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OSTERWALDER–SCHRADER AXIOMS FOR SCALAR QED$_3$ IN THE COULOMB PHASE

github.com/marikgoldstein/scalar-qed-existence*

Abstract. King constructed the continuum, infinite-volume limit of three-dimensional scalar quantum electrodynamics, the $U(1)$ (abelian) Higgs model, along Neumann–Dirichlet cubes at block factor 5 [25, 26], and listed its regularity, rotation invariance and ergodicity as open [26, p. 326]. For his state at fixed photon mass, on his whole coupling range, we prove the Osterwalder–Schrader axioms for the curvature: regularity, invariance under the full Euclidean group, reflection positivity about every plane, symmetry, and the cluster axiom on the ergodic components of its measure. Besides the renormalisation group, the main tools are the correlation inequalities of Brydges, Fröhlich and Seiler [9] and Gaussian domination. Gaussian domination, a diamagnetic bound on the photon field by its free field, gives the regularity of the curvature with absolute constants, independent of the couplings and of the photon mass. Hence King's states converge along subsequences as the photon mass goes to zero; the limits satisfy the same axioms. The correlation inequalities make the cube states monotone in the cube on a cone of observables; they give the infinite-volume limit and the comparisons of cubes, boxes and tori used for reflection positivity and rotation invariance. We prove the existence of King's cube limits from his renormalisation-group analysis, and rotation invariance by comparing two lattices whose orientations differ by $\arctan(3/4)$. At fixed photon mass, with real scalar sources and under a one-sided sign clause on the couplings, which we prove on explicit regions, the joint field of the curvature and the renormalised Wick square satisfies, for Wick amplitudes below a threshold, the axioms of regularity, symmetry, reflection positivity about the coordinate planes and, outside an exceptional set of couplings, Euclidean invariance.

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1. Introduction

1.1. **The model and the problem.** The abelian Higgs model, scalar quantum electrodynamics, is the Euclidean field theory of a real vector field A_μ and a complex scalar field ϕ on $\mathbb{R}^3$ with formal action

$$S(A,\phi) = \int_{\mathbb{R}^3} \Big[\tfrac{1}{4} F_{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}\mu_0^2 A_\mu A_\mu + \tfrac{1}{2}|(\partial_\mu + \mathrm{i}eA_\mu)\phi|^2 + \tfrac{1}{2}m^2|\phi|^2 + \lambda|\phi|^4 \Big]\, dx, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{1}$$

in the normalisation of King [25, (2.1)–(2.4)]; on the lattice the photon is written in Feynman form, with the gradient terms of each of its components, and carries the mass $\mu_0 > 0$ (Section 2). In three dimensions e^2 and λ have the dimension of a mass and the model is superrenormalisable: one mass counterterm δm_ε^2 for the scalar, one number per spacing, renormalises it. For $m^2 < 0$ the scalar condenses and the photon acquires a mass by the Higgs mechanism; for $m^2 > 0$ the scalar is massive and the photon is expected to stay massless. In the language of superconductivity (the abelian Higgs model is the relativistic field-theoretic version of its Ginzburg–Landau theory), the first regime, the *Higgs phase*, is the superconducting state and the second, the *Coulomb phase*, the normal state. The two phases are treated in two papers: the Higgs phase in the companion paper [32], the Coulomb phase here, where the bare mass m is positive throughout and the photon mass $\mu_0 > 0$, the infrared regulator of [1, 2, 3, 4, 25, 26], is kept fixed. None of our

proofs uses the phase. One-loop perturbation theory predicts a fluctuation-induced first-order transition at very small quartic coupling [13, 21]; no claim about the phase diagram is made here.

The gauge field is non-compact: on the lattice A is real-valued on the bonds and the gauge action is a quadratic form, as in Bałaban [1, 2, 3, 4] and King [25, 26]. For observables of integer charge on a finite lattice with free boundary conditions the non-compact model is the compact Villain model with every monopole removed [5]. The choice matters in the Coulomb phase: in the compact pure gauge theory in three dimensions monopoles produce a mass gap and confinement at every coupling [19], while the non-compact pure gauge theory is the free field, whose photon is massless. Masslessness of the photon as $\mu_0 \to 0$ is the expectation for the model with matter; we work at fixed μ_0, except in Sections 30 and 31.

The problem treated here is due to King [25, 26]: with the renormalisation-group method of Bałaban [1, 2, 3, 4] he constructed the continuum limit of the model in finite volume and then its infinite-volume limit, through cubes with Neumann conditions for A and Dirichlet conditions for ϕ, proved its translation invariance and reflection positivity, and addressed its rotation invariance through a comparison of two rotated lattices. The details, and the earlier work on the model, are recalled in Section 1.4. The present paper establishes the Osterwalder–Schrader axioms for his state, rotation invariance included, by a different route.

The question is whether the Schwinger functions of the local gauge-invariant fields, the *curvature* $F = dA$ and the renormalised *Wick square* of ϕ, can be constructed as limits of the lattice regularisation and satisfy the axioms of Osterwalder and Schrader [29, 30]: regularity (OS0), Euclidean invariance (OS1), reflection positivity (OS2), symmetry (OS3) and clustering (OS4). Translations, the cubic group and reflection positivity are properties of the lattice model that have to be carried through the cube limits (King's cube itself is not reflection symmetric, Remark 9.7); invariance under rotations is a symmetry of no lattice theory and has to be recovered in the limit.

This paper proves the Osterwalder–Schrader axioms, with full Euclidean invariance, for the state King constructs in [26], at fixed photon mass, on all of King's coupling range and at his block factor 5. With real scalar sources and under a one-sided sign clause on the couplings, it proves, for the joint field of the curvature and the renormalised Wick square on a tube of small Wick amplitudes, the axioms: regularity, invariance under translations and the lattice point group, Euclidean invariance of the joint field outside an exceptional set of couplings (Theorem 27.13), reflection positivity about the coordinate planes for Wick amplitudes below a threshold, symmetry, and the measure with its ergodic decomposition and the cluster axiom in its weak form, ergodicity of the components under translations.

We call that state *King's state*, Z^K_∞: the infinite-volume state of King's construction [25, 26], taken on his own cube convention, as opposed to the comparison convention of Section 4. The existence of King's cube limits, the existence clause of his Theorem 2.1, is proved here (Section 5; with scalar sources, in the corrected form of Section 6), by King's route in a form in which the vacuum energies cancel.

The passage from the cube limits to the axioms in [26] rests on three steps that we carry out differently. First, King's Neumann–Dirichlet cube is not reflection symmetric about any plane, and its natural symmetrisation is not reflection positive at small charge (Remark 9.7); reflection positivity goes through a modified box state, whose positivity, King's Proposition 3.3, is proved here directly (Lemma 9.3). Second, the inequality by which [26] returns from the boxes to the cube is false for the observables the argument needs, and holds in the reverse direction (Lemma 9.6); it is replaced by a two-sided sandwich between cube states (Lemmas 9.9 and 9.10). Third, where King passes to subsequences of spacings, the subsequences are constructed here, and for King's own state the limits are taken along the full sequence (Remark 14.16).

With scalar sources, King's Theorem 2.1 needs a vacuum counterterm that his Wick ordering does not contain; we work with a *scheme a* that provides it.

The following table arranges the statements above and those of Sections 1.2, 1.3 and 1.4 by subject; the main results of this paper, items (1)–(8), are listed, with the theorems that state them, in Section 1.2.

	In [25, 26]	Proved again here	New here
Cube limits	the generating functionals converge as the spacing goes to zero, on tori and on cubes	the existence of King's cube limits, by King's route in a form in which the vacuum energies cancel; the subsequences of spacings are constructed, and for King's own state the limits are taken along the full sequence	
Infinite volume	through cubes with Neumann conditions for A and Dirichlet conditions for ϕ, monotone in the volume by the correlation inequalities	the cube states are monotone in J on a cone of observables, which gives the infinite-volume limit	
Regularity	listed as open		the regularity axiom, with absolute constants
Reflection positivity	through a modified box state, with an inequality that returns from the boxes to the cube	the positivity of the modified box state, proved directly; the inequality is false for the observables the argument needs and is replaced by a two-sided sandwich between cube states	
Rotations	listed as open; on a fixed torus, the lattice compared with its copy rotated by $\theta_0 = \arctan(3/4)$		invariance under the full Euclidean group
Ergodicity	listed as open		the decomposition of the measure into translation-ergodic components, almost every one of which satisfies the cluster axiom in all its forms; whether the measure is itself ergodic is not decided
Scalar sources	Theorem 2.1 of [26], which needs a vacuum counterterm that its Wick ordering does not contain	in corrected form: with real scalar sources, the cube limits in the reference scheme, for couplings satisfying the sign clause and coefficient vectors whose positive scalar part lies below the threshold of the Wick tube (item (1))	under the sign clause, the axioms on the tube in the form of item (6), with reflection positivity about every coordinate plane for Wick amplitudes below $\sigma_+/8$; the Euclidean clause below the amplitude threshold, outside the exceptional set E_W, and open above that threshold
Photon mass	kept positive		the limit of vanishing photon mass in the curvature sector (item (8))

1.2. **Main results.** In words, the main theorem, Theorem 28.2, says the following; the precise statement, with its domains, is there, and items (1)–(7) below state its clauses.

Curvature sector. On all of King's range, $e > 0$ and $\lambda, m, \mu_0 > 0$ arbitrary, the Schwinger functions of the curvature in King's state are tempered distributions and satisfy regularity bounds with no dependence on μ_0, e, λ, m or M. They are invariant under translations and the cubic group and Euclidean covariant, reflection positive about every plane, real and symmetric. For every even $r \geq 14$ they are the moments of exactly one Borel probability measure on $\mathcal{H}^{-r}(\mathbb{R}^3; \Lambda^2)$. The measure decomposes into translation-ergodic components, and almost every component is invariant under the Euclidean group, reflection positive about every plane and satisfies the cluster axiom in all its forms (Theorem 11.5).

Wick sector. With real scalar sources, for couplings that satisfy the sign clause (T3) and in an admissible scheme, the joint field of the curvature and the renormalised Wick square satisfies the axioms on a tube of small Wick amplitudes: regularity, with constants that are not uniform in μ_0; invariance under translations and the lattice point group; reflection positivity about the coordinate planes for Wick amplitudes below a threshold; symmetry; and the measure with its ergodic decomposition and the cluster axiom in its weak form. Euclidean invariance of the joint field is proved below an amplitude threshold, outside the exceptional set of couplings, in Theorem 27.13.

Item (8) describes the limit of vanishing photon mass in the curvature sector, which is beyond the main theorem.

Sources are real 2-forms $f_j \in C_c^\infty(\mathbb{R}^3; \Lambda^2)$ with complex coefficients z_j and real scalar test functions $h_j \in C_c^\infty(\mathbb{R}^3)$ with coefficients w_j; $F(f)$ is the curvature paired with f, $W^a(h)$ the renormalised Wick square in the scheme a, $g(z) = \sum_j (\operatorname{Re} z_j) f_j$, and the norms $\|\cdot\|_2$, $|\cdot|_4$, $|\cdot|_w$ and the tube $\mathcal{D}_{\sigma_+,H}$ are those of Definitions 2.5 and 2.8. The Schwinger functions S_n, $S_{n,p}$ are the Taylor coefficients of the limit functionals at 0. The limits are iterated on King's cubes $C_J = [-MJ, MJ)^3$: the spacing $\varepsilon_k = 5^{-k} \to 0$ first, then $J \to \infty$; the limit objects do not depend on King's large-block integer M, only the tube of (6) may (Remark 2.4).

(1) Existence of the cube limits. For every finite family of real 2-forms, every cube containing their supports at distance greater than 1 from its complement, and every real coefficient vector, the generating functional of the curvature on King's cube converges as $k \to \infty$, geometrically, to a limit in $(0, \infty)$ (Theorem 5.47); this is the existence clause of [26, Thm. 2.1]. With real scalar sources, for couplings satisfying the sign clause (T3) of (7), in the reference scheme and for coefficient vectors whose positive scalar part lies below the threshold of the Wick tube, the cube limits exist as well (Theorem 6.26).

(2) Regularity. King's state $Z_\infty^K(\sum_j z_j f_j)$ is an entire function of the coefficients and satisfies $|Z_\infty^K(\sum_j z_j f_j)| \leq e^{\|g(z)\|_2^2}$, with no dependence on μ_0, e, λ, m or M. For real test functions $|S_n(f_1, \dots, f_n)| \leq (\sqrt{2}e)^n (n!)^{1/2} \prod_j \|f_j\|_2$; the S_n are tempered distributions and satisfy the regularity axiom E0″ with $s = 4$ (Theorem 7.6).

(3) Euclidean invariance. Z_∞^K is invariant under all translations of $\mathbb{R}^3$ and under the cubic group O_h about every point (Theorem 8.6), and under the full orthogonal group, for all complex coefficients, F transforming as a 2-form; the S_n are Euclidean covariant (Theorem 8.8). The rotations are Theorem 15.1: King's state on curvature sources is $O(3)$-invariant at every point of King's range.

(4) Reflection positivity and symmetry. Reflection positivity about every coordinate plane holds for King's state, in matrix form for exponentials with arbitrary complex curvature coefficients (Theorem 10.5) and in polynomial form (Proposition 10.8); with (3) it holds about every plane $\{x \cdot n = c\}$. The S_n are real and symmetric, and $S_n = 0$ for odd n (Proposition 10.7).

(5) The measure and clustering. For every even $r \geq 14$ there is exactly one Borel probability measure μ_F on $\mathcal{H}^{-r}(\mathbb{R}^3;\Lambda^2)$ with moments S_n and characteristic functional $Z^K_\infty(\mathrm{i}\,\cdot)$, and the localised lattice curvature fields converge in law to those of μ_F. The measure is invariant under the cubic group about every point and its characteristic functional under the Euclidean group. The measure decomposes into translation-ergodic components, and almost every component is $\mathbb{R}^3 \rtimes O(3)$-invariant, reflection positive about every plane, clusters in the mean and under diagonal separation, has a unique joint vacuum in every direction and satisfies OS4 in all its forms for every choice of the time direction (Theorem 11.5). Whether μ_F is itself ergodic is not decided (Section 32).

(6) The Wick tube. Let the couplings satisfy the sign clause (T3) of (7) and let a be an admissible scheme (Definition 2.7). For sources in a fixed box, the joint functional of (F, W^a) on King's cubes converges on the tube $\mathcal{D}_{\sigma_+,H}$ of small Wick amplitudes to a holomorphic limit G^K_∞, King's tube object, with $|G^K_\infty(\zeta)| \leq \exp(\|g\|_2^2 + \kappa\|k_+\|_{L^1} + K_-|k_-|_w)$, $k_\pm$ the positive and negative parts of the scalar source and κ, K_- constants depending only on the couplings, the scheme, M and fixed auxiliary choices. The $S_{n,p}$ satisfy factorial bounds and E0″ with $s = 4$; translations and O_h hold on the tube; reflection positivity about every coordinate plane holds for Wick amplitudes below $\sigma_+/8$; $S_{n,p} = 0$ for odd n; and the measure on $\mathcal{H}^{-r}(\mathbb{R}^3;\Lambda^2 \oplus \mathbb{R})$, with the weak form of OS4 on its components, exists as in (5). The Euclidean clause of the Wick sector is proved on the tube outside the exceptional set E_W (Theorem 27.13) and is open above the amplitude threshold; the constants of the Wick sector are not uniform in μ_0. The tube bound is proved on the comparison convention (Theorem 13.10) and transferred to King's convention, on which the limits are taken (Theorem 28.5, Proposition 28.6); the comparison family enters only through its limit states along continuum subsequences, which coincide with G^K_∞ (Theorem 14.9, Proposition 28.7).

(7) The coupling range. The curvature results (1)–(5) hold on all of King's range, $e > 0$ and $\lambda, m, \mu_0 > 0$ arbitrary: every smallness condition in [25, 26, 1, 2, 3, 4] is a cap on the running couplings at a stopping scale chosen after the couplings and the datum, and no relation between μ_0 and m is used (Lemma 29.3, Remark 29.31). The only coupling condition in the paper is the one-sided sign clause (T3) of the Wick leg (the Wick sector of the main theorem, whose data are collected in Definition 28.1), $\sigma_1(M_1) > 0$ for some auxiliary scalar mass $M_1 \in (0, m)$, where $\sigma_1(M_1)$ is the lower limit, as the spacing goes to zero, of half the difference of King's free-boundary bare mass coefficients at the scalar masses m and M_1. It fails exactly when m minimises, over the scalar mass in $(0, m]$, the continuum limit of the difference of the bare mass coefficients at that mass and at m (only this relative limit exists, Lemma 29.1). To one-loop order it holds if and only if e^2 lies below an explicit threshold $e_c^2(\lambda, m, \mu_0)$ (equation (83)); and beyond one loop it is proved on explicit coupling regions, for every $\mu_0 \in (0, m]$ on the box $0 < \lambda < 2.14398\, m$, $0 < e^2 < 6.30\, m$, uniform in the photon mass (Propositions 29.29 and 29.30). The Wick results hold wherever (T3) holds.

(8) The limit of vanishing photon mass in the curvature sector. Beyond the main theorem, Section 30 takes the limit of vanishing photon mass in the curvature sector at fixed e, λ, m. King's own states at photon masses $\mu_n \downarrow 0$ converge along a subsequence, locally uniformly in the coefficients, to an entire limit which satisfies every clause of the curvature leg of the main theorem except its second, photon-mass-dependent regularity bound and the two sentences on the lattice states, with full Euclidean invariance and reflection positivity about every plane (Theorem 30.2). A second limit, along a family of states with fixed counterterm, is discussed as a further direction in Section 31 (Theorem 31.4, Proposition 31.6).

The main theorem, Theorem 28.2, states (1)–(7) with every clause in full except the Euclidean clause of the Wick sector, which Theorem 27.13 proves below the threshold outside the exceptional set; the proof of the main theorem assembles the sections in between.

1.3. **Methods.** The paper proceeds in five steps, of which the first four, together with the Wick tube, lead to the main theorem and the fifth lies beyond it. Three ideas are used in them. The steps are: the limit of vanishing lattice spacing in a fixed cube; the limit of infinite volume along the cubes; reflection positivity, with the measure and its ergodic components; the passage from the cubic group to all rotations; and, in the curvature sector, the limit of vanishing photon mass. The first idea is the use of the correlation inequalities of [9] (Lemma 2.10), by which the cube states are monotone in J on a cone of observables, and of Gaussian domination (Lemma 2.11), which gives the regularity bounds with absolute constants. Gaussian domination is a diamagnetic bound: at fixed photon field the matter integral is a sum of cosines of linear functionals of the field with non-negative weights, hence largest at zero field; the same holds for its average over the free photon field shifted by a background field, and the exponential moments of the photon field are therefore bounded by those of its free Gaussian field, whatever the matter. The second is the renormalisation-group analysis of Bałaban [1, 2, 3, 4] and King [25, 26], from which the existence of the cube limits is proved. The third is the comparison of two lattices on periodic tori carrying a smooth mass wall, which gives the rotations. With real scalar sources, the renormalised Wick square is added to the curvature on a tube of small Wick amplitudes, under the sign clause (T3). The paragraphs below take up the cubes, the existence of the cube limits, reflection positivity, the Wick tube and the rotations.

Neumann–Dirichlet cubes. King's cube state keeps the photon gradient terms inside the cube (Neumann) and kills the scalar on a boundary layer (Dirichlet). Both features are used: by the correlation inequalities of [9] the cube states are monotone in J on a cone of observables, which gives the infinite-volume limit, and the faces of the cube lie on lattice hyperplanes at every block scale, which is what King's renormalisation group needs. The cube of the existence theorem is the only cube on which his block-spin flow runs (Remark 5.11). Two cube conventions occur: King's own, on which his state lives, and a comparison convention with the scalar alive on the whole cube, on which the Wick tube and the lattice comparisons of the rotation part are written. The two are interlaced in J and have the same infinite-volume limit (Lemma 4.2, Corollary 4.4), while the existence of the fixed-cube limit does not transfer between them and is proved on King's convention.

Existence in ratio form. The existence proof follows King: the generating functional is approximated, up to a small-field remainder, by an integral of the effective action after j renormalisation steps, and two spacings whose effective actions live on the same unit lattice are compared. We write the comparison as a Cauchy property of the normalised functional, in a ratio in which every field-independent constant of the effective action cancels; this removes from King's route the convergence of the vacuum energies, which is not available on the cube (Theorem 5.45).

Reflection positivity through the modified box. Since the natural symmetrisation of King's cube is not reflection positive at small charge, the lattice statement is proved for King's modified box (Lemma 9.3) and carried to the cube by a two-sided sandwich between cube states with a margin of order μ_0^{-1} (Lemmas 9.9 and 9.10). The limits $k \to \infty$ and $J \to \infty$ are taken through uniform local bounds at complex coefficients and Montel's theorem, for the curvature sector and the Wick tube alike (Section 9, Theorem 10.5).

The Wick tube. The joint functional of the curvature and the Wick square is controlled on the tube by three results: reflection positivity and the chessboard estimate on King's periodic tori, the correlation inequalities, and an *extensive bound* for constant positive scalar sources in the fixed-spacing volume limit, derived from Bałaban's stability bounds at two scalar masses and the

sign clause (T3) (Proposition 12.8). The positive part of the scalar source is bounded through the tori, the negative part through a tiling into cells of fixed size, each controlled by one instance of the existence theorem with sources (Theorem 13.10).

Rotations. The ultraviolet analysis needs boundaries that are lattice hyperplanes, and no bounded region has that property for a lattice and its copy rotated by $\theta_0 = \arctan(3/4)$; the rotations by θ_0 in the coordinate planes generate a dense subgroup of $SO(3)$. The two-lattice comparison is therefore placed on periodic tori carrying a smooth *mass wall*, a raise of the scalar mass away from the sources. On each lattice King's cube functional is bounded by walled-torus functionals; the walled tori on the two lattices have the same continuum limit outside the exceptional set E_W (Theorem 18.1; an involution exchanging the two lattices reduces it to one-sided bounds at a single spacing); and King's state is the lower envelope of the walled tori (Theorem 21.4). The heavy-shell estimate behind the last statement, the passage from the generators of the dense subgroup to $O(3)$ and the exceptional set E_W of couplings are the subject of Section 15 (the chain (39)).

The photon mass μ_0 is fixed throughout the main theorem; the limit of vanishing photon mass is taken in Section 30, in the curvature sector (item (8) above). The questions left open, among them the uniqueness and ergodicity of King's state and the Wick sector above the amplitude threshold, are collected in Section 32.

1.4. **Previous work.** Reflection positivity of lattice gauge theories goes back to Osterwalder and Seiler [31] for compact gauge groups and to [9, Thm. 5.3] for the non-compact abelian field. In two dimensions the model was constructed by Brydges, Fröhlich and Seiler [9, 10, 11], whose correlation inequalities are used below. The ultraviolet problem in three dimensions was solved by Bałaban [1, 2, 3, 4]: two-sided bounds $e^{-E_-|T|} \le Z^\varepsilon(T) \le e^{E_+|T|}$ on the partition function with a photon mass on a periodic torus T, with constants independent of the spacing ε and of T [1, (1.14)], [2, (1.3)]. King [25, 26] turned these bounds into limits: the generating functionals converge as $\varepsilon \to 0$ on tori and on cubes, the infinite-volume limit is taken through cubes with Neumann conditions for A and Dirichlet conditions for ϕ, monotone in the volume by the correlation inequalities, and the resulting Schwinger functions are translation invariant and reflection positive. He kept $\mu_0 > 0$ throughout and listed ergodicity, rotation invariance and regularity as open [26, p. 326]. For rotations he compared, on a fixed torus, the lattice with its copy rotated by $\theta_0 = \arctan(3/4)$ [26, Thm. 2.4]. The Higgs phase was treated on the unit lattice in [5] and in the continuum in [6]. Dimock [14] revisited the ultraviolet renormalisation of the model (and of its fermionic counterpart [15]); continuum limits of the compact pure $U(1)$ theory are due to Gross [20] and Driver [16]; for the stochastic quantisation of gauge theories with matter in three dimensions see [12].

The results of King and Bałaban are used in the form stated at their point of use, each with its source: Bałaban's stability bounds and the c-number lemma in Section 12, his torus analysis on King's cube in Section 5, and the statements of the rotation part in Sections 19, 24 and 25. Every statement taken from the renormalisation-group analysis is stated, with its proof, in the section that uses it; Remark 2.18 lists the statements of [25] and [1, 2, 3, 4] used beyond their printed order or case.

The Higgs phase of the abelian Higgs model, in its Villain form, is treated in the companion paper [32] by the same general scheme: gauge-invariant fields on a lattice regularisation, infinite-volume states through boxes with killed matter, and rotation invariance recovered in the limit. There the ultraviolet problem needs no renormalisation group and the rotations are obtained from reflection positivity about nine families of planes; here the rotations come from the walled tori of Section 15. The companion paper also provides ([32, Appendix A]) the construction of the measure and the reconstruction in a coordinate direction used in Section 11.

Added note. The v1 of this manuscript provides a more detailed account of the construction hosted in https://github.com/marikgoldstein/scalar-qed-existence on October 10th, 2026. This line of research is rapidly evolving including parallel and independent efforts by many groups [36, 37, 38] which also approach three-dimensional QFTs.

The v2 of this manuscript adds the limit of vanishing photon mass in the curvature sector: Section 30 and item (8) of Section 1.2, with further directions in Section 31. It also contains a coupling region for the sign clause of the Wick leg that is uniform in the photon mass (Proposition 29.30), and the exposition has been revised throughout.

1.5. **Organisation.** After the setting and the outline, the paper has four parts: the existence part, the axioms, rotation invariance and the main theorem.

Setting and outline. Section 2 fixes the model, King's cube and the comparison cube, the sources and schemes, the coupling region of the Wick leg and the standing lemmas. Section 3 outlines the proof: it states the three theorems on which the main theorem rests, the existence of King's cube limits without and with scalar sources (Theorems A and B) and the rotation theorem (Theorem C), each with the section where it is proved, and Table 1 lists what is proved where. Section 4 interlaces the two cube conventions.

The existence part. The existence part proves King's existence theorem on the cube without (Section 5) and with scalar sources (Section 6).

The axioms. The curvature sector follows in the order of the axioms: regularity (Theorem 7.6), translations, the cubic group and the rotation clause (Theorem 8.8), reflection positivity on the modified box (Section 9) and in the limit (Theorem 10.5), and the measure with its ergodic decomposition (Theorem 11.5). The Wick tube occupies Sections 12 and 13; continuum subsequences form Section 14.

Rotation invariance. The rotation part is Sections 15–27, rows (R0)–(R4) of Table 1: the rotation theorem, Theorem 15.1, is stated at its start, and the results it uses are stated where they are first needed, in Sections 18, 19, 21 and 24.

The main theorem. Theorem 28.2 assembles the main theorem, Section 29 establishes the coupling range, Section 30 takes the limit of vanishing photon mass in the curvature sector, Section 31 records further directions for that limit, and Section 32 collects the open problems.

Throughout the paper, routine verifications are left to the reader.

Guide for the reader

This guide says how the paper is built, in which order it can be read, and what its recurring names and tags mean. Every mathematical statement in it is made, in its precise form, elsewhere in the paper.

The result. The object is King's state Z_∞^K (Section 1): the limit of the lattice model on King's cubes C_J, first as the spacing $\varepsilon_k = 5^{-k}$ goes to zero and then as $J \to \infty$, at a fixed photon mass $\mu_0 > 0$. The main theorem, Theorem 28.2, has four parts: A, the object; B, the curvature leg; C, the Wick leg; D, the comparison family. For the curvature it states the Osterwalder–Schrader axioms, with regularity in the form E0″ and with invariance under the full Euclidean group, on all of King's range. For the joint field of the curvature and the renormalised Wick square, with real scalar sources, it states the axioms on a tube of small Wick amplitudes, in the form of item (6) of Section 1.2, under the sign clause (T3), a one-sided condition on the couplings; Euclidean invariance of the joint field is proved below an amplitude threshold, outside an exceptional set of couplings, in Theorem 27.13. Section 30 treats the limit of vanishing photon mass in the curvature sector, for the family and in the form stated there. Section 1.2 lists the results, items (1)–(8); the questions left open are collected in Section 32.

The parts. After the setting and the outline, the paper has four parts: the existence part, the axioms, rotation invariance and the main theorem.

- *Setting and outline* (Sections 2–4). Section 3 states the three theorems on which the main theorem rests: the cube limits without and with scalar sources, and the rotation theorem. It contains Table 1, which names for each result what it uses and where it is proved. The numbered rows of the table follow the items (1)–(7) of Section 1.2, with (1′) and sub-rows such as (3a), (4b) and (6a), and (R0)–(R4) for the rotation part; the remaining rows carry no number.
- *The existence part* (Sections 5 and 6) proves the cube limits from the renormalisation-group analysis of Bałaban and King.
- *The axioms* (Sections 7–14). The curvature sector (Sections 7–11) proves the axioms in their order: regularity, translations and the cubic group, reflection positivity, and the measure with its ergodic decomposition and the cluster axiom on the components. The Wick tube (Sections 12 and 13) proves the bound on the tube; Section 14 treats continuum subsequences.
- *Rotation invariance* (Sections 15–27), the rotation part, proves the rotation theorem, Theorem 15.1, which is stated at its start. After the model (Section 16), rows (R0)–(R4) of the table are Section 17, Section 18, Section 23, Section 21 and Section 22. The statements on the renormalisation-group flow that these sections use are stated and proved, each with its source, in Section 19 and in Sections 24 and 25; Section 20 is the restricted renormalisation theorem. Section 26 treats the exceptional set E_W of couplings, and Section 27 King's state with scalar sources and its Euclidean invariance.
- *The main theorem* (Sections 28–32): the main theorem (Section 28), the coupling range (Section 29), the limit of vanishing photon mass in the curvature sector (Section 30), further directions for that limit (Section 31) and the open problems (Section 32).

Three ways to read the paper. For the statements and the architecture, read the Introduction, Section 2 as far as Definition 2.8, Section 3 and the statement of Theorem 28.2. For the proof of the axioms in the curvature sector from the cube limits without scalar sources and the rotation theorem (Theorems A and C), add the rest of Section 2, Section 4 and Sections 7–11. For the existence part and the rotation part, Table 1 names what each uses from the other sections; in the rotation part the sections of rows (R0)–(R4) can be read before the statements on which they rest.

Names. The following names recur; the index of notation before the references gives the symbols.

- *King's state* is Z_∞^K. *King's range* is the set of all positive couplings (e, λ, m, μ_0) (Section 29).
- There are two *cube conventions*. On *King's convention* the scalar field is set to zero on the top layer of X_J in each of the three coordinate directions (Definition 2.3); on the *comparison convention* it lives on all of X_J (Definition 2.2). King's state lives on the first. By Lemma 4.2 and Corollary 4.4 the two are interlaced in J and have the same infinite-volume limit, in the sense stated there. Which convention a statement is posed on is part of the statement.
- The *curvature sector*, or *curvature leg*, of a statement concerns curvature sources only. The *Wick sector*, or *Wick leg*, concerns the joint field with scalar sources.
- A *scheme* is the sequence of constants subtracted in the Wick square beside its Wick ordering (Definition 2.5); *admissible* schemes are those of Definition 2.7.
- The *tube* is the set of coefficients on which the Wick sector is controlled (Definition 2.8); the *amplitude* of a family of sources is $\sum_j |\operatorname{Re} w_j| \, \|h_j\|_\infty$, the size of the real parts of its Wick coefficients (Definition 2.5).
- The *cone* is the set of pairs of a real curvature source and a non-negative scalar source, with its two classes of observables (Definition 2.5); the monotonicity in the cube and the sandwich are statements on it.

- The *sign clause* (T3) is the condition $\sigma_1(M_1) > 0$ for some auxiliary scalar mass $M_1 \in (0, m)$, with $\sigma_1(M_1)$ as defined before Definition 2.1; it is proved on the explicit regions of Propositions 29.29 and 29.30.
- The *modified box state* is defined in Definition 9.2 and is reflection positive (Lemma 9.3); the *sandwich* is the pair of comparisons of Lemmas 9.9 and 9.10.
- *Continuum subsequences* and *LIM states* are defined in Definitions 14.1 and 14.2.
- In the rotation part, *walled tori* are periodic tori with the scalar mass raised away from the sources (Definition 16.4, Definition 21.1); the tags (N1)–(N6), (SK), (BAL)$_5$ and (CT) name the statements of Section 19, and the four *shell bounds* are those of Section 23.4.

The literature. The results of King and Bałaban are stated where they are used, each with its source (Section 1.4).

2. The model, King's cube and the standing lemmas

This section fixes the lattice model with King's action and his free-boundary counterterm, King's Neumann–Dirichlet cube and the comparison cube, the sources, schemes and generating functionals, and the standing lemmas. The standing lemmas are the correlation inequalities of [9], with the two inequalities only sketched there proved in full, Gaussian domination, cutting and the elementary facts. Nothing here involves a limit. Every later section uses these definitions. The correlation inequalities drive the interlacing of Section 4, the modified box of Section 9 and the comparisons of the rotation part.

2.1. **Couplings, lattices and the action.** Throughout, $e > 0$ and $\lambda, m, \mu_0 > 0$ are the charge, the quartic coupling, the bare scalar mass and the photon mass. No further condition on the couplings is imposed, since every smallness condition of [25, 26, 1, 2, 3, 4] is a cap on the running couplings at a stopping scale chosen after the point (e, λ, m, μ_0) and after the datum (Section 29).

The block factor is 5: the spacings are $\varepsilon_k := 5^{-k}$, the lattices $\Lambda_k := \varepsilon_k \mathbb{Z}^3$ with the standard frame (e_α), and M is King's large-block integer, "a large integer" [25, §2.3], the integer of [4], with $M \in 50\mathbb{N}$ and $M \geq M_{\min}$.

Here $M_{\min}$ is the least integer such that every integer $M \geq M_{\min}$ satisfies the lower bounds placed on the large-block integer, or on the side of the big or of the large blocks, in four places. The first is the schedule of constants described in Lemma 29.3. In this schedule the constants are fixed in an acyclic order, $L = 5$, $\bar{n}$ and the decay and coercivity constants first, then $M \in 50\mathbb{N}$, then R, then b_0 (with $c_1 p_1(\beta) + p(5\beta) \leq p(\beta)$), then β below Bałaban's and King's caps, then J_K, each step imposing finitely many one-sided or divisibility conditions on one entry, so that every threshold depending on M is imposed on β after M. At $L = 5$ every condition added to Bałaban's parameter list is a lower bound on R ($R \geq D^\flat + 3$, $R \geq 10(M+1)$, $R \geq 10 s_b + 10$, $R \geq 52$) or on b_0, or a downward-closed cap on ε_0, each depending only on data fixed earlier in his order, hence none on the couplings, none on L and none on M. The second is the hypotheses of Bałaban's stability bounds [1, (1.14)], [2, (1.3)] as used in Proposition 12.6. The third is the torus form of his propagator bounds: they hold on every torus with period lattice in $M\mathbb{Z}^d$ and sup-systole at least $3M$, for all x, x' when $\Omega = T$, with his lattice constants δ_0, c_0, R_0 independent of the period lattice; the torus-size condition (T.nw) of Theorem 24.5 implies this. The fourth is the conditions of Theorem 24.5: each is a lower bound on L, satisfied by every integer $L \geq 5$, on M or on b_0, a fixed exponent constraint, or a cap on ε_0 chosen after b_0; the order of choices is acyclic and e, λ, m, μ_0 stay free.

Each of the lower bounds placed on the large-block integer, or on the side of the big or of the large blocks, in these four places holds for all M above a threshold that depends only on the couplings and on constants fixed before M, not on the spacing, the cube, the torus or the sources; so $M_{\min} < \infty$. (The symbols of this schedule are Bałaban's and King's parameters, recalled where

they are used, in Section 5 and among the hypotheses of the rotation part, Section 19; only the conclusion $M_{\min} < \infty$ is used before then.)

Of the divisibility $50 \mid M$, the rotation part uses $M = 5M'$ with M' even, for the grid offset at the box H' of Definition 23.2 (Definition 16.1). M is fixed after the couplings. Where an argument follows [25, 26], a *large block* is King's block of M^3 sites of the lattice of its scale [25, §2.3]. Where it follows [1, 2, 3, 4], these cubes are called *big blocks*, and a large block is a cube of $5M$ sites per side [1, p. 607].

King's cubes are $C_J := [-MJ, MJ)^3$, $J \in \mathbb{N}$, and $X_J := C_J \cap \Lambda_k$ (the spacing understood from the context). A *King torus* is $\mathbb{R}^3/(\ell\mathbb{Z})^3$ with $\ell \in 2M\mathbb{N}$; Λ_k descends to it, with $\ell\, 5^k$ sites per side.

The scalar field is $\phi(x) = r_x e^{\mathrm{i}\vartheta_x} \in \mathbb{C}$ and the photon field a real $A_\alpha(x)$ for every site x and direction α; the bond $\langle x, x + \varepsilon e_\alpha\rangle$ carries $A_\alpha(x)$ and the reversed bond $-A_\alpha(x)$. King's action [26, (2.5)] at spacing ε is

$$\begin{aligned} S = {} & \tfrac{1}{2}\sum_{\alpha,\beta}\sum_{x} \varepsilon \left(A_\alpha(x+\varepsilon e_\beta) - A_\alpha(x)\right)^2 + \frac{\mu_0^2}{2}\sum_x \varepsilon^3 |A(x)|^2 \\ & + \tfrac{1}{2}\sum_{\beta}\sum_{x} \varepsilon \left|e^{\mathrm{i}e\varepsilon A_\beta(x)}\phi(x+\varepsilon e_\beta) - \phi(x)\right|^2 \\ & + \sum_x \varepsilon^3 \Big[\tfrac{1}{2}\big(m^2 + \delta m_\varepsilon^2\big) r_x^2 + \lambda r_x^4\Big], \end{aligned}$$

the photon in Feynman form with mass μ_0, and the hopping term of a bond $\langle x, y\rangle$, $y = x + \varepsilon e_\beta$, equal to $\frac{\varepsilon}{2}(r_x^2 + r_y^2) - \varepsilon r_x r_y \cos(\vartheta_x - e\varepsilon A_\beta(x) - \vartheta_y)$: a *diagonal part* and a *cosine part*. The counterterm δm_ε^2 is King's free-boundary counterterm [26, p. 330], fixed by his two-point graphs of order at most four in (e, λ) at zero momentum with the propagators of the infinite lattice $\varepsilon\mathbb{Z}^3$. It is *one number per spacing*, the same at every site of every box and torus. The Wick constant is the free one, $\mathsf{W}_\varepsilon := (-\Delta_{\varepsilon\mathbb{Z}^3} + m^2)^{-1}(0,0)$. This *free convention* is used on cubes and tori alike unless said otherwise. King's own torus convention appears only inside the torus theorems of the Wick leg.

For a scalar mass $\mathsf{M} > 0$ (the sans-serif letter M denotes a scalar mass throughout, M being King's large-block integer) let $\delta m_\varepsilon^2(\mathsf{M})$ be the same counterterm computed with the bare scalar mass M in place of m, so that $\delta m_\varepsilon^2 = \delta m_\varepsilon^2(m)$, and let

$$\alpha_k^\infty(\mathsf{M}) := \mathsf{M}^2 + \delta m_k^2(\mathsf{M}), \qquad \delta m_k^2(\mathsf{M}) := \delta m_{\varepsilon_k}^2(\mathsf{M}),$$

be King's free-boundary bare coefficient on Λ_k at scalar mass M. For an auxiliary scalar mass $M_1 \in (0, m)$ we put

$$s_k^\infty(M_1) := \tfrac{1}{2}\big(\alpha_k^\infty(m) - \alpha_k^\infty(M_1)\big), \qquad \sigma_1(M_1) := \liminf_{k\to\infty} s_k^\infty(M_1).$$

That $\sigma_1(M_1)$ is finite is Lemma 12.7 (Section 12); each coefficient $\alpha_k^\infty(\mathsf{M})$ alone diverges as $k \to \infty$, only the difference having a limit, given in closed form in Lemma 29.1.

Definition 2.1 ((T3): the coupling region)**.** The couplings (e, λ, m, μ_0) *satisfy the sign clause* (T3) if $\sigma_1(M_1) > 0$ for some $M_1 \in (0, m)$, and the set of couplings satisfying it is the *coupling region* of this paper. By Propositions 29.29 and 29.30 the coupling region contains the explicit regions listed there, among them, for every $\mu_0 \in (0, m]$, the box $0 < \lambda < 2.14398\, m$, $0 < e^2 < 6.30\, m$.

The Wick leg of the main theorem is stated for couplings in the coupling region. The curvature leg carries no condition on the couplings. Membership of the coupling region is the only condition on the couplings anywhere in this paper, and it is one-sided. The set $E_W^<$ of couplings on which

the sign clause fails (Lemma 29.3) and the explicit regions on which it holds (Propositions 29.29 and 29.30) are described in Section 29.

Terminology. Several objects of this paper carry more than one name, and a few names are used for more than one object. We list here what each name refers to, with the defining places. Most of the defining places lie further on, in this section or in the parts named. The list is meant for reference when a name recurs. The two sectors of the main theorem (Theorem 28.2) are the *curvature sector*, also called the curvature leg or Leg C, and the *Wick sector*, also called the Wick leg or Leg W, whose data are collected in Definition 28.1. Within the Wick leg, the *positive leg* and the *negative leg* (Section 12) refer to the sign of the scalar source. In the graph estimates of the existence and rotation parts a "leg" is a half-line of a vertex and a line is a pair of legs (Definition 5.6). In these estimates one speaks of external (unpaired), scalar and background legs. The name *Wick tube* is used for three things: the tube $\mathcal{D}_{\sigma_+,H} = \mathcal{D}_{\sigma_+,H}$ of Definition 2.8, the domain of the Wick leg; the tube $T_{\mathcal{F}} \subset \mathbb{C}^m$ of Definition 10.2, used in the reflection positivity statements of Section 10; and, in the titles of Sections 12–13, in item (6) of the introduction and in Theorem 28.5, the Wick leg as a whole. The symbol tells the two tubes apart. A *scalar source* is a real test function h (Definition 2.5). The scalar source of a coefficient vector ζ, $k(\zeta) = \sum_j (\operatorname{Re} w_j) h_j$ (Definition 2.8), is also called the *Wick source*, as the source of the Wick square: the two names denote the same object. The *lattice point group* named in the abstract and in the introduction is the cubic group O_h (Section 8).

Cubes, conventions and tori. King's RG cube $\mathrm{ND}^K_{k,J}$ (Definition 2.3) is also called King's cube, King's cube state (Remark 2.4), King's Neumann–Dirichlet cube and King's convention. It is the Neumann–Dirichlet state of Definition 2.2 on X_J with the matter killed on the three top face layers T_J. The *comparison cube* $\mathrm{ND}^{(a)}_{k,J}$ of Definition 2.2, the Neumann–Dirichlet state on X_J with no killed sites, is also called the comparison state, the comparison convention or the comparison family. "Neumann–Dirichlet state" or "Neumann–Dirichlet box" alone means the general class of Definition 2.2, to which both belong (Section 4 treats the two conventions together). A *King cube* $C_L = [-ML, ML)^3$, $L \in \mathbb{N}$, is a region, not a state. A *King torus* is $\mathbb{R}^3/(\ell\mathbb{Z})^3$ with $\ell \in 2M\mathbb{N}$ (above, and Definition 12.1), the tori D_J of the rotation part being the case $\ell = 50MJ$. The torus $\mathbb{T}_J$ of Section 12, written T_J in Proposition 12.6, is not the face-layer set T_J. A BFS measure *of King type* is defined in the paragraph on BFS measures of Section 2.2 (the property used in Lemma 2.12). A box or lattice model "of King's type" is one in King's convention (a Neumann–Dirichlet cube state with a killed face layer and a site-independent bare mass). In Lemma 14.7(a) and in Section 31, by contrast, a lattice model "of King's type" is, in the wider sense of that lemma, a Neumann–Dirichlet cube state of either convention with a site-independent real bare coefficient. A bound or a comparison "of King's type" is one in the manner of [25], and in the rotation part a vertex "of King's type" is a vertex of King's vertex list [25, (3.22)].

The shell model. The *shell model* is the model with the shell mass $\kappa \mathbf{1}_U$ inside every matter covariance, named in Section 16 (Definition 16.9), set up in Section 23 and defined at the start of Section 24. Definition 23.2 (shell configurations) fixes its geometry: the torus $D_{J'}$ (written T in Sections 24–25), the box $H = Q_{b'}$ (written H' there), the shell's King cube $C_{L'}$ and the shell U; it is what a citation of the form "the shell model (Definition 23.2)" points to. The set $\mathcal{X}^{\mathrm{cap}}$ of Definition 24.2 is the *capped test set* of the text and the "capped stopping-scale set" of that definition's title and of the introduction of Section 24, also the "capped set" of a subsection title there: one set.

Torus theorems and counterterms. The *torus theorems*, in the plural, are Proposition 12.6 and Lemma 12.7 of Section 12. *The torus theorem*, in the singular, or the corrected torus theorem, is Theorem 18.8 of the rotation part, whose hypotheses (N1)–(N6), also called its statements, are

stated in Section 19. The *counterterm* δm_ε^2 is King's free-boundary counterterm, one number per spacing (above), also called the mass counterterm of the free convention (the torus counterterm below is the mass counterterm of King's torus convention; the bulk and κ-adapted counterterms are the two counterterms of the shell model). The rotation part compares δm_ε^2 with the torus counterterms of [25] (the convention bridge, Remark 16.8), and in the shell model the vertex carrying the difference between the bulk counterterm and the κ-adapted one is the mismatch vertex (Definition 16.9). The vacuum-energy counterterm E_1 of [25, (3.35)] is a different object, a constant.

Flows and the exceptional set. The two flows of the torus comparison (Section 18) are the *standard flow* and the *hybrid flow*. The standard flow is King's block-spin flow in the frame of the lattice it runs on (Y_K in the torus comparison, elsewhere also X_K). The hybrid flow is run on the axis lattice X_K with a first step that averages over rotated blocks (Theorem 19.10); it is also called King's hybrid flow after [26, (4.3)–(4.9)]. The paper also names the two flows by their lattices, "the standard flow of Y_K" and "the hybrid flow of X_K", in proofs simply Y and X. The *exceptional set* is the set E_W of couplings of Proposition 26.8, defined in Section 26, off which the positivity condition of the convention bridge, or its quantitative alternative, holds. It is distinct from the set $E_W^<$ of couplings on which the sign clause (T3) fails (Lemma 29.3, Definition 2.1).

2.2. **The states.** This subsection defines the finite-volume states of the paper: the Neumann–Dirichlet states on boxes, with and without killed sites, King's RG cube and the comparison cube, and the periodic state on a King torus. It closes with the BFS form shared by all of them, on which the correlation inequalities of Lemma 2.10 rest.

Definition 2.2 (Neumann–Dirichlet boxes, killed sites, the comparison cube)**.** For a finite set $X \subset \Lambda_k$ the *Neumann–Dirichlet state* $\langle\cdot\rangle_{k,X}^{\mathrm{ND}}$ is the normalised measure $Z^{-1}e^{-S}\prod_{x\in X} d^2\phi(x)$ $\prod_{x\in X,\alpha} dA_\alpha(x)$ with $\phi = 0$ at every site outside X (Dirichlet condition: a bond leaving X contributes only its diagonal part $\frac{\varepsilon}{2}r_x^2$ at its end $x \in X$), photon gradient terms only for pairs $x, x+\varepsilon e_\beta$ both in X, and the photon mass term on X (Neumann condition). For $U \subset X$ the state *with the matter killed on* U replaces the single-site measure $d^2\phi(x)$ times its weight by the point mass at $\phi(x) = 0$ for $x \in U$; every hopping term with an endpoint in U then reduces to its diagonal part at the other endpoint, and the photon variables are unchanged. The *comparison cube* is $\mathrm{ND}_{k,J}^{(a)} := \langle\cdot\rangle_{k,X_J}^{\mathrm{ND}}$, written $\langle\cdot\rangle_{k,J}^{(a)}$: the scalar is alive on all of X_J, with Dirichlet planes $x_\mu = -MJ - \varepsilon_k$ and $x_\mu = MJ$.

Definition 2.3 (King's RG cube)**.** Let $\mathrm{alive}_J^K := \{x \in X_J : \ x_\mu < MJ - \varepsilon_k \ \forall\mu\}$ and $T_J := X_J \setminus \mathrm{alive}_J^K$, the three top face layers. *King's RG cube* $\mathrm{ND}_{k,J}^K$, written $\langle\cdot\rangle_{k,J}^K$ (the cube with this convention is written C_J^K), is the Neumann–Dirichlet state on X_J with the matter killed on T_J: the photon variables $A_\alpha(x)$, $x \in X_J$, their gradient terms for pairs in X_J and their mass term on X_J are those of the comparison cube; the scalar is alive on alive_J^K and $\phi = 0$ on every other site of Λ_k, in particular on T_J. Its Dirichlet planes are $x_\mu = -MJ-\varepsilon_k$ and $x_\mu = MJ-\varepsilon_k$, at separation $2MJ$; the Neumann image planes of the photon, $x_\mu = -MJ - \varepsilon_k/2$ and $x_\mu = MJ - \varepsilon_k/2$, are the same in both conventions.

The two conventions differ exactly by the matter on T_J; the photon Gaussian is the same. King's RG cube is the box [26, (3.3)–(3.4)] with the Dirichlet planes of [26, (3.9)] and the photon reflection planes of [26, (3.7)], the geometry of his renormalisation group; his Theorem 3.2 and his modified box [26, (3.38)] are written in the comparison convention. The two families are interlaced in Section 4; the existence theorems of this paper are posed on ND^K, while the rotation part and the cell lemmas of the Wick leg are written for the comparison family. The

periodic state $\langle\cdot\rangle_{k,T}$ on a King torus T has the action summed over the finite graph $\Lambda_k/(\ell\mathbb{Z})^3$, in the free convention.

Remark 2.4 (independence of M). King's cube state depends on (M, J) only through MJ (Definition 2.3); the spacing, the action, the counterterm, the Wick constant and the schemes (Definition 2.5) do not involve M. Let M and $\tilde{M}$ both satisfy the conditions above (Section 2.1). Their least common multiple $\check{M}$ satisfies them too, and for $X \in \{M, \tilde{M}\}$ the cubes of $\check{M}$ are cubes of X: $[-\check{M}J, \check{M}J)^3 = [-XJ_X, XJ_X)^3$ with $J_X := (\check{M}/X)J \in \mathbb{N}$. Hence, whenever an iterated limit $\lim_J \lim_k$ of the following sections exists for M and for $\tilde{M}$, the two limits agree with the limit along the cubes of $\check{M}$, so they coincide; for limits on a tube that depends on M, they coincide on the intersection of the two tubes. Thus the limit objects of Theorem 28.2 do not depend on the choice of M among the integers $M \in 50\mathbb{N}$ with $M \geq M_{\min}$: such a choice changes at most the tube on which a limit is claimed, through the constants of the tube, and two such limits agree where both are defined. Moreover, for every integer $M^\sharp \geq 1$ and every $J \in M\mathbb{N}$, $[-M^\sharp J, M^\sharp J)^3 = C_{M^\sharp J/M}$. So along $J \in M\mathbb{N}$ the iterated limits for the cubes $[-M^\sharp J, M^\sharp J)^3$ exist and equal those for M. Along the full sequence $J \in \mathbb{N}$ they are claimed only for $M^\sharp$ satisfying the conditions above (Section 2.1).

BFS measures. A *BFS measure* on a finite set V of sites is a probability measure

$$Z^{-1} \exp\Big(\sum_{\{x,y\}\in B} c_{xy}\, r_x r_y \cos\big(\vartheta_x - \vartheta_y - \ell_{xy}(A)\big)\Big) \prod_{x\in V} d\rho_x(r_x)\, d\vartheta_x\; d\gamma(A),$$

with B a set of pairs of sites, $c_{xy} \geq 0$, each ρ_x a finite measure on $[0,\infty)$ with $\int e^{\alpha r^2} d\rho_x < \infty$ for all $\alpha \geq 0$ (point masses at 0, that is killed sites, included), γ a non-degenerate centred Gaussian measure on $\mathbb{R}^m$ (the photon variables), and linear functionals ℓ_{xy}. Every state above (Section 2.2) is a BFS measure: $c_{xy} = \varepsilon$ for the bonds present, the single-site measures contain $e^{-\varepsilon^3[\frac{1}{2}(m^2+\delta m_\varepsilon^2)r^2+\lambda r^4]}$ and the diagonal parts $\frac{\varepsilon}{2} r_x^2$ of all bonds at x (those leading out of the box or into a killed site included); the photon form is positive definite because $\mu_0 > 0$, and the exponential moments are finite because $\lambda > 0$. So is every measure obtained from such a state by multiplying some c_{xy} or some photon gradient terms by $t \in [0,1]$, adding $s|\ell(A)|^2$ ($s \geq 0$) to the action, multiplying single-site measures by $e^{\varepsilon^3 q(x) r^2}$ with real q, or killing sites. A BFS measure is *of King type* if $c_{xy} \in [0, \varepsilon]$ and each ρ_x is a point mass at 0 or has the density $r\, e^{-\varepsilon^3[\frac{1}{2}(m^2+\delta m_\varepsilon^2)r^2+\lambda r^4]-\varepsilon^3 w(x) r^2 - \frac{\varepsilon}{2} n_x r^2}$ with $w(x) \geq 0$, $n_x \in \{0,\dots,6\}$. Every state above, and every measure obtained from one by the operations just listed with $q \leq 0$, is of King type. Every state above is invariant under the *charge conjugation* $(A, \vartheta) \mapsto (-A, -\vartheta)$, which changes the sign of every field strength and fixes r.

2.3. **Sources, schemes and the objects.** This subsection introduces the curvature and scalar sources, the schemes for the Wick square, the two classes of observables and the generating functionals of the states of Section 2.2; the objects of the paper, King's state and its tube object, are then named as their limits.

Definition 2.5 (sources, schemes, classes, transport)**.** A *curvature source* is a real 2-form $f \in C_c^\infty(\mathbb{R}^3; \Lambda^2)$, a *scalar source* a real $h \in C_c^\infty(\mathbb{R}^3)$. For the plaquette $p = (x; \alpha < \beta)$ with centre $c(p)$ let $F(p) := \varepsilon^{-1}(A_\alpha(x) + A_\beta(x + \varepsilon e_\alpha) - A_\alpha(x + \varepsilon e_\beta) - A_\beta(x))$ and

$$F_{\Lambda_k}(f) := \sum_p \varepsilon^3 f_{c(p)}(e_\alpha, e_\beta) F(p), \qquad \mathsf{S}_{\Lambda_k}(h) := \sum_x \varepsilon^3 h(x),$$

$$:\!\phi^2\!:(h) := \sum_x \varepsilon^3 h(x)\big(|\phi(x)|^2 - 2\mathsf{W}_\varepsilon\big),$$

the sums over the plaquettes (corners in the box), respectively the sites, of the region. A *scheme* is a real sequence $a = (a_k)$, a_k depending only on ε_k and the couplings; $W^a(h) := \sum_x \varepsilon^3 h(x)(|\phi(x)|^2 - 2\mathsf{W}_\varepsilon - a_k)$. For a finite family $\zeta = (z_j, f_j; w_j, h_j)_j$ with complex coefficients put $\Theta(\zeta) := \exp(\sum_j z_j F(f_j) + \sum_j w_j{:}\phi^2{:}(h_j))$ and, for a Neumann–Dirichlet state on a box X (killed sites allowed),

$$Z^a_{k,X}(\zeta) := \Big\langle \exp\Big(\sum_j z_j F(f_j) + \sum_j w_j W^a(h_j)\Big)\Big\rangle^{\mathrm{ND}}_{k,X} = e^{-\sum_j w_j a_k \mathsf{S}_{\Lambda_k}(h_j)} \big\langle \Theta(\zeta)\big\rangle^{\mathrm{ND}}_{k,X};$$

$Z^a_{k,J} := Z^a_{k,X_J}$ for the comparison cube, $Z^{a,K}_{k,J}$ for King's RG cube, and

$$N^a_{k,X}(0,q) := e^{-(a_k + 2\mathsf{W}_\varepsilon)\mathsf{S}(q)} \langle e^{\sum_x \varepsilon^3 q(x) r_x^2}\rangle^{\mathrm{ND}}_{k,X}$$

for a real scalar source. At $h = 0$ the multiplier is 1 and a is dropped: $Z_{k,J}(z) := \langle e^{\sum_j z_j F(f_j)}\rangle^{(a)}_{k,J}$, $Z^K_{k,J}(z) := \langle e^{\sum_j z_j F(f_j)}\rangle^K_{k,J}$ for $z \in \mathbb{C}^n$. The *cone* is $\mathcal{K} := \{(f,g) : f \text{ a real 2-form},\ g \ge 0\}$ with the observable $\Theta(\mathrm{i}f, g)$ (*class* $\mathcal{I}$); *class* $\mathcal{D}$ consists of the observables $\Theta(f,-g)$, $(f,g) \in \mathcal{K}$. Transport: $(O\cdot h)(x) := h(O^{-1}x)$, $(O\cdot f)_x(u,v) := f_{O^{-1}x}(O^{-1}u, O^{-1}v)$ for $O \in O(3)$; $\tau_b h := h(\cdot - b)$, $(\tau_b f)_x := f_{x-b}$. For $\nu \in \{1,2,3\}$, $c \in \mathbb{R}$ let $\rho = \rho_{\nu,c}$, $x \mapsto x - 2(x_\nu - c)e_\nu$; $\theta h := h \circ \rho$, $\theta f := \rho^* f$, $(\rho^* f)_x(u,v) := f_{\rho x}(\rho u, \rho v)$ (the components $f_{\mu\nu}$, $\mu \ne \nu$, change sign); $\theta\zeta^* := (\bar z_j, \rho^* f_j; \bar w_j, h_j \circ \rho)_j$, and $\zeta \sqcup \zeta'$ is the union of two families. The *amplitude* is $\mathsf{a}(\zeta) := \sum_j |\operatorname{Re} w_j|\, \|h_j\|_\infty$. The *reference scheme* is $z = a^\sharp$, $a^\sharp_k := e^2 X_{\varepsilon_k}$, X_ε the coefficient of e^2 in the perturbative expansion of $\langle|\phi(0)|^2\rangle$ on $\varepsilon\mathbb{Z}^3$ at $\lambda = 0$ with the order-e^2 part of δm^2_ε inserted. The letter z names this scheme only as a superscript, $Z^z_{k,J}$, or through its terms z_k; as an argument, $z = (z_j)$ is the vector of curvature coefficients.

By charge conjugation $\langle\Theta(\mathrm{i}f,g)\rangle = \langle \cos F(f)\, e^{:\phi^2:(g)}\rangle$ is real for every state above (Section 2.2), and $Z^a_{k,X}(-z,w) = Z^a_{k,X}(z,w)$.

Remark 2.6 (schemes). The Wick leg is stated for the reference scheme z, or for an admissible scheme a (Definition 2.7). If the fixed-cube limits exist in $(0,\infty)$ for both a and z at a real scalar source, the scheme is admissible: for $0 \le h \in C^\infty_c$ with $\int h > 0$ supported well inside some C_J, $Z^z_{k,J}(0,h)/Z^a_{k,J}(0,h) = e^{-(z_k - a_k)\mathsf{S}_{\Lambda_k}(h)}$ converges to a number in $(0,\infty)$, and $\mathsf{S}_{\Lambda_k}(h) \to \int h > 0$. What the Wick leg uses is only $\sup_k |z_k - a_k| < \infty$.

Definition 2.7 (admissible schemes)**.** A scheme a (Definition 2.5) is *admissible* if $a = z$ or $\lim_k (z_k - a_k)$ exists. For an admissible scheme $\sup_k |z_k - a_k| < \infty$.

Definition 2.8 (norms and the tube)**.** For a family with sources $f_1, \dots, f_n$, $h_1, \dots, h_p$ supported in $Q_R := [-R,R]^3$ and $\zeta = (z,w) \in \mathbb{C}^{n+p}$ put $g(\zeta) := \sum_i (\operatorname{Re} z_i) f_i$ and $k(\zeta) := \sum_j (\operatorname{Re} w_j) h_j = k_+ - k_-$ (pointwise positive and negative parts). Lattice norms: $\|g\|^2_{2,k} := \sum_p \varepsilon_k^3 g_{c(p)}(e_\alpha, e_\beta)^2$ over the plaquettes of Λ_k, $\|q\|_{1,k} := \sum_{x\in\Lambda_k} \varepsilon_k^3 |q(x)|$. Continuum norms: $\|g\|^2_2 := \sum_{\alpha<\beta} \|g_{\alpha\beta}\|^2_{L^2}$; $|q|_w := \sup_x (1+|x|^2)^2 |q(x)|$; $|q|_4 := \sup_{x,|\gamma|\le 4} (1+|x|^2)^2 |D^\gamma q(x)|$ and $|g|_4 := (\sum_{\alpha<\beta} |g_{\alpha\beta}|^2_4)^{1/2}$; on scalar functions in $\mathcal{S}(\mathbb{R}^3)$, $|q|_4 = |q|_4$, where $|q|_p := \sup_{x,|\gamma|\le p} |(1+|x|^2)^{p/2} D^\gamma q(x)|$ is the norm of [30, (2.1), p. 284]. For $\sigma, H > 0$ the *tube* is

$$\mathcal{D}_{\sigma,H} := \big\{\zeta \in \mathbb{C}^{n+p} :\ \|k_+(\zeta)\|_\infty < \sigma/2,\quad \|k_-(\zeta)\|_\infty < H/2\big\}.$$

Since $s \mapsto \|(\sum_j s_j h_j)_\pm\|_\infty = \sup_x (\pm \sum_j s_j h_j(x))_+$ is a supremum of convex functions of $s = \operatorname{Re} w$, hence convex and continuous, $\mathcal{D}_{\sigma,H}$ is the preimage under $\zeta \mapsto \operatorname{Re} w$ of the intersection of two open convex sublevel sets: open, convex, hence connected; it contains 0 and every ζ with purely imaginary w; $\|k(\zeta)\|_\infty < \min(\sigma, H)/2$ implies $\zeta \in \mathcal{D}_{\sigma,H}$; its real points form an open subset of $\mathbb{R}^{n+p}$. The tube of the Wick leg is $\mathcal{D}_{\sigma_+,H}$, with σ_+ and H fixed there.

The objects. For real 2-forms $f_1, \dots, f_n$ let $J_0(f)$ be the least J such that every support lies at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_J$. King's *cube limits* and his *state* are

$$Z_J^K(z) := \lim_{k\to\infty} Z_{k,J}^K(z) \quad (J \ge J_0(f)), \qquad Z_\infty^K\Big(\sum_j z_j f_j, 0\Big) := \lim_{J\to\infty} Z_J^K(z),$$

whenever the limits exist: their existence at real z is Theorem A, and their existence at every $z \in \mathbb{C}^n$, entire in z, is derived from it in Section 7. The object is scheme-free. The limit Z_∞^K is *King's state*, the infinite-volume state of King's construction [25, 26], on his own cube convention. For general sources in a scheme a the *tube object* of King's state is $G_J^K(\zeta) := \lim_k Z_{k,J}^{a,K}(\zeta)$, $G_\infty^K(\zeta) := \lim_J G_J^K(\zeta)$, $\zeta \in \mathcal{D}_{\sigma_+,H}$ (Sections 12–28; its existence is Theorem 28.5). The corresponding limits of the comparison family along a continuum subsequence $\mathbf{k}$ are written $Z_J^{a,\mathbf{k}}$, $Z_\infty^{a,\mathbf{k}}$ (Section 14), and they coincide with G_∞^K on the tube (Proposition 28.7).

2.4. **Elementary facts.** The following lemma collects the elementary facts about lattice sums, the norms of the sources and the sampling of continuum sources that are used throughout the paper.

Lemma 2.9 (elementary facts)**.** (E1) *For $\varepsilon \le 1$ and any translate Γ of $\varepsilon\mathbb{Z}^3$: $\sum_{x\in\Gamma} \varepsilon^3(1+|x|^2)^{-4} \le C_0 := (5/2)^4\pi^2/8 < 49$ and $\sum_{x\in\Gamma} \varepsilon^3(1+|x|^2)^{-2} \le C_0' := (5/2)^2\pi^2 < 62$.* (E2) *$\|q\|_{L^1} \le \pi^2|q|_w$, $\|q\|_{L^2} \le \frac{\pi}{2\sqrt{2}}|q|_4$, $\|q\|_\infty \le |q|_w \le |q|_4$, $\|g\|_2 \le \frac{\pi}{2\sqrt{2}}|g|_4$; $|\cdot|_w$ is a norm, so $|k(\zeta)|_w \le \sum_j |\operatorname{Re} w_j||h_j|_w$ and $|k_\pm|_w \le |k|_w$.* (E3) *Sampling is evaluation at points: it commutes with pointwise positive and negative parts, and a sampled sup is at most the continuum sup, so $\zeta \in \mathcal{D}_{\sigma,H}$ for the continuum sources implies the same bounds for the sampled ones. For continuous compactly supported g, q: $\|g\|_{2,k} \to \|g\|_2$, $\|q\|_{1,k} \to \|q\|_{L^1}$, and for every k: $\|g\|_{2,k}^2 \le C_0|g|_4^2$, $\|q\|_{1,k} \le C_0'|q|_w$.* (E4) *$n^{n/2} \le e^{n/2}(n!)^{1/2}$.*

Proof. (E1) For y in the cube of side ε centred at $x \in \Gamma$, $|x-y|^2 \le \frac{3}{4}$ and $|y|^2 \le 2|x|^2+2|x-y|^2 \le 2|x|^2+\frac{3}{2}$, so $1+|y|^2 \le \frac{5}{2}(1+|x|^2)$ and $(1+|x|^2)^{-r} \le (5/2)^r(1+|y|^2)^{-r}$. Integrate over the cube and sum over the disjoint cubes: $\sum_{x\in\Gamma} \varepsilon^3(1+|x|^2)^{-r} \le (5/2)^r \int_{\mathbb{R}^3}(1+|y|^2)^{-r}dy$, with $\int(1+|y|^2)^{-4} = 4\pi\cdot\frac{\pi}{32} = \frac{\pi^2}{8}$ and $\int(1+|y|^2)^{-2} = 4\pi\cdot\frac{\pi}{4} = \pi^2$. (E2) $\|q\|_{L^1} \le |q|_w \int(1+|y|^2)^{-2}$, $\|q\|_{L^2}^2 \le |q|_4^2 \int(1+|y|^2)^{-4}$; the rest is immediate. (E3) Riemann sums for the convergences; the plaquette centres of one orientation form a translate of $\varepsilon_k\mathbb{Z}^3$, so (E1) bounds $\sum_{p:(\alpha,\beta)} \varepsilon_k^3 g_{c(p)}(e_\alpha, e_\beta)^2$ by $C_0|g_{\alpha\beta}|_4^2$, and the three orientations sum to $C_0|g|_4^2$; the last bound is (E1) with $r = 2$. (E4) is $n^n \le e^n n!$. □

Lattice-to-continuum. On a box X let $\langle A, B\rangle_k := \sum_{x\in X}\sum_\alpha \varepsilon^3 A_\alpha(x)B_\alpha(x)$, and for a real 2-form g whose non-vanishing plaquettes have their corners in X define d_k^*g by $F_{\Lambda_k}(g) = \langle d_k^*g, A\rangle_k$. Each component $(d_k^*g)_\gamma$ is a backward difference quotient of the sampled components $g_{\gamma\beta}$, $\beta \ne \gamma$; for smooth compactly supported g it converges uniformly to $(d^*g)_\gamma$, up to the orientation sign convention of the components, at the rate $O(\varepsilon_k)$ of Taylor's theorem. Hence

$$\|d_k^*g\|_{2,k} \to \|d^*g\|_{L^2}, \qquad \sup_k \|d_k^*g\|_{2,k} < \infty, \qquad \|g\|_{2,k} \to \|g\|_2 \qquad (k \to \infty). \tag{2}$$

2.5. **Correlation inequalities.** For a BFS measure, a real linear functional F of A and $g \ge 0$ on V put $\Theta_{F,g} := e^{\mathrm{i}F}e^{\sum_x g_x r_x^2}$, $\Theta_{F,g}^- := e^F e^{-\sum_x g_x r_x^2}$, and $\langle X; Y\rangle := \langle XY\rangle - \langle X\rangle\langle Y\rangle$. Since $F(f)$ is a real linear functional of A and $e^{:\phi^2:(g)}$ a positive constant times $e^{\sum_x \varepsilon^3 g(x) r_x^2}$, class $\mathcal{I}$ consists of multiples of observables $\Theta_{F,g}$ and class $\mathcal{D}$ of multiples of observables $\Theta_{F,g}^-$.

Lemma 2.10 (correlation inequalities)**.** *Let $\langle\cdot\rangle$ be a BFS measure, F a real linear functional of A, $g \ge 0$, $\Theta = \Theta_{F,g}$, $\Theta^- = \Theta_{F,g}^-$. Then $\langle\Theta\rangle$ is real, and*

(a) $\langle\Theta;|\ell(A)|^2\rangle \le 0$ *for every complex linear functional* ℓ *of* A*; hence adding* $s|\ell(A)|^2$ *to the action,* $s\ge 0$*, does not decrease* $\langle\Theta\rangle$*, and deleting a photon gradient term does not increase it;*

(b) $\langle\Theta; r_x r_y\cos(\vartheta_x-\vartheta_y-\ell_{xy}(A))\rangle\ge 0$*; hence decreasing* c_{xy} *does not increase* $\langle\Theta\rangle$*;*

(c) $\langle\Theta; r_x^2\rangle\ge 0$*; hence multiplying* ρ_x *by* e^{-sr^2}*,* $s\ge 0$*, does not increase* $\langle\Theta\rangle$*, and killing a set* U *of sites does not increase it;*

(d) *for* Θ^- *the three inequalities hold with the opposite signs, and each monotonicity statement holds with the same hypothesis and the opposite conclusion ("does not increase" and "does not decrease" interchanged);*

(e) *if the pairs in* B *and the Gaussian* γ *do not couple the fields at the sites of* $X\subset V$ *(and the photon variables attached to them) to the others, the measure is a product, and for an observable depending only on the fields in* X *the expectation equals the expectation in the* X*-factor.*

Proof. We first derive (a)–(c) from the cited correlation inequalities, obtaining the killing statements by decoupling one site at a time; we then prove the three inequalities of (d) by a duplicate-variable expansion in which every term has a definite sign; (e) is clear.

(a)–(c) are [9, Thms. 6.1, 6.2(1),(2)], whose hypotheses are those of a BFS measure: an arbitrary finite lattice, $c_{xy}\ge 0$, finite single-site measures on $[0,\infty)$ with $\int e^{\alpha r^2}d\rho_x<\infty$ for all $\alpha\ge 0$ (a point mass at 0 is one), an arbitrary centred Gaussian with covariance $C\ge 0$, and $g\ge 0$ (so $g=0$ is included). Theorem 6.1 states $\langle P(r)\cos(n\cdot\vartheta+f\cdot A);Q(r)\cos(m\cdot\vartheta+h\cdot A)\rangle\ge 0$ for polynomials P,Q in the r_x with positive coefficients; (a), (b) follow as in the proof of Theorem 6.2(1),(2) given there: $e^{\sum g_x r_x^2}$ is a limit of such polynomials, $e^{\mathrm{i}F}$ may be replaced by $\cos F$ because the expectation and both observables are invariant under $(A,\vartheta)\mapsto(-A,-\vartheta)$ (which also gives the realness), and $|\ell(A)|^2=\ell_1(A)^2+\ell_2(A)^2$ with $\ell_r(A)^2=\lim_{t\downarrow 0}t^{-2}[1-\cos(t\ell_r(A))]$; (c) is Theorem 6.1 with $Q=r_x^2$ and $m=0$, $h=0$. Each monotonicity statement follows by differentiating in the parameter, every interpolating measure being a BFS measure. For the killing statements of (c) and (d), let $x\in V$ and let $\langle\cdot\rangle_x$ be the measure with ρ_x replaced by the point mass at 0, in which every term $c_{xy}r_xr_y\cos(\cdot)$ vanishes. Decreasing c_{xy} to 0 for every pair $\{x,y\}\in B$ containing x does not increase $\langle\Theta\rangle$, by (b), and does not decrease $\langle\Theta^-\rangle$, by (d). In the resulting measure (r_x,ϑ_x) is coupled to nothing, so by (e) it is the product of $\rho_x(dr_x)\,d\vartheta_x$ and of the law of the other variables, which is that of $\langle\cdot\rangle_x$; the expectations of Θ and Θ^- in it are $m_x^+\langle\Theta\rangle_x$ and $m_x^-\langle\Theta^-\rangle_x$ with $m_x^\pm:=\int e^{\pm g_x r^2}d\rho_x/\int d\rho_x$, so that $m_x^-\le 1\le m_x^+$. Since $\Theta^->0$, $\langle\Theta^-\rangle_x\ge m_x^-\langle\Theta^-\rangle_x\ge\langle\Theta^-\rangle$; and $\langle\Theta\rangle_x\ge 0$ (multiplying the single-site measures by $e^{g_y r_y^2}$ gives a BFS measure, in which the expectation of $e^{\mathrm{i}F}$ is non-negative by identity (b) of Lemma 2.11, whose proof does not use the present lemma), hence $\langle\Theta\rangle_x\le m_x^+\langle\Theta\rangle_x\le\langle\Theta\rangle$. Several sites are killed one at a time; no condition on ρ_x beyond the definition of a BFS measure is used.

(d) is stated in [9, Thm. 6.2(3),(4)] for the photon and the hopping inequality with a sketched proof, and its r^2-inequality is [26, (3.28)], stated there without proof; we prove all three. Let μ be the unnormalised measure and $\omega'=(r',\vartheta',A')$ an independent copy of $\omega=(r,\vartheta,A)$; then $(\int d\mu)^2\langle\Theta^-;Y\rangle=\iint\Theta^-(\omega)(Y(\omega)-Y(\omega'))\,d\mu(\omega)d\mu(\omega')$. Write $r=u-v$, $r'=u+v$ (so $u_x\ge|v_x|$), $\vartheta=\vartheta_++\vartheta_-$, $\vartheta'=\vartheta_+-\vartheta_-$, $A=a_++a_-$, $A'=a_+-a_-$. Then $a_\pm=(A\pm A')/2$ are independent with the common centred Gaussian law of covariance $C/2$, and the radial law $\prod_x d\rho_x(u_x-v_x)d\rho_x(u_x+v_x)$ on $\{u\ge|v|\}$ is invariant under $v_x\mapsto -v_x$ at each site. With $\chi_{xy}:=\vartheta_{+,x}-\vartheta_{+,y}-\ell_{xy}(a_+)$ and $\psi_{xy}:=\vartheta_{-,x}-\vartheta_{-,y}-\ell_{xy}(a_-)$ the unprimed phase is $\chi+\psi$, the primed one $\chi-\psi$, and

$$r_xr_y\cos(\chi+\psi)+r'_xr'_y\cos(\chi-\psi)=2(u_xu_y+v_xv_y)\cos\chi\cos\psi+2(u_xv_y+v_xu_y)\sin\chi\sin\psi,$$
$$r_xr_y\cos(\chi+\psi)-r'_xr'_y\cos(\chi-\psi)=-2(u_xv_y+v_xu_y)\cos\chi\cos\psi-2(u_xu_y+v_xv_y)\sin\chi\sin\psi,$$

$r_x^2 - r_x'^2 = -4u_x v_x$, $\ell(A)^2 - \ell(A')^2 = 4\ell(a_+)\ell(a_-)$ for real ℓ, and $\Theta^-(\omega) = e^{F(a_+)}e^{F(a_-)}$ $e^{-\sum_x g_x(u_x^2+v_x^2)}e^{2\sum_x g_x u_x v_x}$. Expand $\exp(\sum_b c_b[\text{first line}])$ and $e^{2\sum g_x u_x v_x}$ in power series (absolutely convergent at fixed spacing). Every term of the integrand is a sign σ times a monomial in (u,v) with non-negative coefficient, times the even radial factor $e^{-\sum g_x(u_x^2+v_x^2)}$, times $V(\vartheta_+,a_+)V(\vartheta_-,a_-)$ with one real function V (a product of $e^{F(\cdot)}$, of factors $\cos\chi_b$ or $\sin\chi_b$, and of $\ell(\cdot)$ for the photon slot). The photon integral of $V(\vartheta_+,a_+)V(\vartheta_-,a_-)$ factorises because a_+, a_- are independent and identically distributed. The angular integral factorises by the following argument: per site $d\vartheta\, d\vartheta' = 2\, d\vartheta_+ d\vartheta_-$, and $[0,2\pi)^2$ in the (ϑ,ϑ')-variables is a fundamental domain of the lattice L generated by (π,π) and $(\pi,-\pi)$ in the $(\vartheta_+,\vartheta_-)$-plane, which contains $(2\pi\mathbb{Z})^2$ with index 2; $V(\vartheta_+,\cdot)V(\vartheta_-,\cdot)$ is L-invariant, because the simultaneous shift of $\vartheta_{+,x}$ and $\vartheta_{-,x}$ by π at one site changes every χ_b and ψ_b with $b \ni x$ by π and leaves $\cos\chi_b\cos\psi_b$ and $\sin\chi_b\sin\psi_b$ unchanged; hence $\int_{[0,2\pi)^2} VV\, d\vartheta\, d\vartheta' = 2\int_{L\text{-domain}} VV\, d\vartheta_+ d\vartheta_- = \int_{[0,2\pi)^2} VV\, d\vartheta_+ d\vartheta_- = (\int_0^{2\pi} V)^2$. The radial integral of a monomial vanishes if its degree in some v_x is odd, by the $v \mapsto -v$ symmetry, and is non-negative otherwise, since $u_x \geq |v_x|$. So every term has the sign σ: $\sigma = -1$ for $Y = r_x^2$ (from $-4u_x v_x$) and for the hopping term (second line), $\sigma = +1$ for $Y = \ell(A)^2$ (for complex ℓ, real and imaginary parts). Hence $\langle\Theta^-; r_x^2\rangle \leq 0$, $\langle\Theta^-; r_x r_y\cos(\cdot)\rangle \leq 0$, $\langle\Theta^-; |\ell(A)|^2\rangle \geq 0$. Monotonicity and killing follow as in (a)–(c), with (d)'s hopping inequality in place of (b). (e) is clear. □

Lemma 2.11 (current representation and Gaussian domination)**.** *Let $\langle\cdot\rangle$ be a BFS measure whose Gaussian has covariance C. Integrating out the matter at fixed A, the matter integral is $\Psi(A) = \sum_q w(q)\cos\langle\kappa(q), A\rangle$, q running over the integer-valued divergence-free currents on B, $w(q) \geq 0$, $\sum_q w(q) < \infty$, $\langle\kappa(q), A\rangle := \sum_{xy} q_{xy}\ell_{xy}(A)$. Let $\mathbb{E}^\sharp$ be the probability measure on currents with weights proportional to $w(q)e^{-\langle\kappa(q),C\kappa(q)\rangle/2}$. Then for $v \in \mathbb{R}^m$, $s \in \mathbb{R}$ (C is symmetric, so $\langle Cv,\kappa\rangle = \langle v, C\kappa\rangle$):*

$$\text{(a)} \qquad \langle e^{s\langle v,A\rangle}\rangle = e^{s^2\langle v,Cv\rangle/2}\,\mathbb{E}^\sharp\cos\big(s\langle Cv,\kappa\rangle\big) \leq e^{s^2\langle v,Cv\rangle/2},$$

$$\text{(b)} \qquad \langle e^{\mathrm{i}\langle v,A\rangle}\rangle = e^{-\langle v,Cv\rangle/2}\,\mathbb{E}^\sharp\cosh\big(\langle Cv,\kappa\rangle\big) \;\geq\; e^{-\langle v,Cv\rangle/2} > 0.$$

Proof. Each bond factor expands as $e^{c_{xy}r_x r_y\cos t} = \sum_{q\in\mathbb{Z}} I_q(c_{xy}r_x r_y)e^{\mathrm{i}qt}$ with $I_q \geq 0$, $I_{-q} = I_q$, $\sum_q I_q(c) = e^c$; after the radial integrals the multiple series converges absolutely, dominated by $\prod_x e^{\frac{1}{2}\sum_y c_{xy}r_x^2}$, integrable against the single-site measures. Integrating the angles forces $\sum_y q_{xy} = 0$ at every site (at a killed site $r_x = 0$ and $I_q(0) = \delta_{q0}$, so the currents vanish on its bonds); pairing q with $-q$ gives Ψ with $w(q) \geq 0$. For a centred Gaussian γ with covariance C and $a \in \mathbb{C}^m$, $\int e^{\langle a,A\rangle}d\gamma = e^{\langle a,Ca\rangle/2}$; with $a = sv \pm \mathrm{i}\kappa$ and the average of the two signs, $\int e^{s\langle v,A\rangle}\cos\langle\kappa,A\rangle\, d\gamma = e^{s^2\langle v,Cv\rangle/2 - \langle\kappa,C\kappa\rangle/2}\cos(s\langle v,C\kappa\rangle)$. Sum over q with the weights $w(q)$, divide by the same sum at $s = 0$ (positive), and use $|\cos| \leq 1$. For the imaginary exponent take $a = \mathrm{i}v \pm \mathrm{i}\kappa$: $\int e^{\mathrm{i}\langle v,A\rangle}\cos\langle\kappa,A\rangle\, d\gamma = \frac{1}{2}\big[e^{-\langle v+\kappa,C(v+\kappa)\rangle/2} + e^{-\langle v-\kappa,C(v-\kappa)\rangle/2}\big] = e^{-\langle v,Cv\rangle/2 - \langle\kappa,C\kappa\rangle/2}\cosh\langle v,C\kappa\rangle$; the same sum over q and the same division give the second identity, and $\cosh \geq 1$ gives the lower bound. □

The lemma applies to every state of this paper at every k, J, $\mu_0 > 0$, with or without a real scalar source absorbed in the single-site measures (the photon covariance is unchanged).

Lemma 2.12 (killing)**.** *Let $\langle\cdot\rangle$ be a BFS measure of King type, U a set of alive sites, $\chi : U \to (0,\infty)$, and $\langle\cdot\rangle_{U,s}$ the measure with ρ_x multiplied by $e^{-s\varepsilon^3\chi(x)r_x^2}$ for $x \in U$. Let Θ be continuous with $|\Theta(A,\phi)| \leq e^{\ell(A)}e^{C\sum_x r_x^2}$ for a real linear functional ℓ and $C \geq 0$. Then $\langle\Theta\rangle_{U,s} \to \langle\Theta\rangle_U$ as $s \to \infty$, $\langle\cdot\rangle_U$ the measure with ρ_x replaced by the point mass at 0 for $x \in U$ (the state with the matter killed on U).*

Proof. The unnormalised density is

$$e^{-Q(A)/2}dA \prod_{x \text{ alive}} w_x(r_x) r_x dr_x d\vartheta_x \prod_b e^{c_b r_x r_y \cos(\cdot)} \prod_{x\in U} e^{-s\varepsilon^3 \chi(x) r_x^2},$$

Q positive definite, killed sites carrying no variables. Substitute $r_x = (s\varepsilon^3\chi(x))^{-1/2}u_x$ for $x \in U$; the Jacobians cancel between numerator and denominator. *Domination for $s \geq s_0$:* at $x \in U$, $w_x(r_x) \leq e^{\varepsilon^3|\delta m_\varepsilon^2| r_x^2/2}$ (the other exponents in w_x are ≤ 0), $|\Theta|$ contributes $e^{Cr_x^2}$, and each of the at most six bond weights at x is at most $e^{\frac{\varepsilon}{2}r_x^2 + \frac{\varepsilon}{2}r_y^2}$ ($0 \leq c_b \leq \varepsilon$); in the variable u_x these factors together are $\leq e^{u_x^2/2}$ once $s \geq s_0(\varepsilon, m, \delta m_\varepsilon^2, C, \min\chi)$, so the U-factor is at most $e^{-u_x^2/2}u_x$, integrable on $[0,\infty)\times[0,2\pi)$; at an alive $y \notin U$ the factor is at most $w_y(r_y)e^{(C+3\varepsilon)r_y^2}r_y$, integrable since $\lambda > 0$; the photon factor $e^{\ell(A)}e^{-Q(A)/2}$ is integrable since Q is positive definite. *Pointwise limit:* at fixed $(u, \vartheta, A, r_{V\setminus U})$, $r_x \to 0$ for $x \in U$, so $\Theta \to \Theta|_{\phi_U=0}$ by continuity, every bond weight with an endpoint in U and every $w_x(r_x)$, $x \in U$, tends to 1, and the U-factor tends to $e^{-u_x^2}u_x\, du_x d\vartheta_x$, whose integral cancels between numerator and denominator. By dominated convergence the ratio tends to $\int \Theta|_{\phi_U=0} d\mu_U / \int d\mu_U$, where μ_U has the cosines at U removed, the weights w_y (which already contain all diagonals at y) and the photon unchanged, and the point mass at U: that is $\langle\cdot\rangle_U$, with a positive denominator. □

Lemma 2.13 (cutting)**.** *Let $\langle\cdot\rangle_Y$ be a Neumann–Dirichlet box state on Λ_k, possibly with killed sites, or a periodic state on a King torus, and let X be a set of sites of Y such that X and its nearest neighbours lie in one fundamental cell (a clause in torus language: for a box state it has no content beyond $X \subset Y$). Let $\langle\cdot\rangle_X^{\mathrm{ND}}$ be the Neumann–Dirichlet state on X whose single-site measures (killed sites and the diagonal parts of the bonds leaving X included) are those of $\langle\cdot\rangle_Y$. Let $(f, g) \in \mathcal{K}$ be such that every plaquette and every site at which f, respectively g, does not vanish lies in X. Then, at every lattice spacing,*

$$\langle\Theta(\mathrm{i}f, g)\rangle_X^{\mathrm{ND}} \leq \langle\Theta(\mathrm{i}f, g)\rangle_Y, \qquad \langle\Theta(f, -g)\rangle_X^{\mathrm{ND}} \geq \langle\Theta(f, -g)\rangle_Y,$$

and the same for Z^a. The first inequality uses Lemma 2.10(a),(b),(e) only; the second uses (d),(e). If no site of X is killed, $\langle\cdot\rangle_X^{\mathrm{ND}}$ is the state of Definition 2.2.

Proof. Delete from Y every photon gradient term coupling a site of X to a site outside X, and the cosine parts of all hopping terms between X and its complement, keeping their diagonal parts $\frac{\varepsilon}{2}(r_x^2 + r_y^2)$. By Lemma 2.10(a),(b) this does not increase the first expectation, by (d) it does not decrease the second, and every intermediate measure is a BFS measure. The result is a product (Lemma 2.10(e)) whose X-factor is $\langle\cdot\rangle_X^{\mathrm{ND}}$: the diagonal parts of the cut bonds are exactly the Dirichlet terms and the remaining photon terms the Neumann ones of X; the observable depends on the X-variables only. The free Wick constant, δm_ε^2 and the scheme multiplier are the same on both sides. □

With $Y = X_{J+1}$, $X = X_J$ this is King's monotonicity in the volume [26, Thm. 3.2, (3.31)–(3.33)] between comparison cubes, in the form his proof proves: with one Wick constant for all J, the free subtraction King himself switches to at the start of that proof [26, p. 333]. With the Dirichlet Wick constant of [26, (2.13)], which increases with J, the c-number factor moves against the monotone factor and the monotonicity does not follow; that form is not used. The box version with killed sites and the monotonicity of King's RG cubes are in Section 4.

2.6. **Algebra of the scheme, covariance, reflection, continuity.** This subsection collects four standing lemmas on the functionals Z^a: the algebra of the scheme multiplier (Lemma 2.14), passive covariance (Lemma 2.15), the reflection sandwich at fixed spacing on the comparison convention (Lemma 2.16) and the continuity of convex Borel functions (Lemma 2.17).

Lemma 2.14 (algebra of the scheme)**.** *Let a be a scheme and Z^a the functional of a state above (Section 2.2) on a box at spacing ε_k.*

(a) *For real coefficients w the multiplier $e^{-\sum_j w_j a_k \mathsf{S}(h_j)}$ is a positive constant depending only on k, w and $\mathsf{S}_{\Lambda_k}(h_j)$. Hence the inequalities of Lemmas 2.10 and 2.13 between two states on the same lattice with the same sources hold for Z^a in the same direction.*

(b) *$|Z^a(\zeta)| \le Z^a(\operatorname{Re}\zeta)$, and $\ln Z^a$ is convex in the real coefficients.*

(c) *$Z^a(f,h)\, Z^a(-f,-h) \ge 1$ for real (f,h).*

(d) *For real f and real $k_1, k_2 \ge 0$: $Z^a(f, k_1 - k_2) \le Z^a(2f, -2k_2)^{1/2}\, Z^a(0, 2k_1)^{1/2}$.*

Proof. (a) is immediate. (b) $|\Theta(\zeta)| = \Theta(\operatorname{Re}\zeta)$ and the multiplier has modulus $e^{-\sum_j (\operatorname{Re} w_j) a_k \mathsf{S}(h_j)}$; convexity is Hölder's inequality. (c) $\langle e^X\rangle\langle e^{-X}\rangle \ge 1$ by Cauchy–Schwarz, and the multipliers cancel. (d) Cauchy–Schwarz gives $\langle e^{F+:\phi^2:(k_1)-:\phi^2:(k_2)}\rangle \le \langle e^{2F-2:\phi^2:(k_2)}\rangle^{1/2}\langle e^{2:\phi^2:(k_1)}\rangle^{1/2}$, and $e^{-a_k\mathsf{S}(k_1-k_2)} = (e^{2a_k\mathsf{S}(k_2)})^{1/2}(e^{-2a_k\mathsf{S}(k_1)})^{1/2}$. □

Lemma 2.15 (passive covariance)**.** *Let $O \in O(3)$, Λ a lattice of spacing ε with frame (f_α), and X a box or a torus for Λ. Equip $O\Lambda$ with the frame (Of_α) and OX with the transported boundary conditions. Then $\langle\Theta(z\,O\cdot f, w\,O\cdot h)\rangle_{O\Lambda,OX} = \langle\Theta(zf,wh)\rangle_{\Lambda,X}$ and $\mathsf{S}_{O\Lambda}(O\cdot h) = \mathsf{S}_\Lambda(h)$, likewise for families and for Z^a. On a torus the periodic model does not change if the frame (f_α) is replaced by $(\pm f_{\pi(\alpha)})$, π a permutation.*

Proof. The map $\phi'(Ox) = \phi(x)$, $A'_\alpha(Ox) = A_\alpha(x)$, components taken in the frames (f_α) and (Of_α), is a bijection of the configuration spaces preserving Lebesgue measure; the action is written in the frame, and δm_ε^2, W_ε, a_k depend on ε only; plaquettes, centres and field strengths are transported, and $(O\cdot f)_{Oc}(Of_\alpha, Of_\beta) = f_c(f_\alpha, f_\beta)$. A permutation of the frame relabels the components; reversing f_α replaces $A_\alpha(x)$ by $-A_\alpha(x-\varepsilon f_\alpha)$, a bijection of the torus variables preserving the action, which changes the sign of both $F(p)$ and $f_{c(p)}(f_\alpha, f_\beta)$ and preserves the set of plaquette centres. □

Lemma 2.16 (the reflection sandwich at fixed spacing, comparison convention)**.** *Let $\sigma_3 = \operatorname{diag}(1,1,-1)$ and let $(f,g) \in \mathcal{K}$ be supported at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_{J-1}$. Then for every $k \ge 0$*

$$Z^a_{k,J-1}(\mathrm{i}f, g) \le Z^a_{k,J}(\mathrm{i}\,\sigma_3\cdot f, \sigma_3\cdot g) \le Z^a_{k,J+1}(\mathrm{i}f, g),$$

and the reversed inequalities hold for $Z^a(f,-g)$.

Proof. σ_3 preserves Λ_k and S_{Λ_k} and maps X_J onto $X_+ := (C_J + \varepsilon_k e_3)\cap\Lambda_k$. By Lemma 2.15, $Z^a_{k,J}(\mathrm{i}\sigma_3\cdot f, \sigma_3\cdot g)$ is the functional with sources (f,g) of the Neumann–Dirichlet state on X_+ in the frame $(e_1, e_2, -e_3)$. Written in the frame (e_α), with $A'_3(x) = -A_3(x-\varepsilon e_3)$, this is the BFS measure R in which ϕ, A_1, A_2 live on X_+ and A_3 on $X_+ - \varepsilon e_3 = X_J$, with Dirichlet conditions for ϕ on X_+, photon gradient and mass terms within these sets, and the hopping term in direction 3 the standard one of the bond $\langle y - \varepsilon e_3, y\rangle$; the plaquette terms $(x;\alpha,3)$ pick up two sign changes and keep their centres, so the observable is $\Theta(\mathrm{i}f,g)$ in the standard frame. R is obtained from the Neumann–Dirichlet state on X_{J+1} by deleting photon gradient terms and hopping cosines and factorising, since $X_+ \cup X_J \subset X_{J+1}$, and the state on X_{J-1} is obtained from R in the same way, since $X_{J-1} \subset X_J \cap X_+$; the diagonal parts of the cut bonds are the Dirichlet terms of the smaller state, and the sources stay inside X_{J-1}. Lemma 2.10(a),(b),(e), respectively (d),(e), give the directions, and Lemma 2.14(a) carries them to Z^a. □

Lemma 2.17 (continuity of convex functions)**.** *Let U be a non-empty open convex subset of a Fréchet space and $\Phi : U \to \mathbb{R}$ a convex Borel function. Then Φ is continuous on U.*

Proof. U is a Baire space and $U = \bigcup_{j\in\mathbb{N}}\{\Phi \le j\}$, so some $S := \{\Phi \le j\}$ is non-meagre in U; being Borel, S has the Baire property [24]: $O \setminus \mathcal{M} \subset S$ for a non-empty open $O \subset U$ and a

meagre $\mathcal{M}$. Pick $x_0 \in O$ and a balanced open neighbourhood W of 0 with $x_0 + W + W \subset O$. For $w \in W$ and $y := x_0 + w$, $y + W \subset O$, and the point reflection $\iota(x) := 2y - x$ maps $y + W$ onto itself and preserves meagre sets; so $S \cap (y + W)$ and $\iota(S \cap (y + W))$ are comeagre in $y + W$ and intersect: there is $u \in S$ with $2y - u \in S$, and $\Phi(y) \le \frac{1}{2}(\Phi(u) + \Phi(2y - u)) \le j$. Hence $\Phi \le j$ on $x_0 + W$. A convex function on an open convex set bounded above near one point is bounded above near every point $y \in U$ (choose $z \in U$ with y strictly between x_0 and z, and take the convex combinations of $x_0 + W$ with z), and a convex function locally bounded above is continuous [33]. □

2.7. **Results of Bałaban and King.** Bałaban's analysis of the lattice model on a periodic torus [1, 2, 3, 4] is a block-spin renormalisation-group flow at block factor 5; what this paper takes from it is, first of all, the pair of stability bounds $e^{-E_-|T|} \le Z_{k,T} \le e^{E_+|T|}$ on the source-free partition function of a torus T, with constants independent of the spacing and of the torus (Proposition 12.6). We use this analysis in three places. On King's tori (Section 12) the stability bounds at the two scalar masses m and M_1, with the c-number computation of Lemma 12.7 and the sign clause (T3), bound the free energy of a constant positive Wick source uniformly in the spacing; this is the extensive bound (I2) (Proposition 12.8), which is what the tube bound of Section 13 takes from Bałaban's work. On King's cube, with Neumann conditions for the photon and Dirichlet conditions for the scalar (Definition 2.3), the same flow approximates the generating functional by an integral of the effective action and compares two spacings on a common unit lattice; this gives the Cauchy property of the normalised functional and hence the existence of King's cube limits (Theorem A, proved in Section 5). In the rotation part the flow is run in two settings outside the printed cases: a hybrid flow on the axis lattice whose first step averages over rotated blocks, for the comparison with the rotated lattice (Section 19), and a shell model with a heavy scalar mass on a shell inside every matter covariance (Sections 24 and 25); the estimates needed there are stated and proved in those sections.

Bałaban's torus analysis [1, 2, 3, 4] enters this paper in three ways. The first is the torus theorems of Section 12 (his stability bounds, used as stated at two scalar masses, and a c-number computation). The second is the statements of Section 5 in the existence part. The third is the re-runs of this analysis for the hybrid flow in the rotation part, stated with their proofs in Sections 19, 24 and 25. The following remark lists what is used beyond the printed order or case.

The hybrid flow of the torus comparison and the shell model of the heavy-shell decoupling are not the models treated by King and Bałaban. The first step of the hybrid flow averages over rotated blocks; the shell model carries the shell mass inside every covariance, and its effective action carries the face residues, and in the derivative one inserted vertex, as further vertices. The published arguments nevertheless carry over step by step wherever a step depends only on properties which these models share with the published ones, such as the small-field bounds on the scalar legs or the vanishing of the low moments of the vertices. The remark collects the steps used in this way, together with the readings of Bałaban's argument which the shell model requires, so that the extensions, each proved at the place indicated, can be seen together.

Remark 2.18 (statements of Bałaban and King used beyond their printed order or case). The following statements of [25] and [1, 2, 3, 4] are used in a form read from the printed argument rather than from a printed statement, beyond the order, the model or the case treated in print; each is stated in that form, with its proof, at the place indicated. (i) King's constant-background reduction [25, pp. 661–662, (3.43)–(3.55)]: the identities (3.43)–(3.44), (3.47)–(3.48) and (3.50)–(3.53) hold for any background, and in (3.47)–(3.55) the non-constant part of the background enters the bounds only through (3.44)–(3.45); so the reduction runs at the backgrounds of the hybrid flow (Theorem 19.10(ii), Section 19). (ii) King's renormalisation induction [25, pp. 663–668,

(3.66)–(3.70), (3.76)–(3.79)] with Bałaban's enumeration of the divergent families [3, Sects. 2–3]: the same induction is run for the shell model, with its face residues as vertices, and for the derivative of the effective action with one inserted vertex (the renormalisation bounds of the shell model, Theorems 24.5 and 25.11, Sections 24 and 25). (iii) Bałaban's combinatorial estimate [2, p. 592, (3.41)–(3.42)]: the two $O(1)$ of its exponents are read as constants independent of the scale, the spacing, the torus and the family history, the constant condition of the shell model (Sections 23 and 25). (iv) The Taylor step on the external legs [3, p. 435, (3.10); pp. 440–442, (3.26)–(3.30)]: the vanishing of the zeroth and first moments, printed for the vertices (1.8) and (1.10) of Bałaban's model, is used for the seagull and block-averaging members of the shell model (Lemma 25.2(a) and (c)). (v) The consequences drawn in [3, pp. 421–422] from its Proposition 1 for the lower and upper bounds: the same consequences, with the same constants, for the shell model, whose scalar legs obey the same small-field bound (Proposition 24.7(b)–(d)). (vi) The rewriting of a photon two-point class [3, p. 442, (3.30), with pp. 437–445]: the two coefficient clauses of (3.30) are printed; that after the rewriting the explicit block projections sit only inside composite kernels on which no later step acts is a reading of pp. 437–445 carried to every order (Lemma 5.43). (vii) The regularity assumption [1, p. 610, Proposition 2.1, (2.23)], [4, pp. 572–573, (1.7)] and [2, p. 571, Lemma 2.3, (2.59)–(2.60)]: the assumption (1.7) is read as (2.23) with the rescaled charge, so that the three regularity senses coincide and (2.60) implies it once the scale is small; at block factor 5 the bounds (2.59)–(2.60) hold with $O(p(L^k\varepsilon))$ replaced by $O_L(p(L^k\varepsilon))$ (Proposition 19.5).

3. Outline of the proof

In outline, the paper proceeds in five steps; the first four, together with the Wick tube described after them, lead to the main theorem, and the fifth lies beyond it.

The lattice spacing in a fixed cube. King's cube limits exist by Theorem A, and by Theorem B with scalar sources. Both theorems are proved in the existence part (Sections 5 and 6). Without scalar sources the proof follows King, with the comparison of two spacings written in a ratio in which every field-independent constant of the effective action cancels (Theorem 5.45). With scalar sources King's Theorem 2.1 needs a vacuum counterterm that his Wick ordering does not contain, and a scheme a provides it.

The size of the cube. By the correlation inequalities of [9] (Lemma 2.10) the cube states are monotone in J on a cone of observables, which gives the infinite-volume limit. The regularity bounds, with absolute constants, are Theorem 7.6; invariance under translations and the cubic group is Theorem 8.6, to which Theorem 8.8 adds the rotation clause.

Reflection positivity and the measure. Reflection positivity is proved on the lattice for King's modified box (Lemma 9.3) and carried to the cube by a two-sided sandwich between cube states (Lemmas 9.9 and 9.10); for King's state it is obtained in the limits $k \to \infty$ and $J \to \infty$ (Theorem 10.5). The measure, its ergodic decomposition and the cluster axiom on the components are Theorem 11.5.

From the cubic group to all rotations. This step is Theorem C, proved in the rotation part (Sections 15–27). A lattice and its copy rotated by $\theta_0 = \arctan(3/4)$ are compared on periodic tori carrying a smooth mass wall. On each lattice King's cube functional is bounded by walled-torus functionals; the walled tori on the two lattices have the same continuum limit outside the exceptional set E_W (Theorem 18.1); and King's state is the lower envelope of the walled tori (Theorem 21.4).

The photon mass. All of the above is at a fixed photon mass; Section 30 takes the limit of vanishing photon mass in the curvature sector, beyond the main theorem, and Section 31 records a second limit as a further direction.

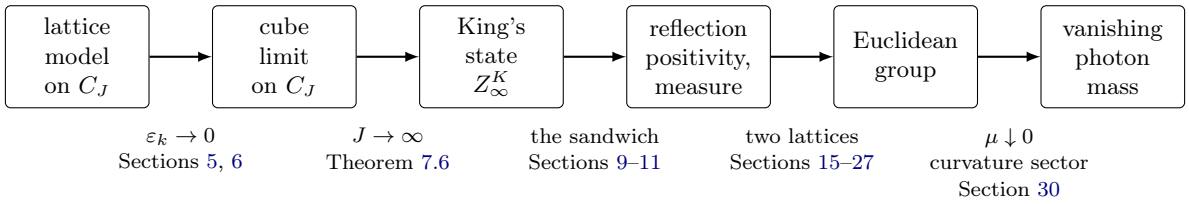

FIGURE 1. The five steps of the paper, with the sections (for the second step, the theorem) that carry them out. The first four enter the main theorem, which rests on rows (1)–(7) of Table 1, among them the Wick tube (Sections 12 and 13, under the sign clause (T3)) and the coupling range (Section 29), and is assembled in Section 28. Continuum subsequences form Section 14; King's state with scalar sources, with Euclidean invariance on a tube, is treated in Section 27 (Theorem 27.13, which is not used by the main theorem). The fifth step is taken in the curvature sector.

With real scalar sources, the renormalised Wick square is added to the curvature under the sign clause (T3), a one-sided condition on the couplings. The joint functional of the curvature and the Wick square is controlled on the tube by three results: reflection positivity and the chessboard estimate on King's periodic tori, the correlation inequalities, and an extensive bound for constant positive scalar sources (Proposition 12.8). The tube bound is proved on the comparison convention (Theorem 13.10) and transferred to King's convention (Theorem 28.5); Section 14 treats continuum subsequences. The main theorem is assembled in Section 28, and Section 29 establishes the coupling range.

Figure 1 shows the five steps.

The main theorem, Theorem 28.2 (Section 28), rests on three theorems of this paper, stated in this section and proved later: the existence of King's cube limits, without and with scalar sources (Theorems A and B; the existence part, Sections 5 and 6), and the rotation theorem (Theorem C; the rotation part, Sections 15–27, whose main theorem is Theorem 15.1). It rests further on the two torus theorems of the Wick leg, stated in Section 12, where they are used: Bałaban's stability bounds (Proposition 12.6), that is [1, (1.14)], [2, (1.3)] used as stated there, read at two scalar masses and in King's torus normalisation of the counterterm, and the c-number lemma (Lemma 12.7), a computation with lattice Feynman integrals. One definition completes the list, the coupling region of the Wick leg (Definition 2.1, Section 2): the set of couplings at which the sign clause (T3) holds, which contains the explicit regions of Proposition 29.29; the Wick leg of the main theorem is stated for couplings in that region, and the curvature leg carries no condition on the couplings. After the two cube conventions, on which every statement is posed, the three theorems are stated, each with the section where it is proved and with what is used of it; the section ends with the route from them to the main theorem.

The two cube conventions. Figure 2 shows them in a plane section. The two cube conventions of Definitions 2.3 and 2.2 are King's convention ND^K (the scalar field killed on the three top face layers T_J of the cube C_J, the convention of the infinite-volume state of King's construction [25, 26]) and the comparison convention $\mathrm{ND}^{(a)}$ (the scalar alive on all of X_J), on which the tube bound (Theorem 13.10), the cell estimates of the Wick leg and the lattice comparisons of the rotation part are written. The two are interlaced by Lemma 4.2. Which convention a statement is posed on is part of it. The existence theorems, Theorems A and B, are posed on King's convention ND^K, on which King's renormalisation-group flow runs (Remark 5.11). By Remark 4.5 the existence of the fixed-cube limits on one convention does not imply it on the other, and the comparison family enters the main theorem through its limit states along

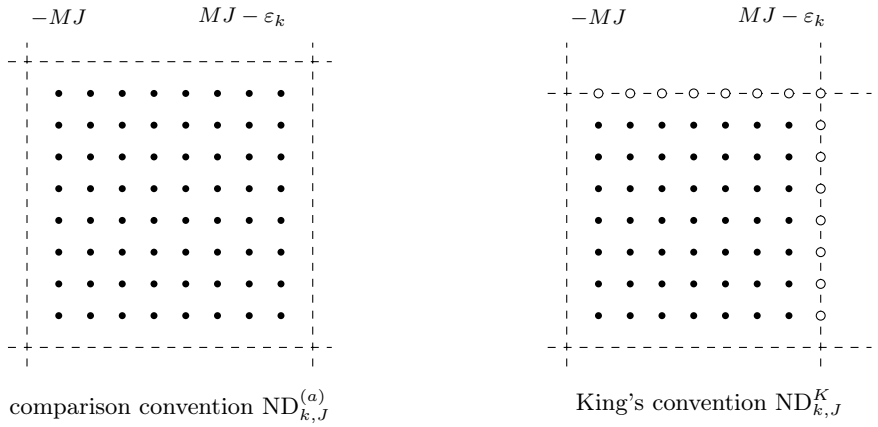

FIGURE 2. The two cube conventions, in a plane section through the cube. Both have the photon variables $A_\alpha(x)$, $x \in X_J = C_J \cap \Lambda_k$ (Definitions 2.2 and 2.3); the marks show the scalar field. On the comparison convention (Definition 2.2) the scalar is alive on all of X_J (filled sites), with Dirichlet planes $x_\mu = -MJ-\varepsilon_k$ and $x_\mu = MJ$ (dashed). On King's convention (Definition 2.3) the scalar is killed on the top face layers T_J (open sites), and the Dirichlet planes are $x_\mu = -MJ-\varepsilon_k$ and $x_\mu = MJ-\varepsilon_k$.

continuum subsequences (Section 14), which exist without any such clause and coincide with King's state (Corollary 14.15, Proposition 28.7). The rotation theorem, Theorem C, is proved in the rotation part for King's state Z^K_∞ on King's convention (Theorem 15.1) and is carried to the limit states of the comparison family through Corollary 4.4. The Euclidean clause of the Wick sector below the amplitude threshold and outside the exceptional set E_W is Theorem 27.13 (in the scheme $\Theta^\sharp$ and in every admissible$_{<s_*}$ scheme on the tube $\Omega_*(h)$, up to the factor of Theorem 27.6(vi)); above the threshold it is open (Section 32). The torus theorems and the sign clause (T3) are statements about King's tori (Definition 12.1) and about c-numbers.

All limit statements of this paper are proved for King's state, whose fixed-cube limits exist by Theorems A and B. Several of them are proved by arguments that use, of the cube functionals, only the existence of the fixed-cube limits at real coefficients, the cube monotonicity on the cone and the bounds established here; a remark in each of the sections concerned notes that these arguments apply verbatim to the comparison family at every family and cube where that family's fixed-cube limits exist. Whether they exist is not decided here (Remark 4.5, Remark 5.48), and no statement of this paper assumes it: the comparison family enters through its limit states along continuum subsequences (Theorem 14.9), which exist unconditionally.

Theorem A (King's cube limits, curvature)**.** *For every finite family $f_1, \dots, f_n$ of real 2-forms in $C^\infty_c(\mathbb{R}^3; \Lambda^2)$, every $J \geq J_0(f)$, where $J_0(f)$ is the least J such that every* $\operatorname{supp} f_j$ *lies at ℓ^∞-distance greater than 1 from $\mathbb{R}^3 \setminus C_J$, and every $z \in \mathbb{R}^n$, the limit*

$$\lim_{k\to\infty} \langle e^{\sum_j z_j F(f_j)} \rangle^K_{k,J}$$

exists and lies in $(0, \infty)$.

Proved in Section 5 (Theorem 5.47), from King's renormalisation-group analysis of the cube. Of Theorem A only the existence of the fixed-cube limit at real coefficients is used: no constants clause, no uniformity in J and no locally uniform convergence in z enter from it, and every bound used on the limit is proved here, with absolute constants, in Theorem 7.6. It is the existence

clause of King's Theorem 2.1 [26] at zero scalar source; the mass clause of [2, Prop. 3.1], upstream of it, is discussed in Remark 29.5 and is not a condition of this paper.

Theorem B (King's cube limits with scalar sources)**.** *Let the couplings lie in the coupling region of Definition 2.1 (the sign clause* (T3)*, under which Proposition 12.8 provides the extensive bound* (I2) *used in the proof), let $z = a^\sharp$ be the reference scheme and let $s_W := s_1$ be the amplitude of Proposition 12.8. For every finite family of real 2-forms $f_j \in C_c^\infty(\mathbb{R}^3;\Lambda^2)$ and real functions $h_i \in C_c^\infty(\mathbb{R}^3)$, every $J \geq J_0(f,h)$, the least J such that every $\operatorname{supp} f_j$ and every $\operatorname{supp} h_i$ lies at ℓ^∞-distance greater than 1 from $\mathbb{R}^3 \setminus C_J$, and every real coefficient vector (z,w) whose positive scalar part $k_+ := \big(\sum_i w_i h_i\big)_+$ satisfies $2\|k_+\|_\infty < s_W$, the condition of the tube $\mathcal{D}_{\sigma_+,H}$ with $\sigma_+ = s_W$ (the negative scalar part is unrestricted), the limit $\lim_{k\to\infty} Z^z_{k,J}(z,w)$ exists and lies in $(0,\infty)$.*

Proved in Section 6 (Theorem 6.26 and Corollary 6.27), by the route of Section 5 with the scheme c-number carried along. At zero Wick coefficient Theorem B is Theorem A. It is *not* King's Theorem 2.1 with scalar sources: in King's Wick ordering that statement fails, and the state with scalar sources exists only after a vacuum counterterm, which the scheme provides (Definition 2.5). The threshold on the positive scalar part is the one at which the positive-source bound of the Wick leg is available (Proposition 13.2); above it nothing is proved (Section 32). By Lemma 2.14(a), Theorem B holds for $Z^a_{k,J}$ in place of $Z^z_{k,J}$ for every admissible scheme a (Definition 2.7).

For a family $f = (f_1,\dots,f_n)$ of real 2-forms and $J \geq J_0(f)$ let L^f_J be the set of locally uniform limits on $\mathbb{C}^n$ of subsequences of $(a^{(a)}_{k,J})_k$, $a^{(a)}_{k,J}(z) = \langle e^{\sum_j z_j F(f_j)}\rangle^{(a)}_{k,J}$ (the comparison family, the superscript (a) marking the comparison convention, on which the scalar is alive on the whole cube; by Lemma 7.3 and Montel's theorem the set is non-empty).

Theorem C (rotation invariance of the comparison family's limit states)**.** *For every finite family f of real 2-forms in $C_c^\infty(\mathbb{R}^3;\Lambda^2)$ and every choice $\zeta_J \in L^f_J$, $J \geq J_0(f)$, the limit $Z^{\mathrm{LIM}}_\infty(\sum_j z_j f_j, 0) := \lim_{J\to\infty} \zeta_J(z)$ exists locally uniformly on $\mathbb{C}^n$, is entire and does not depend on the choices. The functional $Z^{\mathrm{LIM}}_\infty(\cdot,0)$ is invariant under the Euclidean group $\mathbb{R}^3 \rtimes O(3)$: $Z^{\mathrm{LIM}}_\infty(\gamma\cdot f,0) = Z^{\mathrm{LIM}}_\infty(f,0)$ for every Euclidean motion γ and every complex coefficient vector, F transforming as a 2-form.*

Proved in the rotation part: Theorem 15.1 gives the $\mathbb{R}^3 \rtimes O(3)$-invariance of King's state $Z^K_\infty(\cdot,0)$, and by Corollary 4.4(iii), with the fixed-cube limits of Theorem A, every J-limit of subsequential fixed-cube limits of the comparison family exists and equals $Z^K_\infty(\cdot,0)$, which is the statement as posed. King's cubes are compared with periodic tori carrying a smooth mass wall, the walled tori; the route through them is rows (R1)–(R4) of Table 1.

The route to the main theorem. With Theorems A to C in hand, the proof of Theorem 28.2 runs as follows. The curvature sector is treated for King's state in the order of the axioms: the regularity bounds, with absolute constants, in Theorem 7.6 (Section 7); the translations, the cubic group and, from Theorem C, the rotations in Theorems 8.6 and 8.8 (Section 8); reflection positivity about the coordinate planes, proved on King's modified box (Section 9) and carried to the limit together with the symmetry of the Schwinger functions (Theorem 10.5, Section 10); and the measure on the curvature fields, with its ergodic decomposition, the local lattice laws and the clustering of its components (Theorem 11.5, Section 11). The Wick tube is treated on the comparison convention: the torus theorems and the sign clause (T3) give the extensive bound for constant positive scalar sources (Proposition 12.8, Section 12), and a tiling of the negative part of the scalar source into cells of fixed size, each controlled by one instance of Theorem B, gives the uniform tube bound and its limits (Theorem 13.10, Section 13). Continuum subsequences of the comparison family have limit states, in the curvature sector and on the tube,

and these are identified with King's state (Theorem 14.9, Proposition 14.13; Section 14). The tube bound is then transferred to King's convention, on which the limits of the Wick sector are taken (Theorem 28.5), and Theorem 28.2 assembles the axioms (Section 28); Section 29 establishes the coupling range, the explicit regions on which (T3) holds. Section 30 lets the photon mass go to zero in the curvature sector, where the constants are absolute: King's own states at photon masses $\mu_n \downarrow 0$ converge along a subsequence to an entire limit that satisfies clauses (i)–(vi) of Theorem 28.2 B with the first OS0 bound, the lattice sentences excepted, in particular Euclidean invariance and reflection positivity about every plane (Theorem 30.2). Section 31 records a further direction: along the family with the bare scalar coefficient held at the reference photon mass μ_0 (the fixed-counterterm family), the limit states of a common continuum subsequence converge, for every family of curvature sources, as the photon mass goes to zero along a countable set of masses, and the limit satisfies clauses (i)–(vi) of Theorem 14.9 with the same absolute constants, the second OS0 bound and the lattice sentence of clause (v) excepted (Theorem 31.4; the Euclidean clause is not claimed, Remark 31.5). Neither limit is shown to be independent of the choices made (Problem 31.11; Remark 31.5), except conditionally: under two conditions on the counterterm and on King's state in the scalar mass, the limit state of the fixed-counterterm member at a photon mass is King's state at that photon mass and a shifted scalar mass, and if the conditions hold at every photon mass in $(0, \mu_0]$ the limit along the fixed-counterterm family depends neither on the subsequence nor on the set of masses (Proposition 31.6). The Wick leg of the main theorem is the joint field of the curvature and the Wick square on the tube $\mathcal{D}_{\sigma_+,H}$, whose positive Wick amplitudes lie below the threshold of Theorem B: above that threshold neither the existence of the fixed-cube limits nor a uniform positive-source bound is proved, and the Euclidean clause of the Wick sector is proved only below the threshold and outside E_W (Theorem 27.13). These are among the open problems of Section 32, on which no clause of Theorem 28.2 depends. Table 1 lists, for every result, what it uses and where it is proved.

Table 1: What is proved where. "Uses" names what each result uses from the preceding rows and from the literature; a result taken from the renormalisation-group analysis is stated, with its proof, in the section named with it.

Result	Section	Uses	Where proved
The existence part			
(1) Existence of King's cube limits, curvature sources	Section 5	Bałaban's block-factor-5 estimates on tori [1, 2, 3, 4], transferred to King's cube (statements and proofs in Section 5); King's comparison of two spacings at a common block scale [26, Sect. 3.1]	Theorem 5.47; ratio form Theorem 5.45
(1′) Cube limits with scalar sources, reference scheme, below the tube threshold	Section 6	as (1), with the scheme c-number carried along; the sign clause (T3) (through the extensive bound of (6a)); King's localisation bound [25, (3.38)]	Theorem 6.26
Interlacing of the two cube conventions	Section 4	correlation inequalities [9, Thms. 6.1, 6.2], Lemma 2.10; cutting (Lemma 2.13)	Lemma 4.2, Corollary 4.4
The axioms, curvature sector			

Result	Section	Uses	Where proved
(2) Regularity with absolute constants	Theorem 7.6	Gaussian domination (Lemma 2.11); (1)	Theorem 7.6 (Theorem 7.4 on the comparison convention)
(3a) Translations, cubic group	Theorem 8.6	lattice symmetries; interlacing; (2)	Theorem 8.6, Corollary 8.4
(3b) $O(3)$ invariance	Theorem 8.8	rotation theorem (rows (R1)–(R4)); Corollary 4.4(iii)	Theorem 8.8
(4a) Reflection positivity of the modified box; two-sided sandwich	Section 9	[9, Thm. 6.2(1),(2)]; cutting; Gaussian domination	Lemma 9.3; Lemmas 9.9, 9.10
(4b) Reflection positivity in the limit, OS3, polynomial form	Theorem 10.5	(1); (2); (4a); Montel's theorem; Lemma 2.17	Theorem 10.5, Propositions 10.7, 10.8
(5) Measure, ergodic decomposition, clustering, OS4 on the components	Theorem 11.5	(2), (3a), (4b); (3b) for the Euclidean clauses; measure theory on spaces of distributions, ergodic decomposition and reconstruction, taken from the companion paper [32, Appendix A] (see Section 1.4; hypotheses established in Section 11); [29, 30], [18, Thm. 6.1.3], [24]	Theorem 11.5
The Wick tube			
(6a) Torus theorems and the extensive bound	Section 12	Bałaban's stability bounds at two scalar masses (Proposition 12.6); the c-number statement (Lemma 12.7); sign clause (T3); chessboard estimate [17, Thms. 4.1–4.2]	Propositions 12.2, 12.8
(6b) Uniform tube bound, comparison convention	Theorem 13.10	(6a); (1′) one cell at a time; correlation inequalities	Theorem 13.10, Lemma 13.11
Continuum subsequences and the states they define	Section 14	cell bound, scheme bound, independence of the subsequence (Proposition 14.13); (3b)	Theorem 14.9, Corollary 14.15
(6c) Wick tube on King's convention: bound, fixed-cube limits, axioms	Theorem 28.5	(6b); interlacing; (1′)	Theorem 28.5, Proposition 28.6
The rotation part			
(R0) King's state on curvature sources (the object)	Section 17	(1); Lemma 4.2; [26, Thm. 3.2] in the form King's proof proves	Theorem 17.1, Definition 17.2
(R1) Torus comparison on walled tori, two lattices at one spacing	Section 18	corrected form of [26, Thm. 2.4]: one-sided forward bounds on each torus (statement and proof in Section 18) with the large-field clause of King's upper bound; an involution exchanging the lattices	Theorem 18.1; Theorems 18.5, 18.8

Result	Section	Uses	Where proved
(R2) Heavy-shell estimate	Section 23	four estimates on King's block-spin flow with the shell mass inside the matter covariance (statements and proofs in Sections 24 and 25); Bałaban's bounds on his propagator domains	Theorem 23.12
(R3) King's state as the lower envelope of the walled tori	Section 21	(R2); (R1); replacement of the photon beyond a dead shell by King's Neumann photon	Theorem 21.4, Theorem 23.10
(R4) Invariance under the generators; closure to $O(3)$ and to the Euclidean group	Section 22	(R1), (R3); continuity lemma for convex functionals (Lemma 2.17); translations (3a); right-continuity in the scalar mass at the exceptional set	Theorems 22.1, 22.3, 22.4; Theorem 15.1
King's state with scalar sources, Euclidean invariance on a tube	Section 27	cube limits with scalar sources (Section 6); (R1)–(R4) at the scalar source	Theorem 27.13 (not used by Theorem 28.2)
The main theorem and the couplings			
(7) Coupling range: King's range for the curvature leg; (T3) on explicit regions	Section 29	the c-number statement (Lemma 12.7); two-loop majorants of the lattice mass counterterm (statements and proofs in Section 29); the closed form of the limiting bare coefficient (Lemma 29.1); Reisz's power counting [34, 35]	Lemma 29.3, Propositions 29.29 and 29.30
Main theorem	Theorem 28.2	rows (1)–(7)	Theorem 28.2
The limit of vanishing photon mass, curvature sector	Section 30	the main theorem at every point of King's range; the Lipschitz bound of Corollary 8.4 and the extraction argument of Lemma 14.7; the measure and its ergodic decomposition from a limit functional (Theorem 11.5); the cluster property under rotation invariance (Corollary 11.16)	Theorem 30.2
Further directions: the limit along the family with fixed counterterm	Section 31	correlation inequalities (Lemma 2.10); continuum subsequences for countably many models (Lemma 14.7) and the axioms for LIM states (Theorem 14.9); interlacing (Lemma 4.2); the main theorem at every point of King's range	Theorem 31.4; Proposition 31.6 (under Open Problems 31.8–31.10)

4. The two cube conventions: interlacing and margins

King's state lives on King's convention $\mathrm{ND}^{K}_{k,J}$ (Definition 2.3); the rotation part and the Wick-tube estimates of this paper use the comparison convention $\mathrm{ND}^{(a)}_{k,J}$ (Definition 2.2). We show

that the two families are interlaced in J (Lemma 4.2), draw the consequences for the limits (Corollaries 4.3 and 4.4), and show that the comparisons of a cube with a reflected cube, a larger box and a tiling by cells carry over to King's convention, with the margins of the comparison-convention statements unchanged (Lemmas 4.6 to 4.8).

4.1. **The two conventions and the support condition.** This subsection fixes the notation for the two conventions and the support condition (S) under which the sources of this paper lie inside both cubes.

Throughout, $\varepsilon = \varepsilon_k = 5^{-k}$, $X_J = C_J \cap \Lambda_k = \{-MJ, \dots, MJ - \varepsilon\}^3$, $T_J = \{x \in X_J : \ x_\mu = MJ - \varepsilon \text{ for some } \mu\}$ and $\text{alive}_J^K = X_J \setminus T_J = \{-MJ, \dots, MJ - 2\varepsilon\}^3$. Both states carry the same photon variables $A_\alpha(x)$, $x \in X_J$, with the Neumann form of X_J, so their photon Gaussians are identical and every plaquette observable is the same function of the shared variables. They differ exactly by the matter on T_J: alive in the comparison convention (scalar Dirichlet planes $x_\mu = -MJ - \varepsilon$, $x_\mu = MJ$), killed in King's (planes $x_\mu = -MJ - \varepsilon$, $x_\mu = MJ - \varepsilon$, $2MJ$ apart, a multiple of every block scale; cf. Definition 2.3). A bond with one end alive and the other killed or outside the site set contributes its diagonal part $\frac{\varepsilon}{2} r_x^2$ at the alive end, so an alive site carries six diagonals in either convention. In the proof of his Theorem 3.2 and in his box form (3.38), King [26] works in the comparison convention: every site of the region is alive there.

For (f, g) with f a real 2-form and $g \geq 0$ a scalar, both in C_c^∞, we write as in Definition 2.5 $\Theta(\mathrm{i}f, g) = e^{\mathrm{i}F(f) + W(g)}$ (class $\mathcal{I}$), $\Theta(f, -g) = e^{F(f) - W(g)}$ (class $\mathcal{D}$), $W(g) = \sum_x \varepsilon^3 g(x)(|\phi(x)|^2 - 2\mathsf{W}_\varepsilon)$ (free Wick constant), and $Z_{k,J}^a(\mathrm{i}f, g) = e^{-a_k \mathsf{S}(g)} \langle \Theta(\mathrm{i}f, g) \rangle$, $Z_{k,J}^a(f, -g) = e^{a_k \mathsf{S}(g)} \langle \Theta(f, -g) \rangle$, $\mathsf{S}(g) = \sum_x \varepsilon^3 g(x)$; a superscript K ($Z_{k,J}^{a,K}$) marks King's convention. All these expectations are real. The sources lie inside X_J if

(S) every plaquette $p = (x; \alpha, \beta)$ of Λ_k with $f_{c(p)}(e_\alpha, e_\beta) \neq 0$ has its four corners in X_J, and every site $x \in \Lambda_k$ with $g(x) \neq 0$ lies in $\text{alive}_J^K = X_J \setminus T_J$.

If $\operatorname{supp} f \cup \operatorname{supp} g \subset \{y : \operatorname{dist}_\infty(y, \mathbb{R}^3 \setminus C_J) > 1\}$, as every hypothesis of this paper demands of its sources, then (S) holds: a site with $g(x) \neq 0$ has $-MJ + 1 < x_\mu < MJ - 1 \leq MJ - \varepsilon$, so $x \in \text{alive}_J^K$; a plaquette with $f_{c(p)} \neq 0$ has its centre at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_J$ and its corners within $\varepsilon/2 \leq \frac{1}{2}$ of it, hence in X_J. Also (S) for X_{J_0} implies (S) for every $J \geq J_0$.

4.2. **Cut-outs and the interlacing lemma.** This subsection proves the cut-out inequality for boxes with killed sites and derives from it the interlacing of the two cube families (Lemma 4.2), from which the monotonicity and the limits of the K family (Corollaries 4.3 and 4.4) follow in the next subsection.

Lemma 4.1 (cut-outs from a box with killed sites)**.** *Let $\langle \cdot \rangle_Y$, $Y := (V, T)$, be the state of Definition 2.2 on a finite set $V \subset \Lambda_k$ of sites with the matter killed on $T \subset V$, and $X \subset V$ any subset. Let (f, g), $g \geq 0$, satisfy* (S_X)*: every plaquette with $f \neq 0$ has its four corners in X and every site with $g \neq 0$ lies in X (g on $X \cap T$ is immaterial). Then deleting from $\langle \cdot \rangle_Y$ the photon gradient terms coupling X to $V \setminus X$ and the cosine parts of the hopping terms between X and $V \setminus X$ yields a product whose X-factor is $\langle \cdot \rangle_X^{(a), X \cap T}$, the state of Definition 2.2 on X with the matter killed on $X \cap T$; consequently*

$$\langle \Theta(\mathrm{i}f, g) \rangle_X^{(a), X \cap T} \ \leq \ \langle \Theta(\mathrm{i}f, g) \rangle_Y, \qquad \langle \Theta(f, -g) \rangle_X^{(a), X \cap T} \ \geq \ \langle \Theta(f, -g) \rangle_Y,$$

and the same for Z^a. In particular, (a) *if $X \cap T = \emptyset$ the factor is $\langle \cdot \rangle_X^{(a)}$;* (b) *if $X = \prod_\mu \{a_\mu, \dots, b_\mu\}$ is a box and $X \cap T = \{x \in X : \ x_\mu = b_\mu \text{ for some } \mu\}$, the factor is the state of King's convention on X.*

Proof. Each deletion is in the cone direction: deleting a photon gradient term does not increase $\langle\Theta_{F,g}\rangle$ and does not decrease $\langle\Theta^-_{F,g}\rangle$ (Lemma 2.10(a),(d)), and setting a hopping coefficient c_{xy} to 0 likewise (Lemma 2.10(b),(d)); every intermediate measure is a BFS measure. After all deletions nothing couples the X-variables to the rest, so the measure is a product (Lemma 2.10(e)). In the X-factor the photon is the Neumann photon of X; a site of $X\cap T$ carries its point mass; an alive $x\in X\setminus T$ carries the single-site weight times $\frac{\varepsilon}{2}r_x^2$ for each of its six bonds, since in $\langle\cdot\rangle_Y$ every bond at an alive x contributes $\frac{\varepsilon}{2}r_x^2$ at x whether its other end is alive in V (the diagonal part of $\frac{\varepsilon}{2}|e^{\mathrm{i}e\varepsilon A}\phi_y-\phi_x|^2$), killed, or outside V ($\phi=0$ there), and the cut keeps all diagonals; the cosines are those of the bonds within X between alive sites. That is the state of Definition 2.2 on X killed on $X\cap T$. By (S_X) the observable is a function of the X-factor's variables (at $x\in X\cap T$ the factor $e^{\varepsilon^3 g(x)r_x^2}$ is 1 in both states), and the c-numbers $e^{\mp 2\mathsf{W}_\varepsilon\mathsf{S}(g)}$, $e^{\mp a_k\mathsf{S}(g)}$ are the same positive constants on both sides (Lemma 2.14(a); the counterterm is site-independent). Hence $\langle\Theta\rangle_X^{(a),X\cap T}$ is the expectation of Θ in the fully deleted measure, which is $\le\langle\Theta\rangle_Y$ on $\mathcal{I}$ and $\ge$ on $\mathcal{D}$. □

The clause of Lemma 2.13 that X and its neighbours lie in one fundamental cell is a torus clause; for a box it has no content, since a bond leaving V already carries its Dirichlet diagonal in $\langle\cdot\rangle_Y$. Every sandwich $C_{J-1}\subset\cdot\subset C_{J+1}$ of this paper, read in King's convention, is an instance of Lemma 4.1.

Lemma 4.2 (interlacing of the two cube families)**.** *Fix $k\ge 0$, $J\ge 1$ and (f,g), $g\ge 0$, satisfying* (S) *for X_J. Then*

$$\langle\Theta(\mathrm{i}f,g)\rangle^K_{k,J}\ \le\ \langle\Theta(\mathrm{i}f,g)\rangle^{(a)}_{k,J}\ \le\ \langle\Theta(\mathrm{i}f,g)\rangle^K_{k,J+1},$$
$$\langle\Theta(f,-g)\rangle^K_{k,J}\ \ge\ \langle\Theta(f,-g)\rangle^{(a)}_{k,J}\ \ge\ \langle\Theta(f,-g)\rangle^K_{k,J+1},$$

and the same for the scheme functionals $Z^a_{k,J}(\mathrm{i}f,g)$, $Z^a_{k,J}(f,-g)$ of the three states.

Proof. Step 0 (one observable, one multiplier). By (S) the plaquettes with non-zero coefficient have all corners in X_J, whose photon variables are present in all three states, so $F(f)$ is one real linear functional F of the shared variables; the sites with $g\neq 0$ lie in alive^K_J, alive in all three states; W_ε, δm^2_ε, $\mathsf{S}(g)$ are the same for the three regions. Hence, with $g'_x:=\varepsilon^3 g(x)\ge 0$, $\Theta(\mathrm{i}f,g)=e^{-2\mathsf{W}_\varepsilon\mathsf{S}(g)}\Theta_{F,g'}$ and $\Theta(f,-g)=e^{2\mathsf{W}_\varepsilon\mathsf{S}(g)}\Theta^-_{F,g'}$, where $\Theta_{F,g'}=e^{\mathrm{i}F}e^{\sum_x g'_x r_x^2}$, $\Theta^-_{F,g'}=e^{F}e^{-\sum_x g'_x r_x^2}$ are the observables of Lemma 2.10, and the scheme multiplier $e^{\mp a_k\mathsf{S}(g)}$ is one positive constant for all three states (Lemma 2.14(a)). It suffices to treat $\langle\Theta_{F,g'}\rangle$, $\langle\Theta^-_{F,g'}\rangle$.

Step 1 (kill step on $\mathcal{I}$). For $s\ge 0$ let $\langle\cdot\rangle_s$ be $\langle\cdot\rangle^{(a)}_{k,J}$ with the single-site measure at each $x\in T_J$ multiplied by $e^{-s\varepsilon^3 r_x^2}$, a BFS measure, and $\varphi(s):=\langle\Theta_{F,g'}\rangle_s=N(s)/D(s)$, $N(s)=\int\Theta\, e^{-s\varepsilon^3\sum_{T_J}r_x^2}d\mu^{(0)}$, $D(s)=\int e^{-s\varepsilon^3\sum_{T_J}r_x^2}d\mu^{(0)}$, $\mu^{(0)}$ the unnormalised measure at $s=0$. The s-derivative of either integrand is bounded in modulus by $\varepsilon^3\sum_{T_J}r_x^2$ times the integrand at $s=0$, integrable against the quartic weight $e^{-\lambda\varepsilon^3 r^4}$ and the Gaussian; so differentiation under the integral sign is justified, $D(s)>0$, and

$$\varphi'(s)=-\varepsilon^3\sum_{x\in T_J}\big(\langle\Theta\, r_x^2\rangle_s-\langle\Theta\rangle_s\langle r_x^2\rangle_s\big)=-\varepsilon^3\sum_{x\in T_J}\langle\Theta;r_x^2\rangle_s\ \le\ 0$$

by Lemma 2.10(c). So $\varphi(s)\le\varphi(0)=\langle\Theta\rangle^{(a)}_{k,J}$. Since $\Theta_{F,g'}$ is continuous with $|\Theta_{F,g'}|=e^{\sum_x g'_x r_x^2}$ and $\langle\cdot\rangle^{(a)}_{k,J}$ is of King type, Lemma 2.12 gives $\varphi(s)\to\langle\Theta\rangle_{T_J}$, the state with the matter killed on T_J, which is $\langle\cdot\rangle^K_{k,J}$. Hence $\langle\Theta\rangle^K_{k,J}\le\langle\Theta\rangle^{(a)}_{k,J}$.

Step 2 (cut step). Apply Lemma 4.1 with $V := X_{J+1}$, $T := T_{J+1}$ (so $\langle\cdot\rangle_Y = \langle\cdot\rangle^K_{k,J+1}$), $X := X_J$: $X_J \subset X_{J+1}$, and $X_J \cap T_{J+1} = \emptyset$ since every site of T_{J+1} has a coordinate $M(J+1) - \varepsilon \geq MJ > MJ - \varepsilon$. (At $k = 0$, $M = 1$ the outward neighbours of X_J lie on T_{J+1} and are killed in $\langle\cdot\rangle_Y$; harmless, since the bond from an alive $x \in X_J$ to such a neighbour already carries $\frac{\varepsilon}{2} r_x^2$ at x, the Dirichlet diagonal of the comparison cube.) By Lemma 4.1(a) the X_J-factor is $\langle\cdot\rangle^{(a)}_{k,J}$, and (S) is ($\mathrm{S}_{X_J}$): $\langle\Theta(\mathrm{i}f, g)\rangle^{(a)}_{k,J} \leq \langle\Theta(\mathrm{i}f, g)\rangle^K_{k,J+1}$, $\langle\Theta(f, -g)\rangle^{(a)}_{k,J} \geq \langle\Theta(f, -g)\rangle^K_{k,J+1}$.

Step 3 (kill step on $\mathcal{D}$). With $\langle\cdot\rangle_s$ as in Step 1 and $\varphi^-(s) := \langle\Theta^-_{F,g'}\rangle_s$ (differentiation justified as before, now with $|\Theta^-| \leq e^{F(A)}$ integrable against the Gaussian), $(\varphi^-)'(s) = -\varepsilon^3 \sum_{x\in T_J} \langle\Theta^-; r_x^2\rangle_s \geq 0$ by Lemma 2.10(d), so $\varphi^-(s) \geq \varphi^-(0) = \langle\Theta^-\rangle^{(a)}_{k,J}$; as $|\Theta^-_{F,g'}| \leq e^{F(A)}$ with F real linear, Lemma 2.12 (with $\ell = F$, $C = 0$) gives $\varphi^-(s) \to \langle\Theta^-\rangle^K_{k,J}$. Hence $\langle\Theta(f, -g)\rangle^K_{k,J} \geq \langle\Theta(f, -g)\rangle^{(a)}_{k,J}$.

Step 4 (Z^a). The multiplier of Step 0 is one positive constant. □

Nothing in the proof uses a reflection symmetry of either cube (neither has one), the counterterm beyond its site-independence, or the renormalisation group. Since in every kill step of this section the observable does not depend on the fields at the killed sites (its factor there is 1), each such step also follows with no limit: set $c_{xy} = 0$ on every bond at the set U to be killed, one bond at a time (Lemma 2.10(b),(d)); the measure is then a product (Lemma 2.10(e)) whose U-factor cancels, and the other factor is the state with U killed.

4.3. **Monotonicity of the K family and the limits.** The interlacing lemma now gives the monotonicity of King's family in the cube and, with the existence theorem (Theorem A), the relations between the limits of the two families.

Corollary 4.3 (the K family is J-monotone)**.** *Under* (S) *for X_J: $\langle\Theta(\mathrm{i}f, g)\rangle^K_{k,J} \leq \langle\Theta(\mathrm{i}f, g)\rangle^K_{k,J+1}$ and $\langle\Theta(f, -g)\rangle^K_{k,J} \geq \langle\Theta(f, -g)\rangle^K_{k,J+1}$, and the same for $Z^{a,K}$; hence for all $J \geq J_0$ whenever the sources satisfy* (S) *for X_{J_0}.*

Proof. Combine the two inequalities of Lemma 4.2. (For the comparison family this is Lemma 4.1, part (a), with $V = X_{J+1}$, $X = X_J$, i.e. the first inequality of Lemma 2.13.) □

The proof of Theorem 3.2 of [26] first replaces the J-dependent Dirichlet Wick subtraction (2.13) by the free, J-independent one and then proves the monotonicity in J in the comparison convention (Lemma 4.1(a)); by Corollary 4.3 that statement holds between cubes of King's convention too, with the kill step (Step 1), which is absent there. The form with the Dirichlet constant follows in neither convention and is not used here; at $g = 0$ no Wick constant enters.

Corollary 4.4 (the limits)**.** *Let $f = (f_1, \dots, f_N)$ be real 2-forms in C_c^∞ with $\mathrm{dist}_\infty(\mathrm{supp}\, f_j, \mathbb{R}^3 \setminus C_{J_0}) > 1$; for $J \geq J_0$, $z \in \mathbb{C}^N$ put $a^K_{k,J}(z) := \langle e^{\sum_j z_j F(f_j)}\rangle^K_{k,J}$, $a^{(a)}_{k,J}(z) := \langle e^{\sum_j z_j F(f_j)}\rangle^{(a)}_{k,J}$, $g(z) := \sum_j (\mathrm{Re}\, z_j) f_j$. At the cone points $z = \mathrm{i}t$, $t \in \mathbb{R}^N$, both are real with modulus ≤ 1, and* (S) *holds for X_J.*

(i) (One J-limit at fixed k; no hypothesis.) *For every k and t, $J \mapsto a^K_{k,J}(\mathrm{i}t)$ and $J \mapsto a^{(a)}_{k,J}(\mathrm{i}t)$ are nondecreasing on $J \geq J_0$, bounded by 1 and interlaced; hence $\lim_J a^K_{k,J}(\mathrm{i}t) = \lim_J a^{(a)}_{k,J}(\mathrm{i}t) \in [-1, 1]$. The same holds for $\langle\Theta(\mathrm{i}f, g)\rangle$, $g \geq 0$, with a common limit in $(-\infty, \infty]$.*

(ii) (King's cube limits.) *By Theorem A, posed on King's convention, for every such family, every $J \geq J_0$ and every $t \in \mathbb{R}^N$, $\lim_k a^K_{k,J}(t)$ exists in $(0, \infty)$. Then: (α) $a^K_{k,J} \to Z^K_J$ locally*

uniformly on $\mathbb{C}^N$, Z_J^K *is entire and* $|Z_J^K(z)| \le e^{\|g(z)\|_2^2}$; *(β)* $Z_J^K \to Z_\infty^K$ *locally uniformly on* $\mathbb{C}^N$, Z_∞^K *is entire and* $|Z_\infty^K(z)| \le e^{\|g(z)\|_2^2}$; *(γ) for all* $J \ge J_0$ *and* $t \in \mathbb{R}^N$,

$$Z_J^K(\mathrm{i}t) \;\le\; \liminf_k a_{k,J}^{(a)}(\mathrm{i}t) \;\le\; \limsup_k a_{k,J}^{(a)}(\mathrm{i}t) \;\le\; Z_{J+1}^K(\mathrm{i}t),$$

hence $\lim_J \liminf_k a_{k,J}^{(a)}(\mathrm{i}t) = \lim_J \limsup_k a_{k,J}^{(a)}(\mathrm{i}t) = Z_\infty^K(\mathrm{i}t)$.

(iii) (The subsequential limits of the comparison family.) *Let* $\mathbf{k} = (k_m)$ *be an increasing sequence of spacings along which* $a_{k_m,J}^{(a)} \to Z_J^{\mathbf{k}}$ *locally uniformly on* $\mathbb{C}^N$ *for every* $J \ge J_0$ *and every such family (a continuum subsequence of the comparison family, Definition 14.1; they exist with no hypothesis, Lemma 14.7). Then* $Z_J^{\mathbf{k}} \to Z_\infty^K$ *locally uniformly on* $\mathbb{C}^N$ *as* $J \to \infty$*: every subsequential fixed-cube limit of the comparison family, followed by* $J \to \infty$*, is King's state. In particular, if the fixed-cube limits of the comparison family exist along the full sequence,* $\lim_J \lim_k a_{k,J}^{(a)} = Z_\infty^K$.

Proof. (i) Monotonicity is Corollary 4.3 for the K family and Lemma 4.1(a) for the comparison family; Lemma 4.2 interlaces them; two nondecreasing sequences with $a_J \le b_J \le a_{J+1}$ have the same limit, finite or $+\infty$.

(ii)(α) The Montel argument, run on the K family. (1) $a_{k,J}^K$ is entire (Gaussian decay in A, $e^{-\lambda\varepsilon^3 r^4}$ decay in r at the alive sites). (2) The local bound of Lemma 7.3 holds for the K family with the same constant: $\mathrm{Re}\sum_j z_j F(f_j) = F(g(z))$, so $|a_{k,J}^K(z)| \le \langle e^{F(g(z))}\rangle_{k,J}^K$; Lemma 2.11(a) with $s = 1$ and $v := d_k^* g(z)$ (defined by $F_{\Lambda_k}(g(z)) = \langle v, A\rangle_k$) applies to any BFS measure whose Gaussian has covariance C and gives $\langle e^{\langle v,A\rangle}\rangle \le e^{\langle v,Cv\rangle/2}$; the K cube is a BFS measure whose Gaussian is the Neumann photon Gaussian of X_J, the same as the comparison cube's, so its covariance is the C of Lemma 7.1, a statement about this C and the plaquettes with corners in X_J: $\langle v, Cv\rangle_k \le 2\|g(z)\|_{2,k}^2$. Hence $|a_{k,J}^K(z)| \le e^{\|g(z)\|_{2,k}^2} \le e^{3C_0|g(z)|_4^2}$ uniformly in $k \ge 0$, $J \ge J_0$, by (E3) of Lemma 2.9: the plaquette centres of each of the three orientations form a translate of $\varepsilon_k\mathbb{Z}^3$, each contributing at most $C_0|g|_4^2$ to $\|g\|_{2,k}^2$ by (E1), and the three orientations give $3C_0$. Nothing in this argument sees the matter beyond the state being a BFS measure. (3) Theorem A gives convergence at every real point. (4) $\mathbb{R}^N$ is a set of uniqueness for entire functions on $\mathbb{C}^N$; by Montel's theorem every subsequence has a locally uniformly convergent sub-subsequence, all limits are entire and agree on $\mathbb{R}^N$, hence everywhere [22]. So $a_{k,J}^K \to Z_J^K$ locally uniformly, Z_J^K is entire, and the bound passes to the limit, $|Z_J^K(z)| \le \lim_k e^{\|g(z)\|_{2,k}^2} = e^{\|g(z)\|_2^2}$ (Lemma 2.9), a J-free bound.

(β) By Corollary 4.3 at every k and (α), $J \mapsto Z_J^K(\mathrm{i}t)$ is nondecreasing and ≤ 1, so $Z_\infty^K(\mathrm{i}t) := \lim_J Z_J^K(\mathrm{i}t)$ exists; $\{Z_J^K\}_{J\ge J_0}$ is locally uniformly bounded by the J-free bound of (α); $(\mathrm{i}\mathbb{R})^N$ is a maximal totally real subspace, hence a set of uniqueness for entire functions; Montel's theorem and the identity theorem give $Z_J^K \to Z_\infty^K$ locally uniformly, Z_∞^K entire, $|Z_\infty^K(z)| \le e^{\|g(z)\|_2^2}$. (The same Montel argument as in (α), with the uniqueness set $(\mathrm{i}\mathbb{R})^N$ and the cube monotonicity of Corollary 4.3 supplying the convergence there; Theorem 7.6(b) states it.)

(γ) Lemma 4.2 at every k, with $k \to \infty$ at the two ends (limits by (α)) and lim inf, lim sup in the middle, gives the sandwich; (β) gives the limits.

(iii) Lemma 4.2 along $\mathbf{k}$, with the full-sequence limits of (α) at the ends, gives $Z_J^K(\mathrm{i}t) \le Z_J^{\mathbf{k}}(\mathrm{i}t) \le Z_{J+1}^K(\mathrm{i}t)$, so $\lim_J Z_J^{\mathbf{k}}(\mathrm{i}t) = Z_\infty^K(\mathrm{i}t)$ by (β). $\{Z_J^{\mathbf{k}}\}_J$ is locally bounded by the J-free bound $e^{\|g(z)\|_2^2}$ (Lemma 7.3 passed to the limit), so by Montel's theorem and the uniqueness set $(\mathrm{i}\mathbb{R})^N$ it converges locally uniformly to an entire function (Lemma 14.8), which agrees with Z_∞^K on $(\mathrm{i}\mathbb{R})^N$, hence on $\mathbb{C}^N$. If the fixed-cube limits exist along the full sequence, that sequence is a continuum subsequence. □

Statements (ii)–(iii) also hold with the two conventions exchanged, for $J \geq J_0 + 1$, at every family and cube where the comparison family's fixed-cube limits exist at real coefficients (read Lemma 4.2 as $a^{(a)}_{k,J-1} \leq a^{K}_{k,J} \leq a^{(a)}_{k,J}$; whether they exist is not decided here, see Remark 4.5); the exchanged (iii) needs Lemmas 14.7 and 14.8 for the K family, whose proofs use only the local bound and the photon-only bound $\langle F(u-u')^2\rangle \leq 2\|u-u'\|^2_{2,k}$, both transferred by the covariance identity in (α).

Remark 4.5 (what does not follow). At fixed J, Lemma 4.2 yields exactly (γ) and nothing on whether $\lim_k a^{(a)}_{k,J}(\mathrm{i}t)$ exists: the bounds $Z^K_J(\mathrm{i}t)$, $Z^K_{J+1}(\mathrm{i}t)$ differ in general (the cubes differ by M, so the gap does not shrink with k), and nothing constrains the oscillation of $a^{(a)}_{k,J}(\mathrm{i}t)$ in k between them. So the existence of the fixed-cube limits on one convention does not imply it on the other by these results, and likewise with scalar sources: the existence theorem must be proved on the family on which the renormalisation-group construction is carried out; the other is a comparison family with the same infinite-volume state by (ii)–(iii). No counterexample is claimed.

4.4. **Three margins in King's convention.** This subsection carries the three margin statements of the comparison convention (the reflection sandwich, the upper box and the cell bound) over to King's convention, with their margins unchanged.

Let $\sigma_3 = \mathrm{diag}(1,1,-1)$, $X := X_J$, $X_+ := \sigma_3 X = (C_J + \varepsilon e_3) \cap \Lambda_k$ (third coordinates $\{-MJ+\varepsilon, \dots, MJ\}$), $\sigma_3 T_J = \{x \in X_+ : \ x_1 = MJ-\varepsilon \text{ or } x_2 = MJ-\varepsilon \text{ or } x_3 = -MJ+\varepsilon\}$ and $A_\sigma := \sigma_3(\mathrm{alive}^K_J) = \{x : \ x_1, x_2 \in \{-MJ, \dots, MJ-2\varepsilon\},\ x_3 \in \{-MJ+2\varepsilon, \dots, MJ\}\}$.

Lemma 4.6 (the reflection sandwich)**.** *Let $(f,g) \in \mathcal{K}$ be supported at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_{J_0}$.* (a) *If $M \geq 2\varepsilon_k$, which the standing choice $M \in 50\mathbb{N}$ (Section 2.1) guarantees for every k, then for every $J \geq J_0 + 1$*

$$Z^{a,K}_{k,J-1}(\mathrm{i}f, g) \ \leq \ Z^{a,K}_{k,J}(\mathrm{i}\,\sigma_3\cdot f, \sigma_3\cdot g) \ \leq \ Z^{a,K}_{k,J+1}(\mathrm{i}f, g).$$

(b) *For every $k \geq 0$, $M \geq 1$ and $J \geq J_0 + 2$ the same holds with $J \mp 2$ in place of $J \mp 1$.* (c) *Were $M = 1$ allowed, at $k = 0$ the one-step sandwich* (a) *would not be obtainable by deletions and kills: A_σ contains alive sites ($x_3 = MJ$) killed in King's cube of index $J+1$, and* alive^K_{J-1} *contains sites ($x_3 = -MJ+\varepsilon$) killed in the reflected state; this is why the condition in* (a) *is stated.*

Proof. Step 1 (transport). By Lemma 8.5, $Z^{a,K}_{k,J}(\mathrm{i}\sigma_3\cdot f, \sigma_3\cdot g) = e^{-a_k \mathsf{S}(g)}\langle\Theta(\mathrm{i}f,g)\rangle_R$, with R the state on X_+ for the frame $(e_1, e_2, -e_3)$ killed on $\sigma_3 T_J$ ($\mathsf{S}(\sigma_3\cdot g) = \mathsf{S}(g)$). In the frame (e_α), as in Lemma 2.16, R is the BFS measure with A_1, A_2 on X_+ and A_3 on $X_+ - \varepsilon e_3 = X$ (gradient pairs within these sets, masses on them), matter alive exactly on A_σ with point masses on $\sigma_3 T_J$ and $\phi = 0$ outside X_+, all six diagonals at each alive site, and the standard cosine on every bond with both ends in A_σ (the flipped bond with $A'_3(x) = -A_3(x - \varepsilon e_3)$ gives the standard cosine of $\langle x - \varepsilon e_3, x\rangle$); $F(f)$, $W(g)$ are the standard ones (the plaquettes $(x;\alpha,3)$ pick up two sign changes and keep their centres). R is of King type. The sources satisfy (S) for X_{J-1} (resp. X_{J-2}), so by the inclusions below Step 0 of Lemma 4.2 applies to all states compared.

Step 2 (upper end). If $A_\sigma \subset \mathrm{alive}^K_{J'}$ and $X_+ \cup X \subset X_{J'}$: from $\langle\cdot\rangle^K_{k,J'}$ kill every site of $\mathrm{alive}^K_{J'} \setminus A_\sigma$ (Lemma 2.10(c), Lemma 2.12, which allows the already killed $T_{J'}$), leaving exactly the cosines of bonds within A_σ, i.e. R's; then delete every photon gradient term except the pairs within X_+ for A_1, A_2 and within X for A_3 (Lemma 2.10(a); these are gradient pairs of $X_{J'}$). The result is a product (Lemma 2.10(e)) whose factor on R's variables is R (photon, alive set with all six diagonals by the bookkeeping of Lemma 4.1, cosines); the other factor cancels. Hence

$\langle\Theta(\mathrm{i}f,g)\rangle_R \le \langle\Theta(\mathrm{i}f,g)\rangle^K_{k,J'}$. The inclusions hold iff $MJ \le MJ' - 2\varepsilon$ (direction 3; the rest is trivial): for $J' = J+1$ iff $M \ge 2\varepsilon$, for $J' = J+2$ always.

Step 3 (lower end). If $\mathrm{alive}^K_{J''} \subset A_\sigma$ and $X_{J''} \subset X \cap X_+$: from R kill every site of $A_\sigma \setminus \mathrm{alive}^K_{J''}$ (Lemma 2.10(c), Lemma 2.12) and delete every photon gradient term coupling an $X_{J''}$-variable to another variable of R (Lemma 2.10(a)). The $X_{J''}$-factor has the Neumann photon of $X_{J''}$ (gradient pairs within $X_+ \cap X_{J''} = X \cap X_{J''} = X_{J''}$) and the matter alive on $\mathrm{alive}^K_{J''}$ with all diagonals, killed on $T_{J''}$: it is $\langle\cdot\rangle^K_{k,J''}$; the rest cancels. Hence $\langle\Theta(\mathrm{i}f,g)\rangle^K_{k,J''} \le \langle\Theta(\mathrm{i}f,g)\rangle_R$. The inclusions hold iff $-MJ'' \ge -MJ + 2\varepsilon$: for $J'' = J-1$ iff $M \ge 2\varepsilon$, for $J'' = J-2$ always.

Step 4. Steps 2–3 with $J\pm1$ give (a), with $J\pm2$ give (b); the multipliers agree (Lemma 2.14(a)). For (c): at $k=0$, $M=1$ the top of A_σ in direction 3 is J while that of alive^K_{J+1} is $J-1$, and the bottom of alive^K_{J-1} is $-J+1 = -MJ+\varepsilon$, the killed layer of R; deletions and kills never revive a killed site. □

In the curvature sector ($g=0$) the limit sentences of Theorem 8.6 run on (a) or on (b), for all k, with $k\to\infty$ by Corollary 4.4(ii) and $J\to\infty$ by Corollary 4.3; with scalar sources they need the K-family tube limits, established in Theorem 28.5 (Remark 4.10).

Lemma 4.7 (the upper box needs no change)**.** *Let $\Omega \subset B$ be lattice boxes at spacing ε such that B contains every site within ℓ^∞-distance N lattice steps of Ω, $N \ge 1$ (the hypothesis of Lemma 9.10, whose mass condition forces $N \ge 1$), and let $\langle\cdot\rangle^K_B$ be the state of King's convention on B (killed on its top face layers T_B). Then* (a) *$\Omega \cap T_B = \emptyset$;* (b) *the second part of Lemma 9.10 holds with $\langle\cdot\rangle^K_B$ in place of the comparison state on B, with the same proof, and its first part is a photon statement, unchanged;* (c) *consequently the sandwich in the proof of Theorem 10.5 holds for King's cubes with the cubes and margins of (23): for every k and every ζ admitted there, $Z^{a,K}_{k,J}(\zeta) \le \Phi_k(\zeta) \le Z^{a,K}_{k,J+2n}(\zeta)$, Φ_k the functional of the box state of Definition 9.2 on $\Omega_{J,k}$; the lower end is Lemma 9.9 with Step 1 of Lemma 4.2 prepended. (By Corollary 4.3 the cube of index $J+2n+1$ would also do; it is not needed.)*

Proof. (a) For $x \in \Omega$ and each μ, $x + N\varepsilon e_\mu \in B$, so $x_\mu < \max_B x_\mu$ and $x \notin T_B$. (b) By (a) every site of Ω is alive, so once the hopping cosines touching $B \setminus \Omega$ are removed the matter on Ω is the Dirichlet matter of Ω, and the photon law of B is the same in both conventions, so the proof of Lemma 9.10 goes through unchanged. (c) Lower end: Lemma 9.9 gives $\langle Y\rangle^{(a)}_{C_J} \le \langle Y\rangle_{\Omega_{J,k}}$ on $\mathcal{I}$ (its proof lands on the Dirichlet matter of C_J, the comparison convention), and Step 1 of Lemma 4.2 gives $\langle Y\rangle^K_{C_J} \le \langle Y\rangle^{(a)}_{C_J}$, (S) holding since the sources of the enlarged family and their reflections are supported at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_{J_0}$, $C_{J_0} \subset C_J$. Upper end: (b) with $B = X_{J+2n}$, whose hypothesis is the second margin of (23), a statement about photon sites. The multipliers agree (Lemma 2.14(a)). □

For the cell bound we use the cells of Definition 13.4: $\ell_c = 2ML_c \ge 10$; the cells $c = \prod_\mu [w_\mu, w_\mu + \ell_c)$ of the tiling $\mathcal{T}_s$, $s \in \{0,1\}^3$, $w_\mu \in W_\mu(s_\mu)$, $W_\mu(0) = \ell_c\mathbb{Z}$, $W_\mu(1) = ML_c + \ell_c\mathbb{Z}$ (so $w_\mu, w_\mu + \ell_c, MJ \in \mathbb{Z}$); the good sets $G_s = \{x : \mathrm{dist}(x_\mu, W_\mu(s_\mu)) > 2\ \forall\mu\}$ and cut-offs χ_s with $\mathrm{supp}\,\chi_s \subset G_s$; $X_c := c \cap X_J$; $J_1(R) = \min\{J : MJ \ge R + \ell_c\}$ for sources supported in $Q_R = [-R,R]^3$; B_{cell}, defined on the comparison cell C_{L_c}. N^K_{k,C_J} is the normalised scalar-source functional of $\langle\cdot\rangle^K_{k,J}$ and $N^{(a),T}_{k,X}$ that of the comparison state on X killed on $T \subset X$.

Lemma 4.8 (the cell lemmas hold as stated)**.** (a) *For a tiling $\mathcal{T}_s$ and any lattice function $q \ge 0$ on X_J: $N^K_{k,C_J}(0,-q) \le \prod_i N^{(a),X_i\cap T_J}_{k,X_i}(0,-q|_{X_i})$, over the site sets $X_i = c \cap X_J$ of the cells meeting C_J.* (b) *For a full cell $c \subset C_J$: $X_c \cap T_J \ne \emptyset$ iff $w_\mu + \ell_c = MJ$ for some μ.* (c) *Let $q \ge 0$ be any lattice function supported in Q_R, $J \ge J_1(R)$, $s \in \{0,1\}^3$ and $c \in \mathcal{T}_s$ with*

$(\chi_s q)|_{X_c} \not\equiv 0$. *Then* $-MJ < w_\mu$ *and* $w_\mu + \ell_c < MJ - 2$ *for every* μ: c *is a full cell inside* C_J *with* $X_c \cap T_J = \emptyset$, *and* $(\chi_s q)|_{X_c}$ *is supported at* ℓ^∞*-distance* > 2 *from* $\mathbb{R}^3 \setminus c$. (d) *Hence Lemmas 13.6 to 13.8 are unchanged and Proposition 13.9 holds for* $\langle\cdot\rangle^K_{k,J}$ *as stated, under its hypotheses and with its constant* K_-: $\ln N^K_{k,C_J}(0,-q) \le K_-|q|_w$ *for every* k, *every* $J \ge J_1(R)$ *and every lattice* $0 \le q \le H$ *supported in* Q_R. (e) *(Direction.) For a cell* c *and* $\emptyset \ne T \subset X_c$: $N^{(a),T}_{k,c}(0,-\tilde q) \ge N^{(a)}_{k,c}(0,-\tilde q)$ *for* $\tilde q \ge 0$ *vanishing on* T. *So* B_{cell}, *a bound on a comparison cell, does not bound a source-carrying cell of King's type, and* (c) *is essential.*

Proof. (a) The proof of Lemma 13.5 deletes the cross-cell photon gradient terms and hopping cosines, each in the $\mathcal{D}$ direction (Lemma 2.10(d)); by Lemma 2.10(e) and the factor identification of Lemma 4.1 with $V = X_J$, $T = T_J$, $X = X_i$, the product has the factors $N^{(a),X_i\cap T_J}_{k,X_i}$; the observable $e^{-\sum \varepsilon^3 q r^2}$ and the multiplier $e^{a_k \mathsf{S}(q)} = \prod_i e^{a_k \mathsf{S}(q|_{X_i})}$ factorise, and at the sites of T_J the observable factor is 1 in every state. (b) The sites of a full cell have $x_\mu \in \{w_\mu, \dots, w_\mu + \ell_c - \varepsilon\}$, and $x \in T_J$ iff $w_\mu + \ell_c - \varepsilon = MJ - \varepsilon$ for some μ. (c) Let $x \in X_c$ with $(\chi_s q)(x) \ne 0$. Then $x \in G_s$; since $w_\mu, w_\mu + \ell_c \in W_\mu(s_\mu)$ and $w_\mu \le x_\mu < w_\mu + \ell_c$, this gives $w_\mu + 2 < x_\mu < w_\mu + \ell_c - 2$; and $q(x) \ne 0$ gives $|x_\mu| \le R$. Hence $w_\mu + \ell_c < R + \ell_c - 2 \le MJ - 2$ (as $MJ \ge R + \ell_c$) and $w_\mu > -R - \ell_c + 2 \ge -MJ + 2$. So $c \subset (-MJ, MJ)^3$, $w_\mu + \ell_c \ne MJ$, hence $X_c \cap T_J = \emptyset$ by (b), and $\mathrm{dist}_\infty(x, \mathbb{R}^3 \setminus c) = \min_\mu \min(x_\mu - w_\mu, w_\mu + \ell_c - x_\mu) > 2$. (d) In the chain of inequalities of Proposition 13.9 the Hölder step of Lemma 13.8 is an inequality within one state, convention-free; then (a) with the tiling $\mathcal{T}_s$ and $q_s := 8\chi_s q$. A factor with $q_s|_{X_i} \equiv 0$ equals 1 whatever is killed on X_i (this covers the clipped cells at the faces of C_J and any cell flush with them); a factor with $q_s|_{X_i} \not\equiv 0$ is, by (c), the comparison-cell functional $N^{(a)}_{k,c}(0, -q_s|_c)$ of a full cell $c \subset C_J$ with $q_s|_c$ supported at distance > 2 from $\mathbb{R}^3 \setminus c$ and $0 \le q_s \le 8H = H'$: exactly the hypotheses of Lemma 13.7 (hence of Lemma 13.6: translation to C_{L_c}, B_{cell}, Proposition 13.2 on the comparison box), as stated; Lemma 13.8 sums the cells. (A cell flush with a face, $w_\mu + \ell_c = MJ$, can meet $\mathrm{supp}\, q$ but never $\mathrm{supp}(\chi_s q)$; (a) is applied to the cut-off sources, not to q.) (e) Step 3 of Lemma 4.2 applied to the comparison cell with $U = T$: $s \mapsto \langle \Theta^- \rangle_s$ is nondecreasing (Lemma 2.10(d)) and tends to the killed state (Lemma 2.12). □

Remark 4.9 (one-line kill steps). Proposition 13.1 (cube $\le$ torus on $\mathcal{I}$), Proposition 13.2 (box $\le$ cube $\le$ torus), the lower end of Lemma 9.9, and the comparison of the cube with the box state of Definition 9.2 on the cone (Remark 9.8) are each proved for the comparison cube; for King's cube prepend Step 1 of Lemma 4.2 ($\langle\cdot\rangle^K_{k,J} \le \langle\cdot\rangle^{(a)}_{k,J}$ on $\mathcal{I}$), and for the $\mathcal{D}$ versions (cube $\ge$ torus for negative sources) Step 3. The same applies to a box of King's type (a killed face layer) in Proposition 13.2. Nothing beyond Lemma 4.2 is needed.

Remark 4.10 (the Wick sector). With scalar sources $g \ge 0$, Corollary 4.4(i) holds as stated. The analogues of Corollary 4.4(ii)–(iii) on the Wick tube need the fixed-cube limits and the (k, J)-uniform bound (37) for both families. For King's family the bound is the sequence of estimates (37) carried out on ND^K, whose convention-sensitive steps are the cube-versus-torus comparison (Corollary 13.3, via Remark 4.9) and the cell bound (Lemma 4.8); that sequence of estimates and the existence of the K-family tube limits are the content of Theorem 28.5, on which the tube statements for King's state rest (Remark 13.12, Proposition 28.6).

The existence part

5. King's existence theorem on the cube

The existence theorem for King's cube limits (Theorem A), and its form with scalar sources that the Wick leg uses (Theorem B), are proved in this section and the next from estimates on

King's renormalisation-group integrals on the cube C_J^K. The argument is the one King gives for his Theorem 2.1 [26, Sect. 3.1] and [25, Sect. 3.2]: the generating functional is approximated, up to a small-field remainder, by an integral of the effective action after j renormalisation steps, and two spacings whose effective actions live on the same unit lattice are compared. We write it as a Cauchy property of the *normalised* functional, in a ratio form in which every field-independent constant of the effective action cancels; this removes from King's route the convergence of the vacuum energies, which the argument of [26] uses and which is not available on the cube.

Three statements from this analysis recur. (BAL$_5$)[int] (Lemma 5.3) is Bałaban's interaction-term bound at a non-zero background: when the propagator domain and the background field of a renormalisation step are changed, the sum over all localisations of the interaction terms containing at least one difference of propagators or of external fields is at most $C_\kappa t^\kappa$ times the volume of the region, for every $\kappa > 0$, t the scale of the step, with constants independent of the step, of the domains and of the background; every use of Bałaban's upper stability bound and of King's upper bound at block factor 5 passes through it. (REN$_{\mathrm{orb}}$) (Lemma 5.10) is the combinatorial statement behind such bounds: when some lines or legs of a renormalised family are written as differences and the family is expanded, the cut terms regroup, under Bałaban's three completions, into complete renormalised classes of the cut graphs, to which the renormalisation bound of [3, Prop. 1] applies with constants free of the domain and of the background. With scalar sources a third statement, (K$_h$) (Lemma 6.1), gives the localisation bounds for the source-dependent graphs of the effective action on the cube and the convergence of each such graph between the k-model after j steps and the $(k+1)$-model after $j+1$ steps, which live on the same unit lattice.

The results we use are of two kinds. Estimates taken from Bałaban's block-factor-5 analysis [1, 2, 3, 4] on tori, and their transfer to King's cube, are stated as propositions or lemmas, each followed by a description of its content and of the course of its proof, and by the definitions and auxiliary statements that the proof uses. Three of these statements are used at several places and are therefore stated once, with a precise formulation, in Section 5.2 ((BAL$_5$)[int], (REN$_{\mathrm{orb}}$)) and Section 6.1 ((K$_h$)). Everything else in the two sections is proved in place.

5.1. **Renormalisation-group integrals on King's cube.** This subsection sets up King's renormalisation-group flow on the cube and the diagonal pairs of spacings that the Cauchy argument of Section 5.4 compares.

Fix the couplings, a cube C_J and King's RG cube C_J^K of Definition 2.3. In this section and the next the index k is the spacing index ($\varepsilon_k = 5^{-k}$, as throughout), and j counts renormalisation steps, $0 \le j \le k - J_*$ for a fixed J_* (King's "J" of [25, Thm. 3.1]); the block scale after j steps is $\eta := 5^j \varepsilon_k = 5^{j-k} \le 1$, and the unit lattice after j steps is $\eta^{-1}(C_J \cap \eta\mathbb{Z}^3)$. Since the side $2MJ$ of C_J is an integer, the faces of C_J lie on block planes at every scale.

Sources. A real source is $s = (u, h)$ with $u = u_x := \sum_i x_i f_i$ a real combination of 2-forms $f_i \in C_c^\infty(\mathbb{R}^3; \Lambda^2)$ sampled at the plaquette centres in the lattice's own frame, and $h \in C_c^\infty(\mathbb{R}^3)$ real; all supports lie at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_J$ (so $J \ge J_0$). Put $X(s) := F(u) + {:}\phi^2{:}(h)$, with the free Wick constant of Definition 2.5; X is real and linear in s. The unnormalised functional is

$$Z_k(s) \;:=\; \int e^{-S}\, e^{X(s)} \prod dA \prod d^2\phi$$

over the variables of C_J^K at spacing ε_k (any fixed, s-independent normalisation of the measure; it cancels everywhere below), and $N_k(s) := Z_k(s)/Z_k(0) = \langle e^{X(s)} \rangle_{k,J}^K$. At $h = 0$ we write $a_k(x) := N_k(u_x, 0)$, which is the $a_{k,J}^K(x)$ of Corollary 4.4.

Definition 5.1 (King's block spin at zero field on the infinite lattice)**.** On $\Lambda_k = \varepsilon_k\mathbb{Z}^3$ run King's block-spin averaging at $A = 0$ with block factor 5: the uniform photon average, and for the scalar the weighted average [26, (3.1)–(3.2)] with the Dirichlet weight f_D of [26, (3.8)] at the finest step and the uniform weight above it, where f_D is $5\varepsilon_k$-periodic, equal to $(5/4)^3$ on the sites of each $5\varepsilon_k$-block whose coordinates relative to the block's corner lie in $\{0,1,2,3\}\varepsilon_k$ and to 0 on the sites with a residue $4\varepsilon_k$ in some coordinate, so that its block averages are 1 and $\|f_D\|_\infty = (5/4)^3$. Let Q_j be the composite average onto the blocks of side $\eta = 5^j\varepsilon_k$ and Q_j^* its adjoint for the inner products $\sum\varepsilon_k^3$, $\sum\eta^3$. The fluctuation covariance after j steps is

$$G_j := \big(-\Delta_{\varepsilon_k} + m^2 + \mathfrak{a}_j\eta^{-2}Q_j^*Q_j\big)^{-1}, \qquad \mathfrak{a}_j = \mathfrak{a}(1-5^{-2})(1-5^{-2j})^{-1}$$

[25, (2.13)], [26, (3.5)], with $\mathfrak{a} > 0$ the stiffness of King's block-spin kernel; the free propagator splits as $C := (-\Delta_{\varepsilon_k} + m^2)^{-1} = G_j + \Gamma_j$; the photon is the same with m^2 replaced by μ_0^2, the uniform weight and $A = 0$, $D = D_j + \Gamma_j^{\mathrm{ph}}$ with $D_j = \sum_{i<j} G_{(i)}^{\mathrm{ph}}$ [25, (2.17)].

King's flow on the cube. The block-spin transformations of [25, Sect. 2] are run on C_J^K with block factor 5: the photon field is averaged over the $5^j\varepsilon_k$-blocks that tile X_J, with the Neumann condition of Definition 2.3; the scalar field is averaged with King's Dirichlet weight [26, (3.8)–(3.9)] (in the corrected form in which the averaging blocks have their top layers on the Dirichlet planes: the weight is $(5/4)^3$ at the sites of each finest block whose three coordinates have residues in $S = \{0,1,2,3\}$ relative to the block's corner and 0 at the sites with a residue 4 in some coordinate (for a block whose top face lies on a Dirichlet plane, the sites with residue 4 in the coordinate normal to the plane form the layer on that plane), Definition 5.1) at the finest step and with the uniform weight above it. Explicitly (Lemma 5.2 below), at $A = 0$ and zero block fields, with ι the extension by zero from the alive set alive_J^K to $\varepsilon_k\mathbb{Z}^3$, $\ell_i := 5^i\varepsilon_k$ and $Q_j^X := Q_j\iota$ the composite average of Definition 5.1 applied to the field killed off alive_J^K (its range lies on the blocks of X_J, and since f_D vanishes on T_J this is Q_j restricted to the alive set): the scalar fluctuation covariance after $j \geq 1$ steps is $G_j(C_J^K) = (H^X)^{-1}$ with $H^X := \iota^*(-\Delta_{\varepsilon_k} + m^2)\iota + \mathfrak{a}_j\eta^{-2}(Q_j^X)^*Q_j^X = \iota^*G_j^{-1}\iota$, the Dirichlet Laplacian of Definition 2.3 with the block-spin term; the photon covariance after $i \geq 1$ steps is $D_i(C_J^K) = (-\Delta_{X_J}^N + \mu_0^2 + \mathfrak{a}_i\ell_i^{-2}\mathsf{P}_{X_J,\ell_i})^{-1}$ componentwise, with the Neumann Laplacian of Definition 2.3 and $\mathsf{P}_{X_J,\ell}$ the orthogonal projection onto the functions constant on the ℓ-blocks of X_J (the uniform block average), $D_0(C_J^K) := 0$, the slices $G_{(i)}^{\mathrm{ph}}(C_J^K) := D_{i+1}(C_J^K) - D_i(C_J^K) \geq 0$ with $\sum_{i<n} G_{(i)}^{\mathrm{ph}}(C_J^K) = D_n(C_J^K)$; and the background-field profile and the effective quadratic form after j steps are $H_j(C_J^K) = \mathfrak{a}_j\eta^{-2}G_j(C_J^K)(Q_j^X)^*$ and $\Delta^{(j)}(C_J^K) = \mathfrak{a}_j\eta^{-2}\big(\mathbf{1} - \mathfrak{a}_j\eta^{-2}Q_j^XG_j(C_J^K)(Q_j^X)^*\big) > 0$ on the block lattice of C_J, with $\iota^*(-\Delta_{\varepsilon_k} + m^2)\iota\, H_j(C_J^K) = (Q_j^X)^*\Delta^{(j)}(C_J^K)$ and the stiffness $\mathfrak{a}_j$ of Definition 5.1; the Dirichlet weight at the finest step does not change the composite stiffness. The Dirichlet planes of C_J^K are $2MJ$ apart, a multiple of every block side, which is what makes this flow possible on C_J^K and not on the comparison cube (Remark 5.11). Let $S_k^{(j)}(A_j, \phi_j; s)$ be King's effective action after j steps [25, (3.14)], [26, (3.12)], with the source vertices [25, (3.20)–(3.21)]; let χ_j be King's small-field indicator, [25, (3.2)] times the face factor [26, (5.41)], a function of (A_j, ϕ_j) alone; and

$$I_k^{(j)}(s) \;:=\; \int dA_j\, d\phi_j\; \chi_j\; e^{-S_k^{(j)}(A_j,\phi_j;s)}.$$

The *vacuum constant* $v_{k,j} := S_k^{(j)}(0,0;0)$ is finite; it contains King's E_0, E_1, the normalisation constants $\ln Z_j$, $\ln Z_j(0)$ of the block-spin kernels and every field- and source-free graph. The *normalised action* is $S_{N,k}^{(j)}(\cdot; s) := S_k^{(j)}(\cdot; s) - v_{k,j}$ for every s (the field-free, source-dependent part

is kept), and $\hat I_k^{(j)}(s) := \int \chi_j e^{-S^{(j)}_{N,k}(\cdot;s)} = e^{v_{k,j}} I_k^{(j)}(s)$. The flow is a device of the proof: the state $\langle\cdot\rangle^K_{k,J}$ does not depend on it.

Lemma 5.2 (Gaussian data of the flow on C_J^K)**.** *Let the block-spin kernel of step i at $A=0$ be* $\exp(-\frac{\gamma_i}{2}\|\psi_i - Q_{(i)}\psi_{i-1}\|^2_{\ell_i})$ *per real component, up to a field-independent constant (the flow of* [25, Sect. 2] *run on C_J^K as above), with $Q_{(1)}$ the f_D-weighted and $Q_{(i)}$, $i\ge 2$, the uniform average (for the photon the uniform average throughout), with the same stiffnesses γ_i on C_J^K and on Λ_k, and let the Gaussian part of the action at $A=0$ be $\frac12\langle\phi, \iota^*(-\Delta_{\varepsilon_k}+m^2)\iota\phi\rangle$ per real scalar component and $\frac12\langle A_\alpha, (-\Delta^N_{X_J}+\mu_0^2)A_\alpha\rangle$ per photon component (the photon quadratic form of Definition 2.3), with ι, ℓ_i and $Q_j^X = Q_j\iota$ as above. Then $\gamma_i = \mathfrak{a}\ell_i^{-2}$ with the stiffness $\mathfrak{a}$ of Definition 5.1; the composite of $j\ge 1$ steps on C_J^K is a field-independent constant times* $\exp(-\frac{\mathfrak{a}_j\eta^{-2}}{2}\|\psi_j - Q_j^X\phi\|^2_\eta)$, *$\eta=\ell_j$; and the formulas for $G_j(C_J^K)$, $H_j(C_J^K)$, $\Delta^{(j)}(C_J^K)$, $D_i(C_J^K)$ and $G^{\rm ph}_{(i)}(C_J^K)$ displayed above hold, with $\Delta^{(j)}(C_J^K)>0$, $G^{\rm ph}_{(i)}(C_J^K)\ge 0$ and $\iota^*(-\Delta_{\varepsilon_k}+m^2)\iota\, H_j(C_J^K) = (Q_j^X)^*\Delta^{(j)}(C_J^K)$.*

Proof. The proof is the standard Gaussian computation, based on the following facts. The ℓ_i-blocks with corners in $-MJ+\ell_i\mathbb{Z}^3$ tile X_J for $i\le k$ and nest 125-to-one, because the side $2MJ$ of C_J is an integer (Definition 2.3, as noted at the head of this subsection); so $Q_{(i)}Q^*_{(i)} = \mathbf{1}$ for $i\ge 2$ and $Q_i^*Q_i = \mathsf{P}_{X_J,\ell_i}$ for the uniform averages. Integrating out the intermediate block field in the product of two consecutive kernels gives, up to a field-independent constant, a kernel of the same form in the composite average, the stiffnesses adding harmonically, $\Gamma_{i+1}^{-1} = \Gamma_i^{-1} + \gamma_{i+1}^{-1}$ (this uses $Q_{(i+1)}Q^*_{(i+1)} = \mathbf{1}$ only). The recursion holds on Λ_k blockwise, and Definition 5.1 gives $\Gamma_j = \mathfrak{a}_j\eta^{-2}$ there for every j, whence $\gamma_1 = \mathfrak{a}\ell_1^{-2}$ and $\gamma_j^{-1} = \Gamma_j^{-1} - \Gamma_{j-1}^{-1} = \ell_j^2/\mathfrak{a}$ for $j\ge 2$; the same recursion on C_J^K, with the same γ_i, returns $\Gamma_j = \mathfrak{a}_j\eta^{-2}$, the Dirichlet weight at the finest step changing neither the recursion (no condition on $Q_{(1)}$ enters it) nor the composite stiffness. The weight f_D vanishes at the sites with a residue 4, hence on T_J, so $Q_j^X = Q_j\iota$. The form $\iota^*(-\Delta_{\varepsilon_k}+m^2)\iota$ is positive (a function whose Dirichlet gradient form vanishes is constant and vanishes with its values on T_J, and $m^2>0$), and $-\Delta^N_{X_J}+\mu_0^2 \ge \mu_0^2 > 0$. The formulas then follow by completing the square, with $\mathsf{A} = \iota^*(-\Delta_{\varepsilon_k}+m^2)\iota$, $Q = Q_j^X$ and $\Gamma = \mathfrak{a}_j\eta^{-2}$: the fluctuation covariance is $(\mathsf{A}+\Gamma Q^*Q)^{-1} = G_j(C_J^K)$; the minimiser of the exponent at the block field ψ is $\Gamma G_j(C_J^K)Q^*\psi = H_j(C_J^K)\psi$; the quadratic form left in ψ is $\frac12\langle\psi, \Gamma(\mathbf{1}-\Gamma Q G_j(C_J^K)Q^*)\psi\rangle = \frac12\langle\psi, \Delta^{(j)}(C_J^K)\psi\rangle$, and $\Delta^{(j)}(C_J^K)^{-1} = \Gamma^{-1} + Q\mathsf{A}^{-1}Q^*$ shows $\Delta^{(j)}(C_J^K)>0$; and $\mathsf{A}\,H_j(C_J^K) = \Gamma(\mathsf{A}+\Gamma Q^*Q - \Gamma Q^*Q)G_j(C_J^K)Q^* = Q^*\Delta^{(j)}(C_J^K)$ is the identity. For the photon, with $B = -\Delta^N_{X_J}+\mu_0^2$, $\mathsf{P}_i = \mathsf{P}_{X_J,\ell_i}$ and $\Gamma_i = \mathfrak{a}_i\ell_i^{-2}$, which is strictly decreasing in i by the recursion, the same computation gives the covariance $D_i(C_J^K) = (B+\Gamma_i\mathsf{P}_i)^{-1}$ after i steps; since the ranges of the P_i decrease with i, $\Gamma_{i+1}\mathsf{P}_{i+1}\le\Gamma_i\mathsf{P}_i$, so the inverses satisfy $D_{i+1}(C_J^K)\ge D_i(C_J^K)$, which gives the slices $G^{\rm ph}_{(i)}(C_J^K)\ge 0$ and, with $D_0(C_J^K)=0$, $\sum_{i<n} G^{\rm ph}_{(i)}(C_J^K) = D_n(C_J^K)$. □

Diagonal pairs. Fix $\alpha\in(0,\frac12)$ and put $j(k) := \lfloor(1-\alpha)k\rfloor$, $\eta_k := 5^{j(k)-k}$; then $\eta_k\le 5^{-\alpha k}$ and $k-j(k)\le \alpha k+1$. The *diagonal pair* at k consists of the k-model after $j(k)$ steps and the $(k+1)$-model after $j(k)+1$ steps. Both have the block scale η_k ($5^{j+1}\varepsilon_{k+1} = 5^j\varepsilon_k$), hence the same unit lattice and the same indicator $\chi_{j(k)}$; this is the situation of [25, p. 656] ("the fields A_k, ϕ_k have the same bounds in each integral"). No relation between $j(k+1)$ and $j(k)+1$ is used anywhere.

Bałaban's decomposition. In Bałaban's block-spin representation [1, 2], $Z_k(s) = \int d\mu_{k,s}$ with $d\mu_{k,s} = e^{X(s)}d\mu_{k,0}$ a positive measure on the fields and on the block variables of the scales $\le j$, and $P = P_j\in[0,1]$ is the product of all small-field factors of those scales, a function of the block

and fluctuation variables alone. The identity $Z_k(s) = \int P\, d\mu_{k,s} + \int (1-P)\, d\mu_{k,s}$ is exact; the first term is the *small-field part*, the second the *large-field remainder*.

5.2. **Two statements from the renormalisation-group analysis.** This subsection states the two estimates from Bałaban's analysis that recur throughout the existence part, (BAL$_5$)[int] and (REN$_{\mathrm{orb}}$), each followed by a description of its proof.

Lemma 5.3 ((BAL$_5$)[int]: Bałaban's interaction-term bound at a non-zero background)**.** *In the setting of* [2, Sect. 2] *at block factor $L = 5$, in unit-lattice form at the step k, $t = 5^k\varepsilon$: let $\Omega_A = B^k(\Lambda_2^{(k-1)\prime}) \supset \Omega_C = B^k(\Lambda_2^{(k)})$ be the two propagator domains of* [2, Prop. 2.6]*, let $\tilde{B}$ be the background field of* [2, p. 579]*, regular on Ω_A, let $G_k(\Omega, \tilde{B})$ be the scalar propagator of the step on Ω in the background $\tilde{B}$, and write the interaction terms of the stage of* [2, p. 579]*, localised in $B^k(\Lambda_7^{(k)})$, as a sum of per-label full renormalised classes with scalar lines $G_k(\Omega_A, \tilde{B})$, external scalar fields $E_A = a_k G_k(\Omega_A, \tilde{B}) f$ with $\|f\|_\infty \le C p \lambda(t)^{-1/4}$, and other external fields of norms $\le C p^a \lambda(t)^{-b}$ (the setting of Bałaban's domain-change estimate* [2, Prop. 2.6]*, which holds at every regular background). Replace $G_k(\Omega_A, \tilde{B})$ by $G_k(\Omega_C, B^{(k+1),\eta})$ in every scalar line and in every E_A. For every $\kappa > 0$ there are $t_0(\kappa) > 0$ and $C_\kappa < \infty$, depending on κ, on Bałaban's parameters $(R, \varrho, b_0, \tilde{p}, \bar{n}, \alpha_0)$ (in his notation) and on the couplings only, such that for $t \le t_0(\kappa)$ the sum over all localisations of the absolute values of all terms containing at least one difference of propagators or of external fields is at most $C_\kappa\, t^\kappa\, |\Lambda_7^{(k)}|$. The constants do not depend on k, on the domains or on $\tilde{B}$, and $\tilde{B} = 0$ is included.*

The proof writes the difference of the two domains' class sums as a sum over cut skeletons with $(n, m) \neq (0, 0)$ difference lines and bounds the total by $C_\kappa t^\kappa |\Lambda_7^{(k)}|$ for every κ; its course is the following. This is the interaction-term half of Bałaban's domain-change estimate [2, Prop. 2.6], at every background. The terms with exactly one difference line, grouped line by line, carry their counterterm partners but not the complete renormalised classes of the cut graph, and a lone divergent two-photon-leg subgraph is not small per graph; the proof expands every line and every external field in the difference and regroups the terms exactly into contractions of *full* classes of the cut components, to which the renormalisation bound [3, Prop. 1, (1.33)] applies with constants free of the domains and of the background. Every use of the upper stability bound [2, (1.3)] and of [25, Thm. 3.1, (3.4)] at block factor 5 passes through this statement.

The following definitions fix the vocabulary of this proof and of Lemma 5.10: cut terms, cut graphs and their orbits (Definition 5.4), then the renormalised families, their subgraphs, the moves and the classes (Definition 5.6–Definition 5.9), which the first definition already uses.

Definition 5.4 (cut terms, cut graphs and orbits)**.** Let $\mathcal{R} = \sum_{\Gamma \in \mathcal{G}} c_\Gamma E(\Gamma)$ be a renormalised family (Definition 5.6). Fix a set of *decomposed kinds* of objects of the family (for instance all lines and legs on a domain, or all restricted lines) and write every object o of such a kind as $o = o' - \Delta o$ with Δo a symmetric kernel, $\Delta o(x, y) = \Delta o(y, x)^T$, as is every difference of two symmetric resolvents. An *oriented cut term* is a triple $\tau = (\Gamma, S, \theta)$ with $\Gamma \in \mathcal{G}$, S a set of decomposed objects of Γ and θ an orientation of every line of S; its weight is $2^{-|S_L|} c_\Gamma$, S_L the set of lines of S. Since the two orientations of a line add up to Δo by its symmetry, multilinearity gives the exact expansion

$$\mathcal{R} = \sum_\tau 2^{-|S_L|} c_\Gamma \prod_{o \in S} (-\Delta o)_\theta \prod_{o \notin S} o'. \tag{3}$$

The *cut graph* $\Gamma \setminus S$ is Γ with the lines of S removed, their ends carrying the two arguments of Δo (end 0 and end 1 according to θ), each leg of S carrying Δo, and the objects of S labelled; this is Bałaban's "we replace these lines by pairs of external legs" [3, p. 432], with external fields

in the form of functions of several variables [3, p. 419]. A subgraph H of Γ *avoids* S if it contains no line of S. For a move μ at a subgraph H avoiding S put $\mu\cdot\tau := (\mu\Gamma, s_\mu(S), s_\mu(\theta))$: the ends of the S-lines and the S-legs sitting at slots of H are moved to the image slots, and everything else is fixed. The *orbits* are the classes of the equivalence relation generated by $\tau \sim \mu\cdot\tau$.

Lemma 5.5 (properties of the orbits)**.** *By the properties* (A1)–(A5) *of the moves (Definition 5.8): a connected line set of* Γ *avoids* S *exactly when it is a connected line set of* $\Gamma\setminus S$*, with the same vertices, slots, slot types, order and degree in both, so the moves available at it and their results are the same in* Γ *and in* $\Gamma\setminus S$*; when* $\mathcal{R}$ *is closed under the moves,* $\mu\cdot\tau$ *is again an oriented cut term of* $\mathcal{R}$*,* μ^{-1} *acts at* H^**, which avoids* $s_\mu(S)$*, and* $\mu^{-1}\cdot(\mu\cdot\tau)=\tau$*, so that the orbits partition the cut terms, every orbit is finite and the number of* S*-objects of each kind is constant on it; and cutting commutes with the moves,* $(\mu\Gamma)\setminus s_\mu(S) = \mu(\Gamma\setminus S)$*, so that the cut graphs of one orbit form exactly one class of the (possibly disconnected) cut graph, the product of the classes of its connected components, each cut graph being reached from exactly* $r_O \geq 1$ *cut terms of the orbit* O*, where* r_O *is the order of the group of relabellings of identical* S*-objects (same kind, same kernel, same orientation pattern) that preserve the class. These three facts are what the orbit regrouping of Lemma 5.10 rests on.*

Definition 5.6 (renormalised families, labels, lines and legs)**.** A *renormalised family* is a finite sum $\mathcal{R} = \sum_{\Gamma\in\mathcal{G}} c_\Gamma E(\Gamma)$ over a finite set $\mathcal{G}$ of graphs built from the vertices [3, (1.6)–(1.15)], from the mass-counterterm vertices [3, (1.7)] with the coefficients [3, (1.23), (1.29)] and from source vertices; c_Γ collects the couplings, the combinatorial weight and the counterterm values, every counterterm being a vertex of the action with a field- and domain-independent coefficient [3, (1.29)–(1.31), p. 419]. Every vertex carries *labels* [3, p. 420]: each vector-field leg the support box of its partition function, each vertex of type (1.6)/(1.7) its unit cube, each block vertex (1.13)–(1.15) its block; the counterterm vertex that replaces a divergent two-scalar-leg subgraph γ carries the labels of γ, so that a graph and its counterterm partner have the same label multiset [3, pp. 417, 420, 432]. A *line* is a pair of legs with a kernel type (scalar or photon, with its domain and background); an unpaired leg is an *external leg* and carries an external field, possibly a function of several variables [3, p. 419].

Definition 5.7 (subgraphs, degrees and the divergent subgraphs)**.** A *subgraph* H of Γ is a connected set of lines; its *slots* are the legs of its vertices not on lines of H (legs on other lines, external legs, cut ends), its *type* is the number and kind of its slots with the differentiation of the scalar ones, its order is the sum of the orders of its vertices, and its degree $D(H)$ is that of [3, (2.1)–(2.3)], a counterterm vertex counting as the graph it replaces; type and degree depend only on the vertices of H, its lines and which legs are slots. The divergent subgraphs and their completions [3, pp. 420, 429–432] are: a two-scalar-leg one-particle-irreducible H with undifferentiated slots and $D(H)\leq 0$, completed by its *counterterm partner*; a two-vector-leg H with $D(H)\leq 0$, completed by the *two-vector-leg class*, all connected graphs of the considered type, order and label multiset [3, p. 431]; a $\phi\phi A$ subgraph with $D(H)=0$, one-particle-irreducible or reducible, completed by the *permutation* of its two external scalar legs [3, p. 429]. Graphs with a block vertex have $D>0$ [3, Cor. 2.3], so no divergent subgraph contains one.

