. This zero set contains every mass vector at which the polynomial P_{93} of Theorem 6.15 vanishes at the masses of three of the bodies, in particular every mass vector in which three masses are equal (Remark 6.14), and every mass vector in which one mass is the sum of two others (Remark 7.14). Theorem 1.1 says nothing about such masses, among them six equal masses. Finiteness for all positive masses remains open.

The method. We use the framework of [2] in the form of [27]. If there are infinitely many central configurations, the complex solutions of the normalized equations contain an irreducible curve C on which the potential U is a nonzero constant and some mutual distance is not constant. At each singular place P at infinity of C the exponents $x_{kl}(P)$ of the mutual distances form the *type* $x(P) \in \mathbb{Q}^{15}$, and the types balance: $\sum_P d_P\, x(P) = 0$ with $d_P > 0$, the sum running over the singular places, of which there is at least one. Summing the fifteen coordinates shows that some singular place has $\ell(x(P)) = \sum_{k<l} x_{kl}(P) \leq 0$ (Corollary 2.7), that is, the product $\Delta = \prod_{k<l} r_{kl}^2$ of the squared distances does not tend to zero there. It therefore suffices to show that at generic masses no singular place has $\ell(x(P)) \leq 0$. This single inequality already disposes of eleven of the 24 diagrams that Chang and Chen left open (Section 5). The unit Δ, or its inverse, is also used in [3, Thms. 6.1 and 6.3] and in the global argument of [8, §4] for five bodies.

The exponents at a singular place are constrained by combinatorial data: the diagram of the place in the sense of [2], the way the six bodies and the centre of mass cluster, recorded by two ultrametric trees of the exponents, and the terms of least exponent in the equations of the bodies and in two identities for U. These data take one of finitely many values, the *cells*, and each cell confines the exponents to a relatively open polyhedron. We enumerate the cells whose polyhedron meets the half-space $\ell \leq 0$, for every diagram allowed by the rules of [2, §3] that we use (Sections 3 and 4), and exclude all but 43 of them at generic masses (Section 5). In most of the excluded cells an identity has a single term of least exponent, which cannot happen at a place. In the others the leading coefficients satisfy equations that are inconsistent over the field of rational functions of the masses (in some cells after one mass is set to 1), as Gröbner bases show, so the cell can occur only where a nonzero polynomial in the masses vanishes.

The 43 remaining cells lie in seven diagrams, which we treat at the level of places (Sections 6 and 7). In five of them two clusters of bodies decouple at leading order, and the torque of one cluster, as in [2, §5.0] and with a rewriting from [2, §8.6] where needed, either excludes the diagram or gives a condition on the leading configurations of the clusters. These are complex central configurations of three or four bodies, or solutions of a related four-body system, and they are finite in number up to rotation at generic masses; the condition fails at one mass point and is therefore a nonzero polynomial condition on the masses. In the other two diagrams the cluster is a complete graph of z-strokes on four bodies, and its leading system must have a solution. For one diagram it is the system E_4 of complex equilibria of four bodies in both colours, whose mass condition Albouy and Kaloshin mention [2, §5.5] and which Aquino-Michaels showed to have no

solution outside a proper algebraic subset of masses, where he meets it as the core of diagrams 63, 65 and 67 [3, Thm. 5.6]; for the two diagrams treated here he confined only the collinear case [3, Rem. 5.7]. As far as we know, the observation that E_4 is a leading system here is new, and for the other diagram the leading system is a new system C_4 of a four-body cluster with a centre. We show that neither system has a solution outside a proper algebraic subset of masses, for E_4 by hand, up to resultants of integer polynomials, and for C_4 by a certified computation at one mass point.

Every linear program found infeasible has a certificate of infeasibility checked in exact rational arithmetic, and each cell excluded by Gröbner bases also has an explicit identity with integer polynomials that gives the hypothesis of Lemma 5.2. Section 10 describes the computations, their checks and the formal proof in Lean 4 with Mathlib, which follows Sections 2–8 with a few departures.

Organization. Section 2 sets up the equations, the curves and their places, and proves the rules of [2] at places. Sections 3–5 enumerate the diagrams and the cells and exclude the cells at generic masses, Sections 6 and 7 treat the seven diagrams, numbered 64, 66, 88, 89, 93, 94 and 98 in [5], and Section 8 proves Theorem 1.1. Section 9 explains the exclusion of most cells by lemmas proved by hand, independently of the computations by which Section 5 excludes them. Theorem 1.1 needs neither that section nor the treatment of diagrams 81, 82 and 90 in Section 6, so a reader may skip both. Section 10 describes the verification.

2. Curves of normalized central configurations and their places

In this section we set up the six-body versions of the definitions and lemmas of [27, §§2–3]. Since their proofs there do not use the number of bodies, except where noted, we cite them. We use no result of [2] as an input: the rules of [2, §3] that we need are proved at places in Lemma 2.11, and the arguments of [2, §§5.0, 8.2, 8.6] that we adapt will be proved in Sections 6 and 7. Throughout, $m = (m_1, \dots, m_6)$ and $M = m_1 + \cdots + m_6$. Pairs of bodies are written kl.

2.1. **The system and the slice.** Following [2, p. 540], a *normalized central configuration* is a solution $(x, y, \delta) \in \mathbb{C}^6 \times \mathbb{C}^6 \times \mathbb{C}^{15}$ of

(2.1)
$$x_k = \sum_{l \neq k} m_l\, \delta_{kl}^3 (x_k - x_l), \quad y_k = \sum_{l \neq k} m_l\, \delta_{kl}^3 (y_k - y_l), \quad \delta_{kl}^2\big((x_k - x_l)^2 + (y_k - y_l)^2\big) = 1, \quad y_1 = y_2,$$

for $k \neq l$ in $\{1, \dots, 6\}$, with $\delta_{kl} = \delta_{lk}$. For $m \in (\mathbb{R}_{>0})^6$, every central configuration with masses m is similar, by an orientation-preserving similarity, to exactly two real normalized central configurations with positive δ_{kl} [27, Lemma 2.1]. Hence finiteness of the solution set of (2.1) implies finiteness of the central configurations up to similarity.

We put $z = x + iy$, $w = x - iy$, $z_{kl} = z_k - z_l$, $w_{kl} = w_k - w_l$ and $r_{kl}^2 = z_{kl} w_{kl}$. As in [27, §2.2], the unknowns are $u = (u_1, \dots, u_5)$, $v = (v_1, \dots, v_5)$ and $p = (p_{kl})_{k<l}$, with $u_6 = v_6 = 0$, $u_{kl} = u_k - u_l$, $v_{kl} = v_k - v_l$ and $p_{lk} = p_{kl}$. With $F_k(h) = h_k - \sum_{l \neq k} m_l\, p_{kl}^3 (h_k - h_l)$, the 25 *defining polynomials* are

$$D_{kl} = p_{kl}^2 u_{kl} v_{kl} - 1, \qquad Z_k = F_k(u) - F_6(u), \qquad W_k = F_k(v) - F_6(v) \qquad (k \leq 5).$$

Let $X_m \subset \mathbb{C}^{25}$ be their zero set and $Y_m = X_m \cap \{u_1 = 1\}$, the *slice*. On X_m the functions u_k, v_k, u_{kl}, v_{kl} and p_{kl} vanish nowhere.

Lemma 2.1 ([27, Lemma 2.2])**.** (a) *The group $\mathbb{C}^*$ acts freely on X_m by $s \cdot (u, v, p) = (su, s^{-1}v, p)$, the map $\mathbb{C}^* \times Y_m \to X_m$, $(s, y) \mapsto s \cdot y$, is an isomorphism, and* (2.1) *has exactly $2|Y_m|$ solutions.*

(b) *For $t > 0$ there is an isomorphism $Y_m \to Y_{tm}$ that multiplies the p_{kl} by $t^{-1/3}$. Relabelling the bodies together with their masses, followed by a translation and a rotation, is an isomorphism $Y_m \to Y_{m'}$ for the permuted masses m'.*

On X_m we put $z_k = u_k - F_6(u)$ and $w_k = v_k - F_6(v)$ ($k \le 6$). Then $z_k - z_l = u_{kl}$, $w_k - w_l = v_{kl}$, and the equations $D_{kl} = Z_k = W_k = 0$ say that

$$z_k = \sum_{l \neq k} m_l \, Z_{kl}, \qquad w_k = \sum_{l \neq k} m_l \, W_{kl}, \qquad Z_{kl} = p_{kl}^3 z_{kl}, \quad W_{kl} = p_{kl}^3 w_{kl} \tag{2.2}$$

for $k = 1, \dots, 6$. These are the *equations of the bodies*. Summing them with the weights m_k gives $\sum_k m_k z_k = \sum_k m_k w_k = 0$.

Lemma 2.2. *The map $\iota(u, v, p) = (v/v_1,\, v_1 u,\, p)$ is an involution of Y_m.*

Proof. The defining polynomials are permuted by $(u, v, p) \mapsto (v, u, p)$, which fixes the D_{kl} and exchanges Z_k and W_k. Composing with the rotation by $s = v_1^{-1}$ of Lemma 2.1(a) gives a point with $u_1 = 1$. Applying ι twice gives (u, v, p) back, since $u_1 = 1$. □

Let

$$U = \sum_{k<l} m_k m_l \, p_{kl}, \qquad I = \frac{1}{M} \sum_{k<l} m_k m_l \, u_{kl} v_{kl}.$$

Lemma 2.3 ([27, Lemmas 2.3 and 2.4])**.** (a) *On X_m, $U = \sum_k m_k z_k w_k$, and $U = I$ if $M \neq 0$.*

(b) *If $M \neq 0$, U is constant on every irreducible component of Y_m.*

(c) *For $m \in (\mathbb{R}_{>0})^6$, $U > 0$ at the point of Y_m given by a real central configuration.*

Corollary 2.4 ([27, Corollary 2.5])**.** *Let $m \in (\mathbb{R}_{>0})^6$. If the planar six-body problem with masses m has infinitely many central configurations up to similarity, then Y_m contains an irreducible curve C on which U is a nonzero constant and some r_{kl}^2 is not constant.*

The proof in [27] bounds the fibres of $(u, v, p) \mapsto (r_{kl}^2)_{k<l}$ on Y_m. For six bodies the bound is 2^{19} instead of 2^{13}: on a fibre, $u_1 = 1$ determines v_1, each of $u_2, \dots, u_5$ is a root of a quadratic equation, and the fifteen p_{kl} are determined up to sign. We call a curve as in Corollary 2.4 *nontrivial*, and we write E for the value of U on it, so that $E \neq 0$.

2.2. **Places, exponents and balancing.** Let $C \subseteq Y_m$ be an irreducible curve, $\bar{C}$ its smooth projective model, and P a point of $\bar{C}$ not over C, a *place at infinity*. We put

$$\mathcal{Z} = (z_1, \dots, z_6, Z_{12}, \dots, Z_{56}), \qquad \mathcal{W} = (w_1, \dots, w_6, W_{12}, \dots, W_{56}).$$

We call z and w the two *colours*. A function on C is of *z-type* if the rotation $s\cdot$ multiplies it by s (as u_{kl}, z_k, z_{kl}, Z_{kl}), of *w-type* if it multiplies it by s^{-1}, and *neutral* if it is invariant (as p_{kl}, r_{kl}^2). With $\nu = \mathrm{val}_P$ and ν of a vector the minimum over its components, the place is *singular* if $\nu(\mathcal{Z}) + \nu(\mathcal{W}) < 0$. Then we put $d = -\frac{1}{2}(\nu(\mathcal{Z}) + \nu(\mathcal{W})) > 0$, $\gamma = \frac{1}{2}(\nu(\mathcal{W}) - \nu(\mathcal{Z}))$, and define the *exponent*

$$e(f) = \tfrac{2}{d}\big(\nu(f) + \gamma\big), \qquad e(f) = \tfrac{2}{d}\big(\nu(f) - \gamma\big), \qquad e(f) = \tfrac{2}{d}\,\nu(f)$$

of a function f of z-type, w-type or neutral type, with $e(f) = +\infty$ if f vanishes identically on C.

Lemma 2.5 ([27, §3.1 and Lemma 3.1])**.** *Let P be a singular place. Every component of $\mathcal{Z}$ and of $\mathcal{W}$ has exponent at least -2, and each of the two vectors has a component of exponent -2. Let t be a local parameter at P and $\vartheta^2 = t$. After the point of C near P is rotated by $s = \vartheta^{2\gamma}$, every function f of the three kinds that does not vanish identically on C becomes $f = \vartheta^{d\,e(f)} y_f(\vartheta)$ with $y_f \in \mathbb{C}\{\vartheta\}$, $y_f(0) \neq 0$ and $d\,e(f) \in \mathbb{Z}$.*

For a function f of one of the three kinds and a number $e \le e(f)$ we write $[f]_e$ for the coefficient of ϑ^{de} in this expansion. Thus $[f]_{e(f)} = y_f(0) \neq 0$, $[f]_e = 0$ for $e < e(f)$, $[fg]_{e+e'} = [f]_e[g]_{e'}$ and $[f+g]_e = [f]_e + [g]_e$. We write

$$a_k = e(z_k), \quad b_k = e(w_k), \quad \zeta_{kl} = e(z_{kl}), \quad \omega_{kl} = e(w_{kl}), \quad x_{kl} = \tfrac{1}{2}(\zeta_{kl} + \omega_{kl}).$$

Then $e(p_{kl}) = -x_{kl}$, $e(Z_{kl}) = -\frac{1}{2}(\zeta_{kl} + 3\omega_{kl})$, $e(W_{kl}) = -\frac{1}{2}(3\zeta_{kl} + \omega_{kl})$ and $\zeta_{kl} \ge \min(a_k, a_l)$, with equality if $a_k \neq a_l$. The body k is *z-circled* if $a_k = -2$, and kl is a *z-stroke* if $e(Z_{kl}) = -2$, and likewise for w. The vector $x(P) = (x_{kl})_{k<l} \in \mathbb{Q}^{15}$ is the *type* of P.

Lemma 2.6 ([27, §2.4 and (3.1)])**.** *Let C be a nontrivial curve. Then C has at least one singular place, and*

$$\sum_{P \text{ singular}} d_P \, x(P) = 0. \tag{2.3}$$

The proof uses that the p_{kl} are units on C, so that their divisors have degree zero, and that $\nu(p_{kl}) = 0$ at a place that is not singular. We write $\ell(x) = \sum_{k<l} x_{kl}$.

Corollary 2.7. *Let C be a nontrivial curve. Some singular place P of C has $\ell(x(P)) \le 0$.*

Proof. Apply ℓ to (2.3). Since all d_P are positive and there is a singular place, the values $\ell(x(P))$ cannot all be positive. □

Since $x_{kl} = \nu(r_{kl}^2)/d$, we have $\ell(x(P)) = \nu(\Delta)/d$ for the unit $\Delta = \prod_{k<l} r_{kl}^2$ of C, so Corollary 2.7 says that Δ does not tend to zero at some singular place.

Lemma 2.8 ([27, Lemma 3.2])**.** *At a singular place, for every pair kl:*

- (O1) *$\zeta_{kl}, \omega_{kl} \ge -2$, $\zeta_{kl} + 3\omega_{kl} \le 4$ and $3\zeta_{kl} + \omega_{kl} \le 4$. Hence $\zeta_{kl}, \omega_{kl} \le 2$ and $-2 \le x_{kl} \le 1$.*
- (O2) *kl is a z-stroke if and only if $\zeta_{kl} + 3\omega_{kl} = 4$, and a w-stroke if and only if $3\zeta_{kl} + \omega_{kl} = 4$.*
- (O3) *On a z-stroke, $\zeta_{kl} = 3x_{kl} - 2$, $\omega_{kl} = 2 - x_{kl}$ and $0 \le x_{kl} \le 1$. On a w-stroke, $\omega_{kl} = 3x_{kl} - 2$ and $\zeta_{kl} = 2 - x_{kl}$.*
- (O4) *$x_{kl} = 1$ if and only if kl is both a z-stroke and a w-stroke (a zw-edge).*
- (O5) *In every triangle klj the minimum of $\zeta_{kl}, \zeta_{lj}, \zeta_{jk}$ is attained at least twice, and the same holds for ω.*
- (O6) *If k is z-circled and l is not, then $\zeta_{kl} = -2$. If $\zeta_{kl} = 2$, then $\omega_{kl} = -2$ and kl is a w-stroke.*

The same holds with z and w exchanged.

In [27, §3] the masses are positive and the curve is nontrivial, but the proofs of Lemmas 2.5 and 2.8 use neither assumption. We will use the two lemmas for every m and every irreducible curve $C \subseteq Y_m$.

Lemma 2.9. *Let P be a singular place of a curve $C \subseteq Y_m$.*

- (a) *A relabelling of the bodies as in Lemma 2.1*(b) *maps C to a curve in $Y_{m'}$ and P to a singular place whose exponents, circles and strokes are those of P, relabelled.*
- (b) *The involution ι of Lemma 2.2 maps C to a curve in Y_m and P to a singular place at which $a_k, b_k, \zeta_{kl}, \omega_{kl}$ are the values $b_k, a_k, \omega_{kl}, \zeta_{kl}$ at P. The type is unchanged.*

Both maps preserve U and permute the functions r_{kl}^2, so they map nontrivial curves to nontrivial curves.

Proof. (a) The relabelling multiplies $\mathcal{Z}$ and $\mathcal{W}$ by s and s^{-1} for a function s and permutes their components, and it preserves the p_{kl} up to the permutation. (b) Since F_6 is linear, $z_k \circ \iota = w_k/v_1$ and $w_k \circ \iota = v_1 z_k$, so $\mathcal{Z} \circ \iota = v_1^{-1}\mathcal{W}$ and $\mathcal{W} \circ \iota = v_1 \mathcal{Z}$, while the p_{kl} are unchanged. At the image place the numbers d and γ become d and $\nu(v_1) - \gamma$. So the exponent of z_k there is $\frac{2}{d}(\nu(w_k) - \nu(v_1) + \nu(v_1) - \gamma) = e(w_k)$, and likewise for the other functions. □

2.3. **Identities.** The following identities hold on X_m. On a nontrivial curve C we also have $U = E$. For a set J of bodies, J^c is its complement.

- *Subset sums*: $\sum_{k\in J} m_k z_k = \sum_{k\in J,\, l\in J^c} m_k m_l\, Z_{kl}$, and the same in w. For $J = \{k\}$ this is the equation of the body k.
- *Torques*: $T_J := \sum_{k\in J,\, l\in J^c} m_k m_l\, p_{kl}^3 (z_k w_l - z_l w_k) = 0$.
- *Partial virials*: $\sum_{k\in J} m_k z_k w_k - \sum_{k<l\in J} m_k m_l p_{kl} - \frac{1}{2}\sum_{k\in J,\, l\in J^c} m_k m_l p_{kl}^3 (w_k z_{kl} + z_k w_{kl}) = 0$.
- *Differences*: $z_{kl} = \sum_{j\neq k} m_j Z_{kj} - \sum_{j\neq l} m_j Z_{lj}$, and the same in w.
- $U = \sum_k m_k z_k w_k$, $\sum_{k<l} m_k m_l\, z_{kl} w_{kl} = MU$, $p_{kl}^2 z_{kl} w_{kl} = 1$.

The subset sums and the differences follow from (2.2), since the terms of pairs inside J cancel. For the torques, w_k times the z-equation of k minus z_k times its w-equation gives $\sum_{l\neq k} m_l p_{kl}^3 (z_k w_l - z_l w_k) = 0$. Multiplying by m_k and summing over $k \in J$, the pairs inside J cancel because $p_{kl}^3 (z_k w_l - z_l w_k)$ is antisymmetric. For the partial virials, the half sum of w_k times the z-equation of k and z_k times its w-equation gives $m_k z_k w_k = \frac{1}{2}\sum_{l\neq k} m_k m_l p_{kl}^3 (w_k z_{kl} + z_k w_{kl})$, and the pairs inside J contribute $m_k m_l p_{kl}^3 z_{kl} w_{kl} = m_k m_l p_{kl}$. The last line is Lemma 2.3(a) and the equations $D_{kl} = 0$. For $J = \{1, \dots, 6\}$ the partial virial is $\sum_k m_k z_k w_k = U$. The torque identity is the constraint of [2, §5.0] on the external forces of a set of bodies.

Each identity is a sum $\sum_i c_i g_i = 0$ in which the c_i are nonzero polynomials in the masses (or the constant E) and each g_i is a product of the functions $z_k, w_k, z_{kl}, w_{kl}, p_{kl}$, and all g_i are multiplied by the same power of s under the rotation. At a singular place the exponent of g_i is the sum of the exponents of its factors. We omit the terms with a factor that vanishes identically on C. Only z_k and w_k can do so.

Lemma 2.10. *Let $C \subseteq Y_m$ be an irreducible curve and $\sum_i c_i g_i = 0$ an identity on C in which the c_i are nonzero complex numbers, each g_i is a product of functions $z_k, w_k, z_{kl}, w_{kl}, p_{kl}$ that do not vanish identically on C (possibly the empty product 1), and all g_i are multiplied by the same power of s under the rotation. Then at every singular place the least exponent e_0 of the terms is attained by at least two terms, and $\sum_{e(g_i)=e_0} c_i [g_i]_{e_0} = 0$, where $[g_i]_{e_0}$ is the product of the leading coefficients of the factors of g_i.*

Examples are the identities above at masses with $c_i(m) \neq 0$ for all i, and, if $M \neq 0$, the identity $U - c = 0$, where c is the constant value of U on C (Lemma 2.3(b)); if $c = 0$, this identity has no constant term.

Proof. By Lemma 2.5 each term is $\vartheta^{d\, e(g_i)}$ times a unit, after the common rotation. The coefficient of ϑ^{de_0} of the sum is the displayed sum, which vanishes. A single term of least exponent would make it nonzero. □

We call $\sum_{e(g_i)=e_0} c_i [g_i]_{e_0}$ the *initial form* of the identity at the place.

2.4. **Diagrams and the rules of Albouy and Kaloshin.** The *diagram* of a singular place consists of the z-strokes, the w-strokes and the z- and w-circled bodies. A pair is an *edge* if it is a stroke of some colour, and its *kind* is then z, w or zw. The *z-component* of a body is its connected component in the graph of the z-strokes. The following rules are the exponent forms of rules of [2, §3], which are stated there for any number of bodies whose masses have no vanishing partial sum.

Lemma 2.11. *Let $m \in (\mathbb{R}_{>0})^6$ and let P be a singular place of an irreducible curve $C \subseteq Y_m$. The diagram of P satisfies the following rules, and the same rules with z and w exchanged.*

(E1) *There is a z-stroke.*

(E2) *If ab and bc are zw-edges, then ac is a zw-edge.*

(E3) *The three edges of a triangle of edges have the same kind, equal ζ and equal ω.*
(E4) *From each zw-edge ab another z-stroke leaves a or b.*
(E5) *The number of zw-edges is not* 1.
(E6) *If ab, bc, cd and da are edges, then ab and cd have equal ζ, equal ω and the same kind, and so have bc and da.*
(C1) *A body without a z-stroke is not z-circled, and a body with exactly one z-stroke is z-circled.*
(C2) *No w-stroke joins a z-circled body to a body that is not z-circled.*
(C3) *If k is z-circled, some z-circled body of the z-component of k lies outside the w-component of k.*
(C4) *If k and j are z-circled, l is not, and kl and lj are z-strokes, then kj is a z-stroke.*

The rules correspond to Rule 1a (E1, C1), the circling method based on Rule 1b and Estimate 2 (C2), Rule 1d (C3), the corollary of Rule 2c together with Rule 2e (C4), Rules 2b, 2e, 2a (E2, E3, E4), the corollary of Rule 2h (E5) and a strengthening of Rule 2g (E6) of [2]. Rule (C3) implies that a z-component never contains exactly one z-circled body. Besides $m_k \neq 0$, only (E4), (C3) and (E5) use the positivity of the masses. They use it through the nonvanishing of partial sums, and (E5) through $M \neq 0$ in Lemma 2.3(b). Only (E5) uses U, through its constancy on C.

Proof. On a z-stroke that is not a zw-edge, $\zeta = 3x - 2 < 1 < \omega$; on a w-stroke that is not a zw-edge, $\omega < 1 < \zeta$; on a zw-edge, $\zeta = \omega = 1$ (Lemma 2.8). So the kind of an edge is determined by ζ, and on edges $\omega = f(\zeta)$ for the strictly decreasing function $f(\zeta) = (4-\zeta)/3$ ($\zeta \leq 1$), $f(\zeta) = 4 - 3\zeta$ ($\zeta \geq 1$).

(C1) The terms of exponent -2 in the equation $z_k = \sum_l m_l Z_{kl}$ are z_k if k is circled and Z_{kl} for the z-strokes kl. By Lemma 2.10 there cannot be exactly one such term. (E1) Some component of $\mathcal{Z}$ has exponent -2 (Lemma 2.5). If it is z_k, (C1) gives a z-stroke at k.

(C2) On a w-stroke kl, $\zeta_{kl} = 2 - x_{kl} \geq 1 > -2$, so by (O6) the bodies k and l are both z-circled or both not.

(E2) On zw-edges $\zeta = \omega = 1$, so (O5) gives $\zeta_{ac}, \omega_{ac} \geq 1$. Hence $x_{ac} = 1$, and ac is a zw-edge by (O4).

(E3) Let $\zeta_1 = \zeta_2 \leq \zeta_3$ be the values of ζ on the three edges, ordered by (O5). If $\zeta_1 > 1$, all three edges are w-edges. If $\zeta_1 = 1$, the first two are zw-edges, and so is the third by (E2). If $\zeta_1 < 1$, the first two are z-edges with $\omega_1 = \omega_2 = f(\zeta_1) > 1$; if the third were not a z-edge, $\omega_3 \leq 1$ would be the unique minimum of ω, contradicting (O5). In each case the three edges have the same kind. Then $\omega = f(\zeta)$ and (O5) for ω give $\zeta_3 = \zeta_1$, hence equal ω.

(E4) Suppose no other z-stroke leaves a or b. Since $\zeta_{ab} = 1 > -2$, the bodies a and b are both z-circled or both not (O6), and $[z_a]_{-2} = [z_b]_{-2}$. If they are circled, the subset sum for $J = \{a, b\}$ has the terms $m_a z_a, m_b z_b$ of exponent -2 and all other terms of larger exponent, so $(m_a + m_b)[z_a]_{-2} = 0$ by Lemma 2.10, a contradiction. If they are not circled, $m_b Z_{ab}$ is the only term of exponent -2 in the equation of a, contradicting (C1).

(E5) Let c be the constant value of U on C. The terms $m_k m_l p_{kl}$ of $U - c = 0$ have exponents $-x_{kl} \geq -1$, with equality exactly at the zw-edges (O4). A single zw-edge would give a single term of least exponent, contradicting Lemma 2.10.

(E6) The identity $z_{ab} + z_{bc} + z_{cd} + z_{da} = 0$ shows that the minimum of ζ over the four edges is attained at least twice, and the same identity in w, with $\omega = f(\zeta)$, shows that the maximum of ζ is attained at least twice. If the four values are not all equal, two edges carry the minimum α and the other two the maximum $\beta > \alpha$. Suppose ab and bc carry α. Then (O5) in the triangles acd and abc gives $\zeta_{ac} \geq \beta$ and $\omega_{ac} \geq f(\alpha)$. If $\alpha < 1$, then $\zeta_{ac} + 3\omega_{ac} \geq \beta + 4 - \alpha > 4$; if $\alpha \geq 1$,

then $3\zeta_{ac} + \omega_{ac} \geq 3\beta + 4 - 3\alpha > 4$. Both contradict (O1). So opposite edges carry equal values of ζ, hence of ω, and have the same kind.

(C3) Let K be the z-component of k and L its w-component. The subset sum for K has the terms $m_j z_j$, $j \in K$, of exponent -2 exactly when j is circled, and its other terms are $m_j m_i Z_{ji}$ with $j \in K$, $i \notin K$, which are not z-strokes and have exponent greater than -2. So $\sum_j m_j [z_j]_{-2} = 0$ over the circled $j \in K$. On a w-stroke $\zeta \geq 1 > -2$, so all bodies of L have the same coefficient $[z_j]_{-2}$, which is $[z_k]_{-2} \neq 0$. If all circled bodies of K were in L, the sum would be $[z_k]_{-2}$ times a nonempty partial sum of masses, which is not zero.

(C4) By (O6), $\zeta_{kl} = \zeta_{lj} = -2$, so $x = 0$ and $\omega_{kl} = \omega_{lj} = 2$ (O3). By (O5), $\omega_{kj} \geq 2$, hence $\omega_{kj} = 2$, and (O1) gives $\zeta_{kj} \leq -2$. So $\zeta_{kj} + 3\omega_{kj} = 4$, and kj is a z-stroke by (O2). □

If all six pairs of a set of four bodies are edges, (E3) applied to its four triangles shows that the six pairs have the same kind, equal ζ and equal ω. This is the conclusion of Rule 2f of [2] for $p = 4$, under the stronger hypothesis that all six pairs are edges.