Definition 5.8 (the moves)**.** Each completion is a *move* μ at a subgraph H of its source Γ: it replaces H by a graph H^* together with a bijection s_μ from the slots of H to those of H^*, and whatever sits at a slot of H (an outer line end, an external field, a cut end) is re-attached at the image slot; nothing else changes. The moves are: (P), the counterterm partner, which replaces the vertices and lines of H by one counterterm vertex carrying the labels of H, the two slots going to its two legs, and its inverse (P′), the expansion of a counterterm vertex; (W), the two-vector-leg completion $H\to H'$ for every H' in the per-label two-vector-leg class of H

(all connected graphs with the label multiset and order of H whose two free photon legs carry the two slot labels of H, with their counterterm members), the slots mapped by their vector-leg labels, applied at every divergent two-vector-leg subgraph, maximal or not; (X), the exchange of the attachments at the two scalar slots of a divergent $\phi\phi A$ subgraph (a line joining the two slots is mapped to itself with its ends exchanged; two external fields of one kind exchange their arguments). The moves have the five properties on which the regrouping rests: (A1) a move changes only the lines and vertices of H and the attachment of the slots, independently of what is attached there; (A2) its availability at H and its results (H^*, s_μ) depend only on H with its slots, because $D(H)$, the slot types and the per-label two-vector-leg class are intrinsic to H; (A3) every move has an inverse at H^* with slot map s_μ^{-1}; (A4) a move keeps the label multiset, the kind of every slot and the kernel type of every surviving line; (A5) $c_{\mu\Gamma}/c_\Gamma$ depends only on (H, H^*, s_μ).

Definition 5.9 (classes)**.** The *class* $\mathrm{Cl}(\Gamma)$ of a graph is its connected component under the moves, the moves at Γ itself included when Γ is of a primitively divergent type (its one-vertex counterterm partner, its permuted graph, all two-vector-leg graphs of its order). It is Bałaban's class G_{ren} of Γ: "the smallest sets of graphs, symmetric with respect to permutation of external scalar field legs and with each divergent subgraph renormalized" [3, p. 429] are exactly the connected components, the moves being invertible. A *per-label full class* is the set of all connected graphs of one type, one order and one label multiset, with their counterterm members [3, p. 431]; it is closed under every move. A family is *closed under the moves* if $\mathcal{G}$ is a finite union of classes; Bałaban's renormalised families are.

Lemma 5.10 ((REN$_{\mathrm{orb}}$): orbit regrouping of cut renormalised families)**.** *Let* $\mathcal{R} = \sum_{\Gamma\in\mathcal{G}} c_\Gamma E(\Gamma)$ *be a renormalised family of* [3, Sects. 1–3] *with* $\mathcal{G}$ *a finite union of renormalised classes (Definition 5.9), in which the objects (lines or external legs) of certain kinds are decomposed as* $o = o' - \Delta o$ *with symmetric kernels* Δo*, and expand multilinearly,* $\mathcal{R} = \sum_\tau 2^{-|S_L|} c_\Gamma \prod_{o\in S}(-\Delta o) \prod_{o\notin S} o'$ *over the oriented cut terms* $\tau = (\Gamma, S, \theta)$ *of Definition 5.4. Call two cut terms equivalent when one arises from the other by finitely many of Bałaban's three completions of* [3, pp. 420, 429–432] *(the counterterm partner; the two-vector-leg class; the permutation of the external scalar legs of a divergent* $\phi\phi A$ *subgraph) applied at divergent subgraphs containing no line of* S*, the objects of* S *carried along to the image slots. Then:* (i) *the equivalence classes (orbits) partition the cut terms, and every move keeps the label multiset and the number and kinds of the objects of* S*;* (ii) *a connected line set of* Γ *avoiding* S *is a subgraph of the cut graph* $\Gamma\setminus S$ *with the same vertices, slots, slot types, order and degree* [3, (2.1)–(2.3)]*, and cutting commutes with the moves, so that the cut graphs of one orbit form exactly one renormalised class of the cut graph, the product of the classes of its connected components;* (iii) *at fixed labels, after the sum over the separable indices of the cut kernels, the sum of the cut terms of an orbit* O *is* $(-1)^{|S|} 2^{-|S_L|} r_O^{-1} c_* \prod_K E(\mathrm{Cl}(K);$ *its fields*$)$*, the product over the components* K *of the class sums with the cut kernels as external fields,* c_* *the common value on the orbit of* $c_{\Gamma'}/\prod_K c(K')$*,* $c(K')$ *the product of the vertex weights of the component (a constant by (A5) of Definition 5.8), and* $r_O \geq 1$ *the order of the group of relabellings of identical* S*-objects preserving the class (Lemma 5.5);* (iv) *the sum over all orbits is the full expansion;* (v) *if* $\mathcal{G}$ *is a per-label full class, the cut terms with a given cut datum (components with labels and orders, stubs and fields at each component, incidence of the* S*-objects) correspond, with a multiplicity* $s_\delta \geq 1$ *depending only on the cut datum* δ*, to the tuples of members of the per-label full classes of the components, members without internal line and of order* 0 *included; each such group is a union of orbits and is closed under the exchange of the scalar legs of every two-scalar-leg or* $\phi\phi A$ *subgraph.*

The proof partitions the cut terms of Definition 5.4 of a move-closed family into the orbits of the three completions, every vertex carrying Bałaban's labels (support boxes, unit cube, block)

and a counterterm vertex inheriting the labels of the subgraph it replaces, so that the regrouping acts within fixed label multisets; the sum over an orbit is then computed from the class sums of the components of the cut graph, and the count s_δ of (v) from the per-label full classes of the components. This is Bałaban's regrouping step in the proof of [3, Prop. 1, p. 432] ("if we remove a line in any graph of the class, then for the obtained graph we can find in this class all graphs necessary to form a new renormalized class"), made explicit for decomposed objects, with the sharper conclusion that one orbit gives exactly one class of the cut graph. It is needed wherever the renormalisation bound is applied to a family some of whose lines have been cut, in particular in the comparison of two propagator domains and in the telescoping of the cube's large-field remainders; at a non-zero background, where the bound of [3, Prop. 1] is available for per-label full classes, it is used in the form (v).

Remark 5.11 (the cube). The propositions below are stated for King's RG cube C_J^K, on which King's Dirichlet weight is compatible with the image group of the two Dirichlet planes (they are $2MJ$ apart). On the comparison cube $\mathrm{ND}^{(a)}$ the planes are $2MJ+\varepsilon_k$ apart, no partition into consecutive blocks is invariant under the image group, and the flow does not exist; this is why Theorem A is posed on King's convention, and why the fixed-cube limits of the comparison family are reached only through Corollary 4.4. The cube of the existence theorem is C_J^K itself (Definition 2.3).

5.3. **The estimates at zero scalar source.** The propositions of this subsection are the estimates at zero scalar source from which the Cauchy property of Section 5.4 is derived.

Proposition 5.12 ((H0rel)$_C$: the small-field representation on King's cube, zero scalar source)**.** *Fix $J \geq J_0$, a family $f_1, \dots, f_N$ and $x \in \mathbb{R}^N$, and let $s \in \{u_x, 0\}$. There are $\eta_1 > 0$, k_0 and $C, c, \sigma > 0$, depending on the couplings, on J, on f and x, and on nothing else (not on k, j), such that for all $k \geq k_0$ and all j with $J_* \leq j \leq k - J_*$ and $\eta = 5^{j-k} \leq \eta_1$:*

$$e^{-C\eta^\sigma}\, I_k^{(j)}(s) \;\leq\; Z_k(s) \;\leq\; e^{C\eta^\sigma}\, I_k^{(j)}(s) + \rho(\eta)\, Z_k(0), \qquad \rho(\eta) := e^{-c\,p(\eta)^2 + C}, \tag{4}$$

where $p(\eta) = b_0(1 + \ln \eta^{-1})^6$ is Bałaban's small-field threshold (the exponent 6 is compatible with the conditions [1, (3.54)]). Moreover, at $s = 0$ the two terms of the upper line are the two parts of Bałaban's decomposition: $\int P\, d\mu_{k,0} \leq e^{C\eta^\sigma} I_k^{(j)}(0)$ and $\int (1-P)\, d\mu_{k,0} \leq \rho(\eta) Z_k(0)$ ((R$_0$)).

Proposition 5.12 replaces the cube functional $Z_k(s)$ by the integral $I_k^{(j)}(s)$ of the effective action after j renormalisation steps, up to the factors $e^{\pm C\eta^\sigma}$ and a remainder $\rho(\eta) Z_k(0)$, which is exponentially small in the square of Bałaban's small-field threshold $p(\eta)$. Its point is that two spacings whose effective actions, after suitable numbers of steps, live on the same unit lattice (the diagonal pairs of Section 5.1) can be compared through these integrals; this comparison gives the Cauchy property of the normalised functional (Theorem 5.45) and hence the existence of the cube limits (Theorem 5.47). The proof is given below: Lemmas 5.13–5.16 provide the four modifications of the torus argument, Proposition 5.17 the two lines and Theorem 5.18 the clause (R$_0$). This is [25, Thm. 3.1] on the cube, which [26, Sect. 5.2] states ("the proof follows by modifying suitably the material in [Ba 1] and [Ba 2]"), in the form in which the remainder is measured against $Z_k(0)$ rather than against an absolute constant, and with constants uniform in the step j (the form the diagonal pairs need, since $\eta_k \to 0$). The lower line is the weaker of the two readings of the sign in [25, (3.3)]. The statement contains no lower bound on $Z_k(0)$; the proof uses Bałaban's lower bound only to pass from the absolute remainder to the ratio form. Of the statements of Section 5.2, (BAL$_5$)[int] (through Bałaban's torus bounds) and (REN$_{\mathrm{orb}}$) (through the comparison of propagator domains and the telescoping of the large-field remainders) enter.

Proof of Proposition 5.12. The proof is King's proof of [25, Thm. 3.1], that is Bałaban's lower and upper bounds [1, 2] with the modifications of [25, Appendix], run on King's cube C_J^K with King's block-spin flow of Section 5.1. Two things change relative to the torus. First, the scalar field is killed on the Dirichlet planes of the cube and averaged with King's Dirichlet weight; every place where the torus arguments use that the block average is uniform ($QQ^* = 1$, P a projector, the constant reproduced by the average) is replaced by the corresponding fact for the weighted average (Lemma 5.13), and every place where the torus has no boundary acquires a Dirichlet face, at which the scalar background vanishes linearly (Lemma 5.15). Second, the photon has Neumann conditions; every photon object of the cube is the even restriction of the same object on a doubled torus, so every photon estimate of the torus analysis holds on the cube with its constant multiplied by at most 8 (Lemma 5.14). The large-field analysis near the Dirichlet faces uses King's face test [26, (5.41)] as a large-field family at every scale, paid by a Dirichlet-face positivity valid for every gauge field (Lemma 5.16). Throughout, $d = 3$, $L = 5$, the couplings are fixed, $J \geq J_0$, $s \in \{u_x, 0\}$, and constants C, c, δ never depend on the spacing index k, the step j, the field or the domain unless this is said. We work in unit-j units, in which the lattice after j steps has unit spacing, and $\varepsilon_j := 5^{-j}$ denotes the fine spacing.

The weighted average. Let X_{live} be the live scalar sites of the cube at the current scale and K the killed sites (the Dirichlet planes). A *block weight* is a function $w \geq 0$ on the fine sites of a unit block with (W1) $\sum_{x\in B} \varepsilon_j^3 w(x) = 1$, (W2) $w \leq W$, (W3) $w = 0$ on every killed site of the block, (W4) the same w on every block up to translation; King's Dirichlet weight, which vanishes on the top layers of his block, is in the class with $W = 125/64$, and the uniform weight satisfies (W1), (W2), (W4) with $W = 1$ but not (W3) (on a block whose first site plane is a Dirichlet plane its live sites carry total weight at most $1 - \varepsilon_j$), so it is used only on lattices without killed sites: the coarse steps on the block lattice and the photon. The weighted average is $(Q(A)u)(y) := \sum_{x\in B(y)} \varepsilon_j^3 w(x) U(A(\Gamma_{y,x}))^{-1} u(x)$ with King's hierarchical contours $\Gamma_{y,x}$ [25, (2.11)–(2.12)], which start at the live lower corner of the block and stay off the killed set; $\mathsf{P} := Q(A)^* Q(A)$ and $M := -\Delta_A^D + V + \mathfrak{a}_j \mathsf{P}$ is the cube operator, with the Dirichlet Laplacian $-\Delta_A^D$ of the killed planes and the composite stiffness $\mathfrak{a}_j$ of Definition 5.1 (in unit-j units; $\mathfrak{a}$ denotes the bare stiffness).

Lemma 5.13 (the weighted average)**.** *For every weight in the class and every gauge field A, with $W_2 := \sum_{x\in B} \varepsilon_j^3 w(x)^2 \in [1, W]$ ($W_2 = W$ for King's weight): (Q1) $Q(0)\mathbf{1} = \mathbf{1}$ on every block, and $Q(A)$ maps the transported constant to the constant; (Q2) $Q(A)Q(A)^* = W_2 \mathbf{1}$; (Q3) $\mathsf{P} = W_2 \Pi$ with Π an orthogonal projector, $0 \leq \mathsf{P} \leq W_2$ and $\|1 - \mathsf{P}\| = \max(1, |1 - W_2|) \leq 1$ since $W_2 < 2$; (Q4) $|(Qu)(y)| \leq W_2^{1/2} \|\mathbf{1}_{B(y)} u\|_2$ and $\|Q^* e_y\|_2 = W_2^{1/2}$; (Q5) $|(Qu)(y)| \leq \sup_{B(y)} |u|$; (Q6) $|Q^* b|_\infty \leq W |b|_\infty$, $\|\mathsf{P}\|_{\infty\to\infty} \leq W$ and the Schur norm of P is at most W; (Q7) $\sum_x \varepsilon_j^3 |(\mathsf{P}u)(x)| = \sum_y |(Qu)(y)|$; (Q8) $\mathsf{P}(x, x') = 0$ unless x, x' lie in one block, and P vanishes on killed sites, so P never couples a live site to a killed one and every union of blocks is invariant. Every use of $QQ^* = 1$, of P being a projector and of the reproduction of the constant in [1, 2, 25] is replaced by (Q1)–(Q3), the factor W_2 entering the constants only; in particular the block-spin minimiser identity $\langle \phi, \Delta^{(j)} \phi \rangle = \min_\Phi [\langle \Phi, (-\Delta_A^D + V_j)\Phi \rangle + \mathfrak{a}_j \|\phi - Q\Phi\|^2]$ and the exact renormalisation group hold verbatim.*

The photon sector. Let $2X$ be the torus of period $2N_\eta$ in each direction, $N_\eta := 2MJ \cdot 5^{k-j}$ the side of the cube in unit lengths (physical period $4MJ$), the cube embedded as $\{0, \dots, N_\eta - \varepsilon_j\}^3$, and $\mathcal{W} \cong (\mathbb{Z}_2)^3$ the group generated by the reflections in the photon planes $x_\mu = -\varepsilon_j/2$. Since $4MJ$ is a multiple of every block side, the blocks of the cube are blocks of the periodic tiling of $2X$ and $\mathcal{W}$ maps blocks to blocks. For a photon component (a site field) let E be the even extension to $2X$, and E_c the same on the coarse lattices.

Lemma 5.14 (photon objects are even restrictions)**.** *For every photon object O of Bałaban's and King's construction built from the Neumann form $-\Delta^N + \mu_0^2\eta^2$ and the uniform block averages on the cube, and its counterpart O_{2X} on $2X$: (a) the propagators G_j, the block forms $\Delta^{(j)}$, the fluctuation covariances $C^{(j)}$, the background map $A_j \mapsto \mathfrak{a}_j G_j Q_j^* A_j$ and the slices satisfy $O_{2X}\mathsf{E} = \mathsf{E}\,O$; (b) $\langle \mathsf{E}A, \Delta_{2X}^{(j)}\mathsf{E}A\rangle = 8\langle A, \Delta^{(j)}A\rangle$, and likewise for every positive form; (c) for every coarse set Λ and every fine union of blocks Ω the Dirichlet-conditioned compressed operators on Λ and the Neumann operators on Ω intertwine with E and the $\mathcal{W}$-invariant sets $\mathcal{W}\Lambda$, $\mathcal{W}\Omega$; (d) with Bałaban's cut-off kernel ζ of* [2, (2.44)] *applied image-wise on $2X$, the cut-off objects of the cube are even restrictions of those of $2X$ with the torus constants. Consequently every photon estimate of the torus analysis (decay, regularity, the cut-off and localisation lemmas) holds on the cube with its constant multiplied by at most $|\mathcal{W}| = 8$, and the regularity of the photon background is preserved.*

Proof. (a) The Neumann Laplacian of the cube is the periodic Laplacian of $2X$ on the $\mathcal{W}$-even fields, the reflection planes lying between sites: every plane-crossing bond of $2X$ joins a site to its mirror image, where an even field has zero difference, and every bond of the cube appears 8 times. The uniform block averages commute with E (blocks go to blocks) and $Q_{j,2X}^*\mathsf{E}_c = \mathsf{E}Q_j^*$ (the adjoint of a $\mathcal{W}$-covariant uniform average is the uniform block extension). Hence $(-\Delta_{2X}^N + \mu_0^2\eta^2 + \mathfrak{a}_j Q_j^* Q_j)\mathsf{E} = \mathsf{E}(-\Delta^N + \mu_0^2\eta^2 + \mathfrak{a}_j Q_j^* Q_j)$; the torus operator commutes with $\mathcal{W}$, preserves the even subspace and is invertible on it, so $G_{j,2X}\mathsf{E} = \mathsf{E}G_j$, and the block forms, the fluctuation covariances, the background map and the slices follow from the algebra of [25, (2.13)–(2.17)], which is the same on both sides. (b) $\mathsf{E}^*\mathsf{E} = 8$ on the cube, so by (a) $\langle \mathsf{E}A, \Delta_{2X}^{(j)}\mathsf{E}A\rangle = \langle \mathsf{E}A, \mathsf{E}\Delta^{(j)}A\rangle = 8\langle A, \Delta^{(j)}A\rangle$, and likewise for every positive form. (c) Compression to a $\mathcal{W}$-invariant set commutes with $\mathcal{W}$, and $P_{\mathcal{W}\Lambda}\mathsf{E} = \mathsf{E}P_\Lambda$ with P_Λ the projection onto the functions supported on Λ; then (a) on the invariant subspace. For Neumann conditions on Ω, the bonds of $\mathcal{W}\Omega$ are the images of those of Ω plus plane-crossing bonds, on which even fields have zero difference. (d) $\zeta(x, gy)$ depends on $|x - gy|$ only; the image-wise kernel $K(x,y) := \sum_{g\in\mathcal{W}} \zeta(x, gy)K_{2X}(x, gy)$ commutes with $\mathcal{W}$ by construction and restricts to the cube kernel by definition; the support and size statements are statements about $\zeta(x, \cdot)$ on $2X$, and $|x - gy| \geq |x - y|$ for x, y in the cube keeps every support statement (the cut is a device inside the definition of Bałaban's background, so this choice is free). Finally, (a) and (b) give $O = \frac{1}{8}\mathsf{E}^* O_{2X}\mathsf{E}$, that is, since O_{2X} commutes with $\mathcal{W}$, $O(x,y) = \sum_{g\in\mathcal{W}} O_{2X}(x, gy)$ for the kernels: a sum of $|\mathcal{W}| = 8$ torus kernels at the distances $|x-gy| \geq |x-y|$, so every decay, regularity, cut-off and localisation estimate of the torus analysis holds on the cube with its constant multiplied by at most 8; every positivity bound of the photon sector (the photon part of [2, (3.29), Prop. 3.1], the last-step mass bound of [2, (3.11)], the photon exponents of [2, (3.26)]) holds with the constant of $2X$, because by (b) the cube form of an even field is $\frac{1}{8}$ of the torus form and the right-hand sides (gradient sums, mass sums, block sums) scale by the same factor, plane-crossing gradients of even fields vanishing; and the regularity of a photon configuration is preserved by E, since $\mathsf{E}A = A \circ F$ with F the coordinatewise folding map of $2X$ onto the cube, which is 1-Lipschitz, lattice differences across a plane vanishing and the others being copies. □

Dirichlet-face operations. For a sub-domain Ω of the cube (a union of blocks, with the Dirichlet condition of the cube at the killed planes and Neumann cuts elsewhere) and block data b on its blocks, the *leg* of b is $u := M^{-1}\mathfrak{a}_j Q^* b$, the background of the data; $t(x)$ denotes the distance of x to the Dirichlet planes.

Lemma 5.15 (linear vanishing at the Dirichlet face)**.** *For every Ω, every A, block data b and every live $x \in \Omega$: $|u(x)| \leq C_{\mathrm{DV}}|b|_\infty \min(1, t(x))$, and the same for $u = M^{-1}f$ with bounded*

f. Moreover, with $\delta_5 > 0$, the leg $l(\cdot, v)$ of the unit datum on a block v satisfies $|l(x,v)| \le C(\varepsilon_j + t_\mu(x))e^{-\delta_5 \operatorname{dist}(x,B(v))}$ for x in the first-layer slab of the face μ, so that $|u(x)| \le C|b|_\infty(\varepsilon_j + t(x))e^{-\delta_5 \operatorname{dist}(x,\operatorname{supp} b)/2}$ there.

Proof. Fix the face μ with $t_\mu(x) = t(x) \le 1$. Let $F_0 := \mathfrak{a}_j W(1+C)|b|_\infty$ bound $|\mathfrak{a}_j Q^*(b - Qu)|$ by (Q6) and the sup bound on u, let S_0 be that sup bound, and put $\rho(s) := (S_0 + F_0)s - F_0 s^2/2$ on $[0, 1+\varepsilon_j]$, increasing, with $\rho(0) = 0$, $\rho(1) \ge S_0$ and $-\Delta_s \rho = F_0$ exactly. On the first-layer slab $\mathcal{S}_1 := \Omega \cap X_{\text{live}} \cap \{0 < t_\mu \le 1\}$ the barrier $\Phi(x) := \rho(t_\mu(x))$ satisfies $(-\Delta_{0,\Omega}^{N,D} + V_j)\Phi \ge F_0$, which dominates the right-hand side of Kato's inequality $(-\Delta_0^{N,D} + V_j)|u| \le |\mathfrak{a}_j Q^*(b - Qu)|$ (tangential bonds and tangential Neumann faces see Φ constant, a Neumann face at the far side of the slab drops a neighbour with larger Φ, and on face bonds the killed neighbour carries $\Phi = 0 = |u|$); the boundary of $\mathcal{S}_1$ inside the live set lies at $t_\mu \in \{1, 1+\varepsilon_j\}$, where $|u| \le S_0 \le \Phi$. The operator on $\mathcal{S}_1$ with these data is a non-singular M-matrix, so the maximum principle gives $|u| \le \Phi \le (S_0 + F_0)t_\mu$ on $\mathcal{S}_1$. No hypothesis on the faces of Ω is used. The localised form follows by the same barrier with the exponential weight of the Combes–Thomas bound. □

The face test and its payment. Let L_1 be the layer of face blocks (the blocks containing a site of the first live layer F_1).

Lemma 5.16 (Dirichlet-face positivity)**.** *In unit-j units let $y' \in L_1$ be a face block and $\Delta^{(j)}$ the block form of the cube at any gauge field A. Then for every block field ϕ and $\theta \in (0,1)$,*

$$\langle \phi, \Delta^{(j)}\phi\rangle \;\ge\; (1-\theta)\langle \phi, \Delta^{(j)}\phi\rangle + \theta\,\gamma_D\,|\phi(y')|^2, \qquad \gamma_D = \frac{\mathfrak{a}_j}{1 + \mathfrak{a}_j W^2},$$

and the composite multiscale form of [2, (3.23)–(3.25)] *restricted to the fine sites of $B^j(y')$ is at least $(W + \mathfrak{a}_j^{-1})^{-1}\ell_j^{d-2}|\phi_j(y')|^2$, for every gauge field.*

Proof. By the minimiser identity, keep in θF only the gradient terms of the bonds inside $B(y')$, the Dirichlet terms of its face bonds and the block term of y'. With $g := |\Phi|$, unitary transport gives $|D_A\Phi(b)| \ge |g(b_+) - g(b_-)|$, and the discrete Friedrichs inequality along the lines normal to the killed face ($g = 0$ at the killed site) gives $\sum_{B(y')} \varepsilon_j^d g^2 \le E(g)$, the kept energy; then $|(Q\Phi)(y')| \le WE^{1/2}$ by (W2), and $\theta F \ge \theta \min_{s\ge 0}[W^{-2}s^2 + \mathfrak{a}_j(|\phi(y')| - s)_+^2] = \theta\gamma_D|\phi(y')|^2$ (springs in series). The composite form is the same argument scale by scale, with Cauchy–Schwarz at the weighted step and the reverse triangle inequality. This is the extra term $\gamma_0 \sum|\phi|^2$ of [26, Prop. 5.4, (5.26)], here for every gauge field, whereas King's proof [26, (5.28)–(5.29)] expands around a constant field. □

The *face family* at every scale $0 \le i \le K_f$ of the upper bound's large-field decomposition is the set of events $\{|\phi_i(y')| > \sigma_i\}$ at the face blocks $y' \in L_1^{(i)}$ inside the current small-field region (for $i = 0$ on the first live layer itself, where ϕ_0 is the fine field), with $\sigma_i := (2/(\theta_2\gamma_D^{(i)}))^{1/2} p_i \le C_8 p_i$, $\gamma_D^{(0)} := 1$ and $C_8 := \sup_i (2/(\theta_2\gamma_D^{(i)}))^{1/2}$, finite because the stiffnesses $\mathfrak{a}_i$ are bounded below; at the stopping scale it is King's test λ_j, [26, (5.41)] with $C = \sigma_j/p_j$. A violation at scale i is paid by the factor $e^{-p_i^2/4}$ from the θ_2-share of the scale-i form (Lemma 5.16; at $i = 0$ from the face-bond term itself), and violating blocks of different scales are disjoint, so the payments add. On the complement of the face and gradient families, a point within one corridor $r(\ell_i)$ of the face obeys $|\phi_i(x)| \le \sigma_i + r(\ell_i)p_{i-1} \le Cp_i^2$ along a block path to the face layer, and the fluctuation field obeys the same bound; beyond one corridor the Dirichlet image effect on the background is at most $Ce^{-\delta_1 r(\ell_i)}M_i \le p_i$ by Lemma 5.15, so [2, (2.113)] holds there in Bałaban's form; and within distance R_0 of a face a block with $|\phi_i| \ge M_i$ forces either a gradient violation or a face-family element along a shortest block path.

The two lines and the assembly. With $\eta = 5^{j-k}$, $|C_J| = (2MJ)^3$ the volume of the cube $C_J = [-MJ, MJ)^3$ and $p(\eta) = b_0(1 + \ln \eta^{-1})^6$:

Proposition 5.17 (lower and upper lines)**.** *For $\eta \le \eta_1$ and $J_* \le j \le k - J_*$, in the normalisation of Bałaban's block-spin representation,*

$$e^{-a} I_k^{(j)}(s) \le Z_k(s) \le e^{a} I_k^{(j)}(s) + e^{-\frac{1}{2}c_0' p(\eta)^2 + E_+ |C_J|},$$
$$Z_k(0) \ge e^{-E_- |C_J|}, \qquad a = \sum_{i<j} C \ell_i^{\kappa'} |X^{(i)}| = C\eta^{\sigma} |C_J|,$$

with $\sigma = \kappa' - d > 0$, c_0' the constant of the large-field payment, and $E_\pm(M)$ depending on $\varepsilon_0^{\mathrm{Ba}}$ and King's data only; all constants are independent of k and j.

Proof. The lower line is Bałaban's lower-bound induction [1, §§3–4] with King's Appendix class, carried through the scales with a background-gradient clause and a block-mismatch clause in place of King's Laplacian clause; the sum $\sum_{i<j} \ell_i^{\kappa'} |X^{(i)}| = |C_J| \sum_{i<j} \ell_i^{\kappa'-d}$ is geometric and dominated by $i = j - 1$, and the final restriction to King's small-field set is a positivity step inside the class, so the threshold σ_k never enters a constant. The upper line runs Bałaban's large-field decomposition [2, (2.43)] on the cube, his six families at every scale $i \le j$ together with the face family: the term with no large-field block at any scale is bounded by $e^a I_k^{(j)}(s)$, with Bałaban's cut background in every scalar expression and the comparison of the actions at the two backgrounds, and every other term carries the kept factor $e^{-\frac{1}{2}c_0' p_i^2} \le e^{-\frac{1}{2}c_0' p(\eta)^2}$ (p decreases in the scale); continuing the renormalisation group for these terms to the final scale $\ell_f \asymp \varepsilon_0^{\mathrm{Ba}}$, with the face family imposed and paid at each step, the sum over all admissible histories with the remaining half of the payments is at most $e^{E_+ |C_J|}$, $E_+ = O(\ell_f^{-d})$ [2, (2.117), (3.42)–(3.47)]. Each step's constants are independent of k, j and ε_j because no step depends on the spacing except through the lattice regime, which Lemma 5.15 and the first-difference bounds up to the face cover uniformly in $\varepsilon_j \le 1$; the dependence on x enters through the source vertices only. The remaining estimates are those of [1, 2] with the replacements of Lemmas 5.13–5.16, and are routine given them. □

Field-independent constants cancel: if $Z'(s)$, $I'(s)$ are computed with the effective action shifted by a field- and source-independent constant (a different normalisation of the measure, the domain-adapted E_0, E_1 of [25, (3.35)]), then $Z' = e^{-c} Z_k$ and $I' = e^{-c} I_k^{(j)}$ for every s, so the sandwich holds for Z_k and $I_k^{(j)}$ in the normalisation of Section 5.1 with the same a and remainder. Dividing, the remainder is at most $e^{-\frac{1}{2}c_0' p(\eta)^2 + (E_+ + E_-)|C_J|} Z_k(0)$, which is (4) with $c = \frac{1}{2}c_0'$, $\rho(\eta) = e^{-cp(\eta)^2 + C}$ and the volume $|C_J| = (2MJ)^3$ absorbed in C, which may depend on J. The clause (R_0) is the following statement.

Theorem 5.18 (the remainder is the large-field part of Bałaban's decomposition)**.** *Let $P := \prod_{i \le j} P^{(i)}$, where $P^{(i)}$ is the product, over all points, bonds and blocks of the scale-i index sets of the all-small history, of the small-field indicators of the families of scale i (Bałaban's six families, the face family, and at $i = j$ King's χ_j and λ_j), and let $\mu_{k,s}$ be the joint positive measure of the block-spin representation, normalised so that $Z_k(s) = \int d\mu_{k,s}$ as in Section 5.1 (the s-independent normalisation constants of the block-spin kernels are absorbed into $\mu_{k,s}$). Then (a) P is a $\{0,1\}$-valued function on the joint variable space, the same for every source s; (b) $\int P \, d\mu_{k,0}$ is the exact contribution of the all-small history at scale j and $\int P \, d\mu_{k,0} \le e^{C\eta^\sigma} I_k^{(j)}(0)$; (c) $\int (1 - P) \, d\mu_{k,0} \le e^{-\frac{1}{2}c_0' p(\eta)^2 + E_+ |C_J|} \le \rho(\eta) Z_k(0)$.*

Proof. Each block-spin step inserts $1 = N_i^{-1}\int dB\, d\psi\, e^{-\frac{\mathfrak{a}}{2}|B-QA|^2-\frac{\mathfrak{a}}{2}|\psi-Q(A)\phi|^2}$ [2, (2.1)] (the bare stiffness $\mathfrak{a}$, in the units of the step) with N_i field-independent and independent of s, which gives the joint measure, the constants N_i^{-1} being absorbed into it, so that $Z_k(s) = \int d\mu_{k,s}$. The tested quantities are source-free (fields, block fields and fluctuation fields, the backgrounds being linear maps of block fields built from the source-free quadratic forms), the source entering only the vertices [25, (3.20)] and the factor $e^{X(s)}$; this is (a). Expanding $\prod[\chi(\text{small}) + \chi(\text{large})]$ at one scale gives [2, (2.4)–(2.6)]; Bałaban's regrouping and partial resummation [2, (2.9)–(2.10)] replace, for a non-empty large-field region, the indicator of "the true tuple generates this region" by a sum of non-negative indicator products that dominates it, while for the empty region the only admissible tuple is the empty one and its term is exactly the product of the small-field indicators of that scale; the induction over scales inserts the next decomposition where the renormalisation group continues, and the translations inside a term are changes of variables with unit Jacobian. So the all-small history is exactly $\int P\, d\mu_{k,0}$, bounded by the upper line's small-field term, and the large-field histories, which are $\geq \int (1-P)\, d\mu_{k,0}$, are bounded by its remainder. □

Proposition 5.19 ((K-pos)$_N$: one-step bound on the normalised action, zero scalar source)**.** *With J, f, x and $s \in \{u_x, 0\}$ as in Proposition 5.12 there are $\gamma \in (0,1)$, $\beta, \sigma > 0$ and $C_3 < \infty$, depending on the couplings, J, f, x only, such that for all $k \geq k_0$, all $J_* \leq j \leq k - J_*$ with $\eta = 5^{j-k} \leq \eta_1$, and all (A_j, ϕ_j) in the support of χ_j,*

$$\big|S^{(j)}_{N,k}(A_j,\phi_j;s) - S^{(j+1)}_{N,k+1}(A_j,\phi_j;s)\big| \;\leq\; C_3\big(5^{-\gamma j}\eta^{-\beta} + \eta^{\sigma}\big). \tag{5}$$

The proof compares the k-model after j steps and the $(k+1)$-model after $j+1$ steps on one unit lattice, every rate across the pair being $5^{-\gamma j}$ with the exponent γ set by the line comparisons; its course is the following. This is [25, Thm. 3.4] at $n = 1$ on the cube, for the field- and source-dependent part of the effective action: the two members of the pair (k, j), $(k+1, j+1)$ live on one unit lattice with one indicator χ_j, and the statement compares them there. It is assembled from six pieces: the photon quadratic form (an even-restriction identity reduces the cube to the doubled torus, where [25, Prop. 3.10] applies); the scalar quadratic form at zero background (the Dirichlet symbol) and with background (a two-leg tree graph); the background determinant $B_3 := -\ln[Z_j(A^{(j)})/Z_j(0)]$ (King's interpolation [25, (3.94)–(3.98)], of which the middle line of (3.98) is false as stated in [25]: at a constant block field the order-two traces of the two members differ by an amount growing like 5^j times the volume, so no bound order by order at a fixed background exists and only the sum over the orders converges; it is replaced by the scale decomposition of the determinant, $\ln\det G_j(A) - \ln\det G_j(0) = \sum_{i<j}[\ln\det C^{(i)}(A) - \ln\det C^{(i)}(0)]$ with the coarse factors $C^{(i)} = K_i^{-1}$ of Definition 5.21, an identity by Sylvester's and Woodbury's formulas, in which each scale contributes $\sum_{x\in X_i}[g_{i,x}(A) - g_{i,x}(0)]$, $g_{i,x} := \mathrm{Tr}[\ln K_i](x,x)$ and X_i the coarse lattice of the scale, each term of size $O((e_i|dA_i|)^2)$ in a gauge centred at x, charge-conjugation parity removing the first order, with no cancellation between orders; pairing the step i of the k-model with the step $i+1$ of the $(k+1)$-model on the same coarse lattice, the per-site difference is bounded by $C(e_i|dA_i|)^2\delta_{\mathrm{conv}}(i) + C(e_i|dA_i|)(e_i|\partial(A-A')_i|)$, where $\delta_{\mathrm{conv}}(i) \leq C_s p^c(\mu_0\eta)^{-1}(5^{-\gamma' i} + \eta^s)$, for every $s > 0$, is the rate at which the coarse operators of the two members converge and the two unmatched finest steps cost ε_k each; summed over the sites and the scales, with $\sum_{i<j} l_i 5^{-\gamma' i} \leq C\eta 5^{-\gamma' j}$, this gives $|B_3 - B_3'| \leq C_s N e^2 p^{c+2}(1+\mu_0^{-1})\eta^{-3}(5^{-\gamma' j} + 5^{-j} + \eta^s)|C_J|$, which is of the form (5) with $\beta = 3+\epsilon$ and $\sigma = s-3$ for $s > 3$; Theorem 20.11 is the same bound on the torus); the graphs with curvature vertices; the source-free field-dependent graphs away from the faces and near them (the photon face terms and the scalar boundary remainders of [26, pp. 331–332]); and the block-vertex classes. For the curvature-vertex and the source-free field-dependent classes, the localisations whose tree length R exceeds the small-field threshold p are bounded by cutting the localised graph into clusters at mutual distance $\geq R_{\mathrm{cl}} = 3\sqrt{3}$ (the

cut by clusters, Definition 5.24 below) and reducing each cluster in its own box of side $O(p)$, with the cut lines as external lines of both clusters (the degree count row by row at every tuple, which gives King's (3.38) and (3.42) per cluster, and the position sum, which costs $\eta^{-3}|C_J|$ and in the branch $R > p$ leaves the factor $e^{-(\delta-\delta')p(\eta)/2} \le C_N\eta^N$; the zone-B tails by Theorem 5.37(a) and Theorem 5.34(d)). The vacuum constant $v_{k,j}$ is not part of the statement.

Lemma 5.20 (bounds for free)**.** *Put $\beta_f := C_0 \sum_i |f_i|_4^2$ with C_0 as in Lemma 2.9. For all $k \ge 0$, $J \ge J_0$ and $x \in \mathbb{R}^N$,*

$$e^{-\beta_f|x|^2} \;\le\; a_k(x) \;\le\; e^{\beta_f|x|^2}.$$

Proof. By Theorem 7.6(a) (a finite-k Gaussian bound, proved without this section), $a_k(x) \le \exp(\|u_x\|_{2,k}^2)$, and $\|u_x\|_{2,k} \le \sum_i |x_i| \|f_i\|_{2,k} \le |x| (\sum_i \|f_i\|_{2,k}^2)^{1/2} \le |x|\, \beta_f^{1/2}$ by Lemma 2.9(E3). The lower bound is $a_k(x)a_k(-x) \ge 1$, Lemma 2.14(c). □

The rest of this subsection sets up the comparison of the two members of the pair that proves Proposition 5.19. Definition 5.21 fixes the units, the averaging operators and the propagators; Definition 5.22 the localisation of the graphs and the two zones; Definition 5.27 and Lemma 5.28 the rewriting on a box and the lines cut by clusters; Theorems 5.30–5.38 are the estimates, and Lemma 5.41–Theorem 5.44 the statements on the photon two-point components. The same objects are used with scalar sources in Section 6 and in the restricted renormalisation theorem of Section 20.

Definition 5.21 (units, averaging operators and propagators of depth i)**.** Let k be the spacing index, j the number of renormalisation steps and $\eta = 5^{j-k}$ the block scale; the two members of the pair at (k, j) are the k-model after j steps and the $(k+1)$-model after $j+1$ steps (Section 5.1), and quantities of the second member carry a prime. For a depth $0 \le i \le j$ we work in *unit units of depth i*: the lattice X_i of the i-th block variables has spacing 1, the physical unit of length is $l_i := 5^i\varepsilon_k$ (so $l_j = \eta$), the fine lattice has spacing 5^{-i}, and

$$e_i := e\, l_i^{1/2}, \qquad m_i^2 := m^2 l_i^2, \qquad \mu_i^2 := \mu_0^2 l_i^2, \qquad A_i(x) := l_i^{1/2} A(l_i x),$$

so that a unit charge sees the transporter $U(A_b) = \exp(e_i 5^{-i} q A_{i,b})$, q the real antisymmetric generator with $q^2 = -1$. At depth j the slice of index $i \le j$ has the scale $\rho_i := 5^{i-j}$ ($\rho_j = 1$). The $(k+1)$-model at depth $i+1$ lives on the same lattice X_i, with fine spacing 5^{-i-1}.

On the cube C_J^K the scalar field carries the Dirichlet condition of Definition 2.3, the bonds to the Dirichlet planes being kept; $H_A := D_A^* D_A$ is the covariant Laplacian and $M_0(A) := H_A + m_i^2$. The *averaging operator* $Q_i(A)$ averages the fine field over the i-th blocks along the hierarchical staircases of [25, (2.10)–(2.12)], paths of length at most 3 in unit units inside the block, with the weights $w(x) = 5^{-3i} f(x)$, where $f = 1$ above the finest step and $f = f_D$ at the finest step: the *Dirichlet weight* f_D equals $(5/4)^3$ on the sites of a block all of whose coordinates have residues in $\{0, 1, 2, 3\}$ relative to the block's corner and 0 on the others, those with a residue 4 in some coordinate (for a block whose top face lies on a Dirichlet plane these include the layer on that plane), so that $\sum_x w(x) = 1$ and the density of w is at most $\varrho := (5/3)^3$; every bound below holds for every weight of density $\le \varrho$. With the stiffnesses $\mathfrak{a}$ and $\mathfrak{a}_i$ of Definition 5.1 (the bare one and the composite one after i steps, in the units of depth i) we set

$$M(A) := M_0(A) + \mathfrak{a}_i Q_i(A)^* Q_i(A), \qquad G_i(A) := M(A)^{-1},$$
$$H_i(A) := \mathfrak{a}_i G_i(A) Q_i(A)^*, \qquad \Delta^{(i)}(A) := \mathfrak{a}_i - \mathfrak{a}_i Q_i(A) H_i(A),$$

and, with Q the plain one-step block average $X_i \to X_{i+1}$,

$$K_i(A) := \Delta^{(i)}(A) + \mathfrak{a}\, 5^{-2} Q(A)^* Q(A), \qquad C^{(i)}(A) := K_i(A)^{-1},$$

the *coarse factor* of the slice of index i: the slice itself is $G_{(i)} = \mathfrak{a}_i^2 \rho_i^{-4} (G_i Q_i^*) C^{(i)} (Q_i G_i)$ [25, (2.17)], an outer factor on each side of the coarse factor. The photon operators are the same with $f = 1$, m replaced by μ_0, the Neumann condition of Definition 2.3 and blocks tiling C_J; the photon background after j steps is $A^{(j)} := H_j^{\mathrm{ph}} A_j$. On the support of the small-field indicator χ_j ([25, (3.2)] with the face factor of [26, (5.41)]) the field $|dA_j|$ is bounded by $p = p(\eta)$, while $|A_j|$ and $|\phi_j|$ are bounded by p times inverse powers of η: $|A_j| \le Cp/\mu_j$ with $\mu_j = \mu_0 \eta$, and $|\phi_j| \le p\,(\lambda\eta)^{-1/4}$, the quartic coupling in the units of depth j being $\lambda\eta$ (King's clauses [25, (3.2)]); $\varpi_i := e_i \sup |dA_i^{(j)}|$ is the smallness parameter of the expansion at depth i. All statements of this kind hold at every point (e, λ, m^2, μ_0) of King's range; the constants depend on it, on J, on b_0 and p and on the order $\bar{n} = 13$ of the expansion, and on nothing else unless said. The field- and source-free parts of the effective action [25, (3.90)] belong to the vacuum constant $v_{k,j}$ and are dropped wherever they occur.

Definition 5.22 (localisation, boxes and zones at depth j)**.** From here on everything is in the unit units of depth j (Definition 5.21): the *live sites* are $T_\eta := X_j$, the fine spacing is $\vartheta := 5^{-j}$ (the ε_j of the proof of Proposition 5.12; not the phase of ϕ), F denotes the set of scalar Dirichlet planes of ∂T_η, and $t(x) := \mathrm{dist}(x, \mathsf{F}) \ge \vartheta$ for a live site. Every graph of the expansion is localised as in [25, p. 661], by centres: a *tuple* τ of a graph assigns to each vertex a localisation weight χ_z supported within $\frac{2}{3}$ of its centre z in each coordinate, $P(\tau)$ is the set of centres, $n_\tau := |P(\tau)| \le N_*$ with N_* depending on the order $\bar{n}$ of the expansion only, and the *tree length* $R = R(\tau) := \mathrm{dist}(P(\tau))$ is the length of a shortest tree joining the centres (the points of the external fields included, Lemma 5.26); $\square$ is the smallest union of large blocks (side M) containing its vertices, $\square'' \subset \square'$ are the smallest boxes containing the large blocks within p (Bałaban's small-field threshold $p = p(\eta)$ of Proposition 5.12), resp. $2p$, of $\square$, truncated at ∂T_η, and $\Omega := \square' \cap T_\eta$ is a rectangular box with faces on block planes at every level (this is where the integer side $2MJ$ of the cube enters). A localisation is in *zone I* if $\square' \cap \mathsf{F} = \emptyset$ and in *zone B* otherwise; the faces of Ω not on F lie at distance $\ge 2p$ from $\square$. The range of j is $0 \le j \le k - J_*$, with the step threshold J_* of Section 5.1 (King's "J" of [25, Thm. 3.1]) taken so large that $5^{-J_*} \le \delta_*$, where $\delta_*^{-1} > 3p(\delta_*) + 4M + 1$; so $\eta = 5^{j-k} \le \delta_*$ throughout.

Definition 5.23 (the lines on a box and the Dirichlet convention)**.** The internal scalar lines are the slices $G_{(i)}(\square', 0)$ of the f_D flow with the Dirichlet condition on all of $\partial\Omega$, the photon slices carry the Neumann condition, and the lines $\delta G_j(\square', A^{(j)})$ of the exact rewriting $G = G(\square') + \delta G(\square')$ are external in King's sense. The Dirichlet condition is taken in the following convention: if Ω occupies $\prod_\mu [c_\mu^-, c_\mu^+)$ with $c_\mu^\pm \in \mathbb{Z}$, its Dirichlet planes are $x_\mu = c_\mu^\mp - \vartheta$, its live sites are the sites of T_η in $\prod_\mu [c_\mu^-, c_\mu^+ - \vartheta)$, and the matter is killed on the top face layers $x_\mu = c_\mu^+ - \vartheta$; on the faces lying on F this is the convention of T_η itself (Definition 2.3). On every such box the lines obey King's slice bounds [25, (3.63)–(3.65)] up to and including the faces, the scalar and the photon slices being image sums of free kernels over the reflection group of the box, and the external objects obey [26, (5.2)–(5.3)], King's Prop. 5.1 on boxes whose faces lie on block planes, which follows from the image sums and a bootstrap on the sup of $|u| + |D_A u|$ with the commutator localisation of [26, Lemma 5.3], with no a priori sup bound.

Definition 5.24 (the cut by clusters)**.** Put $c_{\mathrm{cl}} := \frac{13}{3}\sqrt{3}$ and $R_{\mathrm{cl}} := 3\sqrt{3}$, the *cluster distance*. The *clusters* K of a tuple τ are the components of the c_{cl}-proximity graph on $P(\tau)$ (two centres adjacent when at distance $\le c_{\mathrm{cl}}$), with $P_K := P(\tau) \cap K$; every line of the graph whose ends lie in different clusters is *cut* and kept whole as an external line of both cluster graphs H_K, and H_K is reduced with its own boxes $\square_K \subset \square_K'' \subset \square_K'$, built from the vertices of K as above, and with the gauge $B_K := A_j(x_0^K)$, x_0^K the centre of $\square_K'$; put $\Omega_K := \square_K' \cap T_\eta$, with the Dirichlet convention

above. A cluster is in zone I if $\Omega_K \cap \mathsf{F} = \emptyset$ (then $\square'_K$ is not truncated and $\Omega_K = \square'_K$) and in zone B otherwise. The geometry is elementary: two vertices in different clusters are at distance $> c_{\rm cl} - \frac{4}{3}\sqrt{3} = R_{\rm cl}$, so every cut line has length $\geq R_{\rm cl}$; $\operatorname{diam} P_K \leq c_{\rm cl}(|P_K| - 1)$; $|\square'_K| \leq C' p^3$; the quotient graph with the clusters as nodes and the cut lines as edges is connected, and $\operatorname{dist}(P(\tau)) \leq \sum_{\ell \in T} |\ell| + c_{\rm cl} n_\tau$ over any spanning tree T of it; and the partition, the boxes, the gauges and the set of cut lines are the same for every member of every family and every position. The two members of the pair live on the same unit lattice and carry the same cut.

Definition 5.25 (the range)**.** Put $C_{\rm box} := \max(c_{\rm cl} N_* + 6M + 4,\, 4M + 7)$; it bounds, in units of p, the side of every cluster box $\square'_K$ and of the one box $\square'$ of a localisation with $R \leq p$ (an elementary count, Lemma 6.3(e)). With $M(A)$ the operator of Definition 5.21 at depth j on a box, $\tilde{A}_K := A^{(j)} - B_K$ the cluster field, $V_j(\tilde{A}_K) := M(\tilde{A}_K) - M(0)$ and $G_0(\Omega_K)$ the zero-background propagator on Ω_K, let $C_\flat$ collect $\sqrt{3}\, C_{\rm box}$, the constant of the difference bound [25, (3.45)] and the constant of the operator bound for the expansion of the reduced propagator in $\tilde{A}_K$, so that $\|V_j(\tilde{A}_K) G_0(\Omega_K)\|_{\infty \to \infty} \leq C_\flat\, e\, \eta^{1/2} p^2$ on every cluster box, from $|\tilde{A}_K|, |\partial \tilde{A}_K| \leq C' p^2$ on Ω_K (the same with Ω_K replaced by the one box Ω). Let $\eta_2 \leq \eta_1$ be so small that for $\eta \leq \eta_2$, with $p = p(\eta)$,

$$\eta^{-1} \;\geq\; 4p + 8M + 2c_{\rm cl} N_*^2 + 8, \qquad p \;\geq\; c_{\rm cl} N_*^2, \qquad C_\flat\, e\, \eta^{1/2}\, p^2 \;\leq\; \tfrac{1}{2}; \tag{6}$$

all three hold for η small because $p(\eta)$ grows slower than every power of η^{-1} and faster than every logarithm. The first condition contains the condition $\delta_*^{-1} > 3p(\delta_*) + 4M + 1$ on the range of j above; where the background is expanded in $\tilde{A}$ on one box, η lies moreover in the range (6), whose third condition is the convergence condition of that expansion on every cluster box $\square'_K$ and on the box $\square'$ of every localisation with $R \leq p$, both of side $\leq C_{\rm box} p$.

Lemma 5.26 (zones)**.** *Let a localisation have the points $\{y\}$ of its external fields and the centres $\{z\}$ of its vertices, and put $R := \operatorname{dist}(\{y\}, \{z\})$, the length of a shortest tree joining them.* (i) *Every localisation with $R \leq p$ that contains a source vertex lies in zone I.* (ii) *Every zone-B localisation containing a source vertex has $R \geq \eta^{-1} - 2p - 4M \geq \eta^{-1}/2$.* (iii) *Every source vertex s has $t(s) \geq \eta^{-1}/2 \geq 1$ and $\operatorname{dist}(s, \partial\square' \setminus \mathsf{F}) \geq 2p - \sqrt{3}$.*

Proof. The sources are supported at continuum ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_J$, that is at $\geq \eta^{-1}$ unit lengths from ∂T_η, and the summed curvature vertex sits on bonds whose weight vanishes within $1 - \varepsilon$ of the faces. In (i) every vertex lies within p of the source vertex, $\square$ within $p + M$ and $\square'$ within $3p + 3M < \eta^{-1}$ of it, by the first condition of (6); (ii) and (iii) are the same arithmetic. □

The next definition and lemma prepare the zone-B bound of Theorem 5.37: the rewriting of a renormalised family on a box, which produces the remainder vertices, and the regularisation of the lines cut by clusters (Definition 5.24). Both are used again for Theorem 5.33 and Theorem 5.37 (Definition 5.32, Definition 5.36) and, with scalar sources, in the proof of Lemma 6.1.

Definition 5.27 (cube rewriting and remainder vertices)**.** Let a renormalised family be localised in zone B (Definition 5.22), its internal lines the cube slices with the Dirichlet condition on $\partial\Omega$ and its vertex positions ranging over the live sites of Ω. Apply to it, step by step, the *free rewriting* of its divergent source-free subgraphs, that is Bałaban's renormalisation procedure [3, Sect. 3] in the form of [25, (3.77)–(3.78)]: Taylor transport of the legs to a reference vertex, Hölder splits of the remainders, resummation over the orderings up to the lowest external index j_0, and evaluation of the resummed coefficients. Each evaluation of a resummed coefficient $c_S(x)$ of a divergent subgraph S rooted at the live site x is a *C-site*. At a C-site write

$$c_S^C(x) = c_S^F(x) + \Delta_S(x),$$

where $c_S^F(x)$ is the value of the same coefficient with the free f_D lines and with the positions summed over $5^{-j}\mathbb{Z}^3$, so that c_S^F is the coefficient of the free rewriting, which vanishes or is a local polynomial exactly as on the torus, and the *remainder* $\Delta_S(x)$ is attached at x as a generalised *remainder vertex* v_S carrying the legs of the evaluated term. Every graph carrying a remainder is *halted*: no later step of the rewriting acts on a stage that contains a remainder vertex. The *moment degree* of S is $D'(S) := D(S)$ when the zeroth moment is evaluated and $D'(S) := D(S) + 1$ for a first moment, $D(S)$ the degree of S [3, (2.1)–(2.3)]. The remainder vertices with $D' \in \{0, -1\}$ are the layered vertices of Theorem 5.37, bounded there by face weights; the free parts c_S^F are governed by the degree count on the free f_D flow (Theorem 5.30).

Lemma 5.28 (regularised cut lines)**.** *Fix a radial $\bar\psi \in C^\infty(\mathbb{R}^3)$ with $0 \le \bar\psi \le 1$, $\bar\psi(w) = 0$ for $|w| \le \frac12$ and $\bar\psi(w) = 1$ for $|w| \ge 1$. For a cut line ℓ of the cut by clusters (Definition 5.24) between an end u in the cluster K and an end v in the cluster K' put*

$$\ell^\flat(u, v) := \bar\psi(u - v)\, \ell(u, v)$$

(for an external line from u to a point y of the external field, $\bar\psi(u - y)\ell(u, y)$), and let

$$\ell^{\flat,\mathrm{ph}} := U\big(B_K(\Gamma_{x_0^K,u})\big)^{-1}\, \ell^\flat\, U\big(B_{K'}(\Gamma_{x_0^{K'},v})\big)$$

be its phase-dressed *form, the transporters along the staircases from the centres x_0^K, $x_0^{K'}$ of the two cluster boxes in the gauges B_K, $B_{K'}$ of the two clusters, which are the unitary factors that King's reduction* [25, (3.46)] *attaches to the ends of a line leaving the box. Then:*

- *(a) Replacing every cut line by $\ell^\flat$ leaves $E^{(j)}(H;\tau)$ unchanged, term by term: for every member of every family, every ordering and every position configuration, each pair (u', v') of evaluation points at which a cut line enters, the lattice differences of the vertex functions at its ends included, satisfies $|u' - v'| \ge R_{\mathrm{cl}} - 4 \cdot 5^{-j} > 1$, where $\bar\psi = 1$; here $R_{\mathrm{cl}} = 3\sqrt{3}$ is the cluster distance (Definition 5.24), the four fine steps account for the differences at the two ends, and when an end lies on a block product of a block vertex the margin of the block product gives $|u' - v'| > 1$ as well.*
- *(b) $\ell^{\flat,\mathrm{ph}}$ is $C^{1,\alpha_{\mathrm{H}}} \otimes C^{1,\alpha_{\mathrm{H}}}$ at unit scale in both arguments, where $\alpha_{\mathrm{H}} \in (0, 1)$ denotes throughout the Hölder exponent of the line bounds, with the decay $e^{-\delta_1|u-v|}$, at all positions and with no distance condition: for all operations O_u, O_v at its two ends of order at most $1 + \alpha_{\mathrm{H}}$ (the identity, a lattice first difference, or a Hölder quotient of a first difference),*

$$|O_u O_v \ell^{\flat,\mathrm{ph}}(u, v)| \le C\, e^{-\delta_1|u-v|},$$

the constant depending on $\bar n$, d, L and α_{H} only, times $(1 + k - j)^5$ for a dressed line, the size of the coarse mass-counterterm blob it carries. This is the line bound [3, (3.1)] *in the form of a cut line with the endpoint localisations: the cut-off $\bar\psi$ removes the short-distance singularity, so the bound holds with one constant at every distance.*

The degree count. Bałaban's renormalised degree count [3, Sect. 3, (3.5)], in the form King uses on the torus [25, (3.77)–(3.78)], says the following. After the rewriting of the divergent subgraphs (Taylor transport of the legs, Hölder splits, resummation over the orderings up to the lowest external index j_0, and evaluation of the resummed coefficients c_S on the free lattice, where they vanish by the counterterm, by sign pairing, by translation invariance or parity, or by the Ward identity of the vacuum polarisation), every ordered ultraviolet stage of a source-free renormalised family has positive degree, and the infrared counterterm lines, which lie in the mass-counterterm units and are attached to the rest of the graph at one vertex, obey [25, (3.78)]: the degree of the whole graph is at most that of every prefix containing the ultraviolet lines. We call this the *degree count* (DEG). In both references it is stated for the flow with the weight

$f=1$; the lines of the zone-I graphs are those of the f_D flow (Definition 5.21), and the transfer rests on the following lemma.

Lemma 5.29 (the two free flows differ by a top-scale kernel)**.** *Let $\Delta G_i := G_i^{f_D} - G_i^1$ on $5^{-i}\mathbb{Z}^3$ at $A=0$, in the units of depth i (Definition 5.21). For $a,b\in\{0,1\}$ and every $\gamma<1$,*

$$|\partial_x^a\partial_y^b \Delta G_i(x,y)| \le C_\gamma 5^{-\gamma i} e^{-\kappa'|x-y|},$$

and $\partial_x\Delta G_i(\cdot,y)$ is α'-Hölder with the constant $C_{\alpha'}5^{-si}$, $s=(1-\alpha')/(1+\alpha')$, times the same decay. In the units of depth j the kernel of level i is the rescaled one, so ΔG_i obeys the slice bounds [25, (3.63)–(3.65)] *of the index i times $5^{-\gamma i}$: it is a line without short-distance singularity.*

Proof. Both operators have the same $C^{-1}=-\Delta+m_i^2$, so $\Delta G_i = G^{f_D}RG^1$ with $R(u,u') = \mathfrak{a}_i[1-w(u)w(u')]\mathbf{1}[B(u)=B(u')]$, $B(u)$ the block of u. Writing $1-w(u)w(u') = (1-w(u)) + w(u)(1-w(u'))$ and using $\sum_{u\in B(y)} 5^{-3i}G^{f_D}(x,u)w(u) = H^{f_D}(x,y)/\mathfrak{a}_i$ we get

$$\Delta G_i(x,x') = \mathfrak{a}_i \sum_y D_0(x,y)g_1(y,x') + \sum_y H^{f_D}(x,y)g_2(y,x'),$$

with $D_0(x,y) := \sum_{u\in B(y)} 5^{-3i}(1-w(u))G^{f_D}(x,u)$, $g_1(y,x') := \sum_{u'\in B(y)} 5^{-3i}G^1(u',x')$ and $g_2(y,x') := \sum_{u'\in B(y)} 5^{-3i}(1-w(u'))G^1(u',x')$. The point is a *dipole property* of the weight: every unit block is a union of finest cells c of side 5^{1-i} on each of which $\sum_{u\in c}(1-w(u))=0$, because $\sum_{u\in c} w = \prod_\mu \sum_{r=0}^4 w_1(r) = 5^3 = |c|$. Hence $\sum_{u\in B} 5^{-3i}(1-w(u))h(u) = \sum_c \sum_{u\in c} 5^{-3i}(1-w(u))[h(u)-h(u_c)]$ is at most $C\sum_c |c|\,\mathrm{osc}_c\, h$, and for $|\partial h|\le C(|x-v|+5^{-i})^{-\mathsf{p}}$ this is $\le C5^{-i}$ when $\mathsf{p}=2$ and $\le C5^{-i}(1+i)$ when $\mathsf{p}=3$. The free kernel bounds $|\partial^a G|\le C(|x-u|+5^{-i})^{-1-a}e^{-\kappa|x-u|}$ then give $|D_0|,|g_2|\le C5^{-i}e^{-\kappa\,\mathrm{dist}}$, $|\partial D_0|,|\partial g_2|\le C5^{-i}(1+i)e^{-\kappa\,\mathrm{dist}}$, $|g_1|,|\partial g_1|\le Ce^{-\kappa\,\mathrm{dist}}$ and $|\partial^a H^{f_D}|\le Ce^{-\kappa|x-y|}$; every product contains one dipole factor and the sums over y converge. For the Hölder clause interpolate the cell oscillation of $h(u)=\partial_x G(x,u)-\partial_x G(\tilde{x},u)$ between the Hölder bound $C|x-\tilde{x}|^{\alpha''}r_c^{-2-\alpha''}$ and $C5^{1-i}r_c^{-3}$, with $\alpha'':=(1+\alpha')/2$. □

Theorem 5.30 (the degree count on the free f_D flow)**.** *Every source-free renormalised prefix component S of the free f_D flow, at zero background, admits the rewriting of* [3, Sect. 3] *with:* (i) *every resummed coefficient that vanishes exactly for $f=1$ vanishes exactly for f_D;* (ii) *$c_a^{f_D} = c_a^1 + c_a(M_S)$, where $|c_a^1|\le C\rho_{j_0}^{D(S)+|a|}$ is the moment of the $f=1$ flow and $|c_a(M_S)|\le C_\gamma 5^{-\gamma i_0}\rho_{j_0}^{D(S)+|a|}$, i_0 the depth of the lowest external index j_0 of S;* (iii) *the remainders have the same shapes as for $f=1$. Hence* (DEG) *holds for the f_D flow with the degree of every stage unchanged, and its infrared half* [25, (3.78)] *holds with the mass-counterterm blob U^{f_D} differing from U^1 by $C(\varepsilon_k/\eta)^\gamma\eta^{1-\varsigma}$ in the units of* [25, (3.78)]*. The $f=1$ statement of* (ii) *on the rows whose coefficient vanishes by the zero-momentum gauge covariance (the Ward rows) is King's reduction* [25, (3.39)–(3.55)]*, run on the cube with the gauge $B=A_j(x_0)$ and, for a cluster of the cut by clusters (Definition 5.24), inside the cluster with its own gauge B_K; on every other row it is the member-wise identity (charge parity, the derivative at a line end, reflection, the η-suppressed tadpole, or the counterterm matching between partners of identical vertex structure).*

Proof. Resum the lines of S to $G_{j_0}^{f_D}$, write $G^{f_D}=G^1+\Delta G$ on every line and expand multilinearly: $F_S^{f_D} = F_S^1 + M_S$, where F_S^1 is the inside family of the same class with $f=1$ lines (the members are the same graphs, since every vertex of a divergent class is weight-free; the counterterms δm^2 and E_1 have the same value in both flows; a counterterm piece all of whose lines are G^1 is the $f=1$ decomposition of the same counterterm) and M_S is the sum over the members and over the non-empty sets T of lines carrying ΔG.

(i). The exact-zero mechanisms (the counterterm per scale assignment, pairing, parity, the telescoping of a summable kernel) are identities for every charge-diagonal line kernel when the partners carry the same lines, so they hold for the f_D lines directly.

(ii). In a term of M_S treat each ΔG line as one line of the fixed index j_0; by Lemma 5.29 it obeys the slice bounds of the index j_0 times $5^{-\gamma i_0}$. In any ordering the first ΔG line forces the index j_0 on every later line, so the free index sums run over G^1 slices only, and the prefixes containing a ΔG line enter no geometric sum. Each prefix component γ with $D(\gamma) \le 0$ before the first ΔG line is a source-free, block-free $f = 1$ subgraph whose complete inside family, partners included, is present at fixed outside, because replacing γ by another member of its class does not touch T; renormalise it by the $f = 1$ rewriting, finest first, with the ΔG ends as legs at the scale ρ_{j_0} (Lipschitz, with an α'-Hölder first difference, which is what the transport and Taylor steps use), the debits landing on a ΔG line being paid at ρ_{j_0}. The doubly differentiated single line between two current vertices whose other ends carry ΔG is rewritten by the free-lattice identity $\sum D_x D_y G^1(\cdot) = 0$ against a constant, whose transport remainder carries $|\Delta G(\cdot, v_2) - \Delta G(\cdot, v_1)| \le C|v_1 - v_2| 5^{-\gamma i_0} \rho_{j_0}^{-2}$. After this every prefix with a non-trivial index sum has positive degree, the lines obey their slice bounds, the moment factors $|x_i - x|^{a_i}$ are distributed along paths of lines (costing $\rho_{j_0}^{a_i}$ on a ΔG line), and the multiscale bound Theorem 5.31(a) with the root held gives $|c_a(M_S)| \le C \rho_{j_0}^{D(S)+|a|} 5^{-\gamma i_0 |T|}$. *(iii)* is the same argument.

The infrared half. The prefix condition of [25, (3.78)] is a statement about the lines and vertices of the blob and is the same for both flows; its proof uses of the lines only their dimensions, the shapes (i)–(iii) of the rewriting at the indices below the infrared ones and the counterterm identities, so it holds for $f = 1$ and for f_D alike. For the value of the blob, $U^{f_D} = U^1 -$ (the terms with at least one soft line ΔG_j), and each such term is a skeleton term of the $f = 1$ argument in which the soft line obeys the soft-line bounds with the constant $C(\varepsilon_k/\eta)^\gamma$ (Lemma 5.29 in physical units), which gives the stated difference. The remaining estimates are routine given the slice bounds of the lines and Theorem 5.31(a). □

Theorem 5.31 (the split degree governs the multiscale bound)**.** *Let $\mathcal{R}$ be a renormalised family of King's expansion of order $\le \bar{n}$ (Definition 5.21), written as a finite sum of graphs G with the source vertices admitted. Write G^{sp} for G with every block vertex* [25, (3.25)–(3.26)] *split into its* points*, each point carrying one scalar leg and the staircase fields of its own staircases, so that the staircase line of a field is a line of its own, the two summation points of a two-point block product being joined by a κ-*line*; weight a vertex v of a connected stage H of G^{sp} by the* split count

$$w_v(H) = 3 + a_v - \tfrac{1}{2} N_v + \tfrac{1}{2} n_v^{\mathrm{ext}}(H) - \nu_v(H) + c_v(H), \qquad D^{\mathrm{sp}}(H) := \sum_{v \in H} w_v(H) - 3,$$

where a_v is the explicit power of the vertex, N_v the number of all its legs, $n_v^{\mathrm{ext}}(H)$ the number of those which are ends of no line of H, $\nu_v(H)$ the number of derivatives on the legs which are ends of lines of H, and the staircase credit *$c_v(H)$ is the number of photon fields of v whose line lies in H if v is a block point and 0 otherwise; so a leg external to H contributes 0, an undifferentiated end of a line of H contributes $-\frac{1}{2}$ and a differentiated one $-\frac{3}{2}$. Assume that, for every member that does not vanish identically at zero background, taken at zero background, and for every term of the expansion in the background field of such a member at a regular background on a box whose faces lie on block planes (C_J^K among them), for each admitted ordering of the lines of G^{sp}: every component of every ultraviolet prefix has current degree $D' > 0$ after the rewriting (the degree count of Theorem 5.30 in the split count); the infrared lines, all on the counterterm lines of the δm^2 units, obey* [25, (3.78)]*; and the lines obey the slice bounds* [25, (3.63)–(3.65)] *up to the faces of the box (for a term at a regular background, its zero-field lines). Let $\alpha_0 > 0$ be the minimum of these degrees. Then, on the torus and on C_J^K, for every such graph G:*

(a) the sum over orderings, compatible indices and positions of $|E^{(j)}(G)|$ *is at most*

$$C\eta^{\rho_0}p^{n_1}(1+k-j)^{p_0}\exp[-\delta\,\mathrm{dist}(\textit{external vertices})],$$

ρ_0 the coupling power of G [25, p. 665]; the same holds with one element weakened by $\theta<\alpha_0$ (a line with $c_l\to c_l\rho_{i_l}^{-\theta}$, a vertex with an explicit power $a_v\ge\theta$, or an external field);

(b) at zero background,

$$|E'^{(j+1)}(G)-E^{(j)}(G)|\le C\eta^{\rho_0'}5^{-\gamma j}\exp[-\delta\,\mathrm{dist}]$$

for $0<\gamma<\min(\alpha_0,\gamma_{\mathrm c},\alpha_{\mathrm H},1-\alpha_{\mathrm H})$, where $\alpha_{\mathrm H}\in(0,1)$ is the Hölder exponent of the line bounds (Lemma 5.28) and $\gamma_{\mathrm c}>0$, depending on $\bar n$ and $\alpha_{\mathrm H}$ only, is the rate at which the resummed local coefficients of the rewritten units of the two members agree;

(c) every rewritten subgraph is source-free, point-free and a component of a prefix of G^{sp}, and $D'(\gamma)\ge D^{\mathrm{sp}}(\gamma)$ for every component γ of every prefix;

(d) the exponent of every stage of the ordered bound is $D'(H_t)$ in the split count, the credit for a staircase field being booked only when its staircase line lies in the stage, unlike King's per-vertex "+m" [25, (3.30)], which credits all m fields of [25, (3.26)] whether or not their lines are present.

At a regular background only (a), (c) and, with "is at least $D'(H_t)$" in place of "is $D'(H_t)$", (d) are claimed, on the boxes named above and not on the torus. The constants depend on $\bar n$, on the point of King's range and on the line constants, not on j, k, η.

Proof. We prove (a) by the multiscale induction on the split graph, (b) by replacing one element of the graph at a time and then pairing the position sums of the two members, and (c) and (d) by the degree count.

(a). Apply the multiscale lemma of [25, (3.66)–(3.70)], a line-by-line induction over the ordering in which each step shrinks one line and sums one of its endpoints, the index sums being geometric by the positive degree, to the connected graph G^{sp} with the slice bounds of its lines. The degree of G^{sp} is D^{sp}. A stage containing κ-lines is not a stage of the split count: if $c_1,\dots,c_r$ are its components after the κ-lines are deleted, each containing a point, its degree is $\sum_i D^{\mathrm{sp}}(c_i)+3(r-1)\ge 2$, since an isolated point has $D^{\mathrm{sp}}=0$ and a component with a point and a line has $D^{\mathrm{sp}}\ge 1$ (a block point weighs $3-\frac{s}{2}+\frac{p}{2}\ge\frac52$, $s\in\{0,1\}$ and $p\ge 0$ the numbers of ends of lines at its scalar leg and at its staircase fields). Infrared stage: points lie in no counterterm unit, so no infrared line ends at a point; the κ-line has index $j\le$ every infrared index, and shrinking it first transfers only the external lines at one summation point to the other and gives $\sum_{x'}5^{-3j}\kappa\le 1$, so that the infrared differences of [25, (3.80)] are unchanged. Fixed lines, remainders and the position sum are as in [25, pp. 663–664], and the weakened element is handled by the same induction with $D'-\theta\ge\alpha_0-\theta>0$. On C_J^K, (a) holds verbatim with the cube line bounds (image sums of free kernels), position sums over C_J being bounded by sums over $\mathbb{Z}^3$; at a regular background the induction runs on the majorant of the expansion in the background field with halved decay rates, which gives (a), (c) and the weak form of (d).

(b). This is [25, pp. 663–665] on G^{sp}. The finer slices of the finer member give $5^{-\gamma j}$ through the finest index sum; otherwise one factor is replaced at a time, the external lines by [25, (3.71)], the sources by sampling ($|h(x')-h(x)|\le\sqrt3\,\varepsilon_k\|\nabla h\|_\infty$), the phase by [25, (3.72)], the plain lines by [25, (3.73)], the staircase lines by [25, (3.74)], the Hölder lines by [25, (3.75)], the local coefficients of the rewritten units at the rate $\gamma_{\mathrm c}$, and an explicit power of the spacing at a vertex by its own smallness, each error being the graph with one element weakened by γ and the prefactor $5^{-\gamma j}$. Then the position sums of the two members are paired: the finer member's cell totals are exact, $\sum_{x'\in B^1(x)}5^{-3(j+1)}f'_D(x')=5^{-3j}$ at every alive coarse site x (on C_J^K the sum runs over the cell

inside the box, f'_D vanishing at the finer plane sites), and the other internal vertices are paired over the alive coarse sites. On C_J^K three further terms arise. A term with one non-point vertex of the finer member restricted to the *top layer*, the sites of the finer lattice in the last fine layer below a Dirichlet plane, which have no coarse partner (one such vertex per term, by the union bound), gives at the step at which that vertex is summed through a line of index $i \geq 0$ the factor $C5^{-\gamma j}\rho_i^{3-\gamma}$ in place of $C\rho_i^3$, that is the graph with that line weakened by γ and the prefactor $5^{-\gamma j}$; a held vertex of the finer member in the top layer has no partner and gives the factor $\frac{12}{5}\,5^{-j}$ per half-open unit cube; and the coarser member's weight f_D differs from the uniform measure by a sum of differences of the graph at points of one level-1 block [26, (3.13)], each $C5^{-\gamma j}$ times the graph with one element at the point weakened by γ, by the Hölder clauses of the cube lines on the closed box (no point is a source vertex, so the sources are not involved). Finally $\kappa' = \kappa$ up to the block parameters, which differ by $|\mathfrak{a}'_{j+1} - \mathfrak{a}_j| \leq C5^{-2j}$ (Definition 5.1). At $j = 0$, where $5^{-\gamma j} = 1$, (b) follows from (a) for the two members. For the comparison bounds the cube lines obey the pair clauses of the two-end lines, of the staircase lines and of their Hölder forms, transferred to the photon staircase lines on the photon box, and the phase of a staircase of length ≤ 3 is compared through $|e^{ia} - e^{ib}| \leq |a - b|$.

(c). A rewritten subgraph is divergent, so it contains no point (a component with a point and a line has $D^{\mathrm{sp}} \geq 1$) and is one of Bałaban's divergent classes, which contain no block vertex [3, pp. 428–431]; debits land on legs of the rewritten subgraph S, and if a leg lies in γ then $S \subset \gamma$, so the net change per component is ≥ 0. *(d)* is the booking of the dimensions: the credit $+1$ for a staircase field of a point is the factor c_l in the dimension of its staircase line, booked at the point for the lines of the stage only, so it enters exactly the stages containing that line. The remaining estimates are routine given the slice bounds and the degree count. □

Definition 5.32 (the setting of the zone-B photon pieces)**.** Let a member family of the cube expansion be localised in zone B (Definition 5.22), every member carrying a source vertex or an external field (so that no member is a field-free vacuum graph), every photon two-point prefix component S' rewritten by the extraction of the zeroth and first moments of [3, (3.26)–(3.31)] with the lines $G_{j_0} = C_D^\xi + G_{j_0}(1 - m_{j_0}^2 - \mathfrak{a}_{j_0}P_{j_0})C_D^\xi$ (in the units of depth j_0), C_D^ξ the Dirichlet covariance of the box, and every other divergent source-free stage rewritten as in Definition 5.27. Inside each collapsed piece write every cube line as the free line plus its image part and every internal sum over Ω as the sum over $5^{-j}\mathbb{Z}^3$ minus the sum outside Ω, and rewrite the free piece by the free rules.

Theorem 5.33 (the zone-B photon pieces)**.** *Let a member family be as in Definition 5.32. Then:*

- *(a) The cube is compared with the free lattice only inside the collapsed pieces and at the two cancellation sites of the bulk structure, the free counterterm against the scalar local parts and the reflection symmetry for the first moment, piece by piece: for the one-loop class the zeroth moment of the C_D^ξ-part vanishes on both sides (an identity for the full kernel of the class, not a per-ordering one), leaving the residual $\Delta c^{S'}$; for a class with $D(S') = 0$ the whole difference $c^\Omega - c^F$ of the zeroth moment is one piece; the* face piece M *comes from the first moment of the one-loop bubble with one leg differentiated; and there are the remaining differences.*
- *(b) Each difference piece is a generalised vertex at the root x of S', at the stage $j_0(S')$, bounded by its free power count times $\omega(x) + \omega_p$, with ω a layered weight of class $\mathcal{L}(\rho_{j_0}, m, \theta)$ (Definition 5.35) and $\omega_p \leq Ce^{-\delta p/\rho_{j_0}}$ a term supported away from F, of any prescribed degree N at the cost $C_N e^{-\delta p/2}$: the face piece M is of class $\mathcal{L}(\rho_{j_0}, 1, 0)$ and of the degree $D(S') + 1$ of the bulk first-moment vertex; the residual satisfies $|\Delta c^{S'}(x)| \leq C\rho_{j_0}^{D(S')}[e^{-\delta t(x)/\rho_{j_0}} + e^{-\delta p/\rho_{j_0}}]$ times the bounds of the legs, one line being replaced by its*

image part, for which $\sum_w 5^{-3j}|\nabla G^{\mathrm{img}}_{j_0}(x,w)| \le C\rho_{j_0}e^{-\delta' t(x)/\rho_{j_0}}$*; it is of class* $\mathcal{L}(\rho_{j_0},0,0)$ *and degree* $D(S')$*, and for* $D(S')=0$ *it is entered together with* $c^{(C)\Omega}-c^{(C)F}$ *as the whole difference* $c^{\Omega}-c^{F}$*, of class* $\mathcal{L}(\rho_{j_0},1,0)$*; the difference of the first moment of the replaced-line part is of class* $\mathcal{L}(\rho_{j_0},0,0)$ *and degree* $D(S')+1$*; and the differences of the Taylor remainders are of class* $\mathcal{L}(\rho_{j_0},0,0)$ *and degree* $D(S')+1+\alpha_{\mathrm{H}}$ *when both moments are extracted* ($D(S')=-1$) *and* $D(S')+\alpha_{\mathrm{H}}$ *when only the zeroth moment is* ($D(S')=0$)*, in both cases* $\ge \alpha_{\mathrm{H}}$*. The members with a block vertex produce no difference piece: they are not collapsed, and every stage containing one of their points has split degree* ≥ 1 *(Theorem 5.31). No piece has the weight* t^{-1}*.*

Proof. We first prove (a) from the exactness of the image representations and of the rewriting rules; we then prove (b) by resumming the internal lines of S' to the cumulative line, splitting it as Bałaban does, Taylor-expanding the second leg at the root, and reading each difference piece as a generalised vertex.

(a). The image representations of the cube lines are exact and the rewriting rules are identities on each side, so the difference of two rewritten pieces is the sum of the differences of their parts; the two cancellation sites are the only places where the free structure of the coefficients is used. For the one-loop class the zeroth moment of the C^{ξ}_D-part vanishes on both sides by the gauge invariance of the scalar functional under constant shifts of the photon field on the simply connected cube.

(b). The component S' is connected, has exactly two photon legs Φ_1 at the root bond $b=(x,\mu)$, of index j_0, and Φ_2 at $b'=(x',\nu)$, of index $j_2\ge j_0$, contains no source vertex and no block point, and has $D(S')\in\{-1,0\}$, with $D(S')=-1$ only for the one-loop class (the bubble and the one-vertex tadpoles). Resum its internal lines to the cumulative line G_{j_0}, which on Ω is the Dirichlet resolvent $(-\Delta^{\Omega}_D+\rho^{-2}\tilde m^2+\rho^{-2}\tilde{\mathfrak{a}}P_{j_0})^{-1}$, $\rho=\rho_{j_0}$ ($\tilde m^2=m^2_{j_0}$ and $\tilde{\mathfrak{a}}=\mathfrak{a}_{j_0}$ the mass and the stiffness of depth j_0, the factor ρ^{-2} converting them to the units of depth j), and split it as Bałaban does [3, p. 437]: $G_{j_0}=C^{\xi}_D+R_{j_0}$ with $C^{\xi}_D=(-\Delta^{\Omega}_D+\rho^{-2})^{-1}$ and $R_{j_0}=G_{j_0}K_{j_0}C^{\xi}_D$, $K_{j_0}=\rho^{-2}(1-\tilde m^2-\tilde{\mathfrak{a}}P_{j_0})$; the kernel $\Pi(b,b')$ of the class, the sum over the inside family with the positions summed over Ω, splits multilinearly as $\Pi^{(C)}+\Pi^{(R)}$, the second collecting the terms with at least one R-line. The rewriting at the root Taylor-expands Φ_2 at b: $\sum_{b'}\Pi(b,b')\Phi_2(b')=\Phi_2(b)c(x)+\partial_s\Phi_2(b)m_s(x)+\sum_{b'}\Pi(b,b')\tau_2(b,b')$ with the zeroth moment $c(x)=\sum_{b'}\Pi(b,b')$, the first moment $m_s(x)=\sum_{b'}\Pi(b,b')(x'-x)_s$ and $|\tau_2|\le C|x'-x|^{1+\alpha_{\mathrm{H}}}\rho_{j_2}^{-1-\alpha_{\mathrm{H}}}\bar\Phi_2$ (for $D(S')=0$ only the zeroth moment is extracted and the remainder is of order α_{H}). Each moment and remainder splits into its (C)- and (R)-parts; a quantity computed on the cube carries the superscript Ω, the same expression with free lines and positions over $5^{-j}\mathbb{Z}^3$ the superscript F, and $\Delta=(\cdot)^{\Omega}-(\cdot)^{F}$. For the one-loop class $c^{(C)\Omega}=c^{(C)F}=0$; for $D(S')=0$ the difference $c^{(C)\Omega}-c^{(C)F}$ is kept and entered together with the residual; by the reflection symmetry [3, (3.27)–(3.28)] $m^{(C)F}=0$, while $m^{(C)\Omega}=M$ is the face piece, the first moment of the bubble with the lines $C^{\xi}_D=\sum_g \mathrm{sgn}(g)C^{\xi}(\cdot,g\,\cdot)$ whose non-identity images have length $\ge t(x)$, which gives the weight of class $\mathcal{L}(\rho,1,0)$, the logarithm coming from the sum over the slices. So the difference pieces are M and $\Delta m^{(R)}$ (for $D(S')=-1$), $\Delta c^{S'}$ with $\Delta c^{(C)}$ when $D(S')=0$, and the remainder differences $\Delta r^{(C)}$, $\Delta r^{(R)}$; each contains an image line of length $\ge t(x)$ or a position sum outside Ω, at distance $\ge p$ from the vertices, which gives the exponentials $e^{-\delta t(x)/\rho}$ and $e^{-\delta p/\rho}$, and the free power count gives the prefactor $\rho^{D(S')}$ or $\rho^{D(S')+1}$. A piece is read as a generalised vertex v at x with a_v its moment degree and $u_v=3+a_v-n_v/2-\epsilon_v$, ϵ_v the number of its differentiated legs internal to the stage; a remainder is read, as in [3, Prop. 2.2, p. 428], as a vertex of explicit power $\rho^{D'}$ joined to x' by a generalised line of index j_0 whose kernel sums to C over x' at the scale ρ, equivalently as the graph S' with a credit $\rho_{i_l}^{\theta_l}$ on its lines, $\sum_l\theta_l=1+\alpha_{\mathrm{H}}$

(resp. $\alpha_{\rm H}$), and the Hölder debit on the leg Φ_2. The members with a block vertex are not part of the C-site and are treated separately: their stages have split degree ≥ 1 and need no layer (Lemma 5.41). The image-part bound in (b) holds because $|x - gw| \geq \max(|x - w|, t(x))$ for every non-trivial reflection g of the box. That the rewriting with the block part left uncollapsed is exact, and that no layered vertex then contains a block vertex or a block point, is the *no-collapse lemma* for the block part, Lemma 5.41 below. □

Theorem 5.34 (the zone-B photon pieces across the pair and at the root)**.** *Let a member family be as in Definition 5.32, with the difference pieces of Theorem 5.33(b). Then:*

- *(c) The degree condition with layered vertices of Theorem 5.37 holds for every difference graph, and for $0 < \gamma < \min(\alpha_0^B, 1 - \theta_{\max}, \gamma_{\rm c})$, $\gamma_{\rm c} \leq \min(\alpha_{\rm H}, 1 - \alpha_{\rm H})$ the rate of Theorem 5.31(b), every difference piece obeys the pair bound $5^{-\gamma j}\rho_{j_0}^{-\gamma} \times$(its bound), its weight weakened to the class $\theta + \gamma$; so, given the pair clauses of the lines and legs, their Hölder quotients included, the ordered bound with layered vertices applies and gives the pair bound with $5^{-\gamma j}$ and the root weight Λ_γ of Theorem 5.37(b). For a member with a line $\delta G_j(\square', \cdot)$, bounded in each model separately, the bound is in the two-term form of Theorem 5.37(b); for these members and for the members with block points the pair rate is proved by the* joint boundary and block-vertex statement*, Theorem 5.44 below, which proves the root-weighted bound and the pair rate together for γ in its own range, with which the range stated here is intersected.*
- *(d) In a host rooted at an interior source vertex ($t(x_0) \geq \eta^{-1}/2$), $\Lambda(x_0) \leq C$ and $\Lambda_\gamma(x_0) \leq C$: the zone-B photon pieces cost a constant. For the source-free field-dependent classes, root-summed, $\sum_{x_0} 5^{-3j}\Lambda(x_0) \leq C$ and $\sum_{x_0} 5^{-3j}\Lambda_\gamma(x_0) \leq C$ per unit face cell, the pair difference carries $5^{-\gamma j}$ at the localisations with $R \leq p$ (in the two-term form for the members with a line $\delta G_j(\square', \cdot)$, by Theorem 5.44(J-b); at $R > p$ each member is bounded separately), and the zone-B total is $\leq C|\partial T^{(j)}| \leq C\eta|T^{(j)}|$ relative to the bulk form.*

Proof. (c). For every remainder graph of Theorem 5.37 and every difference graph here, the hypotheses of the ordered bound with layered vertices hold, the layered vertices being the $D' = 0$ remainder vertices, the charged $D' = -1$ vertices after borrowing and the difference pieces of Theorem 5.33(b), the residual of every one-loop class included, all of classes with $m \leq 1$; every component of a stage that contains a difference piece contains a line at it (so the piece is never a component of its own), and $\alpha_0^B \geq \alpha_* > 0$ with α_* depending on $\bar{n}$ and $\alpha_{\rm H}$ only. The pair hypothesis of the ordered bound holds piece by piece: the residual of a one-loop class by the two-cut-off comparison of the one-loop kernel across the pair; the face piece together with the first moment of the replaced-line part by the comparison of the Dirichlet image sums of the two members; the remainders, and for $D(S') = 0$ the whole zeroth moment $c^\Omega - c^F$, by the comparison of the Taylor remainders and of the cube line differences; in each case the difference is $5^{-\gamma j}\rho^{-\gamma}$ times a weight of the class with θ raised by γ, for $\gamma < \gamma_{\rm c}$. The residuals do not change $\theta_{\max}$ or Λ, and the index j_0 of a residual is the index of its first line and is summed with it.

(d). The root weight Λ of Theorem 5.37 is $\leq C$ at $t(x_0) \geq 1$, and $\sum_x 5^{-3j}\omega(x) \leq C\rho_w$ for every layered weight of width ρ_w and class $\theta < 1$, summed over the face cells; across the pair the root weight is Λ_γ, which is $\leq C$ at $t(x_0) \geq 1$ and root-summable since $\theta_{\max} + \gamma < 1$. This replaces the per-ordering photon boundary step of [26, pp. 331–332]: the cube is never compared with the torus ordering by ordering; the comparison is made once, inside the collapsed pieces, and every remainder is a layered vertex bounded by the ordered bound with layered vertices. The remaining estimates are routine given the slice bounds of the cube lines up to the faces and their pair clauses. □

Definition 5.35 (layered weights)**.** A weight ω on Ω is *layered* of class $\mathcal{L}(\rho, m, \theta)$, where $5^{-j} \le \rho \le 1$, $m \le \bar{n}$, $0 \le \theta < 1$ and $\delta_b > 0$, if

$$|\omega(x)| \le \big(1 + \ln_+(\rho/t(x))\big)^m (\rho/t(x))^\theta e^{-\delta_b t(x)/\rho} \qquad \text{for every live site } x,$$

$t(x) = \operatorname{dist}(x, \mathsf{F})$ as in Definition 5.22. A vertex v is *layered* if its vertex function is a bounded function times $\omega_v(x_v)$ with $\omega_v \in \mathcal{L}(\rho_v, m, \theta)$ and $\rho_v \le \rho_{j(v)}$, $j(v)$ the lowest index of the lines at v.

Definition 5.36 (the setting of the zone-B scalar classes)**.** Let $\mathcal{R}$ be a renormalised family of order $\le \bar{n}$ of King's expansion of $P^{(j),1}$ on C_J^K at zero background, localised in zone B, with at least one external field or at least one source vertex, rewritten by Definition 5.27 (its vacuum-polarisation prefix components by Theorem 5.33), and reduced on one box: the box $\square'$ of a localisation with $R \le p$, or the box $\square'_K$ of a zone-B cluster of a tuple with several clusters, the cut lines being external fields of index j. Fix $0 < \alpha_{\mathrm{H}} < 1$ (the Hölder exponent of the line bounds, Lemma 5.28), $0 < \beta_{\mathrm{H}} < \alpha_{\mathrm{H}}/2$ (the *borrowing exponent* of step (2) of the proof of Theorem 5.37) and $0 < \gamma < \min(\alpha_0^B, \beta_{\mathrm{H}}, \gamma_{\mathrm{c}}, 1 - \alpha_{\mathrm{H}})$, where α_0^B is the minimum positive stage degree over the finitely many graphs and orderings and γ_{c} the rate of Theorem 5.31(b).

Theorem 5.37 (the zone-B scalar classes)**.** *Let $\mathcal{R}$, α_{H}, β_{H} and γ be as in Definition 5.36. Then:*

(a) with one vertex x_0 held (or none), the sum over orderings, compatible indices and positions of $|E^{(j)}(G)|$ over the members of $\mathcal{R}$ is at most

$$C\eta^{\rho_0} p^{n_1} (1 + k - j)^{p_0} \exp[-\delta \operatorname{dist}(V_*)]\, \Lambda(x_0),$$

V_* the vertices with external legs, the held vertex and the source vertices, with the face weight

$$\Lambda(x_0) \le C\big(1 + \ln_+(1/t(x_0))\big)^{\bar{n}+2}\, t(x_0)^{-(1-\beta_{\mathrm{H}})}\, e^{-\delta_\Lambda t(x_0)},$$

$\Lambda = 1$ if no vertex is held, $\Lambda(x_0) \le C$ if $t(x_0) \ge 1$ (in particular at every source vertex), and $\sum_{x_0 \in Q} 5^{-3j} \Lambda(x_0) \le C$ for every unit cube Q;

(b) across the pair, on the box $\square'$ of a localisation with $R \le p$ (on a cluster box only (a) is claimed), $|E'^{(j+1)} - E^{(j)}| \le C5^{-\gamma j} \times$ (the bound (a) with $t^{-(1-\beta_{\mathrm{H}})}$ replaced by $t^{-(1-\beta_{\mathrm{H}}+\gamma)}$) for $0 < \gamma < \gamma_b$, with the threshold $\gamma_b := \min(\alpha_0^B, \beta_{\mathrm{H}}, \gamma_{\mathrm{c}}, 1 - \alpha_{\mathrm{H}}, \alpha_0, \alpha', \gamma_{F3}/2)$ for a fixed $\alpha' \in (0,1)$, independent of j, k, η, J, x_0 (with α_0 and γ_{F3} as in Theorem 5.44); for every member with a line $\delta G_j(\square', \cdot)$ (the remainders of [25, (3.52)–(3.54), p. 665]*, in every class) the right side is $(C5^{-\gamma j} + C_N \eta^N) \times$ (the bound (a)) for every N, which is of the first form on every cone $j \le (1-c)k$, $c > 0$, for $N \ge \gamma(1-c)/c$;*

(c) summed over the zone-B localisations with $R \le p$, the zone-B part of the source-free field-dependent effective-action difference is at most $(C5^{-\gamma j} + C_N \eta^N) \eta^{\rho_0} p^{n_1} (1 + k - j)^{p_0} |\partial T^{(j)}| \le C' 5^{-\gamma' j} \eta^{-\beta'} |C_J| + C'_N \eta^{N + \rho_0 - 2 - \epsilon} |C_J|$ for every $\gamma' \le \gamma$ with $\gamma < \gamma_b$ as in (b), every $\beta' > \max(2 - \rho_0, 0)$, every $\epsilon > 0$ and every N; the second term, from the members with a line $\delta G_j(\square', \cdot)$, is absent on every cone $j \le (1-c)k$ for N large, and for $N > 2 - \rho_0 + \epsilon$ is of the second form of (5); the zone-B localisations with $R > p$, among them every zone-B localisation of a source host, are bounded in each member separately by (a) in each zone-B cluster with no vertex held, with no factor $5^{-\gamma j}$.

The constants depend on the point of King's range, $\bar{n}$, α_{H}, β_{H}, α', γ, the exponents β', ϵ, the order N, the cone parameter c and the line constants, not on j, k, η, J, x_0.

Proof. After recalling the splitting of $\mathcal{R}$ into bulk-type and remainder graphs, the proof has four steps: the layer weight of a remainder vertex, the borrowing rule for the remainder vertices of degree -1, the degree condition with layered vertices, and the ordered bound with layered vertices, which gives (a), (b) and (c).

The free rewriting of Definition 5.27 splits $\mathcal{R}$ exactly into *bulk-type graphs*, in which every resummed coefficient is the free one c_S^F, and *remainder graphs*, which carry at least one remainder vertex v_S with the coefficient $\Delta_S(x) = c_S^C(x) - c_S^F(x)$. The bulk-type graphs obey the degree count of Theorem 5.30 with the cube slice bounds up to the faces, so Theorem 5.31 gives (a) for them with $\Lambda \equiv 1$ and (b) with the cube line differences. The content of the theorem is the treatment of the remainder graphs, in four steps.

The joint statement named in (b) is Theorem 5.44 below, stated after the no-collapse lemma, Lemma 5.41, whose rewriting it uses: of the threshold γ_b of (b), the first four exponents are the range of Definition 5.36, on which the proof below covers the point-free members; the last three complete the range of the *joint boundary and block-vertex statement*, Theorem 5.44, which proves the root-weighted bound and the pair rate together for the zone-B members with block points, with or without layered vertices.

(1) The layer weight of a remainder. Expanding the internal lines of a scalar C-site S into slices of index $\le j_0$ (the lowest index of its legs, $\rho = \rho_{j_0}$) and comparing the cube slices with the free ones, the difference is a sum of image terms, each carrying one reflected line of length $\ge t(x)$; this gives

$$|\Delta_S(x)| \le C\rho^{D'}\big[w_{D'}(t(x)/\rho) + e^{-\delta p/\rho}\big], \qquad w_0(s) = (1+\ln_+(1/s))e^{-\delta s}, \quad w_{-1}(s) = s^{-1}e^{-\delta s},$$

and across the pair $|\Delta'_S(x) - \Delta_S(x)| \le C5^{-\gamma j}\rho^{-\gamma}\times$(the same with the exponent D' lowered by γ). So a $D' = 0$ remainder vertex is layered of class $\mathcal{L}(\rho, 1, 0)$; a $D' = -1$ remainder vertex has the weight t^{-1}, which is not summable over the root, and is treated by borrowing.

(2) Borrowing and the charging rule. Every scalar leg ending at a live site x at distance t from the faces vanishes on F and is Hölder up to the faces, so it carries the factor $m_{\alpha_{\mathrm{H}}}(x;\rho_l) := \min(1, t/\rho_l)^{\alpha_{\mathrm{H}}}$, ρ_l the scale of the leg (half the exponent for a pair difference). In a non-vanishing remainder graph one can choose for each $D' = -1$ remainder vertex one of its two scalar legs so that no line is charged twice (a cut line between two zone-B clusters excepted, which is then charged at both ends and bounded by the two-ended face bound for cut lines, Lemma 6.4(b), proved in Section 6.2 without this theorem): the multigraph on the $D' = -1$ vertices with the shared legs as edges has no component with more edges than vertices, since a cycle would be a closed component made of remainder vertices in a connected graph with an external field or a source vertex. Moving the factor $m_{\beta_{\mathrm{H}}}(x;\rho_l)$, $\beta_{\mathrm{H}} < \alpha_{\mathrm{H}}/2$, from the charged leg to v_S leaves the leg its standard bound and gives v_S the weight $\rho|\Delta_S(x)|m_{\beta_{\mathrm{H}}}(x;\rho_l) \le C[(\rho/t)^{1-\beta_{\mathrm{H}}}e^{-\delta t/\rho} + e^{-\delta p/(2\rho)}]$, of class $\mathcal{L}(\rho, 0, 1-\beta_{\mathrm{H}})$ plus a term of any prescribed degree supported away from F; across the pair the difference is $5^{-\gamma j}\rho^{-\gamma}$ times a weight of class $\mathcal{L}(\rho, 0, 1-\beta_{\mathrm{H}}+\gamma)$, which is why $\gamma < \beta_{\mathrm{H}}$ is required.

(3) The degree condition with layered vertices. The layered vertices are the $D' = 0$ remainder vertices, the $D' = -1$ ones after borrowing, and the difference pieces of Theorem 5.33(b), all of classes with $m \le 1$ and degrees ≥ -1. In every admitted ordering every connected component H of every stage has $D(H) \ge 0$, and $D(H) = 0$ only if H contains a layered vertex: the stages containing a scalar remainder vertex by the census of the scalar remainders (a C-site is the zeroth moment of a scalar self-energy of order 2 or 4, the first moment of the order-2 one-photon exchange, a one-leg-differentiated two-point class or a $\phi\phi A$ class, and a stage through its remainder vertex with an external leg has $D \ge 0$); the stages containing a vacuum-polarisation piece by the integrality of the photon two-point degrees and the presence of an external field or a source vertex; the stages containing a block point by the split count (a block point weighs $3 - \frac{s}{2} + \frac{p}{2} \ge \frac{5}{2}$, so every such stage has $D \ge 1$); the *mixed stages*, which contain a source vertex and a layered vertex, by re-expanding the generalised vertices, after which the source vertex and the standard vertices of the layered vertex's class give $D \ge u_s - 2 \ge 0$, $u_s = 2$ for the scalar source and $\frac{5}{2}$ for the summed curvature vertex (no C-site contains a source vertex, and reserving half

of the decay of every line of such a stage gives moreover a factor $C_N\rho^N$ at its first appearance); and all other stages are bulk-type stages with $D>0$ by Theorem 5.30. Let α_0^B be the minimum positive degree.

(4) The ordered bound with layered vertices. Run the multiscale induction of Theorem 5.31(a) with one change: when the line being shrunk, of scale ρ_i, has a layered endpoint of width $\rho_w\le\rho_i$ that is not held, sum that endpoint against the line, using half of the line's exponential to transfer the layer weight: the factor is $C\rho_i^3(\rho_w/\rho_i)$ instead of $C\rho_i^3$, so an active layered composite gains $+1$ in the exponent at every step until it is merged into the held vertex x_0 or into the final volume sum, where $\sum_x 5^{-3j}\omega(x)\le C\rho_w$ for $\omega\in\mathcal{L}(\rho_w,m,\theta)$, $\theta<1$. Hence every stage has exponent $\ge\min(\alpha_0^B,1)>0$ except the steps at which a $D=0$ component contains x_0 and an inactive layered composite: there the factor $e^{-\delta' t(x_0)/\rho_*}$ forces the merge scale $\rho_*\ge ct(x_0)$ up to a summable tail, the at most $\bar n$ such steps and the merge index give the logarithms $(1+\ln_+(1/t(x_0)))^{\bar n+2}$ and the factor $e^{-\delta_\Lambda t(x_0)}$, and a layered vertex held at x_0 adds its own $t(x_0)^{-\theta}$ with $\theta\le\theta_{\max}=1-\beta_{\mathrm H}$: this is $\Lambda(x_0)$, root-summable because $\theta_{\max}<1$. The infrared stage, which contains no remainder, no source and no layered vertex, gives the factor $(1+k-j)^{p_0}$ as in Theorem 5.31(a). This proves (a). For (b) the same induction is run for the difference of the two members, every line, coefficient and layered weight obeying a pair bound $5^{-\gamma j}\rho^{-\gamma}\times$(its bound), a weakened layered weight being of class $\theta+\gamma<1$, the positions of the two members paired cell by cell as in Theorem 5.31(b), the finer member's held vertices in the top layer, which have no partner, bounded by (a) for the finer member and entering only the sum of (c), where they cost $C5^{-\gamma j}$ per unit cube; the root weight becomes Λ_γ, with $t^{-(1-\beta_{\mathrm H}+\gamma)}$, still root-summable as $1-\beta_{\mathrm H}+\gamma<1$. A member with a line $\delta G_j(\square',\cdot)$ is bounded in each model separately, which gives its term $C_N\eta^N\times$(the bound (a)); for these members and for the members with block points the pair rate is Theorem 5.44(J-b). (c) is (a), (b) and Theorem 5.34(d) summed over the root with $\sum_{x_0}5^{-3j}\Lambda(x_0)\le C$ per unit face cell, $|\partial T^{(j)}|\le C\eta^{-2}|C_J|$ and $p^{n_1}(1+k-j)^{p_0}\le C_\epsilon\eta^{-\epsilon}$. The remaining estimates are routine given the slice bounds of the cube lines up to the faces and their pair clauses. □

The following theorem treats the zone-I graphs that depend on the field or carry a curvature vertex; it is used again, for the shell model, in Section 25.

Theorem 5.38 (the curvature classes and the source-free classes in zone I)**.** *Let H be a graph of King's expansion at step j on C_J^K at $h=0$ that is field-dependent, or contains a curvature vertex, and let it be localised in zone I. Then* [25, (3.39)] *holds for it, and* [25, (3.38)] *with the extra factor $(1+k-j)^{p_0}$; summed over positions, the zone-I contribution to $|S^{(j)}_{N,k}-S^{(j+1)}_{N,k+1}|$ is $\le C[5^{-\gamma' j}\eta^{-\beta'}+\eta^{\sigma'}]|C_J|$, a bound of the form of (5), with C depending on the curvature sources through $\max\|f_{\mu\nu}\|_{C^{2,\alpha}}$ only.*

Proof. At the localisations with $R>p$ the two members are bounded separately by the cut by clusters (Definition 5.24, as in Proposition 6.7) and enter the second term only; at those with $R\le p$ it suffices to treat the free version $\tilde H^{\mathrm{free}}$ of the reduced graph on the free f_D flow, with an error of the form of [25, (3.39)], since the free-boundary counterterm renormalises the free graphs exactly. For the free flow one runs King's proof of [25, Thm. 3.5] and of the step bound [25, Prop. 3.6]: the census of ordered subgraph components (every component containing a source vertex has degree $\ge\frac12$, except the field-independent source-linear family, which is absent at $h=0$ and vanishes for the curvature source at $A_j=\phi_j=0$ by charge parity, and the single-curvature-vertex tadpole, which vanishes by $\operatorname{Tr} q=0$, q the antisymmetric charge matrix of the current vertex); the ordered multiscale bound, Theorem 5.31(a) with the degree count of Theorem 5.30 for the source-free components; the one-factor-at-a-time replacement of [25, pp. 663–665], with the comparison of the cube lines with the free lines and, for the curvature vertex, the comparison of the summed

vertex weights across the pair of spacings, $|w_{\varepsilon_{k+1}}(b') - w_{\varepsilon_k}(b)| \le (2\sqrt{3}+1)\,\varepsilon_k \max\|f_{\mu\nu}\|_{C^2}$ for bonds b' at spacing ε_{k+1} and b at spacing ε_k in the same direction with $|m_{b'} - m_b| \le \sqrt{3}\,\varepsilon_k$ (the mean value theorem), which carries the gain $\varepsilon_k \le 5^{-\gamma j}$ with a bounded summed vertex; and the identities [25, (3.43)–(3.55)], under which the curvature vertex is invariant. The curvature source enters only through the sup and lattice $(1,\alpha)$-norms of the summed vertex weight w, the bond function with $\sum_p g^\varepsilon(p)F(p) = \sum_b w(b)A_b$ exactly for every bond field A on C_J (f a real 2-form of the family, g^ε its centre sampling and $F(p)$ the plaquette curvature), $w(b) = W^\varepsilon_\mu(m_b)$ with $W^\varepsilon_\mu(x) = \sum_{\nu\neq\mu} \varepsilon^{-1}[f_{\mu\nu}(x + \frac{\varepsilon}{2}e_\nu) - f_{\mu\nu}(x - \frac{\varepsilon}{2}e_\nu)]$ and m_b the midpoint of the bond b; these norms are $\le C \max\|f_{\mu\nu}\|_{C^{2,\alpha}}$ uniformly in ε. □

The split degree count for the classes with block vertices. King's expansion of the renormalisation transformation [25, (3.24)–(3.26)] produces, besides the local vertices [25, (3.18)–(3.23)], the *block products*: a vertex at the block $y \in T^{(k)}$ that is the product of two of the expressions

$$\phi_k(y), \qquad -\big(Q_k(A^{(k)})(\phi' + \phi^{(k)})\big)(y),$$
$$-\frac{(e(L^k\varepsilon)^{2-d/2})^m}{m!} \sum_{x\in B^k(y)} \eta^d\, U\big(A^{(k)}(\Gamma^{(k)}_{y,x})\big)\big(e'A'(\Gamma^{(k)}_{y,x})\big)^m q^m(\phi' + \phi^{(k)})(x),$$

one of which is of the third kind, multiplied by $-a_k$ or $-a_k/2$ and summed over y. Each factor of the second or third kind is a sum over a *point* x of the block with the measure η^d, and the m fluctuation fields of the third kind live on the contour $\Gamma^{(k)}_{y,x}$; by [25, (2.19)] the scale-j field of the point x enters only through the sub-contour $\Gamma^{(j+1)}_{x_{j+1},x}$, so a photon line from the contour at slice j is the *contour kernel* $g^{(j)}(x,b) = \sum_{b'\in\Gamma^{(j+1)}_{x_{j+1},x}} \eta\, G_{(j)}(b',b)$ of [25, (3.27)]. A vertex of the expansion that is a block product is a *block vertex*. King's degree [25, (3.30)–(3.31)] credits a block vertex with $+m$ and merges its two summation points into one; Bałaban's degree [3, (2.1)–(2.2)] represents a block product by two vertices and credits a vector leg of a block vertex only when it lies on an internal line of the graph, the factor $L^{j_l}\eta$ of such a leg being booked at the scale of its line. On whole graphs the two agree; on subgraphs King's per-vertex form over-credits a block vertex whose contour fields have their lines outside the subgraph, and the count that governs the bounds is Bałaban's, stated here as the split count.

Definition 5.39 (the split graph)**.** Let $\tilde H$ be a connected graph of King's expansion after the constant-background reduction [25, (3.43)–(3.55)] at zero background, on the torus or on a cube with the slices of the Dirichlet flow, its internal lines sliced as in [25, (3.58)] (fluctuation slices $0 \le j \le k-1$, the finer model's slices $-n \le j \le k-1$, the counterterm lines C^η of the mass-counterterm units with $0 \le j \le K$). The *split graph* G^{sp} (that of Theorem 5.31, recalled here) has as vertices every vertex of $\tilde H$ other than a block product and, for each block product, its summation points x and (when present) x', each with the measure η^d; the fields $\phi_k(y)$, the contour fields of the background and the field $\phi^{(k)}$ are external at the point. Its scale-summed lines are the internal fluctuation lines with their slice index, a photon field on the contour of a point entering through the kernel $g^{(j)}$, and, for each two-point product, the block kernel $\kappa_y(x,x') := \mathbf{1}[x,x' \in B^k(y)]$ times $-a_k$ or $-a_k/2$, read as a line with the single index k. A scale-summed line l has dimension $d_l := 2 - d - \#\{\text{derivatives on } l\} + c_l$, $c_l \in \{0,1,2\}$ the number of its ends on contours (a Hölder factor $|x-y|^\alpha$ adds α), the block kernel has $d_\kappa := 0$, and a vertex carries the weight a_v equal to the number of explicit factors η at v beyond the measure ($m-1$ for a vertex (3.22) of order m, $m-2$ for (3.23), and $a_X := 0$ at every point). For a connected subgraph H of G^{sp} with all lines scale-summed, the *degree of the multiscale bound* is $D^{\mathrm{MS}}(H) := \sum_{l\in H} d_l + \sum_v a_v + d(|V(H)| - 1)$. For a connected component c of the graph γ^{sp}

obtained from a subgraph γ of $\tilde{H}$ by the same splitting (the block kernel is not a line of it), the *split degree* is

$$D^{\mathrm{sp}}(c) := \sum_{v\in c} w_v - d,$$

$$w_v := d + a_v - \tfrac{d-2}{2}\#\{\text{undifferentiated ends of } c\text{-lines at } v\} - \tfrac{d}{2}\#\{\text{differentiated ends}\},$$

where at a point X the weight a_X is replaced by $m_{\mathrm{int}}(X)$, the number of photon fields on the contour of X whose lines are lines of c; and $D^{\mathrm{nat}}(\gamma) := \sum_c D^{\mathrm{sp}}(c)$. For γ without block vertex $D^{\mathrm{nat}}(\gamma)$ is King's degree. For a point X with at least one end of a c-line, $w_X = 3 + \frac{1}{2}m_{\mathrm{int}}(X) - \frac{1}{2}s_{\mathrm{int}}(X) \geq \frac{5}{2}$, $s_{\mathrm{int}}(X) \in \{0,1\}$ recording whether the scalar slot of X is internal, and $D^{\mathrm{sp}}(c)$ is an integer. If $c_1,\dots,c_r$ are the components of H with its κ-lines removed, then $D^{\mathrm{MS}}(H) = \sum_i D^{\mathrm{sp}}(c_i) + d(r-1)$.

On King's vertex list at zero background the weights take the values: a standard vertex other than the source vertices has $w_v \geq \frac{1}{2}$, with equality only for a vertex (3.22) of order one with its derivative internal and all three ends internal; a vertex of a mass-counterterm unit has $w \geq 0$, with $w = 0$ only for its base vertex with all legs and its line inside the subgraph; the scalar-source vertex h has $w = 3 - e/2 \geq 2$ (e its number of internal ends; the shell or insertion vertex of the shell model is of this type); the curvature vertex g has $w = \frac{3}{2}$ with its leg internal or $\frac{5}{2}$ in its summed-by-parts form.

Lemma 5.40 (positivity of block-point components)**.** *Let γ be any connected subgraph of King's expansion at zero background, on the torus or on a cube, with any external legs and with mass-counterterm units of any depth, and let c be a connected component of γ^{sp} containing a point X of a block vertex and at least one line. Then (i) $D^{\mathrm{sp}}(c) \geq 1$; (ii) $D^{\mathrm{sp}}(c) \geq 2$ if c contains two points, an h-type vertex or a curvature vertex. Consequently $D^{\mathrm{nat}}(\gamma) \geq 1$ for every connected γ containing a block vertex, and ≥ 2 if both points of some block vertex of γ carry γ-lines.*

Proof. The lemma is a finite case analysis of the weights: for a unit-free component, $D^{\mathrm{sp}}(c) = w_X + \sum_{v\neq X} w_v - 3$ with $w_X \geq \frac{5}{2}$, and the values $\frac{1}{2}$ and 1 of $\sum_{v\neq X} w_v$ are excluded by the pairing of the scalar and photon ends (a lone vertex (3.22) of order one has one scalar and one photon end left over, two such vertices leave three scalar ends); a component with mass-counterterm units is reduced to a unit-free one by replacing each unit met by a point vertex, each one-loop unit costing one and crediting two. No parity, Furry, Ward or reflection argument and no properness is used. □

The statements that follow concern the photon two-point prefix components in zone B across the pair: the no-collapse lemma (Lemma 5.41), the composite-kernel lemmas (Lemma 5.42 and Lemma 5.43), and the joint statement (Theorem 5.44) named in Theorem 5.37(b). They are used, directly or through Theorem 5.44, in the proofs of Theorem 5.33 and Theorem 5.37 above.

Lemma 5.41 (no collapse of the block part)**.** *Let $\mathcal{R}$ be a renormalised family of King's expansion of order $\leq \bar{n}$, localised in zone B (Definition 5.22), at zero background and zero scalar source, with the lines of the f_D flow of Definition 5.21, in either member of the pair. For a photon two-point prefix component S' split its members as $S' = S'_{\mathrm{hop}} + S'_{\mathrm{blk}}$, where S'_{blk} consists of the members containing at least one block product (of total multiplicity ≥ 1) and S'_{hop} of the others. Let $\mathcal{R}^{C,\mathrm{uc}}$ be the rewriting of Definition 5.27 and Theorem 5.33(a) with one change: no S'_{blk} is collapsed. For every photon two-point prefix component S', at every nesting depth (a photon two-point prefix component may itself lie inside a larger rewritten prefix component), the members of S'_{blk} stay subgraphs of the* host member*, the member of $\mathcal{R}$ of which S' is a prefix component (their lines in slice form, their internal vertices* host vertices*, that is, vertices of the host member*

carrying its localisation); only S'_{hop} is rewritten and compared, with its pieces. Then: (i) $\mathcal{R}^{C,\mathrm{uc}}$ *is an exact rewriting of* $\mathcal{R}$*;* (ii) *every member of* $\mathcal{R}^{C,\mathrm{uc}}$ *is a generalised graph on the split graph* G^{sp} *of Theorem 5.31 whose layered vertices (Definition 5.35) are scalar remainder vertices and* S'_{hop} *difference pieces only, no layered vertex contains a block vertex or a block point, and every block point is a host vertex;* (iii) *every component of every stage of* G^{sp} *has* $D \geq 0$*, with* $D = 0$ *only if it contains a layered vertex:* $D^{\mathrm{sp}} \geq 1$ *with a block point,* $D \geq 0$ *with a scalar remainder vertex (the census of the scalar remainders, step (3) of the proof of Theorem 5.37),* $D \geq 0$ *and layered with an* S'_{hop} *piece (the census of the vacuum-polarisation pieces, same step),* $D > 0$ *otherwise (the degree count of the free* f_D *flow, Theorem 5.30);* (iv) *hence no member of* $\mathcal{R}^{C,\mathrm{uc}}$ *contains a collapsed block piece, Theorem 5.33(b) and Theorem 5.34(c)–(d) hold with the vacuum-polarisation pieces restricted to* S'_{hop} *pieces, and the* S'_{blk} *members are bounded on* C_J^K *by the ordered bound with layered vertices (step (4) of the proof of Theorem 5.37) on* G^{sp}*.*

Proof. (i) The S'_{blk} members are never rewritten: every connected subgraph with a block vertex has $D^{\mathrm{nat}} \geq 1$ (Lemma 5.40: a component of the split graph containing a point of a block vertex and a line has $D^{\mathrm{sp}} \geq 1$, a block point weighing $3 - \frac{s}{2} + \frac{p}{2} \geq \frac{5}{2}$ in the split count, step (3) of the proof of Theorem 5.37), so it is not divergent, and no evaluation, no moment extraction (the extraction of the zeroth and first moments of [3, (3.26)–(3.31)], Theorem 5.33) and no resummation acts on an S'_{blk} member or on a prefix component containing a point. The split $S' = S'_{\mathrm{hop}} + S'_{\mathrm{blk}}$ is a partition of the members, and the rewriting of S'_{hop} uses no S'_{blk} member: the counterterm units of S'_{hop} are summed over its own lines, the bounds of the coefficients $c^{S'_{\mathrm{hop}}}$ use S'_{hop} alone, and the total identity involving S'_{blk} is kept as an identity which no bound uses. In Theorem 5.33(a) the S'_{blk} part of the comparison is the identity $S'^{C}_{\mathrm{blk}} = S'^{\mathrm{free}}_{\mathrm{blk}} + \Delta S'_{\mathrm{blk}}$ followed by a regrouping, and omitting an identity changes no value. Without the collapse the internal scales of the S'_{blk} members are not resummed (undoing the resummation regroups finite sums back into the host's sum over orderings), and their internal vertices carry the host's localisation, as every unrewritten convergent subgraph does [25, p. 661], [26, p. 330]. (ii) By construction, since a C-site is the evaluation of a divergent subgraph and no divergent subgraph contains a block vertex (Lemma 5.40), so that no C-site contains a block point: the census of the C-sites in step (3) of the proof of Theorem 5.37 (a C-site is the zeroth moment of a scalar self-energy of order 2 or 4, the first moment of the order-2 one-photon exchange, a one-leg-differentiated two-point class or a $\phi\phi A$ class) contains no class with a block vertex. (iii) As listed; a stage containing an uncollapsed S'_{blk} member contains its points. (iv) From (ii), (iii) and the ordered bound with layered vertices run on G^{sp}, on which the κ-lines and the staircase lines of the points are lines of the graph and every stage exponent is the split-count degree (Theorem 5.31(c), (d)), so that the contour and block-kernel lines of the points are shrunk as ordinary lines. □

Lemma 5.42 (composite kernels)**.** *In the setting of Lemma 5.41, with* $\vartheta = 5^{-j}$ *the spacing of the coarser member of the pair and* $\vartheta' = \vartheta/5$ *that of the finer one, let* $\rho := \rho_{j''} = 5^{j''}\vartheta$ *with* $1 \leq j'' \leq j$*, and* $K = A\, M_P\, B$*, where* A*,* B *are cumulative (index* $\leq j''$*) lines of the* f_D *flow on the cube,* C_D^{ξ} *(the Dirichlet covariance of Theorem 5.33) or their image parts, each slice obeying* [25, (3.63)–(3.65)] *on the cube with* $\mathsf{q}_A, \mathsf{q}_B \leq 1$ *derivatives at the outer ends* u*,* v *and none at the* P*-ends, and* M_P *is* $\rho^{-2}P_{j''}$*, with* $P_{j''} = Q^*_{j''}Q_{j''}$ *the averaging kernel of* [26, (3.2)] *carrying the weight* f_D *at the finest step* [26, (A.1)–(A.3)]*, or the dipole* $\mathrm{i}[\Lambda, \rho^{-2}P_{j''}]$ *(*Λ *the multiplication by a linear function* λ *with* $B = d\lambda$ *a constant field). Put* $\bar{K} := C\rho^{-1-\mathsf{q}_A-\mathsf{q}_B}e^{-\delta|u-v|/(2\rho)}$*, times* $|B|\rho$ *for the dipole. Let* γ_{F3} *be the rate of the parts of the lines specific to the weight* f_D*: any* $\gamma_{F3} < \min(\alpha_0, s)$*, with* α_0 *the minimum positive stage degree of Theorem 5.31 and* $s = (1-\alpha')/(1+\alpha')$ *the Hölder rate of Lemma 5.29. Then (a)* $|K| \leq \bar{K}$*; (b) across the pair, with* K' *the composite of the finer*

member, $K' - K = (\text{line terms}) + (\mathrm{I}_P) + (\mathrm{ZM}_P)$ with $|(\text{line terms}) + (\mathrm{I}_P)| \le C5^{-\gamma j}\rho^{-\gamma}\bar{K}$ and $|(\mathrm{ZM}_P)| \le C(\vartheta/\rho)^{\alpha'}\bar{K}$, for every $\alpha' < 1$ and every $\gamma < \gamma_{F3}$, and $\gamma < \gamma_{F3}/2$ when image parts occur; so $|K' - K| \le C5^{-\gamma j}\rho^{-\gamma}\bar{K}$ for $\gamma \le \alpha'$ in that range. The bound $\bar{K}$ is the bound of the part $M_P = \rho^{-2}P$; the replaced line $G(1 - m^2 - aP)C_D^{\xi}$ obeys the corresponding bound, and with derivatives at both outer ends its $(1-m^2)GC$ part has the integrable singularity $\rho^{-2}|u-v|^{-1}$ in three dimensions.

Proof. (a). $P(w,\tilde{w}) = \sum_y Q(y;w)Q(y;\tilde{w})$ with $Q(y;w) = c\rho^{-3}f_*(w)1[w \in B^{j''}(y)]$ in the measure ϑ^3, where f_* is the finest-step weight of [26, (3.8)], equal to $(5/4)^3$ on 4^3 sites of each 5ϑ-block and 0 on the remaining layer, so that $\sum_{x\in B(z)} f_*(x) = 5^3$ on every 5ϑ-block, the averaging above the finest step being uniform [26, (A.1)–(A.3)]; use $\sum_w \vartheta^3|A(u,w)| \le C\rho^{2-\mathfrak{q}_A}$, the same for B, and $|w - \tilde{w}| \le c\rho$ on the support.

(b). Telescope $K' - K = (A' - A)M'_P B' + A(M'_P - M_P)B' + AM_P(B' - B)$. The outer terms are cumulative line differences, bounded per factor by $C5^{-\gamma j}\sum_{i\le j''}\rho_i^{-1-\mathfrak{q}-\gamma}e^{-\delta\,\mathrm{dist}\,/\rho_i}$ (the two-member comparison of the cube slices used in Theorem 5.31(b), with the f_D-specific parts of Lemma 5.29; for C_D^{ξ} the lattice Green's function difference), integrated against the other factor and the block kernel; since $2 - \mathfrak{q} - \gamma > 0$ the sums over i are geometric. For the middle term, the two summation points w, $\tilde{w}$ of P enter only through their weights $\vartheta^3 f_*$, through functions constant on each 5ϑ-block and through the factors $A(u,\cdot)$, $B(\cdot,v)$, undifferentiated there; the point-sum identities $\sum_{w'\in B^1(z)}\vartheta'^3 f'_*(w') = \vartheta^3$ [26, (3.13)] and $\sum_{z\in b}(1 - f_*(z)) = 5^3 - 4^3(5/4)^3 = 0$ on every 5ϑ-block b write the difference of the two members' point sums, at w and then at $\tilde{w}$, as $(\mathrm{I}_P) + (\mathrm{ZM}_P)$: (I_P) is the identification $w' \to z$ of the finer sites with the coarser ones, a line-level difference, and (ZM_P) differences $A(u,\cdot)$ at w between z and a reference site z_b of its block, $|z - z_b| \le \sqrt{3}\cdot 5\vartheta$, by the Hölder clause of the slice bounds in the P-end variable (the mixed clause when $\mathfrak{q}_A = 1$), with $\sum_w \vartheta^3|A(u,z) - A(u,z_b)| \le C\sum_{i\le j''}(\vartheta/\rho_i)^{\alpha'}\rho_i^{2-\mathfrak{q}_A} \le C(\vartheta/\rho)^{\alpha'}\rho^{2-\mathfrak{q}_A}$, a geometric sum since $2 - \mathfrak{q}_A - \alpha' \ge 1 - \alpha' > 0$ (this is where the absence of a derivative at the P-end is used). The dipole factor gives $|\lambda(z) - \lambda(z_b)| \le |B|\sqrt{3}\cdot 5\vartheta$ against $|B|c\rho$. Finally $(\vartheta/\rho)^{\alpha'} = 5^{-\alpha' j}\rho^{-\alpha'} \le 5^{-\gamma j}\rho^{-\gamma}$ for $\gamma \le \alpha'$. Image parts keep their decay by $\min(a,b) \le a^{1/2}b^{1/2}$, which halves the admissible rate. □

Lemma 5.43 (composite kernels at every order)**.** *Let X be a cube-side piece carrying an explicit f_D-weighted P of a zone-B photon two-point prefix component S' of order $\le \bar{n}$: the residual $\Delta c^{S'_{\mathrm{hop}}}$ or the replaced-line piece (the difference of the first moment of the replaced-line part) of Theorem 5.33(b), in the representation with the lines $G_{j_0} = C_D^{\xi} + G_{j_0}(1 - m_{j_0}^2 - \mathfrak{a}_{j_0}P_{j_0})C_D^{\xi}$, or an inner resummed coefficient computed with cube lines inside any vacuum-polarisation piece. Let the pair be as in Lemma 5.41, $0 < \alpha' < 1$, and $0 < \gamma < \min(\alpha_0^B, \alpha_0, 1 - \theta_{\max}, \alpha', \gamma_{F3}/2)$, with α_0^B the minimum positive stage degree of Theorem 5.37, $\theta_{\max} = 1 - \beta_{\mathrm{H}}$ the largest class exponent of the layered weights (step (4) of the proof of Theorem 5.37) and γ_{F3} as in Lemma 5.42. Then X has the* composite-kernel structure *(a), (b) and obeys (c):*

- *(a) every explicit P in X sits in exactly one composite $K = A\,M_P\,B$ created by the splitting of a resummed coefficient, with $A = d^{\mathfrak{q}_A}G_{j''}(\cdot,0)$ (a cube cumulative line or its image part), $B = C_D^{\xi''}d^{*\mathfrak{q}_B}$, $\xi'' = 5^{-j''}$, $1 \le j'' \le j_0$; each outer end carries at most one derivative and at most one Hölder quotient, of exponent $\alpha_B \in (0,1)$ (at most one derivative at the classes of* [3, pp. 437–445]*), the P-ends carry no derivative and no other factor, and the P-points enter X only through K;*
- *(b) no step applied after the splitting has a P-point of K as a vertex, a localisation point, a Taylor or Hölder base point, a contour end, a summation-by-parts site, or a slice or ordering variable.*

(c) $|X'(x) - X(x)| \le C5^{-\gamma j}\rho_{j_0}^{-\gamma} \times$ *(the bound of* X*)* $\times\, \omega_\gamma(x)$ *across the pair,* X' *the piece of the finer member, with* ω_γ *a layered weight of class* $\mathcal{L}(\rho_{j_0}, m, \gamma)$ *(Definition 5.35), m the multiplicity of the bound of* X*, provided* $\alpha_B + \alpha' < 1$ *in the point step; the same for the* (j,k)*-member against the* $(j+n, k+n)$*-member, with the constant multiplied by* $(1 - 5^{-\gamma})^{-1}$.

At every order $\le \bar{n}$*, (a) and (b) hold by the argument of* [3, pp. 437–445]*, carried to every order: the rewriting is a finite sequence of steps of four types (the resolvent splitting inside a resummed coefficient, the steps on the lines internal to the coefficient, the steps on the outer legs, and the* f*-free vector averaging), the splitting at a resummed coefficient is the only step that produces an explicit* P*, and no later step has a* P*-point as its target.*

Proof. (a), (b). For the classes treated explicitly in [3, pp. 436–445] they are read off Bałaban's rewriting: the explicit P's outside lines are created only by the resolvent splitting inside a resummed coefficient [3, p. 437, (3.16)], the steps after the splitting act on lines internal to the coefficient and on the outer legs [3, pp. 436–445], and the vector averaging is f-free; on the cube the splitting is an identity on the space with the Dirichlet condition, and $P_{j''}$ at zero background is [26, (3.2)] with f_D at the finest step [26, (A.1)–(A.3)], so that for $j'' \ge 1$ the block indicator $1[w \in B^{j''}(y)]$ is constant on 5ϑ-blocks and w is a summation point with exactly one undifferentiated line factor $A(u,w)$ (and $B(\tilde{w}, v)$ at $\tilde{w}$); an order-$(1+\alpha_B)$ decoration of an outer end arises when a leg of an enclosing graph carries a derivative of that order [25, p. 668] and never reaches a P-end, and the dipole type occurs only at one loop.

(c): pairing. X is built by the same graphs, orderings and steps in both members, except for the finest index $i = -1$ of the finer member and its composite at the lattice scale (absent in the coarser member, which has no P_0); these are bounded at their own scale by (a) and Lemma 5.42(a), where $(\vartheta/\rho)^\gamma \ge c$, and absorbed by $D - \gamma > 0$. Otherwise $X' - X$ telescopes factor by factor, the P-points being differenced only inside their composites by (b), and the plain internal summation points are identified by $\sum_{w' \in B^1(z)} \vartheta'^3 = \vartheta^3$ with the cube two-member line comparison. *Composites.* Lemma 5.42(b), with the outer terms by the line comparison and the lattice Green's function difference, (I_P) a line-level difference, and (ZM_P) by the Hölder clause of A in its P-end variable: the sum $\sum_{i \le j''} (\vartheta/\rho_i)^{\alpha'} \rho_i^{2-(1+\alpha_B)}$ is geometric if and only if $\alpha_B + \alpha' < 1$, which is where that condition enters. The scalar constants of M_P at one physical scale: $|\mathfrak{a}_{j''+1} - \mathfrak{a}_{j''}| \le C5^{-2j''} = C(\vartheta/\rho_{j''})^2$, and the mass $m^2_{j''}$ is the rescaled mass of the Gaussian part, the same in both members at the same physical scale (the mass counterterms are vertices [3, (2.2)–(2.3)]). *Inside the coefficient.* A resummed coefficient is bounded by absolute values of its factors [3, p. 437]; replacing a composite by its difference keeps the form of the bound ($\bar{K}$), and replacing a line adds a factor $|r|^{-\gamma}$ at scale $\le \rho_{j''}$, integrable inside the coefficient for $\gamma < \alpha_0$, the replaced coefficient being convergent with positive room [3, p. 441] (with derivatives at both outer ends the integrand is $\sim |r|^{-1}$, still integrable with room $2 > \gamma$). Nested coefficients are weakened elements of the outer generalised graph, absorbed at their own scale since every ordered stage has $D > 0$. *Layer.* The image part of a line or of a composite's line survives every replacement, by $\min(a,b) \le a^{1/2}b^{1/2}$ in the outer terms and (I_P) and by $|z_b - gu| \ge |z - gu| - \sqrt{3}\cdot 5\vartheta$ in (ZM_P); King's per-scale sum [26, (3.17)–(3.18)] gives $\omega_\gamma \in \mathcal{L}(\rho_{j_0}, m, \gamma)$ (one internal $D = 0$ sub-stage adds 1 to m). For the members at distance n telescope over the n steps. □

Throughout Theorem 5.44 and its proof, $0 < \alpha_{\mathrm{H}} < 1$ and $0 < \beta_{\mathrm{H}} < \alpha_{\mathrm{H}}/2$ are the Hölder and the borrowing exponents of Theorem 5.37, α_0^B is the minimum positive stage degree of Theorem 5.37, α_0 that of Theorem 5.31, and for $0 < \alpha' < 1$, γ_{F3} is the rate of the parts specific to the weight f_D: any $\gamma_{F3} < \min(\alpha_0, s)$, with $s = (1-\alpha')/(1+\alpha')$ the Hölder rate of Lemma 5.29 (as in Lemma 5.42). The S'_{hop} pieces carrying an explicit averaging kernel

$P_{j_0} = Q^*_{j_0} Q_{j_0}$ outside a line are the residual $\Delta c^{S'_{\rm hop}}$, the replaced-line piece of Theorem 5.33(b) and the inner resummed coefficients of the vacuum-polarisation pieces; the *composite-kernel structure* (a), (b) of Lemma 5.43 of such a piece is this: every explicit P sits in exactly one composite $A\,M_P\,B$ created by the splitting of a resummed coefficient, whose outer ends are cube cumulative lines (or their image parts) and Dirichlet covariances carrying at most one derivative and at most one Hölder quotient each, of exponent $\alpha_B \in (0,1)$, whose P-ends carry no derivative and no other factor, and no step applied after the splitting has a P-point as a vertex, a localisation point, a Taylor or Hölder base point, a contour end, a summation-by-parts site, or a slice or ordering variable.

Theorem 5.44 (the boundary bound and the block-vertex difference clause, jointly)**.** *In the setting of Lemma 5.41 let $\mathcal{R}$ be a renormalised family of King's expansion of order $\le \bar n$ with at least one external field, localised in zone B (so source-free, Lemma 5.26), rewritten by $\mathcal{R}^{C,\rm uc}$ (Lemma 5.41) and written on the split graph $G^{\rm sp}$ of Theorem 5.31 with $b \ge 0$ block points and $r \ge 0$ layered vertices. Let $0 < \alpha' < 1$ and $0 < \gamma < \min(\alpha_0^B, \beta_{\rm H}, \alpha_0, \alpha', \gamma_{F3}/2)$. Every $S'_{\rm hop}$ piece of the family carrying an explicit averaging kernel $P_{j_0} = Q^*_{j_0} Q_{j_0}$ outside a line has the composite-kernel structure (a), (b) of Lemma 5.43; for the classes treated explicitly in* [3, pp. 436–445] *this is Bałaban's rewriting, and at every order $\le \bar n$ it holds by the argument of* [3, pp. 437–445]*, carried to every order (Lemma 5.43). Assume $\alpha_B + \alpha' < 1$. Then for every member:*

- *(J-a) the bound of Theorem 5.37(a) holds for $G^{\rm sp}$ with the root weight Λ; the same holds with one element weakened by θ, either a line or vertex in a stage of positive degree with $\theta < \alpha_0^B$ or a layered weight ω_v of class $\mathcal{L}(\rho_v, m, \theta_v)$ (Definition 5.35), θ_v its class exponent, weakened to the class exponent $\theta_v + \theta < 1$, or a factor at a block point with $\theta = \alpha' < 1$;*
- *(J-b) $|E'^{(j+1)} - E^{(j)}| \le C5^{-\gamma j} \times$ (the bound of (J-a) with Λ replaced by Λ_γ); for a member with a line $\delta G_j(\square', A^{(j)})$ the right side is $(C5^{-\gamma j} + C_N \eta^N) \times$ (the bound of (J-a)) for every N, which is of the first form on every cone $j \le (1-c)k$, $c > 0$, for $N \ge \gamma(1-c)/c$, in particular along the diagonal pairs of Section 5.1;*
- *(J-c) summed over the localisations, the zone-B part is $\le C5^{-\gamma j}\eta^{\rho_0} p^{n_1}(1+k-j)^{p_0}|\partial T^{(j)}|$, and the block-vertex part has the form $\le C(5^{-\gamma j}\eta^{-\beta_1} + \eta^\sigma)|C_J|$ for some $\beta_1, \sigma > 0$, the form of the one-step bound (5); here $\eta = 5^{j-k}$ is as in (J-b), and the spacing $\vartheta = 5^{-j}$ of the pair (Lemma 5.42) obeys $\vartheta^{-1} = 5^j \le \eta^{-(1-c)/c}$ on the cone $j \le (1-c)k$.*

In particular Theorem 5.37(b) for the members with block points and the pair rate of Theorem 5.31(b) for the block-vertex families in zone B hold, and neither proof uses the other as a theorem. The constants depend on the point of King's range, $\bar n$, $\alpha_{\rm H}$, $\beta_{\rm H}$, α', α_B, γ, the order N, the cone parameter c and the line constants, not on j, k, η, J, x_0.

Proof. We first collect the unit-level statements on which the proof rests; we then prove (J-a) by the ordered bound with layered vertices, (J-b) by telescoping the difference of the two members over the factors of $G^{\rm sp}$ together with the point step, and (J-c) by summing over the localisations; finally we state the order of the induction.

The proof rests on unit-level statements only, none of them a bound on a whole graph: the slice bounds of the cube lines up to the faces and their pair clauses, with the f_D-specific parts of Lemma 5.29 and Theorem 5.30(ii); the layered vertices, their weights and pair clauses (steps (1)–(2) of the proof of Theorem 5.37, Theorem 5.33(b), Theorem 5.34(c)), and, for the $S'_{\rm hop}$ pieces carrying an explicit averaging kernel P_{j_0}, Lemma 5.42 and Lemma 5.43(c) under the composite-kernel structure (a), (b) of Lemma 5.43, which holds at every order $\le \bar n$ by the argument of [3, pp. 437–445], carried to every order (their range is contained in the one stated, since $1 - \theta_{\max} = \beta_{\rm H}$, and $\alpha_B + \alpha' < 1$ is a hypothesis); the block points, through the split count of Theorem 5.31 and the comparison of the two members' point sums; and the *separation*

of the two mechanisms: no scalar remainder vertex and no S'_{hop} piece contains a block point (Lemma 5.41(ii)), no C-site contains a block point (the census of the C-sites, step (3) of the proof of Theorem 5.37), block points lie in no divergent or remainder unit and no infrared slice ends at one (Lemma 5.40), every connected stage containing a block point has $D \geq 1$ in the split count, also when it contains layered vertices (Lemma 5.40 with Theorem 5.31(c), (d): a block point weighs $\geq \frac{5}{2}$, step (3) of the proof of Theorem 5.37), so the root-weight stages (degree 0, containing x_0 and an inactive layered composite) contain no block point, and every stage containing a line with an end at a block point contains the point, a line lying in a stage only with both ends, hence has $D \geq 1$.

(J-a). Run the ordered bound with layered vertices (step (4) of the proof of Theorem 5.37) on G^{sp}: its lines are slices, contour and block-kernel lines, lines of the split graph of Theorem 5.31(c), (d), on which the multiscale induction runs with the split-count exponents; its layered vertices are the scalar remainder vertices and the S'_{hop} pieces, of widths $\rho_v \leq \rho_{j(v)}$, none of them a block point (Lemma 5.41(ii)); the degree condition is Lemma 5.41(iii). The induction with layered endpoints acts on layered endpoints only; host block points are never layered endpoints, and the points of former S'_{blk} members are host points (Lemma 5.41); a point's contour and block-kernel lines are shrunk as ordinary lines, and the infrared stage is unchanged. Weakened forms: a weakened line in a stage of positive degree costs $\theta < \alpha_0^B$; a weakened layered weight stays admissible since $\theta_v + \theta < 1$; a factor weakened by α' at a block point lies only in stages containing the point, where $D \geq 1 > \alpha'$. The root-weight steps occur only at stages of degree 0 containing x_0 and an inactive layered composite, which contain no block point.

(J-b). Write the difference of the two members as a telescoping sum over the factors of G^{sp} plus the point step, $E' - E = (\mathrm{I}) + (\mathrm{ZM})$. Each term of (I) has exactly one differenced factor and is bounded by that factor's difference bound times the weakened form of (J-a): an internal slice line by the cube two-member line comparison weakened by γ, with its Hölder clause if the line is charged; a contour line by the pair clause of the staircase lines, with the Hölder clause of the slice bounds on each slice of the contour sum if its far end is charged; a layered coefficient or weight by its pair clause (the rate form of the borrowed weight, of class $\mathcal{L}(\rho, 0, 1 - \beta_{\mathrm{H}} + \gamma)$, admissible since $\gamma < \beta_{\mathrm{H}}$; Theorem 5.34(c); Lemma 5.42 and Lemma 5.43(c) for the S'_{hop} pieces carrying an explicit P_{j_0}); external lines and fields by the line comparison, the sources and χ_j by sampling, $\kappa' = \kappa$, no phase at $B = 0$; the finer member's slices of negative index at their own scale (exponents $D - \gamma > 0$ off layered stages, $D + 1 - \gamma$ for active layered ones, $D - \gamma \geq 1 - \gamma$ on point stages). The point step (ZM) at a block point w: w enters only through its weight $\vartheta^3 f_*(w)$, with $\vartheta = 5^{-j}$ the spacing and f_* the finest-step weight of [26, (3.8)] (equal to $(5/4)^3$ on 4^3 sites of each 5ϑ-block and 0 on the remaining layer), through functions constant on each 5ϑ-block (block indicators, κ, block corners of contours) and through at most $\bar{n}$ factors: undifferentiated slices of cube lines, free lines or image parts, contour lines, charged lines (the leg of a remainder vertex of degree -1 carrying the borrowed factor, the borrowing of step (3) of the proof of Theorem 5.37) and external fields through contours. The finer member's point sum is identified block by block with the coarser lattice, $\sum_{w' \in B^1(z)} \vartheta'^3 f'_*(w') = \vartheta^3$ [26, (3.13)] (this is the term (I)), and the coarser member's weight is removed block by block by $\sum_{z \in b}(1 - f_*(z)) = 5^3 - 4^3(5/4)^3 = 0$, which leaves $(\mathrm{ZM}) = \vartheta^3 \sum_b \sum_{z \in b}(1 - f_*(z))[\prod_i F_i(z) - \prod_i F_i(z_b)]$ with a reference site z_b in each block b; telescoping over the factors, each term is G^{sp} with one factor at the point weakened by α', times $C5^{-\alpha' j}$: a slice or contour line loses $\rho_i^{-\alpha'}$ by the Hölder clause of the slice bounds in the point variable (for $|z - z_b| \leq \sqrt{3} \cdot 5\vartheta$ this is $C(\vartheta/\rho_i)^{\alpha'} = C5^{-\alpha' j}\rho_i^{-\alpha'}$); a charged line $G(v, \cdot)$ with its v-end at a remainder vertex of degree -1 is differenced along a lattice path of at most 12 bonds inside the block, each bond by the Hölder clause with the lattice derivative at the far end, which keeps the vanishing factor $\min(1, t(v)/\rho_i)^{\alpha_{\mathrm{H}}}$ of the leg and gives $(|z - z_b|/\rho_i)^{\alpha'}$ (for

the external-line and δG_j leg types the geometric mean of the vanishing bound and of the Hölder bound in the point variable gives the exponents $\alpha_{\mathrm{H}}/2$, $\alpha'/2$, enough since the borrowing uses $\beta_{\mathrm{H}} < \alpha_{\mathrm{H}}/2$); an external field is differenced by its gradient clause; the undifferenced factors keep their bounds up to $e^{5\sqrt{3}\,\delta}$, every decay rate being halved. Every stage containing the weakened factor contains the point, where $D \geq 1 > \alpha'$, so each term is $\leq C5^{-\alpha' j}\times$(J-a). No layered vertex is a host block point (scalar remainder vertices are point-free, S'_{hop} pieces contain no block vertex, and there is no collapsed block piece, Lemma 5.41), so (ZM) never differences a layered weight, and differencing at a host point the end of a line that leads to a piece is a slice or contour line factor. A member with a line $\delta G_j(\Box', A^{(j)})$, an external object [25, p. 661], is bounded in each model separately: by the exponential decay of δG_j and since every vertex lies at distance $\geq 2p$ from $\partial\Omega \setminus \mathsf{F}$, it is $\leq Ce^{-2\delta_0 p}\times$(J-a) with $p = p(\eta)$, and $e^{-2\delta_0 p(\eta)} \leq C_N\eta^N$ for every N, as $p > 1$ [25, (3.41)]; the remainders [25, (3.48), (3.52)] likewise [25, p. 665]. On a cone $j \leq (1-c)k$ one has $\eta = 5^{j-k} \leq 5^{-cj/(1-c)}$, so this is $\leq C5^{-\gamma j}\times$(J-a) once $N \geq \gamma(1-c)/c$. Summing the finitely many terms gives (J-b) for every γ in the stated range.

(J-c). Zone B: the layer sum per unit face cell of Theorem 5.34(d), as in Theorem 5.37(c). The block-vertex form: the position sums of the multiscale induction over C_J^K cost a power of $\vartheta^{-1} = 5^j$, and on the cone $j \leq (1-c)k$ (or along the diagonal pairs) $5^j \leq \eta^{-(1-c)/c}$, which gives the factor $\eta^{-\beta_1}$ in front of the rate $5^{-\gamma j}$ of (J-b); the terms $C_N\eta^N\times$(J-a) of the members with a line $\delta G_j(\Box', A^{(j)})$, summed in the same way, give the second term, with $\sigma = N - \beta_1 > 0$ for $N > \beta_1$.

The induction runs over the ranks (unit statements) $\to$ (J-a) $\to$ (J-b) $\to$ (J-c); inside (J-a) and (J-b) it is the ordered shrinking of the multiscale induction over finitely many lines and prefix components. At no rank is a graph-level rate statement, the boundary bound (b) or the difference clause, used, and the unit statements use neither. For $b = 0$, (J-a)–(J-c) are the boundary bound for the point-free members, and for $r = 0$ they are the difference clause of the block-vertex families. Two further points. The vacuum-polarisation rewriting of Theorem 5.33(a) with every S'_{blk} collapsed into one difference piece is not used: a member of S'_{blk} with two or more two-point block products joined only through the block kernel κ is not one stage-j_0 object of G^{sp} and cannot be collapsed into a layered vertex of width ρ_{j_0}; under Lemma 5.41 these members are host members with block points, treated by (J-a), (J-b). And the pair rate of the members with block points, left to this statement by Theorem 5.34(c), is (J-b). □

5.4. **The Cauchy property in the spacing.** This subsection derives from the two estimates of Section 5.3 the Cauchy property of the normalised functional in the spacing, and from it the existence theorem.

Theorem 5.45 (the Cauchy property, ratio form)**.** *Fix $J \geq J_0$, f, $x \in \mathbb{R}^N$, and let η_1, k_0, C, c, σ and γ, β, C_3 be the constants of (4) and (5) for $s \in \{u_x, 0\}$ (Propositions 5.12 and 5.19). Choose $\alpha \in (0, \min\{\frac{1}{2}, \gamma/(\gamma+\beta)\})$ and the diagonal pairs $j(k) = \lfloor(1-\alpha)k\rfloor$, $\eta_k = 5^{j(k)-k}$. Put*

$$b_k := C\eta_k^{\sigma}, \qquad \theta_k := C_3\big(5^{-\gamma j(k)}\eta_k^{-\beta} + \eta_k^{\sigma}\big),$$
$$\rho_k := \rho(\eta_k), \qquad r_k(0) := 2\rho_k, \qquad r_k(u_x) := 2\rho_k e^{\beta_f|x|^2}.$$

Then there is k_1 such that for all $k \geq k_1$

$$\big|\ln a_{k+1}(x) - \ln a_k(x)\big| \;\leq\; 4b_k + 2\theta_k + r_k(u_x) + r_k(0). \tag{7}$$

The right side is summable in k, and its tail from k on is at most $C'5^{-\kappa k}$ for some $\kappa > 0$. Hence $a_k(x)$ converges, and its limit lies in $[e^{-\beta_f|x|^2}, e^{\beta_f|x|^2}]$.

Proof. The proof has six steps: the remainders relative to Z; the bounds for one model; the comparison across the pair; the ratio of the partition functions; the ratio of ratios, which gives (7); and the summability of the error terms.

Fix $k \geq k_0$ so large that $j(k) \geq J_*$, $k - j(k) \geq J_*$, $\eta_k \leq \eta_1$ and $r_k(u_x) \leq 1$ (possible by Step 6, since $k - j(k) \geq \alpha k$), and put $j := j(k)$, $b := b_k$, $\theta := \theta_k$. For $s \in \{u_x, 0\}$ write $Z := Z_k(s)$, $I := I_k^{(j)}(s)$, $\hat{I} := \hat{I}_k^{(j)}(s)$ for the k-model after j steps, and $Z' := Z_{k+1}(s)$, $I' := I_{k+1}^{(j+1)}(s)$, $\hat{I}' := \hat{I}_{k+1}^{(j+1)}(s)$ for the $(k+1)$-model after $j+1$ steps. Both members have the block scale η_k and the indicator χ_j, and (4) applies to both, since $j + 1 \leq (k + 1) - J_*$; their remainders are $\mathcal{E} := \rho_k Z_k(0)$ and $\mathcal{E}' := \rho_k Z_{k+1}(0)$.

Step 1 (the remainders relative to Z). For $s = 0$, $\mathcal{E}/Z = \rho_k = r_k(0)/2$ and $\mathcal{E}'/Z' = \rho_k$. For $s = u_x$, $Z = a_k(x)Z_k(0) \geq e^{-\beta_f |x|^2} Z_k(0)$ by Lemma 5.20, so $\mathcal{E}/Z \leq \rho_k e^{\beta_f |x|^2} = r_k(u_x)/2$, and likewise $\mathcal{E}'/Z' \leq r_k(u_x)/2$. In both cases $\mathcal{E}/Z,\ \mathcal{E}'/Z' \leq r_k(s)/2 \leq \frac{1}{2}$. No lower bound on $Z_k(0)$ and no bound on $Z_k(u_x)$ beyond Lemma 5.20 is used.

Step 2 (one model). From $e^{-b} I \leq Z \leq e^b I + \mathcal{E}$: $I \leq e^b Z$ and $I \geq e^{-b}(Z - \mathcal{E}) = e^{-b} Z(1 - \mathcal{E}/Z)$. So $I \in [e^{-b} Z(1 - \mathcal{E}/Z),\, e^b Z]$, and in the same way $I' \in [e^{-b} Z'(1 - \mathcal{E}'/Z'),\, e^b Z']$. In particular $I, I' > 0$.

Step 3 (across the pair). On the support of χ_j, (5) gives $|S_{N,k}^{(j)}(\cdot; s) - S_{N,k+1}^{(j+1)}(\cdot; s)| \leq \theta$. The two integrals $\hat{I}$, $\hat{I}'$ run over the same variables with the same χ_j and have non-negative integrands, so $\hat{I}'/\hat{I} \in [e^{-\theta}, e^{\theta}]$. Since $I = e^{-v_{k,j}} \hat{I}$ and $I' = e^{-v_{k+1,j+1}} \hat{I}'$,

$$I'/I \ \in \ e^{\Delta}\,[e^{-\theta}, e^{\theta}], \qquad \Delta := v_{k,j} - v_{k+1,j+1},$$

and Δ does not depend on s: the vacuum constant is the same number in the numerator and in the denominator of a_k.

Step 4 (ratio of partition functions). By Steps 2 and 3,

$$\frac{Z'}{Z} = \frac{Z'}{I'} \cdot \frac{I'}{I} \cdot \frac{I}{Z} \ \in \ e^{\Delta}\Big[e^{-2b-\theta}\big(1 - \tfrac{\mathcal{E}}{Z}\big),\ e^{2b+\theta}\big(1 - \tfrac{\mathcal{E}'}{Z'}\big)^{-1}\Big].$$

With $|\ln(1-t)| \leq 2t$ for $0 \leq t \leq \frac{1}{2}$ and Step 1, $|\ln(1 - \mathcal{E}/Z)|$, $|\ln(1 - \mathcal{E}'/Z')| \leq r_k(s)$, hence

$$\big|\ln(Z'/Z) - \Delta\big| \ \leq \ 2b + \theta + r_k(s).$$

Step 5 (ratio of ratios). $\ln \dfrac{a_{k+1}(x)}{a_k(x)} = \Big[\ln \dfrac{Z_{k+1}(u_x)}{Z_k(u_x)} - \Delta\Big] - \Big[\ln \dfrac{Z_{k+1}(0)}{Z_k(0)} - \Delta\Big]$, and Step 4 at $s = u_x$ and at $s = 0$ gives (7). (If the constants of (4), (5) differ between the two sources, take the larger ones.)

Step 6 (summability). Since $k - j(k) \geq \alpha k$, $\eta_k \leq 5^{-\alpha k}$, so b_k and $C_3 \eta_k^{\sigma}$ decay geometrically. Since $j(k) \geq (1 - \alpha)k - 1$ and $k - j(k) \leq \alpha k + 1$,

$$5^{-\gamma j(k)} \eta_k^{-\beta} \ \leq \ 5^{\gamma + \beta}\, 5^{-(\gamma(1-\alpha) - \beta\alpha)k},$$

which decays geometrically because $\alpha < \gamma/(\gamma + \beta)$. Finally $p(\eta_k)^2 \geq b_0^2 (1 + \alpha k \ln 5)^{12} \geq c_1 k^2$, so $\rho_k \leq e^{-c c_1 k^2 + C}$; thus $r_k(u_x) \leq 1$ for large k, $\sum_k r_k(s) < \infty$, and the tail of $\sum_k \rho_k$ is eventually below any geometric tail. The series in (7) converges with a geometric tail, so $(\ln a_k(x))_k$ is Cauchy, $a_k(x)$ converges, and the limit lies in the closed interval of Lemma 5.20. □

Remark 5.46 (what the argument does not use). (i) No convergence, and no bound, of the vacuum constants $v_{k,j}$ (King's E_0, E_1, the kernel normalisations, the face vacuum energies) enters: they cancel in Step 3. King's route [26, p. 329] goes through the convergence of Z_k itself, which requires exactly those constants to converge. (ii) No upper stability bound, no lower bound on $Z_k(0)$ and no bound on $Z_k(u_x)$ beyond Lemma 5.20 is used; the remainder of (4) is compared with $Z_k(0)$ only. (iii) No reflection positivity, no Ward identity, and no subsequence. (iv) The

all-n form of [25, (3.13)] (comparing the k-model with the $(k+n)$-model at a fixed stopping scale η for every n) also gives the Cauchy property, provided the constants of (4) and (5) are taken at fixed η with no growth in k: then $|\ln a_{k+n} - \ln a_k| \le 4b(\eta) + 2\theta_k(\eta) + 4\rho(\eta)(e^{\beta_f|x|^2}+1)$ for every n, and one lets $k\to\infty$ and then $\eta\to 0$. The diagonal form above uses $n=1$ only, at the price of constants uniform in j; the two forms use the underlying estimates differently, and neither implies the other.

Theorem 5.47 (King's existence theorem on the cube: the curvature field)**.** *For every finite family $f_1,\dots,f_N$ of real 2-forms in $C_c^\infty(\mathbb{R}^3;\Lambda^2)$, every $J\ge J_0(f)$ and every $x\in\mathbb{R}^N$ the limit*

$$Z_J^K(x) = \lim_{k\to\infty} \langle e^{\sum_i x_i F(f_i)}\rangle_{k,J}^K$$

exists and lies in $[e^{-\beta_f|x|^2}, e^{\beta_f|x|^2}] \subset (0,\infty)$, with β_f as in Lemma 5.20; the convergence is geometric. That is, Theorem A holds. Consequently Z_J^K and Z_∞^K exist locally uniformly on $\mathbb{C}^N$ and are entire, with the bounds and the monotonicity of Theorem 7.6(b).

Proof. Propositions 5.12 and 5.19 provide the two estimates (4), (5) that Theorem 5.45 uses, at every x; the theorem gives the limit and its bounds; Theorem 7.6(b) (that is, Corollary 4.4(ii)) gives the rest. The constants k_0, η_1 and the rates may depend on x and on f; existence needs no uniformity in x. □

Remark 5.48 (the comparison family). Theorem 5.47 says nothing about the fixed-cube limits of the comparison family $\mathrm{ND}^{(a)}$ (Remark 5.11); for that family Corollary 4.4(ii)(γ) squeezes the limits inferior and superior at fixed J between Z_J^K and Z_{J+1}^K, and Corollary 4.4(iii) identifies every subsequential limit, followed by $J\to\infty$, with Z_∞^K. At fixed J every inequality available for the comparison family bounds its oscillation in k only by the difference of two distinct limits, which tends to zero only as $J\to\infty$; a Cauchy statement at fixed J for that family would need the small-field analysis at its faces, where no flow of the kind used above exists.

6. King's existence theorem on the cube with scalar sources

This section proves Theorem B, the existence theorem with scalar sources in the form in which the Wick leg uses it: in the reference scheme z, the fixed-cube limits of the generating functional with curvature and scalar sources exist, locally uniformly and holomorphically, on the domain of coefficient vectors whose positive scalar part is below a threshold, and they are positive at real points; the negative scalar part is unrestricted. The route is the Cauchy argument of Section 5 with the scheme c-number carried along (Theorem 6.23), applied at real points with a small non-negative scalar source, followed by a normal-family argument on the domain. The domain is bounded by Hölder's inequality from three one-sided estimates: the curvature bound of Lemma 5.20, a positive-source bound (Proposition 6.13), and a negative-source bound (Theorem 6.25), the last obtained from positive-source statements at a heavier scalar mass by an exact mass shift and a correlation inequality. The notation is that of Section 5.1.

6.1. **Setting; the source-dependent graphs.** Throughout, the scheme is the reference scheme $z=a^\sharp$ of Definition 2.5; write z_k for its k-th term. Put $\mathsf{S}_k(h) := \sum_{x\in\Lambda_k\cap C_J} \varepsilon_k^3 h(x)$ and $c_k(h) := z_k\mathsf{S}_k(h)$, which is linear in h. For a real source $s=(u,h)$ the normalised functional in the scheme is

$$n_k(s) \;:=\; e^{-c_k(h)}\,N_k(s) \;=\; e^{-z_k\mathsf{S}_k(h)}\,\langle e^{F(u)+:\phi^2:(h)}\rangle_{k,J}^K \;=\; Z_{k,J}^{z,K}(s),$$

the object of Theorem B. For a finite family $f_1,\dots,f_n$ of real 2-forms and $h_1,\dots,h_p$ of real functions, all supported at ℓ^∞-distance >1 from $\mathbb{R}^3\setminus C_J$, and $\zeta=(z,w)\in\mathbb{C}^{n+p}$, $n_k(\zeta) := e^{-\sum_j w_j c_k(h_j)}\langle e^{\sum_i z_i F(f_i)+\sum_j w_j:\phi^2:(h_j)}\rangle_{k,J}^K$ (here z_i are coefficients and z_k the scheme term; no

confusion will arise). Every n_k is entire in ζ, $|n_k(\zeta)| \le n_k(\operatorname{Re}\zeta)$, and $\ln n_k$ is convex on real points (Lemma 2.14(b)): a log-Laplace transform minus a linear function.

A second scalar mass $M > 0$ will be used with the *same* e, λ, μ_0, lattice, cube and boundary conditions; $\langle\cdot\rangle^{(M)}$ denotes King's cube state at mass M, $\alpha_k(M) := M^2 + \delta m^2_{\varepsilon_k}(M)$ King's free-boundary bare coefficient at mass M (Section 2.1), $u_k(M) := \frac{1}{2}(\alpha_k(M) - \alpha_k(m))$, and $s_-(M) := \liminf_k u_k(M)$. Every statement of Section 5 and of this section holds at every point of King's range with point-dependent constants, in particular at the point (e, λ, M, μ_0); we use this at $M = m$ and at one heavier M_2.

Lemma 6.1 ((K$_h$): one-step convergence of the source-dependent graphs on King's cube)**.** *Let $P_k^{(j),1}$ be the interaction part of King's effective action after j steps on C_J^K [25, (3.17)], written as a sum over connected renormalised graphs H, and let $\mathcal{N}_h$ be the family of its members that depend on the scalar source h: the field-dependent part of every graph with an h-vertex, and every field-independent h-dependent graph except the h-linear ones of the three lowest orders $(0,0)$, $(0,1)$, $(2,0)$ in (e^2,λ) (the family $\mathcal{T}_h$ of Definition 6.14). Then for every fixed real $h \in C_c^\infty$ supported at distance > 1 from $\mathbb{R}^3 \setminus C_J$ and every member of $\mathcal{N}_h$, for $J_* \le j \le k - J_*$: the localisation bound [25, (3.38)], with the factor $(1+k-j)^{p_0}$ (King's power of $\ln(L^k\varepsilon)^{-1}$, [25, p. 666], a choice of his n_1), holds for its localisations at distance from the faces larger than the small-field scale $2p(\eta)$ of the localisation boxes (zone I of Definition 5.22); the bound [25, (3.38)] with the restricted degree holds with the factor $(1+k-j)^{p_0}$, summed over the localisations sharing the cube of an h-vertex, and for each single localisation at distance > 1 from the faces. Moreover, there are exponents $\gamma', \beta', \sigma' > 0$, depending neither on h nor on the member, such that for every k and every j with $J_* \le j \le k - J_*$ the sum over positions of the difference between the member at (k,j) and the member at $(k+1, j+1)$, on the common unit lattice and on the support of χ_j, is bounded by $C\big[5^{-\gamma' j}\eta^{-\beta'} + \eta^{\sigma'}\big]\,|C_J|$, $\eta = 5^{j-k}$, with C independent of j and k.*

This is the cube form, on C_J^K with King's Dirichlet weight, of the corresponding statement on King's tori; the h-free graphs, those with a curvature vertex and no h-vertex and the source-free ones, are covered on the cube by Proposition 5.19. The localisations of a member of $\mathcal{N}_h$ with tree length $R \le p$ lie at distance > 1 from the faces, where the bounds reduce to those of the free flow on the torus (Proposition 6.5); a localisation with $R > p$ carries the factor $e^{-\delta R}$ once the localised graph is cut into clusters at mutual distance $\ge R_{\mathrm{cl}} = 3\sqrt{3}$ (the cut by clusters of Definition 5.24), each reduced in its own box of side $O(p)$, the clusters containing the source vertex lying away from the faces and the clusters meeting the Dirichlet faces being source-free (Proposition 6.7). The third clause holds at every step of the range; the proof of Proposition 6.9 uses it along the diagonal pairs of Section 5.1. The proof is given in Section 6.2.

6.2. **Proof of Lemma 6.1.** We work in the unit units of depth j of Definition 5.22: the live sites are T_η, F denotes the set of scalar Dirichlet planes, $t(x) := \operatorname{dist}(x, \mathsf{F})$, and a localised graph H carries King's boxes $\square \subset \square'' \subset \square'$ and $\Omega = \square' \cap T_\eta$, in zone I if $\square' \cap \mathsf{F} = \emptyset$ and in zone B otherwise. The localisation is by centres, as in the proof of Proposition 5.19: a tuple τ has the point set $P(\tau)$, $n_\tau = |P(\tau)| \le N_*$, and the tree length $R = R(\tau) = \operatorname{dist}(P(\tau))$ (Definition 5.22). The cut by clusters, its constants $c_{\mathrm{cl}} = \frac{13}{3}\sqrt{3}$ and $R_{\mathrm{cl}} = 3\sqrt{3}$, the clusters K, the cluster graphs H_K with their boxes $\square_K \subset \square''_K \subset \square'_K$, gauges $B_K = A_j(x_0^K)$ and $\Omega_K = \square'_K \cap T_\eta$, and the zone of a cluster are those of Definition 5.24; we use its elementary geometry (cut lines between clusters have length $\ge R_{\mathrm{cl}}$, $\operatorname{diam} P_K \le c_{\mathrm{cl}}(|P_K| - 1)$, $|\square'_K| \le C' p^3$, the quotient graph of clusters and cut lines is connected with $\operatorname{dist}(P(\tau)) \le \sum_{\ell \in T} |\ell| + c_{\mathrm{cl}} n_\tau$ over any spanning tree T, and the partition, the boxes, the gauges and the set of cut lines are the same for every member of every family and every position). The two members of the pair, the k-model at depth j and the

$(k+1)$-model at depth $j+1$, live on the same unit lattice; every bound below holds for each of them separately with the same constants, and the two are compared only in zone I.

The h-vertex is King's vertex [25, (3.21)]: the smooth function h sampled at the points of the lattice's own frame, the coupling $\theta\eta^2$ in the units of depth j, and at most two fields $\phi^{(j)}$. In King's count of the coupling power $\rho(H)=\sum_v \rho_v$ of a graph [25, p. 665], every vertex has $\rho_v - s_v/4 \geq 0$, s_v its number of external scalar fields, and the h-vertex has $\rho_h - s_h/4 \geq 2-\frac{1}{2}=\frac{3}{2}$; hence

$$\rho(H) - \tfrac{1}{4}s(H) \;\geq\; \tfrac{3}{2} \qquad \text{for every member of } \mathcal{N}_h \text{ with external fields,} \tag{8}$$

$s(H)$ its number of external scalar legs, and $\rho(H)\geq 0$ for the field-independent members. The Wick constant of [25, (3.21)] is field-free and h-linear of the lowest order, so it enters only the excluded family $\mathcal{T}_h$ of Definition 6.14 and no member of $\mathcal{N}_h$ depends on it.

Lemma 6.2 (pointwise bounds at a regular background)**.** *In the unit units of depth j (Definition 5.22: live sites T_η, fine spacing ϑ, scalar Dirichlet planes F, $t(x)=\operatorname{dist}(x,\mathsf{F})$) let A be a photon field on the bonds of T_η with $\theta_j := e_j \sup_b |\partial A(b)|$, where $\partial A(b) := (\partial_\nu A_\mu(b))_{\nu=1,2,3}$ is the vector of the lattice derivatives of the bond variable $A(b)=A_\mu(b)$, in the Euclidean norm, and let $G_A := H_A^{-1}$, $H_A := -\Delta_A + \mu + \mathfrak{a}_j Q_j(A)^* Q_j(A)$, $\mu := m^2\eta^2$, be the cube propagator at the background A, on $\mathbb{R}^N$-valued functions on the live sites, with the Dirichlet condition on F and the weight f_D in Q_j, its kernel taken with respect to the measure ϑ^3 and extended by zero on F (H_A is real symmetric, so $G_A(x,z)=G_A(z,x)^T$). An* operation *O_x at a point x is the identity (*order *$o=0$), a covariant bond difference $D_{A,b}$ along a bond b at x of the closed lattice, face bonds included ($o=1$), or an α-Hölder quotient $|x-x''|^{-\alpha}[D_{A,b} - U(A(\Gamma_{x,x''}))D_{A,b''}]$ of two parallel bond differences at points x, x'' with $0<|x-x''|\leq \frac{1}{8}$, transported along an axis path $\Gamma_{x,x''}$ ($o=1+\alpha$); its evaluation set $E(O_x)$ has diameter $\leq \frac{1}{8}+2\vartheta$. A Hölder pair with $|x-x''|>\frac{1}{8}$ is bounded by 8^α times two differences and needs no clause of its own; an operation in the second variable acts through the transposed transport. There are $\theta_{\mathrm{PW}}>0$, C, $\delta_P>0$ and $s_* := 4+2\alpha$, depending on the block factor 5, the point of King's range and α only, such that if $\theta_j \leq \theta_{\mathrm{PW}}$, then for all operations O_x, O'_z and all x, z:*

(PW1) $|O_x G_A(x,z)| \leq C(r+\vartheta)^{-1-o}e^{-\delta_P r}$ *with $r := \operatorname{dist}(E(O_x), z)$, and the same for an operation in the second variable with $r := \operatorname{dist}(x, E(O'_z))$;*

(PW2) $|O_x O'_z G_A(x,z)| \leq C\min(1,r)^{-s_*}e^{-\delta_P r}$ *whenever $r := \operatorname{dist}(E(O_x), E(O'_z)) > 0$;*

(PW3) if $t(x)\leq r/2$ with $r := \operatorname{dist}(x, E(O'_z))$ (O'_z possibly the identity), then $|O'_z G_A(x,z)| \leq C\min(1,t(x))\min(1,r/2)^{-s_*}e^{-\delta_P r/2}$.

At $A=0$ and at a constant A these hold with no condition on θ_j. For the background $A=A^{(j)}$, the interpolation [25, (2.18), (3.43)–(3.44)] of the block field A_j, on $\operatorname{supp}\chi_j$: $|dA^{(j)}|_\infty \leq C_F p$, where C_F is the constant of the interpolation bound, which follows from King's small-field bound $|\partial_\nu A_{j,\mu}(b)|\leq p$ on the block bonds [25, (3.2)] through the interpolation kernel, which reproduces constants and decays exponentially [25, (3.7)], so that C_F depends on the point only through the photon kernel constants of [25, (3.7)]; hence $\theta_j \leq C_{\mathrm{PW}} e_j p \leq C_{\mathrm{PW}}\, e\, \eta^{1/2} p(\eta)$ with $C_{\mathrm{PW}} := \sqrt{3}\, C_F$, and the condition $\theta_j \leq \theta_{\mathrm{PW}}$ holds on $\operatorname{supp}\chi_j$ once $C_{\mathrm{PW}}\, e\, \eta^{1/2} p(\eta) \leq \theta_{\mathrm{PW}}$, a smallness condition of the kind of (6), which is added to the range; the cluster fields $\tilde{A}_K = A^{(j)} - B_K$ of Definition 5.25 have the same lattice derivatives, hence the same θ_j.

Proof. The proof has four steps: we first prove (PW1) and (PW2) at a constant background through the image sums and the slice factorisation; we then compare H_A at each live site with the constant field equal to A there by the resolvent identity; we bootstrap (PW1) and then (PW2) at the background A from this comparison; and finally we obtain the face clause (PW3) by telescoping from the Dirichlet plane.

Constant background. A constant field B on the simply connected cube is pure gauge, $B = d\lambda$, and $G_B = U_\lambda G_0 U_\lambda^*$ with U_λ the pointwise rotation by $\exp(e_j\vartheta q\lambda)$, so B-covariant operations correspond to ordinary ones. At $A = 0$ the propagator is the odd image sum of the free f_D propagator over the group generated by the reflections in the planes F (the image sums of Definition 5.23), the images being lattice isometries with $|x-gw| \geq |x-w|$ coordinatewise and at most $C(1+r)^3$ of them within distance r; each slice factorises as $G_{(i)} = \mathfrak{a}_i^2\rho_i^{-4}(G_iQ_i^*)C^{(i)}(Q_iG_i)$ [25, (2.17)], an operation at each end acting on its own outer factor, and the profile bounds of the free slices give $|O_xO'_wG_{(i)}(x,w)| \leq C\rho_i^{-1-o-o'}e^{-\kappa|x-w|/\rho_i}$; the sum over the slices, $\sum_i \rho_i^{-a}e^{-\kappa r/\rho_i} \leq C(r+\vartheta)^{-a}e^{-\kappa r/2}$, gives (PW1) and (PW2) at a constant background with the rate $\delta_1 := \kappa/4$.

Comparison at each point. For a live site x let B_x be the constant field equal to A on the bonds at x and $V_x := H_{B_x} - H_A$. Writing the hopping terms through covariant differences, $V_x = W_0 + \sum_{\pm\mu} W_{\pm\mu}D_{B_x,\pm\mu} + V_{\mathrm{blk}}$ with $|W_0(w)|, |W_{\pm\mu}(w)|, |\partial W_{\pm\mu}(w)| \leq C\theta_j(1+|w-x|)^2$ and a block-local V_{blk} with $|V_{\mathrm{blk}}(w,w')| \leq C\theta_j(1+|w-x|)$, because $e_j|A(b)-B_x| \leq \theta_j(1+|w-x|)$ on a bond b at w (integrate ∂A along an axis path from x); the resolvent identity $G_A = G_{B_x} + G_{B_x}V_xG_A$ is exact on the finite lattice, and an A-covariant operation at x differs from the B_x-covariant one by at most $C\theta_j(|f| + |Df|)$ on $E(O_x)$.

Bootstrap. Let Φ be the supremum over operations of order ≤ 1 and over x, z of $e^{\delta r}\min(1, r+\vartheta)^{1+o}|O_xG_A(x,z)|$, $\delta := \delta_1/2$, finite on the finite lattice. Inserting the resolvent identity, the middle term is a convolution of the constant-background kernel with V_xG_A, which the definition of Φ bounds by $C\theta_j(1+|w-x|)^3\Phi\min(1, |w-z|+\vartheta)^{-2}e^{-\delta|w-z|}$; the convolution bound $\sum_w \vartheta^3(|x-w| + \vartheta)^{-a}(|w-z| + \vartheta)^{-b}e^{-c|x-w|-c|w-z|} \leq C(r+\vartheta)^{3-a-b}(1 + |\ln(r+\vartheta)|)e^{-cr}$ for $a + b \geq 3$, the exponential $e^{-\delta_1|x-w|}$ of the constant-background kernel absorbing the polynomial growth $(1+|w-x|)^3$, gives $\Phi \leq C_0 + C_1\theta_j\Phi$, hence $\Phi \leq 2C_0$ once $\theta_j \leq \min(1, 1/(2C_1))$; the polynomial weight at large r is recovered from the rate, $\delta_P := \delta_1/4$. An operation of order $1+\alpha$ needs no bootstrap, the right side of its identity involving Φ only. *The mixed bound (PW2).* Let Φ' be the maximum over pairs of operations and over the pairs (x,z) with $r := \mathrm{dist}(E(O_x), E(O'_z)) > 0$ of $e^{\delta_P r}\min(1,r)^{s_*}|O_xO'_zG_A(x,z)|$, finite on the finite lattice. Lattice-scale pairs, $r \leq 8\vartheta$, need no bootstrap: a difference at x costs $\leq 2\vartheta^{-1}$ and a Hölder quotient $\leq 2\vartheta^{-\alpha}$ times differences, so (PW1) in its second-variable form gives $|O_xO'_zG_A(x,z)| \leq C\Phi\vartheta^{-3-2\alpha}$, and $\min(1,r)^{s_*} \leq (8\vartheta)^{4+2\alpha}$ for $\vartheta \leq \frac{1}{8}$ (for $\frac{1}{8} < \vartheta \leq \frac{1}{5}$, $\vartheta^{-3-2\alpha} \leq 5^{3+2\alpha}$): this is where $s_* = 4 + 2\alpha$ enters. For $r > 8\vartheta$ insert the resolvent identity at x and split the middle term by a cut-off $\chi_z(w) := \chi(\mathrm{dist}(w, E(O'_z))/\varrho)$ at the scale $\varrho := \min(\max(\frac{1}{16}, 2\vartheta), r/4)$, so that $2\vartheta \leq \varrho \leq r/4$ and $\varrho \geq \min(1,r)/64$. The far part (the terms of V_x with a difference on G_A, times $1-\chi_z$) is bounded through Φ': the bond of the difference has distance $\geq \varrho/2 \geq \min(1,r)/128$ from $E(O'_z)$, which gives $\leq C\theta_j(128^{s_*}\Phi' + \Phi)\min(1,r)^{-s_*}e^{-\delta_P r}$. The near part (the same terms times χ_z) is summed by parts in w, exactly and with no boundary term, since $g := O'_zG_A(\cdot, z)$ vanishes on F and χ_z has compact support: the difference moves onto $[O_x^{(B_x)}G_{B_x}](x,\cdot)\chi_zW$, whose adjoint difference is $\leq C\theta_j\min(1,r)^{-2-o}e^{-\delta_1 r/4}$ on $\mathrm{supp}\,\chi_z$ (there $\mathrm{dist}(E(O_x), w) \geq r/2$), while $\sum_{w\in\mathrm{supp}\,\chi_z}\vartheta^3|g(w)| \leq C\Phi$ by (PW1), the singularity $(\mathrm{dist}(w, E(O'_z)) + \vartheta)^{-1-o'}$ with $1+o' \leq 2+\alpha < 3$ being integrable over a set of diameter ≤ 1; so the near part is $\leq C\theta_j\Phi\min(1,r)^{-s_*}e^{-\delta_P r}$. The zero-order and block terms of V_x carry no difference on G_A and are bounded by (PW1) and the convolution bound; the term $E_xO'_zG_A$ by (PW1) and Φ'; and the leading term $O_x^{(B_x)}O'_zG_{B_x}$ by the constant-background bound, since $1+o+o' \leq 3+2\alpha \leq s_*$. Altogether $\Phi' \leq C\Phi + C_0 + C\theta_j(128^{s_*}\Phi' + \Phi)$, hence $\Phi' \leq 2(C_0 + C\Phi)$ once $C\theta_j 128^{s_*} \leq \frac{1}{2}$; with $\Phi \leq 2C_0$ this is (PW2), and θ_{PW} is the smaller of the two thresholds.

The face clause (PW3). Let $x_P \in \mathsf{F}$ be the foot of x on the nearest Dirichlet plane, $|x - x_P| = t(x)$. Since $G_A(x_P, \cdot) = 0$, telescoping along the $t(x)/\vartheta$ normal bonds from x_P to x (each a covariant difference times a unimodular transporter) gives $|O'_z G_A(x,z)| \le t(x) \max_b |D_{A,b} O'_z G_A(\cdot, z)|$ over those bonds, each at distance $\ge r - t(x) \ge r/2$ from $E(O'_z)$, so (PW2) (or (PW1) if O'_z is the identity) bounds the maximum by $C \min(1, r/2)^{-s_*} e^{-\delta_P r/2}$; for $t(x) \ge 1$ the plain bounds already have the stated form. At $A = 0$ or at a constant A the constant-background bounds and this telescoping give the three clauses directly, with no condition. □

The range. The constants C_{box}, $C_\flat$ and the range (6) of η, with its threshold $\eta_2 \le \eta_1$, are those of Definition 5.25; all three conditions hold for η small because $p(\eta)$ grows slower than every power of η^{-1} and faster than every logarithm. Since $C_{\mathrm{box}} p$ bounds the side of every cluster box $\square'_K$ and also the side $\le 5p + 4M + 2$ of the one box $\square'$ of a localisation with $R \le p$ (Lemma 6.3(e)), the third condition contains the convergence condition $C e \eta^{1/2} p \sqrt{3}\,(5p + 4M + 2) \le \frac{1}{2}$ of the expansion in $\tilde{A}$ on that one box (the operator bound of Definition 5.25 with Ω_K replaced by Ω), as well as the one on every cluster box. J_* is taken so large that $\eta = 5^{-(k-j)} \le \eta_2$ and $C_{\mathrm{PW}} e \eta^{1/2} p(\eta) \le \theta_{\mathrm{PW}}$, the condition of Lemma 6.2 for the pointwise bounds (PW1)–(PW3) on the background propagator and its first and second operations, whenever $k - j \ge J_*$; the least such integer is the largest of the thresholds of the two geometric conditions of (6), of the convergence conditions of the expansions in the cluster fields and on the one box, and of the smallness of the bounds at the background.

Lemma 6.3 (the source vertices and the clusters)**.** *Let $\eta \le \eta_2$, let H be a member of $\mathcal{N}_h$, s an h-vertex of H with centre z_s, and τ a tuple of H. Then:* (a) $t(z_s) \ge \eta^{-1} - 3$. (b) *If τ is in zone B, then $R(\tau) \ge \eta^{-1}/2$.* (c) *If τ is in zone B, then τ has at least two clusters.* (d) *Every cluster K containing an h-vertex has* $\operatorname{dist}(\square_K, \mathsf{F}) \ge 2p + M + 1$, $\Omega_K = \square'_K$ *and* $\operatorname{dist}(\square_K, \partial\Omega_K) \ge 2p$*: it is a zone-I cluster. Hence every zone-B cluster is source-free.* (e) *Every $\square'_K$ has side $\le c_{\mathrm{cl}} N_* + 4p + 6M$, and the one box $\square'$ of a localisation with $R \le p$ has side $\le R + 4p + 4M + 2 \le 5p + 4M + 2$; both are $\le C_{\mathrm{box}}\, p$, where $C_{\mathrm{box}} = \max(c_{\mathrm{cl}} N_* + 6M + 4,\ 4M + 7)$ is the constant of Definition 5.25. The same holds with a curvature vertex in place of the h-vertex.*

Proof. (a) $h(x) \ne 0$ forces ℓ^∞-$\operatorname{dist}(x, \mathbb{R}^3 \setminus C_J) > 1$ in physical units, that is $> \eta^{-1}$ unit lengths; the planes F lie within one fine spacing $5^{-j} \le 1$ of ∂C_J, and the vertex lies in $\operatorname{supp} \chi_{z_s}$, within $\frac{2}{3}\sqrt{3} < 2$ of its centre. (b) A tree of total length R joins the points of $P(\tau)$, so all of them lie within R of z_s; the vertices lie within $\frac{2}{3}$ of these points in each coordinate, so $\square$ lies in the sup-norm cube of half-side $R + M + 1$ about z_s, and $\square'$, which adds at most $2p + M$ on each face (as in (e)), in that of half-side $R + 2p + 2M + 1 \le R + 2p + 3M$. If $\square'$ meets F, then $t(z_s) \le R + 2p + 3M$, and with (a) and (6) $R \ge \eta^{-1} - 2p - 3M - 3 \ge \eta^{-1}/2$. (c) With one cluster $\operatorname{diam} P(\tau) \le c_{\mathrm{cl}}(n_\tau - 1)$ (the diameter bound of the cut by clusters), so $R(\tau) \le (n_\tau - 1) \operatorname{diam} P(\tau) \le c_{\mathrm{cl}}(N_* - 1)^2 < \eta^{-1}/2$ by (6), against (b). (d) If $s \in K$, every point of P_K lies within $c_{\mathrm{cl}}(n_\tau - 1)$ of z_s, so $\square_K$ lies in the cube of half-side $c_{\mathrm{cl}} n_\tau + M$ about z_s and $\square'_K$ in that of half-side $c_{\mathrm{cl}} n_\tau + 2p + 3M$. By (a) and (6), $t(z_s) - (c_{\mathrm{cl}} n_\tau + 2p + 3M) \ge \eta^{-1} - 3 - c_{\mathrm{cl}} n_\tau - 2p - 3M > 2p + M + 1$. So $\square'_K$ misses F, it is not truncated, it contains every large block within $2p$ of $\square_K$, and $\operatorname{dist}(\square_K, \mathsf{F}) \ge t(z_s) - c_{\mathrm{cl}} n_\tau - M \ge 2p + M + 1$. (e) $\square_K$ has side $\le \operatorname{diam} P_K + \frac{4}{3} + 2M \le c_{\mathrm{cl}} N_* + 2M$ (the vertices lie within $\frac{2}{3}$ of the points in each coordinate, $\operatorname{diam} P_K \le c_{\mathrm{cl}}(N_* - 1)$ by the diameter bound of the cut by clusters, and $c_{\mathrm{cl}} > \frac{4}{3}$) and $\square'_K$ adds at most $2p + M \le 2p + 2M$ per side, as for the one box below; truncation at ∂T_η only shrinks it. For the one box: the points of $P(\tau)$ lie within R of one another, and every vertex lies in the support of its localisation weight, within $\frac{2}{3}$ of its centre in each coordinate; so $\square$ has side $\le R + \frac{4}{3} + 2M \le R + 2M + 2$, and $\square'$ adds on each face at most the large blocks within $2p$ of $\square$, that is at most $2p + M$; hence its side is $\le R + 4p + 4M + 2$. Since $p \ge 1$, $c_{\mathrm{cl}} N_* + 4p + 6M \le (c_{\mathrm{cl}} N_* + 6M + 4)p$ and $5p + 4M + 2 \le (4M + 7)p$. The curvature

source f is supported at the same distance from $\mathbb{R}^3 \setminus C_J$, so the same arithmetic applies to its vertex. □

Lemma 6.4 (cut lines vanish on the Dirichlet planes)**.** *Let ℓ be a scalar, dressed or external cut line of the cut by clusters with its end u in a cluster K, in either model, at the background $A^{(j)}$ or at zero background. Put $m(u') := \min(1, t(u'))$ for a scalar or an external line and $m(u') := m_\alpha(u'; 1) = \min(1, t(u'))^\alpha$ for a dressed line, $m_\alpha(x; \rho) := \min(1, t(x)/\rho)^\alpha$ being the Hölder gain of a slice propagator at a live site x at distance $t(x)$ from a face (half the exponent with a pair difference); in every case $m(u') \le m_\alpha(u'; 1)$.* (a) *For every evaluation point u' of the u-end with $t(u') \le \frac{1}{4}$ and every operation O_v at the far end (a difference or Hölder quotient of order $\le 1+\alpha$, or the identity) whose evaluation set E_v lies at distance $\ge \frac{1}{2}$ from u',*

$$|O_v \ell(u', \cdot)| \;\le\; C\, m(u')\, e^{-\delta \operatorname{dist}(u', E_v)}.$$

(b) *For a scalar or dressed cut line and evaluation points u', v' of its two ends, both undifferentiated, with $|u' - v'| \ge 1$ and $t(u'), t(v') \le \frac{1}{4}$,*

$$|\ell(u', v')| \;\le\; C\, m(u') \min(1, t(v'))\, e^{-\delta |u'-v'|} \;\le\; C\, m_\alpha(u'; 1)\, m_\alpha(v'; 1)\, e^{-\delta |u'-v'|}.$$

The constants depend on those of Lemma 5.28(b) and, for dressed lines, also on the Hölder constants of the slice propagators up to the faces, on the propagator bounds (PW1)–(PW3) of Lemma 6.2 and on the convolution bound for two kernels of the form of (PW1).

Proof. We first prove (a), for scalar and external lines by telescoping along the normal segment from the foot of u', and for dressed lines by a face clause for every slice of the first factor followed by a convolution bound; we then prove (b) by telescoping at the v'-end.

Each of the three kinds of line vanishes when its u-end lies on F: a scalar line by the Dirichlet condition of the cube propagator, a dressed line and an external line through their first factor, which is such a propagator.

(a) for scalar and external lines. Let $u'_{\mathsf{F}} \in \mathsf{F}$ be the foot of u' and telescope along the $t(u')/5^{-j}$ bonds of the normal segment from u'_{F} to u'; the parallel transporters along a bond are unimodular, so $|O_v \ell(u', \cdot)| \le t(u') \max_b |O_v D_b \ell(\cdot, \cdot)|$, the maximum over the bond differences D_b on that segment. Every such bond lies within $\frac{1}{4}$ of u', hence at distance $\ge \frac{1}{4}$ from E_v, and the mixed bound of Lemma 5.28(b) ($C^{1,\alpha} \otimes C^{1,\alpha}$ at unit scale in both arguments, decay $e^{-\delta_1 |u-v|}$, no distance condition) bounds each by $Ce^{-\delta \operatorname{dist}(u', E_v)}$.

(a) for dressed lines. Write the line as $\sum_x \varepsilon_j^3 G_A(u', x)\, c_{\ge j}\, G_A(x, \cdot)$, with $\varepsilon_j = 5^{-j}$ the fine spacing (chains iterated likewise). Its first factor is not cut off near u', so the face factor is taken from the slices of the first factor at all scales. Each slice $G_{(i)}$ obeys the face clause (the Hölder gain of a slice propagator at a live site) $|G_{(i)}(u', x)| \le C\rho_i^{-1} m_\alpha(u'; \rho_i) e^{-\delta |u'-x|/\rho_i}$: its proof uses only the factorisation $G_{(i)} = a_i^2 \rho_i^{-4} (G_i Q_i^*) C^{(i)} (Q_i G_i)$ [25, (2.17)], the Hölder clause of the outer factor $G_i Q_i^*$ up to the faces and its vanishing on F, and the exponential decay of the coarse factor $C^{(i)}$. At the background $A^{(j)}$ the outer-factor clauses hold at every level i by [26, (5.2)–(5.3)] (Definition 5.23), the background being regular at level i. The coarse factor $C^{(i)}(A) = K_i(A)^{-1}$ of Definition 5.21 is the lower right block of the inverse of the block operator with diagonal entries $M(A)$ and $D_i := a_i + a5^{-2} Q^* Q$ and off-diagonal entries $-a_i Q_i^*$ and $-a_i Q_i$, which is positive, its quadratic form being $\langle g_1, M_0(A) g_1 \rangle + a_i \|g_2 - Q_i g_1\|^2 + a5^{-2} \|Q g_2\|^2$. The Schur complement of D_i, with $QQ^* = I$, $QQ_i = Q_{i+1}$ and $a_{i+1}^{-1} = a^{-1} + a_i^{-1} 5^{-2}$, gives $C^{(i)} = D_i^{-1} + a_i^2 D_i^{-1} Q_i G'_{i+1} Q_i^* D_i^{-1}$ with $G'_{i+1} := (M_0(A) + a_{i+1} 5^{-2} Q_{i+1}^* Q_{i+1})^{-1}$, the propagator of level $i+1$ in the units of level i. Since $D_i^{-1} = a_i^{-1}(I - Q^*Q) + (a_i + a5^{-2})^{-1} Q^* Q$ is block-local and bounded, the decay of $C^{(i)}$ is the outer-factor bound at level $i+1$, applied to the block-local bounded sources $Q_i^* D_i^{-1} \delta_y$. Since

$m_\alpha(u';\rho_i) \le \rho_i^{-\alpha} m_\alpha(u';1)$ for $\rho_i \le 1$, the sum over the slices gives

$$|G_A(u',x)| \;\le\; C\, m_\alpha(u';1)\,(|u'-x|+\varepsilon_j)^{2-d-\alpha}\, e^{-\delta|u'-x|/2},$$

a kernel that is locally summable in $d=3$. The sum over x against the constant $c_{\ge j}$ and the last factor $O_v G_A(x,\cdot)$, which (PW1) of Lemma 6.2 bounds, is the convolution of two kernels of the form of (PW1), and the convolution of two such kernels, each locally summable in $d=3$ and exponentially decaying, is bounded by $C\, m_\alpha(u';1)\, e^{-\delta\,\mathrm{dist}(u',E_v)}$, since $\mathrm{dist}(u',E_v) \ge \frac12$.

(b). Put $f(w) := \ell^{\flat}(u',w) = \bar\psi(u'-w)\ell(u',w)$; then $f(v') = \ell(u',v')$, since $|u'-v'| \ge 1$. Let $v'_{\mathsf F} \in \mathsf F$ be the foot of v'; $f(v'_{\mathsf F}) = 0$ by the Dirichlet condition in the second variable (for a dressed line through its last factor). Since $|u'-v'| \ge 1$ (on the supports even $|u'-v'| \ge R_{\mathrm{cl}} - 4\cdot 5^{-j} > 1$, Lemma 5.28(a)) and $t(v') \le \frac14$, every point of the segment from $v'_{\mathsf F}$ to v' lies at distance $\ge \frac34$ from u'. Telescope along its $t(v')/5^{-j}$ bonds, with unimodular transporters: $|f(v')| \le t(v')\max_b |D_b f|$. On each bond, $D_b f$ is $\bar\psi(u'-\cdot)D_b\ell(u',\cdot)$ plus the difference of the fixed smooth cut-off $\bar\psi$, of size $\le \|\nabla\bar\psi\|_\infty$, times $\ell(u',\cdot)$ at an end of the bond; $\bar\psi$ is differentiated once only. Every point involved lies at distance $\ge \frac34 - 5^{-j} \ge \frac12$ from u' and within $\frac14 + 5^{-j}$ of v', so (a), with $O_v := D_b$ and with O_v the identity, bounds both terms by $C\, m(u')e^{-\delta|u'-v'|}$. Hence $|\ell(u',v')| \le C\, m(u')\min(1,t(v'))e^{-\delta|u'-v'|}$, and the second inequality of (b) is $m(u') \le m_\alpha(u';1)$ and $\min(1,t) \le \min(1,t)^\alpha$. Where $t > \frac14$ at an end, the factor of that end is $\ge \frac14$: if $t(v') > \frac14$, (a) with O_v the identity gives the bound; if $t(u') > \frac14$, the telescoping at v' above, with $D_b f = D_b\ell^{\flat}(u',\cdot)$ bounded by Lemma 5.28(b), gives it; if both, Lemma 5.28(b) alone. □

Proposition 6.5 (the branch $R \le p$: zone I)**.** *Let H be a member of $\mathcal N_h$ and τ a localisation of H with $R \le p$; by Lemma 5.26(i) such a localisation lies in zone I. Then, for each model, on* $\operatorname{supp}\chi_j$*,*

$$|E^{(j)}(H;\tau)| \;\le\; C(1+k-j)^{p_0}\,\eta^{\rho(H)}\,p^{n_1}\,e^{-\delta R}, \tag{9}$$

which is [25, (3.38)] *for the renormalised family at the tuple, the factor $(1+k-j)^{p_0}$ being King's power of $\ln(L^k\varepsilon)^{-1}$* [25, p. 666]*, and across the pair (k,j), $(k+1,j+1)$, on the common unit lattice,*

$$|E'^{(j+1)}(H;\tau) - E^{(j)}(H;\tau)| \;\le\; C\,\eta^{\rho(H)-1/2}\,p^{n_1}\big[5^{-\gamma j} + \eta^{3+\sigma+1/2}\big]\,e^{-\delta R}. \tag{10}$$

The constants depend on the point of King's range, J, $\bar n$, $\|h\|_{C^1}$ and, for members with curvature vertices, $\max\|f_{\mu\nu}\|_{C^{2,\alpha}}$*; not on j, k, τ.*

Proof. A zone-I localisation with $R \le p$ is reduced to one King box: King's steps [25, (3.43)–(3.55)] on the box $\square'$ of side $\le 5p+4M+2$, as in Step 1 of the proof of Proposition 6.7 for a single cluster (the constant shift $B = A_j(x_0)$ and $\tilde A = A_j - B$; the gauge transformation [25, (3.46)], which puts the phases at the scalar ends of the external objects and acts trivially on the h-vertex, whose two scalar ends carry cancelling phases; the Taylor expansion of the propagator in $\tilde A$ to a fixed order with the explicit remainder [25, (3.48)–(3.49)], convergent by the third condition of (6); the resolvent identity [25, (3.52)] and the splitting [25, (3.54)], which produce the lines $\delta G_j(\square', A)$). The result is a finite sum over graphs $\tilde H$ with at most $C(\bar n)$ new vertices carrying the common block field $\tilde A$, the same graphs with the same fields for both members of the pair. Each $\tilde H$ equals its free f_D-flow version plus terms carrying an image line of any prescribed degree, of size $\le Ce^{-\delta_0 p/2} \le C\eta^{3+\sigma+1/2}$ times its bound (the exponent of the logarithm in $p(\eta)$ being $6 \ge 2$), and the $\delta G_j(\square', A)$ lines are $O(e^{-\delta_0 p})$ by [26, (5.3)], since $\mathrm{dist}(\square, \partial\square') \ge 2p$. The free version obeys (9) by Theorem 5.31(a) with the root held at the h-vertex, which is [25, (3.38)] with the depth factor $(1+k-j)^{p_0}$: every prefix component has positive degree, the stages containing the h-vertex by the margin (8), the others by the degree count of Theorem 5.30. For (10) the two

members consist of the same elements and are compared element by element as in [25, pp. 663–665]: a line of the finer member of index -1 gives $5^{-\gamma j}$ through the first ultraviolet index sum ($\gamma < \alpha_0$); otherwise the difference is a telescoping sum over the factors of the graph, one factor replaced at a time, the external and internal lines by [25, (3.71), (3.73)–(3.75)] for the free f_D flow, the block-product point sums by Theorem 5.31(b), the local coefficients of the rewritten units at the rate γ_c, the h-vertex by sampling, $|h(x') - h(x)| \le \sqrt{3}\,\varepsilon_k \|\nabla h\|_\infty \le C 5^{-\gamma j} \|h\|_{C^1}$, the only factor that depends on h, and the phases by [25, (3.72)], which costs in addition the factor $\eta^{-1/2}$ of the field bound [25, (3.2)]: this is the power $\rho(H) - \frac{1}{2}$ of [25, (3.56)–(3.57)]; the remainders, the δG_j lines and the image terms are bounded separately in each model and give the term $\eta^{3+\sigma+1/2}$. (A zone-I localisation with $R > p$ has at least two clusters; the localisations with $R > p$ are the subject of Proposition 6.7, the cut by clusters.) □

The following lemma treats a zone-B cluster whose external field is a line cut by clusters; the cluster factor $\mathfrak{E}_K$ is that of Step 3 of the proof of Proposition 6.7, and the lemma is used in Step 4 there.

Lemma 6.6 (the zone-B bound with a cut line as the external field)**.** *Let K be a zone-B cluster of a tuple τ with at least two clusters. Then K is source-free, every connected component of every cluster graph H_K met in the nesting carries a cut-line leg, and the renormalised family of H_K, at $B_K = 0$ and $A' = \tilde{A}_K$ on Ω_K, rewritten by Definition 5.27 with the cut lines as external fields of index j and with the factor $m_\beta(\cdot;1)$ of every cut-line end charged by the charging rule (step (2) of the proof of Theorem 5.37) moved to the remainder vertex before any bound is taken, obeys Theorem 5.37(a) with no vertex held: for either model, on* $\operatorname{supp}\chi_j$*,*

$$\sum_{\text{positions of } K} \mathfrak{E}_K \;\le\; C\,\eta^{\rho_K}\,p^{n_K}\,(1+k-j)^{p_{0,K}}.$$

Proof. K is source-free because the source and curvature vertices lie in zone-I clusters, and every component of H_K carries a cut leg by the degree census of the cut. The zone-B bound, Theorem 5.37, is stated for a renormalised family of King's expansion at zero background with at least one external scalar field; here the family is that of H_K on the box Ω_K of side $O(p)$, its internal lines the zero-background Dirichlet lines, and the external field of every component is a cut leg. It remains to check that the proof of Theorem 5.37 uses an external scalar leg only in ways a cut line satisfies, and the argument has one non-trivial point, the *order of operations*: the rewriting acts on the cut-line legs as functions, entering through their sup and lattice $(1,\alpha)$-norms on Ω_K up to the face bonds in the gauge of $\tilde{A}_K$ (a covariant difference of a cut line is the phase times a difference covariant in $A^{(j)}$ plus a multiplier bounded by $e_j|\tilde{A}_K| \le C$); where the charging rule charges a cut-line leg at an end q, the vanishing factor $m_\beta(q;1)$, $\beta < \alpha/2$, demanded at a remainder vertex of degree -1 is provided by the face factor $m(q) \le m_\alpha(q;1)$ of the cut line with its decay $e^{-\delta\,\mathrm{dist}}$, and it is moved to the remainder vertex before any bound is taken, the leftover $\ell/m_\beta(q;1) \le C m_{\alpha-\beta}(q;1) e^{-\delta\,\mathrm{dist}} \le C e^{-\delta\,\mathrm{dist}}$ keeping the standard bound of the leg. A cut line joining two zone-B clusters may be charged at both ends, by the rules of the two clusters, and then both face factors are provided; equivalently the rule is run once on the multigraph whose nodes are the degree -1 remainder vertices of all zone-B clusters and whose edges are the member and cut lines joining two of them: every node has degree at most 2, every component is a path or a cycle, and orienting each component and charging every node to its left leg end charges every line at most once and no cut line at both ends. A cycle of degree -1 remainder vertices of H_K would use both scalar legs of each of its nodes and so be a whole component of H_K without a cut leg, which does not occur; hence no vacuum component is dropped. In the degree count a cut leg, scalar or photon, is an undifferentiated external leg of index j with no credit, so every source-free stage has an external field and the vacuum exception

is absent; a photon cut line is an even image sum of free lines and has the $C^{1,\alpha}$ regularity up to the Neumann faces that the photon two-point sites need; the infrared lines lie inside free counterterm units within one cluster or on dressed cut lines, the latter costing $(1+k-j)^5$ per unlocalised vertex. No step uses a source vertex, a comparison across the pair or a localisation of diameter larger than $C(\bar n)$. The remaining estimates are those of the proof of Theorem 5.37: the bulk-type graphs by the degree bound with the Ward rows run inside H_K, the remainder graphs by the ordered induction with no vertex held, the last vertex summed over Ω_K with $|\Omega_K| \le Cp^3$, and one factor $C(1+k-j)$ per infrared sum. □

Proposition 6.7 (the branch $R > p$: the cut by clusters)**.** *Let $\eta \le \eta_2$, let H be a member of $\mathcal{N}_h$ and τ a tuple of H with at least two clusters. By Lemma 6.3(c) this covers every zone-B tuple, and by the diameter bound of the cut by clusters and (6) every tuple with $R > p$. Then, for each model separately, on* $\operatorname{supp}\chi_j$*,*

$$|E_N^{(j)}(H;\tau)| \;\le\; C(1+k-j)^{p_0}\,\eta^{\rho(H)}\,p^{n_1}\,e^{-\delta\,\operatorname{dist}(P(\tau))}, \tag{11}$$

E_N the field-dependent part (the whole value for a field-independent member); this is [25, (3.38)] *at the tuple, with the same logarithm. Moreover, for every unit cube y_* and $0 < \delta' \le \delta/2$,*

$$\sum_{\substack{\tau \text{ in zone } B\\ \text{with an } h\text{-vertex in } y_*}} |E_N^{(j)}(H;\tau)|\,e^{\delta' R(\tau)} \;\le\; C(1+k-j)^{p_0}\,\eta^{\rho(H)}\,p^{n_1}\,e^{-\delta/(8\eta)}. \tag{12}$$

The constants depend on the point of King's range, J, $\bar n$, $\|h\|_\infty$ and, for members with curvature vertices, $\max\|f_{\mu\nu}\|_{C^{2,\alpha}}$*.*

Proof. We sketch the argument first. At a tuple with at least two clusters every connected component of every cluster graph carries a cut-line leg, so the vacuum members and the component-wise cancellation of [25, p. 667] do not arise, and $E(H;A^{(j)};\tau)$ and $E(H;0;\tau)$ are bounded separately; at zero background the cut lines are the cube propagators at $A=0$ and $B_K=0$. Fix one of the two terms.

Step 0: regularised cut lines. Replace every cut line ℓ by $\ell^\flat$; by Lemma 5.28(a) this changes nothing.

Step 1: reduction per cluster. Each H_K is reduced by [25, (3.43)–(3.55)] with its own box Ω_K, centre x_0^K, gauge B_K and field $\tilde A_K = A^{(j)} - B_K$ on Ω_K (the cut by clusters of Definition 5.24), its cut lines keeping the background and receiving the unitary factors [25, (3.46)] at their ends. The expansion of the reduced propagator in $\tilde A_K$ converges on Ω_K: $e_j \sup_{\Omega_K}|\tilde A_K| \le Ce_j p \operatorname{diam}\Omega_K \le C_\flat e\eta^{1/2}p^2 \le \frac12$ by the difference bound [25, (3.45)], Lemma 6.3(e) and (6): this is the reduction of the branch $R \le p$ on a box of side $O(p)$, run inside each cluster. By construction of the cut by clusters the partition, the boxes, the gauges and the set of cut lines are the same for every member of every family met in the nesting and for every position, so renormalisation families are formed inside each H_K. A zone-I cluster, in particular every cluster containing a source vertex (Lemma 6.3(d)), satisfies $\operatorname{dist}(\square_K, \partial\Omega_K) \ge 2p$ and $\Omega_K \cap \mathsf{F} = \emptyset$, so H_K is its free f_D-flow version plus terms carrying an image line of any prescribed degree (replacing the cube slices of a zone-I graph by the free f_D slices leaves image-line terms of size $e^{-4\delta_0 p/\rho_i}$, which is $O(\eta^N)$ for every N because $p(\eta)$ is a power ≥ 2 of $\ln\eta^{-1}$), with the cut lines as external legs; the vertices of the expansion in $\tilde A_K$ that lie in $\square'_K \setminus \square_K$ are joined to a localised vertex of K by a path of lines one of which is long, and that line provides a credit of any order, as in King's Ward-row reduction [25, (3.39)–(3.55)] run inside H_K with the cut lines as external legs, where each Ward-row component is zero, a local polynomial of degree $\le -D(S)$ with coefficients $\le C\rho^{D(S)+|a|}$, or a remainder with a line credit ρ^θ against a leg debit $\le\theta$, keeping $D'(S) > 0$. A zone-B cluster is source-free and is rewritten by Definition 5.27 within Ω_K.

Step 2: the cut-line factors. Every evaluation point of a cut-line end in an output term is joined to a localised anchor of its own cluster by at most $C(\bar n)$ lines of that cluster's rewritten units (the Ward-row reduction inside the cluster), and by Lemma 5.28(b) the phase-dressed cut line satisfies $|O_u O_v \ell^{b,\mathrm{ph}}| \le C e^{-\delta_2 d(K,K')} \times$(transfer factors on the lines of those paths), for all operations O_u, O_v of order $\le 1+\alpha$ at its ends, and $\le C$ crudely. The quotient graph (clusters as nodes, cut lines as edges) is connected because H is; bound the cut lines of one spanning tree T of it by the first bound and the others by C.

Step 3: factorisation. The output term is bounded by $\prod_K \mathfrak{E}_K \times \prod_{\ell\in T} C e^{-\delta_2 d(K,K')} \times C^{\#\text{cut lines}}$, where $\mathfrak{E}_K$ depends on the positions of K only; the transfer factors cost one extra decay draw per line of a cluster, within the decay budget of $\bar n + 4$ draws per line of the Ward-row reduction.

Step 4: the cluster factors. Zone-I clusters. The cluster graph is a free f_D-flow graph with its cut lines as undifferentiated external legs of unit scale (Lemma 5.28(b)). For the cluster containing the h-vertex, the census is that of Proposition 6.5; since the cluster carries a cut leg, every component has an external leg and the excepted field-independent family does not occur. For the source-free zone-I clusters, the degree count holds at every tuple by Theorem 5.30 and the Ward-row reduction inside the cluster, and block points by the split count of Theorem 5.31 (a block point with s scalar and p_γ photon fields weighs $3 - s/2 + p_\gamma/2 \ge 5/2$, which makes every block-point component convergent). The multiscale bound Theorem 5.31(a) with the root held, summed over the root in Ω_K (volume $\le Cp^3$ by the cluster geometry), gives $\sum_{\text{positions of } K} \mathfrak{E}_K \le C\eta^{\rho_K} p^{n_K} (1+k-j)^{p_{0,K}}$, with ρ_K the coupling power of the vertices of K. *Zone-B clusters.* The cluster is source-free (Lemma 6.3(d)); apply Theorem 5.37(a) and Theorem 5.33 with no vertex held to the renormalised family of H_K, localised at P_K with the box Ω_K of side $O(p)$; the hypothesis "at least one external field" is the cut leg. That the proof of Theorem 5.37 uses an external scalar leg only in ways a cut line satisfies, the borrowed factor at a $D' = -1$ remainder vertex included, is Lemma 6.6; after the borrowing, the cut-line factor left in the output term is an undifferentiated evaluation bounded by $Ce^{-\delta \,\mathrm{dist}}$ (Lemma 6.4), which Steps 2–3 treat like every other cut-line factor. The resulting bound is again $\le C\eta^{\rho_K} p^{n_K} (1+k-j)^{p_{0,K}}$, the logarithm coming from the infrared sums inside Ω_K.

Step 5: the tree. By the tree bound $\mathrm{dist}(P(\tau)) \le \sum_{\ell\in T} |\ell| + c_{\mathrm{cl}} n_\tau$ of the cut by clusters (Definition 5.24), $\prod_{\ell\in T} e^{-\delta_2 d(K,K')} \le e^{\delta_2 c_{\mathrm{cl}} n_\tau} e^{-\delta_2 \,\mathrm{dist}(P(\tau))}$.

Step 6: exponents. $\rho(H) = \sum_K \rho_K$, the count being vertex-local (an unlocalised mass-counterterm vertex on a cut dressed line keeps its own coupling power); $n_1 := \sum_K n_K$ and $p_0 := \sum_K p_{0,K}$ plus the dressed-line factor of Lemma 5.28(b); the number of clusters is $\le N_*$ and $\delta := \delta_2$. This is (11).

The root sum. On the tuples of (12), $R(\tau) \ge \eta^{-1}/2$ by Lemma 6.3(b). The number of tuples of H with a point in y_* and $R \in [n, n+1)$ is $\le C(n+1)^{3N_*}$, and $e^{\delta' R} e^{-\delta R} \le e^{-\delta R/2}$. So the sum is $\le C(1+k-j)^{p_0} \eta^{\rho(H)} p^{n_1} \sum_{n \ge \eta^{-1}/2 - 1} (n+1)^{3N_*} e^{-\delta n/2} \le C'(1+k-j)^{p_0} \eta^{\rho(H)} p^{n_1} e^{-\delta/(8\eta)}$, since $(n+1)^{3N_*} e^{-\delta n/4} \le C$ and $\eta^{-1} \ge 8$. Members with curvature vertices: the curvature vertex lies in a zone-I cluster (Lemma 6.3(d)), is invariant under the reduction, being gauge invariant, and is bounded through a discrete summation by parts against an explicit weight w with $d^* w = 0$. □

Proof of Lemma 6.1. For a member H of $\mathcal{N}_h$ and $J_* \le j \le k - J_*$ (so that $\eta \le \eta_2$), the proof runs as follows, using the two propositions above (Propositions 6.5 and 6.7). *First clause.* (9) is [25, (3.38)] with the factor $(1+k-j)^{p_0}$ for the localisations in zone I, which are those at distance from the faces larger than the small-field scale $2p(\eta)$ of the localisation boxes (Definition 5.22). *Second clause.* A tuple with one cluster has $R \le c_{\mathrm{cl}}(N_* - 1)^2 \le p$, lies in zone I by Lemma 5.26(i), and obeys (9); a tuple with at least two clusters obeys (11). Summed over the localisations sharing the cube y_* of an h-vertex: the zone-I tuples by $\sum_n (n+1)^{3N_*} e^{-(\delta - \delta')n} \le C$, the zone-B tuples

by (12). This is [25, (3.38)] with the factor $(1+k-j)^{p_0}$. *Third clause.* Here $J_* \le j \le k - J_*$ and $\eta = 5^{j-k}$. Split the localisations at $R = p$. Those with $R \le p$ are in zone I and are reduced on one box of side $\le 5p + 4M + 2$; there (10) is the comparison of [25, pp. 663–665] made element by element, in which the only element depending on h is the coefficient of the h-vertex, replaced by sampling at the cost $C\varepsilon_k\|\nabla h\|_\infty$, the points of the block products being never source vertices. Sum (10) over the localisations, the position sum costing $\eta^{-3}|C_J|$ in the branch $R \le p$: the result is $\le C\eta^{\rho(H)-1/2}p^{n_1}(1+k-j)^{p_0}\eta^{-3}|C_J|[5^{-\gamma j} + \eta^{3+\sigma+1/2}]$, whose first term is $\le C5^{-\gamma' j}\eta^{-\beta'}|C_J|$ for every $\gamma' \le \gamma$ and $\beta' > \frac{7}{2} - \rho(H)$, and whose second term is $\le C\eta^{\rho(H)+\sigma}p^{n_1}(1+k-j)^{p_0}|C_J| \le C\eta^{\sigma'}|C_J|$ for every $\sigma' < \sigma$, since $p^{n_1}(1+k-j)^{p_0}$ is bounded by a constant multiple of a power of $1+\ln\eta^{-1}$, hence $\le C_\epsilon\eta^{-\epsilon}$ for every $\epsilon > 0$. Those with $R > p$ have at least two clusters; bound the two members of the pair separately by (11), with the roots at the h-vertex, $\le C\eta^{-3}|C_J|$ of them. For the zone-I tuples among them the factor $e^{-(\delta-\delta')p/2}$ left by the position sum in the branch $R > p$ is $\le C_N\eta^N$ for every N, in particular $\le C\eta^{3+\sigma}$, since the exponent of the logarithm in $p(\eta)$ is $6 \ge 2$, and the sum is $\le C\eta^{-3}|C_J|\eta^{\rho(H)}p^{n_1}(1+k-j)^{p_0}\eta^{3+\sigma} \le C\eta^{\sigma'}|C_J|$; for the zone-B tuples (12) gives $\le C\eta^{-3}|C_J|\eta^{\rho(H)}p^{n_1}(1+k-j)^{p_0}e^{-\delta/(8\eta)} \le C_N\eta^N|C_J|$ for every N. In both cases $\rho(H) \ge 0$, and the external fields on $\operatorname{supp}\chi_j$ are accounted for in $\rho(H)$ by (8). Adding the three parts gives the third clause, with C independent of j and k. The constants depend on the point of King's range, so the lemma holds at every point of the range, in particular at the second mass M_2 of Section 6.1. □

6.3. **The estimates with scalar sources.** For a source $s = (u, h)$ put

$$\mathsf{S}(s) := \sum_i |x_i| \max_{\mu\nu}\|f_{i,\mu\nu}\|_{C^{2,\alpha}} + \|h\|_\infty,$$

a size; a compact family of sources is contained in $\{\mathsf{S}(s) \le S_0\}$.

Proposition 6.8 ((U$^+$): the small-field lines at a real scalar source)**.** *Fix $J \ge J_0$, $\alpha \in (0, \frac{1}{2})$ and $S_0 > 0$. There are k_0 and $C, \sigma > 0$, depending on the couplings, J, α, S_0 and on nothing else, such that for every real source $s = (u, h)$ with $\mathsf{S}(s) \le S_0$, every $k \ge k_0$ and both members $(k', j') \in \{(k, j(k)), (k+1, j(k)+1)\}$ of the diagonal pair,*

$$Z_{k'}(s) \;\ge\; e^{-C\eta_k^\sigma}\, I_{k'}^{(j')}(s) \qquad \text{and} \qquad \int P\, d\mu_{k',s} \;\le\; e^{C\eta_k^\sigma}\, I_{k'}^{(j')}(s). \tag{13}$$

The proof runs Bałaban's inductive small-field lower bound and the no-large-field term of his upper bound on King's cube with the source inserted through King's single vertex [25, (3.21)], whose coupling carries two powers of the running scale where the other vertices carry at most one and a half; every step of the argument is either blind to the source or a bound on the interaction families which that margin leaves unchanged, and the remaining bookkeeping is routine. These are the two small-field lines of (4) with the scalar source inserted into the small-field analysis through the vertex [25, (3.21)]; the large-field remainder at $s \ne 0$ is not part of the statement, and is provided instead by Lemma 6.21. The source enters the small-field steps only through its vertex, and the sign of h plays no part in them; the statement is applied below at non-negative scalar sources, where the large-field remainder is provided by Lemma 6.21.

Proposition 6.9 ((B^h): one-step bound with the scheme c-number)**.** *Fix $J \ge J_0$ and a compact family Σ of real sources. For $\sigma \in \Sigma \cup \{0\}$ and h_σ its scalar part, there is a sequence $\theta_k(\sigma) \le \theta_k$, with θ_k independent of $\sigma \in \Sigma$ and decaying geometrically, such that for $k \ge k_0$, on the support of $\chi_{j(k)}$,*

$$\big|S_{N,k}^{(j(k))}(\cdot;\sigma) + c_k(h_\sigma) - S_{N,k+1}^{(j(k)+1)}(\cdot;\sigma) - c_{k+1}(h_\sigma)\big| \;\le\; \theta_k(\sigma). \tag{14}$$

The proof is given in Section 6.4. The field-dependent and the h-nonlinear parts are Proposition 5.19, the graphs with curvature vertices at $s=(u,h)$, and (K_h). The field-independent h-linear part of the three lowest orders is a c-number $\bar\beta_k^{(j)}\int h$ up to a localisation error, and $c_k(h)$ is exactly the c-number that makes $\bar\beta_k^{(j)}+z_k$ converge along the diagonal pairs at the rate $5^{-\gamma' j}\eta^{-\beta'}+\eta^{\sigma'}$; this is the sense in which z is the scheme adapted to King's flow. At $\sigma=0$, (14) is (5) with $j=j(k)$.

Lemma 6.10 (c-numbers)**.** (a) $|z_k|\le C(1+k)$. (b) *For every $M>0$, $\Delta_k(M):=2(\mathsf{W}_{\varepsilon_k}(m)-\mathsf{W}_{\varepsilon_k}(M))+z_k(m)-z_k(M)$ is bounded in k, where $\mathsf{W}_\varepsilon(M)=(-\Delta_{\varepsilon\mathbb{Z}^3}+M^2)^{-1}(0,0)$ and $z_k(M)$ is the reference scheme at mass M.*

Both are statements about lattice Feynman integrals, obtained by direct computation: (a) is the logarithmic growth of the order-e^2 coefficient of $\langle|\phi|^2\rangle$ with the counterterm inserted; in (b) the Wick difference is a convergent lattice integral in three dimensions, and the scheme difference converges.

Proposition 6.11 ((NEG_∞): the bare coefficient grows with the mass)**.** *For every point of King's range $(e,\lambda>0,\ m,\mu_0>0)$ and every $M>m$, $u_\infty(M):=\lim_k u_k(M)$ exists and*

$$2u_\infty(M)\ \ge\ (M-m)\Big(M+m+\frac{4\lambda}{\pi}-\frac{e^2}{4\pi}\Big)-\Big(\frac{e^4}{4\pi^2}+\frac{8\lambda^2}{\pi^2}\Big)\ln\frac{M}{m};$$

in particular $s_-(M)=u_\infty(M)\to\infty$ as $M\to\infty$, and $s_-(M_2)\ge\frac12$ for $M_2:=\max\{m,\ \frac{e^2}{4\pi}+(\frac{e^4}{4\pi^2}+\frac{8\lambda^2}{\pi^2})/m\}+1$.

Only $\sup_M s_-(M)=\infty$ is used below; the proof rests on the exact continuum limit of King's free-boundary counterterm at each of its five orders (Lemma 29.1).

Proof of Proposition 6.11. The coefficient $\alpha_k(M)=M^2+\delta m^2_{\varepsilon_k}(M)$ of Section 6.1 is King's free-boundary bare coefficient of Section 2.1, the $\alpha_k^\infty(M)$ of Lemma 29.1; so $u_k(M)=\frac12[\alpha_k^\infty(M)-\alpha_k^\infty(m)]$, and by Lemma 29.1, whose proof is a computation of lattice Feynman integrals that uses nothing of Sections 5–6, the limit $u_\infty(M)=\frac12[\alpha_\infty(M)-\alpha_\infty(m)]$ exists for every $M>0$ and, by (75),

$$2u_\infty(M)=(M^2-m^2)+\lambda D_{01}+e^2D_{20}+e^4D_{40}+\lambda e^2D_{21}+\lambda^2D_{02},\tag{15}$$

where, by (76)–(77) with $\mu:=\mu_0$ as there, the five order-by-order limits are

$$D_{01}=\frac{4}{\pi}(M-m),\qquad D_{02}=-\frac{8}{\pi^2}\ln\frac{M}{m},$$
$$D_{20}=-\frac{(M-m)(Mm+M\mu+m\mu)}{4\pi(M+\mu)(m+\mu)}=-\frac{M-m}{4\pi}\Big(1-\frac{\mu^2}{(M+\mu)(m+\mu)}\Big),$$
$$D_{21}=\frac{2}{\pi^2}\ln\frac{2M+\mu}{2m+\mu}+\frac{\mu}{2\pi^2}\Big(\frac{1}{M+\mu}-\frac{1}{m+\mu}\Big),$$
$$D_{40}=-\frac{1}{4\pi^2}\ln\frac{M}{m}+\Phi_{40}\Big(\frac{\mu}{M}\Big)-\Phi_{40}\Big(\frac{\mu}{m}\Big).$$

Term by term, for $M\ge m$: $\lambda D_{01}=\frac{4\lambda}{\pi}(M-m)$; $e^2D_{20}\ge-\frac{e^2}{4\pi}(M-m)$ by the second form of D_{20}; $\lambda e^2D_{21}\ge0$, since the M^2-derivative y_{21}^∞ of F_{21} is positive, (78); $\lambda^2D_{02}=-\frac{8\lambda^2}{\pi^2}\ln\frac{M}{m}$; and $e^4D_{40}\ge-\frac{e^4}{4\pi^2}\ln\frac{M}{m}$ by Lemma 6.12 below. Adding, with $M^2-m^2=(M-m)(M+m)$, gives the displayed inequality. With $\ln\frac{M}{m}\le\frac{M-m}{m}$ it gives $2u_\infty(M)\ge(M-m)\big[M+m+\frac{4\lambda}{\pi}-\frac{e^2}{4\pi}-\big(\frac{e^4}{4\pi^2}+\frac{8\lambda^2}{\pi^2}\big)/m\big]\ge(M-m)(M-M_2+1)$ for $M\ge M_2$, which is $\ge M_2-m\ge1$ at $M=M_2$ and tends to infinity with M; and $s_-(M)=u_\infty(M)$ since the limit exists. □

The proof above used the following monotonicity of the function Φ_{40} of (77).

Lemma 6.12 (monotonicity of Φ_{40})**.** $\Phi'_{40}(x) < 0$ *for every* $x \geq 0$. *Consequently, for* $M \geq m$, $D_{40} \geq -\frac{1}{4\pi^2}\ln\frac{M}{m}$; *and* $y^{\infty}_{40} > -\frac{1}{8\pi^2 M^2}$ *for all* $M, \mu > 0$.

Proof. Differentiating the closed form (77) gives, for $x \neq 1$,

$$\Phi'_{40}(x) = \frac{N(x)}{32\pi^2(1-x^2)^3(1+x)^2(2+x)(1+2x)}, \qquad N(x) = (x^2-1)Q(x) + P(x)\ln\frac{1+2x}{2+x},$$

with $Q(x) = 16x^7 + 56x^6 + 24x^5 - 100x^4 - 83x^3 + 82x^2 + 109x + 40$ and $P(x) = 8x(x-2)(x+1)^2(x+2)^2(2x+1)$; the denominator has the sign of $1-x$, so the claim is $\operatorname{sign} N(x) = \operatorname{sign}(x-1)$. The substitution $x = \frac{1+\xi}{1-\xi}$, $\xi \in [-1,1)$, gives $\ln\frac{1+2x}{2+x} = 2\operatorname{artanh}\frac{\xi}{3}$ and $\tilde N(\xi) := N(x(\xi))(1-\xi)^9 = A(\xi) + 2B(\xi)\operatorname{artanh}\frac{\xi}{3}$ with the polynomials

$$A(\xi) = 576\xi - 2496\xi^2 + 13264\xi^3 - 3056\xi^4 + 224\xi^5 - 544\xi^6 + 272\xi^7 - 48\xi^8,$$
$$B(\xi) = 32(\xi-3)^2(\xi-1)^2(\xi+1)(\xi+3)(3\xi-1),$$

and $\operatorname{sign}(x-1) = \operatorname{sign}\xi$, so the claim is $\tilde N(\xi)/\xi^3 > 0$ on $[-1,1]$. Expanding $2\operatorname{artanh}\frac{\xi}{3} = \sum_{j\geq 0}\frac{2\xi^{2j+1}}{3^{2j+1}(2j+1)}$, the coefficients of ξ^0, ξ^1, ξ^2 of $\tilde N$ vanish exactly, and $\tilde N(\xi)/\xi^3 = \sum_{k\geq 0} c_k\xi^k$ on $|\xi| \leq 1$ with the exact rational coefficients $c_0 = \frac{34160}{3} > 11386$, $c_1 = -\frac{46448}{9}$, $c_2 = \frac{386528}{135}$, $c_3 = -\frac{397664}{405}$, $c_4 = \frac{256048}{2835}$, $c_5 = -\frac{20144}{8505}$, so that $\sum_{k=1}^{5}|c_k| < 9101$, while for $k \geq 6$ only the series of the logarithm contributes (A has degree 8) and $|c_k| \leq \frac{2\cdot 1100736}{(k-4)\,3^{k+3}}$, so that $\sum_{k\geq 6}|c_k| < 84$. Hence $\tilde N(\xi)/\xi^3 \geq c_0 - \sum_{k\geq 1}|c_k| > 2201 > 0$ on $[-1,1]$, and for $\xi \neq 0$

$$\Phi'_{40}(x(\xi)) = -\frac{\tilde N(\xi)/\xi^3}{2048\pi^2(1-\xi)^3(1+x)^2(2+x)(1+2x)} < 0,$$

with $\Phi'_{40}(1) = -\frac{c_0}{2048\pi^2\cdot 36} < 0$ at $\xi = 0$ by continuity (Φ_{40} is analytic across $x = 1$, Lemma 29.1). For $M \geq m$, $\mu/M \leq \mu/m$, so $\Phi_{40}(\mu/M) \geq \Phi_{40}(\mu/m)$, which is $D_{40} \geq -\frac{1}{4\pi^2}\ln\frac{M}{m}$; and $y^{\infty}_{40} = -\frac{1}{8\pi^2M^2} - \frac{\mu}{2M^3}\Phi'_{40}(\frac{\mu}{M}) > -\frac{1}{8\pi^2M^2}$ by (78). □

Proposition 6.13 ((W$^\sharp$)[s_W]: the positive-source bound)**.** *For couplings in the coupling region of Definition 2.1 (the sign clause* (T3)*), where* (I2) *holds (Proposition 12.8), and with* (I3) *(Proposition 12.2): with* $s_W := s_1$ *(the amplitude of Proposition 13.2) and* $\kappa = C_U/s_1$*: for every* k, J *and every lattice function* $0 \leq q \leq s_W$ *on* X_J,

$$\ln n_k(0,q) \;\leq\; \kappa\,\|q\|_{1,k}.$$

In particular $\sup_k n_k(0,q) < \infty$ *for every* $0 \leq q \in C_c^{\infty}$ *with* $\sup q \leq s_W$ *and* $\operatorname{supp} q$ *at distance* > 1 *from* $\mathbb{R}^3 \setminus C_J$.

Proof. $e^{\sum_x \varepsilon^3 q(x) r_x^2}$ is a class-$\mathcal{I}$ observable, so, since killing the matter on T_J does not increase a class-$\mathcal{I}$ observable (Step 1 of the proof of Lemma 4.2; Remark 4.9), $n_k(0,q) \leq N^{z}_{k,C_J}(0,q)$, the comparison-cube functional, which Proposition 13.2 at $X = X_J$ bounds by $e^{\kappa\|q\|_{1,k}}$; the scheme multiplier is common (Lemma 2.14(a)). For $q \in C_c^{\infty}$, $\|q\|_{1,k} \leq C_0'|q|_w$ by Lemma 2.9(E3). □

Any other threshold with the same property serves below in place of s_1; a second instance can be obtained from Bałaban's torus bounds through the comparison of the cube with King's tori.

6.4. **Proof of Proposition 6.9: the scheme constant makes the h-linear vacuum constant local.** The notation is that of Sections 5.1 and 6.1: k is the spacing index, j counts renormalisation steps, $\eta = 5^{j-k}$ is the block scale, the diagonal pair at k consists of the k-model after $j(k)$ steps and the $(k+1)$-model after $j(k)+1$ steps, both with block scale $\eta_k = 5^{j(k)-k}$, on the same unit lattice and with the same indicator $\chi_{j(k)}$; $S^{(j)}_{N,k}(\cdot;s) = S^{(j)}_k(\cdot;s) - v_{k,j}$ is the normalised action, $\mathsf{S}_k(h) = \sum_{x\in\Lambda_k\cap C_J} \varepsilon_k^3 h(x)$ and $c_k(h) = z_k \mathsf{S}_k(h)$. On the support of χ_j King's effective action after j steps is [25, (3.14)] (cube form [26, (3.12)]), and its whole source dependence sits in $P^{(j),1}$, the sum of the connected renormalised graphs H up to order $\bar n = 13$: the sources enter only through the curvature vertex [25, (3.20)], linear in u, and the scalar-source vertex [25, (3.21)], linear in h. For a graph H let $n_u(H)$, $n_h(H)$ be its numbers of curvature and scalar source vertices, $(\alpha(H),\beta(H))$ its order in (e^2,λ), $E(H;A_j,\phi_j;s)$ its value and $E_N(H;\cdot;s) := E(H;\cdot;s) - E(H;0,0;s)$ its field-dependent part; $E(H;\cdot;(u,h))$ is a homogeneous polynomial of degree $n_u(H)$ in u and $n_h(H)$ in h, and the number of graphs depends on $\bar n$ only.

Definition 6.14 (the three families)**.** Put $P_0 := \{(0,0),(0,1),(2,0)\}$ and

$$\mathcal{T}_h := \{H :\ n_h(H)=1,\ n_u(H)=0,\ (\alpha(H),\beta(H))\in P_0\}, \qquad \mathcal{V}^{(j)}_k(h) := \sum_{H\in\mathcal{T}_h} E(H;0,0;(0,h)),$$

the *h-linear vacuum constant of the three lowest orders*, linear in h and independent of u. The family $\mathcal{N}_h$ consists of the field-dependent parts $E_N(H;\cdot;(u,h))$ of all graphs with $n_h(H)\ge 1$ and of the field-independent values $E(H;0,0;(u,h))$ of all graphs with $n_h(H)\ge 1$ and $H\notin\mathcal{T}_h$. The excepted graphs are exactly those of $\mathcal{T}_h$, which carry no curvature vertex; the h-linear graphs of the orders P_0 which do carry curvature vertices belong to $\mathcal{N}_h$.

Lemma 6.15 (the split)**.** *On the support of χ_j, for every real source (u,h), $S^{(j)}_{N,k}(\cdot;(u,h)) = S^{(j)}_{N,k}(\cdot;(u,0)) + \sum_{T\in\mathcal{N}_h} T + \mathcal{V}^{(j)}_k(h)$.*

Proof. A graph with $n_h(H)=0$ takes the same value at (u,h) and at $(u,0)$, a graph with $n_h(H)\ge 1$ vanishes at $h=0$, and for $H\in\mathcal{T}_h$ one has $n_u(H)=0$, so $E(H;0,0;(u,h)) = E(H;0,0;(0,h))$. □

(K0) and (SRC): the two bounds across the pair. For a diagonal pair write $D[T] := T|_{(k,j(k))} - T|_{(k+1,j(k)+1)}$. For a compact family $\Sigma = \{(u_x,h_y) : (x,y)\in Y\}$, $h_y = \sum_j y_j h_j$, there are k_0 and constants such that for $k\ge k_0$, $j=j(k)$, on the support of χ_j,

$$\big|S^{(j)}_{N,k}(\cdot;(u_x,0)) - S^{(j+1)}_{N,k+1}(\cdot;(u_x,0))\big| \le \theta^{\mathrm{K0}}_k := C^{\mathrm{K0}}\big(5^{-\gamma j}\eta_k^{-\beta} + \eta_k^{\sigma}\big),$$
$$\Big|\sum_{T\in\mathcal{N}_{h_y}} D[T]\Big| \le \theta^{\mathrm{SRC}}_k := C^{\mathrm{SRC}}\big(5^{-\gamma' j}\eta_k^{-\beta'} + \eta_k^{\sigma'}\big) :$$

at a fixed source the first is (5) and the second is the third clause of Lemma 6.1 summed over the finitely many members of $\mathcal{N}_h$, and the uniformity on Σ follows by polarisation, since a function of the source which is, at every field configuration, a homogeneous polynomial of degree $d\le\bar n$ is bounded on Σ by $C(Y)$ times its maximum over the finite set of sources $\sum_{i\le d}\epsilon_i w_i$, $w_i\in\{(f_i,0),(0,h_j)\}$, $\epsilon_i=\pm1$.

(LOW): the h-linear constant is a one-point function. Let $\langle\cdot\rangle^{\mathrm{fl}}_{k,j}$ be King's fluctuation measure [25, (3.17)] on C^K_J at zero block fields $A_j=\phi_j=0$ and $[\,\cdot\,]_{P_0}$ the sum of the Taylor coefficients at the orders P_0 in (e^2,λ). Differentiating $E'_j = -\ln\int\prod d\mu\, e^{V^{(0)}}$ in the source parameter under the finite-dimensional integral, the vertex [25, (3.21)] being $\sum_x \varepsilon_k^3 h(x)(r_x^2 - 2\mathsf{W}_{\varepsilon_k})$ in unscaled

units,

$$\mathcal{V}_k^{(j)}(h) = \sum_{x\in\Lambda_k\cap C_J} \varepsilon_k^3 h(x)\, v_k^{(j)}(x), \qquad v_k^{(j)}(x) = -\Big[\big\langle r_x^2 - 2\mathsf{W}_{\varepsilon_k}\big\rangle_{k,j}^{\mathrm{fl}}\Big]_{P_0}.$$

Definition 6.16 (the bulk density)**.** Let King's block-spin flow at zero field on the infinite lattice $\Lambda_k = \varepsilon_k\mathbb{Z}^3$ be that of Definition 5.1: the Dirichlet weight f_D at the finest step and the uniform weight above it, the composite average Q_j onto the blocks of side $\eta = 5^j\varepsilon_k$, the fluctuation covariance G_j with the stiffness $\mathfrak{a}_j$, the split $C = G_j + \Gamma_j$ of the free propagator and, for the photon, $D = D_j + \Gamma_j^{\mathrm{ph}}$ with $D_j = \sum_{i<j} G_{(i)}^{\mathrm{ph}}$. Let $v_{k,\infty}^{(j)}(x)$ be the density above for this flow on Λ_k (the same graphs, with the free lines on Λ_k and the vertices summed over Λ_k), and

$$\beta_k^{(j)}(x) := v_{k,\infty}^{(j)}(x) + z_k, \qquad \bar\beta_k^{(j)} := \eta^{-3} \sum_{x\in\Lambda_k\cap[0,\eta)^3} \varepsilon_k^3\, \beta_k^{(j)}(x);$$

$\beta_k^{(j)}$ is $\eta\mathbb{Z}^3$-periodic and $\bar\beta_k^{(j)}$ is its block mean, the *bulk density*.

Lemma 6.17 (locality and aliasing)**.** *(a) For every real h supported at ℓ^∞-distance > 1 from $\mathbb{R}^3\setminus C_J$ and $0 \le j \le k - J_0$,*

$$\Big|\mathcal{V}_k^{(j)}(h) + z_k\mathsf{S}_k(h) - \sum_{x\in\Lambda_k} \varepsilon_k^3 h(x)\,\beta_k^{(j)}(x)\Big| \le C\,\|h\|_{L^1}\,\varepsilon_k^{-\nu}\,e^{-c/\eta},$$

with $C, \nu, c > 0$ depending on the couplings and J only. (b) For an $\eta\mathbb{Z}^3$-periodic β on Λ_k with block mean $\bar\beta$, $h \in C_c^\infty(\mathbb{R}^3)$ and every integer $r \ge 2$,

$$\Big|\sum_{x\in\Lambda_k} \varepsilon_k^3 h(x)\beta(x) - \bar\beta\,\mathsf{S}_k(h)\Big| \le C_r\|\beta\|_\infty\|\Delta^r h\|_{L^1}\,\eta^{2r},$$

$$\big|\mathsf{S}_k(h) - \textstyle\int h\big| \le C_r\|\Delta^r h\|_{L^1}\,\varepsilon_k^{2r}.$$

Proof. (a) The density $v_k^{(j)}(x)$ is a finite sum of graphs of King's fluctuation measure with one vertex [25, (3.21)] at x and lines $G_j(C_J^K)$, $G_{(i)}^{\mathrm{ph}}(C_J^K)$; the counterterm $\delta m^2_{\varepsilon_k}$ and $\mathsf{W}_{\varepsilon_k}$ are free-boundary numbers, the same on C_J^K and on Λ_k. The region operators are signed image sums of the free ones over a reflection group ([26, (3.7)] for the photon, [26, (3.9)] for the scalar with the weight f_D, which is why f_D and not $f \equiv 1$ is needed), every non-identity image at distance at least $\mathrm{dist}(x,\partial C_J^K)+\mathrm{dist}(y,\partial C_J^K)$ from the pair (x,y) ([26, Prop. 5.1]). Every free fine line decays at the block scale: for $H = -\Delta_\varepsilon + m^2 + b\eta^{-2}Q_f^*Q_f$ with a weight $f \ge 0$ of block averages 1, a discrete Poincaré inequality on each block gives $H \ge c_0\eta^{-2}$, and the Combes–Thomas argument with the conjugation $e^{\alpha w\cdot x/\eta}$ gives $|H^{-1}(x,y)| \le 2c_0^{-1}\eta^2\varepsilon^{-3}e^{-\alpha_0|x-y|_\infty/\eta}$. Expanding every region line as the free line plus its images and every vertex sum over C_J^K as the sum over Λ_k minus its complement, every term other than the infinite-lattice graph carries a path of fine lines of total length at least $\mathrm{dist}(x,\partial C_J^K) \ge 1$, hence a factor $e^{-\alpha_0/(2\eta)}$, while the singular prefactors are at most $C\varepsilon_k^{-\nu}$; the infinite-lattice graphs give $\sum_x \varepsilon_k^3 h(x) v_{k,\infty}^{(j)}(x)$, and $z_k\mathsf{S}_k(h)$ adds z_k per site. (b) is Poisson summation on Λ_k with the finite Fourier expansion of the periodic β; the non-zero frequencies have modulus at least $2\pi/\eta$ and $|\hat h(p)| \le \|\Delta^r h\|_{L^1}|p|^{-2r}$. □

Lemma 6.18 (the bulk density at $A = 0$)**.** *With $c_\lambda > 0$ the combinatorial tadpole constant of King's order-λ counterterm ($\delta m^{2(\lambda)} = -c_\lambda\lambda\mathsf{W}_{\varepsilon_k}$, [25, (3.33)]), the three orders of $\beta_k^{(j)} = \beta_{(0,0)} + \beta_{(0,1)} + \beta_{(2,0)}$ are*

$$\beta_{(0,0)}(x) = 2\,\Gamma_j(x,x), \qquad \beta_{(0,1)}(x) = -2c_\lambda\lambda\sum_y \varepsilon_k^3 G_j(x,y)^2\,\Gamma_j(y,y),$$

$$\beta_{(2,0)}(x) = e^2\big[\mathfrak{g}_x(C,C,C;D) - \mathfrak{g}_x(G_j,G_j,G_j;D_j)\big] - \mathfrak{a}_x,$$

where $\mathfrak{g}_x(G_1,G_2,G_3;D')$ is the coefficient of e^2 in $\langle r_x^2\rangle$ of the Gaussian matter model with outer lines G_1,G_3, internal scalar line G_2, photon covariance D', King's bond coupling (exchange and seagull) and the free counterterm $\delta m^{2(e^2)}$, so that $e^2\mathfrak{g}_x(C,C,C;D) = z_k$, and $\mathfrak{a}_x$ is the order-e^2 contribution to $\langle r_x^2\rangle^{\mathrm{fl}}$ of the A-dependence of King's averaging vertices [25, (3.24)–(3.26)] *with the contour convention* [25, (2.19), (3.27)].

Proof. Order 1: $\langle r_x^2\rangle^{\mathrm{fl}} = 2G_j(x,x)$ and $G_j(x,x) - \mathsf{W}_{\varepsilon_k} = -\Gamma_j(x,x)$. Order λ: first-order perturbation theory with the vertices [25, (3.18)–(3.19)] at $\phi^{(j)} = 0$. Order e^2: the θ-linear vacuum graphs of order e^2 built from the original vertices are the exchange graph, the seagull graph and the δm^2 insertion with the lines G_j, D_j; the photon tadpole vanishes by charge conjugation; the graphs with averaging vertices form $\mathfrak{a}_x$; z_k has no part of order 1 or λ. □

Lemma 6.19 (the bulk sequence)**.** *Fix $\gamma_b \in (0,\frac12)$ and put $\beta_b := 1$. There is C such that for all k and $1 \le j \le k - J_0$, $|\bar\beta_k^{(j)} - \bar\beta_{k+1}^{(j+1)}| \le C\,5^{-\gamma_b j}\eta^{-\beta_b}$, and along the diagonal pairs, for $k \ge k_0$ and $j = j(k)$, $\|\beta_k^{(j)}\|_\infty \le C\eta_k^{-2}$.*

Proof. The proof has three steps: we first pass to the unit block scale and compare the lattice block-spin covariance with its continuum counterpart in Fourier variables; we then bound the coefficient and its increment across the pair at each of the orders 1, λ and e^2 separately; and finally we add the three orders.

Unit block scale. Put $t := \varepsilon_k/\eta = 5^{-j}$, $m' := m\eta$, $b := \mathfrak{a}_j$; the map $x \mapsto x/\eta$ carries $-\Delta_{\varepsilon_k} + m^2 + \mathfrak{a}_j\eta^{-2}Q_j^*Q_j$ to $\eta^{-2}(-\Delta_t + m'^2 + bQ^*Q)$ on $t\mathbb{Z}^3$ with unit blocks, kernels scaling as $\Gamma_j(x,x') = \eta^{-1}\Gamma_t(x/\eta, x'/\eta)$, so $\bar\beta_{(0,0)} = \eta^{-1}F(t;m',b)$ with $F(t) := 2t^3\sum_{x\in t\mathbb{Z}^3\cap[0,1)^3}\Gamma_t(x,x)$, and a diagonal pair changes (t,b) to $(t/5,\mathfrak{a}_{j+1})$ at fixed m', with $|\mathfrak{a}_j - \mathfrak{a}_{j+1}| \le C5^{-2j}$. *Comparison with the continuum block spin.* Woodbury's identity gives $\Gamma_t = C_tQ^*(b^{-1} + QC_tQ^*)^{-1}QC_t$, hence, in Fourier variables on the unit torus $\mathbb{T} = [-\pi,\pi)^3$ with $p = P + 2\pi n$ running over the aliases $n \in N_t$ (the t^{-3} values with p in the Brillouin zone), the averaging symbol $s_t(p) = t^3\sum_{x\in t\mathbb{Z}^3\cap[0,1)^3} f(x)e^{ip\cdot x}$, $\hat C_t(p) = (\omega_t(p) + m'^2)^{-1}$, where $\omega_t(p) := \sum_\mu 4t^{-2}\sin^2(tp_\mu/2)$ is the symbol of $-\Delta_t$ (and $\omega_0(p) = |p|^2$), and $W_t(P) = \sum_{n\in N_t}|s_t(p)|^2\hat C_t(p)$,

$$\Gamma_t(x,x') = \int_{\mathbb{T}} \frac{\overline{\hat v_x^t(P)}\,\hat v_{x'}^t(P)}{b^{-1} + W_t(P)}\,\frac{dP}{(2\pi)^3}, \qquad \hat v_x^t(P) := \sum_{n\in N_t} s_t(p)\hat C_t(p)e^{-ip\cdot x},$$

and the continuum block-spin covariance Γ_0 is the same formula at $t = 0$ ($N_0 = \mathbb{Z}^3$, $s_0(p) = \int_{[0,1)^3} e^{iu\cdot p}du$). The elementary symbol bounds, $|s_t(p)| \le \prod_\mu \min(1,\pi|\sin(p_\mu/2)|/|p_\mu|)$, $\frac{4}{\pi^2}|p|^2 \le \omega_t(p) \le |p|^2$, $|s_t - s_0| \le C\min(1,t|p|)\prod_\mu\min(1,\pi|\sin(p_\mu/2)|/|p_\mu|)$ and $|\hat C_t - \hat C_0| \le Ct(|p|^2 + m'^2)^{-1/2}$, summed over the aliases, give $0 \le \Gamma_t(x,x) \le C$, $|\Gamma_t(x,x') - \Gamma_0(x,x')| \le Ct$, $|\nabla_x\Gamma_0(x,x)| \le C$ and $|\partial_b\Gamma_t| \le C$ for the uniform weight. With the Dirichlet weight f_D at the finest step the symbol factorises, $s_t^f(p) = s_{5t}(p)\varphi(tp)$ with $|\varphi| \le 1$ and $|\varphi - 1| \le C|tp|$, and the extra aliases $p = q + 2\pi\ell/(5t)$, $\ell \ne 0$, contribute $Ct^2(1 + \ln t^{-1})^3$: the comparison becomes $|\Gamma_t - \Gamma_0| \le Ct(1 + \ln t^{-1})^3 \le C't^{1/2}$, which is why the rate is $5^{-j/2}$ and $\gamma_b < \frac12$. *Order* 1. $|F(t) - F(0)| \le 2\sup|\Gamma_t - \Gamma_0| + Ct$ (the Riemann-sum error, by the gradient bound), so

$$|\bar\beta_{(0,0),k}^{(j)} - \bar\beta_{(0,0),k+1}^{(j+1)}| \le C\eta^{-1}(t(1 + \ln t^{-1})^3 + 5^{-2j}) \le C'\eta^{-1}5^{-j/2}$$

and $\|\beta_{(0,0)}\|_\infty \le C\eta^{-1}$. *Order* λ. At unit block scale $\beta_{(0,1)} = -2c_\lambda\lambda t^3\sum_y G_t(x,y)^2\Gamma_t(y,y)$ is dimensionless; the fine line $G_t = C_t - \Gamma_t$ is compared with $G_0 = C_0 - \Gamma_0$ through the local limit theorem with a rate, $|C_t(r;m') - C_0(r;m')| \le C_*t^2|r|^{-3}$ uniformly in $m' \ge 0$, and

the previous bound, the Combes–Thomas decay of G_t at the block scale controlling the part $|x-y| > R_t := C\ln t^{-1}$; the logarithmic sum of $|r|^{-2}$ over $|r| \le R_t$ gives $\|\beta_{(0,1)}\|_\infty \le C\lambda(1+j)^3$, and the increment is $\le C\lambda t^{1/2}(1+\ln t^{-1})^3 \le C\lambda 5^{-\gamma'' j}$ for every $\gamma'' < \frac{1}{2}$. *Order e^2.* At unit block scale, with $\tau \in \{t, t/5\}$ the spacing of the member of the pair, $C_\tau = (-\Delta_\tau + m'^2)^{-1}$, Γ_τ the coarse line and $G_\tau = C_\tau - \Gamma_\tau$ the fluctuation line, D_τ, $\Gamma^{\mathrm{ph}}_\tau$, $G^{\mathrm{ph}}_\tau = D_\tau - \Gamma^{\mathrm{ph}}_\tau$ their photon analogues at the photon mass, and m_0 an upper bound for the two masses at unit block scale, the matter at $\lambda = 0$ and zero block fields is Gaussian given A with quadratic form $M_A = M_0 + e' M_1 + e'^2(M_2 + \delta) + O(e'^3)$, M_1 and M_2 the first- and second-order terms of the covariant Laplacian in the gauge field and δ King's free order-e^2 counterterm [25, (3.33)], the same number in the full and in the fluctuation measure; with matter covariance G and photon covariance D' the coefficient of e'^2 in $\langle r_x^2\rangle$ is $-2(G\,\Pi[G, D']\,G)(x,x)$ with $\Pi[G, D'] := \mathbb{E}_{D'}[M_2] + \delta - \mathbb{E}_{D'}[M_1 G M_1]$ (seagull plus counterterm minus exchange), so that $\beta_{(2,0)}(x) + \mathfrak{a}_x = -2e^2[(C_\tau \Pi_C C_\tau)(x,x) - (G_\tau \Pi_G G_\tau)(x,x)]$ with $\Pi_C := \Pi[C_\tau, D_\tau]$, which is translation invariant, and $\Pi_G := \Pi[G_\tau, G^{\mathrm{ph}}_\tau]$. The two sides are not compared graph by graph: with $\Gamma = \Gamma_\tau = C_\tau - G_\tau$ one writes the sandwich

$$C\Pi_C C - G\Pi_G G = (\Gamma\Pi_C C + C\Pi_C\Gamma) - \Gamma\Pi_C\Gamma + G(\Pi_C - \Pi_G)G,$$
$$\Pi_C - \Pi_G = \Sigma^{\mathrm{sg}}[\Gamma^{\mathrm{ph}}] - \Sigma^{\mathrm{ex}}[\Gamma, D] - \Sigma^{\mathrm{ex}}[G, \Gamma^{\mathrm{ph}}],$$

where $\Sigma^{\mathrm{ex}}[G, D'] := \mathbb{E}_{D'}[M_1 G M_1]$ and $\Sigma^{\mathrm{sg}}[D'] := \mathbb{E}_{D'}[M_2]$ (bilinearity in (G, D')); the counterterm δ cancels in the last term, and only the first two terms contain the renormalised translation-invariant self-energy Π_C, whose symbol $\hat\Pi_\tau$ vanishes at $p = 0$ (King's counterterm makes the order-e^2 one-particle-irreducible two-point function vanish at zero momentum on the infinite lattice), is bounded by $C(|p| + m_0)$ on the Brillouin zone and differs from -2 times the continuum renormalised one-loop self-energy by at most $C\tau(1+|p|)(|p|+m_0)$, uniformly in $\tau \in (0,1]$ and in the masses in $(0, m_0]$. Every exchange term is a triangle of lines in which, after one summation by parts when both differences of the current vertices fall on the inner matter line, each line carries at most one difference; the fluctuation lines G, ∇G, D, ∇D decay as $(|r|+\tau)^{-1}$, $(|r|+\tau)^{-2}$ and the coarse lines Γ, Γ^{ph} and their differences are bounded, which gives the power counting of each term. Each of the three terms is then compared across the diagonal pair at unit block scale, which keeps η, m', the photon mass and the unit cell fixed and changes (τ, b, weight) from $(t, \mathfrak{a}_j, f$ on $5t$-blocks) to $(t/5, \mathfrak{a}_{j+1}, f$ on t-blocks), with the photon lines D_j against D: the first two terms through the symbol bounds of $\hat\Pi_\tau$ and the alias sums of the coarse lines, the last through the line bounds above; this gives the increment bound $C'''_\gamma e^2 5^{-\gamma''' j}\eta^{-1}$ for every $\gamma''' \in (0,1)$ and the sup bound $Ce^2\eta^{-1}(1+j)^q$ for the original-vertex part; the averaging part $\mathfrak{a}_x$, whose vertices carry the contours of King's block averaging, obeys the same two bounds separately, with $\gamma''' < \frac{1}{2}$. *Conclusion.* Adding the three orders with $\gamma'' = \gamma''' = \gamma_b$ gives the increment bound (as $\eta \le 1$); for the sup bound along the diagonal, $\eta_k^{-1} = 5^{k-j(k)} \ge 5^{\alpha k - 1}$ absorbs $(1+k)^3$ and $(1+k)^q$. The remaining estimates are routine given the symbol bounds and the local limit theorem. □

Proof of Proposition 6.9. Take the diagonal pairs with $0 < \alpha < \min\{\frac{1}{2}, \gamma/(\gamma+\beta), \gamma'/(\gamma'+\beta'), \gamma_b/(\gamma_b+\beta_b)\}$; the Cauchy arguments of Theorems 5.45 and 6.23 hold for every $\alpha \in (0, \frac{1}{2})$ with $\alpha < \gamma/(\gamma+\beta)$, so the two further restrictions cost nothing. Fix $k \ge k_0$, $j = j(k)$, $\sigma = (u_x, h_y) \in \Sigma$ and $h = h_y$. By Lemma 6.15 for both members of the pair, on the support of χ_j,

$$S^{(j)}_{N,k}(\cdot;\sigma) + c_k(h) - S^{(j+1)}_{N,k+1}(\cdot;\sigma) - c_{k+1}(h) = \big[S^{(j)}_{N,k}(\cdot;(u_x,0)) - S^{(j+1)}_{N,k+1}(\cdot;(u_x,0))\big] + \sum_{T\in\mathcal{N}_h} D[T]$$
$$+ \Big(\big[\mathcal{V}^{(j)}_k(h) + c_k(h)\big] - \big[\mathcal{V}^{(j+1)}_{k+1}(h) + c_{k+1}(h)\big]\Big).$$

The first two terms are at most $\theta_k^{\mathrm{K0}}+\theta_k^{\mathrm{SRC}}$. The third is field-independent, linear in h and vanishes at $h=0$; by Lemma 6.17 with $r=2$, $\mathcal{V}_k^{(j)}(h)+c_k(h)=\bar\beta_k^{(j)}\int h+E_k(h)$ with $|E_k(h)|\le C\|h\|_{L^1}\varepsilon_k^{-\nu}e^{-c/\eta_k}+C\|\beta_k^{(j)}\|_\infty\|\Delta^2 h\|_{L^1}(\eta_k^4+\varepsilon_k^4)$, the same for the $(k+1)$-model with $\varepsilon_{k+1}=\varepsilon_k/5$ and the same η_k, so by Lemma 6.19 the third term is at most

$$\theta_k^{\mathrm{LOW}}(h):=C\,5^{-\gamma_b j}\eta_k^{-\beta_b}\Big|\int h\Big|+C\|h\|_{L^1}5^{\nu k}e^{-c5^{\alpha k-1}}+C\|\Delta^2 h\|_{L^1}\big(\eta_k^2+\eta_k^{-2}\varepsilon_k^4\big),$$

bounded on $\{h_y : y\in Y\}$ by a θ_k^{LOW} independent of σ, with $\theta_k^{\mathrm{LOW}}(0)=0$. Along the diagonal pairs $\eta_k\le 5^{-\alpha k}$, $j(k)\ge(1-\alpha)k-1$ and $k-j(k)\le\alpha k+1$, so η_k^{σ}, $\eta_k^{\sigma'}$, η_k^2 and $\eta_k^{-2}\varepsilon_k^4$ decay geometrically, $5^{\nu k}e^{-c5^{\alpha k-1}}$ decays faster than any geometric sequence, and $5^{-\gamma_* j(k)}\eta_k^{-\beta_*}\le 5^{\gamma_*+\beta_*}\,5^{-(\gamma_*(1-\alpha)-\beta_*\alpha)k}$ decays geometrically for each of the three pairs (γ_*,β_*) by the choice of α. Hence $\tilde\theta_k:=\theta_k^{\mathrm{K0}}+\theta_k^{\mathrm{SRC}}+\theta_k^{\mathrm{LOW}}$ decays geometrically and does not depend on σ, and the left side of (14) is bounded on the support of $\chi_{j(k)}$ by $\tilde\theta_k(\sigma):=\theta_k^{\mathrm{K0}}+\theta_k^{\mathrm{SRC}}+\theta_k^{\mathrm{LOW}}(h_\sigma)\le\tilde\theta_k$ for $\sigma\in\Sigma$, and by θ_k^{K0} at $\sigma=0$. □

6.5. **Three lemmas on positive and negative scalar sources.** For a mass M let

$$Y_k^{(M)}(u,h):=e^{-c_k(h)}\langle e^{F(u)+:\phi^2:(h)}\rangle^{(M)}$$

denote the functional of the mass-M state *in the conventions of the mass m* (free Wick constant $\mathsf{W}_{\varepsilon_k}(m)$ and scheme $z_k=z_k(m)$), so that $Y_k^{(m)}=n_k$ and $Y_k^{(M)}(u,h)=e^{-\Delta_k(M)\mathsf{S}_k(h)}n_k^{(M)}(u,h)$, with $n_k^{(M)}$ the functional in the mass-M conventions and $\Delta_k(M)$ as in Lemma 6.10(b). Write $r^2(q):=\sum_x\varepsilon_k^3 q(x)r_x^2$ over the live sites.

Lemma 6.20 (mass shift and the correlation inequality)**.** *Let $M>0$, $k\ge0$, and let $0\le q$ be a lattice function on the live sites with $q(x)\le u_k(M)$ for all x. Then*

$$n_k(0,-q)\;\le\;\frac{1}{Y_k^{(M)}(0,q)},\qquad\text{and}\qquad\frac{1}{n_k(0,q)}\;\le\;n_k(0,-q)\;\le\;\frac{1}{Y_k^{(M)}(0,q)},$$

$$\text{hence}\quad Y_k^{(M)}(0,q)\le n_k(0,q).$$

Proof. Mass shift. In King's action the scalar mass enters only through $\sum_x\varepsilon^3\frac12(M^2+\delta m_\varepsilon^2(M))r_x^2$: δm_ε^2 is one number per spacing and mass (Definition 2.5), the Dirichlet terms $\frac{\varepsilon}{2}r_x^2$ do not depend on M, and the photon does not see M. Hence, with $u:=u_k(M)$ and $r^2(1):=\sum_x\varepsilon_k^3 r_x^2$, $e^{-S^{(m)}}=e^{-S^{(M)}}e^{u\,r^2(1)}$, and for every Φ integrable against both measures $\langle\Phi\rangle^{(m)}=\langle\Phi e^{ur^2(1)}\rangle^{(M)}/\langle e^{ur^2(1)}\rangle^{(M)}$; the quartic term makes $e^{\pm r^2(\cdot)}$ integrable. *Correlation inequality.* Put $g_1:=u\mathbf{1}-q\ge0$ and $g_2:=q\ge0$. The mass-M state is a BFS measure, and so is the state with its single-site measures multiplied by $e^{t\,r^2(g_1)}$, $t\in[0,1]$ (Definition 2.2 and the paragraph on BFS measures following it: the exponential moments stay finite because $\lambda>0$). By Lemma 2.10(c), multiplying the single-site measures by $e^{-tr^2(g_1)}$ does not increase $\langle e^{r^2(g_2)}\rangle$; read from $t=1$ down to $t=0$, $\langle e^{r^2(g_2)}\rangle_{t=1}^{(M)}\ge\langle e^{r^2(g_2)}\rangle_{t=0}^{(M)}$, that is

$$\langle e^{r^2(g_1)}e^{r^2(g_2)}\rangle^{(M)}\;\ge\;\langle e^{r^2(g_1)}\rangle^{(M)}\langle e^{r^2(g_2)}\rangle^{(M)}.$$

Conclusion. With $\Phi=e^{-r^2(q)}$: $\langle e^{-r^2(q)}\rangle^{(m)}=\langle e^{r^2(g_1)}\rangle^{(M)}/\langle e^{r^2(g_1+g_2)}\rangle^{(M)}\le 1/\langle e^{r^2(q)}\rangle^{(M)}$. Multiply both sides by $e^{(2\mathsf{W}_{\varepsilon_k}(m)+z_k)\mathsf{S}_k(q)}$: the left side becomes $n_k(0,-q)$ and the right side $1/Y_k^{(M)}(0,q)$, both in the conventions of the mass m. The left inequality of the second display is Lemma 2.14(c), and the last inequality follows. □

Lemma 6.21 (the large-field remainder at a source, from the one at zero source)**.** *Let s be a real source and $p>1$, and assume (13) at s and (R_0) of Proposition 5.12 for the member (k,j), with $a:=C\eta^\sigma$ and $\rho:=\rho(\eta)$. Then*

$$e^{-a}\,I_k^{(j)}(s)\;\le\;Z_k(s)\;\le\;e^{a}\,I_k^{(j)}(s)+\rho^{1-1/p}\,\langle e^{X(ps)}\rangle_k^{1/p}\,Z_k(0).$$

Proof. The lower line is the first inequality of (13). For the upper line, $Z_k(s)=\int P\,d\mu_{k,s}+\int(1-P)e^{X(s)}d\mu_{k,0}$. The first term is at most $e^aI_k^{(j)}(s)$ by (13). For the second, Hölder's inequality for the positive measure $(1-P)d\mu_{k,0}$ with the functions 1 and $e^{X(s)}$, then $1-P\le1$ in the second factor and $pX(s)=X(ps)$ (linearity), give

$$\begin{aligned}\int(1-P)e^{X(s)}d\mu_{k,0}\;&\le\;\Big[\int(1-P)d\mu_{k,0}\Big]^{1-1/p}\Big[\int e^{X(ps)}d\mu_{k,0}\Big]^{1/p}\\&\le\;\big[\rho Z_k(0)\big]^{1-1/p}\big[Z_k(0)\langle e^{X(ps)}\rangle_k\big]^{1/p},\end{aligned}$$

using (R_0) for the first bracket. □

At a negative scalar source the same step would need a bound on $\langle e^{X(ps)}\rangle$ at a negative source of larger amplitude, which is what is to be proved; Theorem 6.25 avoids this through the mass shift.

Lemma 6.22 (positivity of the limit at a positive source)**.** *Fix a mass M, a function $0\le q\in C_c^\infty$ as in Proposition 6.13, and $\bar w\in(0,1]$. Assume that at mass M, in its own conventions, the hypotheses of Theorem 6.23 below hold at $\sigma=(0,wq)$ for every $w\in[0,\bar w]$ and at $\sigma=0$, with sequences b_k, $\theta_k(wq)$, $r_k(0)$ and $e^{-c_k(wq)}r_k(wq)$ dominated by w-independent summable sequences, for all $k\ge k_0$ with k_0 independent of w. Then $\inf_{k\ge k_0}n_k^{(M)}(0,q)>0$.*

Proof. Steps 1–4 of the proof of Theorem 6.23 use the hypotheses at $\{wq,0\}$ only and give the lower half of (18) with w-independent summable $\bar\delta_k$, $\bar\epsilon_k$: $n_{k+1}(wq)\ge(1-\bar\delta_k)n_k(wq)-\bar\epsilon_k$. By induction, $n_k(wq)\ge\prod_{i=k_1}^{k-1}(1-\bar\delta_i)\,n_{k_1}(wq)-\sum_{i\ge k_1}\bar\epsilon_i$ for $k\ge k_1$. Choose $k_1\ge k_0$ with $\prod_{i\ge k_1}(1-\bar\delta_i)\ge\frac12$ and $\sum_{i\ge k_1}\bar\epsilon_i\le\frac18$. At this fixed k_1, $w\mapsto n_{k_1}(wq)$ is continuous (a finite-dimensional integral) and equals 1 at $w=0$; choose $w_0\in(0,\bar w]$ with $n_{k_1}(w_0q)\ge\frac12$. Then $n_k(w_0q)\ge\frac18$ for all $k\ge k_1$. Since $\ln n_k(wq)$ is convex in w and vanishes at $w=0$, $n_k(w_0q)\le n_k(q)^{w_0}$, so $n_k(q)\ge8^{-1/w_0}$ for $k\ge k_1$; the finitely many $k_0\le k<k_1$ have $n_k(q)>0$. □

6.6. **The Cauchy argument with a c-number.** This subsection carries the Cauchy argument of Section 5.4 to scalar sources, where the scheme c-number $c_k(h)$ enters; the Cauchy property is derived from the two hypotheses (U) and (B) stated in the theorem.

Theorem 6.23 (the Cauchy property with the scheme c-number)**.** *Fix a real source $s=(u,h)$ (notation of Section 5.1 and Section 6.3) and let σ range over $\{s,0\}$. Suppose there are k_0 and non-negative sequences b_k, $r_k(\sigma)$, $\theta_k(\sigma)$ such that for all $k\ge k_0$:*

(U) *for every σ and both members (k',j') of the diagonal pair at k,*

$$e^{-b_k}I_{k'}^{(j')}(\sigma)\le Z_{k'}(\sigma)\le e^{b_k}I_{k'}^{(j')}(\sigma)+r_k(\sigma)Z_{k'}(0),$$

with $\sum_k b_k<\infty$, $r_k(0)\le\frac12$, $\sum_k r_k(0)<\infty$ and $\sum_k e^{C_c(h)(k+2)}r_k(s)<\infty$, where $|c_k(h)|\le C_c(h)(1+k)$ (Lemma 6.10(a));

(B) *on the support of $\chi_{j(k)}$, (14) holds at σ with $\sum_k\theta_k(\sigma)<\infty$.*

Put

$$\Theta_k:=\theta_k(s)+\theta_k(0),\qquad\delta_k:=(4b_k+\Theta_k+2r_k(0))\,e^{4b_k+\Theta_k+2r_k(0)}$$
$$\text{and}\qquad\epsilon_k:=(e^{-c_k(h)}+e^{-c_{k+1}(h)})\,r_k(s).$$

Then for all $k \geq k_0$:

(i) $n_k(s) \leq \mathsf{M}(s) := \exp(\sum_{i\geq k_0}\delta_i)\big(n_{k_0}(s) + \sum_{i\geq k_0}\epsilon_i\big) < \infty$;
(ii) $|n_{k+1}(s) - n_k(s)| \leq \delta_k \mathsf{M}(s) + \epsilon_k$, *so* $n_k(s)$ *converges to a limit* $n(s) \in [0, \mathsf{M}(s)]$;
(iii) *if* (U) *and* (B) *hold at* $-s$ *as well,* $n_k(s) \geq 1/\mathsf{M}(-s)$ *and* $n(s) \in [1/\mathsf{M}(-s), \mathsf{M}(s)] \subset (0,\infty)$.

If b_k, Θ_k *decay geometrically and* $r_k(\sigma) \leq e^{-ck^2+Ck}$, *then* $|n_k(s) - n(s)| \leq C' 5^{-\kappa k}$ *for some* $\kappa > 0$.

Proof. The proof has six steps: the bounds for one model from (U) alone; the removal of the vacuum constant; the comparison across the pair from (B) alone; the two-sided recursion for $n_k(s)$; its boundedness and convergence; and the positivity of the limit.

Fix $k \geq k_0$, $j := j(k)$, and let primes denote the $(k+1)$-model after $j+1$ steps.

Step 1 (one model, (U) *only).* For one member write $Z(\sigma)$, $I(\sigma)$, $b = b_k$, $r(\sigma) = r_k(\sigma)$. From (U) at $\sigma = 0$, $(1 - r(0))Z(0) \leq e^b I(0)$ and $Z(0) \geq e^{-b} I(0)$, so $I(0) > 0$. With $\varrho(\sigma) := I(\sigma)/I(0)$ and $N(\sigma) = Z(\sigma)/Z(0)$,

$$e^{-2b}(1 - r(0))\,\varrho(\sigma) \;\leq\; N(\sigma) \;\leq\; e^{2b}\varrho(\sigma) + r(\sigma)\,: \tag{16}$$

the upper bound from $Z(\sigma) \leq e^b I(\sigma) + r(\sigma)Z(0)$ and $1/Z(0) \leq e^b/I(0)$, the lower one from $Z(\sigma) \geq e^{-b} I(\sigma)$ and $1/Z(0) \geq e^{-b}(1 - r(0))/I(0)$. No lower bound on $Z(\sigma)$ for $\sigma \neq 0$ is used: the remainder enters relative to $Z(0)$ only.

Step 2 (the vacuum constant). $\varrho(\sigma) = \hat{I}(\sigma)/\hat{I}(0)$, since $I = e^{-v}\hat{I}$ with one v for all σ; likewise for the primed member.

Step 3 (across the pair, (B) *only).* On the support of χ_j, (B) at $\sigma = s$ reads $S'_N(\cdot; s) = S_N(\cdot; s) + c_k(h) - c_{k+1}(h) + E$ with $|E| \leq \theta_k(s)$; the integrals $\hat{I}(s)$, $\hat{I}'(s)$ run over the same variables with the same χ_j, so $\hat{I}'(s) \in e^{c_{k+1}(h) - c_k(h)}[e^{-\theta_k(s)}, e^{\theta_k(s)}]\,\hat{I}(s)$. At $\sigma = 0$ ($c_k(0) = 0$), $\hat{I}'(0) \in [e^{-\theta_k(0)}, e^{\theta_k(0)}]\,\hat{I}(0)$. With $\varrho := \varrho(s)$ for the k-model and ϱ' for the $(k+1)$-model,

$$e^{-c_{k+1}(h)}\varrho' \;\in\; [e^{-\Theta_k}, e^{\Theta_k}]\; e^{-c_k(h)}\varrho. \tag{17}$$

Step 4 (two-sided recursion). Write $n := n_k(s)$, $n' := n_{k+1}(s)$, $r_0 := r_k(0)$, $r_s := r_k(s)$, $\Theta := \Theta_k$. Apply (16) to the primed member, multiply by $e^{-c_{k+1}(h)}$, use (17), and then (16) for the k-model in the forms $\varrho \leq e^{2b}(1 - r_0)^{-1} N(s)$ and $\varrho \geq e^{-2b}(N(s) - r_s)$:

$$n' \leq e^{2b}\, e^{-c_{k+1}}\varrho' + e^{-c_{k+1}} r_s \;\leq\; e^{2b+\Theta} e^{-c_k}\varrho + e^{-c_{k+1}} r_s \;\leq\; e^{4b+\Theta}(1 - r_0)^{-1}\, n + e^{-c_{k+1}} r_s,$$

$$n' \geq e^{-2b}(1 - r_0)\, e^{-c_{k+1}}\varrho' \;\geq\; e^{-2b-\Theta}(1 - r_0)\, e^{-c_k}\varrho \;\geq\; e^{-4b-\Theta}(1 - r_0)\big(n - e^{-c_k} r_s\big).$$

For $t \leq \frac{1}{2}$, $(1-t)^{-1} \leq e^{2t}$ and $1 - t \geq e^{-2t}$; with $e^y - 1 \leq y e^y$ and $1 - e^{-y} \leq y$ this gives $e^{4b+\Theta}(1 - r_0)^{-1} - 1 \leq \delta_k$ and $1 - e^{-4b-\Theta}(1 - r_0) \leq \delta_k$, hence

$$(1 - \delta_k)\, n_k(s) - \epsilon_k \;\leq\; n_{k+1}(s) \;\leq\; (1 + \delta_k)\, n_k(s) + \epsilon_k. \tag{18}$$

Step 5 (boundedness and convergence). By induction from the upper half of (18), $n_k \leq \prod_{i=k_0}^{k-1}(1 + \delta_i)\big(n_{k_0} + \sum_{i<k}\epsilon_i\big) \leq \mathsf{M}(s)$. Here $\sum_i \delta_i < \infty$ because b_i, Θ_i, $r_i(0)$ are summable (so bounded, and $\delta_i \leq C(4b_i + \Theta_i + 2r_i(0))$), and $\sum_i \epsilon_i \leq 2\sum_i e^{C_c(h)(i+2)} r_i(s) < \infty$ by (U) and $e^{-c_i(h)} \leq e^{C_c(h)(1+i)}$. Then (18) gives $|n_{k+1} - n_k| \leq \delta_k n_k + \epsilon_k \leq \delta_k \mathsf{M}(s) + \epsilon_k$, summable; so $(n_k(s))$ is Cauchy. This is (i), (ii).

Step 6 (positivity). $X(-s) = -X(s)$ and $c_k(-h) = -c_k(h)$, so $n_k(s) n_k(-s) = \langle e^X\rangle\langle e^{-X}\rangle \geq 1$ (Lemma 2.14(c)). Steps 1–5 at $-s$ give $n_k(-s) \leq \mathsf{M}(-s)$, hence $n_k(s) \geq 1/\mathsf{M}(-s)$, and the limit inherits both bounds. The rate follows by summing the bound of (ii) from k on. □

Remark 6.24 (why the ratio is not taken). At $h = 0$ and $c_k \equiv 0$ this is Theorem 5.45 with Lemma 5.20 replaced by Step 6. The logarithmic form of Theorem 5.45 needs $r_k(\sigma)Z(0)/Z(\sigma)$ small, that is a lower bound on $Z_k(s)$ uniform enough in k; at $h \neq 0$ the only candidates are

Lemma 2.14(c), which needs the upper bound at $-s$ the theorem is to produce, and Jensen's inequality, which needs a renormalised one-point bound that is not available. The additive comparison (16) avoids both: the remainder is compared with $Z(0)$, and the lower bound on $n_k(s)$ is produced at the end. Nothing in the proof uses the cube or any relation between $j(k+1)$ and $j(k)+1$.

6.7. **The negative-source bound.** The next theorem bounds the normalised functional at a negative scalar source uniformly in the spacing; the proof passes to an auxiliary scalar mass M_2, where the normalised functional at a positive source is bounded below (Lemma 6.22), and returns by the mass shift of Lemma 6.20.

Theorem 6.25 ((UB$^-$): uniform bound at a negative scalar source)**.** *Assume* (I2) *and* (I3). *Let $J \geq J_0$ and let $0 \leq q \in C_c^\infty(\mathbb{R}^3)$ be supported at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_J$. Then $\sup_k n_k(0,-q) < \infty$. Moreover, for every real 2-form u with the same support condition, $n_k(u,-q) \leq n_k(2u,0)^{1/2} n_k(0,-2q)^{1/2}$, so $\sup_k n_k(u,-q) < \infty$.*

Proof. By Proposition 6.11 choose $M_2 > m$ with $s_-(M_2) > \sup q$. Propositions 5.12, 6.8 and 6.9 and Lemma 6.10 hold at the point (e, λ, M_2, μ_0) as well; $\Delta := \sup_k |\Delta_k(M_2)| < \infty$.

Step 1. Choose $\bar{w} \in (0, \frac{1}{2}]$ with $2\bar{w} \sup q < s_W$, and $k_* \geq k_0$ with $u_k(M_2) \geq \sup q$ for $k \geq k_*$ (possible since $\sup q < \liminf_k u_k(M_2)$). Then for $k \geq k_*$ and $w \in [0, \bar{w}]$, both $q \leq u_k(M_2)$ and $2wq \leq u_k(M_2)$, so Lemma 6.20 applies at q and at $2wq$.

Step 2 (an a priori bound at M_2). For $k \geq k_*$, $w \in [0, \bar{w}]$: by Lemma 6.20, $Y_k^{(M_2)}(0, 2wq) \leq n_k(0, 2wq)$; by convexity of $t \mapsto \ln n_k(0,tq)$, which is 0 at $t = 0$, $n_k(0,2wq) \leq \max\{1, n_k(0, 2\bar{w}q)\}$; and by Proposition 6.13, $n_k(0, 2\bar{w}q) \leq e^{\kappa \|2\bar{w}q\|_{1,k}} \leq e^{\kappa C_0' |2\bar{w}q|_w}$. Hence, in the conventions of the mass M_2,

$$n_k^{(M_2)}(0,2wq) = e^{\Delta_k(M_2)\mathsf{S}_k(2wq)}\, Y_k^{(M_2)}(0,2wq) \;\leq\; e^{\Delta\, \mathsf{S}_k(2q)} \max\{1, e^{\kappa C_0' |2\bar{w}q|_w}\} \;=:\; \bar{\mathsf{M}},$$

uniformly in w and k ($\mathsf{S}_k(2q) \leq C_0' |2q|_w$).

Step 3 (the hypotheses of Lemma 6.22 at M_2). The family $\{wq : 0 \leq w \leq \bar{w}\}$ is compact. At $\sigma = (0, wq)$: (U) holds by Lemma 6.21 at mass M_2 with $p = 2$, from (13) (Proposition 6.8 at M_2; $wq \geq 0$) and (R$_0$) (Proposition 5.12 at M_2), with $b_k = C\eta_k^\sigma$ and, in the conventions of M_2, $e^{-c_k(wq)} r_k(wq) = \rho_k^{1/2}\, n_k^{(M_2)}(0,2wq)^{1/2} \leq \rho_k^{1/2} \bar{\mathsf{M}}^{1/2}$ by Step 2, a w-independent sequence; since $\rho_k \leq e^{-ck^2 + C}$ and $e^{c_k(wq)} \leq e^{C_c(1+k)}$ with C_c uniform on the family, $\sum_k e^{C_c(k+2)} r_k(wq) < \infty$ uniformly in w. At $\sigma = 0$, (U) is (4) with $r_k(0) = \rho_k \leq \frac{1}{2}$ for $k \geq k_0$. (B) at $\sigma \in \{wq, 0\}$ is Proposition 6.9 at M_2 on the compact family, with w-independent θ_k.

Step 4 (positivity at M_2). Lemma 6.22 gives $c := \inf_{k \geq k_*} n_k^{(M_2)}(0,q) > 0$, hence $Y_k^{(M_2)}(0,q) \geq e^{-\Delta \mathsf{S}_k(q)} c$ for $k \geq k_*$.

Step 5 (the mass shift). By Lemma 6.20 at q, $n_k(0,-q) \leq 1/Y_k^{(M_2)}(0,q) \leq e^{\Delta \mathsf{S}_k(q)}/c \leq e^{\Delta C_0' |q|_w}/c$ for $k \geq k_*$; the finitely many $k < k_*$ are finite numbers. The mixed bound is Cauchy–Schwarz, $\langle e^{F(u) - :\phi^2:(q)} \rangle \leq \langle e^{2F(u)} \rangle^{1/2} \langle e^{-2:\phi^2:(q)} \rangle^{1/2}$, with the c-numbers distributed ($e^{c_k(q)} = (e^{c_k(2q)})^{1/2}$); the first factor is bounded by Lemma 5.20. □

6.8. **Existence on the cube with scalar sources.** The results of this section are now assembled into the existence theorem with scalar sources, Theorem B, in the form the Wick leg uses (Sections 13 and 14).

Theorem 6.26 (King's existence theorem on the cube with scalar sources, reference scheme)**.** *Let the couplings lie in the coupling region of Definition 2.1 (the sign clause* (T3)*); under* (I2) *and* (I3)*, which hold there by Proposition 12.8 and Proposition 12.2, Proposition 6.13 holds with $s_W = s_1$. Let $f_1, \dots, f_n$ be real 2-forms and $h_1, \dots, h_p \geq 0$ real functions, all in C_c^∞ and supported*

at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_J$, $J \geq J_0$. For $\zeta = (z, w) \in \mathbb{C}^{n+p}$ put $k_1(\zeta) := \sum_j (\operatorname{Re} w_j)_+ h_j$, $k_2(\zeta) := \sum_j (\operatorname{Re} w_j)_- h_j$ (both ≥ 0, smooth, with $\sum_j (\operatorname{Re} w_j) h_j = k_1 - k_2$), and

$$D_J(s_W) \;:=\; \big\{\zeta \in \mathbb{C}^{n+p} :\; \sup_x 2k_1(\zeta)(x) < s_W\big\}.$$

Then $D_J(s_W)$ is an open convex tube, containing every ζ with $\operatorname{Re} w_j \leq 0$ for all j and every ζ with w purely imaginary, and

(i) *$n_k \to F_J$ locally uniformly on $D_J(s_W)$, and F_J is holomorphic there;*
(ii) *$F_J(\zeta) \in (0, \infty)$ for every real $\zeta \in D_J(s_W)$.*

Proof. The domain. For each x, $\operatorname{Re} w \mapsto 2k_1(\zeta)(x) = \sum_j 2(\operatorname{Re} w_j)_+ h_j(x)$ is convex, since $h_j(x) \geq 0$ and $t \mapsto t_+$ is convex; so $\operatorname{Re} w \mapsto \sup_x 2k_1(\zeta)(x)$ is a finite convex function on $\mathbb{R}^p$, hence continuous, and $D_J(s_W)$ is the preimage under $\zeta \mapsto \operatorname{Re} w$ of an open convex set: open, convex, connected. Write $D := D_J(s_W)$ and $D_{\mathbb{R}} := D \cap \mathbb{R}^{n+p}$, an open convex subset of $\mathbb{R}^{n+p}$.

Step 1 (local boundedness). Let $Q \subset D$ be compact. $\operatorname{Re} Q$ is a compact subset of the open set $D_{\mathbb{R}}$, so it is covered by finitely many closed coordinate boxes $B \subset D_{\mathbb{R}}$. For $\zeta \in Q$ with $\operatorname{Re}\zeta \in B$, $|n_k(\zeta)| \leq n_k(\operatorname{Re}\zeta) \leq \max_{v \in \mathrm{vert}(B)} n_k(v)$, by Lemma 2.14(b) and the convexity of $\ln n_k$ on B. At a vertex $v = (x, y) \in D_{\mathbb{R}}$ put $k_1 := k_1(v)$, $k_2 := k_2(v)$; then $X(v) = F(u_x) + {:}\phi^2{:}(k_1) - {:}\phi^2{:}(k_2)$ and the c-number is $e^{-c_k(k_1)+c_k(k_2)}$. Hölder's inequality with exponents $(4, 2, 4)$, with the c-number distributed as $e^{-c_k(k_1)} = (e^{-c_k(2k_1)})^{1/2}$, $e^{c_k(k_2)} = (e^{-c_k(-4k_2)})^{1/4}$, gives

$$n_k(v) \;\leq\; a_k(4x)^{1/4}\, n_k(0, 2k_1)^{1/2}\, n_k(0, -4k_2)^{1/4}.$$

The three factors are bounded uniformly in k: the first by Lemma 5.20; the second by Proposition 6.13, since $2k_1$ is a non-negative smooth function with $\sup 2k_1 < s_W$ ($v \in D_{\mathbb{R}}$) and the required support; the third by Theorem 6.25 at $q = 4k_2$. Hence $\sup_k \sup_Q |n_k| < \infty$.

Step 2 (convergence on a real open set). Choose $\delta > 0$ with $4\delta \sup_x \sum_j h_j(x) < s_W$ and put $U := \{\zeta = (x, y) \in \mathbb{R}^{n+p} :\; 0 < y_j < \delta \;\forall j\}$, a non-empty open subset of $\mathbb{R}^{n+p}$ contained in $D_{\mathbb{R}}$. At $\zeta \in U$, $k_2(\zeta) = 0$ and $s := (u_x, k_1(\zeta))$ has a non-negative scalar part; the sources $\{k_1(\zeta) : \zeta \in \bar U\}$ form a compact family. Apply Theorem 6.23(i),(ii) at s. Hypothesis (U) at $\sigma = 0$ is (4) on both members, with $r_k(0) = \rho_k$. Hypothesis (U) at $\sigma = s$ is Lemma 6.21 with $p = 2$, from (13) (Proposition 6.8) and (R_0), with $r_k(s) = \rho_k^{1/2} \langle e^{X(2s)} \rangle_k^{1/2}$; here $\langle e^{X(2s)} \rangle_k = e^{c_k(2k_1)} n_k(2u_x, 2k_1) \leq e^{c_k(2k_1)} a_k(4x)^{1/2} n_k(0, 4k_1)^{1/2}$ by Cauchy–Schwarz, bounded by $e^{C_c(1+k)}$ times a constant (Lemma 5.20; Proposition 6.13, as $\sup 4k_1 < s_W$), so $\sum_k e^{C_c(k+2)} r_k(s) \leq C \sum_k e^{2C_c(k+2)} \rho_k^{1/2} < \infty$ since $\rho_k \leq e^{-ck^2+C}$. Hypothesis (B) at $\sigma \in \{s, 0\}$ is Proposition 6.9. Hence $n_k(\zeta)$ converges for every $\zeta \in U$.

Step 3 (Vitali). By Step 1 the family $\{n_k\}$ is locally bounded on the connected domain D; by Montel's theorem every subsequence has a further subsequence converging locally uniformly to a holomorphic function; any two such limits agree on U (Step 2), hence on D, since a holomorphic function vanishing on a non-empty open subset of $\mathbb{R}^{n+p}$ has all its derivatives equal to zero at a point of it and vanishes on the connected D. So the whole sequence converges locally uniformly on D to a holomorphic F_J: (i).

Step 4 (positivity at real points). On $D_{\mathbb{R}}$ every n_k is positive and $\ln n_k$ is convex, so $F_J \geq 0$ there and $g := \ln F_J : D_{\mathbb{R}} \to [-\infty, \infty)$ satisfies $g(\lambda a + (1-\lambda)b) \leq \lambda g(a) + (1-\lambda) g(b)$ (the limit of the inequalities for $\ln n_k$; all values are $< \infty$). For $\zeta \in U$ the point $-\zeta$ lies in $D_{\mathbb{R}}$ ($k_1(-\zeta) = 0$), both $n_k(\zeta)$ and $n_k(-\zeta)$ converge by (i), and $n_k(\zeta) n_k(-\zeta) \geq 1$ (Lemma 2.14(c)), so $F_J(\zeta) F_J(-\zeta) \geq 1$ and $g > -\infty$ on U. If $g(t) = -\infty$ for some $t \in D_{\mathbb{R}}$, pick $t' \in U$ and, $D_{\mathbb{R}}$ being open, a point $t'' \in D_{\mathbb{R}}$ beyond t' on the ray from t through t', so that $t' = \lambda t + (1-\lambda) t''$ with $\lambda \in (0, 1)$; then $g(t') \leq \lambda g(t) + (1-\lambda) g(t'') = -\infty$, a contradiction. Hence $F_J > 0$ on $D_{\mathbb{R}}$: (ii). $\square$

Corollary 6.27 (King's cube limits with scalar sources in the reference scheme, in the form used later)**.** *Under the hypotheses of Theorem 6.26:*

(a) *For every finite family of real 2-forms f_i and real functions h_j (of either sign) in C_c^∞ with the support condition, and every real coefficient vector (x, y) with $2\|(\sum_j y_j h_j)_+\|_\infty < s_W$, the limit $\lim_k Z_{k,J}^{z,K}(x, y)$ exists and lies in $(0, \infty)$. In particular Theorem B for the reference scheme holds at every real point of the tube $\mathcal{D}_{\sigma_+,H}$ with $\sigma_+ = s_1$ (there $2\|k_+\|_\infty < \sigma_+ = s_W$), and at every real coefficient vector whose scalar source $\sum_j y_j h_j$ is non-positive, with no restriction on its amplitude.*

(b) *For $0 \le q$ as in Theorem 6.25, the comparison-convention functional satisfies*

$$\sup_k N_{k,C_J}^{z}(0, -q) \le \sup_k n_k(0, -q) < \infty.$$

Proof. (a) Put $\mathsf{k} := \sum_j y_j h_j$ and let k_+ be its pointwise positive part. Choose $\chi \in C_c^\infty$ with $0 \le \chi \le 1$, $\chi = 1$ on a neighbourhood of $\operatorname{supp}\mathsf{k}$ and $\operatorname{supp}\chi$ at distance > 1 from $\mathbb{R}^3 \setminus C_J$, and $\delta > 0$ with $2(\|\mathsf{k}_+\|_\infty+\delta) < s_W$; put $h^+ := (\|\mathsf{k}_+\|_\infty+\delta)\chi$ and $h^- := h^+-\mathsf{k}$. Both are smooth, non-negative ($h^+ \ge \mathsf{k}_+ \ge \mathsf{k}$ on $\operatorname{supp}\mathsf{k}$, and $h^+ \ge 0 = \mathsf{k}$ off it), compactly supported with the support condition, and $\mathsf{k} = h^+ - h^-$. Then $Z_{k,J}^{z,K}(x, y)$ for the given family equals $n_k(\zeta')$ for the family $(f_1, \dots, f_n; h^+, h^-)$ at $\zeta' = (x; 1, -1)$, for which $k_1(\zeta') = h^+$ and $\sup 2k_1(\zeta') = 2(\|\mathsf{k}_+\|_\infty + \delta) < s_W$; so $\zeta' \in D_J(s_W)$ and Theorem 6.26 applies. (b) $e^{-:\phi^2:(q)}$ is a class-$\mathcal{D}$ observable; killing the matter on T_J does not decrease its expectation (Step 3 of Lemma 4.2, the $\mathcal{D}$ direction), and the scheme multiplier is common. □

Remark 6.28 (other schemes). For a scheme a, $Z_{k,J}^{a,K}(x, y) = e^{-(a_k-z_k)\sum_j y_j \mathsf{S}_k(h_j)} Z_{k,J}^{z,K}(x, y)$. At a coefficient vector with $\sum_j y_j \int h_j \ne 0$ the limit for a therefore exists if and only if $a_k - z_k$ converges: on the domain of Corollary 6.27(a), Theorem B for a is the convergence of the scheme difference, in agreement with Remark 2.6, and Corollary 6.27(b) holds for a whenever $\sup_k |a_k - z_k| < \infty$.

Remark 6.29 (the cell bound). ($\mathrm{B}_{\mathrm{cell}}$) (Lemma 14.4) is the boundedness of $N_{k,C_{L_c}}^{a}(0, -2H'\chi_0)$ on one comparison-convention cell. When the cell is a King cube, that is $\ell_c = 2MJ'$ for some J' up to a translation by a lattice vector (Lemma 2.15), and χ_0 is supported at ℓ^∞-distance > 1 from the cell's complement, Corollary 6.27(b) and Remark 6.28 give ($\mathrm{B}_{\mathrm{cell}}$) for every scheme with $\sup_k |a_k - z_k| < \infty$, and Corollary 6.27(a) gives the convergence of the King-convention cell functional $Z_{k,C_{L_c}}^{z,K}(0, -2H'\chi_0)$ as well; for the comparison-convention cell functional only the boundedness is claimed (Remark 5.48).

Remark 6.30 (what is and is not claimed). Beyond the statements of Section 5.2 and Lemma 6.1, the only result used in this section is the threshold s_W of Proposition 6.13, provided by (I2) and (I3). Nothing is claimed at a non-negative scalar source of amplitude $\ge s_W/2$: there the positive-source bound is the content of (W2) above the threshold (Problem 32.2, which is open). The negative scalar sources are unrestricted, because the mass shift of Lemma 6.20 turns a negative source of the mass-m model into a positive source of a heavier model, where the small-field analysis applies with no threshold on the amplitude, and Proposition 6.11 provides a heavy enough mass for every amplitude.

The axioms

7. Regularity in the curvature sector

This section proves the regularity axiom E0″ for the curvature field of King's state Z_∞^K (the infinite-volume state of King's construction [25, 26]), which lives on King's convention ND^K

(Definition 2.3); on the comparison convention $\mathrm{ND}^{(a)}$ (Definition 2.2) it proves the finite-volume bound Theorem 7.4 only. Gaussian domination (Lemma 2.11) and a bound on the Neumann photon form of the cube (Lemma 7.1) give a bound on every lattice generating functional, uniform in the spacing, the cube and the scalar sector, with absolute constants (Remark 7.5); by the existence theorem (Theorem A) it passes to the limits of King's state by Montel's theorem, and Cauchy's estimates turn it into E0″ with $s = 4$. The section closes with the curvature-sector properties of the limits of King's state (locally uniform convergence, monotonicity in the cube, positivity, continuity and translations), proved with cone-direction correlation inequalities only (Lemma 7.7), and with a remark on the comparison family (Remark 7.8).

Convention 7.1. Throughout, $f_1, \dots, f_n$ are real 2-forms in $C_c^\infty(\mathbb{R}^3; \Lambda^2)$ with supports at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_{J_0}$, $J \geq J_0$, $z \in \mathbb{C}^n$, and $g(z) := \sum_j (\mathrm{Re}\, z_j) f_j$, so that $\mathrm{Re} \sum_j z_j F(f_j) = F(g(z))$ (Lemma 2.14(b)). For a box X, $\langle A, B\rangle_k$ and $d_k^* g$ are as in (2): $d_k^* g$ is the lattice 1-form with $F_{\Lambda_k}(g) = \langle d_k^* g, A\rangle_k$, defined whenever every plaquette at which $g \neq 0$ has its four corners in X. The support condition implies this for $X = X_J$: the plaquette centre is at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_J$ and the corners are within $\varepsilon/2 \leq \frac{1}{2}$ of it.

7.1. **The Neumann cube photon form.** The first step is a bound on the Neumann photon form of the cube in terms of the source, uniform in the spacing and the cube; with Gaussian domination (Lemma 2.11) it gives the lattice bounds below.

Lemma 7.1 (the Neumann cube photon form)**.** *Let $X = C_J \cap \Lambda_k$, let C be the photon covariance of the cube states on X (the same for $\mathrm{ND}^{(a)}_{k,J}$ and $\mathrm{ND}^{K}_{k,J}$) with respect to $\langle\cdot,\cdot\rangle_k$, let g be a real 2-form on the plaquettes $p = (x; \alpha < \beta)$ all four of whose corners lie in X, and $v := d_k^* g$. Then*

$$\langle v, Cv\rangle_k \;\leq\; \min\big\{\, 2\|g\|_{2,k}^2,\; \mu_0^{-2}\|v\|_{2,k}^2 \,\big\}.$$

Proof. The Gaussian density of the photon field is $e^{-Q(A)/2}$ with

$$Q(A) = \langle A, C^{-1}A\rangle_k = \sum_{\alpha,\beta} \sum_{x,\, x+\varepsilon e_\beta \in X} \varepsilon\, (\partial_\beta A_\alpha(x))^2 + \mu_0^2 \|A\|_{2,k}^2,$$
$$\partial_\beta A_\alpha(x) := A_\alpha(x + \varepsilon e_\beta) - A_\alpha(x)\,:$$

the Neumann rule keeps exactly the gradient terms of pairs inside X. By Legendre duality, $\langle v, Cv\rangle = \sup_A \big[2\langle v, A\rangle - Q(A)\big]$, attained at $A = Cv$. *Second bound.* $Q(A) \geq \mu_0^2 \|A\|^2$, so $C \leq \mu_0^{-2}$ as forms. *First bound.* Let P_g be the set of plaquettes on which $g \neq 0$. For $p = (x; \alpha < \beta) \in P_g$ both gradient terms $(\partial_\alpha A_\beta(x))^2$ (pair $(x, x + \varepsilon e_\alpha)$, component β) and $(\partial_\beta A_\alpha(x))^2$ (pair $(x, x + \varepsilon e_\beta)$, component α) are present in Q, because the pairs lie in X. With $F(p) = \varepsilon^{-1}(\partial_\alpha A_\beta(x) - \partial_\beta A_\alpha(x))$,

$$\|dA\|_{P_g}^2 := \sum_{p \in P_g} \varepsilon^3 F(p)^2 = \sum_{p \in P_g} \varepsilon\, (\partial_\alpha A_\beta(x) - \partial_\beta A_\alpha(x))^2$$
$$\leq 2 \sum_{p \in P_g} \varepsilon \big[(\partial_\alpha A_\beta(x))^2 + (\partial_\beta A_\alpha(x))^2\big] \leq 2Q(A),$$

where the last step is a counting statement: the ordered triple (site x, component β, direction $\alpha \neq \beta$) determines the plaquette $(x; \{\alpha, \beta\})$, so distinct plaquettes use distinct gradient terms of Q and no term is counted twice. Also $\langle v, A\rangle_k = F_{\Lambda_k}(g) = \sum_{p \in P_g} \varepsilon^3 g_p F(p) =: \langle g, dA\rangle_{P_g}$. Hence

$$\langle v, Cv\rangle \leq \sup_A \big[2\langle g, dA\rangle_{P_g} - \tfrac{1}{2}\|dA\|_{P_g}^2\big] \leq \sup_{B \in \mathbb{R}^{P_g}} \big[2\langle g, B\rangle_{P_g} - \tfrac{1}{2}\|B\|_{P_g}^2\big] = 2\|g\|_{2,k}^2,$$

the last supremum being attained at $B = 2g$. The inequality $Q(A) - \frac{1}{2}\|dA\|^2_{P_g} \geq 0$ used in the middle is in fact the identity

$$Q(A) - \tfrac{1}{2}\|dA\|^2_{P_g} = \tfrac{1}{2}\sum_{p\in P_g} \varepsilon\,(\partial_\alpha A_\beta(x) + \partial_\beta A_\alpha(x))^2 + \big(\text{gradient terms of } Q \text{ used by no } p \in P_g\big) + \mu_0^2\|A\|^2_{2,k}. \quad \square$$

Remark 7.2 (on the constant 2). On a King torus (Definition 12.1) the Hodge identity of the periodic lattice gives the constant 1 in place of 2 (a direct computation: the source d^*g is co-exact and the massive photon form commutes with d^*); on the Neumann cube no Hodge argument is available, and whether 2 is the best constant for the exact quantity $\rho := \sup_g \operatorname{Var} F(g)/\|g\|^2_{2,k}$ (which depends on ε and μ_0 only through $\mu_0\varepsilon$) is not known; nothing below depends on it. The second bound is the cube form of the two-point bound $0 \leq S_2(f) \leq \mu_0^{-2}\|d^*f\|^2_{L^2}$; it is the worse of the two at small μ_0.

7.2. **The comparison convention.** On the comparison convention only finite-volume statements are proved: a local bound on the lattice functionals which uses no hypothesis and no correlation inequality, and the finite-volume regularity bound Theorem 7.4.

Lemma 7.3 (a local bound free of hypotheses)**.** *For either convention and every compact $K \subset \mathbb{C}^n$,*

$$\sup_{k\geq 0}\ \sup_{J\geq J_0}\ \sup_{z\in K} \big|\langle e^{\sum_j z_j F_{\Lambda_k}(f_j)}\rangle_{k,J}\big| \ \leq\ \exp\Big(3C_0 \sup_{z\in K}|g(z)|_4^2\Big) < \infty,$$

with C_0 the constant of Lemma 2.9(E1)*. No hypothesis and no correlation inequality is used.*

Proof. $|\langle e^{\sum z_j F(f_j)}\rangle| \leq \langle e^{F(g(z))}\rangle$. Both cube states are BFS measures whose Gaussian is the Neumann photon Gaussian of X_J with covariance C (the matter on T_J does not touch the photon variables), so Lemma 2.11 with $s = 1$ and $v = d_k^* g(z)$ gives $\langle e^{F(g(z))}\rangle \leq e^{\langle v, Cv\rangle_k/2} \leq e^{\|g(z)\|^2_{2,k}}$ by Lemma 7.1. The plaquette centres of each of the three orientations form a translate of $\varepsilon_k\mathbb{Z}^3$, and $|g_{c(p)}(e_\alpha, e_\beta)| \leq (1 + |c(p)|^2)^{-2}|g|_4$, so by (E1) each orientation contributes at most $C_0|g|_4^2$ to $\|g\|^2_{2,k}$ and $\|g(z)\|^2_{2,k} \leq 3C_0|g(z)|_4^2$ for every k (this is the count behind (E3) of Lemma 2.9). Finally $|g(z)|_4 \leq \sum_j |\operatorname{Re} z_j||f_j|_4$ is bounded on K. $\square$

The existence theorem (Theorem A) is posed on King's convention. The finite-volume bound below holds on both conventions and needs no hypothesis; the limit statements are proved for King's state in Theorem 7.6.

Theorem 7.4 (finite-volume regularity, comparison convention)**.** *Let $f_1, \dots, f_n$, J_0, $g(z)$ be as in Convention 7.1. For all $k \geq 0$, $J \geq J_0$, $\mu_0 > 0$, $z \in \mathbb{C}^n$, and in the presence of any real scalar source $h \in C_c^\infty$,*

$$\left|\frac{\langle e^{\sum_j z_j F_{\Lambda_k}(f_j)} e^{:\phi^2:(h)}\rangle^{(a)}_{k,J}}{\langle e^{:\phi^2:(h)}\rangle^{(a)}_{k,J}}\right| \ \leq\ \frac{\langle e^{F_{\Lambda_k}(g(z))} e^{:\phi^2:(h)}\rangle^{(a)}_{k,J}}{\langle e^{:\phi^2:(h)}\rangle^{(a)}_{k,J}} \ \leq\ \exp\Big(\min\big\{\|g(z)\|^2_{2,k},\ \tfrac{1}{2}\mu_0^{-2}\|d_k^* g(z)\|^2_{2,k}\big\}\Big).$$

Nothing beyond the definition of the cube state enters.

Proof. The first inequality is $\operatorname{Re}\sum_j z_j F(f_j) = F(g(z))$. The measure $e^{:\phi^2:(h)} d\langle\cdot\rangle^{(a)}_{k,J}$, normalised, is a BFS measure whose single-site weights carry the extra factor $e^{\varepsilon^3 h(x) r_x^2}$ and whose photon covariance is the same C; Lemma 2.11 with $s = 1$ and $v = d_k^* g(z)$ bounds the middle expression by $e^{\langle v, Cv\rangle_k/2}$, and Lemma 7.1 gives both terms of the minimum. $\square$

Remark 7.5 (the constants are absolute). The bound $e^{\|g\|_2^2}$ of Theorems 7.4 and 7.6 and the constants $(\sqrt{2}\,e)^n(n!)^{1/2}$, $(e\pi/2)^n(n!)^{1/2}$, σ_n of E0″ depend on none of e, λ, m, μ_0 and M: the only ingredients are Gaussian domination, the Neumann form's constant 2 and the elementary facts (E1)–(E4). The second bound of Theorem 7.4 and of Theorem 7.6(a),(b) is the one that carries μ_0^{-2}. The proof of Theorem 7.6(b) uses, besides the existence clause Theorem A, only the cube monotonicity of Corollary 4.3 (cone-direction correlation inequalities), Gaussian domination and Lemma 7.1; it uses neither [9, Thm. 6.2(3),(4)] (Lemma 2.10(d)) nor any local bound resting on that reversed inequality.

7.3. **King's convention.** This subsection proves the regularity axiom for King's state: the lattice bounds, the limits in k and in J, and the bounds on the Schwinger functions (Theorem 7.6).

For the K family write $a^K_{k,J}(z) := \langle e^{\sum_j z_j F(f_j)}\rangle^K_{k,J}$ as in Corollary 4.4, and Z^K_J, Z^K_∞ for its limits.

Theorem 7.6 (regularity of King's state)**.** *Let $f = (f_1, \dots, f_n)$, J_0, $g(z)$ be as in Convention 7.1.*

(a) (No hypothesis.) *For all $k \geq 0$, $J \geq J_0$, $z \in \mathbb{C}^n$,*

$$|a^K_{k,J}(z)| \;\leq\; \langle e^{F(g(z))}\rangle^K_{k,J} \;\leq\; \exp\Big(\min\big\{\|g(z)\|^2_{2,k},\ \tfrac{1}{2}\mu_0^{-2}\|d_k^* g(z)\|^2_{2,k}\big\}\Big),$$

and for every real 2-form u whose plaquettes have their corners in X_J: $\langle F(u)\rangle^K_{k,J} = 0$ and $\langle F(u)^2\rangle^K_{k,J} \leq \langle v, Cv\rangle_k \leq 2\|u\|^2_{2,k}$, $v = d_k^* u$.

(b) (Limits.) *$Z^K_J(z) := \lim_k a^K_{k,J}(z)$ exists locally uniformly on $\mathbb{C}^n$ and is entire; $Z^K_\infty(z) := \lim_J Z^K_J(z)$ exists locally uniformly on $\mathbb{C}^n$ and is entire; $|Z^K_J(z)|, |Z^K_\infty(z)| \leq e^{\|g(z)\|^2_2}$ and also $\leq \exp(\frac{1}{2}\mu_0^{-2}\|d^* g(z)\|^2_{L^2})$; $J \mapsto Z^K_J(\mathrm{i}t)$ is nondecreasing on $J \geq J_0$ with values in $[0,1]$ for $t \in \mathbb{R}^n$; and $Z^K_\infty(x) \geq 1$ at every real $x \in \mathbb{R}^n$.*

(c) (Schwinger functions.) *The Schwinger functions $S^K_n(f_1, \dots, f_n) := \partial_{z_1} \cdots \partial_{z_n} Z^K_\infty(0)$ satisfy $|S^K_n| \leq (\sqrt{2}\,e)^n (n!)^{1/2} \prod_i \|f_i\|_2 \leq (e\pi/2)^n (n!)^{1/2} \prod_i |f_i|_4$ for real f_i; the lattice Schwinger functions $S^{K,(k,J)}_n := \partial_{z_1} \cdots \partial_{z_n} a^K_{k,J}(0)$ satisfy the same bound at every (k, J) with $\|f_i\|_{2,k}$ in place of $\|f_i\|_2$, and $S^{K,(k,J)}_n \to S^K_n$ as $k \to \infty$ and then $J \to \infty$. Each S^K_n extends to a continuous symmetric multilinear form on $\mathcal{S}(\mathbb{R}^3; \Lambda^2_{\mathbb{C}})^n$ with the same bound, carrying the factor 2^n for complex test functions: E0″ holds with $s = 4$ and $\sigma_n = (e\pi)^n (n!)^{1/2} \leq \alpha_\beta (n!)^\beta$ for every $\beta > \frac{1}{2}$.*

Proof. We first prove the bounds of (a) by Gaussian domination; we then prove (b) in four steps, the limit $k \to \infty$ at fixed J, the monotonicity in J on the imaginary axes, the limit $J \to \infty$, and the lower bound at real points; and finally we prove (c) by Cauchy's estimates on a polydisc.

(a) The two inequalities are the bound of Lemma 7.3 for the K family, with Lemma 7.1 kept in both forms: $\mathrm{Re}\sum z_j F(f_j) = F(g(z))$; $\langle\cdot\rangle^K_{k,J}$ is a BFS measure (the killed sites carry point masses, which Lemma 2.11 allows: at a killed site $I_q(0) = \delta_{q0}$ and the currents vanish on its bonds), and its Gaussian is the Neumann photon Gaussian of X_J, the same as the comparison cube's, so Lemma 2.11 with $s = 1$, $v = d_k^* g(z)$ and Lemma 7.1 give $\langle e^{F(g(z))}\rangle^K_{k,J} \leq e^{\langle v, Cv\rangle_k/2} \leq \exp(\frac{1}{2}\min\{2\|g\|^2_{2,k}, \mu_0^{-2}\|v\|^2_{2,k}\})$. *Second-moment clause.* $\langle F(u)\rangle^K_{k,J} = 0$ because the state is invariant under $(A, \vartheta) \mapsto (-A, -\vartheta)$ and $F(u)$ is odd. For real s, $e^x + e^{-x} \geq 2 + x^2$ pointwise and $e^{\pm sF(u)}$ are integrable, so $2 + s^2\langle F(u)^2\rangle^K_{k,J} \leq \langle e^{sF(u)}\rangle^K_{k,J} + \langle e^{-sF(u)}\rangle^K_{k,J} \leq 2e^{s^2\langle v, Cv\rangle_k/2}$ (Lemma 2.11 at $\pm s$; $\pm v$ give the same quadratic form). Hence $\langle F(u)^2\rangle^K_{k,J} \leq 2(e^{s^2\langle v, Cv\rangle_k/2} - 1)/s^2 \to \langle v, Cv\rangle_k$ as $s \downarrow 0$, and Lemma 7.1 applies since the corners lie in X_J.

(b) *Step 1: $k \to \infty$ at fixed $J \geq J_0$.* The functions $a^K_{k,J}$ are entire on $\mathbb{C}^n$: finite-dimensional integrals with Gaussian decay in A ($\mu_0 > 0$) and $e^{-\lambda\varepsilon^3 r^4}$ decay in r at the alive sites, converging

absolutely and locally uniformly in z. By (a) they are locally uniformly bounded, uniformly in k and $J \geq J_0$ ($\|g\|_{2,k}^2 \leq 3C_0|g|_4^2$ by (E3) of Lemma 2.9); by Theorem A they converge at every real point. $\mathbb{R}^n$ is a set of uniqueness for entire functions on $\mathbb{C}^n$ (a holomorphic function vanishing on $\mathbb{R}^n$ has all its Taylor coefficients at 0 equal to 0). By Montel's theorem every subsequence has a locally uniformly convergent sub-subsequence; all subsequential limits are entire and agree on $\mathbb{R}^n$, hence everywhere [22]; so $a^K_{k,J} \to Z^K_J$ locally uniformly on $\mathbb{C}^n$ and Z^K_J is entire (this is Corollary 4.4(ii)(α)). The bounds of (a) pass to the limit: $|Z^K_J(z)| \leq \lim_k e^{\|g(z)\|^2_{2,k}} = e^{\|g(z)\|_2^2}$ by (E3), and $|Z^K_J(z)| \leq \lim_k \exp(\frac{1}{2}\mu_0^{-2}\|d_k^* g(z)\|^2_{2,k}) = \exp(\frac{1}{2}\mu_0^{-2}\|d^* g(z)\|^2_{L^2})$ by (2). Both bounds are free of J.

Step 2: monotonicity in J on the imaginary axes. For $t \in \mathbb{R}^n$ and $f := \sum_j t_j f_j$, $a^K_{k,J}(\mathrm{i}t) = \langle e^{\mathrm{i}F(f)}\rangle^K_{k,J}$ is real (the state is invariant under $(A,\vartheta) \mapsto (-A,-\vartheta)$, which maps $F(f)$ to $-F(f)$) and of modulus ≤ 1, and $J \mapsto a^K_{k,J}(\mathrm{i}t)$ is nondecreasing on $J \geq J_0$ at fixed k: this is the cube monotonicity of Corollary 4.3 for the class-$\mathcal{I}$ observable $\Theta(\mathrm{i}f, 0)$, whose plaquettes lie in X_{J_0} by the support condition ((S) holds for X_{J_0}). It is ≥ 0 by the current representation of Lemma 2.11: with $\Psi(A) = \sum_q w(q)\cos\langle\kappa(q), A\rangle$, $w(q) \geq 0$, and the Gaussian identity at $a = \mathrm{i}v \pm \mathrm{i}\kappa$,

$$\langle e^{\mathrm{i}\langle v, A\rangle}\rangle^K_{k,J} = e^{-\langle v, Cv\rangle_k/2}\,\mathbb{E}^\sharp \cosh\langle Cv, \kappa\rangle \;\geq\; 0, \qquad v = d_k^* \textstyle\sum_j t_j f_j,$$

the killed sites being covered as in (a). Letting $k \to \infty$ (Step 1), $J \mapsto Z^K_J(\mathrm{i}t)$ is nondecreasing with values in $[0,1]$, hence convergent, for every $t \in \mathbb{R}^n$.

Step 3: $J \to \infty$. $\{Z^K_J\}_{J \geq J_0}$ is locally uniformly bounded on $\mathbb{C}^n$ by the J-free bounds of Step 1, and converges on $(\mathrm{i}\mathbb{R})^n$ (Step 2), a totally real subspace of maximal dimension, hence a set of uniqueness for entire functions (at a point $\mathrm{i}t_0$ the restriction to $\mathrm{i}t_0 + \mathrm{i}\mathbb{R}^n$ determines the Taylor series). Montel's theorem and the identity theorem as in Step 1 give $Z^K_J \to Z^K_\infty$ locally uniformly on $\mathbb{C}^n$, Z^K_∞ entire, and both bounds of Step 1 pass to the limit (this is Corollary 4.4(ii)(β)).

Step 4: $Z^K_\infty(x) \geq 1$. For real x, $a^K_{k,J}(x) = \langle e^{F(g(x))}\rangle^K_{k,J} \geq e^{\langle F(g(x))\rangle^K_{k,J}} = 1$ by Jensen's inequality and (a); the inequality is closed under both limits.

(c) Let $\|f_j\|_{2,k} > 0$ for all j (otherwise $F_{\Lambda_k}(f_j) = 0$ and $S_n^{K,(k,J)} = 0$). Cauchy's estimates on the polydisc $|z_j| \leq \rho_j := \rho/(n\|f_j\|_{2,k})$ with the bound (a): there $\|g(z)\|_{2,k} \leq \sum_j |z_j|\|f_j\|_{2,k} \leq \rho$, so $|a^K_{k,J}| \leq e^{\rho^2}$ and $|S_n^{K,(k,J)}| \leq e^{\rho^2}\prod_j \rho_j^{-1} = e^{\rho^2} n^n \rho^{-n} \prod_j \|f_j\|_{2,k}$; at $\rho^2 = n/2$ the prefactor is $e^{n/2}2^{n/2}n^{n/2} \leq e^{n/2}2^{n/2}e^{n/2}(n!)^{1/2} = (\sqrt{2}\,e)^n (n!)^{1/2}$ by (E4). The Taylor coefficients at 0 converge under the locally uniform convergences of (b) (Weierstrass), and $\|f_j\|_{2,k} \to \|f_j\|_2$ by (E3), which gives the bound for S^K_n; (E2) gives the $|\cdot|_4$ form. S^K_n is symmetric and multilinear (derivatives of an entire function of a linear combination) and continuous for the L^2 norms, so it extends by density from C_c^∞ to $\mathcal{S}$; for complex $f_i = f_i' + \mathrm{i}f_i''$ with real parts, multilinearity writes $S^K_n(f_1,\dots,f_n)$ as a sum of 2^n terms $\mathrm{i}^m S^K_n(f_1^\sharp,\dots,f_n^\sharp)$, each bounded by $(e\pi/2)^n(n!)^{1/2}\prod|f_i^\sharp|_4 \leq (e\pi/2)^n(n!)^{1/2}\prod|f_i|_4$, since $|\mathrm{Re}\, f|_4, |\mathrm{Im}\, f|_4 \leq |f|_4$; the kernel theorem gives the tempered distribution. The axiom E0″ of [30, p. 287] asks for $|S_n(f_1,\dots,f_n)| \leq \sigma_n \prod_i |f_i|_s$ on $\mathcal{S}$ with $\sigma_n \leq \alpha(n!)^\beta$; here $s = 4$: on scalar test functions $|\cdot|_4$ is the norm $|\cdot|_4$ of [30, (2.1), p. 284] (Definition 2.8), and a 2-form f whose only non-zero components are $f_{\alpha\beta} = -f_{\beta\alpha} = q$ ($\alpha < \beta$) has $|f|_4 = |q|_4$ by the definition of $|\cdot|_4$ on 2-forms, so the bound, read on such test functions, is E0″ for the Schwinger functions of the three component fields $F_{\alpha\beta}(q) := F(f)$, $\alpha < \beta$, mixed ones included (each slot may carry a different component); and $\sigma_n = (e\pi)^n(n!)^{1/2} \leq \alpha_\beta(n!)^\beta$ exactly when $(e\pi)^n/(n!)^{\beta-1/2}$ is bounded, that is for every $\beta > \frac{1}{2}$. □

7.4. **The limits of King's state, with cone-direction inequalities only.** For King's state and an admissible scheme a, the limit functionals of the Wick sector on the tube have the following

properties, established in Theorem 28.5, Proposition 28.6, and Lemma 13.11: (i) locally uniform convergence of the fixed-cube functionals to a holomorphic G_J^K; (ii) on the cone, $J \mapsto G_J^K$ nondecreasing; (iii) the cube limit $G_\infty^K = \lim_J G_J^K$, locally uniform and holomorphic, positive at real sources, with $G_J^K \le G_\infty^K$ on the cone; (iv) $\ln G_\infty^K$ finite, convex and continuous on the real tube; (v) translation invariance; (vi) the change of scheme. At $h = 0$ these conclusions hold by the existence clause Theorem A alone; the natural local bound, $|Z_J^K(\zeta)| \le Z_J^K(\operatorname{Re}\zeta) \le Z_{J_0}^K(\operatorname{Re}\zeta)$, would use the class-$\mathcal{D}$ form of the cube monotonicity, that is Lemma 2.10(d). The following lemma states the curvature sector with that use removed; the Wick sector on the tube is Lemma 13.11.

Lemma 7.7 (the limits of King's state, curvature sector, with cone-direction inequalities only)**.** *Let $f_1, \dots, f_N$ be a finite family of real 2-forms in C_c^∞ supported at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_{J_0}$, put $Z_{k,J}^K(\sum_j z_j f_j, 0) := \langle e^{\sum_j z_j F(f_j)} \rangle_{k,J}^K$, let Z_J^K, Z_∞^K be its limits of Theorem 7.6, and $g(z) = \sum_j (\operatorname{Re} z_j) f_j$. Then:*

(i) *$Z_{k,J}^K(\cdot, 0) \to Z_J^K(\cdot, 0)$ locally uniformly on $\mathbb{C}^N$ as $k \to \infty$, and $Z_J^K(\cdot, 0)$ is entire, for every $J \ge J_0$;*

(ii) *$J \mapsto Z_J^K(\mathrm{i}f, 0)$ is nondecreasing on $J \ge J_0$ for every real $f = \sum_j t_j f_j$;*

(iii) *for all $k \ge 0$, $J \ge J_0$ and $z \in \mathbb{C}^N$,*

$$\begin{aligned} |Z_{k,J}^K(\textstyle\sum_j z_j f_j, 0)| \le \langle e^{F(g(z))} \rangle_{k,J}^K &\le e^{\frac{1}{2}\langle v, Cv\rangle_k} \\ &\le \exp\left(\tfrac{1}{2}\mu_0^{-2} \|d_k^* g(z)\|_{2,k}^2\right) \\ &\le \exp\left(\tfrac{1}{2}\mu_0^{-2} \sup_{k'} \|d_{k'}^* g(z)\|_{2,k'}^2\right), \quad v = d_k^* g(z), \end{aligned}$$

a bound uniform in k and J and locally uniform in z; consequently $Z_\infty^K(\cdot, 0) = \lim_J Z_J^K(\cdot, 0)$ exists locally uniformly on $\mathbb{C}^N$, is entire, satisfies $Z_\infty^K(f, 0) > 0$ for real f, and $Z_J^K(\mathrm{i}f, 0) \le Z_\infty^K(\mathrm{i}f, 0)$ for real f;

(iv) *for every $\rho > 0$, $\Phi^K(f) := \ln Z_\infty^K(f, 0)$ is finite, convex and continuous on the Fréchet space $C_c^\infty(\bar{B}_\rho; \Lambda^2)$;*

(v) *$Z_\infty^K(\tau_b f, 0) = Z_\infty^K(f, 0)$ for all $b \in \mathbb{R}^3$ and all complex coefficient vectors, τ_b acting on every f_j.*

Clauses (i), (iii)–(v) *use, besides the existence clause Theorem A, only the cube monotonicity of Corollary 4.3 and the translation sandwich of Lemma 8.3 (cone-direction correlation inequalities), Lemma 2.11, Lemma 2.14*(b), (2)*, Lemma 2.17, Montel's theorem and the identity theorem. The nonincreasing clause at $h = 0$, $J \mapsto Z_J^K(f, 0)$ nonincreasing for real f, would follow from the class-$\mathcal{D}$ form of Corollary 4.3, that is Lemma 2.10*(d)*, at $g = 0$ with $k \to \infty$ by* (i)*; it is not used by the curvature sector of this paper. The clause on the change of scheme is void at $h = 0$, no scheme entering.*

Proof. (i) and (ii) are Steps 1 and 2 of the proof of Theorem 7.6(b). (iii) The first inequality is Lemma 2.14(b), the second is Lemma 2.11 with $s = 1$ (King's cube state is a BFS measure, the killed sites carrying point masses, as in the proof of Theorem 7.6(a)), the third is $C \le \mu_0^{-2}$ as forms (the photon form is bounded below by $\mu_0^2 \|A\|_{2,k}^2$, the second bound of Lemma 7.1), and the supremum over k' is finite by (2). The bound passes to Z_J^K by (i) and makes $\{Z_J^K(\cdot, 0)\}_{J \ge J_0}$ locally bounded on $\mathbb{C}^N$; with the convergence on $(\mathrm{i}\mathbb{R})^N$ from (ii) and the bound $|Z_J^K(\mathrm{i}f, 0)| \le 1$, Montel's theorem and the identity theorem give the locally uniform convergence to the entire $Z_\infty^K(\cdot, 0)$, as in Step 3 of the proof of Theorem 7.6(b). Positivity: $Z_{k,J}^K(f, 0) \ge e^{\langle F(f)\rangle_{k,J}^K} = 1$

for real f by Jensen's inequality and $\langle F(f)\rangle^K_{k,J} = 0$ (the symmetry $(A,\vartheta)\mapsto(-A,-\vartheta)$), closed under both limits. The last clause is (ii) in the limit.

(iv) Φ^K is finite by (iii). Convexity: for $f, f' \in C^\infty_c(\bar B_\rho;\Lambda^2)$ and $t\in[0,1]$ apply (i) and (iii) to the family (f,f') at the real coefficients $(t,1-t)$; at fixed k,J, $\ln Z^K_{k,J}(\cdot,0)$ is convex in the coefficients by Hölder's inequality, and convexity survives the two pointwise limits. Borel measurability: fix J with $MJ>\rho+1$, so that every $f\in C^\infty_c(\bar B_\rho;\Lambda^2)$ is supported at ℓ^∞-distance >1 from $\mathbb{R}^3\setminus C_J$; at each k, $\ln Z^K_{k,J}(f,0)$ depends on f only through the finitely many sampled values $f_{c(p)}(e_\alpha,e_\beta)$, continuous functionals of f in the C^∞ topology, and the finite-dimensional integral is continuous in them (dominated convergence); so $\Phi^K=\lim_J\lim_k$ of continuous functions is Borel (of Baire class ≤ 2). $C^\infty_c(\bar B_\rho;\Lambda^2)$ is a closed subspace of $C^\infty(\bar B_{\rho+1};\Lambda^2)$, hence Fréchet. Lemma 2.17 gives the continuity.

(v) is Corollary 8.4, whose proof uses the lattice sandwich of Lemma 8.3 between King's cubes of indices $J-1$ and $J+1$, $k\to\infty$ by (i) (Theorem A covers the translated family), $J\to\infty$ by (iii), the identity theorem with the uniqueness set $(\mathrm{i}\mathbb{R})^N$, the Lipschitz bound from the second-moment clause of Theorem 7.6(a) and the density of $\bigcup_k 5^{-k}\mathbb{Z}^3$. □

The following remark records what the proofs of this section use of the cube functionals, so that they apply to the comparison family $\mathrm{ND}^{(a)}$ (Section 3) wherever its own fixed-cube limits exist, which is not decided here (Remark 4.5); the comparison family itself enters the main theorem only through its limit states along continuum subsequences, which exist without any such clause (Section 14). The remarks of the same name in later sections (Remarks 8.2, 10.10 and 13.13) do the same for the statements of their sections.

Remark 7.8 (the comparison family). The proofs of Theorem 7.6(b),(c) and Lemma 7.7 use, of the K family, only the local bound of Theorem 7.6(a) (which Lemma 7.3 gives for the comparison family with the same constant), the existence of the fixed-cube limits at real coefficients (Theorem A), the cube monotonicity on the cone (Corollary 4.3; for the comparison family the first inequality of Lemma 2.13), the translation sandwich (Lemma 8.3; for the comparison family Lemma 8.1), Montel's theorem and the identity theorem. As noted in Section 3, they apply to the comparison family $\mathrm{ND}^{(a)}$ wherever its fixed-cube limits exist at real coefficients (Remark 4.5). In every clause above the inequalities used are the cone directions of Lemma 2.10(a),(b) and of Lemma 4.2; the reversed inequalities of Lemma 2.10(d) enter the curvature sector of this paper nowhere.

8. Translations, the cubic group, and rotations

This section proves the clause OS1 of King's state (the infinite-volume state of King's construction [25, 26]) as far as the lattice allows: invariance under all translations and under the cubic group O_h about every point; full $O(3)$ invariance comes from the rotation theorem (Theorem C, proved in the rotation part, Theorem 15.1) and is stated at the end. No cube is invariant under a non-trivial translation or a reflection; what replaces exactness is a sandwich between a smaller and a larger cube of the same family, in the direction of the cone inequalities of Lemmas 2.13 and 4.1, followed by the limits $k\to\infty$, $J\to\infty$. Coordinate permutations alone are exact (Lemma 8.5).

Conventions. We use the notation of Definition 2.5 and Corollary 4.4: for a finite family $f=(f_1,\dots,f_N)$ of real 2-forms in $C^\infty_c(\mathbb{R}^3;\Lambda^2)$ and $z\in\mathbb{C}^N$, $a^K_{k,J}(z)=\langle e^{\sum_j z_j F(f_j)}\rangle^K_{k,J}$ and $a^{(a)}_{k,J}(z)=\langle e^{\sum_j z_j F(f_j)}\rangle^{(a)}_{k,J}$ are the lattice functionals, Z^K_J, Z^K_∞ the limits of the first (Theorem 7.6(b)), and $g(z)=\sum_j(\mathrm{Re}\, z_j)f_j$. Since F_{Λ_k} is linear in the source (extended $\mathbb{C}$-linearly), a lattice functional depends on a family only through its total source, and we use the same symbol for a coefficient

vector and for its total source: $a^K_{k,J}(z) = Z^K_{k,J}(\sum_j z_j f_j)$ with $Z^K_{k,J}(u) := \langle e^{F(u)}\rangle^K_{k,J}$ for a complex-valued 2-form u; in particular, for a real 2-form u, $Z^K_{k,J}(\mathrm{i}u)$ is the one-member functional at the coefficient i, and the same convention is carried to Z^K_J, Z^K_∞ (so $Z^K_\infty(\mathrm{i}u_t) = Z^{K,f}_\infty(\mathrm{i}t)$ for $u_t = \sum_j t_j f_j$). When several families are in play we write $a^{K,f}_{k,J}$, $Z^{K,f}_J$, $Z^{K,f}_\infty$. $J_0(f)$ is the least J such that every $\operatorname{supp} f_j$ lies at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_J$; for $J \geq J_0(f)$ the support condition (S) holds for X_J. The transport $(O\cdot f)_x(u,v) = f_{O^{-1}x}(O^{-1}u, O^{-1}v)$ and the translation $(\tau_b f)_x = f_{x-b}$ are those of Definition 2.5; $O\cdot(O'\cdot f) = (OO')\cdot f$, and both are linear in f. Translations and the cubic group act on the Schwinger functions by this signed push-forward of 2-forms: for an improper O no extra sign appears.

8.1. **The translation sandwich on the comparison convention.** This subsection proves the translation sandwich on the comparison convention, where no site is killed and the cut-out lemma applies directly; the version on King's convention, which the limits need, follows in the next subsection.

Lemma 8.1 (translation sandwich, comparison convention)**.** *Let f be a finite family of real 2-forms, $k_0 \geq 0$, $b \in \Lambda_{k_0}$ with $|b|_\infty \leq M$, and $k \geq k_0$. Then, for every $J \geq J_0(f)+1$, the lattice covariance (19) holds for all coefficients, $X_{J-1} \subset X_J - b \subset X_{J+1}$, and for every $t \in \mathbb{R}^N$*

$$a^{(a)}_{k,J-1}(\mathrm{i}t) \;\leq\; a^{(a),\tau_b f}_{k,J}(\mathrm{i}t) \;\leq\; a^{(a)}_{k,J+1}(\mathrm{i}t).$$

Only the first inequality of Lemma 2.13 enters: the sandwich is taken on the cone.

Proof. Step 1 (lattice covariance). Let $k_0 \geq 0$, $b \in \Lambda_{k_0}$ and $k \geq k_0$. Then τ_b maps Λ_k onto itself, and the substitution $A'_\alpha(x) := A_\alpha(x+b)$, $\phi'(x) := \phi(x+b)$ is a bijection of the configurations of the comparison state on X_J onto those of the comparison state on $X_J - b$ (Definition 2.2 on the translated box) which carries the Lebesgue measure and the action to themselves: the action is a sum of local terms with site-independent coefficients (δm^2_ε, W_ε depend on ε only), and the boundary conditions are transported. The plaquette p corresponds to $p-b$ with the same orientation and $F(p)(A) = F(p-b)(A')$, so $F_{\Lambda_k}(\tau_b f)(A) = F_{\Lambda_k}(f)(A')$. Hence, for all coefficients,

$$Z_{k,X_J}\Big(\sum_j z_j \tau_b f_j\Big) = Z_{k,X_J-b}\Big(\sum_j z_j f_j\Big) \qquad (b \in \Lambda_{k_0},\ k \geq k_0), \tag{19}$$

the right side being the functional of the comparison state on $X_J - b$.

Step 2 (interlacing of the translated box). For $|b|_\infty \leq M$: $X_{J-1} \subset X_J - b \subset X_{J+1}$, since $-MJ - b_\mu \leq -M(J-1)$, $MJ - b_\mu \geq M(J-1)$, $-MJ - b_\mu \geq -M(J+1)$ and $MJ - b_\mu \leq M(J+1)$ for every μ.

Step 3 (the sandwich on the imaginary axes). Let $J \geq J_0(f)+1$, $t \in \mathbb{R}^N$ and $u_t = \sum_j t_j f_j$, so that $(u_t, 0) \in \mathcal{K}$ is supported at distance > 1 from $\mathbb{R}^3 \setminus C_{J-1}$. Apply the first inequality of Lemma 2.13 twice: with $Y = X_J - b$ and $X = X_{J-1}$, and with $Y = X_{J+1}$ and $X = X_J - b$. The support condition of the lemma holds because the plaquettes carrying u_t have their corners in X_{J-1}; no site is killed, so each cut-out factor is the comparison state on the smaller box (the Dirichlet terms of the smaller state are the diagonal parts of the cut bonds, and where the boundaries of two of the boxes touch they coincide). With (19),

$$a^{(a)}_{k,J-1}(\mathrm{i}t) \;\leq\; a^{(a),\tau_b f}_{k,J}(\mathrm{i}t) \;\leq\; a^{(a)}_{k,J+1}(\mathrm{i}t) \qquad (k \geq k_0,\ J \geq J_0(f)+1).$$

□

The following remark plays, for the translation argument of this section, the part that Remark 7.8 plays for the curvature sector.

Remark 8.2 (the comparison family). From a lattice sandwich such as Lemma 8.1, the translation invariance of a limit follows by the argument of Corollary 8.4 below: the limits $k \to \infty$ and $J \to \infty$ at the two ends and in the middle, the identity theorem off the imaginary axes, composition over lattice translations and continuity in b. On King's convention, where Theorem A provides the limits, it gives Corollary 8.4; as noted in Section 3, it applies to the comparison family wherever its fixed-cube limits exist, with Lemma 8.1 in place of Lemma 8.3. The translation clause for the joint field on the tube, by the same sandwich at the cone points of the tube, is Lemma 13.11(v).

8.2. **The translation sandwich on King's convention.** On King's convention the translated cube has its killed sites on $T_J - b$, which neither contains nor is contained in $T_{J\pm1}$, so Lemma 2.13 does not compare it with $C^K_{J\pm1}$ directly; one kill step (Lemma 2.10(c) through Lemma 2.12) and one cut-out with killed sites (Lemma 4.1) at each end do.

Lattice covariance on King's convention. For $b \in \Lambda_{k_0}$ and $k \geq k_0$ let $\langle\cdot\rangle^{K,-b}_{k,J}$ denote the comparison state on $X_J - b$ with the matter killed on $T_J - b$ (King's cube translated by $-b$). The substitution of Step 1 in the proof of Lemma 8.1 is a bijection of the configurations on X_J killed on T_J onto those on $X_J - b$ killed on $T_J - b$, carrying the point masses to the point masses and the action to the action (the kill set translates with the box). Hence, for every finite family and all coefficients,

$$Z^{a,K}_{k,J}\Big(\sum_j z_j \tau_b f_j, \sum_j w_j \tau_b h_j\Big) = Z^{a,K,-b}_{k,J}\Big(\sum_j z_j f_j, \sum_j w_j h_j\Big) \qquad (b \in \Lambda_{k_0},\ k \geq k_0), \tag{20}$$

the right side being the scheme functional of $\langle\cdot\rangle^{K,-b}_{k,J}$ with the same multiplier ($\mathsf{S}_{\Lambda_k}(\tau_b h) = \mathsf{S}_{\Lambda_k}(h)$).

Lemma 8.3 (translation sandwich, King's convention)**.** *Let $(f,g) \in \mathcal{K}$ be supported at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_{J_0}$, and let $b \in \Lambda_{k_0}$ with $|b|_\infty \leq M$. Then for every $k \geq k_0$ and every $J \geq J_0 + 1$,*

$$Z^{a,K}_{k,J-1}(\mathrm{i}f, g) \ \leq\ Z^{a,K}_{k,J}(\mathrm{i}\,\tau_b f, \tau_b g) \ \leq\ Z^{a,K}_{k,J+1}(\mathrm{i}f, g),$$

and, for the class-$\mathcal{D}$ observable $\Theta(f,-g)$ (real curvature coefficient), the reversed inequalities $Z^{a,K}_{k,J-1}(f,-g) \geq Z^{a,K}_{k,J}(\tau_b f, -\tau_b g) \geq Z^{a,K}_{k,J+1}(f,-g)$ hold.

Proof. We first reduce to the expectations of $\Theta(\mathrm{i}f, g)$ and $\Theta(f,-g)$; we then describe the geometry of the translated cube and the location of the sources, and prove the upper and the lower end of the sandwich by killing sites and cutting out a sub-cube.

By (20) the middle term is $Z^{a,K,-b}_{k,J}(\mathrm{i}f, g)$; the multipliers of the three terms are the same positive constant (Lemma 2.14(a); S is translation invariant), so it suffices to prove the inequalities for $\langle\Theta(\mathrm{i}f, g)\rangle$, and the reversed ones for $\Theta(f,-g)$; all these expectations are real.

Geometry. Here $b_\mu \in \varepsilon_{k_0}\mathbb{Z}$, $|b_\mu| \leq M$, and all sets are subsets of Λ_k. (G1) $\mathrm{alive}^K_J - b = \prod_\mu\{-MJ - b_\mu, \dots, MJ - 2\varepsilon - b_\mu\} \subset \mathrm{alive}^K_{J+1} = \{-MJ - M, \dots, MJ + M - 2\varepsilon\}^3$, since $-MJ - b_\mu \geq -MJ - M$ and $MJ - 2\varepsilon - b_\mu \leq MJ + M - 2\varepsilon$. (G2) $X_J - b \subset X_{J+1}$: the same arithmetic with ε in place of 2ε. (G3) $\mathrm{alive}^K_{J-1} = \{-MJ + M, \dots, MJ - M - 2\varepsilon\}^3 \subset \mathrm{alive}^K_J - b$, since $-MJ + M \geq -MJ - b_\mu$ and $MJ - M - 2\varepsilon \leq MJ - 2\varepsilon - b_\mu$. (G4) $X_{J-1} \subset X_J - b$, likewise. All four inclusions hold with margin $M - |b_\mu| \geq 0$ on every face, at every spacing, including $k = 0$, $M = 1$, $|b_\mu| = M$; non-strict inclusions are all that Lemma 4.1 needs.

Sources. By (S) for X_{J-1}, every plaquette carrying f has its corners in X_{J-1} and every site carrying g lies in alive^K_{J-1}; by (G1)–(G4) these lie in the site sets and alive sets of all three states. So the observable is one function of variables present and alive in all three states, and its scalar factor equals 1 at every site killed below.

Upper end. Start from $\langle\cdot\rangle^K_{k,J+1}$, the comparison state on X_{J+1} killed on T_{J+1}, a BFS measure of King type (Definition 2.3). (i) Kill $U_+ := \mathrm{alive}^K_{J+1} \setminus (\mathrm{alive}^K_J - b)$. These are alive sites of the state,

$g=0$ on U_+ (by (G3), $U_+\cap \mathrm{alive}^K_{J-1}=\emptyset$), and $\Theta(\mathrm{i}f,g)$ is continuous with $|\Theta|=e^{\sum_x g_x r_x^2}$; so by Lemma 2.10(c), through Lemma 2.12 (which allows the already killed sites T_{J+1}), killing U_+ does not increase $\langle\Theta(\mathrm{i}f,g)\rangle$, and by Lemma 2.10(d) with Lemma 2.12 it does not decrease $\langle\Theta(f,-g)\rangle$. The result is the comparison state on X_{J+1} killed on $T':=T_{J+1}\cup U_+=X_{J+1}\setminus(\mathrm{alive}^K_J-b)$ (by (G1); nothing is revived). (ii) Apply Lemma 4.1 with $V=X_{J+1}$, $T=T'$ and $X=X_J-b\subset V$ (by (G2)); the lemma allows any $T\subset V$ and any $X\subset V$. The cut-out factor is the comparison state on X_J-b killed on $(X_J-b)\cap T'=(X_J-b)\setminus(\mathrm{alive}^K_J-b)=T_J-b$, that is $\langle\cdot\rangle^{K,-b}_{k,J}$; the support condition (S_X) holds for $X=X_J-b$ by *Sources*. Hence $\langle\Theta(\mathrm{i}f,g)\rangle^{K,-b}_{k,J}\le\langle\Theta(\mathrm{i}f,g)\rangle^{(a),T'}_{X_{J+1}}\le\langle\Theta(\mathrm{i}f,g)\rangle^K_{k,J+1}$, and $\ge$ for class $\mathcal{D}$.

Lower end. Start from $\langle\cdot\rangle^{K,-b}_{k,J}$, the comparison state on X_J-b killed on T_J-b (of King type, a translate of a state of Definition 2.3). (i) Kill $U_-:=(\mathrm{alive}^K_J-b)\setminus\mathrm{alive}^K_{J-1}$, alive sites at which $g=0$ by *Sources*; as above, $\langle\Theta(\mathrm{i}f,g)\rangle$ does not increase and $\langle\Theta(f,-g)\rangle$ does not decrease, and the result is the comparison state on X_J-b killed on $T'':=(X_J-b)\setminus\mathrm{alive}^K_{J-1}$ (by (G3)). (ii) Lemma 4.1 with $V=X_J-b$, $T=T''$ and $X=X_{J-1}\subset V$ (by (G4)): the factor is the comparison state on X_{J-1} killed on $X_{J-1}\cap T''=X_{J-1}\setminus\mathrm{alive}^K_{J-1}=T_{J-1}$, that is $\langle\cdot\rangle^K_{k,J-1}$; ($\mathrm{S}_X$) for X_{J-1} is the hypothesis. Hence $\langle\Theta(\mathrm{i}f,g)\rangle^K_{k,J-1}\le\langle\Theta(\mathrm{i}f,g)\rangle^{K,-b}_{k,J}$, and $\ge$ for class $\mathcal{D}$.

There is no edge case because a translation by $b\in\Lambda_{k_0}$, $|b|_\infty\le M$, moves King's cube rigidly, with the alive sets nested at margins $M-|b_\mu|\ge 0$ at every spacing; the reflection of Lemma 4.6 shifts the cube by one lattice layer instead, which is why its one-step form needs $M\ge 2\varepsilon_k$. □

For the two classes of observables of Section 2 (class $\mathcal{I}$: $\Theta(\mathrm{i}f,g)$; class $\mathcal{D}$: $\Theta(f,-g)$), the class-$\mathcal{I}$ clause of the lemma uses Lemma 2.10(a),(b),(c),(e) and Lemma 2.12 only; the class-$\mathcal{D}$ clause uses Lemma 2.10(d) in addition. Only the class-$\mathcal{I}$ clause is used in this section.

Corollary 8.4 (translations of King's state)**.** *For every finite family f of real 2-forms, every $b\in\mathbb{R}^3$ and every $z\in\mathbb{C}^N$, $Z^K_\infty(\sum_j z_j\tau_b f_j)=Z^K_\infty(\sum_j z_j f_j)$. Moreover, for all real 2-forms $u,u'\in C^\infty_c$,*

$$|Z^K_\infty(\mathrm{i}u)-Z^K_\infty(\mathrm{i}u')|\le\sqrt{2}\,\|u-u'\|_2,$$

the L^2 norm of a 2-form being computed in a basis (e_c) of Λ^2 orthonormal for the norm formula.

Proof. Write $u_t=\sum_j t_j f_j$ for $t\in\mathbb{R}^N$. (1) *$b\in\Lambda_{k_0}$, $|b|_\infty\le M$, on the imaginary axes.* For $J\ge J_0(f)+1$ and $k\ge k_0$, Lemma 8.3 at $(u_t,0)\in\mathcal{K}$ gives $a^K_{k,J-1}(\mathrm{i}t)\le a^{K,\tau_b f}_{k,J}(\mathrm{i}t)\le a^K_{k,J+1}(\mathrm{i}t)$. Let $k\to\infty$: the ends converge by Theorem 7.6(b) for f at $J\mp1\ge J_0(f)$; the middle term by Theorem 7.6(b) for the family $\tau_b f$, which is admissible at J since $\operatorname{supp}f_j+b$ lies at distance >1 from $\mathbb{R}^3\setminus C_J$ ($C_{J-1}+b\subset C_J$ for $|b|_\infty\le M$), so that Theorem A, posed for every finite family, applies to it. Hence $Z^K_{J-1}(\mathrm{i}t)\le Z^{K,\tau_b f}_J(\mathrm{i}t)\le Z^K_{J+1}(\mathrm{i}t)$, and $J\to\infty$ (Corollary 4.4(ii)(β) at both ends) gives $Z^{K,\tau_b f}_\infty(\mathrm{i}t)=Z^{K,f}_\infty(\mathrm{i}t)$ for all $t\in\mathbb{R}^N$. (2) *All coefficients, same b.* $D(z):=Z^{K,\tau_b f}_\infty(z)-Z^{K,f}_\infty(z)$ is entire on $\mathbb{C}^N$ (Theorem 7.6(b) for both families) and vanishes on $(\mathrm{i}\mathbb{R})^N$, a totally real subspace of maximal real dimension, so every partial derivative of D at 0 vanishes and $D\equiv0$. (3) *All $b\in\bigcup_k\Lambda_k$.* Such a b lies in some Λ_{k_0} and is a finite sum $b_1+\cdots+b_n$ of vectors $b_i\in\Lambda_{k_0}$ with $|b_i|_\infty\le M$ (each coordinate splits into steps $\pm M$ and a remainder in $\varepsilon_{k_0}\mathbb{Z}$ of modulus $<M$); (1)–(2) hold for every family, in particular for the translated families $\tau_{b_1+\cdots+b_{i-1}}f$, and composing gives the identity for b. (4) *The Lipschitz bound.* For real 2-forms u,u' whose plaquettes have corners in X_J, at every (k,J), $|\langle e^{\mathrm{i}F(u)}\rangle^K_{k,J}-\langle e^{\mathrm{i}F(u')}\rangle^K_{k,J}|\le\langle|F(u-u')|\rangle^K_{k,J}\le\langle F(u-u')^2\rangle^{K\,1/2}_{k,J}\le\sqrt{2}\|u-u'\|_{2,k}$, by $|e^{\mathrm{i}x}-e^{\mathrm{i}y}|\le|x-y|$, the linearity of F_{Λ_k} and the second-moment clause of Theorem 7.6(a). Both sides have limits as $k\to\infty$ and $J\to\infty$ (Theorem 7.6(b) for the one-member families at the coefficient i; $\|u-u'\|_{2,k}\to\|u-u'\|_2$ by (2)). (5) *All $b_0\in\mathbb{R}^3$.*

Pick $b_n \to b_0$ with $b_n \in \bigcup_k \Lambda_k$. For real $u \in C_c^\infty$, $\tau_{b_n} u \to \tau_{b_0} u$ in L^2 (uniform continuity and compact support), so by (4) and (3), $Z_\infty^K(\mathrm{i}\tau_{b_0} u) = \lim_n Z_\infty^K(\mathrm{i}\tau_{b_n} u) = Z_\infty^K(\mathrm{i}u)$. With $u = u_t$ this is $Z_\infty^{K,\tau_{b_0} f}(\mathrm{i}t) = Z_\infty^{K,f}(\mathrm{i}t)$ for all t, and (2) extends it to all $z \in \mathbb{C}^N$. □

Nothing here uses a reflection or any property of the limit beyond Theorem 7.6; King's derivation of translation invariance from the uniqueness of his limit [26, Thm. 2.3(i)] is not used.

8.3. **The cubic group.** Coordinate permutations act exactly on every cube state, reflections do not: Lemma 8.5 extends the passive covariance of Lemma 2.15 to states with killed sites, and together with the reflection sandwich of Lemma 4.6 and the translation sandwich of the previous subsections it gives the cubic symmetry of King's state (Theorem 8.6).

Lemma 8.5 (passive covariance with killed sites)**.** *Let O, Λ, (f_α) and X be as in Lemma 2.15, and let $U \subset X$ be a set of killed sites. Then the identities of Lemma 2.15 hold between the state on X with the matter killed on U and the state on OX (frame (Of_α), transported boundary conditions and wall) with the matter killed on OU:*

$$\big\langle \Theta(z\, O\cdot f, w\, O\cdot h)\big\rangle_{O\Lambda, OX;\, OU} = \big\langle \Theta(zf, wh)\big\rangle_{\Lambda, X;\, U}, \qquad \mathsf{S}_{O\Lambda}(O\cdot h) = \mathsf{S}_\Lambda(h).$$

Proof. The configuration-space bijection $\phi'(Ox) = \phi(x)$, $A'_\alpha(Ox) = A_\alpha(x)$ of Lemma 2.15 carries $\prod_{x\in U} \delta_0(d\phi(x))\, d\nu$ to $\prod_{x\in OU} \delta_0(d\phi'(Ox))\, d\nu$ (the point mass at $\phi(x) = 0$ goes to the point mass at $\phi'(Ox) = 0$), and the diagonal parts $\frac{\varepsilon}{2} r^2$ at the alive ends of the bonds into U are frame-free; the rest of that proof is unchanged. □

Theorem 8.6 (cubic symmetry of King's state)**.** *For all real 2-forms $f_1, \dots, f_N \in C_c^\infty(\mathbb{R}^3; \Lambda^2)$, all $z \in \mathbb{C}^N$, all $O \in O_h$ (the signed permutation matrices) and all $b \in \mathbb{R}^3$,*

$$Z_\infty^K\Big(\sum_j z_j\, \tau_b O\cdot f_j\Big) = Z_\infty^K\Big(\sum_j z_j f_j\Big).$$

Hence the Schwinger functions S_n^K of Theorem 7.6(c) are $\mathbb{R}^3 \rtimes O_h$-covariant, F transforming as a 2-form.

Proof. The proof has seven steps: permutations are exact; the reflection σ_3 is handled on the imaginary axes by a sandwich between two cubes; the reflections σ_1, σ_2 are conjugates of σ_3; the group is generated by these; the identity extends from the imaginary axes to all coefficients; it holds about every point by translation; and it passes to the Schwinger functions.

"(k, J) exact" means an identity of the lattice functionals for all complex coefficients and all k, J for which both families have their plaquettes in X_J.

Step 1 (permutations are exact). Let P be a permutation matrix, $Pe_\alpha = e_{\pi(\alpha)}$. Then $P\Lambda_k = \Lambda_k$, $PX_J = X_J$ and $PT_J = T_J$ (the condition "$x_\mu = MJ - \varepsilon$ for some μ" is permutation invariant), hence P $\mathrm{alive}_J^K = \mathrm{alive}_J^K$. By Lemma 8.5 with $O = P$, $X = X_J$, $U = T_J$, $\langle \Theta(zP\cdot f)\rangle_{P\Lambda_k, PX_J;\, PT_J}$ in the frame $(e_{\pi(\alpha)})$ equals $\langle \Theta(zf)\rangle_{k,J}^K$, and the state on the left is $\langle\cdot\rangle_{k,J}^K$ itself: the Dirichlet rule is a condition on the site set X_J, the kill set is T_J, the Neumann rule keeps the same set of (bond, direction) pairs relabelled, the mass term runs over the same bonds, and the hopping and single-site terms are frame-free. The plaquette set and the centres are the same; for a transposed pair $(x; \beta, \alpha)$ both $F(p)$ and $f_{c(p)}(e_\beta, e_\alpha)$ change sign by antisymmetry, so their product is unchanged. Thus $a_{k,J}^{K, P\cdot f}(z) = a_{k,J}^{K,f}(z)$ is (k, J) exact. The distance-> 1 interior $(-MJ+1, MJ-1)^3$ of C_J is P-invariant, so $J_0(P\cdot f) = J_0(f)$, and Theorem 7.6(b) for both families gives $Z_\infty^{K,P\cdot f} = Z_\infty^{K,f}$ on $\mathbb{C}^N$.

Step 2 (the reflection σ_3 on the imaginary axes). Let $\sigma_3 = \mathrm{diag}(1,1,-1)$, $t \in \mathbb{R}^N$, $u_t = \sum_j t_j f_j$, $J_0 = J_0(f)$. Lemma 4.6(b), for every $k \geq 0$, every $M \geq 1$ and every $J \geq J_0 + 2$, gives

$$a^{K,f}_{k,J-2}(\mathrm{i}t) \;\leq\; a^{K,\sigma_3\cdot f}_{k,J}(\mathrm{i}t) \;\leq\; a^{K,f}_{k,J+2}(\mathrm{i}t),$$

since $\sigma_3 \cdot u_t = \sum_j t_j \sigma_3 \cdot f_j$. The family $\sigma_3 \cdot f$ is admissible at J, the interior being σ_3-symmetric, so $J_0(\sigma_3 \cdot f) = J_0(f)$. Let $k \to \infty$ (Theorem 7.6(b) for f at $J \mp 2$ and for $\sigma_3 \cdot f$ at J), then $J \to \infty$ (Corollary 4.4(ii)(β) at both ends): $Z^{K,\sigma_3\cdot f}_\infty(\mathrm{i}t) = Z^{K,f}_\infty(\mathrm{i}t)$ for all $t \in \mathbb{R}^N$. The one-step sandwich of Lemma 4.6(a) would serve whenever $M \geq 2\varepsilon_k$, that is for all but finitely many k, which the k-limit ignores; the two-step form is used because it needs no case distinction, and the J-limit absorbs the extra step. This is the use King makes of the correlation inequalities in the proof of his Theorem 3.2 [26], with the kill steps added; no reversed inequality enters.

Step 3 (σ_1, σ_2). $\sigma_\nu = P\sigma_3 P^{-1}$ with P the transposition of the axes ν and 3. Since the transport is a left action, for every real 2-form u,

$$Z^K_\infty(\mathrm{i}\sigma_\nu\cdot u) = Z^K_\infty(\mathrm{i}P\cdot(\sigma_3 P^{-1}\cdot u)) = Z^K_\infty(\mathrm{i}\sigma_3\cdot(P^{-1}\cdot u)) = Z^K_\infty(\mathrm{i}P^{-1}\cdot u) = Z^K_\infty(\mathrm{i}u),$$

by Step 1 for the family $\sigma_3 P^{-1} \cdot u$, Step 2 for the family $P^{-1} \cdot u$, and Step 1 again; each step is a statement about every family.

Step 4 (the group, on the imaginary axes). Every $O \in O_h$ is a product of a permutation matrix and some of $\sigma_1, \sigma_2, \sigma_3$, and if $Z^K_\infty(\mathrm{i}O\cdot u) = Z^K_\infty(\mathrm{i}u)$ for all real u holds for O_1 and O_2, it holds for $O_1 O_2$. By Steps 1–3, $Z^{K,O\cdot f}_\infty(\mathrm{i}t) = Z^{K,f}_\infty(\mathrm{i}t)$ for every $O \in O_h$, every family f and every $t \in \mathbb{R}^N$.

Step 5 (all coefficients). $D(z) := Z^{K,O\cdot f}_\infty(z) - Z^{K,f}_\infty(z)$ is entire on $\mathbb{C}^N$ (Theorem 7.6(b); $J_0(O\cdot f) = J_0(f)$) and vanishes on $(\mathrm{i}\mathbb{R})^N$, a set of uniqueness; so $D \equiv 0$. No splitting of sources is needed, there being no scalar source.

Step 6 (about every point). O_h about b is $\tau_b O \tau_{-b}$, and translations are Corollary 8.4 (every family, every $b \in \mathbb{R}^3$, all coefficients): $Z^K_\infty(\sum_j z_j \tau_b O \cdot f_j) = Z^K_\infty(\sum_j z_j O \cdot f_j) = Z^K_\infty(\sum_j z_j f_j)$.

Step 7 (Schwinger functions). Differentiating the identity of entire functions at $z = 0$ gives $S^K_n(\tau_b O \cdot f_1, \dots, \tau_b O \cdot f_n) = S^K_n(f_1, \dots, f_n)$ for all n and $(b, O) \in \mathbb{R}^3 \rtimes O_h$. □

Remark 8.7 (what cubic symmetry needs). King's cube is exactly symmetric under the coordinate permutations (Step 1) but not under the reflections; there the sandwich of Lemma 4.6 replaces exactness, as that of Lemma 2.16 does on the comparison convention. So O_h-invariance of King's state follows from the uniqueness of the monotone cube limit (Corollary 4.3 with Theorem 7.6(b)) together with the cut sandwich, and does not need the uniqueness of the infinite-volume state. Only class-$\mathcal{I}$ clauses enter, and no scheme or Wick constant.

8.4. **Rotations.** This subsection states the full rotation invariance of King's state, in the form in which the later sections use it.

Theorem 8.8 (rotations)**.** *Z^K_∞ is $\mathbb{R}^3 \rtimes O(3)$-invariant: for every finite family f, every $z \in \mathbb{C}^N$ and every $(b, R) \in \mathbb{R}^3 \rtimes O(3)$, $Z^K_\infty(\sum_j z_j \tau_b R \cdot f_j) = Z^K_\infty(\sum_j z_j f_j)$, and the Schwinger functions S^K_n are Euclidean covariant, F transforming as a 2-form.*

Proof. By Theorem 15.1, whose object is King's state $Z^K_\infty(\cdot, 0)$ of Theorem 7.6 (Definition 17.3), at every point of King's range, $Z^K_\infty(\cdot, 0)$ is $O(3)$-invariant for all complex coefficients; translations are Corollary 8.4; the Schwinger functions follow by differentiation at $z = 0$. By Corollary 4.4(iii), with the fixed-cube limits of Theorem A, the limit state $Z^{\mathrm{LIM}}_\infty(\cdot, 0)$ of the comparison family equals $Z^K_\infty(\cdot, 0)$ and is therefore $\mathbb{R}^3 \rtimes O(3)$-invariant as well; this is Theorem C. □

The rotation theorem, Theorem 15.1, asserts that King's state on curvature sources is $O(3)$-invariant at every point of King's range: outside an exceptional set E_W of couplings this follows

from the torus comparison of the rotation part, and at the points of E_W from the right-continuity of the state in the scalar mass.

The rotation theorem is proved in the rotation part (Section 15, Theorem 15.1). Nothing in Corollary 8.4 and Theorem 8.6 depends on it. Its consequences for reflection positivity about every plane and for the components of the measure are drawn in Theorem 11.5(c).

9. Reflection positivity I: the modified box

This section and the next prove reflection positivity about the coordinate planes; in the curvature sector these are rows (4a) and (4b) of Table 1. Everything in this section is on the lattice, at a fixed spacing. Its main results are three lemmas, on which Section 10 builds: reflection positivity of the modified box state (Lemma 9.3), and the two halves of the sandwich, which compares the modified box state with the Neumann–Dirichlet states on a smaller and on a larger box (Lemmas 9.9 and 9.10). Lemma 9.6 and Remark 9.7 explain why the route differs from that of [26].

King's Neumann–Dirichlet cube is not reflection symmetric about any plane, and the natural symmetrisation of its photon form is not reflection positive on gauge-invariant functionals (Remark 9.7). Reflection positivity is therefore proved first for the modified box state of [26, (3.38)] and then carried to the cube states by a two-sided comparison on the cone: the cube C_J sits inside the box (Lemma 9.9), and the box sits inside a larger cube once a margin proportional to μ_0^{-1} is allowed (Lemma 9.10). The passage to the limits is in Theorem 10.5. Throughout, $\varepsilon = \varepsilon_k$, the lattice is $\Lambda_k \subset \mathbb{R}^d$ with $d = 3$ (the certificates of Lemma 9.6 are also given for $d = 2$), a *lattice box* is a product $\Omega = \prod_\mu \{a_\mu, a_\mu + \varepsilon, \dots, b_\mu\} \subset \Lambda_k$, Ω^* is the set of its internal bonds (both ends in Ω) and Ω^{**} the set of its plaquettes (all four corners in Ω).

9.1. The lattice reflection and the modified box.

This subsection fixes the reflection planes available on the lattice (the mid-planes), the reflection of the fields, and King's modified box with its photon form.

Definition 9.1 (mid-planes and the lattice reflection)**.** The *mid-planes* are the planes $\{x_\nu = c\}$ with $c \in D := \bigcup_{k\ge 0} \varepsilon_k(\mathbb{Z} + \frac12)$. If $c \in \varepsilon_{k_0}(\mathbb{Z} + \frac12)$ then $c \in \varepsilon_k(\mathbb{Z} + \frac12)$ for every $k \ge k_0$, because $5^j(m + \frac12) = 5^j m + \frac{5^j - 1}{2} + \frac12$ and 5^j is odd; the same set D is obtained from any tail $k \ge k_1$ of the spacings. D is dense in $\mathbb{R}$ but not all of $\mathbb{R}$. For $c \in D$ let $k_0(c)$ be the least k with $c \in \varepsilon_k(\mathbb{Z} + \frac12)$, and for $\delta > 0$ let $k_0(c, \delta)$ be the least $k \ge k_0(c)$ with $\varepsilon_k < 2\delta$.

Fix ν, a mid-plane $c \in \varepsilon_k(\mathbb{Z} + \frac12)$ and a lattice box $\Omega \subset \Lambda_k$ with $\rho\Omega = \Omega$, $\rho = \rho_{\nu,c}$ the reflection of Definition 2.5. On the fields put $(\hat\rho\phi)(x) := \phi(\rho x)$ and $(\hat\rho A)(b) := A(\rho b)$, the image bond taken with its orientation: $(\hat\rho A)_\alpha(x) = A_\alpha(\rho x)$ for $\alpha \ne \nu$ and $(\hat\rho A)_\nu(x) = -A_\nu(\rho x - \varepsilon e_\nu)$. Then $\hat\rho$ is a signed permutation of the variables attached to Ω and Ω^* (it preserves Lebesgue measure and maps Ω^*, Ω^{**} to themselves); the *crossing bonds* $B_0 := \{\langle x^-, x^- + \varepsilon e_\nu\rangle : x^-_\nu = c - \varepsilon/2\}$ are mapped to their reversals, $(\hat\rho A)(b) = -A(b)$ for $b \in B_0$. Let $\Omega_\pm := \{x \in \Omega : \pm(x_\nu - c) > 0\}$ (no site lies on the plane), B_+ the bonds with both ends in Ω_+, $\mathfrak{B}_+$ the Borel functions of $(\phi|_{\Omega_+}, A|_{B_+})$, and

$$\theta G := \overline{G \circ \hat\rho} \qquad (G \in \mathfrak{B}_+).$$

On the smeared fields, $F_{\Lambda_k}(f) \circ \hat\rho = F_{\Lambda_k}(\rho^* f)$ and $W^a_{\Lambda_k}(h) \circ \hat\rho = W^a_{\Lambda_k}(h \circ \rho)$. Indeed, $F(p) \circ \hat\rho = \sigma_p F(\rho p)$ with $\sigma_p = -1$ exactly when $\nu \in \{\alpha, \beta\}$ (the orientation of the image plaquette, whose base point is shifted back one layer), the centres satisfy $c(\rho p) = \rho c(p)$, and $(\rho^* f)_{c(\rho p)}(e_\alpha, e_\beta) = f_{c(p)}(\rho e_\alpha, \rho e_\beta) = \sigma_p f_{c(p)}(e_\alpha, e_\beta)$; reindexing $p \mapsto \rho p$ gives the first identity. The scalar case is a relabelling of sites, and the c-numbers $2\mathsf{W}_\varepsilon\mathsf{S}(h)$ and $a_k\mathsf{S}(h)$ are invariant because ρ permutes Λ_k

$(2(x_\nu - c) \in \varepsilon\mathbb{Z})$. Consequently, for families ζ_i whose lattice observables $\Theta(\zeta_i)$ lie in $\mathfrak{B}_+$,

$$\theta(\Theta(\zeta_i))\,\Theta(\zeta_j) = \Theta(\theta\zeta_i^* \sqcup \zeta_j), \qquad m(\theta\zeta_i^* \sqcup \zeta_j) = \overline{m(\zeta_i)}\, m(\zeta_j) \tag{21}$$

for the scheme multiplier $m(\zeta) := e^{-\sum_j w_j a_k \mathsf{S}(h_j)}$ (a_k real), so that for any state $\langle\cdot\rangle$ on the box

$$\sum_{i,j} \bar\gamma_i\gamma_j\, m(\theta\zeta_i^* \sqcup \zeta_j)\langle\Theta(\theta\zeta_i^* \sqcup \zeta_j)\rangle = \sum_{i,j} \overline{(\gamma_i m(\zeta_i))}\,(\gamma_j m(\zeta_j))\,\big\langle\theta(\Theta(\zeta_i))\,\Theta(\zeta_j)\big\rangle. \tag{22}$$

Support. If the sources of ζ_i are supported in $\{x_\nu \ge c+\delta\}$ with $\delta > 0$, then for $\varepsilon_k < 2\delta$ every site carrying a sampled h and every corner of every plaquette carrying a sampled f (centre in the support, corners at $x_\nu \ge c+\delta-\varepsilon_k/2 > c$) lies in Ω_+, so $\Theta(\zeta_i) \in \mathfrak{B}_+$. The splitting $h = h^+ - h^-$ of Definition 10.1 is taken with cut-offs supported in $\{x_\nu \ge c+\delta/2\}$, so that $h^\pm$ are supported in $\{x_\nu > c\}$ as well.

Definition 9.2 (the modified box state)**.** Let $\Omega \subset \Lambda_k$ be a lattice box, $F(p) = \varepsilon^{-1}\sum_{b\in\partial p} A(b)$, $\Omega(y)$ the indicator of Ω, and x^* the nearest neighbours of x in Λ_k. King's modified action [26, (3.38)] on Ω is

$$\begin{aligned} S^\varepsilon_\Omega(A,\phi) = {}& \tfrac12 \sum_{p\in\Omega^{**}} \varepsilon^d |F(p)|^2 + \tfrac12\mu_0^2 \sum_{b\in\Omega^*} \varepsilon^d |A(b)|^2 + \tfrac12 \sum_{x\in\Omega} \varepsilon^d \Big| \sum_{y\in x^*} \varepsilon^{-1}\Omega(y) A(\langle x,y\rangle)\Big|^2 \\ & + \tfrac12 \sum_{\langle x,y\rangle} \varepsilon^{d-2}\big[(\Omega r)(x)^2 + (\Omega r)(y)^2 - 2(\Omega r)(x)(\Omega r)(y)\cos(\vartheta_x - e\varepsilon A(\langle x,y\rangle) - \vartheta_y)\big] \\ & + \sum_{x\in\Omega} \varepsilon^d\big[\tfrac12 m^2 r_x^2 + \lambda r_x^4 + \tfrac12 \delta m_\varepsilon^2\, r_x^2\big], \end{aligned}$$

with the *constant* free-boundary counterterm δm_ε^2 of Definition 2.3 at every site, and with the free Wick constant W_ε in the Wick observables. (In [26] the last term carries a site-dependent counterterm; it is not used here, and the statements below are made for the constant one.) The photon variables are the internal bonds Ω^* only. The matter part is exactly the Dirichlet matter of the Neumann–Dirichlet state on $X = \Omega$: cosines on the internal bonds, and the diagonal $\frac12\varepsilon^{d-2} r_x^2$ for every bond leaving Ω. The *modified box state* $\langle\cdot\rangle^{(3.38)}_{k,\Omega}$ is the normalised measure $e^{-S^\varepsilon_\Omega}\prod_{x\in\Omega} d^2\phi_x \prod_{b\in\Omega^*} dA_b$; it is a BFS measure (the Gaussian is non-degenerate since $\mu_0 > 0$, the single-site measures have all exponential moments since $\lambda > 0$). We write $Z^{a,(3.38)}_{k,\Omega}(\zeta) := m(\zeta)\langle\Theta(\zeta)\rangle^{(3.38)}_{k,\Omega}$ with the same observables and multiplier as in Definition 2.5.

The third term is the graph divergence of (Ω,Ω^*): with $d_0 : \mathbb{R}^\Omega \to \mathbb{R}^{\Omega^*}$, $(d_0\chi)(\langle x, x+\varepsilon e_\alpha\rangle) = \varepsilon^{-1}(\chi(x+\varepsilon e_\alpha) - \chi(x))$, one has $\sum_{y\in x^*}\Omega(y)A(\langle x,y\rangle) = -\varepsilon(d_0^T A)(x)$ (only internal bonds contribute). With $d_1 A := (\varepsilon F(p))_{p\in\Omega^{**}}$, the *unitary form* on the box and the modified form are

$$Q^u_\Omega(A) := \tfrac12 \sum_{p\in\Omega^{**}} \varepsilon^d |F(p)|^2 + \tfrac12\mu_0^2 \sum_{b\in\Omega^*} \varepsilon^d |A(b)|^2, \qquad Q^{(3.38)}_\Omega = Q^u_\Omega + \tfrac12\varepsilon^d \|d_0^T A\|^2,$$

i.e. the modified form is the unitary form plus a gauge-fixing term depending on A only through $d_0^T A$. The modified action is invariant under $\hat\rho$ for every mid-plane c with $\rho\Omega = \Omega$: every term is written through $A(\langle x,y\rangle)$ and nearest-neighbour pairs, which ρ permutes, $(d_0^T\hat\rho A)(x) = (d_0^T A)(\rho x)$, and δm_ε^2 is site independent.

The boxes. Fix $c \in D$, $k \ge k_0(c)$, $J \ge J_0$ such that every source of the enlarged family of Definition 10.1 and its reflection is supported at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_{J_0}$, an integer $n \ge 1$,

and put $a_k := c + \varepsilon_k/2 \in \varepsilon_k\mathbb{Z}$ and

$$\Omega_{J,k} := (C_{J+n} + a_k e_\nu) \cap \Lambda_k, \qquad Mn \;\geq\; |c| + 2 + R(\mu_0), \qquad R(\mu_0) := \frac{\pi\sqrt{3}}{2\mu_0}. \tag{23}$$

$C_{J+n} \cap \Lambda_k$ is symmetric in direction ν about $-\varepsilon_k/2$ (sites $x_\nu \in \{-M(J+n), \dots, M(J+n) - \varepsilon_k\}$, and $M(J+n) \in \varepsilon_k\mathbb{Z}$), so $\Omega_{J,k}$ is a lattice box symmetric about $\{x_\nu = c\}$ with an even number $2M(J+n)5^k$ of layers. Under the margin condition of (23), for every k: (i) C_J plus one layer is contained in $\Omega_{J,k}$ (this needs $Mn \geq |c| + \frac{3}{2}\varepsilon_k$); (ii) C_{J+2n} contains every site at ℓ^∞-distance $\leq R(\mu_0) + \varepsilon_k$ from $\Omega_{J,k}$ (this needs $Mn \geq |c| + R(\mu_0) + \frac{3}{2}\varepsilon_k$). We fix such an $n = n(\mu_0, c)$, independent of k and J. The two margins are exactly the hypotheses of Lemmas 9.9 and 9.10 below; by Lemma 4.7 they serve King's convention as they stand.

9.2. **Reflection positivity of the modified box.** For a lattice box symmetric about a mid-plane, the modified box state is reflection positive at every spacing: this is the lattice statement that the sandwich of Lemmas 9.9 and 9.10 carries to the cube states.

Lemma 9.3 (lattice reflection positivity of the modified box)**.** *Let $\mu_0 > 0$, c a mid-plane and Ω a lattice box with $\rho\Omega = \Omega$, $\rho = \rho_{\nu,c}$.*

(a) *For every gauge-invariant $G \in \mathfrak{B}_+$ that is bounded, or of the form $\Theta(\zeta)$ with sources supported in Ω_+, or more generally with $\langle |G|^2 \rangle^{(3.38)}_{k,\Omega} < \infty$: $\langle \theta(G) G \rangle^{(3.38)}_{k,\Omega} \geq 0$.*

(b) *$(G, G') \mapsto \langle \theta(G') G \rangle^{(3.38)}_{k,\Omega}$ is a Hermitian positive semidefinite form on such G; hence $\sum_{i,j} \bar\gamma_i \gamma_j \langle \theta(G_i) G_j \rangle^{(3.38)}_{k,\Omega} \geq 0$, and by (22) and the support clause of Definition 9.1, $\sum_{i,j} \bar\gamma_i \gamma_j Z^{a,(3.38)}_{k,\Omega}(\theta\zeta_i^* \sqcup \zeta_j) \geq 0$ for families ζ_i with sources supported in $\{x_\nu \geq c + \delta\}$, $\varepsilon_k < 2\delta$.*

Proof. The proof has three steps: gauge-invariant expectations of the modified form equal those of the unitary form; the unitary box measure is reflection positive on $\mathfrak{B}_+$, by the positive cross-term criterion with the crossing variables split into two independent Gaussians; and the two are combined and extended to L^2.

Step 1: gauge-invariant expectations of the modified form equal those of the unitary form. The graph (Ω, Ω^*) is connected, $\ker d_0$ consists of the constants, $\mathbb{R}^{\Omega^*} = \operatorname{ran} d_0 \oplus \ker d_0^T$ orthogonally, and $d_1 d_0 = 0$. Write $A = A^T + d_0\chi$ with $d_0^T A^T = 0$ and $\chi \perp 1$; $(A^T, \chi) \mapsto A$ is a linear bijection with constant Jacobian. Then $Q^u_\Omega(A) = Q^u_\Omega(A^T) + \frac{\mu_0^2 \varepsilon^d}{2} \|d_0\chi\|^2$ (orthogonality and $d_1 d_0 = 0$) and $Q^{(3.38)}_\Omega(A) = Q^u_\Omega(A^T) + q(\chi)$ with $q(\chi) = \frac{\mu_0^2\varepsilon^d}{2}\|d_0\chi\|^2 + \frac{\varepsilon^d}{2}\|d_0^T d_0 \chi\|^2$; both χ-forms are positive definite on $1^\perp$ because $\mu_0 > 0$. A gauge transformation is $A \mapsto A + d_0\chi$ together with $\vartheta \mapsto \vartheta - e\chi$: the hopping phase $\vartheta_x - e\varepsilon A_b - e(\chi(y) - \chi(x)) - \vartheta_y$ equals $(\vartheta_x + e\chi_x) - e\varepsilon A_b - (\vartheta_y + e\chi_y)$ and the Dirichlet diagonals carry no A, so for gauge-invariant G the matter integral $\Psi(A) := \int \prod_x d^2\phi_x \, (G e^{-S_m})(A, \phi)$ satisfies $\Psi(A^T + d_0\chi) = \Psi(A^T)$ (rotate ϕ site by site; $\prod d^2\phi$ is invariant). Hence $\int dA\, e^{-Q^{(3.38)}_\Omega} \Psi = \int dA^T e^{-Q^u_\Omega(A^T)} \Psi(A^T) \int d\chi\, e^{-q(\chi)}$, and the same with Q^u_Ω in place of $Q^{(3.38)}_\Omega$ and a different finite χ-factor; the factors cancel in the normalised expectation. Fubini applies, since $|G| \leq C e^{a\sum_b |A(b)|} e^{C\sum_x r_x^2}$ for $G = \Theta(\zeta)$ (trivially for bounded G) is dominated by the Gaussian and by $e^{-\lambda\varepsilon^d r^4}$. This is the mechanism of [9, Thm. 5.1, p. 270], carried to the box graph.

Step 2: the unitary box measure is reflection positive on all of $\mathfrak{B}_+$. Let $S^u_\Omega := Q^u_\Omega + S_m$, write $\omega = (\omega_+, \omega_-, a)$ with $a = (a_b)_{b \in B_0}$ the crossing-bond variables, and put $\mathsf{M} := \mu_0^2 \varepsilon^d$. The terms of S^u_Ω are of four kinds. Single-site terms (local matter and Dirichlet diagonals), hopping cosines on bonds of $B_\pm$, masses of bonds of $B_\pm$, and plaquettes with all corners in Ω_+ or all in Ω_- involve

one side only; by $\hat\rho$-invariance the minus-side terms are the $\hat\rho$-transport of the plus-side terms. The crossing masses are $\frac{\mathsf M}{2}a_b^2$, $b\in B_0$. A crossing plaquette $p=(x^-;\alpha,\nu)$, $\alpha\neq\nu$, is present iff $x^-,x^-+\varepsilon e_\alpha\in\Omega$ (then all four corners are in Ω by symmetry); with $x^+:=x^-+\varepsilon e_\nu$, $b=\langle x^-,x^+\rangle$ and $b'=\langle x^-+\varepsilon e_\alpha,x^++\varepsilon e_\alpha\rangle$ it reads $\varepsilon F(p)=\pm[A_\alpha(x^-)+a_{b'}-A_\alpha(x^+)-a_b]$, the overall sign immaterial after squaring. A crossing hopping term is $-\varepsilon^{d-2}r_{x^-}r_{x^+}\cos(\vartheta_{x^-}-e\varepsilon a_b-\vartheta_{x^+})$, $b\in B_0$.

Split each crossing variable: $\int_{\mathbb R}g(a_b)e^{-\frac{\mathsf M}{2}a_b^2}da_b=\sqrt{2\pi/\mathsf M}\,\mathbb E\,g(\alpha_b-\alpha_b')$ with α_b,α_b' independent $N(0,\frac{1}{2\mathsf M})$, whose difference is $N(0,\frac{1}{\mathsf M})$. Put $\xi:=(\omega_+,\alpha)$ and $\xi':=(\omega_-^\rho,\alpha')$, where ω_-^ρ is the $\hat\rho$-image of the minus data transported to the plus side ($(\omega_-^\rho)_\alpha(x^+)=A_\alpha(x^-)$ for the tangential bonds, $\phi^\rho(x^+)=\phi(x^-)$, the ν-bonds of B_- carried with the sign of Definition 9.1); by $\hat\rho$-invariance the minus-side one-sided terms are the plus-side terms evaluated on ω_-^ρ, and the a priori measures of ξ and ξ' coincide. For a crossing plaquette put $X_p(\xi):=A_\alpha(x^+)-\alpha_{b'}+\alpha_b$, a function of plus variables and α (note $A_\alpha(x^+)\in B_+$ since $x^+,x^++\varepsilon e_\alpha\in\Omega_+$), and the same function on ξ', $X_p(\xi')=A_\alpha(x^-)-\alpha'_{b'}+\alpha'_b$; then $\varepsilon F(p)=\pm[X_p(\xi')-X_p(\xi)]$ and

$$\tfrac{\varepsilon^d}{2}F(p)^2=\tfrac{\varepsilon^{d-2}}{2}X_p(\xi)^2+\tfrac{\varepsilon^{d-2}}{2}X_p(\xi')^2-\varepsilon^{d-2}X_p(\xi)X_p(\xi').$$

For a crossing hopping term set $\psi_b(\xi):=\phi(x^+)e^{\mathrm{i}e\varepsilon\alpha_b}$, $\psi_b(\xi')=\phi(x^-)e^{\mathrm{i}e\varepsilon\alpha'_b}$; then $r_{x^-}r_{x^+}\cos(\vartheta_{x^-}-e\varepsilon(\alpha_b-\alpha'_b)-\vartheta_{x^+})=\mathrm{Re}[\psi_b(\xi')\overline{\psi_b(\xi)}]=\sum_{i=1,2}\psi_b^i(\xi)\psi_b^i(\xi')$. Hence, with S' the action minus the crossing masses,

$$e^{-S'}=\mathcal G(\xi)\,\mathcal G(\xi')\,\exp\Big(\sum_\iota\gamma_\iota Y_\iota(\xi)Y_\iota(\xi')\Big),\qquad \mathcal G(\xi):=e^{-S_+(\xi)-\frac{\varepsilon^{d-2}}{2}\sum_p X_p(\xi)^2},$$

where S_+ collects the one-sided plus terms and $\{(\gamma_\iota,Y_\iota)\}=\{(\varepsilon^{d-2},X_p)\}_p\cup\{(\varepsilon^{d-2},\psi_b^i)\}_{b,i}$: all $\gamma_\iota>0$, all Y_ι real. For $G\in\mathfrak B_+$ we have $G(\omega)=G(\xi)$ and $\theta G(\omega)=\overline{G(\xi')}$, the same function on the transported data (G depends neither on a nor on α,α'). Expanding the exponential,

$$Z\,\langle\theta(G)G\rangle^u_{k,\Omega}=K\sum_{n\in\mathbb N^I}\prod_\iota\frac{\gamma_\iota^{n_\iota}}{n_\iota!}\Big|\int d\xi\,G(\xi)\mathcal G(\xi)\prod_\iota Y_\iota(\xi)^{n_\iota}\Big|^2\ \geq 0,\qquad K>0,$$

the two integrals of each term being complex conjugates because measures and functions coincide on the two copies. The interchange of sum and integral is justified by $\sum_n\prod_\iota\gamma_\iota^{n_\iota}|Y_\iota Y_\iota'|^{n_\iota}/n_\iota!\leq\exp(\frac12\sum_\iota\gamma_\iota(Y_\iota^2+Y_\iota'^2))$: the plaquette parts cancel against $\mathcal G$ exactly, $|\psi_b|=r_{x^+}$, and

$$\int|G|e^{-S_+(\xi)+\frac{\varepsilon^{d-2}}{2}\sum_b r_{x^+}^2}d\xi<\infty$$

by $\lambda>0$ (a quartic term at every plus site), $\mu_0>0$ (a mass on every B_+ bond and the α-Gaussians) and the growth bound on G. The argument is the positive cross-term criterion of [18, 17], with the crossing bond variables split into two independent Gaussians and the Dirichlet diagonals as single-site terms. The Hermitian property $\overline{\langle\theta(G')G\rangle}=\langle(G'\circ\hat\rho)\bar G\rangle=\langle G'\,\overline{G\circ\hat\rho}\rangle=\langle\theta(G)G'\rangle$ follows from the $\hat\rho$-invariance of the measure.

Step 3: combination and L^2 extension. For gauge-invariant $G\in\mathfrak B_+$, bounded or equal to $\Theta(\zeta)$, the product $\theta(G)G$ is gauge invariant ($\hat\rho$ maps gauge transformations to gauge transformations: $\hat\rho\,d_0\chi=d_0(\chi\circ\rho)$), so $\langle\theta(G)G\rangle^{(3.38)}_{k,\Omega}=\langle\theta(G)G\rangle^u_{k,\Omega}\geq0$ by Steps 1–2. For gauge-invariant $G\in\mathfrak B_+$ with $\langle|G|^2\rangle<\infty$ (for instance polynomials in the plus-side smeared fields, Proposition 10.8), $G_n:=G\,\mathbf 1\{|G|\leq n\}$ is bounded, gauge invariant and in $\mathfrak B_+$, so $\langle\theta(G_n)G_n\rangle\geq0$; moreover $|\theta(G_n)G_n|\leq|\theta G||G|$ and $\langle|\theta G||G|\rangle\leq\langle|G|^2\rangle^{1/2}\langle|\theta G|^2\rangle^{1/2}=\langle|G|^2\rangle$ by Cauchy–Schwarz and $\hat\rho$-invariance, so dominated convergence gives (a). The modified measure is itself $\hat\rho$-invariant (Definition 9.2), so the computation of the Hermitian property in Step 2 holds verbatim for it on L^2; the form is sesquilinear, Hermitian and non-negative on the diagonal, hence positive semidefinite (apply (a) to $G=\sum_i\gamma_iG_i$). With (22) and the support clause this is (b). □

Remark 9.4 (on King's Proposition 3.3). Lemma 9.3 is the statement of [26, Prop. 3.3], which King derives from [9, Cor. 5.4]. That corollary (with [9, Lemma 5.2, Thm. 5.3]) concerns the lattice actions of [9, §5], whose photon Gaussian is translation invariant with a reflection-positive boundary term; King's finite-box Gaussian with its masked divergence is not one of them, the obstruction being the box geometry of the gauge-fixing term. So his Proposition 3.3 is true, and it is proved here rather than derived from [9, Cor. 5.4]. The gauge-fixing term $\|d_0^T A\|^2$ is not itself compatible with the mechanism of Step 2 (at a site next to the plane it produces cross terms $Y(\xi)Z(\xi') + Z(\xi)Y(\xi')$ of indefinite sign); on the torus it drops out for gauge-invariant observables, on King's cube it does not reduce to a divergence at the boundary (Lemma 9.5), and this is why the photon form has to be changed before reflecting (Remark 9.7).

9.3. **The same box: the boundary identity and the reversed inequality.** This subsection compares, on one and the same box, the modified box state with the Neumann–Dirichlet state: a boundary identity (Lemma 9.5) identifies the modified photon form as the larger one, so that the second inequality of King's (3.42) is reversed (Lemma 9.6).

In this subsection we use lattice units, $\varepsilon = 1$ (every term of the three photon forms carries the same weight $\frac{1}{2}\varepsilon^{d-2}$), and write $\mu := \mu_0\varepsilon$. For a lattice box Ω and $A \in \mathbb{R}^{\Omega^*}$ let $K_\Omega^{\mathrm{int}}(A) := \sum_\alpha \sum_\nu \sum |A_\alpha(x+e_\nu) - A_\alpha(x)|^2$, the sum over pairs with both bonds $(x,\alpha),(x+e_\nu,\alpha)$ in Ω^* (the componentwise Neumann gradient form on the internal bonds), and $N_\partial(A) := \sum |A_b|^2$, the sum over the normal crossing pairs: for $b = (x,\alpha) \in \Omega^*$ one term for each of $(x \pm e_\alpha, \alpha)$ that is not in Ω^*. Split $N_\partial = N_\partial^{\mathrm{up}} + N_\partial^{\mathrm{low}}$ according to whether the missing partner is $(x+e_\alpha,\alpha)$ (i.e. $x + 2e_\alpha \notin \Omega$) or $(x-e_\alpha,\alpha)$ (i.e. $x - e_\alpha \notin \Omega$).

Lemma 9.5 (the boundary identity)**.** *For every lattice box Ω and every $A \in \mathbb{R}^{\Omega^*}$,*

$$\|d_1 A\|_{\Omega^{**}}^2 + \|d_0^T A\|^2 = K_\Omega^{\mathrm{int}}(A) + N_\partial(A).$$

Consequently $Q_\Omega^{(3.38)} = \frac{1}{2}\big[K_\Omega^{\mathrm{int}} + N_\partial + \mu^2\|\cdot\|_{\Omega^*}^2\big]$: the modified photon form is the internal Neumann–Feynman form plus the positive boundary term N_∂. (In the notation of the proof of Lemma 9.6(a), the marginal of the Neumann–Dirichlet Gaussian of Ω on the internal bonds has the precision $M^{(3.38)} - N_\partial^{\mathrm{low}} - P_{12}P_{22}^{-1}P_{21}$: the modified form minus the squares of the near-normal bonds and minus the Schur term of the far-face variables.)

Proof. Extend A by 0 to a 1-form a on $\mathbb{Z}^d$. For finitely supported 1-forms, $\sum_p |da(p)|^2 + \sum_x |d^*a(x)|^2 = \sum_\alpha \sum_\nu \sum_x |a_\alpha(x+e_\nu) - a_\alpha(x)|^2$ (the Hodge Laplacian on $\mathbb{Z}^d$ is the componentwise Laplacian; summation by parts). On the left, $d^*a(x)$ is the masked divergence for $x \in \Omega$ and 0 outside, since no internal bond has an end outside Ω; a plaquette $p \notin \Omega^{**}$ meeting Ω^* has exactly one internal bond b (two parallel internal edges put all four corners in Ω; two adjacent ones put three corners in Ω, and a product of intervals then contains the fourth), so $|da(p)|^2 = |A_b|^2$. On the right, pairs with both bonds internal give K_Ω^{int}; a pair $(x,\alpha),(x+e_\nu,\alpha)$ with exactly one internal bond contributes $|A_b|^2$; for $\nu \neq \alpha$ the partner is non-internal iff $x + e_\nu \notin \Omega$ (as $x, x+e_\alpha \in \Omega$, and in a box $x + e_\nu \in \Omega \iff x + e_\alpha + e_\nu \in \Omega$), i.e. iff the plaquette spanned by b and e_ν lies outside Ω^{**}; these pairs are in bijection with the plaquettes counted on the left and cancel; for $\nu = \alpha$ they are the terms of N_∂. □

Lemma 9.6 (the same box: the second inequality of King's (3.42) is reversed)**.** *Let Ω be a lattice box, $\langle\cdot\rangle_\Omega^{\mathrm{ND}}$ the Neumann–Dirichlet state on $X = \Omega$ (variables $A_\alpha(x)$, $x \in \Omega$, all outgoing bonds; photon form $Q_\Omega^{\mathrm{ND}} = \frac{1}{2}[K_\Omega + \mu^2\|\cdot\|^2]$ of Definition 2.2) and $\langle\cdot\rangle_\Omega^{(3.38)}$ the modified box state of Definition 9.2 (variables Ω^*); both carry the same Dirichlet matter on Ω (constant δm_ε^2, free Wick constant), coupled to A through the internal bonds only. Let $\Sigma_\Omega^{\mathrm{ND}}$ and Σ_Ω^{u} be the covariances of $F = (d_1 A(p))_{p \in \Omega^{**}}$ under the two photon Gaussians. Then:*

(a) $\Sigma^u_\Omega = \Sigma^{(3.38)}_\Omega \le \Sigma^{\mathrm{ND}}_\Omega$ *on* $\mathbb{R}^{\Omega^{**}}$*;*

(b) *for every gauge-invariant* $Y \in \mathcal{I}$ *with sources in* Ω *and every* $e \ge 0$*:* $\langle Y\rangle^{\mathrm{ND}}_\Omega \le \langle Y\rangle^{(3.38)}_\Omega$*, and* $\langle X\rangle^{\mathrm{ND}}_\Omega \ge \langle X\rangle^{(3.38)}_\Omega$ *for* $X \in \mathcal{D}$*. This is the reverse of the second inequality of* [26, (3.42)]*, whose middle term is the modified box state on* C_{J+1}*; it holds weakly, for the modified state with the constant* δm^2_ε *(nothing is claimed for the site-dependent counterterm of* [26]*);*

(c) *King's inequality fails strictly in the Gaussian sector: at* $e = 0$ *and* $Y = e^{\mathrm{i}t\langle v,F\rangle}$ *it reads* $v^T\Sigma^{\mathrm{ND}}_\Omega v \le v^T\Sigma^u_\Omega v$*, while* $\Delta_\Omega := \Sigma^{\mathrm{ND}}_\Omega - \Sigma^u_\Omega \ge 0$ *is non-zero. In exact rational arithmetic the inertia (negative, zero, positive) of* Δ_Ω *is* $(0,1,8)$ *on the* 4×4 *box (*$d=2$*) for* $\mu^2 \in \{\frac{1}{20}, \frac{1}{4}, 1\}$ *and* $(0,3,13)$ *on the* $4\times2\times2$ *box (*$d=3$*) for* $\mu^2 \in \{\frac{1}{20}, \frac{1}{4}\}$*; single-plaquette certificates are* $\Delta_{pp} = \mathrm{Var}^{\mathrm{ND}} F(p) - \mathrm{Var}^{(3.38)} F(p) = \frac{14285248}{44314767}$ *at* $p = ((0,0);0,1)$ *on* 4×4*,* $\mu^2 = \frac{1}{4}$*, and* $\frac{121897601600}{395580222961}$ *at* $p = ((0,0,0);1,2)$ *on* $4\times2\times2$*,* $\mu^2 = \frac{1}{20}$*.*

Proof. (a) Write the Neumann–Dirichlet precision $P := K_\Omega + \mu^2$ in blocks, the internal variables Ω^* first and the outgoing boundary variables second. Its internal block is $P_{11} = K^{\mathrm{int}}_\Omega + N^{\mathrm{up}}_\partial + \mu^2$: pairs of K_Ω with both bonds internal give K^{int}_Ω; a pair $(x,\alpha),(x+e_\nu,\alpha)$ with exactly one internal bond is impossible for $\nu \ne \alpha$ (box convexity, as in Lemma 9.5) and for $\nu = \alpha$ is (x,α) internal with $(x+e_\alpha,\alpha)$ outgoing, i.e. $x+2e_\alpha \notin \Omega$ (the other assignment would need $x+e_\alpha \notin \Omega$ and $x+e_\alpha \in \Omega$ at once); its diagonal contribution to the internal block is $|A_\alpha(x)|^2$, which is N^{up}_∂. Hence, by Lemma 9.5, $M^{(3.38)} := K^{\mathrm{int}}_\Omega + N_\partial + \mu^2 = P_{11} + N^{\mathrm{low}}_\partial \ge P_{11}$. The marginal covariance of the internal variables under the Neumann–Dirichlet Gaussian is the inverse of the Schur complement, $(P^{-1})_{11} = (P_{11} - P_{12}P_{22}^{-1}P_{21})^{-1} \ge P_{11}^{-1} \ge (M^{(3.38)})^{-1}$, by operator monotonicity of the inverse on positive definite matrices and $P_{12}P_{22}^{-1}P_{21} \ge 0$. Since $F = d_1 A$ on Ω^{**} uses internal bonds only, $\Sigma^{\mathrm{ND}}_\Omega = D_1(P^{-1})_{11}D_1^T \ge D_1(M^{(3.38)})^{-1}D_1^T = \Sigma^{(3.38)}_\Omega$, and $\Sigma^{(3.38)}_\Omega = \Sigma^u_\Omega$ by Step 0 in the proof of Lemma 9.10 (the field strength depends only on the $\ker d_0^T$-component, whose law is the same under both Gaussians, Step 1 of Lemma 9.3). (b) Both states carry the Dirichlet matter of Ω coupled to Ω^*, and their field-strength laws on Ω^{**} are ordered by (a); so the argument of the proof of Lemma 9.10(b) after its removal step (gauge invariance, the realisation of the field-strength law, the δ-regularisation and Lemma 2.10(a),(d)) applies verbatim, with $\Sigma^{(3.38)}_\Omega$ and $\Sigma^{\mathrm{ND}}_\Omega$ in the roles of Σ^{ND}_B and Σ^u_Ω. (c) At $e = 0$ the matter decouples from A and is identical in the two states, so for $Y = e^{\mathrm{i}t\langle v,F\rangle}$ both sides equal $\exp(-\frac{t^2}{2}v^T\Sigma v)$ with the respective Σ; King's $\le$ becomes $v^T\Sigma^{\mathrm{ND}}_\Omega v \le v^T\Sigma^u_\Omega v$, which is false for every v with $v^T\Delta_\Omega v > 0$, for instance $v = e_p$ at the plaquettes listed. The inertias and the certificates are obtained from the two covariance matrices in exact rational arithmetic (symmetric elimination with diagonal pivoting over $\mathbb{Q}$). □

Scope. (i) King's passage from the box states to the cube limit on [26, p. 336] needs the sandwich for the box functionals with purely imaginary curvature coefficients, i.e. for the whole class $\mathcal{I}$; (c) refutes the second inequality on that class exactly. (ii) For the scalar observable $\Theta(0,2h_+)$ of the display in [26], (b) shows that the inequality can hold only with equality; at $e = 0$ it does (identical decoupled matter), and at $e > 0$ it is not decided here whether the equality fails. Both clauses are for the modified state with the constant δm^2_ε: if King's site-dependent counterterm differs from the free-boundary number, the two matter measures differ, the mechanism of (b) does not apply, and neither clause is proved for the objects of [26]; (c) alone is independent of the counterterm, the matter decoupling from A at $e = 0$.