3. The diagrams of six bodies

A *labelled diagram* on six bodies assigns to each pair one of the kinds none, z, w, zw, and to each body a z-circle or not and a w-circle or not. The symmetric group S_6 acts by relabelling, and the exchange of the colours acts by exchanging the kinds z and w and the two sets of circles. By Lemma 2.9, the diagram of the image of a singular place under a relabelling or under ι is the image of its diagram, so the rules of Lemma 2.11 are invariant under both actions. From now on we mark with † the statements whose own proof includes a computation. Statements that only cite them are not marked.

Proposition 3.1. † *Exactly* 82,712 *labelled diagrams on six bodies satisfy the combinatorial parts of the rules* (E1)–(E6) *and* (C1)–(C4) *of Lemma 2.11 in both colours. Up to relabelling and exchange of the colours they form* 191 *classes.*

Here the combinatorial parts are the rules themselves, except that (E3) and (E6) are reduced to their statements on the kinds of the edges, since a diagram does not record ζ and ω.

Proof. A depth-first search assigns kinds to the fifteen pairs. It checks (E2), (E3) and (E6), as conditions on the kinds, as soon as a triangle or a cycle of four pairs is complete. These checks leave 932,677 assignments. At the leaves it checks (E1), (E4) and (E5), and then enumerates the circles of each colour allowed by (C1) and tests (C2)–(C4). Every labelled diagram that satisfies the rules is a leaf of the search. The classes are the orbits of the 1440 symmetries. □

The program passes three checks. For four bodies it finds 41 labelled diagrams in 5 classes, the five four-body diagrams of [2, §4, Fig. 8]. For five bodies it finds 1376 labelled diagrams in 24 classes. This is the count of the search checked in the Lean kernel in [27, §12.4], and its classes are the sixteen diagrams of [2, Fig. 11] and eight others. For six bodies, the 117 diagrams of Chang and Chen [5, Figs. 3–6] satisfy our rules and lie in 117 distinct classes of our 191. Each of the other 74 classes violates one of the further rules of [5, §3] or the following consequence of (E3) and (O6): a triangle of strokes of one colour does not have exactly one circled vertex of that colour. Indeed, if k is the only z-circled vertex of a triangle klj of z-strokes, then $\zeta_{kl} = -2$ by (O6), so $\zeta_{lj} = -2$ by (E3), contradicting $\zeta_{lj} \geq \min(a_l, a_j) > -2$. Eight of the 74 classes violate only this rule. We do not need these rules, because Section 4 treats all 191 classes.

We refer to a diagram by its number in [5, Figs. 3–6] when it has one, with the labelling of the bodies used there. Table 1 and Figure 1 show the ten diagrams of Sections 6 and 7. None of them has a zw-edge.

diagram	z-strokes	z-circles	w-strokes	w-circles
64	all pairs of $\{3,4,5,6\}$	none	12	$1,2$
66	all pairs of $\{3,4,5,6\}$	none	12, 16, 26	$1,2$
81	56	$5,6$	34	$3,4$
82	56	$5,6$	23, 24, 34	$3,4$
88	56	$5,6$	all pairs of $\{1,2,3,4\}$	$1,2,3,4$
89	45, 46, 56	$5,6$	23, 24, 34	$2,3$
90	45, 46, 56	$5,6$	12, 13, 23	$2,3$
93	45, 46, 56	$5,6$	12, 13, 23	$1,2,3$
94	45, 46, 56	$5,6$	all pairs of $\{1,2,3,4\}$	$1,2,3$
98	45, 46, 56	$4,5,6$	12, 13, 23	$1,2,3$

TABLE 1. The diagrams treated in Sections 6 and 7, numbered and labelled as in [5, Figs. 5, 6].

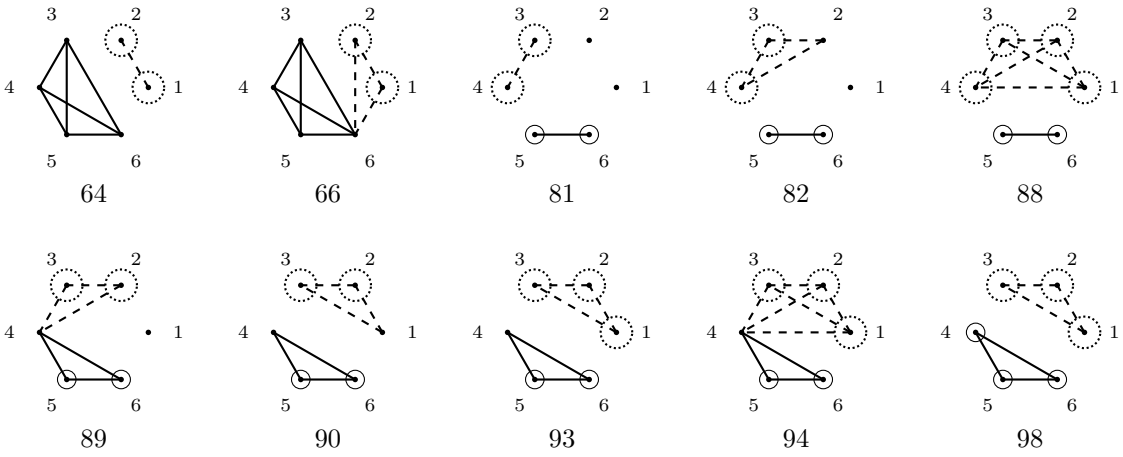


FIGURE 1. The diagrams of Table 1: z-strokes solid, w-strokes dashed, z-circles solid, w-circles dotted.

4. Cells of exponents

In this section we describe the exponents at a singular place with a given diagram by finitely many cells, and we enumerate the cells on which $\ell(x) \le 0$ is possible. Throughout, $m \in (\mathbb{R}_{>0})^6$, $C \subseteq Y_m$ is a nontrivial curve and P is a singular place of C.

4.1. **Trees.** Let O be a further point with $z_O = w_O = 0$, the centre of mass, and let $B = \{1,\dots,6,O\}$. For $p \neq q$ in B the exponent $e(z_p - z_q)$ is finite unless $\{p,q\} = \{k,O\}$ for a body k with $z_k \equiv 0$ on C. At most one body has this property, since $z_k - z_l$ vanishes nowhere on X_m.

Definition 4.1. A *tree* is a rooted tree whose leaves carry the points of B, one point on each leaf, except that the leaf of O may carry one body as well. Every internal node has at least two children and a rational *height*, and the height of an internal node is smaller than the heights of its internal children. For points p,q on different leaves, $h(p,q)$ is the height of the lowest common ancestor of their leaves. For points on the same leaf, $h(p,q) = +\infty$.

Lemma 4.2. *There is a unique tree T_z with $e(z_p - z_q) = h(p,q)$ for all $p \neq q$ in B, and likewise a tree T_w for w. For each internal node v of T_z with children $c_0,\dots,c_r$ there are pairwise distinct*

numbers $X_{v,c_0} = 0, X_{v,c_1}, \dots, X_{v,c_r} \in \mathbb{C}$ *such that*

$$[z_p - z_q]_{h_v} = X_{v,c(p)} - X_{v,c(q)}$$

for all points p, q *below* v *in different children* $c(p)$, $c(q)$ *of* v, *where* h_v *is the height of* v.

Proof. On functions of z-type, e is an increasing affine function of the valuation. Hence $e(z_p - z_r) \geq \min(e(z_p - z_q), e(z_q - z_r))$ for all $p, q, r \in B$, so $-e(z_p - z_q)$ is an ultrametric on the set of leaves, with the convention above for a body at O. The closed balls of a finite ultrametric space form a hierarchy, and the tree is the tree of this hierarchy. Its internal nodes are the balls with at least two leaves, and the height of such a ball is the least exponent $e(z_p - z_q)$ of two of its points. This tree is the only one with the stated property. For the second statement, fix a point o below c_0 and put $X_{v,c(p)} = [z_p - z_o]_{h_v}$ for p below v. This depends only on the child $c(p)$, since points below one child have differences of exponent larger than h_v, and it is 0 for $c(p) = c_0$. For p, q in different children, $z_p - z_q$ has exponent exactly h_v, so the difference $X_{v,c(p)} - X_{v,c(q)}$ is nonzero. □

We call the $X_{v,c}$ and the analogous $Y_{v,c}$ of T_w the *node coordinates*. The exponents at P are read off the trees: $a_k = h_z(k, O)$, $\zeta_{kl} = h_z(k, l)$, and likewise b_k and ω_{kl} from T_w.

Lemma 4.3. *The trees of* P *satisfy the following conditions.*

(a) *The root heights are at least* -2. *For each pair* kl, $\zeta_{kl} + 3\omega_{kl} \leq 4$, *with equality if and only if* kl *is a* z*-stroke, and* $3\zeta_{kl} + \omega_{kl} \leq 4$, *with equality if and only if* kl *is a* w*-stroke. A body* k *has* $a_k = -2$ *if and only if it is* z*-circled, and* $b_k = -2$ *if and only if it is* w*-circled.*

(b) *The root of each tree has at least three children.*

(c) *In each of the twelve equations of the bodies and in each of the identities* $U = E$ *and* $\sum_{k<l} m_k m_l z_{kl} w_{kl} = ME$, *the least exponent of the terms is attained by at least two terms.*

Proof. (a) is Lemma 2.5, (O1), (O2) and the definitions, since the root height is an exponent of a position or of a separation. (b) Let r be the root of T_z and c_O its child containing O. A body below another child c has $a_k = h_r$ and $[z_k]_{h_r} = X_{r,c} - X_{r,c_O}$ (Lemma 4.2), and a body below c_O has $a_k > h_r$. The coefficient of exponent h_r in $\sum_k m_k z_k = 0$ is therefore $\sum_{c \neq c_O} (X_{r,c} - X_{r,c_O}) M_c$, where M_c is the sum of the masses below c. If r had only the two children c_O and c, this would be $(X_{r,c} - X_{r,c_O}) M_c \neq 0$. The same holds for T_w. (c) is Lemma 2.10, since the m_k, E and M are nonzero. □

Note that condition (c) for the second identity contains [27, Lemma 3.3]: if all x_{kl} were positive, the constant term ME would be the only term of least exponent.

4.2. **Cells.** The equations of the bodies, $U = E$ and $\sum_{k<l} m_k m_l z_{kl} w_{kl} = ME$ are the fourteen *base identities*. Their terms are z_k (omitted if k shares the leaf of O) and $m_l Z_{kl}$ in the z-equation of k, likewise in w, the constant and the $m_k m_l p_{kl}$ in $U = E$, and the constant and the $m_k m_l z_{kl} w_{kl}$ in the last identity. Their exponents are linear forms in the heights.

Definition 4.4. Let D be a labelled diagram. A *cell* of D consists of a pair of trees without heights satisfying Lemma 4.3(b), and of a set of at least two terms for each base identity, its *argmin set*, such that the following set Π of height vectors is not empty: the heights increase strictly along the trees, they satisfy Lemma 4.3(a) for D, and in each base identity the terms of the argmin set have equal exponents and all other terms have larger exponents. The cell is *nonpositive* if $\ell(x) \leq 0$ at some point of Π.

The set Π is a relatively open polyhedron. By Lemma 4.3, the trees of P and the sets of terms of least exponent at P form a cell of the diagram of P, and the heights of P lie in its set Π. We then say that P *lies in* this cell. When the diagram forces terms of exponent -2 in a base identity (a circle or a stroke in an equation of a body), the argmin set consists of these terms, by (O1). If the diagram has zw-edges, the argmin set of $U = E$ consists of their terms, of exponent -1, by (O1) and (O4). The relabellings and ι act on the cells of all diagrams compatibly with Lemma 2.9, so it suffices to enumerate the cells of one representative of each class.

Proposition 4.5. † *The enumeration below lists* 91,273 *cells of the representatives of the* 191 *classes of Proposition 3.1, in* 69 *classes. Every nonpositive cell of a representative is in the list.*

Proof. The program starts from the trees with the single leaf O and inserts the bodies one at a time into both trees, in all ways: below an internal node, by subdividing an edge, as a new root, or on the leaf of O. It discards a pair of partial trees if the linear conditions among the points inserted so far are infeasible together with the relaxation of $\ell(x) \le 0$ in which each pair not yet inserted contributes its least possible value of x: -2 in general, 0 on a stroke and 1 on a zw-edge (Lemma 2.8). Restricting two trees to fewer points gives partial trees visited by the search, so no pair of trees is lost. Each tree arises exactly once: without pruning, the insertions produce 55,720 distinct trees, the 39,208 trees with seven leaves and the 16,512 trees with a body on the leaf of O. For complete pairs satisfying Lemma 4.3(b) it chooses the argmin sets one base identity at a time and discards infeasible choices. Apart from linear programs, the search discards a pair or a choice for only four reasons: the root of a complete tree has two children (Lemma 4.3(b)), a body circled in the colour of a tree shares the leaf of O in that tree, a base identity has a single forced term of least exponent, or fewer than two terms remain for an argmin set. Every infeasibility is certified by Lemma 4.6 below. Feasibility is decided by the floating-point solver without a certificate (see Section 10). The search for each class ran in 384 parts. Each part repeats the insertions of the first two bodies, and a counter over the insertions of the third body assigns each of them to one part. This division is consistent because all parts reach the same decisions on the first two insertions: the discarded pairs are certified in every part, and each kept pair was confirmed by an exact rational solution of its conditions. So every node of the search lies in some part. □

A linear system as in the proof has rows $\mathrm{lo}_i \le a_i \cdot h + c_i s \le \mathrm{hi}_i$, where h is the vector of heights, either bound may be absent, and a strict inequality $f > 0$ is written $f - s \ge 0$ with a common slack s. Each row has $c_i \in \{-1, 0, 1\}$, and the rows with $c_i \ne 0$ remain valid when s decreases. The system has a solution with $s > 0$ exactly when the strict system is feasible.

Lemma 4.6. *Let y_i be rational numbers with $y_i > 0$ only if hi_i exists and $y_i < 0$ only if lo_i exists. Put $b_i = \mathrm{hi}_i$ for $y_i > 0$ and $b_i = \mathrm{lo}_i$ for $y_i < 0$, $g = \sum_i y_i c_i$, $\beta = \sum_i y_i b_i$ and $g^- = \max(0, -g)$. Suppose that $\sum_i y_i a_i = 0$ and $\beta + g^- \le 0$, and that $g + g^- > 0$ or $\beta + g^- < 0$. Then the system has no solution with $s > 0$.*

Proof. Let (h, s) be a solution with $s > 0$. Decreasing s keeps the rows valid, so we may assume $s \le 1$. Adding the rows with the weights y_i gives $gs \le \beta$, and adding $g^- s \le g^-$ gives $(g + g^-)s \le \beta + g^- \le 0$. Since $g + g^- \ge 0$ and $s > 0$, both $g + g^-$ and $\beta + g^-$ vanish, contrary to the hypothesis. □

By the transposition theorem of Motzkin, multipliers as in Lemma 4.6 exist for any infeasible system of this form. The program obtains them by rounding the dual solution or the dual ray of the floating-point solver, and checks them in rational arithmetic. When rounding fails, it computes them with an exact simplex method. The counts are in Table 2.

5. Generic exclusion of the cells

We now exclude the listed cells at generic masses, except 43 of them. Let D be the representative of a class and Π the polyhedron of a listed cell of D. Recall from Definition 4.4 that a cell consists of a pair of trees and of an argmin set for each base identity. Throughout, $m \in (\mathbb{R}_{>0})^6$ and $C \subseteq Y_m$ is a nontrivial curve.

For an identity of §2.3 we say that its *argmin set is constant on the cell* if there is a set S of its terms such that at every point of Π the terms in S have equal exponents and all other terms have larger exponents. Besides the base identities, whose argmin sets are part of the cell, we use the subset sums for $2 \le |J| \le 4$, the torques T_J for the sets $J \ni 1$ with $|J| \le 5$, the partial virials for all nonempty $J \neq \{1, \dots, 6\}$, the differences, the identity $U = \sum_k m_k z_k w_k$ and the identity $\sum_k m_k z_k w_k = E$. Whether an argmin set is constant on the cell is decided by linear programs, and each decision on which the conclusion depends is certified by Lemma 4.6.

Lemma 5.1. *If some identity has on the whole of* Π *a single term of least exponent, no singular place of* C *lies in the cell.*

Proof. At such a place, the least exponent of the identity is attained once, contradicting Lemma 2.10. The coefficients $c_i(m)$ are rational multiples of products of masses, or E, or ME, and do not vanish. □

Let the cell have no identity as in Lemma 5.1. Its *initial system* is the following set of polynomials in the node coordinates $X_{v,c}, Y_{v,c}$ ($c \neq c_0$), in variables P_{kl} for the leading coefficients of the p_{kl}, and in E, with coefficients in $\mathbb{Q}[m]$: for each identity whose argmin set S is constant on the cell, the initial form $\sum_{i \in S} c_i\,[g_i]$, where $[g_i]$ is the product of the leading coefficients of the factors of g_i, written through Lemma 4.2 ($[z_p - z_q] = X_{v,c(p)} - X_{v,c(q)}$ and $[p_{kl}] = P_{kl}$); and for each pair kl the polynomial $P_{kl}^2 [z_{kl}][w_{kl}] - 1$. At a place that lies in the cell, the leading coefficients are a common zero of the initial system (Lemma 2.10 and $p_{kl}^2 z_{kl} w_{kl} = 1$) at which the product Θ of the following quantities does not vanish: the node coordinates $X_{v,c}$, $Y_{v,c}$ with $c \neq c_0$ and their pairwise differences at each node, the P_{kl}, and E.

Lemma 5.2. *Let* G *be a subset of the initial system of a cell.*

(a) *If the saturation of the ideal generated by* G *in* $\mathbb{Q}(m_1, \dots, m_6)[X, Y, P, E]$ *by* Θ *is the unit ideal, there is a nonzero* $h \in \mathbb{Z}[m_1, \dots, m_6]$ *such that no singular place of* C *lies in the cell when* $h(m) \neq 0$.

(b) *The same conclusion holds if the saturation is the unit ideal over* $\mathbb{Q}(m_2, \dots, m_6)$ *after the substitution* $m_1 = 1$*, or after the same substitution for another mass.*

Proof. (a) The hypothesis gives $\Theta^N = \sum_i g_i f_i$ with $f_i \in G$ and g_i polynomials over $\mathbb{Q}(m)$. Clearing denominators gives $h\,\Theta^N = \sum_i g_i' f_i$ with $0 \neq h \in \mathbb{Z}[m]$ and g_i' polynomials over $\mathbb{Q}[m]$. At the leading coefficients of a place in the cell the right side vanishes and $\Theta \neq 0$, so $h(m) = 0$. (b) The isomorphism $Y_m \to Y_{m/m_1}$ of Lemma 2.1(b) maps C to a nontrivial curve and preserves exponents, so a place in the cell at m gives one at m/m_1. Part (a), applied over $\mathbb{Q}(m_2, \dots, m_6)$, gives $h_1 \neq 0$ with $h_1(m_2/m_1, \dots, m_6/m_1) = 0$, and $h = m_1^{\deg h_1} h_1(m_2/m_1, \dots, m_6/m_1)$ is a nonzero polynomial. □

A subset G as in Lemma 5.2 is a *core* of the cell. The program searches for cores with the divide-and-conquer procedure QuickXplain of [18], testing with Singular, modulo the prime 32003 and at random masses, whether a subset generates the unit ideal. This search only proposes candidates. Each proposed core is then certified by a standard basis in Singular, over $\mathbb{Q}(m)$ or, in the second pass below, over the field of rational functions of the other masses after one mass that occurs in the core is set to 1. Both steps ran without saturation (Section 10),

so each certified core generates the unit ideal, which is more than Lemma 5.2 needs. Each cell with a core also has an explicit identity $\sum_i c_i f_i = h$, in which the f_i are the elements of a core of the cell, the c_i are polynomials over $\mathbb{Z}$ and $h \in \mathbb{Z}[m]$ is not zero. These identities are checked in Lean (§10.1).

Proposition 5.3. † *Of the* 91,273 *cells of the list,* 52,432 *satisfy the hypothesis of Lemma 5.1. Of the others,* 37,808 *have a core certified as in Lemma 5.2*(a)*, and* 990 *further cells have a core certified as in Lemma 5.2*(b)*, found by repeating the search with random orders of the initial forms. The remaining* 43 *cells belong to the diagrams* 64 (11 *cells*)*,* 66 (11)*,* 88 (9)*,* 89 (1)*,* 93 (1)*,* 94 (9) *and* 98 (1)*.*

The first pass left 1,033 cells without a certified core, and the second pass certified 990 of them (Section 10). Neither pass produced a certified core for the remaining 43 cells. For these we need Sections 6 and 7, which treat their seven diagrams at the level of places and so cover all cells of these diagrams. For diagram 89 the torque of Theorem 6.5 adds nothing to the initial system where $b_4 - 2 < 1 + \frac{1}{2}a_1$, since there its initial form is a multiple of that of the equation of body 4 (Remark 6.6).

Of the 24 diagrams left unresolved in [5], the eleven numbered 5, 6, 65, 67, 69, 71, 73, 74, 75, 79 and 80 have no nonpositive cell, so that a place with one of these diagrams has $\ell(x) > 0$ and is never the place of Corollary 2.7. For the six numbered 8, 52, 63, 81, 82 and 90, Proposition 5.3 excludes all listed cells. The remaining seven are the diagrams of Proposition 5.3 that contain the 43 cells, treated in Sections 6 and 7.

6. Diagrams with a circled cluster of each colour

In this section we treat the diagrams 81, 82, 88, 89, 90, 93, 94 and 98 of Table 1 at the level of places. Each has a cluster of z-strokes with z-circled bodies and a cluster of w-strokes with w-circled bodies, and the constraint comes from the torque of one cluster [2, §5.0]. Throughout, $m \in (\mathbb{R}_{>0})^6$, $C \subseteq Y_m$ is an irreducible curve and P is a singular place of C. We do not assume that $U \not\equiv 0$. We state each result for the labelling of Table 1. By Lemmas 2.1(b) and 2.9, it then holds for all labellings, after the masses are permuted accordingly. For a body k we write $\hat{z}_k = [z_k]_{a_k}$ and $\hat{w}_k = [w_k]_{b_k}$, and for a pair $[p_{kl}] = [p_{kl}]_{-x_{kl}}$.

6.1. Common facts.

Lemma 6.1. (F1) *Let A be a set of two, three or four bodies whose pairs are all z-strokes and none a zw-edge, and suppose that A contains a z-circled body and every z-circled body lies in A. Then $\zeta_{kl} = -2$, $\omega_{kl} = 2$ and $x_{kl} = 0$ for all pairs kl in A.*

(F2) *In the situation of* (F1)*, every z-circled body k has $b_k \geq 2$.*

(F3) *If kl is a w-stroke with $x_{kl} = 0$, then $a_k \geq \min(a_l, 2)$.*

(F4) *If $a_j < a_l$ and $b_l < b_j$, then $\zeta_{jl} = a_j$, $\omega_{jl} = b_l$, $[z_{jl}]_{a_j} = \hat{z}_j$, $[w_{jl}]_{b_l} = -\hat{w}_l$, $e(p_{jl}) = -\frac{1}{2}(a_j + b_l)$ and $[p_{jl}]^2 \hat{z}_j \hat{w}_l = -1$. If moreover $a_l + b_j > a_j + b_l$, the term $m_j m_l p_{jl}^3 (z_j w_l - z_l w_j)$ has exponent $-\frac{1}{2}(a_j + b_l)$ and leading coefficient $-m_j m_l [p_{jl}]$.*

The same holds with z and w exchanged.

Proof. (F1) By (E3) and the remark after Lemma 2.11, the values of ζ, ω and x are the same on all pairs of A, and by (O3), (O4), $\zeta = 3x - 2$ and $\omega = 2 - x$ with $0 \leq x < 1$. If some body of A is not circled, (O6) gives $\zeta = -2$ on its pair with a circled body. If all are circled and $x > 0$, then $\zeta > -2$ on A, so the bodies of A have a common coefficient $c = [z_k]_{-2} \neq 0$, and the coefficient of exponent -2 in $\sum_k m_k z_k = 0$ is $c \sum_{k \in A} m_k \neq 0$. So $x = 0$. (F2) For $l \in A$, $e(W_{kl}) = -\frac{1}{2}(3\zeta_{kl} + \omega_{kl}) = 2$. For $l \notin A$, l is not circled, so $\zeta_{kl} = -2$ by (O6) and $e(W_{kl}) = 3 - \frac{1}{2}\omega_{kl} \geq 2$ by (O1). All terms of the right side of $w_k = \sum_l m_l W_{kl}$ have exponent at

least 2. (F3) $z_k = z_l + z_{kl}$ and $\zeta_{kl} = 2$ by (O3). (F4) The first statements follow from $a_j < a_l$, $b_l < b_j$ and the distance equation at exponent 0. The term $p_{jl}^3 z_j w_l$ has exponent $-\frac{1}{2}(a_j + b_l)$ and coefficient $[p_{jl}]^3 \hat{z}_j \hat{w}_l = -[p_{jl}]$, and the term $p_{jl}^3 z_l w_j$ has exponent $-\frac{3}{2}(a_j + b_l) + a_l + b_j$, which is larger. □

Lemma 6.2. *Let S be a set of bodies, $o \notin S$ a body, $A_z = \sum_{j\in S} m_j Z_{oj}$, $A_w = \sum_{j\in S} m_j W_{oj}$, and $B_z = z_o - A_z$, $B_w = w_o - A_w$, so that $B_z = \sum_{l\notin S\cup\{o\}} m_l Z_{ol}$ and $B_w = \sum_{l\notin S\cup\{o\}} m_l W_{ol}$ by the equations of the body o. Then*

$$\sum_{j\in S} m_j m_o p_{jo}^3 (z_j w_o - z_o w_j) = m_o(z_o A_w - w_o A_z) = m_o(w_o B_z - z_o B_w).$$

Proof. Substitute $p_{jo}^3 z_j = p_{jo}^3 z_o - Z_{oj}$ and $p_{jo}^3 w_j = p_{jo}^3 w_o - W_{oj}$. The terms $p_{jo}^3 z_o w_o$ cancel. Then substitute $A_z = z_o - B_z$ and $A_w = w_o - B_w$. □

This is the rewriting of [2, §8.6, (30)–(31)] through the equations of the body o. Its point is that the individual terms of B_w can have smaller exponent than B_w itself. We will apply it with o the body shared by the two clusters of diagrams 89 and 94.

Lemma 6.3. *Suppose that 5 and 6 are the only z-circled bodies, so that $m_5\hat{z}_5 + m_6\hat{z}_6 = 0$, and let $k \notin \{5, 6\}$ satisfy $a_k > -2$ and $b_k < \min(b_5, b_6)$. Then $m_5[p_{5k}] + m_6[p_{6k}] \neq 0$.*

Proof. The relation $m_5\hat{z}_5 + m_6\hat{z}_6 = 0$ is the coefficient of exponent -2 in $\sum_j m_j z_j = 0$. By (F4), $[p_{jk}]^2 = -1/(\hat{z}_j\hat{w}_k)$ for $j = 5, 6$. If $m_5[p_{5k}] = -m_6[p_{6k}]$, squaring gives $m_5^2/\hat{z}_5 = m_6^2/\hat{z}_6$, and with $\hat{z}_6 = -m_5\hat{z}_5/m_6$ this gives $m_5^3 + m_6^3 = 0$. □

6.2. **Diagrams 90 and 89.** Proposition 5.3 already excludes the listed cells of diagram 90. We treat it first because the proof of Proposition 6.4 is the model for that of Theorem 6.5.

Proposition 6.4. *No singular place of an irreducible curve $C \subseteq Y_m$ has diagram 90.*

Proof. Let $A = \{4, 5, 6\}$ and $B = \{1, 2, 3\}$. By (F1), $x = 0$, $\zeta = -2$ and $\omega = 2$ on the pairs of A, and $x = 0$, $\omega = -2$, $\zeta = 2$ on the pairs of B. By (O6), $\zeta = \omega = -2$ on 52, 53, 62, 63. By (F2), $b_5, b_6 \geq 2$ and $a_2, a_3 \geq 2$, and by (F3), $b_4 \geq 2$ and $a_1 \geq 2$. If $a_4 \geq 2$, then $\zeta_{42} \geq 2$, so $\zeta_{42} = 2$ and 42 would be a w-stroke by (O6). Hence $-2 < a_4 < 2$, and in the same way $-2 < b_1 < 2$.