Remark 9.7 (what is false in King's route). (i) King's Neumann–Dirichlet cube state is not reflection symmetric about any plane: the site set $C_J \cap \Lambda_k$ is symmetric in direction ν about $x_\nu = -\varepsilon_k/2$, the set of ν-bonds about $x_\nu = 0$, and no reflection maps the variable set to

itself (Lemma 2.16 accordingly obtains the reflection symmetry of the limit from a cut sandwich, not from a lattice symmetry). (ii) The natural symmetrisation, the Feynman gradient form on the internal bonds $K_\Omega^{\mathrm{int}} + \mu^2$ (the modified form without N_∂; reflection symmetric), is *not* reflection positive on gauge-invariant functionals. This is exact at $e = 0$: on the 4×4 box at $\mu^2 = \frac{1}{20}$ the reflection kernel $R_{pq} = \sigma_p \Sigma[\rho p, q]$ of the plus-side plaquettes is negative definite (inertia $(3,0,0)$), and on $4 \times 2 \times 2$ at $\mu^2 = \frac{1}{20}$ it has inertia $(2,2,2)$ with $R_{pp} = \mathrm{Cov}(\theta F(p), F(p)) < 0$ for $p = ((2,0,0);0,1)$. Indeed, at $e = 0$ the matter decouples, and the gauge-invariant plus-side functionals $G_1 = e^{\mathrm{i}t\langle v, F_+\rangle}$, $G_2 = 1$ give the matrix $(\langle\theta(G_i)G_j\rangle)$ with determinant $e^{-t^2 v^T \Sigma_{++} v}(e^{t^2 v^T R v} - 1) < 0$ for every $t > 0$ when $v^T R v < 0$, since $\langle\theta(G_1)G_1\rangle = \exp(-t^2 v^T \Sigma_{++} v + t^2 v^T R v)$ and $\langle\theta(G_1)\rangle = \langle G_1\rangle = \exp(-\frac{1}{2} t^2 v^T \Sigma_{++} v)$ by the reflection symmetry $\Sigma_{--} = \Sigma_{++}$. By continuity at fixed spacing the symmetrised cube state is not reflection positive for small $e > 0$ either. A proof must work at every coupling, so reflection positivity of King's own cube form, or of this symmetrisation, at fixed spacing is not available as a route: any route through cube states at fixed spacing has to change the photon form, as the modified box does (a reflection-symmetric, reflection-positive modification whose values converge to the same limit, Theorem 10.5). Nothing is claimed about the symmetrised form at a given larger e or about other proofs; the route taken here needs the comparison of Lemmas 9.9 and 9.10, and its reflecting plane must be a lattice mid-plane for the half-bond mechanism of Lemma 9.3 (King's cubes have an even number of layers). (iii) Accordingly, the second inequality of [26, (3.42)] is false as stated there, in the following sense: false for the class of observables that the argument on [26, p. 336] needs (exactly, Lemma 9.6(c)), and never strictly true for the modified state with the constant counterterm (Lemma 9.6(b)). Its valid first half is Lemma 9.9; the invalid half is replaced by Lemma 9.10 with the margin $R(\mu_0)$ of (23), $n(\mu_0, c)$ cube steps instead of one.

Remark 9.8 (the box form is the larger one). King describes the modified box action as obtained from the cube action by omitting some terms. By Lemma 9.5 the relation is the reverse: on one box the modified photon form is the internal Neumann–Dirichlet form P_{11} plus the positive term $N_\partial^{\mathrm{low}}$ and without the outgoing boundary variables, so the modified state is the one with the *larger* form, the smaller field-strength covariance, and the larger $\mathcal{I}$-expectations: $\langle Y\rangle_C^{\mathrm{ND}} \le \langle Y\rangle_C^{(3.38)}$ on the cone for every box C (Lemma 9.6(b)), the direction that Lemma 2.13 requires for consistency with Lemmas 9.9 and 9.10. For King's convention the same comparison holds with Step 1 of Lemma 4.2 prepended (Remark 4.9): $\langle Y\rangle_{k,J}^{K} \le \langle Y\rangle_{k,J}^{(a)} \le \langle Y\rangle_{k,X_J}^{(3.38)}$ on $\mathcal{I}$.

9.4. **The sandwich.** Figure 3 shows the three boxes of this subsection.

This subsection proves the two halves of the sandwich that carries reflection positivity from the modified box to the cube states: the cube sits inside the box (Lemma 9.9), and the box sits inside a larger cube once a margin proportional to μ_0^{-1} is allowed (Lemma 9.10).

Lemma 9.9 (the lower half)**.** *Let $C \subset \Omega$ be lattice boxes with C plus one layer contained in Ω, so that the variables $A_\mu(x)$, $x \in C$, of the Neumann–Dirichlet state on C (the outgoing bonds) are internal bonds of Ω. Then*

(a) $Q_\Omega^{(3.38)} \ge Q_C^{\mathrm{ND}} \oplus \frac{\mu^2}{2} I_{\mathrm{rest}}$ *as quadratic forms on* $\mathbb{R}^{\Omega^*} = \mathbb{R}^{C\text{-variables}} \oplus \mathbb{R}^{\mathrm{rest}}$;

(b) *for sources such that every plaquette and every site at which they do not vanish lies in C,*

$$\langle Y\rangle_C^{\mathrm{ND}} \le \langle Y\rangle_\Omega^{(3.38)} \quad (Y \in \mathcal{I}), \qquad \langle X\rangle_C^{\mathrm{ND}} \ge \langle X\rangle_\Omega^{(3.38)} \quad (X \in \mathcal{D}),$$

and the same for Z^a (equal positive multipliers, Lemma 2.14(a)). The $\mathcal{D}$ half uses Lemma 2.10(d); the $\mathcal{I}$ half only (a), (b), (e).

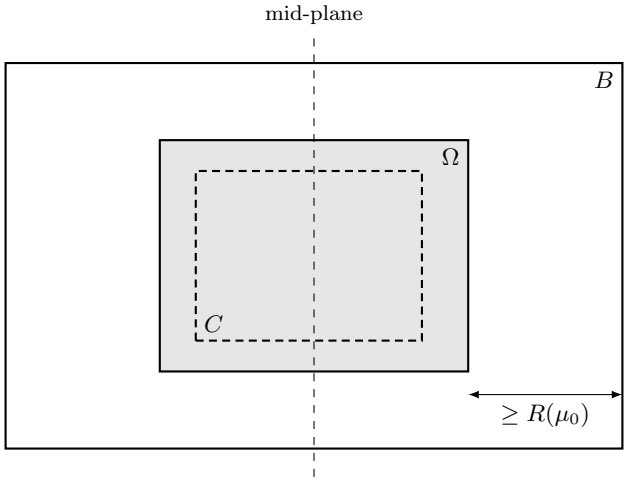

FIGURE 3. The sandwich of Lemmas 9.9 and 9.10; not to scale. The modified box state on the lattice box Ω (Definition 9.2, shaded) is reflection positive, on the gauge-invariant observables of Lemma 9.3, about a mid-plane whose reflection maps Ω to itself. For observables of the class $\mathcal{I}$ with sources in a lattice box C whose one-layer neighbourhood lies in Ω, the expectation in the Neumann–Dirichlet state on C is at most that in the modified box state (Lemma 9.9), which is at most that in the Neumann–Dirichlet state on a lattice box B containing every site within N lattice steps of Ω in the ℓ^∞-distance, where $N\varepsilon_k \geq R(\mu_0) = \pi\sqrt{3}/(2\mu_0)$ (Lemma 9.10).

Proof. (a) K_Ω^{int} contains every gradient pair of K_C (both base points in C, both bonds internal to Ω); the remaining pairs and N_∂ are non-negative; the mass term splits. (b) Start from $\langle\cdot\rangle_\Omega^{(3.38)}$ and remove the hopping cosines across ∂C and all cosines not internal to C; by Lemma 2.10(b),(d) $\langle Y\rangle$ does not increase and $\langle X\rangle$ does not decrease, every intermediate measure being a BFS measure. The matter is now the Dirichlet matter of C: the diagonal parts $\frac{1}{2}r^2$ of the cut bonds stay and are exactly the Dirichlet terms, as in the proof of Lemma 2.13; it is coupled to A only through bonds internal to C, which are C-variables, and the matter of $\Omega \setminus C$ is decoupled and cancels (Lemma 2.10(e)). Then pass from the form $Q_\Omega^{(3.38)}$ to the smaller form $Q_C^{\mathrm{ND}} \oplus \frac{\mu^2}{2} I_{\mathrm{rest}}$: by (a) the difference is positive semidefinite, a finite sum $\sum_i s_i|\ell_i(A)|^2$ with $s_i \geq 0$, and adding each term does not decrease $\langle Y\rangle$ and does not increase $\langle X\rangle$ (Lemma 2.10(a),(d)). So the state with the larger form, the modified one, has the larger $\langle Y\rangle$ and the smaller $\langle X\rangle$. The state with the smaller form factorises as $\langle\cdot\rangle_C^{\mathrm{ND}}$ times an independent Gaussian on the rest, which cancels. □

Lemma 9.9 compares the Neumann–Dirichlet state of a box C with the modified state of a larger region Ω in the direction of the correlation inequalities, and it is stated on the comparison convention. For King's convention the lower end is used in the form $\langle Y\rangle_{k,J}^K \leq \langle Y\rangle_{k,J}^{(a)} \leq \langle Y\rangle_\Omega^{(3.38)}$ on $\mathcal{I}$ (Step 1 of Lemma 4.2, then Lemma 9.9; Remark 4.9), and the upper end needs no change (Lemma 4.7).

Lemma 9.10 (the upper half, with a margin)**.** *Let $\Omega \subset B$ be lattice boxes at spacing ε such that B contains every site within ℓ^∞-distance N lattice steps of Ω, and let $\mu = \mu_0\varepsilon$ satisfy $d(\pi/2N)^2 \leq \mu^2$, i.e.*

$$N\varepsilon \;\geq\; \frac{\pi\sqrt{d}}{2\mu_0} \qquad \big(= R(\mu_0) = \tfrac{\pi\sqrt{3}}{2\mu_0} \text{ in } d = 3\big).$$

Let Σ_X denote the covariance of $(\varepsilon F(p))_{p\in\Omega^{**}}$ under the photon Gaussian of the state on X. Then

(a) $\Sigma_B^{\mathrm{ND}} \le \Sigma_\Omega^{(3.38)} = \Sigma_\Omega^u$ *on* $\mathbb{R}^{\Omega^{**}}$;

(b) *for sources supported in Ω as in Lemma 9.9,*

$$\langle Y\rangle_\Omega^{(3.38)} \le \langle Y\rangle_B^{\mathrm{ND}} \quad (Y\in\mathcal{I}), \qquad \langle X\rangle_\Omega^{(3.38)} \ge \langle X\rangle_B^{\mathrm{ND}} \quad (X\in\mathcal{D}),$$

and the same for Z^a. Only the $\mathcal{I}$ half is used in the sequel.

Proof. We first prove (a) in lattice units, by a variational identity for the covariance, a cut-off outside the box, the Hodge identity on $\mathbb{Z}^d$ and a discrete IMS identity; we then prove (b) by removing the hopping terms touching $B\setminus\Omega$ and comparing the two photon Gaussians through the ordering of their covariances from (a), after a regularisation that is removed at the end.

(a), in lattice units. Step 0: $\Sigma_\Omega^{(3.38)} = \Sigma_\Omega^u$, because $d_1A = d_1A^T$ depends only on the $\ker d_0^T$-component, whose law $\propto e^{-Q_\Omega^u(A^T)}$ is the same under both Gaussians (Step 1 of Lemma 9.3). Step 1 (variational identity): for a Gaussian with density $\propto e^{-\frac12 A^T M_X A}$, $\langle g,\Sigma_X g\rangle = \mathrm{Var}\langle g,d_1A\rangle = \sup_A[2\langle g,d_1A\rangle - A^TM_XA]$, the supremum attained at $A = M_X^{-1}d_1^Tg$. So it suffices to show that for every A on the B-variables there is $A'\in\mathbb{R}^{\Omega^*}$ with $d_1A' = d_1A$ on Ω^{**} and $\|d_1A'\|^2_{\Omega^{**}} + \mu^2\|A'\|^2_{\Omega^*} \le K_B(A) + \mu^2\|A\|_B^2$; take $A' := A|_{\Omega^*}$ (the plaquettes of Ω^{**} use only Ω^*-bonds). Step 2 (cut-off): let $\psi(x) = \prod_\nu \phi_\nu(x_\nu)$ with $\phi_\nu = 1$ on the ν-range of Ω and on the first layer outside, $\phi_\nu(t) = \cos(\pi(t-1)/2N)$ at the t-th layer outside for $1\le t\le N+1$ (so $\phi_\nu(N+1)=0$), and 0 beyond; then $0\le\psi\le1$, $\psi=1$ on Ω and its neighbours, $\operatorname{supp}\psi\subset B$. Put $a := \psi A$ on $\mathbb{Z}^d$ (the bond $\langle x,x+e_\mu\rangle$ weighted by $\psi(x)$; $A:=0$ off the B-variables). Step 3 (Hodge on $\mathbb{Z}^d$): $\|d_1A\|^2_{\Omega^{**}} = \|d_1A'\|^2_{\Omega^{**}} \le \sum_p |da(p)|^2 \le \sum_p|da|^2 + \sum_x|d^*a|^2 = \sum_\mu\sum_\nu\sum_x|a_\mu(x+e_\nu)-a_\mu(x)|^2$, since the bonds of plaquettes in Ω^{**} have base points in Ω, where $\psi=1$. Step 4 (discrete IMS identity): for finitely supported ψf and a direction e,

$$\sum_x|\psi(x+e)f(x+e)-\psi(x)f(x)|^2 = \sum_x\psi(x)\psi(x+e)|f(x+e)-f(x)|^2 + \sum_x\psi(x)\big(2\psi(x)-\psi(x+e)-\psi(x-e)\big)f(x)^2,$$

exactly (per bond the difference of the two sides is $\psi'(\psi'-\psi)f'^2 + \psi(\psi-\psi')f^2$; collect by site). With $f=A_\mu$ and $e=e_\nu$: $\psi\psi'\le1$, and $\psi\psi'>0$ only for pairs with both base points in B, which are gradient pairs of K_B; so the first sums, over μ,ν, total at most $K_B(A)$. The potential is $V(x) = \sum_\nu\big(\prod_{\kappa\neq\nu}\phi_\kappa^2\big)\phi_\nu(-\Delta\phi_\nu)(x_\nu)$. Step 5 (the profile): with $\varpi := \pi/2N$, $\phi(-\Delta\phi) = 0$ on the range of Ω (all neighbours equal 1), $= 1-\cos\varpi \le \varpi^2/2$ at $t=1$, $= 2\cos^2(\varpi(t-1))(1-\cos\varpi)\le\varpi^2$ for $2\le t\le N$ (using $\phi(N+1)=\cos(\varpi N)=0$ at $t=N$), and $=0$ for $t\ge N+1$; hence $V=0$ on Ω and $V\le d\varpi^2$ everywhere. Step 6: $\|d_1A'\|^2_{\Omega^{**}} \le K_B(A) + d\varpi^2\sum_{x\in B\setminus\Omega}|A(x)|^2$; the variables with base point in $B\setminus\Omega$ and those of Ω^* (base point in Ω) are disjoint subsets of the B-variables, so $\|d_1A'\|^2_{\Omega^{**}} + \mu^2\|A'\|^2_{\Omega^*} \le K_B(A) + \mu^2\|A\|_B^2$ as soon as $d\varpi^2\le\mu^2$.

(b). Start from $\langle\cdot\rangle_B^{\mathrm{ND}}$ and remove all hopping cosines touching $B\setminus\Omega$ (Lemma 2.10(b): $\mathcal{I}$ does not increase; (d): $\mathcal{D}$ does not decrease). The matter is now the Dirichlet matter of Ω (the diagonals of the bonds leaving Ω kept; every such bond ends in B since $N\ge1$), coupled to A on Ω^*, and the sites of $B\setminus\Omega$ decouple. This matter is identical to that of the modified box state: the same cosines on Ω^*, the same diagonals $\frac12 r_x^2$ for every bond leaving Ω, the same single-site weights with the constant δm_ε^2 and the same Wick constant. For a gauge-invariant observable with sources in Ω the matter integral at fixed A, $\Psi_Y(A) = \int\prod_{x\in\Omega}d^2\phi_x\, Y\, e^{-S_m}$, is invariant under $A\mapsto A+d_0\chi$, $\chi\in\mathbb{R}^\Omega$ (as in Step 1 of Lemma 9.3), hence a function of the class of $A|_{\Omega^*}$ modulo

$\operatorname{ran} d_0 = \ker d_1$ (a box has $H^1 = 0$: the cycle space of a grid graph is generated by its 4-cycles), hence a function $\tilde\Psi_Y(d_1 A|_{\Omega^{**}})$ of the field strengths on Ω^{**}. So $\langle Y\rangle = \mathbb{E}[\tilde\Psi_Y(F)]/\mathbb{E}[\tilde\Psi_1(F)]$ depends on the photon law only through the centred Gaussian law of $F = (d_1 A(p))_{p\in\Omega^{**}}$, with covariance Σ_B^{ND} on the Neumann–Dirichlet side after the removal and Σ_Ω^u on the modified side (Step 0). Both laws live on $\operatorname{ran} d_1$. Let $T : \mathbb{R}^{\Omega^{**}} \to \mathbb{R}^{\Omega^*}$ be a linear map with $d_1 T = \mathrm{id}$ on $\operatorname{ran} d_1$, extended linearly to all of $\mathbb{R}^{\Omega^{**}}$ (any extension serves, both laws being supported on $\operatorname{ran} d_1$), and realise each law as the photon Gaussian of $A'' := TF \in \mathbb{R}^{\Omega^*}$, covariance $T\Sigma T^T$: then $d_1 A'' = F$ almost surely, the gauge-invariant expectations are reproduced, and $T\Sigma_B^{\mathrm{ND}} T^T \le T\Sigma_\Omega^u T^T$ by (a). These covariances are degenerate; regularise to $T\Sigma T^T + \delta I$, $\delta > 0$, which are non-degenerate, so the regularised states are BFS measures, the ordering of the covariances is preserved, and the law of $F'' = d_1 A''$ is $N(0, \Sigma + \delta d_1 d_1^T)$, still ordered. Lemma 2.10(a) by interpolation (the difference of the precision matrices is positive semidefinite by operator monotonicity of the inverse, hence a finite sum of terms $s|\ell(A)|^2$, $s \ge 0$) gives $\langle Y\rangle_{\Sigma^u+\delta} \le \langle Y\rangle_{\Sigma^{\mathrm{ND}}+\delta}$ for $Y \in \mathcal{I}$, and the reverse for $\mathcal{D}$ by Lemma 2.10(d). *The limit* $\delta \downarrow 0$. For the $\mathcal{I}$ half, $\tilde\Psi_Y$ is bounded and continuous: $|\Psi_Y(A)| \le \int e^{r^2(g)} e^{\sum \varepsilon^{d-2} r_x r_y} \prod_x d\rho_x < \infty$ independently of A ($|e^{\mathrm{i}F}| = 1$; the hopping bound $e^{\varepsilon^{d-2} r_x r_y} \le e^{\frac12 \varepsilon^{d-2}(r_x^2 + r_y^2)}$ cancels the diagonals and $\lambda > 0$ does the rest), and so is $\tilde\Psi_1$, with $\tilde\Psi_1 > 0$; the Gaussians $N(0, \Sigma + \delta d_1 d_1^T)$ converge weakly to $N(0,\Sigma)$, so numerator and denominator converge and the inequality passes to the limit. For the $\mathcal{D}$ half (not used below), $X = \Theta(f, -g)$ gives $\tilde\Psi_X(F) = e^{\langle v, F\rangle}\tilde\Psi_X^0(F)$ with v the plaquette vector of f and $0 < \tilde\Psi_X^0 \le e^{2\mathsf{W}_\varepsilon \mathsf{S}(g)}\tilde\Psi_1 \le C$ bounded and continuous, so $\tilde\Psi_X$ is continuous but unbounded; under $N(0, \Sigma + \delta d_1 d_1^T)$ one has $\mathbb{E}[e^{2\langle v, F''\rangle}] = e^{2v^T(\Sigma + \delta d_1 d_1^T)v} \le e^{2v^T\Sigma v + 2\|d_1^T v\|^2}$ for $\delta \le 1$, so $\{\tilde\Psi_X(F'')\}_{0<\delta\le 1}$ is bounded in L^2, hence uniformly integrable, and weak convergence with uniform integrability and continuity gives $\mathbb{E}[\tilde\Psi_X(F'')] \to \mathbb{E}[\tilde\Psi_X(F)]$. Altogether $\langle Y\rangle_\Omega^{(3.38)} \le \langle Y\rangle^{\text{after removal}} \le \langle Y\rangle_B^{\mathrm{ND}}$, and the reverse for $\mathcal{D}$. □

Remark 9.11 (why a margin, and why a fixed photon mass). Part (a) of Lemma 9.10 fails without a margin: for $B = \Omega$ the comparison is reversed (Lemma 9.6). No fixed margin works at $\mu_0 = 0$: there Σ_Ω^u is the projector onto exact 2-forms, while $\mathrm{Var}_B\langle g, dA\rangle > \|g\|^2$ for a Coulomb potential whose gradient leaks out of B. So Lemma 9.10 is a statement at fixed $\mu_0 > 0$ with $R(\mu_0) \to \infty$ as $\mu_0 \to 0$; a limit of vanishing photon mass inherits reflection positivity, if it exists, as a pointwise limit of positive semidefinite matrices (this is how Theorems 30.2 and 31.4 obtain it). On one box the three states are ordered on $\mathcal{I}$ as $\langle Y\rangle_\Omega^{\mathrm{ND}} \le \langle Y\rangle_\Omega^{(3.38)} \le \langle Y\rangle_B^{\mathrm{ND}}$ (Lemmas 9.6 and 9.10), as the monotonicity of Lemma 2.13 requires. The sandwich with the boxes (23) is assembled in the proof of Theorem 10.5; for King's convention its lower end carries the kill step of Remark 4.9 and its upper end is unchanged (Lemma 4.7).

10. Reflection positivity II: the limit, OS3 and the polynomial form

This section completes the proof of reflection positivity about the coordinate planes. Its results are reflection positivity of King's state about every coordinate plane in the curvature sector (Theorem 10.5) and, under the hypotheses of Theorem 10.6 (the sign clause, an admissible scheme and Wick amplitudes below $\sigma_+/8$), on the Wick tube, together with realness and symmetry (Proposition 10.7) and the polynomial form of reflection positivity (Proposition 10.8). The curvature statements are row (4b) of Table 1; they enter the measure (Section 11) and the main theorem (Theorem 28.2).

The lattice statements of the previous section (Section 9) are now carried to the limits $k \to \infty$ and $J \to \infty$. The scheme of proof is the same for the curvature sector and for the Wick tube: uniform local bounds for the modified-box functionals at complex coefficients (Lemma 10.3), Montel's theorem in k at fixed J, the two-sided sandwich of Lemmas 9.9 and 9.10 on the cone as

$J \to \infty$ with the identity theorem off it, and a translation for the planes that are not mid-planes. The statement specific to each sector is the convergence of the cube functionals themselves: the existence theorem (Theorem A) in the curvature sector, and on the Wick tube the positive-leg bound of Corollary 13.3 together with the existence theorem with scalar sources Theorem B (Proposition 10.4). All statements are for King's state, the infinite-volume state of King's construction [25, 26]; the two places where the cube convention is seen are the ends of the sandwich (26) (Lemma 4.7(c), Remark 4.9), and Remark 10.10 states what the arguments give for the comparison family. What is proved of the Wick sector is its tube form; reflection positivity at arbitrary complex Wick coefficients is not treated, the fixed-cube limits and a uniform positive-source bound above the amplitude threshold of Theorem B being open (Section 32).

10.1. **The enlarged family.** The limits are taken for one finite family of sources at a time, enlarged by the reflected sources and by the positive and negative parts of the scalar sources, so that the matrix elements of the reflection-positivity form are values of a single functional on it; this subsection sets up that family.

Definition 10.1 (the enlarged family, its functionals and its matrix points)**.** Fix $\nu \in \{1,2,3\}$, a mid-plane $c \in D$ (Definition 9.1), and finite families $\zeta_1, \dots, \zeta_N$ of real sources $f_{il} \in C_c^\infty(\mathbb{R}^3; \Lambda^2)$, $h_{il} \in C_c^\infty(\mathbb{R}^3)$ supported in $\{x_\nu \geq c + \delta\}$ for some $\delta > 0$, with complex coefficients z_{il}, w_{il}. Split each scalar source as $h_{il} = h_{il}^+ - h_{il}^-$ with $h_{il}^- := \|h_{il}\|_\infty \chi_{il}$, where $\chi_{il} \in C_c^\infty$, $0 \leq \chi_{il} \leq 1$, $\chi_{il} = 1$ on $\operatorname{supp} h_{il}$ and $\operatorname{supp} \chi_{il} \subset \{x_\nu \geq c + \delta/2\}$, and $h_{il}^+ := h_{il} + h_{il}^- \geq 0$. The *enlarged family* is the family of real sources

$$\mathcal{F} := \{\rho^* f_{il}\} \cup \{f_{il}\} \cup \{h_{il}^\pm \circ \rho\} \cup \{h_{il}^\pm\}, \qquad \text{coefficient space } \mathbb{C}^m,\ m = m_F + m_h,$$

whose scalar members $\tilde{h}_j \geq 0$ are all non-negative. For a real coefficient vector $\xi \in \mathbb{R}^m$ the total source is $(f_\xi, k_1(\xi) - k_2(\xi))$ with $k_1(\xi) := \sum_j (\xi_j)_+ \tilde{h}_j$, $k_2(\xi) := \sum_j (\xi_j)_- \tilde{h}_j$, both smooth and non-negative ($x_\pm := \max(\pm x, 0)$). Let J_0 be the least J such that every member of $\mathcal{F}$ is supported at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_J$; for $J \geq J_0$, $n = n(\mu_0, c)$ and the boxes $\Omega_{J,k}$ of (23) put

$$\Phi_k(\zeta) := Z^{a,(3.38)}_{k,\Omega_{J,k}}(\zeta), \qquad \zeta \in \mathbb{C}^m,$$

an entire function (a finite-dimensional integral with Gaussian decay in A and $e^{-\lambda\varepsilon^3 r^4}$ decay in r). The *matrix points* $\zeta_{ij}^* \in \mathbb{C}^m$ carry $\bar{z}_{il}$ on $\rho^* f_{il}$, z_{jl} on f_{jl}, $\pm\bar{w}_{il}$ on $h_{il}^\pm \circ \rho$, $\pm w_{jl}$ on $h_{jl}^\pm$ and 0 elsewhere; their total source is that of $\theta\zeta_i^* \sqcup \zeta_j$, so $\Phi_k(\zeta_{ij}^*)$ is the (i,j) entry of Lemma 9.3(b). Since reflected and unreflected members have disjoint supports, $\|h^+\|_\infty \leq 2\|h\|_\infty$ and $\|h^-\|_\infty = \|h\|_\infty$,

$$\|k_1(\operatorname{Re}\zeta_{ij}^*)\|_\infty \ \leq\ 2\max\big(\mathsf{a}(\zeta_i), \mathsf{a}(\zeta_j)\big). \tag{24}$$

Every lattice functional, and hence every limit below, depends on a family only through its total sources; the splitting is therefore immaterial at the matrix points and at the Taylor coefficients at 0.

Definition 10.2 (the Wick tube, the bounded domain and the uniqueness set)**.** Let $\zeta_1, \dots, \zeta_N$, $\mathcal{F}$, Φ_k and the matrix points ζ_{ij}^* be as in Definition 10.1.

(i) The *Wick tube* of $\mathcal{F}$ is $T_\mathcal{F} := \{\zeta \in \mathbb{C}^m : \ 2\|k_1(\operatorname{Re}\zeta)\|_\infty < \sigma_+/2\}$, with $\sigma_+ = s_1$ the amplitude of (I2) (Definition 12.5, (36)). It is open, convex ($\xi \mapsto \|k_1(\xi)\|_\infty$ is a supremum of convex functions), connected, invariant under $z \mapsto -z$ on the curvature coordinates, contains 0, and by (24) contains every ζ_{ij}^* as soon as $\mathsf{a}(\zeta_i) < \sigma_+/8 =: \sigma_W$ for every i.

(ii) The *bounded domain*: let $B_1 \subset \mathbb{R}^m$ be the open convex hull of the unit ℓ^∞-balls about 0 and about the real parts $\operatorname{Re}\zeta_{ij}^*$; let $\beta_j := 2\sup_{\xi \in B_1} |\xi_j|$ for the scalar coordinates (so $\beta_j = 2(|\operatorname{Re} w_{il}| + 1)$ for the coordinate of $h_{il}^\pm$ or of $h_{il}^\pm \circ \rho$), $B'' := \prod_j [0, \beta_j]$ over the scalar

coordinates, and $q_v := \sum_j v_j \tilde{h}_j$ for $v \in \mathrm{vert}(B'')$, the *vertex sources*. All q_v are pointwise $\le q_\zeta := \sum_{i,l} 2(|\operatorname{Re} w_{il}| + 1)(h^+_{il} + h^-_{il} + h^+_{il} \circ \rho + h^-_{il} \circ \rho)$. Put $\mathcal{T}_1 := \{\zeta \in \mathbb{C}^m : \operatorname{Re}\zeta \in B_1\}$: open, convex, connected, containing every ζ^*_{ij} and the unit polydisc about 0, so that $\mathcal{T}_1 \cap T_{\mathcal{F}}$ is a non-empty open set. For $\xi \in B_1$ both $2\xi_+$ and ξ_+ lie in B''.

(iii) The *uniqueness set* $U \subset \mathbb{C}^m$ is the set of coefficient vectors whose curvature coefficients are purely imaginary and whose scalar coefficients are positive reals. It is a non-empty, relatively open subset of the totally real subspace $(\mathrm{i}\mathbb{R})^{m_F} \times \mathbb{R}^{m_h}$ of maximal real dimension m (it is not open in $\mathbb{C}^m$), hence a set of uniqueness for holomorphic functions on any connected open set meeting it: a holomorphic function vanishing on a relatively open piece of a maximal totally real subspace has all derivatives zero there [22]. $U \cap \mathcal{T}_1 \neq \emptyset$ and $U \cap T_{\mathcal{F}} \neq \emptyset$ (small positive scalar coefficients). On U the total source is $(\mathrm{i}f_\zeta, g_\zeta) \in \mathcal{K}$, so $\Theta(\zeta) \in \mathcal{I}$; the scheme multipliers are positive reals, equal for the Neumann–Dirichlet and the modified states; and every lattice value is real (charge conjugation, Definition 2.5: $\overline{Z^a(\mathrm{i}f, g)} = Z^a(-\mathrm{i}f, g) = Z^a(\mathrm{i}f, g)$).

10.2. **Uniform local bounds.** The uniform local bounds below are stated for King's state: in the Wick sector the first factor of the Cauchy–Schwarz splitting is controlled through the fixed-cube limits of Theorem B at one cube, the second through the positive-leg bound Corollary 13.3.

Lemma 10.3 (uniform local bounds for the modified-box functionals)**.** *Let $\mathcal{F}$, Φ_k, $\mathcal{T}_1$, $T_{\mathcal{F}}$ be as in Definitions 10.1 and 10.2.*

(T) (The tube.) *Using, of the K-family functionals $Z^{a,K}_{k,J}$ (admissible scheme a, couplings in the coupling region of Definition 2.1), only their fixed-cube limits of Theorem B, together with* (I2), (I3) *and the correlation inequalities of Lemma 2.10(a)–(e): for every compact $Q \subset T_{\mathcal{F}}$ there is $C_Q < \infty$, independent of k and of $J \ge J_0$, with $\limsup_k \sup_Q |\Phi_k| \le C_Q$; and at fixed J, $\sup_k \sup_Q |\Phi_k| < \infty$. No further threshold on J enters.*

(C) (Curvature sector, $h \equiv 0$; no hypothesis.) $|\Phi_k(z)| \le \langle e^{F(f_{\operatorname{Re} z})}\rangle^{(3.38)}_{k,\Omega_{J,k}} \le \exp(\tfrac{1}{2}\operatorname{Var}^{(3.38)} F(f_{\operatorname{Re} z}))$ *and* $\operatorname{Var}^{(3.38)} F(f) \le \sum_p \varepsilon^3 f^2_{c(p)} \to \|f\|^2_{L^2}$, *uniformly in k and J.*

Proof. $|\Phi_k(\zeta)| \le \Phi_k(\operatorname{Re}\zeta)$ by the proof of Lemma 2.14(b) ($|\Theta(\zeta)| = \Theta(\operatorname{Re}\zeta)$ and the modulus of the multiplier), which is valid for any probability measure with the same real multiplier, in particular for the modified box state, which is not one of the states of Definitions 2.2 and 2.3. At a real ξ the proof of Lemma 2.14(d) (Cauchy–Schwarz with the multiplier identity, again valid for any such measure) gives

$$\Phi_k(\xi) \;\le\; Z^{a,(3.38)}_{k,\Omega_{J,k}}(2f_\xi, -2k_2(\xi))^{1/2}\; Z^{a,(3.38)}_{k,\Omega_{J,k}}(0, 2k_1(\xi))^{1/2}. \tag{25}$$

First factor (class $\mathcal{D}$). By the $\mathcal{D}$ half of Lemma 9.9 with $C = C_J \cap \Lambda_k$ (its hypothesis is (23)(i)), read for King's cube with Step 3 of Lemma 4.2 prepended (Remark 4.9), it is $\le Z^{a,K}_{k,J}(2f_\xi, -2k_2) \le Z^{a,K}_{k,J_0}(2f_\xi, -2k_2)$, the second step being the class-$\mathcal{D}$ cube monotonicity of Corollary 4.3 at fixed k. The map $\xi \mapsto \eta(\xi) :=$ (coefficient vector of $(2f_\xi, -2k_2(\xi))$) is continuous, so $\eta(\operatorname{Re} Q)$ lies in a real box B'; $\eta \mapsto \ln Z^{a,K}_{k,J_0}(\eta)$ is convex (Lemma 2.14(b)), so $\sup_\xi Z^{a,K}_{k,J_0}(\eta(\xi)) \le \max_{v \in \mathrm{vert}(B')} Z^{a,K}_{k,J_0}(v)$, finitely many values, each convergent in k by Theorem B (the sources here have negative Wick coefficients, on which that theorem has no amplitude restriction; for the admissible scheme a through the convergent factor of Definition 2.7), and convergence in k implies boundedness in k. So the first factor is bounded uniformly in k, ξ and $J \ge J_0$. *Second factor* (class $\mathcal{I}$). By the $\mathcal{I}$ half of Lemma 9.10 with $B = C_{J+2n} \cap \Lambda_k$ and $N := \lceil R(\mu_0)/\varepsilon_k \rceil$ (hypothesis (23)(ii)), with King's state on B (Lemma 4.7(b): the box $\Omega_{J,k}$ does not meet the killed layers of C_{J+2n}), it is $\le Z^{a,K}_{k,J+2n}(0, 2k_1(\xi))$. For $\xi \in \operatorname{Re} Q$, $Q \subset T_{\mathcal{F}}$, one has $0 \le 2k_1(\xi) \le \sigma_+/2 \le s_1$ on the lattice

(the sampled supremum is at most the continuum one), so Corollary 13.3 on King's convention (Remark 4.9, Remark 13.12), $\ln N^{a,K}_{k,C_J}(0,q) \le \kappa\|q\|_{1,k}$ for $0 \le q \le s_1$ on $C_J \cap \Lambda_k$, uniformly in k and J, applied with $N^{a,K}_{k,X}(0,q) = Z^{a,K}_{k,X}(0,q)$ (the same functional with the same scheme multiplier, so no J-dependent factor intervenes), gives $Z^{a,K}_{k,J+2n}(0,2k_1(\xi)) \le \exp(\kappa\|2k_1(\xi)\|_{1,k}) \le \exp\big(2\kappa\sum_j|\xi_j|\sup_{k'}\|\tilde h_j\|_{1,k'}\big)$, bounded uniformly in k, J and $\xi \in \operatorname{Re} Q$ because $\|\tilde h_j\|_{1,k'} \to \|\tilde h_j\|_{L^1}$ (Riemann sums, (2)). No lower bound on J beyond J_0 enters: Corollary 13.3 holds for every k and J. *(C).* At $h \equiv 0$ the first inequality is Lemma 2.14(b) as above and the second is Lemma 2.11 with $s = 1$, the modified box state being a BFS measure (Definition 9.2). Its photon Gaussian has the unitary form for gauge-invariant observables (Step 1 of Lemma 9.3), and $\operatorname{Var}^u\langle v, d_1A\rangle = \varepsilon^{-1}\langle d_1^Tv, (d_1^Td_1+\mu_0^2\varepsilon^2)^{-1}d_1^Tv\rangle \le \varepsilon^{-1}\|v\|^2$; with $v_p = \varepsilon^2 f_{c(p)}$ this is $\operatorname{Var} F(f) \le \sum_p \varepsilon^3 f^2_{c(p)}$. □

10.3. **King's state on the Wick tube.** On the Wick tube the convergence of the cube functionals is obtained from the existence theorem with scalar sources Theorem B together with the positive-leg bound Corollary 13.3; the next proposition gives the resulting limit object and its translation invariance.

Proposition 10.4 (the Wick-tube object of King's state: existence and translations)**.** *Let the couplings lie in the coupling region of Definition 2.1 and let a be an admissible scheme (Definition 2.7), so that* (I2) *and* (I3) *hold (Propositions 12.2 and 12.8) and the fixed-cube limits* $Z^{a,K}_J := \lim_k Z^{a,K}_{k,J}$ *of the K-family functionals exist, locally uniformly and holomorphic, on every Wick tube below (Theorem B in the form Theorem 6.26(i), whose domain* $\{\sup_x 2k_1 < s_W\}$, $s_W = s_1$, *contains* $T_{\mathcal{F}'}$*; for the admissible scheme a through Definition 2.7). For a finite family* $\mathcal{F}'$ *of real sources with non-negative scalar members* $\tilde h_j \ge 0$ *(any family is brought to this form by the splitting of Definition 10.1) let* $T_{\mathcal{F}'}$ *be its Wick tube, as in Definition 10.2(i).*

(a) $Z^{a,K}_J \to Z^{a,K}_{\infty,T}$ *locally uniformly on* $T_{\mathcal{F}'}$ *as* $J \to \infty$*;* $Z^{a,K}_{\infty,T}$ *is holomorphic on* $T_{\mathcal{F}'}$*;* $Z^{a,K}_{\infty,T} = G^K_\infty$ *on* $T_{\mathcal{F}'} \cap \mathcal{D}_{\sigma_+,H}$ *(Theorem 28.5; the same J-limit of the same* $Z^{a,K}_J$*);* $|Z^{a,K}_{\infty,T}(\zeta)| \le Z^{a,K}_{\infty,T}(\operatorname{Re}\zeta)$*;* $Z^{a,K}_{\infty,T}(\xi) > 0$ *for real* $\xi \in T_{\mathcal{F}'}$ *with* $-\xi \in T_{\mathcal{F}'}$*; and* $Z^{a,K}_{\infty,T}(-z,w) = Z^{a,K}_{\infty,T}(z,w)$.

(b) (Translations.) $Z^{a,K}_{\infty,T}(\tau_b\cdot\zeta) = Z^{a,K}_{\infty,T}(\zeta)$ *for all* $b \in \mathbb{R}^3$ *and* $\zeta \in T_{\mathcal{F}'}$*, where* τ_b *acts on all sources; the translated family has the same tube.*

The proof uses, besides the fixed-cube limits of Theorem B, only Corollary 13.3 on King's convention (Remark 4.9; hence (I2)*,* (I3) *and* [9, Thm. 6.2(1),(2)]*), the cube monotonicity of Corollary 4.3 in both classes (its class-*$\mathcal{D}$ *half resting on Lemma 2.10(d)), the translation sandwich of Lemma 8.3, Lemmas 2.14 and 2.17, Montel's theorem and the identity theorem.*

Proof. We first prove (a), the local boundedness of the family and its convergence on the tube; we then prove (b) for lattice translations by the translation sandwich, for all real translations by the continuity of a convex function, and for complex coefficients by the identity theorem.

(a) *Local boundedness* of $\{Z^{a,K}_J\}_{J\ge J_0}$ on $T_{\mathcal{F}'}$:

$$|Z^{a,K}_J(\zeta)| \le Z^{a,K}_J(\operatorname{Re}\zeta) \le Z^{a,K}_J(2f,-2k_2)^{1/2}Z^{a,K}_J(0,2k_1)^{1/2} \le Z^{a,K}_{J_0}(2f,-2k_2)^{1/2}\exp(\kappa\|k_1\|_{L^1}),$$

by Lemma 2.14(b),(d), the nonincreasing $\mathcal{D}$-factor (the class-$\mathcal{D}$ cube monotonicity of Corollary 4.3 at fixed k, then $k\to\infty$), and Corollary 13.3 on King's convention (Remark 4.9) in the limit $k\to\infty$ for the $\mathcal{I}$-factor, valid since the source $2k_1$ has $2\sup(2k_1) = 4\|k_1\|_\infty < \sigma_+ = s_1$; the first factor is bounded on compacts by the vertex argument of Lemma 10.3 with the fixed-cube limits of Theorem B at C_{J_0} (negative Wick coefficients only). *Convergence on* $U \cap T_{\mathcal{F}'}$:

there the values are real, nondecreasing in J (Corollary 4.3, then $k\to\infty$) and bounded, since $Z_J^{a,K}(\mathrm{i}f_\zeta,g_\zeta)\le Z_J^{a,K}(0,g_\zeta)$ by Lemma 2.14(b) and $Z_J^{a,K}(0,g_\zeta)\le\exp(\kappa\|g_\zeta\|_{L^1})$ by Corollary 13.3 on King's convention in the limit $k\to\infty$ ($0\le g_\zeta\le\sigma_+/4$ there, so $2\sup g_\zeta<s_1$). Montel's theorem on the connected open $T_{\mathcal{F}'}$ with the uniqueness set $U\cap T_{\mathcal{F}'}$ gives locally uniform convergence to a holomorphic $Z_{\infty,T}^{a,K}$; on $T_{\mathcal{F}'}\cap\mathcal{D}_{\sigma_+,H}$ this is the limit G_∞^K of Theorem 28.5, the same argument on the other tube. The bound $|Z_{\infty,T}^{a,K}(\zeta)|\le Z_{\infty,T}^{a,K}(\mathrm{Re}\,\zeta)$ and the evenness in z pass from the lattice (Lemma 2.14(b); the invariance $(A,\vartheta)\mapsto(-A,-\vartheta)$ of every state (charge conjugation, Definition 2.5), which maps $F(f)\mapsto-F(f)$ and fixes r; $T_{\mathcal{F}'}$ is invariant under $z\mapsto-z$). Positivity at symmetric real points is Lemma 2.14(c), $Z^{a,K}(\xi)Z^{a,K}(-\xi)\ge1$, in the limit.

(b) *Lattice translations.* Let $b\in5^{-k_1}\mathbb{Z}^3$, $|b|_\infty\le M$, $k\ge k_1$. τ_b preserves Λ_k and S_{Λ_k}, so $Z_{k,J}^{a,K}(\tau_b\cdot\zeta)=Z_{k,J}^{a,K,-b}(\zeta)$, the functional of King's cube translated by $-b$ ((20)). For $\zeta\in U$ the sources lie in the cone, so the translation sandwich of Lemma 8.3 gives, for $J\ge J_0+1$ (so that every plaquette and site at which the sources do not vanish lies in C_{J-1}), $Z_{k,J-1}^{a,K}(\zeta)\le Z_{k,J}^{a,K}(\tau_b\cdot\zeta)\le Z_{k,J+1}^{a,K}(\zeta)$. Let $k\to\infty$ and then $J\to\infty$ on $U\cap T_{\mathcal{F}'}$, where the outer limits converge by (a): $Z_{\infty,T}^{a,K}(\tau_b\cdot\zeta)=Z_{\infty,T}^{a,K}(\zeta)$ there, hence on all of $T_{\mathcal{F}'}$ by the identity theorem (both sides are holomorphic on the connected $T_{\mathcal{F}'}$, the translated family having the same supremum norms). Iterating, the identity holds for all $b\in\bigcup_k5^{-k}\mathbb{Z}^3$. *All* $b\in\mathbb{R}^3$. For $\rho>0$ let $E_\rho:=C_c^\infty(\bar B_\rho;\Lambda^2)\times C_c^\infty(\bar B_\rho)$ (a Fréchet space) and $O_\rho:=\{(f,h)\in E_\rho:\ 4\|h\|_\infty<\sigma_+/2\}$, open, convex and symmetric under $s\mapsto-s$. For $s=(f,h)\in O_\rho$ the one-member family with its splitting has the coefficient vector $(1,1,-1)$ in its tube ($2\|k_1\|_\infty=2\|h^+\|_\infty\le4\|h\|_\infty<\sigma_+/2$), so $\Phi_T(s):=\ln Z_{\infty,T}^{a,K}(f,h)$ is defined and finite on O_ρ by (a). Φ_T is convex (a pointwise limit of the functions $\ln Z_{k,J}^{a,K}$, convex by Hölder's inequality) and Borel (a pointwise limit of continuous functions of finitely many sampled values). By Lemma 2.17, Φ_T is continuous on O_ρ. For $s\in O_\rho$ and $b\in\mathbb{R}^3$ pick $b_n\to b$ in $\bigcup_k5^{-k}\mathbb{Z}^3$; then $\tau_{b_n}s,\tau_bs\in O_{\rho+|b|+1}$, $\tau_{b_n}s\to\tau_bs$ in $E_{\rho+|b|+1}$ (translations act continuously on C_c^∞), and $\Phi_T(\tau_{b_n}s)=\Phi_T(s)$; so $\Phi_T(\tau_bs)=\Phi_T(s)$. *Complex coefficients.* For a family $\mathcal{F}'$ the two holomorphic functions $\zeta\mapsto Z_{\infty,T}^{a,K}(\tau_b\cdot\zeta)$ and $\zeta\mapsto Z_{\infty,T}^{a,K}(\zeta)$ on $T_{\mathcal{F}'}$ agree at the real points of $\{\xi\in\mathbb{R}^m:4\|h_\xi\|_\infty<\sigma_+/2\}\cap T_{\mathcal{F}'}$, where both limits exist and both equal the one-member value $\exp\Phi_T(f_\xi,h_\xi)$ (the same J-limit of the same $Z_J^{a,K}$). This intersection is open in $\mathbb{R}^m$ and contains 0 (the first set alone need not lie in the tube: with $\tilde h_1=\tilde h_2$ and $\xi=(1,-1)$, $h_\xi=0$ while $k_1=\tilde h_1$), hence is a set of uniqueness in the connected $T_{\mathcal{F}'}$; so the two functions agree on $T_{\mathcal{F}'}$. □

10.4. **The two limits and the passage to every plane.** This subsection carries the reflection-positivity inequality of the modified boxes to the limits $k\to\infty$ and $J\to\infty$, in three steps, and then to every plane $\{x_\nu=c\}$, $c\in\mathbb{R}$; the curvature sector and the tube of King's state are treated together, as two versions of one argument.

Throughout this subsection $\mathcal{F}$, Φ_k, ζ_{ij}^*, $\mathcal{T}_1$, $T_{\mathcal{F}}$, U are those of Definitions 10.1 and 10.2 for a mid-plane $c\in D$, and "version" refers to Lemma 10.3: (C) is the curvature sector of King's state (the uniform local bound being Theorem 7.6(a),(b)), (T) the tube of King's state under Theorem B, (I2), (I3). The cube functionals are those of the K family, with the fixed-cube limits Z_J^K (version (C)) and $Z_J^{a,K}$ (version (T)); $Z_{\infty(,T)}^{a,K}$ denotes $Z_\infty^K(\cdot,0)$ in version (C) and $Z_{\infty,T}^{a,K}$ in version (T).

Step 1 ($k\to\infty$ at fixed J). By the fixed-J bound of Lemma 10.3 and Montel's theorem there is a subsequence $k_m\to\infty$ along which $\Phi_{k_m}\to\Phi^J$ locally uniformly on $\mathbb{C}^m$, with Φ^J entire. No convergence of the full sequence is claimed or needed. By Lemma 9.3(b), $\sum_{ij}\bar\gamma_i\gamma_j\Phi_k(\zeta_{ij}^*)\ge0$ for every $k\ge k_0(c,\delta)$ and $\gamma\in\mathbb{C}^N$; positive semidefiniteness of a Hermitian matrix is closed

under entrywise limits, so $\sum_{ij}\bar\gamma_i\gamma_j\Phi^J(\zeta^*_{ij})\ge 0$. The uniform bounds of Lemma 10.3 pass to Φ^J: $\sup_Q|\Phi^J|\le C_Q$ in (T) and (C), uniformly in $J\ge J_0$, whatever subsequence was taken. (The box $\Omega_{J,k}$ depends on J, so Φ^J does; the subsequence may depend on J.)

Step 2 ($J\to\infty$: sandwich on the cone, identity theorem off it). By the $\mathcal{I}$ halves of Lemmas 9.9 and 9.10, whose hypotheses are (23)(i),(ii), read for King's cubes (Lemma 4.7(c): the lower end with Step 1 of Lemma 4.2 prepended, Remark 4.9; the upper end with King's state on X_{J+2n}, Lemma 4.7(b)), for every k and every $\zeta\in U$,

$$Z^{a,K}_{k,J}(\zeta)\ \le\ \Phi_k(\zeta)\ \le\ Z^{a,K}_{k,J+2n}(\zeta). \tag{26}$$

Let $k\to\infty$ along the subsequence of Step 1. The outer terms converge along the full sequence (version (C): Theorem 7.6(b); (T): the fixed-cube limits of Theorem B, the points of U carrying non-negative scalar sources below the threshold; each with Montel's theorem), so $Z^{a,K}_J\le\Phi^J\le Z^{a,K}_{J+2n}$ on U for every subsequential Φ^J. As $J\to\infty$ the outer terms converge on the respective uniqueness set: in (C) on $U=(\mathrm{i}\mathbb{R})^{m_F}$ to $Z^K_\infty(\cdot,0)$ (Theorem 7.6(b)); in (T) on $U\cap T_{\mathcal{F}}$ to $Z^{a,K}_{\infty,T}$ (Proposition 10.4(a); there $J\mapsto Z^{a,K}_J(\zeta)$ is real, nondecreasing and bounded by $Z^{a,K}_J(0,g_\zeta)\le\exp(\kappa\|g_\zeta\|_{L^1})$, since $0\le g_\zeta\le\sigma_+/4$, so $2\sup g_\zeta<s_1$). Hence $\Phi^J\to Z^{a,K}_{\infty(,T)}$ pointwise on that set. By Step 1 the family $\{\Phi^J\}_J$ is locally bounded on the corresponding connected domain $(\mathbb{C}^m,T_{\mathcal{F}})$, on which the limit functional is holomorphic (Theorem 7.6(b), Proposition 10.4(a) respectively); by Montel's theorem every subsequence of (Φ^J) has a further subsequence converging locally uniformly to a holomorphic function, which agrees with the limit functional on the uniqueness set and hence on the whole domain [22]. So $\Phi^J\to Z^{a,K}_{\infty(,T)}$ locally uniformly on the domain, in particular at the finitely many matrix points ζ^*_{ij} (which lie in $T_{\mathcal{F}}$ by (24) when every $\mathsf{a}(\zeta_i)<\sigma_W$), where $\Phi^J(\zeta^*_{ij})\to Z^{a,K}_{\infty(,T)}(\theta\zeta^*_i\sqcup\zeta_j)$ because the functionals depend on the family only through the total sources. Positivity passes to the limit:

$$\sum_{i,j}\bar\gamma_i\gamma_j\,Z^{a,K}_{\infty(,T)}(\theta\zeta^*_i\sqcup\zeta_j)\ \ge\ 0\qquad\text{for the plane }\{x_\nu=c\},\ c\in D. \tag{27}$$

A side result: the modified-box values converge to the cube state, $\lim_J\lim_{k_m}Z^{a,(3.38)}_{k_m,\Omega_{J,k_m}}(\zeta)=Z^{a,K}_{\infty(,T)}(\zeta)$, on the respective domain and along any k-subsequence on which the fixed-J limits exist; this is the corrected form of King's passage from the modified boxes to the cube limit [26, (3.42)], with the margin $2n$ in place of one cube step (Lemma 9.6 and Remark 9.8). The reversal of King's second inequality is proved, weakly at every coupling and strictly at $e=0$ (Lemma 9.6); no step of the passage to the limit uses that inequality in King's direction.

Step 3 (every plane). Let $c\in\mathbb{R}$ be arbitrary and the sources supported in $\{x_\nu>c\}$. If $c\in D$, (27) is the result. Otherwise pick $c'\in D$ and $b:=(c'-c)e_\nu$. The translated sources $\tau_b\zeta_i$ are supported in $\{x_\nu>c'\}$, and $\theta_{c'}(\tau_b\zeta)^*=\tau_b(\theta_c\zeta^*)$, because $\rho_{c'}(x+b)=\rho_c(x)+b$ for this b and the linear parts of $\rho_c,\rho_{c'}$ agree. By (27) for the plane c' and the translation invariance of the limit functional,

$$\sum\bar\gamma_i\gamma_jZ^{a,K}_{\infty(,T)}(\theta_c\zeta^*_i\sqcup\zeta_j)=\sum\bar\gamma_i\gamma_jZ^{a,K}_{\infty(,T)}\big(\tau_b(\theta_c\zeta^*_i\sqcup\zeta_j)\big)=\sum\bar\gamma_i\gamma_jZ^{a,K}_{\infty(,T)}\big(\theta_{c'}(\tau_b\zeta_i)^*\sqcup\tau_b\zeta_j\big)\ \ge\ 0.$$

The translation invariance is provided in (C) by Corollary 8.4, and in (T) by Proposition 10.4(b) on $T_{\mathcal{F}}$ (the amplitude a is translation invariant, so the matrix points stay in the tube).

10.5. **The theorems.** The pieces are now assembled: reflection positivity of King's state about every coordinate plane, first in the curvature sector and then on the Wick tube.

Theorem 10.5 (reflection positivity of King's state, curvature sector)**.** *Let $\mu_0 > 0$, $\nu \in \{1,2,3\}$, $c \in \mathbb{R}$, and let $\zeta_1, \dots, \zeta_N$ be finite families of real 2-form sources $f_{il} \in C_c^\infty(\mathbb{R}^3; \Lambda^2)$ supported in the open half-space $\{x_\nu > c\}$, with arbitrary complex coefficients z_{il} and no scalar sources. Then for all $\gamma \in \mathbb{C}^N$*

$$\sum_{i,j=1}^{N} \bar\gamma_i \gamma_j \, Z_\infty^K\big(\theta\zeta_i^* \sqcup \zeta_j,\, 0\big) \;\geq\; 0;$$

moreover Propositions 10.7 and 10.8 below hold for $Z_\infty^K(\cdot, 0)$. The results used are Theorem A, [9, Thm. 6.2(1),(2)] *(through the cube monotonicity Corollary 4.3 and the $\mathcal{I}$ halves of Lemmas 9.9 and 9.10), Theorem 7.6(a),(b) with Lemma 7.3, Lemma 4.7(c) and Remark 4.9 for the two places where the cube convention is seen, Corollary 8.4 for the translations, and Lemma 2.11, Lemma 2.14(b),(c), Lemma 2.10(e), Lemma 2.17; no inequality of class $\mathcal{D}$ enters.*

Proof. At $h \equiv 0$ there is no Wick constant and no scheme, and $\mathcal{F}$ consists of the f_{il} and $\rho^* f_{il}$ with $U = (\mathrm{i}\mathbb{R})^m$. What the argument of Steps 1–3 needs of the cube functionals is: the existence of $Z_J^K(\cdot,0) = \lim_k a_{k,J}^K$ locally uniformly on $\mathbb{C}^m$, entire; the monotonicity of $J \mapsto Z_J^K(\mathrm{i}f, 0)$ on the cone; the existence of $Z_\infty^K(\cdot,0) = \lim_J Z_J^K(\cdot,0)$, entire, locally uniformly on $\mathbb{C}^m$; and translation invariance. The first three are Theorem 7.6(b), whose local bound is the μ_0-free bound of Theorem 7.6(a) and Lemma 7.3 rather than a local bound resting on the class-$\mathcal{D}$ half of the cube monotonicity and so on [9, Thm. 6.2(3),(4)]; the fourth is Corollary 8.4. Lemmas 9.3 and 9.5 concern the modified box state alone and the lattice reflection, and Lemma 10.3(C) bounds Φ_k on the modified photon form: these are convention-free, as is Step 1 (Montel in k; positivity passes). The two places where the Neumann–Dirichlet cube states enter are the ends of the sandwich (26), i.e. the $\mathcal{I}$ halves of Lemmas 9.9 and 9.10; on King's convention they read $Z_{k,J}^K(\zeta) \leq \Phi_k(\zeta) \leq Z_{k,J+2n}^K(\zeta)$ for every k and $\zeta \in U = (\mathrm{i}\mathbb{R})^m$: the lower end is Lemma 9.9 with Step 1 of Lemma 4.2 prepended (Remark 4.9: $\langle\cdot\rangle_{k,J}^K \leq \langle\cdot\rangle_{k,J}^{(a)} \leq \langle\cdot\rangle_{k,\Omega_{J,k}}^{(3.38)}$ on $\mathcal{I}$), the upper end is Lemma 9.10 with $\langle\cdot\rangle_{k,J}^K$ on X_{J+2n} in place of the comparison state (Lemma 4.7(b),(c): the box $\Omega_{J,k}$ does not meet the killed layers T_{J+2n}, and part (a) of Lemma 9.10 is photon-only), under the hypotheses of (23). Here J_0 is the J_0 of the enlarged family (the sources and their reflections), which exceeds that of the sources alone when $c \neq 0$; Theorem 7.6(b) is applied to the enlarged family at J and $J+2n$, $J \geq J_0$. Along Step 1's subsequence the ends converge along the full sequence by Theorem 7.6(b), so $Z_J^K \leq \Phi^J \leq Z_{J+2n}^K$ on U for every subsequential Φ^J; as $J \to \infty$ both ends tend to Z_∞^K on U, $\{\Phi^J\}_J$ is locally bounded on $\mathbb{C}^m$ by Lemma 10.3(C), and U is a set of uniqueness, so $\Phi^J \to Z_\infty^K$ locally uniformly on $\mathbb{C}^m$ as in Step 2, in particular at the matrix points; positivity passes to the limit. Step 3 uses Corollary 8.4. Propositions 10.7 and 10.8 use only the existence, holomorphy near 0 and lattice symmetries of the limit, and the locally uniform convergences just established; their proofs below apply with Z_∞^K, the charge-conjugation invariance being exact for $\langle\cdot\rangle_{k,J}^K$. □

Theorem 10.6 (reflection positivity on the Wick tube, King's state)**.** *Let the couplings lie in the coupling region of Definition 2.1 and let a be an admissible scheme, so that* (I2) *and* (I3) *hold (the hypotheses of Corollary 13.3) and the fixed-cube limits of Theorem B are available as in Proposition 10.4; the $\mathcal{D}$-class steps use in addition Lemma 2.10(d). Let $\sigma_+ = s_1$ be the amplitude of* (I2) *and $\sigma_W := \sigma_+/8$. Let $\mu_0 > 0$, ν, $c \in \mathbb{R}$, and let $\zeta_1, \dots, \zeta_N$ be finite families of real sources $f_{il} \in C_c^\infty(\mathbb{R}^3; \Lambda^2)$, $h_{il} \in C_c^\infty(\mathbb{R}^3)$ supported in $\{x_\nu > c\}$, with complex coefficients z_{il}, w_{il} such that $\mathsf{a}(\zeta_i) < \sigma_W$ for every i; in particular, all purely imaginary w_{il} with arbitrary z_{il} are admitted.*

Then the values $Z^{a,K}_{\infty,T}(\theta\zeta_i^* \sqcup \zeta_j) = \lim_J Z^{a,K}_J(\theta\zeta_i^* \sqcup \zeta_j)$ *exist (Proposition 10.4) and*

$$\sum_{i,j=1}^{N} \bar\gamma_i\gamma_j\, Z^{a,K}_{\infty,T}\big(\theta\zeta_i^* \sqcup \zeta_j\big) \;\geq\; 0 \qquad \text{for all } \gamma \in \mathbb{C}^N.$$

On $T_{\mathcal{F}} \cap \mathcal{D}_{\sigma_+,H}$, $Z^{a,K}_{\infty,T} = G^K_\infty$. *No lower bound on* J *beyond* J_0 *enters, and no bound at positive scalar sources beyond the cone* $\sup q \leq s_1$ *of Corollary 13.3 is used.*

Proof. With the splitting of Definition 10.1 the matrix points lie in $T_{\mathcal{F}}$ by (24). Lemma 10.3(T) gives the bounds; Steps 1 and 2 in version (T), with the sandwich (26) in its King form (Lemma 4.7(c)) and Proposition 10.4(a) for the limit on $U \cap T_{\mathcal{F}}$ and its holomorphy on $T_{\mathcal{F}}$, give (27) for $c \in D$; Step 3 with Proposition 10.4(b) gives every plane. The $\mathcal{D}$-class steps are the first factor of (25) (the $\mathcal{D}$ half of Lemma 9.9 with Step 3 of Lemma 4.2 prepended, and the class-$\mathcal{D}$ cube monotonicity of Corollary 4.3) and the nonincreasing factor in Proposition 10.4(a). □

10.6. OS3 **and the polynomial form.** This subsection draws from the limits of this section the symmetry properties of the Schwinger functions, OS3 (Proposition 10.7), and the polynomial form of reflection positivity (Proposition 10.8).

Proposition 10.7 (symmetry, realness and charge conjugation)**.** *Let* Z *be one of* $Z^K_\infty(\cdot,0)$ *(Theorem 7.6),* $Z^{a,K}_{\infty,T}$ *(Proposition 10.4) and* G^K_∞ *(Theorem 28.5), or a limit state of the comparison family along a continuum subsequence, in either sector (Theorem 14.9). The Schwinger functions*

$$S_{n,p}(f_1,\dots,f_n;h_1,\dots,h_p) := \partial_{z_1}\cdots\partial_{z_n}\partial_{w_1}\cdots\partial_{w_p}\, Z\big(\textstyle\sum_i z_i f_i, \sum_l w_l h_l\big)\big|_0$$

(with $p = 0$ *and* S_n *in the curvature sector) are (a) symmetric multilinear in the* f*'s and in the* h*'s; (b) real for real test functions; (c) zero for odd* n*. The proof uses only that* Z *exists and is holomorphic near* 0*, with the stated symmetries of the lattice functionals; the existence is established by the result cited for each.*

Proof. (a) Z depends on a family only through its total sources (so does every lattice functional, hence both limits), so its Taylor coefficients at 0 are symmetric multilinear forms in the f's and in the h's; $Z^{a,K}_{\infty,T}$ is holomorphic on a tube containing 0. The definition of $S_{n,p}$ does not depend on the family used to compute it (restrict a larger family to the given sources). (b) For real coefficients every lattice functional is real (real integrand and multiplier, Lemma 2.14(a)), so Z is real at the real points near 0 (for $Z^{a,K}_{\infty,T}$ on $\mathbb{R}^m \cap T$, an open piece of $\mathbb{R}^m$ containing 0); a holomorphic function real on the real points near 0 has real Taylor coefficients at 0. (c) Every cube state and every modified box state is invariant under $(A,\vartheta) \mapsto (-A,-\vartheta)$ (the photon terms are even in A, the masked divergence included; $\cos(\vartheta_x - e\varepsilon A - \vartheta_y)$ is invariant; W^a depends on r only; Lebesgue measure is preserved), which maps $F_{\Lambda_k}(f) \mapsto -F_{\Lambda_k}(f)$ and fixes $W^a_{\Lambda_k}(h)$. Hence $Z_{k,X}(-z,w) = Z_{k,X}(z,w)$ for all k, X, and in both limits $Z(-z,w) = Z(z,w)$ (on the $z \mapsto -z$ invariant tube for $Z^{a,K}_{\infty,T}$). So $\partial_z^{(n)}\partial_w^{(p)} Z(0) = 0$ for odd n. □

Proposition 10.8 (the polynomial form of reflection positivity)**.** *Let* $P = \sum_\alpha c_\alpha \prod_i F(f_{\alpha i}) \prod_l W^a(h_{\alpha l})$ *be a polynomial in the smeared fields with real sources supported in* $\{x_\nu > c\}$*, and* $\theta P := \sum_\alpha \bar c_\alpha \prod_i F(\rho^* f_{\alpha i}) \prod_l W^a(h_{\alpha l}\circ\rho)$*. Then* $S(\theta(P)P) \geq 0$*, where* S *assigns to a monomial the corresponding Schwinger function of Proposition 10.7: the Taylor coefficient at* 0 *of* $Z^{a,K}_{\infty,T}$ *(Theorem 10.6), resp. of* $Z^K_\infty(\cdot,0)$ *(Theorem 10.5), where at* $h \equiv 0$ *the* W^a *factors are absent.*

Proof. Let all sources of P be supported in $\{x_\nu \geq c+\delta\}$ and let $\mathcal{F}'$ be the family of all sources of P and their reflections. Its scalar members have no sign, so Steps 1–2 are applied to the split family $\mathcal{F}'_{\text{split}}$ (members $h^{\pm}_{\alpha l}$, $h^{\pm}_{\alpha l}\circ\rho$, all ≥ 0); its tube contains 0 ($k_1(0) = 0$), and the unsplit functional

is the composition of the split one with the linear map $w_{\alpha l} \mapsto (w^+_{\alpha l}, w^-_{\alpha l}) = (w_{\alpha l}, -w_{\alpha l})$, so by the chain rule the derivatives at 0 of the unsplit Φ_k, Φ^J, $Z^{a,K}_{\infty(,T)}$ are finite linear combinations of those of the split ones, and locally uniform convergence near 0 passes through the linear map. (i) *Lattice.* For $k \geq k_0$ the sampled P is a gauge-invariant polynomial in the plus-side smeared fields, lies in L^2 of the modified box state (Gaussian moments; $\lambda > 0$) and in $\mathfrak{B}_+$ (Definition 9.1), and $\theta(P) = \overline{P \circ \hat{\rho}}$ by (21). By Lemma 9.3(a) (its L^2 extension), $\langle \theta(P)P \rangle^{(3.38)}_{k,\Omega_{J,k}} \geq 0$. (ii) *Moments as derivatives.* $\langle \theta(P)P \rangle^{(3.38)} = \sum_{\alpha,\beta} \bar{c}_\alpha c_\beta \langle \prod_i F(\rho^* f_{\alpha i}) \prod_l W^a(h_{\alpha l} \circ \rho) \prod_i F(f_{\beta i}) \prod_l W^a(h_{\beta l}) \rangle^{(3.38)}$, and each mixed moment is a mixed partial derivative at 0 of Φ_k for the family $\mathcal{F}'$ (differentiate under the integral; the scheme multiplier $e^{-wa_k \mathsf{S}(h)}$ is the $-a_k$ inside W^a). (iii) *Limits.* By Steps 1–2 for $\mathcal{F}'_{\text{split}}$, $\Phi_{k_m} \to \Phi^J$ and $\Phi^J \to Z^{a,K}_{\infty(,T)}$ locally uniformly on a neighbourhood of 0, so by Weierstrass' theorem all derivatives at 0 converge, and $\sum \bar{c}_\alpha c_\beta \, \partial^{(\alpha,\beta)} Z^{a,K}_{\infty(,T)}(0) \geq 0$; the left side is $S(\theta(P)P)$ by the family-independence of the Schwinger functions (Proposition 10.7(a)). Every plane: Step 3 (translation of the sources; S is translation invariant by Proposition 10.4(b), resp. Corollary 8.4). The curvature sector is the case $h \equiv 0$ with no splitting. This is the analogue, for the curvature and the renormalised Wick square with reflection in the planes $\{x_\nu = c\}$, of the positivity condition [29, Axiom E2, p. 88] in its form for finite sums of tensor products of test functions supported in a half-space. □

Remark 10.9 (which form is used, and the amplitude constants). The main theorem (Theorem 28.2) states reflection positivity of the Wick sector in the tube form of Theorem 10.6, with Propositions 10.7 and 10.8 for the Schwinger functions; the results it uses for this clause are Theorem B, (I2), (I3)and [9, Thm. 6.2(1)–(4)]. This suffices for its use in the proof of Proposition 28.6(d): the measure and the weak form of OS4 for the joint field use reflection positivity for exponentials $e^{\mathrm{i}\Phi(f)}$ of the smeared fields, whose Wick coefficients are purely imaginary, so amplitude 0 only. Three positive, ε-free amplitude constants occur in the Wick sector: $\sigma_W = \sigma_+/8$ here; $H/2 = \sigma_+/32$, the negative-part radius of the tube $\mathcal{D}_{\sigma_+,H}$ of Theorems 13.10 and 28.5 ($H = s_1/16$); and $\sigma_0 = H/4$ in the bound E0″ of Theorem 28.5. On $T_{\mathcal{F}} \cap \mathcal{D}_{\sigma_+,H}$ the objects $Z^{a,K}_{\infty,T}$ and G^K_∞ coincide (Proposition 10.4(a)). The tube statements of this section are stated on King's convention; their convention-sensitive steps are those listed in Remark 13.12 and Proposition 28.6(c), and the bound (37) enters on King's convention as (73) (Theorem 28.5). The margin $R(\mu_0)$ of Lemma 9.10 enters the proof, not the conclusion: reflection positivity is proved at every fixed $\mu_0 > 0$; in the curvature sector the limit of vanishing photon mass is taken in Section 30, where reflection positivity about every plane passes to the subsequential limits of King's states (Theorem 30.2).

The following remark does for the limit statements of this section what Remark 7.8 does for the regularity statements of Section 7: it lists what their proofs use of the cube functionals, so that they apply to the comparison family wherever that family's fixed-cube limits exist; the two cube conventions and the role of the comparison family are described before Remark 7.8.

Remark 10.10 (the comparison family). The proofs of this section use of the cube functionals only their fixed-cube limits (at real coefficients in the curvature sector, at the real points of the tube in the Wick sector), the cube monotonicity on the cone, the positive-leg bound Corollary 13.3 and the bounds of Lemma 10.3; all of these are available for the comparison family as well, with the first inequality of Lemma 2.13 in place of Corollary 4.3 and Lemma 8.1 in place of Lemma 8.3. As noted in Section 3, they apply to the comparison family $\mathrm{ND}^{(a)}$ wherever its fixed-cube limits exist. The limit states of the comparison family along continuum subsequences, which exist without that clause, are treated in Section 14, and they coincide with the objects of King's state (Corollary 14.15, Proposition 28.7).

11. The measure and weak clustering

This section constructs the measure of King's state (the infinite-volume state of King's construction [25, 26]) in the curvature sector and derives from it the consequences of reflection positivity: the ergodic decomposition, clustering, the unique joint vacuum on a component, the growth of the components' moments, and weak OS4 on almost every component. The general results used are those of [32, Appendix A] (the measure on the Hermite–Sobolev space and its uniqueness; the ergodic decomposition, obtained by conditioning on the translation-invariant σ-algebra, almost every component of which inherits the reflections and reflection positivity, the components regrouping into O_h-invariant ones when the measure is O_h-invariant and the moment bounds passing to almost every component; the Osterwalder–Schrader reconstruction in each coordinate direction and the clustering statements), cited where they are used, and the rotation corollary, Corollary 11.16. The joint field with the Wick square is the subject of the tube sections. The measure is constructed from the characteristic functional, not from the lattice laws, which converge only locally (Remark 11.6).

Conventions. $\mathcal{V} := \Lambda^2\mathbb{R}^3$ with its Euclidean fibre norm. For a real $u \in C_c^\infty(\mathbb{R}^3;\Lambda^2)$ put $\mathcal{Z}^K(u) := Z_\infty^K(\mathrm{i}u)$, the one-member functional at the coefficient i (Theorem 7.6(b), with the total-source convention of Theorem 8.6). Fix an even $r \geq 14$. $\mathcal{H}^{-r}(\mathbb{R}^3;\Lambda^2)$ is the Hermite–Sobolev space, the dual of the completion of $\mathcal{S}(\mathbb{R}^3;\Lambda^2)$ under $\|u\|_r^2 = \sum_{\alpha,c}\lambda_\alpha^r\langle h_{\alpha,c},u\rangle^2$, where $(h_{\alpha,c})_{\alpha\in\mathbb{N}^3,\,c\leq 3}$ is the Hermite basis of $L^2(\mathbb{R}^3;\Lambda^2)$ and $\lambda_\alpha = 2|\alpha|+4$ are the eigenvalues of $-\Delta+|x|^2+1$; $|\cdot|_4$ is the weighted C^4 norm of Lemma 2.9, and $|u|_4 \leq C\|u\|_{10}$ on $\mathcal{S}$ ([32, Lemma A.4(b)]). $X := H^{-r}_{\mathrm{loc}}(\mathbb{R}^3;\Lambda^2)$, a Fréchet space with the seminorms $\|\chi_n\Phi\|_{H^{-r}}$ for a fixed sequence of cut-offs $\chi_n \in C_c^\infty(\mathbb{R}^3)$ with $\chi_n = 1$ on the ball of radius n, carries the actions $(\tau_a\Phi)(u) = \Phi(\tau_{-a}u)$, $(g\Phi)(u) = \Phi(g^{-1}\cdot u)$ for $g \in O_h$ (or $O(3)$; with the translations, the push-forward action of the Euclidean group), $(\theta_j\Phi)(u) = \Phi(\sigma_j\cdot u)$ for the coordinate reflections σ_j of $\mathbb{R}^3$ (written ρ_j in Definition 11.4 and in Section 11.3), with the signed transport of 2-forms of Definition 2.5, and the local σ-algebras $\Sigma(U)$ generated by $\Phi \mapsto \Phi(u)$, $u \in C_c^\infty(U;\Lambda^2)$ real; $\theta_{j,c} := \tau_{ce_j}\theta_j\tau_{-ce_j}$ is the reflection in the plane $x_j = c$ and $\rho_{j,c*}u$ the reflected source (Definition 9.1). The *lattice curvature field* of King's cube is the random distribution $\Phi^K_{k,J} : u \mapsto F_{\Lambda_k}(u)$ under $\langle\cdot\rangle^K_{k,J}$. $J_0(u)$ is the least J with $\operatorname{supp} u$ at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_J$.

We first fix the forms of the cluster property OS4 that are proved in this section (Theorem 11.5) and for the LIM states (Theorem 14.9), and the abstract setting (Definition 11.4) in which the ergodic decomposition and the clustering theorems are proved.

Throughout, μ is a translation-invariant Borel probability measure on X, and $U(a)G := G\circ\tau_{-a}$ is the induced unitary representation of the translation group $\mathbb{R}^3$ on $L^2(\mu)$; e_1, e_2, e_3 is the standard basis of $\mathbb{R}^3$, $H^+_{j,0} := \{x \in \mathbb{R}^3 : x_j > 0\}$, and the expectation $\mathbb{E}_\mu[G'G]$ is taken without conjugation.

Definition 11.1 (the cluster property)**.** For $j \in \{1,2,3\}$ consider the two conditions:

(OS4$_j$) (ergodicity of the translations in the direction e_j) every $G \in L^2(\mu)$ with $U(te_j)G = G$ for all $t \in \mathbb{R}$ is μ-almost surely constant;

(OS4$'_j$) (clustering along the axis e_j) $\mathbb{E}_\mu[G'\,U(te_j)G] \to \mathbb{E}_\mu[G']\,\mathbb{E}_\mu[G]$ as $|t| \to \infty$, for all $G, G' \in L^2(\mu)$.

The measure μ satisfies OS4 if (OS4$_j$) holds for $j = 1,2,3$. A family of Schwinger functions satisfies OS4 if it is the family of moments of exactly one translation-invariant probability measure μ on X, and this μ satisfies OS4.

Definition 11.2 (forms of the cluster axiom; the weak form)**.** Let X, μ and U be as in Definition 11.1, and let $j \in \{1,2,3\}$ be the direction chosen as Euclidean time. In (i) assume moreover

that X is a space of distributions with values in a finite-dimensional real vector space $\mathcal{V}$ (for the curvature field $\mathcal{V} = \Lambda^2\mathbb{R}^3$, for the joint field with the Wick square $\mathcal{V} = \Lambda^2\mathbb{R}^3 \oplus \mathbb{R}$), that μ is θ_j-invariant, and let $\mathcal{P}_j^+$ be the set of polynomials in the fields $\Phi(f)$, $f \in C_c^\infty(H_{j,0}^+;\mathcal{V})$ real, that belong to $L^2(\mu)$.

(i) (cluster property, [29, E4, p. 88]) For all $F, G \in \mathcal{P}_j^+$ and every $a \in \mathbb{R}^3 \setminus \{0\}$ with $a \perp e_j$,

$$\lim_{s\to\infty} \mathbb{E}_\mu\big[\overline{F \circ \theta_j}\; U(sa)G\big] = \mathbb{E}_\mu\big[\overline{F}\,\big]\,\mathbb{E}_\mu[G].$$

(ii) (ergodicity of the time translations, [18, (6.1.10), p. 92]) Condition (OS4$_j$) of Definition 11.1 holds.

(iii) (weak form) μ is ergodic for the action of the translation group $\mathbb{R}^3$: every $G \in L^2(\mu)$ with $U(a)G = G$ for all $a \in \mathbb{R}^3$ is μ-almost surely constant.

We call (i) and (ii) the single-axis forms of OS4, and say that μ satisfies OS4 *in the weak form* if (iii) holds.

Remark 11.3 (relations between the forms). (a) Condition (OS4$_j$) for one j implies the weak form (iii), since an $\mathbb{R}^3$-invariant function is invariant under the translations along e_j; the converse does not hold in general, the weak form controlling the joint fixed space of the reconstruction and not the kernel of a single Hamiltonian.

(b) If the coordinate permutations act measurably on X, as in the setting of the field spaces used here, and μ is invariant under them (under the cubic group, as the measure of King's state is), then (OS4$_1$), (OS4$_2$) and (OS4$_3$) are equivalent: for such a permutation g, $G \mapsto G \circ g^{-1}$ is a unitary of $L^2(\mu)$ that preserves the constants and carries the functions invariant along e_j onto those invariant along ge_j, since $g\,\tau_a\,g^{-1} = \tau_{ga}$. In general they are three statements.

(c) Under the hypotheses of the reconstruction in the coordinate directions (translation invariance, local σ-algebras generated by the bounded regions, reflection positivity about the coordinate planes) the conditions (OS4$_j$) and (OS4$'_j$) are equivalent for each j. Writing P_j for the orthogonal projection of $L^2(\mu)$ onto the e_j-invariant functions, (OS4$_j$) says that $P_jG = \mathbb{E}_\mu G$ for every $G \in L^2(\mu)$ (an e_j-invariant G equals P_jG, and P_jG is e_j-invariant), while $\mathbb{E}_\mu[G'\,U(te_j)G] \to \mathbb{E}_\mu[G'\,P_jG]$ as $|t| \to \infty$ by the spectral theorem for the self-adjoint contraction semigroup of the reconstruction in the direction e_j. If $P_jG = \mathbb{E}_\mu G$ for all G, the limit is $\mathbb{E}_\mu[G']\,\mathbb{E}_\mu[G]$. Conversely, if (OS4$'_j$) holds, then $\mathbb{E}_\mu[G'(P_jG - \mathbb{E}_\mu G)] = 0$ for all G', and the choice $G' = \overline{P_jG - \mathbb{E}_\mu G}$ gives $P_jG = \mathbb{E}_\mu G$.

For $j \in \{1,2,3\}$ and $c \in \mathbb{R}$ let $H_{j,c}^{\pm} := \{x \in \mathbb{R}^3 : \pm(x_j - c) > 0\}$ be the open half spaces, $\rho_j x := x - 2x_j e_j$ the reflection in the plane $\{x_j = 0\}$ (the reflection written σ_j in the Conventions of Section 11) and $\rho_{j,c}$ the reflection in $\{x_j = c\}$; K denotes a bounded open subset of $\mathbb{R}^3$ and $K \Subset U$ means $\overline{K} \subset U$. The ergodic decomposition, the clustering theorems and the reconstruction of the vacuum are proved in [32, Appendix A] for an abstract *reflection-positive stationary system*, which we now define.

Definition 11.4 (the abstract setting)**.** A *reflection-positive stationary system* $(X, \tau, \theta, \Sigma, \mu)$ consists of the following data, subject to (H1)–(H3).

(H1) (*Space, actions, measure.*) X is a Polish space with a jointly continuous action $(a, x) \mapsto \tau_a x$ of $\mathbb{R}^3$ and three involutive homeomorphisms $\theta_1, \theta_2, \theta_3$ of X with $\theta_j\tau_a = \tau_{\rho_j a}\theta_j$; μ is a Borel probability measure on X invariant under every τ_a and every θ_j.

(H2) (*Local structure.*) To every open $U \subset \mathbb{R}^3$ is assigned a sub-σ-algebra $\Sigma(U)$ of the Borel σ-algebra $\mathcal{B}$ of X such that $\Sigma(U) \subset \Sigma(U')$ if $U \subset U'$; $\Sigma(U) = \bigvee_{K \Subset U} \Sigma(K)$; $\bigvee_K \Sigma(K) = \mathcal{B}$, the join over all bounded open K; each $\Sigma(U)$ is generated by a countable family of

bounded continuous functions on X; and, with

$$U(a)G := G\circ\tau_{-a},\qquad \Theta_j G := G\circ\theta_j, \tag{28}$$

$U(a)$ maps $\Sigma(U)$-measurable functions to $\Sigma(U+a)$-measurable ones and Θ_j maps $\Sigma(U)$-measurable functions to $\Sigma(\rho_j U)$-measurable ones.

(H3) (*Coordinate reflection positivity.*) For every j, every $c\in\mathbb{R}$ and every $F\in L^2(\Sigma(H^+_{j,c}),\mu)$,

$$\mathbb{E}_\mu\big[\,\overline{F\circ\theta_{j,c}}\,F\,\big]\;\geq\;0,\qquad \theta_{j,c}:=\tau_{ce_j}\theta_j\tau_{-ce_j}. \tag{29}$$

By (H1), $a\mapsto U(a)$ is a unitary representation of $\mathbb{R}^3$ on $L^2(\mu)$, each Θ_j is a unitary involution, and $\Theta_jU(a)=U(\rho_ja)\Theta_j$; in particular $\Theta_jU(te_j)=U(-te_j)\Theta_j$ and $\Theta_jU(y)=U(y)\Theta_j$ for $y\perp e_j$. By translation invariance, (29) for $c=0$ implies it for every c: for $F\in L^2(\Sigma(H^+_{j,c}),\mu)$ the function $F_0:=U(-ce_j)F$ is $\Sigma(H^+_{j,0})$-measurable by (H2), $\overline{F\circ\theta_{j,c}}\,F=(\overline{F_0\circ\theta_j}\,F_0)\circ\tau_{-ce_j}$, and τ_{-ce_j} preserves μ. On a space of distributions we use $(\tau_a\Phi)(u):=\Phi(\tau_{-a}u)$ with $\tau_bu=u(\cdot-b)$, the push-forward of Φ by $x\mapsto x+a$; then $U(a)[\Phi(u)]=\Phi(\tau_au)$, whose test function is supported in $\operatorname{supp}u+a$, as (H2) requires. A Borel probability measure m on X invariant under every τ_a is *ergodic* if $m(B)\in\{0,1\}$ for every Borel set B with $\tau_aB=B$ for all $a\in\mathbb{R}^3$; for a sub-σ-algebra $\mathcal{F}\subset\mathcal{B}$, $L^2(\mathcal{F},m)$ denotes the closed subspace of classes with an $\mathcal{F}$-measurable representative. In the application $X=H^{-r}_{\mathrm{loc}}(\mathbb{R}^3;\mathcal{V})$ with $r\geq14$ even, $\Sigma(U)$ is generated by the evaluations $\Phi\mapsto\Phi(u)$, $\operatorname{supp}u\subset U$, and the actions are the push-forwards of the Euclidean motions; μ is the measure of the curvature field, for which (H1)–(H3) are proved in Section 11.

Theorem 11.5 (the measure, the local lattice laws, weak clustering)**.** *The limits below exist by Theorem A.*

(a) (The measure.) *For every even $r\geq14$ there is exactly one Borel probability measure $\mu^{K,H}$ on $\mathcal{H}^{-r}(\mathbb{R}^3;\Lambda^2)$ whose mixed moments on C_c^∞ are the Schwinger functions S_n^K of King's state. Its characteristic functional on C_c^∞, indeed on $\mathcal{S}$, is $\mathcal{Z}^K$ (extended by continuity); its moments on $\mathcal{S}$ are the continuous extensions of the S_n^K and satisfy the bounds of Theorem 7.6(c). Its image μ_S^K under the restriction map $\iota:\mathcal{H}^{-r}\to X$ is the unique Borel probability measure on X with characteristic functional $\mathcal{Z}^K$ on C_c^∞, and also the unique one whose mixed moments on C_c^∞ are the S_n^K.* Local convergence of the lattice laws: *for every real $\chi\in C_c^\infty(\mathbb{R}^3)$ and every box Q containing $\operatorname{supp}\chi$ in its interior, the laws $\nu^K_{k,J,\chi}$ of the localised lattice fields $\chi\Phi^K_{k,J}$ on $H^{-r}_{\mathrm{per}}(Q;\Lambda^2)$ converge weakly, along the full sequence $k\to\infty$ at every fixed $J\geq J_0(\chi)$, to a law $\nu^K_{J,\chi}$, and $\nu^K_{J,\chi}\Rightarrow\nu^K_\chi$ as $J\to\infty$, where ν^K_χ is the law of $\chi\Phi$ under μ^K_S. Nothing is claimed about the laws of the unlocalised lattice fields at fixed J (Remark 11.6).*

(b) (Invariance and weak clustering.) *μ_S^K satisfies the hypotheses (H1)–(H3) of Definition 11.4, is invariant under $\mathbb{R}^3\rtimes O_h$ about every point, and the general results of* [32, Appendix A] *on the ergodic decomposition and on the Osterwalder–Schrader reconstruction in each coordinate direction from reflection positivity about the coordinate planes apply to it (Steps 3 and 4 of the proof): $\mu_S^K=\int\mu_\xi\,d\nu(\xi)$ with components that are translation-invariant and ergodic, θ_j-invariant, reflection positive about every coordinate plane and satisfy (H2); clustering in the mean and under diagonal separation; the Osterwalder–Schrader reconstruction in each coordinate direction with a unique joint vacuum on every component; the growth bound $|S_N^\xi(u_1,\dots,u_N)|\leq\sigma_N^\xi\prod_i\|u_i\|_{r+2}$, $\sigma_N^\xi\leq\alpha_\beta(\xi)(N!)^\beta$ for every $\beta>1$, for the components' Schwartz moments; and the cubic-invariant decomposition into mixtures of $k(\zeta)\in\{1,2,3,6\}$ ergodic measures. The same holds on $\mathcal{H}^{-r}$ for $\mu^{K,H}$. ν-almost every component satisfies* OS4 *in the weak form, ergodicity under the translation group $\mathbb{R}^3$ (Definition 11.2(iii)).*

(c) (The rotation corollary.) *$\mathcal{Z}^K$ is $O(3)$-invariant, $\mathcal{Z}^K(R\cdot u)=\mathcal{Z}^K(u)$ for all $R\in O(3)$ and real u, by Theorem 8.8 (the rotation theorem, Theorem C); the clauses below use only this invariance. Then for ν-a.e. ξ: μ_ξ is invariant under $\mathbb{R}^3\rtimes O(3)$; $\mathbb{E}_{\mu_\xi}[\overline{F\circ\theta_{n,c}}\,F]\geq 0$ for every unit vector n, every $c\in\mathbb{R}$ and every $F\in L^2(\Sigma(H^+_{n,c}),\mu_\xi)$, $H^+_{n,c}=\{x\cdot n>c\}$; and $\mathbb{E}_{\mu_\xi}[G'\,U(a)G]\to\mathbb{E}_{\mu_\xi}[G']\,\mathbb{E}_{\mu_\xi}[G]$ as $|a|\to\infty$ in any manner, for all $G,G'\in L^2(\mu_\xi)$. Consequently μ_ξ satisfies* OS4 *in each of the forms of Definition 11.1 and Definition 11.2 above and Definition 11.15 below (the cluster property on the positive-time algebra, ergodicity and clustering of the translations along the time direction, ergodicity under $\mathbb{R}^3$) for every choice of the time direction, coordinate or not, and in the reconstruction from μ_ξ in any direction n the joint fixed space is $\ker H_n=\mathbb{C}\Omega_n$ (Step 5 of the proof). The moments of μ_ξ are real and symmetric; the vanishing of its odd curvature moments is* not *claimed for a component.*

The proof has three stages. First (Steps 1, 2 and 2′) the characteristic functional $\mathcal{Z}^K$, King's state at purely imaginary coefficients, is shown to be positive definite and continuous on $\mathcal{S}$, so that the Bochner–Minlos theorem gives a probability measure; the moment bounds of Theorem 7.6(c) and the Lipschitz bound of Corollary 8.4 then place it on the Hermite–Sobolev space $\mathcal{H}^{-r}$, identify its moments with the Schwinger functions, make it unique and give the local convergence of the lattice laws. Second (Steps 3 and 4) translation invariance, the cubic symmetry (Corollary 8.4 and Theorem 8.6) and reflection positivity (Theorem 10.5) make the image measure on X a reflection-positive stationary system in the sense of Definition 11.4, and the general results of [32, Appendix A] for such systems give the ergodic decomposition into translation-invariant components, their clustering in the mean and under diagonal separation, the Osterwalder–Schrader reconstruction with a unique joint vacuum on each component, the growth bound on the components' moments and the cubic decomposition. Third (Step 5) the $O(3)$ invariance of $\mathcal{Z}^K$ (Theorem 8.8) is carried to almost every component by Corollary 11.16, which gives Euclidean invariance, reflection positivity about every plane and clustering in every direction. The following paragraph says which statement of this paper provides each hypothesis at each step.

The results used at each step. Clause (a) uses Theorem 7.6 and Corollary 8.4 only (Steps 1, 2 and 2′). Clause (b) uses also Theorems 8.6 and 10.5, which give (H1)–(H3) in Step 3; the general results of [32, Appendix A] applied in Step 4 are [32, Theorem A.18] (the decomposition), [32, Theorem A.26] and [32, Theorem A.28] (clustering in the mean and under diagonal separation), [32, Proposition A.23] with [32, Corollary A.29(iii)] (the reconstruction and the unique joint vacuum), [32, Theorem A.21] and [32, Corollary A.22] (the growth bound), [32, Theorem A.20] (the cubic decomposition), [32, Lemma A.11] (the passage to $\mathcal{H}^{-r}$) and Definition 11.2(iii) with [32, Corollary A.29(i)] (weak OS4). Clause (c) uses only the $O(3)$-invariance of $\mathcal{Z}^K$ and is Corollary 11.16, applied in Step 5.

11.1. **Proof of (a).** *Step 1 (moment bounds, and where they hold).* For real 2-forms $f_1,\dots,f_n$ at distance >1 from $\mathbb{R}^3\setminus C_{J_0}$, Theorem 7.6(c) gives, at every k and $J\geq J_0$, $|S_n^{K,(k,J)}|\leq(\sqrt{2}\,e)^n(n!)^{1/2}\prod_j\|f_j\|_{2,k}$ with $\|f\|^2_{2,k}\leq 3C_0|f|_4^2$ uniformly in k (Lemma 2.9), and in the limit the bound of Theorem 7.6(c); by Theorem 7.6(a), $\langle F(u)^2\rangle^K_{k,J}\leq 2\|u\|^2_{2,k}$ and hence $S_2^K(u,u)\leq 2\|u\|^2_2$. The lattice Schwinger functions are the Taylor coefficients at 0 of $a^K_{k,J}$, that is the moments of $\Phi^K_{k,J}$, and converge to the S_n^K. These bounds are uniform in k and $J\geq J_0$ for sources at distance >1 inside C_{J_0}; at a fixed J they are not available for sources reaching ∂C_J, and neither is the convergence in k: Theorem A and Theorem 7.6(a) are stated for sources whose plaquettes have their corners in X_J.

Step 2 (the measure). (i) *The functional on $\mathcal{S}$.* For real $u_1,\dots,u_N\in C_c^\infty$ and $c\in\mathbb{C}^N$, at every (k,J) with $J\ge J_0(u)$, $\sum_{l,m}\bar c_l c_m\, Z^K_{k,J}(\mathrm{i}(u_m-u_l))=\langle|\sum_m c_m e^{\mathrm{i}F(u_m)}|^2\rangle^K_{k,J}\ge 0$ by the linearity of F_{Λ_k} in the source. The limits $k\to\infty$, then $J\to\infty$, exist at these points by Theorem 7.6(b) for the family u, so $\mathcal{Z}^K$ is positive definite on C_c^∞ with $\mathcal{Z}^K(0)=1$. By Corollary 8.4's Lipschitz bound and (E2), $|\mathcal{Z}^K(u)-\mathcal{Z}^K(u')|\le\sqrt{2}\|u-u'\|_2\le\frac{\pi}{2}|u-u'|_4$. Since $|\cdot|_4$ is a continuous seminorm on $\mathcal{S}(\mathbb{R}^3;\Lambda^2)$ and C_c^∞ is dense in $\mathcal{S}$, $\mathcal{Z}^K$ extends uniquely to a continuous function on $\mathcal{S}$, still positive definite with $\mathcal{Z}^K(0)=1$. By the Bochner–Minlos theorem [18, 23] ($\mathcal{S}$ is nuclear) there is exactly one Borel probability measure μ' on $\mathcal{S}'(\mathbb{R}^3;\Lambda^2)$, with its cylinder σ-algebra, such that $\int e^{\mathrm{i}\Phi(u)}\,d\mu'=\mathcal{Z}^K(u)$ for all real $u\in\mathcal{S}$.

(ii) *Moments on C_c^∞, and on $\mathcal{S}$ by uniform integrability.* For real $u_1,\dots,u_N\in C_c^\infty$ the characteristic function of $(\Phi(u_l))_l$ under μ' is $\varphi(t)=\mathcal{Z}^K(\sum_l t_l u_l)=Z^{K,u}_\infty(\mathrm{i}t)$, the restriction to $\mathrm{i}\mathbb{R}^N$ of the entire function $Z^{K,u}_\infty$ (Theorem 7.6(b)). A characteristic function real-analytic near 0 has moments of all orders, with $\int\prod_l\Phi(u_l)\,d\mu'=(-\mathrm{i})^N\partial_{t_1}\cdots\partial_{t_N}\varphi(0)$ [28]; the chain rule gives $\partial_{t_1}\cdots\partial_{t_N}\varphi(0)=\mathrm{i}^N\partial_{z_1}\cdots\partial_{z_N}Z^{K,u}_\infty(0)$, so the factors cancel and

(30)
$$\int\prod_{l=1}^N\Phi(u_l)\,d\mu'=S^K_N(u_1,\dots,u_N);\qquad \int\Phi(u)^2\,d\mu'=S^K_2(u,u)\le 2\|u\|_2^2\le\tfrac{\pi^2}{4}|u|_4^2\le C_2''\|u\|_{10}^2$$

by Step 1, (E2) and $|u|_4\le C\|u\|_{10}$. *Extension to $\mathcal{S}$.* By Theorem 7.6(c) each S^K_N extends to a continuous symmetric multilinear form on $\mathcal{S}^N$ with $|S^K_N|\le\sigma_N\prod_l|u_l|_4$. For $u_l\in\mathcal{S}$ take cut-offs $\chi_R=\chi(\cdot/R)$, $\chi\in C_c^\infty$, $\chi=1$ on the unit ball, $0\le\chi\le1$, $R\ge1$: $\chi_R u_l\to u_l$ in $\mathcal{S}$, so $\Phi(\chi_R u_l)\to\Phi(u_l)$ for every $\Phi\in\mathcal{S}'$; and $|\chi_R u|_4\le C_\chi|u|_4$ uniformly in $R\ge1$ with $C_\chi=2^4\|\chi\|_{C^4}$, by Leibniz with $|R^{-|\beta|}(D^\beta\chi)(\cdot/R)|\le\|\chi\|_{C^4}$ (the norm $|\cdot|_4$ carries derivatives up to order 4, so the cut-off does not pass through it with constant 1). Hence $\int\prod_l\Phi(\chi_R u_l)^2\,d\mu'=S^K_{2N}(\chi_R u_1,\chi_R u_1,\dots)\le\sigma_{2N}C_\chi^{2N}\prod_l|u_l|_4^2$ uniformly in R: the products $\prod_l\Phi(\chi_R u_l)$ are bounded in $L^2(\mu')$, hence uniformly integrable, and converge pointwise, so $\int\prod_l\Phi(u_l)\,d\mu'=\lim_R S^K_N(\chi_R u_1,\dots)=S^K_N(u_1,\dots,u_N)$ by the continuity of S^K_N on $\mathcal{S}^N$.

(iii) *The Hermite carrier.* Put $M(\Phi):=\sum_{\alpha,c}\lambda_\alpha^{-r}\Phi(h_{\alpha,c})^2$. By (30) with $\|h_{\alpha,c}\|_{10}^2=\lambda_\alpha^{10}$, $\int M\,d\mu'\le 3C_2''\sum_\alpha\lambda_\alpha^{10-r}<\infty$ as soon as $r-10>3$ (there are $(n+1)(n+2)/2$ multi-indices with $|\alpha|=n$), that is for every even $r\ge14$; so $\mu'(M<\infty)=1$. For $\Phi\in\mathcal{S}'$ and $u\in\mathcal{S}$ the Hermite expansion $u=\sum_{\alpha,c}\langle h_{\alpha,c},u\rangle\,h_{\alpha,c}$ converges in $\mathcal{S}$ ([32, Lemma A.4(c)]), so $\Phi(u)=\sum_{\alpha,c}\langle h_{\alpha,c},u\rangle\,\Phi(h_{\alpha,c})$ and, on $\{M<\infty\}$, $|\Phi(u)|\le M(\Phi)^{1/2}\|u\|_r$ by Cauchy–Schwarz: Φ extends to $\Psi(\Phi)\in\mathcal{H}^{-r}$ with $\|\Psi(\Phi)\|_{-r}^2=M(\Phi)$ and $\Psi(\Phi)=\Phi$ on $\mathcal{S}$. The map $\Phi\mapsto\Psi(\Phi)$ ($:=0$ off $\{M<\infty\}$) is measurable from the cylinder σ-algebra to the Borel σ-algebra of the separable Hilbert space $\mathcal{H}^{-r}$, its coordinates being evaluations. Put $\mu^{K,H}:=\Psi_*\mu'$. Then $\int e^{\mathrm{i}\Psi(u)}\,d\mu^{K,H}=\mathcal{Z}^K(u)$ and $\int\prod_l\Psi(u_l)\,d\mu^{K,H}=S^K_N(u_1,\dots,u_N)$ for all $u,u_l\in\mathcal{S}$, by (ii). This is [32, Lemma A.8], applied with the two-point bound (30) on $\mathcal{S}$.

(iv) *Uniqueness on $\mathcal{H}^{-r}$.* By [32, Proposition A.9(b)] on $\mathcal{H}^{-r}$, $\mu^{K,H}$ is the only Borel probability measure on $\mathcal{H}^{-r}$ whose mixed moments on C_c^∞ exist and equal the S^K_n, provided the moments m_n of $Z=\sum_l t_l\Psi(u_l)$ satisfy $|m_n|\le c_S(n+1)K^n n!$ for constants c_S, K depending on the u_l and t. Our moments give this with $c_S=1$: $m_n=S^K_n(g,\dots,g)$ for $g=\sum_l t_l u_l$, and $|m_n|\le(\sqrt{2}\,e)^n(n!)^{1/2}\|g\|_2^n\le(n+1)K^n n!$ with $K=\sqrt{2}\,e\|g\|_2$.

(v) *The law on X and its two uniqueness statements.* The map $\iota:\mathcal{H}^{-r}\to X$, $\iota\Psi:=\Psi|_{C_c^\infty}$, is continuous, linear and injective ([32, Lemma A.5(d)]): $|\Psi(\chi f)|\le\|\Psi\|_{-r}\|\chi f\|_r\le C_\chi\|\Psi\|_{-r}\|f\|_{H^r}$ for a cut-off χ, and C_c^∞ is dense in $\mathcal{S}$, which is dense in $\mathcal{H}^r$. Put $\mu^K_S:=\iota_*\mu^{K,H}$; its characteristic functional on real C_c^∞ is $\mathcal{Z}^K$. By [32, Proposition A.9(a)] on X (for the countable $(f_n)\subset C_c^\infty$

of [32, Lemma A.5(c)] the sets $\{\Phi : (\Phi(f_1),\dots,\Phi(f_N)) \in B\}$ form a π-system generating the Borel σ-algebra, and the law of each such vector is determined by $t \mapsto \mathcal{Z}^K(\sum_k t_k f_k))$, μ_S^K is the only Borel probability measure on X with this characteristic functional. *Moment uniqueness on X:* if μ'' is any Borel probability measure on X whose mixed moments on C_c^∞ are the S_n^K, [32, Proposition A.9(b)] on X applies with the moment bound of (iv), so $\mu'' = \mu_S^K$. And $\mu^{K,H}$ is the only measure on $\mathcal{H}^{-r}$ with $\iota_*\mu^{K,H} = \mu_S^K$, since ι is a Borel isomorphism onto the Borel set $\iota\mathcal{H}^{-r}$. Nothing in Step 2 sees the lattice: the measure clauses of (a) depend on King's state only through $\mathcal{Z}^K$ and the S_n^K.

Step 2′ (local convergence of the lattice laws). Fix a real $\chi \in C_c^\infty(\mathbb{R}^3)$ and a box Q with $\operatorname{supp}\chi$ compact in the interior of Q, and let $(e_{\alpha,c})_{\alpha\in\mathbb{Z}^3,\,c\le 3}$ be the Fourier basis of $L^2(Q;\Lambda^2)$, $|e_{\alpha,c}(x)| = |Q|^{-1/2}$. For a real Q-periodic smooth 2-form ψ put $\langle \chi\Phi_{k,J}^K, \psi\rangle := \Phi_{k,J}^K(\chi\psi) = F_{\Lambda_k}(\chi\psi)$, where $\chi\psi \in C_c^\infty$ is supported in $\operatorname{supp}\chi$ (for the complex basis vectors, used only in (1), take real and imaginary parts); this identifies $\chi\Phi_{k,J}^K$ with an element of the real Hilbert space $H_\chi := H^{-r}_{\mathrm{per}}(Q;\Lambda^2)$, and $\nu_{k,J,\chi}^K$ denotes its law.

(1) *Second moments, uniformly.* For $J \ge J_0(\chi)$ and every k, every plaquette carrying $\chi e_{\alpha,c}$ has its centre in $\operatorname{supp}\chi$, hence its corners in X_J, so the second-moment clause of Theorem 7.6(a) applies to the real and imaginary parts: $\mathbb{E}_{k,J}^K|F_{\Lambda_k}(\chi e_{\alpha,c})|^2 \le 2\|\chi \operatorname{Re} e_{\alpha,c}\|_{2,k}^2 + 2\|\chi \operatorname{Im} e_{\alpha,c}\|_{2,k}^2 \le 4\|\chi e_{\alpha,c}\|_{2,k}^2$. *Riemann-sum bound:* the plaquettes of one orientation have their centres on a translate of $\varepsilon_k\mathbb{Z}^3$; those carrying $\chi e_{\alpha,c}$ have centres in $\operatorname{supp}\chi$; the cubes of side $\varepsilon_k \le 1$ about these centres are disjoint and lie in the $\frac12$-neighbourhood $N_{1/2}(\operatorname{supp}\chi)$; so $\|\chi e_{\alpha,c}\|_{2,k}^2 \le 3\operatorname{vol}(N_{1/2}(\operatorname{supp}\chi))\,\|\chi\|_\infty^2/|Q| =: K_\chi$, uniformly in α, c, k and $J \ge J_0(\chi)$ (no derivative of χ enters). Hence, for $s_0 > \frac32$,

(31)
$$\mathbb{E}_{k,J}^K\|\chi\Phi_{k,J}^K\|^2_{H^{-s_0}_{\mathrm{per}}(Q)} = \sum_{\alpha,c}(1+|\alpha|^2)^{-s_0}\,\mathbb{E}_{k,J}^K|F_{\Lambda_k}(\chi e_{\alpha,c})|^2 \;\le\; 12K_\chi \sum_{\alpha\in\mathbb{Z}^3}(1+|\alpha|^2)^{-s_0} =: C'_\chi < \infty,$$

uniformly in k and $J \ge J_0(\chi)$ (the constants 4 and $12K_\chi$ are valid but not sharp).

(2) *Tightness.* The embedding $H^{-s_0}_{\mathrm{per}}(Q) \hookrightarrow H^{-r}_{\mathrm{per}}(Q)$ is compact for $r > s_0$ (Rellich on the torus); by Chebyshev, $\nu_{k,J,\chi}^K(\|\cdot\|_{H^{-s_0}} > \rho) \le C'_\chi/\rho^2$, and the H^{-s_0}-ball of radius ρ is compact in H_χ; so $(\nu_{k,J,\chi}^K)_{k\ge0,\,J\ge J_0(\chi)}$ is tight on the separable Hilbert space H_χ, and Prokhorov's theorem [8] gives weak limit points along k at fixed J.

(3) *Identification at fixed J: the full sequence, and where Theorem A enters.* For a real trigonometric polynomial ψ, $\mathbb{E}_{k,J}^K e^{\mathrm{i}\langle\chi\Phi_{k,J}^K,\psi\rangle} = Z_{k,J}^K(\mathrm{i}\chi\psi) \to Z_J^K(\mathrm{i}\chi\psi)$ as $k\to\infty$, along the full sequence, by Theorem 7.6(b) (that is Corollary 4.4(ii)) for the one-member family $(\chi\psi)$ at $J \ge J_0(\chi) \ge J_0(\chi\psi)$: this is Theorem A for that family on King's cube, together with the local bound of Theorem 7.6(a) and Montel's theorem. A Borel probability measure on H_χ is determined by its characteristic functional on the dual $H^r_{\mathrm{per}}(Q;\Lambda^2)$, which is continuous there (dominated convergence), hence by its values on the dense set of real trigonometric polynomials. So every weak limit point of $(\nu_{k,J,\chi}^K)_k$ is the one probability measure $\nu_{J,\chi}^K$ on H_χ with characteristic functional $\psi \mapsto Z_J^K(\mathrm{i}\chi\psi)$, and $\nu_{k,J,\chi}^K \Rightarrow \nu_{J,\chi}^K$ along the whole sequence $k\to\infty$ by tightness and the uniqueness of the limit point. Without Theorem A the same argument gives the convergence along every continuum subsequence (Theorem 14.9 and Corollary 14.15), not along the full sequence; the full-sequence clause is what Theorem A adds for King's state.

(4) $J\to\infty$. $\|\cdot\|^2_{H^{-s_0}_{\mathrm{per}}} = \sum_{\alpha,c}(1+|\alpha|^2)^{-s_0}|\langle\cdot, e_{\alpha,c}\rangle|^2$ is lower semicontinuous on H_χ (a supremum of finite sums of continuous functions), so by the portmanteau theorem [8] the bound (31) passes to $\nu_{J,\chi}^K$: $\int\|\cdot\|^2_{H^{-s_0}}\,d\nu_{J,\chi}^K \le C'_\chi$ for every $J \ge J_0(\chi)$; hence $(\nu_{J,\chi}^K)_J$ is tight on H_χ by the argument

of (2), and $Z_J^K(\mathrm{i}\chi\psi) \to Z_\infty^K(\mathrm{i}\chi\psi) = \mathcal{Z}^K(\chi\psi)$ (Theorem 7.6(b)) identifies every limit point: $\nu_{J,\chi}^K \Rightarrow \nu_\chi^K$, the measure with characteristic functional $\psi \mapsto \mathcal{Z}^K(\chi\psi)$.

(5) *ν_χ^K is the law of $\chi\Phi$ under μ_S^K.* For $\Phi \in X$, $\chi\Phi \in H^{-r}(\mathbb{R}^3;\Lambda^2)$ is supported in $\operatorname{supp}\chi$, compact in the interior of Q, so its periodisation is an element of H_χ with $\langle\chi\Phi,\psi\rangle = \Phi(\chi\psi)$ for Q-periodic smooth ψ; the map $\Phi \mapsto \chi\Phi$ is continuous from X to H_χ, since $|\Phi(\chi\psi)| \le \|\tilde\chi\Phi\|_{H^{-r}}\|\chi\psi\|_{H^r} \le c_\chi\|\tilde\chi\Phi\|_{H^{-r}}\|\psi\|_{H^r_{\mathrm{per}}(Q)}$ for a cut-off $\tilde\chi = 1$ on Q. And $\int e^{\mathrm{i}\langle\chi\Phi,\psi\rangle}\,d\mu_S^K = \int e^{\mathrm{i}\Phi(\chi\psi)}\,d\mu_S^K = \mathcal{Z}^K(\chi\psi)$ by Step 2(v).

(6) *Moments.* For every trigonometric polynomial ψ (indeed every $\psi \in H^r_{\mathrm{per}}(Q;\Lambda^2)$, by the uniform-integrability argument of Step 2(ii) with $\chi\psi_n \to \chi\psi$), all moments of $\langle\chi\Phi_{k,J}^K,\psi\rangle = F_{\Lambda_k}(\chi\psi)$ are bounded uniformly in k and $J \ge J_0(\chi)$ (Step 1; $|\chi\psi|_4 \le d_\chi\|\psi\|_{C^4}$ by Leibniz), so by uniform integrability the moments of $\nu_{J,\chi}^K$ are the Taylor coefficients of Z_J^K at $\chi\psi$, and those of ν_χ^K are the $S_n^K(\chi\psi,\dots,\chi\psi)$, consistently with (30). This proves (a). □

Remark 11.6 (why the convergence clause is local). The convergence $Z_{k,J}^K(\mathrm{i}u) \to Z_J^K(\mathrm{i}u)$ is available only for sources u whose plaquettes have their corners in X_J, that is for u at distance > 1 inside C_J, where Theorem A speaks. For a u reaching ∂C_J the lattice functional is defined only after a convention for the plaquettes with a corner outside X_J (for instance, drop them), and for it no bound uniform in k and no convergence is claimed: the second-moment bound of Theorem 7.6(a) carries the corner hypothesis, and the crude bound $\langle e^{F(\tilde u)}\rangle_{k,J}^K \le e^{\frac12\mu_0^{-2}\|d_k^*\tilde u\|_{2,k}^2}$ (Lemma 2.11 with $C \le \mu_0^{-2}$ as forms) is not uniform in k for a source $\tilde u$ truncated at ∂C_J: $d_k^*\tilde u$ is of order $\|u\|_\infty/\varepsilon_k$ on the $O(\varepsilon_k^{-2}|\partial C_J|)$ bonds of the truncation layer, so $\|d_k^*\tilde u\|_{2,k}^2 = O(\varepsilon_k^{-1})$. In particular the fixed-J laws of the unlocalised lattice field on X are not claimed to be tight; nothing below depends on them.

11.2. **Proof of (b) and (c).** *Step 3 ((H1)–(H3)).* The hypotheses (H1)–(H3) are those of Definition 11.4. On X with the actions of the Conventions, the structural parts of (H1) (a Polish space with a jointly continuous $\mathbb{R}^3$-action and involutive homeomorphisms θ_j with $\theta_j\tau_a = \tau_{\rho_j a}\theta_j$) and all of (H2) hold by [32, Lemma A.6], which uses only the space and the push-forward actions. *Invariance of μ_S^K:* by the uniqueness in Step 2(v) ([32, Proposition A.10(a)]) it suffices that $\mathcal{Z}^K(\tau_a u) = \mathcal{Z}^K(u)$ and $\mathcal{Z}^K(\sigma_j\cdot u) = \mathcal{Z}^K(u)$ for real u; the first is Corollary 8.4 at the coefficient i, the second Theorem 8.6 at $z = \mathrm{i}$ with $O = \sigma_j$, and the same theorem with $O = \tau_b g\tau_{-b}$ gives the O_h-invariance about every point. *(H3):* for $E_a = e^{\mathrm{i}\Phi(u_a)}$ with u_a real in $C_c^\infty(\{x_j > c\};\Lambda^2)$, $\overline{E_a\circ\theta_{j,c}}\,E_b = e^{\mathrm{i}\Phi(u_b - \rho_{j,c*}u_a)}$ and

$$\mathbb{E}_{\mu_S^K}\big[\overline{E_a\circ\theta_{j,c}}\,E_b\big] = \mathcal{Z}^K(u_b - \rho_{j,c*}u_a) = Z_\infty^K(\theta\zeta_a^*\sqcup\zeta_b), \qquad \zeta_a = (\mathrm{i},u_a),$$

the conjugated coefficient $-\mathrm{i}$ sitting on the reflected source as in Definition 9.1. By Theorem 10.5, for the plane $x_j = c$ and every $c \in \mathbb{R}$, this Gram matrix is positive semidefinite. The exponential algebra is dense in $L^2(\Sigma(H_{j,c}^+),\mu_S^K)$ ([32, Lemma A.7]: an algebra of bounded complex functions, closed under conjugation and containing the constants, is dense in L^2 of the σ-algebra it generates), and $\theta_{j,c}$ preserves μ_S^K, so (H3) follows by [32, Proposition A.10(c)]. Finally [32, Lemma A.11] carries (H1)–(H3) and the O_h-invariance about every point from (X,μ_S^K) to $(\mathcal{H}^{-r},\mu^{K,H})$.

Step 4 (weak clustering). The general results of [32, Appendix A] used here, the ergodic decomposition and the Osterwalder–Schrader reconstruction [29, 30], [18, Thm. 6.1.3] in each coordinate direction from reflection positivity about the coordinate planes ($\mathbb{R}^3$-ergodicity makes the joint fixed space $\mathbb{C}\Omega$, and the self-adjoint contraction semigroups then give clustering in the mean and under diagonal separation), assume only that μ satisfies (H1)–(H3) of Definition 11.4. So

[32, Theorem A.18] gives the ergodic decomposition $\mu_S^K = \int \mu_\xi \, d\nu$ with components translation-invariant and ergodic, θ_j-invariant, satisfying (H3) and (H2): the decomposition is the regular conditional probability given the translation-invariant σ-algebra, and every invariant set lies, modulo null sets, in $\Sigma(H_j^+) \cap \Sigma(H_j^-)$ by the local structure, so that reflection positivity applied to $\mathbf{1}_B - \mathbf{1}_B \circ \theta_j$ forces the θ_j-invariance of the invariant functions, and almost every component is reflection invariant and reflection positive. [32, Theorem A.26] and [32, Theorem A.28] give clustering in the mean and under diagonal separation (von Neumann's mean ergodic theorem for the strongly continuous unitary $\mathbb{R}^3$-action on $L^2(\mu)$: the cube averages of $U(a)$ converge strongly to the projection onto the invariant functions, the conditional expectation given the invariant σ-algebra; by the coordinate-direction reconstruction, translation by a_j acts as the contraction $e^{-|a_j|H_j}$ and the transverse components as unitaries, so that as $\min_j |a_j| \to \infty$ the correlation tends to the projection onto the joint fixed space, the conditional expectation given the invariant σ-algebra). [32, Proposition A.23] is the reconstruction in direction j, and on an ergodic component the joint fixed space is $\mathbb{C}\Omega_j$ ([32, Proposition A.24(iii)], [32, Corollary A.29(iii)]: the mean ergodic theorem on $\mathcal{H}_j$ and on $L^2(\mu)$ takes the Cesàro means of $T_j(t)$ and $V_j(y)$ to the class of the conditional expectation on the invariant σ-algebra, which lies in $L^2_{j,+}$, so the joint fixed space is $\mathbb{C}\Omega_j$ if and only if μ is ergodic). The hypothesis of [32, Theorem A.21], a moment bound on the Hermite basis, is provided for $\mu^{K,H}$ by Theorem 7.6(c) through (30): $\int \Psi(h_{\alpha,c})^{2n} \, d\mu^{K,H} = S^K_{2n}(h_{\alpha,c}, \dots, h_{\alpha,c}) \le (e\pi/2)^{2n}((2n)!)^{1/2} |h_{\alpha,c}|_4^{2n} \le (e\pi/2)^{2n}(2n)! \, |h_{\alpha,c}|_4^{2n}$, and by [32, Lemma A.4], with the inequality $|u|_4 \le C\|u\|_{10}$ of the Conventions, that bound holds with $c_S = 1$ and $C_0 = Ce\pi/2$. So [32, Theorem A.21] gives the growth bound of (b) for the components of $\mu^{K,H}$ (the factorial growth passes to almost every component by Chebyshev's inequality and Borel–Cantelli in ν: from the Hermite bound $\mathbb{E}_\mu \|\Psi\|^{2N}_{-r} \le c_S(4C_1)^N (N!)^2$, $C_1 = w_r C_0^2$ with the Hermite weight sum $w_r := 3\sum_{\alpha \in \mathbb{N}^3} \lambda_\alpha^{10-r}$, finite for $r \ge 14$, obtained by Minkowski's inequality in L^N over the Hermite expansion of the H^{-r} norm, $\int (\sigma_N^\xi)^2 \, d\nu \le \mathbb{E}_\mu \|\Psi\|^{2N}_{-r} \le c_S(4C_1)^N (N!)^2$ (the constant $\sigma_N^\xi := \mathbb{E}_{\mu_\xi} \|\Psi\|^N_{-r}$ serves, as $|\Psi(u)| \le \|\Psi\|_{-r} \|u\|_r$, and $(\mathbb{E}_{\mu_\xi} \|\Psi\|^N_{-r})^2 \le \mathbb{E}_{\mu_\xi} \|\Psi\|^{2N}_{-r}$ by Jensen's inequality), so $\nu\{\sigma_N^\xi > N c_S^{1/2} (4C_1)^{N/2} N!\} \le N^{-2}$, whence the bound for $N \ge n_0(\xi)$, ν-almost everywhere), and [32, Corollary A.22], with the restriction map ι of Step 2(v), for those of μ_S^K; the translation invariance of $\mu^{K,H}$ and the joint continuity of the translations on $\mathcal{H}^{-r}$ assumed there are (H1) for $(\mathcal{H}^{-r}, \mu^{K,H})$, from Step 3. The norm $\|u_i\|_{r+2}$ of (b) dominates $\|u_i\|_r$. [32, Theorem A.20], the ergodic decomposition for $\mathbb{R}^3 \rtimes O_h$, in which a component is the O_h-orbit average of an $\mathbb{R}^3$-ergodic measure invariant under the coordinate reflections, so that the orbit is an S_3-orbit of size 1, 2, 3 or 6, assumes in addition that O_h acts on X by homeomorphisms with $\rho_j = \theta_j$, $g\tau_a g^{-1} = \tau_{ga}$, μ invariant under every g, and $G \mapsto G \circ g^{-1}$ mapping $\Sigma(U)$-measurable functions to $\Sigma(gU)$-measurable ones: structural for the push-forward action ($(g\Phi)(u) = \Phi(g^{-1} \cdot u)$ is a linear homeomorphism of X, the relations are identities on test functions, $\operatorname{supp} g \cdot u = g \operatorname{supp} u$) together with the invariance of Step 3; it gives the cubic-invariant decomposition with $k(\zeta) \in \{1, 2, 3, 6\}$. Weak OS4 is Definition 11.2(iii), ergodicity for the action of $\mathbb{R}^3$, the ergodicity clause of the decomposition for ν-a.e. component.

Step 5 (the rotation corollary). Corollary 11.16 assumes (H1)–(H3) and an admissible action of $O(3)$ on X (Definition 11.10: homeomorphisms $x \mapsto Rx$, continuous in R for each x, with $\rho_j x = \theta_j x$ for the coordinate reflections, $R\tau_a R^{-1} = \tau_{Ra}$, and $G \mapsto G \circ R^{-1}$ mapping $\Sigma(U)$-measurable functions to $\Sigma(RU)$-measurable ones), under which μ is invariant. (H1)–(H3) hold by Step 3. The push-forward $(R\Phi)(u) = \Phi(R^{-1} \cdot u)$ is an admissible action on $X = H^{-r}_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^2)$ by Lemma 11.11(i), with $\mathcal{V} = \Lambda^2$. By Lemma 11.11(ii), the invariance of μ_S^K under every $R \in O(3)$ is the identity $\mathcal{Z}^K(R \cdot u) = \mathcal{Z}^K(u)$ for real u, the hypothesis of (c), that is the $O(3)$-invariance of Z_∞^K at imaginary coefficients, provided by Theorem 8.8; this section does

not derive it. So Corollary 11.16 applies to μ_S^K and its ergodic decomposition of Step 4, and gives the conclusions quoted in (c): the Euclidean invariance and the reflection positivity about every plane of almost every component (Corollary 11.16(a),(b)), clustering as $|a| \to \infty$ in any manner (Corollary 11.16(c)), every form of OS4 for every time direction, coordinate or not, and $\ker H_n = \mathbb{C}\Omega_n$ in every direction n (the final clause of Corollary 11.16). They contain nothing on OS3: the vanishing of the odd curvature moments of a component is not among them, and is not known for a single component.

Step 6 (curvature alone). Everything above runs on $X = H^{-r}_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^2)$ for the curvature field alone: the regularity statement is Theorem 7.6, the reflection positivity statement Theorem 10.5, the invariance statement Theorem 8.6, the $O(3)$ statement that of (c). The result is an ergodic decomposition of the law of the curvature field; nothing is claimed about the components of a joint law with the Wick square. □

The general results of [32, Appendix A] used in Steps 3 and 4 concern a reflection-positive stationary system (Definition 11.4): a probability measure on a Polish space with (H1) actions of the translations and of the three coordinate reflections under which the measure is invariant, (H2) a local σ-algebra attached to every open set, compatible with these actions, and (H3) reflection positivity about every coordinate plane. Some of these results carry further hypotheses: a two-point bound, a moment bound on the Hermite basis, the Borel structure of X, the O_h action and, for Corollary 11.16, an admissible $O(3)$ action. The following remark matches each of them to the step of the proof that establishes it.

Remark 11.7 (the hypotheses of the general results, matched). For each general result of [32, Appendix A] and each statement from the literature used above we list the step that establishes its hypotheses. (H1)–(H3) of Definition 11.4: Step 3 (structure from [32, Lemma A.6]; invariance from Corollary 8.4 and Theorem 8.6 through [32, Proposition A.10(a)]; positivity from Theorem 10.5, [32, Lemma A.7] and [32, Proposition A.10(c)]). [32, Lemma A.4]: no hypothesis; Step 2(iii). The two-point bound $\mathbb{E}\Phi(f)^2 \le C\|f\|_{10}^2$ of [32, Lemma A.8]: Step 2(ii). The moment bound of [32, Proposition A.9(b)], $r \ge 14$ even: Step 2(iv), with $K = \sqrt{2}\,e\|g\|_2$ and $c_S = 1$. The uniqueness of a measure on X with a given characteristic functional ([32, Proposition A.9(a)], with the Borel structure of [32, Lemma A.5]): Step 2(v). The transfer between X and $\mathcal{H}^{-r}$ ([32, Lemma A.11]): Steps 2(v) and 3. The standing hypothesis "μ satisfies (H1)–(H3)" of [32, Theorem A.18], [32, Theorem A.26], [32, Theorem A.28], [32, Proposition A.23] and [32, Proposition A.24]: Step 3; the ergodicity needed in [32, Proposition A.24(iii)] holds on a component by [32, Theorem A.18]. [32, Theorem A.20] (the O_h action and invariance): Step 4 and Theorem 8.6. The moment bound on the Hermite basis assumed in [32, Theorem A.21]: Theorem 7.6(c) through Step 2(ii), with $c_S = 1$ and $C_0 = Ce\pi/2$ (Step 4). Corollary 11.16 (the admissible $O(3)$ action and the invariance): Step 5, the structure from Lemma 11.11(i), the invariance from the hypothesis of (c) through Lemma 11.11(ii). The forms of OS4 (Definition 11.1, Definition 11.2): definitions, Steps 4 and 5; Definition 11.2(iii) is the ergodicity clause of [32, Theorem A.18]. The same matching is made for the joint field with the Wick square in the proof of Proposition 28.6(d) and for the limit states of the comparison family in the proof of Theorem 14.9(v),(vi); for Corollary 11.16 it is made in the proof of Theorem 14.9(vi), where the admissible push-forward action (Lemma 11.11(i)) and the $O(3)$-invariance of the characteristic functional are established. The vanishing of the odd moments is not transferred to the components (clause (c)).

Remark 11.8 (the comparison family). The same conclusions hold for the limit states of the comparison family $\mathrm{ND}^{(a)}$: for every continuum subsequence $\mathbf{k}$ (Definition 14.2) the limit state $Z^{\mathbf{k}}_\infty$ carries a measure with the properties of Theorem 11.5(a) and (b), the local lattice laws converging along $\mathbf{k}$ instead of the full sequence, and (c) (Theorem 14.9, clauses (v)–(vii)). By

Theorem A every such limit state equals Z^K_∞ (Corollary 4.4(iii)), so for King's state the two routes to the measure agree. Whether the comparison family's own fixed-cube limits exist is not decided here (Remark 4.5, Proposition 14.13(b)); nothing beyond Corollary 4.4(iii) is claimed about its iterated limit.

11.3. **Rotation invariance: reflection positivity about every plane and clustering in every direction.** A law with the properties (H1)–(H3) of Definition 11.4 which is moreover invariant under $O(3)$ has almost all of its ergodic components invariant under the Euclidean group, reflection positive about every plane, and clustering as the separation tends to infinity in any manner; hence every form of OS4 holds on almost every component for every choice of the time direction, coordinate or not. Reflection positivity is essential: without it an $O(3)$-average of rotated copies of a translation-ergodic law is $O(3)$-invariant with components that are not. The two ideas are that a translation-invariant function is fixed by every reflection (Lemma 11.13), which passes the $O(3)$-invariance of μ to its components, and that an $O(3)$-invariant ergodic law clusters along every sequence of separations, because the sequence can be rotated onto the diagonal, where clustering is known (Lemma 11.9).

Setting. μ is a Borel probability measure on a Polish space X satisfying (H1)–(H3) (as in Theorem 11.5(b)), with the translations τ_a, the coordinate reflections θ_j, the local σ-algebras $\Sigma(U)$, and $U(a)G := G\circ\tau_{-a}$, $\Theta_jG := G\circ\theta_j$ on $L^2(\mu)$; the functions in $L^2(\Sigma(K),\mu)$, K bounded, are dense in $L^2(\mu)$. $\mu=\int\mu_\xi\,d\nu(\xi)$ is its ergodic decomposition and $\mathcal{J}$ the σ-algebra of translation-invariant Borel sets completed by the μ-null sets; we use that $\xi\mapsto\mathbb{E}_{\mu_\xi}[G]$ is a version of $\mathbb{E}_\mu[G\mid\mathcal{J}]$, that is $\mathbb{E}_\mu[WG]=\int W(\xi)\,\mathbb{E}_{\mu_\xi}[G]\,d\nu(\xi)$ for bounded $\mathcal{J}$-measurable W, and that for ν-almost every ξ the component μ_ξ is translation invariant, ergodic and satisfies (H1)–(H3). For a coordinate direction j, $L^2_{j,+}:=L^2(\Sigma(H^+_{j,0}),\mu)$, and $\mathcal{H}_j$, $[\,\cdot\,]$, Ω_j, H_j and the transverse translations $V_j(y)$ are the objects of the Osterwalder–Schrader reconstruction in direction j. For a unit vector $n\in\mathbb{R}^3$ and $c\in\mathbb{R}$ put $H^+_{n,c}:=\{x: x\cdot n>c\}$, $H^-_{n,c}:=\{x: x\cdot n<c\}$, and let $\rho_n:=1-2nn^{\mathsf T}$ be the reflection in the plane $n^\perp$; for $n=e_j$ these are the half spaces $H^\pm_{j,c}$ and the reflection ρ_j. For $R\in O(3)$, $R\,H^\pm_{n,c}=H^\pm_{Rn,c}$ and $R\rho_nR^{-1}=\rho_{Rn}$.

Lemma 11.9 (clustering under diagonal separation)**.** *Let μ satisfy (H1)–(H3). For all $G,G'\in L^2(\mu)$,*

$$\lim_{\min_j|a_j|\to\infty}\mathbb{E}_\mu\big[G'\,U(a)G\big]=\mathbb{E}_\mu\big[\mathbb{E}_\mu[G'\mid\mathcal{J}]\,\mathbb{E}_\mu[G\mid\mathcal{J}]\big];$$

for an ergodic μ the right side is $\mathbb{E}_\mu[G']\,\mathbb{E}_\mu[G]$.

This is the diagonal clustering theorem [32, Theorem A.28], quoted for the measure of King's state in Theorem 11.5(b): by the reconstruction in each coordinate direction, the translation by a_je_j acts on the positive-time space as the contraction $e^{-|a_j|H_j}$ and the transverse components as unitaries, so that as $\min_j|a_j|\to\infty$ the correlation tends to the projection onto the joint fixed space of the translations, which is the conditional expectation given $\mathcal{J}$.

Definition 11.10 (admissible action of $O(3)$)**.** An action of $O(3)$ on X is *admissible* if it is an action $(R,x)\mapsto Rx$ by homeomorphisms of X, with $R\mapsto Rx$ continuous for each $x\in X$, such that $\rho_jx=\theta_jx$ for the three coordinate reflections, $R\,\tau_a\,R^{-1}=\tau_{Ra}$ for all $R\in O(3)$, $a\in\mathbb{R}^3$, and $\Gamma_RG:=G\circ R^{-1}$ maps $\Sigma(U)$-measurable functions to $\Sigma(RU)$-measurable ones for every open $U\subset\mathbb{R}^3$.

For an admissible action $R\mapsto\Gamma_R$ is a representation, $\Gamma_R\Gamma_S=\Gamma_{RS}$, and $R\tau_aR^{-1}=\tau_{Ra}$ gives

$$\Gamma_R\,U(a)=U(Ra)\,\Gamma_R\qquad(R\in O(3),\ a\in\mathbb{R}^3).\tag{32}$$

If μ is invariant under R, then Γ_R is unitary on $L^2(\mu)$, multiplicative, commutes with conjugation, and $\mathbb{E}_\mu[\Gamma_RG]=\mathbb{E}_\mu[G]$ for $G\in L^1(\mu)$.

Lemma 11.11 (the push-forward action)**.** *Let $\mathcal{V}$ be $\Lambda^2\mathbb{R}^3$ or $\Lambda^2\mathbb{R}^3 \oplus \mathbb{R}$, let $r \geq 2$ be even, and let $X = H^{-r}_{\mathrm{loc}}(\mathbb{R}^3;\mathcal{V})$ be the space of the Conventions of Section 11, with $\mathcal{V}$ in place of $\Lambda^2\mathbb{R}^3$: the translations $(\tau_a\Phi)(f) = \Phi(\tau_{-a}f)$, the coordinate reflections $(\theta_j\Phi)(f) = \Phi(\sigma_j \cdot f)$ with the signed transport of each summand, and the local σ-algebras $\Sigma(U)$ generated by $\Phi \mapsto \Phi(f)$, $f \in C^\infty_c(U;\mathcal{V})$ real. Let $O(3)$ act on X by the push-forward $(R\Phi)(f) := \Phi(R^{-1}\cdot f)$, where $R\cdot f$ is the transport of Definition 2.5 on each summand, $(R\cdot f)(x) = \pi(R)f(R^{-1}x)$ with $\pi(R)$ the orthogonal map of $\mathcal{V}$ induced by R.*

(i) $\Phi \mapsto R\Phi$ is an admissible action of $O(3)$ on X, each R acting by a linear homeomorphism.

(ii) A Borel probability measure μ on X with characteristic functional $\mathcal{Z}(f) := \mathbb{E}_\mu[e^{\mathrm{i}\Phi(f)}]$ is invariant under $R \in O(3)$ if and only if $\mathcal{Z}(R\cdot f) = \mathcal{Z}(f)$ for every real $f \in C^\infty_c(\mathbb{R}^3;\mathcal{V})$.

Part (i) is the restriction to $O(3)$ of the Euclidean action of the Conventions, a group action by linear homeomorphisms with ρ_j acting as θ_j, $R\tau_aR^{-1} = \tau_{Ra}$ and $\operatorname{supp}(R\cdot f) = R\operatorname{supp} f$; part (ii) is the uniqueness of a Borel probability measure on X from its characteristic functional on real test functions ([32, Proposition A.9(a)], as in Step 2(v) of the proof of Theorem 11.5) applied to $R_*\mu$, whose characteristic functional is $f \mapsto \mathcal{Z}(R^{-1}\cdot f)$.

Lemma 11.12 (rotated frames)**.** *Let μ satisfy (H1)–(H3), let $O(3)$ act admissibly on X, and let μ be invariant under every $R \in O(3)$. Then:*

(a) for every unit vector n, every $c \in \mathbb{R}$ and every $F \in L^2(\Sigma(H^+_{n,c}),\mu)$,

$$\mathbb{E}_\mu\big[\overline{F\circ\theta_{n,c}}\,F\big] \geq 0, \qquad \theta_{n,c} := \tau_{cn}\,\rho_n\,\tau_{-cn};$$

(b) for every $R \in O(3)$ the system $(X,\tau^R,\theta^R,\Sigma^R,\mu)$ with $\tau^R_a := \tau_{Ra}$, $\theta^R_j := \rho_{Re_j} = R\theta_jR^{-1}$ and $\Sigma^R(U) := \Sigma(RU)$ satisfies (H1)–(H3); its translations are $U^R(a) = U(Ra)$, its half spaces $\Sigma^R(H^\pm_{j,c}) = \Sigma(H^\pm_{Re_j,c})$ and its reflections in the planes $\{x_j = c\}$ are $\theta_{Re_j,c}$.

Proof. (a) Let first $c = 0$ and choose $R \in SO(3)$ with $Re_1 = n$, so that $\rho_n = R\rho_1R^{-1}$ and $RH^+_{1,0} = H^+_{n,0}$. For $F \in L^2(\Sigma(H^+_{n,0}),\mu)$ the function $F' := \Gamma_{R^{-1}}F = F\circ R$ is $\Sigma(H^+_{1,0})$-measurable and lies in $L^2(\mu)$ by R-invariance, $F = \Gamma_RF'$ and $F\circ\rho_n = \Gamma_R\Theta_1F'$; hence $\mathbb{E}_\mu[\overline{F\circ\rho_n}\,F] = \mathbb{E}_\mu[\Gamma_R(\overline{\Theta_1F'}\,F')] = \mathbb{E}_\mu[\overline{\Theta_1F'}\,F'] \geq 0$ by (H3). For $c \neq 0$, $F_0 := U(-cn)F$ is $\Sigma(H^+_{n,0})$-measurable by (H2), and translation invariance reduces the claim to $c = 0$. (b) (H1) and (H2) for the new system are the relations $R\theta_jR^{-1}\tau_{Ra} = \tau_{R\rho_ja}R\theta_jR^{-1}$, $U(Ra)\Sigma(RU) \subset \Sigma(RU + Ra)$ and $\Theta^R_j = \Gamma_R\Theta_j\Gamma_{R^{-1}}$, which follow from admissibility; (H3) for the new system is (a) for the unit vector Re_j, since $\theta^R_{j,c} = \tau_{cRe_j}\rho_{Re_j}\tau_{-cRe_j} = \theta_{Re_j,c}$. □

The reconstruction in a direction n. For a unit vector n choose R with $Re_1 = n$ and apply the reconstruction in direction 1 to the system of Lemma 11.12(b). Its ingredients depend only on n: the space $L^2(\Sigma(H^+_{n,0}),\mu)$, the form $\mathbb{E}_\mu[\overline{F\circ\rho_n}\,G]$, the semigroup $[G] \mapsto [U(tn)G]$, $t \geq 0$, and the transverse translations $[G] \mapsto [U(y)G]$, $y \perp n$. We write $\mathcal{H}_n$, Ω_n, H_n and $V_n(y)$ for the objects so obtained, and P_n for the orthogonal projection of $L^2(\mu)$ onto $\{G : U(tn)G = G\ \forall t \in \mathbb{R}\}$.

Lemma 11.13 (invariant functions are fixed by every reflection)**.** *Let μ satisfy (H1)–(H3), let $O(3)$ act admissibly on X, and let μ be invariant under every $R \in O(3)$.*

(a) If n is a unit vector and $W \in L^2(\mu)$ satisfies $U(tn)W = W$ for all $t \in \mathbb{R}$, then $W \in L^2(\Sigma(H^+_{n,c}),\mu) \cap L^2(\Sigma(H^-_{n,c}),\mu)$ for every $c \in \mathbb{R}$, and $\Gamma_{\rho_n}W = W$.

(b) If $W \in L^2(\mu)$ satisfies $U(a)W = W$ for all $a \in \mathbb{R}^3$, then $\Gamma_RW = W$ for every $R \in O(3)$.

Proof. (a) Let $G_K \in L^2(\Sigma(K),\mu)$, K bounded, with $G_K \to W$ in $L^2(\mu)$. For s large, $K + sn \subset H^+_{n,c}$, so $U(sn)G_K \in L^2(\Sigma(H^+_{n,c}),\mu)$ by (H2), and $U(sn)G_K \to U(sn)W = W$; thus W lies in the closed subspace $L^2(\Sigma(H^+_{n,c}),\mu)$, and likewise in $L^2(\Sigma(H^-_{n,c}),\mu)$. Put $D := W - \Gamma_{\rho_n}W$.

Since Γ_{ρ_n} maps $\Sigma(H^-_{n,0})$-measurable functions to $\Sigma(H^+_{n,0})$-measurable ones, $D \in L^2(\Sigma(H^+_{n,0}),\mu)$, and $\Gamma_{\rho_n} D = -D$; Lemma 11.12(a) gives $0 \le \mathbb{E}_\mu[\overline{\Gamma_{\rho_n} D}\, D] = -\mathbb{E}_\mu|D|^2$, so $D = 0$. (b) By (a), $\Gamma_{\rho_n} W = W$ for every unit vector n, and every $R \in O(3)$ is a product of at most three reflections ρ_n: if $R \neq 1$, choose v with $Rv \neq v$ and put $n := (Rv - v)/|Rv - v|$; then ρ_n exchanges Rv and v, so $\rho_n R$ fixes v and acts on $v^\perp$ as an element of $O(2)$, a product of at most two reflections in lines of $v^\perp$. Since Γ is a representation, $\Gamma_R W = W$. □

Proposition 11.14 (ergodicity along n and the zero-energy space)**.** *Let μ satisfy (H1)–(H3), let $O(3)$ act admissibly on X, let μ be $O(3)$-invariant and let n be a unit vector. Then every $U(tn)$-invariant $G \in L^2(\mu)$ is μ-almost surely constant if and only if* $\ker H_n = \mathbb{C}\Omega_n$ *in the reconstruction in direction n.*

Proof. Write $L^2_{n,+} := L^2(\Sigma(H^+_{n,0}),\mu)$. The constants lie in $L^2_{n,+}$, $\Omega_n = [1]$ is a unit vector with $H_n\Omega_n = 0$, and $\{[G] : G \in L^2_{n,+}\}$ is dense in $\mathcal{H}_n$. For $G \in L^2_{n,+}$ the invariant function P_nG lies in $L^2_{n,+}$ by Lemma 11.13(a), and $[P_nG]$ is the projection of $[G]$ onto $\ker H_n$, since $[U(tn)G] \to [P_nG]$ in the mean as $t \to \infty$ and e^{-tH_n} converges strongly in the mean to the projection onto $\ker H_n$. If invariant functions are constant, then $P_nG = \mathbb{E}_\mu[G]$, so the projection of the dense set $\{[G]\}$ onto $\ker H_n$ lies in $\mathbb{C}\Omega_n$, and $\ker H_n = \mathbb{C}\Omega_n$. Conversely, if $\ker H_n = \mathbb{C}\Omega_n$, then $[P_nG] = c\,\Omega_n$ with $c = \langle \Omega_n, [P_nG]\rangle = \mathbb{E}_\mu[\overline{\Gamma_{\rho_n} 1}\, P_nG] = \mathbb{E}_\mu[G]$; the invariant function $W := P_nG - \mathbb{E}_\mu[G]$ satisfies $\Gamma_{\rho_n} W = W$ (Lemma 11.13(a)), so $0 = \|[W]\|^2 = \mathbb{E}_\mu[\overline{\Gamma_{\rho_n} W}\, W] = \mathbb{E}_\mu|W|^2$, and $P_nG = \mathbb{E}_\mu[G]$ on $L^2_{n,+}$. For $G \in L^2(\Sigma(K),\mu)$ with K bounded, $U(sn)G \in L^2_{n,+}$ for s large, and P_n commutes with $U(sn)$, so $P_nG = P_nU(sn)G = \mathbb{E}_\mu[G]$; by density $P_nG = \mathbb{E}_\mu[G]$ on $L^2(\mu)$, and an invariant G equals P_nG, a constant. □

Definition 11.15 (the forms of the cluster property along a direction)**.** Let μ satisfy (H1), let $O(3)$ act admissibly on X, let n be a unit vector chosen as Euclidean time, and let μ be invariant under ρ_n. When X is a space of $\mathcal{V}$-valued distributions as in Lemma 11.11, let $\mathcal{P}^+_n$ be the set of polynomials in the fields $\Phi(f)$, $f \in C^\infty_c(H^+_{n,0};\mathcal{V})$ real, that belong to $L^2(\mu)$. We consider:

- (E_n) (ergodicity along n) every $G \in L^2(\mu)$ with $U(tn)G = G$ for all $t \in \mathbb{R}$ is μ-almost surely constant;
- (C_n) (clustering along n) $\mathbb{E}_\mu[G'\,U(tn)G] \to \mathbb{E}_\mu[G']\,\mathbb{E}_\mu[G]$ as $|t| \to \infty$, for all $G, G' \in L^2(\mu)$;
- (OS_n) (the cluster property E4 of [29, p. 88] with time direction n) for all $F, G \in \mathcal{P}^+_n$ and every $a \neq 0$ with $a \perp n$, $\lim_{\lambda\to\infty} \mathbb{E}_\mu\big[\overline{F \circ \rho_n}\, U(\lambda a)G\big] = \mathbb{E}_\mu[\overline{F}]\,\mathbb{E}_\mu[G]$.

For $n = e_j$ these are the forms of OS4 with the coordinate time direction e_j used in Theorem 11.5; in [29] the axiom E4 is stated for one time direction and holds for every time direction by Euclidean covariance [29, p. 88, and Remark 2, p. 89], and (OS_n) for all n is its form for a measure.

Corollary 11.16 (the cluster property under rotation invariance)**.** *Let μ satisfy (H1)–(H3), let $O(3)$ act admissibly on X (Definition 11.10), let μ be invariant under every $R \in O(3)$, and let $\mu = \int \mu_\xi\, d\nu(\xi)$ be its ergodic decomposition. Then for ν-almost every ξ:*

- *(a) μ_ξ is invariant under the Euclidean group $\mathbb{R}^3 \rtimes O(3)$;*
- *(b) $\mathbb{E}_{\mu_\xi}[\overline{F \circ \theta_{n,c}}\, F] \ge 0$ for every unit vector n, every $c \in \mathbb{R}$ and every $F \in L^2(\Sigma(H^+_{n,c}),\mu_\xi)$;*
- *(c) for all $G, G' \in L^2(\mu_\xi)$, $\mathbb{E}_{\mu_\xi}[G'\,U(a)G] \to \mathbb{E}_{\mu_\xi}[G']\,\mathbb{E}_{\mu_\xi}[G]$ as $|a| \to \infty$ in any manner.*

Consequently, for ν-almost every ξ and every unit vector n, μ_ξ is ergodic under $\mathbb{R}^3$ and satisfies (E_n) and (C_n), and (OS_n) when X is a space of distributions as in Lemma 11.11; in the reconstruction from μ_ξ in any direction n, $\ker H_n = \mathbb{C}\Omega_n$, *so the joint fixed space of the e^{-tH_n}, $t \ge 0$, and the $V_n(y)$, $y \perp n$, is $\mathbb{C}\Omega_n$.*

Proof. Step 1: (a) and (b). Let d be a complete metric for X and (x_k) a dense sequence. For $k,m\in\mathbb{N}$ and rational $q>0$ let $\varphi_{k,q,m}(x):=\min\big(1,m\,d(x,X\setminus B(x_k,q))\big)$, and let $\mathcal{D}_0$ be the countable set of finite products of these bounded continuous functions. As $m\to\infty$, $\varphi_{k,q,m}\uparrow \mathbf{1}_{B(x_k,q)}$, so two Borel probability measures that agree on $\mathcal{D}_0$ agree on the finite intersections of the balls $B(x_k,q)$, a π-system generating the Borel sets, and are equal. Let $\mathcal{R}_{\mathrm{d}}\subset O(3)$ be a countable dense subgroup; for instance the group generated by the coordinate reflection ρ_1 together with the countable group G_0 of Section 15, which is generated by the finite set $\mathcal{R}_0$ of the rotations $R_{\mu\nu}(\pm\theta_0)$ introduced there and is dense in $SO(3)$. Fix $R\in\mathcal{R}_{\mathrm{d}}$, $G\in\mathcal{D}_0$ and a bounded $\mathcal{J}$-measurable W. Then W agrees μ-a.e. with a translation-invariant Borel function, so $U(a)W=W$ in $L^2(\mu)$ for every a, and $\Gamma_R W=W$ by Lemma 11.13(b). By R-invariance of μ,

$$\mathbb{E}_\mu[W\,\Gamma_R G]=\mathbb{E}_\mu[(\Gamma_R W)(\Gamma_R G)]=\mathbb{E}_\mu[\Gamma_R(WG)]=\mathbb{E}_\mu[WG].$$

Disintegrating both sides, $\int W(\xi)\,D(\xi)\,d\nu(\xi)=0$ with $D(\xi):=\mathbb{E}_{\mu_\xi}[\Gamma_R G]-\mathbb{E}_{\mu_\xi}[G]$, which is bounded, real and $\mathcal{J}$-measurable; $W=\mathbf{1}_{\{D>0\}}$ and $W=\mathbf{1}_{\{D<0\}}$ give $D=0$ ν-a.e. The pairs (R,G) are countably many, so off one ν-null set the image of μ_ξ under R^{-1} agrees with μ_ξ on $\mathcal{D}_0$ for all $R\in\mathcal{R}_{\mathrm{d}}$, and μ_ξ is invariant under $\mathcal{R}_{\mathrm{d}}$. For such ξ and bounded continuous G, $R\mapsto\mathbb{E}_{\mu_\xi}[G\circ R^{-1}]$ is continuous on $O(3)$ (continuity of $R\mapsto R^{-1}x$ and dominated convergence) and equals $\mathbb{E}_{\mu_\xi}[G]$ on the dense set $\mathcal{R}_{\mathrm{d}}$, hence everywhere: μ_ξ is $O(3)$-invariant, and with its translation invariance this is (a). For ν-a.e. ξ the component satisfies (H1)–(H3), so Lemma 11.12(a) applied to μ_ξ gives (b).

Step 2: (c). Fix such a ξ and write $m:=\mu_\xi$: ergodic under $\mathbb{R}^3$, $O(3)$-invariant, with (H1)–(H3). Each Γ_R is unitary on $L^2(m)$, and $R\mapsto\Gamma_R$ is strongly continuous: for bounded continuous G by dominated convergence, in general by density and $\|\Gamma_R\|=1$. Put $n_0:=(1,1,1)/\sqrt{3}$, let $|a^{(l)}|\to\infty$, and choose $R_l\in O(3)$ with $R_l a^{(l)}=b_l:=|a^{(l)}|\,n_0$. By R_l-invariance and (32),

$$\mathbb{E}_m\big[G'\,U(a^{(l)})G\big]=\mathbb{E}_m\big[\Gamma_{R_l}\big(G'\,U(a^{(l)})G\big)\big]=\mathbb{E}_m\big[(\Gamma_{R_l}G')\,U(b_l)(\Gamma_{R_l}G)\big].$$

Every subsequence has a further subsequence along which $R_l\to R\in O(3)$, and along it, since $|\mathbb{E}_m[A\,U(b)B]|\le\|A\|_2\|B\|_2$,

$$\begin{aligned}\big|\mathbb{E}_m\big[(\Gamma_{R_l}G')\,U(b_l)(\Gamma_{R_l}G)\big]-\mathbb{E}_m\big[(\Gamma_R G')\,U(b_l)(\Gamma_R G)\big]\big|&\\ \le\|\Gamma_{R_l}G'-\Gamma_R G'\|_2\|G\|_2+\|G'\|_2\|\Gamma_{R_l}G-\Gamma_R G\|_2\to 0.&\end{aligned}$$

Every coordinate of b_l equals $|a^{(l)}|/\sqrt{3}\to\infty$, so Lemma 11.9 for the ergodic law m gives $\mathbb{E}_m[(\Gamma_R G')\,U(b_l)(\Gamma_R G)]\to\mathbb{E}_m[\Gamma_R G']\,\mathbb{E}_m[\Gamma_R G]=\mathbb{E}_m[G']\,\mathbb{E}_m[G]$. The limit does not depend on the subsequence, so (c) follows.

Step 3: the forms of the cluster property. Fix m as in Step 2 and a unit vector n. (C_n) is (c) with $a=tn$. If $U(tn)G=G$ for all t, then (c) with $G'=\overline{G}$ gives $\mathbb{E}_m|G|^2=\mathbb{E}_m[\overline{G}\,U(tn)G]\to|\mathbb{E}_m G|^2$, so G is a.s. constant: (E_n) holds, and $\ker H_n=\mathbb{C}\Omega_n$ by Proposition 11.14; the joint fixed space contains Ω_n and lies in $\ker H_n$. For (OS_n), let $F,G\in\mathcal{P}_n^+$ and $a\perp n$, $a\neq 0$; then $\overline{F\circ\rho_n}\in L^2(m)$, and (c) along λa gives the limit $\mathbb{E}_m[\overline{F\circ\rho_n}]\,\mathbb{E}_m[G]=\mathbb{E}_m[\overline{F}]\,\mathbb{E}_m[G]$ by the ρ_n-invariance of m. □

12. The Wick tube I: torus theorems and the extensive bound

This section and the next prove, under the sign clause (T3), the bound on the tube for the joint field of the curvature and the renormalised Wick square (Theorem 13.10); they are rows (6a) and (6b) of Table 1. This section works on King's tori, at fixed spacing; no continuum limit is taken here. Its results are lattice reflection positivity on King's tori (Proposition 12.2), the chessboard estimate it yields (Lemma 12.4), and the extensive bound for constant positive scalar

sources in the fixed-spacing volume limit (Proposition 12.8), which is derived from the two torus theorems (Proposition 12.6 and Lemma 12.7) and the sign clause (T3).

The Wick leg controls the joint functional of the curvature and the Wick square on the tube $\mathcal{D}_{\sigma_+,H}$ of Definition 2.8. Its positive leg (sources $q \geq 0$) and negative leg (sources $-q$) are proved in the next section from three statements on *King's tori*: lattice reflection positivity (Proposition 12.2), the chessboard estimate it yields (Lemma 12.4), and an *extensive bound* (I2) for constant positive scalar sources in the fixed-spacing volume limit (Definition 12.5). This section proves the first two and derives the third from the torus theorems (Proposition 12.6 and Lemma 12.7) and the sign clause (T3) (Proposition 12.8); a variant with the auxiliary mass on either side of m (Proposition 12.13) serves the discussion of the coupling range; no continuum limit is taken here.

12.1. **King's tori, the two conventions and mid-plane pairs.** This subsection defines King's periodic tori, the two conventions for their counterterm (the free convention of the cube states and King's torus convention), and the mid-plane pairs in which they are reflected.

Definition 12.1 (King's tori)**.** A *King torus* is $T = \mathbb{R}^3/(\ell\mathbb{Z})^3$ with $\ell \in 2M\mathbb{N}$; $|T| := \ell^3$ is its volume, and $\mathbb{T}_J := \mathbb{R}^3/(2MJ\mathbb{Z})^3$, $J \in \mathbb{N}$, is the torus with fundamental cell C_J. Since $\ell\mathbb{Z}^3 \subset \mathbb{Z}^3 \subset \Lambda_k$, the lattice Λ_k descends to T, with $\ell\, 5^k$ sites per side, an even number, and $|T|/\varepsilon_k^3$ sites in all. The *periodic state* $\langle\cdot\rangle_{k,T}$ is King's action of Section 2 summed over the finite graph $\Lambda_k/(\ell\mathbb{Z})^3$, with periodic conditions for both fields [26, (4.1)], [25, (2.21)]: Feynman-form gradient terms for every pair of parallel neighbouring bonds, the photon mass μ_0^2, hopping terms on every bond, the single-site weights $e^{-\varepsilon^3[\frac{1}{2}(m^2+\delta m^2)r^2+\lambda r^4]}$. Two normalisations of the counterterm are used:

(i) the *free convention* of Section 2 ($\delta m^2 = \delta m_\varepsilon^2$, Wick constant W_ε; the convention of every cube state of this paper), with $N_{k,T}^a(0,q) := e^{-(a_k+2\mathsf{W}_\varepsilon)\mathsf{S}(q)}\langle e^{\sum_x \varepsilon^3 q(x) r_x^2}\rangle_{k,T}$ for a real function q on the sites of T, exactly as on boxes (Definition 2.5), and

$$\alpha_{k,T}^a(u) := |T|^{-1} \ln N_{k,T}^a(0, u\mathbf{1}_T) \qquad (u \in \mathbb{R}),$$

the *free energy density* of a constant source;

(ii) *King's torus convention* [25, (2.4), (3.33), (3.35)]: the counterterm $\delta m_{k,T}^2(\mathsf{M})$ and the Wick constant $c_{k,T}(\mathsf{M})$ computed with the periodic propagators of T at scalar mass M, together with the field-independent constants $E(\mathsf{M}) = E_0(\mathsf{M}) + E_1(\mathsf{M})$ of King's torus action. $Z_{k,T}^{(\mathsf{M})} := \int e^{-S}$ is the source-free partition function of this model at scalar mass M (bare coefficient $\mathsf{M}^2 + \delta m_{k,T}^2(\mathsf{M})$; Proposition 12.6 is a statement about these numbers), and $N_{k,T}^{z,\mathrm{tor}}(0,q)$ the normalised source functional of the torus model at mass m in the reference scheme z, with $c_{k,T} = c_{k,T}(m)$ in place of W_ε.

In either convention the coefficients are the same at every site of T. A *mid-plane pair* is $\pi = \{x_\nu = c\} \cup \{x_\nu = c + \ell/2\}$ with $\nu \in \{1,2,3\}$ and $c \in \varepsilon_k(\mathbb{Z} + \frac{1}{2})$ (both planes lie between lattice planes, since $\ell 5^k$ is even); θ denotes the reflection of T in π, which exchanges the two slabs of $T \setminus \pi$ ("the two sides of π"), acting on the fields as in Definition 9.1.

12.2. **Reflection positivity of King's tori.** Lattice reflection positivity of the periodic state about a mid-plane pair is the first of the three torus statements; it yields the chessboard estimate of the next subsection.

Proposition 12.2 (lattice reflection positivity of King's tori)**.** *Let T be a King torus, $k \geq 0$, π a mid-plane pair and θ the reflection in π. For every finite family $G_1, \dots, G_N$ of gauge-invariant*

Borel functions of the fields on one side of π with $\langle |G_i|^2\rangle_{k,T} < \infty$ and every $\gamma \in \mathbb{C}^N$,

$$\sum_{i,l=1}^{N} \bar\gamma_i \gamma_l \, \langle \theta(G_i)\, G_l\rangle_{k,T} \;\geq\; 0.$$

The periodic state is invariant under θ and under all lattice translations. This holds in both conventions of Definition 12.1: reflection positivity does not see the value of the (site-independent) counterterm.

Proof. The proof is the mechanism of Lemma 9.3 with a pair of planes in place of one and without Dirichlet diagonals, in three steps.

Step 1: from the Feynman form to the Proca form. On the torus write $A = Z + d_0\chi$ with $d_0^T Z = 0$, $\chi \perp 1$ (Hodge decomposition on the finite periodic graph). The Feynman form equals $\frac12\langle Z, (d_1^T d_1 + \mu_0^2)Z\rangle$ plus a quadratic form in χ, and the Proca form $\frac12\langle A, (d_1^T d_1 + \mu_0^2)A\rangle$ equals the same Z-form plus another χ-form; both χ-forms are positive definite on $1^\perp$ because $\mu_0 > 0$. For gauge-invariant G the matter integral at fixed A is invariant under $A \mapsto A + d_0\chi$ (rotate ϕ site by site), so the χ-integrals factor and cancel in the normalised expectation: gauge-invariant expectations in the Feynman form equal those in the Proca form. Fubini's theorem applies by $\mu_0 > 0$, $\lambda > 0$ and a growth bound $|G| \leq Ce^{a\sum_b |A(b)|} e^{C\sum_x r_x^2}$, satisfied by bounded G and by the observables $\Theta(\zeta)$; so this step uses $\mu_0 > 0$, gauge invariance, $\lambda > 0$ and the growth bound, and nothing about the value of the counterterm.

Step 2: the Proca torus measure is reflection positive about π. In the Proca form the action couples the two sides of π only through the plaquettes and hopping terms crossing one of the two planes of π (no gradient terms between parallel bonds across a plane; the mass term is local). Write each crossing bond variable as $a_b = \alpha_b - \alpha'_b$ with α_b, α'_b independent centred Gaussians of variance $(2\mu_0^2\varepsilon^3)^{-1}$, attached to the two sides. Then θ becomes the swap of the two data sets $\xi = (\omega_+, \alpha)$, $\xi' = (\omega_-^\theta, \alpha')$, whose a priori measures coincide by the θ-invariance of the one-sided terms, and exactly as in Step 2 of Lemma 9.3 every crossing plaquette term is $\frac{\varepsilon}{2}X_p(\xi)^2 + \frac{\varepsilon}{2}X_p(\xi')^2 - \varepsilon X_p(\xi)X_p(\xi')$ and every crossing hopping term is $-\varepsilon\sum_{i=1,2}\psi_b^i(\xi)\psi_b^i(\xi')$, with the same real functions on both copies. So the Boltzmann factor is $\mathcal{G}(\xi)\mathcal{G}(\xi')\exp(\sum_\iota \gamma_\iota Y_\iota(\xi)Y_\iota(\xi'))$ with all $\gamma_\iota = \varepsilon > 0$ and all Y_ι real, and for G a function of one side $\theta(G)G = \overline{G(\xi')}G(\xi)$. Expanding the exponential, $Z\langle\theta(G)G\rangle$ is a sum with non-negative coefficients of squared moduli $|\int d\xi\, G(\xi)\mathcal{G}(\xi)\prod_\iota Y_\iota(\xi)^{n_\iota}|^2$, the interchange of sum and integral being justified as in Lemma 9.3 ($\lambda > 0$, $\mu_0 > 0$ and the growth bound). The Hermitian property $\overline{\langle\theta(G')G\rangle} = \langle\theta(G)G'\rangle$ follows from the θ-invariance of the measure, so $(G, G') \mapsto \langle\theta(G')G\rangle$ is a positive semidefinite Hermitian form on bounded gauge-invariant one-sided G.

Step 3: L^2 observables. For gauge-invariant one-sided G with $\langle|G|^2\rangle < \infty$ apply Step 2 to $G_n := G\mathbf{1}\{|G| \leq n\}$; since $|\theta(G_n)G_n| \leq |\theta G||G|$ and $\langle|\theta G||G|\rangle \leq \langle|G|^2\rangle$ by Cauchy–Schwarz and θ-invariance, dominated convergence gives the inequality for $G = \sum_i \gamma_i G_i$, hence the form statement.

Invariance. Under θ and under every lattice translation each term of the periodic action, in either convention, goes to a term of the same kind with the same coefficient, and the a priori measure is invariant; the value of the site-independent bare coefficient enters nowhere above. □

Remark 12.3 (reflection positivity in every gauge, and the chessboard estimate). (i) [9, Cor. 5.4] (reflection positivity "in every gauge", for covariances $C_F = C_u + dFd^* \geq 0$ with F of either sign) is the mechanism of the proof above: on the torus the Feynman covariance has this form with $F = -\mu_0^{-2}(\Delta_0 + \mu_0^2)^{-1} \leq 0$ on functions orthogonal to the constants. It is not a literal instance (the region of [9, §5] is a bounded subset of the lattice with a boundary condition, and reflection positivity of periodic conditions is claimed there in a remark), and Proposition 12.2 does not rest

on it. (ii) The chessboard estimate uses Proposition 12.2 for functions of the r_x only; nothing in this section uses reflection positivity of a cube or of a limit state.

12.3. **The chessboard estimate on King's tori.** The chessboard estimate bounds the functional of a scalar source on a torus by the free energy density of constant sources, which is the quantity the extensive bound (I2) of the next subsection controls.

Lemma 12.4 (chessboard estimate)**.** *Let T be a King torus at spacing $\varepsilon = \varepsilon_k$, a a scheme, and q a real function on the sites of T. In the free convention,*

$$\ln N^a_{k,T}(0,q) \;\le\; \sum_x \varepsilon^3\, \alpha^a_{k,T}(q(x)), \tag{33}$$

and $u \mapsto \alpha^a_{k,T}(u)$ is convex on $\mathbb{R}$ with $\alpha^a_{k,T}(0) = 0$. In King's torus convention

$$u \mapsto |T|^{-1} \ln N^{z,\mathrm{tor}}_{k,T}(0, u\mathbf{1}_T)$$

is likewise convex with value 0 at $u = 0$.

Proof. We apply the chessboard estimate [17, Thm. 4.2] (lattice-gauge form: [9, Thm. 2.10]), whose hypotheses we verify: the site set of T is a rectangular lattice box with sides of even length $\ell 5^k$; the periodic state is invariant under translations modulo the sides and reflection positive with respect to the reflections in all planes perpendicular to the coordinate axes running midway between neighbouring sites (Proposition 12.2; on the torus such a reflection is the reflection in a mid-plane pair, the antipodal plane $c + \ell/2$ being again a mid-plane). Take as the "spin at x" the gauge-invariant variable r_x and

$$G_x := \exp\left(\varepsilon^3 q(x)(r_x^2 - 2\mathsf{W}_\varepsilon - a_k)\right), \qquad \text{so that} \qquad \Big\langle \prod_x G_x \Big\rangle_{k,T} = N^a_{k,T}(0,q).$$

The G_x are positive with all moments ($\lambda > 0$), and the Schwarz inequality for such unbounded gauge-invariant functions, all that [17, Thm. 4.1] uses beyond cyclic invariance, follows from the bounded case by truncation and monotone convergence. The conclusion is $|\langle \prod_x G_x\rangle| \le \prod_x \langle \prod_y G_x^{(y)}\rangle^{1/|\Lambda|}$, $G_x^{(y)}$ the copy of G_x carried to y by reflections, $|\Lambda| = |T|/\varepsilon^3$ the number of sites. Because W_ε, a_k and δm^2_ε are the same at every site, $\prod_y G_x^{(y)} = \exp(q(x)\sum_y \varepsilon^3(r_y^2 - 2\mathsf{W}_\varepsilon - a_k))$ has expectation $N^a_{k,T}(0, q(x)\mathbf{1}_T) = e^{|T|\alpha^a_{k,T}(q(x))}$ in the same state; taking logarithms gives (33). Convexity: $|T|\alpha^a_{k,T}(u)$ is the logarithm of a Laplace transform in the linear parameter u plus a linear term, and vanishes at $u = 0$; the same computation gives the convexity statement in the torus convention. □

12.4. **Constant sources on large tori: the extensive bound.** The extensive bound (I2) concerns constant positive scalar sources in the fixed-spacing volume limit; this subsection states it and derives it from Bałaban's stability bounds at two scalar masses and the sign clause (T3) (Proposition 12.8).

Definition 12.5 (the extensive bound (I2))**.** We say that (I2) (one-sided, in the volume limit at fixed spacing) holds for the scheme a in the free convention if there are $s_1 > 0$ and $C_U \in [0,\infty)$ such that

$$\sup_{k\ge 0}\ \limsup_{J\to\infty}\ \alpha^a_{k,\mathbb{T}_J}(s_1) \;\le\; C_U. \tag{34}$$

The positive leg of the next section uses exactly this, with $\kappa := C_U/s_1$ (by convexity and $\alpha(0) = 0$, $\limsup_J \alpha^a_{k,\mathbb{T}_J}(u) \le \kappa u$ for $0 \le u \le s_1$). Throughout, $M_1 \in (0, m)$ is an auxiliary scalar mass, fixed at the end of this paragraph. We recall from Definition 2.1 King's free-boundary bare coefficient $\alpha^\infty_k(\mathsf{M}) = \mathsf{M}^2 + \delta m^2_k(\mathsf{M})$ on Λ_k at scalar mass M (it is the fixed-k volume limit of the

torus coefficient $\mathsf{M}^2+\delta m^2_{k,\mathbb{T}_J}(\mathsf{M})$, Lemma 12.11), the difference $s^\infty_k(M_1)=\frac12\big(\alpha^\infty_k(m)-\alpha^\infty_k(M_1)\big)$ and $\sigma_1=\sigma_1(M_1)=\liminf_{k\to\infty}s^\infty_k(M_1)$, and we put, in the reference scheme z,

$$B^z_{k,T}(\mathsf{M}):=E(\mathsf{M})-E(m)-\tfrac12\big(2c_{k,T}+z_k\big)\big(\alpha^{\mathrm{tor}}_{k,T}(m)-\alpha^{\mathrm{tor}}_{k,T}(\mathsf{M})\big)|T|,$$

$$b^\infty_k(M_1):=\lim_{J\to\infty}|\mathbb{T}_J|^{-1}B^z_{k,\mathbb{T}_J}(M_1),\qquad \beta_1=\beta_1(M_1):=\limsup_{k\to\infty}b^\infty_k(M_1),$$

where $\alpha^{\mathrm{tor}}_{k,T}(\mathsf{M}):=\mathsf{M}^2+\delta m^2_{k,T}(\mathsf{M})$ is the bare coefficient of King's torus convention and the limit defining b^∞_k exists by Lemma 12.11. That $\sigma_1(M_1)$ and $\beta_1(M_1)$ are finite is Lemma 12.7 below; (T3) says that $\sigma_1(M_1)>0$ for some $M_1\in(0,m)$ (Definition 2.1), and we fix M_1 as such a mass, so that $\sigma_1(M_1)>0$ and the interval $(0,\sigma_1)$ of Proposition 12.8 is not empty.

Proposition 12.6 (torus stability at two scalar masses)**.** *For each scalar mass $\mathsf{M}\in\{m,M_1\}$, M_1 the auxiliary mass fixed after Definition 12.5 (the sans-serif letter M denotes a scalar mass throughout, M being King's large-block integer), there are constants $E_\pm(\mathsf{M})<\infty$, a tail K_1 and, for each $k\ge K_1$, a $J_*(k)$ such that the source-free partition functions of King's tori $T=T_J=\mathbb{R}^3/(2MJ\mathbb{Z})^3$ (Definition 12.1; $J\in\mathbb{N}$ is the torus parameter) at spacing ε_k and at scalar mass M satisfy, in King's torus convention,*

$$e^{-E_-(\mathsf{M})|T|}\le Z^{(\mathsf{M})}_{k,T}\le e^{E_+(\mathsf{M})|T|}\qquad(k\ge K_1,\ J\ge J_*(k)).$$

These are Bałaban's lower and upper stability bounds, [1, (1.14)] and [2, (1.3)], at a fixed scalar mass, with block factor 5 and constants uniform in the spacing and in the torus, used as stated there and read at the two masses m and M_1 and in King's torus normalisation of the counterterm. Proposition 3.1 of [2], behind the upper bound, is stated under a cap on the dimensionless scalar mass at the scales at which it is applied; it is cited under that cap, one more one-sided cap on the stopping scale and no condition on the bare mass (Remark 29.5).

Lemma 12.7 (the c-number statement)**.** *For the auxiliary mass M_1 fixed after Definition 12.5, the quantities $\sigma_1(M_1)$ (Definition 2.1) and $\beta_1(M_1)$ defined above are finite.*

The proof is a computation with lattice Feynman integrals, carried out in Section 29. For $\sigma_1(M_1)$ the quantity is the difference of two mass counterterms at the same photon mass, whose divergent parts ($1/\varepsilon$ at one loop, $\ln(1/\varepsilon)$ at two loops) do not depend on the scalar mass and cancel in the difference, and the remaining lattice sums converge as Riemann sums of massive lattice Feynman integrals, by Reisz's power-counting theorems (Remark 19.15). Only the relative limit $\alpha_\infty(\mathsf{M})-\alpha_\infty(m)=\lim_k[\alpha^\infty_k(\mathsf{M})-\alpha^\infty_k(m)]$ exists, each coefficient alone carrying the $1/\varepsilon$ and $\ln(1/\varepsilon)$ divergences of the counterterm; this relative limit, its description as a function of the couplings and the coupling region it determines are given in Lemmas 29.1 and 29.3, the first of which gives the limit in closed form and with it the finiteness of $\sigma_1(M_1)$. The quantity $b^\infty_k(M_1)$ is of a different kind. By its definition and Lemma 12.11 it is the difference, per unit volume and in the volume limit at fixed spacing, of King's field-independent constants E_0+E_1 at the two scalar masses, less the multiple $(2\mathsf{W}_{\varepsilon_k}+z_k)\,s^\infty_k(M_1)$ of the difference of the bare coefficients by the Wick and scheme constants; each of these pieces diverges as $k\to\infty$, and the divergences cancel between the vacuum energies and that multiple, not between the two masses. That $b^\infty_k(M_1)$ is bounded in k, so that $\beta_1(M_1)$ is finite, is Lemma 29.2, which rests on an exact identity for the derivative of the vacuum energy with respect to the squared scalar mass.

Proposition 12.8 ((I2) from the torus theorems)**.** *Let the couplings lie in the coupling region of Definition 2.1 (the sign clause (T3)). Then (I2) holds in the reference scheme z for every $s_1\in(0,\sigma_1)$, with a constant $C_U=C_U(s_1)<\infty$ depending on (e,λ,m,μ_0,M), on M_1 and on s_1. For a scheme $a\ne z$ with $\sup_k|z_k-a_k|<\infty$ (an admissible scheme, Definition 2.7) it holds with the constant $C_U+s_1\sup_k|z_k-a_k|$.*

The proof uses the torus theorems (Proposition 12.6), at the two masses m and M_1, and Lemma 12.7, and assembles four lemmas on King's torus model at fixed spacing, stated in the form used and with their short proofs where elementary. Notation at fixed k (spacing ε) on $T=\mathbb{T}_J$: $f^{\mathrm{tor}}(u):=\ln N^{z,\mathrm{tor}}_{k,T}(0,u\mathbf{1})$ (King's torus model at mass m, Wick constant $c_{k,T}$, scheme z) and $f^F(u):=\ln N^{z}_{k,T}(0,u\mathbf{1})$ (free convention); both are convex with $f(0)=0$ (Lemma 12.4). Write $\alpha^{\mathrm{tor}}:=m^2+\delta m^2_{k,T}(m)$, $\alpha^F:=m^2+\delta m^2_{\varepsilon}$, $\sigma_{k,T}:=\frac12(\alpha^{\mathrm{tor}}-\alpha^F)$ and $\Delta c_T:=\mathsf{W}_\varepsilon-c_{k,T}$ (the two *convention offsets*), and $s_{k,T}(M_1):=\frac12(\alpha^{\mathrm{tor}}_{k,T}(m)-\alpha^{\mathrm{tor}}_{k,T}(M_1))$.

Lemma 12.9 (mass shift)**.** *If $M''>0$ satisfies $\alpha^{\mathrm{tor}}_{k,T}(M'')=\alpha^{\mathrm{tor}}_{k,T}(m)-2u$, then*

$$N^{z,\mathrm{tor}}_{k,T}(0,u\mathbf{1})=e^{B}\,Z^{(M'')}_{k,T}/Z^{(m)}_{k,T},\qquad B=E(M'')-E(m)-(2c_{k,T}+z_k)\,u\,|T|.$$

Proof. The source $u\sum_x\varepsilon^3r_x^2$ shifts the bare coefficient of the torus model at mass m by $-2u$, that is to the bare coefficient of the model at mass M''; beyond the fixed terms, King's torus action [25, (2.4)] contains only the counterterm and the field-independent constants E_0,E_1, so the two Boltzmann factors differ by the constant $e^{E(M'')-E(m)}$, and the Wick and scheme multipliers $e^{-(2c_{k,T}+z_k)u|T|}$ of $N^{z,\mathrm{tor}}$ are bookkeeping. □

Lemma 12.10 (chord)**.** *If f is convex on $\mathbb{R}$ with $f(0)=0$, then $f(u)\le(u/s)f(s)$ for $0\le u\le s$, $s>0$.*

Proof. $u=(1-u/s)\cdot0+(u/s)\cdot s$ and $f(0)=0$. □

Lemma 12.11 (the volume limit at fixed spacing)**.** *Fix k. As $J\to\infty$: $\alpha^{\mathrm{tor}}_{k,\mathbb{T}_J}(\mathsf{M})\to\alpha^{\infty}_k(\mathsf{M})$ uniformly for $\mathsf{M}\in[M_1,m]$; $s_{k,\mathbb{T}_J}(M_1)\to s^{\infty}_k(M_1)$; $|\mathbb{T}_J|^{-1}B^{z}_{k,\mathbb{T}_J}(M_1)\to b^{\infty}_k(M_1)$, a finite limit; and the offsets vanish, $\sigma_{k,\mathbb{T}_J}\to0$ and $\Delta c_{\mathbb{T}_J}\to0$. Moreover (the* boundedness clause*) $\sup_{J\ge1}\sup_{\mathsf{M}\in[M_1,m]}\big(|\delta m^2_{k,\mathbb{T}_J}(\mathsf{M})|+|c_{k,\mathbb{T}_J}(\mathsf{M})|\big)<\infty$; in particular $\delta m^2_{k,\mathbb{T}_J}(m)$ and $c_{k,\mathbb{T}_J}$ are bounded in $J\ge1$.*

The proof is the convergence of Riemann sums: at fixed k the torus quantities are sums over the dual lattice of $\mathbb{T}_J$ of smooth periodic functions of the momentum (the propagators are massive lattice covariances at fixed spacing), bounded uniformly in J and convergent as $J\to\infty$ to the corresponding Brillouin-zone integrals; this gives the limits, the boundedness clause and the vanishing of the offsets. No rate and no uniformity in the source are claimed or used. Of the clauses listed, Step 2 of the proof of Proposition 12.8 uses the convergence of $s_{k,\mathbb{T}_J}(M_1)$ and the $\limsup_J$ of $|\mathbb{T}_J|^{-1}B^{z}_{k,\mathbb{T}_J}(M_1)$, and Step 3 the two offset clauses.

Lemma 12.12 (the fixed-spacing extensive bound)**.** *Fix k and $v>0$. There is $C_k(v)<\infty$ such that for every King torus $T=\mathbb{T}_J$, $J\ge1$, in either convention, $f(\pm v)\le C_k(v)|T|$; hence, by convexity, $|f(w)|\le C_k(v)\,|w|\,|T|/v$ for $|w|\le v$.*

Proof. Let $I(b):=\int_{\mathbb{C}}e^{-\varepsilon^3[\frac12 b|\phi|^2+\lambda|\phi|^4]}d^2\phi<\infty$ for real b ($\lambda>0$), α the bare coefficient and c the Wick constant of the convention in use, and Z_γ the photon Gaussian integral, finite because $\mu_0>0$ (at $\mu_0=0$ the constant mode is in the kernel of the Feynman form). The hopping term is ≥0, so dropping it bounds the numerator: $\int e^{-S+w\sum\varepsilon^3r^2}\le Z_\gamma I(\alpha-2w)^{|T|/\varepsilon^3}$. The hopping term of a bond is $\le\varepsilon(r_x^2+r_y^2)$, and each site lies on six bonds, so the hopping part of the action is $\le6\varepsilon^{-2}\sum_x\varepsilon^3r_x^2=\sum_x\varepsilon^3\cdot\frac12(12\varepsilon^{-2})r_x^2$ and $\int e^{-S}\ge Z_\gamma I(\alpha+12\varepsilon^{-2})^{|T|/\varepsilon^3}$. Hence $\ln\langle e^{w\sum\varepsilon^3r^2}\rangle\le\varepsilon^{-3}|T|[\ln I(\alpha-2w)-\ln I(\alpha+12\varepsilon^{-2})]$ and

$$f(w)\le\big\{\varepsilon^{-3}[\ln I(\alpha-2w)-\ln I(\alpha+12\varepsilon^{-2})]+(2|c|+|z_k|)|w|\big\}|T|.$$

In the free convention α and c do not depend on T; in the torus convention they converge as $J\to\infty$ and are bounded in $J\ge1$ (the boundedness clause of Lemma 12.11). So $C_k(v)$, the

supremum over the sign and over $J \geq 1$ of the brace at $w = \pm v$, is finite. The last claim is convexity with $f(0) = 0$: $f(w) \leq (|w|/v) f(\pm v)$ and $f(w) \geq -f(-w) \geq -(|w|/v) f(\mp v)$. □

Proof of Proposition 12.8. Step 1: the exact convention bridge. On the same torus the free-convention action differs from the torus-convention action by

$$\tfrac{1}{2}(\alpha^{F} - \alpha^{\mathrm{tor}}) \sum_{x} \varepsilon^{3} r_x^2 = -\sigma_{k,T} \sum_{x} \varepsilon^{3} r_x^2$$

plus field-independent constants. Hence $\langle e^{u\sum \varepsilon^3 r^2}\rangle^{F} = \langle e^{(u+\sigma)\sum \varepsilon^3 r^2}\rangle^{\mathrm{tor}} / \langle e^{\sigma \sum \varepsilon^3 r^2}\rangle^{\mathrm{tor}}$ with $\sigma = \sigma_{k,T}$, and inserting the Wick and scheme multipliers,

$$f^{F}(u) = f^{\mathrm{tor}}(u+\sigma) - f^{\mathrm{tor}}(\sigma) - 2\,\Delta c_T\, u\,|T|. \tag{35}$$

Step 2: the bound in the torus convention. Let $0 \leq u < \sigma_1$ and $\eta > 0$. By the definitions of σ_1 and β_1 there is $K_1 = K_1(u,\eta)$, which we take at least as large as the tail of Proposition 12.6, such that $s_k^{\infty}(M_1) > u$ and $b_k^{\infty}(M_1) < \beta_1 + \eta$ for all $k \geq K_1$. Fix such a k. By Lemma 12.11 there is $J_* \geq J_*(k)$ with $s := s_{k,\mathbb{T}_J}(M_1) > u$ for $J \geq J_*$. The chord (Lemma 12.10) at this s gives $f^{\mathrm{tor}}(u) \leq (u/s) f^{\mathrm{tor}}(s) \leq [f^{\mathrm{tor}}(s)]_+$. Lemma 12.9 with $M'' = M_1$ applies exactly, since $\alpha^{\mathrm{tor}}_{k,\mathbb{T}_J}(M_1) = \alpha^{\mathrm{tor}}_{k,\mathbb{T}_J}(m) - 2s$ by the definition of s:

$$f^{\mathrm{tor}}(s) = B^{z}_{k,\mathbb{T}_J}(M_1) + \ln Z^{(M_1)}_{k,\mathbb{T}_J} - \ln Z^{(m)}_{k,\mathbb{T}_J} \leq B^{z}_{k,\mathbb{T}_J}(M_1) + \big(E_+(M_1) + E_-(m)\big)|\mathbb{T}_J|$$

by Proposition 12.6, used as an upper bound at M_1 and as a lower bound at m. Divide by $|\mathbb{T}_J|$ and let $J \to \infty$ (Lemma 12.11 for the first term):

$$\limsup_{J\to\infty} |\mathbb{T}_J|^{-1} f^{\mathrm{tor}}_{k,\mathbb{T}_J}(u) \leq \big[\beta_1 + \eta + E_+(M_1) + E_-(m)\big]_+ \qquad (k \geq K_1).$$

For each of the finitely many $k < K_1$, Lemma 12.12 (any $v \geq u$) gives $\limsup_J |\mathbb{T}_J|^{-1} f^{\mathrm{tor}}_{k,\mathbb{T}_J}(u) \leq C_k(v) u/v < \infty$. Fix $\eta := 1$ (any $\eta > 0$ serves); then

$$C^{\mathrm{tor}}(u) := \sup_{k\geq 0} \limsup_{J\to\infty} |\mathbb{T}_J|^{-1} f^{\mathrm{tor}}_{k,\mathbb{T}_J}(u) < \infty \qquad \text{for every } u \in [0,\sigma_1).$$

Step 3: transfer to the free convention. Fix $u \in [0,\sigma_1)$ and $p > 1$ with $pu < \sigma_1$, and let $p' = p/(p-1)$. Since $u+\sigma = \frac{1}{p}(pu) + \frac{1}{p'}(p'\sigma)$, convexity gives $f^{\mathrm{tor}}(u+\sigma) \leq \frac{1}{p} f^{\mathrm{tor}}(pu) + \frac{1}{p'} f^{\mathrm{tor}}(p'\sigma)$, whatever the sign of $\sigma = \sigma_{k,\mathbb{T}_J}$. Once $p'|\sigma_{k,\mathbb{T}_J}| \leq v$, which holds for all large J by Lemma 12.11, Lemma 12.12 gives $|f^{\mathrm{tor}}(p'\sigma)|, |f^{\mathrm{tor}}(\sigma)| \leq C_k(v) p' |\sigma_{k,\mathbb{T}_J}| |\mathbb{T}_J|/v$. Inserting this into (35),

$$|\mathbb{T}_J|^{-1} f^{F}_{k,\mathbb{T}_J}(u) \;\leq\; \tfrac{1}{p} |\mathbb{T}_J|^{-1} f^{\mathrm{tor}}_{k,\mathbb{T}_J}(pu) + \tfrac{2C_k(v)p'}{v} |\sigma_{k,\mathbb{T}_J}| + 2|\Delta c_{\mathbb{T}_J}|\, u \;\xrightarrow[J\to\infty]{\limsup}\; \tfrac{1}{p} C^{\mathrm{tor}}(pu),$$

because $\sigma_{k,\mathbb{T}_J} \to 0$ and $\Delta c_{\mathbb{T}_J} \to 0$ at fixed k (Lemma 12.11; the factor $\frac{1}{p'} \leq 1$ in the middle term has been dropped). Hence $\sup_k \limsup_J \alpha^{z}_{k,\mathbb{T}_J}(u) \leq \frac{1}{p} C^{\mathrm{tor}}(pu) < \infty$ for every $u \in [0,\sigma_1)$. Take $u = s_1 \in (0,\sigma_1)$ and $C_U := [\inf_{p>1:\, ps_1 < \sigma_1} \frac{1}{p} C^{\mathrm{tor}}(ps_1)]_+$; this is (34) in the scheme z. For a scheme $a \neq z$, $N^{a}_{k,T}(0,q) = e^{-(a_k - z_k)\mathsf{S}(q)} N^{z}_{k,T}(0,q)$ gives $\alpha^{a}_{k,T}(u) = \alpha^{z}_{k,T}(u) + (z_k - a_k)u \leq \alpha^{z}_{k,T}(u) + |z_k - a_k|\, u$, so (34) holds with $C_U + s_1 \sup_k |z_k - a_k|$. □

Proposition 12.13 (the auxiliary mass on either side of m)**.** *Let $M_1 > 0$, $M_1 \neq m$, be a scalar mass for which $s_k^{\infty}(M_1) = \frac{1}{2}(\alpha_k^{\infty}(m) - \alpha_k^{\infty}(M_1))$, of either sign, has $\sigma_1(M_1) = \liminf_k s_k^{\infty}(M_1) > 0$, the quantity being defined by the formula above with this M_1 (the computation of Lemma 12.7 is at a fixed scalar mass; the finiteness of $\beta_1(M_1)$, defined likewise with this M_1, holds for every such M_1 by Lemma 29.2). Then, with the stability bounds of Proposition 12.6 at m and at this M_1 (Bałaban's bounds hold at every fixed scalar mass; Proposition 12.6 states them at m and at the mass $M_1 \in (0,m)$ fixed after Definition 12.5),* (I2) *holds in the scheme z for every $s_1 \in (0,\sigma_1(M_1))$, and for $a \neq z$ with the shifted constant, exactly as in Proposition 12.8.*

Proof. The proof of Proposition 12.8 is repeated with Lemma 12.11 stated on the interval $I := [\min(M_1, m), \max(M_1, m)]$, and we verify that no step uses $M_1 < m$. Lemma 12.9 is an identity for any M'' with $\alpha^{\mathrm{tor}}(M'') = \alpha^{\mathrm{tor}}(m) - 2u$, whatever the order of M'' and m. Lemma 12.10 needs $0 \le u \le s$ only. Lemma 12.11: its proof uses only that the propagators are translation-invariant massive lattice covariances at fixed ε with masses in $I \cup \{\mu_0\}$, bounded below by $\min(M_1, m, \mu_0) > 0$, so it holds on I word for word; Step 2 uses it at the two masses m and M_1 only. Lemma 12.12 carries no mass order, and (35) is an identity. In Step 2 the sign clause is $s_k^{\infty}(M_1) > u \ge 0$, the chord is applied at $s = s_{k,\mathbb{T}_J}(M_1) > u \ge 0$, Lemma 12.9 at $M'' = M_1$ is exact by the definition of s, and Proposition 12.6 is used as an upper bound at M_1 and a lower bound at m; none of this uses $M_1 < m$. The finiteness of $\beta_1(M_1)$, which Step 2 uses to choose K_1 with $b_k^{\infty}(M_1) < \beta_1(M_1) + \eta$ for $k \ge K_1$ and in its final bound, and which Proposition 12.8 takes from Lemma 12.7 at the mass $M_1 \in (0, m)$ fixed there, is Lemma 29.2 for the present M_1. Step 3 carries no mass. □

The two-sided variant matters exactly on the set $E_W^{<} \setminus E_W$ of couplings at which m minimises α_∞ on $(0, m]$ but not on $(0, \infty)$ (here $\alpha_\infty(\mathsf{M}) := \lim_k \alpha_k^{\infty}(\mathsf{M})$, and $E_W \subseteq E_W^{<}$ are the coupling sets of Lemma 29.3; for instance when $\alpha_\infty'(m) < 0$ and m minimises α_∞ on $(0, m]$): there (T3) fails for every $M_1 < m$ while some $M_1 > m$ has $\sigma_1(M_1) > 0$. The main results use Proposition 12.8 with $M_1 \in (0, m)$, as (T3) is posed.

Remark 12.14 (the convention of Proposition 12.6 and how much of it is used). (i) Proposition 12.6 is stated in King's torus convention (counterterm and Wick constant computed with the periodic propagators of the torus itself). For the source-free partition function this is also the convention of Bałaban's torus model [1, (1.1)–(1.13)], which is the same action: his E_0 is the Gaussian normalisation and his E_1 [1, (1.13)] is King's field-independent constant on the torus, the same connected vacuum graphs with the same counterterm vertices [3, (1.24)]; so the partition function of Proposition 12.6 is exactly the quantity bounded in [1, (1.14)] and [2, (1.3)], and no change of normalisation is involved in reading those bounds as Proposition 12.6. This is what the proof needs: the offsets $\sigma_{k,\mathbb{T}_J}$, $\Delta c_{\mathbb{T}_J}$ between King's torus convention and the free convention vanish in the volume limit at fixed spacing (Lemma 12.11), and Step 3 of the proof of Proposition 12.8 transfers the bound through them; between other normalisations they need not vanish. (ii) Only a tail of Proposition 12.6 is used: Step 2 of that proof invokes it for $k \ge K_1$ and $J \ge J_*(k)$, the finitely many $k < K_1$ being handled by Lemma 12.12 at fixed spacing, and the constants $E_{\pm}(\mathsf{M})$ are independent of k and of the torus. (iii) In the scheme z, C_U depends on the couplings, on M and on the auxiliary choices M_1, s_1, and on nothing else; neither it nor any other bound of this section is uniform in μ_0.