For $j \in A$ and $l \in B$ we have $a_j < 2 \leq a_l$, $b_l < 2 \leq b_j$ and $a_l + b_j \geq 4 > a_j + b_l$, so (F4) applies to the nine pairs between A and B. In the torque T_A the term of the pair jl has exponent $E_{jl} = -\frac{1}{2}(a_j + b_l)$ and a nonzero leading coefficient: $E_{41} = -\frac{1}{2}(a_4 + b_1)$, $E_{4l} = 1 - \frac{1}{2}a_4$ for $l = 2, 3$, $E_{j1} = 1 - \frac{1}{2}b_1$ for $j = 5, 6$, and $E_{jl} = 2$ for $j = 5, 6$, $l = 2, 3$. Since $E_{4l} - E_{41} = 1 + \frac{1}{2}b_1$, $E_{j1} - E_{41} = 1 + \frac{1}{2}a_4$ and $2 - E_{41} = 2 + \frac{1}{2}(a_4 + b_1)$ are positive, E_{41} is attained by a single term, contradicting Lemma 2.10. □

Theorem 6.5. *No singular place of an irreducible curve $C \subseteq Y_m$ has diagram 89.*

Proof. Here $A = \{4, 5, 6\}$ carries the z-strokes with circles at 5, 6, and $W = \{2, 3, 4\}$ the w-strokes with circles at 2, 3. Body 1 is *free*: it has no stroke and no circle.

Exponents. By (F1), $x = 0$ on the six strokes, with $\zeta = -2$, $\omega = 2$ on 45, 46, 56 and $\omega = -2$, $\zeta = 2$ on 23, 24, 34. By (O6), $\zeta = \omega = -2$ on 25, 26, 35, 36, $\zeta_{15} = \zeta_{16} = -2$ and $\omega_{12} = \omega_{13} = -2$. By (F2) and (F3), $b_5, b_6, b_4 \geq 2$ and $a_2, a_3, a_4 \geq 2$. If $a_1 \geq 2$, then $\zeta_{12} = 2$ and 12 would be a w-stroke. If $b_1 \geq 2$, then $\omega_{15} = 2$ and 15 would be a z-stroke. So $a_1, b_1 \in (-2, 2)$, and $\zeta_{1l} = a_1$ for $l = 2, 3, 4$, $\omega_{1j} = b_1$ for $j = 4, 5, 6$.

$a_1 + b_1 = 0$. By Lemma 2.3(b), U is constant on C. In $U = \sum m_k m_l p_{kl}$ the exponents $-x_{kl}$ are 0 on the strokes, 2 on 25, 26, 35, 36, $1 - \frac{1}{2}a_1 > 0$ on 12, 13, $1 - \frac{1}{2}b_1 > 0$ on 15, 16, and $-\frac{1}{2}(a_1 + b_1)$ on 14. If $a_1 + b_1 > 0$, the term of 14 is the only term of negative exponent. In

$\sum_k m_k z_k w_k = U$ the exponents $a_k + b_k$ are at least 0 for $k \neq 1$, so if $a_1 + b_1 < 0$, the term of $k = 1$ is the only one of negative exponent. Both contradict Lemma 2.10.

The torque of $S = \{5, 6\}$. The pairs (j, l) with $j \in S$, $l = 2, 3$ give terms of exponent at least 2. The pairs $(j, 1)$ satisfy (F4), since $a_j = -2 < a_1$, $b_1 < 2 \leq b_j$ and $a_1 + b_j > -2 + b_1$. Their terms have exponent $1 - \frac{1}{2}b_1 = 1 + \frac{1}{2}a_1$ and coefficients $-m_j m_1 [p_{j1}]$. By Lemma 6.2 with $o = 4$, the pairs $(j, 4)$ contribute $m_4(w_4 B_z - z_4 B_w)$ with $B_z = m_1 Z_{41} + m_2 Z_{42} + m_3 Z_{43}$ and $B_w = w_4 - A_w$. Here $e(Z_{42}) = e(Z_{43}) = 2$ and $e(Z_{41}) = -\frac{1}{2}(a_1 + 3b_1) = a_1$, so $e(B_z) \geq a_1$. Also $e(A_w) \geq 2$, so $e(B_w) \geq 2$, although W_{42} and W_{43} have exponent -2. Since $a_4, b_4 \geq 2$, this contribution has exponent at least $\min(2 + a_1, 4)$, which exceeds $1 + \frac{1}{2}a_1$ for $-2 < a_1 < 2$, and so does 2. The coefficient of exponent $1 + \frac{1}{2}a_1$ in T_S is therefore $-m_1(m_5[p_{51}] + m_6[p_{61}])$, which is not zero by Lemma 6.3. This contradicts $T_S = 0$. □

Remark 6.6. The terms of the pairs $(j, 4)$ in $T_{\{5,6\}}$ have exponent $b_4 - 2$, which can be smaller than $1 + \frac{1}{2}a_1$. Their initial form $m_4 \hat{w}_4 \sum_j m_j [p_{j4}]^3 \hat{z}_j$ is a multiple of the initial form of the z-equation of body 4 at exponent -2, which reads $\sum_j m_j [p_{4j}]^3 (-\hat{z}_j) = 0$. So when $b_4 - 2 < 1 + \frac{1}{2}a_1$, the initial form of the torque is implied by that of the equation of body 4. Lemma 6.2 removes the cancellation before the exponents are compared. In five bodies the corresponding diagram is diagram 6 of [2, Fig. 11], without the free body, and the same torque gives the conditions $m_1 = m_2$, $m_3 = m_4$ of [2, (32)].

6.3. **Diagrams 82 and 81.** Proposition 5.3 also excludes all listed cells of these two diagrams. We include them because the arguments are short and hold at every positive mass.

Proposition 6.7. *No singular place of an irreducible curve* $C \subseteq Y_m$ *has diagram* 82.

Proof. Exponents. By (F1), $x = 0$ on 56 and on 23, 24, 34, where $\zeta = 2$, $\omega = -2$. By (O6), $\zeta = \omega = -2$ on $\{5, 6\} \times \{3, 4\}$, $\zeta_{j1} = \zeta_{j2} = -2$ for $j = 5, 6$, and $\omega_{l1} = -2$ for $l = 3, 4$. By (F2) and (F3), $b_5, b_6 \geq 2$ and $a_3, a_4, a_2 \geq 2$. As in Theorem 6.5, $a_1, b_1, b_2 \in (-2, 2)$, since otherwise 13 would be a w-stroke or 15, 25 a z-stroke. Hence $\omega_{jk} = b_k$ for $j = 5, 6$, $k = 1, 2$, and $\zeta_{1l} = a_1$ for $l = 2, 3, 4$.

$b_1 = b_2$. In $T_{\{5,6\}}$ the pairs (j, l), $l = 3, 4$, give terms of exponent at least 2, and the pairs (j, k), $k = 1, 2$, satisfy (F4) and give terms of exponent $1 - \frac{1}{2}b_k < 2$ and coefficients $-m_k m_j [p_{jk}]$. If $b_1 \neq b_2$, the body k with the larger b_k gives the coefficient $-m_k(m_5[p_{5k}] + m_6[p_{6k}])$ of least exponent, which is not zero by Lemma 6.3. So $b_1 = b_2 =: b$.

$a_1 = -b$. All pairs other than 12 have $x \leq 0$. Since U is constant, the term of 12 cannot be the only one of negative exponent, so $2x_{12} = a_1 + \omega_{12} \leq 0$, and $\omega_{12} \geq b$ gives $a_1 + b \leq 0$. In $\sum_k m_k z_k w_k = U$ the exponents $a_k + b_k$ are at least 0 for $k \neq 1$, so $a_1 + b \geq 0$. Hence $a_1 = -b$, $\omega_{12} = b$, and $e(Z_{21}) = -\frac{1}{2}(\zeta_{12} + 3\omega_{12}) = -b$.

Body 2. In $z_2 = m_1 Z_{21} + m_3 Z_{23} + m_4 Z_{24} + m_5 Z_{25} + m_6 Z_{26}$ the exponents are $-b$, 2, 2, $1 - \frac{3}{2}b$, $1 - \frac{3}{2}b$ on the right and at least 2 on the left. For $-2 < b < 2$ the exponent $-b$ is attained by a single term, contradicting Lemma 2.10. □

Proposition 6.8. *If a singular place of an irreducible curve* $C \subseteq Y_m$ *has diagram* 81*, then* $m_1 = m_2$*,* $m_3 = m_4$ *and* $m_5 = m_6$*.*

Proof. Exponents. By (F1), $x_{56} = x_{34} = 0$, and by (O6), $\zeta = \omega = -2$ on $\{5, 6\} \times \{3, 4\}$. By (F2), $b_5, b_6, a_3, a_4 \geq 2$, and as before $a_k, b_k \in (-2, 2)$ for $k = 1, 2$. So $\zeta_{jk} = -2$, $\omega_{jk} = b_k$ for $j \in \{5, 6\}$, and $\omega_{lk} = -2$, $\zeta_{lk} = a_k$ for $l \in \{3, 4\}$.

Equal exponents. As in Proposition 6.7, the torque of $\{5, 6\}$ gives $b_1 = b_2 =: b$ and the torque of $\{3, 4\}$ gives $a_1 = a_2 =: a$. Since U is constant and all pairs other than 12 have $x \leq 0$, $a + b \leq \zeta_{12} + \omega_{12} = 2x_{12} \leq 0$.

The pair 12. The equation of body 1 in z has the terms z_1 (exponent a), m_2Z_{12} (exponent $e_Z = -\frac{1}{2}(\zeta_{12}+3\omega_{12})$), $m_5Z_{15}+m_6Z_{16}$ (exponent $1-\frac{3}{2}b$ exactly, since its leading coefficient is $(m_5[p_{15}]+m_6[p_{16}])/\hat{w}_1 \neq 0$ by (F4) and Lemma 6.3), and $m_3Z_{13}+m_4Z_{14}$ (exponent at least $3-\frac{1}{2}a > a$). The least exponent is attained twice among the first three. If $a = 1-\frac{3}{2}b \leq e_Z$, then $a+b \leq 0$ gives $b \geq 2$. If $e_Z = 1-\frac{3}{2}b < a$, then $e_Z \leq -\frac{1}{2}(a+3b)$ gives $a \leq -2$. Both are impossible, so $a = e_Z$, and in the same way $b = -\frac{1}{2}(3\zeta_{12}+\omega_{12})$. Adding, $a+b = -2(\zeta_{12}+\omega_{12}) = -4x_{12} \geq 0$. Hence $a+b = 0$, $x_{12} = 0$, $\zeta_{12} = a$ and $\omega_{12} = -a$.

The binary $\{1,2\}$. At exponent a in z and $-a$ in w, the equations of bodies 1 and 2 contain only z_k, w_k and their mutual terms, since $1+\frac{3}{2}a > a$ and $3-\frac{1}{2}a > a$ for $-2 < a < 2$. Hence $m_1\hat{z}_1 + m_2\hat{z}_2 = 0$ and $m_1\hat{w}_1 + m_2\hat{w}_2 = 0$.

The torques. In $T_{\{5,6\}}$ the two bodies $k = 1,2$ tie at exponent $1-\frac{1}{2}b$, below the terms of 3, 4. Write $\hat{z}_5 = cm_6$, $\hat{z}_6 = -cm_5$ with $c \neq 0$. By (F4), $m_5[p_{5k}]+m_6[p_{6k}] = (c\hat{w}_k)^{-1/2}(m_5m_6)^{-1/2}u_k$ with $u_k \in \{\pm m_6^{3/2} \pm i m_5^{3/2}\}$ for fixed branches of the square roots. The coefficient of least exponent gives $m_1\hat{w}_1^{-1/2}u_1 + m_2\hat{w}_2^{-1/2}u_2 = 0$. Since $|u_1| = |u_2|$ and $\hat{w}_1/\hat{w}_2 = -m_2/m_1$, this gives $(u_1/u_2)^2 = -m_2^3/m_1^3$, so $m_1 = m_2$ and $u_1/u_2 = \pm i$, which forces $m_5^{3/2} = m_6^{3/2}$. The torque of $\{3,4\}$ gives $m_3 = m_4$ in the same way. □

6.4. **Complex central configurations of three bodies.** For $\mu = (\mu_1,\mu_2,\mu_3)$ with $\mu_k \neq 0$ and $M_\mu = \mu_1+\mu_2+\mu_3 \neq 0$, let $N(\mu)$ be the system

$$z_k = \sum_{l\neq k} \mu_l p_{kl}^3 (z_k - z_l), \quad w_k = \sum_{l\neq k} \mu_l p_{kl}^3 (w_k - w_l), \quad p_{kl}^2 (z_k - z_l)(w_k - w_l) = 1$$

in $z, w \in \mathbb{C}^3$ and p_{12}, p_{13}, p_{23}. It implies $\sum \mu_k z_k = \sum \mu_k w_k = 0$, and the rotations $(z,w) \mapsto (sz, s^{-1}w)$ act on its solutions. With the slice $z_1 - z_2 = 1$ and the *shape* $\tau = z_1 - z_3$, the positions are $z_k = \ell_k(\tau)/M_\mu$, where $\ell_1(t) = \mu_2 + \mu_3 t$, $\ell_2 = \ell_1 - M_\mu$ and $\ell_3 = \ell_1 - M_\mu t$. Put $\Lambda_k = \prod_{k'\neq k} \ell_{k'}$,

$$N_{01} = t^4 - 5t^3 + 9t^2 - 5t + 1, \quad N_{10} = t^4 + t^3 - 2t + 1, \quad N_{11} = t^4 - 2t^3 + t + 1,$$

and, for the four classes $\sigma = (\sigma_{12}, \sigma_{13}, \sigma_{23}) \in \{(1,1,1), (1,1,-1), (1,-1,-1), (1,-1,1)\}$ of signs modulo ± 1, let $Q_\sigma(x) = x^2(1+x)^2\big(x(F_1 - F_2) - (F_2 - F_3)\big)$, where $F_k = \sum_{l\neq k} \mu_l \sigma_{kl} z_{kl}^{-2}$ with $z_{12} = 1$, $z_{23} = x$, $z_{13} = 1+x$ and $\sigma_{lk} = -\sigma_{kl}$. Let

$$S_\mu(t) = (t^2 - t + 1)\, N_{01}(t)\, N_{10}(t)\, N_{11}(t) \prod_\sigma Q_\sigma(t-1),$$

a polynomial of degree 34 with leading coefficient $(\mu_1+\mu_2)^4$.

Lemma 6.9. *Every solution of* $N(\mu)$ *is of* Lagrange type, $p_{12}^3 = p_{13}^3 = p_{23}^3 = 1/M_\mu$, *or of* collinear type, $w = \lambda z$ *with* $\lambda \neq 0$. *The shape of each solution is a root of* S_μ.

Proof. Let L be the matrix with entries $L_{kl} = \mu_l p_{kl}^3$ for $k \neq l$ and zero row sums. The equations say $(L+I)z = (L+I)w = 0$, and $\mu^{\mathsf{T}} L = 0$, so z and w lie in the plane $H = \{v : \sum \mu_k v_k = 0\}$. If z and w are independent, $L+I$ vanishes on H, so $L + I = u\mu^{\mathsf{T}}$. Applied to $(1,1,1)$, this gives $u = M_\mu^{-1}(1,1,1)$, and the off-diagonal entries give $p_{kl}^3 = 1/M_\mu$. Otherwise $w = \lambda z$, since $z \neq 0$.

In the Lagrange case, put $R_{kl} = p_{kl}^{-2} = M_\mu^{2/3} \mathrm{j}^{a_{kl}}$ with $\mathrm{j} = e^{2\pi i/3}$. With $z_{12} = 1$, $z_{13} = \tau$, $w_{12} = R_{12}$ and $w_{13} = R_{13}/\tau$, the distance equation of 23 becomes $R_{12}\tau^2 - (R_{12} + R_{13} - R_{23})\tau + R_{13} = 0$, that is $t^2 - (1 + \mathrm{j}^b - \mathrm{j}^c)t + \mathrm{j}^b = 0$ for some $(b,c) \in (\mathbb{Z}/3)^2$. The classes $(0,0)$, $(1,2)$, $(2,1)$ give $t^2 - t + 1$, and the three pairs of conjugate classes give the products N_{01}, N_{10}, N_{11} of conjugate quadratics. In the collinear case $p_{kl} = \sigma_{kl}\kappa/z_{kl}$ with $\kappa^2 = 1/\lambda$ and σ antisymmetric, so $z_k = \kappa^3 F_k$, and $z_{12} = 1$, $z_{23} = x$ give $x(F_1 - F_2) = F_2 - F_3$, the vanishing of $Q_\sigma(x)$ with $x = \tau - 1$. The patterns σ and $-\sigma$ give the same solutions. □

The real solutions are the Lagrange and Euler configurations. The dichotomy is the one of [27, Lemma E.2], and the factors of S_μ are those of the complex three-body shapes listed in the appendix of [2] (S_1–S_8 there, written in the w-coordinates of the bodies): t^2-t+1 is the equilateral factor, the N's are the three other Lagrange factors, three Q_σ are the Euler quintics, and the fourth is divisible by x^2+x+1. For $\mu=(1,2,7)$, $(1,2,8)$ and $(2,3,11)$, an elimination in Singular of all unknowns except τ gives an eliminant whose square-free part equals that of S_μ, of degree 32, the number of distinct shapes, and the system with the slice $z_1-z_2=1$ has 102 solutions, 114 with multiplicity.†

6.5. **Diagram 98.** Let $A=\{4,5,6\}$ be the z-circled triangle and $B=\{1,2,3\}$ the w-circled one. In a general labelling A is any set of three bodies and B its complement, and the exchange of the colours exchanges the roles of A and B.

Lemma 6.10. *At a singular place with diagram* 98*:*

(a) $\zeta=-2$, $\omega=2$, $x=0$ *on the pairs of* A, *and* $\omega=-2$, $\zeta=2$, $x=0$ *on the pairs of* B*;*
(b) $\zeta=\omega=-2$ *and* $x=-2$ *on the nine pairs* jl *with* $j\in A$, $l\in B$*;*
(c) $a_j=-2$ *and* $b_j\ge 2$ *for* $j\in A$, *and* $b_l=-2$ *and* $a_l\ge 2$ *for* $l\in B$.

Proof. (a) is (F1), (b) is (O6) in both colours, and (c) is (F2). □

For $j,j'\in A$ and $l,l'\in B$ put $\hat z_j=[z_j]_{-2}$, $\tilde w_j=[w_j]_2$, $\hat p_{jj'}=[p_{jj'}]_0$, $\hat w_l=[w_l]_{-2}$, $\tilde z_l=[z_l]_2$, $\hat p_{ll'}=[p_{ll'}]_0$ and $\hat p_{jl}=[p_{jl}]_2$.

Lemma 6.11. *At a singular place with diagram* 98*:*

(a) $(\hat z_A,\tilde w_A,\hat p_A)$ *is a solution of* $N(m_A)$, *and* $\hat z_j\ne 0$ *for* $j\in A$*;*
(b) $(\hat w_B,\tilde z_B,\hat p_B)$ *is a solution of* $N(m_B)$, *and* $\hat w_l\ne 0$ *for* $l\in B$*;*
(c) $\hat p_{jl}^2\hat z_j\hat w_l=-1$ *for* $j\in A$, $l\in B$*;*
(d) $\sum_{j\in A,\,l\in B} m_jm_l\hat p_{jl}=0$.

Proof. (a) In the z-equation of $j\in A$ the terms $Z_{jj'}$, $j'\in A$, have exponent -2 and the terms Z_{jl}, $l\in B$, exponent 4. At exponent -2 it gives $\hat z_j=\sum_{j'} m_{j'}\hat p_{jj'}^3(\hat z_j-\hat z_{j'})$, where $\hat z_j-\hat z_{j'}=[z_{jj'}]_{-2}\ne 0$. The w-equation of j has terms of exponent 2 ($W_{jj'}$) and 4 (W_{jl}), and $b_j\ge 2$. At exponent 2 it gives the w-equation of $N(m_A)$. The distance equation of jj' at exponent 0 gives $\hat p_{jj'}^2(\hat z_j-\hat z_{j'})(\tilde w_j-\tilde w_{j'})=1$. (b) is (a) with the colours exchanged. (c) is (F4). (d) In T_A all terms have exponent at least 2 by (F4), and the coefficient of exponent 2 is $-\sum m_jm_l\hat p_{jl}$. □

Theorem 6.12. *There is a nonzero polynomial* $P\in\mathbb{Z}[\mu_1,\mu_2,\mu_3,\mu_1',\mu_2',\mu_3']$ *such that for every singular place with diagram* 98, $P(m_A,m_B)=0$, *where* m_A *and* m_B *list the masses of* A *and* B *in increasing order of the labels. Hence diagram* 98 *occurs at no singular place for masses with* $P_{98}(m)=\prod_{|A|=3}P(m_A,m_{A^c})\ne 0$.

Proof. Let $\Phi(Y)=\prod_\varepsilon\sum_{j,l=1}^3\varepsilon_{jl}\sqrt{Y_{jl}}$, the product over the 256 sign matrices $\varepsilon\in\{\pm1\}^{3\times3}$ with $\varepsilon_{11}=1$. A sign change of one $\sqrt{Y_{jl}}$ permutes the factors up to sign, and 256 is even, so Φ is a polynomial in the Y_{jl} with integer coefficients, homogeneous of degree 128. Put

$$\Psi(t,t';\mu,\mu')=\Phi\Big(\big(\mu_j^2\mu_l'^2\Lambda_j(t;\mu)\Lambda_l(t';\mu')\big)_{j,l}\Big),$$
$$R(t)=\operatorname{Res}_{t'}\big(S_{\mu'}(t'),\Psi\big),\qquad P=\operatorname{Res}_t\big(S_\mu(t),R(t)\big).$$

Resultants of polynomials with integer coefficients have integer coefficients.

Take a place with diagram 98. By Lemmas 6.11(a) and 6.9, a rotation of the A-data has shape τ, a root of S_{m_A}, with $\hat z_j=s\,\ell_j(\tau)/M_A$ for some $s\ne 0$. Likewise the w-shape τ' of the

B-data is a root of S_{m_B}, with $\hat{w}_l = s'\ell_l(\tau'; m_B)/M_B$. By Lemma 6.11(c), $c_{jl} = m_j m_l \hat{p}_{jl}$ satisfies $c_{jl}^2 = \kappa\, m_j^2 m_l^2 \Lambda_j(\tau)\Lambda_l(\tau')$ with the common factor $\kappa = -M_A M_B/(ss' \prod_j \ell_j(\tau) \prod_l \ell_l(\tau')) \neq 0$. By Lemma 6.11(d), $\sum c_{jl} = 0$, so one factor of $\Phi(c^2)$ vanishes. Since Φ is homogeneous, $\Psi(\tau, \tau') = \kappa^{-128}\Phi(c^2) = 0$. Hence $R(\tau) = 0$ and then $P(m_A, m_B) = 0$, since a resultant lies in the ideal of its two arguments. The 20 choices of A cover all labellings and the exchange of the colours.

$P \neq 0$: at points with $(\mu_1 + \mu_2)(\mu_1' + \mu_2') \neq 0$, specialization commutes with the resultants and

$$P(\mu, \mu') = (\mu_1 + \mu_2)^{4D_1} (\mu_1' + \mu_2')^{4\cdot 34\cdot D_2} \prod_{S_\mu(t)=0} \prod_{S_{\mu'}(t')=0} \Psi(t, t'; \mu, \mu')$$

over the roots with multiplicity, where D_1 and D_2 are the formal degrees of R in t and of Ψ in t'. By Lemma 6.13 below, $P((1,2,40),(1,2,40)) \neq 0$. □

For a root t of S_μ put $A_j(t) = \mu_j |\Lambda_j(t)|^{1/2}$ $(j = 1, 2, 3)$ and, if $A_3(t) \neq 0$, $\rho(t) = A_3/(A_1 + A_2 + A_3)$.

Lemma 6.13. *For $\mu = (1, 2, 40)$ every root t of S_μ satisfies $|t| > \frac{1}{10}$ and $|t - 1| > \frac{1}{10}$, and $\rho(t) > \frac{9}{10}$. Consequently $\Psi(t, t') \neq 0$ for all roots t, t' of S_μ, and $P((1,2,40),(1,2,40)) \neq 0$.*

Proof. The four Lagrange factors $t^2 - t + 1$, N_{01}, N_{10}, N_{11} of S_μ take the value 1 at $t = 0$ and at $t = 1$. If p is one of them, $c \in \{0, 1\}$ and $|h| \leq \frac{1}{10}$, then $|p(c + h) - 1| \leq \sum_{k \geq 1} |a_k| 10^{-k} \leq 0.5951$, where the a_k are the coefficients of $p(c + h)$ as a polynomial in h. So p has no root in the two disks. For the collinear factors put $t = 1 + x$, so that $x = 0$ and $x = -1$ correspond to $t = 1$ and $t = 0$. At $\mu = (1, 2, 40)$,

$$Q_\sigma = 40\, C_\sigma + D_\sigma, \qquad C_\sigma = \sigma_{13} x^3 - \sigma_{23}(1 + x)^3,$$
$$D_\sigma = 3\sigma_{12} x^3 (1 + x)^2 + \sigma_{12} x^2 (1 + x)^2 - \sigma_{13} x^2 - 2\sigma_{23}(1 + x)^2.$$

For $c \in \{0, -1\}$ we have $|C_\sigma(c)| = 1$, and on the disk $|x - c| \leq \frac{1}{10}$ the triangle inequality, applied to the coefficients of the expansions in powers of $x - c$, gives $|C_\sigma - C_\sigma(c)| \leq 0.332$ and $|D_\sigma| \leq 2.446$. So $40|C_\sigma| > |D_\sigma|$ there, and Q_σ has no root in the disk. This proves the first statement.

Now $\ell_1 = 2 + 40t$, $\ell_2 = 40(t - 1) - 1$ and $\ell_3 = 2 - 3t$. Hence $|\ell_1| \geq 20|t|$, $|\ell_2| \geq 20|t - 1|$, $|\ell_3| \leq 2 + 3|t|$ and $|\ell_3| \leq 1 + 3|t - 1|$. In particular $\Lambda_3 = \ell_1 \ell_2 \neq 0$, and

$$\frac{A_1}{A_3} = \frac{1}{40}\Big|\frac{\ell_3}{\ell_1}\Big|^{1/2} \leq \frac{1}{40}\Big(\frac{2 + 3|t|}{20|t|}\Big)^{1/2} < 0.03, \qquad \frac{A_2}{A_3} = \frac{1}{20}\Big|\frac{\ell_3}{\ell_2}\Big|^{1/2} \leq \frac{1}{20}\Big(\frac{1 + 3|t - 1|}{20|t - 1|}\Big)^{1/2} < 0.05.$$

So $\rho(t) > 1/1.08 > \frac{9}{10}$.

For roots t, t' the moduli of the nine numbers $\sqrt{Y_{jl}}$ in $\Psi(t, t')$ are $A_j(t) A_l(t')$. Each signed sum therefore satisfies

$$\Big|\sum \varepsilon_{jl} \sqrt{Y_{jl}}\Big| \geq |\sqrt{Y_{33}}| - \sum_{(j,l) \neq (3,3)} |\sqrt{Y_{jl}}| = (2\rho(t)\rho(t') - 1) \sum |\sqrt{Y_{jl}}| > 0.$$

So $\Psi(t, t') \neq 0$, and $P((1,2,40),(1,2,40)) \neq 0$ by the formula in the proof of Theorem 6.12. □

The bound says that the heaviest body of each triangle has the dominant cross coefficient in each shape. It is a two-triangle form of the T-test of [27, §E.9]. We take a heavy third body because it makes the proof short: ℓ_3 does not depend on μ_3, while ℓ_1 and ℓ_2 grow like μ_3, and for large μ_3 the shapes stay away from the collisions $t = 0$ and $t = 1$ of the third body with the other two. At $\mu = (1, 2, 7)$ we have checked the same conclusion by computation. At this mass point S_μ has 34 roots counted with multiplicity, 32 distinct, all with $\ell_1 \ell_2 \ell_3 \neq 0$, and the minimum of ρ over them is $0.7641079385\ldots > 1/\sqrt{2}$, attained at a root of N_{11}, $t \approx -0.47356 + 0.44477i$.

Two independent programs enclose the roots and bound ρ, one in ball arithmetic in Arb with certified root isolation and one in exact rational arithmetic with Rouché disks of radius 2^{-120}. A direct evaluation of all 1024×256 signed sums at $\mu=\mu'=(1,2,7)$ gives $|\sum\varepsilon c|/\sum|c|\geq0.4537$, and the bound $\rho\rho'>\frac12$ also holds at $\mu=(1,2,7)$, $\mu'=(1,2,8)$.†

Remark 6.14. A sign matrix $\varepsilon_{jl}=\delta_j\eta_l$ splits the signed sum, for suitable branches, into

$$\Big(\sum_j\delta_j\mu_j\sqrt{\Lambda_j(t)}\Big)\Big(\sum_l\eta_l\mu'_l\sqrt{\Lambda_l(t')}\Big).$$

Hence, for positive μ and μ', $P(\mu,\mu')=0$ whenever $P_{93}(\mu)=0$ or $P_{93}(\mu')=0$, with P_{93} as in Theorem 6.15. Since P_{93} vanishes at equal masses, $P_{98}(m)=0$ for every m in which three masses are equal.

6.6. **Diagram 93.** Here $B=\{1,2,3\}$ is the fully w-circled triangle and $A=\{4,5,6\}$ the z-triangle, in which 4 is not z-circled. Let $\Phi_3=Y_1^2+Y_2^2+Y_3^2-2(Y_1Y_2+Y_2Y_3+Y_3Y_1)$, the product of the four sums $\sqrt{Y_1}\pm\sqrt{Y_2}\pm\sqrt{Y_3}$, and put $P_{93}(\mu)=\operatorname{Res}_t\big(S_\mu(t),\Phi_3(t;\mu)\big)$ with $Y_l=\mu_l^2\Lambda_l(t;\mu)$.

Theorem 6.15. † *For every singular place with diagram* 93, $P_{93}(m_B)=0$. *Up to a constant factor and a power of* $\mu_1+\mu_2+\mu_3$, P_{93} *is the product* $L_1L_2L_3L_4(L_5L'_5)^2L_6L_7L_8$ *of the polynomials of* [2, §8.2.1], *and* $P_{93}(1,2,40)\neq0$.

Proof. By (F1), $x=0$ on the pairs of A and B. By (O6), $\zeta=\omega=-2$ on the pairs (j,l) with $j\in\{5,6\}$, $l\in B$, and $\omega_{4l}=-2$. By (F2) and (F3), $b_5,b_6,b_4\geq2$ and $a_l\geq2$ for $l\in B$, and $-2<a_4<2$, since otherwise $4l$ would be a w-stroke. So $\zeta_{4l}=a_4$. In the torque T_A, the pairs $(4,l)$ satisfy (F4) with exponent $1-\frac12a_4<2$, and the pairs (j,l) with $j=5,6$ give exponent 2. Hence $\sum_{l\in B}m_l[p_{4l}]=0$, and with $[p_{4l}]^2=-1/(\hat z_4\hat w_l)$,

$$\sum_{l\in B}\varepsilon_l m_l\hat w_l^{-1/2}=0$$

for some signs, the relation [2, (24)]. The leading system of B in both colours is complete, in the sense that the cross terms in the w-equations of B at exponent -2 and in the z-equations at exponent 2 have exponents at least $1-\frac32a_4>-2$ and $3-\frac12a_4>2$. So $(\hat w_B,[z_B]_2,\hat p_B)$ solves $N(m_B)$, its shape is a root of S_{m_B} by Lemma 6.9, and the displayed relation makes Φ_3 vanish there. As in Theorem 6.12, $P_{93}(m_B)=0$.

The factorization of P_{93} is computed in $\mathbb{Q}[\mu_1,\mu_2,\mu_3]$ by Singular, one factor of S_μ at a time.† The factor t^2-t+1 of S_μ gives $L_5L'_5$ and a power of M_μ; each of the three Lagrange quartics gives a factor of degree 28 with 433 terms (L_6, L_7, L_8); each Euler quintic gives a factor of degree 36 with 565 terms (L_1, L_2, L_3); and the fourth quintic gives $L_5L'_5$ and a factor of degree 22 with 264 terms (L_4). The two terms of each factor printed in [2, p. 574] agree with ours. By the proof of Lemma 6.13, at $\mu=(1,2,40)$ every root t of S_μ has $A_3(t)>A_1(t)+A_2(t)$. Since $|\sqrt{Y_l}|=A_l(t)$, no sum $\sqrt{Y_1}\pm\sqrt{Y_2}\pm\sqrt{Y_3}$ vanishes, and $P_{93}(1,2,40)$, a nonzero multiple of the product of $\Phi_3(t;\mu)$ over the roots t of S_μ, is not zero. An exact evaluation shows that $P_{93}(1,2,7)\neq0$ as well. □

The factor $L_5=\frac12\sum(\mu_i-\mu_j)^2$ vanishes at equal masses, where the equilateral shape satisfies the displayed relation. Numerical minimization, which the proof does not use, finds further positive zeros of P_{93} on twelve rays in μ-space besides $(1,1,1)$, but we do not know whether there are others.

6.7. **Diagrams 88 and 94.** In diagram 88 the binary $\{5,6\}$ is z-circled and the four bodies of $K=\{1,2,3,4\}$ are w-circled and pairwise joined by w-strokes. In diagram 94 the triangle $\{4,5,6\}$ carries z-strokes with circles at 5, 6, and $Q=\{1,2,3,4\}$ carries w-strokes on all pairs with circles at 1, 2, 3.

For masses $\mu=(\mu_1,\dots,\mu_4)$, let $N_4(\mu)$ be the system of §6.4 for four bodies, in unknowns $X,Y\in\mathbb{C}^4$ and p_{kl}:

$$X_k=\sum_{l\neq k}\mu_l p_{kl}^3(X_k-X_l),\qquad Y_k=\sum_{l\neq k}\mu_l p_{kl}^3(Y_k-Y_l),\qquad p_{kl}^2(X_k-X_l)(Y_k-Y_l)=1.$$

The *centred system* $\mathrm{KC}(\mu)$ consists of $X_4=0$, the X-equations of $N_4(\mu)$ for $k=1,2,3$, the equation $0=\sum_{l\le3}\mu_l p_{4l}^3(X_4-X_l)$, the Y-equations for $k=1,2,3$, and the six distance equations. It has no Y-equation for body 4.

Lemma 6.16. (a) *At a singular place with diagram* 88, *put* $X_l=\hat w_l$, $Y_l=[z_l]_2$, $\hat p_{ll'}=[p_{ll'}]_0$ *for* $l,l'\in K$ *and* $\hat p_{jl}=[p_{jl}]_2$ *for* $j\in\{5,6\}$, $l\in K$. *Then* $m_5\hat z_5+m_6\hat z_6=0$, $(X,Y,\hat p_K)$ *solves* $N_4(m_K)$ *with all* $X_l\neq0$, $\hat p_{jl}^2\hat z_jX_l=-1$, *and*

$$\sum_{j\in\{5,6\},\,l\in K}m_jm_l\hat p_{jl}=0. \tag{6.1}$$

(b) *At a singular place with diagram* 94, *put* $X_l=[w_l]_{-2}$, $Y_l=[z_l]_2$ *for* $l\in Q$ *and* $\hat p$ *as in* (a). *Then* $m_5\hat z_5+m_6\hat z_6=0$, $(X,Y,\hat p_Q)$ *solves* $\mathrm{KC}(m_1,\dots,m_4)$ *with* $X_1X_2X_3\neq0$, $\hat p_{jl}^2\hat z_jX_l=-1$ *for* $l\le3$, *and*

$$\sum_{j\in\{5,6\},\,l\le3}m_jm_l\hat p_{jl}=0. \tag{6.2}$$

Proof. (a) By (F1), $x=0$ on 56 and on the pairs of K, with $\omega=-2$, $\zeta=2$ there. By (O6), $\zeta=\omega=-2$ on the eight pairs between $\{5,6\}$ and K, and by (F2), $b_5,b_6\ge2$ and $a_l\ge2$ for $l\in K$. The equations of the bodies of K at exponent -2 in w and 2 in z, and their distance equations at exponent 0, form $N_4(m_K)$, since the cross terms have exponent 4. The relation $m_5\hat z_5+m_6\hat z_6=0$ is the coefficient of exponent -2 in $\sum_k m_kz_k=0$, the relation for $\hat p_{jl}$ is (F4), and the sum over j and l is the coefficient of exponent 2 in $T_{\{5,6\}}$. (b) By (F1) and (O6), $x=0$ on all strokes and $\zeta=\omega=-2$ on $\{5,6\}\times\{1,2,3\}$. By (F2) and (F3), $b_5,b_6\ge2$, $a_1,a_2,a_3\ge2$, $a_4\ge2$ ($z_4=z_1-z_{14}$) and $b_4\ge2$ ($w_4=w_5-w_{54}$). So $X_4=0$. The w-equations of $1,2,3$ at exponent -2, their z-equations at exponent 2 (with $[z_{l4}]_2=Y_l-Y_4$) and the distance equations form KC. The w-equation of body 4 at exponent -2 has the terms W_{4l}, $l\le3$, of exponent -2 and the terms W_{4j}, $j=5,6$, of exponent 2, and gives the equation of the centre. The z-equation of body 4 at exponent 2 contains subleading terms of the triangle and is not used. In $T_{\{5,6\}}$ the pairs (j,l), $l\le3$, give exponent 2 by (F4). By Lemma 6.2 with $o=4$, the pairs $(j,4)$ contribute $m_4(w_4B_z-z_4B_w)$ with $B_z=\sum_{l\le3}m_lZ_{4l}$ and $B_w=w_4-A_w$ of exponent at least 2, although each W_{4l} has exponent -2. So this contribution has exponent at least 4. □

Lemma 6.17. (a) *In* (a) *of Lemma 6.16, if* $m_k|X_k|^{-1/2}>\sum_{l\in K,\,l\neq k}m_l|X_l|^{-1/2}$ *for some* $k\in K$, *then* (6.1) *fails for every choice of the signs of the* $\hat p_{jl}$ *and all* $m_5,m_6>0$.

(b) *In* (b), *with* $\beta_l=m_l/\sqrt{X_l}$ *for fixed branches and* $\rho=(m_6/m_5)^{3/2}$, (6.2) *holds for some signs of the* $\hat p_{jl}$ *if and only if* $\sum_{l\le3}\varepsilon_l\beta_l(1+is_l\rho)=0$ *for some* $\varepsilon,s\in\{\pm1\}^3$.

Proof. Write $\hat z_5=cm_6$, $\hat z_6=-cm_5$ with $c\neq0$. Then $\hat p_{6l}/\hat p_{5l}=\pm(\hat z_5/\hat z_6)^{1/2}=\pm i(m_6/m_5)^{1/2}$, so $m_l(m_5\hat p_{5l}+m_6\hat p_{6l})=m_lm_5\hat p_{5l}(1\pm i\rho)$, with $|\hat p_{5l}|=|cm_6X_l|^{-1/2}$. (a) The l-th group has modulus $K'm_l|X_l|^{-1/2}$ with $K'=|c|^{-1/2}(m_5m_6)^{-1/2}(m_5^3+m_6^3)^{1/2}$, independent of l and of the signs, and the dominant group cannot be cancelled. (b) The sum is $m_5(-cm_6)^{-1/2}\sum_l\varepsilon_l\beta_l(1+is_l\rho)$. □

The inequality in (a) is the T-test of [27, §E.9] for a cluster of four bodies, and (b) is the reduction of [2, §8.2]. The test in (a) does not involve the binary masses and is invariant under rotation. For the computations we use the following systems. For N_4: the slice $X_1 - X_2 = 1$, the unknowns $v = (X_2, X_3, Y_1, Y_2, Y_3, p_{12}, \dots, p_{34})$, and X_4, Y_4 eliminated by the centre of mass. The system S_{12} consists of the X- and Y-equations of bodies $1, 2, 3$ and the six distance equations, and S_{11} omits the Y-equation of body 3. Let $\mathrm{Xeq}_k = X_k - \sum_{l \neq k} \mu_l p_{kl}^3 (X_k - X_l)$ and Yeq_k likewise. Since $\sum_k \mu_k (Y_k \,\mathrm{Xeq}_k - X_k \,\mathrm{Yeq}_k) = 0$ identically, $\mu_3 (X_4 - X_3) \mathrm{Yeq}_3$ lies in the ideal of S_{11}, and $X_4 - X_3$ is invertible modulo the distance equation of 34. So S_{11} and S_{12} generate the same ideal when μ_3 and μ_4 are units. For KC: the slice $X_1 = 1$, the twelve unknowns $X_2, X_3, Y_1, \dots, Y_4, p_{12}, \dots, p_{34}$, and the system KC_{12} of the four X-equations, the Y-equations of bodies $1, 2$ and the six distances, which generates the same ideal as KC when μ_3 is a unit, by the same torque identity. In the coordinates $U_k = Y_k - Y_4$ ($k \leq 3$), with $U_4 = 0$, the Y-equation of body 1 reads $Y_4 = -U_1 + \sum_{l \neq 1} \mu_l p_{1l}^3 (U_1 - U_l)$. Substituting this into the Y-equations of bodies 2 and 3 gives a system KC' of twelve equations in the eleven unknowns $X_2, X_3, U_1, U_2, U_3, p_{12}, \dots, p_{34}$: the four X-equations, these two Y-equations and the six distances. Its coordinate ring is isomorphic to that of KC.

Lemma 6.18. *Let p be a prime, $\mu_0 \in \mathbb{Z}^4$, $\mathcal{O} = \mathbb{Z}[\mu]_{\mathfrak{m}}$ with $\mathfrak{m} = (p, \mu - \mu_0)$, and let $I \subseteq \mathcal{O}[v]$ be an ideal. Let $\xi_1, \dots, \xi_N$ be polynomials in v over $\mathcal{O}$ that are units modulo I, such that each v_i is congruent modulo I to a polynomial in ξ over $\mathcal{O}$, and let $G \subseteq I$ be a finite set of polynomials in ξ. Suppose that all coefficients of the polynomials in G are units of $\mathcal{O}$. Let Σ be a fan with support the tropical prevariety of G that contains all faces of its cones and on the relative interior of each cone of which the initial form of each $g \in G$ is constant. Suppose that for each cone Γ of Σ that meets $\{w : \min_i w_i < 0\}$, the initial forms of G at a relative interior point of Γ, reduced modulo $\mathfrak{m}$, that is, specialized at $\mu = \mu_0$ and reduced modulo p, have no common zero in $(\bar{\mathbb{F}}_p^*)^N$. Then $\mathcal{O}[v]/I$ is a finite $\mathcal{O}$-module.*

Proof. It suffices that each v_i is integral over $\mathcal{O}$ modulo each minimal prime $\mathfrak{q}$ of I, since a product of monic relations modulo the minimal primes is nilpotent modulo I. Let $\mathcal{O}'$ be the image of $\mathcal{O}$ in $R' = \mathcal{O}[v]/\mathfrak{q}$, a local domain with maximal ideal $\mathfrak{m}'$, and suppose $y = v_i$ is not integral over $\mathcal{O}'$. Then $\mathfrak{m}'\mathcal{O}'[1/y] + (1/y)$ is a proper ideal: a relation $1 = a_0 + a_1 y^{-1} + \cdots + a_r y^{-r}$ with $a_0 \in \mathfrak{m}'$ would make y integral, since $1 - a_0$ is a unit. A maximal ideal $\mathfrak{n}$ containing it gives a Noetherian local domain $\mathcal{O}'[1/y]_{\mathfrak{n}}$ that dominates $\mathcal{O}'$, and a discrete valuation ν of the fraction field of R' dominating it [12, II, 7.1.7]. Then $\nu(y) < 0$, $\nu \geq 0$ on $\mathcal{O}'$, the residue field κ of ν is an extension of $\mathbb{F}_p$, and the masses have residues μ_0 modulo p. The point ξ of the generic point of R' has $w = \nu(\xi) \in \mathbb{Z}^N$ with some $w_i < 0$, because $\nu(y) < 0$ and y is congruent modulo $I \subseteq \mathfrak{q}$ to a polynomial in ξ over $\mathcal{O}$. Each $g \in G$ vanishes at ξ and has unit coefficients, so its terms of least weight w cancel. Hence w lies in the prevariety, and the initial forms of G at w, reduced modulo $\mathfrak{m}$, vanish at the residues in κ of the $\xi_i \pi^{-w_i}$, where π is a uniformizer of ν. These residues are nonzero. So the ideal of $\mathbb{F}_p[\xi, t]$ generated by the reduced initial forms and $1 - t \prod_i \xi_i$ has a zero over κ, hence is proper, and by the Nullstellensatz it has a zero over $\bar{\mathbb{F}}_p$: the reduced initial forms have a common zero in $(\bar{\mathbb{F}}_p^*)^N$. The point w lies in the relative interior of a cone of Σ that meets $\{\min w_i < 0\}$, and the initial forms are constant on that relative interior. This contradicts the hypothesis. □

Lemma 6.19. † *Let $\mu_0 = (1, 2, 3, 30)$ and $p = 65521$.*

(a) *The hypotheses of Lemma 6.18 hold for the ideal of S_{12} and for the ideal of* KC_{12}*, with* 17 *coordinates ξ (the differences of the positions in each colour and the p_{kl}), with G of* 30 *and of* 101 *polynomials, respectively, and with prevarieties of* 2149 *and* 224 *cones, of which* 2148 *and* 223*, respectively, meet $\{\min w_i < 0\}$.*

(b) $\dim_{\mathbb{F}_p} \mathbb{F}_p[v]/\langle S_{12}(\mu_0)\rangle = 20{,}316$ *and* $\dim_{\mathbb{F}_p} \mathbb{F}_p[v']/\langle \mathrm{KC}'(\mu_0)\rangle = 4788$, *where* v' *are the unknowns of* KC'.

(c) *There are* 20,316 *pairwise disjoint boxes in* $\mathbb{C}^{11}$, *each containing exactly one zero of* $S_{11}(\mu_0)$, *at which the Jacobian is invertible and* $X_3 - X_4 \neq 0$, *and on each of which the test of Lemma 6.17*(a) *holds for some* k *with* $m_k|X_k|^{-1/2} - \sum_{l\neq k} m_l|X_l|^{-1/2} \geq 0.1756 \sum_l m_l|X_l|^{-1/2}$.

(d) *There are* 4788 *pairwise disjoint boxes in* $\mathbb{C}^{12}$, *each containing exactly one zero of* $\mathrm{KC}_{12}(\mu_0)$, *at which the Jacobian is invertible and* $X_3 \neq 0$, *and on each of which, for* $(m_5, m_6) = (1, 2)$, *all* 32 *sums of Lemma 6.17*(b) *have modulus at least* $1.47 \cdot 10^{-6}(1 + \rho^2)^{1/2} \sum_l |\beta_l|$.

Description of the computations. (a) The polynomials of G are the linear relations among the differences, the equations of the bodies, the distance equations, torques T_J, subset sums and the identity of Lemma 2.3(a), written in the coordinates ξ, and for KC also the eliminations of Y_4 through the Y-equation of each of the bodies $1, 2, 3$. Each lies in the ideal by an explicit identity over $\mathbb{Z}[\mu, 1/M, 1/\mu_4]$. A symbolic check shows that no coefficient vanishes at μ_0 modulo p (354 and 1133 terms). The prevarieties are computed by gfan. The listed cones are all faces of the common refinement, so the initial forms are constant on their relative interiors. Since the distance equations force $2w_{p_{kl}} + w_{X_{kl}} + w_{Y_{kl}} = 0$, the prevariety meets the nonnegative orthant only at 0. For each cone that meets $\{\min w_i < 0\}$, the initial system at a relative interior point is saturated by the product of the coordinates and found to be the unit ideal over $\mathbb{F}_p$ in Singular. (b) Gröbner bases modulo p in msolve, and the count of standard monomials of the leading ideal. (c), (d) Krawczyk's test [19] in ball arithmetic (Arb), at 256 and 384 bits, on boxes of radius below $6 \cdot 10^{-47}$ and $5 \cdot 10^{-54}$, respectively, around approximate zeros. The inequalities are lower bounds in ball arithmetic. The Arb programs do not test that the boxes are pairwise disjoint, but the two checks of Table 3 do. □

Section 10 lists the checks of these computations. By (a) and Lemma 6.18, the coordinate rings of $S_{12}(\mu_0)$ and $\mathrm{KC}_{12}(\mu_0)$ over $\mathbb{Z}_{(p)}$, obtained by the base change $\mu \mapsto \mu_0$ from $\mathcal{O}$, are finite $\mathbb{Z}_{(p)}$-modules. Their fibres over $\mathbb{F}_p$ have the dimensions in (b), since μ_3 is a unit modulo p, so that KC_{12}, KC and KC' have isomorphic coordinate rings. By Nakayama's lemma the two systems have at most 20,316 and 4788 complex zeros at μ_0, respectively, so the boxes of (c) and (d) contain all of them.

Theorem 6.20. *There are nonzero polynomials* $P_{88}, P_{94} \in \mathbb{Z}[\mu_1, \dots, \mu_4; \mu_5, \mu_6]$ *with the following properties. If a singular place has diagram* 88 *with z-binary* $\{a, b\}$ *and w-cluster* $K = \{1, \dots, 6\} \setminus \{a, b\}$, *then* $P_{88}(m_K; m_a, m_b) = 0$. *If a singular place has diagram* 94 *with the labelling of Table 1, then* $P_{94}(m_1, m_2, m_3, m_4; m_5, m_6) = 0$.

Proof. Let $V \subseteq \{\mu \in \mathbb{C}^4 : \mu_4 \neq 0\} \times \mathbb{C}^{11}$ be the zero set of S_{12}, whose coefficients involve $1/\mu_4$, over the space of the masses μ. By Lemmas 6.18 and 6.19(a), $R = \mathcal{O}[v]/\langle S_{12}\rangle$ is a finite $\mathcal{O}$-module, so by Nakayama's lemma $\dim_{\mathbb{Q}(\mu)} R \otimes \mathbb{Q}(\mu) \leq \dim_{\mathbb{F}_p} R/\mathfrak{m}R = 20{,}316$ (Lemma 6.19(b)). Hence there are $d \leq 20{,}316$ and a dense Zariski-open set $W \subseteq \{\mu_4 \neq 0\}$ over which all fibres of V have exactly d points. By Lemma 6.19(c) and the implicit function theorem for S_{11}, there are a neighbourhood N of μ_0 and 20,316 disjoint analytic branches of zeros of S_{12} over N. For $\mu \in N \cap W$ the fibre contains these branches and has $d \leq 20{,}316$ points, so $d = 20{,}316$ and the fibre consists of the branches. The test of Lemma 6.17(a) holds on each branch at μ_0 with a positive margin, hence on a smaller neighbourhood N'. By Lemma 6.17(a), for $\mu \in N' \cap W \cap \mathbb{R}^4$ and all $m_5, m_6 > 0$ the leading data of Lemma 6.16(a) do not exist.

Let $\mathrm{Bad} \subseteq \mathbb{C}^6$ be the set of (μ, m_5, m_6) for which the leading data of Lemma 6.16(a) exist, with any signs. It is the projection of a constructible set defined over $\mathbb{Q}$, hence constructible. It

misses $(N' \cap W \cap \mathbb{R}^4) \times \mathbb{R}^2_{>0}$, which contains a nonempty open subset of $\mathbb{R}^6$, since the real points of $\mathbb{C}^4 \setminus W$ are nowhere dense in $\mathbb{R}^4$. A constructible set that is Zariski dense in $\mathbb{C}^6$ contains a nonempty Zariski-open set, whose real points are dense in $\mathbb{R}^6$. So the Zariski closure of Bad is a proper subvariety defined over $\mathbb{Q}$. We take for P_{88} a nonzero integer polynomial vanishing on it. Other labellings follow by permuting the masses, and the exchange of the colours gives the same leading system with z and w exchanged.

For diagram 94 the same argument applies to KC_{12} with the count 4788 and Lemma 6.19(d): the 32 sums of Lemma 6.17(b) are continuous along the branches and bounded away from zero at $(\mu_0, 1, 2)$, so they do not vanish on a neighbourhood of $(\mu_0, 1, 2)$ in $\mathbb{C}^4 \times \mathbb{R}^2_{>0}$, intersected with $W \times \mathbb{R}^2_{>0}$. □

Note that for μ_0 the test of Lemma 6.17(a) holds at every solution of N_4, whereas the same test over the bodies $1, 2, 3$ fails at 1092 of the 4788 solutions of KC.† For this reason we test the sums of Lemma 6.19(d) at a single ratio of the binary masses. At 36 solutions of KC the smallest of the 32 sums has relative margin below 10^{-4} in the sense of Lemma 6.19(d). These solutions have $|Y_4| > 5 \cdot 10^3$, and there $|Y_l/Y_4| < 3 \cdot 10^{-5}$ for $l \le 3$.† At these solutions the equation of the centre, which reads $\sum_{l\le 3} \varepsilon_l m_l X_l^{-1/2}(1 - Y_l/Y_4)^{-3/2} = 0$ for suitable signs, is close to $\sum_{l\le 3} \varepsilon_l m_l X_l^{-1/2} = 0$, which says that a sum of Lemma 6.17(b) with $s_1 = s_2 = s_3$ vanishes.

7. Diagrams with a cluster of four bodies without circles

In this section we treat diagrams 64 and 66. In both, the bodies of $K = \{3, 4, 5, 6\}$ are pairwise joined by z-strokes and none is z-circled. In diagram 64 the w-strokes form the binary 12, and in diagram 66 the triangle $\{1, 2, 6\}$. In both, 1 and 2 are the w-circled bodies. As in Section 6, $m \in (\mathbb{R}_{>0})^6$, $C \subseteq Y_m$ is an irreducible curve, P is a singular place, the results are stated for the labelling of Table 1, and $U \not\equiv 0$ is not assumed.

7.1. **Leading systems.**

Lemma 7.1. *At a singular place with diagram* 64*:*

(a) *on the six pairs of* K, $\zeta = 3x - 2$ *and* $\omega = 2 - x$ *with a common* x, $0 < x < 1$*;*
(b) $\zeta_{12} = 2$, $\omega_{12} = -2$, $x_{12} = 0$*;*
(c) $\omega_{jk} = -2$ *and* $\zeta_{jk} = a_k < 2$ *for* $j \in \{1, 2\}$, $k \in K$*;*
(d) $b_1 = b_2 = -2$, $a_1, a_2 \ge 2$, *and for* $k \in K$, $b_k > -2$ *and* $3x - 2 \le a_k < 2$, *with equality for at least three* $k \in K$.

Proof. (a) By (E3) and the remark after Lemma 2.11, ζ, ω and x are equal on the six pairs, and (O3), (O4) give $\zeta = 3x - 2$, $\omega = 2 - x$, $0 \le x < 1$. If $x = 0$, then $\zeta_{kl} = -2$, so k or l would be z-circled. (b) is (F1) with the colours exchanged. (c), (d) The bodies 1, 2 are the w-circled ones, so $\omega_{jk} = -2$ by (O6). Since jk is not a w-stroke, $\zeta_{jk} < 2$ by (O6) and (O1). In the z-equation of j the term Z_{12} has exponent 2 and the terms Z_{jk} have exponent $3 - \frac{1}{2}\zeta_{jk} > 2$, so $a_j \ge 2$, and then $a_k \le \zeta_{jk} < 2 \le a_j$ gives $\zeta_{jk} = a_k$. Let $a_* = \min_K a_k$. If $a_* < 3x - 2$, all bodies of K have exponent a_* and a common leading coefficient $c \ne 0$, since $\zeta_{kl} = 3x - 2 > a_*$ on K, and the coefficient of exponent a_* in $\sum_k m_k z_k = 0$ is $c \sum_{k\in K} m_k \ne 0$. So $a_k \ge 3x - 2$, and two bodies with $a_k, a_l > 3x - 2$ would give $\zeta_{kl} > 3x - 2$. □

Lemma 7.2. *At a singular place with diagram* 64, $a_k = 3x - 2$ *for all* $k \in K$.

Proof. Suppose $a_{k_0} > 3x - 2$ for some $k_0 \in K$. By Lemma 7.1(d) it is the only such body. For $j \in \{1, 2\}$ and $k \in K$, (F4) applies to the pair (k, j), since $a_k < a_j$, $b_j < b_k$ and $a_j + b_k > a_k + b_j$. In the torque $T_{\{1,2\}}$ the term of the pair jk therefore has exponent $1 - \frac{1}{2}a_k$ and leading coefficient $m_j m_k [p_{jk}]$. The least exponent, $1 - \frac{1}{2}a_{k_0}$, is attained only by the pairs (j, k_0), and its coefficient

is $m_{k_0}(m_1[p_{1k_0}] + m_2[p_{2k_0}])$. This is not zero by Lemma 6.3 with the colours exchanged and the binary $\{1,2\}$ in place of $\{5,6\}$. □

For four masses $\mu = (\mu_1, \dots, \mu_4)$, let $E_4(\mu)$ be the system

$$\sum_{l\neq k} \mu_l P_{kl}^3(\xi_k - \xi_l) = 0, \qquad \sum_{l\neq k} \mu_l P_{kl}^3(\eta_k - \eta_l) = 0 \quad (k = 1, \dots, 4), \qquad P_{kl}^2(\xi_k - \xi_l)(\eta_k - \eta_l) = 1,$$

in $\xi, \eta \in \mathbb{C}^4$ and P_{kl}. Its solutions are complex equilibria of four bodies in both colours.