13. The Wick tube II: the uniform bound and its limits

This section proves the (k, J)-uniform bound (37) on the general-source functionals of the comparison cubes on the tube $\mathcal{D}_{\sigma_+,H}$ from the extensive bound (I2) (Definition 12.5): the positive part of the Wick source (the scalar source $k(\zeta)$ of Definition 2.8, so called as the source of the Wick square) is controlled through King's tori, the negative part through a tiling into cells of fixed size, each controlled by one instance of Theorem B and the positive bound. Then come the limits on the tube, E0″, and the continuity and translation invariance of the tube object; these limit clauses, Theorem 13.10(i),(ii) and Lemma 13.11, are proved in Steps 7–11 of the proof of Theorem 28.5 in Section 28 and only assembled here. States are those of the comparison convention ND$^{(a)}$ (Definition 2.2) unless King's convention is named; $N_{k,X}^{a}(0, q)$ is the functional of Definition 2.5 on a finite $X \subset \Lambda_k$ or a King torus (Definition 12.1), the torus with fundamental cell C_J being $\mathbb{T}_J$ as there (T_J is the top-layer set of King's RG cube), and

$\alpha^a_{k,\mathbb{T}}(u) = |\mathbb{T}|^{-1} \ln N^a_{k,\mathbb{T}}(0, u\mathbf{1})$. $s_1 \in (0, \sigma_1)$ and C_U are the amplitude and constant of (I2), $\kappa := C_U/s_1$; the sources $f_1, \dots, f_n$, $h_1, \dots, h_p$ are in C^∞_c with supports in $Q_R = [-R, R]^3$; $g(\zeta)$, $k(\zeta) = k_+ - k_-$ and the norms are as in Definition 2.8. The scheme is the reference scheme z, or an admissible scheme a (Definition 2.7, so that $\sup_k |z_k - a_k| < \infty$); then C_U stands for $C_U + s_1 \sup_k |z_k - a_k|$ throughout (Proposition 12.8).

13.1. **The positive leg on cubes.** The positive leg compares the cube with the torus of the same fundamental cell and then applies the chessboard estimate and the extensive bound; the result is a bound on the positive part of the Wick source linear in its L^1 norm, uniform in the spacing and the cube.

Proposition 13.1 (cube $\le$ torus at one bare mass)**.** *For every k, J and every lattice function $q \ge 0$ on X_J: $N^a_{k,C_J}(0,q) \le N^a_{k,\mathbb{T}_J}(0,q)$, both in the free convention.*

Proof. The periodic state on $\mathbb{T}_J$ has the sites of X_J with opposite faces identified; its action is the Neumann–Dirichlet action on C_J plus, for every wrap-around pair $\{x, y\}$, the photon gradient terms $\varepsilon(A_\alpha(y) - A_\alpha(x))^2$ and the hopping term $\frac{\varepsilon}{2}(r_x^2 + r_y^2) - \varepsilon r_x r_y \cos(\cdot)$, whose diagonal part is exactly the pair of Dirichlet terms the cube action carries at x, y for their outward bonds; so the cube state is the torus state with the wrap-around gradient terms and cosine parts deleted. For the class-$\mathcal{I}$ observable $e^{\sum \varepsilon^3 q r^2}$ ($\Theta_{F,g}$, $F = 0$, $g = \varepsilon^3 q \ge 0$) deleting a gradient term, a quadratic $|\ell(A)|^2$, does not increase the expectation (Lemma 2.10(a)), nor does deleting a cosine part (Lemma 2.10(b)); each intermediate measure is a BFS measure, and δm^2_ε, W_ε, a_k, $\mathsf{S}(q)$ agree on both sides. □

Proposition 13.2 (the positive leg in L^1 form, on every Neumann–Dirichlet box)**.** *Let* (I2) *hold (Definition 12.5); lattice reflection positivity, Proposition 12.2, enters through the chessboard estimate Lemma 12.4. Let $X \subset \Lambda_k$ be finite, $\langle\cdot\rangle^{\mathrm{ND}}_{k,X}$ the Neumann–Dirichlet state of Definition 2.2 on X, and $0 \le q \le s_1$ a lattice function on X. Then*

$$\ln N^a_{k,X}(0,q) \;\le\; \kappa\, \|q\|_{1,k}, \qquad \kappa = C_U/s_1.$$

Proof. Choose J' with X and all its nearest neighbours in $C_{J'}$. (1) *Box $\le$ cube:* the first inequality of Lemma 2.13 with $Y = X_{J'}$ and $(f, g) = (0, q)$ (Lemma 2.10(a),(b)); its X-state is the state of Definition 2.2 on X, since all six bonds at each site of X are present in Y and no site is killed; the scheme multiplier is common (Lemma 2.14(a)). (2) *Cube $\le$ torus:* Proposition 13.1 on $C_{J'}$, q extended by 0. (3) *Chessboard:* Lemma 12.4 on $\mathbb{T}_{J'}$, $\ln N^a_{k,\mathbb{T}_{J'}}(0,q) \le \sum_x \varepsilon^3 \alpha^a_{k,\mathbb{T}_{J'}}(q(x))$. (4) *Chord:* $\alpha = \alpha^a_{k,\mathbb{T}_{J'}}$ is convex with $\alpha(0) = 0$, so $\alpha(u) \le (u/s_1)[\alpha(s_1)]_+$ for $0 \le u \le s_1$, and (3) is at most $s_1^{-1}[\alpha(s_1)]_+ \|q\|_{1,k}$ (the ℓ^1 norm of q is the same on X, $C_{J'}$, $\mathbb{T}_{J'}$). (5) The left side does not depend on J'; let $J' \to \infty$ at fixed k and use (I2): $\limsup_{J'} \alpha^a_{k,\mathbb{T}_{J'}}(s_1) \le C_U$ for every k. □

Corollary 13.3 (the positive leg on the cubes)**.** *Under the hypotheses of Proposition 13.2, for $0 \le q \le s_1$ on X_J: $\ln N^a_{k,C_J}(0,q) \le \kappa \|q\|_{1,k}$, uniformly in k and J. Consequently, for every $0 \le q \in C^\infty_c$ with $\sup q \le s_1$ and every J with $\operatorname{supp} q$ at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_J$, $\limsup_k Z^a_{k,J}(0,q) \le e^{\kappa \|q\|_{L^1}}$, uniformly in J. On King's convention the same bounds hold for $N^{a,K}_{k,C_J}(0,q)$ and $\limsup_k Z^{a,K}_{k,J}(0,q)$ (Remark 4.9: Step 1 of Lemma 4.2 prepended to Proposition 13.2), and for $2 \sup q < s_1$ the fixed-cube limits exist by Theorem B, so that*

$$\sup_J \lim_k Z^{a,K}_{k,J}(0,q) \le e^{\kappa \|q\|_{L^1}}.$$

Proof. Proposition 13.2 at $X = X_J$. The sampled source satisfies $0 \le q \le s_1$ (Lemma 2.9(E3)) and $\|q\|_{1,k} \to \|q\|_{L^1}$, which gives the lim sup bound. On King's convention the observable

$e^{\sum_x \varepsilon^3 q(x) r_x^2}$ is of class $\mathcal{I}$, so the kill step of Remark 4.9 gives $N^{a,K}_{k,C_J}(0,q) \le N^a_{k,C_J}(0,q)$ with a common multiplier (Lemma 2.14(a)); the source q is the family $h_1 = q$ at the coefficient $w = 1$, whose positive scalar part is q, so Theorem B gives the limit when $2\|q\|_\infty < s_W = s_1$, and there the limit equals the lim sup. □

13.2. **Cells and the constants of the negative leg.** For the negative leg the cube is tiled into cells of a fixed size, each a translate of one King cube, so that one instance of the existence theorem with scalar sources controls every cell; this subsection fixes the tilings and the constants of the Wick leg.

Definition 13.4 (cells, tilings, cut-offs)**.** Let $L_c := \min\{L \in \mathbb{N} : ML \ge 5\}$, $\ell_c := 2ML_c \ge 10$; the *cell* $C_{L_c} = [-ML_c, ML_c)^3$ is a King cube. For $s \in \{0,1\}^3$ let $W_\mu(0) := \ell_c\mathbb{Z}$, $W_\mu(1) := ML_c + \ell_c\mathbb{Z}$, and let the tiling $\mathcal{T}_s$ consist of the cells $\prod_\mu [w_\mu, w_\mu + \ell_c)$, $w_\mu \in W_\mu(s_\mu)$; every cell is $C_{L_c} + b$ with $b \in ML_c\mathbb{Z}^3 \subset \Lambda_k$. The good sets $G_s := \{x : \operatorname{dist}(x_\mu, W_\mu(s_\mu)) > 2\ \forall \mu\}$ are open and cover $\mathbb{R}^3$ (the 2-neighbourhoods of $W_\mu(0)$, $W_\mu(1)$ are disjoint as $ML_c \ge 5 > 4$). Fix a smooth partition of unity $0 \le \chi_s \le 1$, $\sum_s \chi_s = 1$, $\operatorname{supp}\chi_s \subset G_s$; so χ_s vanishes within distance 2 of every wall of $\mathcal{T}_s$. Fix $\chi_0 \in C_c^\infty$, $0 \le \chi_0 \le 1$, $\chi_0 = 1$ on $\{\operatorname{dist}_\infty(x, \mathbb{R}^3 \setminus C_{L_c}) \ge 2\}$, $\operatorname{supp}\chi_0 \subset \{\operatorname{dist}_\infty(x, \mathbb{R}^3 \setminus C_{L_c}) > 1\}$, and put $\chi_c := \tau_b \chi_0$ for the cell $c = C_{L_c} + b$. For sources supported in Q_R let $J_1(R) := \min\{J \in \mathbb{N} : MJ \ge R + \ell_c\}$; a cell meeting Q_R has $w_\mu \in [-R - \ell_c, R]$, hence lies in $[-R - \ell_c, R + \ell_c)^3 \subset C_J$ for $J \ge J_1(R)$, so every cell meeting Q_R is a full cell inside C_J and the cells clipped by the faces of C_J carry no source.

The constants of the Wick leg are

$$\sigma_+ := s_1, \quad H' := \frac{s_1}{2}, \quad H := \frac{H'}{8} = \frac{s_1}{16}, \quad \kappa := \frac{C_U}{s_1}, \quad B_{\text{cell}} := \sup_k \ln N^a_{k,C_{L_c}}(0, -2H'\chi_0),$$
$$C_{\text{cell}} := \tfrac{1}{2} B_{\text{cell}} + \kappa H' \ell_c^3, \qquad C_\Sigma := \frac{(2 + 6\ell_c^2)^2 \pi^2}{\ell_c^3}, \qquad K_- := \frac{8\,[C_{\text{cell}}]_+\, C_\Sigma}{H'}. \tag{36}$$

B_{cell} is finite by Lemma 14.4, that is by Corollary 6.27(b): at the single cube C_{L_c} and the single real source $-2H'\chi_0$ the comparison-convention functional is bounded by the King-convention one (the $\mathcal{D}$ direction of Lemma 4.2), which converges by Theorem B ($\operatorname{supp}\chi_0$ is at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_{L_c}$ as Theorem B requires, and the theorem puts no amplitude restriction on a negative Wick source, a position-dependent increase of the scalar mass squared). This boundedness statement is (B_{cell}) (Lemma 14.4) and the only place where the existence part (Sections 5 and 6) enters the bound (37); no fixed-cube limit of the comparison family is used.

13.3. **The negative leg on cubes.** The negative leg is proved by inserting walls along the cells of a tiling (Lemma 13.5), which does not decrease the functional of a negative source, and bounding each cell separately.

Lemma 13.5 (wall insertion)**.** *Let $X_1, \dots, X_N$ be the site sets $c \cap X_J$ of the cells c of one tiling $\mathcal{T}_s$ that meet C_J, and $q \ge 0$ a lattice function on X_J. Then $N^a_{k,C_J}(0,-q) \le \prod_{i=1}^N N^a_{k,X_i}(0, -q|_{X_i})$, with N^a_{k,X_i} the functional of the Neumann–Dirichlet state of Definition 2.2 on X_i.*

Proof. $e^{-\sum \varepsilon^3 q r^2}$ is the class-$\mathcal{D}$ observable $\Theta^-_{F,g}$, $F = 0$, $g = \varepsilon^3 q \ge 0$. Delete, for every nearest-neighbour pair with ends in different cells, the three photon gradient terms and the cosine part of the hopping term, one at a time: by Lemma 2.10(d) (the inequalities [9, Thm. 6.2(3),(4)], proved in Section 2) neither deletion decreases $\langle \Theta^- \rangle$, and each intermediate measure is a BFS measure. The result is a product over the cells (Lemma 2.10(e)) whose factors are the Neumann–Dirichlet states on the X_i: the kept diagonal parts of the deleted bonds are their Dirichlet terms, the

remaining gradient terms their Neumann terms, and δm_ε^2, W_ε, a_k are site-independent. The observable and the multiplier $e^{a_k \mathsf{S}(q)} = \prod_i e^{a_k \mathsf{S}(q|_{X_i})}$ factorise. □

Lemma 13.6 (one cell)**.** *Let* (I2) *hold (Definition 12.5), and let $c = C_{L_c} + b$ be a cell ($b \in ML_c\mathbb{Z}^3$) and $0 \le \tilde q \le H'$ a lattice function supported in $\{x : \operatorname{dist}_\infty(x, \mathbb{R}^3 \setminus c) \ge 2\}$. Then $\ln N^a_{k,c}(0, -\tilde q) \le C_{\mathrm{cell}}$ for every k.*

Proof. $\chi_c = 1$ on $\operatorname{supp}\tilde q$, so $-\tilde q = k_1 - k_2$ with $k_1 := H'\chi_c - \tilde q \ge 0$, $k_2 := H'\chi_c$, and Lemma 2.14(d) at $f = 0$ gives $N^a_{k,c}(0, -\tilde q) \le N^a_{k,c}(0, -2H'\chi_c)^{1/2} N^a_{k,c}(0, 2k_1)^{1/2}$. *First factor:* exact lattice translation covariance ($b \in \Lambda_k$; the change of variables $x \mapsto x - b$ transports the boundary conditions, and the action, δm_ε^2, W_ε, a_k are site-independent) gives $N^a_{k,c}(0, -2H'\chi_c) = N^a_{k,C_{L_c}}(0, -2H'\chi_0) \le e^{B_{\mathrm{cell}}}$. *Second factor:* Proposition 13.2 on the box $c \cap \Lambda_k$ with $q' := 2k_1$, $0 \le q' \le 2H' = s_1$, gives at most $\exp(\kappa \|q'\|_{1,k}) \le \exp(2\kappa H' \ell_c^3)$, the cell having volume ℓ_c^3. Hence $\ln N^a_{k,c}(0, -\tilde q) \le \frac{1}{2} B_{\mathrm{cell}} + \kappa H' \ell_c^3$. □

Lemma 13.7 (chord on the cell)**.** *Under the hypotheses of Lemma 13.6:* $\ln N^a_{k,c}(0, -\tilde q) \le ([C_{\mathrm{cell}}]_+ / H') \sup \tilde q$.

Proof. Let $t := \sup \tilde q / H' \in (0, 1]$, $\hat q := \tilde q / t$ ($0 \le \hat q \le H'$, same support). $\varphi(u) := \ln N^a_{k,c}(0, -u\hat q)$ is convex (Lemma 2.14(b)), $\varphi(0) = 0$, $\varphi(1) \le C_{\mathrm{cell}}$ (Lemma 13.6); so $\varphi(t) \le t\varphi(1) \le t[C_{\mathrm{cell}}]_+$. □

Lemma 13.8 (eight families and the sum over cells)**.** (a) *For a lattice function $q \ge 0$ on X_J: $N^a_{k,C_J}(0, -q) \le \prod_{s \in \{0,1\}^3} N^a_{k,C_J}(0, -8\chi_s q)^{1/8}$. If q is supported in Q_R and $J \ge J_1(R)$, then for each s every cell of $\mathcal{T}_s$ meeting $\operatorname{supp}(\chi_s q)$ is a full cell inside C_J, on which $\chi_s q$ is supported at ℓ^∞-distance > 2 from the complement of the cell.* (b) *For any tiling of $\mathbb{R}^3$ by cubes of side ℓ_c: $\sum_c \sup_{x \in c} (1 + |x|^2)^{-2} \le C_\Sigma$.*

Proof. (a) Hölder's inequality with eight exponents 8 in the one state $\langle \cdot \rangle^{\mathrm{ND}}_{k,C_J}$ for $e^{-\sum \varepsilon^3 q r^2} = \prod_s e^{-\sum \varepsilon^3 \chi_s q r^2}$; the multipliers match since $\mathsf{S}(q) = \sum_s \mathsf{S}(\chi_s q)$ at lattice points. Support: every cell meeting Q_R is a full cell inside C_J for $J \ge J_1(R)$ (Definition 13.4), and a point of G_s in the cell $\prod_\mu [w_\mu, w_\mu + \ell_c)$ has $\operatorname{dist}(x_\mu, \{w_\mu, w_\mu + \ell_c\}) > 2$ for every μ. (b) For x, y in one cell $|y| \le |x| + \sqrt{3}\ell_c$, so $1 + |y|^2 \le (2 + 6\ell_c^2)(1 + |x|^2)$ and $\sup_{x \in c} (1 + |x|^2)^{-2} \le (2 + 6\ell_c^2)^2 \ell_c^{-3} \int_c (1 + |y|^2)^{-2} dy$; sum over the disjoint cells, with $\int_{\mathbb{R}^3} (1 + |y|^2)^{-2} dy = \pi^2$. □

Proposition 13.9 (the negative leg)**.** *Let* (I2) *hold (Definition 12.5). For every k, every $J \ge J_1(R)$ and every lattice function $q \ge 0$ supported in Q_R with $\sup q \le H$:*

$$\ln N^a_{k,C_J}(0, -q) \;\le\; K_- \, |q|_w .$$

Proof. By Lemma 13.8(a), then Lemma 13.5 for each tiling $\mathcal{T}_s$ (the clipped cells carry no source and contribute $\ln N^a_{k,X_i}(0,0) = 0$), then Lemma 13.7, applicable since $8\chi_s q \le 8 \sup q \le H'$ and by the support clause of Lemma 13.8(a):

$$\begin{aligned}
\ln N^a_{k,C_J}(0, -q) &\le \frac{1}{8} \sum_s \sum_{c \in \mathcal{T}_s} \ln N^a_{k,c}(0, -8\chi_s q|_c) \\
&\le \frac{1}{8} \sum_s \sum_c \frac{[C_{\mathrm{cell}}]_+}{H'} \, 8 \sup_c q \le \frac{[C_{\mathrm{cell}}]_+}{H'} \, |q|_w \sum_s \sum_c \sup_{x \in c} (1 + |x|^2)^{-2} \\
&\le \frac{8[C_{\mathrm{cell}}]_+ C_\Sigma}{H'} \, |q|_w ,
\end{aligned}$$

using $\sup_c q \le |q|_w \sup_{x \in c} (1 + |x|^2)^{-2}$ (a sampled supremum is at most the continuum one, Lemma 2.9(E3)) and Lemma 13.8(b) for each of the eight tilings. □

The output norm is $|q|_w$: the chord bound on a cell sees only $\sup_c q$, so no L^1 information inside a cell survives, and the tube bound carries $K_-|k_-|_w$ rather than an L^1 norm of k_-.

13.4. **The uniform bound on the tube and its limits.** This subsection combines the two legs into the tube bound and records its consequences for King's state: the limits on the tube and E0″ (Theorem 13.10), then the continuity and translation invariance of the tube object (Lemma 13.11).

For a family with supports in Q_R and $\zeta = (z,w) \in \mathbb{C}^{n+p}$ put $G^{(k,C_J)}(\zeta) := Z^a_{k,J}(\sum_i z_i f_i, \sum_j w_j h_j)$, and let $\mathcal{D}_{\sigma_+,H}$ be the tube of Definition 2.8 with $\sigma_+ = s_1$, $H = s_1/16$. Two facts from several complex variables [22] are used: (C2) (the identity theorem) a holomorphic function on a connected open $D \subset \mathbb{C}^m$ vanishing on a non-empty relatively open subset of $D \cap \Sigma$, Σ an affine totally real subspace of real dimension m (such as $\mathbb{R}^m$ or $(\mathrm{i}\mathbb{R})^{m_1} \times \mathbb{R}^{m_2}$), vanishes on D (all its derivatives vanish at such a point); (C3) (Montel–Vitali) a locally bounded family of holomorphic functions on D is normal, and if it converges on a set as in (C2) it converges locally uniformly on D to a holomorphic limit.

Theorem 13.10 (the tube bound)**.** *Let the couplings lie in the coupling region of Definition 2.1, so that* (I2) *holds (Proposition 12.8, from the sign clause and the torus theorems, Proposition 12.6 and Lemma 12.7), for the reference scheme z or for an admissible scheme a (Definition 2.7); here and below* (I3) *and* (I4) *denote the proved statements Proposition 12.2 and Lemma 2.10. The constants σ_+, H, κ, K_- of (36) depend on $(e, \lambda, m, \mu_0, M, a)$ and on the fixed auxiliary choices M_1, χ_0, $s_1 \in (0, \sigma_1)$, and on nothing else; no uniformity in μ_0 is claimed. For every family with supports in Q_R, every $k \geq 0$, $J \geq J_1(R)$ and $\zeta \in \mathcal{D}_{\sigma_+,H}$:*

$$|G^{(k,C_J)}(\zeta)| \;\leq\; G^{(k,C_J)}(\operatorname{Re}\zeta) \;\leq\; \exp\Big(\|g(\zeta)\|^2_{2,k} + \kappa\|k_+(\zeta)\|_{1,k} + K_-|k_-(\zeta)|_w\Big). \tag{37}$$

On King's convention the bound holds in the form (73) for the K-family functionals $Z^{a,K}_{k,J}$ (Theorem 28.5), and their fixed-cube limits exist on the tube (Theorem B); consequently, for King's state (the infinite-volume state of King's construction [25, 26]*):*

(i) (Limits on the whole tube.) *$G^K_J := \lim_k Z^{a,K}_{k,J}$ exists locally uniformly on $\mathcal{D}_{\sigma_+,H}$ for every $J \geq J_1(R)$, $G^K_\infty := \lim_J G^K_J$ exists locally uniformly on $\mathcal{D}_{\sigma_+,H}$ (the limit along $J \geq J_1(R+\frac{1}{2})$, hence along any tail), both are holomorphic there, and*

$$|G^K_\infty(\zeta)| \leq \exp\big(\|g(\zeta)\|^2_2 + \kappa\|k_+(\zeta)\|_{L^1} + K_-|k_-(\zeta)|_w\big) \qquad (\zeta \in \mathcal{D}_{\sigma_+,H}).$$

(ii) (E0″.) *With $\sigma_0 := H/4$, $K' := \kappa\pi^2 + K_-$, the mixed Schwinger functions $S_{n,p}(f;h) := \partial_z^{(n)}\partial_w^{(p)} G^K_\infty(0)$ satisfy, for real test functions,*

$$\begin{aligned}|S_{n,p}(f_1,\dots,f_n;h_1,\dots,h_p)| &\leq e^{K'\sigma_0}(\sqrt{2}e)^n (n!)^{1/2} (e/\sigma_0)^p\, p! \prod_i \|f_i\|_2 \prod_j |h_j|_w \\ &\leq \sigma_{n+p} \prod_i |f_i|_4 \prod_j |h_j|_4,\end{aligned}$$

$\sigma_N := e^{K'\sigma_0} C_1^N N!$, $C_1 := 2\max(e\pi/2, e/\sigma_0)$; for complex test functions the first bound carries the factor $2^{n+p} \leq 2^N$, $N = n+p$, absorbed in C_1. This is E0″ *with $s = 4$ and every $\beta > 1$; the $S_{n,p}$ are tempered distributions.*

σ_1, C_U depend on $(e, \lambda, m, \mu_0, M_1)$ through the torus stability constants of Proposition 12.6 at m, M_1, the c-number β_1 of Lemma 12.7 and finitely many fixed-spacing constants; B_{cell} depends on everything, including μ_0, M, a. The curvature term of (37) rests on the constant 2 of Lemma 7.1, which is not known to be sharp (Remark 7.2).

Proof. The bound (37). The first inequality is Lemma 2.14(b). Write $g = g(\zeta)$, $k = k(\zeta)$ (real). *Step 1* (curvature off): Theorem 7.4 in the presence of the real scalar source k gives $N^a(g,k) \le e^{\|g\|^2_{2,k}} N^a(0,k)$; the supports of the f_i are at distance > 1 from $\mathbb{R}^3 \setminus C_J$ since $MJ \ge R + \ell_c > R + 1$. *Step 2*: $N^a(0, k_+ - k_-) \le N^a(0, 2k_+)^{1/2} N^a(0, -2k_-)^{1/2}$ (Lemma 2.14(d) at $f = 0$). *Step 3* (positive leg): $\zeta \in \mathcal{D}_{\sigma_+,H}$ and (E3) give $0 \le 2k_+ \le s_1$ on the lattice, so $\frac{1}{2} \ln N^a(0, 2k_+) \le \kappa \|k_+\|_{1,k}$ by Corollary 13.3. *Step 4* (negative leg): $0 \le 2k_- \le H$, supported in Q_R, $J \ge J_1(R)$, so $\frac{1}{2} \ln N^a(0, -2k_-) \le \frac{1}{2} K_- |2k_-|_w = K_- |k_-|_w$ by Proposition 13.9.

(i), (ii). The bound (73) on King's convention is Steps 1–6 of the proof of Theorem 28.5, the sequence of estimates above carried out on ND^K with the kill step of Remark 4.9 in the positive leg and the cell statement of Lemma 4.8 in the negative leg. From it, from the fixed-cube limits of Theorem B at the real points of the tube (stated on King's convention) and from the cube monotonicity of Corollary 4.3 on the cone, Steps 7–11 of that proof give (i) and (ii): Montel's theorem in k on the tube with the real points as the uniqueness set (C3), the monotone cube limit on a sub-tube of the split family and the identity theorem (C2) on the whole tube, and Cauchy's estimates on a polydisc inside the tube, with the extension to $\mathcal{S}$, to complex test functions and to tempered distributions as in Theorem 7.6(c). □

Lemma 13.11 (the tube object of King's state: limit, continuity, translations)**.** *Let the couplings lie in the coupling region of Definition 2.1 and let a be an admissible scheme, so that the bound* (73) *and the fixed-cube limits G^K_J of Theorem 13.10(i) are available. Write $\mathcal{D} = \mathcal{D}_{\sigma_+,H}$ for the tube of a family with supports in Q_R; the items are numbered in continuation of Theorem 13.10(i),(ii).*

(iii) (The J-limit on the tube.) *For $J \ge J_1(R + \frac{1}{2})$, which exceeds the least J_0 with the supports at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_J$ (as $MJ_1(R') - R' \ge \ell_c \ge 10 > 1$), $G^K_\infty = \lim_J G^K_J$ exists locally uniformly on $\mathcal{D}$, is holomorphic on $\mathcal{D}$, satisfies the bound of Theorem 13.10*(i)*, is > 0 at every real point of $\mathcal{D}$, and $G^K_J \le G^K_\infty$ at the cone points of $\mathcal{D}$ ($z \in (\mathrm{i}\mathbb{R})^n$, $w \in \mathbb{R}^p$, $k(\zeta) \ge 0$: the real scalar source $k(\zeta) \ge 0$ sits in the slot of the cone).*

(iv) (Continuity on the real tube.) *Let $E_\rho := C^\infty_c(\bar{B}_\rho; \Lambda^2) \times C^\infty_c(\bar{B}_\rho)$ with its C^∞ topology, a Fréchet space, and $U_\rho := \{(f,h) \in E_\rho : \|h_+\|_\infty < \sigma_+/2,\ \|h_-\|_\infty < H/2\}$, open and convex ($h \mapsto \|h_\pm\|_\infty$ is continuous and convex). Then $\Phi^{a,K}(f,h) := \ln G^K_\infty(f,h)$ is finite, convex and continuous on U_ρ.*

(v) (Translations on the tube.) *For every family and every $b \in \mathbb{R}^3$: $G^K_\infty(\tau_b \cdot \zeta) = G^K_\infty(\zeta)$ for all $\zeta \in \mathcal{D}$; the tube of the translated family is that of the family, since $k(\tau_b \cdot \zeta) = \tau_b k(\zeta)$ has the same supremum norms.*

Besides the fixed-cube limits of Theorem B and the bound (73) *the proofs use only the cube monotonicity of Corollary 4.3 on the cone, the translation sandwich of Lemma 8.3, Lemmas 2.14 and 2.17, lattice covariance and (C2), (C3); Lemma 2.10(d) enters only inside* (73).

Proof. We prove (iii) by the steps of the proof of Theorem 28.5 together with a convexity argument for the positivity at real points; (iv) from (iii) and the convexity of the logarithm; and (v) first for lattice translations at cone points by the translation sandwich, then for all points of the tube and all translations by the identity theorem and continuity.

(iii) Existence, holomorphy and the bound are Steps 8–9 of the proof of Theorem 28.5, which use (73), the fixed-cube limits of Theorem B and the cube monotonicity of Corollary 4.3. *Positivity at real points.* For real $\zeta \in \mathcal{D}$ pick $\epsilon > 0$ with $-\epsilon\zeta \in \mathcal{D}$ ($\mathcal{D}$ is an open neighbourhood of 0). $\varphi_J(t) := \ln G^K_J(t\zeta)$ is convex on $[-\epsilon, 1]$ (Lemma 2.14(b) in the limit) with $\varphi_J(0) = 0$, so $0 \le \frac{\epsilon}{1+\epsilon}\varphi_J(1) + \frac{1}{1+\epsilon}\varphi_J(-\epsilon)$, i.e. $\varphi_J(1) \ge -\varphi_J(-\epsilon)/\epsilon \ge -C_\epsilon/\epsilon$, C_ϵ the limit bound of (73) at $-\epsilon\zeta$; hence $G^K_J(\zeta) \ge e^{-C_\epsilon/\epsilon}$ uniformly in J and $G^K_\infty(\zeta) > 0$. *Cone points.* There $G^K_J \le G^K_{J+1}$

(Corollary 4.3 with Lemma 2.14(a) at fixed k, then $k \to \infty$; it needs $J \geq J_0$, which the range of (iii) ensures), so $G_J^K \leq G_\infty^K$.

(iv) *Finite:* (iii) for the two-source family $(f; h)$ at $\zeta = (1, 1)$, in $\mathcal{D}$ by the definition of U_ρ; the value is the pointwise iterated limit and does not depend on the family used. *Convex:* for $s, s' \in U_\rho$, $t \in [0, 1]$, use the family $(f, f'; h, h')$ at the real point $(t, 1-t, t, 1-t)$, in $\mathcal{D}$ because $(th + (1-t)h')_\pm \leq th_\pm + (1-t)h'_\pm$ pointwise; the convexity of $\ln Z_{k,J}^{a,K}$ in the real coefficients (Lemma 2.14(b)) passes to the limits. *Borel:* a pointwise limit of functions each depending continuously on finitely many values of the sources. *Continuous:* Lemma 2.17 on the open convex U_ρ.

(v) *Lattice translations at cone points.* Let $b \in 5^{-k_0}\mathbb{Z}^3$, $|b|_\infty \leq M$, and ζ a cone point of $\mathcal{D}$ ($k_- = 0$, $q := k(\zeta) \geq 0$, $\|q\|_\infty < \sigma_+/2$). The translation sandwich of Lemma 8.3 at the cone point $(\mathrm{i}f_\zeta, q)$, with the real scalar source q in the slot of the cone (lattice covariance (20): τ_b preserves Λ_k and S_{Λ_k} for $k \geq k_0$, and King's cube translated by $-b$ is interlaced between King's cubes of indices $J \mp 1$ by the kill steps and cut-outs of its proof), gives $Z_{k,C_{J-1}}^{a,K}(\zeta) \leq Z_{k,C_J}^{a,K}(\tau_b \cdot \zeta) \leq Z_{k,C_{J+1}}^{a,K}(\zeta)$ for $J \geq J_0 + 1$. Let $k \to \infty$ by Theorem 13.10(i) for the three families (the translated family is supported at distance > 1 from $\mathbb{R}^3 \setminus C_{J_0+1}$), then $J \to \infty$ by (iii), the three terms being cone points of their tubes: $G_\infty^K(\tau_b \cdot \zeta) = G_\infty^K(\zeta)$ at the cone points of $\mathcal{D}$. *All $\zeta \in \mathcal{D}$.* For the split family of Step 8 of the proof of Theorem 28.5 and its translate (with the sets of that step: $\mathcal{D}''$ the convex tube of the split family, $\Omega''_+ \subset \mathcal{D}''$ its points with imaginary curvature coefficients and positive scalar coefficients, and $\mathcal{D}' \subset \mathcal{D}_{\sigma_+,H}$ the sub-tube on which the diagonal $w^+ = w$, $w^- = -w$ recovers the original family), $F_\infty^{K,\tau_b} - F_\infty^K$ is holomorphic on the connected $\mathcal{D}''$ and vanishes on Ω''_+, whose points are cone points of the split family's tube and which is an open subset of the maximal totally real subspace $(\mathrm{i}\mathbb{R})^n \times \mathbb{R}^{2p}$; by (C2) it vanishes on $\mathcal{D}''$, and the diagonal gives $G_\infty^{K,\tau_b} = G_\infty^K$ on $\mathcal{D}'$. Then $G_\infty^{K,\tau_b} - G_\infty^K$ is holomorphic on the connected $\mathcal{D}$ ((iii) for both families) and vanishes on the open set $\mathcal{D}'$, hence on $\mathcal{D}$. *All $b \in \bigcup_k 5^{-k}\mathbb{Z}^3$:* iterate; the tube is the same at every step. *All $b \in \mathbb{R}^3$ at real points.* For real $\zeta \in \mathcal{D}$ the source pair $s := (\sum_i z_i f_i, \sum_j w_j h_j)$ lies in some U_ρ, $\tau_b s \in U_{\rho'}$ for $|b| \leq \rho' - \rho$, and $b \mapsto \tau_b s$ is continuous into $E_{\rho'}$; by (iv) and the density of $\bigcup_k 5^{-k}\mathbb{Z}^3$, $\Phi^{a,K}(\tau_b s) = \Phi^{a,K}(s)$. *Complex $\zeta \in \mathcal{D}$.* $G_\infty^{K,\tau_b} - G_\infty^K$ is holomorphic on $\mathcal{D}$ and vanishes on the real points of $\mathcal{D}$, a non-empty open subset of $\mathbb{R}^{n+p}$; (C2) gives the claim. □

Remark 13.12 (King's convention). On King's RG cube $\mathrm{ND}_{k,J}^K$ the negative leg holds as stated: by Lemma 4.8, the wall insertion of Lemma 13.5 runs on $\mathrm{ND}_{k,J}^K$ with factors that are comparison box states killed on $X_i \cap T_J$, and every cell on which a cut-off source $8\chi_s q$ does not vanish is a full cell inside C_J meeting no killed site, because χ_s vanishes within distance 2 of every cell wall; so Lemmas 13.6 to 13.8 and Proposition 13.9 apply with the constant K_- unchanged. The direction matters: killing sites of a cell increases a class-$\mathcal{D}$ functional (Lemma 4.8(e)), so B_{cell} does not control a source-carrying cell with killed sites. For the positive leg, Remark 4.9 prepends the interlacing inequality $\langle\cdot\rangle_{k,J}^K \leq \langle\cdot\rangle_{k,J}^{(a)}$ on class $\mathcal{I}$ (Step 1 of Lemma 4.2) to Propositions 13.1 and 13.2, so Corollary 13.3 holds for $N_{k,C_J}^K(0, q)$ at every (k, J) and, for the fixed-cube limits, in the form $\limsup_k Z_{k,J}^{a,K}(0, q) \leq e^{\kappa\|q\|_{L^1}}$. The assembly of (37) on ND^K from these pieces, with the check that every step of its proof runs in the right direction on King's convention, is Steps 1–6 of the proof of Theorem 28.5 (the bound (73)); with the fixed-cube limits of Theorem B it gives the limit statements Theorem 13.10(i),(ii) and Lemma 13.11 for King's state, on which the Wick sector of the main theorem is stated (Proposition 28.6).

The following remark does for the tube bound and its limit statements what Remark 7.8 does for the regularity statements of Section 7; the two cube conventions and the role of the comparison family are described before that remark.

Remark 13.13 (the comparison family). The bound (37) is proved for the comparison convention, and the arguments behind Theorem 13.10(i),(ii) and Lemma 13.11 use of the cube functionals only such a bound, the fixed-cube limits at the real points of the tube and the cube monotonicity on the cone (for the comparison family the first inequality of Lemma 2.13 and the sandwich of Lemma 8.1). As noted in Section 3, they apply to the comparison family wherever its fixed-cube limits exist at the real points of the tube. Its limit states along continuum subsequences, which exist without that clause, are treated in Section 14 and coincide with G^K_∞ (Proposition 28.7).

14. Continuum subsequences and LIM states

The axioms of the preceding sections rest on the existence of the fixed-cube limits $\lim_k$: on Theorem A in the curvature sector, on Theorem B on the Wick tube, both proved in the existence part (Sections 5 and 6) on King's convention. This section shows that the axioms hold without the existence theorems, for the limit state along any continuum subsequence of spacings, and identifies every such state with the iterated limit. The k-uniform bounds of Lemma 7.3 and (37) make the fixed-cube functionals normal families, so one subsequence of spacings serves every cube and every family at once. Along it the cube limit $J \to \infty$ exists because the cone functionals are monotone in the cube at fixed k (Lemma 2.13). The limit state of such a subsequence (a *LIM state*) satisfies every axiom proved above, with no fixed-cube limit used in the curvature sector and, on the Wick tube, for every admissible scheme at couplings in the coupling region of Definition 2.1 (through two properties of the scheme, the cell bound and the scheme bound, proved below). To this, Theorem A (resp. Theorem B) adds exactly the identification with the iterated limit and the independence of the subsequence. The section is written on the comparison convention ND$^{(a)}$ (Definition 2.2), on which the tube bound (37) and the rotation part are written. King's convention is Corollary 14.15, and the Wick-sector LIM states are identified with King's tube object in Proposition 28.7. Notation is that of Definitions 2.5, 2.8 and 13.4 and (36): $a^{(a)}_{k,J}(z) = \langle e^{\sum_j z_j F(f_j)}\rangle^{(a)}_{k,J}$ for a family $f = (f_1, \dots, f_N)$ of real 2-forms; $Z^a_{k,J}(\zeta)$ for $\mathcal{F} = (f_1, \dots, f_n; h_1, \dots, h_p)$ and $\zeta = (z, w) \in \mathbb{C}^{n+p}$ in the scheme a; $g(\zeta)$, $k(\zeta) = k_+ - k_-$, $\mathsf{a}(\zeta)$; $U_J := \{x : \mathrm{dist}_\infty(x, \mathbb{R}^3 \setminus C_J) > 1\}$, $J_0(f)$ the least J with all supports in U_J, and $R(\mathcal{F})$ the least R with all supports in $Q_R = [-R, R]^3$. Two standard facts [22] are used by name, the *identity theorem* and *Montel–Vitali*: they are the facts (C2) and (C3) stated before Theorem 13.10.

14.1. Definitions, the cell bound and the scheme bound.

This subsection defines continuum subsequences and their LIM states, and isolates the two properties of the scheme, the cell bound and the scheme bound, under which the Wick sector is treated.

Definition 14.1 (continuum subsequences)**.** An increasing sequence $\mathbf{k} = (k_m)_{m\ge1} \subset \mathbb{N}$ of spacings is a *continuum subsequence* (of the comparison family) if for every finite family f of real 2-forms and every $J \ge J_0(f)$ the sequence $(a^{(a)}_{k_m,J})_m$ converges locally uniformly on $\mathbb{C}^N$; the limit is written $Z^{\mathbf{k}}_J(\sum_j z_j f_j, 0)$ and depends on the family only through the total complex source, as each $a^{(a)}_{k,J}$ does. Fix a scheme a. $\mathbf{k}$ is a *continuum subsequence for the Wick sector* (Form (B)) if it is a continuum subsequence and, for every finite family $\mathcal{F}$ of real 2-forms and scalars and every J with $MJ > R(\mathcal{F}) + \ell_c$, the sequence $(Z^a_{k_m,J})_m$ converges locally uniformly on the tube $\mathcal{D}_{\sigma_+,H}$ of $\mathcal{F}$; the limit is $Z^{a,\mathbf{k}}_J$. (The strict inequality replaces the threshold $J_1(R)$ of Theorem 13.10 by at most one cube step; only tails in J are used.) Convergence on all of $\mathbb{C}^{n+p}$ for the same J is not used here.

Definition 14.2 (LIM states)**.** The *LIM state* of a continuum subsequence $\mathbf{k}$ is $Z^{\mathbf{k}}_\infty := \lim_J Z^{\mathbf{k}}_J$ on $\mathbb{C}^N$ (curvature sector), resp. $Z^{a,\mathbf{k}}_\infty := \lim_J Z^{a,\mathbf{k}}_J$ on the tube (Wick sector); Lemma 14.8 gives existence. A subsequence of a continuum subsequence is a continuum subsequence with the same fixed-cube limits, hence the same LIM state.

Definition 14.3 (the limit state of the rotation theorem)**.** For a family f and $J \geq J_0(f)$ let L^f_J be the set of locally uniform limits on $\mathbb{C}^N$ of subsequences of $(a^{(a)}_{k,J})_k$. The rotation theorem (Theorem C) gives an entire $Z^{\mathrm{LIM}}_\infty(\cdot, 0)$ to which *every* choice $\zeta_J \in L^f_J$ converges locally uniformly as $J \to \infty$, the subsequence being allowed to depend on J. With J-dependent subsequences one only has $\liminf_k a^{(a)}_{k,J}(\mathrm{i}u) \leq \zeta_J(\mathrm{i}u) \leq \limsup_k a^{(a)}_{k,J}(\mathrm{i}u)$, and the J-limit is pinned only if the J-limits of the two bounds agree, which is the content of that theorem. A LIM state in the sense of Definition 14.2 uses one $\mathbf{k}$ common to all J and all families, along which the fixed-cube limits are monotone in J (Lemma 14.8), and uses no fixed-cube limit; Proposition 14.13(a) relates the two.

The limit states along continuum subsequences rest on bounds uniform in the spacing, not on fixed-cube limits. In the curvature sector the uniform bound is the finite-volume regularity bound of Theorem 7.4, which comes from Gaussian domination (Lemma 2.11) and needs no existence statement. On the Wick tube the bound (37) of Theorem 13.10 is used. It rests on the sign clause (T3) and the torus theorems (Proposition 12.6, Lemma 12.7, and Definition 2.1, which give the extensive bound (I2) with amplitude $s_1 \in (0, \sigma_1)$ and constant C_U through Proposition 12.8; (I3), lattice reflection positivity of King's tori, is Proposition 12.2, and (I4), the correlation inequalities, is Lemma 2.10) and on one instance of Theorem B, isolated as follows.

We say that the scheme a has the *cell bound* ($\mathrm{B}_{\mathrm{cell}}$) if $B_{\mathrm{cell}} := \sup_k \ln N^a_{k,C_{L_c}}(0, -2H'\chi_0) < \infty$, for the one cell C_{L_c} (side $\ell_c \geq 10$) and the one negative Wick source $-2H'\chi_0$ of Definition 13.4. We say that the scheme a has the *scheme bound* (SCH) if either $a = z$, the reference scheme in which Lemma 12.7 is computed, or $\sup_k |z_k - a_k| < \infty$. The cell bound is the boundedness form of Theorem B at one cube and one source, and the only place where that theorem enters the proof of (37) (through Lemmas 13.6 and 13.7 and Proposition 13.9; K_- is built from B_{cell}). Both properties hold for the schemes of the main theorem:

Lemma 14.4 ($(\mathrm{B}_{\mathrm{cell}})$, the cell bound)**.** *Let the couplings lie in the coupling region of Definition 2.1 and let a be an admissible scheme (Definition 2.7). Then a has the cell bound: $B_{\mathrm{cell}} = \sup_k \ln N^a_{k,C_{L_c}}(0, -2H'\chi_0) < \infty$.*

Proof. By Definition 13.4 the cell $C_{L_c} = [-ML_c, ML_c)^3$ is a King cube and $\operatorname{supp}\chi_0$ lies at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_{L_c}$; the source $q := 2H'\chi_0$ is a non-negative function of $C^\infty_c(\mathbb{R}^3)$ with this support condition. At couplings in the coupling region the hypotheses (I2) and (I3) of the existence theorem with scalar sources hold (Proposition 12.8, Proposition 12.2), so Corollary 6.27(b) applies and gives $\sup_k N^z_{k,C_{L_c}}(0, -q) < \infty$ for the comparison-convention functional in the reference scheme z. For a scheme a with $\sup_k |a_k - z_k| < \infty$ the two functionals differ by the factor $e^{-(a_k - z_k)\mathsf{S}_k(-q)}$ (Remark 2.6), bounded uniformly in k since $\mathsf{S}_k(q) \to \int q$; this is Remark 6.28, and an admissible scheme has $\sup_k |z_k - a_k| < \infty$ by Definition 2.7. Hence $\sup_k N^a_{k,C_{L_c}}(0, -2H'\chi_0) < \infty$, and $B_{\mathrm{cell}} < \infty$ because each term is positive (Remark 6.29). □

Lemma 14.5 ((SCH), the scheme bound)**.** *An admissible scheme has the scheme bound. If $a \neq z$ has the scheme bound, the constants of (37) hold with C_U replaced by $C_U + s_1 \sup_k |z_k - a_k|$.*

Proof. For an admissible scheme either $a = z$ or $\lim_k(z_k - a_k)$ exists, so $\sup_k |z_k - a_k| < \infty$ (Definition 2.7). The replacement of the constant is the last clause of Proposition 12.8, carried through Theorem 13.10. □

Proposition 14.6 ((UNIQ), independence of the subsequence)**.** *Any two continuum subsequences* $\mathbf{k}, \mathbf{k}'$ *have the same LIM state:* $Z^{\mathbf{k}}_\infty(\cdot,0) = Z^{\mathbf{k}'}_\infty(\cdot,0)$ *for every family.*

Proof. This is Proposition 14.13(a) below, from the rotation theorem (Theorem C): every continuum subsequence's LIM state is the limit state $Z^{\mathrm{LIM}}_\infty(\cdot,0)$ of that theorem. It also follows from Theorem A through Corollary 4.4(iii) (Corollary 14.15(b)): every LIM state of the comparison family is King's state Z^K_∞, the infinite-volume state of King's construction [25, 26]. □

The proofs of the axioms below hold for each LIM state separately and do not use (UNIQ). It enters only in the Euclidean clause (vii) of Theorem 14.9, through Proposition 14.13(a). Without the rotation theorem and the existence theorem one would only know that every LIM state lies on $(\mathrm{i}\mathbb{R})^N$ between $\lim_J \liminf_k a^{(a)}_{k,J}$ and $\lim_J \limsup_k a^{(a)}_{k,J}$ (both exist, by monotonicity in J), with equality of the LIM states if these coincide at every real argument, an entire function being determined on $(\mathrm{i}\mathbb{R})^N$.

14.2. **Existence of continuum subsequences and the cube limit.** The uniform bounds make the fixed-cube functionals normal families, so every increasing sequence of spacings has a subsequence along which the cube functionals converge for every family and cube. The cube limit $J \to \infty$ then exists by the monotonicity on the cone.

Lemma 14.7 (continuum subsequences exist)**.**

(a) (Curvature sector; no fixed-cube limit used.) *Every increasing sequence of spacings has a continuum subsequence. For every continuum subsequence* $\mathbf{k}$*, family* f *and* $J \geq J_0(f)$*:* $Z^{\mathbf{k}}_J(\cdot,0)$ *is entire,* $|Z^{\mathbf{k}}_J(\sum_j z_j f_j, 0)| \leq e^{\|g(z)\|_2^2}$*,* $Z^{\mathbf{k}}_J(x,0)$ *is real and* ≥ 1 *at real* x*, and* $0 \leq Z^{\mathbf{k}}_J(\mathrm{i}t,0) \leq 1$ *at imaginary* $\mathrm{i}t$*.* Several models at once: *for every countable set* $\mathcal{M}$ *of lattice models of King's type (Neumann–Dirichlet cube states with a site-independent real bare scalar coefficient, photon mass* $\mu > 0$*,* $\lambda > 0$*; for instance the models at the photon masses of a countable set, with the counterterm fixed) one* $\mathbf{k}$ *is a continuum subsequence for every member of* $\mathcal{M}$*.*

(b) (Wick sector, Form (B).) *Let the couplings lie in the coupling region of Definition 2.1 and let the scheme a have the cell bound* ($\mathrm{B}_{\mathrm{cell}}$) *and the scheme bound* (SCH) *(both hold for every admissible scheme, Lemmas 14.4 and 14.5). Then every increasing sequence of spacings has a continuum subsequence for the Wick sector, and for every such* $\mathbf{k}$*, family* $\mathcal{F}$ *and* J *with* $MJ > R(\mathcal{F}) + \ell_c$*,* $Z^{a,\mathbf{k}}_J$ *is holomorphic on* $\mathcal{D}_{\sigma_+,H}$ *and*

$$|Z^{a,\mathbf{k}}_J(\zeta)| \leq \exp\left(\|g(\zeta)\|_2^2 + \kappa\|k_+(\zeta)\|_{L^1} + K_-|k_-(\zeta)|_w\right) \qquad (\zeta \in \mathcal{D}_{\sigma_+,H}), \tag{38}$$

with the constants of (36)*.* Several models at once: *for a countable set of models all satisfying* (37) *with one set of constants, one* $\mathbf{k}$ *serves all.*

The subsequence is chosen in Step 2 below and nowhere else (Remark 14.16).

Proof. For each of (a) and (b) the proof has five steps: a countable core of families, a diagonal extraction by Montel's theorem, the extension to every family at imaginary coefficients by a Lipschitz bound and density, the extension to all complex coefficients (to the whole tube in (b)) by the Montel–Vitali theorem, and the case of several models.

(a). Statements used: Lemma 7.3 ($\sup_{k\geq 0,\, J\geq J_0(f)} \sup_K |a^{(a)}_{k,J}| < \infty$ for every family and compact $K \subset \mathbb{C}^N$); Lemma 7.1 at second order ($\langle F(u)^2\rangle^{(a)}_{k,J} \leq 2\|u\|^2_{2,k}$ for real u with plaquettes in X_J); charge conjugation $(A,\vartheta) \mapsto (-A,-\vartheta)$, a symmetry of every Neumann–Dirichlet state, so $\langle F(u)\rangle^{(a)}_{k,J} = 0$ and $\langle e^{\mathrm{i}F(u)}\rangle^{(a)}_{k,J}$ is real. *Step 1 (a countable core).* Let $\mathcal{B}$ be the countable set of finite unions of closed boxes with rational corners, for $B \in \mathcal{B}$ let $S_0(B)$ be a countable

$\|\cdot\|_\infty$-dense subset of $C_c^\infty(\operatorname{int} B;\Lambda^2)$, and let $\mathcal{P}$ be the countable set of pairs (J,u'), u' a finite family from $\bigcup_B S_0(B)$, $J\ge J_0(u')$. *Step 2 (Montel and the diagonal).* For $(J,u')\in\mathcal{P}$, $(a^{u'}_{k,J})_k$ is locally bounded on $\mathbb{C}^N$ uniformly in k (Lemma 7.3), hence normal; enumerate $\mathcal{P}$, extract successively, and let $\mathbf{k}=(k_m)$ be the diagonal sequence: $(a^{u'}_{k_m,J})_m$ converges locally uniformly for every $(J,u')\in\mathcal{P}$. *Step 3 (every family, imaginary coefficients).* Let f be a family, $J\ge J_0(f)$, and $B\in\mathcal{B}$ with $\bigcup_j\operatorname{supp} f_j\subset\operatorname{int} B\subset B\subset U_{J_0(f)}$ (so every $u'\in S_0(B)$ is admissible at every $J\ge J_0(f)$). For real 2-forms u,u',g supported in B and every k with $\varepsilon_k\le 1$,

$$|\langle e^{\mathrm{i}F(u)}\rangle^{(a)}_{k,J}-\langle e^{\mathrm{i}F(u')}\rangle^{(a)}_{k,J}|\le\langle F(u-u')^2\rangle^{(a)}_{k,J}{}^{1/2}\le\sqrt{2}\|u-u'\|_{2,k},$$
$$\|g\|_{2,k}^2\le V_B\|g\|_\infty^2,\quad V_B:=3\,\mathrm{vol}(N_{1/2}(B)),$$

the first by $|e^{\mathrm{i}x}-1|\le|x|$ and the second-order bound, the second because the cubes of side ε_k about the centres of the plaquettes of one orientation on which $g\ne0$ are disjoint and lie in the closed ℓ^∞-neighbourhood $N_{1/2}(B)$. Given $t\in\mathbb{R}^N$ and $\eta>0$ pick $u'\in S_0(B)$ with $\|\sum_j t_jf_j-u'\|_\infty\le\eta$; then $|a^f_{k_m,J}(\mathrm{i}t)-a^f_{k_{m'},J}(\mathrm{i}t)|\le 2\sqrt{2V_B}\,\eta+|\langle e^{\mathrm{i}F(u')}\rangle^{(a)}_{k,J_{k_m}}-\langle e^{\mathrm{i}F(u')}\rangle^{(a)}_{k,J_{k_{m'}}}|$, and the last term tends to 0 by Step 2. So $(a^f_{k_m,J}(\mathrm{i}t))_m$ converges for every t. *Step 4 (all complex coefficients).* $(a^f_{k_m,J})_m$ is locally bounded on $\mathbb{C}^N$ and converges on the maximal totally real $(\mathrm{i}\mathbb{R})^N$; Montel–Vitali gives locally uniform convergence to an entire $Z^{\mathbf{k}}_J(\cdot,0)$. The bound $|a^{(a)}_{k,J}(z)|\le e^{\|g(z)\|^2_{2,k}}$ (Theorem 7.4) passes to the limit with $\|g\|_{2,k}\to\|g\|_2$ (2); realness at real points passes; $a^{(a)}_{k,J}(x)=\langle e^{F(u_x)}\rangle^{(a)}_{k,J}\ge e^{\langle F(u_x)\rangle^{(a)}_{k,J}}=1$ by Jensen, and $[1,\infty)$ is closed; $0\le a^{(a)}_{k,J}(\mathrm{i}t)\le1$ by the current representation of Lemma 2.11(b), $\langle e^{\mathrm{i}\langle v,A\rangle}\rangle^{(a)}_{k,J}=e^{-\langle v,Cv\rangle/2}\,\mathbb{E}^\sharp\cosh\langle Cv,\kappa\rangle\ge0$, and $|e^{\mathrm{i}F}|=1$. Dependence only on the total source holds at every (k,J), hence in the limit. *Step 5 (several models).* The constant of Lemma 7.3, the bound $e^{\|g\|^2_{2,k}}$ and the Lipschitz constant $\sqrt{2}$ are model-free: Theorem 7.4 is Lemma 2.11(a) for a measure of the class of Lemma 2.10 plus Lemma 7.1 for the Neumann–Dirichlet photon covariance, valid for every real site-independent bare coefficient (the matter weight does not enter) and every $\mu>0$. For countable $\mathcal{M}$ run Step 2 over $\mathcal{M}\times\mathcal{P}$ and Steps 3–4 member by member.

(b). Statements used: under (I2)–(I4) and $B_{\mathrm{cell}}<\infty$, for every family with supports in Q_R, every k, $J\ge J_1(R)$ and $\zeta\in\mathcal{D}_{\sigma_+,H}$, (37) gives $|Z^a_{k,J}(\zeta)|\le Z^a_{k,J}(\operatorname{Re}\zeta)\le\exp(\|g\|^2_{2,k}+\kappa\|k_+\|_{1,k}+K_-|k_-|_w)$, with (I2) from (T3) by Proposition 12.8 and (SCH) for the scheme; on compacts of the tube the right side is bounded uniformly in k (Lemma 2.9). *Steps 1–2.* As in (a), with $S_0^{\mathsf{W}}(B)$ a countable $\|\cdot\|_\infty$-dense subset of $C_c^\infty(\operatorname{int}B;\Lambda^2)\times C_c^\infty(\operatorname{int}B)$ and $\mathcal{P}^{\mathsf{W}}$ the countable set of pairs $(J,\mathcal{F}')$, $\mathcal{F}'$ a finite family from $\bigcup_B S_0^{\mathsf{W}}(B)$, $MJ>R(\mathcal{F}')+\ell_c$. For such a pair $J\ge J_1(R)$ for every $R\in(R(\mathcal{F}'),MJ-\ell_c]$, so (37) holds for $\mathcal{F}'$ at every k and $(Z^a_{k,J})_k$ is normal on the tube of $\mathcal{F}'$; the diagonal over $\mathcal{P}^{\mathsf{W}}$, together with that of (a), gives $\mathbf{k}$. *Step 3 (every family, imaginary coefficients).* Let $\mathcal{F}=(f_i;h_j)$, $MJ>R(\mathcal{F})+\ell_c$, $R_J:=MJ-\ell_c>R(\mathcal{F})$, so that all supports lie in the open cube $Q^\circ_{R_J}$ and $J\ge J_1(R_J)$; choose $B\in\mathcal{B}$ with the supports in $\operatorname{int}B\subset B\subset Q^\circ_{R_J}$, so that $(J,\mathcal{F}')\in\mathcal{P}^{\mathsf{W}}$ for every $\mathcal{F}'$ from $S_0^{\mathsf{W}}(B)$. For $(s,s')\in\mathbb{R}^{n+p}$ put $\sigma:=(\sum_i s_if_i,\sum_j s'_jh_j)$, so that $Z^a_{k,J}(\mathrm{i}s,\mathrm{i}s')=\langle e^{\mathrm{i}\Phi^a_k(\sigma)}\rangle^{(a)}_{k,J}$ with $\Phi^a_k(g,h):=F_{\Lambda_k}(g)+W^a_{\Lambda_k}(h)$. *Lipschitz bound.* For a real pair $q=(g_q,h_q)$ supported in Q_{R_J}, $(t_1,t_2)\mapsto Z^a_{k,J}(t_1g_q,t_2h_q)$ is holomorphic, and on the polydisc $|t_1|\le\|g_q\|^{-1}_{2,k}$, $|t_2|\le\sigma_0/|h_q|_w$ one has $\|k_\pm\|_\infty\le|\operatorname{Re}t_2|\,\|h_q\|_\infty\le\sigma_0<H/2<\sigma_+/2$, so the polydisc lies in the tube and (37) gives $|Z^a_{k,J}|\le e^{1+K'_{\mathrm{lat}}\sigma_0}$ there, $K'_{\mathrm{lat}}:=\kappa C'_0+K_-$ ($\|k_+\|_{1,k}\le C'_0|k_+|_w$ by Lemma 2.9). The Taylor coefficients at 0 are the second moments of

$F(g_q)$ and $W^a(h_q)$, and Cauchy's estimates give

$$\langle \Phi^a_k(q)^2\rangle^{(a)}_{k,J}{}^{1/2} \le C\,(\|g_q\|_{2,k}+|h_q|_w), \qquad C:=\sqrt{2}\,e^{(1+K'_{\rm lat}\sigma_0)/2}\max(1,\sigma_0^{-1}),$$

uniformly in k, J and R_J; hence $|Z^a_{k,J}(\mathrm{i}\sigma)-Z^a_{k,J}(\mathrm{i}\sigma')|\le C(\|g_{\sigma-\sigma'}\|_{2,k}+|h_{\sigma-\sigma'}|_w)$ for real pairs supported in Q_{R_J}. *Density.* Given $\eta>0$ pick $\sigma'\in S_0^{\mathsf W}(B)$ with $\|g_\sigma-g_{\sigma'}\|_\infty, \|h_\sigma-h_{\sigma'}\|_\infty\le\eta$; with V_B and $|h|_w\le(1+3R_J^2)^2\|h\|_\infty$ for h supported in Q_{R_J}, $|Z^a_{k,J}(\mathrm{i}\sigma)-Z^a_{k,J}(\mathrm{i}\sigma')|\le C(\sqrt{V_B}+(1+3R_J^2)^2)\eta$ for every k, while $Z^a_{k_m,J}(\mathrm{i}\sigma')$, the core functional at the point (i,i) of its tube, converges by Step 2. So $(Z^a_{k_m,J})_m$ converges on $(\mathrm{i}\mathbb{R})^{n+p}$. *Step 4 (the tube).* $(Z^a_{k_m,J})_m$ is locally bounded on the connected open $\mathcal{D}_{\sigma_+,H}$ and converges on the maximal totally real $(\mathrm{i}\mathbb{R})^{n+p}\subset\mathcal{D}_{\sigma_+,H}$; Montel–Vitali gives locally uniform convergence to a holomorphic $Z^{a,\mathbf{k}}_J$, and (37) passes to the limit with $\|g\|_{2,k}\to\|g\|_2$, $\|k_+\|_{1,k}\to\|k_+\|_{L^1}$ (Riemann sums; k_+ is continuous with compact support) and sampled $|k_-|_w\le$ continuum $|k_-|_w$: this is (38). *Step 5 (several models).* With one set of constants in (37) for all members, C is model-free and Step 2 runs over the product of the model set with $\mathcal{P}^{\mathsf W}$. □

Lemma 14.8 (the cube limit along $\mathbf{k}$ is a full limit)**.** *Let* $\mathbf{k}$ *be a continuum subsequence.* (Curvature sector.) *For every family, $Z^{\mathbf{k}}_J(\cdot,0)\to Z^{\mathbf{k}}_\infty(\cdot,0)$ locally uniformly on $\mathbb{C}^N$ as $J\to\infty$; the limit is entire, $|Z^{\mathbf{k}}_\infty(\sum_j z_jf_j,0)|\le e^{\|g(z)\|_2^2}$, real and ≥ 1 at real points, in $[0,1]$ at imaginary points; $J\mapsto Z^{\mathbf{k}}_J(\mathrm{i}t,0)$ is nondecreasing, so $Z^{\mathbf{k}}_J\le Z^{\mathbf{k}}_\infty$ on $(\mathrm{i}\mathbb{R})^N$.* (Wick sector, Form (B).) *Under the hypotheses of Lemma 14.7(b), $Z^{a,\mathbf{k}}_J\to Z^{a,\mathbf{k}}_\infty$ locally uniformly on $\mathcal{D}_{\sigma_+,H}$; the limit is holomorphic, obeys (38), is >0 at the real points of the tube, and $Z^{a,\mathbf{k}}_J\le Z^{a,\mathbf{k}}_\infty$ at the cone points ($z\in(\mathrm{i}\mathbb{R})^n$, w real, $k(\zeta)\ge 0$).*

Proof. We first treat the curvature sector, where the monotonicity in J on the imaginary axes and Montel's theorem give the limit; we then treat the Wick sector by passing to a split family with non-negative scalar sources, proving monotone convergence on its positive cone, recovering the original family on the diagonal, and finally proving the positivity at real tube points and the monotonicity at cone points.

Curvature sector. For $t\in\mathbb{R}^N$, $J\ge J_0(f)$ and every k: $a^{(a)}_{k,J}(\mathrm{i}t)\le a^{(a)}_{k,J+1}(\mathrm{i}t)\le 1$ (first inequality of Lemma 2.13 with $Y=C_{J+1}\cap\Lambda_k$, $X=X_J$; the cut-out factor is the Neumann–Dirichlet state on C_J because δm_ε^2 is site-independent; realness by charge conjugation). Both sides converge along $\mathbf{k}$, so $J\mapsto Z^{\mathbf{k}}_J(\mathrm{i}t,0)$ is nondecreasing, bounded by 1, convergent. $\{Z^{\mathbf{k}}_J\}_J$ is locally bounded on $\mathbb{C}^N$ by the J-free bound $e^{\|g(z)\|_2^2}$ and converges on $(\mathrm{i}\mathbb{R})^N$; Montel–Vitali gives the entire limit with the same bound; realness, ≥ 1 at real points and $[0,1]$ at imaginary points pass to the limit.

Wick sector. (1) *Split family.* Write $h_j=h_j^+-h_j^-$, $h_j^-:=\|h_j\|_\infty\chi_j$, $h_j^+:=h_j+h_j^-\ge 0$, with $\chi_j\in C_c^\infty$, $0\le\chi_j\le 1$, $\chi_j=1$ on $\operatorname{supp}h_j$, $\operatorname{supp}\chi_j\subset N_{1/2}(\operatorname{supp}h_j)$; put $F_{k,J}(z,w^+,w^-):=Z^a_{k,J}(\sum_i z_if_i,\sum_j(w_j^+h_j^++w_j^-h_j^-))$ and $\mathcal{D}'':=\{|\operatorname{Re}w_j^\pm|<\delta_j\}$ with $\sum_j\delta_j(\|h_j^+\|_\infty+\|h_j^-\|_\infty)<H/2$, a convex subset of the split family's tube. By Lemma 14.7(b) for the split family, $F_{k_m,J}\to F^{\mathbf{k}}_J$ locally uniformly on $\mathcal{D}''$ for $MJ>R(\mathcal{F})+\frac12+\ell_c$, with (38). (2) On $\Omega_+:=\mathcal{D}''\cap((\mathrm{i}\mathbb{R})^n\times(0,\infty)^{2p})$ the observable has imaginary curvature coefficients and non-negative total scalar source, so $J\mapsto F_{k,J}$ is nondecreasing at every k (first inequality of Lemma 2.13 with Lemma 2.14(a), the scheme multiplier being common); along $\mathbf{k}$, $J\mapsto F^{\mathbf{k}}_J$ is nondecreasing and bounded on Ω_+, hence convergent. (3) Ω_+ is a non-empty open subset of the maximal totally real $(\mathrm{i}\mathbb{R})^n\times\mathbb{R}^{2p}$ and $\mathcal{D}''$ is connected: $F^{\mathbf{k}}_J\to F^{\mathbf{k}}_\infty$ locally uniformly on $\mathcal{D}''$. (4) At $w^+=w$, $w^-=-w$ the lattice functionals equal $Z^a_{k,J}(\zeta)$ exactly (equal total sources), so $Z^{a,\mathbf{k}}_J$ converges locally uniformly on $\mathcal{D}':=\{|\operatorname{Re}w_j|<\delta_j\}\times\mathbb{C}^n\subset\mathcal{D}_{\sigma_+,H}$; $\{Z^{a,\mathbf{k}}_J\}_J$ is locally bounded on $\mathcal{D}_{\sigma_+,H}$ by the J-free (38) and converges on the open $\mathcal{D}'$; Montel–Vitali on the connected tube gives $Z^{a,\mathbf{k}}_J\to Z^{a,\mathbf{k}}_\infty$, holomorphic,

with (38). (5) *Positivity at real tube points.* For real $\zeta \in \mathcal{D}_{\sigma_+,H}$ pick $\epsilon > 0$ with $-\epsilon\zeta \in \mathcal{D}_{\sigma_+,H}$ and let $C_\epsilon := \sup_k(\|g\|^2_{2,k}+\kappa\|k_+\|_{1,k}+K_-|k_-|_w)|_{-\epsilon\zeta} < \infty$, the exponent of (37) at $-\epsilon\zeta$ (bounded in k by Lemma 2.9, free of J). At every k, $t \mapsto \ln Z^a_{k,J}(t\zeta)$ is convex on $\mathbb{R}$ (Hölder, Lemma 2.14(b)) and vanishes at $t = 0$, so $1 \le Z^a_{k,J}(\zeta)^{\epsilon/(1+\epsilon)} Z^a_{k,J}(-\epsilon\zeta)^{1/(1+\epsilon)} \le Z^a_{k,J}(\zeta)^{\epsilon/(1+\epsilon)} e^{C_\epsilon/(1+\epsilon)}$, i.e. $Z^a_{k,J}(\zeta) \ge e^{-C_\epsilon/\epsilon}$ for every k and J; the bound passes along $\mathbf{k}$ and then in J: $Z^{a,\mathbf{k}}_\infty(\zeta) \ge e^{-C_\epsilon/\epsilon} > 0$. (The product inequality of Lemma 2.14(c) is not used: the tube is asymmetric, $H = \sigma_+/16$, and for real $\xi \in \mathcal{D}_{\sigma_+,H}$ with $\|k_+(\xi)\|_\infty \ge H/2$ the point $-\xi$ lies outside it, where Definition 14.1 gives no limit.) (6) At a cone point of the original family $J \mapsto Z^a_{k,J}(\zeta)$ is nondecreasing at every k (first inequality of Lemma 2.13 with Lemma 2.14(a)); pass along $\mathbf{k}$. □

14.3. **The axioms for LIM states.** This subsection states and proves the axioms for the LIM state of an arbitrary continuum subsequence: in the curvature sector with no fixed-cube limit used, in the Wick sector under the cell bound and the scheme bound.

Throughout this subsection, let $e > 0$, $\lambda, m, \mu_0 > 0$, $\varepsilon_k = 5^{-k}$, $C_J = [-MJ, MJ)^3$, and let $\mathrm{ND}^{(a)}_{k,J}$ be the states of Definition 2.2. For a continuum subsequence $\mathbf{k}$ (Definition 14.1) and a finite family of real 2-forms the LIM state (Definition 14.2) is

$$Z^{\mathbf{k}}_\infty\Big(\sum_j z_j f_j, 0\Big) = \lim_{J\to\infty}\lim_{m\to\infty}\Big\langle e^{\sum_j z_j F_{\Lambda_{k_m}}(f_j)}\Big\rangle^{(a)}_{k_m,J}$$

with the Schwinger functions $S^{\mathbf{k}}_n(f_1,\dots,f_n) := \partial_{z_1}\cdots\partial_{z_n} Z^{\mathbf{k}}_\infty(\sum_j z_j f_j, 0)|_{z=0}$. In the Wick sector, $Z^{a,\mathbf{k}}_\infty$ is the LIM state of $\mathbf{k}$ for a family $\mathcal{F} = (f_1,\dots,f_n; h_1,\dots,h_p)$ of real 2-forms and real scalars on its tube $\mathcal{D}_{\sigma_+,H}$, with the joint Schwinger functions $S^{\mathbf{k}}_{n,p}(f;h) = \partial^{(n)}_z \partial^{(p)}_w Z^{a,\mathbf{k}}_\infty(0)$, and $\sigma_0 = H/4$, $K' = \kappa\pi^2 + K_-$ with the constants of (36).

Theorem 14.9 (LIM states of the comparison family at fixed photon mass)**.** ***Curvature sector (no fixed-cube limit used).*** *Every increasing sequence of spacings has a continuum subsequence. For every continuum subsequence* $\mathbf{k}$ *and finite family of real 2-forms,* $Z^{\mathbf{k}}_\infty(\sum_j z_j f_j, 0)$ *exists locally uniformly in* $z \in \mathbb{C}^N$*, is entire, and depends on the family only through the total source, and its Schwinger functions* $S^{\mathbf{k}}_n$ *satisfy:*

(i) (OS0.) $|Z^{\mathbf{k}}_\infty(\sum_j z_j f_j, 0)| \le e^{\|g(z)\|^2_2}$ *and* $\le \exp(\frac{1}{2}\mu_0^{-2}\|\sum_j(\mathrm{Re}\, z_j) d^* f_j\|^2_2)$ *(the first bound independent of* μ_0, e, λ, m*);* $|S^{\mathbf{k}}_n| \le (\sqrt{2}e)^n (n!)^{1/2} \prod_j \|f_j\|_2 \le (e\pi/2)^n (n!)^{1/2} \prod_j |f_j|_4$*; the* $S^{\mathbf{k}}_n$ *are tempered distributions, and* E0″ *holds with* $s = 4$ *and every* $\beta > \frac{1}{2}$*.*

(ii) (OS1.) $S^{\mathbf{k}}_n$ *is invariant under all translations and under* O_h *about every point,* F *transforming as a 2-form.*

(iii) (OS2.) *Reflection positivity about every coordinate plane* $\{x_\nu = c\}$*: for exponentials* $E_a = e^{\sum_l z_{al} F(f_{al})}$ *with arbitrary complex* z_{al} *and* f_{al} *supported in* $\{x_\nu > c\}$*,* $(S(\overline{\theta_c E_a} E_b))_{ab}$ *is positive semidefinite, and* $S(\theta_c(P)P) \ge 0$ *for polynomials* P *in such fields.*

(iv) (OS3.) $S^{\mathbf{k}}_n$ *is real, symmetric, and zero for odd* n*.*

(v) (The measure.) *For every even* $r \ge 14$ *there is exactly one Borel probability measure* $\mu^{\mathbf{k}}_F$ *on* $\mathcal{H}^{-r}(\mathbb{R}^3;\Lambda^2)$ *whose moments on* C^∞_c *are the* $S^{\mathbf{k}}_n$*; its characteristic functional is* $Z^{\mathbf{k}}_\infty(\mathrm{i}\,\cdot, 0)$ *extended by continuity to* $\mathcal{S}$*, and its image on* H^{-r}_{loc} *is the unique measure there with this characteristic functional and the unique one with these moments. For every* $\chi \in C^\infty_c$ *and box* Q *containing* $\mathrm{supp}\,\chi$ *in its interior, the laws of* $\chi F_{\Lambda_{k_m}}$ *under* $\mathrm{ND}^{(a)}_{k_m,J}$ *converge weakly on* $H^{-r}_{\mathrm{per}}(Q;\Lambda^2)$*,* first as $m \to \infty$ along $\mathbf{k}$ *at every fixed* $J \ge J_0(\chi)$ *and then as* $J \to \infty$*, to the law of* $\chi\Phi$ *under* $\mu^{\mathbf{k}}_F$*.*

(vi) (Weak OS4.) $\mu_F^{\mathbf{k}}$ *is* $\mathbb{R}^3 \rtimes O_h$*-invariant and satisfies the clauses of Theorem 11.5(b) with* $\mu_F^{\mathbf{k}}$ *in place of* μ_S^K.

(vii) (Euclidean clause.) $Z_\infty^{\mathbf{k}}$ *does not depend on* $\mathbf{k}$*, equals the limit state* Z_∞^{LIM} *of Definition 14.3, is* $\mathbb{R}^3 \rtimes O(3)$*-invariant for all complex coefficients (Theorem C), and the Euclidean forms of (ii) (full* $\mathbb{R}^3 \rtimes O(3)$*), (iii) (reflection positivity about every plane) and (vi) (almost every component* $\mathbb{R}^3 \rtimes O(3)$*-invariant, reflection positive about every plane, clustering in any manner, with a unique vacuum in every direction; the vanishing of a component's odd moments is not claimed) hold.*

Wick sector, Form (B). *Let the couplings lie in the coupling region of Definition 2.1 and let the scheme a have the cell bound* ($\mathrm{B_{cell}}$) *and the scheme bound* (SCH) *(Lemmas 14.4 and 14.5; every admissible scheme does). Every increasing sequence of spacings has a continuum subsequence for the Wick sector, and for every such* $\mathbf{k}$ *and family the LIM state* $Z_\infty^{a,\mathbf{k}}$ *exists locally uniformly on* $\mathcal{D}_{\sigma_+,H}$*, is holomorphic there, obeys (38), and (i)–(vi) hold for the joint Schwinger functions* $S_{n,p}^{\mathbf{k}}$ *with: in (i) the mixed bounds* $|S_{n,p}^{\mathbf{k}}| \le e^{K'\sigma_0}(\sqrt{2}e)^n (n!)^{1/2}(e/\sigma_0)^p\, p! \prod_i \|f_i\|_2 \prod_j |h_j|_w$*, and* E0″ *with* $s = 4$ *and every* $\beta > 1$*; in (ii)* W^a *a scalar, the identity holding on the tube; in (iii) exponentials with arbitrary complex curvature coefficients and Wick coefficient vectors* $\zeta_1, \dots, \zeta_N$ *of* amplitude $\mathsf{a}(\zeta_i) < H/2$ *(in particular all purely imaginary Wick coefficients), and the polynomial form; in (iv)* $S_{n,p}^{\mathbf{k}} = 0$ *for odd* n*; in (v) the measures on* $\mathcal{H}^{-r}(\mathbb{R}^3; \Lambda^2 \oplus \mathbb{R})$ *and* H^{-r}_{loc}*, the localised lattice fields* $\chi(F_{\Lambda_k}, W^a_{\Lambda_k})$ *converging along* $\mathbf{k}$ *and then in* J*; (vi) unchanged; and* no Euclidean clause.

Proof. Each clause is the corresponding proof for King's state (Theorems 7.6, 8.6, 10.5, 10.6 and 11.5, Corollary 8.4, and Proposition 10.7), read for the comparison family with the cone inequalities of Lemma 2.13 in place of the cube monotonicity of Corollary 4.3 (Remarks 7.8, 8.2, 10.10 and 13.13), with exactly one of three replacements: (α) a fixed-(k, J) identity or inequality between Neumann–Dirichlet functionals of explicit finite families is passed along $\mathbf{k}$ (both sides converge by Lemma 14.7, which covers *every* family); (β) Montel on a k-uniform bound, the identification of the limit by convergence at real points being replaced by the choice of $\mathbf{k}$; (γ) a step about the limit object only, unchanged. The only uniqueness used is that of the monotone cube limit along $\mathbf{k}$ (Lemma 14.8).

(i). Lemmas 14.7 and 14.8 give existence, entireness and the first bound; the second,

$$\exp(\tfrac{1}{2}\mu_0^{-2}\|d_k^* g\|_{2,k}^2)$$

at every (k, J) (Theorem 7.4), passes along $\mathbf{k}$ and J with $\|d_k^* g\|_{2,k} \to \|d^* g\|_2$. The Schwinger bounds, the extension to $\mathcal{S}$, the kernel theorem and E0″ are the arguments of Theorem 7.6(c) (Wick sector: Theorem 13.10(ii), on the polydisc inside the tube) run on the limit function with its bound (γ).

(ii), translations. For $b \in 5^{-k_0}\mathbb{Z}^3$, $|b|_\infty \le M$, $k \ge k_0$: $Z^a_{k,C_J}(\tau_b \cdot \zeta) = Z^a_{k,C_J - b}(\zeta)$ (exact lattice covariance) and $C_{J-1} \cap \Lambda_k \subset (C_J - b) \cap \Lambda_k \subset C_{J+1} \cap \Lambda_k$, so for a cone point s supported in U_{J_0} and $J \ge J_0 + 1$, $Z^a_{k,C_{J-1}}(s) \le Z^a_{k,C_J}(\tau_b \cdot s) \le Z^a_{k,C_{J+1}}(s)$ (first inequality of Lemma 2.13, twice). The three terms are functionals of the families s and $\tau_b \cdot s$ (supports in Q_{R+M}, the same tube), so they converge along $\mathbf{k}$ (α), and Lemma 14.8 sends both ends to $Z_\infty^{\mathbf{k}}(s)$: $Z_\infty^{\mathbf{k}}(\tau_b \cdot s) = Z_\infty^{\mathbf{k}}(s)$ at cone points, with no uniqueness of the k-limit. All coefficients (γ): the difference of the two limit functions is entire and vanishes on $(\mathrm{i}\mathbb{R})^N$ (curvature), resp. for the split family is holomorphic on the connected $\mathcal{D}''$ and vanishes on Ω_+, so the identity theorem, the diagonal (4) of Lemma 14.8 and the identity theorem on the connected tube give it on $\mathcal{D}_{\sigma_+,H}$. All $b \in \bigcup_k 5^{-k}\mathbb{Z}^3$ by iteration; all $b \in \mathbb{R}^3$ at imaginary coefficients because the Lipschitz bounds of Lemma 14.7, Step 3, pass to the limits $(|Z_\infty^{\mathbf{k}}(\mathrm{i}u, 0) - Z_\infty^{\mathbf{k}}(\mathrm{i}u', 0)| \le \sqrt{2}\|u - u'\|_2$; $|Z_\infty^{a,\mathbf{k}}(\mathrm{i}\sigma) - Z_\infty^{a,\mathbf{k}}(\mathrm{i}\sigma')| \le C(\|g_{\sigma-\sigma'}\|_2 + |h_{\sigma-\sigma'}|_w))$ and $\tau_{b_n} s \to \tau_{b_0} s$ in L^2 and in $|\cdot|_w$ for $s \in C_c^\infty$; complex coefficients by the identity theorem.

(ii), the cubic group. Coordinate permutations are exact at every (k,J) (Lemma 2.15); pass along $\mathbf{k}$ and J (α). For the reflection σ_3 and a cone point (f,g) supported in U_{J-1}, at every k, $Z^a_{k,C_{J-1}}(\mathrm{i}f,g) \le Z^a_{k,C_J}(\mathrm{i}\sigma_3\cdot f,\sigma_3\cdot g) \le Z^a_{k,C_{J+1}}(\mathrm{i}f,g)$ (the frame flip of Lemma 2.15 with the one-site shift $X_+ = (C_J+\varepsilon_k e_3)\cap\Lambda_k$ and two deletions in the cone direction, Lemma 2.10(a),(b)); the middle term is the functional of the reflected family (same tube), so (α) and Lemma 14.8 give $Z^{\mathbf{k}}_\infty(\mathrm{i}\sigma_3\cdot f,\sigma_3\cdot g) = Z^{\mathbf{k}}_\infty(\mathrm{i}f,g)$ on the cone (all of it at $g=0$; in Form (B) at cone points with $\|g\|_\infty < \sigma_+/2$): the reflections are not exact lattice symmetries; the sandwich replaces exactness and survives along $\mathbf{k}$. Conjugation $\sigma_\nu = P\sigma_3P^{-1}$, group closure on the cone, the identity theorem off the cone (on $\mathbb{C}^N$ with $(\mathrm{i}\mathbb{R})^N$; on $\mathcal{D}''$ with Ω_+), invariance about every point by the translations, and differentiation are (γ), as in Theorem 8.6.

(iii), reflection positivity. Fix ν, a mid-plane $c\in D$ (Definition 9.1) and the enlarged family $\mathcal{F}$ of Definition 10.1 with the modified-box functionals $\Phi_k(\zeta) = Z^{a,(3.38)}_{k,\Omega_{J,k}}(\zeta)$, $\zeta\in\mathbb{C}^m$, and support radius $R_{\mathcal{F}}$. The fixed-k steps (reflection positivity of the modified box, Lemma 9.3; the sandwich lemmas, Lemmas 9.9 and 9.10) concern the photon form and the inequalities of Lemma 2.10 only and are unchanged. *Curvature sector.* At $h=0$, $|\Phi_k(z)| \le e^{\|f_{\mathrm{Re}\,z}\|^2_{2,k}}$ by Lemma 2.11(a) and the modified photon form, uniformly in k and J. (β) Extract from $\mathbf{k}$ a further $\mathbf{k}'$ along which $\Phi_k \to \Phi^J$ locally uniformly on $\mathbb{C}^m$; positive semidefiniteness of the Gram matrices at every $k\ge k_0(c,\delta)$ passes to Φ^J. (α) On U (imaginary curvature coefficients) $a^{(a)}_{k,C_J}(\zeta) \le \Phi_k(\zeta) \le a^{(a)}_{k,C_{J+2n}}(\zeta)$ at every k (Lemmas 9.9 and 9.10, cone halves); along $\mathbf{k}'$ the ends converge to $Z^{\mathbf{k}}_J, Z^{\mathbf{k}}_{J+2n}$, so $Z^{\mathbf{k}}_J \le \Phi^J \le Z^{\mathbf{k}}_{J+2n}$ on U, and Lemma 14.8 sends both ends to $Z^{\mathbf{k}}_\infty$. $\{\Phi^J\}_J$ is locally bounded on $\mathbb{C}^m$ (J-free bound) and U is a set of uniqueness, so $\Phi^J \to Z^{\mathbf{k}}_\infty$ locally uniformly, in particular at the matrix points, where $\Phi^J(\zeta^*_{ij}) \to Z^{\mathbf{k}}_\infty(\theta\zeta^*_i \sqcup \zeta_j)$ (dependence through the total source only); positivity passes. $\mathbf{k}'$ may depend on J; this is harmless, since the sandwich ends converge along all of $\mathbf{k}$ and the bound on Φ^J is J-free, so neither $\lim_J \Phi^J$ nor the positivity depends on the choice. Every plane: translate by $(c'-c)e_\nu$ and use (ii). The polynomial form: Weierstrass along $\mathbf{k}'$ and J (Proposition 10.8). *Wick sector, amplitude $< H/2$, without Theorem B.* Let $\zeta_1,\dots,\zeta_N$ be families of real sources supported in $\{x_\nu > c\}$ with $\mathsf{a}(\zeta_i) < H/2$; the (i,j) Gram entry is $\Phi_k(\zeta^*_{ij})$, with total scalar source $\sum_l \bar w_{il}\, h_{il}\circ\rho + \sum_l w_{jl}h_{jl}$. The bound of Lemma 10.3, which depends on Theorem B, is replaced by Lemma 14.11 below, after which the argument is completed.

(iv). Symmetry: Taylor coefficients of a function of the total source. Realness: real at real points at every (k,J), closed under limits (Form (B): at the real points of the tube). Charge conjugation gives $Z^a_{k,J}(-z,w) = Z^a_{k,J}(z,w)$ at every (k,J), hence for $Z^{\mathbf{k}}_J$ and $Z^{\mathbf{k}}_\infty$ (the tube is invariant under $z\mapsto -z$); odd z-derivatives at 0 vanish.

(v), (vi). The moment bounds at (k,J) (Theorem 7.4, resp. (37), with Cauchy's estimates) are uniform in k and J above the thresholds, and the lattice Schwinger functions converge along $\mathbf{k}$ then J (Weierstrass). The measure is built from the limit functional (γ): $\mathcal{Z}^{\mathbf{k}}(f) := Z^{\mathbf{k}}_\infty(\mathrm{i}f)$ on real $f\in C^\infty_c$ is normalised, positive definite (at every (k,J) the matrix $(Z^a_{k,J}(\mathrm{i}(f_m-f_l)))_{lm}$ is a Gram matrix; the entries converge along $\mathbf{k}$ then J) and Lipschitz for $\sqrt{2}\|\cdot\|_2$, resp. $C(\|\cdot\|_2+|\cdot|_w)$, hence continuous on $\mathcal{S}$; Bochner–Minlos, moments by Lukacs [28] and uniform integrability, the Hermite carrier on $\mathcal{H}^{-r}$, $r\ge 14$ even, and the two uniqueness statements are the steps of Theorem 11.5(a) with [32, Lemma A.8], [32, Proposition A.9] and [32, Lemma A.5]. *Local laws along* $\mathbf{k}$ (β; the one weakened clause): tightness of the laws of $\chi\Phi^a_{k,J}$ on $H^{-r}_{\mathrm{per}}(Q;\mathcal{V})$ is uniform in k and $J\ge J(\chi)$ from the second-moment bounds (Prokhorov [8]); the characteristic functionals $Z^a_{k,J}(\mathrm{i}\chi\psi)$ converge *along* $\mathbf{k}$ to $Z^{a,\mathbf{k}}_J(\mathrm{i}\chi\psi)$; a probability measure on H^{-r}_{per} is determined by its characteristic functional on trigonometric polynomials; so the laws converge along $\mathbf{k}$, then in J (Lemma 14.8), to the law

of $\chi\Phi$ under the measure just built (the periodisation identity of Theorem 11.5). (vi): $\mathbb{R}^3 \rtimes O_h$-invariance of the measure is (ii) at imaginary coefficients plus uniqueness of the characteristic functional; (H3) of Definition 11.4 is (iii) at purely imaginary Wick coefficients ($\mathsf{a} = 0 < H/2$) with the density of the exponential algebra ([32, Lemma A.7]) and [32, Proposition A.10(c)]; then [32, Theorem A.18], [32, Proposition A.23], [32, Corollary A.29], [32, Theorem A.21], [32, Corollary A.22] and [32, Theorem A.20], the mean-ergodic and diagonal-clustering statements [32, Theorem A.26] and [32, Theorem A.28] (von Neumann's mean ergodic theorem for the strongly continuous unitary $\mathbb{R}^3$-action on $L^2(\mu)$: the cube averages of $U(a)$ converge strongly to the projection onto the invariant functions, the conditional expectation given the invariant σ-algebra; by the coordinate-direction reconstruction, translation by a_j acts as the contraction $e^{-|a_j|H_j}$ and the transverse components as unitaries, so that as $\min_j |a_j| \to \infty$ the correlation tends to the projection onto the joint fixed space, the conditional expectation given the invariant σ-algebra) and Definition 11.2(iii), stated for a measure satisfying (H1)–(H3), apply.

(vii). By Proposition 14.13(a), $Z^{\mathbf{k}}_\infty = Z^{\mathrm{LIM}}_\infty$, which is $\mathbb{R}^3 \rtimes O(3)$-invariant for all $z \in \mathbb{C}^N$ by Theorem C. (ii) in Euclidean form follows by differentiation; for (iii), a plane P is $E\{x_\nu = c\}$ for a Euclidean motion E, $\theta_P = E\theta_{\nu,c}E^{-1}$, and $Z^{\mathbf{k}}_\infty(\theta_P\zeta_i^* \sqcup \zeta_j) = Z^{\mathbf{k}}_\infty(\theta_{\nu,c}(E^{-1}\zeta_i)^* \sqcup E^{-1}\zeta_j)$ by invariance at complex coefficients, the $E^{-1}\zeta_i$ being supported in $\{x_\nu > c\}$; (vi) is Corollary 11.16, with the admissible push-forward action of Lemma 11.11(i) and the $O(3)$-invariance of the measure, which by Lemma 11.11(ii) is that of $\mathcal{Z}^{\mathbf{k}}$, as in Theorem 11.5(c), with its proviso on the odd moments of a component. □

Proposition 14.10 (LIM states of two schemes)**.** *Let the couplings lie in the coupling region of Definition 2.1 and let the schemes a, a' both have the cell bound* ($\mathrm{B}_{\mathrm{cell}}$) *and the scheme bound* (SCH)*. Then every continuum subsequence for the Wick sector in the scheme a has a further subsequence $\mathbf{k}'$ along which $a'_k - a_k \to \kappa^{\mathbf{k}'}$, and $Z^{a',\mathbf{k}'}_\infty(\zeta) = e^{-\kappa^{\mathbf{k}'}\sum_j w_j \int h_j} Z^{a,\mathbf{k}'}_\infty(\zeta)$: the two LIM states differ by the shift $W \mapsto W - \kappa^{\mathbf{k}'}\int(\cdot)$ of the Wick component, and joint cumulants of order ≥ 2 are scheme-free.*

Proof. $\sup_k |a'_k - a_k| \leq \sup_k |a'_k - z_k| + \sup_k |z_k - a_k| < \infty$ by (SCH) for both. For a continuum subsequence $\mathbf{k}$ for a, Bolzano–Weierstrass gives $\mathbf{k}' \subset \mathbf{k}$ with $a'_k - a_k \to \kappa^{\mathbf{k}'} \in \mathbb{R}$. At every (k, J), $Z^{a'}_{k,J}(\zeta) = e^{-\sum_j w_j (a'_k - a_k)\mathsf{S}_{\Lambda_k}(h_j)} Z^a_{k,J}(\zeta)$ (Remark 2.6) with $\mathsf{S}_{\Lambda_k}(h_j) \to \int h_j$; so along $\mathbf{k}'$ the a'-functionals converge wherever the a-functionals do, $\mathbf{k}'$ is a continuum subsequence for a' with $Z^{a',\mathbf{k}'}_J = e^{-\kappa^{\mathbf{k}'}\sum_j w_j \int h_j} Z^{a,\mathbf{k}'}_J$, and then $J \to \infty$. (For admissible a, a', $a'_k - a_k$ converges along the full sequence and $\kappa^{\mathbf{k}'} = \kappa$.) The cumulant statement is that of Theorem 28.2. ($\mathrm{B}_{\mathrm{cell}}$) transfers between schemes with bounded difference: $B^{a'}_{\mathrm{cell}} \leq B^a_{\mathrm{cell}} + \sup_k \big(|a'_k - a_k|\, \|2H'\chi_0\|_{1,k}\big)$. □

Lemma 14.11 (a k- and J-uniform bound for the modified-box functionals on the tube of the enlarged family)**.** *For couplings in the coupling region of Definition 2.1, under* ($\mathrm{B}_{\mathrm{cell}}$) *and* (SCH)*, for every k, every $J \geq \max(J_0, J_1(R_{\mathcal{F}}))$ and every $\zeta \in \mathcal{D}^{\mathcal{F}}_{\sigma_+,H}$ (the tube of $\mathcal{F}$ in the sense of Definition 2.8),*

$$|\Phi_k(\zeta)| \leq \exp\big(2\|g(\zeta)\|^2_{2,k} + \kappa\|k_+(\zeta)\|_{1,k} + K_-|k_-(\zeta)|_w\big).$$

The curvature constant is 2, not the 1 of (37), because the Cauchy–Schwarz split below doubles the curvature source and Theorem 7.4 is quadratic in it; nothing depends on its value.

Proof. $|\Phi_k(\zeta)| \leq \Phi_k(\xi)$, $\xi := \mathrm{Re}\,\zeta$ (Lemma 2.14(b)). Let $k(\xi)$ be the total real scalar source at ξ and $k_1 := k(\xi)_+$, $k_2 := k(\xi)_-$ its *pointwise* parts, sampled on the lattice. Cauchy–Schwarz at the lattice level (Lemma 2.14(d); the multiplier $e^{-a_k\mathsf{S}(k_1-k_2)}$ splits by linearity of S): $\Phi_k(\xi) \leq Z^{a,(3.38)}(2f_\xi, -2k_2)^{1/2} Z^{a,(3.38)}(0, 2k_1)^{1/2}$. *First factor* (class $\mathcal{D}$): the $\mathcal{D}$-half of Lemma 9.9, which

needs only that the sources vanish off X_J ($J \geq J_0$), gives $\leq Z^a_{k,C_J}(2f_\xi, -2k_2)$; Theorem 7.4 with the real scalar source $-2k_2$ gives $\leq e^{\|2f_\xi\|^2_{2,k}} N^a_{k,C_J}(0,-2k_2)$ (the same scheme multiplier on both sides); Proposition 13.9 with $q := 2k_2 \geq 0$, supported in $Q_{R_\mathcal{F}}$, $J \geq J_1(R_\mathcal{F})$ and $\sup q = 2\|k(\xi)_-\|_\infty < H$ (this is $\zeta \in \mathcal{D}^\mathcal{F}_{\sigma_+,H}$; the proposition allows $\sup q \leq H$) gives $\ln N^a_{k,C_J}(0,-2k_2) \leq K_-|2k_2|_w$. Hence $\frac{1}{2}\ln(\text{first factor}) \leq 2\|g(\zeta)\|^2_{2,k} + K_-|k_-(\zeta)|_w$, as $f_\xi = g(\zeta)$ and $k_2 = k_-(\zeta)$. *Second factor* (class $\mathcal{I}$): the $\mathcal{I}$-half of Lemma 9.10 with $B = C_{J+2n} \cap \Lambda_k$ gives $\leq Z^a_{k,C_{J+2n}}(0,2k_1)$, and Corollary 13.3, uniform in k and in the cube, with $0 \leq 2k_1 \leq s_1$ on the lattice ($\|k_+(\zeta)\|_\infty < \sigma_+/2$, sampled $\leq$ continuum) gives $\ln Z^a_{k,C_{J+2n}}(0,2k_1) \leq \kappa\|2k_1\|_{1,k}$, so $\frac{1}{2}\ln(\text{second factor}) \leq \kappa\|k_+(\zeta)\|_{1,k}$. Add. On compacts of the tube the right side is bounded uniformly in k (Lemma 2.9). □

The lemma is (37) (with curvature constant 2) for the modified-box functional, transported through Lemmas 9.9 and 9.10; the two factors live on C_J and C_{J+2n}, which is harmless as Proposition 13.9 and Corollary 13.3 are uniform in the cube, and the amplitude $H/2$ is exactly the tube condition $\|k(\xi)_-\|_\infty < H/2$ on the total source. *The matrix points lie in the tube:* at $\xi^*_{ij} = \operatorname{Re}\zeta^*_{ij}$ the two sums of the total scalar source have disjoint supports, so $\|k(\xi^*)_\pm\|_\infty \leq \max(\mathsf{a}(\zeta_i), \mathsf{a}(\zeta_j)) < H/2 \leq \sigma_+/2$, and by continuity a neighbourhood of the finitely many matrix points lies in $\mathcal{D}^\mathcal{F}_{\sigma_+,H}$. *End of the proof of (iii), Wick sector.* (β) By Lemma 14.11, $(\Phi_k)_k$ is locally bounded on the open connected $\mathcal{D}^\mathcal{F}_{\sigma_+,H}$; extract $\mathbf{k}' \subset \mathbf{k}$ with $\Phi_k \to \Phi^J$ locally uniformly there; Gram positivity passes ($\mathbf{k}'$ may depend on J; harmless as in the curvature sector). (α) Let U be the set of coefficient vectors with purely imaginary curvature coefficients and positive real scalar coefficients on the non-negative $\tilde{h}_j$; $U \cap \mathcal{D}^\mathcal{F} \neq \emptyset$ ($k_- = 0$, k_+ small), and it is a relatively open subset of the maximal totally real $(\mathrm{i}\mathbb{R})^{m_F} \times \mathbb{R}^{m_h}$, a set of uniqueness in $\mathcal{D}^\mathcal{F}$. On U, $Z^a_{k,C_J}(\zeta) \leq \Phi_k(\zeta) \leq Z^a_{k,C_{J+2n}}(\zeta)$ at every k (Lemmas 9.9 and 9.10, $\mathcal{I}$-halves); the ends are functionals of $\mathcal{F}$ at points of its tube, converging along $\mathbf{k}'$ (Lemma 14.7(b), J above the threshold of $\mathcal{F}$) to $Z^{a,\mathbf{k}}_J, Z^{a,\mathbf{k}}_{J+2n}$, which tend to $Z^{a,\mathbf{k}}_\infty$ (Lemma 14.8, locally uniformly on the tube of $\mathcal{F}$, in particular on $U \cap \mathcal{D}^\mathcal{F}$). $\{\Phi^J\}_J$ is locally bounded on $\mathcal{D}^\mathcal{F}$ (J-free), so $\Phi^J \to Z^{a,\mathbf{k}}_\infty$ locally uniformly there, in particular at the matrix points, where the limit is $Z^{a,\mathbf{k}}_\infty(\theta\zeta^*_i \sqcup \zeta_j)$; positivity passes. Every plane by (ii) (the amplitude is translation invariant); the polynomial form by Weierstrass.

Remark 14.12 (amplitudes in $[H/2, \sigma_+/8)$). The proof of Lemma 10.3 bounds the first factor on the Wick tube $T_{\mathcal{F}'}$ of Definition 10.2 by the vertex argument at the cube C_{J_0}, which needs the fixed-cube limits in their boundedness form only. For amplitudes in $[H/2, \sigma_+/8)$ the matrix points lie in $T_{\mathcal{F}'}$ but in general outside $\mathcal{D}^\mathcal{F}_{\sigma_+,H}$, the only domain on which Definition 14.1 gives convergence along $\mathbf{k}$. For King's state these amplitudes are covered by Theorem 10.6, whose fixed-cube limits on $T_{\mathcal{F}'}$ are those of Theorem B. For a LIM state of the comparison family, OS2at amplitudes in $[H/2, \sigma_+/8)$ along $\mathbf{k}$ itself is not decided here; on $\mathcal{D}^\mathcal{F}_{\sigma_+,H} \cap \mathcal{D}_{\sigma_+,H}$ the LIM state is King's tube object (Proposition 28.7).

14.4. **Identification, and King's state.** Finally the LIM states are identified: by the rotation theorem (Theorem C) they all coincide in the curvature sector, and the fixed-cube limits, where they exist, make every LIM state the iterated limit.

Proposition 14.13 (identification)**.** (a) *For every continuum subsequence* $\mathbf{k}$ *and every family,* $Z^\mathbf{k}_\infty(\cdot,0) = Z^{\mathrm{LIM}}_\infty(\cdot,0)$*, the limit state of the rotation theorem (Theorem C); in particular all LIM states coincide,* (UNIQ) *holds, and they are* $\mathbb{R}^3 \rtimes O(3)$*-invariant. Theorem C does* not *give Theorem A.*

(b) *The following are equivalent: (i) the fixed-cube limits of the comparison family exist in* $(0,\infty)$ *at real coefficients, for every family and every* $J \geq J_0(f)$*; (ii) for every* J *and family,* $a^{(a)}_{k,J} \to Z_J$ *locally uniformly on* $\mathbb{C}^N$ *along the full sequence; (iii) every increasing sequence of spacings is a continuum subsequence. When they hold,* $Z^{\mathbf{k}}_J = Z_J$ *and* $Z^{\mathbf{k}}_\infty = Z_\infty := \lim_J Z_J$ *for every* $\mathbf{k}$*: every LIM state is the iterated limit of the comparison family. Whether they hold is not decided in this paper (Remark 4.5).*

(c) *Let the couplings lie in the coupling region of Definition 2.1 and let* a *be admissible. Then* (SCH) *and* ($\mathrm{B}_{\mathrm{cell}}$) *hold (Lemmas 14.4 and 14.5). On King's convention, where Theorem B provides the fixed-cube limits on the tube, every increasing sequence of spacings is a continuum subsequence of the K family for the Wick sector, and every K-family LIM state on* $\mathcal{D}_{\sigma_+,H}$ *is King's tube object* G^K_∞ *of Theorem 28.5, which equals* $Z^{a,K}_{\infty,T}$ *of Proposition 10.4 on* $T_{\mathcal{F}'} \cap \mathcal{D}_{\sigma_+,H}$*; the LIM states of the comparison family on* $\mathcal{D}_{\sigma_+,H}$ *are* G^K_∞ *as well (Proposition 28.7).*

Proof. (a) Let $\mathbf{k}$ be a continuum subsequence, f a family, $J \geq J_0(f)$. By Definition 14.1, $Z^{\mathbf{k}}_J(\cdot,0)$ is the locally uniform limit of $(a^{(a)}_{k_m,J})_m$, so $Z^{\mathbf{k}}_J(\cdot,0) \in L^f_J$. By Theorem C every choice $\zeta_J \in L^f_J$, $J \geq J_0(f)$, converges locally uniformly to $Z^{\mathrm{LIM}}_\infty(\cdot,0)$; take $\zeta_J := Z^{\mathbf{k}}_J(\cdot,0)$, which by Lemma 14.8 converges to $Z^{\mathbf{k}}_\infty(\cdot,0)$, and limits are unique. Hence any two continuum subsequences have the same LIM state, and invariance is the second clause of Theorem C. That theorem pins the J-limit of the subsequential limits, not the fixed-cube limits themselves, so it does not give Theorem A.

(b) (i)⇒(ii): $(a^{(a)}_{k,J})_k$ is locally bounded on $\mathbb{C}^N$ (Lemma 7.3) and converges at every real point; $\mathbb{R}^N$ is a set of uniqueness for entire functions, and Montel–Vitali gives locally uniform convergence of the whole sequence to an entire Z_J. (ii)⇒(iii): every subsequence converges to the same limit. (iii)⇒(i): the full sequence is itself an increasing sequence, hence a continuum subsequence, so $a^{(a)}_{k,J}$ converges locally uniformly, and at real x the limit lies in $[1,\infty) \subset (0,\infty)$ (Lemma 14.7, Step 4). The last clause is Lemma 14.8.

(c) (SCH) and ($\mathrm{B}_{\mathrm{cell}}$) are Lemmas 14.4 and 14.5; so (37) holds, and on King's convention (73). $(Z^{a,K}_{k,J})_k$ is locally bounded on the tube by (73) and converges at the real points of the tube by Theorem B, a non-empty open subset of $\mathbb{R}^{n+p}$ and a set of uniqueness in the connected tube; Montel–Vitali: the whole sequence converges locally uniformly on the tube (Step 7 of the proof of Theorem 28.5), so every increasing sequence is a continuum subsequence of the K family for the Wick sector, with fixed-cube limits G^K_J, and its LIM state is G^K_∞ by the monotone cube limit (Lemma 14.8 for the K family, Corollary 14.15; Steps 8–9 of the proof of Theorem 28.5); on $T_{\mathcal{F}'} \cap \mathcal{D}_{\sigma_+,H}$ this is the same J-limit of the same G^K_J as $Z^{a,K}_{\infty,T}$ (Proposition 10.4(a)). The comparison family's LIM states are identified in Proposition 28.7. □

Remark 14.14 (the iterated limits of the comparison family). The proofs of this section use of the lattice functionals only the k-uniform bounds, the cone monotonicity at fixed k and the convergence along $\mathbf{k}$. At every family and cube where the comparison family's fixed-cube limits exist along the full sequence (Proposition 14.13(b)), every increasing sequence is a continuum subsequence, the LIM state is the iterated limit of the comparison family, and the lattice laws of (v) converge along the full sequence; on King's convention this is the case by Theorems A and B (Corollary 14.15, part D of Theorem 28.2). Its Wick-sector LIM states are King's tube object in any case (Proposition 28.7).

Theorem B is the scheme-a analogue, with scalar sources, of the existence clause of King's theorem; it is not that theorem, which fails with scalar sources in King's Wick ordering. The fixed-cube limits at arbitrary Wick amplitudes and a Euclidean clause for the Wick sector above the threshold are open (Problems 32.2 and 32.3).

Corollary 14.15 (King's convention)**.** *Define continuum subsequences and LIM states of the K-family* $a^K_{k,J}(z) = \langle e^{\sum_j z_j F(f_j)}\rangle^K_{k,J}$ *(Definition 2.3) by Definitions 14.1 and 14.2 with* $\langle\cdot\rangle^K_{k,J}$ *in place of* $\langle\cdot\rangle^{(a)}_{k,J}$.

(a) (No fixed-cube limit used.) *Every increasing sequence of spacings has a continuum subsequence of the K-family; for every such* $\mathbf{k}$*, family and* $J \geq J_0(f)$ *the fixed-cube limits* $Z^{K,\mathbf{k}}_J$ *and the LIM state* $Z^{K,\mathbf{k}}_\infty = \lim_J Z^{K,\mathbf{k}}_J$ *exist, are entire and obey* $|Z^{K,\mathbf{k}}_J|, |Z^{K,\mathbf{k}}_\infty| \leq e^{\|g(z)\|_2^2}$*, with the other clauses of Lemma 14.7(a) and Lemma 14.8. Nothing beyond this is claimed for a K-family LIM state from the bounds alone.*

(b) (With Theorem A.) *Every increasing sequence of spacings is a continuum subsequence of the K-family, every K-family LIM state is King's state* Z^K_∞*, every LIM state of the comparison family is* Z^K_∞ *as well (Corollary 4.4(iii)), and the curvature-sector axioms of Theorem 28.2 hold for it, with the lattice laws of Theorem 11.5(a) converging along the full sequence.*

With scalar sources, the corresponding statements on King's convention are provided by Theorem 28.5 and Proposition 28.6.

Proof. (a) The proof of Lemma 14.7(a) uses only the k- and J-uniform local bound of Lemma 7.3 (Step 2), the second-order bound $\langle F(u)^2\rangle \leq 2\|u\|^2_{2,k}$ of Lemma 7.1 (Step 3), and the current representation and Jensen's inequality (Step 4); that of Lemma 14.8 uses in addition the first inequality of Lemma 2.13. All of these hold on King's convention: the bounds by Theorem 7.6 (the second-order bound is photon-only and transfers by the covariance identity of Lemma 4.2), the cut inequality by Corollary 4.3. So the two lemmas hold for the K-family by the same proofs; the argument makes no use of Theorem A. (b) By Theorem A, Corollary 4.4(ii)(α) gives, for every $J \geq J_0$ and family, locally uniform convergence of $a^K_{k,J}$ along the full sequence, so every increasing sequence is a continuum subsequence of the K-family, and by (β) every K-family LIM state is Z^K_∞; Corollary 4.4(iii) is the statement for the comparison family; the axioms are the curvature sector of Theorem 28.2, and the full-sequence lattice laws are Theorem 11.5(a). □

Remark 14.16 (on the construction of the subsequences). Every subsequence of spacings used in this paper is constructed in Step 2 of the proof of Lemma 14.7 from the k-uniform bounds of Lemma 7.3 and (37) and Montel's theorem, or is a further extraction from one so constructed, and the identification of its limits uses only the monotone cube limit of Lemma 14.8; no subsequence statement is cited from [26].

Rotation invariance

15. Rotation invariance of King's state

This and the following twelve sections (Sections 16–27) form the rotation part of the paper. They prove that King's state on curvature sources, the infinite-volume limit along King's cubes of Theorem A (Theorem 5.47), is invariant under every Euclidean motion (Theorem 15.1), using from the earlier sections only the setting of Section 2 (the cubes, the sources and the correlation inequalities, Definitions 2.2, 2.3 and 2.5 and Lemmas 2.10 and 4.2) and King's cube limit. The result is used as the rotation theorem (Theorem C): it gives the $O(3)$ clause of the curvature sector (Theorem 8.8), the Euclidean clause of the ergodic components (Theorem 11.5) and the Euclidean clauses of Theorem 28.2.

15.1. **Background.** The lattice $U(1)$ Higgs model couples a two-component scalar field $\phi = re^{i\vartheta}$ to a non-compact abelian gauge field A; in three dimensions King constructed its continuum limit [25, 26], for the version of the model in which the photon is written in Feynman form and carries a mass $\mu_0 > 0$ [26, (2.5)], using the ultraviolet stability bounds of Bałaban [1, 2,

3, 4]. Rotation invariance is not automatic, since all lattices in the construction have the same orientation. Theorem 2.4 of [26] states that on the tori D_J the lattices $\varepsilon_k\mathbb{Z}^3$ and $R(\varepsilon_k\mathbb{Z}^3)$, with R the rotation by $\theta_0 = \arctan(3/4)$ in a coordinate plane, have the same continuum limit; since θ_0/π is irrational these rotations generate a dense subgroup of $SO(3)$. The statement as given in [26, Thm. 2.4] is false in two ways (Remark 20.12): the source it pairs with the plaquettes is one scalar function on both lattices, which couples the field to two different 2-forms, and the Wick constant it uses misses an order-e^2 logarithm. We therefore call the statement of [26, Thm. 2.4] *King's torus conjecture*, and we use instead the corrected statement, for 2-form sources sampled in each lattice's own frame and for the two lattices at the same spacing, which is proved in Theorem 18.8 with Theorem 18.9. The ultraviolet analysis of [25, 26] needs boundaries that are lattice hyperplanes [26, p. 326], and no bounded region has that property for both lattices [26, p. 326]. Where to place the two-lattice comparison is therefore the whole question. This part places it on periodic tori carrying a smooth mass wall, and proves $O(3)$ invariance of King's state in the curvature sector, on all of King's range. The results used in the proof that are not theorems of the cited literature are stated, each with its proof, where first used, and the corrections to the cited sources are Remark 20.12.

15.2. **The torus route.** On each lattice separately King's cubes are compared with tori carrying a *smooth mass wall*. Fix R in the generating set $\mathcal{R}_0 = \{R_{\mu\nu}(\pm\theta_0)\}$ and a real 2-form f, smooth with compact support. For a walled torus $\mathfrak{c} = (J', w)$, the periodic model on $D_{J'}$ in which the scalar mass squared is raised by $2w$ with $w \geq 0$ smooth and vanishing near the support of f, the chain is

$$Z_{C_L}(\mathrm{i}\,R\cdot f, 0) \;=\; Z^{R^{-1}\Lambda}_{R^{-1}C_L}(\mathrm{i}f, 0) \;\leq\; T^{R^{-1}\Lambda}_{\mathfrak{c}}(f) \;\xrightarrow[\varepsilon\to 0]{}\; \mathcal{W}_{\mathfrak{c}}(f,0) \;\xleftarrow[\varepsilon\to 0]{}\; T^{\Lambda}_{\mathfrak{c}}(f). \tag{39}$$

Here Z_{C_L} is King's cube functional on the axis lattice Λ, the first equality is a passive rotation, the inequality is a correlation inequality on the rotated lattice $R^{-1}\Lambda$, for which $R^{-1}C_L$ is a cube with faces on lattice hyperplanes, and the two continuum limits agree by the torus comparison. What is missing is an upper bound for walled tori on the axis lattice by King's state:

$$\text{for every } \eta > 0 \text{ there is an admissible } \mathfrak{c} \text{ with } \mathcal{W}_{\mathfrak{c}}(f,0) \leq Z_\infty(\mathrm{i}f, 0) + \eta. \tag{40}$$

This is the statement (TIGHT$_0$) of Section 21. It involves one lattice and one orientation, and the reverse inequality holds in the limit (Lemma 21.2), so (TIGHT$_0$) says that King's state is the lower envelope of the walled tori. Granting it, letting $L \to \infty$ in (39) gives $Z_\infty(\mathrm{i}\,R\cdot f, 0) \leq Z_\infty(\mathrm{i}f, 0)$, and applying this to $R^{-1} \in \mathcal{R}_0$ gives equality. The passage from the dense subgroup to $O(3)$ uses a continuity lemma for convex functionals (Lemma 2.17), so no regularity of the Schwinger functions is needed.

We isolate the local estimate that implies (TIGHT$_0$). In the walled torus let the source sit in a box H and let U be a thick shell around H inside the heavy plateau, where the wall is at least 1. Raise the scalar mass squared on the shell further by $2K$, through a second smooth profile, and let $\Psi_k(n, K)$ be the resulting expectation at lattice spacing ε_k, with $\Psi_k(n, \infty)$ the expectation with the matter removed on the shell. The statement (RES$_0$) of Section 23 asks that for one such configuration and one finite K

$$\liminf_{k\to\infty} \big[\Psi_k(n, K) - \Psi_k(n, \infty)\big] \leq \eta,$$

with the width of the shell chosen so that a photon boundary term is below η as well. At every fixed spacing the bracket tends to 0 as $K \to \infty$ (Lemma 23.6); the content of (RES$_0$) is that the heavy-shell limit and the continuum limit can be interchanged, at one geometry and with no rate. A two-line sandwich (Lemma 23.8) shows that the heavy shell is bounded below by a higher wall. Once the matter on the shell is dead, the photon on the far side can be pinned and then replaced

by King's Neumann photon on a larger cube at a cost exponentially small in the width of the shell, uniformly in the spacing (Lemma 23.1); so (RES$_0$) implies (TIGHT$_0$) (Theorem 23.10). In the curvature sector (RES$_0$) at one configuration is itself proved (Theorem 23.12): the shell mass κ is put inside the matter covariance of King's block-spin flow. Four estimates on that flow, two-sided bounds on the stopping-scale integral uniformly in κ, a disc bound for the heavy shell, a κ-Cauchy property of the reduced small-field action, and a threshold clause, give the tail bound at one configuration. Each of the four is proved in Theorems 24.5 and 25.11, with the shell-model statements they rest on (Proposition 24.7, Definition 24.2); the statements used inside those proofs are stated in Sections 24 and 25.

The torus comparison is reduced to one-sided bounds at a single lattice spacing (Section 18). The passive map $S = R^{-1}\sigma_\nu$ is an involution that exchanges the two lattices and maps the torus D_J to the torus $R^{-1}D_J$. A forward bound of King's type on each of the two tori therefore gives both directions of the comparison on D_J. The forward bounds and the existence of the axis-lattice limits are the content of the corrected torus theorem (Theorem 18.8; Theorem 18.9), whose proof establishes them with the large-field clause of King's upper bound in place of King's torus bound [26, (4.2)], which is vacuous at a non-zero wall; we state the theorem and its hypotheses. An alternative placement of the comparison, on a Neumann–Dirichlet box aligned with one lattice and a staircase for the other, is possible in principle and is not used here.

15.3. **Guide to the proof.** The route is rows (R0)–(R4) of the roadmap table (Table 1), and the sections of this part prove them in that order. Section 16 fixes the lattices, the states, the sources and the standing lemmas. Section 17 defines the object, King's state on curvature sources (Theorem 17.1, Definition 17.2), from King's cube limit [26, Thm. 2.1] (Theorem 5.47). Section 18 proves the torus comparison (K$_0$) (Theorem 18.1) by the corrected torus theorem (Theorem 18.8), whose hypotheses (N1)–(N6) are stated, each with its proof, in Section 19, and whose positivity condition is Proposition 26.8 (Section 26). Section 21 introduces the walled tori and the closing statement (TIGHT$_0$) (Theorem 21.4). Section 23 proves it: the heavy-shell estimate (RES$_0$) at one configuration (Theorem 23.12) rests on the four shell estimates on King's block-spin flow with the shell mass inside the covariance (Theorems 24.5 and 25.11, Lemma 25.13, Corollary 25.12; Sections 24 and 25), and Theorem 23.10 derives (TIGHT$_0$) from (K$_0$), (RES$_0$) and the dead-shell bound (Theorem 23.7). The invariance under the generators is obtained by letting $L \to \infty$ in the chain of inequalities (39) (Theorem 22.1), the passage from the dense subgroup to $O(3)$ is Theorem 22.3, and Theorem 22.4 states the result for the curvature sector (Section 22). Every statement used is proved or cited where it is first used.

15.4. **The main theorem.** We fix $e > 0$, $\lambda, m, \mu_0 > 0$ in King's range and King's block factor 5 (King's L; below, L denotes only the index of a King cube C_L). The proof of the theorem of this part runs through two statements about King's walled tori, (K$_0$) and (TIGHT$_0$) (Theorems 18.1 and 21.4):

(K$_0$) *The torus comparison, curvature sector*: for every $R \in \mathcal{R}_0$, every walled configuration $\mathfrak{c}$ of the class $\mathcal{C}$ (Definition 16.6) and every real smooth periodic 2-form G, $\lim_k \langle e^{F(G)} \rangle_{\mathfrak{c},\Lambda_k}$ and $\lim_k \langle e^{F(G)} \rangle_{\mathfrak{c},R^{-1}\Lambda_k}$ exist and are equal.

(TIGHT$_0$) *The closing statement, curvature sector*: $\mathcal{W}^{\mathrm{env}}(f,0) \le Z_\infty(\mathrm{i}f,0)$ for every real 2-form $f \in C_c^\infty$; that is, King's state is the lower envelope of the walled tori.

Both are proved: (K$_0$) is proved from the hypotheses (N1)–(N6) of Section 19, on the class $\mathcal{C}$, by the corrected torus theorem (Theorem 18.8); (TIGHT$_0$) is proved from (K$_0$), the heavy-shell estimate (RES$_0$) at one configuration and the dead-shell bound (Theorem 23.7) at that configuration (Theorem 23.10), and (RES$_0$) is proved from the four shell estimates (Theorems 24.5 and 25.11, Lemma 25.13, Corollary 25.12) by Theorem 23.12.

Let $Z_\infty(\cdot,0)$ denote King's state on curvature sources (Definition 17.2, Theorem 17.1): the entire function $z \mapsto Z_\infty(\sum_j z_j f_j, 0)$ of the coefficients of real 2-forms $f_j \in C_c^\infty$, whose derivatives at $z = 0$ are the curvature Schwinger functions $\langle F(f_1)\cdots F(f_N)\rangle$. For $R \in O(3)$ and $b \in \mathbb{R}^3$, sources are transported as 2-forms, $R \cdot f$ and $\tau_b f$ (Definition 2.5).

Theorem 15.1 ($O(3)$ invariance of the curvature sector)**.** *Let $e > 0$ and $\lambda, m, \mu_0 > 0$ be any point of King's range, at block factor* 5. *Then* (K_0) *and* (TIGHT_0) *hold outside the set E_W of Proposition 26.8, and King's state on curvature sources (Definition 17.2) is $O(3)$-invariant at every point of King's range, at the points of E_W by right-continuity in the scalar mass: for every $R \in O(3)$, all real 2-forms $f_1, \dots, f_N \in C_c^\infty$ and all $z \in \mathbb{C}^N$,*

$$Z_\infty\Big(\sum_j z_j\, R\cdot f_j,\, 0\Big) = Z_\infty\Big(\sum_j z_j f_j,\, 0\Big),$$

so all curvature Schwinger functions are $O(3)$-covariant; with the translations (Theorem 17.1(v)) the same holds for every Euclidean motion $x \mapsto Rx + b$, with $\tau_b R \cdot f_j$ in place of $R \cdot f_j$, at the points of E_W again by right-continuity in the scalar mass (Theorem 26.7): the curvature sector is $E(3)$-invariant. For the curvature two-point function the weaker closing condition (TIGHT_2) *suffices (Corollary 22.5).*

Theorem 15.1 has no hypothesis beyond King's range and the block factor 5. Its proof is the following assembly of the results of Sections 18–23. The implication (N1)–(N6) $\Rightarrow$ (K_0), on the class $\mathcal{C}$, is Theorem 18.8, its hypotheses (N1)–(N6) being the statements of Section 19. The implication from the four shell estimates (Theorems 24.5 and 25.11, Lemma 25.13, Corollary 25.12) to (RES_0) is Theorem 23.12. Then (RES_0), (K_0) and the dead-shell bound (Theorem 23.7) give (TIGHT_0) by Theorem 23.10; and (K_0) and (TIGHT_0) give the invariance by Theorem 22.4. King's cube limit [26, Thm. 2.1] (Theorem 5.47) defines the object, King's state on curvature sources on King's cubes (Definition 17.2, Theorem 17.1). At the points of E_W, where the positivity condition of Proposition 26.8 fails, the argument for (K_0) does not apply and the invariance follows from the nearby scalar masses by right-continuity of King's state in the scalar mass (Theorem 26.7). In one sentence: (K_0) is a one-sided comparison of effective actions at equal spacing of the kind sketched in [26, §4] for King's torus conjecture, carried out in Theorem 18.8 and Theorem 18.9 from the hypotheses (N1)–(N6) of Section 19, on the class $\mathcal{C}$, outside the set E_W; (RES_0) is the uniform-in-the-spacing decoupling of a heavy weakly coupled shell from a light core across a heavy plateau, proved from the four shell estimates (Theorems 24.5 and 25.11, Lemma 25.13, Corollary 25.12) on King's block-spin flow with the shell mass inside the covariance. Every statement used is proved in this part or in the existence part (Theorem 5.47), or is cited from the literature at its point of use.

16. The model

This section fixes the objects of the rotation part: the generating rotations and the large-block integer (Definition 16.1), with the geometry that lets the rotated lattice descend to the torus (Lemma 16.2); the lattices with their frames, the tori and the walled tori (Definitions 16.3 and 16.4) and the configuration class on which the torus comparison is proved (Definition 16.6); the two conventions for the counterterm and the mismatch vertex of the shell model (Remark 16.8, Definition 16.9); two facts about King's sources that fix the form of the statements, the logarithm in the Wick square and the failure of the source rotated as a scalar (Propositions 16.13 and 16.14); and the standing lemmas of the setting in the walled forms used below (Lemmas 16.15, 16.16, 16.17 and Corollary 16.19). Apart from Definition 16.9, whose objects are those of the shell model of Section 23, and the two propositions, which are computations in perturbation theory

and for the free field, nothing here goes beyond the setting of Section 2 and the correlation inequalities of Lemma 2.10.

16.1. **Geometry.** The rotations of the generating set and the block integer are chosen so that the rotated lattice descends to the torus D_J; this subsection fixes them and proves the geometric facts used later (Lemma 16.2).

Definition 16.1. Let $\theta_0 \in (0, \pi/2)$ with $\cos\theta_0 = \frac{4}{5}$, $\sin\theta_0 = \frac{3}{5}$. For $1 \le \mu < \nu \le 3$ let $R_{\mu\nu}(\theta)$ be the rotation with $Re_\mu = \cos\theta\, e_\mu + \sin\theta\, e_\nu$, $Re_\nu = -\sin\theta\, e_\mu + \cos\theta\, e_\nu$, $Re_\rho = e_\rho$ for $\rho \neq \mu, \nu$. Put $\mathcal{R}_0 := \{R_{\mu\nu}(\pm\theta_0) : 1 \le \mu < \nu \le 3\}$ and let G_0 be the group generated by $\mathcal{R}_0$. Let $\varepsilon_k := 5^{-k}$ and $\Lambda_k := \varepsilon_k \mathbb{Z}^3$, $k \ge 0$. Let M be the large-block integer of [25, §2.3], fixed throughout, chosen as in the setting: $M \in 50\mathbb{N}$ and $M \ge M_{\min}$ (Section 2). Then $M = 5M'$ with M' divisible by the prime $\mathfrak{p} := 2$ and $M' \ge 2\mathfrak{p}$, that is, M' even and $M' \ge 4$ (indeed $M' = M/5 \in 10\mathbb{N}$), for instance $M = 20 \cdot 5^b$ with b large (the prime $\mathfrak{p}$ of the grid offset of the class $\mathcal{C}$ (Definition 16.6) is 2: the box H' of Definition 23.2 is centred, so the offset $c \equiv M'/\mathfrak{p} \pmod{M'}$ at both corners $\pm c$ forces $\mathfrak{p} = 2$, and for an odd prime the class would be empty). This is admissible: every condition that the constructions of [25, 26, 1, 2, 3, 4] place on M is a lower bound, apart from divisibility conditions on the sides of King's cubes and tori, such as that the side $2MJ$ of C_J be a multiple of every block side [26, (3.4)], which hold for every M by the definitions of C_J and D_J below; it does not change King's state (Definition 17.3), and it is used, with the grid offset of the box H' in the class $\mathcal{C}$, by the sub-domain form of the perturbative bound (Proposition 24.7). The *King cubes* are $C_J := \{x \in \mathbb{R}^3 : -MJ \le x_\mu < MJ,\ \mu = 1, 2, 3\}$, $J \in \mathbb{N}$ [26, (2.7)], and the *King tori* are $D_J := \mathbb{R}^3/(50MJ\,\mathbb{Z})^3$ with fundamental cell $\Gamma_J := [-25MJ, 25MJ)^3$ [26, (4.1)]. We write $Q_b := [-b, b]^3$, $N_r(X) := \{y : \mathrm{dist}_\infty(y, X) \le r\}$, and $\bar{B}_\rho$ for the closed Euclidean ball of radius ρ about 0.

Lemma 16.2. *(a) For $R \in \mathcal{R}_0$ the matrix $5R$ has integer entries, $R^{-1} \in \mathcal{R}_0$, and $\sigma_\nu R_{\mu\nu}(\theta)\sigma_\nu = R_{\mu\nu}(-\theta)$, where σ_ν is the reflection $x_\nu \mapsto -x_\nu$.*

(b) For $R \in \mathcal{R}_0$, $k \ge 0$ and $J \ge 1$ the lattices $R\Lambda_k$ and $R^{-1}\Lambda_k$ are invariant under translations by $50MJ\,\mathbb{Z}^3$, so they descend to D_J.

(c) For $R \in \mathcal{R}_0$, $|Rx|_\infty \le \frac{7}{5}|x|_\infty$, and $Q_{5b/7} \subset R^{-1}Q_b \subset Q_{7b/5}$ for $b > 0$.

(d) θ_0/π is irrational, and G_0 is dense in $SO(3)$.

Proof. (a) The entries of R are $0, 1, \pm\frac{3}{5}, \pm\frac{4}{5}$, and $R_{\mu\nu}(\theta)^{-1} = R_{\mu\nu}(-\theta)$; conjugation by σ_ν reverses the orientation of the (x_μ, x_ν)-plane and fixes the third axis. (b) For $p \in 50MJ\,\mathbb{Z}^3$ we have $R^{\pm1}p = \frac{1}{5}(5R^{\pm1})p \in 10MJ\,\mathbb{Z}^3 \subset \Lambda_k$. Hence $R^{-1}\Lambda_k + p = R^{-1}(\Lambda_k + Rp) = R^{-1}\Lambda_k$, and likewise for $R\Lambda_k$. (c) The sums of the absolute values of the entries in each row of R are at most $\frac{7}{5}$. If $Rx \in Q_b$ then $|x|_\infty = |R^{-1}(Rx)|_\infty \le \frac{7}{5}b$, since $R^{-1} \in \mathcal{R}_0$; if $|x|_\infty \le \frac{5}{7}b$ then $|Rx|_\infty \le b$. (d) If $\theta_0 \in \pi\mathbb{Q}$, then $e^{\mathrm{i}\theta_0} = (4 + 3\mathrm{i})/5$ is a root of unity, hence an algebraic integer. Its minimal polynomial over $\mathbb{Q}$ is $x^2 - \frac{8}{5}x + 1$, which does not have integer coefficients; this is a contradiction (compare [26, pp. 337–338]). Hence $\{n\theta_0\}_{n\in\mathbb{Z}}$ is dense modulo 2π, and the closure of $\{R_{\mu\nu}(\theta_0)^n\}_{n\in\mathbb{Z}}$ is the group of all rotations in the (x_μ, x_ν)-plane. Every element of $SO(3)$ is a product $R_{12}(\alpha)R_{23}(\beta)R_{12}(\gamma)$ (Euler angles), so $\overline{G_0} = SO(3)$. □

16.2. **Lattices, states, sources and the standing lemmas.** This subsection fixes the lattices with their frames, the states on tori and walled tori and the configuration class of the torus comparison, describes the two counterterm conventions and the mismatch vertex of the shell model, proves two facts about King's sources (the logarithm in the Wick square and the failure of the source rotated as a scalar). It also restates the standing lemmas of the setting (the passive covariance, the current representation with Gaussian domination, the cutting inequality and the

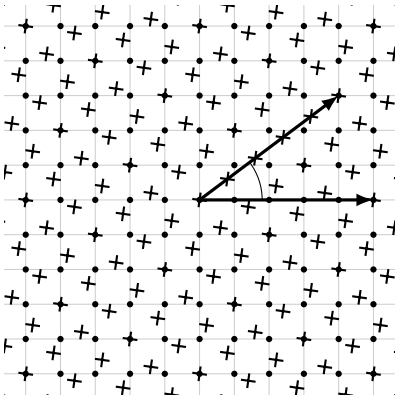

FIGURE 4. The lattice Λ_k (dots) and its rotated copy $R\Lambda_k$ (crosses) for $R = R_{12}(\theta_0)$, in the plane of the rotation; $\cos\theta_0 = \frac{4}{5}$ and $\sin\theta_0 = \frac{3}{5}$ (Definition 16.1). By Lemma 16.2 the matrix $5R$ has integer entries, the lattices $R\Lambda_k$ and $R^{-1}\Lambda_k$ are invariant under the translations by $50MJ\,\mathbb{Z}^3$ and descend to the King tori D_J, and the rotations of $\mathcal{R}_0$ generate a dense subgroup of $SO(3)$.

comparison of King's cubes with the boxes) in the walled forms that the torus comparison and the heavy shell use.

Figure 4 shows the lattice Λ_k and one of its rotated copies.

Definition 16.3 (lattices and fields)**.** A *lattice of spacing* ε_k is $\Lambda = O\Lambda_k$ with $O \in O(3)$, equipped with the frame $f_\alpha := Oe_\alpha$, $\alpha = 1,2,3$, and with the nearest-neighbour bonds $\langle x, x+\varepsilon_k f_\alpha\rangle$. A field configuration on a set X of sites consists of $\phi(x) = r_x e^{\mathrm{i}\vartheta_x} \in \mathbb{C}$ and $A_\alpha(x) \in \mathbb{R}$, $x \in X$; the bond $\langle x, x+\varepsilon f_\alpha\rangle$ carries $A_{\langle x,x+\varepsilon f_\alpha\rangle} := A_\alpha(x)$ and the reversed bond carries $-A_\alpha(x)$ (Feynman form of the vector field, [26, (2.5)]). A plaquette is $p = (x;\alpha,\beta)$ with $\alpha < \beta$; its centre is $c(p) := x + \frac{\varepsilon}{2}(f_\alpha + f_\beta)$ and its field strength is

$$F(p) := \varepsilon^{-1}\big(A_\alpha(x) + A_\beta(x+\varepsilon f_\alpha) - A_\alpha(x+\varepsilon f_\beta) - A_\beta(x)\big) \qquad [26,\ (2.11)].$$

The action on Λ is the action of the setting (Section 2) written in the frame (f_α), the gradient and hopping terms running along the bonds $\langle x, x+\varepsilon f_\beta\rangle$, with the free-boundary mass counterterm δm_ε^2 (King's [25, (3.32)–(3.34)], a polynomial in e^2 and λ whose coefficients depend only on ε, m and μ_0) and the free Wick constant W_ε of the setting (Section 2); King's field-independent constants E_0, E_1 [25, (2.4), (3.35)], [26, p. 325] cancel in every normalised expectation and are omitted. The sources $F_\Lambda(f)$, $\mathsf{S}_\Lambda(h)$, $:\!\phi^2\!:(h)$ and the observables Θ of Definition 2.5 are formed on Λ with the plaquettes and the frame of Λ: $F_\Lambda(f) = \sum_p \varepsilon^3 f_{c(p)}(f_\alpha, f_\beta)F(p)$ (Remark 16.12); on a torus smooth periodic sources are used and the sums run over one fundamental cell. The cone $\mathcal{K}$ of Definition 2.5 is invariant under $O(3)$, under translations, under $(f,g) \mapsto (tf, tg)$, $t \geq 0$, and under sums.

Definition 16.4 (boxes, tori, walled tori and killed sites)**.** Let Λ be a lattice of spacing ε and $d\nu := \prod_x \prod_\alpha dA_\alpha(x)\, d^2\phi(x)$ over the sites in question.

(a) *Boxes and King's cubes.* For a finite set X of sites of Λ the Neumann–Dirichlet box $\langle\cdot\rangle^{\mathrm{ND}}_{\Lambda,X}$ is the state of Definition 2.2 on Λ, with the frame (f_β) of Λ in place of (e_β) (for $\Lambda = \Lambda_k$ and $X = C_J \cap \Lambda_k$ written $\langle\cdot\rangle^{\mathrm{ND}}_{k,J}$), and King's cube C_J^K with its state $\langle\cdot\rangle^K_{k,J}$ is Definition 2.3.

(b) *Tori.* If Λ descends to D_J, $\langle\cdot\rangle_{\Lambda,D_J}$ is the normalised measure with the action S summed over the sites and bonds of the finite graph $\Lambda/(50MJ\mathbb{Z})^3$ (periodic conditions for both fields).

(c) *Walled tori.* A *walled configuration* is $\mathfrak{c} = (J', w)$ with $J' \in \mathbb{N}$ and $w \in C^\infty(D_{J'}; [0,\infty))$, the *wall*. If Λ descends to $D_{J'}$, $\langle\cdot\rangle_{\mathfrak{c},\Lambda}$ is the torus state of (b) with the density multiplied by $\prod_x \exp(-\varepsilon^3 w(x)|\phi(x)|^2)$. The wall raises the scalar mass squared by $2w$ and acts in the single-site measures. We write $(J', \psi, n) := (J', n\psi)$ for $n \geq 0$.

(d) *Killed sites.* For any of the states above and a set U of sites, the state *with the matter killed on* U is obtained by replacing the single-site measure $d^2\phi(x)$ times its weight, for $x \in U$, by the point mass at $\phi(x) = 0$. All hopping terms with an endpoint in U then reduce to their diagonal parts at the other endpoint; the photon variables are unchanged.

Definition 16.5 (the unnormalised functional and the walled expectation)**.** For a lattice Λ descending to a torus T we also use the *unnormalised* generating functional $\mathcal{Z}_{\Lambda,T}(G,h) := \int e^{-S(A,\phi)+F_\Lambda(G)+:\phi^2:_\Lambda(h)}\, d\nu$ ($:\phi^2:_\Lambda(h) = :\phi^2:(h)$ formed on Λ, Definition 2.5), with King's constants E_0, E_1 omitted. Two lattices of the same spacing descending to T have the same number of sites in a fundamental cell, so any field-independent constant per site multiplies both by the same factor. All the states of Definition 16.4 are BFS measures of King type in the sense of the setting (Section 2.2; the wall multiplies the single-site measure at x by $e^{-\varepsilon^3 w(x) r^2}$ with $w \geq 0$), and the charge conjugation $(A,\vartheta) \mapsto (-A,-\vartheta)$ leaves each of them invariant; for a walled configuration $\mathfrak{c} = (J', w)$, a lattice Λ descending to $D_{J'}$ and a real 2-form f supported in one fundamental cell we write $T_{\mathfrak{c},\Lambda}(f) := \langle\Theta(\mathrm{i}f, 0)\rangle_{\mathfrak{c},\Lambda} = \langle e^{\mathrm{i}F_\Lambda(f)}\rangle_{\mathfrak{c},\Lambda} = \mathcal{Z}_{\Lambda,D_{J'}}(\mathrm{i}f, -w)/\mathcal{Z}_{\Lambda,D_{J'}}(0,-w)$ for the *walled expectation* of f (the Wick constants cancel in the ratio); it is real.

Definition 16.6 (the configuration class)**.** Let $R \in \mathcal{R}_0$, let C_L, $L \in \mathbb{N}$, be a King cube, and let $X_K := \Lambda_K$ be the axis lattice of spacing ε_K. The *configuration class* $\mathcal{C}$, on which the torus comparison (Theorem 18.8) is proved and over which the infimum of the closing statement (TIGHT$_0$) (Theorem 21.4) is taken, consists of the walled configurations $\mathfrak{c} = (J, \psi, n)$ on the King tori $T := D_J$ with the following properties: $J \in 5^{k_*}\mathbb{N}$ with $J \geq \max(J_0(\varepsilon_0), J_{\mathrm{pos}})$ (k_* the number of large-field scales of the rotated flow, $J_0(\varepsilon_0)$ the non-wrapping threshold of the skew torus at the coarsest scale ε_0, J_{pos} the threshold of the positivity condition of Proposition 26.8, above which its margin holds); $\psi \geq 0$ smooth and D_J-periodic, $\psi = 0$ on the 2-neighbourhood of $R^{-1}C_L$, and $n > 0$ (a mass-increasing wall); the wall condition $n\langle\psi\rangle_{X_K} > \Delta_*|T|/2$ for every K, with Δ_* the bound of the convention bridge (CT)(b) (Remark 16.8, Theorem 19.6); block alignment: T and the box H' (Definition 23.2) are unions of blocks of the final scale, whose side we denote ℓ_f; and the grid offset: with $M = 5M'$ and the prime $\mathfrak{p} = 2$ dividing M' of Definition 16.1, every corner coordinate of H' is $c\,\ell_f$ with $c \equiv M'/\mathfrak{p} \pmod{M'}$ (the sides $50MJ$ of D_J are multiples of $5M\ell_f$). The class carries the cap of Section 26 on the coarsest scale, $\varepsilon_0 \leq \varepsilon_{\mathrm{sys}}(M, R, \rho_{\mathrm{B}}, \delta_0)$ (ρ_{B} Bałaban's radius and δ_0 the finest block size, as there), which does not depend on J and is closed downward, and under it the non-wrapping threshold at that spacing equals 1 for every J of the residue class, so that no further restriction of J is needed.

Remark 16.7 (the thresholds of the class). The threshold J_{pos} of Proposition 26.8 belongs to the class because the margin of the positivity condition is proved for $J \geq J_{\mathrm{pos}}$ only. The large-field clause of Theorem 19.3 at the wall on the skew torus takes one step at the final spacing of Bałaban's fixed-mass bounds; the cap on the coarsest scale that the class carries is a proviso of the torus comparison (Theorem 18.8) on its parameters (Section 26). Theorem 23.10 produces configurations in this class (Remark 21.5).

Remark 16.8 (the counterterm). King's δm_ε^2 is the free-boundary counterterm of the setting (Section 2, (84)), one number per spacing; it is the convention under which King's cut of a

cube out of a larger cube is "exactly" the smaller Neumann–Dirichlet state [26, p. 334]. The theorems of [25] are proved for the torus conventions of [25, (3.33), (2.8), (3.21), (3.35)]; the bridge between the two conventions is the statement (CT) (the torus counterterms of [25] and the free-boundary ones of [26] differ by $\Delta_T = \delta m^2 - \delta m_T^2$, ΔE_T and Δc_T, so that the torus model is the free-boundary model with the mass term shifted by $\frac{1}{2}\Delta_T \sum s^3 |\phi|^2$ (s the spacing) and multiplied by $e^{-\Delta E_T |T| - N \Delta c_T \langle h \rangle}$ (N the number of sites of T), exactly; the bound Δ_* on $|\Delta_T(s)|$, $s \le s_0$, and the decay of the difference terms are Theorem 19.6) on the King side and $(\mathrm{CT})_{\mathrm{sh}}$ (Definition 16.11 and Lemma 24.1: the offset Δ_T is a constant added to the counterterm-mismatch vertex; no cross-convention equality of states is used) on the shell side (Lemma 24.1 and Theorem 25.11 of Sections 24 and 25, respectively).

Definition 16.9 (the mismatch vertex)**.** The shell model keeps the bulk mass counterterm $\delta m^2_{\mathrm{bulk}}$, King's counterterm [25, (3.32)–(3.33)] at zero shell mass and zero wall, which is κ-free. Relative to the κ-adapted local subtraction $\delta m^2_\kappa(x)$, the counterterm number solving $-\delta m^2(x) + \sum_{x'} \eta^d \Gamma(x,x') = 0$ order by order with the two-leg graphs Γ of order at most four built from the full fine-lattice lines $C^{\eta,\kappa} = (-\Delta_A + V)^{-1}$ at zero background, the action carries the *mismatch vertex*

$$\mathcal{M} = -\tfrac{1}{2}\ell_k^2 \sum_x \eta^d \, \Delta\delta(x) \, |\phi'(x) + \phi^{(k)}_\kappa(x)|^2, \qquad \Delta\delta(x) = \delta m^2_{\mathrm{bulk}} - \delta m^2_\kappa(x),$$

where ϕ' is the fluctuation field and $\phi^{(k)}_\kappa$ the background of the block data. Only the matter lines differ between the two counterterm numbers ($\Delta\delta$ is a finite sum, over King's two-point graphs of orders at most four in (e,λ) [25, (3.32)], of differences of bulk and shell-model graphs, in which every member without a matter line, the e^2 tadpole, the tadpole part of the order-two seagull and the e^4 double tadpole, cancels exactly), so $\Delta\delta = [\delta m^2_{\mathrm{bulk}} - \delta m^2_0(x)] + [\delta m^2_0(x) - \delta m^2_\kappa(x)]$, the first bracket the wall term at zero shell mass and the second minus the zeroth moment of $\Gamma^\kappa - \Gamma^0$ summed over the counterterm graphs; at one loop, with $c_\lambda = 4\lambda(N+2)$, $\Delta\delta_\lambda = -c_\lambda [C^{\eta,\mathrm{bulk}} - C^{\eta,\kappa}](x,x) \le 0$, but no step uses a sign of $\Delta\delta$, only $|\Delta\delta|$.

Definition 16.10 (the size convention and the bounds on the mismatch)**.** Let $\Delta\delta$ be as in Definition 16.9. The size convention is, with $t(x)$ the distance of x to ∂U and the wall floor $C_n \le C\sqrt{n}$ coming from the κ-independent part $n\psi$ of the profile,

$$W(x) := \min\{\varepsilon^{-1}, \sqrt{\kappa}, t(x)^{-1}\} + C_n \quad (x \in H'), \qquad W(x) := \min\{\varepsilon^{-1}, \sqrt{\kappa}\} + C_n \quad (x \in U)$$

(no decay deep in U, and none is needed), the one-loop part obeying $|\Delta\delta_\lambda(x)| \le C\lambda\, W(x)$ and the full counterterm difference

$$|\Delta\delta(x)| \le C(\lambda + e^2)\, W(x) + C(\lambda + e^2)^2\, \mathcal{L}(x)^{q_F},$$
$$\mathcal{L} = 1 + \log_+(1/t) \text{ on } H', \quad \mathcal{L} = 1 + \log\big(1 + \min(\kappa, \varepsilon^{-2})/m^2\big) \text{ on } U,$$

with $q_F \in \{1,2\}$ the logarithm exponent of the two-loop family ($q_F = 1$ for the λ^2, $e^2\lambda$ and nested e^4 families, $q_F = 2$ for the crossed e^4 family).

Definition 16.11 (the counterterm offset and the bridge $(\mathrm{CT})_{\mathrm{sh}}$)**.** The *counterterm offset*: if the counterterm of Definition 16.9 is $\delta m^2 + c(\varepsilon)$ with $c = \sum_{1 \le \alpha+\beta \le 4} c(\alpha,\beta)$ graded in (e,λ) as δm^2 is, with $\Delta_c < \infty$ as in Theorem 24.5, then the κ-adapted subtraction is unchanged (c is bounded uniformly in ε, so a c-vertex lowers the superficial degree by two and occurs in no divergent two-leg graph), $\Delta\delta \to \Delta\delta + c$ exactly, every statement about the shell model holds with constants depending also on Δ_c, and $\partial_\kappa c = 0$. This is the bridge *(CT)*$_{\mathrm{sh}}$ of Remark 16.8 between the torus convention of the counterterm and the shell model's: the offset used is the free-boundary offset $c = \Delta_T(\varepsilon)$ of that remark, bounded uniformly in ε by Δ_*; a constant added

to the counterterm-mismatch vertex, with no sign of Δ_T and no equality of states across the two conventions used.

Remark 16.12 (the frame convention). In [25, 26] the curvature source is written

$$F(g) = \sum_p \varepsilon^3 g(p) F(p)$$

with one function g on all plaquettes [26, (2.12)]. On one lattice a 2-form sampled as above is a function on plaquettes, and King's proofs treat the source terms componentwise ("the sources $g_{\mu\nu}(x'), h(x')$ ... can be replaced by $g_{\mu\nu}(x), h(x)$", [25, p. 665]; the sentence is claimed there without an argument). We use King's single-lattice statements at $h = 0$ for 2-form sources in this sense: Theorem 2.1 of [26] holds, on King's cubes C_J^K (Definition 2.3), for real smooth 2-form sources sampled at plaquette centres in the lattice's own frame, supported at distance > 1 from $\mathbb{R}^3 \setminus C_J$ ([26, Thm. 2.1], stated there without a complete argument; proved as Theorem 5.47); the statements of [26, Thms. 2.2, 2.3, (3.40)–(3.44)] that we use, all at $h = 0$, are proved here from it and from Lemmas 16.17 and 4.2 (Section 17; clauses (i)–(iii) of Theorem 17.1 are Theorem 7.6(b) with Corollary 4.3), and [26, (3.42)–(3.43)] are not used, and the torus statement [25, Thm. 2.1] at a wall is Theorem 19.17. The two-lattice statement is different: Proposition 16.14 shows that it fails for a single scalar function on all plaquettes, and Theorem 18.1 is formulated for 2-forms sampled in each lattice's own frame.

Proposition 16.13 (the logarithm)**.** *In formal perturbation theory, King's model on $\varepsilon\mathbb{Z}^3$ with the free Wick constant W_ε alone satisfies, at order $e^2\lambda^0$,*

$$\big\langle :\phi^2:_{\varepsilon\mathbb{Z}^3}(h) \big\rangle = \frac{e^2}{4\pi^2}\,\ln(1/\varepsilon) \int h + O(1).$$

The coefficient does not depend on m or μ_0.

Proof. In the normalisation of the action used throughout, each real component of ϕ has the propagator $(p^2+m^2)^{-1}$ at $e = 0$. Write the one-loop correction to the inverse propagator as $\Sigma(p)$; the mass counterterm removes $\Sigma(0)$. The photon-exchange part is $-e^2 \int \frac{d^3k}{(2\pi)^3} \frac{(2p-k)^2}{((p-k)^2+m^2)(k^2+\mu_0^2)}$; writing $(2p-k)^2 = 2p^2 + 2(p-k)^2 - k^2$ and using $\int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2(p-k)^2} = \frac{1}{8|p|}$, one finds $\Sigma(p) - \Sigma(0) = -\frac{e^2}{4}|p| + O(m+\mu_0)$ for $1 \ll |p| \ll \varepsilon^{-1}$. Inserting,

$$\langle |\phi(0)|^2 \rangle - 2\mathsf{W}_\varepsilon = -2 \int \frac{d^3p}{(2\pi)^3} \frac{\Sigma(p) - \Sigma(0)}{(p^2+m^2)^2} + O(e^4) = \frac{e^2}{2} \cdot \frac{1}{2\pi^2} \ln(1/\varepsilon) + O(1)$$

at order e^2. The lattice regularisation changes only momentum-independent terms and terms of order εp^2; carrying it through the two integrals is routine. □

Two consequences. (a) For h with $\int h \neq 0$ King's generating functionals with scalar sources diverge as $\varepsilon \to 0$ in perturbation theory, beyond the curvature sector: for $h \geq 0$, $h \not\equiv 0$, Jensen's inequality gives $\langle e^{:\phi^2:(h)} \rangle \geq e^{\langle :\phi^2:(h) \rangle} \to \infty$. This affects [25, Thm. 2.1] and [26, Thms. 2.1–2.3, (3.43), (4.2)] for scalar sources with $\int h \neq 0$, and the statement of [25, p. 677] that the sources are proportional to $(5^k\varepsilon)^\alpha$, $\alpha > 0$, in the small-field region does not hold for the marginal graph that produces the logarithm. Walled ratios and the curvature sector are unaffected, since the h-linear c-numbers cancel in them; for scalar sources a further vacuum counterterm, linear in h, is needed (the scheme $\Theta^\sharp$ of Section 27). (b) King's proof of [26, Thm. 2.4] compares the effective action generated from spacing ε by a first block-spin step with rotated blocks with the one generated from the rotated lattice of spacing 5ε [26, (4.3)–(4.9)]. If the logarithm is removed by a vacuum counterterm $a_\varepsilon \mathsf{S}(h)$, linear in h, the two generating functionals carry the multipliers $e^{-a_\varepsilon \mathsf{S}(h)}$

and $e^{-a_{5\varepsilon}\mathsf{S}(h)}$ with $a_\varepsilon - a_{5\varepsilon} \to \frac{e^2}{4\pi^2}\ln 5$, so the comparison [26, (4.6)] must carry the c-number $\frac{e^2}{4\pi^2}\ln 5 \int h$, and as stated in [26] it cannot hold for $\int h \neq 0$. Since a walled ratio contains the wall source $-w$ with $\int w > 0$, this c-number enters the proof of walled comparisons even though their statement is free of the counterterm. In the equal-spacing formulation of Section 18 the two lattices carry the same counterterm, and the issue does not arise.

Rotating a source given by one function on all plaquettes, as in Theorem 2.4 of [26], fails already for the free field.

Proposition 16.14 (the source rotated as a scalar)**.** *Let $e = 0$ and $F^{sc}_\Lambda(g) := \sum_p \varepsilon^3 g(c(p))F(p)$, with $c(p)$ the centre of the plaquette p and a single real function g on all plaquettes. Let $R = R_{12}(\theta_0)$ be the rotation about the 3-axis by the angle $\theta_0 = \arctan(3/4)$, let $\ell = 50MJ$ be the side of the torus D_J, and let $g(x) = \cos(2\pi x_1/\ell)$. Then*

$$\lim_k \langle e^{F^{sc}(g)}\rangle_{\Lambda_k, D_J} \neq \lim_k \langle e^{F^{sc}(g)}\rangle_{R\Lambda_k, D_J}.$$

Proof. At $e = 0$ the photon is the centred Gaussian field with covariance $\delta_{\alpha\beta}(-\Delta_\varepsilon + \mu_0^2)^{-1}$ on the torus, so $\langle e^{F^{sc}(g)}\rangle = e^{\frac{1}{2}\operatorname{Var} F^{sc}(g)}$. On a lattice with orthonormal frame (f_α) the sum of the three plaquette components is $\sum_{\alpha<\beta} F(f_\alpha, f_\beta) = \langle u, B\rangle$ for a continuum field strength $F = dA$ with $B = \star F$, where $u := \sum_{\alpha<\beta} f_\alpha \times f_\beta$. For the plane wave g only the momenta $\pm k_0$, $k_0 = (2\pi/\ell)e_1$, contribute, and the lattice symbols converge, so

$$\lim_k \operatorname{Var} F^{sc}(g) = \frac{\ell^3}{2}\,\frac{|k_0|^2\big(|u|^2 - u_1^2\big)}{|k_0|^2 + \mu_0^2}.$$

For Λ_k one has $u = (1,-1,1)$ and $|u|^2 - u_1^2 = 2$; for $R\Lambda_k$, $u = R(1,-1,1) = (\frac{7}{5}, -\frac{1}{5}, 1)$ and $|u|^2 - u_1^2 = \frac{26}{25}$. The limiting variances differ by the factor $\frac{25}{13}$. □

With 2-form sources sampled in each lattice's own frame (Definition 2.5) the limit on either lattice is the variance of $\int\langle G, F\rangle$, which does not depend on the frame, and the obstruction disappears. This form is consistent with King's proof, which treats the source components separately, and it is the form of Theorem 18.1 and of Theorem 18.8. The second error in King's statement concerns the Wick constant (Proposition 16.13). Since the statement is false on both counts, we refer to it in that form as *King's original torus comparison* (Remark 20.12(ii)). Theorem 2.4 of [26] is not, by itself, a statement about King's infinite-volume state: the passage from tori to the state [26, (2.24)–(2.25)] is conjectural there, and it is not used here.

We now restate the standing lemmas of the setting in the walled forms used below. The first is the passive covariance: transporting the lattice with its frame, the boundary conditions and the wall by an orthogonal map leaves the generating functionals unchanged.

Lemma 16.15 (passive covariance)**.** *Let $O \in O(3)$, let Λ be a lattice of spacing ε with frame (f_α), and let X be a box, a torus or a walled torus for Λ. Equip $O\Lambda$ with the frame (Of_α) and OX with the transported boundary conditions and wall $O\cdot w$. Then, for all coefficients z, u,*

$$\big\langle \Theta(z\,O\cdot f, u\,O\cdot h)\big\rangle_{O\Lambda, OX} = \big\langle \Theta(zf, uh)\big\rangle_{\Lambda, X}, \qquad \mathsf{S}_{O\Lambda}(O\cdot h) = \mathsf{S}_\Lambda(h),$$

and likewise for finite families. Moreover, on every torus the periodic model on Λ does not change if the frame (f_α) is replaced by $(\pm f_{\pi(\alpha)})$, for a permutation π and any choice of signs.

Proof. The map $\phi'(Ox) = \phi(x)$, $A'_\alpha(Ox) = A_\alpha(x)$, with components taken in the frames (f_α) and (Of_α), is a bijection of the configuration spaces preserving $d\nu$. The action is written in the frame, and δm_ε^2 and W_ε depend on ε only. Plaquettes, their centres and field strengths are transported, and $(O\cdot f)_{Oc}(Of_\alpha, Of_\beta) = f_c(f_\alpha, f_\beta)$. For the second statement, a permutation of the frame relabels the components; reversing f_α replaces $A_\alpha(x)$ by $-A_\alpha(x - \varepsilon f_\alpha)$, a bijection of the torus

variables that preserves the action, and changes the sign of both $F(p)$ and $f_{c(p)}(f_\alpha, f_\beta)$ while preserving the set of plaquette centres. For a box or a torus without wall this is Lemma 2.15. □

The second is the current representation of the matter integral, with the Gaussian domination bounds it yields; it is stated for a general BFS measure, so that it applies to every state of Definition 16.4.

Lemma 16.16 (current representation and Gaussian domination)**.** *Let $\langle\cdot\rangle$ be a BFS measure whose Gaussian has covariance C, in the notation of Lemma 2.11 (B its set of bonds, $\ell_{xy}(A)$ its bond functionals of the vector field and $\mathbb{R}^m$ the space of the vector-field configurations). Integrating out the matter at fixed A, the matter integral is $\Psi(A) = \sum_q w(q)\cos\langle\kappa(q), A\rangle$, where q runs over integer-valued divergence-free currents on B, $w(q) \geq 0$, $\sum_q w(q) < \infty$, and $\kappa(q) \in \mathbb{R}^m$ is defined by $\langle\kappa(q), A\rangle := \sum_{xy} q_{xy}\ell_{xy}(A)$. Let $\mathbb{E}^\sharp$ be the probability measure on currents with weights proportional to $w(q)e^{-\langle\kappa(q),C\kappa(q)\rangle/2}$. Then for $v \in \mathbb{R}^m$:*

(a) $\langle e^{\mathrm{i}\langle v,A\rangle}\rangle = e^{-\langle v,Cv\rangle/2}\,\mathbb{E}^\sharp\cosh\langle Cv,\kappa\rangle$ *and*

$$\langle e^{s\langle v,A\rangle}\rangle = e^{s^2\langle v,Cv\rangle/2}\,\mathbb{E}^\sharp\cos(s\langle Cv,\kappa\rangle) \leq e^{s^2\langle v,Cv\rangle/2}$$

for $s \in \mathbb{R}$;

(b) $\langle\langle v,A\rangle^2\rangle = \langle v,Cv\rangle - \mathbb{E}^\sharp\langle Cv,\kappa\rangle^2 \in [0, \langle v,Cv\rangle]$.

Proof. The representation and the two formulas of (a) are Lemma 2.11(b) and (a). Replacing v by tv in the first formula of (a) and comparing the terms of order t^2 (all sums converge absolutely at fixed lattice spacing) gives (b). □

The third is the cutting inequality: the Neumann–Dirichlet state on a set of sites bounds, in the two directions stated, any of the states above in which that set sits, walls and killed sites included.

Lemma 16.17 (cutting)**.** *Let $\langle\cdot\rangle_Y$ be a box, torus or walled-torus state on a lattice Λ, possibly with killed sites, and let X be a set of sites of Λ such that X and its nearest neighbours lie in one fundamental cell. Let $\langle\cdot\rangle_X^{\mathrm{ND}}$ be the Neumann–Dirichlet state on X whose single-site measures (wall, killed sites and diagonal parts of the bonds leaving X included) are those of $\langle\cdot\rangle_Y$. Let f, g be sources with $g \geq 0$ such that every plaquette and site at which they do not vanish lies in X. Then, at every lattice spacing,*

$$\langle\Theta(\mathrm{i}f, g)\rangle_X^{\mathrm{ND}} \leq \langle\Theta(\mathrm{i}f, g)\rangle_Y, \qquad \langle\Theta(f, -g)\rangle_X^{\mathrm{ND}} \geq \langle\Theta(f, -g)\rangle_Y.$$

If the wall of Y vanishes on X and no site of X is killed, $\langle\cdot\rangle_X^{\mathrm{ND}} = \langle\cdot\rangle_{\Lambda,X}^{\mathrm{ND}}$ is the state of Definition 2.2. For a box of Λ_k or a King torus without wall this is Lemma 2.13.

Proof. Delete from Y every photon gradient term coupling a site of X to a site outside X, and the cosine parts of all hopping terms between X and its complement, keeping their diagonal parts $\frac{\varepsilon}{2}(r_x^2 + r_y^2)$. By Lemma 2.10(a),(b),(d) this does not increase the first expectation and does not decrease the second. The resulting measure is a product (Lemma 2.10(e)); its X-factor is $\langle\cdot\rangle_X^{\mathrm{ND}}$, since the diagonal parts of the cut bonds are exactly the Dirichlet terms and the remaining photon terms are the Neumann ones. The free Wick constants and δm_ε^2 are the same in both states. □

Remark 16.18 (King's cube in King's coordinates). Let $X_J := C_J \cap \Lambda_k$ and let T_J be its three top face layers (Definition 2.3). The translation by $(MJ + \varepsilon_k)(1,1,1)$ carries X_J onto the box $\{0 < x_\mu \leq 2MJ\}$ of [26, (3.3)] with $L_\mu = 2MJ$, the Neumann images of the photon onto those of [26, (3.7)], about the planes half a spacing outside the box, and the scalar Dirichlet condition onto that of [26, (3.9)], on the planes $x_\mu = 0$ and $x_\mu = L_\mu$, the second of which is a layer of

the box; we interpret (3.9) as setting the scalar field to zero on these planes. The side $2MJ$ is a multiple of every block side, as [26, (3.4)] requires, and King's block-spin flow [26, §3.1] is defined on C_J^K with the Dirichlet weight f_D for the block averages of the scalar field, the weight for which the image representation [26, (3.9)] holds (it does not hold for the weight given in [26, (3.8)]), and with the weight 1 and blocks tiling C_J for the photon. For the box X_J itself no flow with Dirichlet images is defined (its scalar Dirichlet planes are at distance $2MJ+\varepsilon_k$, a multiple of no block side); the box and King's cube C_J^K differ by the matter on T_J only (Remark 5.11).

The last of the standing lemmas compares King's cubes with the Neumann–Dirichlet boxes: King's cubes are monotone in the volume, and at every fixed spacing, at zero scalar source, the cube and box functionals have the same limit as the volume grows.

Corollary 16.19 (King's cubes in the volume)**.** *Let f be a real 2-form and $g \geq 0$, both in C_c^∞ and supported at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_{J_0}$.*

(a) For all $k \geq 0$ and $J \geq J_0$,

$$\langle \Theta(\mathrm{i}f, g)\rangle_{k,J}^K \leq \langle \Theta(\mathrm{i}f, g)\rangle_{k,J+1}^K \quad \text{and} \quad \langle \Theta(f, -g)\rangle_{k,J}^K \geq \langle \Theta(f, -g)\rangle_{k,J+1}^K :$$

King's monotonicity in the volume [26, Thm. 3.2]*, in the form his proof proves, with one J-independent Wick constant, holds on King's cubes.*

(b) At $g = 0$ and every fixed k, the families $J \mapsto \langle e^{\mathrm{i}F(f)}\rangle_{k,J}^K$ and $J \mapsto \langle e^{\mathrm{i}F(f)}\rangle_{k,J}^{\mathrm{ND}}$ are nondecreasing on $J \geq J_0$, bounded by 1, and have the same limit as $J \to \infty$.

(c) If $Z_J^K(\mathrm{i}f, 0) := \lim_k \langle e^{\mathrm{i}F(f)}\rangle_{k,J}^K$ exists for every $J \geq J_0$, then $\lim_J \liminf_k \langle e^{\mathrm{i}F(f)}\rangle_{k,J}^{\mathrm{ND}} = \lim_J \limsup_k \langle e^{\mathrm{i}F(f)}\rangle_{k,J}^{\mathrm{ND}} = \lim_J Z_J^K(\mathrm{i}f, 0)$; if instead $\lim_k \langle e^{\mathrm{i}F(f)}\rangle_{k,J}^{\mathrm{ND}}$ exists for every $J \geq J_0$, the same holds with the two families exchanged.

Proof. (a) The hypothesis of Lemma 4.2 holds for every $J \geq J_0$; combine its two inequalities (this is Corollary 4.3). (b) Both families are nondecreasing, by (a) and by Lemma 16.17 with $Y = C_{J+1} \cap \Lambda_k$, $X = C_J \cap \Lambda_k$; they are real and bounded by 1; and two nondecreasing sequences with $a_J \leq b_J \leq a_{J+1}$ (Lemma 4.2) have the same limit. (c) Lemma 4.2 at every k gives $Z_J^K \leq \liminf_k \langle e^{\mathrm{i}F(f)}\rangle_{k,J}^{\mathrm{ND}} \leq \limsup_k \langle e^{\mathrm{i}F(f)}\rangle_{k,J}^{\mathrm{ND}} \leq Z_{J+1}^K$, and $J \mapsto Z_J^K(\mathrm{i}f, 0)$ is nondecreasing and bounded by (a); for the exchanged statement use $\langle\cdot\rangle_{k,J-1}^{\mathrm{ND}} \leq \langle\cdot\rangle_{k,J}^K \leq \langle\cdot\rangle_{k,J}^{\mathrm{ND}}$, $J \geq J_0 + 1$ ((b) is Corollary 4.4(i); under Theorem A the first statement of (c) is Corollary 4.4(ii)(γ)). □

Remark 16.20 (what the interlacing gives). Lemma 4.2 relates the two families at every spacing and every J, so that they have the same infinite-volume state (Corollary 16.19), but it gives no information on the fixed-J continuum limit of one family from that of the other: the two bounds $\langle\cdot\rangle_{k,J}^K$ and $\langle\cdot\rangle_{k,J+1}^K$ differ macroscopically. King's cube limit (Theorem 5.47) is therefore posed on the cubes on which King's renormalisation group is run, and every other comparison of cubes in this part is transferred to them through Lemma 4.2, at the cost of one index at each end: the comparisons of a cube with walled tori (Theorems 22.4 and 23.7), the monotonicity and the translations (Theorem 17.1), and the reflection (Lemma 22.2).

17. King's state on curvature sources

This section defines the object of Theorem 15.1. From King's cube limit [26, Thm. 2.1] (Theorem 5.47), Theorem 17.1 gives the infinite-volume limit of the cube functionals along King's cubes as an entire function of the coefficients, with its monotonicity in J, its positivity at real sources, the convexity and continuity of its logarithm and its translation invariance; Definition 17.2 fixes King's state on curvature sources. The torus comparison of Section 18 and the closing statement of Section 21 (the inequality (TIGHT$_0$), which bounds this state from below by the lower envelope of the walled tori) are statements about this object.

For $\rho > 0$ let $E_\rho := C_c^\infty(\bar{B}_\rho; \Lambda^2)$ be the space of smooth real 2-forms supported in the closed ball $\bar{B}_\rho$, with the C^∞ topology; it is a Fréchet space and it is invariant under $O(3)$. For a finite family of real 2-forms $f_1, \dots, f_N \in C_c^\infty$ and $z \in \mathbb{C}^N$ write

$$Z_{k,J}^K\Big(\sum_j z_j f_j, 0\Big) := \Big\langle \exp\Big(\sum_j z_j F_{\Lambda_k}(f_j)\Big)\Big\rangle_{k,J}^K$$

for the functional of King's cube C_J^K (Definition 2.3) at spacing ε_k, and $Z_{k,J}^{\mathrm{ND}}(\cdot, 0)$ for that of the box $C_J \cap \Lambda_k$ of Definition 2.2. If the sources are supported at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_J$, King's cube limit (Theorem 5.47) gives $Z_J^K(\sum_j z_j f_j, 0) := \lim_k Z_{k,J}^K(\sum_j z_j f_j, 0)$ in $(0, \infty)$ for real z.

Theorem 17.1 (King's state on curvature sources)**.** *Let $f_1, \dots, f_N \in C_c^\infty$ be real 2-forms supported at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_{J_0}$.*

- *(i) For $J \geq J_0$, $Z_{k,J}^K(\sum_j z_j f_j, 0) \to Z_J^K(\sum_j z_j f_j, 0)$ locally uniformly on $\mathbb{C}^N$ as $k \to \infty$, and $z \mapsto Z_J^K(\sum_j z_j f_j, 0)$ is entire.*
- *(ii) For a real 2-form f supported at distance > 1 from $\mathbb{R}^3 \setminus C_{J_0}$, $J \mapsto Z_J^K(\mathrm{i}f, 0)$ is nondecreasing and $J \mapsto Z_J^K(f, 0)$ is nonincreasing for $J \geq J_0$.*
- *(iii) The limit $Z_\infty(\sum_j z_j f_j, 0) := \lim_{J\to\infty} Z_J^K(\sum_j z_j f_j, 0)$ exists locally uniformly on $\mathbb{C}^N$ and is entire; $Z_\infty(f, 0) > 0$ for every real 2-form f; and $Z_J^K(\mathrm{i}f, 0) \leq Z_\infty(\mathrm{i}f, 0)$ for $J \geq J_0$.*
- *(iv) For every $\rho > 0$, $\Phi(f) := \ln Z_\infty(f, 0)$ is finite, convex and continuous on E_ρ.*
- *(v) $Z_\infty(\sum_j z_j\, \tau_b f_j, 0) = Z_\infty(\sum_j z_j f_j, 0)$ for all $b \in \mathbb{R}^3$ and $z \in \mathbb{C}^N$.*

Proof. We take the clauses in turn: (i)–(iii) are read off from the regularity theorem for King's cube state, (iv) follows from convexity and the continuity lemma for convex functions (Lemma 2.17), and (v), translation invariance, is proved by sandwiching the translated cube between two boxes and continuing analytically.

(i)–(iii) are Theorem 7.6(b): the two limits exist locally uniformly on $\mathbb{C}^N$ and are entire, $J \mapsto Z_J^K(\mathrm{i}f, 0)$ is nondecreasing, and $Z_\infty(f, 0) \geq 1 > 0$ at real f; the nonincreasing clause of (ii) is Corollary 4.3 at $g = 0$ and every k, passed to the limit $k \to \infty$ by (i); and $Z_J^K(\mathrm{i}f, 0) \leq Z_\infty(\mathrm{i}f, 0)$ follows from (ii).

(iv) Finiteness and positivity by (iii); convexity as a pointwise limit of convex functions (Lemma 2.14(b)). For J so large that $N_1(\bar{B}_\rho) \subset C_J$, and each k, the function $f \mapsto \ln Z_{k,J}^K(f, 0)$ on E_ρ depends continuously on finitely many sampled values of f, hence is continuous; so $\Phi = \lim_J \lim_k$ of continuous functions is Borel. Apply Lemma 2.17.

(v) Let $b \in 5^{-k_0}\mathbb{Z}^3$ with $|b|_\infty \leq M$ and $k \geq k_0$, so that τ_b preserves Λ_k. Write $\langle\cdot\rangle_{k,J}^{\mathrm{ND},b}$ and $\langle\cdot\rangle_{k,J}^{K,b}$ for the states of the box and of King's cube translated by $-b$, which are again states on Λ_k, so that $\langle\Theta(\tau_b \cdot s)\rangle_{k,J}^K = \langle\Theta(s)\rangle_{k,J}^{K,b}$, and let $s = \mathrm{i}\sum_j t_j f_j$ with $t \in \mathbb{R}^N$, the sources being supported at distance > 1 from $\mathbb{R}^3 \setminus C_{J_0}$. As sets of sites, $C_{J-1} \cap \Lambda_k \subset (C_J - b) \cap \Lambda_k \subset C_{J+1} \cap \Lambda_k$, so Lemma 16.17 (with $X = C_{J-1} \cap \Lambda_k$ in the state on $(C_J - b) \cap \Lambda_k$, then $X = (C_J - b) \cap \Lambda_k$ in the state on $C_{J+1} \cap \Lambda_k$) gives, for $J \geq J_0 + 1$ (so that, the corners of a plaquette lying within ℓ^∞-distance $\varepsilon/2 \leq \frac{1}{2}$ of its centre, every plaquette at which s does not vanish lies in C_{J-1}), $\langle\Theta(s)\rangle_{k,J-1}^{\mathrm{ND}} \leq \langle\Theta(s)\rangle_{k,J}^{\mathrm{ND},b} \leq \langle\Theta(s)\rangle_{k,J+1}^{\mathrm{ND}}$. Lemma 4.2, applied to the cubes and to their translates, and this sandwich at J and at $J - 1$ give for $J \geq J_0 + 3$

$$\begin{aligned}\langle\Theta(s)\rangle_{k,J-2}^K \leq \langle\Theta(s)\rangle_{k,J-2}^{\mathrm{ND}} \leq \langle\Theta(s)\rangle_{k,J-1}^{\mathrm{ND},b} &\leq \langle\Theta(s)\rangle_{k,J}^{K,b} \\ &\leq \langle\Theta(s)\rangle_{k,J}^{\mathrm{ND},b} \leq \langle\Theta(s)\rangle_{k,J+1}^{\mathrm{ND}} \leq \langle\Theta(s)\rangle_{k,J+2}^K.\end{aligned}$$

Let $k \to \infty$ by Theorem 5.47 and then $J \to \infty$ by (iii): $Z_\infty(\tau_b \cdot s, 0) = Z_\infty(s, 0)$ on $(\mathrm{i}\mathbb{R})^N$. The identity extends to all $z \in \mathbb{C}^N$, since $(\mathrm{i}\mathbb{R})^N$ is a set of uniqueness for entire functions (Corollary 4.4(ii)(β)), to all $b \in \bigcup_k 5^{-k}\mathbb{Z}^3$ by iteration, and to all $b \in \mathbb{R}^3$ at real coefficients by (iv), since $b \mapsto \tau_b f \in E_{\rho'}$ is continuous; then to complex coefficients by the identity theorem. (King states translation invariance as [26, Thm. 2.3(i)]; his argument is not used.) □

Theorem 17.1 provides the two limits; the object of Theorem 15.1 can now be fixed.

Definition 17.2 (the object of Theorem 15.1: the infinite-volume state of King's construction [25, 26])**.** Fix $e > 0$, $\lambda, m, \mu_0 > 0$ in King's range, the block factor 5 and the large-block integer M of Definition 16.1 ($M \in 50\mathbb{N}$, $M \geq M_{\min}$), of the form $M = 5M'$ with M' divisible by the prime $\mathfrak{p} = 2$ and $M' \geq 2\mathfrak{p}$, and let $\Lambda_k = \varepsilon_k\mathbb{Z}^3$, $\varepsilon_k = 5^{-k}$. For $J \in \mathbb{N}$ let $\langle\cdot\rangle^K_{k,J}$ be the state of King's cube C_J^K at spacing ε_k (Definition 2.3): the Neumann photon on $C_J \cap \Lambda_k$, and the scalar field with Dirichlet conditions on the planes $x_\mu = -MJ - \varepsilon_k$ and $x_\mu = MJ - \varepsilon_k$, that is, the cube of [26, (3.3), (3.7), (3.9), p. 329] with $L_\mu = 2MJ$, where (3.9) is interpreted, after the translation of Remark 16.18, as setting the scalar field to zero on the planes $x_\mu = 0$ and $x_\mu = L_\mu$, for the action in the free convention (Section 2), whose mass counterterm $\delta m^2_{\varepsilon_k}$ is the free-boundary one, computed on the infinite lattice $\varepsilon_k\mathbb{Z}^3$ and the same at every site (Remark 16.8). For finite families of real 2-forms $f_j \in C_c^\infty$, sampled at plaquette centres in the lattice's own frame (Definition 2.5), and $z \in \mathbb{C}^N$, *King's state on curvature sources* is

$$Z_\infty\Big(\sum_j z_j f_j, 0\Big) := \lim_{J\to\infty}\lim_{k\to\infty}\Big\langle \exp\Big(\sum_j z_j F_{\Lambda_k}(f_j)\Big)\Big\rangle^K_{k,J}.$$

The inner limit is King's cube limit [26, Thm. 2.1] at $h = 0$ along the spacings 5^{-k}, at fixed μ_0 and fixed J, taken for the J at which the sources have ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_J$; King's proof runs his block-spin flow on C_J^K, whose block averages of the scalar field carry the Dirichlet weight f_D, and the limit is Theorem 5.47. The outer limit runs along the family $(C_J^K)_J$. Both limits exist, locally uniformly in z, given Theorem 5.47 (Theorem 17.1).

Definition 17.3 (the box limits, the choice of M and the identification of the object)**.** Let $Z_\infty(\cdot, 0)$ be King's state on curvature sources (Definition 17.2). On imaginary coefficients the lower and upper iterated limits of the functionals $\langle e^{\mathrm{i}F(f)}\rangle^{\mathrm{ND}}_{k,J}$ of the boxes $C_J \cap \Lambda_k$ of Definition 2.2, $\lim_J \liminf_k$ and $\lim_J \limsup_k$, both equal $Z_\infty(\mathrm{i}f, 0)$ (Corollary 16.19(c)); whether the fixed-cube limits $\lim_k$ of the box functionals exist is not decided (Remark 4.5, Proposition 14.13(b)). The state does not depend on the admissible choice of M: the states $\langle\cdot\rangle^K_{k,J}$ depend on M only through MJ, so if M_1 divides M_2 the cubes of M_2 are the cubes $C^K_{JM_2/M_1}$ of M_1, and the limits along the two families agree, on imaginary coefficients by Corollary 16.19(a) and then everywhere by Theorem 17.1(iii); two admissible values divide their least common multiple, which is again admissible. This is King's state [26, (2.15)] at $h = 0$, the state $Z^K_\infty(\cdot, 0)$ of Theorem 7.6 (the same cubes, action and iterated limit, for the same M), and it is the object of Theorem 15.1; the admissible values of M are those of the setting, and the independence of M just shown is Remark 2.4.

We do not address the regularity of the Schwinger functions (they are distributions), the limit $\mu_0 \to 0$, or the uniqueness of the vacuum.

18. The torus comparison

This section proves (K_0), the equality of the continuum limits of a walled torus on the axis lattice and on its rotated copy (Theorem 18.1). The involution of Lemma 18.3 reduces the comparison to one-sided bounds at a single spacing (Theorem 18.5). The bounds are provided

by the corrected torus theorem, Theorem 18.8, whose hypotheses (N1)–(N6) are the statements of Section 19 on the class $\mathcal{C}$. (K_0) is used by the closing statement (Theorem 21.4: the inequality (TIGHT_0), which bounds King's state from below by the lower envelope of the walled tori) and by the invariance theorem (Theorem 22.1).

18.1. **The statement.** The theorem states (K_0): for a walled configuration of the class $\mathcal{C}$ and a real smooth periodic 2-form, the continuum limits of the curvature functional on the axis lattice and on its copy rotated by $R \in \mathcal{R}_0$ exist and coincide, for couplings outside the exceptional set E_W of Proposition 26.8. The remaining subsections prove it.

Theorem 18.1 ((K_0): the torus comparison, curvature sector)**.** *Let the couplings lie outside the set E_W of Proposition 26.8. For every $R \in \mathcal{R}_0$, every walled configuration $\mathfrak{c}$ of the class $\mathcal{C}$ (the class on which (K_0) enters Theorem 15.1; Definition 16.6, Remark 21.5) and every real smooth periodic 2-form G, $\lim_k \langle e^{F(G)} \rangle_{\mathfrak{c},\Lambda_k}$ and $\lim_k \langle e^{F(G)} \rangle_{\mathfrak{c},R^{-1}\Lambda_k}$ exist and are equal.*

Proof. (K_0) asserts that, for a walled configuration $\mathfrak{c}$ of the class $\mathcal{C}$ and a smooth periodic 2-form G, the walled expectations of $e^{F(G)}$ on the axis lattice Λ_k and on its rotated copy $R^{-1}\Lambda_k$ converge, as the spacing tends to zero, to the same limit. This is the case $z = 1$ of the corrected torus theorem, Theorem 18.8, which compares the two lattices at a common spacing through two block-spin flows, the standard one on the rotated lattice and King's hybrid flow, whose first step averages over rotated blocks, on the axis lattice: its part (ii) says that the two families $\langle e^{zF(G)} \rangle$ converge, locally uniformly in $z \in \mathbb{C}$, to the same entire function, and its part (iii) that the same holds for every rotation of the finite set $\mathcal{R}_0$. The hypotheses of that theorem are the statements (N1)–(N6) of Section 19 on the class $\mathcal{C}$, together with the large-field clause of Bałaban's estimates at block factor five (Theorem 19.3) at the wall; the last of them, (N6), is the existence of the walled axis-lattice limit (Theorem 19.17), which holds for $J \geq J_{\mathrm{pos}}$ when the couplings lie outside the exceptional set E_W, off which the positivity condition of Proposition 26.8 holds; this is where the condition on E_W enters.

The claim is therefore Theorem 18.8(ii) and (iii), proved from the statements of Section 19 on the class $\mathcal{C}$, the large-field clause of Theorem 19.3 at the wall and Theorem 19.17 holding outside E_W for $J \geq J_{\mathrm{pos}}$ (Proposition 26.8, Remark 18.11). □

Below, "under (K_0)" means: for the configurations for which the two limits exist and agree.

Here $R^{-1}\Lambda_k$ carries the frame $(R^{-1}e_\alpha)$ and descends to D_J by Lemma 16.2(b); both lattices have spacing ε_k and carry the same $\delta m^2_{\varepsilon_k}$. Since $\mathcal{R}_0 = \mathcal{R}_0^{-1}$, ($\mathrm{K}_0$) for all $R \in \mathcal{R}_0$ is the same as the statement for the lattices $R\Lambda_k$ used in [26]. Formulated with a single scalar function on all plaquettes, as in [26, (2.12), Thm. 2.4], the comparison is false already at $e = 0$ (Proposition 16.14); the 2-form formulation is consistent with King's proof, which treats the source components separately (Remark 16.12).

Lemma 18.2. *Under (K_0) the convergence holds locally uniformly in $z \in \mathbb{C}$: the two limits $\lim_k \langle e^{zF(G)} \rangle_{\mathfrak{c},\Lambda_k}$ and $\lim_k \langle e^{zF(G)} \rangle_{\mathfrak{c},R^{-1}\Lambda_k}$ exist and are the same entire function of z. In particular, for every walled configuration $\mathfrak{c}$ for which (K_0) holds (on the class $\mathcal{C}$, by Theorem 18.1) and every real 2-form $f \in C_c^\infty$ supported in one cell, $\lim_k T_{\mathfrak{c},\Lambda_k}(f)$ and $\lim_k T_{\mathfrak{c},R^{-1}\Lambda_k}(f)$ exist and are equal.*

Proof. As in Theorem 17.1(i): local boundedness from $|\langle e^{zF} \rangle| \leq \langle e^{\mathrm{Re}\, z\, F} \rangle$ and convexity, Montel's theorem, and $\mathbb{R}$ as a set of uniqueness. For the second claim take G the periodised f and $z = \mathrm{i}$. □

18.2. **Reduction to one-sided bounds at one lattice spacing.** King's proof of his Theorem 2.4 compares the lattice of spacing ε with the rotated lattice of spacing 5ε, and obtains the reverse inequality by "switching around the argument" [26, p. 337]. We compare the two lattices at the *same* spacing, and obtain the reverse inequality from an exact symmetry.

Lemma 18.3 (the involution)**.** *Let $R = R_{\mu\nu}(\pm\theta_0) \in \mathcal{R}_0$, let $\sigma := \sigma_\nu$ be the reflection $x_\nu \mapsto -x_\nu$, and put $S := R^{-1}\sigma$. Then $S^2 = 1$, $S\Lambda_k = R^{-1}\Lambda_k$ and $S(R^{-1}\Lambda_k) = \Lambda_k$, and S maps $D_J = \mathbb{R}^3/P_J$, $P_J = (50MJ\mathbb{Z})^3$, onto the torus $R^{-1}D_J := \mathbb{R}^3/R^{-1}P_J$. Both lattices descend to both tori, and for all real smooth D_J-periodic sources $s = (G, h)$,*

$$\mathcal{Z}_{R^{-1}\Lambda_k, D_J}(s) = \mathcal{Z}_{\Lambda_k, R^{-1}D_J}(S\cdot s), \qquad \mathcal{Z}_{\Lambda_k, D_J}(s) = \mathcal{Z}_{R^{-1}\Lambda_k, R^{-1}D_J}(S\cdot s),$$
$$\mathsf{S}_{R^{-1}\Lambda_k, D_J}(h) = \mathsf{S}_{\Lambda_k, R^{-1}D_J}(S\cdot h).$$

Proof. By Lemma 16.2(a), $\sigma R\sigma = R^{-1}$, so $S^2 = R^{-1}(\sigma R^{-1}\sigma) = R^{-1}R = 1$; $\sigma\Lambda_k = \Lambda_k$ gives $S\Lambda_k = R^{-1}\Lambda_k$, and then $S(R^{-1}\Lambda_k) = S^2\Lambda_k = \Lambda_k$. Also $\sigma P_J = P_J$, so $SP_J = R^{-1}P_J$. The lattice $R^{-1}\Lambda_k$ is invariant under $R^{-1}P_J$ trivially, and Λ_k is invariant under $R^{-1}P_J$ because $R^{-1}p \in 10MJ\mathbb{Z}^3$ for $p \in P_J$ (Lemma 16.2(b)). The map S carries the frame $(R^{-1}e_\alpha)$ to $(SR^{-1}e_\alpha) = (\sigma e_\alpha)$ and the frame (e_α) to $(R^{-1}\sigma e_\alpha)$; these differ from the frames of Λ_k and $R^{-1}\Lambda_k$ only by signs, which do not change the periodic models (Lemma 16.15). Lemma 16.15 with $O = S$, applied to the unnormalised integrals (the bijection of configurations preserves $d\nu$), gives the identities. □

Definition 18.4 ((F) and (F$_0$): one-sided bounds at one spacing)**.** Let $R \in \mathcal{R}_0$ and let $J \in \mathbb{N}$. The following two named bounds are the abstract form to which the torus comparison is reduced below; (F$_0$) is established, on the class $\mathcal{C}$ and outside E_W, by the corrected torus theorem (Theorem 18.8, Remark 18.7), and (F) at general sources is not used. Write $T_1 := D_J$, $T_2 := R^{-1}D_J$. **(F)**: for $i = 1, 2$ there are real numbers $v_k(T_i)$ and $c_k(T_i)$, $k \geq k_0(J)$, independent of the fields and the sources, with $|c_k(T_i)| \leq C\varepsilon_k^{-q}$ for some $q < \infty$ and $v_k(T_i)$ bounded, such that for every real smooth T_i-periodic source $s = (G, h)$,

$$\mathcal{Z}_{R^{-1}\Lambda_k, T_i}(s) \leq \big(1 + \delta_k(T_i, s)\big)\exp\big(-v_k(T_i) - c_k(T_i)\,\mathsf{S}_{\Lambda_k, T_i}(h)\big)\,\mathcal{Z}_{\Lambda_k, T_i}(s), \qquad \lim_{k\to\infty}\delta_k(T_i, s) = 0.$$

(F$_0$) is (F) restricted to sources with $h = 0$ and to the sources (G, h_0), where $h_0 = -w \leq 0$ is a wall, in total form ($v_k \equiv 0$), on T_1 for these sources and on T_2 for their images under the involution S of Lemma 18.3, together with **(C$_0$)**: $\limsup_k c_k(T_1) \leq 0$ and $c_k(T_2) - c_k(T_1) \to 0$. The sources $(G, -h_0)$, with a mass-decreasing wall, are not needed.

On each torus, (F) bounds the functional of the rotated lattice from above by that of the axis lattice. The rotated lattice is the one whose flow is King's standard block-spin flow in its own frame, and the axis lattice is the one run through King's hybrid flow; this is the direction of [26, (4.7)–(4.9)]. Since smooth periodic functions have rapidly convergent Riemann sums, $|\mathsf{S}_{\Lambda,T}(h) - \int_\Gamma h| \leq C_N(h)\varepsilon_k^N$ for every N and every lattice Λ of spacing ε_k descending to T, Γ a fundamental cell; with $N > q$ the error $c_k(\mathsf{S}_{\Lambda,T}(h) - \int h)$ tends to 0.

Theorem 18.5 (closing at one spacing)**.** *Let $R \in \mathcal{R}_0$, $J \in \mathbb{N}$, and assume* (F). *Write $v = v_k(T_1)$, $v' = v_k(T_2)$, $c = c_k(T_1)$, $c' = c_k(T_2)$, and $\rho_k(s) := \mathcal{Z}_{R^{-1}\Lambda_k, D_J}(s)/\mathcal{Z}_{\Lambda_k, D_J}(s)$.*

- *(a) For every real smooth D_J-periodic source $s = (G, h)$, $e^{v'+c'\int h}(1 + o(1)) \leq \rho_k(s) \leq e^{-v-c\int h}(1 + o(1))$ as $k \to \infty$.*
- *(b) If (F) holds in total form ($v = v' = 0$) on T_1 for the sources $s_t = (tG, h_0)$, $t \in \mathbb{R}$, with $\int h_0 < 0$, and on T_2 for their S-images, and $\limsup_k(c + c') \leq 0$, then $c + c' \to 0$ and $\rho_k(s_t)e^{c\int h_0} \to 1$ for every real t; in particular, for $s_0 = (0, h_0)$ and $s_1 = (tG, h_0)$,*

$$\frac{\mathcal{Z}_{R^{-1}\Lambda_k, D_J}(s_1)}{\mathcal{Z}_{R^{-1}\Lambda_k, D_J}(s_0)} = \frac{\mathcal{Z}_{\Lambda_k, D_J}(s_1)}{\mathcal{Z}_{\Lambda_k, D_J}(s_0)}\,(1 + o(1)).$$

No convergence of either family is assumed.

Proof. (a) The upper bound is (F) on T_1, with $c\,\mathsf{S}_{\Lambda_k,D_J}(h) = c\int h + o(1)$. Apply (F) on T_2 to the source $S\cdot s$ and use Lemma 18.3: $\mathcal{Z}_{R^{-1}\Lambda_k,T_2}(S\cdot s) = \mathcal{Z}_{\Lambda_k,D_J}(s)$, $\mathcal{Z}_{\Lambda_k,T_2}(S\cdot s) = \mathcal{Z}_{R^{-1}\Lambda_k,D_J}(s)$ and $\mathsf{S}_{\Lambda_k,T_2}(S\cdot h) = \mathsf{S}_{R^{-1}\Lambda_k,D_J}(h) = \int h + O(\varepsilon_k^N)$. This is the lower bound. (b) By (a) at s_t the interval is non-empty, so $(c+c')\int h_0 \le o(1)$; as $\int h_0 < 0$, $c + c' \ge -o(1)$, and with $\limsup_k(c+c') \le 0$ this gives $c + c' \to 0$. The two ends of the interval in (a) then differ by a factor $e^{o(1)}$, so $\rho_k(s_t)e^{c\int h_0} \to 1$; dividing the statement for s_1 by the one for s_0 cancels the prefactor. □

The proof uses (F) only at the sources named. In particular, under (F_0) the conclusions (a) and (b) hold for the sources (tG, h_0) (where $\int h_0 < 0$ unless $w = 0$) and, with no condition on c, c', (a) holds in total form for the sources with $h = 0$.

Corollary 18.6 (walled curvature ratios)**.** *Assume* (F_0) *for R and J'. Then for every walled configuration $\mathfrak{c} = (J', w)$ whose wall w is one for which* (F_0) *holds at $h_0 = -w$, and every real smooth periodic 2-form G, $\langle e^{zF(G)}\rangle_{\mathfrak{c},R^{-1}\Lambda_k} - \langle e^{zF(G)}\rangle_{\mathfrak{c},\Lambda_k} \to 0$ locally uniformly in $z \in \mathbb{C}$. If moreover $\lim_k \langle e^{tF(G)}\rangle_{\mathfrak{c},\Lambda_k}$ exists for real t, then* (K_0) *holds for $\mathfrak{c}$ and G.*

Proof. With $h_0 = -w$, $\langle e^{tF(G)}\rangle_{\mathfrak{c},\Lambda} = \mathcal{Z}_\Lambda(tG, h_0)/\mathcal{Z}_\Lambda(0, h_0)$ (the Wick constants cancel), a balanced ratio. By Theorem 18.5(b) (at $w = 0$ directly by (a) in total form),

$$u_k(t) := \langle e^{tF(G)}\rangle_{\mathfrak{c},R^{-1}\Lambda_k} / \langle e^{tF(G)}\rangle_{\mathfrak{c},\Lambda_k} \to 1$$

for real t. By Lemma 16.16(a) and the symmetry $(A,\vartheta) \mapsto (-A,-\vartheta)$, $1 \le \langle e^{tF(G)}\rangle \le e^{t^2\sigma_\Lambda/2}$ for real t, where $\sigma_\Lambda = \langle v_G, Cv_G\rangle$ is the variance of $F(G)$ under the free periodic photon on the lattice, v_G the vector with $\langle v_G, A\rangle = F(G)$ and C the photon covariance; $\sigma_\Lambda \le \mu_0^{-2}\|\delta_\varepsilon G\|^2$ is bounded uniformly in k, δ_ε the lattice codifferential. Hence the difference of the two functions tends to 0 on $\mathbb{R}$, and it is bounded on every strip $|\operatorname{Re} z| \le r$ uniformly in k, because $|\langle e^{zF}\rangle| \le \langle e^{\operatorname{Re} z\, F}\rangle$. Vitali's theorem gives locally uniform convergence to 0 on $\mathbb{C}$. The last statement follows by the same argument. □

Remark 18.7 (what this subsection proves). Lemma 18.3, Theorem 18.5 and its corollary are a reduction of the torus comparison to the abstract one-sided bound (F): they are proved, and they provide none of the bounds. In the curvature sector the bound (F_0) that Corollary 18.6 uses, the one-sided forward bound in total form on the mass-increasing family with its source-linear constant on D_J of upper limit at most zero and the constant on the rotated torus $R^{-1}D_J$ equal to it up to $o(1)$, comes from the proof of the corrected torus theorem itself (Proposition 18.10, Section 18.4), in which King's torus bound [26, (4.2)] is not used, because at a non-zero wall it is vacuous; its place is taken by the large-field clause of King's upper bound at blocking factor 5, Theorem 19.3, and at a mass-increasing wall the polynomial lower bound (N5) is the wall instance Proposition 19.16.

18.3. **The corrected torus theorem.** The bound (F_0) and the existence of the axis-lattice limits for walled curvature ratios are the content of the corrected torus theorem, which we now state in the form used. Write $X_K := \Lambda_K$ and $Y_K := R\Lambda_K$, $R = R_{12}(\theta_0)$, so that (K_0) for R^{-1} is the statement for the pair (X_K, Y_K). In this subsection and in Proposition 18.10 the spacing index is written K, $\varepsilon_K = 5^{-K}$ (it is the index k of the other sections, not the shell mass of Sections 15, 21 and 23). Configurations are taken in the class $\mathcal{C}$ of Definition 16.6: $J \in 5^{k_*}\mathbb{N}$ with $J \ge \max(J_0(\varepsilon_0), J_{\mathrm{pos}})$, $\mathfrak{c} = (J, \psi, n)$ with $n > 0$, ψ vanishing on the 2-neighbourhood of $R^{-1}C_L$, the wall condition $n\langle\psi\rangle_{X_K} > \Delta_*|T|/2$, block alignment and the grid offset of the corners of H'.

Theorem 18.8 (the corrected torus theorem, curvature sector)**.** *Let the couplings lie outside the set E_W of Proposition 26.8. Let $e^2, \lambda, m^2, \mu_0^2 > 0$ be in King's range, the block factor 5, and*

$\mathfrak{c} = (J, \psi, n)$ a walled configuration of the class $\mathcal{C}$ with wall source $h_0 = -n\psi$. Then for every real smooth D_J-periodic 2-form G:

(i) $\langle e^{zF(G)}\rangle_{\mathfrak{c},Y_K} - \langle e^{zF(G)}\rangle_{\mathfrak{c},X_K} \to 0$ *as* $K \to \infty$, *locally uniformly in* $z \in \mathbb{C}$; *in particular* $\langle F(G)^p\rangle_{\mathfrak{c},Y_K} - \langle F(G)^p\rangle_{\mathfrak{c},X_K} \to 0$ *for every moment order* $p \in \mathbb{N}$;

(ii) both families converge, locally uniformly on $\mathbb{C}$, *to the same entire function* $W_{\mathfrak{c}}(zG)$, *and all moments converge to the same limits (this uses (N6), the existence of* $\lim_K \langle e^{tF(G)}\rangle_{\mathfrak{c},X_K}$ *for real* t, *Theorem 19.17);*

(iii) the same holds with R *replaced by any* $R' \in \mathcal{R}_0$.

At $e = 0$ *the statement holds without (N1)–(N6) and without the condition on* E_W.

The hypotheses (N1)–(N5) of the forward bound (Proposition 18.10, Section 18.4) are the statements of Section 19, for the sources (tG, h_0), $t \in \mathbb{R}$, on D_J, for their S^*-images on RD_J, and for their images under the coordinate reflections and permutations on D_J together with the S^*-images of those on RD_J, where $S^* := R\sigma$ is the involution of Lemma 18.3 for R^{-1} in place of R (it exchanges X_K and Y_K and maps D_J onto RD_J). The proof of Theorem 18.8 is Section 18.4. Its steps are the following. One step is the involution of Lemma 18.3 and exact passive-rotation reductions. Another is an equal-spacing comparison, in total form (every field-independent constant included), of the effective actions of the two block-spin flows on King's small-field domain, the standard flow of Y_K and King's hybrid flow of X_K, whose first step averages over rotated blocks. Another is a one-sided forward bound of the form (F) on each torus, from King's upper bound [25, Thm. 3.1] for Y_K and the hybrid lower bound [26, Prop. 4.1] for X_K; the additive large-field term, in the form of the large-field clause of Theorem 19.3, is absorbed on the left by the polynomial lower bound (N5), which at a mass-increasing wall is the wall instance Proposition 19.16. The remaining step is the closing by Theorem 18.5(b) on the mass-increasing family, where the vacuum constants are absent (the comparison is in total form) and the source-linear constants of the pair differ by $o(1)$, the axis-lattice one tending to zero. King's torus bound [26, (4.2)] is not used anywhere in this argument.

The sector in which the sources are coupled through a single scalar function on all plaquettes is false (Proposition 16.14) and is not what the theorem states. The scalar (Wick-source) sector of the torus theorem, which needs the convergence on the torus of the Wick-source functionals with a renormalised Wick square (Section 32.6), is outside Theorem 15.1. Its walled form on the cone, at $g \neq 0$ and outside E_W, is proved in Section 27.3 (Theorem 27.9).

18.4. **Proof of the corrected torus theorem.** The proof compares, in total form, the effective actions produced by the two block-spin flows at one common spacing. We first bound the field-independent part of that comparison, the vacuum constants of the two flows.

For the vacuum families, the connected source-free graphs of the effective action together with the vacuum-energy counterterm E_1 of [25, (3.35)], the two flows are compared in the C/Γ *form*: each line $G_k^F = C - \Gamma^F$ is split into the common free propagator C and the flow-dependent part $\Gamma^F = H^F B^F H^{F*}$ (H^F the external-line kernels of the flow, B^F its block kernel; $\sup|\Gamma^F| \le C_1$), and the renormalised family of each flow is written, exactly, as the sum over connected vacuum graphs G and non-empty line subsets $S \subset G$ of $(-1)^{|S|} G[S \mapsto \Gamma^F,\ G \setminus S \mapsto C]$, plus the graphs with contour vertices: the all-C monomials ($S = \emptyset$) are cancelled exactly by E_1 within each flow, and the Γ-lines play the roles of legs and recipients in Theorem 19.7. The vacuum constant of the effective-action comparison on T splits as $v_K(T) = v_K^{\mathrm{G}}(T) + v_K^{\mathrm{I}}(T)$, the Gaussian part and the interaction part, with $v_K^{\mathrm{I}}(T) = |T|\Delta p_K + r_K(T)$, Δp_K the difference of the two flows' vacuum pressures per coarse site, and $v_K^{\mathrm{G}}(T) = \ln N_k^Y - \ln N_k^X + \ln N_k^Y(0) - \ln N_k^X(0)$ the difference of the Gaussian normalisations of the two flows' effective actions at zero fields ([25, (3.89)] at $A_k = \phi_k = 0$), $\ln N_k = -\frac{3}{2}\sum_p \ln(\Delta_{\mu_0}^{(k)}(p)/2\pi)$ and $\ln N_k(0) = -\frac{N}{2}\sum_p \ln(\Delta_m^{(k)}(p)/2\pi)$ being the

sums over the dual of the common coarse torus of the logarithms of the flow's coarse symbol at the photon and the scalar mass. Here Y is the standard and X the hybrid flow, and $k = k(K)$ is the level of the construction: throughout, $k(K) := \lfloor (1-\upsilon)K \rfloor$ with a fixed $\upsilon \in (0, \upsilon_0)$,

$$\upsilon_0 := \min\Big\{ \frac{\theta_1}{\theta_1 + \frac{7}{2} + \beta_G},\ \frac{\gamma}{\gamma + \beta} \Big\},$$

where θ_1 and β_G are the exponents of Theorems 19.7 and 18.9 and γ, β those of the two-cut-off estimate in step (S4) of the proof of Theorem 19.17, and $K \geq K_0(\upsilon)$ is so large that $k(K) \geq \max(2, k_0)$, k_0 as in Definition 19.2, and that $k(K)$ lies under the cap $k_X(K, S_0)$ of Proposition 19.13. Then $K - k(K) \in [\upsilon K, \upsilon K + 1)$, so $\delta_{k(K)} = 5^{k(K)-K} \in (5^{-\upsilon K - 1}, 5^{-\upsilon K}]$ and, by Convention 19.1, $p(\delta_{k(K)})^2 = b_0^2(1 + (K - k(K)) \ln 5)^{12} \geq b_0^2(1 + \upsilon K \ln 5)^{12}$. Hence, along $k = k(K)$, every power δ_k^σ with $\sigma > 0$ beats every polynomial in K, $e^{-cp(\delta_k)^2}$ with $c > 0$ beats every fixed power of ε_K and every factor $e^{o(K^2)}$, and the quantities $5^{-\theta_1 k}\delta_k^{-7/2-\beta_G}$, $5^{-\gamma k}\delta_k^{-\beta}$, $5^{-2K}\delta_k^{-5}$, $5^{-2k}\delta_k^{-5}$, $\varepsilon_K \delta_k^{-3}$, $5^{-\gamma_L k}\delta_k^{-7/2}$ and $5^{-\gamma_b k}\delta_k^{-1}$ decay geometrically in K, the rates $\gamma_L < \frac{1}{2}$ of Theorem 20.11 and $\gamma_b < \frac{1}{2}$ of Lemma 6.19 being taken in $[\frac{1}{4}, \frac{1}{2})$; here $\theta_1 < \frac{1}{16}$, so that $\upsilon < \frac{1}{57}$. In the unit scale of the level k (the physical spacing $\delta_k = 5^k \varepsilon_K$ of the common coarse lattice is the unit, so $\delta_k = 5^{k-K}$ in the unit scale of the whole construction) the fine spacing is $\eta' := 5^{-k}$ and $\eta := 5\eta' = 5^{1-k}$ is the spacing of the first-step coarse lattice, the letters of the hybrid kernels (Definition 19.8 and the lines before it); τ is the bijection between the first-step cells of the two lattices (so that τ-matched points are corresponding cell points), θ_1 the exponent of Theorem 19.7, $m_0 := \min(m, \mu_0)$, and ϑ_K the rate of the forward bound, Proposition 18.10, which tends to 0.

Theorem 18.9 (the vacuum-family two-flow comparison)**.** *There are $p = p(\bar n)$, $p' = p'(\bar n)$, $\beta_G = \beta_G(\bar n) \leq 3 \cdot$ (number of lines fixed at the top scale) and $C = C(a, m, \mu_0, N, \bar n)$ such that:*

(i) (per coarse site) *for every order at most $\bar n$, every renormalised vacuum family R in the C/Γ form, every $\theta \leq \theta_1$, at τ-matched points of the two lattices,*

$$|E^X(R) - E^Y(R)| \leq C\, p_k^2 \delta_k^{-1/2}\, \eta'^{\theta}\, (1+K)^p\, (m_0\delta_k)^{-\beta_G} \cdot \mathcal{B}_R,$$

where $\mathcal{B}_R := e_k^{d_v} \lambda_k^{d_s}$ times the per-flow constants of [3, (1.33)] is the coupling factor of the family, d_v and d_s the numbers of its charge and quartic vertices, and the prefactor $p_k^2\delta_k^{-1/2}$ is present only when a contour element is switched ($e > 0$);

(ii) (per torus) *$|\Delta p_K| \leq C\, 5^{-\theta k}(m_0\delta_k)^{-\beta_G}(1+K)^{p'}$ per coarse site, and $|v_K^{\mathrm{I}}(T)| \leq \vartheta_K |T|$ for every $T \in \{D_J, RD_J\}$;*

(iii) (the form used) *$|v_K^{\mathrm{G}}(T)| \leq 12.5\, a\, 5^{3K-5k}|T| = 12.5\, a\, 5^{-2K}\delta_k^{-5}|T|$, which tends to 0 at the level $k = k(K)$, since $K - k(K) < \upsilon K + 1$ with $\upsilon < \frac{2}{5}$, so that $5^{-2K}\delta_k^{-5} < 5^5\, 5^{-(2-5\upsilon)K}$; hence, by (ii) and the choice of υ ($5^{-\theta_1 k}\delta_k^{-\beta_G} \to 0$), $\Delta p_K \to 0$ and $v_K(T) \to 0$ on each torus $T \in \{D_J, RD_J\}$ as $K \to \infty$; in particular $\liminf_K [v_K(D_J) + v_K(RD_J)] \geq 0$ for every admissible J.*

The constants are not uniform in the masses: they depend on m, μ_0 through $(m_0\delta_k)^{-\beta_G}$, the price of the lines fixed at the top scale, whose decay is at the rate $m_0\delta_k$ only.

Only (iii) is used: the forward bound of Proposition 18.10 uses ϑ_K through $\delta_K \to 0$, and Theorem 18.5 needs only $v_K(D_J) + v_K(RD_J) \geq -o(1)$, since in the comparison in total form the vacuum constants are absent from the ratio.

Proof. The proof has four parts: the interaction part of the vacuum constant, its Gaussian part, the background determinants, and finally the deduction of clause (iii).

The interaction part, (i)–(ii). The bound is Theorem 19.7 in its vacuum instance: the bound $|\Gamma^X - \Gamma^Y \circ \tau| \le C\eta^{1/2}$ at matched pairs, pointwise, is combined with each Γ^F's decay at the rate $m_0\delta_k$ (the kernel bounds (N2): [25, Thm. 3.3] for the standard flow and Theorem 19.10 for the hybrid flow) into the decaying form $|\Delta\Gamma(u)| \le C\eta^{1/4}e^{-m_0\delta_k|u|/2}$ (the geometric mean), the all-C components of the vacuum monomials carry at least two Γ-ends, and the zero-degree all-C subgraphs are enumerated by the census of Theorem 19.7; the orders λ and $\lambda^2 e^0$ are treated directly. The per-torus bound $|v_K^{\mathrm{I}}(T)| \le \vartheta_K|T|$ follows by summing the per-site bound over the $|T|\delta_k^{-3}$ coarse sites, the finitely many families and ordering classes per order; the torus remainder $r_K(T)$ consists of the winding images of the infinite-lattice kernels, each torus kernel being the periodisation of its infinite-lattice kernel, and is at most $C\varepsilon^{-q}e^{-c\delta_k^{-1}}$ because the shortest period of the torus is at least $50\delta_k^{-1}$ in unit scale; Δp_K is a difference of infinite-lattice vacuum densities, the same for both tori.

The Gaussian part. By [25, (3.89)] at $A_k = \phi_k = 0$ the field-independent part of each flow's effective action is $\ln N_k + \ln N_k(0) + P^{(k)}(0)$, and the counterterm E_1 is the same c-number for both lattices. The coarse symbols $A^X(p)$, $A^Y(p)$ of the two flows obey $|A^X(p) - A^Y(p)| \le 0.20\, a\,\eta^2 A^X(p)A^Y(p)$ at every momentum p and every mass (Woodbury's formula for the composite block average, whose symbol is the product of the two averaging symbols; the terms η^2/a cancel in the difference, and the sum over the aliases of the squared averaging symbol is one), so $|\ln(A^X(p)/A^Y(p))| \le 0.20\, a\eta^2$; summing over the $|T|\delta_k^{-3}$ momenta of the dual coarse torus and the $3+N=5$ components with the factor $\frac12$ gives $|v_K^{\mathrm{G}}(T)| \le \frac12\, a\,\eta^2\delta_k^{-3}|T|$ (the factor is $0.20\cdot 5\cdot\frac12 = \frac12$); with $\eta = 5^{1-k}$ and $\delta_k = 5^{k-K}$ one has $\eta^2\delta_k^{-3} = 25\cdot 5^{-2k}\cdot 5^{3(K-k)} = 25\cdot 5^{3K-5k}$, which gives the bound $12.5\, a\, 5^{3K-5k}|T|$ of (iii).

The background determinants. Besides the perturbative families, the effective actions of the two flows differ in the normalisations of their fluctuation integrals at the background including the Cameron–Martin shift of the curvature source; given the kernel bounds of both flows ([25, Thm. 3.3], Theorem 19.10), the logarithms of the two determinant ratios differ on $\operatorname{supp}\chi_k$, for $T \in \{D_J, RD_J\}$ and at the level $k = k(K)$, by at most $\vartheta'_K|T|$ with $\vartheta'_K \to 0$ along $k = k(K)$, uniformly in J and the torus, ϑ'_K being the gain $5^{-\theta_1 k}$ of the kernel difference bounds (as in Theorem 19.7) against the δ_k^{-3} sites per unit volume, the price $(m_0\delta_k)^{-\beta_G}$ of the lines fixed at the top scale and polynomial factors in K, which the choice of v makes small (each ratio is split over the depths and sites of the flow and written as a second-order Duhamel integral in the background, the first order vanishing by charge-conjugation parity, and the difference of the two flows is telescoped through the resolvents, the kernel difference bounds supplying the gain), a difference which the forward bound of Proposition 18.10 absorbs.

Clause (iii). With $v_K^{\mathrm{G}} \to 0$, $r_K \to 0$, Δp_K the same for both tori and $|D_J| = |RD_J|$, $v_K(D_J) + v_K(RD_J) = 2|D_J|\Delta p_K + o(1)$, so $\liminf_K[v_K(D_J) + v_K(RD_J)] \ge 0$ is equivalent to $\liminf_K \Delta p_K \ge 0$, hence, with the upper half $v + v' \le o(1)$ of Theorem 18.5(a), to $\Delta p_K \to 0$ and to $v_K(T) \to 0$ on each torus. □

The corrected torus theorem is proved by comparing, at one common spacing and depth by depth, the effective action of the standard block-spin flow on the rotated lattice with that of King's hybrid flow on the axis lattice, in total form, every field-independent constant included, and by closing the one-sided bounds so obtained through the exact symmetry of Lemma 18.3 (Theorem 18.5). The theorems named below are the estimates on which this comparison rests. Theorem 19.7 bounds the difference between the values of a renormalised family under the two flows, and Theorem 19.10 gives the hybrid block-spin kernels and their bounds at a regular background. The upper bound (N3), the hybrid lower bound (N4) (Proposition 19.13) and the polynomial lower bound (N5) at the wall (Proposition 19.16) are the hypotheses of the forward

bound, Proposition 18.10, in which the large-field term is absorbed on the left. Theorem 19.17 is the existence of the walled axis-lattice limit, (N6). The vacuum-family comparison (Theorem 18.9) controls the field-independent constants of the two flows. The bound clause of the convention bridge (Theorem 19.6) bounds the difference between the effective actions in King's torus convention and in the free-boundary convention of the counterterm, so that bounds proved in the former apply to the latter. The cumulant bound (Theorem 19.12) controls, at each depth of the flow, the integral of the exponential of a small perturbation against a cut-off measure by its first cumulants.

Theorem 18.8 is proved below from Theorems 19.7–19.17, with Theorems 18.9 and 19.6 and the cumulant bound (Theorem 19.12) applied at each depth of the flow. King's statements enter as citations ([25, Thm. 3.1, Thm. 3.3, Prop. 3.7 within its stated scope, Prop. 3.10, Lemma 4.5]), except the two proved in Section 19 in the forms used (Theorems 19.17 and 20.4). Bałaban's steps left without argument are Theorems 19.3 and 19.4. The two-lattice content of the proof is the following one-sided bound, in which the large-field term is absorbed on the left.

Proposition 18.10 (the forward bound)**.** *Let $T \in \{D_J, RD_J\}$ and let $s = (zG, h)$ be a real source at which the hypotheses (N3), (N4), (N5) of Section 19 and the comparison of the two effective actions on* $\operatorname{supp}\chi_k$ *hold, the latter in the form $e^{-S^{(k)}} \le e^{-\hat{s}^{(k)} - c_K\langle h\rangle_{X_K} + \vartheta_K|T|}$ with $\vartheta_K \to 0$ and a constant c_K independent of T and of the source. Then*

$$Z_{Y_K}(T;s) \;\le\; (1+\delta_K)\, e^{-c_K\langle h\rangle_{X_K}}\, Z_{X_K}(T;s), \qquad \delta_K = \delta_K(T,s) \to 0.$$

No bound on c_K is used.

Proof. Write $I_Y := \int \chi_k e^{-S^{(k)}}$ and $I_X := \int \chi_k e^{-\hat{s}^{(k)}}$. By (N3), King's upper bound with the large-field clause of Theorem 19.3, $Z_Y \le e^{C\delta_k^{\sigma}|T|} I_Y + B$ with $B := \exp(-c_* p(\delta_k)^2 + C|T| + R_K)$, $c_* > 0$ and $0 \le R_K = o(K^2)$, in the form of (N3); the large-field clause of Theorem 19.3 is stated at the stopping depth $r(k) \le k$ of the flow, and $\delta_{r(k)} \le \delta_k$ gives $p(\delta_{r(k)}) \ge p(\delta_k)$, so that the clause implies its form at the depth k used here and in (N3). By (N5), $B \le b\, Z_Y$ with $b := \exp(-c_* p(\delta_k)^2 + (C + C_1)|T| + R_K)\, \varepsilon_K^{-q}$; since $p(\delta_k)^2 \ge b_0^2(1 + \upsilon K \ln 5)^{12}$ along $k = k(K)$ (Convention 19.1) while $R_K + qK \ln 5 = o(K^2)$, $b \to 0$, so $Z_Y \le (1-b)^{-1} e^{C\delta_k^{\sigma}|T|} I_Y$. The large-field term is thus absorbed on the *left*, which is why no bound on c_K and no lower bound for the axis lattice is needed. On $\operatorname{supp}\chi_k$ the comparison of the effective actions gives $I_Y \le e^{-c_K\langle h\rangle + \vartheta_K|T|} I_X$, and by (N4), Proposition 19.13, $I_X \le e^{C(1+K)^{p'}\delta_k^{\sigma}|T|} Z_X$. So $\delta_K = (1-b)^{-1} e^{(2C(1+K)^{p'}\delta_k^{\sigma} + \vartheta_K)|T|} - 1 \to 0$, since $(1+K)^{p'}\delta_k^{\sigma} \to 0$. □

Proof of Theorem 18.8. The proof has six steps: we first compare the effective actions of the two flows, then establish the forward bounds on both tori, sandwich the two partition functions through the involution, pass to complex coefficients and moments, deduce the existence of the limit, and finally treat the other generators. Fix $\mathfrak{c} = (J, \psi, n)$ and G, and let $\mathcal{S}^- := \{(tG, h_0) : t \in \mathbb{R}\}$, $h_0 = -n\psi$, the *mass-increasing family*; the sources $(tG, +n\psi)$ do not occur.

Step 1 (comparison of the effective actions). The two flows, the standard flow of Y_K and the hybrid flow of X_K, are run to one common spacing, and their effective actions on $\operatorname{supp}\chi_k$ are compared term by term in total form, every field-independent constant included: by the two-flow comparison bound (N1), Theorem 19.7, applied to every renormalised family of [25, §3.5] in per-flow normal form, by the kernel bounds (N2) of both flows and by the difference bounds of Theorem 19.10, $e^{-S^{(k)}} \le e^{-\hat{s}^{(k)} - c_K\langle h\rangle_{X_K} + \vartheta_K|T|}$ on D_J for $s \in \mathcal{S}^-$ and on RD_J for S^*s (Lemma 16.15 transports the own-frame sources, and the S^*-image of a wall is a wall on RD_J), with one constant c_K for both tori; the h-linear field-independent families, excepted from Theorem 19.7, are exactly the terms collected in $c_K\langle h\rangle$, and the vacuum constants are bounded by Theorem 18.9(iii).

Step 2 (forward bounds). Proposition 18.10 on D_J and on RD_J for these sources. Its hypotheses (N3), (N4), (N5) hold on both tori: on D_J every Y-frame step is a skew step, so (N3) is the instance [SKEW-U] and (N4), (N5) are instances of [SKEW-L]; on RD_J only the first hybrid step is skew, with image terms $O(e^{-c\,50MJ/\varepsilon})$. The wall instance of (N5) at $h_0 = -n\psi$ is Proposition 19.16.

Step 3 (sandwich). Pull the forward bound on RD_J back to D_J by the involution S of Lemma 18.3 ($S^{*2} = 1$): $Z_{X_K}(D_J; s) \le (1 + o(1))e^{-c'_K\langle h\rangle_{Y_K}} Z_{Y_K}(D_J; s)$ with $c'_K = c_K + o(1)$. Together with the forward bound on D_J this gives $e^{c'_K \int h}(1+o(1)) \le Z_{Y_K}(D_J; s)/Z_{X_K}(D_J; s) \le e^{-c_K \int h}(1+o(1))$ for $s \in \mathcal{S}^-$, the lattice sums $\langle h\rangle_\Lambda$ differing from $\int h$ by $O(\varepsilon_K^N)$. The axis-lattice constant has upper limit at most zero, $\limsup_K c_K \le 0$ (Section 18.3), so $\limsup_K(c_K + c'_K) \le 0$, which is the hypothesis of Theorem 18.5(b): since $\int h_0 < 0$ the interval is non-empty only if $c_K + c'_K \to 0$, and then the ratio identity of the same clause, applied to the balanced pair $s_1 = (tG, h_0)$, $s_0 = (0, h_0)$, gives

$$u_K(t) := \langle e^{tF(G)}\rangle_{\mathfrak{c},Y_K} - \langle e^{tF(G)}\rangle_{\mathfrak{c},X_K} \to 0 \qquad \text{for every real } t;$$

no closure of $\mathcal{S}^-$ under $h \mapsto -h$ is needed, and for $n = 0$ there is no h and no c.

Step 4 (complex coefficient, moments). By Gaussian domination, $1 \le \langle e^{tF}\rangle \le e^{t^2 q_\Lambda/2}$ with $q_\Lambda \to q(G)$ on both lattices, and $|\langle e^{zF}\rangle| \le \langle e^{\mathrm{Re}\, z\, F}\rangle$; so u_K is bounded on strips uniformly in K and tends to 0 on $\mathbb{R}$, Vitali's theorem gives (i), and Cauchy's estimates the moments.

Step 5 (existence). Under (N6), Theorem 19.17, the axis family converges for real t; by the same domination it converges locally uniformly on $\mathbb{C}$ to an entire function $W_{\mathfrak{c}}(zG)$, and by (i) so does the rotated family, to the same limit. This is (ii).

Step 6 (other generators). $\sigma X_K = X_K$, $\sigma D_J = D_J$ and $\sigma R^{-1}(\varepsilon\mathbb{Z}^3) = R(\varepsilon\mathbb{Z}^3)$, and $R_{\mu\nu}(\pm\theta_0) = \Pi R^{\pm1}\Pi^{-1}$ for a permutation matrix Π preserving X_K and D_J; by Lemma 16.15 with $O = \sigma$, resp. $O = \Pi$, the statement for (R, D_J) and all sources is equivalent to the statement for (R', D_J) and all sources, the hypotheses for the images being those the theorem assumes. This is (iii). At $e = 0$ the hopping term does not contain A, so $\langle e^{zF(G)}\rangle_{\mathfrak{c},\Lambda} = e^{z^2 q_\Lambda(G)/2}$ exactly for both lattices, with $q_\Lambda(G) \to q(G) := \int_{D_J}\langle d^*G, (-\Delta + \mu_0^2)^{-1} d^*G\rangle\, dx$, a frame-free quantity. □

The closing uses no reverse step, no comparison of two spacings, no Wick scheme, no convergence of the rotated family on its own and no infinite-volume limit. Its two-lattice content is Steps 1–4, and existence is a one-lattice question, Step 5. The group generated by $\mathcal{R}_0$ and the cube group is dense in $O(3)$, because $\theta_0/2\pi$ is irrational.

Remark 18.11 (the content of (K_0)). Theorem 18.1 is the curvature sector of [26, Thm. 2.4] with the corrections of Remark 20.12, for walled tori and with free-boundary counterterms; by Corollary 18.6 its two-lattice content is the one-sided bound (F_0), and its one-lattice content is the existence of the walled limits on the axis lattice. By Theorem 18.8, (K_0) holds for every $R \in \mathcal{R}_0$ and every configuration of the class $\mathcal{C}$, by the statements of Section 19 with Propositions 26.2 and 26.8 of Section 26, outside the set E_W there; the interpretations (B1)–(B6) enter as clauses of Theorem 19.3. Thus (K_0) is proved in this section (Section 18) alone, through Theorem 18.8, from the statements of Section 19 and the proof in Section 18.4; nothing from Sections 21–23 enters it. The rotation enters the whole argument of this part only here.

Remark 18.12 (what the torus comparison alone does not give). Given (K_0), the walled values of a configuration $\mathfrak{c}$ on D_J agree on the two lattices, and by passive covariance the walled value of $R \cdot \mathfrak{c}$ on the rotated torus $R\,D_J$ for the axis lattice equals the walled value of $\mathfrak{c}$ on D_J for the rotated lattice. Rotation invariance of a limit along tori would therefore follow from (K_0) if the limit did not depend on the shape and orientation of the tori, which is the content of King's expected identity [26, (2.24)] between torus and cube limits. That identity is a uniqueness

statement for the infinite-volume limit, and we do not use it. The walls of Section 21 replace it by a one-sided statement about one lattice, (TIGHT$_0$), whose local form (RES$_0$) is the hypothesis of Theorem 23.10.

19. The hypotheses of the torus theorem

This section contains the statements behind the torus comparison of Section 18, row (R1) of Table 1. They are statements of the renormalisation-group analysis of Bałaban and King, stated and proved here, each with its source, in the forms in which the proof of Theorem 18.8 uses them. A reader who first follows the sections of rows (R0)–(R4) of Table 1 may take them as given.

Theorem 18.8 is proved in Section 18.4 from the exact reductions, Theorem 18.9 and Proposition 18.10, and the statements of this section: the hypotheses (N1)–(N6), the skew-torus instances (SK) of King's and Bałaban's bounds, Bałaban's estimates at blocking factor five, (BAL)$_5$, and the convention bridge (CT), stated and proved in the order in which each rests on the ones before it; the corrections to the cited sources are collected in Remark 20.12. Throughout, $\mathfrak{c} = (J, \psi, n)$ is a configuration of the class $\mathcal{C}$ (Definition 16.6), k the depth of a flow, K the number of scales, and [SKEW-U], [SKEW-L] the skew-torus instances of (N3)–(N5); the shell-side estimates are in Sections 24 and 25.

Convention 19.1 (the exponent of the small-field function)**.** The small-field function of King's block-spin bounds [25, Thm. 3.1] is $p(\delta) = b_0(1 + \log \delta^{-1})^p$ with a fixed integer exponent, $p = 6$ throughout; any fixed integer $p \geq 4$ would do, the lower bounds in the literature being $p > 1$ ([25, (3.41)], King's standing inequality), $p \geq 2$ ([1, (3.54)], kept in [25, p. 676]) and $p \geq 4$ ([2, pp. 557, 594]: "$p > 2$" and "$2p \geq (d+1)r$" with $r \geq 2$, $d = 3$). Every large-field bound of this part uses p only through $p(\delta_{r(k)})^2$, which grows at least quadratically in k, and [25, (3.41)] uses only $p > 1$.

19.1. **Bałaban's estimates at blocking factor five.** The two block-spin flows compared in Theorem 18.8 have block factor 5 because $5R$ is an integer matrix: one step of side 5 brings the rotated lattice Y_K and the axis lattice X_K to a common coarse lattice (see the subsection on the hybrid kernels below). Every estimate taken from Bałaban must therefore hold at this block factor, on the skew tori D_J, RD_J as well as on aligned tori, and with constants that do not depend on the number of steps already taken; this is what the two theorems below assert. What the torus theorem needs from these estimates is of three kinds: the large-field clause of King's upper bound, which is the additive term of (N3), with the constant $c_* > 0$ that governs the rate at which (N5) absorbs that term; the propagator theorem of [4] and the regularity of the gauge background along the flow, on which the kernel bounds (N2) rest; and the steps of Bałaban's lower-bound argument, on which the lower bounds (N4) and (N5) rest.

The estimates of [1, 2, 3, 4] enter at block factor $L = 5$ ($d = 3$, $a = 1$, $r = 2$, $p = 6$), on the aligned tori and on the skew tori D_J, RD_J of Section 16; since the proofs of [2, 3, 4] leave steps to the reader and several of Bałaban's statements are used in an interpretation that his text does not fix, the following definition fixes the estimates meant, the stopping depth and the large-field constant, and the two theorems after it collect what is used.

Definition 19.2 (the estimates used, the stopping depth and the large-field constant)**.** The *estimates used* are the estimates of [1, 2, 3, 4] that are claimed there without a complete argument and that the proofs of this part use: the four steps of [2, 3] left without argument, namely the cube step [3, p. 433], the cut-off $\zeta^{(k)}$ of [2, (2.44)], the boxes of [2, Lemma 2.4] and the cube partition [2, (3.43)–(3.45)]; the steps behind [25, Thm. 3.3], [1, (1.14)], [25, Prop. 3.6] and [25, Thm. 3.1] named in Theorem 19.4(l), in Proposition 19.16, in Theorem 20.4 and in Theorem 19.6(c); the

completions of [3, pp. 433–445] and [2, §3C] named in the proof of Theorem 19.3; and the interpretations (a)–(f) of Bałaban's statements in Theorem 19.3. The *stopping depth* $r(k)$ of a flow of depth $k \geq k_0$ is such that $k - r(k)$, the number of steps left to the flow, is a fixed fraction, less than one ninth, of k, rounded down. Throughout, $p(\delta) = b_0(1 + \log \delta^{-1})^p$ at the fixed integer p of Convention 19.1 (the function $p(\cdot)$ and the integer p are told apart by the argument). The *large-field constant* is a constant $c_* > 0$: $c_* = c_0'/2$ at zero source, c_0' the constant of Bałaban's large-field bound [2, (3.41)] at block factor 5 (at most $\min\{a/8, \gamma_*/4, 1\}$), any $c_* < c_0'/2$ at a non-zero curvature source (the constants of the bound then depending on c_*) and $c_* = (c_0'/2)(1 - 1/p_H)$ at a non-zero wall (every use needs only $c_* > 0$: $p(\delta_{r(k)})^2$ grows at least quadratically in k).

Theorem 19.3 ((BAL)$_5$: Bałaban's estimates at blocking factor five)**.** *Every estimate used (Definition 19.2) holds as stated at blocking factor* 5*, with per-step constants that depend on the step only through the running scale (scale-local), hence uniformly in the initial spacing, including the large-field clause of King's upper bound* [25, (3.4)] *on the skew torus: the large-field contribution to the partition function at the stopping depth $r(k)$ of the flow, $k \geq k_0$ (and at every admissible depth for the standard flow on D_J), is at most* $\exp(-c_* p(\delta_{r(k)})^2 + C|T| + o(K^2))$ *(Remark 20.12(i)), with the large-field constant $c_* > 0$ of Definition 19.2. In particular the following hold.*

*(a) (*Vector-field propagator bounds on the periodic torus*,* [3, (2.10)–(2.12), p. 426]*.) The torus kernels of Bałaban's vector propagators obey the pointwise bound $C l_j^{-1} e^{-\delta_1 |x-x'|_T / l_j}$, the bounds with one and two differences (l_j^{-1} per difference) and the Hölder clause in the quotient metric; a leg integrated along a contour is bounded by the sum of the pointwise bounds along it.*

(b) ([3, p. 437]*.) The massive nearest-neighbour Green's function and its first differences decay exponentially, at a rate $c_{\mathrm{G}} > 0$ depending only on the mass and the dimension (its value is not used).*

*(c) (*The local term and the first-moment step*,* [3, (3.33)–(3.38), (3.26)–(3.28)]*.) The local term of the primitively divergent classes with two scalar legs and one vector leg is bounded, window by window, and its cube sum converges at every window; and with the wall in the covariance, at $A = 0$, on any rectangular torus and at zero shell mass, the first moment $\Pi[G](y)$ of the vacuum-polarisation bubble satisfies $\sup_y |\Pi[G](y)| \leq C \operatorname{Tr} q^2$ uniformly in the spacing, the torus and the wall.*

*(d) (*The integral over the vector fields in the completion of [2, §3C, p. 591]*.) Given the scalar half of* [2, (3.39)]*, the vector half of the passage to* [2, (3.40)] *closes without vector characteristic functions, with $\gamma_* = \min(\gamma_0^s, \gamma_v)$ in* [2, (3.41)]*, at the cost $(d/2) \ln 2$ per site, the Gaussian renormalisation cost, and with no remainder; the constants of the completion of* [2, §3C] *at block factor* 5 *are explicit, and the bounds that* [2] *writes as $O(1)$ without qualification are uniform in the scale index. The count* [2, (3.45)] *and the first inequality of* [2, (3.46)] *are false as stated in* [2]*, at every block factor, and are not used: a corrected count of the cubes replaces them.*

*(e) (*Regularity of the gauge background*,* [1, Prop. 2.1]*,* [4, (1.7)]*,* [2, Lemma 2.3]*.) On the propagator domain the gauge background is regular at block factor* 5 *(Proposition 19.5), and the uncut background is regular within one block of it: one block beyond the propagator domain $|\partial_\mu^\eta A_\nu^{(k)}| \leq C_Z p_k$ with $C_Z = C_1 c_1 (1 + \log 5)^p + 1$, so that King's regularity* [25, (3.5)] *holds at scale k on the margin with a constant free of k, ε, Ω and κ.*

(f) *(*Carrying the scalar block-field bound to the next step, [2].*)* *The cap is inherited, and the gradient line* $|D_{\bar A^{(k)}}\phi_k(b)| \le c_1^{\mathrm{sc}} p_{k-1}$ *holds on the unit-k bonds b of* $\Lambda_2^{(k-1)\prime}$ *at* $k \ge 2$*, with* $c_1^{\mathrm{sc}} = 3d\cdot 5^{3/2} + 5C'_\Delta e(1+\lambda^{-1/4})\kappa_*$, $\kappa_* = \sup_{\ell\le 1}\ell^{1/4}p(\ell)$*, scale-free.*

Proof. We first describe how the four steps of Bałaban's proofs are repeated at block factor 5, then prove clauses (a)–(f) one by one, and finally treat the large-field clause. The proofs of [1, 2, 3, 4] are repeated at block factor 5, each scale treated as the first in its running spacing (whence scale-local constants), with the four steps written out: the cube step of [3, p. 433] by the end-functional bound of the shell-model repetition at zero shell mass (Theorem 25.5, resting on clauses (c), (e), (f) of this theorem; this is the completion of [3, pp. 433–445]), after which the bounds hold on the torus and on every propagator domain with domain-independent constants; the cut-off $\zeta^{(k)}$ at $\theta = 2/5$, where Proposition 19.5 gives the regularity of the background; the boxes of [2, Lemma 2.4] and the cube partition with the corrected cube count of (d) (the completion of [2, §3C]); and the identification step, in which the per-step determinants compose exactly, by [2, (2.105)], into King's matter determinant, bounded through the first-order clause of Theorem 24.6, King's action at Bałaban's cut and at the uncut background differing on the small-field set by at most $Ct_k^s|T^{(k)}|$ (Proposition 24.7(b)). Clause by clause: (a) is periodisation of the infinite-lattice kernels of [1, 2]; (b) is Combes–Thomas; (c) follows from the ℓ^1 row sum of the window resolvent and the ℓ^2 bound of the photon line, from Theorem 25.1 with the Ward identities (Lemma 25.2(b), (c)) for the cube sum, and from the reflection $\Pi[G] = -\Pi[G^I]$ up to the reflected kernel and the two wrap-around planes; (d) follows from (a), Proposition 19.5 and the completions named in (d)–(f); (e) is Proposition 19.5 on the propagator domain and, one block beyond, the McShane extension of the restricted field with [2, Lemma 2.3] at $\Lambda_0 = T$ through the three-layer containment [2, (2.7)–(2.8)]; (f) compares the transport $U(e_k a)$ along the bond with Bałaban's line $\bar A_b^{(k)}$. In the large-field clause the bound on the free massive lattice propagator and its first three differences, uniform in the spacing and the mass, is standard, Hölder's inequality separates the wall term (whence $c_* = (c_0'/2)(1-1/p_H)$), the propagators of [2, Prop. 2.6] are replaced on the skew tori through Theorem 19.4(k), the forms for the contour families of the flows and at the walls follow from Propositions 26.2 and 26.8 outside E_W under the final-scale cap of the class $\mathcal{C}$ (Proposition 26.2 also through (N4) and the comparison of the effective actions), and Bałaban's bulk estimate [2, (2.113)] on King's cube with the Dirichlet weight acquires a face correction and a first-scale lift form by the proof of [2, Lemma 2.7] run from the exact identity $aL^{-2}(C_\Lambda f) - f = -C_\Lambda \Lambda\Delta^{(k)}\Lambda f$. □

Theorem 19.4 ((BAL)$_5$, continued: the completions on the skew tori, at the wall and across domains, and the propagator theorem of [4])**.** *The following hold at blocking factor* 5*, with per-step constants that are scale-local as in Theorem 19.3, whose lettering continues.*

(g) *(*[3, p. 412].*)* *The coarse-covariance decay bound holds in product form.*

(h) *Bałaban's steps behind the estimates of Theorem 19.3 and of this theorem and behind the kernel bounds of Theorem 19.10 hold on the skew tori and in the boundary boxes.*

(i) *The Hölder clause of the shell estimates (Theorem 24.5) and of the renormalisation bound for the hybrid flow (Proposition 19.13) holds.*

(j) *(*The renormalisation bound with the scalar wall, [3, Prop. 1, (1.33)].*)* *Bałaban's renormalisation bound holds for the operator with the scalar wall* $0 \le v \le \bar v$*, with constants polynomial in* $\bar v$*.*

(k) *(*The change of domain, [2, Prop. 2.6].*)* *On the skew tori the change of domain of Bałaban's propagators between nested domains decays with the depth of its two points, exponentially in both depths.*

(l) *(*The propagator theorem of [4].*) The Theorem of* [4]*, (1.9)–(1.12) there (equivalently* [1, Prop. 2.1] *and* [25, Thm. 3.3]*), holds at $L = 5$, for every $\tilde{a} \in [a(1 - L^{-2}), a]$ and every constant mass $m^2 \geq 0$, in its restricted scope: at points at distance at least R_0 from the complement of the domain, with no restriction on a torus, and with δ_0, c_0^{Ba}, R_0 independent of A, k and the domain (the unrestricted clause of* [4] *for rectangular parallelepipeds at a general regular A is neither claimed nor used). The three steps that* [4] *states without argument hold: (i) the renormalisation-group identity* [4, (2.34)]*, which is* [1, (2.43)]*,*

$$G_k(\Box) = C^{(0),\eta}(\Box) + \sum_{j=1}^{k-1} a_j^2 (L^j \eta)^{-4}\, G_j^{\eta} Q_j^* C^{(j),L^j\eta} Q_j G_j^{\eta},$$

the iterate of the one-step identity $G_{j+1} = G_j + a_j^2 G_j Q_j^* C^{(j)} Q_j G_j$ with $C^{(j)} = (\Delta^{(j)} + aL^{-2}P)^{-1}$ and $\Delta^{(j)} = a_j - a_j^2 Q_j G_j Q_j^*$, holds exactly on every domain Ω (Neumann conditions preserved at all scales), for every regular A and every constant $m^2 \geq 0$, in any normalisation in which $P_k = Q_k^* Q_k$ and $P = Q^* Q$ are projections and $Q_{k+1} = Q Q_k$; (ii) the analyticity neighbourhood in [4, Lemma 2.4] *(p. 586) is uniform in j: the symbol of the block-averaged propagator is analytic and at least $\frac{1}{2} a_*(2/\pi)^{2d}$ in modulus on a strip of width κ_* independent of j and L, which gives the decay $e^{-\kappa_* |x-y|}$ and the Hölder clause of* [4, (2.35)–(2.36)] *with $\delta_0 = \kappa_*$; (iii) the gap "π^2" of* [4, (2.27)] *is false for a finite block, the lattice Neumann gap being $4N^2 \sin^2(\pi/2N) \in [8, \pi^2)$ for a block of N points per side, and the lower bound 4 suffices for every use. With these three steps the proof of* [4, §§2–3] *runs as written, its constants depending on d, a, M, L (c_0^{Ba} also on the Hölder exponent α) and not on A, k or Ω; for $\tilde{a}$ off Bałaban's orbit of coefficients, the bare value $a' = \tilde{a}(1 - L^{-2k})/(1 - L^{-2})$ has $a'_k = \tilde{a}$ with all $a'_j \in [0.9216\, a, 1.0417\, a]$.*

Proof. (g)–(i) are the completions of Bałaban's steps on the skew tori and in the boundary boxes, with the Hölder constant kept explicit; (j) holds since the wall enters [3, (1.33)] only through the scalar propagators, whose bounds improve under a non-negative potential, and the walled counterterms; (k) is the resolvent identity through the boundary of the inner domain with Bałaban's Green-function and block-operator decay. In (l) the renormalisation-group identity is the Woodbury identity for $H_{k+1} = H_k - Q_k^* W Q_k$, $W = a_k - a_{k+1} L^{-2} P$, the j-uniformity is Paley–Wiener and the gap is the explicit lattice Neumann eigenvalue; the bounds of [4, §§2–4] pass to the admissible lattice tori and to unions of large blocks by the random-walk transfer, only the restricted clause holding within R_0 of the complement. □

Proposition 19.5 (the regular background on the propagator domain at block factor five)**.** *Let $L = 5$, $r = 2$, let the localisation radius satisfy $R \geq \max\{5(2M + 1), C_Z(M + s)\}$ with C_Z an absolute constant and $R \geq 10(M + 1)$, and let Bałaban's smoothing cut-off ζ_θ of* [2, (2.44)] *be taken at $\theta = 2/5$ (the cut-off condition (Z_θ)). At every step k of the upper bound, on the propagator domain $\Omega = B^k(\Lambda_2^{(k-1)\prime})$ of that scale, the gauge background obeys* [2, Lemma 2.3, (2.59)–(2.60)] *with $O(p(L^k\varepsilon))$ replaced by $O_L(p(L^k\varepsilon))$: for $x \in B^k(y)$, $y \in \Lambda_2^{(k-1)\prime}$,*

$$A^{(k)}(x) = A(y) + O_L(p(L^k\varepsilon)) = (Q_k^* A)(x) + O_L(p(L^k\varepsilon)), \quad (\partial_\mu^{\eta} A^{(k)})(x) = O_L(p(L^k\varepsilon)) \quad (x \in \Omega),$$

where the constant contains the factor $(1+\ln L)^p = 315.7$ ($p = 6$) and otherwise only the constants of [2, (2.55)] *and of* [2, Prop. 2.2]*. The cut field $\theta_k A^{(k)}$ obeys the same bounds on Ω, since $\theta_k = 1$ there, together with their Hölder clause. Consequently the configuration $A^{(k)}$ on Ω satisfies the regularity assumption of* [1, Prop. 2.1]*, which is what the propagator bounds of the step require. On the tori of the construction the same holds in the quotient metric.*

Proof. For $y \in \Lambda_2^{(k-1)\prime}$ and $x \in B^k(y)$ the ζ_θ-ball lies in the rescaled $\Lambda_{-1}^{(k-1)\prime}$, where the restrictions [2, (2.55)] hold (this needs $\theta < 3/L$), and there the argument of [2, Lemma 2.3] runs with ζ_θ, the previous scale's restrictions carrying $p(t_{k-1}) \le (1+\ln L)^p p(t_k)$, the origin of the constant. The split [2, (2.89)] is exact with $\zeta_\theta^{(k+1)}$ only if $L\theta \le 2$, so $\theta = 2/5$ is the largest admissible value at $L = 5$, and the lower bounds on R give the transition width $\frac{1}{2}\theta r(t_k) - 2M \ge 2$; with the cut-off as written in [2] the lemma fails at $L = 5$ (the restrictions reach only depth $3r(t_{k-1})/L - 2$, which exceeds $r(t_k) - 2M$ at every scale only if $L \le 3$), and the repair costs $(1 + \ln L)^p$, absorbed into the class constants fixed before M. □

19.2. **The convention bridge (CT).** King's torus paper [25], from which the upper bound (N3) and King's small-field expansion are taken, fixes the counterterms by conditions on the torus, whereas the model of this paper, like King's free-boundary paper [26], is written with free-boundary counterterms; the two models differ by a constant shift of the mass term and by a field-independent factor. The convention bridge makes this difference exact and bounds its effect on the effective actions of both flows at level k, on the small-field set, by $C_\sigma \delta_k^\sigma |T|$; this is what allows King's upper bound and the hybrid lower bound, proved in the torus conventions, to be applied to the free-boundary partition functions that Theorem 18.8 compares.

(CT) (Remark 16.8) bridges the torus counterterm conventions of [25] and the free-boundary conventions of [26, p. 330]: (a) exactly, the torus model is the free-boundary model with the mass term shifted by $\frac{1}{2}\Delta_T \sum s^3 |\phi|^2$ and multiplied by $e^{-\Delta E_T |T| - N \Delta c_T \langle h \rangle}$; (b) the bound clause, Theorem 19.6, the decay of the difference terms, carries King's upper bound and the hybrid lower bound from the torus-convention model to the free-boundary model. The bridge constants enter through a single bound Δ_* on $|\Delta_T(s)|$ for $s \le s_0$ (also the constant of the wall condition of the class $\mathcal{C}$, and all that $(\mathrm{CT})_{\mathrm{sh}}$, Lemma 24.1, uses on the shell side) and a bound on the shift $\Delta c_T(s)$ of the Wick constant for $s \le s_0$; clause (c) of the theorem is used by (N3) and (N4), the rate clause (d) only through (N6). In the effective action at level k the mass shift generates the *difference terms* D_k^F: the renormalised families of King's expansion (King order at most 13) with $j \ge 1$ vertices $u^{(\alpha,\beta)}$ carrying the order-(α,β) part of Δ_T, $|u^{(\alpha,\beta)}| \le u_0 := \sup_{s \le 1} |\Delta_T(s)|/2$; counted with the u-vertices, their free-boundary order $\sum_i t_i + n_h + n_\lambda + m_\Sigma$ is at least 14, where $m_\Sigma := \sum_v m_v$ is the sum of the photon orders m_v of the charge vertices.

Theorem 19.6 (the bound clause of the convention bridge)**.** *For $K \ge K_0$, $k = k(K)$, both flows $F \in \{X, Y\}$, both tori $T \in \{D_J, RD_J\}$, every configuration in* $\operatorname{supp} \chi_k$ *and every $|u^{(\alpha,\beta)}| \le u_0$,*

$$\sup_{\operatorname{supp} \chi_k} |D_k^F| \;\le\; C_\sigma \, \delta_k^\sigma \, |T| \qquad \text{for every } \sigma < 1,$$

two-sided, the field-free (A_k-dependent, ϕ_k-free) part included; C_σ is independent of K, T and the configuration, and uniform in the source through u_0 and S_0. In particular the upper bound (N3) and the hybrid lower bound (N4) below, proved for the torus-convention model, hold for the free-boundary model, the forward bound of Section 18.4 tolerating any fixed $\sigma > 0$ and any polynomial prefactor in K. In addition:

- *(c) (*Uniformity under a constant source shift.*) The constant of King's upper bound* [25, Thm. 3.1] *is uniform under a shift of the scalar source by a bounded constant (King's statement that the constant depends on g, h and $|T|$), and King's statement that the small-field analysis includes the sources* [25, p. 677] *holds in the form used: the small-field half of* [25, Thm. 3.1]*, in King's vertex form* [25, (3.14)–(3.21)]*, holds at every fixed source of bounded size, on King's aligned tori and on the rotated torus, for King's flow and for the centred flow, and for the hybrid flow in its lower half, with a cap depending on the number of scales.*

*(d) (*The rate.*)* $|\Delta_T(s) - \Delta_T(0)| \le Cs$ *for spacings* $s \le s_0$*, where* $\Delta_T(s)$ *is the constant of the convention bridge at spacing* s *and* $\Delta_T(0)$ *its limit; the rate is read off the counterterm differences* Δ_T*,* ΔE_T*,* Δc_T *of the two conventions.*

Proof. In the unit scale of level k a connected graph of the difference terms with $j \ge 1$ vertices u, n_h source vertices, n_λ quartic vertices, charge vertices of total photon order m_Σ and s external scalar legs carries δ_k^P, $P = 2j + 2n_h + n_\lambda + m_\Sigma/2$ ([25, p. 657], [3, (1.33)]), the position sums against the external legs cost $\delta_k^{-1/4}$ each and the volume $\delta_k^{-3}|T|$, and connectivity with the order bound $\sum_i t_i + n_h + n_\lambda + m_\Sigma \ge 14$ gives $Q = P - s/4 - 3 \ge j \ge 1$; the logarithms go into δ_k^σ, the uniformity in K, T and the source is that of King's small-field bounds given u_0 and S_0, the position sum of the families with external legs is the renormalisation bound [3, (1.33)] at depth k (slice lines through Theorem 20.4), and the field-free families of the hybrid flow need a vacuum localisation with one u-vertex held fixed, which gives $C(1+K)^p\delta_k^4|T|$, the form that (N4) needs. Clause (c) is the block-factor-five repetition of the proof of [25, Thm. 3.1] with Theorems 19.3 and 19.4 and King's statements [25, Thm. 3.3, Props. 3.7–3.10, Lemma 4.5], the small-field half in King's vertex form absorbing the source at every scale. Clause (d) is the Lipschitz dependence on the spacing s of the constant of the convention bridge, read off the counterterm differences of the two conventions by Reisz's power counting for the Wick-ordered torus one-point graphs through order 13, uniformly in the scalar mass on a compact interval of King's range, only the coefficient of order $e^2\lambda^0$ being logarithmic. □

19.3. **(N1) The two-flow comparison bound. (N1)** *(the two-flow comparison bound).* Let $T \in \{D_J, RD_J\}$, and let the standard flow of Y_K and the hybrid flow of X_K be run on T to a common depth k, in the unit scale of level k. Let $\alpha \in (0, 1/\bar n)$ be a Hölder exponent for which the kernel bounds (N2) hold, and put $\theta_1 := \alpha^2/(4(1+\alpha)^2)$. There are $q_1 = q_1(\bar n)$, $\beta' = \beta'(\bar n) \ge 0$ and C such that for every renormalised family $\mathcal{R}$ of King's expansion of order at most $\bar n$, in per-flow normal form, of one of the three kinds that the comparison meets (the h-linear families other than the field-independent ones, the families of King's constant-background reduction at $e > 0$, and the vacuum families in their C/Γ form with the Γ-lines as legs and recipients), and for every $0 < \theta \le \theta_1$,

$$|E^X(\mathcal{R}) - E^Y(\mathcal{R})| \;\le\; C\, n_{\mathcal{R}}\, p_k^2\, \delta_k^{-1/2}\, \eta'^{\theta}\, (1+K)^{q_1}\, \delta_k^{-\beta'} \cdot \big[\text{the right side of [3, (1.33)] for } \mathcal{R}\big],$$

uniformly in k, K and the torus, where $E^F(\mathcal{R})$ is the value of the family on the flow $F \in \{X, Y\}$, $\beta' = 0$ for the h-families and the families of the constant-background reduction, and $\beta' \le \beta_G = \beta_G(\bar n) \le 3 \cdot$ (number of lines fixed at the top scale) for the vacuum families, whose constants depend on the masses through $m_0^{-\beta_G}$, $m_0 := \min(m, \mu_0)$; the other constants are uniform in the masses.

The two flows are different renormalisation-group transformations: the standard flow of the rotated lattice Y_K averages over blocks of its own frame, while the hybrid flow of the axis lattice X_K averages in its first step over rotated blocks and then continues as the standard flow. At a common depth k both effective actions are sums of the same renormalised families of King's expansion, each evaluated with the lines, propagators and block-spin kernels, of its own flow, and (N1) says that the two values of any one family differ by a positive power η'^{θ} of the fine-to-coarse scale ratio, up to polynomial factors in K and inverse powers of δ_k, relative to the size that Bałaban's bound [3, (1.33)] gives the family. The torus theorem needs it in the first step of its proof, the comparison of the two effective actions on the small-field set, term by term and in total form, every field-independent constant included: with the kernel bounds (N2) and the difference bounds of Theorem 19.10(ii), summed over the families and the coarse sites of the torus, (N1) yields $e^{-S^{(k)}} \le e^{-\hat s^{(k)} - c_K\langle h\rangle_{X_K} + \vartheta_K|T|}$ with $\vartheta_K \to 0$ and one constant c_K for both tori,

the field-independent h-linear families, which (N1) excepts, being the terms collected in $c_K\langle h\rangle_{X_K}$ and the vacuum constants being bounded by Theorem 18.9. Without (N1) the small-field parts of the two partition functions would be unrelated.

Sources and proofs. (N1) is the comparison of the values on the standard and on the hybrid flow of the renormalised families of [25, §3.5] with lines taken from the two flows, in per-flow normal form, Theorem 19.7; its one-flow instance is King's, in the proof of [25, Prop. 3.6], p. 665 (Theorem 20.4).

Theorem 19.7 (the two-flow comparison bound)**.** *Let $\alpha \in (0, 1/\bar{n})$ be a Hölder exponent for which the kernel bounds of Theorem 19.10(ii) hold (any exponent smaller than King's serves, the Hölder clause at a given exponent implying it at every smaller one on pairs at distance at most 1), and put*

$$\theta_1 := \frac{\alpha^2}{4(1+\alpha)^2}.$$

There are $q_1 = q_1(\bar{n})$, $\beta' = \beta'(\bar{n}) \geq 0$ and C such that for every renormalised family $\mathcal{R}$ of order at most $\bar{n}$ in per-flow normal form that the comparison meets, that is, for the three kinds of family that occur in it (the h-linear families other than the field-independent ones, the families of King's constant-background reduction at $e > 0$, and the vacuum families in their C/Γ form with the Γ-lines as legs and recipients), for $0 < \theta \leq \theta_1$, uniformly in k, K and the torus,

$$|E^X(\mathcal{R}) - E^Y(\mathcal{R})| \;\leq\; C\, n_{\mathcal{R}}\, p_k^2\, \delta_k^{-1/2}\, \eta'^{\theta}\, (1+K)^{q_1}\, \delta_k^{-\beta'} \cdot \big[\text{the right side of [3, (1.33)] for } \mathcal{R}\big],$$

with $\beta' = 0$ for the h-families and the families of the constant-background reduction, and $\beta' \leq \beta_G \leq 3 \cdot$ (number of lines fixed at the top scale) for the vacuum families, whose constants depend on the masses through $m_0^{-\beta_G}$, $m_0 := \min(m, \mu_0)$; the other constants are uniform in the masses. θ_1 does not depend on $\bar{n}$, and any $\theta > 0$ with any polynomial factor suffices for the applications below.

Proof. Fix an ordering of the scale-summed lines and telescope the difference of the two flows element by element, each insertion bundled with the earliest of its legs; a restricted subtraction rule removes from every connected component of degree ≤ 0 formed in an initial segment its common zero- and first-order moments, which agree for the two flows by isotropy and the zero-momentum counterterm conditions [25, (3.32)–(3.33)] (for the vacuum polarisation by the Ward identities and [3, (2.18)–(2.28), Prop. 2.1]), and a census by exact integer arithmetic over the graphs of order at most $\bar{n}$ finds no component of non-positive degree that the rule does not handle. The difference elements carry the positive power of the scale ratio of Theorem 19.10(iii), the scale sums give the gain η'^{θ_1} with the margin $\min(\alpha, 1) - \theta_1 > 0$, a switched contour element costs $p_k^2 \delta_k^{-1/2}$ ([25, (3.46)]), the sums over slices and localisations the polynomial factor, and the top-scale lines of the vacuum families $(m_0\delta_k)^{-\beta_G}$. The remaining estimates use Theorem 19.10(ii), the census of the zero-degree all-C subgraphs and the local limit law for the simple random walk [27, Chap. 2]. □

19.4. **(N2) Decay and regularity of the block-spin operators. (N2)** *(decay and regularity of the block-spin operators).* Let $F \in \{X, Y\}$ and $2 \leq k \leq K$, in the unit scale of level k; let A be a background regular on Λ_F, or on Λ_F/P for a torus P fitted by both lattices, in particular for the period lattices P_J, RP_J of Lemma 19.9; and let $e_k^\beta \leq c(a_{\mathrm{bs}})$ (backgrounds, regularity and the operators A^F, $C^{(k),F}$, G_k^F, H^F, B^F, Γ^F of the flow F are defined before Theorem 19.10 below). Then, with constants C and $\delta_0 > 0$ independent of k, K, the torus and the background (for the hybrid flow they depend on $d = 3$, $L = 5$, a_{bs}, M and α only; for the standard flow they are

King's), except the Hölder constant, which is $C_\alpha \log^{3\alpha}(1/\eta')$,

$$|A^F(A)(z,w)|,\ |C^{(k),F}(A)(z,w)| \le Ce^{-\delta_0|z-w|},$$
$$|(G_k^F(A)f)(x)|,\ |(D_{A,\mu}G_k^F(A)f)(x)| \le Ce^{-\delta_0 \operatorname{dist}(x,\operatorname{supp} f)}\|f\|_\infty,$$
$$|x-y|^{-\alpha}\big|U(A(\Gamma_{x,y}))(D_{A,\mu}G_k^F(A)f)(y) - (D_{A,\mu}G_k^F(A)f)(x)\big| \le Ce^{-\delta_0 \operatorname{dist}(\{x,y\},\operatorname{supp} f)}\|f\|_\infty \quad (0<\alpha<1),$$

together with the bounds $|H^F(A)(x,z)| \le Ce^{-\delta_0|x-z|}$ and $|D_A H^F(A)(x,z)| \le Ce^{-\delta_0|x-z|}$ and their Hölder clause, $|B^F(A)(z,w)| \le C(1+|z-w|)^{-1}$ and $|\Gamma^F(A)(x,y)| \le C(1+|x-y|)^{-1}$ with one covariant derivative and the Hölder clause in the fine variables, B^F and Γ^F decaying only at the coarse mass; that is, [25, (3.6)–(3.8)] for both flows.

The block-spin operators of a flow at level k, the pinned propagator $G_k^F(A)$, King's effective Laplacian $A^F(A)$, the next fluctuation covariance $C^{(k),F}(A)$ and the lift $H^F(A)$, are the lines from which King's expansion for that flow is built, and (N2) says that at every regular background they decay exponentially on the unit scale of the level, with one covariant derivative and a Hölder clause, with constants that depend neither on the level nor on the number of scales, the torus or the background. For the standard flow this is King's Theorem 3.3; for the hybrid flow, whose first step averages over rotated blocks and whose later contours are only rotation-paired with those of the standard flow, it has to be proved afresh, and that is Theorem 19.10(ii). The torus theorem needs the bounds for both flows in the comparison of the two effective actions, the first step of its proof and the vacuum-family comparison of Theorem 18.9: there the pointwise bound on the difference of the two flows' lines at matched points is combined with the decay of each line into a difference bound that decays, which the sums over the coarse sites of the torus require, and the two background determinants are compared by means of the kernel bounds of both flows; the difference bounds of Theorem 19.10(ii) are used alongside and are not part of (N2). Without the decay the sums over positions behind (N1) and Theorem 18.9 would not be bounded uniformly in K.

Sources and proofs. (N2) is the decay and regularity of the block-spin operators, [25, Thm. 3.3]. For the standard flow it is proved in [4, Lemmas 2.1, 2.2, 2.4, Cor. 2.3] up to the three steps claimed there without argument, the coefficients $a_k \in [a(1-5^{-2}), a]$, the uniformity in a constant $m^2 \ge 0$ and the transfer to the admissible lattice tori (Theorem 19.4(l), with the random-walk transfer named in its proof), and up to the photon clause of [25, Thm. 3.3] (p. 656, "of course with $A = 0$"); King's steps of the torus construction stated without a complete argument enter by citation of [25, Thm. 3.1, Thm. 3.3, Prop. 3.7 (within its stated scope, the K-slice treated separately), Prop. 3.10, Lemma 4.5], King's conventions for the sources (pp. 664–665, 677) are Theorem 20.4 and Theorem 19.6(c), and the coercivity bound (4.33) behind [25, Lemma 4.5] is proved in Section 20.1, item (iv). For the hybrid flow (N2) is Theorem 19.10 (periodic form, two-torus clause in the smaller quotient metric; the Neumann variant on digitised rotated domains is neither claimed nor used), whose first step averages over rotated blocks, a block-spin step by Lemma 19.9.

19.5. **The hybrid kernels at a regular background.** Fix K and $2 \le k \le K$ and work in the unit scale of level k: the physical spacing $\delta_k = 5^k\varepsilon_K$ is the unit, the fine spacing is $\eta' := 5^{-k}$, $\eta := 5\eta'$ and $\ell_j := 5^j\eta'$ (so $\ell_k = 1$). With R the rotation of the frame and $f_\alpha = Re_\alpha$, the coarse lattices are $\Gamma_j := R(\ell_j\mathbb{Z}^3)$, $0 \le j \le k$; $\Lambda_Y := \Gamma_0 = R\eta'\mathbb{Z}^3$ is the fine lattice of the standard flow Y and $\Lambda_X := \eta'\mathbb{Z}^3$ that of the hybrid flow X; since $5R \in \mathbb{Z}^{3\times3}$, $\Gamma_1 \subset \Lambda_X \cap \Lambda_Y$. For $z \in \Gamma_j$ the *level-j block* of the flow $F \in \{X,Y\}$ is $B_j^F(z) := \Lambda_F \cap (z + R(-\frac{\ell_j}{2}, \frac{\ell_j}{2})^3)$, a cube of 5^{3j} points for $F = Y$ and the digitised rotated cube for $F = X$ (no lattice point lies on its boundary;

$B_1^X(z)$ is the tile); blocks are nested and *centred*, the root of every block being its centre, and for $x \in B_k^F(z)$ we write x_j for the centre of the level-j block containing x ($x_k = z$). The matching $\tau\colon \Lambda_X \to \Lambda_Y$ maps the tile $B_1^X(x)$ onto the cube $B_1^Y(x)$ for every $x \in \Gamma_1$, with $|\tau x' - x'| \le c_\tau \eta'$, $c_\tau = \sqrt{13} + 2\sqrt{3} < 7.1$.

Definition 19.8 (the two contour systems)**.** (Y) For $x \in B_k^Y(z)$ the *composite contour* is King's [25, (2.12)], $\Gamma^{Y,(k)}_{z,x} = \Gamma^Y_{z,x_{k-1}} \cup \Gamma^Y_{x_{k-1},x_{k-2}} \cup \cdots \cup \Gamma^Y_{x_1,x}$, where $\Gamma^Y_{u,v}$ is the L-path from u to v that changes the f_3-coordinate first, then f_2, then f_1, each leg a straight run of fine Λ_Y-bonds.

(X) For $x' \in B_k^X(z)$ the *composite contour* is $\Gamma^{X,(k)}_{z,x'} = \Gamma^X_{z,x_{k-1}} \cup \cdots \cup \Gamma^X_{x_2,x_1} \cup \Gamma^X_{x_1,x'}$, where for $i \ge 1$ the piece $\Gamma^X_{x_{i+1},x_i}$ is the L-path $\Gamma^Y_{x_{i+1},x_i}$, whose legs are concatenations of η-bonds $(u, u \pm \eta f_\alpha)$ with $u \in \Gamma_1$, in which every η-bond $(u, u + \eta f_\alpha)$ is replaced by the *fixed* fine axis staircase $u + \eta'\gamma_\alpha$ (and $(u, u - \eta f_\alpha)$ by its reverse), with

$$\gamma_1 :\ e_1, e_2, e_1, e_2, e_1, e_2, e_1 \text{ (from 0 to } (4,3,0)),$$
$$\gamma_2 :\ e_2, -e_1, e_2, -e_1, e_2, -e_1, e_2 \text{ (to } (-3,4,0)), \qquad \gamma_3 :\ e_3^5 \text{ (to } (0,0,5)),$$

and $\Gamma^X_{x_1,x'}$ is a fine axis path inside the tile $B_1^X(x_1)$ of at most 5 bonds, chosen once as a function of $x' - x_1$ only, so that the block and contour system is invariant under the translations of Γ_1, in particular under the periods of the tori. The gauge field along a composite contour is read in King's convention [25, (2.19)]: the stage-j fluctuation is integrated along $\Gamma^{(j+1)}_{x_{j+1},x}$ only, the lifted coarse background along the whole contour.

The first step of the hybrid flow averages over the *tiles* $B(y) := y + \varepsilon B_0$, $y \in Y_{K-1}$, where $B_0 := \mathbb{Z}^3 \cap R(-\frac{5}{2}, \frac{5}{2})^3$ is the set of integer points of the open rotated cube of side 5; the frame of Y_K is $f_\alpha := Re_\alpha$. The block averages of the step are $(\hat{Q}A)_\alpha(y) := \frac{1}{125}\sum_{x \in B(y)} f_\alpha \cdot A(x)$ on bond fields and $(Q(A)\phi)(y) := \frac{1}{125}\sum_{x \in B(y)} U(A(\Gamma_{yx}))\phi(x)$ on scalar fields, Γ_{yx} the first-step contour from y to x and U the parallel transport along it, $|U| = 1$; the pairings are $\langle u, v\rangle_\Lambda := \sum_x s^3 u(x) \cdot v(x)$ on a lattice of spacing s.

Lemma 19.9 (the tile average is a block-spin average)**.** *(i) B_0 is a complete residue system of $\mathbb{Z}^3/5R\mathbb{Z}^3$ with $|B_0| = 125$; the tiles $B(y)$, $y \in Y_{K-1}$, partition X_K, and for each of the period lattices $P \in \{P_J, RP_J\}$, which satisfy $P \subset Y_{K-1}$, the tiles $B(y)$, $y \in Y_{K-1}/P$, partition X_K/P.*

(ii) $B_0 = -B_0$, $\max_{x \in B_0} |x|^2 = 13$, $|x|_\infty \le 3$ on B_0, and every $x \in B_0$ is joined to 0 by a nearest-neighbour path inside B_0, monotone in each coordinate, of length $|x|_1 \le 5$. So a contour inside the tile, of at most 5 fine bonds, exists for every point of the tile; it is the first-step contour Γ_{yx}.

(iii) $\hat{Q}\hat{Q}^* = \mathbf{1}$ on coarse bond fields and $Q(A)Q(A)^* = \mathbf{1}$ on coarse scalar fields, for every bond field A. Hence $\hat{Q}^*\hat{Q}$ and $Q(A)^*Q(A)$ are orthogonal projections, the Gaussian kernels $\exp[-\frac{a_{\rm bs}}{2}\|A_1 - \hat{Q}A\|^2]$ and $\exp[-\frac{a_{\rm bs}}{2}\|\phi_1 - Q(A)\phi\|^2]$, $a_{\rm bs} > 0$ the constant of King's block-spin kernels (the subscript keeps it apart from the scheme $a = (a_k)$ of Definition 2.5), have the normalisations of [25, (2.10)–(2.14)], and the first hybrid step is King's renormalisation transformation with $B(y)$ in place of the cube.

(iv) The composite of the first step with the $j-1$ standard steps that follow it is the equal-weight average over the composite blocks $B^{(j)}(y)$, $y \in Y_{K-j}$, unions of 125^{j-1} tiles, which partition X_K and have diameter at most $C5^j\varepsilon$; the composite Gaussian kernel has King's coefficient $a_{{\rm bs},j}$ ([25, p. 653]), as for the standard flow.

Proof. (i): the open rotated cube is a fundamental domain of $5R\mathbb{Z}^3$ whose boundary planes carry no integer point, so B_0 is a complete residue system with $|B_0| = |\det 5R| = 125$, and the partition descends to X_K/P since $P \subset Y_{K-1}$; (ii) is a finite check over the 125 points. (iii): with respect to the pairings $(\hat{Q}^*\psi)(x) = \sum_\alpha \psi_\alpha(y(x))f_\alpha$ and $(Q(A)^*\chi)(x) = \overline{U(A(\Gamma_{y(x)x}))}\chi(y(x))$, since $(5\varepsilon)^3/125 = \varepsilon^3$, so $f_\alpha \cdot f_\beta = \delta_{\alpha\beta}$ and $U\bar{U} = 1$ give the identities, and the Gaussian integrals over A_1, ϕ_1 are then field-independent constants, all that [25, (2.10)–(2.14)] use of the block. (iv): equal-weight averages over successive partitions compose, and the coefficient is King's computation [25, p. 653], which uses only the form of the kernels. □

A *background* is a real bond field A on Λ_F, $U(A(\Gamma)) = \exp[ie_kA(\Gamma)]$ with $A(\Gamma) = \sum_{b\in\Gamma}\eta' A_b$ and the running charge $e_k = e\,\delta_k^{1/2}$; it is *regular* if $|\partial_\mu A_\nu| \le Ce_k^{\beta-1}$ for some $\beta > 0$ [25, Def. 3.2]. The pinned operators of the flow F are

$$Q_F(A)\phi(z) := \sum_{x\in B_k^F(z)} \eta'^3\, U(A(\Gamma_{z,x}^{F,(k)}))^{-1}\phi(x),$$

$$G_k^F(A) := \big(-\Delta_A + m_k^2 + a_{\mathrm{bs},k}Q_F(A)^*Q_F(A)\big)^{-1},$$

$m_k = m\delta_k$, $a_{\mathrm{bs},k} = a_{\mathrm{bs}}(1-5^{-2})(1-5^{-2k})^{-1}$ (a_{bs} the constant of King's block-spin kernels, as in Lemma 19.9), and with $G_A := (-\Delta_A + m_k^2)^{-1}$ we put $A^F(A) := (a_{\mathrm{bs},k}^{-1} + Q_FC_AQ_F^*)^{-1}$ (King's effective Laplacian $\Delta^{(k)}(A)$), $B^F := (A^F)^{-1}$, $H^F := a_{\mathrm{bs},k}G_k^FQ_F^*$ (the lift operator), $\Gamma^F := G_A - G_k^F = H^FB^FH^{F*}$, $P(A) := Q^{(1)}(A)^*Q^{(1)}(A)$ the pin of the next step and $C^{(k),F}(A) = (A^F(A) + a_{\mathrm{bs}}L^{-2}P(A))^{-1}$ the next fluctuation covariance; these identities are exact for every A by the Woodbury identity. The photon kernels are three copies of the scalar kernels at $A = 0$ and mass $\mu_0\delta_k$: no photon kernel sees a background. A *comparable pair* consists of two tori, or of a torus and the infinite lattice, both fitted by both lattices, of side at least $10L$ in unit scale, compared on a common fundamental region F that is a union of level-$(k+1)$ blocks, with boundary ∂ and the same background on it; $d_\wedge$ denotes the smaller of the two quotient metrics. For a subset Λ of the coarse lattice we put $\hat{C}_\Lambda^{(k)} := ((A^X(A) + a_{\mathrm{bs}}L^{-2}P(A))|_\Lambda)^{-1}$.

Theorem 19.10 (the hybrid block-spin kernels at a regular background)**.** *Let $k \ge 2$.*

(i) (Pairing; block-spin algebra.) *The hybrid composite contours lie inside their level-k blocks, every level-i piece inside its level-$(i+1)$ block, and have length at most* 2.1*; for every background A, $Q_X(A)Q_X(A)^* = \mathbf{1}$, $Q_X(A)^*Q_X(A)$ is an orthogonal projection, gauge covariance holds, and $Q_X(A) = Q_{k-1}^{\mathrm{rot}}(A)\circ\hat{Q}(A)$. The contours are* rotation-paired*: for $z \in \Gamma_k$, $x' \in B_k^X(z)$, $x'' = \tau x'$, the Hausdorff distance between $\Gamma^X_{x_{i+1},x_i}$ and $\Gamma^Y_{x_{i+1},x_i}$ is at most η' for $i \ge 1$ and at most $2\sqrt{13}\,\eta'$ for $i = 0$, and the closed polygon $\Gamma^{X,(k)}_{z,x'} \cup [x',x''] \cup (\Gamma^{Y,(k)}_{z,x''})^{-1}$ bounds a piecewise planar surface $\Sigma(z,x')$ with* $\mathrm{area}(\Sigma) \le 1.5\,\eta' + 33\,\eta'^2$. *The roots of the two flows are matched.*

(ii) (Decay and regularity; periodic form.) *Let A be regular on Λ_X or on Λ_X/P for a torus P fitted by both lattices, and let $e_k^\beta \le c(a_{\mathrm{bs}})$. Then, with constants depending on $d = 3$, $L = 5$, a_{bs}, M and α only, except the Hölder constant, which is $C_\alpha\log^{3\alpha}(1/\eta')$,*

$$|A^X(A)(z,w)|,\ |C^{(k),X}(A)(z,w)| \le Ce^{-\delta_0|z-w|},$$

$$|(G_k^X(A)f)(x)|,\ |(D_{A,\mu}G_k^X(A)f)(x)| \le Ce^{-\delta_0\,\mathrm{dist}(x,\mathrm{supp}\,f)}\|f\|_\infty,$$

$$|x-y|^{-\alpha}\big|U(A(\Gamma_{x,y}))(D_{A,\mu}G_k^X(A)f)(y) - (D_{A,\mu}G_k^X(A)f)(x)\big| \le Ce^{-\delta_0\,\mathrm{dist}(\{x,y\},\mathrm{supp}\,f)}\|f\|_\infty \quad (0<\alpha<1),$$

that is, [25, (3.6)–(3.8)] *for the hybrid flow. The differences between the two members of a comparable pair obey the same bounds with* $\mathrm{dist}(\cdot,\mathrm{supp}\,f)$ *taken in* $d_\wedge$ *and the additional factor* $e^{-\delta(\mathrm{dist}(\{x,y\},\partial)+\mathrm{dist}(\mathrm{supp}\,f,\partial))}$, $\delta=(2\sqrt{3}M)^{-1}$; *for* A^X *and* $C^{(k),X}$ *the decay of the difference is in* $d_\wedge$, *with the factor* $e^{-\delta(\mathrm{dist}(z,\partial)+\mathrm{dist}(w,\partial))}$. *For every subset* Λ *of the coarse lattice, in particular every sum of big blocks,* $\hat{C}^{(k)}_\Lambda$ *obeys the bound* [26, (5.54)]. *In addition* $|H^X(A)(x,z)|, |D_A H^X(A)(x,z)|\le Ce^{-\delta_0|x-z|}$ *with the Hölder clause (constant* $C_\alpha\log^{3\alpha}(1/\eta')$*),* $|B^X(A)(z,w)|\le C(1+|z-w|)^{-1}$ *and* $|\Gamma^X(A)(x,y)|\le C(1+|x-y|)^{-1}$, *with one covariant derivative and the Hölder clause in the fine variables;* B^X *and* Γ^X *decay only at the coarse mass. The same statements hold for the standard flow* [25, Thm. 3.3], [4], *there with King's uniform constants.*

(iii) (Gain at $A\neq 0$.) *Let* A_k *be in the small-field class and* A^F *its lifts. After King's constant-background reduction* [25, (3.43)–(3.55)], *applied to both flows on their infinite lattices with the torus lines as exact periodisations, every* X–Y *difference element at matched points carries a positive power of the scale ratio: the* $A=0$ *kernel differences gain* $\eta'^{1/2}$ *(slice-weighted), the contour lines gain* $(\eta'/\ell_j)^{1/2}$ *on slice* j *(the shape part of it* η'/ℓ_j*), the double contour lines the same against their size* $C\ell_j$, *the contour vertices and the lifted external fields gain* $\eta^{1/2}$ *(the former times* $e\,p_k^2\delta_k^{1/2}$*), and the constant-background phases gain* η' *(times* $e\,p_k\mu_0^{-1}\delta_k^{-1/2}$*); the remainders are bounded per flow by (ii). Hence the comparison of the two flows holds at* $e>0$ *as at* $e=0$, *with the gain exponent* $\theta=\frac12$ *and a prefactor polynomial in* p_k *and* $\delta_k^{-1/2}$.

Part (ii), in the form used by the hybrid lower bound at its first step, holds also at the levels $k\in\{0,1\}$.

Proof. (i) is the geometry of Definition 19.8: each staircase lies within η' of its segment and bounds with it an area at most $5\eta'^2$, the pieces stay in their blocks since the η-bonds keep distance at least $\frac52\eta'$ from the faces, $Q_X(A)Q_X(A)^*=\mathbf{1}$ follows from $|U|=1$ and the unit block volume, and $\Sigma(z,x')$ is a union of at most $0.3/\eta'$ slivers and one polygon in the ball of radius $\sqrt{13}\eta'$ about x_1. (ii) is proved by a coercivity lemma that localises on axis cubes of side M and gauges the constant part of A away cube by cube, by Combes–Thomas decay and by Bałaban's random-walk assembly [4, (2.6)] with periodic cells fitted by both lattices in the form of [26, §5.3], whose $\log(1/\eta')$ scales give the logarithm in the Hölder constant; the two-tori clause compares the random walks of the two cells on their common fundamental region, the bounds of [26, Prop. 5.6, (5.54)] apply verbatim to the hybrid operator, Theorem 19.4(h) gives the standard-flow bounds in the same form (a lift-distance clause for the domain difference is false and not used), and the same proofs give the levels $k\in\{0,1\}$, since they use $k\ge 2$ nowhere. (iii): after King's constant-background reduction every compared line is an $A=0$ line and the new difference elements are the contour vertices and the one-ended and double contour lines [25, (3.26)], whose mismatch is the flux through the sliver between the paired contours, of area $O(\eta')$ by (i); with the cubic-Hermite cardinal interpolation of the lattice kernels as the shared continuum field, both flows have, after the isometry, one fine lattice, propagator, vertices and counterterms, the zero-degree all-C subgraphs of the C/Γ normal form are enumerated and bounded by a multiscale bound, and the gains carry the small-field bound [25, (3.2)]. □

19.6. **(N3) King's upper bound. (N3)** *(King's upper bound with the additive large-field term).* Let $T\in\{D_J, RD_J\}$, let s be one of the sources (tG,h_0), $t\in\mathbb{R}$, on D_J, or the S^*-image of one of them on RD_J (Section 18.3), and let the standard flow of Y_K on T be run to the depth $k=k(K)$ of the construction. There are $C<\infty$, $\sigma>0$, $q<\infty$ and $c_*>0$, independent of K (they may

depend on T and on s, at which the forward bound is applied), such that for all $K \geq K_0$

$$Z_{Y_K}(T;s) \;\leq\; e^{C\delta_k^\sigma|T|}\int \chi_k\, e^{-S^{(k)}} \;+\; \exp\big(-c_*p(\delta_k)^2 + C|T| + R_K\big), \qquad 0 \leq R_K = o(K^2),$$

where χ_k is King's small-field indicator at depth k, $S^{(k)}$ the effective action of the standard flow at that depth, source included (for curvature sources in the Cameron–Martin form the integral is posed on the common set $\chi_k(\cdot + a_k^Y) \subset \chi_k^{(1+\varepsilon_k)}$ and carries the factor $e^{z^2 q_F/2}$, as in Proposition 19.13), $p(\cdot)$ the small-field function of Convention 19.1 and c_* the large-field constant of Definition 19.2; $R_K = qK\ln 5$ on RD_J (King's bound, [25, Thm. 3.1]) and $R_K = o(K^2)$ on D_J (the large-field clause of Theorem 19.3, stated at the stopping depth $r(k) \leq k$, where $p(\delta_{r(k)}) \geq p(\delta_k)$); only $R_K = o(K^2)$ is used.

King's upper bound [25, Thm. 3.1, (3.4)] bounds the partition function of the rotated lattice by its small-field part, the integral over King's small-field set of the exponential of the effective action after k block-spin steps, up to the factor $e^{C\delta_k^\sigma|T|}$, plus an additive term for the configurations outside the small-field set. The additive term is absolute, not relative to the partition function: it is at most $e^{-c_*p(\delta_k)^2}$, which at the depth $k = k(K)$ of Section 18.4 decays faster than any fixed power of the spacing ε_K, times $e^{C|T|}$ and a factor e^{R_K} with $R_K = o(K^2)$: a fixed power of ε_K^{-1} on RD_J, the large-field clause of Theorem 19.3 on D_J, and this is the form in which that clause gives it on both tori. In the forward bound, Proposition 18.10, (N3) is the first step: $Z_{Y_K}(T;s)$ is replaced by its small-field integral $\int \chi_k e^{-S^{(k)}}$, which the comparison of the two effective actions then bounds by the hybrid small-field integral, while the additive term is absorbed with the help of (N5). The factor $e^{C\delta_k^\sigma|T|}$ covers, in particular, the passage from King's torus conventions to the free-boundary model, Theorem 19.6; any fixed $\sigma > 0$ and any polynomial factor in K in front of δ_k^σ would be tolerated.

Sources and proofs. (N3) is King's upper bound [25, Thm. 3.1, (3.4)] for Y_K with the additive large-field term in the form of the large-field clause of Theorem 19.3: on RD_J it follows from Theorems 19.3 and 19.4 and the statements of King cited with (N2), given (CT), carried from the axis lattice on D_J by the rotation (Lemma 16.15 with $O = R$); on D_J it is the instance [SKEW-U], proved relative to the aligned bounds with the propagators of [2, Prop. 2.6] replaced through Theorem 19.4(k); for the contours of the flow of Y_K it uses Proposition 26.2, and at the wall Proposition 26.8 for $J \geq J_{\mathrm{pos}}$, on D_J under the final-scale cap of the class $\mathcal{C}$, through the large-field clause of Theorem 19.3 (Section 26); Theorem 19.6 carries it to the free-boundary model, clause (c) giving the uniformity of its constant at the shifted constant source.

19.7. **(N4) The hybrid lower bound. (N4)** *(the hybrid lower bound).* Let $T \in \{D_J, RD_J\}$ and let the hybrid flow of X_K on T be run to the level $k \leq k_X(K, S_0)$, where the depth $k_X(K, S_0)$ and the size $S(s)$ of a source are defined before Proposition 19.13 below (the level $k = k(K)$ of Section 18.4 lies below this cap once $K \geq K_0$, as stated there). There are $p' = p'(p_{\mathrm{pb}}, \bar{n})$, $\sigma > 0$ and, for every $S_0 > 0$, a constant $C(S_0)$ such that for every source $s = (G, h)$ with $S(s) \leq S_0$

$$Z_{X_K}(T;s) \;\geq\; e^{-C(S_0)(1+K)^{p'}\delta_k^\sigma|T|}\int \chi_k\, e^{-\hat{s}_T^{(k)}},$$

uniformly in K, T and the source, where χ_k is King's small-field indicator and $\hat{s}_T^{(k)}$ the hybrid effective action at level k (for curvature sources in the Cameron–Martin form the integral is posed, like King's upper bound (N3), on the common set $\chi_k(\cdot + a_k^Y) \subset \chi_k^{(1+\varepsilon_k)}$ and carries the factor $e^{z^2 q_F/2}$); $(1+K)^{p'}\delta_k^\sigma \to 0$ along $k = k(K)$.

(N4) is the lower-bound half of the comparison: the partition function of the axis lattice X_K on T is at least its small-field part, the integral over King's small-field set of the exponential of

the hybrid effective action, up to the factor $e^{-C(S_0)(1+K)^{p'}\delta_k^{\sigma}|T|}$, which is $e^{-o(|T|)}$ along $k=k(K)$; in other words, the configurations outside the small-field set and the fluctuation integrals of the steps cost at most that factor. It is the analogue for the hybrid flow of King's lower bound [26, Prop. 4.1, (4.5)], and it has to be proved along the hybrid flow, by Bałaban's lower-bound argument run with the hybrid kernels (N2) and with a cumulant bound for the fluctuation integral of one step (Theorem 19.12) whose constants are free of the spacing and hold on the skew tori. In the forward bound it is the closing step: the comparison of the two effective actions gives $\int \chi_k e^{-S^{(k)}} \le e^{-c_K\langle h\rangle_{X_K}+\vartheta_K|T|}\int \chi_k e^{-\hat{s}^{(k)}}$, and (N4) turns the hybrid small-field integral back into the partition function $Z_{X_K}(T;s)$. Without (N4) the comparison of the effective actions would say nothing about Z_{X_K}, the small-field integral being only a part of the partition function.

Sources and proofs. (N4) is the hybrid lower bound, the analogue of [26, Prop. 4.1, (4.5)] for the hybrid flow, Proposition 19.13, proved from Theorems 19.3 and 19.4, (CT), Theorem 19.10 and, on D_J, [SKEW-L], with Proposition 26.2 for the composite contours; its argument needs a cumulant bound for the fluctuation integral of one step, for a Gaussian field with exponentially decaying precision on $\mathbb{Z}^3$, on cubic tori and on the skew tori, with constants free of the spacing and of the shell mass, which the cluster-expansion lemma of [7] (the nearest-neighbour Markov field, thresholds growing with the volume) does not give, and which is used again in Theorem 24.5.

Definition 19.11 (the setting of the cumulant bound)**.** Let $d\ge 1$, let $\mathbb{L}=\mathbb{Z}^d$ or $\mathbb{L}=(\mathbb{Z}/N\mathbb{Z})^d$ with its Euclidean, resp. torus Euclidean, distance $|x-y|$, let $\Lambda\subset\mathbb{L}$ be finite and let $z=(z_x)_{x\in\Lambda}$, $z_x\in\mathbb{R}^{n_c}$, be the field; operators on $(\mathbb{R}^{n_c})^\Lambda$ are written through their $n_c\times n_c$ block kernels $P(x,y)$, with the block operator norm $\|P(x,y)\|$ (a complex-Hermitian operator on $\mathbb{C}^\Lambda$ is read in its real form, which has the same spectrum and the same block norms). The conditions on the Gaussian measure μ and on the interaction V are

- (Pr) $\mu=N(0,C)$ on $(\mathbb{R}^{n_c})^\Lambda$, $C=P^{-1}$, P real symmetric with $\gamma_P\le P\le\gamma_1$;
- (Dec) $\|P(x,y)\|\le c_P e^{-\delta_P|x-y|}$ for all $x,y\in\Lambda$;
- (V) $V=\sum_{Y\subset J}V_Y$ with $J\subset\Lambda$, every V_Y a real polynomial in $(z_y)_{y\in Y}$ of degree $\le D$ without constant term, and $\mathsf{A}:=\|V\|_\nu:=\sup_{x\in\mathbb{L}}\sum_{Y\ni x}e^{\nu\ell(Y)}|V_Y|_1$, where $\ell(Y)$ is the length of a shortest connected graph containing Y and $|V_Y|_1$ the sum of the moduli of the coefficients of V_Y;

and the cut-off is $\chi_b:=\prod_{x\in\Lambda}\mathbf{1}\{|z_x|\le b\}$, or $\chi_b^K:=\prod_{x\in\Lambda}\mathbf{1}\{z_x\in bK\}$ with $K\subset\mathbb{R}^{n_c}$ symmetric, convex, $B(1)\subset K\subset B(c_K)$ (Bałaban's cut-off [1, (3.43), (3.50)], which restricts the photon and the matter components separately, is of this form with $c_K=\sqrt{2}$; this K is a convex body, not the scale index). $\mathcal{E}_n^T(V)$ are the cumulants of V under μ, with no cut-off, and with $\tilde{c}(\theta):=\sum_{z\in\mathbb{Z}^d}e^{-\theta|z|}$ we put

$$\mu_*:=\min\Big(\frac{\delta_P}{2},\ \frac{\gamma_P}{2c_P\sum_{z\in\mathbb{Z}^d}|z|e^{-\delta_P|z|/2}}\Big),\qquad M_1:=\frac{2c_P}{\gamma_P}\,\tilde{c}(\mu_*/2)^2,$$
$$\gamma:=\min\{\tfrac12,(2M_1)^{-1}\},\qquad \gamma_K:=\min\{\tfrac12,(2c_KM_1)^{-1}\},$$

and $\bar{\mu}:=\min(\mu_*,\nu)$; $\bar{n}\ge 1$ is the order of the expansion.

Theorem 19.12 (the cumulant bound)**.** *In the setting of Definition 19.11, assume (Pr), (Dec) and (V). There are $b_*<\infty$ and $S<\infty$, depending only on $(d,n_c,D,\bar{n},\nu,\gamma_P,c_P,\delta_P)$ (and on c_K for the cut-off χ_b^K, for which γ_K replaces γ), such that for every finite $\Lambda\subset\mathbb{L}$, every $J\subset\Lambda$,*

every $b \geq b_*$ and every V with $\mathsf{A}\, b^{D+2d} \leq 1$,

(CU)

$$\int d\mu\, \chi_b\, e^{V} \leq \exp\Big\{ \sum_{n=1}^{\bar n} \frac{\mathcal{E}_n^T(V)}{n!} + S|J|\big[(\mathsf{A}\, b^{D+2d})^{\bar n+1} + e^{-(\bar\mu/16)b^{3/2}}\big]\Big\},$$

(CL)

$$\int d\mu\, \chi_b\, e^{V} \geq \exp\Big\{ \sum_{n=1}^{\bar n} \frac{\mathcal{E}_n^T(V)}{n!} - S|J|\big[(\mathsf{A}\, b^{D+2d})^{\bar n+1} + e^{-(\bar\mu/16)b^{3/2}}\big] - 4n_c|\Lambda| e^{-\gamma_P \gamma^{2(d+1)} b^2/(2n_c)}\Big\}.$$

The constants do not depend on Λ, J, the torus side, the lattice spacing, the step or the background fields of the application; translation invariance is not used, and the upper bound γ_1 of (Pr) is not used. The same bounds hold, with the same constants, on every finite Λ of a skew torus $T = \mathbb{Z}^3/G$ ($d = 3$), $G = mG_+$ or $G = mG_-$ with $m \in \mathbb{N}$ and $G_\pm := \langle (40, \mp 30, 0), (\pm 30, 40, 0), (0, 0, 50)\rangle$ (the tori $R^{-1}D_J$ and RD_J in the units of the step in which they are skew), with the quotient Euclidean distance $|x-y|_T := \min_{g\in G} |\tilde x - \tilde y - g|$ ($\tilde x, \tilde y$ lifts to $\mathbb{Z}^3$) in place of $|x-y|$ in (Dec) and in $\ell(Y)$.

Proof. The proof of the cluster-expansion lemma of Benfatto et al. [7, p. 152, (4.6)–(4.7)] is rerun for a Gaussian measure whose precision is only bounded below and exponentially decaying: the Markov factorisation across corridors is replaced by a decoupling estimate across corridors of width $w = \lceil b^{3/2} \rceil$ ((Dec) and the Schur test), the nearest-neighbour formula for the conditional mean by Combes–Thomas decay of the conditioned Gaussian, and the superstability bound by the Khatri–Šidák bound $\mu(\chi_{\gamma^{d+1}b}) \geq \exp(-4n_c|\Lambda| e^{-\gamma_P\gamma^{2(d+1)}b^2/(2n_c)})$, the last term of (CL). On each cell the cumulant expansion of the bounded interaction to order $\bar n$, the replacement of the conditioned law by μ at the cost $O(e^{-\mu_* w/8})$ and the connected Wick expansion (anchored tree sums at the rate $\bar\mu$, where $\|V\|_\nu$ enters) telescope over the $d+1$ shifted pavements to $\sum_{n\leq \bar n} \mathcal{E}_n^T(V)/n!$. On the skew tori the pavement is built coordinatewise on the cycles of the period lattice, and of the quotient metric only the 1-Lipschitz distance functions, the domination of its sums by the $\mathbb{Z}^3$ sums, $\ell(Y) \geq |x-y|_T$ and the existence of the pavement are used; translation invariance is used nowhere. □

For a source $s = (G, h)$ we write $S(s) := \max_{\mu\nu}(\|G_{\mu\nu}\|_\infty + \|\nabla G_{\mu\nu}\|_\infty) + \|h\|_\infty$ for its *size*, and for $S_0 > 0$ we let $k_X(K, S_0)$ be the largest k such that $(1+K)^{p_{\rm pb}} \ell_j^{\beta_0} p_1(\ell_j)^{D+2d} \leq c_{\rm B}$ for all $j \leq k$, where $\ell_j = 5^j \varepsilon_K$, $p_1(t) = b_1(1 + \ln t^{-1})^{\varpi_1}$, with a fixed exponent $\varpi_1 \geq 1$, is Bałaban's fluctuation cut-off ([1, (3.43), (3.50)]), $\beta_0 = \frac14 - \beta$, and D, d are the degree and the dimension of the cumulant bound, Theorem 19.12, as it is used in (L3) below, and $c_{\rm B}$ the constant for which the cap implies the norm condition $\mathsf{A}\, b^{D+2d} \leq 1$ of that theorem for the step interaction (D is not a degree of a graph). The cap contains the level $k = k(K)$ of the construction: for every fixed $\upsilon > 0$ it holds at $k = k(K)$ once $K \geq K_0(\upsilon)$.

Proposition 19.13 (the hybrid lower bound)**.** *Let $T \in \{D_J, RD_J\}$ and let the hybrid flow of X_K on T be run to the level $k \leq k_X(K, S_0)$. There are $p' = p'(p_{\rm pb}, \bar n) \leq 14 p_{\rm pb} + c_{\bar n}$ ($c_{\bar n}$ depending only on $\bar n$), $\sigma > 0$ and, for every $S_0 > 0$, a constant $C(S_0)$ such that for every source $s = (G, h)$ with $S(s) \leq S_0$*

$$Z_{X_K}(T; s) \;\geq\; e^{-C(S_0)(1+K)^{p'}\delta_k^\sigma |T|} \int \chi_k\, e^{-\hat s_T^{(k)}},$$

uniformly in K, T and the source; for curvature sources in the Cameron–Martin form the integral is posed, like King's upper bound (N3), on the common set $\chi_k(\cdot + a_k^Y) \subset \chi_k^{(1+\varepsilon_k)}$ and carries the factor $e^{z^2 q_F/2}$. Here $p_{\rm pb}$ is the exponent of the whole-torus perturbative bound for the hybrid

flow (the hybrid form of [3, Prop. 1]*; it is not the exponent p of Convention 19.1), χ_k is King's small-field indicator and $\hat{s}_T^{(k)}$ the hybrid effective action; $(1+K)^{p'}\delta_k^\sigma \to 0$ along $k = k(K)$.*

Proof. Bałaban's lower-bound argument [1, §3] is run along the hybrid flow on four facts: (L1) the first step is a block-spin step and the later composite contours are rotation-paired (Lemma 19.9, Theorem 19.10(i)); (L2) the hybrid analogues of [1, Props. 2.1–2.3] are Theorem 19.10(ii) at every level $k \geq 2$ and, by the last clause of that theorem, at the levels 0 and 1; (L3) the cumulant lower bound for the step Gaussian is Theorem 19.12 for Bałaban's two-component cut-off ($c_K = \sqrt{2}$, γ_K), with (Pr), (Dec) from (L2) and (V) from the coefficient bounds of the step interaction [26, §5.3] under the cap on $k_X(K, S_0)$, its additional term at the step j (threshold $b_j = p_1(t_j)$) being at most $C_\kappa t_j^\kappa |T_1^{(j)}|$ for every κ, with sum at most $C'_\sigma \delta_k^\sigma |T|$, and its pavement on the skew tori built coordinatewise on the cycles of the period lattice; (L4) Bałaban's renormalisation bound [3, Prop. 1] is proved again for the hybrid kernels on the whole torus, with the unweakened degree count and the Hölder clause of Theorem 19.4(i), whose Hölder constant $C_\alpha \log^{3\alpha}(1/\eta')$ is the origin of $C(1+K)^{p_{\mathrm{pb}}}$. On D_J the steps $2, \dots, k$ are the skew-torus instance [SKEW-L], on RD_J only step 1 is skew, on the torus with period lattice $RP_J = R(50MJ\,\mathbb{Z}^3)$, whose blocks tile RD_J (Lemma 19.9(i)), whose kernels have torus images $O(e^{-c\,50MJ/\varepsilon})$ and which lands on $Y_{K-1} \subset RD_J$, and with curvature sources the bound is used on the common set named in the statement, through Theorem 19.6(c) and the Cameron–Martin form of the effective action. Given (L1)–(L4) the remaining estimates are those of [1, §3], with $C(S_0)$ uniform over $S(s) \leq S_0$ because the source enters the step interaction only through its sup norms. □

19.8. **(N5) The polynomial lower bound. (N5)** *(the polynomial lower bound).* Let $T \in \{D_J, RD_J\}$ and let h be either 0 or the wall $h_0 = -n\psi$ of the configuration (on RD_J, the S^*-image of the wall). There are $C_1 < \infty$, $q' < \infty$ and K_0, independent of K and of t (q' may depend on $|T|$), such that

$$Z_{Y_K}(T; tG, h) \;\geq\; e^{-C_1|T|}\,\varepsilon_K^{q'} \qquad \text{for every real } t \text{ and all } K \geq K_0;$$

Proposition 19.16 gives $q' = q + C|T|/\ln 5$, q the exponent of [1, (1.14)]; the forward bound tolerates any fixed exponent (its proof writes one letter q for the larger of q' and the power in (N3)).

The large-field term of King's upper bound is additive, so before it can be dropped it has to be compared with the partition function itself, and that comparison is what (N5) provides: $Z_{Y_K}(T; s)$ is at least $e^{-C_1|T|}$ times a fixed power of the spacing. Since $p(\delta_k)^2$ grows at least quadratically in K at the depth $k = k(K)$ of Section 18.4 ($K - k(K) \geq \upsilon K$, so $p(\delta_k) \geq b_0(1 + \upsilon K \ln 5)^6$) while $\ln \varepsilon_K^{-1} = K \ln 5$ grows only linearly, the additive term is a vanishing fraction $b \to 0$ of $Z_{Y_K}(T; s)$, and the forward bound absorbs it on the left, into the factor $(1-b)^{-1}$. This is why the forward bound needs neither a lower bound for the partition function of the axis lattice nor any bound on the constant c_K. At $h = 0$ the bound is Bałaban's [1, (1.14)]; the instance the torus theorem uses is the one at the wall $h_0 = -n\psi$ of the configuration, Proposition 19.16, where the bound has to hold for the walled model and for every real t.

Sources and proofs. (N5) is a polynomial lower bound $Z_{Y_K}(T; s) \geq e^{-C_1|T|}\varepsilon_K^{q'}$: at $h = 0$ Bałaban's lower bound [1, (1.14)], the steps of whose proof [1, §§3–4] are completed by Theorems 19.3 and 19.4, given (CT), on RD_J directly and on D_J relative to it; at the mass-increasing wall source $h_0 = -n\psi$ Proposition 19.16, from the source-free bounds at two other fixed masses, the torus one-point Wick bound (Remark 19.15), the skew instances under the final-scale cap and exact identities. The Wick bound rests on Reisz's theorems [34, Thm. 1], [35, Renormalization Theorem, p. 92] in the corrected three-dimensional form of Theorem 19.14 below, in which a naturalness condition proved for the graphs used replaces Reisz's naturalness clause, false as

stated, and the forest identity [35, Lemma 5.1] is Zimmermann's forest formula (R_Γ below) with its hypothesis [35, (2-24)] checked.

Reisz's power counting in three dimensions. In the notation of [34, 35], with lattice spacing $a > 0$, $\mathcal{C}^c$ is the class of finite sums of functions $V(u, w; a) = a^{-m} F(au, aw)$, F smooth, possessing a continuum limit as $a \to 0$, and the integrand class $\mathcal{F}$ consists of the quotients V/C with $V \in \mathcal{C}^c$ (coefficients polynomial in the masses) and $C = \prod_i [e_i(l_i; a) + \mu_i^2]$, $\mu_i > 0$, where $e_i(l; a) = a^{-2}\eta_i(al)$ with η_i smooth, 2π-periodic, positive on the Brillouin zone off its single zero at 0, and $l_i \neq 0$ linear in the loop and external momenta. A Zimmermann subspace H of the loop momenta $k \in \mathbb{R}^{3m}$ is given by a basis (u, v) of a set of line momenta, u its variable part, and degr_u is Reisz's lattice ultraviolet degree. For a subgraph γ with $m(\gamma)$ loops, $\omega_{\mathrm{loc}}(\gamma)$ is the sum of the degrees of its propagators and vertex functions and $\omega_3(\gamma) := \omega_{\mathrm{loc}}(\gamma) + 3m(\gamma)$; a *forest* U is a set of one-particle-irreducible subgraphs, pairwise nested or disjoint, and $\bar\gamma(U)$ is γ with the maximal elements γ_i of U below γ contracted. The subtraction operators t^δ_γ, $\delta \in \mathbb{N}_0$, are Reisz's periodic lattice Taylor operators in the external momenta of γ, $t^\delta := 0$ for $\delta < 0$, and $R_\Gamma F := \sum_U \prod_{\gamma\in U}(-t^{\delta(\gamma)}_\gamma)F$ is the forest formula, the internal momenta k^γ of a subgraph being fixed by Reisz's chord rule for a line ordering (L_j is a chord of γ if the lines $L_j, L_{j+1}, \dots$ of γ contain a cycle through L_j). A set of line momenta is *natural* if every maximal linearly independent subset expresses the others integrally, and $L_{\mathrm{eff}}(\Gamma) := \{l_L : L \in \Gamma\} \cup \bigcup_{\gamma:\,\delta(\gamma)\geq 0}\{k^\gamma_L : L \in \gamma\}$. The degree conditions on the integer labels $\delta(\gamma)$ are

$$\delta(\gamma) \geq \omega_3(\gamma) \qquad \text{and} \qquad \delta(\gamma) \geq \omega_3(\bar\gamma(U)) + \sum_i \delta(\gamma_i) \ \text{ for every forest } U, \tag{41}$$

the three-dimensional form of [35, (4-3), (4-4)]; they hold with equality for $\delta \equiv \omega_3$ and, by additivity and the Euler identity, for every label scheme $w \geq \omega$ on lines and vertices with $\delta(\gamma) = \sum_{L\in\gamma} w(L) + \sum_{B\in\gamma} w(B) + 3m(\gamma)$, in particular for the mass derivatives with the labels of the undifferentiated graph. The family Fam to which the torus one-point Wick bound (Remark 19.15) applies consists of the rings through the Wick vertex at zero external momentum, the insertion graphs of that bound, the two-point graphs at zero momentum with one scalar line doubled (the counterterm derivatives), and the photon two-point graphs at the fixed momenta of a basis of the dual lattice P^* of the torus, up to order 13, each with its derivatives ∂^b, $|b| \leq 3m + 2$, m the number of loops of the graph, with the labels $\delta(\gamma) := \omega_3(\gamma)$ on every one-particle-irreducible γ, Γ included (for a ring the top label is $1 - n \leq -1$, n its order, so $t^\delta = 0$ at the top), which satisfy (41) with equality, so that the subtracted set $\{\gamma : \delta(\gamma) \geq 0\}$ consists of one-loop subgraphs and of two-loop two-point subgraphs only, all with $\omega_3 \leq 1$; the family is restricted to the graphs with no one-particle-irreducible subgraph having no external scalar leg and an odd number of photon legs (insertion vertices contracted), the omitted graphs contributing 0 to every torus amplitude and counterterm at fixed ε, by reversing the charge arrows of one closed scalar component.

Theorem 19.14 (the renormalisation theorem in three dimensions)**.** *Let Γ be a one-particle-irreducible lattice graph with m loops in three dimensions: propagators that are products of regular denominators $e_{Lj}(la) + \mu^2_{Lj}$, $\mu_{Lj} > 0$, with numerators in $\mathcal{C}^c$; periodic vertex functions in $\mathcal{C}^c$; internal momenta of subgraphs by the chord rule; lines and vertices may carry indices, the degrees being the maxima over the index values. Let integers $\delta(\gamma)$, for every one-particle-irreducible γ including Γ, satisfy (41), and let R_Γ be the forest formula with the subtraction operators t^δ (components of the q^γ in $\mathbb{R}^3$). Then:*

- *(a) $R_\Gamma = V/(B_1B_2)$ as in [35, (5-1)–(5-3)]: $R_\Gamma \in \mathcal{F}$, $(2\pi/a)$-periodic in $k_1, \dots, k_m$, all masses positive;*
- *(b) for every Zimmermann subspace H of the set of all line and internal momenta, parametrised by $t = (t_1, \dots, t_h)$, $t_j \in \mathbb{R}^3$: $\operatorname{degr}_t R_\Gamma \leq -3h - 1$, i.e. $3h + \operatorname{degr}_t R_\Gamma < 0$;*

(c) if $L_{\mathrm{eff}}(\Gamma)$ *is natural for the ordering used, then* $\lim_{a\to 0}\int_{B_a^m} R_\Gamma$ *exists and equals* $\int_{\mathbb{R}^{3m}} \lim_{a\to 0} R_\Gamma$, $B_a = [-\pi/a,\pi/a)^3$, *and* $\sup_{0<a\le a_0}\int_{B_a^m} |\partial_k^\alpha R_\Gamma|\, d^{3m}k < \infty$ *for* R_Γ *and each of its momentum derivatives.*

*(d) (*Naturalness for the family.*) For every* $\Gamma \in \mathsf{Fam}$ *there is a line ordering obeying Reisz's chord rule such that* $L_{\mathrm{eff}}(\Gamma)$ *is natural, so that (c) applies to* Γ*; and the torus sums and Brillouin-zone integrals of every forest term of* $R_\Gamma F_\Gamma$, F_Γ *the Feynman integrand of* Γ, *do not depend on the line ordering obeying the chord rule.*

Proof. The degree bound of (b) is $-3h-1$ in place of Reisz's $-4h-1$, the dimension entering his induction only through the rank identities for M_U, the rank bound $M_U(\Gamma)\ge 3h$ and the degree lemma, and the labels of unsubtracted subgraphs are $\omega_3(\gamma)<0$, never $-\infty$, the second condition of (41) being an inequality of integers used also where $t^\delta = 0$. The naturalness of L_{eff}, stated in [35, p. 96] for every ordering obeying the chord rule, is a hypothesis of (c): it fails for the crossed order-e^4 scalar self-energy inserted in a ring through the Wick vertex, and [34] needs it for the integer translation in its sector decomposition. For (d) one ordering per graph suffices (with suitable trees of the two-loop subtracted subgraphs every internal momentum is, up to sign, a line momentum of the graph or of the graph with one vertex split, a natural set), a change of chords is a unimodular integer change of the loop momenta, and a change of trees of a subtracted γ translates its internal momenta by a P^*-valued linear function of the entering momenta, which leaves a $\delta(\gamma)=0$ term invariant by periodicity and adds, for a one-loop element with $\delta(\gamma)=1$, an odd function whose loop sum vanishes; the remaining steps are Reisz's, read in three dimensions, with structural lemmas on the two-point subgraphs of the family. □

Remark 19.15 (membership in Reisz's class; the torus one-point Wick bound). In the notation of [34, Defs. 2.2–2.5], Reisz's class consists of the quotients V/C with $V=\sum_i P_i(\mu)V_i$, $V_i\in\mathcal{C}^c_{m_i}$ ($\mathcal{C}^c_m$: the functions $a^{-m}F(xa)$ of the momenta, F smooth, with a limit as $a\to 0$ at every x), P_i polynomials in the masses, and $C=\prod_i(e_i(l_i;a)+\mu_i^2)$ with $e_i\in\mathcal{C}^c_2$ as in the preamble, $\lim e_i = l_i^2$. For the graphs of Fam and the operators used with them (evaluation of external momenta at 0, Reisz's periodic first-order subtraction operator t^1, a Taylor operator in $P(q;a)=a^{-1}\sin(qa)$ and not in q, and derivatives in the masses and in the external momenta) every forest term of the subtracted integrand, and each of its derivatives, lies in Reisz's class and is $2\pi/a$-periodic in every loop momentum, the loop momenta of each subgraph chosen as in [35, (2-24)]: these operations preserve the classes (∂_j maps $\mathcal{C}^c_m$ to $\mathcal{C}^c_{m-1}$, $a^{-1}\sin(qa)\in\mathcal{C}^c_1$), the subtraction operators turn $e(k^\gamma+p)+\mu^2$ into $e(k^\gamma)+\mu^2$ with $k^\gamma\neq 0$ by one-particle irreducibility, and induction over the finitely many forests gives the claim. *The torus one-point Wick bound.* For the Wick-ordered torus one-point graphs through order 13 (the order of King's Appendix [25, (A.1)]), Reisz's power counting, Theorem 19.14 applied to the graphs of Fam, bounds the coefficients uniformly in $\varepsilon\le\varepsilon_0$ and in the scalar mass μ^2 in a compact interval I of King's range, only the coefficient x_{20} of the order $e^2\lambda^0$ being logarithmic, $x_{20}=O(1+\ln(1/\varepsilon))$, all others bounded. This bound enters Proposition 19.16 and Theorem 19.17.

Proposition 19.16 (the polynomial lower bound for the rotated lattice)**.** *Let* m^2 *be a* King point*, that is, a squared scalar mass with* (e,λ,m,μ_0) *in King's range, as in Proposition 26.8 (the model at a King point* M^2 *is King's model with scalar mass* M *and everything else equal), let* $T\in\{D_J, RD_J\}$, $\Lambda\in\{Y_K, X_K\}$, *and let* h *be a real* T*-periodic function, possibly depending on* K*, with* $\|h\|_\infty\le H_0$ *uniformly in* K*. There are* $C_1<\infty$, $q'<\infty$ *and* K_0*, where* q' *may depend on* $|T|$ *(the proof gives* $q'=q+C|T|/\ln 5$*,* q *the exponent of* [1, (1.14)]*; the statements that use the bound allow this), such that*

$$Z^{m^2}_\Lambda(T;tG,h)\ \ge\ e^{-C_1|T|}\,\varepsilon_K^{\,q'}\qquad \textit{for every real } t \textit{ and all } K\ge K_0.$$

At $h = 0$ this is Bałaban's lower bound [1, (1.14)] *for Y_K on RD_J, aligned in the frame of Y_K (the source tG costs nothing, by Jensen's inequality and charge conjugation, $\langle e^{tF(G)}\rangle \geq e^{t\langle F(G)\rangle} = 1$). At the wall $h = h_0 = -n\psi$ it is the instance used by Theorem 18.8; the sign of h is used nowhere, so the bound holds for every bounded real wall. On D_J, which is skew in the frame of Y_K, both clauses use the skew-torus instances [SKEW-L] of Bałaban's lower and upper lines under the final-scale cap of the class $\mathcal{C}$ (Definition 16.6), which these absolute bounds need because Bałaban's flow ends at the final spacing $t_K \in (\varepsilon_0/5, \varepsilon_0]$.*

Proof. Only the wall clause needs an argument, which uses no stability bound at m^2 itself, no bound on the one-point function at m^2 and no two-sided Wick bound. With $a_s := s + \delta m^2(s;\varepsilon)$ the bare mass at the King point s, $\hat{Z}(\gamma)$ the convention-free integral at bare mass γ and $w_K(s)$ the one-point Wick function of the source-free model at s: the *floor shift* moves $h_{\min} = \min h \geq -H_0$ into the bare mass, $\gamma' := a_m - 2h_{\min}$, Jensen's inequality bounding the rest below by $e^{\sum s^3(h-h_{\min})w(\gamma')}$; a *higher King point* M^2 with $\liminf_{\varepsilon\to 0}[a_M - a_m] > 2H_0$ exists at every coupling and wall height, the coefficients of $\partial_s\delta m^2$ being non-negative at order λ, $O(s^{-1/2})$ at order e^2 and $O(s^{-1})$ at two loops, uniformly in $\varepsilon \leq \varepsilon_0$ ([34, Thm. 1] and, for the subtracted two-loop integrands, Theorem 19.14, as in the proof of Lemma 29.1), and since $-\partial_\gamma \ln \hat{Z}(\gamma) = \frac{1}{2}\langle \sum s^3|\phi|^2\rangle_\gamma$ is non-increasing, $\ln \hat{Z}(\gamma') \geq \ln \hat{Z}(a_M) + u|T|(2c_M + w_K(M))$ with $u := \frac{1}{2}(a_M - \gamma') \in [0, C]$ and $\ln \hat{Z}(a_M) \geq -C|T| - qK \ln 5$ by Bałaban's lower line [1, (1.14)] at M^2; the *c-numbers* collapse, by the integrated counterterm identity $P_M - P_m = -|T| \int_{m^2}^{M^2} c_s\, \partial_s a_s\, ds +$ $\mathcal{R}_{m,M}$, to $2u(c_M - c_m)|T| = O(|T|)$ plus a term $\geq -C|T|$ and the perturbative remainder $\mathcal{R}_{m,M} \geq -C(1+K)|T|$ (King's vacuum-energy counterterm [25, (3.32)–(3.35)], of which only the order-e^2 coefficient is logarithmic, by the torus one-point Wick bound of Remark 19.15); and the *mirror bound* $w_K(M) \geq -C(1+K)$ follows by the same monotonicity argument run upwards to a King point M_+^2, from the lower line at M^2 and the upper line [2, (1.3)] at M_+^2. Altogether $\ln Z_\Lambda^{m^2}(T; 0, h) \geq -C|T| - C(1+K)|T| + q \ln \varepsilon_K$, the source tG is added as in the statement, and on D_J the skew instances [SKEW-L] replace the aligned lines. □

19.9. **(SK) The skew-torus instances.** The rotated lattice $Y_K = R\Lambda_K$ is aligned with the torus RD_J but not with D_J, which is skew in its frame, and King's and Bałaban's bounds are proved for tori aligned with the lattice. The bounds (N3)–(N5) for the standard flow of Y_K on D_J are therefore separate statements, the skew-torus instances [SKEW-U] (the upper bound) and [SKEW-L] (the lower bounds), obtained from the aligned bounds by controlling the change of domain and the fluctuation integrals on the skew torus; on RD_J only the first step of the hybrid flow is skew. They are what the second step of the proof of Theorem 18.8 invokes when it applies the forward bound on both tori.

[SKEW-U] and [SKEW-L] are proved relative to the aligned bounds at blocking factor 5 (Theorem 19.3), given the skew-torus clause of Theorem 19.12 and, for [SKEW-U], the change of domain, Theorem 19.4(k). The last hypothesis, (N6), the existence of the axis-lattice limit $\lim_K \langle e^{tF(G)}\rangle_{\mathfrak{c},X_K}$, is Theorem 19.17, which rests on the statements that follow.

19.10. **(N6) The walled axis-lattice limit. (N6)** *(the walled axis-lattice limit).* Let $\mathfrak{c} = (J, \psi, n)$ be a configuration of the class $\mathcal{C}$ (Definition 16.6) with $n > 0$ and wall source $h_0 = -n\psi$, let the scalar mass be a point of King's range at which the positivity condition of Proposition 26.8 holds, with $J \geq J_{\text{pos}}$ (the couplings outside the set E_W of that proposition, as Theorem 18.8 assumes), and let G be a real smooth P_J-periodic 2-form. Then for every real t the limit

$$W_{\mathfrak{c}}(tG) \;:=\; \lim_{K\to\infty} \langle e^{tF(G)}\rangle_{\mathfrak{c},X_K}$$

exists along the axis lattices X_K of spacing $\varepsilon_K = 5^{-K}$, and $1 \le W_{\mathfrak{c}}(tG) \le e^{C_G t^2}$ with $C_G := \sup_K q_{X_K}(G)/2$, $q_{X_K}(G)$ the variance of $F(G)$ under the free photon Gaussian of X_K on T; here $\langle e^{tF(G)}\rangle_{\mathfrak{c},X_K} = Z_{X_K}(T;tG,h_0)/Z_{X_K}(T;0,h_0)$ is the walled expectation in the free-boundary conventions of this paper.

The hypotheses (N1)–(N5) give the two-lattice statement of the torus theorem, that the walled expectations of $e^{tF(G)}$ on the rotated lattice Y_K and on the axis lattice X_K have the same limit points: their difference tends to zero, for every real t and then, by Gaussian domination and Vitali's theorem, locally uniformly in $z \in \mathbb{C}$. They say nothing about the convergence of either family by itself. (N6) is the one-lattice statement that completes the theorem: along the axis lattices the walled expectation converges, for every real t, so that the rotated family converges too, to the same entire function $W_{\mathfrak{c}}(zG)$, which is part (ii) of Theorem 18.8 and the existence half of the torus comparison (K_0). It is proved for the axis lattice only, where King's renormalisation theorem is available in the restricted form of Section 20; the bound $1 \le W_{\mathfrak{c}}(tG) \le e^{C_G t^2}$ is the Gaussian domination that the passage to complex z uses.

Sources and proofs. (N6) is the existence of $\lim_K \langle e^{tF(G)}\rangle_{\mathfrak{c},X_K}$ along the axis lattices X_K of spacing ε_K (Section 16), King's Theorem 2.1 of [25] at a wall for the normalised ratio; its proof uses the large-field clause of the walled envelope at the wall (Theorem 26.11), the wall instance of (N5) (Proposition 19.16), the rate clause of the convention bridge (Theorem 19.6(d)) and the restricted form of King's renormalisation theorem, Section 20.

Theorem 19.17 (the walled axis-lattice limit)**.** *Let $\mathfrak{c} = (J,\psi,n)$ be a configuration of the class $\mathcal{C}$ (Definition 16.6) with $n > 0$ (the renormalisation bound for the operator with the scalar wall and the mismatch-vertex bound, Theorem 19.4(j) and Proposition 25.4(b), hold for every such configuration), let the scalar mass be a point of King's range at which the positivity condition of Proposition 26.8 holds, with $J \ge J_{\mathrm{pos}}$, and let G be a real smooth P_J-periodic 2-form. Put $r_K(z) := Z_{X_K}(T;zG,h_0)/Z_{X_K}(T;0,h_0)$, the normalised ratio at the wall $h_0 = -n\psi$ in the free-boundary conventions of this paper, so that $r_K(t) = \langle e^{tF(G)}\rangle_{\mathfrak{c},X_K}$ for real t, and let $C_G := \sup_K q_{X_K}(G)/2$, $q_{X_K}(G)$ the variance of $F(G)$ under the free photon Gaussian of X_K on T, finite by the convergence of the lattice variances. Then:*

- *(i) for every real t the limit $W_{\mathfrak{c}}(tG) := \lim_{K\to\infty} r_K(t)$ exists along $\varepsilon_K = 5^{-K}$, and $1 \le W_{\mathfrak{c}}(tG) \le e^{C_G t^2}$;*
- *(ii) the limit is the same number in King's torus conventions at the shifted wall: $W_{\mathfrak{c}}(tG) = \lim_K Z^{\mathrm{tor}}_{X_K}(T;tG,h_\infty)/Z^{\mathrm{tor}}_{X_K}(T;0,h_\infty)$ with $h_\infty = h_0 - \frac{1}{2}\Delta_T(0)$, a constant shift of the wall of size $O(e^{-c\,50MJ})$;*
- *(iii) r_K converges locally uniformly on $\mathbb{C}$ to an entire function $W_{\mathfrak{c}}(zG)$, and*

$$\sup_K \sup_{|z|\le R} |r_K(z)| \le e^{C_G R^2}$$

for every $R > 0$; in particular $r_K(\mathrm{i}) \to W_{\mathfrak{c}}(\mathrm{i}G)$.

The statement is at fixed $\mathfrak{c}$, G and t, with $K \to \infty$ first, and does not depend on the later infimum over configurations or on the wall-height index of the closing estimate. The free-boundary limit and the torus-convention limit at the same wall h_0 are not claimed equal, nor is the case $n = 0$ claimed; the positivity condition and $J \ge J_{\mathrm{pos}}$ enter only through the large-field clause of the walled envelope at the wall, and at the points of King's range where the condition fails Theorem 15.1 is obtained by right-continuity in the scalar mass without the present theorem.

Proof. In either convention $1 \le Z_{X_K}(T;tG,h)/Z_{X_K}(T;0,h) \le \exp[t^2 q_{X_K}(G)/2] \le e^{C_G t^2}$ (Jensen's inequality and the free photon variance) and $|r_K(z)| \le r_K(\mathrm{Re}\, z)$, so (iii) follows from (i) by Vitali's theorem. By the convention bridge, $r_K(t) = Z_K(t;h_K)/Z_K(0;h_K)$

with $Z_K(t;h) := Z_{X_K}^{\mathrm{tor}}(T;tG,h)$ and the moving wall $h_K := h_0 - \frac{1}{2}\Delta_T(\varepsilon_K)$, the h-linear c-numbers cancelling in the ratio. With a depth $k = k(K) \to \infty$ so slow that $e^{-c\,p(\delta_k)^2}$ beats every fixed power of ε_K, and $\rho_K(t;h) := I_K(t;h)/I_K(0;h)$ the ratio of the small-field integrals $I_K(t;h) := \int (dA_k)(d\phi_k)\,\chi_k\, e^{-S_K^{(k),1}(tG,h)}$ of King's effective action at depth k, for $h \in \{h_K, h_\infty\}$: (S1) $e^{-C\delta_k^\sigma|T|} I_K(t;h) \le Z_K(t;h) \le e^{C\delta_k^\sigma|T|} I_K(t;h) + e^{-c_* p(\delta_k)^2 + C|T|}\varepsilon_K^{-q}$ with constants independent of K and of h in the family (King's upper bound [25, Thm. 3.1] in its walled envelope form, Theorem 26.11, and the lower bound by the small-field integral); (S2) $Z_K(0;h) \ge e^{-C_1|T|}\varepsilon_K^{q_1}$ (Proposition 19.16), whence $Z_K(t;h)/Z_K(0;h) - \rho_K(t;h) \to 0$; (S3) $|S_K^{(k),1}(tG,h_K) - S_K^{(k),1}(tG,h_\infty)| \le C(1+k)|h_K - h_\infty| p(\delta_k)^{n_1}\delta_k^{-3}|T| + 2C\delta_k^\sigma|T| \to 0$ on $\operatorname{supp}\chi_k$, since $h_K - h_\infty = \frac{1}{2}(\Delta_T(0) - \Delta_T(\varepsilon_K)) = O(\varepsilon_K)$ by the rate clause of the convention bridge (Theorem 19.6(d), used here and nowhere else), whence $\rho_K(t;h_K) - \rho_K(t;h_\infty) \to 0$; (S4) King's two-cut-off estimate at the wall h_∞, $|S_K^{(k),1}(tG,h_\infty) - S_{K+1}^{(k+1),1}(tG,h_\infty) - d_K| \le C(5^{-\gamma k}\delta_k^{-\beta} + \delta_k^\sigma)|T|$ on $\operatorname{supp}\chi_k$ with a field- and t-independent constant d_K, the step $n = 1$ of [25, Thm. 3.4] from X_K at depth k to X_{K+1} at depth $k+1$, whose hypotheses at a wall are the restricted estimates (Theorem 20.8), the routing of the sources through the renormalised classes with Theorem 20.9, the torus one-point Wick bound beyond one loop, and Lemma 20.10 with Theorem 20.11 for the non-polynomial terms; (S5) by (S4) $s_K(t) := Z_K(t;h_\infty)/Z_K(0;h_\infty)$ is Cauchy, d_K cancelling in the ratio, and by (S2), (S3) $r_K(t) - s_K(t) \to 0$, so $\lim_K r_K(t) = \lim_K s_K(t) =: W_{\mathfrak{c}}(tG)$, which is (i) and (ii). □

20. The restricted renormalisation theorem

King's renormalisation theorem is used in this paper only in a restricted form. This section recalls the theorem, shows that in its full generality it fails on one family of graphs (Proposition 20.6), and proves the restricted form, which excludes that family (Theorem 20.8). The restricted form is used in Section 19 and in Section 25.

King's renormalisation theorem [25, Thm. 3.5] is stated for every graph of his expansion, whatever its degree, scalar and curvature sources included. With the notation of [25, (3.37)], the contribution of a graph H with s external scalar legs to the effective action after k steps is $E^{(k)}(H) = \sum_{\{y_i\},\{z_q\}\subset T^{(k)}} \phi_k(y_1)\cdots\phi_k(y_s)\, E^{(k)}(H, A^{(k)}; \{y_i\}, \{z_q\})$; $\operatorname{dist}(\{v_i\})$ is the length of the shortest tree connecting the points, $\delta_k := L^k\varepsilon$ with $L = 5$, and p abbreviates $p(\delta_k)$, the small-field function of Convention 19.1 at δ_k. The theorem asserts that for some $n_1, \delta, \gamma, \sigma > 0$ depending only on the order cap $\bar n$ and on d, and for every $\rho \ge s/4$,

(42)
$$|E^{(\theta)}(H, A^{(\theta)}; \{y_i\}, \{z_q\})| \le C\delta_k^\rho\, p^{n_1} \exp[-\delta \operatorname{dist}(\{y_i\}, \{z_q\})], \qquad \theta \in \{k, k+n\},$$

(43)
$$|E^{(k+n)}(H, A^{(k+n)}; \cdot) - E^{(k)}(H, A^{(k)}; \cdot)| \le C\delta_k^{\rho - 1/2} p^{n_1}\big[L^{-\gamma k} + \delta_k^{d+\sigma+1/2}\big] e^{-\delta \operatorname{dist}} \quad \text{when } \operatorname{dist} \le p,$$

King's (3.38) and (3.39), which [25, (3.40)–(3.42)] sum over the positions to $|E^{(k+n)}(H) - E^{(k)}(H)| \le C[L^{-\gamma k}\delta_k^{-\beta} + \delta_k^{\sigma'}]|T|$. In that generality the theorem is false, failing on the order-e^2 h-linear vacuum family with its counterterm partner (Proposition 20.6); all that is used is its restricted form, Theorem 20.8, which excludes that family and rests on Theorems 20.1, 20.3, 20.4 and 20.5 below and on the census, in its proof, of the subgraphs of non-positive degree that contain a source vertex; the proof of (N6) uses besides Theorem 20.9, Lemma 20.10 and Theorem 20.11. Throughout, the lines of a graph carry slice indices [25, (3.58)], an ordering lists them, H_i is the prefix of the first i lines, the degree of a connected subgraph is Bałaban's [3, (2.1)–(2.2)], equal to King's [25, (3.30)–(3.31)] on subgraphs without block vertices, the *strict*

ordered degree condition (DEG) is $D(H_i) > 0$ for every prefix of every admitted ordering ([25, (3.77)], [3, (3.5)]), and $\varrho_j := L^j\eta$ is the range of a line of index j in the units of level k (η the spacing).

The growth form for the mass-counterterm blobs. The mass counterterm vertex [25, (3.19)] is $-\frac{1}{2}(L^k\varepsilon)^2\sum_x \eta^d\,\delta m^2(x;e'e,\lambda'\lambda)|\phi'+\phi^{(k)}|^2$ with $\delta m^2 = \sum_{2\le\alpha+2\beta\le4} e^\alpha\lambda^\beta\,\delta m^2(x;\alpha,\beta)$ [25, (3.32)–(3.33)], built recursively from the one-particle-irreducible two-point graphs with two undifferentiated external scalar lines, whose lines C^η are decomposed as $\sum_{j=0}^K G^\eta_{(j)}$; a term with some index $j \ge k$ is a *coarse* term of the blob, for which no partition of unity is introduced [25, p. 667]. With $\delta m^2_{\ge k}$ the sum of the coarse terms, their coefficient in the effective interaction $P^{(k),1}$ in the units of level k is $c^{(k)}_{\ge k}(x) := \frac{1}{2}\ell^2\sum_{\alpha,\beta} e^\alpha\lambda^\beta\,\delta m^2_{\ge k}(x;\alpha,\beta)$, $\ell := L^k\varepsilon$, and by scaling $\frac{1}{2}\ell^2 e^\alpha\lambda^\beta\,\delta m^2(x;\alpha,\beta) = e_k^\alpha\lambda_k^\beta\,\Gamma^{(k)}_{\alpha,\beta}(x)$ with the running couplings $e_k = e\ell^{1/2}$, $\lambda_k = \lambda\ell$ and $\Gamma^{(k)}_{\alpha,\beta}$ the same graph in the units of level k with the couplings stripped.

Theorem 20.1 (the growth form for every mass-counterterm blob)**.** *For every order (α,β) and every $k \le K$, uniformly in x,*

$$|\Gamma^{(k)}_{\alpha,\beta,\ge k}(x)| \le C\,(K-k+1)^5\,\ell^{-\max(0,D_{\max}(\alpha,\beta))},$$
$$D_{\max} = -1 \text{ for } \alpha+2\beta = 2, \quad D_{\max} = +1 \text{ for } \alpha+2\beta = 4;$$

hence

$$|c^{(k)}_{\ge k}(x)| \le C\,\ell\,(K-k+1)^5 \le C'\,\ell\,(1+\ln\ell^{-1})^5, \tag{44}$$

Bałaban's bound [3, (1.29)] *up to a power of the logarithm. Moreover: (a) in a graph of $P^{(k),1}$ the counterterm vertex splits as $c_{<k} + c^{(k)}_{\ge k}$, the fine terms being the counterterm partners of the ultraviolet half and the coarse part a constant point vertex whose index sums factorise off, so that* [25, (3.78)] *is vacuous for the host and* [25, Prop. 3.6] *applies to the host with the point vertex $c^{(k)}_{\ge k}$ of size (44); (b)* [25, (3.38)] *holds with the exponent $\rho \ge s/4$, the logarithm being absorbed in the factor $p(L^k\varepsilon)^{n_1}$; (c) for every $n \ge 1$ and $0 < \gamma < \gamma_{\mathrm{IR}} := \min(\alpha_1, \gamma_{3.9}, 1)$, $\alpha_1 > 0$ the least positive ordered degree over the finitely many rewritten shapes of the blobs and $\gamma_{3.9}$ the exponent of* [25, Prop. 3.9]*, $|c^{(k+n)}_{\ge k} - c^{(k)}_{\ge k}| \le C\,L^{-\gamma k}\,\ell\,(1+\ln\ell^{-1})^5$ with C independent of n, k, K and ε; (d) the bounds hold on the skew torus and on the aligned torus alike.*

Proof. A finite census of the blobs (at most five lines each) gives the maximal quotient degree over the fine sets F and the coarse prefix sets S as -1 at order two and $+1$ at order four, so every positive quotient needs a fine or earlier-coarse divergent sub-piece; with the lines ordered as in [25, (3.76)], each divergent fine component is rewritten by the renormalisation theorem in its free-sum form (the zero-momentum gauge covariance identity inside the order-e^2 vacuum polarisation being applied with the fine indices summed, never to a fixed fine ordering), and the coarse sum is at most the raw sum over the coarse indices, $(K-k+1)^m\ell^{-\max(0,D_{\max})}$ with $m \le 5$, by [25, Prop. 3.7], which gives (44) since $e_k^\alpha\lambda_k^\beta\ell^{-D^+_{\max}} = O(\ell)$. For (c) the difference of the two models telescopes one factor at a time, each factor carrying $L^{-\gamma k}$ (the shrinking induction, the difference form of the renormalisation theorem, [25, Prop. 3.9]), the weakening $\gamma < \alpha_1$ being absorbed by the geometric sums; the census is torus-free, which gives (d). □

The split degree count. For the classes with block vertices the count that governs the bounds is the split count of Definition 5.39 (a block product represented by its two summation points, a vector leg of a block vertex credited only on an internal line of the subgraph), in which every component with a block point has degree at least one (Lemma 5.40); Theorem 20.3 below says that this count, and not King's per-vertex credit, is what the multiscale bound sees.

Definition 20.2 (the condition (DEG^{nat}))**.** Let $\mathcal{R}$ be a renormalised family of King's expansion of order at most $\bar{n}$, with the source vertices [25, (3.20), (3.21)] admitted, written as a finite sum of graphs G. The family satisfies the condition (DEG^{nat}) if, for each admitted ordering of the scale-summed lines of the split graph G^{sp}, every component of every ultraviolet prefix (indices $j < k$) has current degree $D' > 0$ after the rewriting of Theorem 20.3(iii) and the infrared counterterm lines (indices $k \le j \le K$) obey the growth form of Theorem 20.1; $\alpha_0 > 0$ is then the minimum of these degrees over the finitely many graphs and orderings.

Theorem 20.3 (the split count governs the multiscale bound)**.** *Let $\mathcal{R}$ be a renormalised family as in Definition 20.2 satisfying* (DEG^{nat})*, with the minimal degree $\alpha_0 > 0$ of that definition, and let the lines obey King's bounds* [25, (3.63)–(3.65)]*, the infrared slices* [25, p. 666] *included, the contour and contour-Hölder lines with the contour factor ϱ_{j_l} booked on the line, for the finer model's slices as well. Then:*

- *(i) the sum over orderings, compatible integer assignments and positions of $|E^{(k)}(G)|$ is at most $C\,(L^k\varepsilon)^{\rho}\,p(L^k\varepsilon)^{n_1}\,(1+K)^{p_0}\exp[-\delta_0\,\mathrm{dist}(\textit{external vertices})]$, the bound* [25, (3.38)] *with King's exponents, decay rate and prefactor, and the same holds with one element weakened by $\theta < \alpha_0$ (a line l with $c_l \mapsto c_l\varrho_{j_l}^{-\theta}$, a vertex with an explicit η-power $a_v \ge \theta$ reduced to $a_v - \theta$, or an external field replaced by one of the same $(1,\alpha)$-norm);*
- *(ii) for $0 < \gamma < \alpha_0$, $|E^{(k+n)}(G) - E^{(k)}(G)| \le C\,(L^k\varepsilon)^{\rho'}L^{-\gamma k}\exp[-\delta_0\,\mathrm{dist}]$, uniformly in n;*
- *(iii) every rewritten unit is source-free, point-free and a component of a prefix of G^{sp}, and $D'(\gamma) \ge D^{\mathrm{sp}}(\gamma)$ for every component of every prefix;*
- *(iv) the exponent of every stage of the ordered bound before the κ-lines is D'^{nat} of the stage, raised by d per merging κ-line from a κ-line on (immaterial, since $\varrho_k = 1$); no per-vertex credit $+m$ enters.*

The constants depend on $\bar{n}$, d, α, θ (resp. γ), δ_0 and King's constants only. Parts (i), (iii), (iv) hold verbatim on a cube with the slices of the Dirichlet flow, and part (ii) holds there for the diagonal pair (K,k) against $(K+1,k+1)$ on the common unit lattice, at $h = 0$ and on the small-field set, in the form that the sum over positions of $|E^{(k+1)}_{K+1}(G) - E^{(k)}_{K}(G)|$ is at most $C(L^{-\gamma k}\eta^{-\beta} + \eta^{\sigma})$ times the volume of the cube, for some $\gamma, \sigma > 0$ and $\beta \ge 0$ (King's two-term bracket [25, (3.56)]*).*

Proof. (i) is the shrinking induction [25, (3.66)–(3.70)], read per ordering, on G^{sp} with the dimensions and weights of Definition 5.39: a connected all-scale-summed subgraph containing a κ-line has $D^{\mathrm{MS}} = \sum_i D^{\mathrm{sp}}(c_i) + d(r-1) \ge 1$ by Lemma 5.40, so the hypothesis of the induction is exactly (DEG^{nat}), and shrinking the kernel κ leaves the infrared differences [25, (3.80)] and the growth form of Theorem 20.1 unchanged. (ii) is King's replacement scheme [25, pp. 663–665, (3.70)–(3.75), (3.88)] on G^{sp}, each error term being the same graph with one element weakened by γ and the prefactor $L^{-\gamma k}$, bounded by the weakened clause of (i); (iii) holds because a rewritten unit is divergent, hence point-free by Lemma 5.40 and a graph of Bałaban's divergent classes [3, pp. 428–429]; (iv) is the telescoping identity of the induction. On the cube the same holds through the split-graph forms of the line bounds, (ii) as the diagonal-pair comparison with King's two-term bracket, a contour line being a weakened scale-summed line with gain $(\eta'/\varrho_j)^{1/2}$ whose degree requirement holds with margin $\frac{1}{2}$ by Lemma 5.40. □

Bałaban's renormalisation rewriting in King's scheme. The plain degree of a connected component γ is $D(\gamma) = \sum_{v\in\gamma} u_v - 3 + \frac{1}{2}E_\gamma$, with $u_v := 3 + a_v - \frac{1}{2}n_v - \delta_v$ for a vertex with n_v legs, a_v extra powers of η beyond η^d and δ_v differentiated internal ends, and E_γ the number of legs at vertices of γ that are not ends of lines of γ ($u = 2$ for the h-vertex, $u = \frac{5}{2}$ for the summed curvature vertex). A *unit* is a connected prefix component S of an admitted ordering

with ordered degree $D'(S) \le 0$, D' being the plain degree with the credits and debits of the earlier rewritings; a unit contains no h-, g- or source vertex. Units are rewritten finest first, on their admitted set A_S (all internal indices at most j_0, the lowest index among the legs of S) and with their inside family F_S (all connected members of the same order and profile with the same legs at the same outside points, δm^2- and Ward–Takahashi partners included); ϱ_j^θ, $\theta > 0$, is a *credit* on a line of index j, where $\varrho_j := L^j\eta$ is the range of a line of index j in unit-k units (η the spacing, so $\varrho_j = \ell_j/\ell_k$ with $\ell_j = L^j\varepsilon$ the physical block scales). The rewriting of a unit, its sum over F_S and A_S, is an exact identity whose output terms are of three shapes: (U0) zero; (U1) collapsed local terms $\sum_x \eta^d c_a(x) \prod \partial^a \Phi(x)$, $|a| \le n := -D(S)$, times fixed localisation weights (Φ the external fields of S); (U2) remainders with one credit $\varrho_{j_c}^\theta$ on a line of S and one debit of total order at most θ on one leg of S or on a smooth weight, $\theta = n + \alpha$, with $D'(S) > 0$. The long-line units (zero debit, $\theta = 3$) and the face units ($\theta = 1$) are units.

Theorem 20.4 (the strict degree property, componentwise)**.** *Run the construction of* [3, §§2–3] *in King's scheme, with h and g counted as vertices. Then:*

- *(i) (*unit shapes*) the rewriting of each unit S is an exact identity with output terms of the shapes (U0)–(U2); in (U1), $|c_a(x)| \le C\varrho_{j_0}^{D(S)+|a|}$ uniformly in x, j_0, k, η, with $c_0 \equiv 0$ for the one-particle-irreducible two-scalar-leg units and for the tree* [3, (2.4)]*; in (U2), $D'(S) \ge \alpha$ and $D'(\gamma) \ge D(\gamma)$ for every prefix component $\gamma \subsetneq S$. The constants depend only on $\bar n = 13$, $d = 3$, $L = 5$, $\alpha \in (0,1)$ and King's constants* [25, Thm. 3.3, Props. 3.7–3.9]*;*
- *(ii) (*strictness, componentwise*) for every output term and every stage i, every source-free connected component γ of H_i has $D'(\gamma) > 0$, and the minimum α_0 of these values, the* minimal ordered degree*, is positive and depends only on $\bar n$, d, α ($\alpha_0 \ge \alpha$ for $\alpha < 1/N(\bar n)$, $N(\bar n)$ the maximal leg count of a component). Every component meeting h or g has $D'(\gamma) \ge 1$ off the excluded shapes of the census (the whole vacuum graph and the two shapes that vanish identically, in the proof of Theorem 20.8). Hence $D'(H_i) = \sum_\gamma D'(\gamma) \ge \min(\alpha_0, 1) > 0$ for every prefix: the strict ordered degree condition (DEG), with this α_0;*
- *(iii) (*one-particle-reducible components*) (i)–(ii) include the one-particle-reducible source-free components; for the scalar chains with one-particle-irreducible pieces the zeroth moment vanishes exactly for every scale assignment;*
- *(iv) (*embedded form; far ends*) the legs of a unit may be any lines of the graph (h-, g-, counterterm-, block- and δG-lines included) or external fields; the rewriting uses a leg only through its index, its value and its Taylor or Hölder quotient in its S-end variable (uniformly in the far end) and the cube of its far end, and never acts on the far-end vertex; h and the summed g put no derivative on their lines;*
- *(v) (*King's scheme*) the counterterm partners are King's δm^2* [25, (3.32)–(3.34)] *and E_1* [25, (3.35)]*, which on the ultraviolet slices coincide graph by graph with Bałaban's δm_k^2, E_k* [3, (1.26)–(1.29)]*;*
- *(vi) (*generic legs*) (i)–(ii) hold with the product of the external fields of S replaced by any function Ψ of the leg positions, the bounds depending on Ψ through sup norms and the norm* [3, (1.32)] *in each variable only; in the form used, h and $w = \eta^{-1}d^*g$ enter only as far-end vertex functions, through $\|h\|_\infty$, $\|w\|_\infty$;*
- *(vii) (*difference form*) for the two lattices of* (43)*, $|c_a^{(k+n)} - c_a^{(k)}| \le CL^{-\gamma k}\varrho_{j_0}^{D(S)+|a|-\gamma}$;*
- *(viii) (*cube*) on King's cube C_J^K, in the zone of the free flow and under the range condition $L^k\varepsilon \le \varepsilon_0$, (i)–(vi) hold with the cube form of the renormalised-degree structure (Theorem 5.30) in place of its torus form.*

Proof. (i) is the renormalisation rewriting of a unit (the resummation of its inside family F_S, the transport and Taylor identities in the debited variable, the moment computation of the collapsed coefficients), whose hypotheses hold for every unit by the prefix criterion and the single-debit clause of Theorem 20.5 and by the split count (Lemma 5.40, Theorem 20.3), units being rewritten finest first. (ii): by the stage bookkeeping of Theorem 20.5 a rewriting never lowers D' of a prefix component, a source-free component of current degree ≤ 0 would itself be a unit, already rewritten, $\alpha_0 = \min D'$ is a minimum over a finite subset of $\mathbb{Z} + \alpha\mathbb{Z}_{\ge 0}$, and the components meeting h or g have $D' \ge D_{\text{plain}} \ge \frac{1}{2}$ by the census, hence ≥ 1 by integrality, off the whole vacuum graph and the shapes that vanish by $\operatorname{Tr} q = 0$ (neither the bookkeeping nor the census uses a strict degree property, so there is no circularity). (iii)–(vi) are read off the rewriting, which sees a leg only through its index, its value, its quotient at the S-end and the cube of its far end, (vii), (viii) are its difference form for the two lattices of (43) and its cube form through Theorem 5.30, and the componentwise form is Bałaban's own [3, pp. 425, 428, 433], every step of [3, §3] stated there without argument being replaced by a lemma of the rewriting, the Ward identity being applied only to the C^{ξ} replacement of the resummed two-photon family, and the vector-propagator clause of [3, p. 426] being the vector-propagator bound of Theorem 19.3. □

Bałaban's construction with the source vertices counted. The restricted estimates rest on four statements. (I1) King's two-cut-off estimate for renormalised graphs, [25, Prop. 3.6] in its one-flow instance: for the graphs $\tilde{H}$ of [25, (3.55)] satisfying (DEG), read on the split graph (source vertices included, g at $\frac{5}{2}$), with the growth form of the coarse counterterm attachments (Theorem 20.1),

$$|E^{(k+n)}(\tilde{H};\{y\},\{z\},\{w\}) - E^{(k)}(\tilde{H};\cdot)| \le C\delta_k^{\rho-1/2} p^{n_2}\left[L^{-\gamma k} + \delta_k^{d+\sigma+1/2}\right] \exp[-\delta \operatorname{dist}(\{y\},\{z\},\{w\})], \tag{45}$$

King's (3.56), with γ smaller than the minimal degree (for hosts with host-level divergent units the collapsed coefficients differ between the two models by Theorem 20.4(vii), an admissible weakened element). (I2) The multiscale bound of type (42) for a single model on the split graph (points as vertices, contour lines with $d_l = -1 + \#\{\text{contour ends}\}$, the block kernel a line of index k), read per ordering, with the infrared logarithm $C(1 + K - k)$ per divergent subgraph. (I3) The strict degree property of the source-free renormalised components under Bałaban's rewriting, Theorem 20.4. (I4) King's propagator and remainder bounds [25, Thm. 3.3, Props. 3.7–3.9] on the torus slices $j \le k-1$ and the infrared slices $k \le j \le K$ [25, p. 666]; the clauses of [25, (3.71), (3.73)–(3.75), Lemma 4.5] that King states without proof hold with a smaller exponent $\gamma' > 0$, and every use needs only $0 < \gamma' < \alpha_0$. The rewriting of (I3) acts only on source-free divergent subgraphs (x and g carry no counterterm off P_0, and the Wick constant of [25, (3.21)] is a c-number entering only the $(0,0)$ tadpole) and does not lower the degree of an ordered subgraph containing x or g; this *rewriting property* (RW) is not automatic and is a consequence of the following theorem, in which the construction is run on the whole graph with x and g counted as vertices.

Theorem 20.5 (Bałaban's construction with source vertices)**.** *Let $\tilde{H}$ be a connected graph of King's family* [25, (3.37), (3.55)] *at $B = 0$, with an external leg or a g-vertex; x the h-vertex ($u_x = 2$), g the curvature vertex with the curl summed onto g ($u_g = \frac{5}{2}$), the counterterm graphs* [25, (3.34)] *with their merged attachment vertex x_S; all lines (x-, g- and counterterm lines included) carry slice indices, and the degrees are the plain degrees by legs. Run Bałaban's construction* [3, Ch. 3] *on $\tilde{H}$: "divergent" means a connected prefix component with current degree ≤ 0; the first divergent component is rewritten by Bałaban's rule for its type; degree-zero single lines* [3, (2.4)] *are postponed into a containing divergent component. Then (i) every rewritten S is*

source-free and is a prefix component, i.e. it is rewritten only in configurations where every S-index is at most every external-leg index (x- and g-lines included); (ii) for every output term, stage i and component γ of H_i: $D'(\gamma) \geq D_{\mathrm{plain}}(\gamma) \geq \frac{1}{2}$ if γ meets x or g, and $D'(\gamma) > 0$ if γ is source-free; (iii) hence $D'(H_i) \geq \alpha_0 > 0$ at every stage with a strict index increase, $\alpha_0 = \min(\frac{1}{2}, \alpha,$ Bałaban's source-free margins$)$; (iv) the Hölder exponent $\alpha \in (0,1)$ is free. The same holds, with the same four statements, for a connected vacuum graph $\tilde{H}$ at $B = 0$ with $m_h \geq 1$ h-vertices, no g-vertex and either $m_h \geq 2$ or total order $\alpha_e + 2\beta_\lambda \geq 4$ in the order $(\alpha_e, \beta_\lambda)$ in (e, λ) (the orders of the counterterm units included), i.e. for the $E = 0$, G-free members outside $P_0 := \{(0,0), (0,1), (2,0)\}$.

Proof. A connected S is a component of some prefix if and only if every adjacent outside line is later than every line of S (the prefix criterion), so a component containing a debited leg contains all of S and $D'(\gamma) \geq D_{\mathrm{plain}}(\gamma)$ for every prefix component, wherever the credit sits (the stage bookkeeping); a leg receives at most one Taylor debit $1 + \alpha$ plus one α per unit having it as an external leg, and a source line at most one debit (the single-debit lemma); scalar-closed subgraphs with an odd number of photon legs vanish by $\operatorname{Tr} q^{\mathrm{odd}} = 0$ (the parity lemma). Inducting over units, finest first, every x- or g-component has $D_{\mathrm{plain}} \geq \frac{1}{2}$ by the census in the proof of Theorem 20.8 (its case (T) needs $E = 0$, impossible with an external leg or a g, and in the vacuum case the order is off P_0 or $m_h \geq 2$; its two degree-zero exceptions kill the graph), so the first divergent component is source-free and Bałaban's type list applies, whence (i); for a source-free component the prefix criterion makes the construction Bałaban's embedded one, so $D' > 0$ is Theorem 20.4(ii) componentwise, whence (ii); (iii) and (iv) follow by summing components. □

Proposition 20.6 (the failing class)**.** *The bounds (42)–(43) fail on the order-e^2 h-linear vacuum family $T_h^{(2,0)}$: the photon exchange on the scalar loop with one h-insertion, the seagull tadpole and the two-photon current loop, with their $\delta m^2(2,0)$ partner, which have degree 0 as whole graphs and no counterterm in King's scheme. Precisely, with $c_Z := e^2/(4\pi^2)$ the coefficient of the logarithmic growth of the order-e^2 Wick constant: (a) at fixed $\delta = \delta_k$ the position sum of the family grows like $c_Z k \ln 5 \int h$, while (42) summed over positions is at most $C(\delta)|T|$ uniformly in k; (b) (42) and (43) cannot both hold there: with King's position sum [25, (3.40)–(3.42)] the per-step difference would be at most $C(1+K-k)^{p_0}[5^{-\gamma k}\delta^{-\beta} + \delta^{\sigma'}]|T| \to C'\delta^{\sigma'}|T|$ as $k \to \infty$ at fixed δ, while it tends to $c_Z \ln 5 \int h \neq 0$.*

Proof. The order-e^2 h-linear field-free part of the effective action changes per step by an amount tending to $c_Z \ln 5 \int h$, the logarithmic growth $z_\varepsilon - z_{5\varepsilon} \to c_Z \ln 5$ of the order-e^2 Wick constant (Proposition 16.13; the schemes of Definition 27.1 absorb it), so its value after k steps is $c_Z k \ln 5 \int h + O(1)$: the whole graph is the one stage of degree 0 of the ordered bound, whose index sum $\sum_{j<k}(L^{j-k})^0$ grows like k. □

The next lemma reduces King's two-cut-off bound (43) for a graph H to the bound (45) for the graphs $\tilde{H}$ that King's expansion produces from H.

Lemma 20.7 (graph-local reduction)**.** *Fix one graph H of [25, (3.37)] and one ρ. If (45) holds, with this ρ, for every graph $\tilde{H}$ that the expansion [25, (3.43)–(3.55)] produces from H, and $E^{(k+n)}(\tilde{H})$ obeys the multiscale bound (I2) (needed only for the term in which a_k is replaced by a_{k+n}), then (43) holds for H with $n_1 = n_2 + 2r$, r the number of $\tilde{A}$ legs.*

Proof. The steps [25, (3.43)–(3.55)] are identities applied to H alone, and the same graphs $\tilde{H}$ occur for $\theta = k$ and $\theta = k + n$ because $L^k \varepsilon_K = L^{k+n}\varepsilon_{K+n}$; so [25, (3.57)] is the triangle inequality termwise, plus the term in which $\tilde{A}_k(w)$ is compared between the two models, of size

$O(L^{-2k}\delta_k^2 p)$ since $a_{k+n}-a_k=O(L^{-2k})$, times a bound of type (42) for $E^{(k+n)}(\tilde H)$; the w-sum with $|\tilde A_k(w)|\le Cp(|w-x_0|+1)$ and the exponential tree decay gives p^{2r}. □

We can now state the restricted form of King's renormalisation theorem that the proof of (N6) uses. Its three parts cover the field- and curvature-dependent parts of every family, the field-independent graphs depending on a fixed scalar source except the h-linear ones of order in P_0, and the one-cut-off bound for the constant-source shift.

Theorem 20.8 (the restricted estimates)**.** *(i) (*Field- and curvature-dependent parts.*) For a graph H of* [25, (3.37)] *in the renormalised family $P^{(k),1}(tG,h_\infty)$ ($h_\infty=h_0-\frac12\Delta_T(0)$ the shifted wall of Theorem 19.17(ii); curl summed onto g) write $E_N(H):=E(H)$ if H has an external scalar leg ($s\ge1$) or a curvature vertex, and $E_N(H):=E(H;A_k)-E(H;A_k=0)$ otherwise. For every H,* (43) *holds for $E_N(H)$,* (42) *holds for it with the extra factor $(1+K-k)^{p_0}$, and*

$$|E_N^{(k+n)}(H)-E_N^{(k)}(H)|\le C\big[5^{-\gamma' k}\delta_k^{-\beta'}+\delta_k^{\sigma'}\big]|T|$$

for any $\gamma'<\gamma$, $\beta'>d+\frac12$, $\sigma'<\sigma$, with C independent of k, K, n. The classes covered are the field-dependent h-families, the families with an h- and a curvature vertex, the h-independent families (source-free and with g), and the field-dependent members of every other family. Not covered are the field-independent, G-independent constants (the h-linear vacuum values at zero field, the h-free vacuum energy, the h-nonlinear g-free vacuum graphs); they are t- and G-independent.

*(ii) (*Fixed scalar source, curvature source zero.*) At $G=0$ and a fixed smooth source h', let $N_{\rm ii}$ consist of every $E_N(H)$ of (i) and of every field-independent graph H depending on h' except the h-linear ones of order in P_0; it contains the h-linear vacuum classes of order $\alpha_e+2\beta_\lambda\ge4$ (every order outside P_0, α_e being even for vacuum classes at $G=0$) and the h-nonlinear vacuum graphs (two or more h-vertices). For every member,* (43) *holds,* (42) *holds with $(1+K-k)^{p_0}$, and the position sum is at most $C[5^{-\gamma' k}\delta_k^{-\beta'}+\delta_k^{\sigma'}]|T|$.*

*(iii) (*The one-cut-off bound for the constant-source shift.*) (a) For every graph H with at least one h-vertex other than the $D=0$ members of the order-$(2,0)$ h-linear family $T_h^{(2,0)}$, and for the $(0,0)$, $(0,1)$ members,*

$$|E^{(k)}(H;\{y\},\{z\})|\le C(1+K-k)^{p_0}\delta_k^{\rho}p^{n_1}e^{-\delta\,\mathrm{dist}}$$

with C independent of k, K. The order-$(2,0)$ h-linear field-free members with a block-spin vertex belong to (a)*: every prefix component has split degree at least 2, and the plain multiscale bound bounds them by $O(1)$. (b) For $T_h^{(2,0)}$ the position-summed bound $\sum_{\text{positions, indices}}|E'|\le C(1+k)$ of Theorem 20.9 holds; the factor $1+k$ is necessary (Proposition 20.6), and the per-position exponential-decay clause of* (a) *is not claimed for this family and is not used. For a graph H at non-zero background,* (42) *is read through* [25, (3.55)]*: the core $\tilde H_0$ is the $A_k=0$ value,* (b) *if $H\in T_h^{(2,0)}$ and* (a) *otherwise (the torus value and King's box-localised core differ by graphs with a ∂G_k line, which* (a) *covers); every other $\tilde H$ has an external leg, so* (a) *applies, with the w-sum of Lemma 20.7. The factor $(1+K-k)^{p_0}$ is a power of $\log\delta_k^{-1}$, absorbed by enlarging n_1.*

Proof. The proof has two parts: a census of the connected graphs of non-positive degree, and then the bounds, which follow from the census and the properties (I1)–(I3). *The census.* With the degree by legs $D(\gamma)=\sum_{v\in\gamma}u_v-3+\frac12E_\gamma$ and the vertex values $u=2$ (h-vertex), 1 (quartic vertex), $\frac12$ or $\frac32$ (one-photon current vertex), ≥1 (current and seagull vertices with more photon legs), $\frac52$ (curvature vertex, after the exact summation by parts onto g) and $\ge\frac52$ (block point

carrying a line), a connected field-independent h-linear vacuum graph without curvature vertex has $D = \frac{1}{2}(\alpha_e + 2\beta_\lambda) - 1$, so $D \le 0$ exactly for the orders in P_0 (the subtracted tadpole, $D = -1$; the figure-eight and the family $T_h^{(2,0)}$, $D = 0$), of which $(0,0)$ and $(0,1)$ have exact counterterms and $(2,0)$ none; a connected γ with exactly one h-vertex x has $D(\gamma) = -1 + \sum_{v \ne x} u_v + \frac{1}{2}E_\gamma \le 0$ only in the case (T), $E_\gamma = 0$ and γ a whole h-linear vacuum graph of order in P_0 or one of its counterterm partners, and in the case (F), $\gamma = \gamma_F$ the h-vertex joined by both lines to a one-photon current vertex whose photon leg is external ($D = 0$); a connected γ with a curvature vertex and no h-vertex has $D \ge \frac{1}{2}$ except γ_G, the curvature vertex joined to a one-photon current vertex with a scalar loop ($D = 0$); two or more h-vertices, or an h- and a curvature vertex, give $D \ge \frac{1}{2}$; every graph of [25, (3.55)] in which γ_F or γ_G occurs with internal lines vanishes identically, its closed scalar loop carrying $\operatorname{Tr} q = 0$ (Furry's theorem at graph level); and a counterterm insertion is a unit, treated by the census as a point vertex. *The bounds.* Expanding $E(H)$ by [25, (3.43)–(3.55)] at A_k (and at $A_k = 0$ when $s = 0$ and H has no g-vertex), the core graph cancels in $E_N(H)$, so every $\tilde{H}$ entering $E_N(H)$ is connected with an external leg or a curvature vertex, and every connected component of an ordered subgraph of $\tilde{H}$ has positive degree: with a block point by Lemma 5.40; with an h- or a curvature vertex by the census, case (T) being excluded by the external leg or g-vertex (in (ii) by the order off P_0 or $m_h \ge 2$) and the cases (F), γ_G vanishing; and source-free by (I3), whose rewriting does not lower the degrees used in the other cases (the rewriting property (RW), by the stage bookkeeping of Theorem 20.5, on the $E = 0$, G-free members through its vacuum form); so every non-vanishing $\tilde{H}$ satisfies (DEG) with a positive minimum α_0 over the finitely many graphs and orderings, the infrared indices occurring only on counterterm lines, and (I1) gives (45) with $\gamma < \alpha_0$, Lemma 20.7 gives (43), (I2) gives (42) with $(1 + K - k)^{p_0}$, $p_0 \le \bar{n}$, and King's position sum [25, (3.40)–(3.42)] gives the volume bound, with $\beta = d + \frac{1}{2}$ and $\sigma' = \sigma$ worsened by an arbitrarily small amount to absorb p^{n_1} and the logarithms. In (iii)(a) the $(0,0)$ member is the subtracted tadpole at zero coarse field and the $(0,1)$ member the figure-eight minus its δm_1^2 partner, both bounded directly; in (iii)(b) only the whole graph of a member of $T_h^{(2,0)}$ has $D = 0$, so the last geometric sum of the multiscale bound, $\sum_{j \le k-1} (L^j \eta)^0 = k$, is the factor $1 + k$, and the Taylor step with the $\delta m^2(2,0)$ partner and the lifting of the degree-zero line are Theorem 20.9; the h-free, field-free vacuum constants cancel in the ratio of partition functions, and (ii) makes no claim about them. □

The position-summed bound on the failing family. For the family $T_h^{(2,0)}$, on which the per-position bound (42) fails, the proof of (N6) uses a position-summed bound. The members of the family at $A_k = 0$, with the source vertex x carrying the two scalar legs $l_e, l_{e'}$, are the self-energy insertion (E) (a scalar line s and a photon line p between the two legs, the *unit* $S = \{s, p\}$, in the four derivative placements dd, du, ud, uu of the two current derivatives), the two-photon-current tadpole (T2), the seagull tadpole (T3) and the counterterm member (P), in which the mass counterterm δm^2 of [25, (3.33)] sits between the legs and which splits into the self-energy unit P_E and the seagull-tadpole unit P_{T3}; every line carries a slice index $j < k$, and $\mathbf{j} = (j_e, j_{e'}, j_s, j_p)$ is the index tuple. A *generalised graph* is a graph with a finite number of lines bounded as in [25, (3.63)–(3.65)], with $d_l = -1 - \#\{\text{derivatives on } l\}$, at most one line credited by a power $\theta \in \{\alpha, 1 + \alpha\}$ of $|y' - y|/l_j$ and at most one weakened by the same power, with vertex weights $a_v \ge 0$ and one fixed vertex x; its degrees are Bałaban's [3, (2.1)–(2.2)], and the *multiscale bound* is the statement that a generalised graph all of whose proper prefix components, in a fixed ordering of its lines by index, have positive degree, and whose whole-graph degree is 0, has its sum over indices and positions bounded by $C(1 + k)$, one factor $(1 + k)$ per zero-degree step (the telescoping of [25, (3.69)–(3.70)]); the bound does not depend on the routing of the momenta, two chord choices differing by a unimodular integer change of the loop variables.

Theorem 20.9 (the position-summed bound on the order-$(2,0)$ h-linear vacuum family)**.** *Let $F_k(x)$ be the sum of the renormalised values $E^{(k)}(H; A_k = 0)$ over the members* (E)*,* (T2)*,* (T3)*,* (P)*, all positions other than x summed. Then $F_k(x)$ is a finite sum, with a number of terms depending on the graph list only, of generalised expressions $E'(G')$, each with*

$$\sum_{\text{positions, indices}} |E'(G')| \le C(1+k),$$

C independent of k, K and x; without the summation by parts at the degree-zero scalar line the same holds with $(1+k)^2$. Every generalised graph G' that is index-summed under the multiscale bound has, in each of its admitted orderings, all proper prefix components of degree at least $\alpha > 0$ (at least 1 for the components containing x) and whole-graph degree 0.

Proof. Slice matching. The unit of (E) with slices (j_s, j_p), $j_s, j_p < k$, carries the zero-momentum value of the kernel $\Sigma^{dd}_{j_s j_p}$ that sits between the legs of the counterterm member ([25, (2.17), (2.20), (3.33)]), so that

$$E_{dd}(\mathbf{j}) + E_P(\mathbf{j}) = c_{\mathrm{P}} \sum_{y,y'} \eta^{2d} \operatorname{Tr}\Big[G_{j_e}(x,y)\, \Sigma^{dd}_{j_s j_p}(y,y') \big(G_{j_{e'}}(y',x) - G_{j_{e'}}(y,x) \big) \Big], \tag{46}$$

(T3) and P_{T3} cancel pairwise, and the pieces of (P) with $\max(j_s, j_p) \ge k$ are bounded by $C(1+k)$ directly, by the K-slice propagator bounds [25, p. 666]. *The Taylor step and the degree-zero line.* On the unit-first index set $A = \{\max(j_s, j_p) < \min(j_e, j_{e'})\}$ Bałaban's exact lattice Taylor formula [25, (3.83)–(3.87)] splits (46) into a remainder, a generalised graph with the credit $1+\alpha$ on the finest unit line and the leg $l_{e'}$ weakened by the same power (the Hölder clause [25, (3.65)]), and a linear term whose two-leg vertex is bounded uniformly in the index by the scalar resolvent identity at unit scale and King's propagator bounds [25, Thm. 3.3]; the placements du, ud, uu and (T2) are summable directly ([3, (3.24)], [25, (3.63)]); off A the only proper prefix component of degree ≤ 0 is the scalar line $\{s\}$ first, in the placement dd (and $\{s'\}$ in P_E), which the exact summation by parts at a current vertex [3, (2.8)] lifts to degree 1 without lowering any prefix component, skipping it costing a second zero-degree step, hence $(1+k)^2$. This is Bałaban's general step [3, pp. 434–435, 444–445] with the final whole-graph renormalisation omitted, King's scheme having no counterterm for H [25, p. 659], and every resulting generalised graph satisfies the hypotheses of the multiscale bound with one zero-degree step (a credit inside S and a debit on a leg of S never lower a prefix component containing the debited leg, the components containing x have degree at least 1, and the proper source-free components are lifted to at least α), so $|F_k(x)| \le C(1+k)$; no photon Ward identity is used, and the per-position exponential decay is neither claimed for this family nor used. □

The non-polynomial terms between two cut-offs. Let $T = D_J$ carry King's standard flow on X_K (block factor 5, corner-rooted blocks aligned with T) at the depth $k = k(K)$, with $\delta = \delta_k = 5^k \varepsilon_K$ the block scale and $p = p(\delta)$; the two cut-offs are the K-model at depth k and the $(K+1)$-model at depth $k+1$, which live on the same unit lattice $T^{(k)}$ and carry the same block fields A_k, ϕ_k, quantities of the second carrying a prime. King's effective action [25, (3.90)] is the sum of the polynomial $P^{(k),1}$ and of the three *non-polynomial terms*

$$B_1 := \tfrac{1}{2}\langle A_k, \Delta^{(k)}_{\mathrm{ph}} A_k \rangle, \qquad B_2 := \tfrac{1}{2}\langle \phi_k, \Delta^{(k)}(A^{(k)})\phi_k \rangle, \qquad B_3 := -\ln \frac{Z_k(A^{(k)})}{Z_k(0)},$$

the photon quadratic form, the scalar quadratic form at the background $A^{(k)} = H^{\mathrm{ph}}_k A_k$ and the background determinant, in the notation of Definition 5.21 with the torus in place of the cube. Step (S4) of the proof of Theorem 19.17 needs their change between the two cut-offs at the fixed

wall [25, (3.14)] in the form $|B_i - B_i'| \le Cp^c\delta^{-a}(5^{-\gamma_L k} + \delta^s)|T|$ on $\operatorname{supp}\chi_k$ for each of the three, at some rate $\gamma_L > 0$.

Lemma 20.10 (the bound on B_1)**.** *On* $\operatorname{supp}\chi_k$, $|B_1 - B_1'| \le \frac{1}{2}\|\Delta_T^{(k)} - \Delta_T^{\prime(k+1)}\|\,\|A_k\|^2 \le C\mu_0^{-2}p^2\,5^{-2k}\delta^{-5}|T|$.

Proof. Both operators are King's zero-field photon operators on the unit torus $T^{(k)}$, diagonal in Fourier space, whose free symbols differ by at most $Ca5^{-2k}$ uniformly in the unit mass $\mu_k^2 = \mu_0^2\delta^2$; $\|A_k\|^2 \le C\mu_0^{-2}p^2\delta^{-5}|T|$ is the field bound [25, (3.2)] in unit-lattice counting. □

Theorem 20.11 (the bounds on B_2 and B_3)**.** *On* $\operatorname{supp}\chi_k$, *for every* $\gamma_L < \frac{1}{2}$ *and* $s > 0$*:*

(a) $|B_2 - B_2'| \le \frac{1}{2}\|\Delta_T^{(k)}(A^{(k)}) - \Delta_T^{\prime(k+1)}(A^{\prime(k+1)})\|\,\|\phi_k\|^2 \le C_s\lambda^{-1/2}p^{c+2}\delta^{-7/2}(5^{-\gamma_L k} + \delta^s)|T|$;
(b) $|B_3 - B_3'| \le C_sNe^2(1 + \mu_0^{-1})p^{c+2}\delta^{-3}\big(5^{-\gamma_L k} + 5^{-k} + \delta^s\big)|T|$.

Proof. On the universal cover $\Lambda := \varepsilon_K\mathbb{Z}^3$ of the fine torus $T_\varepsilon = \Lambda/P_J$, periodisation commutes with products and inverses and is bounded on ℓ^2 by the ℓ^1 kernel norm (Schur's test), every operator of Definition 5.21 on T is the periodisation of its cover version (block-local coercivity and the Combes–Thomas bound on $\operatorname{supp}\chi_k$), and the per-site traces $g_{j,x}^{\Lambda} := \operatorname{Tr}_N[\ln K_{j,\Lambda}](x,x)$ are invariant under the gauge transformations $B\cdot(z - x_0)$ of the cover; the two models' lifts $A^{(k)} = HA_k$ and $A^{\prime(k+1)} = H'A_k$ of the same unit-lattice field differ by at most $Cp(5^{-\gamma_L k} + \mu_0\delta\,5^{-2k})$ at every common fine site (the constant mode exactly, the rest by the kernel comparison of [25, (3.71)] at zero field against the first-difference bound of [25, (3.2)]). *(a)* is the field bound on $\|\phi_k\|^2$ [25, (3.2)] times the operator bound $\|\Delta_T^{(k)}(A^{(k)}) - \Delta_T^{\prime(k+1)}(A^{\prime(k+1)})\| \le C_sp^c(5^{-\gamma_L k} + \delta^s)$, proved on the cover by gauging both models at a common coarse site and expanding the gauged kernels in the coupling to an order $n_T \ge 2s + 2$, compared chain by chain through the zero-field infinite-lattice lines at the rate $5^{-\gamma_L k}$, with Taylor remainders at most $C\delta^s$. *(b)* $B_3 = \frac{1}{2}\sum_{j<k}\sum_{x\in X_j/P_J}[g_{j,x}^T(A^{(k)}) - g_{j,x}^T(0)]$ factorises over the scales; pairing the K-model's step j with the $(K+1)$-model's step $j+1$ at the same physical scale (the three unpaired steps cost $\varepsilon_K = 5^{-k}\delta$), with the image terms summing to $C_s\delta^s|T|$ and, per site, the per-site gauge, charge-conjugation parity and the Taylor expansion in the coupling, the paired differences are at most $CN(e_j|\partial A_j|)^2\delta_{\mathrm{conv}}(j) + CN(e_j|\partial A_j|)(e_j|\partial(A - A')_j|)$ with $\delta_{\mathrm{conv}}(j) \le C_sp^c(\mu_0\delta)^{-1}(5^{-\gamma_L j} + \delta^s)$, and the sum over sites and scales ($e_j|\partial A_j| = el_j^2F_{\mathrm{ph}}$, $F_{\mathrm{ph}} \le Cp\delta^{-3/2}$, $\sum_{j<k} l_j5^{-\gamma_L j} \le C\delta5^{-\gamma_L k}$) gives (b) for $K \ge K_0$. □

20.1. **Corrections to the cited sources.** The two remarks of this subsection list the points at which the cited sources are corrected or restricted before they are used in this part; the proofs refer to them by clause.

Remark 20.12 (corrections to the literature). The following points of the literature affect its use in the rotation part; the corrected or restricted forms are used where indicated. None of them changes King's cube limit at $h = 0$, Theorem 5.47.

(i) *The large-field term of* [25, Thm. 3.1]. The upper bound (3.4) there adds to the small-field integral $I^{(k)}$ after k block-spin steps the remainder $\exp[-p(\eta)^2 + C|T|]$, $\eta = 5^k\varepsilon_K$, with C depending on $|T|$, and even with C independent of $|T|$ it gives no volume-uniform two-sided representation of the free energy per unit volume by a small-field integral at a fixed coarse scale ($\log(x + y) \le \max(\log x, \log y) + \log 2$). None is used: wherever these bounds enter, the volume is fixed (the one-sided bounds (F) of Definition 18.4 are posed on one torus at a time) and the large-field term is absorbed by a polynomial lower bound for the walled partition function (Remark 18.7); the constant of [25, Thm. 2.1(ii)] likewise

depends on $|T|$, so no bound on torus functionals uniform in the torus is taken from [25]. The exponent p is fixed by Convention 19.1.

(ii) *King's torus comparison* [26, Thm. 2.4] is false as stated in [26] on two counts: the source is rotated as a scalar (Proposition 16.14), and the Wick constant omits the logarithm of Proposition 16.13; we refer to the statement in that form as *King's original torus comparison*; the statement that is true is Theorem 18.8.

(iii) *The Wick square.* With the free Wick constant alone, $\langle{:}\phi^2{:}(h)\rangle = \frac{e^2}{4\pi^2}\ln(1/\varepsilon)\int h + O(1)$ at order $e^2\lambda^0$ (Proposition 16.13), so King's normalisation must absorb this logarithm, and his comparison between the spacings ε and 5ε acquires the c-number $\frac{e^2}{4\pi^2}\ln 5\int h$. The normalisation used here is the scheme $\Theta^\sharp$ of Section 27, with its vacuum counterterm linear in h; in the equal-spacing comparison of Section 18 no such c-number arises.

(iv) *The bound (4.33) of* [25] is derived there through the step (4.36)–(4.37) in the proof of [25, Lemma 4.5], which treats the block average Q^*Q as a Fourier multiplier and is false as stated (Q^*Q commutes with the translations by $L\mathbb{Z}^d$ only; for $d=1$, $L=5$ there is a ψ with $Q\psi\equiv 0$ for which the left side of (4.36) vanishes while its right side is positive). The conclusion holds with explicit constants: at the unit scale of step k, with a and $a_j \geq \frac{24}{25}a$ King's coefficients of [25, (4.5)], Ω a periodic box of the unit lattice that is a union of blocks of side 5 and $Q^*Q = P$ the orthogonal projection onto block-constant functions, $X(s) := s\Delta^{(k)} + (1-s)\Delta^{(k+n)} + \frac{a}{25}Q^*Q \geq \min\left(4c_\Delta \sin^2\frac{\pi}{10}, \frac{a}{25}\right)$ on $\ell^2(\Omega)$ for all $s\in[0,1]$, k, n and $m_k^2 \geq 0$, $c_\Delta := \left(\frac{25\cdot 3\pi^2}{24a} + \frac{\pi^2}{4}\right)^{-1}$, by the symbol bound $\Delta^{(j)}(p) \geq c_\Delta \sum_\mu 4\sin^2(p_\mu/2)$ and the Poincaré inequality $\|(1-P)\phi\|^2 \leq (4\sin^2\frac{\pi}{10})^{-1}\langle\phi, -\Delta\phi\rangle$ of the Neumann Laplacian of a block of side 5.

(v) *Further points of* [25] are collected in Remark 20.13.

(vi) *Minor points.* King's vertex list [25, (3.23)] is stated for more than two photon legs, while the action [25, (2.1)–(2.4)] gives it for every even number ≥ 2: the seagull vertex of order two belongs to the model, and without it the zeroth moment of the one-loop photon two-point function does not vanish [3, (2.23)–(2.28)]. The inequalities [26, (3.24)–(3.29)] are used in the form of [9, Thms. 6.1, 6.2], Lemma 2.10, which also covers killed sites and the two-level walls; the state named in the proof of [26, Thm. 3.2] is that of the box $C_J \cap \Lambda_k$ of Definition 2.2, the inequalities (3.32)–(3.33) between King's cubes follow from Lemma 4.2 (Corollary 16.19), and the parts $h_\pm$ of a scalar source are taken smooth (Definition 27.3).

Remark 20.13 (further corrections to the literature). Each of the following is false as stated in [25] or is used there beyond what is proved there; we name the corrected or restricted form used.

(i) [25, Thm. 3.5(i), (3.38)] fails on the h-linear vacuum family of order $(2,0)$. Theorem 3.5, Proposition 3.6 and (3.42) of [25] are stated for every graph of the expansion, the scalar-source vertex [25, (3.21)] included, while the proof [25, p. 664] assumes a renormalised graph in which every subgraph has positive degree; the order-e^2 members of the h-linear field-free family have degree 0 as whole graphs and no counterterm, since [25, (3.35)] is built at $h=0$, so the last stage of the ordered bound [25, (3.66)–(3.70)] is the whole graph at degree 0, whose index sum grows like k, and the growth is real: the value of the family at a constant source is the drift of Proposition 16.13 (Proposition 20.6). So (3.38) fails for every choice of its exponents, and (3.39) and Proposition 3.6 fail with it. The bounds hold on the families whose non-vanishing ultraviolet ordered subgraphs have positive degree after Bałaban's rewriting and whose infrared lines lie in mass-counterterm units, and only in that restricted form are they used, Theorem 20.8. [25, Thm. 3.4], the convergence theorem, is false as stated and is not used: only its one-step clause, from the

depth k to the depth $k+1$, enters, at a fixed wall and with the restricted forms as its hypotheses, which is Theorem 19.17.

(ii) [25, (3.78)] fails on certain vacuum graphs, whose contribution is a removed constant (the graph of five quartic vertices $a,\dots,e$ with the lines $ab, ab, ac, ad, bc, bd, ce, ce, de, de$, with the two lines ab ultraviolet, has the prefix $\{ab, ab\}$ of degree $1 < 2$, while the checkpoints of (3.78) need $D(H_m) \le D(H_t)$ for every such prefix). The strict degree property of King's renormalised graphs is used in the componentwise form of [3, (3.5)], Theorem 20.4, and the infrared bound in its growth form, Theorem 20.1, not as (3.78).

(iii) [25, (4.21)] is false as stated for $m > 0$: the factor $|p'|^2$ must read $|p'|^2 + m^2(L^k\varepsilon)^2$, since at $p' = 0$ the right side as stated vanishes while the left side is positive for King's massive scalar; with this correction Propositions 3.8 and 3.9 of [25] go through, only the truncation of (4.21) to the alias-free term being used, and (4.28) is not used. The step (4.36)–(4.37) in the proof of [25, Lemma 4.5] is replaced by the explicit bound of Remark 20.12(iv).

(iv) The statement [25, p. 677] that the sources are proportional to $(5^k\varepsilon)^\alpha$ in the small-field region does not hold for the marginal graph of Proposition 16.13, and the sentence there that the inclusion of the sources in Theorem 3.1 is straightforward carries no argument; the inclusion of the sources is used only as the small-field half of [25, Thm. 3.1] at fixed sources, in the form of Theorem 19.6(c), and at a source with a positive part the walled envelope takes its place on the cone $2\sup h_+ < s_*(e)$ and nowhere beyond.

(v) The statement [25, p. 666] that the divergences cancel for each ordering of the renormalised graphs is false as a statement about each ordering separately, the symmetry-cancelled degree-0 classes [25, (3.84)] not being cancelled per ordering (Theorem 25.10(a)); the identities that hold are those for full propagators [3, (2.24)–(2.27)] and Bałaban's resummed organisation, and no identity for a fixed ordering or scale assignment of King's graphs is used anywhere (Theorem 20.8).

21. Walled tori and the closing estimate

This section introduces the admissible walls and the walled values (Definition 21.1) and states the closing statement (TIGHT$_0$): King's state is the lower envelope of the walled tori (Theorem 21.4), with the weaker (TIGHT$_2$) for the two-point function. Lemma 21.2 gives the free direction of the inequality and Lemma 21.3 the independence of the orientation under (K$_0$). The statement is proved in Section 23; Section 22 uses it.

Figure 5 shows a walled torus.

Definition 21.1 (admissible walls and walled values)**.** For a bounded set $X \subset \mathbb{R}^3$ let $\mathcal{A}(X)$ be the set of walled configurations $\mathfrak{c} = (J', w)$, $w = n\psi$, of the class $\mathcal{C}$ for every $R \in \mathcal{R}_0$ and the cube C_1 (Definition 16.6: $J' \in 5^{k_*}\mathbb{N}$ with $J' \ge \max(J_0(\varepsilon_0), J_{\mathrm{pos}})$, $n > 0$, $\psi = 0$ on the 2-neighbourhood of $R^{-1}C_1$ for every $R \in \mathcal{R}_0$, the wall condition $n\langle\psi\rangle_{X_K} > \Delta_*|T|/2$ for every K, the block alignment and the grid offset of the corners of H'; the conditions other than the vanishing of ψ near $R^{-1}C_1$ do not depend on R, J_{pos} being the maximum over the finite set $\mathcal{R}_0$ and the three torus pairs, Proposition 26.8) with $N_1(X) \subset (-25MJ', 25MJ')^3$ and $w = 0$ on $N_1(X)$; the cell $(-25MJ', 25MJ')^3$ is a fundamental domain of $D_{J'}$ for both lattices, $R^{-1}\Lambda_k$ descending to $D_{J'}$ by Lemma 16.2(b), and sources supported in X are periodised from it. Put $b_\circ := \frac{7}{5}M + 2$; for $X \supset Q_{b_\circ}$ the vanishing of ψ near $R^{-1}C_1$ follows from $w = 0$ on $N_1(X)$, since $R^{-1}C_1 \subset Q_{7M/5}$ (Lemma 16.2(c)), so that $\mathcal{A}(X)$ then consists of all walled configurations of the class $\mathcal{C}$ vanishing on $N_1(X)$. If $X \subset Y$ then $\mathcal{A}(Y) \subset \mathcal{A}(X)$. Assume (K$_0$), which holds for every $\mathfrak{c} \in \mathcal{A}(X)$ when the couplings lie outside the set E_W of Proposition 26.8 (Theorem 18.1). For a real 2-form $f \in C_c^\infty$ supported in X and $\mathfrak{c} \in \mathcal{A}(X)$, the source restricted to the cell and

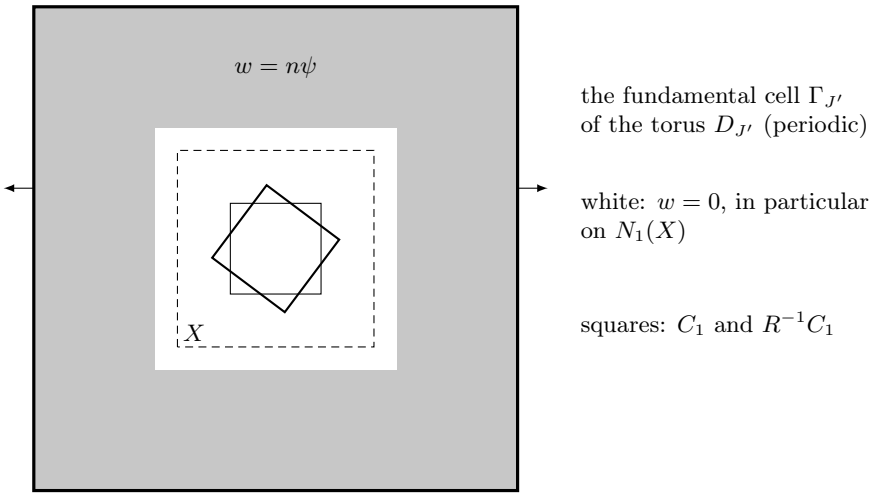

FIGURE 5. A walled torus (Definition 16.4(c), Definition 21.1); not to scale. The outer square is the fundamental cell $\Gamma_{J'} = [-25MJ', 25MJ')^3$ of the King torus $D_{J'} = \mathbb{R}^3/(50MJ'\mathbb{Z})^3$ (Definition 16.1), to which the lattices $R^{-1}\Lambda_k$, $R \in \mathcal{R}_0$, descend (Lemma 16.2). In a walled configuration the density of the torus state is multiplied by $\prod_x \exp(-\varepsilon^3 w(x)|\phi(x)|^2)$, and the wall $w = n\psi$ (grey) raises the scalar mass squared by $2w$ (Definition 16.4(c)). For the configurations of Definition 21.1 the wall vanishes on the neighbourhood $N_1(X)$ of the set X in which the sources are supported, and ψ vanishes on the 2-neighbourhood of the rotated cubes $R^{-1}C_1$, $R \in \mathcal{R}_0$.

periodised is smooth, and by Lemma 18.2 the *walled value*

$$\mathcal{W}_{\mathfrak{c}}(f,0) := \lim_{k\to\infty} T_{\mathfrak{c},\Lambda_k}(f) = \lim_{k\to\infty} T_{\mathfrak{c},R^{-1}\Lambda_k}(f) \qquad (R \in \mathcal{R}_0)$$

exists. For $b > 0$ and f supported in Q_b let $I_b(f) := \inf_{\mathfrak{c}\in\mathcal{A}(Q_b)} \mathcal{W}_{\mathfrak{c}}(f,0)$, which is nondecreasing in b, and

$$\mathcal{W}^{\mathrm{env}}(f,0) := \lim_{b\to\infty} I_b(f).$$

The inequality opposite to (TIGHT$_0$) is free: every admissible walled value dominates the functional of the King cube on which it is admissible, and the envelope dominates King's state.

Lemma 21.2 (the free direction)**.** *Assume (K_0).*

(a) If f is a real 2-form supported at distance > 1 from $\mathbb{R}^3 \setminus C_L$ and $\mathfrak{c} \in \mathcal{A}(\bar{C}_L)$, then $\mathcal{W}_{\mathfrak{c}}(f,0) \geq Z_L^K(\mathrm{i}f,0) := \lim_k \langle e^{\mathrm{i}F(f)}\rangle_{k,L}^K$.

(b) $\mathcal{W}^{\mathrm{env}}(f,0) \geq Z_\infty(\mathrm{i}f,0) > 0$ for every real 2-form $f \in C_c^\infty$.

Proof. (a) At each k, Lemma 16.17 with $X = C_L \cap \Lambda_k$ and Lemma 4.2 give $T_{\mathfrak{c},\Lambda_k}(f) \geq \langle e^{\mathrm{i}F(f)}\rangle_{k,L}^{\mathrm{ND}} \geq \langle e^{\mathrm{i}F(f)}\rangle_{k,L}^K$; let $k \to \infty$, the limit of the right side existing by King's cube limit [26, Thm. 2.1] (Theorem 5.47) with Theorem 17.1(i). (b) If $\bar{C}_L \subset Q_b$ then $\mathcal{A}(Q_b) \subset \mathcal{A}(\bar{C}_L)$, so $I_b \geq Z_L^K(\mathrm{i}f,0)$; let $b \to \infty$ and then $L \to \infty$ (Theorem 17.1(iii)). Positivity: by Lemma 16.16(a), $\langle e^{\mathrm{i}F(f)}\rangle_{k,L}^K \geq e^{-\langle v,Cv\rangle/2}$ with v the vector with $\langle v, A\rangle = F(f)$ and C the Neumann photon covariance of the cube, and $\langle v, Cv\rangle \leq \mu_0^{-2}\|d_\varepsilon^* f\|^2$, with $\|d_\varepsilon^* f\|$ the lattice $L^2(\varepsilon_k)$ norm of the lattice coderivative $d_\varepsilon^* f$ of f, is bounded uniformly in k and L. □

The envelope can also be taken over the configurations admissible for a rotated cube; this is the form in which the chain (39) is closed in Section 22.

Lemma 21.3 (independence of the orientation)**.** *Assume (K_0). For $R \in \mathcal{R}_0$ and every real 2-form $f \in C_c^\infty$, $\lim_{L\to\infty} \inf_{\mathfrak{c}\in\mathcal{A}(R^{-1}Q_{ML})} \mathcal{W}_\mathfrak{c}(f,0) = \mathcal{W}^{\mathrm{env}}(f,0)$, the infimum being defined as soon as* $\operatorname{supp} f \subset Q_{5ML/7}$.

Proof. By Lemma 16.2(c), $Q_{5ML/7} \subset R^{-1}Q_{ML} \subset Q_{7ML/5}$, hence $\mathcal{A}(Q_{7ML/5}) \subset \mathcal{A}(R^{-1}Q_{ML}) \subset \mathcal{A}(Q_{5ML/7})$, and

$$I_{5ML/7}(f) \le \inf_{\mathcal{A}(R^{-1}Q_{ML})} \mathcal{W}_\mathfrak{c}(f,0) \le I_{7ML/5}(f).$$

Both bounds tend to $\mathcal{W}^{\mathrm{env}}(f,0)$. □

With the walled values and the envelope in hand, the closing statement reads as follows.

Theorem 21.4 ((TIGHT$_0$) and (TIGHT$_2$): the closing statement)**.** *Let the couplings lie outside the set E_W of Proposition 26.8, so that (K_0) holds for every configuration of the class $\mathcal{C}$, hence for every member of every $\mathcal{A}(X)$ (Theorem 18.1, Definition 21.1). Then: **(TIGHT$_0$)**: $\mathcal{W}^{\mathrm{env}}(f,0) \le Z_\infty(\mathrm{i}f,0)$ for every real 2-form $f \in C_c^\infty$. Equivalently: for every $b > 0$, every real 2-form f supported in Q_b and every $\eta > 0$ there is $\mathfrak{c} \in \mathcal{A}(Q_b)$ with $\mathcal{W}_\mathfrak{c}(f,0) \le Z_\infty(\mathrm{i}f,0)+\eta$; it involves only ($K_0$) and King's state (Definition 17.2). For $b \ge \frac{7}{5}ML+2$ such a configuration has $\psi = 0$ on the 2-neighbourhood of $R^{-1}C_L$ for every $R \in \mathcal{R}_0$ (Lemma 16.2(c)), so it lies in the class $\mathcal{C}$ for R and L as well. **(TIGHT$_2$)**: for every real 2-form $f \in C_c^\infty$, $\lim_{b\to\infty} \sup_{\mathfrak{c}\in\mathcal{A}(Q_b)} \sigma_\mathfrak{c}^2(f) \ge S_2(f)$, where $\sigma_\mathfrak{c}^2(f) := \lim_k \langle F_{\Lambda_k}(f)^2\rangle_{\mathfrak{c},\Lambda_k}$ and $S_2(f) := \lim_J \lim_k \langle F_{\Lambda_k}(f)^2\rangle_{k,J}^K$, on King's cubes (Definition 2.3).*

Proof. (TIGHT$_0$) says that, in the limit of large source boxes, the walled values lie at or below King's state: $\mathcal{W}^{\mathrm{env}}(f,0)$, the limit over b of the infimum of the walled values $\mathcal{W}_\mathfrak{c}(f,0)$ over the admissible configurations for Q_b, is at most $Z_\infty(\mathrm{i}f,0)$. It is proved in Section 23 in three steps. The dead-shell bound (Theorem 23.7) says that the two-level expectation with the matter killed on a shell around the source box is at most the expectation of $e^{\mathrm{i}F(f)}$ in King's cube state on a cube one step larger than the shell's cube plus $(e^{\tau(\mathfrak{s})}-1)^{1/2}$, hence in the limit at most $Z_\infty(\mathrm{i}f,0)$ plus that term, where $\tau(\mathfrak{s})$ depends only on the source box and the width of the shell and is small when the shell is wide; this holds at every sufficiently small spacing. The heavy-shell estimate (RES$_0$) at one configuration (Theorem 23.12) says that replacing the killed matter on the shell by a large finite shell mass K changes the expectation, in the continuum limit, by at most η, for one configuration and one K. Theorem 23.10 combines the two: the single-level configuration $(J',\psi,n+K)$ is admissible, (K$_0$) (Theorem 18.1) makes its walled value exist as the common limit of its expectations on the axis lattice and on its rotated copies, and the two bounds give $\mathcal{W}_{\mathfrak{c}'}(f,0) \le Z_\infty(\mathrm{i}f,0) + (e^\eta-1)^{1/2} + \eta$.