Lemma 7.3. *At a singular place with diagram* 64*, put* $\xi_k = [z_k]_{3x-2}$*,* $\eta_{kl} = [w_{kl}]_{2-x}$ *and* $P_{kl} = [p_{kl}]_{-x}$ *for* $k, l \in K$*. Then* $\eta_{kl} = \eta_k - \eta_l$ *for some* $\eta \in \mathbb{C}^K$*, all* $\xi_k - \xi_l$*,* η_{kl} *and* P_{kl} *are nonzero, and* (ξ, η, P) *solves* $E_4(m_K)$*.*

Proof. By Lemma 7.2, $\zeta_{kl} = 3x - 2$ exactly, so $\xi_k - \xi_l = [z_{kl}]_{3x-2} \neq 0$. Also $\omega_{kl} = 2 - x$ exactly, so $\eta_{kl} \neq 0$, and $\eta_{kl} + \eta_{lj} = \eta_{kj}$. The distance equations at exponent 0 give $P_{kl}^2(\xi_k - \xi_l)\eta_{kl} = 1$. In the z-equation of $k \in K$, the terms Z_{kl}, $l \in K$, have exponent -2 and coefficient $P_{kl}^3(\xi_k - \xi_l)$, the terms Z_{kj}, $j = 1, 2$, have exponent $4 - \frac{3}{2}x > -2$, and $a_k > -2$. At exponent -2 this gives $\Phi_k^\xi := \sum_l m_l P_{kl}^3(\xi_k - \xi_l) = 0$. In the w-equation of k, the terms W_{kl}, $l \in K$, have exponent $2 - 4x$ and coefficient $P_{kl}^3\eta_{kl}$, and the terms W_{kj} have exponent $(2 - 4x) + (2 - \frac{1}{2}x)$. So $[w_k]_{2-4x} = \Phi_k^\eta := \sum_l m_l P_{kl}^3 \eta_{kl}$. If $b_k > 2 - 4x$ for all $k \in K$, then $\Phi_k^\eta = 0$. Otherwise some $b_k \leq 2 - 4x < 2 - x = \omega_{kl}$, so all bodies of K have the same exponent b_k and the same leading coefficient $c' \neq 0$. If $b_k < 2 - 4x$, the w-equation at exponent b_k gives $c' = 0$. If $b_k = 2 - 4x$, then $\Phi_l^\eta = c'$ for all $l \in K$, while $\sum_{k\in K} m_k \Phi_k^\eta = 0$ by antisymmetry, so $c' \sum_{k\in K} m_k = 0$. Both are impossible. □

The leading system uses both colours, each at its own order: z at exponent -2, w at exponent $2-4x$, the distances at 0. A count of equations and unknowns suggests that E_4 is overdetermined by one equation. By the two momentum identities and the torque identity, at most five of its eight force equations are independent, so it has at most eleven independent equations. Its 14 unknowns modulo the symmetries $(\xi, \eta, P) \mapsto (\lambda s\xi, \lambda s^{-1}\eta, \lambda^{-1}P)$ and the translations of ξ and η leave ten. We will show in Proposition 7.8 that E_4 has no solution at generic masses.

Lemma 7.4. *At a singular place with diagram* 66*, let* $S = \{3, 4, 5\}$*. Then* (a) *of Lemma 7.1 holds. On* 12*,* 16*,* 26*,* $\omega = -2$*,* $\zeta = 2$ *and* $x = 0$*. For* $j \in \{1, 2\}$ *and* $k \in S$*,* $\omega_{jk} = -2$ *and* $\zeta_{jk} = a_k = 3x - 2$*. Moreover* $b_1 = b_2 = -2$*,* $b_k > -2$ *for* $k \in K$*, and* $a_1, a_2, a_6 \geq 2$*. Hence* $[z_6]_{3x-2} = 0$ *and* $\sum_{k\in S} m_k[z_k]_{3x-2} = 0$*.*

Proof. (a) as before. On the w-triangle $\{1, 2, 6\}$, (F1) with the colours exchanged gives $x = 0$, $\omega = -2$, $\zeta = 2$. As in Lemma 7.1, $\omega_{jk} = -2$, $\zeta_{jk} < 2$ and $a_j \geq 2$ for $j = 1, 2$, $k \in S$, and $\zeta_{jk} = a_k$. By (F3), $a_6 \geq 2$. For $k \in S$, $\zeta_{k6} = 3x - 2$ and $a_k < 2 \leq a_6$, so $a_k = 3x - 2$. At exponent $3x - 2$ only the bodies of S appear in $\sum_k m_k z_k = 0$. □

Lemma 7.5. *At a singular place with diagram* 66*, put* $\xi_k = [z_k]_{3x-2}$ *(*$\xi_6 = 0$*),* $\eta_{kl} = [w_{kl}]_{2-x} = \eta_k - \eta_l$ *and* $P_{kl} = [p_{kl}]_{-x}$ *for* $k, l \in K$*. Then the differences* $\xi_k - \xi_l$*, the* η_{kl} *and the* P_{kl} *are nonzero, and:*

(i) $P_{kl}^2(\xi_k - \xi_l)\eta_{kl} = 1$ *for the six pairs of* K*;*
(ii) $\Phi_k^\xi = 0$ *for all* $k \in K$*;*
(iii) $\Phi_k^\eta = c$ *for* $k \in S$*, with* $c \in \mathbb{C}$ *independent of* k*, possibly* 0*.*

Proof. (i) as in Lemma 7.3. (ii) For $k \in S$ as there. For $k = 6$ the terms Z_{6j}, $j = 1, 2$, have exponent 2 and $a_6 \geq 2$, so the coefficient of exponent -2 is $\Phi_6^\xi = 0$. (iii) For $k \in S$, $[w_k]_{2-4x} = \Phi_k^\eta$. If $b_k > 2 - 4x$ for all $k \in S$, then $c = 0$. Otherwise, as in Lemma 7.3, all

bodies of K have exponent $b_k = 2 - 4x$ and a common coefficient c', and $\Phi_k^\eta = c'$ for $k \in S$. The w-equation of body 6 at exponent $2 - 4x$ contains subleading terms of W_{61}, W_{62} and is not used. □

Renumber K as $1, 2, 3, 4$ with the centre 6 last, and let $\mu = (m_3, m_4, m_5, m_6)$. We call $C_4(\mu)$ the system of (i), (ii), (iii), $\xi_4 = 0$ and $\sum_{k\le 3} \mu_k \xi_k = 0$, in the unknowns ξ, η, P, c: a cluster of four bodies with body 4 at its centre, in equilibrium in the colour z and with the same force c on the three other bodies in the colour w. If $c = 0$, then $\Phi_4^\eta = 0$ as well, by antisymmetry, and the solution solves $E_4(\mu)$.

7.2. **The four-body equilibria.** The system E_4 is the leading system of the fully edged four-body diagram of [2, §5.5]. In that section Albouy and Kaloshin call its mass condition complicated and do not discuss it. Aquino-Michaels [3, Lemmas 5.3–5.5 and Theorem 5.6] proves that the fully balanced four-body core of diagrams 63, 65 and 67, which is the system E_4, is confined to a proper algebraic subset of masses. His reduction is of the same kind as Lemma 7.6 below: a rank-one equilibrium stress in the non-collinear case, where he then uses the Cayley–Menger relation, and sign classes in the collinear case. For diagrams 64 and 66, whose cores he describes as not fully balanced, he confines the collinear case and leaves the non-collinear case open [3, Rem. 5.7], and he asks for centred next-scale models of them [3, Problem 8.2]. Lemmas 7.3 and 7.5 give their leading systems. We give the reduction in the form we use.

Lemma 7.6. *For each of the 3^6 symmetric patterns $\varepsilon = (\varepsilon_{kl})$ of cube roots of unity, the powers of $\mathrm{j} = e^{2\pi i/3}$, and each of the 2^6 antisymmetric patterns $\sigma = (\sigma_{kl})$ of signs, let $A_k = \sum_{l\ne k} \varepsilon_{kl}\mu_l q_l$ and*

$$\mathcal{F}_\varepsilon = \Big\{ q_j^2 A_k - q_k^2 A_j \ (j<k),\ \sum_k \mu_k q_k^3 \Big\}, \qquad \mathcal{G}_\sigma = \Big\{ \sum_{l\ne k} \sigma_{kl}\mu_l \prod_{j\ne k,l} (\xi_k - \xi_j)^2 \ (k = 1, \dots, 4) \Big\},$$

seven cubic forms in $q \in \mathbb{P}^3$ and four quartic forms in $(\xi_2 : \xi_3 : \xi_4) \in \mathbb{P}^2$ with $\xi_1 = 0$. If $\mu_1\mu_2\mu_3\mu_4 \ne 0$ and $E_4(\mu)$ has a solution, then for some pattern the forms $\mathcal{F}_\varepsilon$ or $\mathcal{G}_\sigma$ have a common zero.

Proof. Let L be the matrix with $L_{kl} = \mu_l P_{kl}^3$ for $k \ne l$ and zero row sums. The equations of E_4 say $L\xi = L\eta = 0$, and $L\mathbf{1} = 0$. The span V of $\mathbf{1}, \xi, \eta$ has dimension at least 2, since ξ is not constant.

If $\dim V = 3$, then $\operatorname{rank} L \le 1$, and $L \ne 0$. The matrix $L \operatorname{diag}(\mu)^{-1}$ is symmetric of rank one, so $P_{kl}^3 = u_k u_l$ for $k \ne l$ with all $u_k \ne 0$. Then $L\mathbf{1} = 0$, $L\xi = 0$ and $L\eta = 0$ give $\sum \alpha_k = \sum \alpha_k \xi_k = \sum \alpha_k \eta_k = 0$ with $\alpha_k = \mu_k u_k$. For $d_{kl} = (\xi_k - \xi_l)(\eta_k - \eta_l) = P_{kl}^{-2}$ this gives $\sum_l \alpha_l d_{kl} = \sum_l \alpha_l \xi_l \eta_l =: -\beta$ for each k. With $q_k^3 = u_k$ we have $P_{kl} = \varepsilon_{kl} q_k q_l$ for cube roots of unity ε_{kl}, and $d_{kl} = \varepsilon_{kl} q_k^{-2} q_l^{-2}$. So $A_k = -\beta q_k^2$ for all k, and $\sum_k \mu_k q_k^3 = \sum_k \alpha_k = 0$: q is a common zero of $\mathcal{F}_\varepsilon$.

If $\dim V = 2$, then $\eta = \lambda\xi + \theta\mathbf{1}$ with $\lambda \ne 0$, so $P_{kl}(\xi_k - \xi_l) = \sigma_{kl}\kappa$ with $\kappa^2 = 1/\lambda$ and σ antisymmetric. The ξ-equations become $\kappa^3 \sum_{l\ne k} \sigma_{kl}\mu_l (\xi_k - \xi_l)^{-2} = 0$. After a translation $\xi_1 = 0$ and multiplication by $\prod_{j\ne k}(\xi_k - \xi_j)^2$, the point $(\xi_2 : \xi_3 : \xi_4)$ is a common zero of $\mathcal{G}_\sigma$. □

Lemma 7.7. † *At $\mu = (1, 1, 1, 1)$ no $\mathcal{F}_\varepsilon$ has a common zero in $\mathbb{P}^3$, and no $\mathcal{G}_\sigma$ has a common zero in $\mathbb{P}^2$.*

Proof. The forms $\mathcal{F}_\varepsilon$. At equal masses $A = Eq$, where $E = (\varepsilon_{kl})$ has zero diagonal. At a common zero q of $\mathcal{F}_\varepsilon$ the vectors A and $q \circ q = (q_k^2)_k \ne 0$ are proportional, so $A = -\beta\, q \circ q$ for some $\beta \in \mathbb{C}$, and $\sum_k q_k^3 = 0$. If $\beta = 0$, then $Eq = 0$. If $\beta \ne 0$, then $p = -\beta q \ne 0$ satisfies $Ep = p \circ p$ and $\sum_k p_k^3 = 0$. Write $\varepsilon_{kl} = \mathrm{j}^{e_{kl}}$. The substitution $q_l \mapsto \mathrm{j}^{c_l} q_l$ maps the common zeros

of $\mathcal{F}_\varepsilon$ to those of $\mathcal{F}_{\varepsilon'}$ with $e'_{kl} = e_{kl} - c_k - c_l$. These substitutions act freely on the 729 patterns, so they have 9 orbits. The triple $(e_{12}+e_{34}, e_{13}+e_{24}, e_{14}+e_{23})$ modulo $(1,1,1)$ is constant on the orbits and takes 9 values, so it separates them. Relabelling the bodies permutes its entries, and complex conjugation, which maps common zeros to common zeros, negates them. So it suffices to treat three patterns, with the triples $(0,0,0)$, $(0,0,1)$ and $(0,1,2)$:

(I) all $e_{kl} = 0$. Then $E = J - I$ is invertible. If $Ep = p \circ p$, then $p_k^2 + p_k = S := \sum_l p_l$ for all k, so each p_k is a root r_1 or r_2 of $r^2 + r - S$, and $r_1 + r_2 = -1$. If all p_k equal r, then $r(r-3) = 0$ and $r \neq 0$, so $\sum p_k^3 = 108$. If three of them equal r_1 and one equals r_2, then $r_1 r_2 = -S$ gives $r_1^2 - r_1 + 1 = 0$ and $\sum p_k^3 = 3r_1^3 + r_2^3 = -3 \mp 3\sqrt{3}\, i$. If two equal r_1 and two equal r_2, then $S = -2$, $r_1 r_2 = 2$ and $\sum p_k^3 = 2(r_1^3 + r_2^3) = 10$.

(II) $e_{14} = 1$ and all other $e_{kl} = 0$. Then E is invertible, and $Ep = p \circ p$ reads $p_2+p_3+\mathrm{j}p_4 = p_1^2$, $p_1+p_3+p_4 = p_2^2$, $p_1+p_2+p_4 = p_3^2$, $\mathrm{j}p_1+p_2+p_3 = p_4^2$. Subtracting gives $(p_2-p_3)(p_2+p_3+1) = 0$ and $(p_1-p_4)(p_1+p_4+\mathrm{j}) = 0$. If $p_2 = p_3 = a$ and $p_1 = p_4 = b$, then $a^3 + b^3 = 0$, and $b \neq 0$ since $p \neq 0$, so $a = -\zeta b$ with $\zeta^3 = 1$. The second and the first equation give $b = (2-\zeta)\zeta$ and $b = \mathrm{j} - 2\zeta$, so $\zeta^2 - 4\zeta + \mathrm{j} = 0$, which fails for $\zeta = 1, \mathrm{j}, \mathrm{j}^2$. If $p_2 = p_3 = a$ and $p_1 + p_4 = -\mathrm{j}$, then $a^2 - a + \mathrm{j} = 0$ and $p_1 p_4 = \mathrm{j}^2 - 2a$, so $\sum p_k^3 = 2a^3 + 2 - 6\mathrm{j}a$, which equals $2a(1-4\mathrm{j}) + 2 - 2\mathrm{j}$ because $a^2 = a - \mathrm{j}$, and vanishes only for $a = (\mathrm{j}-1)/(1-4\mathrm{j})$. For this value $a^2 - a + \mathrm{j} = (21-3\mathrm{j})/(1-4\mathrm{j})^2 \neq 0$. The case $p_2 + p_3 = -1$, $p_1 = p_4 = b$ is the same with $b = (\mathrm{j}-1)/(\mathrm{j}^2-4)$ and $b^2 - \mathrm{j}b + 1 = (21-3\mathrm{j})/(\mathrm{j}^2-4)^2 \neq 0$. If $p_2 + p_3 = -1$ and $p_1 + p_4 = -\mathrm{j}$, then $p_2 p_3 = -\mathrm{j}^2$, $p_1 p_4 = -\mathrm{j}$ and $\sum p_k^3 = 4 + 6\mathrm{j} \neq 0$.

(III) $e_{12} = e_{34} = 0$, $e_{13} = e_{24} = 2$, $e_{14} = e_{23} = 1$. Then E has the eigenvectors $\chi_0 = (1,1,1,1)$, $\chi_1 = (1,1,-1,-1)$, $\chi_2 = (1,-1,1,-1)$, $\chi_3 = (1,-1,-1,1)$, with eigenvalues 0, 2, $\lambda_2 = -2 - 2\mathrm{j}$, $\lambda_3 = 2\mathrm{j}$. If $Eq = 0$, then $q \in \mathbb{C}\chi_0$ and $\sum q_k^3 = 4q_1^3$, so $q = 0$. Write $p = \frac{1}{4}\sum_i s_i \chi_i$. Since $\chi_i \circ \chi_k = \chi_{i \oplus k}$ for the group law $\oplus$ of $\{0,1,2,3\} \cong (\mathbb{Z}/2)^2$, the equation $Ep = p \circ p$ reads

$$s_0^2 + s_1^2 + s_2^2 + s_3^2 = 0, \quad s_1(4 - s_0) = s_2 s_3, \quad s_2(2\lambda_2 - s_0) = s_1 s_3, \quad s_3(2\lambda_3 - s_0) = s_1 s_2,$$

and then $\sum p_k^3 = \frac{1}{8}(3 s_1 s_2 s_3 - s_0^3)$. If $s_1 s_2 s_3 \neq 0$, the last three equations give $s_1^2 = (2\lambda_2 - s_0)(2\lambda_3 - s_0)$, $s_2^2 = (4 - s_0)(2\lambda_3 - s_0)$ and $s_3^2 = (4 - s_0)(2\lambda_2 - s_0)$. Since $4 + 2\lambda_2 + 2\lambda_3 = 0$ and $8\lambda_2 + 8\lambda_3 + 4\lambda_2\lambda_3 = 0$, the first equation becomes $4s_0^2 = 0$, and then $\sum p_k^3 = \frac{3}{8} s_1 s_2 s_3 \neq 0$. If $s_1 s_2 s_3 = 0$, then $\sum p_k^3 = -\frac{1}{8} s_0^3$, so a solution has $s_0 = 0$. Then the last three equations show that two of s_1, s_2, s_3 vanish, and the first that the third vanishes too, so $p = 0$.

In each case $Eq = 0$ and $\sum_k q_k^3 = 0$ only for $q = 0$, and $\sum p_k^3 \neq 0$ unless $p = 0$, so $\mathcal{F}_\varepsilon$ has no common zero.

The forms $\mathcal{G}_\sigma$. Let $(\xi_2 : \xi_3 : \xi_4)$ be a common zero, with $\xi_1 = 0$. Not all four ξ_k coincide. If exactly two coincide, $\xi_a = \xi_b$, every term of the form for a except the one with $l = b$ contains $(\xi_a - \xi_b)^2$, so this form equals $\sigma_{ab} \prod_{j \neq a,b} (\xi_a - \xi_j)^2 \neq 0$. The same holds if two pairs coincide. If $\xi_a = \xi_b = \xi_c \neq \xi_d$, the form for d equals $(\xi_d - \xi_a)^4 (\sigma_{da} + \sigma_{db} + \sigma_{dc}) \neq 0$. So the ξ_k are distinct, and $\sum_{l \neq k} \sigma_{kl} (\xi_k - \xi_l)^{-2} = 0$ for all k. These equations are invariant under relabelling and under $\sigma \mapsto -\sigma$. We read σ as a tournament, with $k \to l$ when $\sigma_{kl} = 1$. A tournament on four vertices is determined up to isomorphism by its score sequence, $(3,2,1,0)$, $(3,1,1,1)$, $(2,2,2,0)$ or $(2,2,1,1)$, and reversal exchanges the second and the third. So it suffices to treat the transitive tournament $\mathcal{T}_1$, the tournament $\mathcal{T}_2$ with a source and a 3-cycle, and the strongly connected tournament $\mathcal{T}_3$, with $\sigma_{kl} = 1$ for $k < l$ except $\sigma_{24} = -1$ in $\mathcal{T}_2$ and $\sigma_{14} = -1$ in $\mathcal{T}_3$. After a translation and a scaling, $\xi = (0, 1, t, s)$. Let e_k be the numerator of $\sum_{l \neq k} \sigma_{kl} (\xi_k - \xi_l)^{-2}$, and let $R_1 = \mathrm{Res}_s(e_1, e_2)$ and $R_2 = \mathrm{Res}_s(e_1, e_3)$, which vanish at the t of a solution. In all three

cases $R_2 = t^4 R_2'$ with

$$\begin{aligned}
\mathcal{T}_1: &\quad R_1 = 5t^8 - 20t^7 + 26t^6 - 20t^5 + 19t^4 - 4t^3 + 2t^2 + 1,\\
\mathcal{T}_2: &\quad R_1 = 5t^8 - 20t^7 + 30t^6 - 36t^5 + 43t^4 - 28t^3 + 22t^2 - 8t + 1,\\
\mathcal{T}_1, \mathcal{T}_2: &\quad R_2' = t^8 + 14t^6 - 28t^5 + 47t^4 - 52t^3 + 42t^2 - 20t + 5,\\
\mathcal{T}_3: &\quad R_1 = -3t^8 + 12t^7 - 18t^6 + 20t^5 - 17t^4 + 4t^3 + 2t^2 + 1,\\
\mathcal{T}_3: &\quad R_2' = t^8 - 10t^6 + 28t^5 - 37t^4 + 36t^3 - 26t^2 + 12t - 3.
\end{aligned}$$

Here $R_1(0) = 1$, and $\mathrm{Res}_t(R_1, R_2')$ is $2^{16}3^4 5^4 7^2 \cdot 3881$, $2^{16}3^8 5^{10}$ and $-2^{16}5^4 \cdot 23 \cdot 433$ for $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$. Thus R_1 and R_2 have no common zero, so there is no solution. □

Proposition 7.8. *Let $B_E \subset \mathbb{C}^4$ be the union of $\{\mu_1\mu_2\mu_3\mu_4 = 0\}$ and of the set of μ for which some $\mathcal{F}_\varepsilon$ or some $\mathcal{G}_\sigma$ has a common projective zero. Then B_E is a proper Zariski-closed subset of $\mathbb{C}^4$, defined over $\mathbb{Q}$, and $E_4(\mu)$ has no solution for $\mu \notin B_E$.*

Proof. The second set is the projection of a closed subset of $\mathbb{C}^4 \times \mathbb{P}^3$, resp. $\mathbb{C}^4 \times \mathbb{P}^2$, so it is closed. It is a union over the patterns of sets defined over $\mathbb{Q}(\mathrm{j})$, since the forms have coefficients in $\mathbb{Z}[\mathrm{j}][\mu]$, and the union is stable under $\mathrm{j} \mapsto \mathrm{j}^2$, which permutes the patterns. Therefore B_E is defined over $\mathbb{Q}$. By Lemma 7.7, $(1,1,1,1) \notin B_E$. The last statement is Lemma 7.6. □

We have also checked by computation that a second mass point lies outside B_E. Let $\mu_0 = (3,5,7,11)$ and $p = 2147483629$. For each of the 729 patterns ε, the Macaulay matrix of $\mathcal{F}_\varepsilon$ in degree 7, with entries in $\mathbb{Z}[\mathrm{j}]$ reduced by $\mathrm{j} \mapsto 882503254$ (a root of $t^2 + t + 1$ modulo p), has full rank 120 modulo p, and for each of the 64 patterns σ the Macaulay matrix of $\mathcal{G}_\sigma$ in degree 8 has full rank 45 modulo p. A nonzero maximal minor modulo p is a nonzero element of $\mathbb{Z}[\mathrm{j}]$, since $\mathrm{j} \mapsto 882503254$ defines a ring map $\mathbb{Z}[\mathrm{j}] \to \mathbb{F}_p$. Thus these Macaulay matrices have full rank over $\mathbb{Q}(\mathrm{j})$, every monomial of degree 7 (resp. 8) lies in the ideal of the forms, and $\mu_0 \notin B_E$. The ranks are computed with python-flint, and Gröbner bases over $\mathbb{Q}(\mathrm{j})$ and over $\mathbb{Q}$ in Singular, of dimension zero, give the same conclusion.†

7.3. **The system with a centre.** Eliminate η from $C_4(\mu)$ by $\eta_{kl} = 1/(P_{kl}^2(\xi_k - \xi_l))$, and write the solutions of $\xi_4 = 0$, $\sum_{k\le 3} \mu_k \xi_k = 0$ as $\xi = (\mu_3 s, \mu_3 t, -\mu_1 s - \mu_2 t, 0)$, which is possible when $\mu_3 \neq 0$. With $a_{kl} = P_{kl}^2(\xi_k - \xi_l)$, the equations become

$$\begin{aligned}
&\textstyle\sum_{l\neq k} \mu_l P_{kl}^3 (\xi_k - \xi_l) = 0 && (k = 1, \dots, 4),\\
&\textstyle\sum_{l\neq k} \mu_l P_{kl} \prod_{j\neq k,l} (\xi_k - \xi_j) - c \prod_{l\neq k} (\xi_k - \xi_l) = 0 && (k = 1,2,3),\\
&a_{ln}a_{nk} + a_{kl}a_{nk} + a_{kl}a_{ln} = 0 && (\text{the four triangles } \{k,l,n\}).
\end{aligned}$$

The second line is (iii) multiplied by $\prod_{l\neq k}(\xi_k - \xi_l)$, and the third is the relation $\eta_{kl} + \eta_{ln} + \eta_{nk} = 0$ multiplied by $a_{kl}a_{ln}a_{nk}$. The equations are bihomogeneous of bidegrees $(1,3)$, $(2,1)$ and $(2,4)$ for the grading $\deg s = \deg t = (1,0)$, $\deg P_{kl} = (0,1)$, $\deg c = (-1,1)$. Let X be the quotient of $(\mathbb{C}^2 \setminus 0) \times (\mathbb{C}^7 \setminus 0)$, with coordinates (s,t) and $(P_{12}, \dots, P_{34}, c)$, by the action $(\lambda_1, \lambda_2) \cdot (s,t,P,c) = (\lambda_1 s, \lambda_1 t, \lambda_2 P, \lambda_1^{-1}\lambda_2 c)$ of $(\mathbb{C}^*)^2$. It is a $\mathbb{P}^6$-bundle over $\mathbb{P}^1$, hence projective. Let $Z(\mu) \subseteq X$ be the zero set of the equations. A solution of $C_4(\mu)$ with $\mu_3 \neq 0$ gives a point of $Z(\mu)$.

Lemma 7.9. † *Let $\mu_0 = (3,5,7,11)$. On each of the 14 affine charts $\{s = 1 \text{ or } t = 1\} \times \{\text{one of } P_{12}, \dots, P_{34}, c \text{ equal to } 1\}$, which cover X, the equations at μ_0 generate the unit ideal over $\mathbb{F}_{32003}$ and over $\mathbb{F}_{1073741789}$.*

The first prime is used by Singular and the second by msolve, from two programs that build the equations and produce identical polynomials.

Lemma 7.10. *Let $\mu \in \mathbb{Z}^4$. If the equations of $Z(\mu)$ generate the unit ideal over $\mathbb{F}_p$ on the 14 charts, then $Z(\mu) = \emptyset$.*

Proof. Let $I_p \subseteq \mathbb{F}_p[s,t,P,c]$ be the ideal of the equations at μ and $B = (s,t)\cdot(P,c)$. Every point of $\bar{\mathbb{F}}_p^9$ outside the zero set of B maps to a point of $X(\bar{\mathbb{F}}_p)$ in one of the charts, so the zero set of I_p lies in that of B, and $B^N \subseteq I_p$ for some N by the Nullstellensatz. A monomial of bidegree (a,b) has (s,t)-degree $a + \deg_c \geq a$ and (P,c)-degree b, so for $a,b \geq N$ it lies in B^N. Hence the Macaulay matrix of the equations in bidegree (N,N), an integer matrix with finitely many columns, has full rank modulo p, hence over $\mathbb{Q}$, and every monomial of bidegree (N,N) lies in the ideal of the equations at μ. At a point of X one of s,t, say s, is nonzero, and either some $P_{kl} \neq 0$, in which case $s^N P_{kl}^N \neq 0$, or $c \neq 0$, in which case $s^{2N} c^N \neq 0$, and both monomials have bidegree (N,N). Therefore $Z(\mu) = \emptyset$. □

Proposition 7.11. *Let $B_C \subset \mathbb{C}^4$ be the union of $\{\mu_3 = 0\}$ and of the set of μ with $Z(\mu) \neq \emptyset$. Then B_C is a proper Zariski-closed subset of $\mathbb{C}^4$, defined over $\mathbb{Q}$, and $C_4(\mu)$ has no solution for $\mu \notin B_C$.*

Proof. Since X is projective, the projection to $\mathbb{C}^4$ of the closed set $\{(\mu, y) : y \in Z(\mu)\}$ is closed. The equations have coefficients in $\mathbb{Z}[\mu]$, so this projection, and hence B_C, is defined over $\mathbb{Q}$. By Lemmas 7.9 and 7.10, $\mu_0 \notin B_C$. A solution of $C_4(\mu)$ with $\mu_3 \neq 0$ gives a point of $Z(\mu)$. □

7.4. **The diagrams.**

Theorem 7.12. *If a singular place has diagram 64 with w-binary $\{a,b\}$ and z-cluster $K = \{1,\dots,6\} \setminus \{a,b\}$, then $m_K \in B_E$.*

Proof. By Lemmas 7.2 and 7.3, $E_4(m_K)$ has a solution. Apply Proposition 7.8. □

Theorem 7.13. *If a singular place has diagram 66 with z-cluster K, centre $o \in K$ and w-triangle $\{a,b,o\}$, then $(m_{K\setminus\{o\}}, m_o) \in B_C$, where $m_{K\setminus\{o\}}$ lists the masses in increasing order of the labels.*

Proof. By Lemmas 7.4 and 7.5, $C_4(m_{K\setminus\{o\}}, m_o)$ has a solution. Apply Proposition 7.11. □

Both theorems hold for all labellings. By Lemma 7.7, $(1,1,1,1) \notin B_E$. The computation of Lemma 7.9 gives the same result at $\mu = (1,1,1,1)$, so $(1,1,1,1) \notin B_C$ by Lemma 7.10.† Theorems 7.12 and 7.13 therefore exclude diagrams 64 and 66 at six equal masses. We leave open whether E_4 or C_4 has solutions at positive masses.