(TIGHT$_0$) is Theorem 23.10, from (K$_0$) (Theorem 18.1), the heavy-shell estimate (RES$_0$) at one configuration (Theorem 23.12) and the dead-shell chain (Theorem 23.7) at that configuration; (TIGHT$_2$) follows from (TIGHT$_0$) by Lemma 21.6. □

Below, "assume (TIGHT$_0$)" refers to this statement.

The equivalence in (TIGHT$_0$) holds because I_b is nondecreasing in b. By Lemma 21.2, (TIGHT$_0$) is equivalent to $\mathcal{W}^{\mathrm{env}}(f,0) = Z_\infty(\mathrm{i}f,0)$ for every real 2-form f: *King's state is the limit of the lower envelopes of walled tori, the continuum limit being taken first.* The limits defining $\sigma_\mathfrak{c}^2$ and S_2 exist, by Lemma 18.2 under (K$_0$) and by Theorem 17.1 respectively, since second derivatives of locally uniformly convergent entire functions converge.

Remark 21.5 (the configuration class). (K$_0$) is proved by Theorem 18.8 for the configurations of the class $\mathcal{C}$ of Definition 16.6, of which the sets $\mathcal{A}(X)$ of Definition 21.1 consist. The configuration that Theorem 23.10 produces lies in it: its wall is $(n+K)\psi$ with $n+K \ge n_* > 0$, mass-increasing; the wall condition is linear in the height and holds for K large at fixed T, which Remark 23.11(iii) allows; J (the J' of Theorem 23.12), the block alignment of H', T and the grid offset of the corners

of H' are choices of the torus and of the box that Theorem 23.12 makes ($J' \geq \max(J_0, J_{\mathrm{pos}})$ among them), the last moving H' by less than $M'\ell_f$ (ℓ_f the side of the blocks of the final scale, Definition 16.6).

The two-point statement (TIGHT$_2$) is a consequence of (TIGHT$_0$); the next lemma proves this and shows that the inequality in (TIGHT$_2$) is in fact an equality.

Lemma 21.6. *Under (K$_0$), (TIGHT$_0$) implies (TIGHT$_2$), and* $\lim_b \sup_{\mathcal{A}(Q_b)} \sigma^2_{\mathfrak{c}}(f) \leq S_2(f)$ *always; so (TIGHT$_2$) is an equality.*

Proof. Fix f, b and $\eta' > 0$. All functions of t below are real and even. Since $\cos x \geq 1 - \frac{x^2}{2}$ and $\cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$ at every lattice level, and derivatives converge in the limits,

$$\mathcal{W}_{\mathfrak{c}}(tf, 0) \geq 1 - \tfrac{t^2}{2}\sigma^2_{\mathfrak{c}}(f), \qquad Z_\infty(\mathrm{i}tf, 0) \leq 1 - \tfrac{t^2}{2} S_2(f) + \tfrac{t^4}{24} M_4(f),$$

with $M_4(f) := \partial_t^4 Z_\infty(tf, 0)|_{t=0} < \infty$. By (TIGHT$_0$) applied to tf, for every $t > 0$ and $\eta > 0$ there is $\mathfrak{c} \in \mathcal{A}(Q_b)$ with $\mathcal{W}_{\mathfrak{c}}(tf, 0) \leq Z_\infty(\mathrm{i}tf, 0) + \eta$, hence $\sigma^2_{\mathfrak{c}}(f) \geq S_2(f) - \frac{t^2}{12} M_4(f) - \frac{2\eta}{t^2}$. Choosing t with $t^2 M_4(f)/12 < \eta'/2$ and then $\eta = t^2\eta'/4$ gives $\sup_{\mathcal{A}(Q_b)} \sigma^2_{\mathfrak{c}}(f) \geq S_2(f) - \eta'$. For the second claim, let $\mathfrak{c} \in \mathcal{A}(\bar{C}_L)$ with f supported well inside C_L. At every k, Lemma 4.2 and Lemma 16.17, as in the proof of Lemma 21.2(a), give $T_{\mathfrak{c},\Lambda_k}(tf) \geq \langle e^{\mathrm{i}tF(f)}\rangle^K_{k,L}$; so $\mathcal{W}_{\mathfrak{c}}(tf, 0) \geq Z^K_L(\mathrm{i}tf, 0) = \lim_k \langle e^{\mathrm{i}tF(f)}\rangle^K_{k,L}$ for all real t, with equality at $t = 0$, and both sides are even and smooth, so $\sigma^2_{\mathfrak{c}}(f) \leq \langle F(f)^2\rangle_{C^K_L} := \lim_k \langle F(f)^2\rangle^K_{k,L}$. Hence $\sup_{\mathcal{A}(Q_b)} \sigma^2_{\mathfrak{c}}(f) \leq \langle F(f)^2\rangle_{C^K_L}$ whenever $\bar{C}_L \subset Q_b$. By Theorem 17.1(ii), comparing second derivatives at $t = 0$, the cube variances are nonincreasing in L, and they converge to $S_2(f)$. □

22. Rotation invariance

This section derives the invariance of King's state from (K$_0$) and (TIGHT$_0$): Theorem 22.1 gives the invariance under the generating rotations by letting $L \to \infty$ in the chain (39), Lemma 22.2 adds a reflection, Theorem 22.3 passes from the dense subgroup to $O(3)$ by the continuity lemma for convex functionals, and Theorem 22.4 states the result for the curvature sector, with the translations of Theorem 17.1(v). Theorem 15.1 is this theorem once (K$_0$) and (TIGHT$_0$) are proved (Sections 18 and 23).

Theorem 22.1 (invariance under the generators)**.** *Assume (K$_0$) and (TIGHT$_0$) (Theorems 18.1 and 21.4). Then for every $R \in \mathcal{R}_0$ and every real 2-form $f \in C^\infty_c$,*

$$Z_\infty(\mathrm{i}\, R \cdot f, 0) = Z_\infty(\mathrm{i}f, 0),$$

and consequently the same holds for every $R \in G_0$.

Proof. The proof has six steps: a geometric set-up, cutting on the rotated lattice, the passive rotation onto King's cube, the continuum limits, the infimum over configurations with the volume limit, and finally the inverse rotation. Fix $R \in \mathcal{R}_0$ and a real 2-form f supported in Q_ρ.

Step 0 (geometry). Take L with $ML \geq \frac{7}{5}\rho + 2$. By Lemma 16.2(c), $R \cdot f$ is supported in $Q_{7\rho/5}$, at distance > 1 from $\mathbb{R}^3 \setminus C_L$, and f is supported in $Q_{5ML/7} \subset R^{-1}Q_{ML}$. Let $\mathfrak{c} \in \mathcal{A}(R^{-1}Q_{ML})$. Then $R^{-1}\Lambda_k$ descends to $D_{J'}$ (Lemma 16.2(b)), and $B'_k := R^{-1}(C_L \cap \Lambda_k)$ is a set of sites of $R^{-1}\Lambda_k$ which, with its nearest neighbours, lies in the open cell, and on which $w = 0$. Since $R \cdot f$ has Euclidean distance > 1 from $\mathbb{R}^3 \setminus C_L$, every plaquette of $R^{-1}\Lambda_k$ at which f does not vanish lies in B'_k.

Step 1 (cutting on the rotated lattice). By Lemma 16.17, at every k,

$$\langle e^{\mathrm{i}F(f)}\rangle^{\mathrm{ND}}_{R^{-1}\Lambda_k, B'_k} \leq T_{\mathfrak{c}, R^{-1}\Lambda_k}(f).$$

Step 2 (passive rotation, and King's cube). By Lemma 16.15 with $O = R$, $\langle e^{\mathrm{i}F(f)}\rangle^{\mathrm{ND}}_{R^{-1}\Lambda_k,B'_k} = \langle e^{\mathrm{i}F(R\cdot f)}\rangle^{\mathrm{ND}}_{k,L}$, and by Lemma 4.2 the functional of King's cube C^K_L for the source $R\cdot f$ is bounded by that of the box: $\langle e^{\mathrm{i}F(R\cdot f)}\rangle^{K}_{k,L} \le \langle e^{\mathrm{i}F(R\cdot f)}\rangle^{\mathrm{ND}}_{k,L}$.

Step 3 (continuum limits). As $k\to\infty$ the left side converges by King's cube limit [26, Thm. 2.1] (Theorem 5.47) with Theorem 17.1(i) and the right side by (K_0) (Definition 21.1):

$$Z^K_L(\mathrm{i}\,R\cdot f,0) \le \mathcal{W}_{\mathfrak{c}}(f,0) \qquad \text{for every } \mathfrak{c}\in\mathcal{A}(R^{-1}Q_{ML}).$$

Step 4 (infimum, volume limit, closing estimate). Take the infimum over $\mathfrak{c}\in\mathcal{A}(R^{-1}Q_{ML})$ and let $L\to\infty$. By Theorem 17.1(iii), Lemma 21.3 and (TIGHT_0),

$$Z_\infty(\mathrm{i}\,R\cdot f,0) = \lim_{L\to\infty} Z^K_L(\mathrm{i}\,R\cdot f,0) \le \mathcal{W}^{\mathrm{env}}(f,0) \le Z_\infty(\mathrm{i}f,0).$$

Step 5 (the inverse). $R^{-1}\in\mathcal{R}_0$ and $R\cdot f$ is a real 2-form, so Steps 0–4 for $(R^{-1},R\cdot f)$ give $Z_\infty(\mathrm{i}f,0)\le Z_\infty(\mathrm{i}\,R\cdot f,0)$. Since the real 2-forms are G_0-invariant, the claim extends to G_0. □

The order of limits is: at fixed R, f, L and $\mathfrak{c}$, first $k\to\infty$; then the infimum over $\mathfrak{c}$; then $L\to\infty$. The continuum limit is always taken first, at fixed volume and fixed wall, as in King's construction [26, p. 336], and (K_0) is used only as an equality of continuum limits, not at fixed lattice spacing.

The generating rotations lie in $SO(3)$; to reach all of $O(3)$ one reflection is needed, which the next lemma supplies by comparing the reflected box with its neighbours.

Lemma 22.2 (reflection)**.** *Let $\sigma_3 = \mathrm{diag}(1,1,-1)$. Then $Z_\infty(\mathrm{i}\,\sigma_3\cdot f,0) = Z_\infty(\mathrm{i}f,0)$ for every real 2-form $f\in C^\infty_c$.*

Proof. σ_3 preserves Λ_k, and maps $C_J\cap\Lambda_k$ onto $X_+ := (C_J+\varepsilon_k e_3)\cap\Lambda_k$. By Lemma 16.15, $\langle e^{\mathrm{i}F(\sigma_3\cdot f)}\rangle^{\mathrm{ND}}_{k,J}$ equals the corresponding expectation in the Neumann–Dirichlet state on X_+ defined with the frame $(e_1,e_2,-e_3)$. Written in the frame (e_α) (with $A'_3(x) = -A_3(x-\varepsilon e_3)$), this is the BFS measure in which ϕ, A_1, A_2 live on X_+ and A_3 on $X_+ - \varepsilon e_3 = C_J\cap\Lambda_k$, with Dirichlet conditions for ϕ on X_+ and photon gradient terms within these sets. It is obtained from the Neumann–Dirichlet state on $C_{J+1}\cap\Lambda_k$ by deleting photon gradient terms and cosine parts of hopping terms, since $X_+\cup(X_+-\varepsilon e_3)\subset C_{J+1}$; and the state on $C_{J-1}\cap\Lambda_k$ is obtained from it in the same way, since $C_{J-1}\subset C_J\cap(C_J+\varepsilon e_3)$. As in Lemma 16.17, for J large and every k, $\langle e^{\mathrm{i}F(f)}\rangle^{\mathrm{ND}}_{k,J-1} \le \langle e^{\mathrm{i}F(\sigma_3\cdot f)}\rangle^{\mathrm{ND}}_{k,J} \le \langle e^{\mathrm{i}F(f)}\rangle^{\mathrm{ND}}_{k,J+1}$ (this is Lemma 2.16 at zero scalar source).

On King's cubes: this sandwich at J and at $J-1$, and Lemma 4.2, applied at its two ends and twice to the source $\sigma_3\cdot f$ in the middle, give for J large and every $k\ge 0$

$$\begin{aligned}\langle e^{\mathrm{i}F(f)}\rangle^{K}_{k,J-2} \le \langle e^{\mathrm{i}F(f)}\rangle^{\mathrm{ND}}_{k,J-2} \le \langle e^{\mathrm{i}F(\sigma_3\cdot f)}\rangle^{\mathrm{ND}}_{k,J-1} \le \langle e^{\mathrm{i}F(\sigma_3\cdot f)}\rangle^{K}_{k,J}\\ \le \langle e^{\mathrm{i}F(\sigma_3\cdot f)}\rangle^{\mathrm{ND}}_{k,J} \le \langle e^{\mathrm{i}F(f)}\rangle^{\mathrm{ND}}_{k,J+1} \le \langle e^{\mathrm{i}F(f)}\rangle^{K}_{k,J+2};\end{aligned}$$

let $k\to\infty$ by Theorem 5.47 and then $J\to\infty$ (Theorem 17.1(iii)). The comparison of C^K_J with $C^K_{J\mp1}$ is also obtained by deletions and kills when $M\ge 2\varepsilon_k$, which holds at every k for the M of Definition 16.1 (Lemma 4.6(a)); the two-step form used here needs no such condition. □

The passage from the dense subgroup G_0 to the whole of $O(3)$ uses the continuity of the convex functional $\Phi = \ln Z_\infty(\cdot,0)$ of Theorem 17.1(iv) and the identity theorem for entire functions.

Theorem 22.3 (closure: from the generators to $O(3)$)**.** *Suppose $Z_\infty(\mathrm{i}\,R\cdot f,0) = Z_\infty(\mathrm{i}f,0)$ for all real 2-forms $f\in C^\infty_c$ and $R\in\mathcal{R}_0$. Then for every $R\in O(3)$, every N, all real 2-forms $f_j\in C^\infty_c$ and all $z\in\mathbb{C}^N$,*

$$Z_\infty\Big(\sum_j z_j\,R\cdot f_j,0\Big) = Z_\infty\Big(\sum_j z_j f_j,0\Big).$$

Proof. Complex coefficients. The hypothesis passes to G_0, since the real 2-forms are invariant under $O(3)$ and $\mathcal{R}_0 = \mathcal{R}_0^{-1}$. For $R \in G_0$ and a finite family, the difference of the two sides, as a function of $z \in \mathbb{C}^N$, is entire (Theorem 17.1(iii)) and vanishes on $(\mathrm{i}\mathbb{R})^N$, where $\sum_j z_j f_j = \mathrm{i}\sum_j t_j f_j$ with $t \in \mathbb{R}^N$ and the hypothesis applies to the real 2-form $\sum_j t_j f_j$; hence it vanishes identically.

Closure in $SO(3)$. At real coefficients, $\Phi(R \cdot f) = \Phi(f)$ for $f \in E_\rho$ and $R \in G_0$. The map $R \mapsto R \cdot f \in E_\rho$ is continuous on $O(3)$, and Φ is continuous on E_ρ by Theorem 17.1(iv). Hence $G := \{R \in O(3) : \Phi(R \cdot f) = \Phi(f) \text{ for all } \rho \text{ and } f \in E_\rho\}$ is a closed subgroup of $O(3)$ containing G_0, so $G \supset SO(3)$ by Lemma 16.2(d).

Reflection. Lemma 22.2 and the first step (with σ_3 in place of R) give $\sigma_3 \in G$, so $G = O(3) = SO(3) \cup \sigma_3 SO(3)$.

Conclusion. For $R \in O(3)$ both sides of the claim are entire in $z \in \mathbb{C}^N$ and agree on $\mathbb{R}^N$. □

Theorem 22.4 (the curvature sector)**.** *Assume (K_0) and (TIGHT$_0$) (Theorems 18.1 and 21.4). Then King's state on curvature sources is $O(3)$-invariant: for every $R \in O(3)$, all real 2-forms $f_j \in C_c^\infty$ and all $z \in \mathbb{C}^N$, $Z_\infty(\sum_j z_j\, R \cdot f_j, 0) = Z_\infty(\sum_j z_j f_j, 0)$, and all curvature Schwinger functions are $O(3)$-covariant. Moreover (Theorem 17.1(v)), for every Euclidean motion $x \mapsto Rx + b$, $Z_\infty(\sum_j z_j\, \tau_b R \cdot f_j, 0) = Z_\infty(\sum_j z_j f_j, 0)$.*

Proof. Theorem 22.1 gives invariance under $\mathcal{R}_0$ on imaginary coefficients; Theorem 22.3 gives $O(3)$ and the entire extension. Translations are Theorem 17.1(v), proved there from Theorem 5.47, Lemma 4.2 and the cut sandwich; King's argument [26, Thm. 2.3(i)] is not used. □

For the curvature two-point function alone the weaker closing condition (TIGHT$_2$) suffices.

Corollary 22.5 (the curvature two-point function)**.** *Assume (K_0) and (TIGHT$_2$). Then*

$$S_2(f_1, f_2) := \lim_J \lim_k \langle F_{\Lambda_k}(f_1) F_{\Lambda_k}(f_2)\rangle^K_{k,J}$$

satisfies $S_2(R \cdot f_1, R \cdot f_2) = S_2(f_1, f_2)$ for all $R \in O(3)$ and real 2-forms $f_1, f_2 \in C_c^\infty$.

Proof. Let $R \in \mathcal{R}_0$. Steps 0–3 of the proof of Theorem 22.1, with source tf, give $Z_L^K(\mathrm{i}t\, R \cdot f, 0) \le \mathcal{W}_{\mathfrak{c}}(tf, 0)$ for all real t and every $\mathfrak{c} \in \mathcal{A}(R^{-1}Q_{ML})$. Both sides are even and smooth in t and equal 1 at $t = 0$, so $\langle F(R \cdot f)^2\rangle_{C_L^K} \ge \sigma_{\mathfrak{c}}^2(f)$. Since $\mathcal{A}(Q_{7ML/5+2}) \subset \mathcal{A}(R^{-1}Q_{ML})$ (Lemma 16.2(c)) and $b \mapsto \sup_{\mathcal{A}(Q_b)} \sigma_{\mathfrak{c}}^2(f)$ is nonincreasing,

$$\langle F(R \cdot f)^2\rangle_{C_L^K} \ge \sup_{\mathcal{A}(Q_{7ML/5+2})} \sigma_{\mathfrak{c}}^2(f) \ge \lim_{b\to\infty} \sup_{\mathcal{A}(Q_b)} \sigma_{\mathfrak{c}}^2(f) \ge S_2(f)$$

by (TIGHT$_2$). The cube variances converge to S_2 (proof of Lemma 21.6), so $S_2(R \cdot f) \ge S_2(f)$, and the inverse generator gives equality; hence S_2 is G_0-invariant. By Lemma 16.16, $\langle F(f)^2\rangle^K_{k,J} \le \langle v, Cv\rangle_\varepsilon$ with $v = d^*_\varepsilon f$, where C is the photon covariance of the cube, the Neumann photon of $C_J \cap \Lambda_k$; the photon form is bounded below by μ_0^2, so $0 \le S_2(f) \le \mu_0^{-2}\|d^* f\|^2_{L^2}$. Since S_2 is a positive semidefinite quadratic form, $|S_2(f) - S_2(f')| \le S_2(f - f')^{1/2} S_2(f + f')^{1/2}$, and $R \mapsto S_2(R \cdot f)$ is continuous on $O(3)$. Density of G_0 gives $SO(3)$, Lemma 22.2 (expanded to second order) gives σ_3, and polarisation gives the bilinear form. □

23. The heavy-shell decoupling

This section proves the closing statement (TIGHT$_0$), the lower bound on King's state by the lower envelope of the walled tori. Lemma 23.1 switches the photon boundary condition across a matter-free collar at a cost exponentially small in the width of the collar, uniformly in the spacing; Theorem 23.7 bounds the expectation with the matter dead on a shell by King's state

plus the collar term; Theorem 23.10 derives (TIGHT$_0$) from the heavy-shell decoupling (RES$_0$) at one configuration; and Theorem 23.12 proves (RES$_0$) from the four shell estimates (SW1°)$_{\mathrm{int}}$, (SW2^{min}), (SW3°)$_{\mathrm{int}}$ and (SWT) on King's block-spin flow with the shell mass inside the matter covariance, which are stated and proved in Sections 24 and 25.

Throughout this section $\Lambda = \Lambda_k$ is the axis lattice, $\varepsilon = \varepsilon_k \le 1/\mu_0$, and $m_\varepsilon := \frac{2}{\varepsilon}\operatorname{arcsinh}(\varepsilon\mu_0/2)$; since $t \mapsto \operatorname{arcsinh}(t)/t$ is decreasing, $m_\varepsilon \ge 0.96\,\mu_0$. Let $G_{\mathbb{Z}^3}$ be the Green's function of simple random walk on $\mathbb{Z}^3$ and

$$\mathsf{a} := \sup_{z\in\mathbb{Z}^3\setminus\{0\}} |z|_\infty\, G_{\mathbb{Z}^3}(0,z),$$

which is finite by the Green's function asymptotics of simple random walk in three dimensions [27, Chap. 4]. For a set $X \subset \mathbb{R}^3$ put $\mathrm{vol}_\varepsilon(X) := \varepsilon^3 \#(X \cap \Lambda)$.

23.1. **Photon boundary conditions across a matter-free collar.** The photon in Feynman form is Gaussian, so the comparison of two boundary conditions across a region without matter is a statement about Gaussian measures and lattice Green's functions; this subsection makes it quantitative, uniformly in the spacing (Lemma 23.1), which is what allows a walled torus to be compared with King's Neumann cube.

The photon in Feynman form consists of three independent scalar Gaussian fields A_α, $\alpha = 1,2,3$, and on a finite graph $\mathcal{G}$ of sites of Λ each has the precision $K_{\mathcal{G}} = \varepsilon L_{\mathcal{G}} + \mu_0^2\varepsilon^3$, with $L_{\mathcal{G}}$ the graph Laplacian of $\mathcal{G}$ with unit conductances and $\langle u, v\rangle = \sum_x u(x)v(x)$; King's Neumann photon on a cube is the case of the induced subgraph [26, (2.5), p. 333], and the periodic photon the case of the torus graph.

Lemma 23.1 (photon boundary switch)**.** *Let $H = Q_{b'}$ be a closed cube and C_L a King cube with $\mathrm{dist}_\infty(H, \mathbb{R}^3 \setminus C_L) = D + 3$, where $D \ge \max(1, 3/\mu_0)$ and $\varepsilon \le \min(1/\mu_0, 1/4)$. Let $\langle\cdot\rangle_\natural$ and $\langle\cdot\rangle_N$ be BFS measures with the same matter part: the sites of $H \cap \Lambda$, BFS single-site measures (arbitrary; wall and killed sites included), and hopping terms only between neighbouring sites of H, coupled to the photon variables of the bonds of H as in King's action. Their photon parts are:*

- *(♮) the variables $A_\alpha(x)$ with $x, x + \varepsilon e_\alpha \in C_L$, with the law of the periodic photon of a torus whose cell contains $N_1(C_L)$, conditioned on $A_\beta(y) = 0$ for all other (β, y);*
- *(N) King's Neumann photon on $C_L \cap \Lambda$.*

Let $\Theta = e^{\mathrm{i}F(f)} e^{\sum_x \varepsilon^3 g(x) r_x^2 - c}$ with f supported on plaquettes of H, $g \ge 0$ supported in H, and $c \in \mathbb{R}$; both expectations of Θ below are real, since the charge conjugation $(A, \vartheta) \mapsto (-A, -\vartheta)$ leaves each measure invariant and carries Θ to its complex conjugate. Then

$$0 \le \langle\Theta\rangle_\natural - \langle\Theta\rangle_N \le \langle|\Theta|^2\rangle_N^{1/2} \big(e^\tau - 1\big)^{1/2}, \qquad \tau \le C_{\mathrm{GB}}\, \mathsf{a}\, \mathrm{vol}_\varepsilon(N_2(H))\, D^{-1} e^{-\frac{3}{2} m_\varepsilon D},$$

with an absolute constant C_{GB}.

Proof. The proof has four steps: we first order the two Gaussian marginals, then bound the difference of the expectations by a Cauchy–Schwarz argument in terms of a trace, then estimate that trace by the diagonal of the covariance difference near H, and finally bound this diagonal by a random-walk estimate.

Step 1 (marginals and order). Fix a component α. Let $X_\alpha := \{x \in C_L \cap \Lambda : x + \varepsilon e_\alpha \in C_L\}$; conditioning a Gaussian on the vanishing of some coordinates leaves the principal submatrix of the precision on the others, so in (♮) the variables $A_\alpha(x)$, $x \in X_\alpha$, have precision $K_\natural = \varepsilon(6 - \mathrm{Adj}_{X_\alpha}) + \mu_0^2\varepsilon^3$, where Adj_{X_α} is the adjacency matrix of X_α. In (N) the variables are $A_\alpha(x)$, $x \in C_L \cap \Lambda$, with precision $K_N = \varepsilon L_{C_L} + \mu_0^2\varepsilon^3$. The principal submatrix of K_N on X_α is $\varepsilon(\deg_{C_L} - \mathrm{Adj}_{X_\alpha}) + \mu_0^2\varepsilon^3 \le K_\natural$, and the Schur complement of K_N on X_α is at most its principal submatrix; hence $C_\natural := K_\natural^{-1} \le C_N|_{X_\alpha}$, $C_N := K_N^{-1}$. Let $\hat{X} := \{(\alpha, x) : x, x + \varepsilon e_\alpha \in H\}$. The

integrands depend on A only through $A_{\hat X}$, whose laws are $N(0, C_\natural|_{\hat X})$ and $N(0, C_N|_{\hat X})$, with $C_\natural|_{\hat X} \le C_N|_{\hat X}$. Put $M := (C_\natural|_{\hat X})^{-1} - (C_N|_{\hat X})^{-1} \ge 0$. The $\natural$-measure is obtained from the N-measure, after integrating out the photon variables off $\hat X$, by adding $\frac12\langle A, M\rangle A$ to the action; by Lemma 2.10(a) along $s \mapsto \frac{s}{2}\langle A, M\rangle A$, $\langle\Theta\rangle_\natural \ge \langle\Theta\rangle_N$.

Step 2 (Cauchy–Schwarz). With $Q := \langle A, M\rangle A$ and $\varsigma := e^{-Q/2}$, $\langle\Theta\rangle_\natural = \langle\Theta\varsigma\rangle_N/\langle\varsigma\rangle_N$, so

$$\langle\Theta\rangle_\natural - \langle\Theta\rangle_N = \frac{\langle(\Theta - \langle\Theta\rangle_N)(\varsigma - \langle\varsigma\rangle_N)\rangle_N}{\langle\varsigma\rangle_N} \le \langle|\Theta|^2\rangle_N^{1/2}\Big(\frac{\langle e^{-Q}\rangle_N}{\langle e^{-Q/2}\rangle_N^2} - 1\Big)^{1/2} \le \langle|\Theta|^2\rangle_N^{1/2}\big(e^{\langle Q\rangle_N} - 1\big)^{1/2},$$

by $\langle e^{-Q}\rangle_N \le 1$ and Jensen's inequality. Writing $M = \sum_j \lambda_j v_j v_j^T$, Lemma 16.16(b) in the N-measure gives $\langle Q\rangle_N \le \sum_j \lambda_j \langle v_j, C_N v_j\rangle = \operatorname{tr}(M C_N|_{\hat X}) =: \tau$.

Step 3 (the trace). Let $\Delta := C_N|_{X_\alpha} - C_\natural \ge 0$, for each α. Then $M C_N|_{\hat X} = (C_\natural|_{\hat X})^{-1}\Delta_{\hat X}$, so $\tau = \mathbb{E}\langle Y, (C_\natural|_{\hat X})^{-1} Y\rangle$ with $Y \sim N(0, \Delta_{\hat X})$. Since the inverse of a principal block of the covariance is the minimum of the precision form over extensions, $\langle a, (C_\natural|_{\hat X})^{-1} a\rangle = \min\{\langle u, K_\natural u\rangle : u|_{\hat X} = a\}$. Let $\tilde Y \sim N(0, \Delta)$ on X_α and $\eta(x) := (1 - \operatorname{dist}_\infty(x, H))_+$; then $\eta\tilde Y$ is an extension of $\tilde Y|_{\hat X}$, so $\tau \le \sum_\alpha \mathbb{E}\langle \eta\tilde Y, K_\natural \eta\tilde Y\rangle$. Let int be the set of $x \in X_\alpha$ whose six neighbours lie in X_α; it contains $N_2(H) \cap \Lambda$. At $x \in \mathrm{int}$ the rows of $K_\natural$, K_N and of the infinite-lattice precision K_∞ coincide, so $(K_\infty\Delta)(x, w) = (K_N C_N)(x, w) - (K_\natural C_\natural)(x, w) = 0$ for all $w \in X_\alpha$, hence $\operatorname{Var}((K_\infty\tilde Y)(x)) = 0$ and $K_\infty\tilde Y = 0$ on int almost surely. For u with $K_\infty u = 0$ on $\mathrm{int} \supset \operatorname{supp}\eta$, summation by parts and the identity $(\eta_x u_x - \eta_y u_y)^2 - (u_x - u_y)(\eta_x^2 u_x - \eta_y^2 u_y) = (\eta_x - \eta_y)^2 u_x u_y$ give $\langle \eta u, K_\natural \eta u\rangle = \langle \eta u, K_\infty \eta u\rangle = \varepsilon \sum_{\langle xy\rangle} (\eta_x - \eta_y)^2 u_x u_y$. Since $|\eta_x - \eta_y| \le \varepsilon$ and $|\Delta(x, y)| \le \frac12(\Delta(x, x) + \Delta(y, y))$,

$$\tau \le \sum_\alpha \varepsilon^3 \sum_{\langle xy\rangle \subset N_2(H)} |\Delta(x, y)| \le 9\,\mathrm{vol}_\varepsilon(N_2(H)) \sup_{\alpha,\ x \in N_2(H)} \Delta(x, x).$$

Step 4 (decay of Δ). For $K = \mathcal{D} - \varepsilon \mathrm{Adj}_{\mathcal G}$ with $\mathcal D$ diagonal, $K^{-1}(x, y) = \sum_\omega \prod_{(u\to v)\in\omega} (\varepsilon/\mathcal D(u))\cdot \mathcal D(y)^{-1}$, the sum over walks ω from x to y in $\mathcal G$. For $K_\natural$ and K_N the weights of steps from sites of int coincide. Decomposing walks from $x \in N_2(H)$ at their first visit to $\Sigma := X_\alpha \setminus \mathrm{int}$, $K^{-1}(x, y) = G^{\mathrm{int}}(x, y) + \sum_{z\in\Sigma} \mathsf H(x, z) K^{-1}(z, y)$ with G^{int} and $\mathsf H$ the same for both precisions. Hence

$$\Delta(x, x) = \sum_{z\in\Sigma} \mathsf H(x, z)\big(K_N^{-1}(z, x) - K_\natural^{-1}(z, x)\big) \le \Big(\sum_{z\in\Sigma} \mathsf H(x, z)\Big) \sup_{z\in\Sigma} K_N^{-1}(z, x).$$

The first factor is the probability that simple random walk from x, killed with probability $\mu_0^2\varepsilon^2/(6 + \mu_0^2\varepsilon^2)$ per step, reaches Σ, which lies at ℓ^∞-distance at least D from x. With $\cosh\lambda = 1 + \mu_0^2\varepsilon^2/2$, that is $\lambda = \varepsilon m_\varepsilon$, $e^{\pm\lambda S_n^\beta}\mathbf 1_{\{\text{alive at } n\}}$ is a martingale for each coordinate S^β; optional stopping and a union over the six half-spaces give $\sum_z \mathsf H(x, z) \le 6e^{-m_\varepsilon D}$. The same argument, with the decomposition at the first exit from the ℓ^∞-ball of radius $|y|_\infty/2$ and the bound $K^{-1}_{\varepsilon\mathbb Z^3, \mu_0} \le K^{-1}_{\varepsilon\mathbb Z^3, 0} = (6\varepsilon)^{-1} G_{\mathbb Z^3}(\cdot/\varepsilon)$ entrywise, gives the free bound

$$G_\varepsilon(y) := K^{-1}_{\varepsilon\mathbb Z^3}(0, y) \le 4\mathsf a\,|y|_\infty^{-1} e^{-m_\varepsilon|y|_\infty/2} \qquad (|y|_\infty \ge 4\varepsilon).$$

King's Neumann precision on the box $C_L \cap \Lambda$ has the Green's function $K_N^{-1}(z, x) = \sum_{x'} G_\varepsilon(z - x')$, the sum over the images x' of x under the reflections of $\varepsilon\mathbb Z^3$ in the planes half a lattice spacing outside the faces of the box (method of images; the mass term is uniform). Every image other than x lies outside C_L at ℓ^∞-distance at least D from every site of C_L, and so does x from $z \in \Sigma$. If $s \ge 2D$ is the side of C_L, at most $8(j + 3)^3$ images satisfy $|z - x'|_\infty < D + (j + 1)s$; since

$m_\varepsilon D \geq 2$,

$$K_N^{-1}(z,x) \leq \sum_{j\geq 0} 8(j+3)^3 \, \frac{4\mathsf{a}\, e^{-m_\varepsilon (D+js)/2}}{D+js} \leq C_1\, \mathsf{a}\, D^{-1} e^{-m_\varepsilon D/2}$$

with an absolute constant C_1. Combining, $\Delta(x,x) \leq 6C_1 \mathsf{a} D^{-1} e^{-\frac{3}{2} m_\varepsilon D}$, and Step 3 gives the claim with $C_{\mathrm{GB}} = 54 C_1$. □

The Gaussian measures in Lemma 23.1 are compared only on the bonds of H, where no live matter crosses the region in which the two photon boundary conditions differ; this is what makes the bound uniform in ε. A comparison in which the photon is pinned or cut on a surface that live matter crosses is not of this type: a one-loop computation suggests that the order-e^2 difference of such states diverges logarithmically as $\varepsilon \to 0$, because the free-boundary mass counterterm compensates the photon self-energy of the matter only where the photon has its bulk covariance; nothing below depends on this.

23.2. **Shell configurations, two-level walls, and the dead-shell bound.** This subsection fixes the geometry of the shell and the two-level walls (Definitions 23.2, 23.3 and 23.5), gives the elementary properties of the tail (Lemma 23.6), and proves the dead-shell bound (Theorem 23.7): once the matter on the shell is removed, the walled expectation is bounded by King's state up to the collar term of Lemma 23.1.

Definition 23.2 (shell configurations)**.** Let $b > 0$. A *shell configuration* for Q_b is $\mathfrak{s} = (\mathfrak{c}, b', L')$, where $\mathfrak{c} = (J', \psi, n)$ with $\psi \in C^\infty(D_{J'}; [0,\infty))$ and $n \geq 0$ is a walled configuration (Definition 16.4(c); $\mathfrak{c} = (J', n\psi)$ in the notation of Definition 21.1) with $N_1(Q_b) \subset (-25MJ', 25MJ')^3$ and $\psi = 0$ on $N_1(Q_b)$, $H := Q_{b'}$ with $b' \geq b+3$, and $C_{L'}$ is a King cube with $\mathrm{dist}_\infty(H, \mathbb{R}^3 \setminus C_{L'}) = D(\mathfrak{s}) + 3$, $D(\mathfrak{s}) \geq \max(1, 3/\mu_0)$, and $N_1(\bar{C}_{L'}) \subset (-25MJ', 25MJ')^3$. Its *shell* is a set U with $C_{L'} \setminus H \subset U \subset T \setminus H$, where $T = D_{J'}$ is the torus of the configuration and $C_{L'}$ is understood in its cell (the two cases used are the cubic shell $U = C_{L'} \setminus H$ and the whole exterior $U = T \setminus H$, Section 23.4), and we require the *plateau condition* $\psi \geq 1$ on $N_1(U)$ (compatible with $\psi = 0$ on $N_1(Q_b)$, since $\mathrm{dist}_\infty(U, Q_b) \geq 3$). Put

$$\tau(\mathfrak{s}) := C_{\mathrm{GB}}\, \mathsf{a}\, \mathrm{vol}(N_3(H))\, D(\mathfrak{s})^{-1} e^{-1.44\, \mu_0 D(\mathfrak{s})}.$$

Definition 23.3 (shell profiles and two-level configurations)**.** Let $\mathfrak{s}$ be a shell configuration for Q_b with the shell U (Definition 23.2). A *shell profile* for $\mathfrak{s}$ is $\chi \in C^\infty(D_{J'}; [0,1])$ with $\chi = 1$ on U and $\mathrm{supp}\, \chi \subset N_1(U)$, or the *sharp profile* $\chi = \mathbf{1}_U$ (Remark 23.11(ii): the lattice statements of this section hold for it verbatim); then $\chi \leq \psi$ automatically, and $\chi = 0$ on $N_1(Q_b)$. Its *dead set* is $D_\chi := \{\chi > 0\} \subset N_1(U)$; every site of $C_{L'}$ within ε of $\mathbb{R}^3 \setminus C_{L'}$, and every site of U, lies in D_χ. We call $(\mathfrak{s}, \chi)$ a *two-level configuration* for Q_b.

Definition 23.4 (conventions for shell configurations)**.** In Definition 23.2, the conditions on $\mathfrak{c}$ are the cell and support conditions of Definition 21.1; the conditions of the class $\mathcal{C}$ (Definition 16.6) are imposed, in Theorem 23.9, on the single-level configuration $(J', \psi, n + K)$, not on $\mathfrak{c}$. On the lattice, H is the site set of the half-open box $[-b', b')^3$, a union of blocks as in Theorem 24.5: the sites of the faces $x_\mu = +b'$ belong to the shell U and those of the faces $x_\mu = -b'$ to H, ∂U being the midplane between the last U-layer and the first H-layer of sites, as there; no clause of Definition 23.2 depends on this convention, which fixes the site set of the sharp shell profile $\chi = \mathbf{1}_U$ of Remark 23.11(ii) and of the shell mass in the shell estimates (Sections 24 and 25) as the one of Theorem 24.5. The King cube $C_{L'}$ is the shell's cube, with its own index L', written $C_{L'}$ to keep it apart from the source cube C_L of the class $\mathcal{C}$ and of Theorems 22.1 and 22.4; Lemma 23.1 is applied with its cube equal to $C_{L'}$.

Definition 23.5 (two-level states and the tail)**.** Let $(\mathfrak{s},\chi)$ be a two-level configuration for Q_b, f a real 2-form in C_c^∞ supported in Q_b, and k with $\varepsilon_k \le \min(1/\mu_0, 1/4)$. For $K \in [0,\infty)$ let

$$\Psi_k(n,K) := T_{(J',\, n\psi + K\chi),\Lambda_k}(f)$$

be the walled expectation of the configuration with the two-level wall $n\psi + K\chi$, which vanishes on $N_1(Q_b)$ (for the sharp profile the wall is a lattice function and the state is defined by the same single-site weights, Definition 16.4(c)); and let $\Psi_k(n,\infty) := \langle e^{\mathrm{i}F(f)}\rangle$ in the walled torus state of $\mathfrak{c} = (J', n\psi)$ with the matter killed on $D_\chi \cap \Lambda_k$ (Definition 16.4(d)). Put $\mathrm{tail}_k(K) := \Psi_k(n,K) - \Psi_k(n,\infty)$.

Lemma 23.6 (elementary properties)**.** *(a) $K \mapsto \Psi_k(n,K)$ is nonincreasing on $[0,\infty)$, $\Psi_k(n,K) \ge \Psi_k(n,\infty)$, and $\Psi_k(n,K) \to \Psi_k(n,\infty)$ as $K \to \infty$. Hence $\mathrm{tail}_k(K) \ge 0$ is nonincreasing in K and tends to 0 as $K \to \infty$, at every fixed k.*

(b) (two-level sandwich) For $n, K \ge 0$: $\Psi_k(n+K,0) \le \Psi_k(n,K) \le \Psi_k(n,0)$.

Proof. (a) Increasing K multiplies the single-site measures on D_χ by $e^{-(K'-K)\varepsilon^3\chi(x)r_x^2}$; Lemma 2.10(c). Killing D_χ is the limit (Lemma 2.12 with the rates $\chi > 0$ on D_χ), and it does not increase the expectation. (b) The wall of $\Psi_k(n+K,0)$ exceeds that of $\Psi_k(n,K)$ by $K(\psi-\chi) \ge 0$, since $\chi \le \psi$; the wall of $\Psi_k(n,K)$ exceeds that of $\Psi_k(n,0)$ by $K\chi \ge 0$. Apply Lemma 2.10(c). □

Part (b) is the two-level sandwich (Lemma 23.8 below in the form used); it is the only place where $\chi \le \psi$ is needed. In words: raising the matter mass on the shell alone, by any amount, gives a state above the one in which the whole plateau is raised by the same amount, and below the one with the shell untouched.

Theorem 23.7 (the dead-shell bound)**.** *For every two-level configuration $(\mathfrak{s},\chi)$ for Q_b, every real 2-form $f \in C_c^\infty$ supported in Q_b and every k with $\varepsilon_k \le \min(1/\mu_0, 1/4)$ and $\mathrm{vol}_{\varepsilon_k}(N_2(H)) \le \mathrm{vol}(N_3(H))$,*

$$\Psi_k(n,\infty) \le \langle e^{\mathrm{i}F(f)}\rangle^{\mathrm{ND}}_{k,L'} + \left(e^{\tau(\mathfrak{s})} - 1\right)^{1/2} \le \langle e^{\mathrm{i}F(f)}\rangle^{K}_{k,L'+1} + \left(e^{\tau(\mathfrak{s})} - 1\right)^{1/2}, \tag{47}$$

$C_{L'}$ the shell's cube, and hence $\limsup_k \Psi_k(n,\infty) \le Z_\infty(\mathrm{i}f, 0) + (e^{\tau(\mathfrak{s})} - 1)^{1/2}$.

Proof. The proof has four steps: pinning the photon outside the box, switching the photon law by Lemma 23.1, passing to the box and to King's cube, and finally taking the limits. Fix such a k, and write $\varepsilon = \varepsilon_k$ and $\Theta = e^{\mathrm{i}F(f)}$. Let E be the set of sites of the torus outside $C_{L'}$. The dead set enters only through $C_{L'} \setminus H \subset U \subset D_\chi$: the matter being dead on all of D_χ rather than on the cubic shell $C_{L'} \setminus H$ alone does not increase $\Psi_k(n,\infty)$, since killing is the limit of raising the mass (Lemma 2.12) and raising the mass does not increase the expectation (Lemma 2.10(c)); the steps below are run for D_χ directly and use of it only that it contains U and every site of $C_{L'}$ adjacent to E.

Step 1 (pinning). In the state with the matter killed on $D_\chi \cap \Lambda_k$, add $s|A_\beta(y)|^2$ for every photon variable on a bond with an endpoint in E, and let $s \to \infty$. By Lemma 2.10(a) the expectation of Θ does not decrease, and in the limit the photon is the conditioned law ($\natural$) of Lemma 23.1. In this state the hopping terms within E do not involve A, and there are no hopping terms between E and live sites: every site of $C_{L'}$ adjacent to E lies in D_χ and is dead. So the matter on E is a factor independent of A and cancels (Lemma 2.10(e)), or is dead altogether when $E \subset U$. What remains is the $\natural$-measure of Lemma 23.1, with the matter on H (dead on $D_\chi \cap H$), the wall, and Dirichlet conditions at ∂H, since $U \subset D_\chi$ is dead. Hence $\Psi_k(n,\infty) \le \langle\Theta\rangle_\natural$.

Step 2 (photon switch). By Lemma 23.1, $\langle\Theta\rangle_\natural \le \langle\Theta\rangle_N + \langle|\Theta|^2\rangle_N^{1/2}(e^{\tau(\mathfrak{s})} - 1)^{1/2}$, and $|\Theta|^2 = 1$; here $m_\varepsilon \ge 0.96\mu_0$ and the assumption $\mathrm{vol}_\varepsilon(N_2(H)) \le \mathrm{vol}(N_3(H))$ give $\tau \le \tau(\mathfrak{s})$.

Step 3 (to the box $C_{L'}$, and to King's cube). The N-measure is the limit $s \to \infty$ of the walled Neumann–Dirichlet state on $C_{L'} \cap \Lambda_k$ with the single-site measures on D_χ multiplied by $e^{-s\varepsilon^3 \chi(x) r^2}$ (Lemma 2.12). By Lemma 2.10(c) the expectation of Θ does not decrease as s decreases to 0, nor when the wall is then removed. The result is the box state $\langle\cdot\rangle^{\mathrm{ND}}_{k,L'}$ of Definition 2.2: this is the first inequality of (47). The second is Lemma 4.2, since f is supported in H, at distance > 1 from $\mathbb{R}^3 \setminus C_{L'}$.

Step 4 (limits). The limit of $\langle e^{\mathrm{i}F(f)}\rangle^K_{k,L'+1}$ as $k \to \infty$ exists by Theorem 17.1(i) (Theorem 5.47, Lemma 2.14(b), Montel) and is at most $Z_\infty(\mathrm{i}f, 0)$ (Theorem 17.1(iii)). This gives the lim sup bound. □

23.3. **The main statement.** This subsection states the heavy-shell decoupling (RES$_0$) (Theorem 23.9) and proves that, together with (K$_0$), it implies the closing statement (TIGHT$_0$) (Theorem 23.10), by the two-level sandwich of Lemma 23.8 and the dead-shell bound.

Lemma 23.8 (two-level sandwich)**.** *For every two-level configuration, every real 2-form $f \in C^\infty_c$ supported in Q_b, every k and $n, K \geq 0$,*

$$\Psi_k(n+K, 0) \;\leq\; \Psi_k(n, K) \;=\; \Psi_k(n, \infty) + \mathrm{tail}_k(K) \;\leq\; \Psi_k(n, 0), \qquad \mathrm{tail}_k(K) \geq 0.$$

Proof. Lemma 23.6. □

Theorem 23.9 ((RES$_0$): heavy-shell decoupling at one configuration)**.** *For every $b > 0$, every real 2-form $f \in C^\infty_c$ supported in Q_b and every $\eta > 0$ there are a two-level configuration $(\mathfrak{s}, \chi)$ for Q_b with the shell $U = T \setminus H$, the whole exterior of the box $H = Q_{b'}$ in the torus $T = D_{J'}$ of the configuration, the sharp profile $\chi = \mathbf{1}_U$ and $\tau(\mathfrak{s}) \leq \eta$, and a shell mass $K \in [0, \infty)$ such that the single-level configuration $(J', \psi, n + K)$ lies in $\mathcal{A}(Q_b)$ and*

$$\liminf_{k\to\infty} \mathrm{tail}_k(K) = \liminf_{k\to\infty} \big[\Psi_k(n, K) - \Psi_k(n, \infty)\big] \leq \eta.$$

Proof. Theorem 23.12, from the shell estimates (Remark 23.13), at one two-level configuration with the shell $U = T \setminus H'$, $H' = H$; the quantifiers are met in the order stated there, and $(J', \psi, n + K)$ is placed in $\mathcal{A}(Q_b)$ by the choice of K (Remark 21.5). For $b < b_\circ$, $b_\circ$ as in Definition 21.1, the configuration for $Q_{b_\circ}$ serves: it is a two-level configuration for Q_b, and $\mathcal{A}(Q_{b_\circ}) \subset \mathcal{A}(Q_b)$. □

Below, "(RES$_0$)" refers to this statement.

In words: in a walled torus, once the scalar mass squared on a thick shell of the heavy plateau has been raised by $2K$, removing the matter on the shell altogether changes the expectation of the observable by at most η in the continuum limit, for one geometry and one K, with no rate in K and no uniformity in anything. Since $\tau(\mathfrak{s})$ depends only on H and the width $D(\mathfrak{s})$, the condition $\tau(\mathfrak{s}) \leq \eta$ only asks the shell to be wide enough, logarithmically in b' and $1/\eta$. By Lemma 23.6(a) the bracket tends to 0 as $K \to \infty$ at every fixed k; the condition says that the heavy-shell limit and the continuum limit may be interchanged at one configuration. The quantifiers are the weakest that Theorem 23.10 accepts: one configuration, one K, the lim inf along the spacings, and no uniformity in n, L', D or the torus.

Theorem 23.10 ((RES$_0$) implies (TIGHT$_0$))**.** *Under* (K$_0$), (RES$_0$) *implies* (TIGHT$_0$)*. More precisely, if $(\mathfrak{s}, \chi)$ and K are as in* (RES$_0$) *for f and η, then the single-level configuration $\mathfrak{c}' := (J', \psi, n + K)$, which lies in $\mathcal{A}(Q_b)$ by* (RES$_0$)*, satisfies*

$$\mathcal{W}_{\mathfrak{c}'}(f, 0) \leq Z_\infty(\mathrm{i}f, 0) + \big(e^\eta - 1\big)^{1/2} + \eta.$$

Proof. By Lemma 23.8, $T_{\mathfrak{c}',\Lambda_k}(f) = \Psi_k(n+K,0) \le \Psi_k(n,\infty) + \mathrm{tail}_k(K)$ at every k. The left side converges to $\mathcal{W}_{\mathfrak{c}'}(f,0)$ by (K_0), since $\mathfrak{c}' \in \mathcal{A}(Q_b)$ (Definition 21.1, Lemma 18.2); hence

$$\begin{aligned}\mathcal{W}_{\mathfrak{c}'}(f,0) = \liminf_k \Psi_k(n+K,0) &\le \limsup_k \Psi_k(n,\infty) + \liminf_k \mathrm{tail}_k(K)\\ &\le Z_\infty(\mathrm{i}f,0) + \big(e^{\tau(\mathfrak{s})}-1\big)^{1/2} + \eta\end{aligned}$$

by Theorem 23.7 and (RES_0), using $\liminf(x_k+y_k) \le \limsup x_k + \liminf y_k$. With $\tau(\mathfrak{s}) \le \eta$ this is the displayed bound; as η is arbitrary and $\mathfrak{c}' \in \mathcal{A}(Q_b)$, $I_b(f) \le Z_\infty(\mathrm{i}f,0)$ for every b, which is (TIGHT_0). □

Remark 23.11 (on the proof). (i) The two-level state is only an intermediate inequality; the configuration that enters (TIGHT_0) is the single-level wall $(n+K)\psi$, and the only limit statement used is the existence of its walled value, which is part of (K_0) (the hypothesis (N6) of Theorem 18.8: King's Theorem 2.1 of [25] with the wall as King's scalar source, formulated for the normalised ratio). No existence statement is needed for the two-level state or for the dead-shell state. (ii) The lattice statements of this subsection hold verbatim for the sharp profile $\chi = \mathbf{1}_U$ (then $D_\chi = U$ and $\chi \le \psi$ needs only $\psi \ge 1$ on U); smoothness of χ would be needed only where the two-level walled value must have a continuum limit, which, by (i), the argument never needs. (iii) Since $\mathrm{tail}_k(K)$ is nonincreasing in K, (RES_0) for one K implies it for every larger K; so K may be taken as large as one likes, which is used to place $n+K$ in any prescribed window of wall heights.

23.4. **The heavy-shell estimate from the shell bounds.** This subsection proves (RES_0) in the curvature sector, with the shell $U := T \setminus H'$, the whole exterior of the source box $H' := H$ in the torus T of the configuration, a shell admitted by Definition 23.2: $\partial U = \partial H'$ is a single convex surface without re-entrant edges (at which the derivative clause of King's bound on the block propagators [25, Thm. 3.3] fails), the plateau condition reads $\psi \ge 1$ on $N_1(U)$, compatible with $\psi = 0$ on $N_1(Q_b)$ since $H' = Q_{b'}$ with $b' \ge b+3$, the shell profile is the sharp one, $\chi = \mathbf{1}_U$, with dead set $D_\chi = U$, and the King cube $C_{L'}$ of $\mathfrak{s}$ (any King cube with $H \subset C_{L'}$, $N_1(\bar{C}_{L'})$ inside the cell and $\mathrm{dist}_\infty(H, \mathbb{R}^3 \setminus C_{L'}) = D(\mathfrak{s})+3$) has its shell $C_{L'} \setminus H$ inside U, so that Definition 23.5 and the dead-shell chain of Theorem 23.7 apply as stated. For $\kappa \ge 0$, $\Psi_k(n,\kappa)$ is the two-level expectation of Definition 23.5 with the shell mass $\kappa\mathbf{1}_U$ added to the wall, which raises the scalar mass squared by 2κ on U (Definition 16.4(c)), so that $\Psi_k(n,K)$ and $\Psi_k(n,\infty)$ in (RES_0) are the cases $\kappa = K$ and $\kappa = \infty$, and $Z_\kappa(zf, g')$ the corresponding partition function with sources; King's block-spin flow [25, §3] is run with $\kappa\mathbf{1}_U$ in the matter covariance at every scale, $\eta_0 \in 5^{-\mathbb{N}}$ is the stopping scale, $\kappa_0 := \max(c_0,1)/\eta_0^2$ the threshold above which the shell is heavy in block units at the stopping scale and both $(\mathrm{SW1}^\circ)_{\mathrm{int}}$ (Theorem 24.5, stated for $\kappa \ge c_0/\eta_0^2$) and $(\mathrm{SW3}^\circ)_{\mathrm{int}}$ (Theorem 25.11) hold, I_κ^{cap} the stopping-scale integral over the capped test set, and $\Delta_z(b)$ the difference of the reduced small-field actions (the capped reduced actions of Theorem 25.11) at the shell masses κ' and κ at the block field b and source coefficient z. The shell estimates are stated in Theorems 24.5 and 25.11 for King's wall profile ($\psi\colon T \to [0,1]$, $\psi = 0$ on the source region, $\psi = 1$ off its 1-neighbourhood, so $\psi \equiv 1$ on U and its η_0-neighbourhood), at a fixed plateau height $n \ge 1$ and in the normalisation in which V enters $\frac{1}{2}\langle\phi, V\phi\rangle$, so that the wall $n\psi + \kappa\mathbf{1}_U$ is their $2n\psi + 2\kappa\mathbf{1}_U$; as they hold uniformly in $\kappa \ge \kappa_0$ at every fixed plateau height, the dictionary changes the threshold κ_0 by a factor 2 and nothing else, with $n_* := 1$ every $n \ge n_*$ has plateau height $2n \ge 1$, as $(\mathrm{SW1}^\circ)_{\mathrm{int}}$, $(\mathrm{SW3}^\circ)_{\mathrm{int}}$ and (SWT) require, and the two-level configurations of Theorem 23.12 carry this profile. The notation is that of Theorem 24.5, Proposition 24.7, Definition 24.2 and Theorem 25.11.

Theorem 23.12 (the heavy-shell estimate from the shell bounds)**.** *Let $g = 0$ and let $(\mathfrak{s}, \chi)$ be a two-level configuration with the shell $U = T \setminus H'$ and the sharp profile $\chi = \mathbf{1}_U$, the torus $T = D_{J'}$*

with $J' \in 5^{k_*}\mathbb{N}$ and $J' \geq \max(J_0, J_{\mathrm{pos}})$, $J_0 = J_0(\varepsilon_0)$ the non-wrapping threshold of the class $\mathcal{C}$ (Definition 16.6), King's wall profile ($\psi \in [0,1]$, $\psi = 0$ on $N_1(Q_b)$, $\psi = 1$ off $N_2(Q_b)$), plateau height $n \geq n_*$, H', T unions of blocks of the final scale and the corners of H' in the residue class of Definition 16.6; let $(\mathrm{SW1}^\circ)_{\mathrm{int}}$, $(\mathrm{SW2}^{\min})$, $(\mathrm{SW3}^\circ)_{\mathrm{int}}$ *and* (SWT) *be Theorem 24.5, Lemma 25.13, Theorem 25.11 and Corollary 25.12 of Sections 24 and 25. Then for every $\eta > 0$ there are a stopping scale η_0 and a shell mass $K < \infty$ with $\limsup_k \mathrm{tail}_k(K) \leq \eta$, in the quantifier order of* (RES_0)*: given b, a source f supported in Q_b and $\eta > 0$, the configuration $(\mathfrak{s}, \chi)$ is chosen first, with $D(\mathfrak{s})$ so large that $\tau(\mathfrak{s}) \leq \eta$, and then η_0 and K; so* (RES_0) *holds at $(\mathfrak{s}, \chi)$ with this K, the single-level configuration $(J', \psi, n + K)$ lying in $\mathcal{A}(Q_b)$ for $b \geq b_\circ$ once K is taken large enough for the wall condition of the class $\mathcal{C}$ (Remark 23.11(iii); the vanishing of ψ near $R^{-1}C_1$ follows from $b \geq b_\circ$, Definition 21.1), and under* (K_0), (TIGHT_0) *holds (Theorem 23.10).*

Proof. We first compare the normalised partition functions with their capped small-field counterparts, then show that the latter vary little in the shell mass above a threshold, which bounds the tail on the real segment $[-1,1]$; the two-constants theorem transfers this bound to the point i, and a choice of the constants in the right order gives (RES_0), whence (TIGHT_0) by Theorem 23.10. Fix $(\mathfrak{s}, \chi)$, f and $g = 0$, put $\Psi_\kappa(z) := Z_\kappa(zf, 0)/Z_\kappa(0,0)$ for $\kappa \in [0, \infty]$ and $z \in \mathbb{C}$ (Z_∞ the partition function of the state with the matter killed on D_χ), so that, the shell mass $\kappa \mathbf{1}_U$ being the second level $\kappa\chi$ of the wall, Definition 23.5 gives $\Psi_K(\mathrm{i}) = \Psi_k(n, K)$, $\Psi_\infty(\mathrm{i}) = \Psi_k(n, \infty)$ and $\mathrm{tail}_k(K) = \Psi_K(\mathrm{i}) - \Psi_\infty(\mathrm{i})$, put $J_\kappa(z) := I^{\mathrm{cap}}_\kappa(zf, 0)/I^{\mathrm{cap}}_\kappa(0,0)$ for $\kappa \geq \kappa_0$ and real z, and let $a = a(\eta_0)$, $b = b(\eta_0)$, η_1, η_2, $\varepsilon_0(\eta_0)$, c_0 and $r(\kappa; \eta_0)$ be the constants of Theorems 24.5 and 25.11, every bound below being uniform in $\varepsilon \leq \varepsilon_0(\eta_0)$. By $(\mathrm{SW2}^{\min})$, $|\Psi_\kappa(z)| \leq B_\Psi$ on $|z| \leq 2$ uniformly in $\kappa \in [0, \infty]$ and in ε, and $\Psi_\kappa(z) \to \Psi_\infty(z)$ as $\kappa \to \infty$ at fixed ε for real z (the substitution and dominated convergence of the proof of Lemma 2.12, $e^{zF(f)}$ being integrable against the photon Gaussian by the same bound); for $\eta_0 \leq \min(\eta_1, \eta_2)$ with $b(\eta_0) \leq \frac{1}{2}$, $\kappa \geq \kappa_0$ and $z \in [-1,1]$, $(\mathrm{SW1}^\circ)_{\mathrm{int}}$ at $(0,0)$ and at $(zf, 0)$ gives $e^{-2a}(1-b)J_\kappa(z) \leq \Psi_\kappa(z) \leq e^{2a}J_\kappa(z) + b$, hence $J_\kappa(z) \leq 2e^{2a}B_\Psi =: \bar{J}$ and $|\Psi_\kappa(z) - J_\kappa(z)| \leq \bar{J}(e^{2a} - e^{-2a} + b) + b =: m_1(\eta_0)$, which tends to 0 with η_0 because $a, b \to 0$; for $\kappa_0 \leq \kappa \leq \kappa' < \infty$, $J_\kappa(z)/J_{\kappa'}(z)$ is the quotient of the integrals of $e^{-\tilde{S}_\kappa(b;z) - \tilde{S}_{\kappa'}(b';0)}$ and of $e^{-\tilde{S}_{\kappa'}(b;z) - \tilde{S}_\kappa(b';0)}$ over $\tilde{\chi} \times \tilde{\chi}$, $\tilde{S}_\kappa(\cdot; z)$ the reduced small-field action at shell mass κ and $\tilde{\chi}$ the capped test set, the same for all $\kappa \geq \kappa_0$ (this is where (SWT) enters: no threshold term arises between κ and κ'), whose integrands differ pointwise by the factor $e^{\Delta_z(b) - \Delta_0(b')} \in [e^{-2r}, e^{2r}]$, $r := r(\kappa; \eta_0)$, by $(\mathrm{SW3}^\circ)_{\mathrm{int}}$, so that $|J_\kappa(z) - J_{\kappa'}(z)| \leq \bar{J}(e^{2r} - 1)$ and, with $\bar{r}(K; \eta_0) := \sup_{\kappa \geq K} r(\kappa; \eta_0)$, which tends to 0 as $K \to \infty$ at fixed η_0,

$$|\Psi_\kappa(z) - \Psi_{\kappa'}(z)| \leq m(K, \eta_0) := 2m_1(\eta_0) + \bar{J}\big(e^{2\bar{r}(K;\eta_0)} - 1\big) \qquad (\kappa' \geq \kappa \geq K \geq \kappa_0,\ z \in [-1,1]),$$

and letting $\kappa' \to \infty$ at fixed ε, $|\Psi_K(z) - \Psi_\infty(z)| \leq m(K, \eta_0)$ for $z \in [-1,1]$. Thus $D := \Psi_K - \Psi_\infty$ is entire with $|D| \leq 2B_\Psi$ on $\{|z| \leq 2\}$ and $|D| \leq m(K, \eta_0)$ on $[-1,1]$, so that the two-constants theorem on $\{|z| < 2\} \setminus [-1,1]$ ($\log|D|$ subharmonic, dominated by the harmonic function with boundary values $\log(2B_\Psi)$ on $|z| = 2$ and $\log m(K, \eta_0)$ on the slit) gives $|D(\mathrm{i})| \leq (2B_\Psi)^{1-\omega} m(K, \eta_0)^\omega$, $\omega \in (0,1)$ the harmonic measure of the slit $[-1,1]$ at i, an absolute constant; the map $(A, \phi) \mapsto (-A, \bar{\phi})$ preserves the action, the wall, the shell mass, the gauge fixing and the measure and reverses the sign of $F(f)$, so $\Psi_\kappa(-z) = \Psi_\kappa(z) = \overline{\Psi_\kappa(\bar{z})}$ and $\Psi_\kappa(\mathrm{i})$ is real, whence $0 \leq \mathrm{tail}_k(K) = D(\mathrm{i}) \leq (2B_\Psi)^{1-\omega}\, m(K, \eta_0)^\omega$ for every $\varepsilon_k \leq \varepsilon_0(\eta_0)$; given $\eta > 0$, take $\delta > 0$ with $(2B_\Psi)^{1-\omega}\delta^\omega \leq \eta$, then $\eta_0 \leq \min(\eta_1, \eta_2)$ with $b(\eta_0) \leq \frac{1}{2}$ and $2m_1(\eta_0) \leq \delta/2$, then $K \geq \kappa_0$ with $\bar{J}(e^{2\bar{r}(K;\eta_0)} - 1) \leq \delta/2$, and only then $\varepsilon \leq \varepsilon_0(\eta_0)$: then $m(K, \eta_0) \leq \delta$ and $\mathrm{tail}_k(K) \leq \eta$ for all $\varepsilon_k \leq \varepsilon_0(\eta_0)$, so $\limsup_k \mathrm{tail}_k(K) \leq \eta$, which is (RES_0) at $(\mathfrak{s}, \chi)$ with this K in its quantifier order (one configuration, one K, no rate, the limit along the spacings last), and since tail_k is nonincreasing in K (Lemma 23.6(a)) K may be enlarged at will (Remark 23.11(iii)), which

places $(J', \psi, n+K)$ in the class $\mathcal{C}$, hence in $\mathcal{A}(Q_b)$ for $b \geq b_\circ$ (Remark 21.5, Definition 21.1), and Theorem 23.10 then gives (TIGHT$_0$) under (K$_0$). □

The four shell estimates are statements about King's block-spin flow run with the shell mass $\kappa \mathbf{1}_U$ inside every matter covariance, for a two-level configuration with the sharp profile. $(\mathrm{SW1}^\circ)_{\mathrm{int}}$ (Theorem 24.5) is a two-sided bound between the partition function $Z_\kappa(zf, g')$ of the walled torus with sources and the stopping-scale integral I_κ of the small-field effective action: $e^{-\mathsf{a}} I_\kappa \leq Z_\kappa \leq e^{\mathsf{a}} I_\kappa + b\, Z_\kappa(0,0)$, with a and b tending to zero with the stopping scale η_0, uniformly in the shell mass $\kappa \geq \kappa_0$ and in the spacing $\varepsilon \leq \varepsilon_0(\eta_0)$. $(\mathrm{SW2}^{\min})$ (Lemma 25.13) bounds the generating function $z \mapsto Z_\kappa(zf,0)/Z_\kappa(0,0)$ on the disc $|z| \leq 2$ uniformly in $\kappa \in [0,\infty]$ and in the spacing; at zero scalar source it is Gaussian domination. $(\mathrm{SW3}^\circ)_{\mathrm{int}}$ (Theorem 25.11), the κ-Cauchy property, says that on the capped test set the reduced small-field actions, the capped reduced actions of the theorem, at two shell masses $\kappa_0 \leq \kappa \leq \kappa'$ differ, up to a constant independent of the block field, by at most $2r(\kappa;\eta_0)$, which tends to zero as $\kappa \to \infty$ at fixed η_0, uniformly in the spacing. (SWT) (Corollary 25.12) says that the capped test set is the same for every $\kappa \geq \kappa_0$, so that no threshold term arises between two shell masses. Together they show that the heavy-shell limit $\kappa \to \infty$ and the continuum limit can be interchanged at one configuration, which is (RES$_0$) (Theorem 23.12).

Remark 23.13 (the shell bounds). $(\mathrm{SW1}^\circ)_{\mathrm{int}}$, $(\mathrm{SW2}^{\min})$, $(\mathrm{SW3}^\circ)_{\mathrm{int}}$ and (SWT) are Theorem 24.5, Lemma 25.13, Theorem 25.11 and Corollary 25.12 of Sections 24 and 25. Theorem 23.12 uses Theorem 25.7 only through the decay $r(\kappa;\eta_0) \to 0$ of the bound in $(\mathrm{SW3}^\circ)_{\mathrm{int}}$ as $\kappa \to \infty$, so a change of the rate of that decay, its exponent or a logarithmic factor, leaves the theorem unchanged; the dead-shell bound (Theorem 23.7) enters only through Theorem 23.10 at the configuration of Theorem 23.12, and none of the auxiliary statements of Sections 24 and 25 enters the statement of (RES$_0$) that Theorem 23.10 uses.

24. The shell model with the shell mass inside the covariance: the two-sided bound

This section and the next give the four shell bounds from which Section 23.4 derives the heavy-shell estimate (Theorem 23.12), row (R2) of Table 1. A reader who follows the main line of the rotation part first may take the four bounds as given. This section sets up the shell model and derives the first of them (Theorem 24.5) from Bałaban's estimates for the shell model, which are given in the first part of Section 25; the other three are obtained in the second part of that section.

The shell estimates $(\mathrm{SW1}^\circ)_{\mathrm{int}}$, $(\mathrm{SW2}^{\min})$, $(\mathrm{SW3}^\circ)_{\mathrm{int}}$ and (SWT) of Section 23.4 are King's renormalisation bounds [25, Thm. 3.1] and Bałaban's lower and upper bounds [1, 2, 3] run for the *shell model*, in which the shell mass $\kappa \mathbf{1}_U$ sits inside every matter covariance. This section sets the model up and proves the first of the four. We fix the model, its two stopping-scale tests and the notation; we then bound the mismatch vertex, which carries the difference between the counterterms of the shell model and those of the bulk, introduce the capped stopping-scale set, one set of block fields for all heavy shell masses, and state the exact mass shift on a finite lattice. With these we prove the two-sided bound (Theorem 24.5) and the perturbative bound for the shell model on which it rests (Proposition 24.7). Bałaban's estimates for the shell model and the κ-Cauchy property (Theorem 25.11), with the threshold clause (Corollary 25.12) and the disc bound (Lemma 25.13), follow in Section 25.

The shell model and the stopping-scale tests. Throughout, T, $\varepsilon = L^{-K}$ $(L=5)$ and $\ell_i := L^i \varepsilon$ are the torus, the spacing and the block scales of Definitions 16.3 and 16.4, $a_i := a(1-L^{-2})(1-L^{-2i})^{-1} \in [a(1-L^{-2}), a]$ the averaging constants of King's block-spin transformation [25, (2.10)–(2.17)], and *unit-i units* means the lattice rescaled so that i-blocks have side one. Bałaban's *final*

scale is $K_f := \max\{i : \ell_i \le \varepsilon_0^{\mathrm{Ba}}\}$, $\varepsilon_0^{\mathrm{Ba}}$ his fixed final spacing [1, (3.66)], [2, (2.116)]; the *final block scale* $\ell_f := \ell_{K_f} \in (\varepsilon_0^{\mathrm{Ba}}/L, \varepsilon_0^{\mathrm{Ba}}]$ is a fixed length independent of ε, η_0 and κ, and all scales below are $\le K_f \le K$. The box $H' \subset T$, the *shell* $U := T \setminus H'$, the wall profile $\psi\colon T \to [0,1]$ (0 on the source region $V \subset H'$, 1 off a neighbourhood of V, hence on U) and the sources f, g supported in V at distance $\ge \eta_0$ from U are those of Section 23. The shell model with shell mass $\kappa \in [0,\infty)$ is King's model with the profile

$$V(x) := m^2 + n\psi(x) + \kappa \mathbf{1}_U(x) \ \ge m^2 > 0, \qquad n \ge 1 \text{ fixed}, \tag{48}$$

inside every matter covariance: each block-spin step is run with $-\Delta_A + V$, so that κ sits in no vertex and enters the effective actions only through the lines, the legs $\phi_\kappa^{(i)} = a_i \ell_i^{-2} G_i^\kappa Q_i^* \phi_i$ of the block fields and the determinants. In unit-i units the shell mass is $w_i := \kappa \ell_i^2$; the scale i is *heavy* when $w_i \ge c_0$ and *light* otherwise, where $c_0 \in (0, c_0^*]$ is a fixed threshold, $c_0^* := 1/(2C_{\mathrm{far}})$ with C_{far} the constant of the modulus form $|\phi_\kappa^{(i)}(x)| \ge (1 - C_{\mathrm{far}} c_0)|\phi_i(y)| - C_1 p_i$ of [2, (2.65)], so that $1 - C_{\mathrm{far}} c_0 \ge \frac12$. The stopping scale k, $\ell_k = \eta_0$, is heavy for $\kappa \ge c_0/\eta_0^2$; as in Section 23, $\kappa_0 := \max(c_0, 1)/\eta_0^2$, so that on $[\kappa_0, \infty)$ the stopping scale is heavy with $w_k \ge \max(c_0, 1)$ (every statement below on $[\kappa_0, \infty)$ uses only $w_k \ge c_0$ and holds on $[c_0/\eta_0^2, \infty)$ as well). We write $p(t) = b_0(1 + \log t^{-1})^p$ [25, (3.1)] (b_0, p King's constants), $p_i := p(\ell_i)$, $M_i := p_i(\lambda \ell_i)^{-1/4}$, $U^{(i)}$ for the i-blocks in U and $L_1^{(i)}$ for the *first light layer*, the i-blocks of H' adjacent to $U^{(i)}$.

Block alignment and torus size. The *block-alignment condition* (BA): T and H' are unions of ℓ_f-blocks with the grid offset of the class $\mathcal{C}$ (Definition 16.6: the large-block size is $M = LM'$ with M' having a prime factor $q_0 \nmid L$ and $M' \ge 2q_0$, and every corner coordinate of H' is $c\,\ell_f$ with $c \equiv M'/q_0 \pmod{M'}$, counted from the physical origin, the anchor of Bałaban's block grids), each side of H' at least $2\ell_f$ and the shell at least $3\ell_f$ thick. Under (BA), H' and U are unions of i-blocks at every scale $i \le K_f$, and every face of every union of large blocks of scale $i-1$ or i lies at least two unit-i blocks from every parallel face of ∂U. The *torus-size condition* (T.nw): each side of T exceeds the fixed length $2\big(8r(\varepsilon_0^{\mathrm{Ba}}) + s_b\big)\varepsilon_0^{\mathrm{Ba}}$, where $r(t) = R(1 + \log t^{-1})^\varrho$ is Bałaban's localisation radius (in blocks), R his radius constant, $\varrho = 2$, and $s_b = LM$ the side of his large blocks in block units; like (BA) it is chosen after $\varepsilon_0^{\mathrm{Ba}}$, and it gives torus sides $N_\mu > 16\, r(\varepsilon_0^{\mathrm{Ba}}) + 2s_b$ in unit-i units at every scale $i \le K_f$, so that no localisation box wraps the torus.

Besides King's small-field indicator, the shell model carries two tests on the block fields, one on the shell blocks at each scale and one on the first light layer at the stopping scale, both imposed on the small-field integral at the stopping scale k; their thresholds depend on the shell mass through the coefficients γ, γ_2 defined now. Fix shares $\vartheta_1 \in (0, \frac12)$ and $\vartheta_2 \in (0, \vartheta_1/4]$ with $\vartheta_1 + \vartheta_2 < 1$, and put $\gamma(w) := a_i w/(a_i + w)$, $\gamma_2(w) := (a_k^{-1} + c_P^{-1} + w^{-1})^{-1}$, c_P the Poincaré constant of a unit block. The *near-U test* at scale i and the *face-layer test* at the stopping scale k are

$$\begin{aligned} \tau_i &:= \prod_{y \in U^{(i)}} \chi\big(|\phi_i(y)| \le \bar\tau_i\big), \quad \bar\tau_i := \Big(\frac{2}{\vartheta_1 \gamma(w_i)}\Big)^{1/2} p_i; \\ \lambda_k &:= \prod_{y' \in L_1^{(k)}} \chi\big(|\phi_k(y')| \le \sigma_k\big), \quad \sigma_k := \Big(\frac{2}{\vartheta_2 \gamma_2(w_k)}\Big)^{1/2} p_k, \end{aligned} \tag{49}$$

Both thresholds decrease in κ; at a heavy scale $\bar\tau_i \le C_7 p_i$ and $\sigma_k \le C_8 p_k$ with the constants C_7, C_8 of Definition 24.2. Let χ_k be the indicator of King's small-field set [25, (3.2)] (the block-sup clause, the covariant gradient clause $|D_{A_k}\phi_k(b)| \le p_{k-1}$ on every block bond $b = (y, y')$ of $T^{(k)}$, where $D_{A_k}\phi_k(b) := \phi_k(y') - U(A_k(b))\,\phi_k(y)$ with $U(A_k(b))$ the parallel transporter of A_k along b, in unit-k units, and the photon clauses), $\chi_k^\flat$ the same with the gradient clause imposed only on bonds with both ends in $H'^{(k)}$, $S^{(k),\kappa}(A_k, \phi_k, zf, g')$ the effective action after k steps with the

profile inside, and

$$
\begin{aligned}
I_\kappa(z,g') &:= \int (dA_k)(d\phi_k)\, \chi_k^\flat \tau_k \lambda_k\, e^{-S^{(k),\kappa}}, \\
I_\kappa^\sharp(z,g') &:= \int (dA_k)(d\phi_k)\, \chi_k \tau_k \lambda_k\, e^{-S^{(k),\kappa}} \;\le\; I_\kappa(z,g').
\end{aligned}
\tag{50}
$$

At $\kappa = 0$ both thresholds are $+\infty$, so $I_0^\sharp$ is King's small-field integral.

Unit-k notation. Write $\eta := \varepsilon/\eta_0$ for the spacing in unit-k units, $\ell_j = L^j\varepsilon$ for the block scales (written δ_j in the rest of this part), $\hat\kappa := w_k = \kappa\eta_0^2$, $V = V_0 + \kappa\mathbf{1}_U$ with $V_0 = m^2 + n\psi$ its κ-free part and $m_V^2 := \inf_{H'} V_0 \ge m^2$ the plateau gap; $Z_\kappa(zf, g')$ is the partition function with the sources zf (real z) and $g' \in \{0, g\}$, supported in H' at distance $\ge \eta_0$ from U, $A^{(k)}$ the lifted background, $P^{(k),\kappa}$ the renormalised perturbative part of $S^{(k),\kappa}$, $M_k := p_{k-1}(\lambda\ell_{k-1})^{-1/4}$ King's block sup threshold [25, (3.2)], and $|U|_k$, $|\partial H'|_k$ the numbers of unit-k cubes in U and meeting ∂U. All constants C depend on $(L, a, N, \lambda, e, m, \mu_0, \alpha_0, \bar n, \vartheta_1, \vartheta_2, c_0)$ and King's constants unless a further dependence is displayed; none depends on κ, ε or k.

24.1. **The mismatch vertex, the capped set and the mass shift.** This subsection collects three objects of the shell model on which the proofs of this section and the next rest: the mismatch vertex, which carries the difference between the counterterms of the shell model and those of the bulk (Lemma 24.1); the capped test set and the reduced action on it (Definition 24.2); and the exact mass shift on a finite lattice (Lemma 24.3). The shell model subtracts its counterterms κ-adaptedly (Proposition 24.7); the difference to the bulk counterterm is the mismatch vertex $\mathcal{M}$ of Definitions 16.9 and 16.11, with the offset Δ_T of the shell-side convention bridge $(\mathrm{CT})_{\mathrm{sh}}$, $|\Delta_T(s)| \le \Delta_*$ for $s \le s_0$, treated as an additional two-leg vertex. In unit-k units, $C_{\mathcal{M}} := C(\lambda + e^2)\ell_k$ is the coupling of one mismatch vertex, $r_j := L^{j-k}$ the range of a line of scale j ($r_j = \varrho_j$ in the notation of Theorem 20.4) and $r_* := r_{j_*}$ that of the finer line j_* of a vertex.

The mismatch vertex enters the perturbative bound as an additional vertex of the graphs (Proposition 24.7). The following lemma collects the three bounds on it that are used: a bound on its tree term (M1), its one-line form (M2) and a lossless form of its two-line density (M3), in which each line keeps the square root of its own amplitude. The lossless form is needed because the lossy bound $\rho_{j_*}^{1/2}\rho_{j_1}^{1/2} \le C\rho_{j_*}$ would leave a deficit $L^{k-j_*}\min(\kappa_{j_*}^{1/2}, \kappa_{j_*}^{-1/2})$ deep in U, not summable over the scales, whereas the deep part $\hat\kappa^{1/2} m(j_*)m(j_1)$ of the lossless density is absorbed in Theorem 25.6.

Lemma 24.1 (the three clauses of the mismatch vertex)**.** *For King's legs of block data b on the small-field set, $|b|_\infty \le p_{k-1}(\lambda\ell_{k-1})^{-1/4}$, and for internal lines with their half-amplitude bounds, uniformly in $\kappa \in [0,\infty)$ and ε where no scale is named, in unit-k units:*

(M1) at a heavy scale ($\hat\kappa \ge c_0$) the tree term $\frac12\ell_k^2\sum_x \eta^d |\Delta\delta(x)||\phi_\kappa^{(k)}(x)|^2$ is at most

$$
\begin{aligned}
&C(\lambda+e^2)\lambda^{-1/2}\ell_k^{1/2}p_{k-1}^2\big[|\partial H'|_k(1+\hat\kappa^{-1}\log(2+\hat\kappa)) + |U|_k\hat\kappa^{-1}\big] \\
&\qquad + C(\lambda+e^2)^2\lambda^{-1/2}\ell_k^{3/2}(1+\log(1/\ell_k))^{q_F}p_{k-1}^2|T^{(k)}|
\end{aligned}
$$

plus the non-decaying wall-floor term $CC_n(\lambda+e^2)\lambda^{-1/2}L^{1/2}\ell_k^{3/2}p_{k-1}^2|T^{(k)}|$; at a light scale ($\hat\kappa < c_0$) it is at most $C(\lambda+e^2)\lambda^{-1/2}\ell_k^{1/2}p_{k-1}^2\hat\kappa^{1/2}|T^{(k)}|$ plus the wall-floor term;

(M2) a mismatch vertex with one leg and one internal line is a plain one-line vertex with the cube-summed coefficient $CC_{\mathcal{M}}|b|_\infty$;

(M3) (lossless form) for a mismatch vertex at x joining two lines of scales $j_* \le j_1 \le k-1$, the density $g(x) := \ell_k^2 |\Delta\delta(x)| \rho_{j_*}(x)^{1/2} \rho_{j_1}(x)^{1/2}$ obeys

$$g(x) \le C C_{\mathcal{M}} \Big[1 + \min\big(r_*^{-1}, t(x)^{-1}\big) + \mathbf{1}_U(x)\, \hat\kappa^{1/2} m(j_*) m(j_1) \Big],$$
$$m(j) := \min(1, \kappa_j^{-1/2}), \ \ \kappa_j := \hat\kappa r_j^2,$$

with the exact identity $\hat\kappa^{1/2} m(j_*) m(j_1) = L^{k-j_1} \min(1, \kappa_{j_*}^{-1/2}) \min(1, \kappa_{j_1}^{1/2})$; hence $g \le C C_{\mathcal{M}} r_*^{-1}$ pointwise, and over a unit-k cube Q, $\sum_{x\in Q} \eta^d g(x) \le C C_{\mathcal{M}} [1 + (k - j_*) \log L]$ if Q meets ∂U from H', $\le C C_{\mathcal{M}} / t(Q)$ if $Q \subset H'$ away from ∂U ($t(Q)$ the distance of Q to ∂U), and $\le C C_{\mathcal{M}} [1 + L^{k-j_1} \min(1, \kappa_{j_*}^{-1/2}) \min(1, \kappa_{j_1}^{1/2})]$ if Q meets U, the cases read additively for a cube meeting both sides of ∂U.

Under the counterterm offset of Definition 16.11, (M1)–(M3) hold with $C(\lambda + e^2)$ enlarged by a constant multiple of Δ_c (the c-part of each coefficient being bounded by the sup bound of the legs) and with $C_n(\lambda + e^2) + \Delta_c$ in place of $C_n(\lambda + e^2)$ in the wall-floor term of (M1).

Proof. The resolvent identity $C^{\eta,\mathrm{bulk}} - C^{\eta,\kappa} = C^{\eta,\mathrm{bulk}} (n\psi + \kappa \mathbf{1}_U) C^{\eta,\kappa}$, the entrywise positivity of the kernels at zero background and $\kappa \sum_{y\in U} \varepsilon^3 C^{\eta,\kappa}(x,y) \le N$ give every graph of $\Delta\delta$ the profile $\min(\sqrt{\hat\kappa}, t^{-1})$ in the distance t to ∂U, up to a logarithm for the exchange and two-loop graphs; (M1) and (M2) are the arithmetic of this profile against the amplitude of the legs and the Yukawa decay of the background into U. In (M3) each line keeps the square root of its own amplitude, so that the heavy-side profile splits into a Compton part of face type and a deep part κ_j^{-1}, the product of the two deep parts being $\hat\kappa^{1/2} m(j_*) m(j_1)$. □

The lemma holds at every shell mass; for heavy shell masses the tests themselves simplify. On $[\kappa_0, \infty)$ the thresholds of the two tests are dominated by κ-free caps, and the small-field object becomes one set of block fields for all heavy shell masses. The following definition fixes this set and the reduced action on it, in whose terms the κ-Cauchy property is stated in Section 25.

Definition 24.2 (the capped stopping-scale set and the reduced action)**.** Let $\hat c_0 := \max(c_0, 1)$ and, as in Section 23.4, $\kappa_0 := \hat c_0 / \eta_0^2$, so that $\kappa \ge \kappa_0$ means $\hat\kappa \ge \hat c_0$, hence $w_k \ge c_0$ and $\hat\kappa \ge 1$: the stopping scale is heavy throughout. The *cap thresholds* are the κ-free numbers $C_7 p_k$ and $C_8 p_k$,

$$C_7 := \Big(\frac{2\,(a(1 - L^{-2}) + c_0)}{\vartheta_1\, a(1 - L^{-2})\, c_0} \Big)^{1/2}, \qquad C_8 := \Big(\frac{2\,(1/(a(1 - L^{-2})) + 1/c_P + 1/c_0)}{\vartheta_2} \Big)^{1/2},$$

which dominate the thresholds $\bar\tau_k$ and σ_k of (49) for every $w_k \ge c_0$. The *capped test set* is

$$\mathcal{X}^{\mathrm{cap}} := \mathrm{supp}\,\big(\chi_k^\flat\, \tau_k^{\mathrm{cap}}\, \lambda_k^{\mathrm{cap}}\big), \qquad \tau_k^{\mathrm{cap}}(\phi_k) := \prod_{y \in U^{(k)}} \chi(|\phi_k(y)| \le C_7 p_k),$$
$$\lambda_k^{\mathrm{cap}}(\phi_k) := \prod_{y' \in L_1^{(k)}} \chi(|\phi_k(y')| \le C_8 p_k),$$

the same set for every $\kappa \ge \kappa_0$. On it $|\phi_k| \le M_k$ on H'-blocks (the block-sup bound of King's small-field set [25, (3.2)]), $|\phi_k| \le C_7 p_k$ on U-blocks, $|\phi_k| \le C_8 p_k$ on $L_1^{(k)}$, the gradient test holds on H'-internal bonds, and A_k obeys King's (3.2); these five bounds are the only properties of the set that are used. The standing smallness $\eta_0 \le L/(\lambda C_7^4)$ gives $C_7 p_k \le M_k$, so that the block-sup clause of $\chi_k^\flat$ on U-blocks is implied by the cap. The thresholds $\kappa_j := \hat c_0 L^{2j} / \eta_0^2$ and the *Compton intervals* $\bar I_j := [\kappa_j, \kappa_{j+1}]$ ($\kappa \le \kappa' \le L^2 \kappa$ for every pair in $\bar I_j$) are a bookkeeping of the κ-integral only: no threshold, test or representation changes at κ_j, and the intervals beyond c_0/ε^2 are treated like the others.

Write $b := (A_k, \phi_k|_{H'^{(k)}})$ for the exterior block data and $\phi_U := \phi_k|_{U^{(k)}}$ for the shell block fields; under the standing smallness, $\tilde\chi^{\mathrm{cap}} := \{b : \chi_k^\flat \lambda_k^{\mathrm{cap}}(b) = 1\}$ is a set of exterior data and $\mathcal{X}^{\mathrm{cap}} = \tilde\chi^{\mathrm{cap}} \times \{\phi_U : |\phi_k(y)| \le C_7 p_k \ \forall y \in U^{(k)}\}$. The *capped reduced action* and the capped small-field integral are

$$\tilde S_\kappa^{\mathrm{cap}}(b; zf, g') := -\ln \int \prod_{y\in U^{(k)}} d\phi_k(y)\ \tau_k^{\mathrm{cap}}(\phi_U)\, e^{-S^{(k),\kappa}(A_k,\phi_k,zf,g')},$$

$$I_\kappa^{\mathrm{cap}}(z,g') = \int_{\tilde\chi^{\mathrm{cap}}} e^{-\tilde S_\kappa^{\mathrm{cap}}(b;zf,g')}\, db,$$

and for $\kappa_0 \le \kappa \le \kappa' < \infty$, $z \in [-1,1]$, $g' \in \{0, g\}$,

$$\Delta_z(b) := \tilde S_\kappa^{\mathrm{cap}}(b; zf, g') - \tilde S_{\kappa'}^{\mathrm{cap}}(b; zf, g'), \qquad \Delta_0(b') := \tilde S_\kappa^{\mathrm{cap}}(b'; 0, 0) - \tilde S_{\kappa'}^{\mathrm{cap}}(b'; 0, 0),$$

$$J_\kappa(z) := \frac{I_\kappa^{\mathrm{cap}}(z,g)}{I_\kappa^{\mathrm{cap}}(0,0)}.$$

The two-sided bound for I_κ^{cap} is the cap form of Theorem 24.5; the test set being the same for all $\kappa \ge \kappa_0$, the threshold clause (SWT), which compares the small-field integrals of two adjacent representations, holds with $t_j \equiv 0$, and the exact identity

$$\frac{J_\kappa}{J_{\kappa'}} = \frac{\iint e^{-\tilde S_\kappa^{\mathrm{cap}}(b;z) - \tilde S_{\kappa'}^{\mathrm{cap}}(b';0)}\, db\, db'}{\iint e^{-\tilde S_{\kappa'}^{\mathrm{cap}}(b;z) - \tilde S_\kappa^{\mathrm{cap}}(b';0)}\, db\, db'} \qquad (\text{both over } \tilde\chi^{\mathrm{cap}} \times \tilde\chi^{\mathrm{cap}}),$$

is the reason the bound $(\mathrm{SW3}^\circ)_{\mathrm{int}}$ (Theorem 25.11) is posed as a *mixed* difference $\Delta_z(b) - \Delta_0(b')$ with no constant exhibited: a field-free constant in the action cancels in it.

The set $\mathcal{X}^{\mathrm{cap}}$ is called the capped test set in the text, the capped stopping-scale set in the title of Definition 24.2 and in the introduction of Section 24, and the capped set in the title of this subsection; the three names denote one and the same set.

Two exact statements on King's torus model enter the proof of the two-sided bound, both in the torus convention: the mass shift of Lemma 24.3 below, and the fixed-torus moment of the scalar source (Theorem 24.4), which the upper bound needs at $g' = g$ and with which the next subsection opens. Fix K (spacing $s = \varepsilon_K$) and a King torus T. In the *torus convention* the mass counterterm is $\delta m^2_{T,s}(m)$ of [25, (3.33)] with $\Omega = T$ and the torus propagators, and the Wick constant is $c_{K,T} = (-\Delta^{s,T} + m^2)^{-1}(x,x)$ of [25, (2.8)]. For a real source h on T put

$$N(0,h) := N_{K,T}^{\mathrm{tor},z}(0,h) := e^{-z_K \mathsf{S}_K(h)}\, Z_{K,T}^{\mathrm{tor}}(0,h)/Z_{K,T}^{\mathrm{tor}}(0,0), \qquad \mathsf{S}_K(h) = \sum_x s^3 h(x),$$

with $z_K = a_K^\sharp$ the scheme constant of Definition 27.1 at the spacing ε_K, and $f(u) := \ln N(0, u\cdot 1)$; $f(0) = 0$, and $h \mapsto \ln N(0,h)$ is convex by Hölder's inequality. For a scalar mass M (M is King's large-block integer of Definition 16.1) let $\alpha_s^T(\mathsf{M}) = \mathsf{M}^2 + \delta m^2_{T,s}(\mathsf{M})$ be the coefficient of $\frac12 \sum_x s^3 |\phi(x)|^2$ in the action and $s_{K,T}(\mathsf{M}) := \frac12(\alpha^T_{\varepsilon_K}(m) - \alpha^T_{\varepsilon_K}(\mathsf{M}))$; the threshold $s_*(e)$ of Definition 27.2 is the supremum over $\mathsf{M} > 0$ of the free-convention limit $s_{\inf}(\mathsf{M})$ of this coefficient (Proposition 26.3).

On a finite lattice the mass shift is exact: in the action [26, (2.5)] the scalar mass enters only through $\frac12 \sum_x s^3(\mathsf{M}^2 + \delta m^2(\mathsf{M}))|\phi(x)|^2$ and the field-independent constants. The proof of Theorem 24.4 uses the lemma to write the moment of the scalar source as a ratio of torus-convention functionals, and the threshold proposition of Section 26 (Proposition 26.3) and the proof of $(\mathrm{W}^\sharp 2)$ in Section 27.2 use it again.

Lemma 24.3 (the exact mass shift on a finite lattice)**.** *Let $\langle\cdot\rangle^{(\mathsf{M})}$ be the state of the same lattice, boundary condition and convention at the scalar mass M in place of m, and $u := \frac{1}{2}(\alpha(\mathsf{M}) - \alpha(m))$. Then, for every F integrable against both measures, $\langle F\rangle^{(m)} = \langle F\, e^{u\, r^2(1)}\rangle^{(\mathsf{M})} / \langle e^{u\, r^2(1)}\rangle^{(\mathsf{M})}$, $r^2(1) := \sum_x s^3 |\phi(x)|^2$, and $f(s_{K,T}(\mathsf{M})) = B^z_{K,T}(\mathsf{M}) + \ln Z^{(\mathsf{M})} - \ln Z^{(m)}$ with $B^z_{K,T}(\mathsf{M}) := E(\mathsf{M}) - E(m) - (2c_{K,T} + z_K)\, s_{K,T}(\mathsf{M})\, |T|$, where $E(\mathsf{M}) = E_0(\mathsf{M}) + E_1(\mathsf{M})$ are the field-free constants of King's action at the mass M [25, (2.6), (3.35)].*

Proof. By the form of the action, the two Gibbs densities differ by the factor $e^{u\, r^2(1)}$ and a constant, $e^{\pm r^2(\cdot)}$ being integrable by the quartic term; the second identity is the first at $F = e^{r^2(s_{K,T}(\mathsf{M})\cdot 1)}$ with the constants written out. □

24.2. **The two-sided bound and the perturbative bound.** This subsection proves the two-sided bound $(\mathrm{SW1}^\circ)_{\mathrm{int}}$ of Theorem 24.5, King's bound [25, Theorem 3.1] for the shell model, whose shell mass $\kappa \mathbf{1}_U$ sits inside every matter covariance, with constants uniform in $\kappa \in [0, \infty)$, in the spacing and in the source coefficient. The mechanism is one: Bałaban's lower and upper bounds are run with the profile (48) in every covariance, and at each step the shell mass is handled by an exact Gaussian identity, by a positivity or by one of the bounds on the shell model proved below, Bałaban's four replaced objects being those of the shell model (the notation is collected in the closing paragraph of this subsection). The bounds behind the replacements are three, each stated here in the form in which the proof uses it, and proved here, except the mismatch factor, which is proved in Section 25: the first-order determinant term (Theorem 24.6), the perturbative bound with the mismatch factor and the vacuum sector (Proposition 24.7), and the class of small fields carried through the lower bound (Lemma 24.8); the first, and the two displayed bounds of the vacuum sector, hold up to the endpoint $\kappa = \infty$. We begin with the fixed-torus moment of the scalar source (Theorem 24.4), which the upper bound needs at $g' = g$, then fix the standing data and state the two-sided bound, prove the three bounds, and close with the notation for Bałaban's steps, which Section 25 runs for the shell model (Theorems 25.5 and 25.1).

The upper bound of Theorem 24.5 at $g' = g$ carries, through Hölder's inequality, a normalised moment of the Wick-ordered scalar source on the torus, and its constant b_g in (54) is finite exactly when that moment is. The following theorem provides the finiteness. It is a by-product of the proof of Theorem 27.8(ii) run on the bare torus: by the mass-shift lemma the normalised functional of the source is a ratio of two torus-convention functionals, times convergent prefactors, and both functionals converge in $(0, \infty)$ along the spacings, so the ratio is bounded. Only the bound is used; no limit value, rate or uniformity in T enters.

Theorem 24.4 (the fixed-torus moment of the scalar source)**.** *Let $T = D_{J'}$ with $J' \in 5^{k_*}\mathbb{N}$, and let $0 \le q \in C^\infty_c$ with $2 \sup q < s_*(e)$, at a point of King's range outside E_W, with $J' \ge J_0(\varepsilon_0, \sup q + \bar{\Delta}/2)$ and $J' \ge J_{\mathrm{pos}}(q)$. Here $J_{\mathrm{pos}}(q)$ is the threshold of Theorem 26.11 at $\Gamma_q = \sup q$; $\bar{\Delta} := \sup_J \Delta_*(J) < \infty$, where $\Delta_*(J) := \sup_{s \le s_0} |\Delta_{D_J}(s)|$ is the bound Δ_* of the convention bridge at the torus D_J (finite, and tending to 0 as $J \to \infty$ by Lemma 26.4); $J_0(\varepsilon_0, \Gamma)$ is the non-wrapping threshold $J_0(\varepsilon_0)$ of the class $\mathcal{C}$ (Definition 16.6) at a torus source with positive part Γ (a finite lower bound on J', equal to $J_0(\varepsilon_0)$ wherever the threshold carries no amplitude); and*

$$N^z_{K,T}(0, q) := e^{-a^\sharp_K \mathsf{S}_K(q)}\, \frac{Z^{\mathrm{F}}_{K,T}(0, q)}{Z^{\mathrm{F}}_{K,T}(0, 0)}, \qquad \mathsf{S}_K(q) := \sum_x \varepsilon_K^3 q(x),$$

is the normalised Wick functional of King's torus model on T at spacing ε_K in the free convention (the single-site terms of King's cubes: the mass counterterm $\delta m^2_{\varepsilon_K}(m)$ and the Wick constant $\mathsf{W}_{\varepsilon_K}$), with no wall and no shell, in the scheme $\Theta^\sharp$, $Z^{\mathrm{F}}_{K,T}(0, q)$ being its partition function with

the Wick square of the source q inserted. Then

$$M_T(q) := \sup_K N^z_{K,T}(0,q) < \infty \qquad \text{along } \varepsilon_K = 5^{-K}.$$

Proof. Run the proof of Theorem 27.8(ii) on the bare torus ($n = 0$, $\psi \equiv 0$, $\beta = 1$) for the sources q and 0, with the moving families $h_K = q - \frac{1}{2}\Delta_T(\varepsilon_K)$, $h_{0,K} = -\frac{1}{2}\Delta_T(\varepsilon_K)$ of Definition 27.10 and, for $J' \geq J_{\mathrm{pos}}(q)$, the envelope of Theorem 26.11(A) at $\psi \equiv 0$ as its upper line. By Lemma 24.3, $N^z_{K,T}(0,q)$ is a ratio of torus-convention functionals at h_K and $h_{0,K}$ times convergent prefactors (the scheme bridge $\lim_K(a^\sharp_K - z_K)$ of Section 27.3, $z_K = e^2 X_{\varepsilon_K}(m;\mu_*)$), and the telescoping step of that proof makes both functionals converge in $(0,\infty)$; hence $N^z_{K,T}(0,q)$ converges in $(0,\infty)$ and is bounded. □

Standing data. Fix $a, N, \lambda > 0$, e, $m^2 > 0$, $\mu_0^2 > 0$ and $L = 5$; $d = 3$, $a = 1$ (King's averaging constant), $\varrho = 2$ and $p = 6$ are the values at which $(\mathrm{BAL})_5$ is stated and its four steps are repeated (Theorem 19.3). Fix the shares ϑ_1, ϑ_2, a geometry (H', T) satisfying (BA) with its grid offset and (T.nw), the wall height $n \geq 1$, the wall profile ψ, the sources f, g and the Wick-ordered source scheme of Section 27, and let the counterterm be $\delta m^2_\varepsilon + c(\varepsilon)$ with $c = \sum_{1\leq\alpha+\beta\leq 4} c_{\alpha\beta}$, $c_{\alpha\beta}$ of order $e^\alpha\lambda^\beta$, and $\Delta_c := \sum_{\alpha,\beta} \sup_{\varepsilon\leq\varepsilon_0} |c_{\alpha\beta}(\varepsilon)| < \infty$. Throughout, $\varepsilon_0 \in (0,1]$ is the fixed upper bound on the spacings of the class $\mathcal{C}$ (Definition 16.6), and every threshold $\varepsilon_0(\eta_0)$ below is at most ε_0. Three constants recur: $M'_k = L^{1/4}(p_{k-1}/p_k)M_k$ is the block-sup threshold of [25, (3.2)]; $B_\varepsilon(f)$, half the variance of the plaquette source $F(f)$ under the free photon measure at spacing ε, the constant of the Gaussian domination $\langle e^{tF(f)}\rangle \leq e^{t^2 B_\varepsilon(f)}$ of Lemma 16.16(a), with $B := \sup_{\varepsilon\leq\varepsilon_0} B_\varepsilon(f) \leq C(f)/(2\mu_0^2)$; and $M_T(q)$, the fixed-torus moment of Theorem 24.4 at the amplitude q ($s_*(e)$, $\bar\Delta$, $D_{J'}$ and k_* as in Section 27). The provisions on the choice of the constants are those of Definition 25.3.

The two-sided bound compares the partition function $Z_\kappa(zf, g')$ of the shell model with the stopping-scale integral $I_\kappa(z, g')$ and its variant $I^\sharp_\kappa(z, g')$: a lower bound by $e^{-\mathsf{a}} I^\sharp_\kappa$ at every κ, a lower bound by $e^{-\mathsf{a}} I_\kappa$ once the shell mass is heavy at the stopping scale, and an upper bound by $e^{\mathsf{a}} I_\kappa$ up to a large-field remainder $b_{g'} Z_\kappa(0,0)$. What is used later is that a, b and b_g depend neither on κ nor on the spacing nor on z, and vanish as the stopping scale η_0 tends to zero: in this form the bound is the shell estimate $(\mathrm{SW1}^\circ)_{\mathrm{int}}$, one of the four shell estimates on which the heavy-shell decoupling (Theorem 23.12) rests. The statement also gives the bounds with the capped stopping-scale integral of Definition 24.2 in place of I_κ, and the order in which the constants are chosen.

Theorem 24.5 (two-sided bound with the shell mass inside: $(\mathrm{SW1}^\circ)_{\mathrm{int}}$)**.** *Under the standing data there are $\eta_1 > 0$ (depending on n as well as on the constants listed below, through the condition $(m^2 + n)\ell_k^2 M'_k \leq p_k$ ($M'_k = L^{1/4}(p_{k-1}/p_k)M_k$ the block-sup threshold of [25, (3.2)]) at the stopping scale and through the threshold of the walled model at the fixed wall height), $\sigma > 0$, $c_1 > 0$, for every $\eta_0 \in L^{-\mathbb{N}}$ with $\eta_0 \leq \eta_1$ a threshold $\varepsilon_0(\eta_0) > 0$, and a constant $C < \infty$ depending on $(L, a, N, \lambda, e, m, \mu_0, |T|, n, f, g, H', \vartheta_1, \vartheta_2, \Delta_c, c_0)$ but not on η_0, κ or ε, such that for all $\kappa \in [0,\infty)$, $\varepsilon = L^{-K} \leq \varepsilon_0(\eta_0)$, $z \in [-1,1]$ and $g' \in \{0, g\}$,*

$$Z_\kappa(zf, g') \;\geq\; e^{-\mathsf{a}}\, I^\sharp_\kappa(z, g') \qquad (\kappa \in [0,\infty)), \tag{51}$$

$$Z_\kappa(zf, g') \;\geq\; e^{-\mathsf{a}}\, I_\kappa(z, g') \qquad (\kappa \in [\kappa_0,\infty)), \tag{52}$$

$$Z_\kappa(zf, g') \;\leq\; e^{\mathsf{a}}\, I_\kappa(z, g') + b_{g'}\, Z_\kappa(0,0) \qquad (\kappa \in [0,\infty)), \tag{53}$$

with $\mathsf{a} = C\eta_0^{\sigma}|T|$ *and* $b = e^{-c_1\, p(\eta_0)^2 + C|T|}$*; in* (53)*,* $b_{g'}$ *denotes* b *at* $g' = 0$ *and* b_g *at* $g' = g$*, where*
(54)
$$b_g := b^{1/q}\, e^{q'B}\, M_T(q'g)^{1/q'}, \qquad q' = q'(g) \in (1,2],\ 2q' \sup g < s_*(e),\ q = \tfrac{q'}{q'-1},\ B := \sup_{\varepsilon \le \varepsilon_0} B_\varepsilon(f),$$

where $M_T(q'g)$ *is finite under its torus proviso, which at* $g' = g$ *is part of the hypotheses: the side* J' *of* $T = D_{J'}$ *lies in* $5^{k_*}\mathbb{N}$ *with* $J' \ge J_0(\varepsilon_0, q' \sup g + \bar{\Delta}/2)$ *and* $J' \ge J_{\mathrm{pos}}(q' \sup g)$*, and the couplings lie off the exceptional set* E_W*; otherwise* $b_g = \infty$ *and* (53) *is void at* $g' = g$*. In particular* a*,* b *and* b_g *(written* $a(\eta_0)$*,* $b(\eta_0)$ *where their dependence on the stopping scale is displayed) do not depend on* κ*,* ε *or* z*, and* $\mathsf{a}, b, b_g \to 0$ *as* $\eta_0 \to 0$ *at fixed* $(n, |T|)$*, uniformly in* $\kappa \in [0,\infty)$ *and* $\varepsilon \le \varepsilon_0(\eta_0)$*.*

Cap form. *For* $\kappa \in [\kappa_0, \infty)$*,* (52) *and* (53) *hold with the capped stopping-scale integral* $I_\kappa^{\mathrm{cap}}(z, g')$ *of Definition 24.2, the integral over the capped test set* $\mathcal{X}^{\mathrm{cap}}$ *of Definition 24.2, in place of* I_κ *and with the same* a*,* b *and* b_g*:* $e^{-\mathsf{a}} I_\kappa^{\mathrm{cap}}(z, g') \le Z_\kappa(zf, g') \le e^{\mathsf{a}} I_\kappa^{\mathrm{cap}}(z, g') + b_{g'} Z_\kappa(0,0)$*.*

Choice of the constants. *The constants are chosen in the acyclic order* $L \to M \to R \to b_0 \to n \to \varepsilon_0^{\mathrm{Ba}} \to (f,g), \psi, H' \to \eta_1 \to \eta_0 \to K$*: every condition on* M*,* R *or* b_0 *is a lower bound, every condition on* η_1 *a cap chosen after* b_0*, and* e*,* λ*,* m*,* μ_0 *stay free. The theorem is unconditional at the values* $d = 3$*,* $a = 1$*,* $\varrho = 2$*,* $p = 6$*,* $L = 5$ *of the standing data; at other values of* (a, ϱ, p)*, or at another integer* $L \ge 5$*, it holds given* $(\mathrm{BAL})_L$*, the same four steps repeated at those values; every condition that the proof places on* L *alone reduces to* $L^{3/4} \ge 3$*,* $L^{3/2} \ge 3$ *and* $L^{5/2} \ge 6$ *(the room in the three clauses of the class of Lemma 24.8), which hold for every integer* $L \ge 5$*, and none involves the* κ*-uniform operator constants.*

The one new point of the proof is uniformity in the shell-mass parameter κ: at each step the independence of κ is an exact Gaussian identity, a positivity, or one of the bounds proved in this section and the next, namely the first-order determinant term (Theorem 24.6), the perturbative bound for the shell model with its mismatch clause (Proposition 24.7, Lemma 24.1, Theorem 25.6), the pointwise bounds on the step lines (Theorem 25.1), the form of $(\mathrm{BAL})_5$ in Theorem 25.5 for the L-dependent steps, and the class carried through the lower bound (Lemma 24.8). The proof names the statements of King and Bałaban which these replace.

Proof. We first describe how the proofs of King and Bałaban are repeated with the shell profile and which of our statements replace theirs, then prove the lower bounds, and finally the upper bound. The proof is King's [25, Theorem 3.1] and Bałaban's lower and upper bounds [1, (3.26)–(3.69)], [2, (2.2)–(3.54)], [3, Prop. 1, (1.33)], [4, (1.8), (1.15)–(1.16), (1.20), (2.5)–(2.33)], repeated with the profile (48) in every covariance. At each step the independence of κ is an exact Gaussian identity (the composition law, the scale decomposition (61), the minimiser identity $a_i(\phi_i - Q_i\phi_\kappa^{(i)}) = Q_i[(-\Delta_A + V_i)\phi_\kappa^{(i)}]$), a positivity, or one of the bounds stated below. In place of King's [25, Theorem 3.3], whose heavy-side Hölder clause is never used, stand Theorem 25.1 and the amplitude bound $|\phi_\kappa^{(j)}(x)| \le C\rho_j(x)|b|_\infty$; in place of [3, Proposition 1] stands Proposition 24.7, with Lemma 24.1, Theorem 25.6, Lemma 25.2 and Theorem 24.6; Theorem 25.5 covers the four L-dependent steps; and Lemma 24.8, with King's near-face analysis [26, (5.41)–(5.49)] at the shell face, gives the class carried through the lower bound.

The tests are paid by positivity: the Schur complement of $a_i|\phi_i(y) - Q\phi|^2 + w_i|\phi|^2$ over the fine field is $\gamma(w_i)|\phi_i(y)|^2$, so ϑ_1 pays $e^{-p_i^2}$ for the near-U test and $\vartheta_2 \le \vartheta_1/4$, with the Poincaré inequality on two adjacent unit blocks, pays $e^{-p_k^2/4}$ for the face-layer test, uniformly in κ. In the lower bounds the tests are inserted at the stopping scale. There $\mathrm{supp}(\chi_k^\flat \tau_k \lambda_k)$ lies in the class only at a heavy stopping scale, whence the range of (52); $\mathrm{supp}(\chi_k \tau_k \lambda_k)$ lies in the class for every κ, whence (51); and the cap set lies in the class (Lemma 24.8(d)(i)) with $I_\kappa \le I_\kappa^{\mathrm{cap}}$ for $w_k \ge c_0$, whence the cap form.

In the upper bound the large-field remainder is $bZ_\kappa(0,0)$ because the field-independent part is recentred by the shell model's own constant. At $g' = g$ the Wick-ordered source is carried inside the Gaussian domination of Lemma 16.16(a) (a radial single-site weight keeps the state a BFS measure, Section 2), and Hölder's inequality with the exponents of (54) turns $bZ_\kappa(0,0)$ into $b_g Z_\kappa(0,0)$: the normalised moment of the source is at most $M_T(q'g)$ by monotonicity in κ and n ([26, Prop. 3.1, Thm. 3.2], [9, Thm. 6.1], Lemma 2.10(d)) and finite by Theorem 24.4. The families with King's source vertex [25, (3.21)] carry $(1+\|g\|_\infty)^{n_h}$. □

The first of the three bounds controls the first-order term of the step determinants per site, through a uniform mass shift, without a per-ordering Ward identity. In Bałaban's upper bound the Gaussian factors of the earlier steps compose, on each of the two domains of the kept terms or on the torus, into one determinant, and the term of first order in the fluctuation field of the resulting polynomial has to be bounded block by block at the current scale; with the shell mass inside the operator the bound must be uniform in κ, up to the endpoint $\kappa = \infty$. The theorem gives such a bound through the first differences of the background and of the fluctuation field alone, with no Hölder quotient, together with a bound on the shell-mass residual of the current at the blocks of the vertices; in the proof of Proposition 24.7 it is used for the first-order term of the step determinants. The uniformity comes from form monotonicity in the shell mass, the diamagnetic bound and the coercivity $-\Delta_B^N + a_i Q^*Q \geq \gamma_0$ at regular backgrounds [4, (1.8)], which gives Agmon-type decay of the resolvent columns, uniformly in κ, in the spacing, in Ω and in the regular background.

Theorem 24.6 (the first-order determinant term, uniformly in the shell mass: (DOM))**.** *Let $i \leq K_f$ be a scale of the flow, work in the units of the blocks of scale i and put $e_i^2 := e^2\ell_i$. Let Ω be a union of blocks of scale i with Neumann conditions on $\partial\Omega$, one of the two domains $B^i(\Lambda_0^{(i)})$, $B^i(\Lambda_2^{(i)})$ of the kept terms of Bałaban's upper bound at scale i [2, (2.104)–(2.107)] or the torus T, whose sides are at least M/ℓ_i blocks. Let $M_\Omega^\kappa(A)$ (an operator, never the block integer) be the operator of the composed step on Ω at the photon field A, that is, the covariant Laplacian with Neumann conditions on $\partial\Omega$, the scalar potential with the wall and the shell mass, and the block term of scale i, and put $F_\Omega^\kappa := \ln\det M_\Omega^\kappa$. Let the first differences of the background B be at most $Cp(\ell_i)$ [2, (2.98)], with no bound on $|B|$, and let the fluctuation field a obey $|a| \leq Cp(\ell_i)$ with first differences at most $Cp(\ell_i)$ [2, (2.86)], [3, p. 412].*

Then the factors $Z^{(j)}$, $j < i$, of the earlier steps compose on Ω, and their product equals $c\,(\det M_\Omega^\kappa(A))^{-1/2}$ for a constant c independent of A and κ. The term of first order in the fluctuation field a of the kept polynomial at the background B is $-\frac{1}{2}DF_\Omega^\kappa(B)[a]$; here (B,a) is $(\tilde B, \theta_{i+1}A'^{(i)})$ on $B^i(\Lambda_0^{(i)})$ and $(B^{(i+1),\eta}, g_i A'^{(i)})$ on $B^i(\Lambda_2^{(i)})$ [2, p. 579]. Bałaban's final block size being small enough, there are $C, C_\sigma, \delta > 0$, independent of $\kappa \in [0,\infty]$, of the spacing, of i, of Ω and of B (depending on the parameters of the model, among them m, the wall height n and μ_0, and C_σ on σ), such that:

(a) $DF_\Omega^\kappa(B)[a] = \sum_{x\in\Omega} d_x^\kappa$ with $d_x^\kappa := -\int_0^\infty Dg_x^{\kappa,s}(B)[a]\,ds$, where

$$g_x^{\kappa,s}(B) := (\varepsilon/\ell_i)^3\,\mathrm{Tr}(M_\Omega^\kappa(B)+s)^{-1}(x,x)$$

is invariant under every gauge transformation of B on the sites of Ω; and for every block Q of scale i in Ω (the blocks at the boundary of Ω, next to its holes and at its re-entrant edges included), every $r \geq 1$ and every $\sigma > 0$,

$$\sum_{x\in Q}|d_x^\kappa| \leq Ce_i^2\Big[\sup_{Q_r}|\partial B|\sup_{Q_r}|\partial a| + e^{-\delta r/2}\sup_{\Omega}|\partial B|\sup_{\Omega}|\partial a|\Big] + C_\sigma e_i^2 p(\ell_i)^2\ell_i^\sigma,$$

Q_r the set of points of Ω joined to Q by a path in Ω of length $\le r$; in particular the left side is $\le Ce_i^2 p(\ell_i)^2$, and the same holds for the shell-mass residual $d_x^\kappa - d_x^0$; no Hölder quotient of B or a enters;

(b) on the two domains of the kept terms write $DF_\Omega^\kappa(B)[b] = \sum_{y\in\Omega} b(y)\cdot j_B^\kappa(y)$; at every block Q carrying a vertex of the kept terms (which lies deep inside the domain, at distance $> r(\ell_i) - 1$, respectively $\ge 5r(\ell_i) - 5$, from its complement, $r(\cdot)$ Bałaban's localisation function [2, (2.7)]), if the Hölder quotients of the first differences of B are $\le Cp(\ell_i)$ at the points at distance $< r(\ell_i)$ from Q, then $\sum_{y\in Q}|j_B^\kappa(y) - j_B^0(y)| \le Ce_i^2 p(\ell_i) + C_\sigma \ell_i^\sigma$ for every $\sigma > 0$, and the zero-shell-mass current j_B^0 obeys Bałaban's per-step bound [1, Prop. 3.2, (3.58)], [2, pp. 578–579];

(c) on the torus T, besides (a), the bound of (b) holds at every block of T at King's backgrounds, whose size is bounded by [25, (3.2), (A.1)].

Proof. The Gaussian normalisations of the earlier steps telescope, and the one-shot form $\Delta^{(i)} = a_i - a_i^2 Q_i G_i Q_i^*$ gives $\prod_{j<i} Z^{(j),\kappa}(\Omega, A) = c \det M_\Omega^\kappa(A)^{-1/2}$ with $M_\Omega^\kappa = -\Delta_{A,\Omega}^N + V^\kappa + a_i Q_i^* Q_i$; Jacobi's formula gives $DF_\Omega^\kappa(B)[a] = -\lim_{S\to\infty}\int_0^S \sum_x Dg_x^{\kappa,s}(B)[a]\,ds$, and the shell mass, a scalar potential, enters $F_\Omega^\kappa - F_\Omega^0$ locally.

For (a), since $g_x^{\kappa,s}$ is gauge invariant with $Dg_x(0)[a] = 0$, after per-site gauge transformations $Dg_x^{\kappa,s}(B)[a]$ is controlled by the first differences of B and a alone, through the resolvent column $(M_\Omega^\kappa(B) + s)^{-1} e_x$ and its covariant differences, whose Agmon-type decay is uniform in κ, the spacing, Ω and the regular background, as recalled before the statement. Integration over s and summation over $x \in Q$ give (a). Clauses (b) and (c) follow by comparing the currents on the domain and on the torus at the blocks of the vertices and, at $\kappa = 0$, from Bałaban's per-step bound. □

The second bound is the perturbative bound. Its statement needs the perturbative part of the effective action of the shell model, a norm on the external scalar fields adapted to the shell mass, and some of Bałaban's notation, which we introduce first.

The perturbative part. The perturbative part $P^{(k),\kappa}$ of $S^{(k),\kappa}$ is the sum of the derivatives up to coupling order $\bar n = 13$ of King's generating function [25, (3.17)], built from King's vertices [25, (3.18)–(3.26)] unchanged, with the single-scale slices $G_{(j)}^\kappa$, $j < k$, of the shell model as matter covariances, King's photon covariances and the external legs $\phi_\kappa^{(k)} = a_k \ell_k^{-2} G_k^\kappa Q_k(A)^* \phi_k$: a sum over connected graphs with vertices localised in unit-k cubes $\square(v)$, internal matter and photon lines, and external fields, scalar ones $\Phi_{\rm ext}$ (legs, or internal lines of a larger graph treated as external fields) and photon ones $A_{\rm ext}$ (Bałaban's background fields), grouped by order into Bałaban's renormalised families $G_{\rm ren}$ (Definition 5.6, [3, §§2–3]). $E^\kappa(G_{\rm ren}, \{\square(v)\}, \Phi_{\rm ext}, A_{\rm ext})$ denotes the coefficient of a family with its vertices localised in the cubes $\square(v)$, $E^\kappa(H_{\rm ren}, A^{(k)}; \{y_i\}, \{z_q\})$ King's block-data form [25, (3.38)] (external scalar legs the block fields at the blocks y_i, external photon legs at z_q), and for a vacuum family F (no external field) $F(B; Q)$ the family total with the root vertex localised in the unit-k cube Q, every matter line the full step-k line at the background B. The *recentring constant* E_1^κ is King's [25, (3.35)] for the shell model, the constant whose addition to the action makes the Taylor coefficients to order $\bar n$ in (e, λ) of $\ln \int (dA)(d\phi)\, e^{-S_\kappa^\varepsilon + E_1^\kappa}$ vanish, and C_k King's regularity constant for the background at scale k, $|\partial^\eta A^{(k)}| + [\partial^\eta A^{(k)}]_\alpha \le C_k p_{k-1}$ on his small-field set [25, (3.2)]. The counterterms are subtracted κ-adaptedly: for each two-scalar-leg subgraph G_0 of King's list [25, (3.32)] the counterterm vertex of the shell model is $\delta m^2_{\kappa,G_0}(x) := \sum_{x'} \eta^d \Gamma^\kappa_{G_0}(x, x')$, the zeroth moment of the same graph with the shell-model propagators at zero background on the torus, so that the local subtraction is exact; the difference to the bulk counterterm, $\Delta\delta(x) = \delta m^2_{\rm bulk} - \delta m^2_\kappa(x)$, is a finite sum of

differences of graphs, each carrying a factor $w_j \mathbf{1}_U$, $w_j = \kappa \ell_j^2$ the shell mass in unit-j units, by the resolvent identity $G_j^\kappa - G_j^0 = -G_j^0\, w_j \mathbf{1}_U\, G_j^\kappa$, and is carried by the mismatch vertex $\mathcal{M}$ (Lemma 24.1).

Amplitude and half-amplitude norm. At a heavy scale j set, in unit-j units, with η_j the spacing and fixed $C_d \geq 8$, $\delta_j^\flat > 0$,

$$\rho_j(x) := \begin{cases} \min\{1,\ \operatorname{dist}(x, \partial U) + w_j^{-1/2}\}, & x \in H', \\ \dfrac{w_j^{-1/2} + \eta_j}{1 + w_j \eta_j^2}\, C_d\, e^{-\delta_j^\flat (\operatorname{dist}(x,\partial U) - 3\eta_j/2)_+} + w_j^{-1}, & x \in U, \end{cases} \tag{55}$$

and $\rho_j \equiv 1$ at a light scale: the *amplitude*, so that every leg u of block data b obeys $|u(x)| \leq C|b|_\infty \rho_j(x)$ (the linear vanishing of the minimiser at the heavy shell); $\rho_j \leq C_\rho$ everywhere, and on U in the continuum regime $w_j \eta_j^2 \leq 1$, $\rho_j(x) \leq C w_j^{-1/2} e^{-c\sqrt{w_j}\operatorname{dist}(x,\partial U)} + w_j^{-1}$. With the Hölder exponent $\alpha \in (0, \frac{1}{2}]$ of the norm, fixed once for all (we take $\alpha = \frac{1}{3}$; α_0 denotes the minimal ordered degree of Theorem 20.4), and a fixed rate $\delta_4 > 0$, the *half-amplitude norm* of a function Φ at scale j relative to a source set S ($S = T^{(j)}$ for true external legs, where the decay weight is 1) is

$$\begin{aligned} \|\Phi\|_{\kappa,j;S} := \sup_x \frac{e^{\delta_4 \operatorname{dist}(x,S)} |\Phi(x)|}{\rho_j(x)^{1/2}} + \sup_b e^{\delta_4 \operatorname{dist}(b,S)} |D_A \Phi(b)| \\ + \sup_{0<|x-y|\leq 1} e^{\delta_4 \operatorname{dist}(x,S)} \rho_j(x)^{1/2}\, \frac{|U(A(\Gamma_{x,y}))D_A\Phi(y) - D_A\Phi(x)|}{|x-y|^\alpha}. \end{aligned} \tag{56}$$

At light scales this is Bałaban's norm [3, (1.32)] with decay. Its two properties are the *leg bounds*, $\|a_j \ell_j^{-2} G_j^\kappa Q_j^* b\|_{\kappa,j;\operatorname{supp} b} \leq C|b|_\infty$ and $\|G_{(j)}^\kappa(\cdot, z)\|_{\kappa,j;\{z\}} \leq C\rho_j(z)^{1/2}$ uniformly in $\kappa \in [0,\infty)$, and the *pairing property*: in a product of two legs the Hölder quotient of one, carrying the weight $\rho_j^{1/2}$, is paired with the half-amplitude $\rho_j^{1/2}$ of the other, so that the pair is bounded by the product of the two norms times the full amplitude ρ_j, in place of every heavy-side Hölder use of [3].

Notation of the bound. In Bałaban's notation h, h' are the localisations, d_v, d_s the orders in e and λ and $d(\cdot)$ the length of the shortest tree through the cubes; the source term $zF(f)$ enters as an external photon field, with f-dependent constants. On the photon side nothing is converted: $A^{(k)}$ enters every family of $P^{(k),\kappa}$ only through lines, transporters, covariant differences, block averages, legs and the plaquette curl of the source vertex, so it is the regular line field B of the norm form below, and no sup bound $|A_k| \leq Cp$ uniform in the scale is used. $P_0^{(k),\kappa}$ denotes the field-independent part of $P^{(k),\kappa}$, its value at zero block fields, every term of which carries a line of scale $\geq k$, and $\operatorname{gen}_j(\square)$ the constants generated at step j on a unit-j cube $\square$.

The perturbative bound comes in two forms, King's block-data form (a), for legs of block data in King's small-field set, and Bałaban's norm form (b), in which the scalar external fields are measured in the half-amplitude norm (56), with the leg bounds and the pairing property stated after it; clause (c) is the gain of a factor $C(\lambda + e^2)\ell_k$ at every mismatch vertex, and clause (d) controls the field-independent part per step and through its dependence on the background, no absolute bound per block being claimed. In the proof of Theorem 24.5 the proposition, with Theorem 24.6, stands in place of [3, Proposition 1]; its leg estimate enters the class of Lemma 24.8(c), and the estimates of Section 25 draw on it. The proof repeats Bałaban's [3, Proposition 1] for the shell model; the bound differs from his [3, (1.33)] only in the norm on

the scalar external fields, and the wall profile enters only through the scalar propagators, whose bounds improve under a non-negative potential.

Proposition 24.7 (the perturbative bound for the shell model)**.** *There are $\delta_0 > 0$ and constants $O(1)$ depending on α, $\bar{n}$, d, L, a, N, on (m, n, μ_0) through King's constants at the one fixed wall height, and, for the families containing a source vertex* [25, (3.20)–(3.21)]*, on the sources through $\|f\|_{C^1}$ and $\|g\|_\infty$ (a factor $(1+\|g\|_\infty)^{n_h}$ for n_h vertices* [25, (3.21)]*), but* not *on ε, k, $\kappa \in [0,\infty)$, the domains $\Omega, \Omega_1, \Omega_2$ of the construction, or the regular background field B, such that:*

(a) (block-data form) For every renormalised family H_{ren} of $P^{(k),\kappa}$ of order ≤ 13 with at least one external field, with the shell-model propagators and the legs $\phi_\kappa^{(k)}$ of block data in King's small-field set [25, (3.2)]*, of whose photon clauses only the derivative clause $|\partial^\eta A^{(k)}| + [\partial^\eta A^{(k)}]_\alpha \le C_k p_{k-1}$ is used, through the regularity of the line field,*

$$|E^\kappa(H_{\mathrm{ren}}, A^{(k)}; \{y_i\}, \{z_q\})| \le O(1)\,(L^k\varepsilon)^\rho\, p(L^k\varepsilon)^{n_1}\, e^{-\delta_0 \operatorname{dist}(\{y_i\},\{z_q\})}, \qquad \rho \ge s/4, \tag{57}$$

with King's exponents ρ, n_1, s [25, (3.38)]*, in King's ℓ^1-in-localisations form; the vacuum family is excluded, the field-independent part being recentred by E_1^κ (clause (d)).*

(b) (norm form) On the torus, for generalised graphs of any finite order, with constants depending on their size, with external photon fields satisfying Bałaban's conditions, their Hölder clause included, in Bałaban's form [3, (1.33), Proposition 2.2] *with the half-amplitude norm on the scalar external fields and the norm $\|h' A_{\mathrm{ext}}\|_{1,\alpha}$ on the photon ones,*

$$\begin{aligned} |E^\kappa(G_{\mathrm{ren}}, \{\Box(v)\}, \Phi_{\mathrm{ext}}, A_{\mathrm{ext}})| \le O(1)\,(e\ell_k^{1/2})^{d_v} (\lambda\ell_k)^{d_s}\, e^{-\delta_0 d(\{\Box(v)\})} \\ \times \prod \|h\Phi_{\mathrm{ext}}\|_{\kappa,k;S_{\mathrm{ext}}} \prod \|h' A_{\mathrm{ext}}\|_{1,\alpha}. \end{aligned} \tag{58}$$

For King's legs $\|h\phi_\kappa^{(k)}\|_{\kappa,k} \le C|\phi_k|_\infty$, so (b) implies (a) for true legs.

(c) (mismatch factor) On the torus, at King's backgrounds $B = uA^{(k)}$, $u \in [0,1]$, A_k in the small-field set, every graph containing $r \ge 1$ mismatch vertices $\mathcal{M}$ (Definition 16.9) and at least one external field obeys the bound (58) with the additional factor $(C(\lambda+e^2)\ell_k)^r$, C free of $\kappa \in [0,\infty)$, ε, k and A_k (with a counterterm offset c, $C(\lambda+e^2)$ is enlarged by a constant multiple of Δ_c, as in Lemma 24.1), for the family total over the scale assignments of the graph, not per member.

(d) (vacuum sector) $P_0^{(k),\kappa}$ is controlled per step: $|\mathrm{gen}_j(\Box)| \le C\ell_j^{1/2}$ on every unit-j cube, uniformly in κ and ε, so that $|P_0^{(k),\kappa}(X) - P_0^{(K_f),\kappa}(X)| \le C'\ell_k^{1/2}|X|_k$ for every union X of unit-k cubes, and the common constant $P_0^{(K_f),\kappa}(T)$ cancels between the two lines of Theorem 24.5; no absolute bound on $P_0^{(k),\kappa}$ per block, free of the final spacing, is claimed or used. At a regular block configuration A_k (only derivative bounds of $A^{(k)}$ enter), every renormalised vacuum family of order ≤ 13 with $r \ge 0$ mismatch vertices obeys, per unit-k root cube $\Box$,

$$\begin{aligned} |E^\kappa(H_{\mathrm{ren}}, A^{(k)}; \Box) - E^\kappa(H_{\mathrm{ren}}, 0; \Box)| &\le O(1)(e\ell_k^{1/2})^{d_v}(\lambda\ell_k)^{d_s}\big(C(\lambda+e^2)\ell_k\big)^r\big(1+C^\flat p(\ell_k)\big)^{n_1} \\ &\le O(1)\,\ell_k^{\beta_0}, \end{aligned} \tag{59}$$

uniformly in $\kappa \in [0,\infty]$, ε and k, $C^\flat$ fixed by the cube side of the cube step. For a source-free renormalised vacuum family F of order ≤ 13 with $r \ge 1$ mismatch vertices, a unit-k cube Q and two backgrounds B_0, B_1 of King's class $R_k(C_R)$, with $b := B_1 - B_0$

and $N_k(b) := \|\partial b\|_\infty + [\partial b]_\alpha + e^{-\delta_0 N_T/16}\|b\|_\infty$, N_T *the smallest side of* $T^{(k)}$,

$$|F(B_1;Q) - F(B_0;Q)| \le C_{\mathrm{vac}}\left(C(\lambda+e^2)\ell_k\right)^r e_k\, N_k(b),$$

with C_{vac} *depending on* C_R, *King's constants* (μ_0 *included*), $\bar n$, L *and* α, *and not on* $\kappa \in [0,\infty]$, ε, k, B_0, B_1, Q *or the torus.*

Proof. For King's legs $\|h\phi_\kappa^{(k)}\|_{\kappa,k} \le C|\phi_k|_\infty \le Cp(\ell_{k-1})(\lambda\ell_{k-1})^{-1/4}$ on the small-field set, Bałaban's bound on his class [3, p. 421]. Since the bound (58) differs from Bałaban's [3, (1.33)] only in the norm on the scalar external fields, every consequence of (1.33) holds for the shell model with the same constants: the coefficients of [1, Propositions 3.1–3.2] are $O(\ell_k^{\beta_0})e^{-\delta_0 d}$ with $\beta_0 = \frac14 - \beta$, $\beta \in (0, \frac{1}{28})$ the exponent of King's regularity condition [25, (3.5)], fixed once for all, and the remainders $O(1)\ell_k^{\kappa_*}|T^{(k)}|$ with $\kappa_* = \min(\frac{13}{4} - \beta, 14\beta_0) > 3$. The proposition is thus [3, Proposition 1] run for the shell model with the leg bounds and the pairing property of (56) wherever the shell mass enters a line or leg bound, the Taylor step at a divergent two-scalar-leg subgraph (its linear term is the κ-adapted counterterm), Lemma 25.2 for the photon and mixed classes, the mismatch vertex as an additional vertex (Lemma 24.1, Theorem 25.6) and the step determinants per step [1, (3.46)–(3.47)], [2, (2.99)–(2.107)], their first-order term by Theorem 24.6, their second-order term as a photon two-leg class, their third-order term by the gauge identity of the full determinant in a slot of the fluctuation field, and the absolute ratio only at the final scale [1, (3.66)–(3.69)].

Clause (c) is proved as Theorem 25.6: the closed matter blocks with an odd number of photon-type legs are first rewritten by charge conjugation on their full step-k lines, no Ward identity being used for a class containing $\mathcal{M}$. Finally, on the propagator sub-domains the torus form passes to the vertices at depth $\ge R_{\mathrm{Ba}}$ [3, pp. 411–412], [2, pp. 570, 579–580], with the boundary-difference clause [4, (1.20)] through the layer-source identity $\delta G_k^\kappa(\Omega,\Omega_2)F = G_k^\kappa(\Omega_2)\,\sigma$. □

The third bound is the class of small fields carried through the lower bound. Bałaban's lower bound [1, §3] is an induction over the scales on such a class, King's class (A.1) of [25, p. 676]; with the shell mass inside the covariance its clause on the covariant Laplacian of the background fails near ∂U at heavy scales, and we replace it by the two consequences that the power counting uses, a bound on the covariant gradient of the background and a bound on the mismatch between the block field and the block average of its background. The lemma defines the class and shows that it is what the induction needs: the backgrounds compose exactly from one scale to the next; the class at scale $i+1$ together with a small fluctuation gives the class at scale i, under lower bounds on its constants and caps on Bałaban's final spacing that give the remainder of the expansion of the background in the fluctuation gauge field its share of the room, as his compatibility [1, (3.53)–(3.54)] requires; Bałaban's coefficient bounds hold on it; and the supports of the stopping-scale cut-offs of Theorem 24.5, the capped set included, lie in it, which is what fixes the ranges of (52) and (51) and gives the cap form. The room in the three clauses comes from the ratio of their thresholds at consecutive scales, and the clause on the stopping scale is King's near-face analysis at the shell face.

Lemma 24.8 (the class for the shell model, with the gradient and mismatch clauses)**.** *Let* G_i^κ *be King's operator* [25, (2.13)–(2.17)] *at scale* i *with the profile* V *of* (48) *in place of* m^2, $\phi_\kappa^{(i)} := a_i\ell_i^{-2}G_i^\kappa Q_i^*\phi_i$ *the* background *(the minimiser of the one-step functional) of the block field* ϕ_i, $Q_i = Q_i(A^{(i)})$ *the covariant block average,* $p_1(t) := b_1(1+\log t^{-1})^{p_{\mathrm{fl}}}$, $b_1 \ge b_0$, $p_{\mathrm{fl}} \le p-1$, *Bałaban's bound on the fluctuation field* [1, (3.50), (3.54)], $\varkappa_i := aL^{-2}/(a_i + aL^{-2}) \le L^{-2}$, *and* $C_{\sup}$, C_D, C_{Q4} *the constants, independent of* κ, ε *and* i, *in* $|\phi_\kappa^{(i)}|_\infty \le C_{\sup}|\phi_i|_\infty$, $\sup_b |D_A u(b)| \le C_D|b|_\infty$ *and* $|Q_i(A)u|_\infty \le C_{Q4}|b|_\infty$ *for the leg* $u = a_i\ell_i^{-2}G_i^\kappa Q_i^* b$ *of block data* b. *For* $i \le K_f$ *let*

$(\mathrm{A}.1)_\kappa^{(i)}$ *be the class of pairs* $(A^{(i)},\phi_i)$ *such that, in unit-i units,*

$$\sup_x|\phi_\kappa^{(i)}(x)|\le C_1M_i,\qquad \sup_{y\in T^{(i)}}|\phi_i(y)|\le C_1M_i,$$
$$(\mathrm{G})_i:\ \sup_b\big|D_{A^{(i)}}\phi_\kappa^{(i)}(b)\big|\le C_1'p_i,\qquad (\mathrm{E})_i:\ \sup_{y\in T^{(i)}}\big|\phi_i(y)-Q_i(A^{(i)})\phi_\kappa^{(i)}(y)\big|\le C_1''p_i,$$
$$(\Gamma)_i:\ |\partial A^{(i)}|_\infty\le C_\Gamma' p_i,$$

the supremum in $(\mathrm{G})_i$ *over all fine bonds* b *on both sides of* ∂U*, with no clause on* $\Delta_A\phi_\kappa^{(i)}$ *and no near-U clause;* $(\mathrm{G})_i$ *is the* background-gradient clause *and* $(\mathrm{E})_i$ *the* block-mismatch clause*, both linear in* ϕ_i*, and* $(\mathrm{E})_i$ *is the block average of the Laplacian clause, since* $a_i(\phi_i-Q_i\phi_\kappa^{(i)})=Q_i[(-\Delta_A+V)\phi_\kappa^{(i)}]$ *exactly. Then, in unit-i units:*

(a) (exact composition) Under the translation $\phi_i=aL^{-2}C^{(i),\kappa}Q^*\psi+\phi_i'$ *of the* $(i+1)$*-st fluctuation step,* $\psi=\phi_{i+1}$*, the backgrounds compose exactly,* $\phi_\kappa^{(i)}(\phi_i)=\phi_\kappa^{(i+1)}(\psi)+a_i\ell_i^{-2}G_i^\kappa Q_i^*\phi_i'$*, and* $|a_i\ell_i^{-2}G_i^\kappa Q_i^*\phi_i'|_\infty\le C_{\sup}|\phi_i'|_\infty$*.*

(b) (compatibility with coefficient one) If $\psi\in(\mathrm{A}.1)_\kappa^{(i+1)}$ *and* $|\phi_i'|_\infty\le p_1(\ell_i)$*, then* $\phi_i\in(\mathrm{A}.1)_\kappa^{(i)}$*, provided* $L^{3/4}\ge 3$ *(true for every integer* $L\ge5$*),*

$$C_1\ge 3\max(C_{\sup},1)\max(1,b_1/b_0),\qquad C_1'\ge 3C_D\max(1,b_1/b_0),$$
$$C_1''\ge\max\{2(1+C_{Q4})\max(1,b_1/b_0),\,C_1'\},$$

and the three caps on Bałaban's final spacing $\varepsilon_0^{\mathrm{Ba}}$ *(chosen after* C_1*,* C_1'*,* b_1*,* p_{fl}*) for the remainder of the expansion of the background in the fluctuation gauge field, with* C_{rem} *the remainder constant of* (60)*,*

$$C_{\mathrm{rem}}\,e\,C_1\lambda^{-1/4}(\varepsilon_0^{\mathrm{Ba}})^{1/4}p_1(\varepsilon_0^{\mathrm{Ba}})\le\tfrac16C_1',\qquad C_{\mathrm{rem}}\,e\,(\varepsilon_0^{\mathrm{Ba}})^{1/2}p_1(\varepsilon_0^{\mathrm{Ba}})\le\tfrac16,$$
$$C_{\mathrm{rem}}\,e\,p_1(\varepsilon_0^{\mathrm{Ba}})\big(\lambda^{-1/4}(\varepsilon_0^{\mathrm{Ba}})^{1/4}+(C_1'/C_1)(\varepsilon_0^{\mathrm{Ba}})^{1/2}\big)\le\tfrac13,$$

together with $\varepsilon_0^{\mathrm{Ba}}\le e^{1-4p_{\mathrm{fl}}}$*, so that by monotonicity they hold at every scale* $\ell_i\le\varepsilon_0^{\mathrm{Ba}}$*.*

(c) (the fluctuation expansion closes) On $(\mathrm{A}.1)_\kappa^{(i)}$ *the external legs satisfy* $\|\phi_\kappa^{(i)}\|_{\kappa,i}\le CC_1M_i$ *in the half-amplitude norm of Proposition 24.7,* $\lambda\ell_i|\phi_\kappa^{(i)}|^4\le C_1^4p_i^4$ *pointwise, and at every block* y*, every* $x\in B(y)$ *and every fine bond* b*, (P-grad)* $|D_{A^{(i)}}\phi_\kappa^{(i)}(b)|\le C_1'p_i$ *and (P-block)* $|\phi_\kappa^{(i)}(x)-U(A^{(i)}(\Gamma_{x,y}))\phi_i(y)|\le C_Pp_i$*,* $C_P:=C_1''+2dC_1'$*,* $U(A^{(i)}(\Gamma_{x,y}))$ *the transport along a path from* y *to* x *inside the block. These are the conditions under which Bałaban's coefficient bounds* [1, (3.58)], [3, p. 421] *hold for the vertex families with external scalar legs (the current vertex through (P-grad), the kernel vertices through (P-block), every other family through the sup clause alone), hence under which* [1, Propositions 3.1–3.2]*, taken to order* $\bar n=13$*, hold for the shell model with remainder terms* $O(1)\ell_i^{\kappa_*}|T^{(i)}|$*,* $\kappa_*=\min(\frac{13}{4}-\beta,14\beta_0)>3$*, and coefficients* $O(1)\ell_i^{\beta_0}$*,* $\beta_0=\frac14-\beta$*, with* $\beta<\frac{1}{28}$ *the exponent of King's regularity condition* [25, (3.5)]*, as in Proposition 24.7.*

(d) (final restriction) At the stopping scale k*, with* $M_k'=L^{1/4}(p_{k-1}/p_k)M_k$ *the block-sup threshold of* [25, (3.2)]*, as in Theorem 24.5: (i) for* $w_k\ge c_0$*,* $\mathrm{supp}(\chi_k^\flat\tau_k\lambda_k)\subset(\mathrm{A}.1)_\kappa^{(k)}$*, and the same for the larger cap set* $\mathrm{supp}(\chi_k^\flat\tau_k^{\mathrm{cap}}\lambda_k^{\mathrm{cap}})$ *of Definition 24.2, once* $C_1\ge\max(C_{\sup},1)L^{1/4}(1+\log L)^p$*,* $C_1'\ge C_{\mathrm{G}}(1+\log L)^p$ *and* $C_1''\ge C_E(1+\log L)^p$*, with constants* C_{G}*,* C_E *free of* κ*,* ε *and* k*, provided* $(m^2+n)\ell_k^2M_k'\le p_k$ *and* $e_kM_k\le1$ *at* $\ell_k=\eta_0$ *(both are conditions on* η_1*); (ii) for every* $\kappa\in[0,\infty)$*,* $\mathrm{supp}(\chi_k\tau_k\lambda_k)\subset(\mathrm{A}.1)_\kappa^{(k)}$ *under the conditions of (i), those of the w-linear bound on the* κ*-difference of the background (the*

derivative clause for small-field data, $\ell_k \le \varepsilon_0^{\mathrm{Ba}}$, Bałaban's radius constant $R \ge R_*$, the largest of the lower bounds on R of the cut-off listed in Definition 25.3) and with C_1' at least the constant of that bound; (iii) at the final scale, $\{|\phi_{K_f}(y)| \le 1\ \forall y\} \subset (\mathrm{A}.1)_\kappa^{(K_f)}$.

Proof. Clause (a) is the composition law of the kernels $\exp(-\frac{a}{2}\ell^{d-2}|\psi - Q\phi|^2)$, V entering only through $-\Delta_A + V$, with the κ-uniform sup bound of the legs.

For (b), in unit-i units the thresholds of $(\mathrm{A}.1)_\kappa^{(i+1)}$ are at most $L^{-3/4}$, $L^{-3/2}$ and $\varkappa_i L^{-1/2} \le L^{-5/2}$ times those of $(\mathrm{A}.1)_\kappa^{(i)}$ for the sup clauses, (G) and (E) (the last by the push-down identity $aL^{-2}C^{(i),\kappa}Q^*\psi = Q_i\Phi_\psi + \varkappa_i Q^*(\psi - Q_{i+1}\Phi_\psi)$, $\Phi_\psi = \phi_\kappa^{(i+1)}(\psi)$), and each room is split into three shares, for the scale-$(i+1)$ clause, the fluctuation leg and the remainder δ of the expansion of the background in the fluctuation gauge field $A' = A'^{(i)}$,

$$
\begin{aligned}
&G_i^\kappa(A'+B) - G_i^\kappa(B) = G_i^\kappa(A'+B)\, D_B^*\big(U(A')-1\big) D_B\, G_i^\kappa(B),\\
&|U(A')-1| \le e\eta_i|A'|, \qquad |\delta|_\infty \le C_{\mathrm{rem}}\, e\, p_1(\ell_i)\big(C_1\lambda^{-1/4}\ell_i^{1/4} + C_1'\ell_i^{1/2}\big)p_i,
\end{aligned}
\tag{60}
$$

with $|A'|_\infty, |\partial A'|_\infty \le C_\gamma p_1(\ell_i)$ on Bałaban's fluctuation set.

In (c) the norm bound is the leg estimate of Proposition 24.7, (P-grad) is $(\mathrm{G})_i$ and (P-block) follows from $(\mathrm{E})_i$ and $(\mathrm{G})_i$ along a path inside the block. Clause (d) is King's near-face analysis [26, (5.41)–(5.49)], [26, §5.2] at the shell face, with $\phi_\kappa^{(k)} = v + \Delta\phi$, v the background at $\kappa = 0$ controlled by [2, Lemma 2.4, (2.65)] and an interior estimate, $|\Delta\phi| + |D_A\Delta\phi| \le C_\Delta p_{k-1}$ on every block, whence $C_{\mathrm{G}} := C_{G0} + C_\Delta$ and $C_E := C + C_\Delta$; (ii) follows by the w-linear route $\Delta\phi = -(M_k^0)^{-1}(w_k \mathbf{1}_U \phi_\kappa^{(k)})$ for $w_k < c_0$, and (iii) from $1 \le M_{K_f}$ and $C_{\mathrm{sup}} \le C_1$. □

Bałaban's four replaced objects. With the three bounds in place, Section 25 runs Bałaban's sequence of estimates at block factor 5 [2, 3] on the shell model (Definition 23.2; the *walled operator* is the operator of the model at $\kappa = 0$) with the four replaced objects, in the following notation. The steps of his upper and lower bounds [2, §§2–3] are numbered D5 (translation and splitting of the fields [2, (2.18)–(2.19)]), D6 (expansion and cumulants [2, (2.22)–(2.28), Prop. 2.6]), D7 ([2, Prop. 2.1, (2.57), (2.79)], the collar quartic), D8 ([2, Lemmas 2.3–2.4, Prop. 2.2], the backgrounds), D9 ([2, Lemmas 2.5, 2.7], the fluctuation fields), D10 (the Z-factors and the tail removal [2, (2.92)–(2.109)]), D11 ([2, (3.11)–(3.12)]), D12 ([2, Prop. 3.1, (3.26)]), D13 ([2, (3.32)–(3.38)]), D14 (the combinatorics [2, §3C, (3.51)–(3.54)]) and D15 ([2, (2.116)–(2.118)]). The properties (Z1)–(Z7) are those of the cut-off ζ_θ of [2, (2.44)] at $L = 5$ with $\theta = \frac{2}{5}$ under $R \ge \max\{5(2M+1), C_Z(M+s_b)\}$ and $R \ge 10(M+1)$ ($M_\theta = 2M/3$ the thickness of the transition slice of his cut-off θ_k [2, (2.88)–(2.90)]): the ζ_θ-ball of every point of $B^k(\Lambda_2^{(k-1)\prime})$ lies where [2, (2.55)] holds; [2, Lemma 2.3] and its other uses hold with ζ_θ, its bounds [2, (2.59)–(2.60)] with $O_L(p(\ell_k))$ in place of $O(p(\ell_k))$; the terms carrying $1-\zeta_\theta$ in [2, (2.61), (2.64), (2.88)–(2.90)] are $O(\ell_k^s)$ for every s; the transition width $\frac{1}{2}\theta r(\ell_k) - 2M$ is at least 1, giving $|\partial^\eta\zeta_\theta| \le 1$, and at least 2 under $R \ge 10(M+1)$, which $|\partial_\nu^\eta\partial_\mu^\eta\zeta_\theta| \le 2$ needs; $L\theta \le 2$, under which the split [2, (2.89)] is exact; and $\delta A := a_k(1-\zeta_\theta)G_k Q_k^* A_k$ obeys $\|\delta A\|_\infty + \|\partial^\eta\delta A\|_\infty \le \ell_k^s$ for every s, so that the cut-off changes the all-small-field effective action of [25, Thm. 3.1, (3.4)] by $O(\ell_k^{s'})|T^{(k)}|$. R_{Ba} is Bałaban's constant R_0 of [4, (2.19)] in block units, a fixed number ≥ 2, the depth from the complement of a propagator sub-domain at which his bounds are stated and at which his localisation convention [3, pp. 411–412] keeps the vertices of a family (R_0 itself denotes below the near-layer constant of the seventh family of D7). Every exponential rate used at the four replaced objects is one of finitely many κ-uniform rates of the operator bounds of the shell model (the weighted coercivity, the rates δ_2, δ_3 of the kernels and of their first differences, δ_6 of the heavy-side difference bounds, δ_4 of the norm (56), the photon rates and the rate of the walled background in [2, Lemma 2.4]); δ_{nat} denotes their minimum.

25. Bałaban's estimates for the shell model and the κ-Cauchy property

This section completes the proof of the four shell bounds. Its first part gives Bałaban's estimates for the shell model, on which the two-sided bound of Section 24 rests. Its second part proves the κ-Cauchy property (Theorem 25.11), the threshold clause (Corollary 25.12) and the disc bound (Lemma 25.13).

The proof of the two-sided bound of Section 24 (Theorem 24.5) rests on Bałaban's lower and upper bounds [1, 2, 3] run for the shell model, in which the shell mass sits inside every matter covariance. The first subsection proves them in the form in which that proof uses them, with constants independent of the shell mass: the pointwise bounds on the step lines on both sides of ∂U, the bounds on the classes with photon external legs, Bałaban's sequence of steps for the shell model with its four replaced objects, and the mismatch clause of the perturbative bound. The second subsection proves the κ-Cauchy property $(\mathrm{SW3}^\circ)_{\mathrm{int}}$ of the capped reduced action (Theorem 25.11): at fixed spacing the capped reduced actions at two shell masses differ, up to a constant free of the fields, by a quantity which tends to zero as the shell mass grows, uniformly in the spacing; the mechanism is a bound on the derivative in the shell mass, through the tail of the block-free determinant and King's mechanism. The disc bound ($\mathrm{SW2}^{\min}$) and the threshold clause (SWT) follow there. Together with Theorem 24.5 these are the four shell bounds from which Section 23.4 derives the heavy-shell estimate (Theorem 23.12).

25.1. **Bałaban's estimates for the shell model and the step lines.** This subsection proves the bounds of Bałaban's argument which the proof of Theorem 24.5 uses for the shell model, with constants independent of the shell mass; one mechanism carries it. The shell mass enters the matter covariance as a non-negative scalar potential, $w\mathbf{1}_U$ with $w = \kappa\ell_k^2$ in unit-k units, so every step of Bałaban's and King's arguments that is reduced to resolvent kernels, their B-derivatives, block averages and the composed background of the walled operator holds at every $w \geq 0$ with the constants of $w = 0$. The shell mass changes the argument in three places, the bounds taken near ∂U, Bałaban's pointwise statements deep inside U, which are not uniform in w, and the mismatch vertex, and each is handled below. We proceed from the bottom up: the step lines (Theorem 25.1), the classes with photon external legs (Lemma 25.2), Bałaban's sequence of steps (Definition 25.3, Proposition 25.4, Theorem 25.5) and the mismatch clause of the perturbative bound (Theorem 25.6).

We begin with the step lines. In the notation of the shell model, $d = 3$, T, H' and $U = T \setminus H'$ are unions of j-blocks for every scale $j \leq K_f$, the shell at least three and the box at least two j-blocks thick; in unit-j units $V_j = V_j^0 + w_j\mathbf{1}_U$ with $0 \leq V_j^0 \leq c_V$, $M_j^\kappa(\Omega) = -\Delta_{A,\Omega}^N + V_j + a_j P_j(A)$ on a union Ω of j-blocks with Neumann (or, where said, Dirichlet) faces, $P_j(A) = Q_j(A)^* Q_j(A)$, $G_j^\kappa(\Omega) = M_j^\kappa(\Omega)^{-1}$, and the *leg* of block data b is $a_j G_j^\kappa(\Omega) Q_j(A)^* b$. The gauge field A is *regular* at scale j in the sense of [25, (3.5)], $|\partial A| \leq C(e\ell_j^{1/2})^{\beta-1}$ in unit-j units, $\beta \in (0,1)$ King's regularity exponent fixed in the proof of Proposition 24.7; regularity at scale j implies regularity at every scale $i \leq j$ with the same constants. For every $j \geq 1$, $V \geq 0$, A and κ the law of total covariance for the block-spin Gaussian law of (ϕ, ψ) with $H = -\Delta_A + V$ gives the exact scale decomposition

$$G_j^\kappa = G_1^\kappa + \sum_{i=1}^{j-1} G_{(i)}^\kappa, \qquad G_{(i)}^\kappa = a_i^2 \ell_i^{-4}\, G_i Q_i^* C^{(i)} Q_i G_i, \qquad C^{(i)} = \big(\Delta^{(i)} + a\ell_{i+1}^{-2} Q^* Q\big)^{-1}, \tag{61}$$

$\Delta^{(i)} = a_i\ell_i^{-2} - a_i^2\ell_i^{-4} Q_i G_i Q_i^*$ (King's (2.17) [25]), which uses neither the regularity of A, the block alignment nor κ. Fix $j \leq K_f$, let $\eta = L^{-j}$ be the fine spacing in unit-j units and $G := G_j^\kappa(T)$ the step-j line on the torus, its kernel taken with respect to the measure η^d; D_A is the covariant

forward difference on a fine bond, D^x_A and D^y_A act on the first and on the second variable (on the second through the conjugate transport), $|x-y|_\eta := \max(|x-y|,\eta)$ with the torus distance, and a bond b at x has base point x.

The first theorem bounds the step line G and its covariant differences pointwise, at the scaling of the fine lattice and with exponential decay, with constants free of the shell mass, the spacing, the scale and the gauge field, at every position relative to ∂U. In the proof of Theorem 24.5 these bounds, together with the amplitude bound on the legs, stand in place of King's [25, Theorem 3.3]; in this subsection they carry the line–line part of the photon classes (Lemma 25.2) and, through the bounds on single slices, the cube step of Theorem 25.5, and they are one of the completed steps of Proposition 25.4(g). The proof writes each slice of (61) as a pair of legs, bounds the legs on both sides of ∂U uniformly in the shell mass, and sums the slices.

Theorem 25.1 (pointwise bounds on the step lines)**.** *There are $C, c > 0$ depending only on d, L, a, c_0, c_V and King's constants, and not on κ, η, j or A, such that for every $\kappa \in [0,\infty]$, every scale $j \le K_f$, every A regular at scale j, all fine sites x, y of T (on either side of ∂U, in any position relative to the faces, edges and corners of H') and all bonds b at x, b' at y,*

$$|G(x,y)| \le C|x-y|_\eta^{-1} e^{-c|x-y|}, \tag{62}$$

$$|D^x_A G(b,y)| + |D^y_A G(x,b')| \le C|x-y|_\eta^{-2} e^{-c|x-y|}, \tag{63}$$

$$|D^x_A D^y_A G(b,b')| \le C|x-y|_\eta^{-3} e^{-c|x-y|}. \tag{64}$$

The constants depend on c_0 only through $W_* := \max(64, c_0)$, and no lower bound on c_0 is used. The same bounds hold for every partial sum of the slices of the scale decomposition (61), with the same constants, and for every single slice $G^\kappa_{(i)}$ with constants of the same form (any rate $c < \delta_6$, the rate of the single-scale bounds). No Hölder clause and no second difference in the same variable is claimed: $D^x D^x G$ is not uniform in κ near ∂U, its normal part jumping by about $w_j G$ across the face.

Proof. We first bound one slice, in unit-i units. Every slice is a pair of legs,

$$G'_{(i)}(x,y) = \sum_{u,v \in T^{(i)}} \mathsf{I}(x,u)\, C'(u,v)\, \overline{\mathsf{I}(y,v)}, \qquad \mathsf{I}(\cdot,u) := a_i (M_i^\kappa)^{-1} Q_i(A)^* e_u,$$

so that each covariant difference acts on one leg. The legs obey decaying sup and first-difference bounds on both sides of ∂U, uniform in i, κ, η_i and A: for $w_i > W_*$ from the first-difference bound of the shell model, for $w_i \le W_*$ from the interior estimate on $B_{1/4}(x)$; and $|C'(u,v)| \le C_2 e^{-\delta_1|u-v|}$. Hence

$$|D_x^{n_x} D_y^{n_y} G'_{(i)}(x,y)| \le C_3 e^{-\delta_6|x-y|} \quad (n_x, n_y \in \{0,1\}), \qquad \delta_6 := \tfrac{1}{4}\min(\delta_0,\delta_1,\delta_2,\delta_3).$$

The finest term G^κ_1 obeys the same bound by the Combes–Thomas estimate.

Next we sum over the slices. Rescaled by the side $s_i := L^{i-j}$ of an i-block, the slice bound reads $C_4 s_i^{-1-n_x-n_y} e^{-\delta_6|x-y|/s_i}$, and $\sum_{s\in\Lambda} s^{-\nu} e^{-\delta r/s} \le C \max(r,\eta)^{-\nu} e^{-\delta r/2}$ over $\Lambda = \{L^m\eta \le 1/L\}$ sums the slices to (62)–(64) with $c = \delta_6/2$; the same sum over any set of slices gives the bounds for the partial sums, and the slice bound gives them for a single slice with any rate $c < \delta_6$. At $\kappa = \infty$ the bounds pass to the limit, $(M_j^\kappa)^{-1}$ being decreasing in the Loewner order.

It remains to pass to sub-domains. On a union Ω of j-blocks whose faces lie at least two blocks from the parallel faces of ∂U the same holds in sub-domain form: the heavy-scale amplitude comes from the barrier of the shell model periodised by an image sum, the light scales from the interior estimate, and the gauge field reflected at Neumann faces is regular after a gauge transformation. On Bałaban's scale-dependent propagator domains the per-scale bound is applied domain by domain. □

The step-line bounds do not by themselves control the classes with photon external legs. These are bounded only after the Ward–Takahashi combination of the seagull and exchange graphs, step by step, never for a fixed ordering or scale assignment of the lines: the face seagull term $e^2\sum_x \eta^d[C^{\eta,\kappa}-C^{\mathrm{bulk}}](x,x)|L(x)|^2$ of a class with photon legs L is bounded only after the combination, cancelled against the face part of the exchange graph. The next lemma collects what the perturbative bound of Proposition 24.7 needs from these classes, uniformly in κ: the Ward–Takahashi identities of the shell model (a), the pointwise bound and the moment form behind the Taylor step at a divergent photon two-leg subgraph (b), and the first-moment bounds of the two-photon and mixed kernels with the transport bound (c). The identities follow from gauge invariance alone; the bounds rest on the step-line bounds of Theorem 25.1 on both sides of ∂U, on Bałaban's bulk statements read at $\kappa=0$ with the wall and on heavy suppression on the shell side.

Lemma 25.2 (the photon two-leg and mixed classes of the shell model)**.** *(a) (Ward identities) The matter action of the shell model, $\frac12\langle\phi,(-\Delta_A+V)\phi\rangle+\sum\varepsilon^3(\frac12\delta m^2|\phi|^2+\lambda|\phi|^4)$, and the block kernel $a_kQ_k(A)^*Q_k(A)$ are invariant under $\phi\to e^{eq\theta}\phi$, $A\to A+\partial\theta$, and the lattice Ward–Takahashi identities of* [3, §2] *hold for the vacuum-polarisation families of the shell model verbatim: at each step j the step-j effective action, in particular the logarithm of the determinant of the step-j operator with its block vertices, is exactly gauge invariant, the seagull and exchange graphs combine into transverse families, and the* zeroth moment *of the combined two-photon kernel Π_j^κ of the step-j family, built with the full step-j lines, vanishes up to the torus zero mode, $\sum_y\Pi_j^\kappa(x,y)=0$ up to a holonomy response of size $O(e^{-\delta|T|^{1/3}/\eta_0})$.*

(b) (Taylor step) The combined two-photon kernel obeys $|\Pi_j^\kappa(x,y)|\le C|x-y|^{-4}$ and the moment form $\sup_x\sum_{y\neq x}\eta_j^3|x-y|^{1+\alpha}\big[|D_A^xG|\,|D_A^yG|+|G|\,|D_A^xD_A^yG|\big](x,y)\le C$, uniformly in κ, α the Hölder exponent of (56); hence the Taylor step at a divergent photon two-leg subgraph, which acts on the photon *legs, whose norms $\|\cdot\|_{1,\alpha}$ do not see κ, has a remainder of degree α with κ-uniform constants.*

(c) (first moment) Split $\Pi_j^\kappa=\Pi_j^0+(\Pi_j^\kappa-\Pi_j^0)$, Π_j^0 the kernel at $\kappa=0$ with the wall, and likewise the mixed kernel with two scalar and one photon leg. For the first moment of the two-photon kernel [3, (3.26)–(3.30)] *and of the mixed kernel* [3, (3.36)–(3.38)]*, the first moment of Π_j^0 vanishes, and the residual, of degree* 0*, obeys two bounds, uniform in κ, ε and k, with no translation invariance on U and no constant-mass kernel. On the light side, for $x\in H'$ at unit-j distance $d\le1$ from ∂U,*

$$\Big|\sum_y(\Pi_j^\kappa-\Pi_j^0)(x,y)(y-x)_\mu\Big|\le C\min(1,w_j)\big(1+\log_+\tfrac{1}{\max(d,w_j^{-1/2},\eta_j)}\big)^{1/2},$$

exponentially small beyond, so that, summed over the layer of unit-j volume $\ell_j|\partial H'|$ against $\sup_Q|L_1|\sup_Q|\partial L_2|\le C_{\mathcal L}^2p^2(1+d_Q)^2$ for the photon legs on each unit-k cube Q (d_Q the distance of Q from the gauge base point) and over the scales, $\sum_j\ell_j\min(1,w_j)\le C$. On the shell side, at a heavy scale, the full kernel is bounded by heavy suppression: $|\Pi_j^\kappa(x,y)|\le Ce_j^2w_j^{-1}e^{-\delta'|x-y|}$ at depth ≥1 and $|c_\mu(x)|:=|\sum_y\eta^d\Pi_j^\kappa(x,y)(y-x)_\mu|\le Ce_j^2w_j^{-1}$ on every deep unit-j cube, $|c_\mu(x)|\le Ce_j^2$ on the at most $C|\partial H'|_j$ cubes of the skin, so that $\sum_{x\in U}\eta^d|c_\mu(x)|\le Ce_j^2[\,|\partial H'|_j+|U|_jw_j^{-1}]$ at every heavy scale and the sum over the heavy scales is $\le Ce^2p^2(|\partial H'|+|U|)$ in unit-k units, while at light scales the relative size is w_j and $\sum_{w_j<c_0}w_j\le C$. The same two bounds apply to the mixed class. The zeroth-moment coefficient of the mixed class, summed over a unit-k cube, is bounded by C times the coupling, uniformly in κ and ε (the conclusion used by Proposition 24.7),

and the transport terms of the exchange-dressed member [3, (2.21)] *and of its dressings by renormalised two-scalar-leg units are bounded at heavy formation scales on the cubes next to ∂U, uniformly in κ (the* transport bound*). Hence the bound* (58) *for the two-photon-leg and mixed classes holds with the κ-free norms of the photon external fields.*

Proof. We prove the three clauses in turn; for (c) we first treat the two-photon kernel, then the mixed class and the transport bound. For (a), the profile V is a scalar potential, $\kappa\mathbf{1}_U$ is diagonal and $Q_k(A)$ is covariant, so the action and the block kernel are gauge invariant; gauge invariance is the only ingredient of the Ward–Takahashi identities of [3, §2], which therefore hold verbatim.

For (b), the line–line part of Π_j^κ rests on (62)–(64) for the step-j lines, on both sides of ∂U and for every $\kappa \in [0,\infty]$; the diagonal seagull term and the terms in which the derivative falls on the block vertices $Q_j(A)$ rest on the sup and first-difference bounds of the legs. The moment form follows, since the lattice shell $|x-y| = m\eta_j$ contributes at most $C\eta_j^\alpha m^{-1+\alpha}$.

For (c), consider first the two-photon kernel. The bulk part is [3, pp. 439–441] read at $\kappa = 0$ with the wall, as a step-by-step statement at the window scale; this gives the vanishing of the first moment of Π_j^0. The light-side bound is the κ-difference of the lines with derivatives. For the shell-side bound every matter line is written in leg form with each end at depth ≥ 1, and each end is bounded by the geometric mean $Cw_j^{-1/2}e^{-\delta_2 \operatorname{dist}(x,B(u))/2}$ of the amplitude w_j^{-1} and the decaying sup bound.

It remains, for (c), to treat the mixed class and to prove the transport bound. The mixed class enters only through the sup and first-difference clauses of its scalar legs and the transport $(gA)(x) - (gA)(x')$ of its photon variable; it is split on U by its maximal matter scale and treated by Bałaban's integration by parts at the photon vertex [3, (2.8), p. 425], an exact torus identity. Its zeroth-moment coefficient is bounded for the members [3, (2.18)–(2.20)] by the bounds just proved for the two-photon kernel and the bounded window coefficient at $\kappa = 0$, and for the exchange-dressed member [3, (2.21)] and its dressings by the transport bound, which we now prove. Below a window $J \leq k$ the lines regroup into the matter and photon window lines $(-\Delta + V + aP)^{-1}$, $(-\Delta + \mu^2 + a'P)^{-1}$ [25, (2.13)]. The window coefficient of the exchange unit is bounded uniformly in κ, the spacing and the position by the ℓ^1 row sum of the window resolvent ($\|Vg\|_1 \leq 1 + a\|Pg\|_1$ for $g = G(z,\cdot)$, from $\operatorname{sgn} g\,(-\Delta g) \geq -\Delta|g|$ summed over the torus, with (62)) and the ℓ^2 bound of the photon line. The transport term, symmetrised in the two ends of the exchange kernel, is bounded at every formation scale j_1 by $Ce^2\rho_{j_1} \sup_{B(x,R\rho_{j_1})} |D\Phi| + Ce^2\rho_{j_1}e^{-cR} \sup |D\Phi|$, through the second absolute moment of the exchange kernel against the ℓ^1 norm of the difference of the host line (the slice identity $G_{(j)} = G_{j+1}(A_j - A_{j+1})G_j^\kappa$, in which the potential and the shell mass cancel). The dressings by one-particle-irreducible two-scalar-leg units of orders 6, 8, 10 are bounded from the energy space of the window line to its dual, uniformly in κ, by one induction over the orders. Every bound used is a step-line bound of Theorem 25.1, the photon bound or an exact torus identity. □

With the step lines and the photon classes in hand we run Bałaban's sequence of steps, his lower and upper bounds [1, 2, 3], for the shell model, in three statements. The shell model replaces four of his objects, in the partition of the torus into boxes, in the cut-off, in his Lemma 2.4 and in the cube step of the perturbative bound, and adds a few conditions of its own on the constants; the definition fixes the order in which the constants are chosen, so that these conditions are imposed without a cycle, and introduces the four replaced objects, and Theorem 24.5 takes its provisions on the choice of the constants from it. The proposition collects the further bounds of his argument which the shell model needs, each with a short proof of its own, and the theorem states that his steps run for the shell model with constants independent of the shell mass.

Definition 25.3 (the setting of (BAL)$_5^\kappa$)**.** The constants are chosen in the order $L \to M \to R \to b_0 \to n \to \varepsilon_0^{\mathrm{Ba}}$ of Theorem 24.5, with four provisions. First, every rate-dependent condition of Bałaban's sequence of steps, each a lower bound on R or on R_0 or a downward-closed cap on the final block size, is imposed at the rate δ_{nat}, fixed before M and R, so that the order of the choices stays acyclic. Second, M is chosen after the rates, D7 needing $dR_0 < s_\flat = LM$. Third, $R_{\mathrm{Ba}} \geq 3$. Fourth, the boxes of the cube step are placed face-adapted: each face of the box of large blocks is moved outward by 0, 1 or 2 large blocks so that it lies at distance ≥ 2 from both planes of ∂U parallel to it, under the condition $R \geq D^\flat + 2s_b + 3$, where $D^\flat := \frac{3}{2}r_\flat + 2s_b + \ell_\sharp$ is the depth of the cube step, $r_\flat$ the side of its inner cube and $\ell_\sharp$ the bound on the diameter of the old-vertex classes at order $\bar{n}$; this is a lower bound on R fixed after M, $\bar{n}$ and R_{Ba}. The *product partition* of $T^{(k)}$ is its partition into boxes that are unions of large blocks with sides $> r_k := r(\ell_k)$ and $\leq 2r_k + 3s_b$, of volume $\leq (3r_k)^d$ ($r_k \geq R \geq 10s_b + 10$) and with corridor $22r_k + 3s_b$; it is defined from $(L, M, R, \ell_k, \bar{n})$ and the regions of the run alone, so that it contains no κ. The *lower bounds on R of the cut-off* are $R \geq 5(2M+1)$, $R \geq 10$, $R \geq M_\theta + 2$, $R \geq (M_\theta + 6)/11.6$, together with $R \geq \max\{5(2M+1), C_Z(M + s_b)\}$ and $R \geq 10(M+1)$ above. The *cut-off background* is $A^{(k)} = a_k \ell_k^{-2} \zeta_\theta^{(k)} G_k^{\mathrm{vec}} Q_k^* A$ ([2, (2.44)] with the cut-off ζ_θ), and $\tilde{B}^{(k)}$, $\theta_{k+1}B^{(k+1)}$ are the fields of [2, (2.88)–(2.90)]; these three are photon objects, formed from the photon block field, the vector operator G_k^{vec} (Bałaban's G_k for the vector operator, with the photon mass μ_0^2 in place of the scalar mass m^2), the vector covariance $C^{(k)}$, the averages at zero background [1, p. 608], the cut-offs and the sets $\Lambda_0^{(k)\prime}$, none of which contains κ. The *regularity cap* is $\sup_{t \leq \varepsilon_0^{\mathrm{Ba}}} C_Z b_0 (1 + \ln(1/t))^6 (et^{1/2})^{1-\beta} \leq C$, a downward-closed cap on the final block size (satisfiable, the function tending to 0 as $t \to 0$; β the exponent of King's regularity condition, as in Proposition 24.7), with the constant C_Z of Theorem 25.5(ii). For [2, Lemma 2.4] let $i \leq K_f$, in unit-i units, $y \in \Lambda_7^{(i-1)\prime}$, $x \in B^i(y)$, $\Omega = B^i(\Lambda_2^{(i-1)\prime})$ (the propagator domain of [2, (2.56)]) or T, ϕ_i the block field with the restrictions [2, (2.55)] on $\Lambda_{-1}^{(i-1)\prime}$ ($|D_{\tilde{A}}\phi_i| \leq c_1 p_{i-1}$ and $|\phi_i| \leq \Upsilon_i := c_1 p_{i-1}(\lambda \ell_{i-1})^{-1/4}$, c_1 Bałaban's constant), $\phi_\kappa^{(i)} := a_i G_i^\kappa(\Omega, A^{(i)}) Q_i^* \Lambda_6 \phi_i$ its background and $\phi_0^{(i)}$ the walled background at $\kappa = 0$; the *edge conditions* are a lower bound on R and a cap on the final block size. In the cube step the cube $\square$ is replaced by the *face-adapted box* $\square^F \supset \square$ of large blocks; the lines $G_k^\kappa(\Omega, \tilde{B})$ of a renormalised class are split as $G_k^\kappa(\square^F, \tilde{B}) + \delta G_k^\kappa(\Omega, \square^F, \tilde{B})$ on $\square^F$, and $B_0 = \tilde{B}(c)$ is the constant field at the centre c.

Proposition 25.4 (further bounds of Bałaban's argument for the shell model)**.** *In the setting of Definition 25.3, under the hypotheses of Theorem 24.5 with (BA) and (T.nw), the following bounds of Bałaban's argument hold for the shell model, uniformly in $\kappa \in [0, \infty)$, ε and k:*

- *(a) (the scale-free sub-domain bound) on sub-domains whose faces lie two blocks from ∂U the heavy-scale leg bound holds scale-free, with decay from ∂U at the rate $\delta_j^\flat = \Lambda_*(\kappa\varepsilon^2)/(4\eta_j)$, $\Lambda_*(\xi) = \max\{\lambda_*(\xi), \log(1 + \xi/d)\}$, $\lambda_*(\xi)$ determined by $\lambda \max(\lambda, 4\eta_{\max}) + \frac{1}{2}\lambda^2(e^\lambda + 1) \leq \xi/d$, and $C_d = (2e^{5/2})^d$ in the amplitude of Proposition 24.7;*
- *(b) (vacuum families with the mismatch vertex) at King's backgrounds, on the torus and on the propagator domains, the renormalised vacuum families with $r \geq 0$ mismatch vertices obey the bounds of Proposition 24.7(d): a vacuum family differs from its zero-background value by $O(1)\ell_k^{\beta_0}$, each mismatch vertex costing $C(\lambda + e^2)\ell_k$, and between two regular backgrounds it changes by at most $Ce_k(C(\lambda + e^2)\ell_k)^r$ times $\|\partial b\|_\infty + [\partial b]_{\alpha_0}$, plus the constant part of b weighted by $e^{-\delta_0 N_T/16}$;*
- *(c) (the odd sector at higher orders) Bałaban's enumeration runs block by block at the orders up to $2\bar{n}$ for the graphs of the odd sector of Theorem 25.6, and his inductive bound* [3,

pp. 424–428], *in the form in which the proof of Proposition 24.7(b) runs it, applies to their subgraphs free of mismatch vertices;*

(d) (the sub-domain form at order fourteen) the sub-domain form of the perturbative bound holds at order fourteen for the removal of the tail of the background [2, (2.89)–(2.90)]*;*

(e) (the photon external fields) the cut-off photon slice lines and King's photon legs are Bałaban's external fields in his localised C^{1,α_0} norm [3, (1.32)]*, with exponential decay in the block distance, componentwise and uniformly in the spacing, the scale and the scale mass; no moment bound of* [25, (A.2)–(A.6)] *is used;*

(f) (uniformity in the scale) the strip bounds of [9, §3.3] *hold uniformly in the scale;*

(g) (the completed steps of [9, §3.3]*) the L^1 block-data clause, the photon two-leg kernel bounds (Theorem 25.1), the zeroth-moment identity in resummed form (Lemma 25.2(a)), the light-side localisation near edges (clause (j)), the external field of the class of Lemma 24.8, and the site-dependent diamagnetic inequality $\log\det M_U \geq \log\det M_1$ for the Gaussian determinant at a unitary connection U hold for the shell model;*

(h) (the random-walk steps) the ballot bound next to the half-line, the lifted-walk identification and the martingale moment bound behind the strip bounds hold as stated there;

(j) (light-side localisation, (LOC-R)*) the light-side estimates hold on sub-domains at deep points, edges and corners of H' included: Bałaban's constant-mass bounds* [25, (3.7)], [2, (2.65), (2.66), (2.113), (2.114)] *hold for $(M^{(m)}+N)^{-1}$ with the wall profile $N = n\psi\ell_j^2 \in [0, n\ell_j^2]$ added to the mass, and the difference $u_\Omega - u_T$ between the sub-domain and the torus legs is bounded by $C_R B e^{-\delta_R \rho_{\mathrm{B}}}$ uniformly in κ, Ω and A, ρ_{B} Bałaban's radius.*

Proof. Clause (a) is a Combes–Thomas argument: conjugating the heavy-scale operator by $e^{\lambda\,\mathrm{dist}(\cdot,\partial U)}$ perturbs it by at most $\lambda\max(\lambda, 4\eta_{\max}) + \frac{1}{2}\lambda^2(e^\lambda+1)$ per unit of mass, which the shell mass $\xi = \kappa\varepsilon^2$ dominates for $\lambda \leq \lambda_*(\xi)$.

For (b), at a regular background only the derivative bounds of $A^{(k)}$ enter King's per-graph estimate; the second bound is the difference of two backgrounds read through Proposition 24.7(d).

For (c) and (d), each block is set at its own background by interpolating the backgrounds over $u \in [0,1]^{n_o}$, each parameter raising the order by one (order $\leq 2\bar{n}$).

Clause (e) follows from the vector-field propagator bounds on the periodic torus (Theorem 19.3) by the pointwise and Hölder bounds of the propagators, the bilinear sums closing by the slice-line decay; (f) is the Montel–Vitali step of [9, §3.3].

For (g), the diamagnetic inequality is the Dyson expansion of $\mathrm{Tr}\, e^{-tM_U}$, a sum over closed paths whose matrix traces are maximal at $U \equiv 1$; the photon two-leg kernel bounds, the zeroth-moment identity, the light-side localisation and the external field of the class are Theorem 25.1, Lemma 25.2(a), clause (j) and Lemma 24.8, each proved where it stands; the L^1 block-data clause is the completed step of [9, §3.3]. (h) consists of the three random-walk steps of [9, §3.3], which hold as stated there.

Finally, for (j), the walled bounds follow from $\phi_0 = \phi_m - (M^{(m)})^{-1}N\phi_0$ with $0 \leq N \leq n\ell_j^2$. For the localisation, $u_\Omega - u_T$ solves the homogeneous equation on the cube of radius $4\rho_{\mathrm{B}}$ about x_0, so a lattice Caccioppoli inequality iterated over shells, with the decaying torus leg bounds, gives $C_R B e^{-\delta_R\rho_{\mathrm{B}}}$. □

The theorem below states Bałaban's estimates in the form in which the proof of Theorem 24.5 uses them. Its proof rests on the mechanism of the subsection, which we now make precise. Every step at which the shell model differs from King's is reduced to resolvent kernels, their B-derivatives, block averages and the composed background of the walled operator $M_\Omega^w(B) = D_B^* D_B + V^0 + V_D + w\mathbf{1}_{U\cap\Omega} + a_k P_\Omega(B)$ on a sub-domain Ω (V_D the Dirichlet face term, $w = \kappa\ell_k^2$, $P_\Omega(B)$ the block projection). For this operator the shell mass is a non-negative scalar potential,

so coercivity, decay and derivative bounds at $w = 0$ hold at every $w \geq 0$ with the same constants. Bałaban's statements at the replaced objects, by contrast, are not uniform in w: deep in U [2, (2.65)] fails for $|\phi_0|$ of the size $\Upsilon_k \gg p(\ell_k)$; under the cap $|\phi_0| \leq \bar{\tau}_k$ of the near-U test the defect is $\frac{w}{a+w}\bar{\tau}_k \leq (2/(\vartheta_1 a_k))^{1/2} p_k$, and the uncapped blocks are paid by the test. The proof therefore treats the four replaced objects one by one.

Theorem 25.5 ((BAL)$_5^\kappa$ in the form in which it is used)**.** *In the setting of Definition 25.3, under the hypotheses of Theorem 24.5 with (BA) and (T.nw), the shell model runs the steps D5–D9 and D11–D15 of Bałaban's lower and upper bounds (every step except D10, the step of the upper bound that uses the tail removal, treated separately) and the cube step of the perturbative bound on the four replaced objects, with constants independent of $\kappa \in [0, \infty)$, ε, k and T; the case $\kappa = \infty$ follows by the endpoint of the heavy-shell estimates. In detail, for every $\kappa \in [0, \infty)$ and every scale $k \leq K_f$:*

- *(i) (the partition) D14 holds in the shell model with the product partition and κ-uniform constants; the shell mass enters D14 only through the paid families and their coefficients, fixed before the count by D7, D12 and the face-adapted placement, with constants uniform in κ there;*
- *(ii) (the cut-off) under the lower bounds on R of the cut-off, the cut-off background $A^{(k)}$ and the fields $\tilde{B}^{(k)}$ and $\theta_{k+1}B^{(k+1)}$ obey the bounds (Z1)–(Z7) in the shell model verbatim, with Bałaban's constants and for every admissible sequence of large-field regions; on $\Omega = B^k(\Lambda_2^{(k-1)\prime})$ one has $|\partial^\eta A^{(k)}| \leq C_Z p(\ell_k)$ with $C_Z = O((1 + \ln 5)^6)$, so that under the regularity cap $A^{(k)}$ is regular at scale k in the sense of [25, (3.5)] for every $\ell_k \leq \varepsilon_0^{\mathrm{Ba}}$, and every operator statement of the shell model applies with the background $A^{(k)}$ on Ω with its κ-uniform constants, which do not depend on κ, η_k, k, Ω or A;*
- *(iii) (Bałaban's Lemma 2.4) D7 and D8 hold with [2, Lemma 2.4], with its box, applied to the walled background at $\kappa = 0$ only, in the following form, with constants free of κ, ε and i. (a) Under the edge conditions the lemma holds for $\phi_0^{(i)}$: $|\phi_0^{(i)}(x) - U(A(\Gamma_{x,y}))\phi_i(y)| \leq C_0 p_i$ and $|D_A \phi_0^{(i)}(b)| \leq C_0 p_i$ for bonds $b \subset B^i(\Lambda_7^{(i-1)\prime})$, independently of the position of y relative to $\partial\Lambda_7$. (b) At light scales ($w_i < c_0$), on either side of ∂U, $|\phi_\kappa^{(i)}(x) - \phi_0^{(i)}(x)| \leq C_\Delta w_i(|\phi_i(y)| + c_1\sqrt{d}\, p_{i-1}) + C_\Delta w_i \Upsilon_i e^{-\delta'(s_i/2 - c_d)}$, s_i the sup-radius of the box of the lemma, the last term $\leq \ell_i^s$ for every s under the edge cap; hence, under $c_0 \leq 1/(2C_\Delta)$ and the downward-closed cap $C_1 \leq \frac{1}{4}(1 - 2^{-1/4})\lambda^{-1/4}(\varepsilon_0^{\mathrm{Ba}})^{-1/4}$ on the final block size, $|\phi_\kappa^{(i)}(x)| \geq (1 - C_\Delta c_0)|\phi_i(y)| - C_1 p_i$ and, on blocks with $|\phi_i(y)| > M_i$, $|\phi_\kappa^{(i)}(x)|^4 \geq \frac{1}{2}(1 - C_\Delta c_0)^4 |\phi_i(y)|^4$, the light-scale clause of D7. (c) At heavy scales, for x in the collar with $\mathrm{dist}(x, U) > R_0 \geq 3$ and at depth $\geq R_{\mathrm{Ba}}$ in Ω, $|\phi_\kappa^{(i)}(x) - \phi_0^{(i)}(x)| \leq C e^{-\delta_6 R_0} \Upsilon_i$ (D7), and for every fine bond b with $\mathrm{dist}(b, \Omega^c) \geq \max(3, R_{\mathrm{Ba}})$ and every block y with unit datum δ_y, $|a_i D_A [G_i^w - G_i^0] Q_i^* \delta_y(b)| \leq C e^{-\delta_6(\mathrm{dist}(b,U) + \mathrm{dist}(y,U) + d(b,y))}$ (D8);*
- *(iv) (the cube step) the cube step inside the perturbative bound and its sub-domain form hold on T and on every propagator domain of the sub-domain form of Proposition 24.7 (the torus, $B^k(\Lambda_2^{(k-1)\prime})$, $B^k(\Lambda_2^{(k)})$ and the splitting sub-domains of D5), faces at distance ≤ 1 from ∂U included, with the face-adapted box $\square^F$ in place of the cube $\square$ and the torus kernel at zero background $G_k^\kappa(T, 0)$, carrying the wall, in place of Bałaban's $G_k(\eta\mathbb{Z}^d, 0)$; on $\square^F$ one has the kernel bounds $|D_x^a D_{x'}^b G_k^w(\square^F, 0)(x, x')| \leq C_K(\eta + \rho)^{-p(a,b)} e^{-\delta_K \rho}$ at every pair of points of $\square^F$, $\rho = |x - x'|$, $a, b \in \{0, 1\}$, $p(a, b) = 1 + a + b$, with $C_K, \delta_K > 0$ depending only on d, L, a, the Hölder exponent of the norm (56), c_0, an upper bound c_V for $(m^2 + n)\ell_k^2$ and King's constants, free of κ, η, k, the box and the positions; the*

difference lines contribute κ-free factors in the norm (56), at base points and sources at depth $\geq r_\flat \geq R_{\mathrm{Ba}} + 12$ in $\square^F$;

(v) *(no new containment across scales) every geometric condition that the steps of the shell model add to Bałaban's is a condition between quantities of one scale, a constant measured against the amplitude profile ρ_k, or a cap on the final block size, never a containment of a block of one scale in a region of another scale: the shell model adds only constants (depth 3, two large blocks per face, $R_{\mathrm{Ba}} \geq 3$), so none of its conditions can fail as $\varepsilon \to 0$ at $L = 5$, and the conditions on L are those of Bałaban's sequence of steps.*

Proof. For (i), D14 uses the partition only through the covering, the minimal arc $\geq r_k + s_b$, the volume bound and the corridor step [2, p. 593], and the product partition contains no κ (Definition 25.3).

For (ii), the three photon objects are bounded by [2, Prop. 2.2], the photon restrictions [2, (2.55)] and arithmetic, none of them containing κ; the factor $(1 + \ln 5)^6$ is absorbed by the cap $e\ell_f^{1/2} \leq e_0$ on the final block size. Regularity is the only hypothesis on the gauge field in the operator bounds of the shell model, so every operator statement applies with the background $A^{(k)}$ on Ω.

For (iii), the defect of the reproduction of constants is $O(p_i)$ by the cap $(m^2 + n)\ell_i^2 M_i' \leq p_i$ of Theorem 24.5. At heavy scales D7 and D8 use the split $\phi_\kappa^{(i)} = \phi_0^{(i)} + \Delta\phi$, $\Delta\phi = -G_i^0 w_i \mathbf{1}_U \phi_\kappa^{(i)}$, with (a) for $\phi_0^{(i)}$ and (c) for $\Delta\phi$.

For (iv), the constant field B_0 is gauged out exactly ($\square^F$ does not wrap); the lines are expanded in $\tilde{B} - B_0$ at Bałaban's order with King's cut-off, the line-created vertices lying at depth $\geq R_{\mathrm{Ba}} + 2$ in $\square^F$; the chains are bounded by [3, Proposition 2.1] on $\square^F$, from the slice bounds of the lines or from the kernel bounds of (iv).

Finally, the shell model adds to Bałaban's conditions only constants (depth 3, two large blocks per face, $R_{\mathrm{Ba}} \geq 3$), which gives (v). With (i)–(v) and the bounds of Proposition 25.4 the steps D5–D9 and D11–D15 and the cube step run for the shell model. The case $\kappa = \infty$ is the endpoint of the heavy-shell estimates. □

Beyond (i)–(v) and the bounds of Proposition 25.4, the proof uses only $(\mathrm{BAL})_5$ for the walled operator at $\kappa = 0$ (Theorems 19.3 and 19.4(j)), the operator lemmas of the shell model on the torus and on sub-domains, and the perturbative bound with its sub-domain form; no bound on the face-localised difference under a flat-face hypothesis on U, no Hölder bound on the heavy side and no Ward identity for a single ordering enters.

The last statement of the subsection is the mismatch clause of the perturbative bound, Proposition 24.7(c): the perturbative bound for the graphs with at least one external field and $r \geq 1$ mismatch vertices, with the additional factor $C_{\mathcal{M}}^r$ of the couplings of the mismatch vertices, uniformly in κ. Its proof is a degree count in Bałaban's induction: a mismatch vertex counts as a plain vertex of degree 2, so that every stage of the even sector containing one has degree exceeding the deficit of its deep part, except for the stages of effective degree zero, whose gap sums, taken first, are uniform in κ and k.

The frame is Bałaban's induction [3, pp. 426–428]: his degree [3, (2.1)–(2.3)] of a connected graph S is

$$D(S) = \sum_{v \in S} [p_v - \frac{1}{2}\#(\text{legs of } v \text{ on lines of } S) - \mathbf{1}(\text{the derivative of } v \text{ acts on a line of } S)] - 3,$$

p_v the number of η-factors of v, a mass counterterm of a graph G_0 adding $D(G_0)$, and $\mathcal{M}$ counts as a plain vertex of the quartic type ($p = 3$, no derivative; $D(\mathcal{M}) = 2$). For an ordering $l(1), \ldots, l(m)$ of the lines with scales $j(1) \leq \cdots \leq j(m) \leq k-1$, the stage i is $G_i = l(1) \cup \cdots \cup l(i)$

with its components, D_i the sum of the degrees of the components of G_i and $d_i := D_i - D_{i-1}$; after the localisation sums the summand is $O(1)\prod_i r_{j(i)}^{d_i}$ [3, (2.15)–(2.16)], summable iff every $D_i > 0$, and in the gap variables $n_i := j(i+1) - j(i)$, $\prod_i r_{j(i)}^{d_i} = L^{-\sum_i n_i D_i}$. For the $\mathcal{M}$-free divergent subgraphs the frame's transformations (integration by parts for the photon class, the Taylor step for the two-scalar-leg class, the Ward identity for the photon and mixed classes) leave every stage with an effective degree at least $c_{\mathrm{frame}} > 0$, depending on α_0 and the order only.

Theorem 25.6 (the mismatch clause of the perturbative bound)**.** *On the torus, for $\kappa \in [0,\infty)$, every graph of $P^{(k),\kappa}$ with at least one external field and $r \geq 1$ mismatch vertices obeys the perturbative bound with the additional factor $(C(\lambda + e^2)\ell_k)^r$, C free of κ, ε and k (with the counterterm offset, $C(\lambda{+}e^2)$ replaced by $C(\lambda{+}e^2{+}\Delta_c)$), for the total over those scale assignments in which no photon-closed one-slot stage contains a mismatch vertex (the* even sector*). The statement holds for classes of any finite order with constants depending on the class only through its size, and its field-independent part is a constant absorbed in the normalisation; the bound is Proposition 24.7(c). The members of the* odd sector*, those with a photon-closed one-slot stage containing a mismatch vertex, are bounded by rewriting the closed odd matter blocks by charge conjugation on full lines before the scale decomposition, which yields families of order at most $14 + n_o$, n_o the number of rewritten blocks, to which the even-sector bound applies.*

Proof. We first determine the degree gained at $\mathcal{M}$ and its deficit. Inserting $\mathcal{M}$ raises the plain degree of every subgraph containing it, by 2 if both its lines lie in the subgraph and by $2 - n_{\mathrm{der}} \in \{1,2\}$ if one does ($n_{\mathrm{der}} = 1$ iff the derivative of the far vertex acts on the line of $\mathcal{M}$). Against this, the deep part $\hat{\kappa}^{1/2} m(j_*) m(j_1)$ of the lossless density of Lemma 24.1(M3) is a power of L brought into the stages between the entry of the finer line and the closing of the host, and a lone $\mathcal{M}$ closing a one-loop core of degree -1 gives a stage of effective degree 0.

Next we bound the degrees of the stages of the even sector. By the census of the $\mathcal{M}$-free graphs [3, (2.17), p. 429], a stage S of the even sector with n_S mismatch vertices has $D(S) \geq 1$ if $n_S = 1$ (with equality for a two-line $\mathcal{M}$ iff the core is a degree-(-1) graph with an internal matter line), and $D(S)$ exceeds its deficit (the number of two-line vertices carrying their deep part plus the number of face weights integrated in S) by at least 1 if $n_S \geq 2$; here the hypothesis on S excludes the one-photon current tadpole with $\mathcal{M}$ on its loop ($D = 0$), under which a one-photon-leg core occurs only with a source vertex [25, (3.20)], of degree ≥ 0.