Remark 7.14. The set B_E contains every μ in which one mass is the sum of two others. Let $\mu_k = \mu_i + \mu_j$ with i,j,k distinct, let l be the fourth index, and take $\xi_i = \xi_j = \xi_k$ and $\xi_l = \xi_i + 1$, translated so that $\xi_1 = 0$. Each term of the form of $\mathcal{G}_\sigma$ for one of the bodies i,j,k contains the square of the difference of two coinciding points, so these three forms vanish, and the form for l is $\sigma_{li}\mu_i + \sigma_{lj}\mu_j + \sigma_{lk}\mu_k$, which vanishes when $\sigma_{li} = \sigma_{lj} = -\sigma_{lk}$. Hence the polynomial F of Section 8 vanishes at every $m \in (\mathbb{R}_{>0})^6$ in which one mass is the sum of two others. These common zeros have coinciding points and do not come from solutions of E_4. At $\mu = (1,1,1,2)$, $(5,2,11,3)$, $(1,2,3,4)$ and $(2,3,5,7)$ the construction applies to exactly 36, 16, 28 and 28 of the 64 patterns σ. An exact computation over $\mathbb{Q}$ finds common zeros of $\mathcal{G}_\sigma$ for exactly these numbers of patterns, but none of them survives the saturation by the product of the differences $\xi_k - \xi_l$.†

8. Proof of Theorem 1.1

Let $F_0 \in \mathbb{Z}[m_1,\dots,m_6]$ be the product of the following nonzero polynomials:

- for each listed cell that has a core in Proposition 5.3, the polynomial h of Lemma 5.2;
- P_{98} of Theorem 6.12, and $P_{93}(m_1, m_2, m_3)$ with P_{93} as in Theorem 6.15;
- $P_{88}(m_1, \dots, m_4; m_5, m_6)$ and $P_{94}(m_1, \dots, m_4; m_5, m_6)$ of Theorem 6.20;
- a nonzero $h_E \in \mathbb{Z}[\mu]$ vanishing on B_E, evaluated at (m_3, m_4, m_5, m_6), and a nonzero $h_C \in \mathbb{Z}[\mu]$ vanishing on B_C, evaluated at (m_3, m_4, m_5, m_6) (Propositions 7.8 and 7.11).

Such h_E and h_C exist because B_E and B_C are proper Zariski-closed subsets defined over $\mathbb{Q}$. Let $F(m) = \prod_{g \in S_6} F_0(g \cdot m)$, where g permutes the masses. Then $F \neq 0$.

Proof of Theorem 1.1. We argue by contradiction. Let $m \in (\mathbb{R}_{>0})^6$ with $F(m) \neq 0$, and suppose that the planar six-body problem with masses m has infinitely many central configurations up to similarity. By Corollary 2.4, Y_m contains a nontrivial curve C, and by Corollary 2.7, C has a singular place P with $\ell(x(P)) \leq 0$. By Lemma 2.11 and Proposition 3.1, the diagram of P is the image of the representative D of its class under a relabelling, possibly followed by the exchange of the colours. By Lemma 2.9, the inverse relabelling and, if needed, the involution ι give a nontrivial curve $C' \subseteq Y_{m'}$, where m' is a permutation of m, and a singular place P' of C' with diagram D whose type is the type of P with the pairs permuted. In particular $\ell(x(P')) \leq 0$, so the trees and argmin sets of P' form a nonpositive cell of D, which is in the list of Proposition 4.5.

If the cell is one of the 43 cells of Proposition 5.3, then D is one of the diagrams 64, 66, 88, 89, 93, 94, 98, up to relabelling and exchange of the colours. Theorem 6.5 excludes P' for diagram 89. For the other six diagrams, Theorems 6.12, 6.15, 6.20, 7.12 and 7.13 show that a polynomial of the list above vanishes at a permutation of m, contradicting $F(m) \neq 0$. Otherwise, by Proposition 5.3 and Lemmas 5.1 and 5.2, either no place of C' lies in the cell, contradicting the fact that P' lies in it, or $h(m') = 0$ for the polynomial h of the cell, again contradicting $F(m) \neq 0$. This contradiction proves the theorem. □

The proof shows that Y_m contains no nontrivial curve when $m \in (\mathbb{R}_{>0})^6$ and $F(m) \neq 0$. But it does not show that Y_m is finite, since curves on which U vanishes identically are not excluded; by Lemma 2.3(c) they contain no point given by a real central configuration.

The positivity of the masses enters the proof in the following places.

- Lemma 2.3(c) and Corollary 2.4.
- The nonvanishing of partial sums of the masses: of M in Lemma 2.3(a), (b) and in the identity $\sum_{k<l} m_k m_l z_{kl} w_{kl} = ME$, of the sums M_A, M_B of the two triangles when Lemma 6.9 is applied in Theorems 6.12 and 6.15, and of other partial sums in Lemmas 2.11, 4.3, 6.1, 7.1 and 7.3.
- $m_i^3 + m_j^3 \neq 0$ in Lemma 6.3.
- $t = 1/m_1 > 0$, when Lemma 5.2(b) uses Lemma 2.1(b).
- The moduli in Lemma 6.17.
- The real points of the open sets in the proof of Theorem 6.20.

Elsewhere only $m_k \neq 0$ is used.

9. Proofs by hand for the cells

Proposition 5.3 is a computation: it excludes 52,432 cells by certified linear programs and 38,798 by Gröbner bases. In this section we explain the exclusion of most cells by lemmas whose hypotheses are conditions on the trees and the argmin sets of a cell and whose proofs are by hand. Each lemma either gives a contradiction at every positive mass or gives a polynomial in the masses that vanishes wherever the cell occurs. The polynomial is nonzero by the statement of the lemma or, for Lemmas 9.13, 9.14 and 9.21, by a check in each cell where the lemma is

used. Proposition 9.23 counts the cells to which the lemmas apply. The list of cells is that of Proposition 4.5, and a program decides which lemma applies to which cell.

As in Section 5, $m \in (\mathbb{R}_{>0})^6$ and $C \subseteq Y_m$ is a nontrivial curve. We fix a listed cell of a representative D, with trees T_z, T_w and polyhedron Π. Recall from §4.2 that a place lies in the cell if its diagram is D and its trees and its sets of terms of least exponent are those of the cell, and that its heights then lie in Π. A lemma *excludes* the cell if it shows that no singular place of C lies in it.

9.1. **Notation.** We write c for one of the colours z, w and c' for the other one, c_k for the position z_k or w_k, Z^c_{kl} for Z_{kl} or W_{kl}, T_c for the tree of colour c, and $X^c_{v,u}$ for its node coordinates ($X^z = X$, $X^w = Y$ in Lemma 4.2). For a body k, A^c_k is the argmin set of the c-equation $c_k = \sum_{l\neq k} m_l Z^c_{kl}$ of k. The *position* of k is *minimal* if $c_k \in A^c_k$, and the *force* of l on k is *minimal* if $m_l Z^c_{kl} \in A^c_k$. If T is a set of bodies and $k \in T$, then k is *balanced in colour c with respect to T* if A^c_k consists of the forces of the bodies of $T \setminus k$ on k.

For points $p, q \in B$, $\mathrm{meet}_c(p,q)$ is the lowest common ancestor of their leaves in T_c. For nodes u, v we write $u \preceq v$ if u lies on the path from the root to v. Then $h_u \le h_v$, and $h_u < h_v$ on all of Π unless $u = v$. If O lies below a node v, the *O-child* of v is the child below which O lies. We choose it as the child c_0 of Lemma 4.2, so that its node coordinate is 0. We write $\hat f = [f]_{e(f)}$ for the leading coefficient of a function f, and $P_{kl} = \hat p_{kl}$. Lemma 4.2 gives the following. If $v = \mathrm{meet}_c(k, O)$ and k lies below the child u of v, then $e(c_k) = h_v$ and $\hat c_k = X^c_{v,u} \neq 0$. If moreover l is a body with $\mathrm{meet}_c(k,l) = v$, then

$$[c_k - c_l]_{h_v} = \begin{cases} \hat c_k & \text{if } l \text{ lies below the } O\text{-child of } v \text{ or shares the leaf of } O, \\ \hat c_k - \hat c_l & \text{otherwise,} \end{cases} \tag{9.1}$$

and in the second case $\hat c_l \neq 0$. If v is the root of T_c, the coefficient of exponent h_v in $\sum_k m_k c_k = 0$ is

$$\textstyle\sum_u M_u X^c_{v,u} = 0, \tag{9.2}$$

the sum over the children u of v other than the O-child, where M_u is the sum of the masses below u (proof of Lemma 4.3(b)).

Lemma 9.1. *Let K be a set of bodies whose c-equations have the same least exponent, and suppose that every minimal force on a body of K is the force of a body of K. Then $\sum m_k \hat c_k = 0$, the sum over the bodies $k \in K$ whose position is minimal.*

Proof. Let e be the common least exponent. The initial form of the c-equation of $k \in K$ is $\delta_k \hat c_k - \sum_l m_l [Z^c_{kl}]_e$, where $\delta_k = 1$ if the position of k is minimal and $\delta_k = 0$ otherwise, and the sum runs over the bodies l whose force on k is minimal. These l lie in K. Since $e(Z^c_{lk}) = e(Z^c_{kl}) = e$, the force of k on l is then minimal as well, and $m_k m_l [Z^c_{kl}]_e + m_l m_k [Z^c_{lk}]_e = 0$. So the sum of the initial forms, weighted by the masses m_k, is $\sum_k \delta_k m_k \hat c_k$. □

9.2. **Lemmas at every positive mass.** The lemmas of this subsection exclude a cell at every positive mass. In the first three, an identity has on Π a single term of least exponent, or terms of least exponent whose coefficients add up to a positive multiple of one leading coefficient.

Let J be a nonempty proper subset of the bodies. The terms of the torque T_J (§2.3) correspond to the ordered pairs (p,q) of bodies with exactly one of p, q in J, the *cross pairs*. The term of (p,q) is $\pm m_p m_q p^3_{pq} z_p w_q$, of exponent $(a_p - \frac{3}{2}\zeta_{pq}) + (b_q - \frac{3}{2}\omega_{pq})$, and it is absent if z_p or w_q vanishes identically, i.e., if p shares the leaf of O in T_z or q does in T_w.

Lemma 9.2. *Let (p,q) be a cross pair of J whose term is present, and suppose:*

(Z) $\mathrm{meet}_z(p,O) \preceq \mathrm{meet}_z(p,q)$, *and* $\mathrm{meet}_z(k,l) \preceq \mathrm{meet}_z(p,q)$ *for every cross pair (k,l);*

(W) $\mathrm{meet}_w(q, O) \preceq \mathrm{meet}_w(p, q)$, *and* $\mathrm{meet}_w(k, l) \preceq \mathrm{meet}_w(p, q)$ *for every cross pair* (k, l)*;*
(S) *no other cross pair* (k, l) *whose term is present has the same nodes* $\mathrm{meet}_z(k, l)$, $\mathrm{meet}_z(k, O)$, $\mathrm{meet}_w(k, l)$ *and* $\mathrm{meet}_w(l, O)$ *as* (p, q).

Then the term of (p, q) *is the only term of* T_J *of least exponent on* Π*, and Lemma 5.1 excludes the cell.*

Proof. Let $(k, l) \neq (p, q)$ be a cross pair whose term is present, and let $t = \mathrm{meet}_z(p, q)$ and $t' = \mathrm{meet}_z(k, l)$, so that $t' \preceq t$ by (Z). If O lies below t', then $t' \preceq \mathrm{meet}_z(k, O)$, so $a_k \geq \zeta_{kl}$ and

$$a_k - \tfrac{3}{2}\zeta_{kl} \;\geq\; -\tfrac{1}{2}\zeta_{kl} \;\geq\; -\tfrac{1}{2}\zeta_{pq} \;\geq\; a_p - \tfrac{3}{2}\zeta_{pq},$$

where the last step is $a_p \leq \zeta_{pq}$, which holds by (Z). If O does not lie below t', then $\mathrm{meet}_z(k, O) = \mathrm{meet}_z(p, O)$, since k and p both lie below t'. So $a_k = a_p$ and again $a_k - \frac{3}{2}\zeta_{kl} \geq a_p - \frac{3}{2}\zeta_{pq}$. Each inequality used is of the form $h_u \leq h_v$ with $u \preceq v$, which is strict on Π unless $u = v$. So equality at a point of Π forces $t' = t$ and $\mathrm{meet}_z(k, O) = \mathrm{meet}_z(p, O)$. In the same way (W) gives $b_l - \frac{3}{2}\omega_{kl} \geq b_q - \frac{3}{2}\omega_{pq}$, with equality only if $\mathrm{meet}_w(k, l) = \mathrm{meet}_w(p, q)$ and $\mathrm{meet}_w(l, O) = \mathrm{meet}_w(q, O)$. By (S), the term of (k, l) has a larger exponent than the term of (p, q) at every point of Π. □

For a pair kl we say that $x_{kl} > 0$ is *forced* if one of the following holds: (x1) kl is a zw-edge of D; (x2) kl is a z-stroke of D and neither k nor l is z-circled, or the same with w; (x3) the term $m_k m_l p_{kl}$ belongs to the argmin set of $U = E$ and the constant E does not; (x4) the force of l on k is minimal in the z-equation of k, the position of k is not, and $\mathrm{meet}_z(k, O) = \mathrm{meet}_z(k, l)$, or the same with w in place of z or with k and l exchanged. Each of these implies $x_{kl} > 0$ on Π: (x1) by (O4); (x2) since $3x_{kl} - 2 = \zeta_{kl} \geq \min(a_k, a_l) > -2$ by (O3); (x3) since $-x_{kl}$ is then the least exponent of $U - E$, which is smaller than $e(E) = 0$; (x4) since $-\frac{1}{2}(\zeta_{kl} + 3\omega_{kl}) = e(Z_{kl}) < a_k = \zeta_{kl}$.

Lemma 9.3. *Suppose that for every body* k *with* $z_k, w_k \not\equiv 0$ *there is a body* l *such that* O *lies below* $\mathrm{meet}_z(k, l)$ *and below* $\mathrm{meet}_w(k, l)$ *and* $x_{kl} > 0$ *is forced. Then the constant* E *is the only term of least exponent of the identity* $\sum_k m_k z_k w_k = E$ *on* Π*, and Lemma 5.1 excludes the cell.*

Proof. If O lies below $\mathrm{meet}_z(k, l)$, then $\mathrm{meet}_z(k, l) \preceq \mathrm{meet}_z(k, O)$, so $a_k \geq \zeta_{kl}$. Likewise $b_k \geq \omega_{kl}$. So $e(z_k w_k) = a_k + b_k \geq 2x_{kl} > 0 = e(E)$ on Π. □

For a set J of bodies, each term of the subset sum $\sum_{k \in J} m_k c_k = \sum_{k \in J, l \notin J} m_k m_l Z^c_{kl}$ is m_k times a term of the c-equation of one body $k \in J$: its position, or the force of a body outside J.

Lemma 9.4. *Let* J *be a set of at least two bodies, and suppose:*

(C1) *the graph on* J *in which* k *and* l *are joined if the forces of* l *on* k *and of* k *on* l *are minimal, or if the positions of* k *and* l *are minimal and* $\mathrm{meet}_c(k, O) = \mathrm{meet}_c(l, O)$*, is connected;*
(C2) *the set* X *of the minimal terms of the* c*-equations of the bodies of* J *that are positions or forces of bodies outside* J *is not empty, and it consists of one term, or of the positions of the bodies of a set* I *that lie below one child of a node* v *with* $\mathrm{meet}_c(k, O) = v$ *for all* $k \in I$*.*

Then the cell is excluded.

Proof. Two bodies joined in (C1) have c-equations with the same least exponent, since $e(Z^c_{kl}) = e(Z^c_{lk})$, and since positions that meet O at the same node have the same exponent. So these least exponents have a common value μ. Each term of the subset sum of J has exponent at least μ, with equality exactly for the elements of X. If X has one element, Lemma 5.1 applies. Otherwise all $\hat{c}_k$ with $k \in I$ equal the node coordinate $X^c_{v,u} \neq 0$ of the child u of v below which they lie, and the initial form of the subset sum is $(\sum_{k \in I} m_k) X^c_{v,u} \neq 0$, contradicting Lemma 2.10. □

Lemma 9.5. *Let* $k \neq l$ *be bodies, and suppose:*

(i) A^c_l *consists of the position of* l *and the force of* k *on* l, $\mathrm{meet}_c(l,k) = \mathrm{meet}_c(l,O)$, *and* k *lies below the* O*-child of this node or shares the leaf of* O*;*
(ii) $A^{c'}_l$ *consists of the position of* l *and the force of* k *on* l, $\mathrm{meet}_{c'}(l,k) = \mathrm{meet}_{c'}(l,O)$, *and* k *lies below a child of this node other than the* O*-child.*

Then the cell is excluded.

Proof. Let $P = P_{kl}$. By (i) and (9.1), the leading coefficient of $c_l - c_k$ is $\hat{c}_l$, so the initial form of the c-equation of l is $\hat{c}_l(1 - m_k P^3) = 0$, so $m_k P^3 = 1$. By (ii) and (9.1), the leading coefficient of $c'_l - c'_k$ is $\hat{c}'_l - \hat{c}'_k$ with $\hat{c}'_k \neq 0$. So the initial form of the c'-equation of l is $\hat{c}'_l - m_k P^3(\hat{c}'_l - \hat{c}'_k) = \hat{c}'_k$, which is not zero. □

Lemma 9.6. *Let* $k \neq l$ *be bodies, and suppose:*

(i) k, l *and* O *lie below three distinct children of one node of* T_c*;* A^c_k *consists of the position of* k *and the force of* l *on* k*, and* A^c_l *of the position of* l *and the force of* k *on* l*;*
(ii) k, l *and* O *lie below three distinct children of one node of* $T_{c'}$*; for a labelling* $\{p,q\} = \{k,l\}$, $A^{c'}_p$ *consists of the position of* p *and the force of* q *on* p*, and* $A^{c'}_q$ *of the position of* q*, the force of* p *on* q *and the force of one body* $j \notin \{k,l\}$ *on* q*.*

Then the cell is excluded.

Proof. Let $P = P_{kl}$. In colour c, Lemma 9.1 for $\{k,l\}$ gives $m_k\hat{c}_k + m_l\hat{c}_l = 0$. With (9.1), the initial form of the c-equation of k becomes $\hat{c}_k - m_l P^3(\hat{c}_k - \hat{c}_l) = \hat{c}_k(1 - (m_k + m_l)P^3)$. Hence $(m_k + m_l)P^3 = 1$. In colour c', the initial form of the equation of p is $\hat{c}'_p(1 - m_q P^3) + m_q P^3 \hat{c}'_q = P^3(m_p\hat{c}'_p + m_q\hat{c}'_q)$, so $m_p\hat{c}'_p + m_q\hat{c}'_q = 0$. Then the initial form of the equation of q is $P^3(m_q\hat{c}'_q + m_p\hat{c}'_p) - m_j[Z^{c'}_{qj}] = -m_j[Z^{c'}_{qj}]$, which is not zero. □

Lemma 9.7. *Let* $T = \{a,b,s\}$ *be three bodies, and suppose:*

(L1) a, b, s *lie below three distinct children of one node* v *of* T_c*;* O *lies below* v *but not below these children; and for each* $k \in T$, A^c_k *consists of the position of* k *and the forces of the other two bodies of* T *on* k*;*
(L2) a, b, s *lie below three distinct children of one node* v' *of* $T_{c'}$*;* s *lies below the* O*-child of* v' *or shares the leaf of* O*;* $A^{c'}_s$ *consists of the forces of* a *and* b *on* s*; and for* $\{k,l\} = \{a,b\}$, $A^{c'}_k$ *consists of the position of* k *and the forces of* s *and* l *on* k*.*

Then the cell is excluded.

Proof. Put $X_k = \hat{c}_k$ for $k \in T$ and $Y_k = \hat{c}'_k$ for $k = a,b$. By (9.1), the leading coefficients of $c_k - c_l$, $c'_k - c'_s$ and $c'_a - c'_b$ are $X_k - X_l$, Y_k and $Y_a - Y_b$. The three pairs of T have the same ζ and the same ω. So the forces between the bodies of T have a common exponent in each colour, and it is the least exponent of each of the six equations in (L1) and (L2).

Lemma 9.1 in colour c' gives $m_aY_a + m_bY_b = 0$, and the initial form of the c'-equation of s gives $m_aP^3_{sa}Y_a + m_bP^3_{sb}Y_b = 0$. Hence $P^3_{sa} = P^3_{sb} =: Q$. Lemma 9.1 in colour c gives $m_aX_a + m_bX_b + m_sX_s = 0$. So the initial form of the c-equation of s is $X_s - Q(m_a(X_s - X_a) + m_b(X_s - X_b)) = X_s(1 - QM_T)$, where $M_T = m_a + m_b + m_s$, and $Q = 1/M_T$ since $X_s \neq 0$. The initial form of the c'-equation of a is $Y_a(1 - m_sQ) - m_bP^3_{ab}(Y_a - Y_b) = (m_a + m_b)Y_a(1/M_T - P^3_{ab})$, so $P^3_{ab} = 1/M_T$ as well. Hence $P^{-2}_{kl} = \varepsilon_{kl}M_T^{2/3}$ with cube roots of unity ε_{kl}. The distance equations give $X_k - X_l = P^{-2}_{kl}[c'_k - c'_l]^{-1}$. We insert this and $Y_b = -m_aY_a/m_b$ into $(X_a - X_b) + (X_b - X_s) + (X_s - X_a) = 0$ and multiply by $m_a(m_a + m_b)Y_aM_T^{-2/3}$. This gives

$$m_am_b\,\varepsilon_{ab} = m_b(m_a + m_b)\,\varepsilon_{bs} + m_a(m_a + m_b)\,\varepsilon_{sa}.$$

If $\varepsilon_{bs} = \varepsilon_{sa}$, the right side has modulus $(m_a+m_b)^2$. Otherwise its squared modulus is $\beta^2+\gamma^2-\beta\gamma \geq \beta\gamma$, where β and γ are its two coefficients, so the modulus is at least $(m_a+m_b)\sqrt{m_a m_b}$. Both bounds exceed $m_a m_b$. □

9.3. **Triangles.** Let K be a set of bodies that lie below distinct children of one node v_z of T_z and of one node v_w of T_w. Then all pairs of K have $\zeta = h_{v_z}$ and $\omega = h_{v_w}$, and so the same value x. Let $\xi_t = X_{v_z,u}$ and $\eta_t = Y_{v_w,u'}$ be the node coordinates of the children u and u' below which $t \in K$ lies, and $P_{tt'} = [p_{tt'}]_{-x}$. The differences $\xi_t - \xi_{t'}$ and $\eta_t - \eta_{t'}$ are the leading coefficients of $z_t - z_{t'}$ and $w_t - w_{t'}$, and they are not zero for $t \neq t'$. The distance equations give

$$P_{tt'}^2(\xi_t - \xi_{t'})(\eta_t - \eta_{t'}) = 1. \tag{9.3}$$

In each colour c the forces between bodies of K have a common exponent e_K^c. We put

$$\Phi_t^z = \sum_{t' \in K\setminus t} m_{t'} P_{tt'}^3(\xi_t - \xi_{t'}), \qquad \Phi_t^w = \sum_{t' \in K\setminus t} m_{t'} P_{tt'}^3(\eta_t - \eta_{t'}).$$

If t is balanced in colour c with respect to K, the initial form of its c-equation is $-\Phi_t^c$, so $\Phi_t^c = 0$. When $K = \{a, b, s\}$ has three bodies, we call it a *triangle*.

Lemma 9.8. *Identically in ξ, η and the symmetric P, $\sum_t m_t\Phi_t^z = \sum_t m_t\Phi_t^w = 0$ and $\sum_t m_t(\xi_t\Phi_t^w - \eta_t\Phi_t^z) = 0$.*

Proof. The terms $m_t m_{t'} P_{tt'}^3(\xi_t - \xi_{t'})$ and $m_t m_{t'} P_{tt'}^3(\xi_t(\eta_t - \eta_{t'}) - \eta_t(\xi_t - \xi_{t'})) = m_t m_{t'} P_{tt'}^3(\xi_{t'}\eta_t - \xi_t\eta_{t'})$ are antisymmetric in (t, t'), and so is the first one with η in place of ξ. □

Theorem 9.9. *Let $K = \{a, b, s\}$ be a triangle.*

(i) *If $\Phi^z = \Phi^w = 0$, then one of $m_a^{-1/2}$, $m_b^{-1/2}$, $m_s^{-1/2}$ is the sum of the other two.*
(ii) *If $\Phi^w = 0$ and $\Phi_a^z = \Phi_b^z \neq 0$, then $m_s^2(m_a+m_b)^4 = m_a^3 m_b^3$.*
(iii) *If $\Phi^w = 0$ and $\Phi_a^z = 0$, then $\Phi^z = 0$.*

The same holds with z and w exchanged.

Proof. Let L be the 3×3 matrix with $L_{tt'} = m_{t'} P_{tt'}^3$ for $t \neq t'$ and zero row sums. Then $\Phi^z = -L\xi$, $\Phi^w = -L\eta$, $L\mathbf{1} = 0$, and $L\,\mathrm{diag}(m)^{-1}$ is symmetric with nonzero entries off the diagonal. We write $\xi_{tt'} = \xi_t - \xi_{t'}$, $\eta_{tt'} = \eta_t - \eta_{t'}$ and $n_t = m_t^{-1/2}$.

(i) If $\mathbf{1}$, ξ, η were linearly independent, L would vanish. Since ξ and η are not constant, $\eta = \lambda\xi + \nu\mathbf{1}$ with $\lambda \neq 0$. Then $(P_{tt'}\xi_{tt'})^2 = 1/\lambda$, so $P_{tt'}\xi_{tt'} = \sigma_{tt'}\psi$ with $\psi^2 = 1/\lambda$ and antisymmetric signs $\sigma_{tt'}$, and $\Phi_t^z = \psi^3 \sum_{t' \neq t} \sigma_{tt'} m_{t'} \xi_{tt'}^{-2}$. The equations $\Phi_a^z = \Phi_b^z = 0$ give $\xi_{as}^2/\xi_{ab}^2 = \pm m_s/m_b$ and $\xi_{bs}^2/\xi_{ab}^2 = \pm m_s/m_a$. Hence $\xi_{as} = \alpha\xi_{ab}n_b/n_s$ and $\xi_{bs} = \beta\xi_{ab}n_a/n_s$ with $\alpha, \beta \in \{\pm 1, \pm i\}$, and $\xi_{as} = \xi_{ab} + \xi_{bs}$ gives $n_s + \beta n_a = \alpha n_b$. Since the n_t are positive, α and β are real, and one of n_a, n_b, n_s is the sum of the other two.

(ii) Since $L\eta = L\mathbf{1} = 0$ and η is not constant, L has rank 1, so $L = uv^{\mathsf{T}}$. The symmetry of $L\,\mathrm{diag}(m)^{-1}$ gives $v = \lambda\, m \circ u$, so $P_{tt'}^3 = \lambda u_t u_{t'}$ with all $u_t \neq 0$, $\sum_t m_t u_t = 0$ and $\Phi^z = -\lambda\langle m \circ u, \xi\rangle u$. As $\Phi_a^z = \Phi_b^z \neq 0$, we get $u_a = u_b$ and $u_s = -u_a(m_a+m_b)/m_s$. So $P_{bs} = \mathrm{j}^k P_{as}$ for some $k \in \{0, 1, 2\}$, where $\mathrm{j} = e^{2\pi i/3}$, and $(P_{ab}/P_{as})^3 = -m_s/(m_a+m_b)$. The equation $\Phi^w = 0$ means $m_a\eta_{as} + m_b\eta_{bs} = 0$, so $\eta_{as} = m_b\nu$, $\eta_{bs} = -m_a\nu$ and $\eta_{ab} = (m_a+m_b)\nu$. The distance equations of as and bs give $\xi_{bs} = -\xi_{as}m_b/(\mathrm{j}^{2k}m_a)$, so $\xi_{ab} = \xi_{as}(\mathrm{j}^{2k}m_a + m_b)/(\mathrm{j}^{2k}m_a)$, and the distance equation of ab divided by that of as gives $(P_{ab}/P_{as})^2 = \mathrm{j}^{2k}m_a m_b/((\mathrm{j}^{2k}m_a + m_b)(m_a+m_b))$. Comparing sixth powers, $m_a^3 m_b^3 = m_s^2(m_a+m_b)(\mathrm{j}^{2k}m_a + m_b)^3$. If $k \neq 0$, the argument of $\mathrm{j}^{2k}m_a + m_b$ lies strictly between 0 and $\pm 2\pi/3$, so its cube is not positive. Hence $k = 0$.

(iii) As in (ii), $\Phi^z = -\lambda\langle m \circ u, \xi\rangle u$ with $u_a \neq 0$, and $\Phi_a^z = 0$ forces $\Phi^z = 0$. □

The two conditions are those that [2, (16), (20)] find for the isolated triangle and for the kite among the diagrams of four bodies. Here, however, both come from one three-body system. For a triangle $T = \{a, b, s\}$ let

$$Q_T = m_a^2 m_b^2 + m_b^2 m_s^2 + m_s^2 m_a^2 - 2m_a m_b m_s (m_a + m_b + m_s) = m_a^2 m_b^2 m_s^2 \prod (n_a \pm n_b \pm n_s),$$

the product over the four choices of signs. At positive masses $Q_T = 0$ exactly when one of n_a, n_b, n_s is the sum of the other two.

Corollary 9.10. *Let T be a triangle. If two bodies of T are balanced in one colour and one body of T is balanced in the other colour, with respect to T, then $Q_T(m) = 0$ at every place in the cell.*

Proof. Let a and b be balanced in colour c. Then $\Phi_a^c = \Phi_b^c = 0$, and $\Phi_s^c = 0$ by Lemma 9.8. If $t \in T$ is balanced in colour c', then $\Phi_t^{c'} = 0$, and Theorem 9.9(iii), with t in the role of a, gives $\Phi^{c'} = 0$. Now Theorem 9.9(i) applies. □

Suppose that three bodies of D are pairwise joined by zw-edges, that none of them is circled, and that no other stroke meets them. At every singular place with diagram D they lie below distinct children of one node in each tree, since $\zeta = \omega = 1$ on their pairs. In their equations the forces between them have exponent -2 and all other terms have larger exponents. So each of the three is balanced in both colours, and Corollary 9.10 applies in every cell of D. Seven of the 191 classes have such a triangle, among them diagrams 7 and 12 of [5]. Six of them have listed cells, 50,399 in all.

A triangle is *centred* in colour c at s if, at its node v in T_c, the body s lies below the O-child or shares the leaf of O, while a and b lie below two other children $A \neq B$ of v.

Lemma 9.11. *Let $T = \{a, b, s\}$ be a triangle centred in colour c at s, with node v and children A, B as above, and suppose that a and b are balanced in colour c with respect to T. Suppose further that one of the following holds, and let M_A, M_B be the sums of masses that it names:*

(a) *v is the root of T_c and has exactly three children; M_A and M_B are the sums of the masses below A and B;*
(b) *v is the root of T_c; every body below a child of v other than A, B and the O-child has a minimal position, and these bodies form a disjoint union of sets K as in Lemma 9.1; M_A and M_B are as in* (a)*;*
(c) *for some set $J \supseteq \{a, b\}$, the argmin set of the subset sum of J in colour c is constant on the cell and consists of the terms $m_a c_a$ and $m_b c_b$; $M_A = m_a$ and $M_B = m_b$.*

Then $N(m_a, m_b, m_s, M_A, M_B) = 0$ at every place in the cell, where

$$N = \operatorname{Res}_\kappa \left(M_A m_b \kappa^3 - M_B m_a,\ m_s^2 (M_A + M_B)(m_a + m_b \kappa)^3 - m_a^3 m_b^2 M_B \right),$$

and N is a nonzero polynomial in m. If $M_A = m_a$ and $M_B = m_b$, then $m_s^2 (m_a + m_b)^4 = m_a^3 m_b^3$ at every place in the cell.

Proof. We write the proof for $c = w$. The case $c = z$ follows by exchanging the colours. Let $G_{kl} = P_{kl}^3 (\eta_k - \eta_l)$. Since a and b are balanced, $\Phi_a^w = \Phi_b^w = 0$, i.e., $m_b G_{ab} = m_s G_{sa}$ and $m_a G_{ab} = m_s G_{bs}$. So $(G_{ab}, G_{bs}, G_{sa}) = \lambda (m_s, m_a, m_b)$ with $\lambda \neq 0$. As $\eta_s = 0$, the node coordinates of A and B are $Y_A = \eta_a = -\lambda m_b / P_{sa}^3$ and $Y_B = \eta_b = \lambda m_a / P_{bs}^3$. We claim that $M_A Y_A + M_B Y_B = 0$. In case (a) this is (9.2). In case (b) it is (9.2) minus the relations of Lemma 9.1 for the sets K, since each body k of these sets has a minimal position and $\hat{w}_k$ is the coordinate of its child of v. In case (c) it is the initial form of the subset sum, as $\hat{w}_a = Y_A$ and $\hat{w}_b = Y_B$. Hence $\kappa = P_{bs} / P_{sa}$ satisfies $M_A m_b \kappa^3 = M_B m_a$. The relation $(\eta_a - \eta_b) + (\eta_b - \eta_s) + (\eta_s - \eta_a) = 0$ reads $m_s / P_{ab}^3 + m_a / P_{bs}^3 + m_b / P_{sa}^3 = 0$, so $m_s (P_{sa} / P_{ab})^3 = -m_b (M_A + M_B) / M_B$. By (9.3),

$\xi_k - \xi_l = P_{kl}/G_{kl}$, and the same relation for ξ reads $P_{ab}/m_s + P_{bs}/m_a + P_{sa}/m_b = 0$, so $P_{ab}/P_{sa} = -m_s(m_a + m_b\kappa)/(m_a m_b)$. Comparing cubes gives $m_s^2(M_A + M_B)(m_a + m_b\kappa)^3 = m_a^3 m_b^2 M_B$. So the two polynomials of the resultant have the common root κ, and $N = 0$.

If the masses below A and B other than m_a and m_b are set to zero, N becomes $(m_a m_b)^3 \prod_\kappa \left(m_s^2(m_a + m_b)(m_a + m_b\kappa)^3 - m_a^3 m_b^3\right)$, the product over the cube roots of unity κ. At $m_s = 0$ this is $-m_a^{12} m_b^{12} \neq 0$, so $N \neq 0$. If $M_A = m_a$ and $M_B = m_b$, then $\kappa^3 = 1$ and $(m_a + m_b\kappa)^3 = m_a^3 m_b^3/(m_s^2(m_a + m_b)) > 0$. For $\kappa \neq 1$ the argument of $m_a + m_b\kappa$ lies strictly between 0 and $\pm 2\pi/3$, so $\kappa = 1$. □

Lemma 9.12. *Let $T = \{a, b, s\}$ be a triangle whose node v in T_c is the root of T_c, with exactly three children: the leaves of a and of b, and a child below which all other points lie. Suppose that s is balanced in colour c and a in colour c', with respect to T. Then at every place in the cell*

$$m_a m_b^5 = m_s^2(m_a + m_b)^3(m_a + \omega m_b)$$

for a cube root of unity ω. The product of the differences of the two sides over the three values of ω is a nonzero polynomial, and at positive masses $\omega = 1$, that is, $m_s^2(m_a + m_b)^4 = m_a m_b^5$.

Proof. We write the proof for $c = w$. By (9.2), $m_a\eta_a + m_b\eta_b = 0$, and $\eta_s = 0$. The initial form of the w-equation of s gives $m_a P_{sa}^3\eta_a + m_b P_{sb}^3\eta_b = 0$, so $P_{sb} = \omega P_{sa}$ with $\omega^3 = 1$. By (9.3), $\xi_k - \xi_l = 1/(P_{kl}^2(\eta_k - \eta_l))$. With $\eta_a - \eta_b = (m_a + m_b)\eta_a/m_b$, the relation $m_b P_{ab}^3(\xi_a - \xi_b) + m_s P_{as}^3(\xi_a - \xi_s) = 0$, which is the initial form of the z-equation of a, becomes $m_b^2 P_{ab}/(m_a + m_b) + m_s P_{as} = 0$ after multiplication by η_a, so $P_{ab}^2/P_{sa}^2 = m_s^2(m_a + m_b)^2/m_b^4$. The relation $(\xi_a - \xi_b) + (\xi_b - \xi_s) + (\xi_s - \xi_a) = 0$, multiplied by $\eta_a P_{ab}^2$, reads

$$\frac{m_b}{m_a + m_b} - \omega\,\frac{m_b}{m_a}\,\frac{P_{ab}^2}{P_{sa}^2} - \frac{P_{ab}^2}{P_{sa}^2} = 0,$$

which is the stated relation. At $m_s = 0$ the product over ω is $(m_a m_b^5)^3 \neq 0$. At positive masses the left side of the stated relation is positive, so $m_a + \omega m_b$ is real and $\omega = 1$. □

Lemma 9.13. *Let $T = \{a, b, s\}$ be a triangle with node v in T_c, and suppose that O lies below v but not below the children of v that contain a, b and s, and that two bodies of T are balanced in colour c with respect to T. Suppose further that there are $e_a, e_b, e_s \in \mathbb{Z}[m]$ and a nonzero $g \in \mathbb{Z}[m]$ such that, at every place in the cell with $g(m) \neq 0$, the node coordinates of these three children are proportional to $(e_a(m), e_b(m), e_s(m)) \neq 0$. Let $u_1 = m_s^2(e_a - e_b)$, $u_2 = m_a^2(e_b - e_s)$, $u_3 = m_b^2(e_s - e_a)$ and*

$$H_W = (u_1u_2 + u_2u_3 + u_3u_1)^3 - 27(u_1u_2u_3)^2.$$

Then $g(m)H_W(m) = 0$ at every place in the cell.

Proof. Suppose that $g(m) \neq 0$, and write the proof for $c = w$. Then $(\eta_a, \eta_b, \eta_s) = \tau(e_a, e_b, e_s)$ at m with $\tau \neq 0$, and the $e_k(m)$ are pairwise distinct. By Lemma 9.8, Φ^w vanishes at all three bodies, and as in the proof of Lemma 9.11, $(G_{ab}, G_{bs}, G_{sa}) = \lambda(m_s, m_a, m_b)$ with $\lambda \neq 0$. So $P_{ab}^3 = \lambda m_s/(\tau(e_a - e_b))$, and cyclically. By (9.3), $\xi_k - \xi_l = P_{kl}/G_{kl}$, and $(\xi_a - \xi_b) + (\xi_b - \xi_s) + (\xi_s - \xi_a) = 0$ becomes $\alpha + \beta + \gamma = 0$ for $\alpha = P_{ab}/m_s$, $\beta = P_{bs}/m_a$ and $\gamma = P_{sa}/m_b$. Here $\alpha^3 = \lambda/(\tau u_1)$, $\beta^3 = \lambda/(\tau u_2)$ and $\gamma^3 = \lambda/(\tau u_3)$. From $\alpha + \beta + \gamma = 0$ we get $\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma$, and cubing gives $(u_1^{-1} + u_2^{-1} + u_3^{-1})^3 = 27/(u_1u_2u_3)$, that is, $H_W = 0$. □

In the cells where we use Lemma 9.13, the coordinates of the children of v other than the O-child satisfy the linear relations (9.2) (if v is the root) and those of Lemma 9.1 at v, with coefficients in $\mathbb{Z}[m]$. Over $\mathbb{Q}(m)$ their solutions, restricted to the three children of T, form a line. Linear algebra over $\mathbb{Q}(m)$ then gives e and g as in the lemma. The relations of Lemma 9.1 at v are those of the following sets. Call a body k a *candidate* if $\mathrm{meet}_c(k, O) = v$ and the position

of k is minimal. For a candidate k, let K be the set of candidates reached from k by chains of minimal forces between candidates. If every minimal force on a body of K is the force of a body of K, Lemma 9.1 gives $\sum_{k\in K} m_k X^c_{v,u(k)} = 0$, where k lies below the child $u(k)$ of v. The counts of Proposition 9.23 use exactly these relations, and these sets in Lemma 9.22(ii).

Lemma 9.14. *Let $T = \{a, b, s\}$ be a triangle with nodes v in T_c and v' in $T_{c'}$. Suppose that s lies below the O-children of v and of v' or shares the leaf of O, and a and b do not; that s is balanced in colour c with respect to T; that A^c_a and $A^{c'}_a$ consist of the position of a and the forces of s and b on a; and that there are $M_A, M_B \in \mathbb{Z}[m]$ and a nonzero $g \in \mathbb{Z}[m]$ such that $M_A\hat{c}_a + M_B\hat{c}_b = 0$ at every place in the cell with $g(m) \neq 0$. Then $g(m)\big(m_a^2M_A^4 - m_b^2M_B^4\big) = 0$ at every place in the cell.*

Proof. Suppose that $g(m) \neq 0$, and put $Y_k = \hat{c}_k$ and $X_k = \hat{c}'_k$ for $k = a, b$. By (9.1), the leading coefficients of $c_a - c_s$, $c_a - c_b$ and $c_s - c_b$ are Y_a, $Y_a - Y_b$ and $-Y_b$, and in colour c' they are the same with X in place of Y. The initial forms of the two equations of a give $Y_a(1 - m_sP^3_{sa}) = m_bP^3_{ab}(Y_a - Y_b)$ and $X_a(1 - m_sP^3_{sa}) = m_bP^3_{ab}(X_a - X_b)$. Since $Y_a \neq Y_b$, the factor $1 - m_sP^3_{sa}$ is not zero, and dividing gives $X_bY_a = X_aY_b$. If $M_B(m) = 0$, then $M_A(m)Y_a = 0$, so $M_A(m) = 0$ and the claim holds. Otherwise $Y_b = -rY_a$ and $X_b = -rX_a$ with $r = M_A/M_B$. The initial form of the c-equation of s is $m_aP^3_{sa}Y_a + m_bP^3_{sb}Y_b = 0$, so $\rho = P_{sb}/P_{sa}$ satisfies $\rho^3 = m_a/(m_br)$. The distance equations $P^2_{sa}X_aY_a = P^2_{sb}X_bY_b = 1$ give $\rho^2 = r^{-2}$. Hence $\rho = \rho^3/\rho^2 = m_ar/m_b$, and $\rho^2 = r^{-2}$ becomes $m_a^2r^4 = m_b^2$. □

At positive masses $m_a^2M_A^4 = m_b^2M_B^4$ says $m_aM_A^2 = m_bM_B^2$. In the cells where we use Lemma 9.14, M_A, M_B and g come from (9.2) and Lemma 9.1 at v, as after Lemma 9.13.

The last lemma of this subsection concerns a body that sees a cluster of bodies as one point.

Lemma 9.15. *Let s be a body and K a set of at least two bodies such that A^c_s consists of the forces of the bodies of K on s. Suppose that in each tree there is a node v with $\operatorname{meet}(s, k) = v$ for all $k \in K$, and that the bodies of K lie below one child of v. Then $\sum_{k\in K} \varepsilon_k m_k = 0$ at every place in the cell, for some signs $\varepsilon_k = \pm 1$.*

Proof. In each colour the leading coefficient of $c_s - c_k$ is the same for all $k \in K$, say γ in colour c and γ' in colour c', and so are the exponents. By the distance equations, $P^2_{sk}\gamma\gamma' = 1$, so $P_{sk} = \varepsilon_kP$ for a common P and signs ε_k. The initial form of the c-equation of s is $-P^3\gamma\sum_k \varepsilon_k m_k$. □

The product of the sums $\sum_k \varepsilon_k m_k$ over all choices of signs is a nonzero polynomial. At positive masses the signs are not all equal. For example, for $K = \{a, b\}$ the lemma says $m_a = m_b$.

9.4. **Cliques.** A set K of at least three bodies is a *zw-clique* if all its pairs are zw-edges, and a *z-clique* if all its pairs are z-strokes and none is a zw-edge, and likewise for w. For cliques the hypotheses of Theorem 9.9 can be read off the argmin sets in another way, which also covers four bodies.

Lemma 9.16. (a) *On the pairs of a zw-clique, $\zeta = \omega = x = 1$.*

(b) *On the pairs of a z-clique K, $\zeta = 3x_K - 2$ and $\omega = 2 - x_K$ for a common $x_K \in [0, 1)$, and $x_K > 0$ if no body of K is z-circled.*

In both cases the bodies of K lie below distinct children of one internal node of T_z, and of one of T_w.

Proof. (a) is (O4) and (O1). (b) By (E3), applied to the triangles of K, the values of ζ, ω and x are the same on all pairs of K, and (O3), (O4) give $\zeta = 3x_K - 2$, $\omega = 2 - x_K$ with $0 \leq x_K < 1$. If $x_K = 0$, then $\zeta = -2$ on the pairs, and $\zeta_{kl} \geq \min(a_k, a_l)$ makes one body of each pair z-circled. The last statement holds because ζ and ω are constant on the pairs of K (Lemma 4.2). □

So a clique K is a set as in §9.3. Its internal order e^c_K is -2 in both colours for a zw-clique, and $e^z_K = -2$, $e^w_K = 2 - 4x_K$ for a z-clique. We call K *c-clean* if $e(Z^c_{tX}) \geq e^c_K$ for all $t \in K$ and all bodies $X \notin K$. A body $t \in K$ is *c-forced* if $e(Z^c_{tX}) = e^c_K$ for some $X \notin K$. We put $f^c_t = \sum_X m_X [Z^c_{tX}]_{e^c_K}$, the sum over these X. A zw-clique is clean in both colours, a z-clique is z-clean, and in these cases t is c-forced if and only if a c-stroke joins t to a body outside K.

Lemma 9.17. *Let K be a clique that is clean in both colours, with $x_K > 0$ if it is a z- or w-clique. Then $e(c_t) \geq e^c_K$ for $t \in K$, the coefficient $\kappa^c = [c_t]_{e^c_K}$ does not depend on t, and*

$$\Phi^z_t = \kappa^z - f^z_t, \qquad \Phi^w_t = \kappa^w - f^w_t.$$

Proof. By cleanliness, every term on the right of the c-equation of t has exponent at least e^c_K, and the internal terms have exponent exactly e^c_K with the coefficients of Φ^c_t. If $e(c_t) < e^c_K$, the coefficient of exponent $e(c_t)$ would be nonzero on the left and zero on the right. So $e(c_t) \geq e^c_K$, and the coefficient of exponent e^c_K gives $[c_t]_{e^c_K} = \Phi^c_t + f^c_t$. The separations inside K have exponents larger than e^c_K, since $1 > -2$, $3x_K - 2 > -2$ and $2 - x_K > 2 - 4x_K$, so $[c_t]_{e^c_K}$ does not depend on t. □

Corollary 9.18. *Let K be a zw-clique, or a z- or w-clique none of whose bodies is circled in the colour of its strokes, of three or four bodies, in a cell of a diagram. Suppose that for each colour c and each $t \in K$ the terms $Z^c_{tt'}$, $t' \in K \setminus t$, belong to the argmin set of the c-equation of t. Let L^c be the span of the vectors e_t of the bodies $t \in K$ whose c-equation has a term Z^c_{tX} with $X \notin K$ in its argmin set, together with the vector $\mathbf{1}$ if the position c_t belongs to the argmin set of the c-equation of some $t \in K$. If K has three bodies, $\dim L^z, \dim L^w \leq 2$ and $\dim L^z + \dim L^w \leq 3$, then at every place in the cell the masses of K satisfy, for some labelling $K = \{a, b, s\}$, the conclusion of Theorem 9.9(i) or (ii). If K has four bodies, $\dim L^z, \dim L^w \leq 1$ and a place lies in the cell, then $m_K \in B_E$ (Proposition 7.8).*

Proof. At a place in the cell the hypotheses say that K is clean in both colours, that the c-forced bodies are those of the vectors e_t in L^c, and that $\kappa^c = 0$ unless $\mathbf{1} \in L^c$. So $\Phi^c = \kappa^c \mathbf{1} - f^c \in L^c$, and $\sum_t m_t \Phi^c_t = 0$ by Lemma 9.8. If $\dim L^c \leq 1$, this forces $\Phi^c = 0$. If K has four bodies, $\Phi^z = \Phi^w = 0$ and (9.3) say that (ξ, η, P) is a solution of $E_4(m_K)$. If K has three bodies and $\dim L^z = 2$, then $\Phi^w = 0$, and either L^z is spanned by $\mathbf{1}$ and e_s, so that $\Phi^z_a = \Phi^z_b$, or L^z is spanned by e_b and e_s, so that $\Phi^z_a = 0$. In the first case Theorem 9.9(ii) applies if $\Phi^z_a \neq 0$, and otherwise Φ^z is a multiple of e_s and vanishes. In the second case $\Phi^z = 0$ by Theorem 9.9(iii). So either Theorem 9.9(ii) applies or $\Phi^z = \Phi^w = 0$. The same holds with the colours exchanged if $\dim L^w = 2$, and $\Phi^z = \Phi^w = 0$ if $\dim L^z, \dim L^w \leq 1$. Whenever $\Phi^z = \Phi^w = 0$, Theorem 9.9(i) applies. □

Proposition 9.19. † *Of the* 91,273 *cells of Proposition 4.5,* 86,938 *have a clique as in Corollary 9.18. They include all cells of* 18 *classes, among them all* 44,247 *cells of diagram* 7 *and all* 32,864 *cells of diagram* 11 *of* [5].

The zw-cliques are read off the diagram, and the other cliques off the argmin sets of each cell. Of the 43 cells of Proposition 5.3, the 11 cells of diagram 64 have a clique of four bodies as in Corollary 9.18, which gives the system E_4 as in Section 7, and the other 32 have no clique as in Corollary 9.18. Of the ten cells whose cores Macaulay2 does not confirm (Section 10), nine have a clique as in Corollary 9.18, while the one of diagram 16 has none.

9.5. **Lemmas with further identities.** The hypotheses of Lemmas 9.20 and 9.21 are relations among leading coefficients at every place in the cell. In the cells where we use them, the relations are initial forms of base identities and of identities of §2.3 whose argmin sets are constant on the cell (Section 5), certified by Lemma 4.6. Lemma 9.22 assumes such an argmin set for a torque.

Lemma 9.20. *Let a, b, s be bodies, and suppose that at every place in the cell the following hold, where $Y_k = [c_k - c_s]$ for $k = a, b$, at a common exponent:*

(i) $m_a Y_a + m_b Y_b = 0$;
(ii) $m_a P_{sa}^3 Y_a + m_b P_{sb}^3 Y_b = 0$;
(iii) $[c'_s - c'_a] = [c'_s - c'_b]$, *at a common exponent.*

Then $m_a^3 + m_b^3 = 0$ at every place in the cell, so the cell is excluded.

Proof. Let ξ be the common value in (iii). By the distance equations, $P_{sk}^2 = -1/(\xi Y_k)$ for $k = a, b$, so $P_{sk}^3 Y_k = -P_{sk}/\xi$, and (ii) becomes $m_a P_{sa} + m_b P_{sb} = 0$. The computation in the proof of Lemma 6.3 now gives $m_a^3 + m_b^3 = 0$. □

In the cells where we use Lemma 9.20, (i) is the initial form of the centre of mass or of a subset sum, (ii) is that of the equation of s, of a torque or of a subset sum, and (iii) holds because a and b lie below one child of the node $\mathrm{meet}_{c'}(s, a) = \mathrm{meet}_{c'}(s, b)$.

Lemma 9.21. *Suppose that at every place in the cell, nonzero leading coefficients $y_1, \dots, y_n$ satisfy r relations $\sum_{j=1}^n M_{ij} y_j = 0$ with $M_{ij} \in \mathbb{Z}[m]$. Then every $n \times n$ minor of the matrix (M_{ij}) vanishes at m if a place lies in the cell.*

Proof. The vector (y_j) is a nonzero vector in the kernel of $(M_{ij}(m))$. □

In the cells where we use Lemma 9.21, the relations are initial forms of subset sums or of the centre of mass whose terms of least exponent are positions of one colour, y_j is the common leading coefficient $X_{v,u}^c$ of the positions below one child u of a node v, and M_{ij} is the sum of their masses in the i-th relation. For example, for two relations with single masses as coefficients, a minor reads $m_a m_b = m_c m_d$.

Lemma 9.22. *Let l_1, l_2, k_1, k_2 be bodies, and suppose:*

(i) *the root of T_c has exactly three children: the leaves of l_1 and of l_2, and the O-child, below which k_1 and k_2 lie;*
(ii) *k_1, k_2 and O lie below three distinct children of the node $v' = \mathrm{meet}_{c'}(k_1, k_2)$ of $T_{c'}$, l_1 and l_2 lie below the O-child of v', and $\{k_1, k_2\}$ is a set as in Lemma 9.1 for the colour c', with both positions minimal;*
(iii) *the argmin set of the torque $T_{\{l_1,l_2\}}$ is constant on the cell and consists of the four terms $\pm m_{l_i} m_{k_j} p_{l_i k_j}^3 c_{l_i} c'_{k_j}$ with $i, j \in \{1, 2\}$.*

Then $m_{l_1} = m_{l_2}$ and $m_{k_1} = m_{k_2}$ at every place in the cell.

Proof. Put $Y_i = \hat{c}_{l_i}$, $X_j = \hat{c}'_{k_j}$ and $P_{ij} = P_{l_i k_j}$. By (9.2) and Lemma 9.1, $m_{l_1} Y_1 + m_{l_2} Y_2 = 0$ and $m_{k_1} X_1 + m_{k_2} X_2 = 0$. By (9.1), the leading coefficients of $c_{k_j} - c_{l_i}$ and of $c'_{k_j} - c'_{l_i}$ are $-Y_i$ and X_j. So the distance equations give $P_{ij}^2 = -1/(X_j Y_i)$, and the initial form of the torque is $\pm \sum_{i,j} m_{l_i} m_{k_j} P_{ij}^3 Y_i X_j = \mp \sum_{i,j} m_{l_i} m_{k_j} P_{ij}$. With $u = -X_1 Y_1$ we get $P_{11}^2 = 1/u$, $P_{12}^2 = -m_{k_2}/(m_{k_1} u)$, $P_{21}^2 = -m_{l_2}/(m_{l_1} u)$ and $P_{22}^2 = m_{k_2} m_{l_2}/(m_{k_1} m_{l_1} u)$. Multiplied by $(m_{k_1} m_{l_1} u)^{1/2}$, the initial form becomes $\varepsilon_{11} A_{11} + i\varepsilon_{12} A_{12} + i\varepsilon_{21} A_{21} + \varepsilon_{22} A_{22}$ with signs ε_{ij} and $A_{ij} = (m_{l_i} m_{k_j})^{3/2} > 0$. Its real and imaginary parts vanish, so $A_{11} = A_{22}$ and $A_{12} = A_{21}$, i.e., $m_{l_1} m_{k_1} = m_{l_2} m_{k_2}$ and $m_{l_1} m_{k_2} = m_{l_2} m_{k_1}$. Hence $m_{l_1} = m_{l_2}$ and $m_{k_1} = m_{k_2}$. □

9.6. **Coverage.**

Proposition 9.23. † *Of the* 91,273 *cells of Proposition* 4.5*:*

(a) 70,406 *are excluded at every positive mass by the following lemmas:* 70,233 *by Lemmas* 9.2–9.7, 172 *by Lemmas* 9.20 (171 *cells*) *and* 9.21 (1), *and one by Lemma* 5.1, *applied to a torque;*

(b) 90,943 *satisfy the hypotheses of one of Lemmas* 9.2–9.7*, Corollary* 9.10*, Lemmas* 9.11(a), (b)*,* 9.12*,* 9.13*,* 9.14 *and* 9.15*, and Corollary* 9.18 *for a clique of three bodies, whose hypotheses are conditions on the trees and the argmin sets of the cell;* 287 *others satisfy those of Lemmas* 9.11(c) (1 *cell*)*,* 9.22 (1)*,* 9.20 (169) *or* 9.21 (116)*; and the remaining* 43 *belong to the diagrams* 64 (10 *cells*)*,* 66 (11)*,* 88 (9)*,* 94 (9)*,* 93 (2)*,* 89 (1) *and* 98 (1)*.*

In each cell of the first two groups of (b) *the lemma gives a contradiction or a nonzero polynomial in the masses that vanishes wherever the cell occurs. So every cell is excluded at all masses outside a proper algebraic subset, by this section or, for the last* 43*, by Sections* 6 *and* 7*.*

Proof. A program checks the hypotheses of the lemmas on each listed cell, reading its trees and argmin sets, and Proposition 9.19 gives the cliques (Table 5). Table 6 lists the results that apply most often. Without Corollary 9.18, this program and a second one, written independently, both find the split 90,919, 311 and 43. Corollary 9.18 adds 24 cells, each with a clique of three bodies and covered by no other lemma of the first group of (b): 22 cells of diagram 48 and one cell each of diagrams 31 and 95. The argmin sets of the subset sums and torques in Lemmas 9.11(c) and 9.22 and in the torque of (a) are certified by Lemma 4.6, and so are those of the identities whose initial forms give the hypotheses of Lemmas 9.20 and 9.21. Lemmas 9.2–9.7 and 9.20 give a contradiction at every positive mass. The polynomials of Corollaries 9.10 and 9.18 and Lemmas 9.11, 9.12, 9.15 and 9.22 are nonzero by their statements. Those of Lemmas 9.13 and 9.14 depend on the cell. In each of the 74 cells in which one of these two lemmas is the only one of the first group of (b) that applies, 16 for Lemma 9.13 and 58 for Lemma 9.14, the polynomial H_W, resp. $m_a^2 M_A^4 - m_b^2 M_B^4$, is not zero at $m = (1, 2, 3, 4, 5, 6)$. The minors of Lemma 9.21 are nonzero polynomials, and in the cell of (a) the minor has positive coefficients. The 43 cells of the last group are excluded by Theorems 6.5, 6.12, 6.15, 6.20, 7.12 and 7.13. They include the 10 cells of diagram 64 in which Corollary 9.18 applies with a clique of four bodies, since its conclusion there rests on Proposition 7.8. □

Unlike Corollary 9.18, Corollary 9.10 needs no strokes: one node in each tree and three balance conditions suffice. The program applies Lemmas 9.20 and 9.21 only to cells in which no lemma of the first group of (b) applies, but these two lemmas also exclude further cells at every positive mass, so the count 70,406 of (a) is a lower bound. Lemma 5.1 excludes at every positive mass the cells outside (a) with an identity that has a single term of least exponent on Π (Proposition 5.3), and Theorem 6.5 and Proposition 6.7 exclude at every positive mass the cells of diagram 89 and further cells of diagram 82, none of them in (a) or among the cells excluded by Lemma 5.1. Table 6 counts these cells: Sections 5, 6 and 9 exclude at least 71,235 cells at every positive mass.

Proposition 9.23 rests on four computations: the list of cells with their argmin sets (Proposition 4.5), the programs that check the hypotheses of the lemmas on each cell (Table 5), the certified argmin sets of further identities for the 287 cells and, in (a), for three cells of diagram 82, and the computations of Sections 6 and 7 for the 43 cells (Tables 3 and 4). These 43 cells, however, differ from those of Proposition 5.3 in two cells: Lemma 9.4 excludes one cell of diagram 64 at every positive mass, and no lemma here was found to apply to one cell of diagram 93, which Proposition 5.3 excludes by a core.

10. Verification

The proofs of the statements marked † include computations. All other proofs are by hand. Tables 2–5 list the computations, the software on which they depend and the checks made of them. The software is Singular [9], msolve [4], gfan [15], FLINT [10] and its ball arithmetic Arb

[17] through python-flint, Macaulay2 [11], HiGHS [14], SymPy [20] and mpmath, and for §10.1 Lean 4 with Mathlib. The programs, certificates and logs are in the supplementary material, available at `https://doi.org/10.5281/zenodo.23472605`, which also gives the versions of the software. It holds the programs of Sections 3–5 at one pinned version, with the list of changes made after the runs, none of which affects their results.

Three remarks qualify the tables. First, as noted in Section 5, the cores of Proposition 5.3 were computed and certified without saturation: the saturations that the programs request, by E and the leading coefficients of the positions in the search and by the factors of Θ in the variables of the core in the certification, were not carried out. There are 2,413 certified cores: 1,423 from the first pass, counted as distinct within each class, and 990 from the second. The cells with a core are checked in Lean through Lemma 5.2(a) (37,824 cells) or Lemma 5.2(b) (974 cells). Macaulay2 confirms all but ten of the cores, which are among those of the second pass and belong to eight cells of diagram 7 and to one cell each of diagrams 14 and 16. The identities checked in Lean (§10.1) remove the Gröbner basis computations from the trusted base of the exclusion of the 38,798 cells with a core. They include the ten cells above, six of them through another subset of the same initial system.

Second, msolve prints the leading monomials of the Gröbner bases of Lemma 6.19(b), not the bases, so the upper bounds for the dimensions cannot be checked from its output. They are checked by Lean's kernel, from traces of the runs of an F4 algorithm written in Lean (§10.1). The matching lower bounds need no Gröbner basis: the boxes of (c) and (d) give 20,316 and 4788 distinct zeros at μ_0, respectively, and the fibre dimension of the finite module in Lemma 6.18 is upper semicontinuous.

Third, the programs that generate the identities, the exponents of their terms and the initial forms, and those that check the certificates of Lemma 4.6, are tested as in Table 2. The former compute the exponents in binary floating point, which is exact here because all their coefficients are small dyadic rationals, and the latter use exact rational arithmetic. Apart from these exponents and from decisions of feasibility, the computations of the proof use floating point only to propose certificates or approximate zeros, which are then checked in exact or ball arithmetic. Feasibility is not certified in the runs of the linear programs, but an error there could only add a cell to the list or remove an identity from an initial system.

10.1. **The formal proof in Lean.** We have formalized Theorem 1.1 in Lean 4 with Mathlib. The formal proof is the theorem `CC6Compose.V4.main_theorem_v4`, whose statement is `CC6Full.MainTheorem`. This Lean theorem has no hypotheses. The definitions are those of the file `CC6Full/Statement.lean`, shown here without comments and with one line break changed (namespace `CC6Full`, with `open Finset`):

```
abbrev Conf (n : ℕ) := Fin n → ℂ
def CollisionFree {n : ℕ} (q : Conf n) : Prop := ∀ i j, i ≠ j → q i ≠ q j
def IsCC {n : ℕ} (m : Fin n → ℝ) (q : Conf n) : Prop :=
  CollisionFree q ∧ ∃ (lam : ℝ) (c : ℂ), ∀ i,
    ((lam * m i : ℝ) : ℂ) * (q i - c) =
      ∑ j ∈ univ.erase i, ((m i * m j / ‖q i - q j‖ ^ 3 : ℝ) : ℂ) * (q i - q j)
def Similar {n : ℕ} (q q' : Conf n) : Prop :=
  ∃ a b : ℂ, a ≠ 0 ∧
    ((∀ i, q' i = a * q i + b) ∨ (∀ i, q' i = a * starRingEnd ℂ (q i) + b))
def FinitelyManyCC {n : ℕ} (m : Fin n → ℝ) : Prop :=
  ∃ F : Finset (Conf n), ∀ q, IsCC m q → ∃ q' ∈ F, Similar q' q
def GenericFiniteness (n : ℕ) : Prop :=
  ∃ F : MvPolynomial (Fin n) ℤ, F ≠ 0 ∧
    ∀ m : Fin n → ℝ, (∀ i, 0 < m i) → MvPolynomial.aeval m F ≠ 0 → FinitelyManyCC m
def MainTheorem : Prop := GenericFiniteness 6
```

statement	computation and trusted software	checks
Prop. 3.1	exhaustive search (Python)	41 diagrams in 5 classes for $n = 4$; 1376 in 24 classes for $n = 5$, as in the kernel-checked search of [27, §12.4]; contains the 117 of [5]. An enumeration that assembles the diagrams from their connected components, written from the statements of the rules independently of the program, finds the same diagrams for $n = 4, 5, 6$, class by class
Prop. 4.5	depth-first search in 384 parts per class; linear programs solved in floating point (HiGHS); each of the 76,087,734 prunes certified by Lemma 4.6 in rational arithmetic, from a dual solution, a dual ray or an exact simplex method in python-flint. The count includes the repetitions of the first two insertions in every part	a rerun of the same programs on seven classes (33,502 cells) reproduces the cells exactly. Deliberately invalid certificates are rejected. The smallest slack is 0.0055 over all programs judged feasible and 0.0345 over the listed cells. The largest optimal slack of a pruned program is $1.4 \cdot 10^{-14}$
Prop. 5.3, argmin sets	184,997,938 decisions on argmin sets, each certified by Lemma 4.6	the generators of the identities and initial forms used by the programs, evaluated at three complex solutions for six bodies, sum to less than $6 \cdot 10^{-14}$, and the exponents of 784,228 terms agree with the sums of the exponents of their factors; in Lean (§10.1): each single term and each argmin set used by a core certified again
Prop. 5.3, cores	cores searched modulo 32003 (not trusted) and certified by standard bases over $\mathbb{Q}(m)$, in the second pass with one mass set to 1, without saturation (Singular)	Macaulay2 confirms 2,403 of the 2,413 cores: 2,394 with all six masses as parameters and 5 with one mass set to 1, both with the saturation by Θ through an extra variable t and the polynomial $t\Theta - 1$, 3 by checking in polynomial arithmetic cofactors, computed in Singular with one mass set to 1, that express 1 in the ideal generated by the core and $t\Theta - 1$, and 1 by showing, with one mass set to 1, that the core itself generates the unit ideal. It corroborates the other ten at two random rational mass points each. For 40 random cores of the first pass, cofactors of the same kind with all six masses as parameters, at most 1.7 MB, pass the same check. For five bodies, the initial system of the cell of diagram 5 of [2, Fig. 11] that contains a place of the continuum of [25] does not generate the unit ideal at the masses $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{8})$, proportional to the masses $(1, 1, 1, 1, -\frac{1}{4})$ of the continuum, modulo 32003 and over $\mathbb{Q}$, and generates it at random masses
Prop. 5.3, identities	identities $\sum_i c_i f_i = h$ with integer polynomials for 1,928 distinct cores (1,900 of the 1,935 distinct stored cores and 28 other subsets of initial systems, §10.1), computed by Singular (`lift`, `liftstd` or elimination) and checked in Lean	in Lean (§10.1): each identity checked by expansion, with the initial forms recomputed from the trees

TABLE 2. Computations of Sections 3–5 and their checks.

statement	computation and trusted software	checks
Lemma 6.9	hand proof	eliminations in Singular at $(1,2,7)$, $(1,2,8)$ and $(2,3,11)$, compared with S_μ in SymPy, and in a second formulation at $(1,2,7)$, $(2,9,4)$ and $(3,5,13)$
Lemma 6.13	hand proof	at $(1,2,40)$: the bounds of the first statement in exact arithmetic, and the roots and ρ numerically (SymPy, mpmath); at $(1,2,7)$: root enclosures and bounds in Arb (python-flint) and in exact rational arithmetic with Rouché disks; a third program in mpmath; a direct test of all signed sums, in the Arb program
Thm. 6.15	factorization in $\mathbb{Q}[\mu]$ (Singular)	the terms printed in [2, p. 574], and a nonzero value at $(1,2,7)$ (SymPy)
Lemma 6.19(a)	initial forms and coefficients (SymPy); prevarieties (gfan 0.7); cone tests (Singular)	a separate program computes the prevarieties with exact certificates and finds the same 2149 and 224 face vectors, with all cone tests redone in msolve. As a check that the tests can find nonempty cones, both find six for S_{12} at the masses $(1,1,2,2)$, and the separate program finds one for KC at $(1,1,1,5)$
Lemma 6.19(b)	systems written by SymPy; Gröbner bases modulo p (msolve)	in Lean (§10.1): the upper bounds, checked by the kernel from traces of runs of an F4 algorithm written in Lean, with bases that have the same leading monomials as those of msolve. Outside Lean: the same leading ideal modulo $2^{31}-1$ from msolve, 20,316 and 4788 from Macaulay2 (F4), and 4788 from Singular
Lemma 6.19(c), (d)	Krawczyk test in Arb	interval arithmetic in mpmath with forward differentiation (for (d) the square roots are bounded with a safety factor instead of outward rounding); a third, rigorous implementation from separately derived equations. Both check that the boxes are pairwise disjoint
§6.7, last paragraph	the test of Lemma 6.17(a) over the bodies $1,2,3$ on the boxes of Lemma 6.19(d), in interval arithmetic (mpmath); $\lvert Y_4\rvert$ and $\max_{l\le 3}\lvert Y_l/Y_4\rvert$ at the 36 solutions with relative margin below 10^{-4}, at the centres of the boxes	the same count in floating point at the centres of the boxes

TABLE 3. Computations of Section 6 and their checks.

The plane is $\mathbb{C}$, and body k of the paper is `k - 1 :  Fin n`. `IsCC m q` is (1.1) for a configuration of distinct points, with $\lambda \in \mathbb{R}$ and $c \in \mathbb{C}$: the sum over `univ.erase i` runs over $j \neq i$, and $\|q_i - q_j\|$ is the distance. Since `starRingEnd` $\mathbb{C}$ is complex conjugation, `Similar` admits reflections. This does not change finiteness, because the complex conjugate of a central configuration is again one. `MvPolynomial.aeval m F` is the value $F(m)$, so `MainTheorem` is Theorem 1.1. The theorem `exists_pos_aeval_ne_zero` shows that a nonzero $F \in \mathbb{Z}[m_1,\dots,m_n]$ does not vanish at some positive mass vector, so the statement is not vacuous. The formal statement does not

statement	computation and trusted software	checks
Lemma 7.7	hand proof; the resultants R_1, R_2, $\mathrm{Res}_t(R_1, R_2')$ of the proof (SymPy)	at $(1,1,1,1)$: standard bases over $\mathbb{Q}(\mathrm{j})$ and $\mathbb{Q}$ (Singular), ranks modulo p (python-flint)
E_4 at $(3,5,7,11)$ (§7.2)	ranks modulo p (python-flint)	standard bases over $\mathbb{Q}(\mathrm{j})$ and $\mathbb{Q}$ (Singular); Macaulay2
Lemma 7.9	unit ideals modulo 32003 (Singular)	msolve modulo 1073741789; Macaulay2 modulo 1000003
C_4 at $(1,1,1,1)$ (§7.4)	unit ideals modulo 32003 (Singular)	msolve modulo 1073741789
Rem. 7.14	standard bases over $\mathbb{Q}$, with saturation (Singular)	ranks of Macaulay matrices modulo p (python-flint) are deficient for the same numbers of patterns

TABLE 4. Computations of Section 7 and their checks.

statement	computation and trusted software	checks
Prop. 9.19	count of the cliques of Corollary 9.18 in the stored cells (Python)	rerun from the stored cells; checked in Lean (§10.1)
Prop. 9.23	checks of the hypotheses of the lemmas of Section 9 on the stored cells (Python); argmin sets of further identities certified by Lemma 4.6; the polynomials of Lemmas 9.13 and 9.14 in the 74 cells where they are used (SymPy); the minors of Lemma 9.21 at three mass points	a second program, written from the statements of the lemmas independently of the first, puts every cell in the same group as the first for the lemmas other than Corollary 9.18 and recertifies the argmin sets of the further identities in rational arithmetic. On the polyhedra of the second program, a linear program in floating point confirms the conclusions of Lemmas 9.2–9.4 in every cell where they apply. Eliminations with saturation in Singular give the polynomials of the triangle lemmas

TABLE 5. Computations of Section 9 and their checks.

result	cells
Corollary 9.18	86,938
Corollary 9.10	86,875
Lemma 9.11(a), (b)	79,318
Lemma 9.12	54,208
Lemma 9.4	46,289
Lemma 9.2	40,084
Corollaries 9.18 and 9.10, both	84,155
at least one of Corollaries 9.18 and 9.10	89,658
cells excluded at every positive mass, at least	71,235
by Proposition 9.23(a), a lower bound	70,406
by Lemma 5.1, other cells	771
by Theorem 6.5, diagram 89	49
by Proposition 6.7, further cells of diagram 82	9

TABLE 6. The results of Section 9 that apply most often, with the numbers of cells of Proposition 9.23 to which they apply, and the cells that Sections 5, 6 and 9 exclude at every positive mass.

describe F. In particular, what Section 1 and Remarks 6.14 and 7.14 say about the zero set of F, the polynomial of Section 8, is not formalized.

The formal proof has three parts. The first is the framework `CC6Full`, which formalizes the theory of Sections 2–8. For any number of bodies it constructs, from infinitely many central configurations, a *germ* with $\ell \leq 0$, following §§2.1 and 2.2. A germ is a solution of the equations of the bodies and the distance equations in Hahn series, with $U = E \neq 0$ and with exponents normalized as in Lemma 2.5, and here it is given by the rotated expansions at a singular place. For germs, again for any number of bodies, the framework proves the identities of §2.3, Lemma 2.10, the relations (O1)–(O6), the rules of Lemma 2.11, the existence part of Lemma 4.2, Lemma 4.3(b) and (c), and Lemmas 5.1 and 5.2. For six bodies it proves the results of Sections 6 and 7 that Section 8 uses, with the computations at single mass points as hypotheses, and the reduction of Section 8 to the classes of diagrams. The second part consists of the two certificate projects, which check the computations of Sections 3–5 and of Sections 6 and 7, respectively. The third is the package `CC6Compose`. It proves the bridges from the statements checked in the certificate projects to the hypotheses of the framework, which are statements about germs and polynomial systems, and it composes the main theorem. The bridges cover Propositions 3.1, 4.5 and 5.3, the check that the representatives of the seven classes with the 43 remaining cells are diagrams of Table 1, Lemmas 6.13, 6.19 and 7.9, and the computation at $(3, 5, 7, 11)$ in §7.2. The README of the framework lists, for each of these results, the certificate check and the bridge. Fourteen modules of `CC6Compose` that the formal proof does not import contain other assemblies of the theorem through checks by compiled code; two of them use the tactic `native_decide`.

The formal proof departs from the proofs in this paper at the following places.

- Theorems 6.12, 6.15 and 6.20 rest on a lemma of integral elimination instead of resultants, constructible sets and the implicit function theorem. If the unknowns of a polynomial system over $\mathbb{Q}[\mu]$ are integral over $\mathbb{Q}[\mu][1/g]$ modulo the system, and the system has no complex solution at a rational point μ_0 with $g(\mu_0) \neq 0$, then some $h \in \mathbb{Q}[\mu]$ with $h(\mu_0) \neq 0$ vanishes wherever the system has a solution and $g \neq 0$. For diagrams 98 and 93 the integrality comes from the roots of S_μ and the square roots c_{jl}, respectively $\sqrt{Y_l}$, so the polynomials Φ and Φ_3 and the resultants are not used. For diagrams 88 and 94 it comes from Lemma 6.18, and the empty fibre at the mass point $(\mu_0; m_5, m_6)$, with $\mu_0 = (1, 2, 3, 30)$ and with $(m_5, m_6) = (1, 1)$ for diagram 88 and $(1, 2)$ for diagram 94, replaces the implicit function theorem, the continuity of the tests and the density of real points.
- Propositions 7.8 and 7.11 rest on a Macaulay transfer instead of the closedness of projections along projective space. If at μ_0 every monomial of some degree, or bidegree, lies in the ideal of the forms, then some maximal minor $h(\mu)$ of the Macaulay matrix is nonzero. Wherever $h \neq 0$, every monomial of that degree then lies in the ideal, so the forms have no common projective zero. This gives the polynomials h_E and h_C of Section 8 without showing that B_E and B_C are closed. The minor h is not computed.
- In place of Lemma 7.7, which the certificate project also checks, the formal proof uses the computation at $(3, 5, 7, 11)$ reported after Proposition 7.8: the forms $\mathcal{F}_\varepsilon$ and $\mathcal{G}_\sigma$ have no common nonzero zero there, and the Nullstellensatz gives the monomials for the Macaulay transfer.
- Lemma 6.18 is proved with valuation rings instead of discrete valuation rings, since Mathlib provides valuation subrings that dominate a local ring but not discrete valuation rings that dominate a Noetherian local domain. The value group is then an arbitrary ordered abelian group. The tropical certificates of Lemma 6.19(a) enter in torus form:

the initial forms have no common zero in the torus over any field of characteristic p. The certificate project proves this for every ordered value group.

- The identities of §2.3, Lemma 2.10, the relations (O1)–(O6) and the rules of Lemma 2.11 are proved for germs, not for the expansions at a place.
- Lemma 7.9 is used over $\mathbb{F}_{32003}$ only. The lemma states the unit ideals on the 14 charts over $\mathbb{F}_{32003}$ and over $\mathbb{F}_{1073741789}$, but Lemma 7.10 needs one prime, and the computation over $\mathbb{F}_{1073741789}$ is an independent check that the formal proof does not use.
- The checks by the kernel take their inputs as Lean literals rather than as data files: the 191 representatives of the classes, the certificates of the cores, and the sets G and the cofactors of Lemma 6.19(a). Checks by compiled code outside the formal proof relate these literals to the data files. No statement or proof of the formal proof reads text.
- Propositions 4.5 and 5.3 are proved as one statement, checked by the kernel (see the list below). The cells of Proposition 4.5 appear only inside that check.
- The test of Lemma 6.17 uses $|X_l|^{-1/2}$. In Lean $\|0\|^{-1/2} = 0$, so $X_l \neq 0$ is a separate hypothesis, which the certificate project checks at the zeros of Lemma 6.19(c) and (d).

The formal proof depends only on the three standard axioms `propext`, `Classical.choice` and `Quot.sound`, as `#print axioms` shows. There is no other axiom and no `sorry`. In particular the formal proof does not use the tactic `native_decide`, which would add an axiom. So Lean's kernel checks the whole proof. It checks the computations by evaluation (the tactic `decide +kernel`).

A script of the framework builds the framework and the composition from scratch on top of the kernel checks of the two certificate projects. It checks that `CC6Compose.V4.main_theorem_v4` depends on exactly these three axioms and that 47 main theorems of the framework, `exists_pos_aeval_ne_zero` among them, depend on no others, and it scans the sources, with the exemptions listed in the README of the framework, for keywords that could add an assumption or bypass the kernel. The kernel checks it uses come from earlier builds, which were verified against the committed sources and hashes before the script ran; the README of the framework describes these builds and their cost.

Each certificate project has a file `LEAN_CHECKS.md` that lists the Lean theorems by the statements of this paper that they check, together with what each theorem assumes and what it costs. A check has two parts: a program on integers, rationals, lists and polynomials, and a theorem proved in Lean that turns the output of the program into a statement over $\mathbb{C}$, $\mathbb{Z}/p$ or a ring of polynomials. In these checks floating-point code is untrusted: it supplies centres, pivots, cofactors and certificates, and every test is exact. For the kernel the programs are written as structural recursions on lists and numbers, the data are given as Lean values rather than strings, and the untrusted parts of a computation, such as the rows and inverses of the Macaulay checks and the cofactors of the unit ideals, are given as data. The formal proof uses the following checks.

- Proposition 3.1: the search, split into parts that the kernel checks one by one, lists every labelled diagram that satisfies the rules, and the 82,712 diagrams are the disjoint union of the orbits of the 191 representatives, which the check takes as a literal list.
- Lemma 4.6, proved over every ordered field.
- Propositions 4.5 and 5.3 together: for each class, and each pair of trees with heights that satisfies the conditions of Lemma 4.3 for the representative and $\ell(x) \leq 0$, either the class is one of the seven classes that contain the 43 remaining cells, or some identity has a single term of least exponent, as in Lemma 5.1, or the initial forms f_i at the pair, written from the trees, satisfy one of 1,928 identities $\sum_i c_i f_i = h$ with integer polynomials, as in Lemma 5.2, where h is a polynomial in the masses that is not zero at an explicit

point. The kernel runs a form of the search of Proposition 4.5 written as total functions and proved in Lean to find every such cell, decides its linear programs by certificates accepted by Lemma 4.6, checks the claims in the cells it finds, and checks the identities by expansion. For 35 cells the identity is for another subset of the initial system of the same cell, since Singular did not lift the stored core within the limits of time and memory tried. The framework and the bridges prove these conditions for germs, so this applies to the trees of a germ with $\ell \leq 0$.

- The representatives of the seven classes that contain the 43 remaining cells are diagrams of Table 1, by evaluation in the kernel on the literal list of representatives.
- Section 6: Lemma 6.13 at $(1, 2, 40)$, in the form that the signed sums do not vanish; the hypotheses of Lemma 6.18 in Lemma 6.19(a): that G lies in the ideal and that its coefficients are units, by evaluation in the kernel, and that the initial forms have no common zero in the torus for all weight vectors with values in any ordered abelian group, from the covering of the space of weights by polyhedral cells, checked by the kernel, with the system of each cell ruled out by cofactors checked by the kernel, except for the cell of S_{12} along which the Y-positions escape, where a trace of the Buchberger algorithm gives a chain of 2,044 identities over earlier polynomials, which the kernel checks; the boxes of Lemma 6.19(c) and (d), with $X_l \neq 0$ on each box, checked by the kernel.
- Section 6, the upper bounds in Lemma 6.19(b): an F4 algorithm written in Lean was run modulo 65521 on $S_{12}(\mu_0)$ and on $\mathrm{KC}'(\mu_0)$, and the traces of the runs were pruned to what the final polynomials need. Each new polynomial of a trace comes with a combination of monomial multiples of earlier polynomials, and the kernel checks that the two are equal. The inputs of each trace are, up to scalars, the polynomials of its system, as the kernel also checks. So the final polynomials of the first trace lie in $\langle S_{12}(\mu_0)\rangle$, and those of the second in $\langle \mathrm{KC}'(\mu_0)\rangle$. The kernel then checks that their leading monomials leave 20,316 standard monomials for S_{12} and 4788 for KC'.
- Section 7: the computation at $(3, 5, 7, 11)$ after Proposition 7.8, a Macaulay check arranged for the kernel, and Lemma 7.9 over $\mathbb{F}_{32003}$, from cofactor certificates with 17,721,624 terms on the 14 charts, checked by the kernel.

The certificate projects also contain checks of the same computations by compiled code, among them Lemma 7.9 over both primes, and they check computations of our proofs outside the formal proof. A replica of the search of Proposition 4.5 in Lean writes exactly the 191 stored lists of cells, with a certificate accepted by Lemma 4.6 for each of the 76,087,734 prunes and an exact rational point in each listed cell. Each stored cell has an identity with a single term of least exponent, has a core, or is one of the 43. Also checked are the counts of diagrams for four and five bodies and the comparison with the 117 diagrams of [5], the census of Proposition 9.19, Lemma 6.9 and its check at three mass points, the check of Lemma 6.13 at $(1, 2, 7)$, the value $P_{93}(1, 2, 7) \neq 0$ in Theorem 6.15, the counts at the end of §6.7, Lemma 7.7, Remark 7.14 and the computation at $(1, 1, 1, 1)$ in §7.4. So are algebraic identities and finite computations inside the proofs by hand in Sections 2–4, 6, 7 and 9, as lemmas over rings and fields. Of these, only the counts of patterns and tournaments and the resultants $\mathrm{Res}_t(R_1, R_2')$ inside the proof of Lemma 7.7 use `native_decide`. In these checks the Lean functions are written from the definitions of the paper, without a bridge to the framework.

The formalization omits Section 9 and the treatment of diagrams 81, 82 and 90 in Section 6.[1]

[1]Neither checked in Lean nor used in the proof of Theorem 1.1 are the counts of Proposition 9.23 and the checks in its proof, the comparison of the other 74 classes with the further rules of [5] and with the triangle rule in Section 3, the kinds of prune certificates and the slacks in Table 2, the number of decisions on argmin sets and the checks other than those in Lean in the rows of Proposition 5.3 of that table, the factorization of P_{93} and its comparison with [2], the eliminations in the check of Lemma 6.9 (whose results the Lean check compares with

Data availability

The supplementary material, with the programs, the Lean projects, the certificates, the data and the logs, is available at `https://doi.org/10.5281/zenodo.23472605`. The programs are also in the repository `https://github.com/tejasvi/smale6-six-bodies-generic`, at commit `84bbcf276d029fdf31375c4a21c6a7e136955e51`. The DOIs `https://doi.org/10.5281/zenodo.23433406` (paper) and `https://doi.org/10.5281/zenodo.23433409` (supplementary material) refer to all versions and resolve to the latest one.

Acknowledgements

Generative AI tools were used to write code, run analyses and draft text; the author directed the work and takes full responsibility for its content.

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