We now sum over the stages. With the composite vertex containing an $\mathcal{M}$ as the root of each stage at which it is joined to a composite without one [3, p. 427], the density of (M3) is summed exactly over the subcubes and split into its floor, face and deep parts. This bounds the localised summand by $O(1)^m C_{\mathcal{M}}^r L^{-\sum_i c_i n_i}$ times the deep factors and the face polynomials, with $c_i \geq \min(1, c_{\mathrm{frame}})$ outside the zero stages (one two-line deep $\mathcal{M}$ with a core of degree -1 per component), whose gap sums, taken first, have a deep factor equal to 1 on a window of values and geometric outside, hence are uniform in κ and k.

Finally, the (M1)- and (M2)-vertices are plain vertices with stages $D \geq 1$ and no deficit, so Bałaban's [3, (2.13)] gives the perturbative bound times $C_{\mathcal{M}}^r$, and $\kappa = \infty$ follows by continuity on the finite lattice. □

25.2. **The determinant tail, King's mechanism and the κ-Cauchy property.** This subsection proves the κ-Cauchy property $(\mathrm{SW3}^\circ)_{\mathrm{int}}$ of the capped reduced action, Theorem 25.11: at fixed spacing, the capped reduced actions at two shell masses $\kappa \leq \kappa'$ differ, up to a constant free of the fields, by at most a quantity $R(\kappa,\kappa')$ which tends to zero as $\kappa \to \infty$, uniformly in the spacing. The proof bounds the κ-derivative of the capped reduced action, one insertion of the two-valent vertex $\mathbf{1}_U$, whose three terms are a quadratic form of the background, the derivative of the determinant and the renormalised graphs carrying the inserted vertex. The estimates behind

these terms come first: the tail in κ of the block-free determinant (Theorem 25.7), Bałaban's all-orders enumeration with one inserted vertex (Theorem 25.8) and King's mechanism for the graphs carrying the insertion (Theorem 25.10). The threshold clause (SWT) (Corollary 25.12) and the disc bound (SW2$^{\min}$) (Lemma 25.13) close the subsection.

Standing assumptions. We work in the shell model of Section 23 (Definition 23.2): T is the torus, $H' \subset T$ the box, $U := T \setminus H'$ the shell and $V_0 = m^2 + n\psi \geq m^2$ the scalar potential with the wall; $\eta_0 = \ell_k = L^k\varepsilon$ is the stopping scale. Fixed throughout are $L = 5$, a, N, $\lambda > 0$, e, $m^2 > 0$, $\mu_0^2 > 0$, $\alpha_0 \in (0, \frac{1}{2}]$ (a Hölder exponent; not the minimal ordered degree α_0 of Theorem 20.4), the shares ϑ_1, ϑ_2, the heavy-scale threshold $c_0 > 0$ (no lower bound on c_0 is assumed) with $\hat{c}_0 = \max(c_0, 1)$, the torus T and the box H', block-aligned and with the grid offset of the corners of H' and the torus-side condition of the stopping-scale bounds, $n \geq 1$, the wall profile ψ, the sources f, g supported in H' at distance at least η_0 from U, and the scheme of the Wick constant. Here $\eta = \varepsilon/\ell_k$ is the lattice spacing in unit-k units and ρ_j the length of scale j; $\kappa_0 = \hat{c}_0/\eta_0^2$ and $w_k = \kappa\eta_0^2$ are as in Definition 24.2, and the threshold κ_0, the Compton scales κ_j and intervals $\bar{I}_j$, the stopping-scale bounds and King's constants are those of Definition 24.2 and Section 23.4. With η_1 the threshold of the stopping-scale bounds and $m_V^2 = \inf_{H'} V_0$ the plateau gap, let

$$\eta_2 := \min\big(\eta_1,\ L/(\lambda C_7^4),\ (\hat{c}_0/(4m_V^2))^{1/2}\big),$$

$c_L := 1 + (L^2 - 1)(1 + 2\log L)^2/(1 - \alpha_0)$ and $c_L' := L^{1-\alpha_0}\sum_{i\geq 0} L^{-(1-\alpha_0)i}(1 + 2i\log L)^2$. The *inserted vertex* is the two-valent vertex $\mathbf{1}_U$ produced by the derivative in the shell mass κ, placed on one internal line or on one leg.

The first estimate concerns the block-free factor of the determinant, $\det H_\kappa(A)$ with $H_\kappa(A) = -\Delta_A + V_0 + \kappa\mathbf{1}_U$. The theorem bounds the tail $T(\kappa; A)$ of its dependence on the gauge field, the part that the shell mass has not yet killed, by an ε-free function of κ decaying like $\kappa^{-1/2}$ up to a logarithm. In the proof of Theorem 25.11 the higher orders in A of the κ-derivative of this factor are bounded through the tail, read at $\kappa/2$: this is the monotone term of the derivative, and the origin of the term $\tau_{\mathrm{bf}}(\kappa)$ in (68), which does not vanish as $\kappa' \downarrow \kappa$. The proof of the theorem expands the inverse of the block-free operator in random walks, so that the gauge field enters only through the holonomies around closed walks rooted in the shell, and the three terms of (65) bound three classes of walks: those confined to the shell, those touching the box without traversing the shell, and those traversing a shell arc. The hypothesis $\varepsilon_0 \leq \ell_H/2$ enters only once, for the third class, through the distance $\min(w_U, \ell_H/2)$ in the return Green's function.

Theorem 25.7 (the tail of the block-free determinant on the torus)**.** *Let T be the torus, $H' \subset T$ the box, $U = T \setminus H'$ the shell, of width w_U in each coordinate direction, ℓ_H the smallest side of H', $V_0 \geq m^2$ on T with $V_0 \geq m_V^2$ on H', $H_\kappa(A) = -\Delta_A + V_0 + \kappa\mathbf{1}_U$ the block-free operator at lattice spacing ε (physical units), and*

$$\Phi^{\mathrm{bf}}(\kappa) := \ln\det H_\kappa(A) - \ln\det H_\kappa(0), \qquad T(\kappa; A) := \Phi^{\mathrm{bf}}(\kappa) - \Phi^{\mathrm{bf}}(\infty).$$

Let A be any lattice gauge field on T with $|\partial_\mu A_\nu| \leq C_A$ (no bound on A itself and no condition on its torus holonomy), and let $\varepsilon_0 \leq \min(w_U, \ell_H/2)$, as at the value $\varepsilon_0 = \min(\varepsilon_0(\eta_0), \eta_0)$ at which Theorem 25.11 uses the theorem, T and H' being block-aligned with each side of H' at least two blocks of the final scale $\geq \eta_0$. Then for $\kappa \geq 2m_V^2$ and all $\varepsilon \leq \varepsilon_0$

$$\begin{aligned} 0 \leq T(\kappa; A) \leq{}& C\, e^2 C_A^2\, |U|\, \kappa^{-1/2} + C(T, H')\, e^2 C_A^2\, m_V^{-1}\kappa^{-1/2}\big(1 + \log(\kappa/m_V^2)\big)^{c_\star} \\ &+ C(T, H', m_V)\Big[e^{-c\, w_U\sqrt{\kappa}} + \Big(1 + \frac{\varepsilon_0^2\kappa}{8d}\Big)^{-w_U/(2\varepsilon_0)}\Big], \end{aligned} \tag{65}$$

with an absolute exponent $c_\star \le 5$, $C(T,H')$ proportional to the cross-sections of the torus, and every constant free of ε, κ and A. Consequently

$$0 \le T(\kappa;A) \le C(T,H',m_V,\varepsilon_0)\,(1+e^2C_A^2)\,\kappa^{-1/2}\big(1+\log(\kappa/m_V^2)\big)^{c_\star} \qquad (\kappa \ge 2m_V^2), \tag{66}$$

an ε-free function of κ dominated by $C_\delta\kappa^{-1/2+\delta}$ for every $\delta>0$. The same bound, with the third term replaced by $e^2(C_a^2K_a + C_A^2K_A)e^{-\Lambda/2}$ for fields with $|A_\nu| \le C_a$ in the given gauge, $\Lambda := (2w_U/\varepsilon_0)\operatorname{arcsinh}(\varepsilon_0\sqrt{\kappa}/2)$ and K_a, K_A depending on (T,H',m_V,ε_0) only, holds for the second-order tail $T_2(\kappa;A) := \lim_{s\to0} s^{-2}T(\kappa;sA)$.

Proof. The random-walk expansion of $G_\kappa = H_\kappa^{-1}$ in $\partial_\kappa\Phi^{\mathrm{bf}} = \operatorname{Tr}\mathbf{1}_U[G_\kappa(A)-G_\kappa(0)]$ writes $T(\kappa;A)$ as a sum over closed walks γ rooted in U of $1-\cos\Phi(\gamma)\ge0$ times a positive weight, $\Phi(\gamma)$ the holonomy of eA around γ. In particular $T\ge0$ (Kato's inequality).

For the walks confined to U, the Lévy-area bound $E_x[\Phi^2;\gamma\subset U, Y_n=x] \le Ce^2C_A^2\varepsilon^4n^2$, against the return probability of the walk killed at rate κ, gives the first term of (65). For the walks touching H' without traversing the shell, Hölder's inequality with $q \asymp \log(\kappa/m_V^2)$ against a free-bridge moment bound of order $(Cq)^{2q}(e^2C_A^2\varepsilon^4n^2)^q$ gives the second term, whence the logarithm. For the walks traversing a shell arc, winding walks included, $1-\cos\Phi\le2$ removes A; the discounted probability of crossing the slab before returning is $\le C(\varepsilon/w_U)e^{-\Lambda/2}$, and the strong Markov property at the exit sends the remainder into the return Green's function at sup-distance $\ge \min(w_U,\ell_H/2)$. This gives the third term and proves (65).

Since $w_U\ge\varepsilon_0$, the bracket in (65) is at most $C\varepsilon_0^{-1}\kappa^{-1/2}$, which gives (66). The second-order tail follows with $\Phi^2/2$ in place of $1-\cos\Phi$. □

We turn to the graphs. The derivative of the graph part $P^{(k),\kappa}$ of the explicit action consists of renormalised graphs of King's kind with one extra two-valent vertex $\mathbf{1}_U$, the inserted vertex, and these have to be bounded uniformly in κ and ε at the single value κ of the shell mass. Two theorems do this. The first says that Bałaban's all-orders enumeration [3, §§2–3] survives both the inserted vertex and our setting of block factor 5, tori and the seagull vertex. The graph pass of the proof of Theorem 25.11 uses (i)–(iv), and (iv) in turn rests on the census, engine and pinning bounds of the second theorem, King's mechanism (Theorem 25.10). The idea of the proof is that the inserted vertex, being two-valent and of King's degree two, can take the place of the source vertex in the degree count of Theorem 20.8, so that the census without the insertion carries over; the renormalisation is done at zero shell mass, so that no scale is both renormalised and heavy, and on the heavy side the graphs with the insertion are bounded at the single value κ.

Theorem 25.8 (Bałaban's all-orders enumeration with one inserted vertex)**.** *Bałaban's all-orders enumeration* [3, §§2–3] *to order* 13 *with one inserted vertex, as used in the proofs of* [25, Thms. 3.4, 3.5]*, holds at block factor* 5 *on tori, with King's vertex list including the seagull vertex of order two:*

- *(i)* (order cap) *$\bar n = 13$ suffices for the cumulant expansion with King's field class;*
- *(ii)* (census) *the classification of the divergent classes, with and without the inserted vertex, which does not involve the shell mass;*
- *(iii)* (renormalisation) *the renormalisation of every class in every ordering at zero shell mass;*
- *(iv)* (bound) *in the shell model at $\hat\kappa = w_k \ge \hat c_0$, the bound of the renormalised graphs with the insertion in the form used by the κ-derivative estimate of the shell model, the census, engine and pinning bounds of Theorem 25.10;*
- *(v)* (cubes) *on King's cubes C_J^K, at zero shell mass and without the inserted vertex, (i)–(iii) hold at every localisation and (iv) is replaced by the cube bound $(iv)_C$: for the classes (i), (ii), (iii)(a) of Theorem 20.8, the two-field classes 2F-R (the classes with a curvature*

source g and the mixed classes with g and h) and the source-free field-dependent classes, at every localisation tuple τ with $R = \operatorname{dist}(P(\tau)) > p(\delta_k)$, on the scales $0 \le k \le K - J_*$ and for each of the two models $\theta = (K, k)$ and $\theta = (K + n, k + n)$ of Theorem 20.8,

$$|E_N^{(\theta)}(H_{\mathrm{ren}}; \tau)| \le C\,(1 + K - k)^{p_0}\,\delta_k^{\rho(H)}\,p(\delta_k)^{n_1}\,e^{-\delta \operatorname{dist}(P(\tau))}. \tag{67}$$

Here the localisation tuple τ, its point set $P(\tau)$ and $\operatorname{dist}(P(\tau))$ are as in Definition 5.22, $\delta_k = 5^{k-K}$, $p(\cdot)$ is the small-field function of Convention 19.1 (as in Proposition 26.1) and J_* the step offset of Section 5.1; H_{ren} is the renormalised graph, E_N is as in Theorem 20.8 ($E_N := E$ for the field-independent members of the class (ii), for the class (iii)(a) and for the g- and source-free classes), $\rho(H)$, n_1 and p_0 are the exponents of King's bound [25, (3.38)] *in the restricted form of Theorem 20.8 (with its factor $(1 + K - k)^{p_0}$, now on King's cube at $R > p(\delta_k)$), and the constants depend on the point of King's range, on J, on the bound on the sources, on $\bar{n}$, d, L and the Hölder exponent α only, uniformly in K. The same bound holds for the finite family $\mathcal{F}_{14}$ of the families of order at most 14 with one inserted external photon field (a vertex of King's list with one more external photon leg), on the scales $0 \le k \le K - J_*(\mathcal{F}_{14})$ and with constants depending on $\mathcal{F}_{14}$ in place of $\bar{n}$, where $J_*(\mathcal{F}_{14}) \ge J_*$ is the finite threshold, not quantified here, at which the geometric ranges of the cube chain hold for $\mathcal{F}_{14}$: the range row (ZB), read as the convergence condition (ZB$^{\Sigma}$) of (iii) (the renormalisation paragraph of the proof below) with $C_1 = C_1(\mathcal{F}_{14})$, and the zone-B constant C_1^B in zone B of Definition 5.22. At $R \le p(\delta_k)$ the bound of the renormalised classes on the cube is King's one-box route (Theorems 5.38 and 5.37, Proposition 6.5) under its own condition (ZB$_{\le}$), the convergence condition of King's expansion in the block field on the one box of a localisation with $R \le p(\delta_k)$ (the third condition of the range (6) of Definition 5.25), and the scales $k > K - J_*$ lie outside the statement; so $J_0 \ge \max(J_*, J(\mathrm{ZB}_{\le}))$ is required of the cube index wherever the bound is used, $J(\mathrm{ZB}_{\le})$ being the least offset such that (ZB$_{\le}$) holds on every scale $k \le K - J(\mathrm{ZB}_{\le})$.*

Proof. For the order cap (i), the coefficients $O_\beta(\ell_j^{1/4-\beta})$ of the effective interaction at scale j make the remainder of the cumulant expansion truncated at $\bar{n} = 13$ of order $\ell_j^{14(1/4-\beta)}|T|$, and $14(\frac{1}{4} - \beta) > 3$ (Lemma 24.8; Proposition 5.12 on the cubes).

For the census (ii), the degree-by-legs count of Theorem 20.8 with the inserted vertex in the place of the source vertex gives degree $\ge \frac{1}{2}$ to every connected subgraph containing it, except the field-free vacuum components and the parity shape, which vanishes at zero shell mass.

The renormalisation (iii) is Bałaban's Taylor and Ward steps and King's reduction [25, (3.37)], which rewrite each divergent class in every ordering into zero, collapsed local terms and remainders of positive degree. King's expansion [25, (3.52)–(3.53)] on the box Ω of a localisation converges absolutely class by class under the condition (ZB$^{\Sigma}$): $C_1\theta_\Omega \le \frac{1}{2}$, where $\theta_\Omega := C_{ZB}\,e_k\,p\,\operatorname{diam}(\Omega)$ is the norm of one term [25, (3.49)] and $C_1 = C_1(\mathcal{F}) \ge 1$ a constant of the finite family.

For the bound (iv), on the heavy side the graphs with the insertion are a finite sum of renormalised graphs of King's kind at the single value κ, and they are bounded by Theorem 25.10(a)–(c) with the counterterms of zero shell mass.

Finally, the cube bound (v) is the cut by clusters of Proposition 6.7 with the restricted Theorem 20.8 as the per-cluster bound and the tree bound of Definition 5.24. The inserted leg of $\mathcal{F}_{14}$ is one more external field admitted by every leg clause, the three numerical uses of $\bar{n} = 13$ hold at order 14, and the row (ZB) at $C_1(\mathcal{F}_{14})$ fixes $J_*(\mathcal{F}_{14})$. □

King's mechanism, Theorem 25.10, is the engine of the graph pass. For each member of the κ-derivative and each ordering of its lines it gives the census (a), the multiscale bound (b) with its pinned form (c), the gains (d), the position sum (e) and the window moments (f), the coefficient bounds of the local vertices in (a); clauses (a)–(c) are what Theorem 25.8(iv) and the graph pass of the proof of Theorem 25.11 use. The statement needs the vocabulary of the definition that precedes it: the members and their degrees; the prefix components of an ordering; the odd-block step, the fundamental theorem of calculus in the background along the full lines of a closed matter block with an odd number of photon ends (which vanishes at zero field, by Furry's theorem), inserting one background leg; the extended list of small divergent subgraphs, which enter not as factors of positive degree but as local vertices at their window scale; and the window kernel, whose moments are their coefficients. The last sentence of the theorem marks a limitation: the bounds are uniform in the shell mass but carry no smallness in w_k; in the graph pass the decay in κ comes from the inserted line, through its heavy-kernel bounds per unit w. The proof runs the degree count of Theorem 20.8 with the inserted vertex as the source vertex, proves the multiscale bound in ordered form, one prefix component at a time, and reads the window moments off the gauge invariance and the Ward identity of the window class; it uses no Ward identity per ordering, no Hölder clause on the heavy side and no lower bound on c_0, the split degree count (Lemma 5.40, Theorem 20.3) replacing King's per-vertex credit [25, (3.68)–(3.70)].

Definition 25.9 (members, degrees, the odd-block step, the extended list and the window kernel)**.** A *member* is a graph of the expansion of the κ-derivative of the shell functional after the regroupings of the κ-Cauchy estimate, with the inserted vertex u (the κ-derivative of $\kappa\mathbf{1}_U$) on the *inserted line* l_0; the members form the *graph pass* (III), the term $\partial_{\hat{k}}P^{(k),\kappa}$ of the derivative formula in the proof of Theorem 25.11, and the *source-dependent dressings* are the members with a curvature source g' attached. M_u is the matter block containing u, *open* if a vertex of it carries an external scalar end and *closed* otherwise. The *degree* of a connected subgraph γ is the integer $D(\gamma) = 3(|V(\gamma)| - 1) + \sum_v a_v - \#\text{lines}(\gamma) - \sum_v \delta_v$ (a_v the vertex weights, δ_v the differenced ends), the *prefix component* H_i of an ordering $l(1), \dots, l(n)$ of the lines compatible with the scale assignment is the connected component of $\{l(1), \dots, l(i)\}$ containing $l(i)$, and the *window* of a divergent subgraph is the smallest scale J of the fluctuation photon lines with exactly one end on it. c_l is the clause constant of a line l (on the inserted line l_0: the propagator bounds per unit w, the bound of the inserted line differenced at both ends and the bounds of the inserted leg per unit w, as re-posed in the κ-Cauchy estimate); in unit-k units, $t(y) := \operatorname{dist}(y, U)$, $\tau_0 := \max(w_k^{-1/2}, \eta)$, $\tau_U(y) := \tau_0 + t(y)$ on H' and $\tau_U := \tau_0$ on U.

The *odd-block step*: a closed matter block with an odd number of photon-type ends vanishes at zero field, so $V_\beta(A_k) = \int_0^1 V_\beta^{(1)}(tA_k)[A_k]\,dt$ with one background leg inserted; it is performed once per closed odd block, before the ordering is chosen. The *extended list* consists of the divergent two-scalar-leg subgraphs of orders ≤ 4, the two-photon-leg subgraphs of orders ≤ 4 with two line slots or with one line slot and one background leg (nested chains included), and the mixed subgraphs of order 3 with a fluctuation photon end or with the background leg. For a renormalised family of the extended list with one background leg, attached at the vertex x of the family: its *slot line* s is its fluctuation photon line, of index J; its *background leg* B'' is the background A_k re-gauged to vanish at the cube of x, with base point x_0; its *window kernel* $\Pi_{[J]}(x, y)$ is the kernel of the *window class*, the sum of the family's members over the internal scale assignments with all indices below J, the matter lines at the background tA_k of the odd-block step, read as the bilinear form $\sum_{x,y} \eta^3\eta^3\, s(x)\Pi_{[J]}(x, y)B''(y)$. With $e_J^2 := e^2\rho_J$ and R_J a quarter of the side of the torus, both in the units of step J, N the side of the torus in unit-k units and p the small-field bound of the block fields, the Taylor expansion of B'' about x splits the

family into the *zeroth moment* $\zeta_J(x) := \sum_y \eta^3 \Pi_{[J]}(x,y)$ of the window kernel in the background slot, the *first moment* $c_\nu(x) := \sum_y \eta^3 \Pi_{[J]}(x,y)(y-x)_\nu$ and the *second-order Taylor remainder* $\sum_y \eta^3 \Pi_{[J]}(x,y)[B''(y) - B''(x) - (y-x)\cdot \partial A_k(x)]$.

Theorem 25.10 (King's mechanism for the κ-model)**.** *Let $w_k \geq \hat{c}_0$, $\varepsilon \leq \varepsilon_0(\eta_0)$, $(A_k, \phi_k) \in \mathcal{X}^{\mathrm{cap}}$, the capped test set of Definition 24.2, $|z| \leq 2$ and $g' \in \{0, g, 2g\}$. For every member of (III), or of the source-dependent dressings, after the regroupings, and every ordering compatible with its scale assignment, the following hold.*

- *(a) (census) Every prefix component containing the insertion u has $D \geq 1$ if M_u is open. If M_u is closed, the only exceptions are (T) the whole graph and (F) the subgraph $\gamma_F(u)$ formed by u, by the vertex y of King's type* [25, (3.22)] *at which both u-lines end, and by these two lines, with its photon slot external; both are bounded separately within the class of the members whose insertion lies on a closed matter component, which the proof of Theorem 25.11 bounds as a class. After the odd-block step (Definition 25.9), which raises the degree of every containing subgraph by at least one, and Bałaban's summation by parts* [3, (2.8)] *per ordering, every prefix component not containing u is a factor of integer degree ≥ 1 or lies inside a renormalised family of the extended list. The latter are not factors of positive degree: their linear and first-moment parts, King's symmetry-cancelled degree-0 classes* [25, (3.84)]*, are not cancelled per ordering in the κ-model and enter only as* local vertices at their window scale *with the κ-uniform coefficient bounds of (f), pointwise or averaged over the cubes of that scale, plus remainder members whose stages have positive degree, the containing prefix components keeping their census degree or gaining. The two-scalar-leg classes are paired at step k with the part of King's κ-free δm^2 whose indices are all $\leq k-1$.*
- *(b) (engine) The member is bounded by the multiscale bound: the sum over the positions but one held vertex and over the scales is $\leq C \prod_l c_l \prod_v (\text{vertex factors})\, e^{-\delta \operatorname{dist}(\text{held})} (1 + \log_L \eta_0^{-1})^{p_0}$, with $C = C(\text{graph}, \delta)$ free of w_k, ε and k. The clause constants are κ-uniform for undifferenced lines and legs, for one difference and for a difference at each end, and the local vertices of (a) enter with their coefficient bounds. The bound is linear in each clause constant and may be evaluated on configuration sets of one vertex and with any fixed fraction of the decay rates.*
- *(c) (fixed insertion scale, pinning) With l_0 at a fixed scale i, the sum over the other scales is $\leq C_{d'} \rho_i^{d'} \prod c_l$ for every $d' < d_* := \min D$ over the prefix components containing l_0, or $\leq C \rho_i^{d_*} (1 + k - i)^{n_{d_*}} \prod c_l$, n_{d_*} the number of those components with $D = d_*$. Pinning at an end of l_0 replaces its clause constants by their supremum over the other end.*
- *(d) (amplitudes and gains, in the bound (b)) At a held vertex y, a line of scale j or a true leg in a set W at y carries $C \min(1, \tau_U(y)/\rho_j)^s$, $s \in [0,1]$, as a dimension shift, provided $D(H) - s|H \cap W| \geq 0$ on every prefix component H. The factor $\min(1, |x_b - u|/\rho_s)$ at the attachment b of a line s, u and b being joined before s enters, raises the exponent by one on the prefix components that contain the joining path and not s, and lowers none.*
- *(e) (position sum) The factor $(1 + K - k)^{p_0}$ with $1 + K - k = 1 + \log_L \eta_0^{-1}$ arises from the infrared units of both models, scalar and photon lines sliced, and sits inside local vertex values; p_0 is raised by at most one. At one loop the coarse tail is a bounded κ-uniform constant with no scale-sum logarithm; at orders 2 to 4 it carries $(1 + K - k)$.*
- *(f) (window moments) For every renormalised family of the extended list with one background leg inside the member (Definition 25.9): the zeroth moment $\zeta_J(x)$ is the torus zero-mode tail, $|\zeta_J(x)| \leq C e_J^2 R_J^{-1} \leq C e^2$. The first moment obeys $|c_\nu(x)| \leq C e^2$ uniformly in κ,*

ε, k, t and x on either side of ∂U, the window's internal scales summed. The second-order Taylor remainder is bounded in the near region by $C[dA_k]_{\alpha_0}$, the Hölder seminorm of the curl of the background, times the moment $\sup_x \sum_y \eta^3 |x-y|^{1+\alpha_0} |\Pi_{[J]}(x,y)| \le C$, *$\kappa$-uniformly, and in the far region* $\operatorname{dist}(y,x_0) \ge N/4$ *by the norm* $\le Cp\,(N+1/(\mu_0 \ell_{k-1}))$ *of the leg there times the decay $e^{-\delta N/8}$ of half a line of the path from x to y, uniformly in k, ε and κ.*

No smallness in w_k is claimed: a rate per graph with κ-uniform constants is false in general.

Proof. We begin with the census (a). The degree is an integer and u a two-valent scalar vertex with coefficient $\mathbf{1}_U$ and no counterterm unit, so the degree count of Theorem 20.8 with u as the source vertex gives $D \ge 1$ for every connected $\gamma \ni u$ except (T) and (F). The prefix components not containing u have the census of the uninserted expansion (Theorem 20.5, Lemma 5.40, Theorem 20.3): Bałaban's special graph (2.4) is removed per ordering by the κ-free summation by parts [3, (2.8)], the odd-block step produces one background vertex, and the extended list enters as local vertices at the window scale by the Taylor and Ward steps in their κ-model forms.

The same Taylor and Ward steps give the window moments (f): the gauge invariance of the window class under $A \mapsto A + d\lambda$ and the decay of $\Pi_{[J]}$ bound the zeroth moment, its Ward identity the first, and the moment of $\Pi_{[J]}$ (near region) or the decay along half a line of the path from x to y (far region) the remainder.

We come to the multiscale bound (b). For a fixed ordering and scale assignment the member is dominated by a product of line kernels with κ-uniform clause constants [3, (2.8), (2.10)], vertex weights and the local vertices of (a). The multiscale bound is proved in ordered form, step i treating exactly H_i; the degree-zero index sums give $(1+k)^{n_0}$, n_0 the number of degree-zero prefix components (0 for M_u open), and the members with M_u closed other than (T), (F) are bounded inside the class bound of the proof of Theorem 25.11 by one gauge identity for the product of the full matter lines of M_u and the exact count of the commutator terms of King's path convention [25, (2.19)]. In the ordered bound, the pinning (c) is the inequality $\mathbf{1}_{j_{l_0}=i} \le (\ell_i/\ell_{j_{l_0}})^t \mathbf{1}_{j_{l_0}\ge i}$ with $t = \theta'$, respectively $t = \theta$.

It remains to see (d) and (e). The leg bounds $C \min(1, \tau_U(y)/\rho_j)$ and $C|b|\tau_U(y)$ at a held vertex are dimension shifts in the ordered bound, with a path inequality from u to b after halving the rates on the path, which gives (d); and (e) counts the integers $k \le j \le K$ of the infrared units of both models. □

We can now state and prove the κ-Cauchy property. The theorem compares the capped reduced actions at two shell masses $\kappa \le \kappa'$ on the capped test set: up to a constant free of the block fields, of z and of g', they differ by at most $R(\kappa,\kappa')$, the integral of a profile θ per unit $\hat\kappa$ plus the tail $\tau_{\mathrm{bf}}(\kappa)$ of the determinant, and R is dominated by $r(\kappa;\eta_0)$, which tends to zero as $\kappa \to \infty$ uniformly in the spacing. This is the shell estimate $(\mathrm{SW3}^\circ)_{\mathrm{int}}$: Theorem 23.12 assumes it together with the three other shell estimates, and of the present theorem it uses only the decay $r(\kappa;\eta_0) \to 0$. The proof writes the difference of the two actions as the integral in κ of the derivative. Since the shell mass enters the explicit action only through the matter Gaussian, the derivative is a single shell-mass insertion with three terms: the quadratic form of the background, bounded directly; the determinant, whose block-free factor is bounded by its one-loop bound and the tail of Theorem 25.7; and the graph pass, to which Theorems 25.8 and 25.10 apply. The tail is read at $\kappa/2$ and needs $\kappa/2 \ge 2m_V^2$; this is what the third entry in the definition of η_2 provides, since $\eta_0 \le \eta_2$ gives $\kappa_0 = \hat c_0/\eta_0^2 \ge 4m_V^2$.

Theorem 25.11 (the κ-Cauchy property of the capped reduced action, $(\mathrm{SW3}^\circ)_{\mathrm{int}}$)**.** *For every $\eta_0 \in L^{-\mathbb{N}}$ with $\eta_0 \le \eta_2$ there is a constant $C_{\mathfrak{c}} = C_{\mathfrak{c}}(\eta_0) < \infty$, depending on $(T, H', n, \lambda, e, L, a, N, m, \mu_0, \alpha_0, c_0, \eta_0)$, on f and g (through the one-loop z^2-term and the g'-tadpole) and on King's*

constants, but not on ε, κ, z or g', such that for every $\varepsilon = L^{-K} \le \varepsilon_0(\eta_0)$, every $\kappa_0 \le \kappa \le \kappa' < \infty$ (any ratio κ'/κ), every real $z \in [-1,1]$ and $g' \in \{0, g\}$:

(68)

$$\sup_{b,\,b' \in \tilde{\chi}^{\mathrm{cap}}} \big|\Delta_z(b) - \Delta_0(b')\big| \;\le\; 2R(\kappa,\kappa') \;\le\; 2\,r(\kappa;\eta_0), \qquad R(\kappa,\kappa') := \int_{\kappa\eta_0^2}^{\kappa'\eta_0^2} \theta(\hat\kappa)\,d\hat\kappa + \tau_{\mathrm{bf}}(\kappa),$$

$$\theta(\hat\kappa) := C_{\mathfrak{c}}\,\hat\kappa^{-1}\,\hat\kappa^{-(1-\alpha_0)/2}\big(1+\log\hat\kappa\big)^2, \qquad \tau_{\mathrm{bf}}(\kappa) := C_{\mathfrak{c}}\,(\kappa\eta_0^2)^{-\frac12+\frac{\alpha_0}{4}},$$

where θ is the per-unit-$\hat\kappa$ profile of the derivative bound and τ the integrated tail of the block-free factor of the determinant, a decreasing power majorant of the law of Theorem 25.7, which does not vanish as $\kappa' \downarrow \kappa$. For every $\bar\kappa \ge \kappa_0$

$$R(\bar\kappa,\infty) = \int_{\bar\kappa\eta_0^2}^{\infty} \theta + \tau_{\mathrm{bf}}(\bar\kappa) \;\le\; r(\bar\kappa;\eta_0) := c_L c_L' C_{\mathfrak{c}}\,(\bar\kappa\eta_0^2)^{-(1-\alpha_0)/2}\big(1+\log(\bar\kappa\eta_0^2)\big)^2 \;\xrightarrow[\bar\kappa\to\infty]{}\; 0, \tag{69}$$

uniformly in $\varepsilon \le \varepsilon_0(\eta_0)$. The bound holds for all $|z| \le 2$ and for $g' = 2g$ with the same constants. More precisely, there is a function $\kappa \mapsto c_\kappa$ on $[\kappa_0,\infty)$, independent of the block fields, of z and of g', with

$$\sup_{(A_k,\phi_k)\in\mathcal{X}^{\mathrm{cap}}} \big|S^{(k),\kappa}(A_k,\phi_k,zf,g') - S^{(k),\kappa'}(A_k,\phi_k,zf,g') - (c_\kappa - c_{\kappa'})\big| \;\le\; R(\kappa,\kappa'), \tag{70}$$

and (70) passes to every reduced action cut by a measurable family of U-block sections inside the capped fibre, with the same c_κ. The finite endpoints $\kappa \le \kappa'$ may lie on either side of the crossover $\kappa \asymp \varepsilon^{-2}$ or in the lattice-heavy regime $\kappa \ge c_0/\varepsilon^2$, and the endpoint $\kappa' = \infty$ is never part of the hypothesis.

Proof. The explicit action is real-analytic in κ at fixed ε, so $S^{(k),\kappa} - S^{(k),\kappa'} = \int_{\kappa\eta_0^2}^{\kappa'\eta_0^2} \partial_{\hat\kappa} S\,d\hat\kappa$, and the derivative is one shell-mass insertion; in unit-k units the derivative formula reads

$$\partial_{\hat\kappa} S^{(k),\kappa} = \tfrac12 \sum_{x\in U} \eta^d |\phi_\kappa^{(k)}(x)|^2 + \tfrac{N}{2} \sum_{x\in U} \eta^d \big[G_k^\kappa(A^{(k)})(x,x) - C^{\eta,\kappa}(0)(x,x)\big] + \partial_{\hat\kappa} P^{(k),\kappa},$$

$\phi_\kappa^{(k)}$ the background of the block data, G_k^κ the block operator and $\partial_{\hat\kappa}P^{(k),\kappa}$ the graphs of $P^{(k),\kappa}$ with one κ-dependent factor differentiated; we call the three terms on the right the quadratic form (I), the determinant (II) and the graph pass (III). We show that on $\mathcal{X}^{\mathrm{cap}}$ the right side is c_κ', free of the fields, of z and of g', plus one term monotone in κ with integrated tail τ and a remainder of modulus $\le \theta(\hat\kappa)$, and then integrate.

The quadratic form (I) is non-negative and of order $\hat\kappa^{-3/2}$ per unit $\hat\kappa$. The block part of the determinant (II) is bounded in the same way. Its block-free factor is bounded by the one-loop bound per unit $\hat\kappa$ at the stopping scale for $|\partial_\mu A_\nu| \le C_A$, $|A_\nu| \le C_a$ and, in the higher orders in A, by the tail of Theorem 25.7 read at $\kappa/2$; this is the monotone term with the tail τ.

In the graph pass (III), each term of $\partial_{\hat\kappa}P^{(k),\kappa} - c_\kappa'$ is a convergent renormalised graph of King's kind at the single value κ (the inserted vertex has King's degree 2). The graph pass per unit κ is King's bookkeeping of the renormalised graphs [25, (3.38), Thm. 3.5] restricted to graphs whose every subgraph containing the insertion is convergent, with the enumeration of Theorem 25.8, the census, engine and pinning bounds (a)–(c) of Theorem 25.10, the κ-uniform shell-model bounds on the undifferentiated lines and legs and, on the inserted line, the heavy-kernel clauses $|G(x,y)| \le C|x-y|_\eta^{-1}e^{-\delta|x-y|}$, $\sum_{u\in U}\eta^d|G(u,z)| \le C/w$ and $\sum_{u\in U}\eta^d|G(x,u)|^2 \le Cw^{-1/2}$, uniform in κ and in both regimes. An insertion at scale i carries a shell factor $\propto w_i$ at light and $Cw_i^{-1/2}$ at heavy scales, so the scale sums are geometric with no $\log(1/\varepsilon)$; the insertions on a closed matter component of a deep cluster are bounded as a class by one gauge identity for the

product of its full matter lines. Theorem 25.10 holds for $|z| \le 2$ and $g' \in \{0, g, 2g\}$, so the bound holds on that range with the same constants.

Integrating the three bounds gives (70) with $c_\kappa = \int c'$, and the triangle inequality the factor 2 in (68). The Taylor-remainder terms with an insertion on a scalar leg give the rate $\bar\kappa^{-(1-\alpha_0)/2}$, the Gaussian half and the matter-tadpole sector the $\bar\kappa^{-1/2}$ law, and summing θ Compton interval by Compton interval gives (69). The bound passes to $\tilde S_\kappa^{\mathrm{cap}}$ and to the sections inside the capped fibre pointwise on each section, since e^{-F_κ} lies between $e^{\mp r - c} e^{-F_{\kappa'}}$. □

Two further shell estimates follow with little effort. The first is the threshold clause (SWT): the comparison of two adjacent representations of the stopping-scale integral carries threshold terms at the Compton thresholds κ_j, which vanish when neither the test set nor the representation changes there, and for the capped object there is a single test set for every $\kappa \ge \kappa_0$. The second is the disc bound (SW2$^{\min}$): at zero scalar source the generating function $z \mapsto \langle e^{zF(f)}\rangle_\kappa$ of the two-level state is entire and bounded on the disc $|z| \le 2$ uniformly in the shell mass, $\kappa = \infty$ included, and in the spacing. At $g = 0$ it is the Gaussian domination of Lemma 16.16(a) alone, which is the form in which Theorem 23.12 uses it; the disc bound with the scalar source is proved in Section 27.

Corollary 25.12 (the threshold clause (SWT))**.** *The threshold clause between adjacent representations of the stopping-scale integral is empty for the capped object: there is one test set on $[\kappa_0, \infty)$, $\kappa_0 = \max(c_0, 1)/\eta_0^2$ the threshold of Section 23.4, namely the capped test set of Definition 24.2, and no threshold term arises between two shell masses $\kappa_0 \le \kappa \le \kappa' < \infty$.*

Proof. By Definition 24.2 the capped test set $\mathcal{X}^{\mathrm{cap}}$ is the same set of block fields for every $\kappa \ge \kappa_0$, and the cap clause of Theorem 24.5 gives the two-sided bound (SW1°)$_{\mathrm{int}}$ on this one set for every $\kappa \in [\kappa_0, \infty)$ with the same constants, so that the representations of the stopping-scale integral at two such masses are integrals over the same set. Theorem 25.11 compares the capped reduced actions at any $\kappa_0 \le \kappa \le \kappa' < \infty$ on this set, the Compton thresholds κ_j being a bookkeeping of the κ-integral at which no test or representation changes, so the threshold terms t_j of the comparison of two adjacent representations vanish. □

Lemma 25.13 (the disc bound (SW2$^{\min}$))**.** *Let $(\mathfrak{s}, \chi)$ be a two-level configuration for Q_b with the shell U and the sharp profile $\chi = \mathbf{1}_U$ (Definitions 23.2 and 23.3), f a real 2-form in C_c^∞ supported in Q_b, $g = 0$, and k with $\varepsilon_k \le \min(1/\mu_0, 1/4)$. For $\kappa \in [0, \infty)$ let $\langle\cdot\rangle_\kappa := \langle\cdot\rangle_{(J', n\psi + \kappa\chi), \Lambda_k}$ be the two-level state of Definition 23.5, in which the shell mass $\kappa\mathbf{1}_U$ is the second level of the wall (it raises the scalar mass squared by 2κ on U, Definition 16.4(c)), let $\langle\cdot\rangle_\infty$ be the walled torus state of $\mathfrak{c} = (J', n\psi)$ with the matter killed on $D_\chi \cap \Lambda_k$ (Definition 16.4(d)), let $Z_\kappa(zf, 0)$, $z \in \mathbb{C}$, be the partition function of $\langle\cdot\rangle_\kappa$ with the source zf and no scalar source, and put*

$$\Psi_\kappa(z) := Z_\kappa(zf, 0)/Z_\kappa(0, 0) = \langle e^{zF(f)}\rangle_\kappa, \qquad \kappa \in [0, \infty],$$

so that $\Psi_\kappa(\mathrm{i}) = \Psi_k(n, \kappa)$ and $\mathrm{tail}_k(K) = \Psi_K(\mathrm{i}) - \Psi_\infty(\mathrm{i})$; let C be the photon covariance of these states and v the vector with $\langle v, A\rangle = F(f)$, as in Lemma 16.16. Then at fixed spacing $z \mapsto \Psi_\kappa(z)$ is entire, and the disc bound for the heavy shell at zero scalar source holds:

$$\sup_{|z| \le 2} |Z_\kappa(zf, 0)/Z_\kappa(0, 0)| \le B_\Psi := 2e^{2\langle v, Cv\rangle} < \infty$$

uniformly in $\kappa \in [0, \infty]$ and in ε ($\langle v, Cv\rangle$, the variance of $F(f)$ under the free massive photon, is bounded uniformly in ε for smooth f). At $g = 0$ this is the Gaussian domination of Lemma 16.16(a) alone, and it is all that Theorem 23.12, stated at $g = 0$, uses; the disc bound with the scalar source g, with the constant $2e^{4B} M_T(g)$, is proved in Section 27 from Theorems 24.5 and 25.11 with the scalar source.

Proof. At fixed spacing $z \mapsto \Psi_\kappa(z)$ is a finite-dimensional integral of $e^{zF(f)}$ against a measure with Gaussian photon tails, hence entire. For $|z| \le 2$, $|e^{zF(f)}| \le e^{2F(f)} + e^{-2F(f)}$, and Lemma 16.16(a) with $s = \pm 2$ gives $\langle e^{\pm 2F(f)} \rangle_\kappa \le e^{2\langle v, Cv\rangle}$ in every state with the photon covariance C, the shell mass and the killing acting on the matter only; hence $|\Psi_\kappa(z)| \le 2e^{2\langle v,Cv\rangle} = B_\Psi$ for all $\kappa \in [0,\infty]$. □

26. The exceptional set, the convention bridge and right-continuity in the mass

This section treats the exceptional set E_W of couplings. Proposition 26.8 defines the set; at its points the rotation theorem for curvature sources (Theorem 15.1) is obtained by right-continuity in the scalar mass (Theorem 26.7). The section also proves a bound on the block vertices at long trees (Proposition 26.1).

The convention bridge of Section 19 relates two conventions for the counterterms, which differ by a constant shift of the mass term and by a field-independent factor; its constant $\Delta_T(s)$ is subject to a positivity condition in the proof of the rotation theorem. Proposition 26.8 shows that this condition, or a quantitative alternative to it, holds for every large J at every point of King's range outside an exceptional set E_W of couplings, a finite set of scalar masses at fixed (e, λ, μ_0). The section also proves a bound on the block vertices at long trees (Proposition 26.1), in the notation of the cube limits with scalar sources.

The statements of this section enter the proof of Theorem 15.1 through the torus comparison (K_0), the walled limit (Theorem 19.17) and the cube limits with scalar sources (Section 27), in the forms used there; their proofs rest on Sections 19, 24 and 25. Proposition 26.8, the positivity condition on the constant of the convention bridge, defines the exceptional set E_W of Theorem 15.1, at whose points the theorem is obtained by right-continuity in the scalar mass (Theorem 26.7).

Proposition 26.1 is stated in the notation of Section 27 and of the split degree count (Theorem 20.3) and uses two bounds of King's cube construction, which also enter Section 27: King's cut of the long lines [25, (3.38), p. 666], as (R1) the bound on the sum of a renormalised family on King's cube at a localisation τ whose tree has length $d_\tau > p(\delta_k)$ in units of the blocks of scale k ($p(\delta)$ the length threshold of that passage), in the zone of the faces of the cube and away from it, and (R2) its summed form (the weights $e^{-\delta \operatorname{dist}(P(\tau))}$ of the zone-B tuples summed with $e^{\delta' R}$, $\delta' \le \delta/2$); and King's long-tree bound [25, (3.41)], as (R3) the bound $C(5^{-\gamma k}\delta_k^{-\beta} + \delta_k^{\sigma})|C_J|$ on the long trees in the difference clause of the split degree count on King's cubes at the two cut-offs of a diagonal pair. The localisations τ, the zone of the faces and the diagonal pair of cut-offs below are those of (R1)–(R3).

Proposition 26.1 (block vertices at long trees in the families with one inserted field)**.** *On King's cube C_J^K with the flow of weight f_D, let $\varepsilon = 5^{-K}$, $\delta_k := 5^{k-K}$, $0 \le k \le K - J_*$ and $p(\delta)$ be as in (R1), and let the scalar source be zero. Consider the families of the perturbative part at a background $B = B' + \sigma X$ on C_J^K, $\sigma \in [0,1]$, whose first differences and their Hölder quotients of exponent α_0 are bounded by $C_1 p(\delta_k)$ (the class of* [3, pp. 412–413]*), among them the backgrounds $A_\theta + \sigma X_0$, with A_θ Bałaban's cut-off photon background and X_0 its difference from King's background, that carry one inserted vertex whose additional leg is the external field X (the field X_0 in that case), and whose graphs, with every block vertex split into its two summation points as in the split count (Theorem 20.3), contain a block vertex. Then at every localisation τ of such a family whose tree has length $d_\tau > p(\delta_k)$ in units of the blocks of scale k, in the zone of the faces of the cube and away from it, the sum $E(\tau)$ of the renormalised family at τ obeys at each of the two cut-offs of a diagonal pair the bound of (R1) multiplied by $\|X\|_{1,\alpha_0}$, the supremum of X, of its first differences and of their Hölder quotients of exponent α_0, with constants independent*

of k, K, B' and X (within the class, at fixed C_1); so that, by [25, (3.41)]*, these localisations contribute at most $C_N\delta_k^N\|X\|_{1,\alpha_0}|C_J|$ for every N.*

Proof. The bound is (R3) for these families, with the three further rows that the inserted field requires (one cluster for each product of block vertices, the cut lines paired at the ends of the products, the cut photon lines with an end on a contour, and the ordering bound with the split count), each at the per-tuple bound of (R1) times $\|X\|_{1,\alpha_0}$ and summed by [25, (3.41)]. □

Proposition 26.2 (the fluctuation estimates for centre-rooted and hybrid in-block contours)**.** *Bałaban's construction transports the fields inside each block along a contour $\Gamma_{r,x}$ from a point x of the block to its reference point r* [1, (2.1)–(2.3)]*, and King's flow uses axial contours rooted at a corner of the block* [25, (2.12)]*; the flows of the torus theorem use in-block contours of two other families, the axial contours rooted at the centre of each block, of length at most $\frac{3}{2}$ times the side of the block, the coarse variable being attached to the centre, and the composite contours of the hybrid flow, of length at most* 2.1 *times the side of the block of each level. Every estimate of the fluctuation integrals of* [1, 2, 3]*, among them* [3, Prop. 1]*, and King's upper bound* [25, Thm. 3.1]*, in the forms that the torus theorem uses (Section 18.3), holds for these two families in place of King's, with constants that depend on the family through the bound $c_\Gamma \in \{\frac{3}{2}, 2.1\}$ on the length of its contours in units of the block side and, for the hybrid family, also through the hybrid kernel bounds (Theorem 19.10) and the geometry of the tiles.*

Proof. Bałaban's and King's proofs use the contour family only through properties shared by the three families ($QQ^* = \mathbf{1}$ for the block average, unitarity of the contour phases, gauge covariance with the phase at the root, path independence at a constant field, length bounds, containment in the block, King's tail convention [25, (2.19)], a segment property used once at [2, p. 560, (2.16)], and the Green's-function bounds of [4]), with constants at most King's. The Green's-function bounds pass to the centre-rooted family by a change of root, a diagonal unitary on the coarse variables leaving $P_k = Q_k^*Q_k$, the Green's functions and the moduli of all kernels invariant, and for the hybrid family (Definition 19.8, Lemma 19.9) they are the hybrid kernel bounds (Theorem 19.10(ii)); the segment property is not used there, and Bałaban's bound for the fluctuation field [2, Lemma 2.4, (2.65)–(2.66)] follows by the steps of his proof with the hybrid lengths. □

The threshold proposition. In the notation introduced before Lemma 24.3 (a King torus T at spacing $s = \varepsilon_K$ in the torus convention; $N(0,h) = N_{K,T}^{\mathrm{tor},z}(0,h)$ with $z_K = a_K^\sharp$ the scheme constant of Definition 27.1, the reference scheme $z = a^\sharp$; the convex $h \mapsto \ln N(0,h)$ and $f(u) = \ln N(0, u\cdot 1)$; the full torus coefficient $\alpha_s^T(\mathsf{M})$ of a scalar mass M and the relative coefficient $s_{K,T}(\mathsf{M})$), let $\alpha_\varepsilon(\mathsf{M}) = \mathsf{M}^2 + \delta m_\varepsilon^2(\mathsf{M})$ be the full coefficient of $\frac{1}{2}\sum_x s^3|\phi(x)|^2$ on $\varepsilon\mathbb{Z}^3$ in the free convention and $s_\varepsilon^{\mathrm{f}}(\mathsf{M}) := \frac{1}{2}(\alpha_\varepsilon(m) - \alpha_\varepsilon(\mathsf{M}))$. Its limit $s_{\inf}(\mathsf{M}) := \lim_{\varepsilon\to 0} s_\varepsilon^{\mathrm{f}}(\mathsf{M})$, which exists by part (i) below, is the relative coefficient of Proposition 26.8, with $\alpha_{\inf}(\mathsf{M}) - \alpha_{\inf}(m) = -2s_{\inf}(\mathsf{M})$ and $s_{\inf} = s_\infty$ of Section 29; the *threshold* $s_*(e) = \sup_{\mathsf{M}>0} s_{\inf}(\mathsf{M}) \in [0,\infty]$ is that of Definition 27.2 (equal to s_* of (81)) and depends on (λ, m, μ_0) as well.

Proposition 26.3 (the threshold proposition)**.** *(i) For every scalar mass $\mathsf{M} > 0$ the limit $\lim_{\varepsilon\to 0}\big(\delta m_\varepsilon^2(m) - \delta m_\varepsilon^2(\mathsf{M})\big)$ of the free-boundary counterterm differences on $\varepsilon\mathbb{Z}^3$ exists; hence $s_{\inf}(\mathsf{M}) = \lim_{\varepsilon\to 0} s_\varepsilon^{\mathrm{f}}(\mathsf{M})$ exists, is a polynomial of degree at most two in (e^2,λ) with finite coefficients, and is Lipschitz in M on compact subsets of $(0,\infty)$, with $s_{\inf}(m) = 0$. This is a statement about the free-boundary coefficient on $\varepsilon\mathbb{Z}^3$ alone.*

(ii) Let $0 \le q \in C_c^\infty(\mathbb{R}^3)$ with $\bar q := \sup q < s_*(e)$. There are $l_0 \in 2M\mathbb{N}$ (M the large-block integer of Definition 16.1) such that $\operatorname{supp} q$ lies in one cell $c + [0,l_0)^3$, K_0 and $C(q) < \infty$ such that on every King torus T with sides in $l_0\mathbb{N}$, with $Q := \sum_{b\in l_0\mathbb{Z}^3/T} q(\cdot - b)$ the

l_0-periodisation of q,

$$\sup_{K\ge K_0} \ln N^{\mathrm{tor},z}_{K,T}(0,Q) \le C(q)\,|T|.$$

Proof. (i) The derivative in M^2 of each of the five orders λ, e^2, λ^2, $e^2\lambda$, e^4 of δm^2_ε is a Brillouin-zone integral of the subtracted lattice integrand, convergent as $\varepsilon\to 0$ by Reisz's theorem in its periodic form [35] (Theorem 19.14) uniformly in the mass on compact sets [34]; integration from M^2 to m^2 gives the limit, its polynomial form and the Lipschitz bound. (ii) The torus coefficient deviates from the free one by $O(\ell(T)^{-4})$ uniformly in K and in the mass on compact sets, so, as $s_{\inf}$ is continuous with $s_{\inf}(m)=0$ and $\sup s_{\inf}>\bar q$, there are $l_0\in 2M\mathbb{N}$ ($\operatorname{supp} q$ in one cell), K_0 and two masses whose torus coefficients satisfy $\bar q<s_1<s_2$ uniformly on the tori with sides in $l_0\mathbb{N}$ for $K\ge K_0$, where $|f(s_i)|\le A|T|$ by Lemma 24.3 and the stability bounds of Theorem 19.3. With $P:=\bar q\cdot 1-Q\ge 0$, the second Griffiths inequality [9, Thm. 6.1] gives $\ln N(0,Q)\le f(\bar q)-\ln N(0,P)$, translation averaging and convexity give $\ln N(0,P)\ge f(\bar p)$ for some $\bar p\in[0,\bar q]$, and the chord and slope inequalities of the convex f on $0\le\bar p\le s_1<s_2$ bound both terms by a constant times $|T|$. □

Lemma 26.4 (the bridge constant decays in the period)**.** *Let $M>0$, let $\Delta_\tau(s;M)$ be the constant of the convention bridge of Proposition 26.8, computed at the scalar mass M in place of m, on the pair τ of a lattice and a torus of that proposition (so that $\Delta_\tau(s;m)$ is its constant $\Delta_T(s)$ on the torus T of the pair), and put*

$$\kappa_M:=\tfrac12\min(M,\mu_0,2),\qquad \Phi_\kappa(l):=\big(1+(\kappa l)^{-1}\big)^6 l^{-2}e^{-\kappa l/\sqrt3},\qquad l_0(M):=2\sqrt3\,\pi/\kappa_M.$$

There is $K_\Delta(M)<\infty$, depending on (e,λ,μ_0,M) and on nothing else (not on k, on J or on the frame), such that for each of the three pairs τ, with torus D_J or RD_J of side $l=l_J:=50M_{\mathrm b}J\ge l_0(M)$ (here $M_{\mathrm b}$ denotes the block integer, written M elsewhere in this paper, since in this lemma M is the scalar mass), and every admissible spacing $s=\varepsilon_k\le 1$,

$$|\Delta_\tau(s;M)|\le K_\Delta(M)\,\Phi_{\kappa_M}(l); \tag{71}$$

the limit $\Delta_\tau(0;M):=\lim_{s\to0}\Delta_\tau(s;M)$ exists and obeys the same bound. The function $\Phi_\kappa(l)$ is strictly decreasing in l and non-increasing in κ, and tends to 0 as $l\to\infty$.

Proof. The convention bridge ((CT)(b), with the rate clause of Theorem 19.6(d)) uses the scalar mass only through the scalar propagators, the strip width of the strip coercivity and the condition $s\kappa\le1$, so it holds at the mass M with the strip width κ_M for every $s\le1$, frame by frame: $\sup_{s\le1}|\Delta_\tau(s;M)|<\infty$, $\Delta_\tau(0;M)$ exists and $|\Delta_\tau(s;M)-\Delta_\tau(0;M)|\le Cs$. Up to King's overall sign, $\Delta_\tau(s;M)$ is a finite sum over the one- and two-loop graphs of King's counterterm of differences (torus value) − (free value), each an image sum over the period lattice P of an integrand analytic in a strip of width κ_M, a Riemann sum over the dual lattice of an outer factor times such an image sum (bounded for $l\ge l_0(M)$, where the dual Voronoi cells have circumradius $\sqrt3\,\pi/l\le\kappa_M/2$), or an exact cancellation. Poisson summation bounds each image sum by a constant free of l, k, J and the frame times $\sum_{\gamma\in P\setminus0}e^{-\kappa_M|\gamma|_\infty}|\gamma|_\infty^{-2}\le 81\,\Phi_{\kappa_M}(l)$, a bound that uses only the shortest vector l of P; it passes to $s\to0$, and the monotonicity of Φ_κ is read off its factors. □

Right-continuity in the scalar mass at E_W. Fix (e,λ,μ_0); for a scalar mass $M>0$ the *King point* $\tau_M:=(e,\lambda,M,\mu_0)$ is King's model with scalar mass M and everything else equal, and $\tau_*:=\tau_m$. On King's cube C^K_J (Definition 2.3) at spacing ε_K the lattice measures of τ_M and τ_m differ, up to field-independent constants, only by the single-site factors $\exp(-\frac{\varepsilon_K^3}{2}t_K(M)r_x^2)$ at the live sites, $t_K(M):=\alpha_K(M)-\alpha_K(m)$ with the bare coefficient $\alpha_K(M)=M^2+\delta m^2_{\varepsilon_K}(M)$; as $K\to\infty$, $t_K(M)\to t_\infty(M):=\alpha_\infty(M)-\alpha_\infty(m)$ for every $M>0$, which defines α_∞ up to an additive

constant as a function real-analytic on $(0,\infty)$ (the closed form of Section 29; Remark 28.3), and E_W of (81) is the set of couplings at which m is a global minimiser of α_∞. A *source* u is a real smooth compactly supported 2-form, $F(u)$ the curvature source and $J_0(u)$ as in Theorem 5.47; for $J \geq J_0(u)$ put $G_{K,J}(M) := \langle e^{\mathrm{i}F(u)}\rangle^K_{K,J;\tau_M} = \langle \cos F(u)\rangle^K_{K,J;\tau_M} \in [-1,1]$ (real by charge conjugation), $\Gamma_J(M) := \lim_K G_{K,J}(M)$, which exists by Theorem 5.47 at τ_M, and $\Gamma(M) := \lim_J \Gamma_J(M)$, which exists by Lemma 26.5(b); $\Gamma(M;u)$ is King's state at τ_M on $e^{\mathrm{i}F(u)}$. For a finite family $f=(f_1,\dots,f_N)$ of sources and $z\in\mathbb{C}^N$ put $a_{K,J}(z;M) := \langle \exp\sum_j z_j F(f_j)\rangle^K_{K,J;\tau_M}$, $Z^f_J(z;M) := \lim_K a_{K,J}(z;M)$ and $Z^f_\infty(z;M) := \lim_J Z^f_J(z;M)$, King's curvature functional at τ_M (the object Z^K_∞ of Theorem 7.6); by Lemma 5.20, $|a_{K,J}(z;M)| \leq a_{K,J}(\operatorname{Re} z;M) \leq e^{\beta_f|\operatorname{Re} z|^2}$ with $\beta_f = C_0\sum_j |f_j|_4^2$ free of K, J and the couplings, and the bound passes to the limits.

Lemma 26.5 (the two monotonicities)**.** *(a) (in the bare coefficient) For every K, every $J \geq J_0(u)$ and every M with $t_K(M)\geq 0$: $G_{K,J}(M) \leq G_{K,J}(m)$.*

(b) (in the cube) For every K and $J\geq J_0(u)$: $G_{K,J}(M) \leq G_{K,J+1}(M)$. Hence at every King point $J\mapsto \Gamma_J(M)$ is non-decreasing, $\Gamma(M)=\sup_J\Gamma_J(M)$ exists, and $Z^f_\infty(\,\cdot\,;M)$ exists, is entire and obeys $|Z^f_\infty(z;M)| \leq e^{\beta_f|\operatorname{Re} z|^2}$.

Proof. (a) The measures with single-site factors $\exp(-\frac{\varepsilon_K^3}{2}\theta r_x^2)$ at the live sites, $0\leq\theta\leq t_K(M)$, interpolate between τ_m and τ_M within the class of Lemma 2.10, whose part (c) says that such a factor does not increase $\langle\cos F(u)\rangle$. (b) Passing from C^K_{J+1} to C^K_J removes photon-gradient and hopping terms, none of which increases $\langle\cos F(u)\rangle$ (Lemma 2.10(a),(b); [26, Thm. 3.2, (3.32)] at $h=0$), the single-site measures being the same on both cubes. Hence $Z^f_J(\mathrm{i}t;M) = \Gamma_J(M;\sum_j t_j f_j)$ converges for $t\in\mathbb{R}^N$, and the family $Z^f_J(\,\cdot\,;M)$, normal by the bound $e^{\beta_f|\operatorname{Re} z|^2}$, converges locally uniformly on $\mathbb{C}^N$ to an entire limit with the same bound, $(\mathrm{i}\mathbb{R})^N$ being a set of uniqueness. □

Theorem 26.6 (continuity in the offset, uniformly in the spacing)**.** *In the geometry of the proof of Theorem 26.7(a) below (torus T, box H' with the dead set $U := T\setminus H'$, wall $n\psi$, source u supported in $V\subset H'$), let the* dead-shell model with offset $c\in[0,c_{\max}]$ *be the endpoint $\kappa=\infty$ of the heavy-shell model whose counterterm is King's free-boundary $\delta m^2_{\varepsilon_K}(m)$ plus the constant c, declared of order λ (in King's complexified counterterm vertex [25, (3.19)] it reads $\delta m^2(x;e'e,\lambda'\lambda)+\lambda' c$): periodic photon on T, matter alive on H' with Dirichlet condition at $\partial H'$, single-site coefficient $\alpha_K(m)+c+n\psi(x)$; and let $X^{[c]}_K$ be its expectation of $\cos F(u)$ at the spacing ε_K. Then*

$$\lim_{c\downarrow 0}\ \limsup_{K\to\infty}\ \big|X^{[c]}_K - X^{[0]}_K\big| = 0.$$

Proof. Let $a^{[c]}_K(x) := \langle e^{xF(u)}\rangle$ in the offset model, entire in x with $X^{[c]}_K = a^{[c]}_K(\mathrm{i})$ by charge conjugation, and $I^{[c]}_D(x)$ the small-field integral of the dead-shell effective action at a stopping scale $\eta_0\leq\eta_1$ over its c-independent test set T_k; the two-sided heavy-shell bound at the endpoint (Theorem 24.5 at $\kappa=\infty$), whose offset clause covers the constant c declared of order λ as the single graded summand $c_{01}=c$, holds with one set of constants for all $c\in[0,c_{\max}]$: for $\varepsilon_K\leq\varepsilon_0(\eta_0)$,

(72)
$$e^{-a'}\,\frac{I^{[c]}_D(x)}{Z^{[c]}_\infty(0)} \ \leq\ \frac{Z^{[c]}_\infty(x)}{Z^{[c]}_\infty(0)} \ \leq\ e^{a}\,\frac{I^{[c]}_D(x)}{Z^{[c]}_\infty(0)} + b, \qquad a = C\eta_0^\sigma|T|,\quad a'\leq 2a,\quad b = e^{-c_{\mathrm{lf}}\,p(\eta_0)^2+C|T|},$$

with $c_{\mathrm{lf}}>0$ the large-field constant of the heavy-shell estimates. The offset enters only as the factor $\lambda' c$ of one vertex, so the two effective actions differ on T_k by a polynomial of degree at

most 13 in c with coefficients bounded on T_k uniformly in K, which Markov's inequality bounds by $C_W(\eta_0)\,c$ up to a constant cancelling in the ratios $I_D^{[c]}(x)/I_D^{[c]}(0)$, which Gaussian domination on the dead-shell torus (Lemma 2.11) bounds uniformly in c and K. The sandwich at x and at 0 gives $\sup_{|x|\le 1}|a_K^{[c]}(x)-a_K^{[0]}(x)|\le \Phi(c,\eta_0)$ with $\lim_{c\downarrow 0}\Phi(c,\eta_0)\to 0$ as $\eta_0\to 0$, the left side being free of η_0, and a normal-family argument carries the convergence to $z=\mathrm{i}$. □

Theorem 26.7 (right-continuity of King's state in the scalar mass at E_W)**.** *Let $\tau_*=(e,\lambda,m,\mu_0)\in E_W$.*

(a) (the lower half) For every source u, $\liminf_{M\downarrow m}\Gamma(M;u)\ge\Gamma(m;u)$.

(b) For every finite family $f=(f_1,\dots,f_N)$ of sources,

$$Z^f_\infty(\,\cdot\,;M)\longrightarrow Z^f_\infty(\,\cdot\,;m)\qquad \text{locally uniformly on } \mathbb{C}^N \text{ as } M\downarrow m;$$

in particular each smeared curvature Schwinger function converges, $S_n(u_1,\dots,u_n;\tau_M)\to S_n(u_1,\dots,u_n;\tau_*)$.

(c) The nearby King points τ_M, $0<|M-m|<\delta$, lie off E_W.

Proof. (c) At $\tau_*\in E_W$, $t_\infty=\alpha_\infty-\alpha_\infty(m)$ is real-analytic, non-negative, vanishes at m and is not constant ($t_\infty(M)=M^2+O(M)$ as $M\to\infty$ by the closed form of Section 29), so $t_\infty>0$ on $0<|M-m|<\delta$ and $t_K(M)>0$ for $K\ge K_1(M)$. (a) Fix u, $J\ge J_0(u)$ and an admissible geometry of the heavy-shell estimates: a torus T, a box $H'\subset T$ containing a source region $V\supset C_J$ with $\operatorname{supp}u\subset V$ at distance ≥ 1 from the dead set $U:=T\setminus H'$, a wall $n\psi$ ($n\ge 1$, $\psi\ge 0$ vanishing on V) and a King cube $C_{J''}\supset H'$ with collar width $w_c:=\operatorname{dist}(H',\partial C_{J''})\ge 1$ whose translates by the periods of T have disjoint interiors; let $X_K^{[c]}$ be $\langle\cos F(u)\rangle$ in the *dead-shell model* at spacing ε_K (periodic photon on T, matter alive on H' with Dirichlet condition at $\partial H'$, single-site coefficient $\alpha_K(m)+c+n\psi(x)$, $c\ge 0$ declared of order λ as in Theorem 26.6). The photon switch across the matter-free collar (Neumann images) changes expectations of $e^{\mathrm{i}F(u)}$ by at most $\delta_g(w_c)\to 0$ as $w_c\to\infty$, uniformly in the single-site terms and in T, and the monotonicities of Lemma 2.10 give $G_{K,J}(m)\le X_K^{[0]}+\delta_g(w_c)$ and $G_{K,J''}(M)\ge X_K^{[c]}-\delta_g(w_c)$ whenever $0\le t_K(M)\le c$; with $c=2t_\infty(M)$, $M\in(m,m+\delta)$, and $K\to\infty$, $\Gamma(M)\ge\Gamma_{J''}(M)\ge\Gamma_J(m)-2\delta_g(w_c)-\limsup_K|X_K^{[c]}-X_K^{[0]}|$ (Lemma 26.5(b)), the last term tending to 0 as $M\downarrow m$ by Theorem 26.6 at the fixed geometry; then $w_c\to\infty$ at fixed J, then $J\to\infty$. (b) Lemma 26.5(a) and (a) give $\Gamma(M;u)\to\Gamma(m;u)$, that is, convergence of $Z^f_\infty(\mathrm{i}t;M)=\Gamma(M;\sum_j t_jf_j)$ on the set of uniqueness $(\mathrm{i}\mathbb{R})^N$; the bound $e^{\beta_f|\operatorname{Re}z|^2}$ of Lemma 26.5(b), uniform in M, makes the family normal, so the convergence is locally uniform on $\mathbb{C}^N$ and the Taylor coefficients S_n converge. □

Any identity between smeared curvature Schwinger functions holding at the nearby King points transfers to τ_*: if $\Gamma(M;g\cdot u)=\Gamma(M;u)$ for a Euclidean motion g, every source u and all M with $0<M-m<\delta$, then $\Gamma(m;g\cdot u)=\Gamma(m;u)$ by Theorem 26.7(b), and the Schwinger functions follow with $\sum_j t_ju_j$.

Throughout, for a scalar mass $m'>0$ let $\alpha_k(m'):=m'^2+\delta m^2_{T,k}(m')$, where $\delta m^2_{T,k}(m')$ is King's mass counterterm [25, (3.32)–(3.33)] at the mass m', computed with the propagators of the torus T at spacing ε_k, and let $\Delta_T(s)$, $s\le s_0$, be the constants of the convention bridge (CT)(b) on T, with $s_0=1$ (so that every spacing $\varepsilon_k\le 1$ is covered).

Proposition 26.8 (the positivity condition on the constant of the convention bridge)**.** *For each $R\in\mathcal{R}_0$ and each of the three pairs of a lattice and a torus used, the axis lattice on D_J and the lattice rotated by R on D_J and on RD_J, there are $p_H>1$, a mass $m_1>0$ with (e,λ,m_1,μ_0) in King's range and a threshold, depending on the couplings, the block integer M, p_H, m_1 and R (equal to 1 if $\Delta_T(s)\ge 0$ for all $s\le s_0$ and every J), such that for every J above it either*

$\Delta_T(s) \geq 0$ *for all* $s \leq s_0$, *or*

$$\liminf_{k\to\infty} \big(\alpha_k(m) - \alpha_k(m_1)\big) > p_H \sup_{s\leq s_0} \max\{0, -\Delta_T(s)\};$$

J_{pos} *denotes the maximum of these thresholds over the finite set* $\mathcal{R}_0$ *and the three pairs. This holds at every point of King's range,* $e > 0$, *outside the set* E_W *of the points at which the scalar mass* m *minimises, at fixed* (e, λ, μ_0), *the function* α_∞ *of the mass defined, up to an additive constant, by the limits* $\alpha_\infty(m') - \alpha_\infty(m) = \lim_k \big(\alpha_k^{\mathrm{f}}(m') - \alpha_k^{\mathrm{f}}(m)\big)$, *where* α_k^{f} *is* α_k *with King's counterterm computed in the free-boundary conventions;* E_W *is the set of the masses minimising that function: for fixed* (e, λ, μ_0) *these masses form a non-empty finite set, and for fixed* (λ, m, μ_0) *the set* E_W *contains at most two values of* e^2 *(both read off the closed form of* α_∞*: the four definitions of the free-boundary coefficient coincide, and its mass differences converge to the closed form* $\alpha_{\mathrm{inf}}(m') - \alpha_{\mathrm{inf}}(m) = -2s_{\mathrm{inf}}(m')$, *where* $s_{\mathrm{inf}}(m') := \lim_k \frac{1}{2}\big(\alpha_k^{\mathrm{f}}(m) - \alpha_k^{\mathrm{f}}(m')\big)$ *is a polynomial of degree at most two in* (e^2, λ) *(Proposition 26.3), real-analytic in the mass (Remark 28.3), so that* α_∞ *and* α_{inf} *differ by a mass-independent constant and* E_W *is the set of global minimisers of* α_{inf}*).*

Proof. Fix $R \in \mathcal{R}_0$ and a pair τ with torus T, and write $\Delta_\tau(s; m')$ for the bridge constant at the scalar mass m' (Lemma 26.4), $\Delta_T(s) = \Delta_\tau(s; m)$; if $\Delta_T(s) \geq 0$ for all $s \leq s_0$ and every J the threshold is 1, and otherwise the exact bridge $\alpha_k(m') = \alpha_k^{\mathrm{f}}(m') - \Delta_\tau(\varepsilon_k; m')$ (Remark 16.8) gives

$$\alpha_k(m) - \alpha_k(m_1) = \big(\alpha_k^{\mathrm{f}}(m) - \alpha_k^{\mathrm{f}}(m_1)\big) - \Delta_\tau(\varepsilon_k; m) + \Delta_\tau(\varepsilon_k; m_1),$$

where the bracket tends to $2s_{\mathrm{inf}}(m_1)$ (Proposition 26.3, a result on $\varepsilon_k\mathbb{Z}^3$ alone) and $|\Delta_\tau(s; m')| \leq K_\Delta(m')\, \Phi_{\kappa_{m'}}(50MJ)$ for $s \leq 1$ once $50MJ \geq l_0(m')$ (Lemma 26.4). Outside E_W there is $m_1 > 0$ with $\sigma_1 := s_{\mathrm{inf}}(m_1) > 0$; with $\kappa_1 := \frac{1}{2}\min(m, m_1, \mu_0, 2)$, $K_1 := \max\{K_\Delta(m), K_\Delta(m_1)\}$ and J_1 the least J with $\ell = 50MJ \geq 2\sqrt{3}\,\pi/\kappa_1$ and $(2 + p_H)K_1\Phi_{\kappa_1}(\ell) \leq \sigma_1$, for $J \geq J_1$ the $\liminf_k$ of the left side exceeds $p_H \sup_{s\leq s_0} \max\{0, -\Delta_T(s)\} \leq p_H K_1 \Phi_{\kappa_1}(\ell)$ by at least σ_1, where κ_1, K_1, σ_1 are free of τ, R and J, so one m_1 serves all pairs, and no sign of Δ_T is used. The statements on E_W follow from the closed form: α_{inf} is real-analytic and not constant (Remark 28.3), with $\alpha_{\mathrm{inf}}(m') = m'^2 + O(m')$ as $m' \to \infty$ (Section 29), so its global minimisers are isolated and finitely many, and for fixed (λ, m, μ_0) the condition $\partial_{m'}\alpha_{\mathrm{inf}}(m) = 0$ is a polynomial equation of degree at most two in e^2. □

The scalar-mass variable is written m' above and M or M, M_1 in Remark 26.9, Proposition 26.3, Theorem 26.7 and Lemma 26.4. The constant c_* of the large-field clause of Theorem 19.3 at a non-zero wall is $(c_0'/2)(1 - 1/p_H)$ ($c_* = c_0'/4$ at $p_H = 2$ if $\Delta_T \geq 0$), with $c_0' = \min\{c_0'^{\mathrm{AS}}, c_0'^{\mathrm{D}}\} \in (0, 1]$, where $c_0'^{\mathrm{AS}} = \min\{a/8, \gamma_*/4, 1\}$ (a the block-spin coefficient of [25], $\gamma_* = \min(\gamma_0^s, \gamma_v)$ the exponent of [2, (3.41)]) is the constant of Bałaban's kept convergence factors [2, (3.41)] at block factor 5 and $c_0'^{\mathrm{D}}$ the constant of the upper-bound estimates of the shell model.

Remark 26.9 (the exceptional set and the scope of the general-source results). The torus coefficients $\alpha_k^\tau(M)$ enter Proposition 26.8 only through its $\liminf$ clause, where the exact bridge $\alpha_s^\tau(M) = \alpha_s^{\mathrm{f}}(M) - \Delta_T(s; M)$ between the torus and the free-boundary coefficient at the spacing s is used with the decay bound on the bridge constants Δ_T, as a bound and not as an identity; the free-boundary coefficient is one number however it is computed (as $\alpha_k^{\mathrm{f}}(M)$ of Proposition 26.8, at the spacing ε_k in the torus conventions, through the closed form of α_∞, or on the infinite lattice $\varepsilon_k\mathbb{Z}^3$), its mass differences converge to $\alpha_\infty(M) - \alpha_\infty(m) = -2s_{\mathrm{inf}}(M)$ in the notation of Definition 27.2, so that $\alpha_\infty - \alpha_{\mathrm{inf}}$ is independent of the mass, $E_W = \{m \in \operatorname{argmin}_M \alpha_\infty\} = \{s_*(e) = 0\}$ exactly, and $s_*(e) > 0$ at every other point of King's range. On E_W the cone, the tube and the

split domain of Definitions 27.2 and 27.3 are empty, their defining inequalities being strict, and nothing of the general-source results is claimed there, the rotation theorem being obtained by right-continuity of King's state in the scalar mass; E_W does meet King's range (a numerical evaluation of the closed form of α_∞ finds points of E_W there, for instance at $\lambda = 0.05$, $m = \mu_0 = 1$, $e^2 \approx 12.086$), for fixed (λ, m, μ_0) it contains at most two values of e^2, since $E_W \subset \{\alpha'_\infty(m) = 0\}$ and $e^2 \mapsto \alpha'_\infty(m)$ is a polynomial of degree at most two (Lemma 29.4(iii)), and what happens at its points beyond the right-continuity argument is problem 1 of Section 32.6.

Proposition 26.10 (the positivity margin)**.** *Fix (e, λ, m, μ_0) in King's range, $p \geq 1$ and $\Gamma \geq 0$ with $p\Gamma < s_*(e) = \sup_{M>0} s_{\inf}(M)$, $s_{\inf}$ as in Definition 27.2 (so outside E_W). Pick a King point M_1 with $\sigma_1 := s_{\inf}(M_1) > p\Gamma$ and put $\kappa_1 := \min(m, M_1, \mu_0, 2)/2$ and $K_1 := \max\{K_\Delta(m), K_\Delta(M_1)\}$, where $K_\Delta(M)$ is the constant of the decay bound*

$$|\Delta_\tau(s; M)| \leq K_\Delta(M)\Phi_{\kappa(M)}(l)$$

of Lemma 26.4 on the bridge constants of Proposition 26.8 at the scalar mass M in the torus side $l \geq l_0(M)$, with Φ_κ as there and the rate $\kappa(M) = \frac{1}{2}\min(M, \mu_0, 2) \geq \kappa_1$ for $M \in \{m, M_1\}$. Let the margin threshold $J_{\mathrm{mar}}(\Gamma)$ *be the least J with $l_J \geq 2\sqrt{3}\pi/\kappa_1$ (l_J the side of D_J, Definition 16.1; then $l_J \geq l_0(m)$ and $l_J \geq l_0(M_1)$) and $(2+p)K_1\Phi_{\kappa_1}(l_J) \leq \sigma_1 - p\Gamma$. Write $-2u_k(M) := \alpha_k(m) - \alpha_k(M)$ for the torus coefficients of Proposition 26.8 and*

$$2V_+^{(p)}(\tau) := p \sup_{s\in S} \max\{0, -\Delta_\tau(s; m)\},$$

S the set of spacings $s \leq s_0$ of that proposition and τ one of its three pairs of a lattice and a torus. Then for $J \geq J_{\mathrm{mar}}(\Gamma)$, on each of the three pairs,

$$\liminf_{k\to\infty} \big(-2u_k(M_1)\big) \;\geq\; 2V_+^{(p)}(\tau) + 2p\Gamma + (\sigma_1 - p\Gamma),$$

that is, the positivity condition of Proposition 26.8 holds at the one point M_1 with the margin $2p\Gamma$ to spare; with $\bar V := \frac{1}{2}\sup_{s\in S}|\Delta_\tau(s; m)|$ in place of $V_+^{(1)}(\tau)$, both signs of the bridge constant being admitted, the same bound holds with $2p\bar V$ in place of $2V_+^{(p)}(\tau)$.

Proof. $\Phi_\kappa(l)$ decreases strictly to 0, so $J_{\mathrm{mar}}(\Gamma)$ exists. The bridge $\alpha_k(M) = \alpha_k^{\mathrm{f}}(M) - \Delta_\tau(\varepsilon_k; M)$ at m and M_1 with the decay bound of Lemma 26.4 ($\kappa_1 \leq \kappa(m), \kappa(M_1)$) gives $-2u_k(M_1) \geq 2s^{\mathrm{f}}_{\varepsilon_k}(M_1) - 2K_1\Phi_{\kappa_1}(l_J)$ with $s^{\mathrm{f}}_{\varepsilon_k}(M_1) := \frac{1}{2}(\alpha_k^{\mathrm{f}}(m) - \alpha_k^{\mathrm{f}}(M_1)) \to \sigma_1$ (Remark 26.9), and $2V_+^{(p)}(\tau)$, $2p\bar V \leq pK_1\Phi_{\kappa_1}(l_J)$. Subtracting, $\liminf_k(-2u_k(M_1)) - 2V_+^{(p)}(\tau) \geq 2\sigma_1 - (2+p)K_1\Phi_{\kappa_1}(l_J) \geq \sigma_1 + p\Gamma$ for $J \geq J_{\mathrm{mar}}(\Gamma)$, with M_1, K_1, κ_1 free of τ and J. □

Here and below $\Delta_T(s; M)$ and $\Delta_\tau(s; M)$ are the bridge constants of Lemma 26.4 on the torus T and on the pair τ, $\Delta_T(s) := \Delta_T(s; m)$, $\Delta_T(0; M) := \lim_{s\to 0}\Delta_T(s; M)$, and S, $V_+^{(p)}(\tau)$ and $-2u_k^\tau(M)$ (the coefficient $-2u_k(M)$ on the pair τ) are as in Proposition 26.10. The source families of the walled torus comparison at a configuration $\mathfrak{c} = (J, \psi, n)$ and a source g (Definition 27.10) are the moving families $h_K = g - n\psi - \frac{1}{2}\Delta_T(\varepsilon_K)$ and $h_{0,K} = -n\psi - \frac{1}{2}\Delta_T(\varepsilon_K)$ with their limits at $s = 0$, the sources $(tG, g - n\psi + u\cdot 1)$ and $(0, -n\psi + u\cdot 1)$ with $u = u(s) := -\frac{1}{2}\Delta_\tau(s; m)$, $s \in S$, their images under the reflections and the rotation of the pair (which preserve sup norms and positive parts), and the mirrors $-\beta h_K$, $-\beta h_{0,K}$ with their limits, $\beta \in (0, 1]$; $\Gamma_g := \sup g^+$, $\Delta_* \geq \sup_{s\in S}|\Delta_T(s; m)|$ is the two-sided bound on the bridge constants, and $\bar H := \|g\|_\infty + n\sup\psi + \frac{1}{2}\Delta_*$.

Theorem 26.11 (the choice rule and the envelope on the cone)**.** *Fix a point of King's range outside E_W, $p_H > 1$, and g with $p_H\Gamma_g < s_*(e)$; fix once, per value of Γ_g, a King point $M_1 =$*

$M_1(\Gamma_g)$ with $\sigma_1 := s_{\inf}(M_1) > p_H\Gamma_g$ ($M_1(0)$ the King point of Proposition 26.8), let κ_1, K_1 be as in Proposition 26.10, and set

$$J_{\rm pos}(g) := \max\{J_{\rm pos},\ J_{\rm mar}(\Gamma_g)\},$$
$$J_{\rm mar}(\Gamma_g) = \min\{J:\ l_J \ge 2\sqrt{3}\pi/\kappa_1,\ (2+p_H)K_1\Phi_{\kappa_1}(l_J) \le \sigma_1 - p_H\Gamma_g\},$$

$J_{\rm pos}$ the threshold of Proposition 26.8 and $J_{\rm mar}(\Gamma_g)$ the margin threshold of Proposition 26.10 at $p = p_H$, $\Gamma = \Gamma_g$; this is the definition of $J_{\rm pos}(g)$ used throughout Section 27. Then $J_{\rm pos}(g)$ depends on g through Γ_g only, not on n, ψ, β, τ or R; it is finite, so that the admissible J are cofinal; and $J_{\rm pos}(0) = J_{\rm pos}$. Let $\mathfrak{c} = (J,\psi,n)$ with $J \ge J_{\rm pos}(g)$, $J \in 5^{k_*}\mathbb{N}$ and $J \ge J_0(\varepsilon_0)$. Then on each of the three pairs:

(A) (margin) $\liminf_k(-2u_k^\tau(M_1)) - 2V_+^{(p_H)}(\tau) \ge 2\sigma_1 - (2+p_H)K_1\Phi_{\kappa_1}(l_J) \ge \sigma_1 + p_H\Gamma_g > 2p_H\Gamma_g$. Hence the positivity condition of Proposition 26.8 holds at M_1 with the margin $p_H(\Gamma_g + V_+^{(1)}(\tau))$, and the walled large-field envelope, King's upper bound [25, Thm. 3.1, (3.3)–(3.4)] in the envelope form of (BAL)$_5$ (Theorem 19.3), holds at every source of the direct families and their images and at $\{h_K\}\cup\{h_\infty\}$, with the large-field constant $c_* = (c_0'/2)(1-1/p_H)$ and an exponent q free of K; no β enters.

(B) (mirrors) For the same M_1 and J, and every $\beta \in (0,1]$ with

$$(\beta)\qquad 2p_H\,\beta\bar H < \sigma_1 + p_H\Gamma_g \qquad (p_H = 2:\ 4\beta\bar H < \sigma_1 + 2\sup g^+),$$

the positivity condition holds at M_1 with the margin $p_H\beta\bar H$, hence the envelope holds at $-\beta h_K$, $-\beta h_{0,K}$ and their limits, which is the upper line of clause (a) of ($EX_W^\sharp{}'(\beta)$) (Definition 27.11), also at $(K+1,k+1)$. The admissible β form $(0,1]\cap(0,\beta_0)$ with $\beta_0 := (\sigma_1 + p_H\Gamma_g)/(2p_H\bar H) > \Gamma_g/\bar H$, a non-empty set also at $\Gamma_g = 0$.

Proof. (A) is Proposition 26.10 at $p = p_H$, $\Gamma = \Gamma_g$ for $J \ge J_{\rm pos}(g) \ge J_{\rm mar}(\Gamma_g)$, with the same K_1, κ_1 on every pair, and $\sigma_1 > p_H\Gamma_g$. By $(a+b)^+ \le a^+ + b^+$ the direct families and their images have positive part at most $\Gamma_g + V_+^{(1)}(\tau)$ and every member is $\ge -\bar H$ with oscillation $\le 2\bar H$, so each direct source at the amplitude p_H is $v\cdot 1 - q'$ with $-p_H\bar H \le v \le p_H(\Gamma_g + V_+^{(1)}(\tau))$ and $0 \le q' \le 2p_H\bar H$, its positive part is paid from the margin (A), and King's envelope at a non-positive source applies with an exponent $q = q(G,\Gamma_h,V_-,|T|,p_H)$ independent of the shape of the source. (B) The mirrors have positive part at most $\beta\bar H$ and (A) gives $\liminf_k(-2u_k^\tau(M_1)) \ge \sigma_1 + p_H\Gamma_g > 2p_H\beta\bar H$ under (β); as $J_{\rm pos}(g)$ is free of n, ψ and β, the configuration $(n+K)\psi$ stays in the class for every height K with any $\beta < \min(1,\beta_0(K))$, and the admissible J are cofinal. □

The final-scale cap. Bałaban's lower and upper bounds [1, (1.14)], [2, (1.3)] for the partition functions at the fixed masses end at a final spacing $t_K \in (\varepsilon_0/5, \varepsilon_0]$, at which the large-field clause of Theorem 19.3 at the wall on the skew torus D_J takes one step; so that the skew tori do not wrap at that scale, the coarsest scale ε_0 of the flow is at most a threshold $\varepsilon_{\rm sys}(M,R,\rho_{\rm B},\delta_{\rm fb})$ fixed by the large-block integer M, the rotation R, Bałaban's radius $\rho_{\rm B}$ and the finest block size $\delta_{\rm fb}$, independent of J and of the scalar mass and closed downward, under which the non-wrapping threshold at the final scale equals 1 for every J in the residue class of Definition 16.6, so that every configuration of the class $\mathcal{C}$ meets it and the wall condition holds on every skew torus $R^{-1}D_J$. The cap is a proviso of the torus theorem (Theorem 18.8) on its parameters, carried by the class $\mathcal{C}$ (Definition 16.6, Remark 21.5), not an assumption of Theorem 15.1.

The odd-block clause of the degree census. The degree census of the κ-Cauchy estimate (Theorem 25.11) has a clause for the closed matter blocks of degree -2 or -1 with one photon end: after the rewriting they appear as classes with two photon-type ends, a background leg and a line slot, of degree exactly -1 or 0, entering the enumeration as black boxes, and after the summation by parts on the trees with one background slot every proper prefix sub-component of a

renormalised member has integer degree at least one or lies inside a black box, which enters the enumeration as a local vertex at the window scale. This is Theorem 25.10(a), proved from Furry's theorem with the fundamental theorem of calculus in the background and the Ward identity for the window class, which controls the zeroth and first moments of the two-photon kernel in the background slot (Theorem 25.10(f)); Theorem 25.10(e) bounds the tail of the coarse counterterm at orders two to four that its one-loop form leaves.

27. King's state with scalar sources, and its Euclidean invariance

This section extends Theorem 15.1 to neutral scalar sources: outside the exceptional set E_W, King's state with one further vacuum counterterm, the scheme $\Theta^\sharp$, exists on an explicit tube of complex coefficients around the origin and is invariant under every Euclidean motion there (Theorem 27.13). It uses the argument of Sections 18–23 carried out on a cone of scalar sources, together with the cube limits with scalar sources of the existence part; nothing in it is used by Theorem 15.1, and at $h = 0$ every statement below reduces to the corresponding statement of Sections 17–23.

For scalar sources with $\int h \neq 0$ King's Wick ordering has no finite continuum limit beyond the curvature sector: with the free Wick constant alone the expectation of the Wick square of Definition 2.5 is $(e^2/4\pi^2)\ln(1/\varepsilon)\int h + O(1)$ at order $e^2\lambda^0$ in formal perturbation theory, so it diverges at order e^2 (Proposition 16.13), and the Wick constant must absorb this logarithm, which grows by $(e^2/4\pi^2)\ln 5$ per step of the block factor 5. The object with scalar sources is therefore King's state with one further vacuum counterterm, linear in h and depending on the lattice spacing only, which we call a *scheme*; the state is constructed, in the scheme $\Theta^\sharp$ below, on a tube of complex coefficients around the origin whose real points are the sources whose mass-decreasing part is below a threshold fixed by the couplings (Definition 27.2). The torus comparison in its walled form is provided by the torus theorem on a mixed source family (Theorem 27.8; Theorem 27.9), and the closing statement on the cone (the lower bound of the state by the envelope of the walled values, Section 27.4) is obtained from the heavy-shell estimate at one configuration, which the shell estimates at the scalar source provide.

27.1. Schemes, the threshold, the cone and the tube.

This subsection fixes the vocabulary of the section: the schemes, real sequences indexed by the spacing, of which $\Theta^\sharp$ is one; the threshold $s_*(e)$; the cone of sources on which the walled comparison is run; and the tube $\Omega_*(h)$ of complex coefficients around the origin on which the theorem is stated.

Definition 27.1 (schemes)**.** A *scheme* is a real sequence $a = (a_k)_{k\geq 0}$ such that a_k depends only on ε_k and on (e, λ, m, μ_0). For a state $\langle\cdot\rangle_{\Lambda,X}$ of Definition 16.4 on a lattice of spacing ε_k put

$$Z^a_{\Lambda,X}\Big(\sum_j z_j f_j, \sum_j w_j h_j\Big) := e^{-\sum_j w_j a_k \mathsf{S}_\Lambda(h_j)}\Big\langle \Theta\Big(\sum_j z_j f_j, \sum_j w_j h_j\Big)\Big\rangle_{\Lambda,X},$$

that is, the expectation of $\exp(\sum_j z_j F(f_j) + \sum_j w_j \sum_x \varepsilon^3 h_j(x)[|\phi(x)|^2 - 2\mathsf{W}_\varepsilon - a_k])$. We write Θ^a for the corresponding observable. For walled tori $\mathfrak{c} = (J', w)$ and (f, g) with $w = 0$ on the support of g put

$$T^a_{\mathfrak{c},\Lambda}(f, g) := \langle\Theta^a(\mathrm{i}f, g)\rangle_{\mathfrak{c},\Lambda} = \frac{Z^a_{\Lambda,D_{J'}}(\mathrm{i}f, g - w)}{Z^a_{\Lambda,D_{J'}}(0, -w)}.$$

Examples. (i) King's scheme $a \equiv 0$. (ii) The scheme $\Theta^\sharp$: $a^\sharp_k := e^2 X_{\varepsilon_k}$, where X_ε is the coefficient of e^2 in the perturbative expansion of $\langle|\phi(0)|^2\rangle$ for the model on $\varepsilon\mathbb{Z}^3$ at $\lambda = 0$, with the order-e^2 part of δm^2_ε inserted; we write $\Theta^\sharp := \Theta^{a^\sharp}$. The scheme and the reference scheme $z = a^\sharp$ are those of Definition 2.5 (the functional $Z^a_{k,X}$ there is $Z^a_{\Lambda_k,X}$ here), so $\Theta^\sharp$ is the reference scheme of Definition 28.1.

The threshold is defined from the function α_∞ of Proposition 26.8; the curvature sector carries no threshold.

Definition 27.2 (the threshold, the cone and the tube)**.** Fix (e,λ,m,μ_0) in King's range and let α_∞ be the function of the scalar mass of Proposition 26.8 (defined up to an additive constant). Put

$$s_*(e) := \tfrac{1}{2}\sup_{M>0}\big(\alpha_\infty(m)-\alpha_\infty(M)\big) \in [0,\infty];$$

$s_*(e)=0$ exactly when m is a global minimiser of α_∞, that is, exactly at the points of the set E_W of Proposition 26.8, and $s_*(e)>0$ at every other point of King's range ($s_*(e)$ is the threshold s_* of (81), and E_W is the set defined there). The *cone* is

$$\mathcal{K}_{<s_*} := \{(f,g)\in\mathcal{K}:\ \sup_x 2g(x) < s_*(e)\},$$

which is invariant under $O(3)$ and under the translations. For a finite family of real 2-forms $f_j\in C_c^\infty$ and real $h_j\in C_c^\infty$, $j\le N$, the *tube* is the open convex set

$$\Omega_*(h) := \Big\{(z,w)\in\mathbb{C}^N\times\mathbb{C}^N:\ \sup_{x\in\mathbb{R}^3} 2\Big(\sum_j \operatorname{Re} w_j\,h_j(x)\Big)_+ < s_*(e)\Big\} \ni 0,$$

which contains $\mathbb{C}^N\times(\{u\in\mathbb{R}^N:\sum_j u_jh_j\le 0\}+\mathrm{i}\mathbb{R}^N)$ and is cut only on the mass-decreasing side; it is invariant under $(f_j,h_j)\mapsto(\tau_bR\cdot f_j,\tau_bR\cdot h_j)$ for every Euclidean motion.

Definition 27.3 (the split of a scalar source and the split domain)**.** Let f_j and h_j, $j\le N$, be as in Definition 27.2. Split each $h_j=h_j^+-h_j^-$ with $h_j^-:=\|h_j\|_\infty\chi_j$ and $h_j^+:=h_j+h_j^-$, where $\chi_j\in C_c^\infty$ takes values in $[0,1]$, equals 1 on $\operatorname{supp}h_j$ and is supported in $N_{1/2}(\operatorname{supp}h_j)$, so that $h_j^\pm\ge 0$ are smooth. With coefficients $(z,w^+,w^-)\in\mathbb{C}^{3N}$ of the curvature source $\sum_j z_jf_j$ and of the scalar source $\sum_j(w_j^+h_j^+ + w_j^-h_j^-)$, let $k_1:=\sum_j[(\operatorname{Re}w_j^+)_+h_j^+ + (\operatorname{Re}w_j^-)_+h_j^-]$, the sum of the positive parts of the terms of the split source at the real parts of the coefficients (an upper bound for the positive part of the split source), and let the *split domain* $D(h)\subset\mathbb{C}^{3N}$ be the open convex set of coefficients with $\sup 2k_1 < s_*(e)$: the bound k_1 on the positive part of the split source is below half the threshold, and the negative part is arbitrary. Its trace $\{w^+=w,\ w^-=-w\}$ lies in $\Omega_*(h)$ and contains a neighbourhood of 0. At the points of E_W the cone, the tube and the split domain are empty (strict inequalities), and only the curvature sector remains there.

The value $s_*(e)=+\infty$, which would occur if α_∞ were unbounded below, is admitted: every statement of this section uses $s_*(e)$ only through strict inequalities "$<s_*(e)$", through $s_*(e)>0$ and, in Section 27.4, through the exponent $q'(g)$ of (54) and the threshold $J_1(g)$ defined there, both taken in $[0,\infty]$ ($q'(g)=2$ and $J_1(g)=1$ at $s_*(e)=+\infty$), so nothing below depends on its finiteness. The function α_∞ is King's counterterm [25, (3.32)–(3.33)] with the propagators of $\varepsilon\mathbb{Z}^3$ and King's free boundary condition [26, pp. 329–330]; that $E_W=\{s_*(e)=0\}$ exactly follows from the closed form of α_∞ (the four definitions of the free-boundary coefficient coincide, and the mass differences of α_∞ converge, as the spacing tends to zero, to the closed form $\alpha_{\inf}(M)-\alpha_{\inf}(m)=-2s_{\inf}(M)$, where $s_{\inf}(M)$ is the limit of the free-boundary counterterm difference $\frac{1}{2}(m^2-M^2)+\frac{1}{2}[\delta m_\varepsilon^2(m)-\delta m_\varepsilon^2(M)]$; so α_∞ and $\alpha_{\inf}$ differ by a mass-independent constant, $s_*(e)=\sup_M s_{\inf}(M)$, and E_W, the set of global minimisers of $\alpha_{\inf}$, is exactly $\{s_*(e)=0\}$, with $s_*(e)>0$ off E_W; the torus form of α_∞ and the scope of the statements with scalar sources are the subject of Remark 26.9).

27.2. **The condition (W$^\sharp$) and King's state on the cone.** This subsection defines the condition (W$^\sharp$) on a scheme, shows that the scheme $\Theta^\sharp$ is admissible below the threshold (Theorem 27.5) and derives King's state with scalar sources on the cone (Theorem 27.6).

Definition 27.4 ((W$^\sharp$): the scheme condition on the cone)**.** Let a be a scheme and write $Z^{a,K}_{k,J}$ for the functionals Z^a of Definition 27.1 in the state $\langle\cdot\rangle^K_{k,J}$ of King's cube C^K_J (Definition 2.3).

(W$^\sharp$1) For every King cube C^K_J, every finite family of real 2-forms $f_j \in C^\infty_c$, $j \le N$, sampled at plaquette centres in the lattice's own frame, and of non-negative scalar sources $0 \le g_i \in C^\infty_c$, $i \le N'$, all supported at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_J$, the limit $\lim_k Z^{a,K}_{k,J}(\sum_j z_j f_j, \sum_i v_i g_i)$ exists locally uniformly on the open convex set $\{(z,v) \in \mathbb{C}^N \times \mathbb{C}^{N'} : \sup_x 2\sum_i (\mathrm{Re}\, v_i)_+ g_i(x) < s_*(e)\}$, and it lies in $(0,\infty)$ at the real points of that set. For a family $(f_j; h_j)$ of real scalar sources split as in Definition 27.3, applied to the non-negative family $\{h^+_j\} \cup \{h^-_j\}$, this is the existence of $\lim_k Z^{a,K}_{k,J}(\sum_j z_j f_j, \sum_j (w^+_j h^+_j + w^-_j h^-_j))$ locally uniformly on the split domain $D(h)$, in $(0,\infty)$ at the real points of $D(h)$.

(W$^\sharp$2) For every real $0 \le q \in C^\infty_c$ with $\sup q < s_*(e)$, $\sup_{J \ge J_q} \limsup_k Z^a_{\Lambda_k, C_J \cap \Lambda_k}(0,q) < \infty$, where J_q is the least J such that $\mathrm{supp}\, q$ has distance > 1 from $\mathbb{R}^3 \setminus C_J$.

A scheme a with (W$^\sharp$1) and (W$^\sharp$2) is called *admissible$_{<s_*}$*; Theorem 27.5 below proves both for the scheme $\Theta^\sharp$.

At $h = 0$ the limit of (W$^\sharp$1) is King's cube limit [26, Thm. 2.1] (Theorem 5.47); (W$^\sharp$2) is the analogue, below the threshold and in the lim sup form, of the bound [26, (3.43)], which in King's Wick ordering fails for $\int h_+ > 0$ at order e^2.

Theorem 27.5 (the scheme $\Theta^\sharp$ is admissible$_{<s_*}$)**.** *At every point of King's range, the scheme $\Theta^\sharp$ of Definition 27.1 satisfies* (W$^\sharp$1)*, for every King cube and every family as stated there, and* (W$^\sharp$2)*. Hence $\Theta^\sharp$ is admissible$_{<s_*}$.*

The proof uses no statement beyond those used for Theorem 15.1. (W$^\sharp$1) is the cube construction of the existence part run with a non-negative scalar source below the threshold, on King's cubes, with the cut of the long lines (R1), the bound at long trees in the families with one inserted field at a scalar source and the rate of the split degree count at a non-zero scalar source; the limit on the split domain is then a normal-family argument: on $D(h)$ the cube functionals are locally bounded, by (W$^\sharp$2) for the positive parts below the threshold $s_*(e)$ and by the bounds at non-positive sources for the negative parts, and they converge at small real sources by the cube construction, so by Vitali's theorem they converge locally uniformly on $D(h)$ to a holomorphic limit. (W$^\sharp$2) is proved in the displayed lim sup form by a convention bridge between the counterterm conventions, from the ultraviolet stability bounds: the exact mass shift between the conventions on a finite lattice is Lemma 24.3; the chessboard estimate [17, Thm. 4.1] on a torus with sides in $l_0\mathbb{N}$ localises the extensive bound for the periodised source pq to its single cell, the Hölder exponent $p > 1$ absorbing the passage to the free convention; and the sign of the threshold is a lattice computation: $\delta m^2_\varepsilon(m) - \delta m^2_\varepsilon(M_1)$ is a finite sum of massive lattice integrals with at most two loops, Reisz's absolute bound [34] gives its $\varepsilon \to 0$ limit, so that $s_{\inf}(M_1)$ exists and is Lipschitz in M_1, with a torus correction $O(l^{-4})$, and an M_1 with $s_{\inf}(M_1) > 0$ exists exactly when m is not a global minimiser of $\alpha_{\inf}$.

Theorem 27.6 (King's state in an admissible$_{<s_*}$ scheme, on the cone)**.** *Let a be admissible$_{<s_*}$ (Definition 27.4). For a family of real 2-forms f_j and real h_j in C^∞_c, $j \le N$, let J_0 be such that all supports have ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_{J_0}$, and write $Z^{a,K}_J(\zeta) := \lim_k Z^{a,K}_{k,J}(\zeta)$ for $\zeta = (z,w)$.*

(i) For $J \ge J_0 + 1$, $Z^{a,K}_{k,J} \to Z^{a,K}_J$ locally uniformly on $\Omega_*(h)$; $Z^{a,K}_J$ is holomorphic on $\Omega_*(h)$ and lies in $(0,\infty)$ at its real points. For a one-member family $(f; g)$ with $g \ge 0$ the same holds already for $J \ge J_0$.

(ii) *For $(f,g)\in\mathcal{K}_{<s_*}$ supported at ℓ^∞-distance >1 from $\mathbb{R}^3\setminus C_{J_0}$, $J\mapsto Z_J^{a,K}(\mathrm{i}f,g)$ is nondecreasing for $J\geq J_0$; if $s_*(e)>0$, then for every $(f,g)\in\mathcal{K}$ so supported, $J\mapsto Z_J^{a,K}(f,-g)$ is nonincreasing for $J\geq J_0$.*

(iii) *$Z_\infty^a(\zeta):=\lim_{J\to\infty}Z_J^{a,K}(\zeta)$ exists locally uniformly on $\Omega_*(h)$ and is holomorphic there; $Z_\infty^a(f,h)\in(0,\infty)$ at every real point of $\Omega_*(h)$; and $Z_J^{a,K}\leq Z_\infty^a$ on $\mathcal{K}_{<s_*}$.*

(iv) *For every $\rho>0$, $\Phi^a(s):=\ln Z_\infty^a(s)$ is finite, convex and continuous on the open convex set $U_\rho:=\{(f,h)\in E_\rho\times C_c^\infty(\bar B_\rho):\sup 2h_+<s_*(e)\}$.*

(v) *$Z_\infty^a(\tau_b\cdot\zeta)=Z_\infty^a(\zeta)$ for all $b\in\mathbb{R}^3$ and $\zeta\in\Omega_*(h)=\Omega_*(\tau_b\cdot h)$.*

(vi) *If the scheme a' has $a'_k-a_k\to\kappa\in\mathbb{R}$, then a' is admissible$_{<s_*}$ and $Z_\infty^{a'}=e^{-\kappa\sum_j w_j\int h_j}Z_\infty^a$ on $\Omega_*(h)$; if $s_*(e)>0$ and a' is admissible$_{<s_*}$, then a'_k-a_k converges.*

The proof re-centres the family at a point of the tube, where (W$^\sharp$1) on the split domain gives the cube limits, and then runs the argument of Theorem 17.1: the monotonicity in J of (ii) from the cutting lemma at a non-negative source, the limit $J\to\infty$ with its holomorphy from the bounds of (W$^\sharp$2), convexity and continuity as in Lemma 2.17, and the translations as in Theorem 17.1(v).

27.3. **The walled torus comparison on the cone.** The comparison on the cone rests on the torus theorem of Section 18.3 run on a family of sources that mixes the 2-form source with the scalar one; we state this form first.

Definition 27.7 (the mixed source family and the scheme of the torus theorem)**.** Let e^2,λ,m^2,μ_0^2 be in King's range, off the exceptional set E_W of Proposition 26.8, let $J\in 5^{k_*}\mathbb{N}$ satisfy the final-scale cap, let $\mathfrak{c}=(J,\psi,n)$ be a walled configuration with $n>0$ (the wall condition of the class $\mathcal{C}$ of Definition 16.6 is not assumed), G a real smooth D_J-periodic 2-form and g a real smooth D_J-periodic function, with $J\geq J_{\mathrm{pos}}(g)$, the threshold of Theorem 26.11 defined before Theorem 27.9 (a function of $\sup g^+$ alone), and, when g has a positive part, $2\sup g<s_*(e)$. Let $h:=g-n\psi$, $h_0:=-n\psi$ and

$$\mathcal{S}_W:=\{(tG,h):t\in\mathbb{R}\}\cup\{(0,h_0)\},$$

the *mixed source family*. The scheme is that of the torus theorem, $z_K=e^2X_{\varepsilon_K}(m;\mu_*)$ along $\varepsilon_K=5^{-K}$ ($X_\varepsilon(m;\mu)$ the coefficient X_ε of Definition 27.1 at the scalar mass m and the photon mass μ, and $\mu_*>0$ a *fixed reference photon mass*, independent of K, at which the scheme constant is computed; it need not equal the photon mass μ_0 of the model, and z_K differs from $a_K^\sharp=e^2X_{\varepsilon_K}(m;\mu_0)$ by $e^2[X_{\varepsilon_K}(m;\mu_0)-X_{\varepsilon_K}(m;\mu_*)]$, so that $a_K^\sharp-z_K$ converges, by statement (4) of Section 27.3).

Theorem 27.8 (the torus comparison on the mixed source family)**.** *In the setting of Definition 27.7, assume the hypotheses (N1)–(N5) of Section 19 at the sources of $\mathcal{S}_W$ on D_J, at their S^*-images on RD_J, and at their coordinate-reflected and permuted images on D_J with the S^*-images of these on RD_J. Then, along $\varepsilon_K=5^{-K}$ and in the scheme z_K:*

(i) *(*two-lattice comparison, scheme-free*) $Z_{Y_K}(D_J;s)/Z_{X_K}(D_J;s)\to 1$ for every $s\in\mathcal{S}_W$. Consequently $\langle e^{zF(G)}\rangle_{\mathfrak{c},g,Y_K}-\langle e^{zF(G)}\rangle_{\mathfrak{c},g,X_K}\to 0$ locally uniformly in $z\in\mathbb{C}$, all moments $\langle F(G)^m\rangle_{\mathfrak{c},g}$ agree in the limit, $T_{\mathfrak{c}}^{Y_K}(tG,g)/T_{\mathfrak{c}}^{X_K}(tG,g)\to 1$ for every real t, and for complex z, $T_{\mathfrak{c}}^{Y_K}(zG,g)-T_{\mathfrak{c}}^{X_K}(zG,g)=\rho_K\cdot o(1)$ locally uniformly, with $\rho_K:=T_{\mathfrak{c}}^{X_K}(0,g)$.*

(ii) *(*existence on the axis lattice*) If $\lim_K\langle e^{tF(G)}\rangle_{\mathfrak{c},g,X_K}$ exists for every real t (the walled existence (N6$_W$) at the mixed source), both families converge, locally uniformly on $\mathbb{C}$, to the same entire function $W_{\mathfrak{c},g}(zG)$, with all moments. If moreover the single-torus hypotheses (EX$_W^\sharp{}'(\beta)$) of Definition 27.11 hold for one amplitude $\beta\in(0,1]$, then*

$$\rho_K=e^{-z_K\langle g\rangle_{X_K}}Z_{X_K}(D_J;0,h)/Z_{X_K}(D_J;0,h_0)\to\rho_\infty\in(0,\infty),$$

hence $T^{X_K}_{\mathfrak{c}}(zG,g)$ *and* $T^{Y_K}_{\mathfrak{c}}(zG,g)$ *converge, locally uniformly in* $z\in\mathbb{C}$, *to* $T_{\mathfrak{c}}(zG,g):=\rho_\infty W_{\mathfrak{c},g}(zG)$; *at* $zG=\mathrm{i}f$ *this is the axis-lattice limit of the walled ratio of Definition 27.1.*

(iii) (generators) *(i) and (ii) hold with* R *replaced by any* $R'\in\mathcal{R}_0$.

The order of limits is $\varepsilon\to 0$ *along* ε_K *at fixed* $\mathfrak{c}$, g, G, *and* β *is chosen at fixed* $\mathfrak{c}$ *and* g.

In the application $g\ge 0$ and $\psi=0$ on the 1-neighbourhood of $\operatorname{supp} g$; neither the sign of g nor any support property of g is used in the theorem. The proof is that of Theorem 18.8 (the forward bound and the closing of Section 18.3) run on the family $\mathcal{S}_W$ in place of $\{(tG,h_0)\}$: the instance of (N5) at the mixed source is Proposition 19.16, whose wall bound holds for every bounded real wall, $\|h\|_\infty\le\sup|g|+n\sup\psi$, with no sign of h, no condition on $\min h$ and no condition on $\langle h\rangle$; the source-linear constant c_K tends to zero, so no closure of $\mathcal{S}_W$ under $(G,h)\mapsto(G,-h)$ is needed; and (ii) adds the identification of the limit at the limiting source $h_\infty=h-\frac12\Delta_T(0)$ of the convention bridge ($\Delta_T(s)=\Delta_T(s;m)$ the bridge constant of Proposition 26.8 on $T=D_J$, Definition 27.10) with the Vitali upgrade to complex coefficients. In (i) the scheme factors differ by $e^{-z_K(\langle g\rangle_{Y_K}-\langle g\rangle_{X_K})}=1+O((1+K)\varepsilon_K^N)$; no Wick scheme enters (i) or (iii) beyond the growth bound $|z_K|\le C(1+K)$.

For $0\le g\in C_c^\infty$ let $J_{\mathrm{pos}}(g):=\max\{J_{\mathrm{pos}},J_{\mathrm{mar}}(\sup g)\}$ be the threshold of Theorem 26.11: the larger of the threshold J_{pos} of Proposition 26.8 and the least J at which the positivity margin of Proposition 26.10 exceeds $2p_H\sup g$, $p_H=2$ (the right side of the display of Proposition 26.8 enlarged by $2p_H\sup g$), at a King point $M_1=M_1(\sup g)$ (a scalar mass with (e,λ,M_1,μ_0) in King's range, as in Proposition 26.8) with $\frac12(\alpha_\infty(m)-\alpha_\infty(M_1))>p_H\sup g$, which exists exactly when $p_H\sup g<s_*(e)$. For a bounded $X\subset\mathbb{R}^3$ let $\mathcal{A}_W(X;g)$ be the set of the walled configurations $\mathfrak{c}=(J',n\psi)\in\mathcal{A}(X)$ of Definition 21.1 (so $\psi=0$ on $N_1(X)$) with $\psi\ge 0$ smooth, $\psi\not\equiv 0$ and $D_{J'}$-periodic, $n>0$, $J'\in 5^{k_*}\mathbb{N}$ and $J'\ge\max(J_0(\varepsilon_0),J_{\mathrm{pos}}(g))$, under the cap of Section 26 imposed for every $R\in\mathcal{R}_0$ at once; the class $\mathcal{C}$ of Definition 16.6 for R and L is contained in $\mathcal{A}_W(X;0)$ whenever $X\subset R^{-1}C_L$. The *walled value* $\mathcal{W}^a_{\mathfrak{c}}(f,g)$ is the common limit of the walled ratios $T^a_{\mathfrak{c},\Lambda_k}(f,g)$ and $T^a_{\mathfrak{c},R^{-1}\Lambda_k}(f,g)$ of Definition 27.1, when it exists, and the *envelope* $\mathcal{W}^{a,\mathrm{env}}(f,g)$ is the limit in b of $\inf_{\mathfrak{c}\in\mathcal{A}_W(Q_b;g)}\mathcal{W}^a_{\mathfrak{c}}(f,g)$ for $(f,g)\in\mathcal{K}_{<s_*}$ supported in Q_b.

Theorem 27.9 (the walled torus comparison holds on the cone)**.** *At every point of King's range outside* E_W, *for every* $R\in\mathcal{R}_0$, *every bounded* $X\subset\mathbb{R}^3$, *every* $(f,g)\in\mathcal{K}_{<s_*}$ *with* $g\ne 0$ *supported in* X *and every* $\mathfrak{c}\in\mathcal{A}_W(X;g)$, *the limits* $\lim_k T^{a^\sharp}_{\mathfrak{c},\Lambda_k}(f,g)$ *and* $\lim_k T^{a^\sharp}_{\mathfrak{c},R^{-1}\Lambda_k}(f,g)$ *exist and are equal; the same holds for every scheme* a *with* $a_k-a^\sharp_k$ *convergent. At* $g=0$ *the same holds for every* $\mathfrak{c}$ *of the class* $\mathcal{C}$ *(Theorem 18.1; the walled ratio does not depend on the scheme there).*

Definition 27.10 (the moving families, their limits and their mirrors)**.** Let $\mathfrak{c}=(J,\psi,n)$, $n\ge 0$, g, $h=g-n\psi$ and $h_0=-n\psi$ be as in Definition 27.7 (there $n>0$; $n=0$, the bare torus, is admitted here), let $\Delta_T(s)$ be the constant of the convention bridge of Proposition 26.8 on the torus $T=D_J$ at spacing s, with $\Delta_T(0):=\lim_{s\to0}\Delta_T(s)$ (Lemma 26.4), and put $h_K:=h-\frac12\Delta_T(\varepsilon_K)$, $h_{0,K}:=h_0-\frac12\Delta_T(\varepsilon_K)$ (the *moving families*) and $h_\infty:=h-\frac12\Delta_T(0)$, $h_{0,\infty}:=h_0-\frac12\Delta_T(0)$ (their limits), so that $h_K-h_{0,K}=h_\infty-h_{0,\infty}=g$; fix $\beta\in(0,1]$, write $a^\sharp_K=e^2X_{\varepsilon_K}(m;\mu_0)$ for the scheme constant of Definition 27.1 at spacing ε_K ($X_\varepsilon(m;\mu)$ the coefficient X_ε of that definition at the scalar mass m and the photon mass μ, as in statement (4) below), and let $H':=\|g\|_\infty+n\sup\psi+\frac12\Delta_*$ be the sup-norm bound of the families with their bridge shift, Δ_* the bound of the convention bridge of the class $\mathcal{C}$ of Definition 16.6. The four moving families h_K, $h_{0,K}$, $-\beta h_K$, $-\beta h_{0,K}$ and their limits lie in one bounded set of King's class, of sup norm at most H', the *mirrors* $-\beta h_K$, $-\beta h_{0,K}$ having sup norm at most $\beta H'$.

Definition 27.11 (($\mathrm{EX}^\sharp_W{}'(\beta)$): the single-torus statements with a mirror of amplitude β)**.** Let the moving families h_K, $h_{0,K}$, their limits h_∞, $h_{0,\infty}$, the mirrors of amplitude $\beta \in (0,1]$, the scheme constant $a^\sharp_K$ and the bound H' be as in Definition 27.10. The condition *($EX^\sharp_W{}'(\beta)$)* is the conjunction of the following four statements for King's torus-convention model on the axis lattice X_K and the torus D_J, along $\varepsilon_K = 5^{-K}$:

(a) the two-sided single-torus bound at the four families h_K, $h_{0,K}$, $-\beta h_K$, $-\beta h_{0,K}$, at the depth $k(K)$ and at $(K+1, k+1)$: its upper line is the walled large-field envelope, King's upper bound [25, Thm. 3.1] in the envelope form of Theorem 19.3, with the large-field coefficient $c_* = (c_0'/2)(1 - 1/p_H)$; its lower line is the lower bound by the small-field integral, the small-field half of King's bound with sources [25, Thm. 3.1] at the fixed source size $S_0 = H'$, for both signs of the source and at any amplitude (Theorem 19.6(c));
(b) King's low-field positivity [25, Thm. 3.4] with $n = 1$ at the four limiting sources h_∞, $h_{0,\infty}$, $-\beta h_\infty$, $-\beta h_{0,\infty}$, for the effective action minus only its field-independent h-linear constant at the orders 1, λ, e^2, in the restricted form of Remark 20.13(i);
(c) King's vacuum-energy bound [25, Thm. 2.1] at zero source, in torus conventions;
(d) the bulk bound at order one: the source-linear field-independent constant of the effective action at the orders 1, λ, e^2, in the free conventions and for the constant weight, taken together with the scheme term $a^\sharp_K$, is bounded by $C\delta_k^{-2}$, and its increments between the depth k at spacing ε_K and the depth $k+1$ at spacing ε_{K+1} are summable along $k(K)$ (Lemma 6.19).

The mirrors enter the proof of Theorem 27.8(ii) only through its telescoping step. On the cone, for $J \geq J_{\mathrm{pos}}(g)$ and outside E_W, the upper line of (a) is provided by Theorem 26.11, (A) at the direct families and (B) at the mirrors for every β with the clause (β) there; (b), (c) and (d) involve β only through the choice of the sources.

The statement of Theorem 27.9 at $g \neq 0$ is the torus theorem on the *mixed source family* $\{(tG, g - n\psi) : t \in \mathbb{R}\} \cup \{(0, -n\psi)\}$ (Theorem 27.8), for the scheme $z_K^{\mu_*} = e^2 X_{\varepsilon_K}(m; \mu_*)$ of that theorem (the sequence written z_K in its statement; the superscript distinguishes it from the reference scheme $z = a^\sharp$ of Definition 27.1, which is $e^2 X_{\varepsilon_K}(m; \mu_0)$; $\mu_* > 0$ is the fixed reference photon mass of that scheme, Definition 27.7), evaluated at the point $(\mathrm{i}; 1)$: its two-lattice comparison and the existence of the axis-lattice limit give the two limits and their equality. Its hypotheses are provided at every such configuration by four statements, each given in the form used, with its proof: (1) Theorem 27.8 itself on the mixed family, with its composition step with the two-flow comparison (N1*) in the form of a bound in the sup norms of h and its derivatives; (2) (N1)–(N5) at the family and its images (notation of Section 18.3), (N6$_W$), that is, the convergence of the mixed-source walled ratio along $\varepsilon_K = 5^{-K}$, which the walled torus theorem gives at the shifted source $h_\infty = h - \frac{1}{2}\Delta_T(0)$, uniform bounds on compact t-sets upgrading it, by Vitali's theorem, to an entire limit with all moments, and the clauses of the single-torus statement ($\mathrm{EX}^\sharp_W{}'(\beta)$) of Definition 27.11, proved by the method of the torus theorem (the passive-rotation reductions, the equal-spacing comparison of the hybrid and the standard flow, the one-sided forward bound and the final step, the involution at $\theta_0 = \arctan(3/4)$; Theorem 18.9) from the single-torus statements off E_W: for $2 \sup g < s_*(e)$ and $J \geq J_{\mathrm{pos}}(g)$ the curvature theorem at $p_H = 2$ gives the two-sided bound $e^{\mp C_1 \delta_k^\sigma |T|} I_K^{(k)}$ with the large-field tail $e^{-(c_0'/4)p(\delta_k)^2 + C_1|T| + C_2 K}$, the amplitude β being fixed after $(\mathfrak{c}, g)$; (3) the walled large-field envelope at a source with a positive part (Theorem 26.11 and Proposition 26.10; the positive part $\sup h^+$ of the source is paid from the positivity margin of Proposition 26.8 with the exponent p_H, after which King's envelope bound applies with an exponent depending on $\sup h^+$): King's upper bound [25, Thm. 3.1] in the envelope form of Theorem 19.3 at the sources $h = g - n\psi$,

$-n\psi$ and their mirrors $-\beta(g-n\psi)$, $-\beta(-n\psi)$, on $p_H \sup h_+ < s_*(e)$ and for $J' \geq J_{\mathrm{pos}}(g)$, the amplitude $\beta \in (0,1]$ of the mirrors being chosen after $\mathfrak{c}$ and g with $2p_H\beta H' < \sigma_1 + p_H \sup g$, $H' := \|g\|_\infty + n \sup\psi + \frac{1}{2}\Delta_*$ and $\sigma_1 = \frac{1}{2}(\alpha_\infty(m) - \alpha_\infty(M_1))$; it takes the place of King's convention for the sources [25, p. 677], which carries no argument and is not used; (4) the scheme bridge $a_k^\sharp - z_k^{\mu_*} \to d \in \mathbb{R}$ (a lattice-integral asymptotic: $X_\varepsilon(m;\mu) = X_1(\varepsilon m, \varepsilon\mu)$ by exact scaling, and $X_1(tm,t\mu) = (4\pi^2)^{-1}\ln(1/t) + \kappa_{\mathrm{lat}} + \kappa_c(m,\mu) + o(1)$ with a mass-free lattice integral κ_{lat} and a continuum integral κ_c, so that $z_\varepsilon^{\mu_*} - z_{5\varepsilon}^{\mu_*} \to (e^2/4\pi^2)\ln 5$ and $z_\varepsilon^{\mu_*} \to +\infty$), under which the statements for z^{μ_*} and for $a^\sharp$ are equivalent and $Z_\infty^{a^\sharp} = e^{-d\sum_j w_j \int h_j} Z_\infty^{z^{\mu_*}}$ on $\Omega_*(h)$ by Theorem 27.6(vi), a factor invariant under every Euclidean motion since $\int \tau_b R \cdot h = \int h$. Statement (3) is available at the mass-decreasing sources on the cone and nowhere else: at $e = \lambda = 0$ the walled partition function at a mass-decreasing source above a threshold is infinite.

27.4. **The heavy shell on the cone and the closing statement.** The *closing statement on the cone* for an admissible$_{<s_*}$ scheme a is $\mathcal{W}^{a,\mathrm{env}} \leq Z_\infty^a$ on $\mathcal{K}_{<s_*}$, with the envelope over the classes $\mathcal{A}_W(\cdot;g)$ (at $g=0$ understood on the class $\mathcal{C}$ of Definition 16.6); equivalently, for every $b>0$, $(f,g) \in \mathcal{K}_{<s_*}$ supported in Q_b and $\eta > 0$ there is $\mathfrak{c} \in \mathcal{A}_W(Q_b;g)$ with $\mathcal{W}_\mathfrak{c}^a(f,g) \leq Z_\infty^a(\mathrm{i}f,g) + \eta$.

Proposition 27.12 (the closing statement on the cone)**.** *The closing statement on the cone holds for the scheme $\Theta^\sharp$ at every $(f,g) \in \mathcal{K}_{<s_*}$, in the pointwise form at the box Q_b of the source: for every $\eta > 0$ there is a single-level configuration $\mathfrak{c} = (J', \psi, n+K) \in \mathcal{A}_W(Q_b;g)$ with $\mathcal{W}_\mathfrak{c}^{a^\sharp}(f,g) \leq Z_\infty^{a^\sharp}(\mathrm{i}f,g) + \eta$.*

Proof. We first reduce the statement to the heavy-shell decoupling at one two-level configuration, then obtain the four shell estimates at the scalar source g by repeating their proofs with the source vertex, and finally run the proof of Theorem 23.12 at that configuration. The statement is obtained, as in Section 23, from the heavy-shell decoupling at one configuration: for every such b, (f,g) and η there are a two-level configuration $(\mathfrak{s},\chi)$ for Q_b with $\tau(\mathfrak{s}) \leq \eta$ and a shell mass $K < \infty$ with $\liminf_k[\Psi_k(n,K) - \Psi_k(n,\infty)] \leq \eta$, where $\Psi_k(n,K) := T^a_{(J',n\psi+K\chi),\Lambda_k}(f,g)$ is the walled ratio of the two-level configuration (wall height n, shell mass K) at spacing ε_k and $\Psi_k(n,\infty)$ its value with the matter removed on the shell, as in Definition 23.5, the states now carrying the source (zf,g) with the multiplier $e^{-a_k^\sharp \mathsf{S}(g)}$ of the scheme. The two-level sandwich and the dead-shell bound on the cone, whose constant $M_W(2g) := \sup_J \limsup_k Z^a_{\Lambda_k, C_J\cap\Lambda_k}(0,2g)$ is finite by (W$^\sharp$2), give the closing statement from the decoupling, under the walled comparison of Theorem 27.9. The decoupling is proved from the shell estimates at the scalar source $g' = g$: the two-sided bounds (SW1°)$_{\mathrm{int}}$ at the source g on the capped stopping-scale integral, with the exponent $q'(g)$ of (54) in place of the constants of (SW1°)$_{\mathrm{int}}$; the disc bound (SW2^{min}) at g, with the constant $2e^{4B}M_T(g)$, where $M_T(q)$ is the supremum over the spacings of the Wick-source expectation of the bare torus, finite at $2\sup q < s_*(e)$ by Theorem 24.4 (the fixed-torus moment); the κ-Cauchy property (SW3°)$_{\mathrm{int}}$ at g; and the threshold clause (SWT), all proved from Theorems 24.5 and 25.11, whose proofs are repeated with the scalar source: the source vertex $l_k^2 g(l_k x)$, with two scalar legs, is a two-leg vertex of coefficient at most $l_k^2\|g\|_\infty$, so the power counting of the difference bounds applies with the factor $(1+\|g\|_\infty)^{n_h}$ and the per-vertex power $3/2$; and the dependence on the background is controlled by interpolating $B + t\tilde{B}$, $t \in [0,1]$ (regular, as the hypotheses of Theorem 23.12 require), the t-derivative of the h-part being a family with one extra background leg, bounded by the source-vertex difference bounds, so that $\|\tilde{B}\| \leq C_q l_i^q$ for all q yields any power of l_i. The heavy-shell estimate at a scalar source then runs the proof of Theorem 23.12 at a two-level configuration with the sharp shell profile and the torus $T = D_{J'}$ above the thresholds of the class and those at which (SW1°)$_{\mathrm{int}}$ at g and the bare-torus theorem are used at the amplitudes g and $q'g$ (that is, with

$\Delta_*(J') < s_*(e) - 2q' \sup g$, where $\Delta_*(J')$ is the bound Δ_* of the convention bridge evaluated at the torus $D_{J'}$; since $\Delta_*(J') \to 0$ as $J' \to \infty$ by Lemma 26.4, there is a least $J_1(g)$ such that this holds for every $J' \geq J_1(g)$, the threshold $J_1(g)$ named in Section 27.1), with the prime $\mathfrak{p} = 2$ of Definition 16.1; it gives, for every $\eta > 0$, a stopping scale and a shell mass K with $\limsup_k \mathrm{tail}_k(K) \leq \eta$, in the quantifier order of Theorem 23.12. Consequently the closing statement holds for $\Theta^\sharp$ at every $(f, g) \in \mathcal{K}_{<s_*}$, in the pointwise form at the box Q_b of the source, with the single-level configuration $(J', \psi, n + K) \in \mathcal{A}_W(Q_b; g)$ as the witness. □

27.5. **The theorem.** The following theorem is the main result of the section, the Euclidean invariance of King's state with general uncharged sources on the tube; its proof combines the statements of the preceding subsections, and the paragraph after the proof lists what it uses beyond the curvature sector.

Theorem 27.13 ($E(3)$ invariance of King's state with general uncharged sources)**.** *Let $e > 0$ and $\lambda, m, \mu_0 > 0$ be a point of King's range outside the set E_W of Proposition 26.8, at block factor 5, so that $s_*(e) > 0$. Let $f_1, \dots, f_N$ be real 2-forms and $h_1, \dots, h_N$ real functions in C_c^∞, with $\int h_j$ arbitrary, and let $\Omega_*(h)$ be the tube of Definition 27.2. Then:*

(i) King's state in the scheme $\Theta^\sharp$,

$$Z_\infty^\sharp\Big(\sum_j z_j f_j, \sum_j w_j h_j\Big) = \lim_J \lim_k Z_{k,J}^{a^\sharp, K}\Big(\sum_j z_j f_j, \sum_j w_j h_j\Big),$$

exists locally uniformly on $\Omega_*(h) \ni 0$, is holomorphic there, and lies in $(0, \infty)$ at the real points of $\Omega_*(h)$.

(ii) For every Euclidean motion $x \mapsto Rx + b$, $R \in O(3)$, $b \in \mathbb{R}^3$, and all $(z, w) \in \Omega_*(h) = \Omega_*(\tau_b R \cdot h)$,

$$Z_\infty^\sharp\Big(\sum_j z_j\, \tau_b R \cdot f_j, \sum_j w_j\, \tau_b R \cdot h_j\Big) = Z_\infty^\sharp\Big(\sum_j z_j f_j, \sum_j w_j h_j\Big).$$

(iii) Consequently every Schwinger function with curvature and Wick-square insertions,

$$\partial_{z_1} \cdots \partial_{z_p} \partial_{w_1} \cdots \partial_{w_q} Z_\infty^\sharp|_{(z,w)=0},$$

is $E(3)$-covariant, and the state at real sources (f, h) with $\sup 2h_+ < s_*(e)$ (every $h \leq 0$ with $\int h \neq 0$ included, and every $g \geq 0$ with $\sup 2g < s_*(e)$) is $E(3)$-invariant.

At $h = 0$ this is Theorem 15.1 outside E_W. The statement holds for every admissible$_{<s_*}$ scheme in place of $\Theta^\sharp$, with the factor of Theorem 27.6(vi).

Proof. We first dispose of (i) and of the case $h = 0$, then prove the invariance under the generating rotations on the cone of non-negative scalar sources by the argument of Theorem 22.1 and close it to $O(3)$ by that of Theorem 22.3, extend it to complex coefficients with sign changes by the identity theorem on the split domain, and finally pass to all Euclidean motions and read off (iii). Let $a := a^\sharp$, admissible$_{<s_*}$ by Theorem 27.5. (i) is Theorem 27.6(iii) for a. If $h = 0$, (ii) and (iii) are Theorem 15.1 outside E_W, the multiplier of the scheme being 1 at $w = 0$. Let some $h_j \neq 0$. On the set of $(f, g) \in \mathcal{K}_{<s_*}$ with $g \neq 0$, which is invariant under the group generated by $\mathcal{R}_0$, the walled comparison holds by Theorem 27.9 and the closing statement holds pointwise at the box of the source (Proposition 27.12); the argument of Theorem 22.1, run on the cone with the walled values of the scheme, gives $Z_\infty^a(\mathrm{i}\, R \cdot f, R \cdot g) = Z_\infty^a(\mathrm{i} f, g)$ for these (f, g) and every $R \in \mathcal{R}_0$; the closure argument of Theorem 22.3, with the threshold $s_*(e)$ and the convexity of Theorem 27.6(iv), gives (ii) for $b = 0$ and every $R \in O(3)$ on $\Omega_*(h)$. The passage from the invariance at $(\mathrm{i} f, g)$, $g \geq 0$, to the complex coefficients of a family with sign changes goes through the split of Definition 27.3: for R in the group generated by $\mathcal{R}_0$, the difference of the two

sides of (ii), as a function of (z, w^+, w^-) for the non-negative $2N$-member family $(f_j; h_j^+, h_j^-)$, is holomorphic on the connected split domain $D(h)$ (Theorem 27.6(iii) for that family, whose tube contains $D(h)$; the family $(R \cdot h_j^\pm)$ is the split of $(R \cdot h_j)$ with the same split domain, $\sup_x$ being invariant under R) and vanishes on $(\mathrm{i}\mathbb{R})^N \times \{u \in (0,\infty)^{2N} : \sup 2k_1(u) < s_*(e)\}$, a non-empty open subset of a maximal totally real subspace on which every source lies in $\mathcal{K}_{<s_*}$ with $g \neq 0$ (as $h_j^- \neq 0$); so it vanishes on $D(h)$, hence on the trace $\{w^+ = w,\ w^- = -w\}$, an open subset of $\Omega_*(h)$, hence on the connected set $\Omega_*(h)$. The closure from this dense group to $O(3)$ runs at real coefficients, on the $O(3)$-invariant sets U_ρ on which Φ^a is continuous, and the return to complex coefficients is the identity theorem again, the real points of $\Omega_*(h)$ being a non-empty open subset of the maximal totally real subspace $\mathbb{R}^{2N}$. Theorem 27.6(v) gives (ii) for $R = 1$; and the composition of the two gives (ii) for every Euclidean motion, $\Omega_*(h)$ being invariant under $h_j \mapsto \tau_b R \cdot h_j$. (iii): 0 is an interior point of $\Omega_*(h)$, on which both sides of (ii) are holomorphic and equal, so all their mixed derivatives at 0 agree; the real points of $\Omega_*(h)$ are the real sources with $\sup 2h_+ < s_*(e)$. The scheme sentence is Theorem 27.6(vi). □

Theorem 27.13 extends the Euclidean invariance of King's state from curvature sources to general uncharged sources, in the scheme $\Theta^\sharp$ and on the tube $\Omega_*(h)$ of Definition 27.2, at the points of King's range outside the exceptional set E_W, where the threshold $s_*(e)$ is positive. The statements used in the proof of Theorem 27.13 beyond those of Theorem 15.1 are the identification $E_W = \{s_*(e) = 0\}$ of Definition 27.2, the statements used in Theorem 27.5, the four statements given after Theorem 27.9 and the shell estimates at the scalar source of Section 27.4, each stated where it is first used (Sections 24, 25, 26, 27.3), with its proof, or a cited theorem of the literature. Both sides of (ii) being holomorphic on $\Omega_*(h)$, the identity theorem, applied through the split family as in the proof (with the split domain cut at s', whose trace is an open subset of $\Omega_*(h)$ containing 0), would give (ii) from the invariance at $(\mathrm{i}f, g)$, $g \neq 0$, on any sub-cone $s' \in (0, s_*(e)]$, so that the walled comparison and the closing statement at scalar sources are used only at arbitrarily small positive amplitudes while (W$^\sharp$2) on the whole cone fixes the domain; the proof above applies $s' = s_*(e)$ and claims no more. What the theorem does not state, the points of E_W and the sources above the threshold, is stated with the open problems (Section 32.6).

The main theorem

28. The main theorem for King's state

We state the main theorem in full and prove it by assembling Sections 4–14. The object is King's state, the infinite-volume state of King's construction [25, 26],

$$Z^K_\infty(f, 0) = \lim_J \lim_k \langle e^{F(f)} \rangle^K_{k,J}$$

on King's RG cube (Definition 2.3), block factor 5, at a fixed photon mass $\mu_0 > 0$ and all positive couplings e, λ, m (Lemma 29.3). Both sectors are proved on King's convention ND^K, on which the existence theorems (Theorems A and B) are posed. In the Wick sector (Leg W), proved in its tube form, Form (B), the tube bound (37) is proved on the comparison convention ND$^{(a)}$ (Definition 2.2). Its transfer to King's convention, the bound (73), and the fixed-cube limits on the tube for the K-family functionals $Z^{a,K}_{k,J}$ are Theorem 28.5, on which the limit statements Theorem 13.10(i),(ii), Lemma 13.11, Proposition 10.4 and Theorem 10.6 for King's state rest. After that the Wick sector is assembled on ND^K (Proposition 28.6), and the comparison family enters through its LIM states, which coincide with King's tube object (Proposition 28.7). The torus theorems are Proposition 12.6 and Lemma 12.7, and the couplings of the Wick leg lie in the coupling region of Definition 2.1 (the set on which the sign clause (T3) holds), which contains the

explicit regions of Proposition 29.29; $\mathcal{K} = \{(\mathrm{i}f, g) : g \geq 0\}$ is the cone. The theorem announced in the introduction is Theorem 28.2 below.

Throughout this section, let $e > 0$, $\lambda, m, \mu_0 > 0$, $\varepsilon_k = 5^{-k}$, $C_J = [-MJ, MJ)^3$ with King's large-block integer $M \in 50\mathbb{N}$, $M \geq M_{\min}$ (the objects below do not depend on such M, the tube limits of part C coinciding where both are defined, Remark 2.4), and let $\mathrm{ND}^K_{k,J}$ be King's cube state. For a finite family $f = (f_1, \dots, f_N)$ of real 2-forms in $C^\infty_c(\mathbb{R}^3; \Lambda^2)$ and $z \in \mathbb{C}^N$ we write $g(z) = \sum_j (\operatorname{Re} z_j) f_j$, and $S^K_n(f_1, \dots, f_n) = \partial_{z_1} \cdots \partial_{z_n} Z^K_\infty\Big(\sum_j z_j f_j, 0\Big)\Big|_{z=0}$ for the Schwinger functions of King's state.

Definition 28.1 (the data of the Wick leg)**.** Let the couplings lie in the coupling region of Definition 2.1, the set on which (T3) holds, which contains the explicit regions of Proposition 29.29, and fix an admissible scheme a (Definition 2.7; $z = a^\sharp$ the reference scheme). Let $\sigma_+ = s_1$, $H = s_1/16$, $\kappa = C_U/s_1$, K_- be the constants of (36) (K_- built from B_{cell}; for $a \neq z$ with C_U replaced by $C_U + s_1 \sup_k |z_k - a_k|$), $\sigma_0 = H/4$ and $K' = \kappa\pi^2 + K_-$. For a finite family $\mathcal{F} = (f_1, \dots, f_n; h_1, \dots, h_p)$ of real 2-forms and real scalars and $\zeta = (z, w) \in \mathbb{C}^{n+p}$ (the notation of Definition 2.8) the tube is

$$\mathcal{D}_{\sigma_+, H} = \{\|k_+(\zeta)\|_\infty < \sigma_+/2,\ \|k_-(\zeta)\|_\infty < H/2\},$$

the tube object is

$$G^K_\infty(\zeta) = \lim_{J\to\infty} \lim_{k\to\infty} Z^{a,K}_{k,J}\Big(\sum_i z_i f_i, \sum_j w_j h_j\Big)$$

wherever the limit exists, and the joint Schwinger functions of the pair (F, W^a) are

$$S_{n,p}(f_1, \dots, f_n; h_1, \dots, h_p) = \partial^{(n)}_z \partial^{(p)}_w G^K_\infty(0).$$

Theorem 28.2 (the axioms for King's state at fixed photon mass)**.** ***A. The object.*** *For every finite family $f = (f_1, \dots, f_N)$ of real 2-forms in $C^\infty_c(\mathbb{R}^3; \Lambda^2)$ the limit*

$$Z^K_\infty\Big(\sum_j z_j f_j, 0\Big) = \lim_{J\to\infty} \lim_{k\to\infty} \langle e^{\sum_j z_j F_{\Lambda_k}(f_j)} \rangle^K_{k,J}$$

exists locally uniformly in $z \in \mathbb{C}^N$, is entire, depends on the family only through the total source, is real and ≥ 1 at real z, lies in $[0, 1]$ at purely imaginary z, and satisfies $|Z^K_\infty(\sum_j z_j f_j, 0)| \leq e^{\|g(z)\|^2_2}$ with no dependence on μ_0, e, λ, m or M.

B. Leg C (the curvature field). *The Schwinger functions S^K_n satisfy:*

(i) OS0. *$|Z^K_\infty(\sum_j z_j f_j, 0)| \leq e^{\|g(z)\|^2_2}$ for all $z \in \mathbb{C}^N$, and also $\leq \exp(\frac{1}{2}\mu_0^{-2} \|\sum_j (\operatorname{Re} z_j) d^* f_j\|^2_2)$; the same two bounds, with the lattice norms $\|\cdot\|_{2,k}$ and d^*_k, hold for every $k \geq 0$ and $J \geq J_0(f)$ for the lattice functionals $\langle e^{\sum_j z_j F_{\Lambda_k}(f_j)} \rangle^K_{k,J}$. For real test functions*

$$|S^K_n(f_1, \dots, f_n)| \leq (\sqrt{2}\, e)^n (n!)^{1/2} \prod_{i=1}^n \|f_i\|_2 \leq (e\pi/2)^n (n!)^{1/2} \prod_{i=1}^n |f_i|_4;$$

each S^K_n extends to a continuous symmetric multilinear form on $\mathcal{S}(\mathbb{R}^3; \Lambda^2_{\mathbb{C}})^n$, a tempered distribution, the bound carrying the factor 2^n for complex test functions; so E0″ *holds with $s = 4$, $\sigma_n = (e\pi)^n (n!)^{1/2}$ and every $\beta > \frac{1}{2}$.*

(ii) OS1. *The S^K_n are invariant under all translations of $\mathbb{R}^3$ and under the cubic group O_h about every point, F transforming as a 2-form. Moreover Z^K_∞ is invariant under the Euclidean group $\mathbb{R}^3 \rtimes O(3)$ and the S^K_n are Euclidean covariant.*

(iii) OS2. *Reflection positivity about every coordinate plane* $\{x_\nu = c\}$, $\nu \in \{1,2,3\}$, $c \in \mathbb{R}$: *for finite families* $\zeta_1, \dots, \zeta_N$ *of real* 2*-form sources supported in* $\{x_\nu > c\}$ *with arbitrary complex coefficients,* $\sum_{i,j} \bar\gamma_i \gamma_j Z^K_\infty(\theta\zeta_i^* \sqcup \zeta_j, 0) \geq 0$ *for all* $\gamma \in \mathbb{C}^N$ *(the matrix form for exponentials), and* $S(\theta(P)P) \geq 0$ *for every polynomial* P *in the smeared curvature with sources supported in* $\{x_\nu > c\}$ *(the polynomial form). The same holds about every plane* $\{x \cdot n = c\}$, n *a unit vector.*

(iv) OS3. S^K_n *is real for real test functions, symmetric, and zero for odd* n.

(v) The measure. *For every even* $r \geq 14$ *there is exactly one Borel probability measure* $\mu^{K,H}$ *on* $\mathcal{H}^{-r}(\mathbb{R}^3; \Lambda^2)$ *whose moments on* C^∞_c *are the* S^K_n*; its characteristic functional is* $Z^K_\infty(\mathrm{i}\,\cdot, 0)$, *extended by continuity to* $\mathcal{S}$*; its image* μ^K_S *on* $H^{-r}_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^2)$ *is the unique Borel probability measure there with this characteristic functional, and the unique one with these moments. For every real* $\chi \in C^\infty_c(\mathbb{R}^3)$ *and every box* Q *containing* $\operatorname{supp}\chi$ *in its interior, the laws of the localised lattice curvature fields* $\chi\Phi^K_{k,J}$ *under* $\mathrm{ND}^K_{k,J}$ *on* $H^{-r}_{\mathrm{per}}(Q; \Lambda^2)$ *converge weakly, along the full sequence* $k \to \infty$ *at every fixed* $J \geq J_0(\chi)$ *and then as* $J \to \infty$, *to the law of* $\chi\Phi$ *under* μ^K_S. *Nothing is claimed about the laws of the unlocalised lattice fields at fixed* J.

(vi) Weak OS4. μ^K_S *is* $\mathbb{R}^3 \rtimes O_h$*-invariant and is an integral* $\int \mu_\xi \, d\nu(\xi)$ *of translation-ergodic, reflection-positive,* θ_j*-invariant components which cluster in the mean and under diagonal separation, have a unique joint vacuum in each coordinate direction, and satisfy the factorial growth bound* $|S^\xi_N(u_1, \dots, u_N)| \leq \sigma^\xi_N \prod_i \|u_i\|_{r+2}$, $\sigma^\xi_N \leq \alpha_\beta(\xi)(N!)^\beta$ *for every* $\beta > 1$, *with component constants (no tube bound); there is an* O_h*-invariant decomposition into mixtures of* $k \in \{1,2,3,6\}$ *ergodic measures; and* ν*-almost every component satisfies* OS4 *in the weak form (ergodicity under* $\mathbb{R}^3$*). Moreover* ν*-almost every component is* $\mathbb{R}^3 \rtimes O(3)$*-invariant, reflection positive about every plane, and satisfies* OS4 *in each of its forms for every time direction, with* $\ker H_n = \mathbb{C}\Omega_n$ *in every direction* n*; its moments are real and symmetric. The vanishing of a component's odd moments is not claimed.*

C. Leg W, Form (B), on King's convention. *Let the couplings lie in the coupling region of Definition 2.1 and let* a *be an admissible scheme, with the constants* $\sigma_+, H, \kappa, K_-, \sigma_0, K'$ *and the tube* $\mathcal{D}_{\sigma_+,H}$ *of Definition 28.1. Then for every finite family* $\mathcal{F} = (f_1, \dots, f_n; h_1, \dots, h_p)$ *of real* 2*-forms and real scalars the tube object* G^K_∞ *exists locally uniformly on the tube and is holomorphic there; on the Wick tube of Definition 10.2 the limit* $Z^{a,K}_{\infty,T} = \lim_J \lim_k Z^{a,K}_{k,J}$ *exists and equals* G^K_∞ *where both are defined. The joint Schwinger functions* $S_{n,p}$ *of the pair* (F, W^a) *satisfy:*

(i) OS0. $|G^K_\infty(\zeta)| \leq \exp(\|g(\zeta)\|_2^2 + \kappa\|k_+(\zeta)\|_{L^1} + K_-|k_-(\zeta)|_w)$ *on* $\mathcal{D}_{\sigma_+,H}$, *and the same bound with lattice norms holds for* $Z^{a,K}_{k,J}$ *uniformly in* $k \geq 0$ *and* $J \geq J_1(R)$ *when the supports lie in* Q_R*; for real test functions*

$$|S_{n,p}(f_1, \dots, f_n; h_1, \dots, h_p)| \leq e^{K'\sigma_0}(\sqrt{2}\,e)^n (n!)^{1/2} (e/\sigma_0)^p \, p! \prod_{i=1}^n \|f_i\|_2 \prod_{j=1}^p |h_j|_w,$$

with the factor 2^{n+p} *for complex test functions; the* $S_{n,p}$ *are tempered distributions and* E0″ *holds with* $s = 4$ *and every* $\beta > 1$. *The constants* σ_+, H, κ, K_- *depend on* $(e, \lambda, m, \mu_0, M, a)$, *on the fixed auxiliary choices* M_1, χ_0 *and the amplitude* $s_1 \in (0, \sigma_1)$ *of* (I2), *and on nothing else; they are not uniform in* μ_0.

(ii) OS1. $G^K_\infty(\tau_b R \cdot \zeta) = G^K_\infty(\zeta)$ *on* $\mathcal{D}_{\sigma_+,H}$ *for every* $b \in \mathbb{R}^3$ *and* $R \in O_h$, F *transforming as a* 2*-form and* W^a *as a scalar (the transformed family has the same tube); hence the* $S_{n,p}$ *are invariant under translations and under* O_h *about every point. No Euclidean clause is claimed.*

(iii) OS2. *For finite families* $\zeta_1, \dots, \zeta_N$ *of real sources supported in* $\{x_\nu > c\}$ *with complex coefficients of amplitude* $\mathsf{a}(\zeta_i) < \sigma_W := \sigma_+/8$ *(in particular all purely imaginary Wick*

coefficients with arbitrary complex curvature coefficients), $\sum_{i,j}\bar\gamma_i\gamma_j Z^{a,K}_{\infty,T}(\theta\zeta_i^*\sqcup\zeta_j)\ge 0$ *about every coordinate plane; and the polynomial form* $S(\theta(P)P)\ge 0$ *for polynomials in the smeared* F *and* W^a *with sources in* $\{x_\nu > c\}$.

(iv) OS3. *The* $S_{n,p}$ *are symmetric multilinear in the* f*'s and in the* h*'s, real for real test functions, and zero for odd* n.

(v),(vi) The measure and weak OS4. *Clauses (v) and (vi) of Leg C, without the Euclidean sentence of (vi) (the sentence beginning "Moreover"; that is, Theorem 11.5(a),(b) for the joint field), hold for the joint field on* $\mathcal{H}^{-r}(\mathbb{R}^3;\Lambda^2\oplus\mathbb{R})$ *and* $H^{-r}_{\mathrm{loc}}(\mathbb{R}^3;\Lambda^2\oplus\mathbb{R})$, *every even* $r\ge 14$, *with moments* $S_{n,p}$, *characteristic functional* $(u,v)\mapsto G^K_\infty(\mathrm{i}u,\mathrm{i}v)$ *on real test functions (purely imaginary coefficients have* $k(\zeta)=0$ *and lie in the tube), the component growth bound with the constants of (i), and the local convergence of the laws of the scheme-shifted lattice fields* $\chi(F_{\Lambda_k},W^a_{\Lambda_k})$ *under* $\mathrm{ND}^K_{k,J}$ *along the full sequence* $k\to\infty$ *at every fixed* $J\ge\max(J_0(\chi),J_1(R_\chi))$, *then* $J\to\infty$. *No Euclidean clause is claimed. Two admissible schemes* a,a' *give objects differing by the shift* $W\mapsto W-\kappa_{a'a}\int(\cdot)$ *of the Wick component,* $\kappa_{a'a}=\lim_k(a'_k-a_k)$; *joint cumulants of order* ≥ 2 *are scheme-free.*

D. The comparison family. *Under the conditions of C, every increasing sequence of spacings has a continuum subsequence* $\mathbf{k}$ *for the Wick sector of the comparison family* $\mathrm{ND}^{(a)}$ *(Definition 14.1), and for every such* $\mathbf{k}$ *and every family the LIM state* $Z^{a,\mathbf{k}}_\infty=\lim_J\lim_{k\in\mathbf{k}}Z^a_{k,J}$ *exists on* $\mathcal{D}_{\sigma_+,H}$ *and equals* G^K_∞ *(Proposition 28.7). Hence clauses (i)–(vi) of C hold for the Wick-sector LIM states of the comparison family, which coincide with King's tube object and do not depend on* $\mathbf{k}$; *in (v) the laws of the localised lattice fields* $\chi(F_{\Lambda_k},W^a_{\Lambda_k})$ *under* $\mathrm{ND}^{(a)}_{k,J}$ *converge along* $\mathbf{k}$ *at every fixed* J *with* $MJ>R(\chi)+\ell_c$, *then as* $J\to\infty$, *to the law of the joint field under the measure of C(v).*

Proof. The proof follows the four parts of the statement: A, the object itself; B, the axioms of the curvature field clause by clause; C, the Wick sector under (T3); D, the comparison family.

A. Theorem 7.6(b): by Theorem A the two limits exist locally uniformly on $\mathbb{C}^N$ and are entire, $|Z^K_J|,|Z^K_\infty|\le e^{\|g(z)\|_2^2}$, $J\mapsto Z^K_J(\mathrm{i}t)$ is nondecreasing with values in $[0,1]$, and $Z^K_\infty(x)\ge 1$ at real x; realness at real z holds because the integrand $e^{\sum_j z_j F_{\Lambda_k}(f_j)}$ is then real under the probability measure $\mathrm{ND}^K_{k,J}$, and realness passes to the limits; the symmetry $(A,\vartheta)\mapsto(-A,-\vartheta)$ of every $\mathrm{ND}^K_{k,J}$ gives $Z^K_\infty(-x)=Z^K_\infty(x)$ at real x. Each lattice functional depends on the family only through its total source, hence so do the limits. The bound is absolute (Remark 7.5).

Part B collects the axioms of the curvature sector. Regularity (i) is the bound $|Z^K_\infty|\le e^{\|g(z)\|_2^2}$ with its lattice and Schwinger-function forms, which come from Gaussian domination on the lattice and carry no dependence on the couplings. Invariance (ii) has three sources: translations, through a lattice sandwich and the identity theorem; the cubic group about every point, Theorem 8.6; and rotations, from the rotation theorem of the rotation part, off the exceptional set E_W directly and at the points of E_W by right-continuity in the scalar mass. Reflection positivity (iii) about the coordinate planes and the symmetry of the Schwinger functions (iv) come from the modified-box argument of Sections 9 and 10, and (iii) about an arbitrary plane then follows by (ii). The measure (v) and its ergodic decomposition with weak OS4 (vi) are the content of Section 11, whose $O(3)$ hypothesis is the Euclidean clause of (ii).

B. (i) is Theorem 7.6: the finite-volume bound (a) with $\min\{\|g\|^2_{2,k},\frac12\mu_0^{-2}\|d_k^*g\|^2_{2,k}\}$ at every (k,J); the two limit bounds are (b), the second carried from the lattice by $\|d_k^*g\|_{2,k}\to\|d^*g\|_2$, (2); the Schwinger bounds, the extension to $\mathcal{S}$, the factor 2^n and E0″ (c). (ii): translations are Corollary 8.4 (the lattice sandwich Lemma 8.3, the two limits, the identity theorem, density); the cubic group about every point is Theorem 8.6; the Euclidean clause is Theorem 8.8, proved

from Theorem 15.1 and Corollary 4.4 (the rotation theorem Theorem C at every point of King's range: Theorem 22.4 off the exceptional set E_W of Proposition 26.8 and Definition 27.2, Theorem 26.7 and Remark 28.3 at the points of E_W) with Corollary 8.4. (iii): the matrix form about every coordinate plane and the polynomial form are Theorem 10.5 with Proposition 10.8; a plane $\{x\cdot n=c\}$ is the image of a coordinate plane under a Euclidean motion γ, and by (ii) $Z^K_\infty(\theta_\gamma\zeta_i^*\sqcup\zeta_j,0)=Z^K_\infty(\gamma^{-1}\cdot(\theta_\gamma\zeta_i^*\sqcup\zeta_j),0)$ is a Gram entry about the coordinate plane for the sources $\gamma^{-1}\cdot\zeta_i$, supported in the corresponding coordinate half-space. (iv) is Proposition 10.7 for $Z^K_\infty(\cdot,0)$ (through Theorem 10.5; charge conjugation gives the vanishing for odd n). (v) is Theorem 11.5(a), including the local convergence of the lattice laws along the full sequence, with Remark 11.6 for the unlocalised fields; (vi) is Theorem 11.5(b) and (c), whose $O(3)$ hypothesis is (ii)'s Euclidean clause; the disclaimer on odd moments is that of Theorem 11.5(c).

C. Under (T3), Proposition 12.8 gives (I2) with amplitude $s_1\in(0,\sigma_1)$ and constant C_U from the torus theorems (Proposition 12.6 and Lemma 12.7); for $a\neq z$ the difference z_k-a_k converges by admissibility, so $\sup_k|z_k-a_k|<\infty$ and C_U is replaced as stated in Definition 28.1 (Lemma 14.5); (I3) is Proposition 12.2, the correlation inequalities are Lemma 2.10, and the cell constant $B^{(a)}_{\mathrm{cell}}$ is finite (Lemma 14.4). *(i)*: Theorem 28.5 gives the bound (73) for every $k\geq0$ and $J\geq J_1(R)$, the fixed-cube limits G^K_J and the cube limit G^K_∞ locally uniformly on $\mathcal{D}_{\sigma_+,H}$, their holomorphy, the bound in the limit, the Schwinger bounds, the factor 2^{n+p} and E0″, with the stated dependence of the constants and no uniformity in μ_0. On the Wick tube $T_{\mathcal{F}'}$ of the enlarged family of Definitions 10.1 and 10.2, whose scalar members are non-negative and whose points have $2\|k_1\|_\infty<\sigma_+/2$, the fixed-cube limits of $Z^{a,K}_{k,J}$ exist locally uniformly by Theorem 6.26(i), whose domain $\{\sup_x 2k_1<s_W\}$, $s_W=s_1$, contains $T_{\mathcal{F}'}$, for the reference scheme and hence for a (Definition 2.7); their cube limit $Z^{a,K}_{\infty,T}$ is Proposition 10.4(a) (Proposition 28.6(c)), and it equals G^K_∞ on $T_{\mathcal{F}'}\cap\mathcal{D}_{\sigma_+,H}$ as the same J-limit of the same fixed-cube limits. *(ii)*: Proposition 28.6(b). *(iii)*: Proposition 28.6(c), that is Theorem 10.6 with Proposition 10.4, and the polynomial form Proposition 10.8. *(iv)*: Proposition 10.7 for $Z^{a,K}_{\infty,T}$ and G^K_∞ (Proposition 28.6(c)). *(v),(vi)*: Proposition 28.6(d). The Gram points of (v)'s characteristic functional have purely imaginary coefficients and lie in the tube, and the reflection positivity used there is (iii) at amplitude 0 (Remark 28.8). *The scheme clause*: for admissible a,a', $Z^{a',K}_{k,J}(\zeta)=e^{-(a'_k-a_k)\sum_j w_j\mathsf{S}_{\Lambda_k}(h_j)}Z^{a,K}_{k,J}(\zeta)$ (Definition 2.5), where $a'_k-a_k=(z_k-a_k)-(z_k-a'_k)\to\kappa_{a'a}$ and $\mathsf{S}_{\Lambda_k}(h_j)\to\int h_j$; the factor converges locally uniformly in ζ, so the fixed-cube limits and then the cube limits satisfy $G^{K,a'}_\infty(\zeta)=e^{-\kappa_{a'a}\sum_j w_j\int h_j}G^{K,a}_\infty(\zeta)$ on the tube, the shift $W\mapsto W-\kappa_{a'a}\int(\cdot)$ of the Wick component; a factor $e^{\ell(w)}$ with ℓ linear changes only the first cumulants in w, so the joint cumulants of order ≥2 are scheme-free. No bound at positive scalar sources beyond the cone $\sup q\leq s_1$ of Corollary 13.3 is used.

D. Proposition 28.7; the clauses of C then hold for $Z^{a,\mathbf{k}}_\infty=G^K_\infty$, and the convergence of the laws of the localised comparison-cube fields along $\mathbf{k}$ is clause (v) of the Wick sector of Theorem 14.9, whose hypotheses ($\mathrm{B}_{\mathrm{cell}}$) and (SCH) hold for every admissible scheme under (T3) (Lemmas 14.4 and 14.5). □

Remark 28.3 (monotonicity, semicontinuity and right-continuity at E_W). Let $\tau_*=(e,\lambda,m,\mu_0)$ be a point of E_W (the exceptional set of Proposition 26.8). The scalar mass m is then a global minimiser of the limiting bare coefficient α_∞, and τ_* is, near m, the lightest King point:

$$t_\infty(M):=\alpha_\infty(M)-\alpha_\infty(m)>0$$

for $0<|M-m|<\delta$, because α_∞ is real-analytic and not constant, so that its other global minimisers, if any, are isolated. Both monotonicities of Lemma 26.5 raise $\langle\cos F(u)\rangle$ when terms are added, cube growth adding hopping and photon gradient terms and a smaller bare coefficient

adding weight to r^2; so along King points near m King's state is a non-increasing function of the bare coefficient. What is used in finite volume is the one-sided statement of Theorem 26.7(a), proved through the dead-shell torus. On E_W right-continuity is exactly what is needed, and nothing to the left of m is used; only the limit $M \downarrow m$ is claimed (on E_W, $t_\infty > 0$ on both sides of m, so both one-sided limits approach m from the side of the larger bare coefficient).

Remark 28.4 (the Euclidean clause). The Euclidean clauses of B(ii), (iii), (vi) rest on the rotation theorem (Theorem C), proved in the rotation part (Theorem 15.1): the walled tori on the two lattices have the same continuum limits (Theorem 18.1), King's state is the lower envelope of the walled tori (Theorem 21.4), and that closing estimate (the lower-envelope inequality) comes from the heavy-shell estimates on King's block-spin flow (Theorem 23.9 and Remark 23.13), both the comparison and the closing estimate holding off the exceptional set E_W of Proposition 26.8 and Definition 27.2, the points of E_W being reached by right-continuity in the scalar mass (Theorem 26.7, Remark 28.3); the Euclidean clause of the Wick sector below the threshold is Theorem 27.13; above it, Problem 32.3 is open.

The tube bound of the Wick sector is proved on the comparison convention; the next theorem carries it to King's convention and gives the fixed-cube limits there.

Theorem 28.5 (the Wick tube on King's convention)**.** *Let the couplings lie in the coupling region of Definition 2.1 and let a be an admissible scheme. Let $\sigma_+ = s_1$, H, κ be the constants of* (36) *and let K_- be the constant there, computed from the cell constant $B^{(a)}_{\mathrm{cell}}$ of the comparison convention (Lemma 14.4), which is finite. Then for every finite family of real sources with supports in Q_R, every $k \geq 0$, every $J \geq J_1(R)$ and every $\zeta \in \mathcal{D}_{\sigma_+,H}$,*

$$|Z^{a,K}_{k,J}(\zeta)| \;\leq\; Z^{a,K}_{k,J}(\operatorname{Re}\zeta) \;\leq\; \exp\big(\|g(\zeta)\|^2_{2,k} + \kappa\|k_+(\zeta)\|_{1,k} + K_-|k_-(\zeta)|_w\big), \tag{73}$$

the bound (37) *for the K-family functionals with the constants of the comparison convention; and the fixed-cube limits $G^K_J(\zeta) = \lim_{k\to\infty} Z^{a,K}_{k,J}(\zeta)$ exist for every $\zeta \in \mathcal{D}_{\sigma_+,H}$, locally uniformly, which gives Theorem 13.10(i),(ii): G^K_J and $G^K_\infty = \lim_J G^K_J$ exist locally uniformly on $\mathcal{D}_{\sigma_+,H}$ and are holomorphic, the bound* (73) *passes to the limits, and the Schwinger bounds and* E0″ *of Theorem 28.2C(i) hold for G^K_∞; moreover $G^K_\infty > 0$ at the real points of $\mathcal{D}_{\sigma_+,H}$ and $G^K_J \leq G^K_\infty$ at its cone points.*

Proof. Throughout, $\mathcal{F} = (f_1, \dots, f_n; h_1, \dots, h_p)$ is a family of real sources with supports in Q_R, $J \geq J_1(R)$, so that $MJ \geq R + \ell_c \geq R + 10$: every source lies at ℓ^∞-distance > 1 from $\mathbb{R}^3 \setminus C_J$, the support condition (S) holds for X_J, and $J \geq J_0(\mathcal{F})$. For $\zeta \in \mathcal{D}_{\sigma_+,H}$ write $g = g(\zeta)$ and $k = k(\zeta) = k_+ - k_-$ (Definition 2.8); by Lemma 2.9(E3) the sampled sources satisfy $0 \leq 2k_+ \leq s_1$ and $0 \leq 2k_- \leq H$ on Λ_k. The statements used are: (I2) with amplitude $s_1 \in (0, \sigma_1)$ and constant C_U, from the sign clause (T3) of the coupling region and the torus theorems Proposition 12.6 and Lemma 12.7 through Proposition 12.8, for $a \neq z$ with the constant $C_U + s_1 \sup_k |z_k - a_k|$, finite by Definition 2.7 (Lemma 14.5); (I3), which is Proposition 12.2; the correlation inequalities (I4), which are Lemma 2.10; and the cell constant $B^{(a)}_{\mathrm{cell}} < \infty$ of the comparison convention (Lemma 14.4). Under these, Corollary 13.3 and Proposition 13.9 hold on the comparison convention and Lemma 4.8(d) on King's.

Step 1 (modulus). Lemma 2.14(b) holds for every state of Definition 2.2, killed sites included: $|Z^{a,K}_{k,J}(\zeta)| \leq Z^{a,K}_{k,J}(\operatorname{Re}\zeta)$. It therefore suffices to bound $Z^{a,K}_{k,J}$ at the real point $\operatorname{Re}\zeta$, where $Z^{a,K}_{k,J}(\operatorname{Re}\zeta) = N^{a,K}(g,k) := e^{-a_k \mathsf{S}(k)} \langle e^{F(g) + :\phi^2:(k)} \rangle^K_{k,J}$.

Step 2 (curvature off). Let $\langle\cdot\rangle_{K,k}$ be King's cube with each single-site measure at an alive site multiplied by $e^{\varepsilon^3 k(x) r_x^2}$; it is a BFS measure (the factor keeps $\int e^{\alpha r^2} d\rho_x < \infty$ because $\lambda > 0$, and the killed sites are unchanged), and $N^{a,K}(g,k) = N^{a,K}(0,k)\, \langle e^{F(g)} \rangle_{K,k}$. Its Gaussian is

the Neumann photon Gaussian of X_J, the same for both conventions, with covariance C as in Lemma 7.1. By (S) every plaquette with $g \neq 0$ has its corners in X_J, so $F(g) = \langle v, A\rangle_k$ with $v = d_k^* g$, and Lemma 2.11 (with $s = 1$) and Lemma 7.1 give $\langle e^{F(g)}\rangle_{K,k} \le e^{\langle v, Cv\rangle_k/2} \le e^{\|g\|^2_{2,k}}$. This is the argument of Theorem 7.6(a) in the presence of the real scalar source k. Hence $N^{a,K}(g,k) \le e^{\|g\|^2_{2,k}} N^{a,K}(0,k)$.

Step 3 (the split). Lemma 2.14(d) at $f = 0$ with $k_1 = k_+$, $k_2 = k_-$: $N^{a,K}(0, k_+ - k_-) \le N^{a,K}(0, 2k_+)^{1/2} N^{a,K}(0, -2k_-)^{1/2}$.

Step 4 (the positive leg on King's convention). Put $q := 2k_+$, a lattice function with $0 \le q \le s_1$ supported in $Q_R \cap \Lambda_k \subset \mathrm{alive}_J^K$. The observable $e^{\sum_x \varepsilon^3 q(x) r_x^2}$ is $\Theta_{0,q'}$ with $q' = \varepsilon^3 q \ge 0$, of class $\mathcal{I}$, and (S) holds for X_J; Step 1 of the proof of Lemma 4.2 (the kill step on $\mathcal{I}$) gives $\langle \Theta_{0,q'}\rangle^K_{k,J} \le \langle \Theta_{0,q'}\rangle^{(a)}_{k,J}$, and the multiplier $e^{-(a_k + 2\mathsf{W}_\varepsilon)\mathsf{S}(q)}$ is one positive constant for both states (Lemma 2.14(a)). Hence

$$N^{a,K}_{k,C_J}(0,q) \;\le\; N^{a}_{k,C_J}(0,q) \;\le\; e^{\kappa \|q\|_{1,k}}$$

by Corollary 13.3, whose hypotheses (I2), (I3) hold (this is the cone instance of Remark 4.9, as in Remark 13.12). So $\frac{1}{2}\ln N^{a,K}(0,2k_+) \le \kappa \|k_+\|_{1,k}$.

Step 5 (the negative leg on King's convention). Put $q := 2k_-$, a lattice function with $0 \le q \le H$ supported in Q_R, and $J \ge J_1(R)$. Lemma 4.8(d) gives $\ln N^{a,K}_{k,C_J}(0,-q) \le K_- |q|_w$ with the constant K_- of (36), built from the cell constant $B^{(a)}_{\mathrm{cell}}$ of the comparison cell, which is finite by Lemma 14.4: by Lemma 4.8(c) every cell on which one of the cut-off sources $8\chi_s q$ does not vanish is a full cell inside C_J containing no killed site, so each factor of the wall insertion of Lemma 4.8(a) that carries a source is a comparison-cell functional, to which Lemmas 13.6 and 13.7 apply as stated, and the sum over cells is Lemma 13.8; the scheme multiplier factorises over the cells (Lemma 2.14(a)). The direction of the kill matters here and is the content of Lemma 4.8(e). Hence $\frac{1}{2}\ln N^{a,K}(0,-2k_-) \le \frac{1}{2} K_- |2k_-|_w = K_- |k_-|_w$.

Step 6 (assembly). Steps 1–5 give, for every $k \ge 0$, $J \ge J_1(R)$ and $\zeta \in \mathcal{D}_{\sigma_+,H}$,

$$|Z^{a,K}_{k,J}(\zeta)| \le Z^{a,K}_{k,J}(\mathrm{Re}\,\zeta) \le \exp\big(\|g(\zeta)\|^2_{2,k} + \kappa \|k_+(\zeta)\|_{1,k} + K_- |k_-(\zeta)|_w\big),$$

which is (73). The constants are those of (36), with $B^{(a)}_{\mathrm{cell}}$ in K_- and, for $a \neq z$, $C_U + s_1 \sup_k |z_k - a_k|$ in κ.

Step 7 (the fixed-cube limits). Each $Z^{a,K}_{k,J}$ is entire in $\zeta \in \mathbb{C}^{n+p}$: it is a finite-dimensional integral with Gaussian decay in A ($\mu_0 > 0$) and $e^{-\lambda\varepsilon^3 r^4}$ decay in r at every alive site, the killed sites carrying no variables, and differentiation under the integral sign is justified. On a compact $Q \subset \mathcal{D}_{\sigma_+,H}$ the family $\{Z^{a,K}_{k,J}\}_{k\ge 0}$ is uniformly bounded, by (73) and Lemma 2.9(E1)–(E3): $\|g\|^2_{2,k} \le C_0 |g|^2_4$, $\|k_+\|_{1,k} \le C_0' |k_+|_w$ and $|k_\pm|_w \le \sum_j |\mathrm{Re}\, w_j| |h_j|_w$, all continuous in ζ. At every real point of $\mathcal{D}_{\sigma_+,H}$ the sequence converges: there $2\|k_+\|_\infty < \sigma_+ = s_1 = s_W$ and the negative part is unrestricted, so Theorem B, which is posed on King's convention (Corollary 6.27(a)), gives the limit in $(0,\infty)$ for the reference scheme; for the admissible scheme a, $Z^{a,K}_{k,J} = e^{-(a_k - z_k)\sum_j w_j \mathsf{S}_k(h_j)} Z^{z,K}_{k,J}$ with $a_k - z_k$ convergent and $\mathsf{S}_k(h_j) \to \int h_j$ (Definition 2.7, Remark 6.28). The real points of $\mathcal{D}_{\sigma_+,H}$ form a non-empty open subset of $\mathbb{R}^{n+p}$ (Definition 2.8), a set of uniqueness in the connected open $\mathcal{D}_{\sigma_+,H}$; by Montel's and Vitali's theorems (the fact (C3) stated before Theorem 13.10), $Z^{a,K}_{k,J} \to G^K_J$ locally uniformly on $\mathcal{D}_{\sigma_+,H}$, G^K_J is holomorphic there, and (73) passes to the limit with $\|g\|_{2,k} \to \|g\|_2$, $\|k_+\|_{1,k} \to \|k_+\|_{L^1}$ (k_+ is continuous with compact support) and the sampled $|k_-|_w$ at most the continuum one (E3):

$$|G^K_J(\zeta)| \le \exp\big(\|g(\zeta)\|^2_2 + \kappa \|k_+(\zeta)\|_{L^1} + K_- |k_-(\zeta)|_w\big) \qquad (\zeta \in \mathcal{D}_{\sigma_+,H},\ J \ge J_1(R)), \tag{74}$$

a bound free of J. At real points $G^K_J > 0$ (Theorem B).

Step 8 (the cube limit on a sub-tube). Split $h_j = h_j^+ - h_j^-$ with $h_j^- := \|h_j\|_\infty \chi_j$, $\chi_j \in C_c^\infty$, $0 \le \chi_j \le 1$, $\chi_j = 1$ on $\operatorname{supp} h_j$, $\operatorname{supp}\chi_j \subset Q_{R+1/2}$, and $h_j^+ := h_j + h_j^- \ge 0$. For the split family $\mathcal{F}'' = (f; h^+, h^-)$ put $F^K_{k,J}(z, w^+, w^-) := Z^{a,K}_{k,J}(\sum_i z_i f_i, \sum_j (w_j^+ h_j^+ + w_j^- h_j^-))$ and $\mathcal{D}'' := \{(z, w^+, w^-) \in \mathbb{C}^{n+2p} : |\operatorname{Re} w_j^\pm| < \delta_j\}$ with $\sum_j \delta_j(\|h_j^+\|_\infty + \|h_j^-\|_\infty) < H/2$. On $\mathcal{D}''$ the scalar source k'' of the split family has $\|k''_\pm\|_\infty \le \|k''\|_\infty < H/2 \le \sigma_+/2$, so $\mathcal{D}''$ lies in the tube of $\mathcal{F}''$, whose supports lie in $Q_{R+1/2}$; Steps 1–7 for $\mathcal{F}''$ and $J \ge J_1(R + \frac12)$ give $F^K_{k,J} \to F^K_J$ locally uniformly on $\mathcal{D}''$ with the J-free bound (74) for $\mathcal{F}''$. On $\Omega''_+ := \mathcal{D}'' \cap ((\mathrm{i}\mathbb{R})^n \times (0,\infty)^{2p})$ the curvature coefficients are imaginary and the total scalar source $\sum_j (w_j^+ h_j^+ + w_j^- h_j^-)$ is non-negative, so the observable is of class $\mathcal{I}$ and (S) holds for X_J ($MJ \ge R + \frac12 + \ell_c$); Corollary 4.3 gives $F^K_{k,J} \le F^K_{k,J+1}$ at every k (the scheme multiplier is common, Lemma 2.14(a)), and $k \to \infty$ gives $F^K_J \le F^K_{J+1}$ on Ω''_+. Being nondecreasing and bounded there, $J \mapsto F^K_J$ converges on Ω''_+, a non-empty open subset of the maximal totally real subspace $(\mathrm{i}\mathbb{R})^n \times \mathbb{R}^{2p}$; $\mathcal{D}''$ is convex, hence connected, and the Montel–Vitali fact (C3) stated before Theorem 13.10 gives $F^K_J \to F^K_\infty$ locally uniformly on $\mathcal{D}''$. At $w_j^+ = w_j$, $w_j^- = -w_j$ the total scalar source is $\sum_j w_j(h_j^+ - h_j^-) = \sum_j w_j h_j$, so $F^K_{k,J}(z, w, -w) = Z^{a,K}_{k,J}(\zeta)$ exactly at every k; hence $G^K_J \to G^K_\infty$ locally uniformly on $\mathcal{D}' := \{\zeta : |\operatorname{Re} w_j| < \delta_j\}$, which lies in $\mathcal{D}_{\sigma_+,H}$ because there $\|k_\pm\|_\infty \le \|k\|_\infty < \sum_j \delta_j \|h_j\|_\infty < H/2$.

Step 9 (the whole tube). The family $\{G^K_J\}_{J \ge J_1(R+1/2)}$ is locally uniformly bounded on $\mathcal{D}_{\sigma_+,H}$ by (74); by Montel's theorem every subsequence has a further subsequence converging locally uniformly to a holomorphic function, any two such limits agree on the open set $\mathcal{D}'$ by Step 8, and $\mathcal{D}_{\sigma_+,H}$ is connected, so all subsequential limits coincide: $G^K_J \to G^K_\infty$ locally uniformly on $\mathcal{D}_{\sigma_+,H}$, G^K_∞ is holomorphic, and (74) passes to G^K_∞. The finitely many indices $J_1(R) \le J < J_1(R + \frac12)$ do not affect the limit.

Step 10 (positivity and the cone). For a real $\zeta \in \mathcal{D}_{\sigma_+,H}$ pick $\epsilon > 0$ with $-\epsilon\zeta \in \mathcal{D}_{\sigma_+,H}$. The function $\varphi_J(t) := \ln G^K_J(t\zeta)$ is finite and convex on $[-\epsilon, 1]$ (a pointwise limit of the convex functions $\ln Z^{a,K}_{k,J}(t\zeta)$, Lemma 2.14(b), with positive limits at real points) and $\varphi_J(0) = 0$, so $\varphi_J(1) \ge -\varphi_J(-\epsilon)/\epsilon \ge -C_\epsilon/\epsilon$, C_ϵ the exponent of (74) at $-\epsilon\zeta$; hence $G^K_\infty(\zeta) \ge e^{-C_\epsilon/\epsilon} > 0$. At the cone points of $\mathcal{D}_{\sigma_+,H}$ ($z \in (\mathrm{i}\mathbb{R})^n$, $w \in \mathbb{R}^p$, $k(\zeta) \ge 0$), Corollary 4.3 at every k and $k \to \infty$ give $G^K_J \le G^K_{J+1} \le G^K_\infty$.

Step 11 (the Schwinger bounds and E0″*).* Put $S^K_{n,p}(f; h) := \partial_z^{(n)} \partial_w^{(p)} G^K_\infty(0)$, $\sigma_0 = H/4$, $K' = \kappa\pi^2 + K_-$. Normalise $\|f_i\|_2 = 1$ and let $\rho_j := \sigma_0/(p|h_j|_w)$. On the polydisc $\{|w_j| \le \rho_j\} \times \mathbb{C}^n$ we have $\|k\|_\infty \le \sum_j |\operatorname{Re} w_j| \|h_j\|_\infty \le \sum_j \rho_j |h_j|_w = \sigma_0 < H/2 \le \sigma_+/2$ (Lemma 2.9(E2)), so the polydisc lies in $\mathcal{D}_{\sigma_+,H}$, where G^K_∞ exists with the bound (74), whose exponent is at most $\|g(z)\|_2^2 + \kappa\pi^2 |k_+|_w + K_- |k_-|_w \le \|g(z)\|_2^2 + K'|k|_w \le \|g(z)\|_2^2 + K'\sigma_0$ (E2). Cauchy's estimate for the mixed first derivative in the w's at $w = 0$, z fixed, gives $|\partial_w^{(p)} G^K_\infty(z, 0)| \le \prod_j \rho_j^{-1} \sup |G^K_\infty| \le (p/\sigma_0)^p \prod_j |h_j|_w \, e^{K'\sigma_0} e^{\|g(z)\|_2^2}$, and $p^p \le e^p p!$ (E4). The function $z \mapsto \partial_w^{(p)} G^K_\infty(z, 0)$ is entire ($\mathbb{C}^n \times \{w\} \subset \mathcal{D}_{\sigma_+,H}$ for $|\operatorname{Re} w_j|$ small) and bounded by $A\, e^{\|g(z)\|_2^2}$ with $A := e^{K'\sigma_0}(e/\sigma_0)^p p! \prod_j |h_j|_w$; on the polydisc of radius $r = (2n)^{-1/2}$ about 0, $\|g(z)\|_2 \le \sum_i |\operatorname{Re} z_i| \le nr$, so Cauchy's estimate for the mixed first z-derivative gives $|S^K_{n,p}| \le A\, r^{-n} e^{n^2 r^2} = A\,(2n)^{n/2} e^{n/2} \le A\,(\sqrt{2}e)^n (n!)^{1/2}$ by (E4). Restoring the normalisation by multilinearity,

$$|S^K_{n,p}(f_1, \dots, f_n; h_1, \dots, h_p)| \le e^{K'\sigma_0} (\sqrt{2}e)^n (n!)^{1/2} (e/\sigma_0)^p \, p! \prod_i \|f_i\|_2 \prod_j |h_j|_w .$$

By (E2), $\|f_i\|_2 \le \frac{\pi}{2\sqrt{2}}|f_i|_4$ and $|h_j|_w \le |h_j|_4$, and $(n!)^{1/2}p! \le (n+p)!$, so the right side is at most $\sigma_{n+p}\prod_i|f_i|_4\prod_j|h_j|_4$ with $\sigma_N = e^{K'\sigma_0}C_1^N N!$, $C_1 = 2\max(e\pi/2, e/\sigma_0)$; for complex test functions each argument is split into real and imaginary parts, which gives the factor $2^{n+p} \le 2^N$ absorbed in C_1. The $S^K_{n,p}$ are therefore continuous multilinear forms in the norms $|\cdot|_4$, extend to $\mathcal{S}(\mathbb{R}^3;\Lambda^2_{\mathbb{C}})^n \times \mathcal{S}(\mathbb{R}^3;\mathbb{C})^p$ and are tempered distributions, as in Theorem 7.6(c), and $C_1^N/(N!)^{\beta-1} \to 0$ for every $\beta > 1$: this is E0″ with $s = 4$ and every $\beta > 1$, with the constants of Theorem 28.2C(i). □

With Theorem 28.5 in hand, the Wick sector is assembled on King's convention.

Proposition 28.6 (the Wick sector on King's convention)**.** *Let the couplings lie in the coupling region of Definition 2.1 and let a be an admissible scheme. Then, with the constants of Definition 28.1:*

(a) *$G^K_J := \lim_k Z^{a,K}_{k,J}$ exists locally uniformly on $\mathcal{D}_{\sigma_+,H}$ for every $J \ge J_1(R)$, $G^K_\infty := \lim_J G^K_J$ exists locally uniformly on $\mathcal{D}_{\sigma_+,H}$, both are holomorphic, $|G^K_\infty(\zeta)| \le \exp(\|g(\zeta)\|_2^2 + \kappa\|k_+(\zeta)\|_{L^1} + K_-|k_-(\zeta)|_w)$, and the Schwinger bounds and E0″ of Theorem 28.2C(i) follow.*

(b) *$G^K_\infty(\tau_b R\cdot\zeta) = G^K_\infty(\zeta)$ on $\mathcal{D}_{\sigma_+,H}$ for all $b \in \mathbb{R}^3$, $R \in O_h$, W^a a scalar.*

(c) *For every family $\mathcal{F}'$ with non-negative scalar members the Wick-tube object*

$$Z^{a,K}_{\infty,T} := \lim_J \lim_k Z^{a,K}_{k,J}$$

exists locally uniformly on $T_{\mathcal{F}'}$ and equals G^K_∞ on $T_{\mathcal{F}'} \cap \mathcal{D}_{\sigma_+,H}$; reflection positivity about every coordinate plane holds in the matrix form at amplitude $< \sigma_W$ and in the polynomial form, and OS3*, for $Z^{a,K}_{\infty,T}$, G^K_∞ and their Schwinger functions.*

(d) The measure and weak OS4 for the joint field. *For every even $r \ge 14$ there is exactly one Borel probability measure on $\mathcal{H}^{-r}(\mathbb{R}^3;\Lambda^2\oplus\mathbb{R})$ whose moments on C_c^∞ are the $S_{n,p}$; its characteristic functional is $(u,v) \mapsto G^K_\infty(\mathrm{i}u, \mathrm{i}v)$ on real test functions, and its image on $H^{-r}_{\mathrm{loc}}(\mathbb{R}^3;\Lambda^2\oplus\mathbb{R})$ is the unique Borel probability measure there with this characteristic functional, and the unique one with these moments. For every real $\chi \in C_c^\infty(\mathbb{R}^3)$ the laws of the localised scheme-shifted lattice fields $\chi(F_{\Lambda_k}, W^a_{\Lambda_k})$ under $\mathrm{ND}^K_{k,J}$ converge weakly along the full sequence $k \to \infty$ at every fixed $J \ge \max(J_0(\chi), J_1(R_\chi))$, then as $J \to \infty$, to the law of the localised joint field under this measure. The measure is $\mathbb{R}^3 \rtimes O_h$-invariant and is an integral of translation-ergodic, reflection-positive, θ_j-invariant components which cluster in the mean and under diagonal separation, have a unique joint vacuum in each coordinate direction and satisfy the factorial growth bound of Theorem 28.2B(vi) with the constants of (a); there is an O_h-invariant decomposition into mixtures of $k \in \{1,2,3,6\}$ ergodic measures, and almost every component is ergodic under $\mathbb{R}^3$ (weak* OS4*). No Euclidean clause is claimed. These are clauses (v) and (vi) of Theorem 28.2C.*

Proof. Parts (a) and (c) are proved below. Parts (b) and (d) are proved by running the cited proofs of the curvature sector on the joint field; the paragraphs (b) and (d) below give the steps of these proofs and the result each step uses.

(a) is Theorem 28.5, whose proof contains the fixed-cube limits (Step 7), the cube limit on the connected tube with the cube monotonicity of Corollary 4.3 in place of the first inequality of Lemma 2.13 (Steps 8–9), the positivity at real points and the cone monotonicity (Step 10), and the Schwinger bounds by Cauchy's estimates on a polydisc in the tube (Step 11).

(b) The steps of Theorem 8.6 with scalar sources. Coordinate permutations are exact on ND^K for (f,h) by Lemma 8.5. The reflection sandwich is Lemma 4.6(b) at the cone points of the tube: for $(f,g) \in \mathcal{K}$ with $\|g\|_\infty < \sigma_+/2$ the point $(\mathrm{i},1)$ lies in $\mathcal{D}_{\sigma_+,H}$, so (a) provides the three k-limits and the two J-limits of the two-step sandwich $C^K_{J\mp2}$. Conjugation and group closure on the cone

are unchanged. Off the cone, the identity theorem on the split family's tube of Lemma 13.11, where (a) gives the limits, extends the identity from the cone points to the connected tube; so Lemma 4.6 is needed on $\mathcal{K}$ only, and no class-$\mathcal{D}$ form of it enters (the proof splits $h = h^+ - h^-$ and never applies the sandwich to a negative scalar source). Translations are Lemma 8.3, stated on $\mathcal{K}$ and applied at the cone points of the tube with (a) for the limits, then the continuity route of Corollary 8.4 with the Wick-sector second-moment bound, which Cauchy's estimates give from (37) on ND^K; the translated family needs $J \geq \max(J_0+1, J_1(R+M+\frac{1}{2}))$. Invariance about every point and differentiation concern the limit object.

(c) is Theorem 10.6 with Proposition 10.4, both stated for King's state. The convention-sensitive steps used there are: in the sandwich of the J-limit the upper box is Lemma 4.7 (the boxes C_J, C_{J+2n} as stated there) and the lower end carries the kill step of Remark 4.9; the positive-leg bound Corollary 13.3 holds on ND^K by Remark 4.9 prepended to Proposition 13.2 (Remark 13.12); the cube monotonicity in both classes is Corollary 4.3; the fixed-cube limits on $T_{\mathcal{F}'}$ are Theorem 6.26(i), whose domain $\{\sup_x 2k_1 < s_W\}$, $s_W = s_1$, contains $T_{\mathcal{F}'}$ (for the reference scheme, hence for a by Definition 2.7), and on $\mathcal{D}_{\sigma_+,H}$ they are (a); the cube limit on $T_{\mathcal{F}'}$ is Montel's theorem with the uniqueness set of Definition 10.2(iii) (Proposition 10.4(a)); the translations to the planes $c \notin D$ are Proposition 10.4(b), that is (b) restricted to translations. OS3 and the polynomial form use only the existence and holomorphy of the limit near 0 and charge conjugation (Propositions 10.7 and 10.8).

(d) The steps of Theorem 11.5 on $\mathcal{V} = \Lambda^2 \oplus \mathbb{R}$ with the scheme-shifted field: (37) on ND^K in the moment bounds, (a) for the limit functionals and the lattice laws, (b) and (c) for the invariance and the reflection positivity, which give (H1)–(H3) of Definition 11.4 as in Step 3 of the proof of Theorem 11.5, with the joint field in place of the curvature field. The second moments of the lattice field under $\mathrm{ND}^K_{k,J}$ come from (37) on ND^K by Cauchy's estimates, so tightness and the identification of the limit law run as in the curvature sector, with the fixed-cube limits of (a) along the full sequence, for $J \geq \max(J_0(\chi), J_1(R_\chi))$; the remaining steps concern the limit measure. □

Proposition 28.7 (the comparison family on the Wick tube)**.** *Let the couplings lie in the coupling region of Definition 2.1 and let a be an admissible scheme.*

(a) *Every increasing sequence of spacings has a continuum subsequence $\mathbf{k}$ for the Wick sector of the comparison family (Definition 14.1).*

(b) *For every such $\mathbf{k}$, every family $\mathcal{F}$ of real sources with supports in Q_R and every $\zeta \in \mathcal{D}_{\sigma_+,H}$, the LIM state of Definition 14.2 is King's tube object:*

$$Z^{a,\mathbf{k}}_\infty(\zeta) \;:=\; \lim_{J\to\infty}\lim_{m\to\infty} Z^a_{k_m,J}(\zeta) \;=\; G^K_\infty(\zeta) \qquad (\zeta \in \mathcal{D}_{\sigma_+,H}).$$

In particular the Wick-sector LIM state does not depend on $\mathbf{k}$, and the cube limit of every subsequential fixed-cube limit of the comparison family on the tube is G^K_∞.

Proof. (a) By Lemmas 14.4 and 14.5 the scheme has the cell bound and the scheme bound, so Lemma 14.7(b) applies.

(b) Fix $\mathbf{k} = (k_m)$ and $\mathcal{F}$. By Lemma 14.8 and the Wick sector of Lemma 14.7(b), $Z^{a,\mathbf{k}}_\infty$ exists locally uniformly on $\mathcal{D}_{\sigma_+,H}$ and is holomorphic there; by Theorem 28.5 so is G^K_∞. Take the split family $\mathcal{F}'' = (f; h^+, h^-)$ of Step 8 of the proof of Theorem 28.5, with supports in $Q_{R+1/2}$, the domain $\mathcal{D}''$ and the set $\Omega''_+ = \mathcal{D}'' \cap ((\mathrm{i}\mathbb{R})^n \times (0,\infty)^{2p})$ defined there; write $F^K_{k,J}$ and $F^{(a)}_{k,J}$ for the functionals of $\mathcal{F}''$ on King's and on the comparison cube. Let J be so large that $MJ > R + \frac{1}{2} + \ell_c$. At a point of Ω''_+ the curvature coefficients are purely imaginary and the total scalar source is non-negative, so the observable is of class $\mathcal{I}$, and the support condition (S) holds

for X_J; Lemma 4.2 (with its clause for the scheme functionals) gives, at every k,

$$F^K_{k,J} \;\le\; F^{(a)}_{k,J} \;\le\; F^K_{k,J+1} \qquad \text{on } \Omega''_+.$$

Along $\mathbf{k}$ the two ends converge, by Theorem 28.5 for the family $\mathcal{F}''$ (whose fixed-cube limits exist along the full sequence, hence along $\mathbf{k}$), to the fixed-cube limits F^K_J, F^K_{J+1} of Step 8 of that proof, and the middle term converges to the fixed-cube limit $F^{(a),\mathbf{k}}_J$ of the comparison family along $\mathbf{k}$ (Definition 14.1 for $\mathcal{F}''$, whose tube contains $\mathcal{D}''$ and whose threshold is $MJ > R(\mathcal{F}'') + \ell_c$, $R(\mathcal{F}'') \le R + \frac{1}{2}$). Letting $J \to \infty$, the two ends tend to F^K_∞ (Steps 8–9 of the proof of Theorem 28.5) and the middle term to the LIM state $F^{(a),\mathbf{k}}_\infty$ of $\mathcal{F}''$ (Lemma 14.8). Hence $F^{(a),\mathbf{k}}_\infty = F^K_\infty$ on Ω''_+, a non-empty open subset of the maximal totally real subspace $(\mathrm{i}\mathbb{R})^n \times \mathbb{R}^{2p}$, and both functions are holomorphic on the connected open set $\mathcal{D}''$; by the identity theorem (the fact (C2) stated before Theorem 13.10) they agree on $\mathcal{D}''$. At $w^+_j = w_j$, $w^-_j = -w_j$ the lattice functionals of $\mathcal{F}''$ coincide exactly with those of $\mathcal{F}$ at ζ, on both conventions and at every k (equal total sources); so $Z^{a,\mathbf{k}}_\infty = G^K_\infty$ on $\mathcal{D}' = \{\zeta : |\operatorname{Re} w_j| < \delta_j\} \subset \mathcal{D}_{\sigma_+,H}$. Both are holomorphic on the connected $\mathcal{D}_{\sigma_+,H}$ and agree on the open set $\mathcal{D}'$, hence on $\mathcal{D}_{\sigma_+,H}$. The last clause is Definition 14.2: every subsequential fixed-cube limit on the tube is the fixed-cube limit along some continuum subsequence. □

Remark 28.8 (the three amplitude constants). Three amplitudes occur in the Wick sector, all built from s_1: the matrix form of OS2 is proved at amplitude $\mathsf{a}(\zeta_i) < \sigma_W = \sigma_+/8$ on the Wick tube (Theorem 10.6); the tube $\mathcal{D}_{\sigma_+,H}$ has half-widths $\sigma_+/2$ for the positive and $H/2 = \sigma_+/32$ for the negative part of the real Wick source, and the form of OS2 proved for LIM states (Theorem 14.9) takes $\mathsf{a}(\zeta_i) < H/2 < \sigma_W$, so that every Gram point lies in $\mathcal{D}_{\sigma_+,H}$, where those limits exist; and $\sigma_0 = H/4$ is the radius of the polydisc in Cauchy's estimates for E0″. The use of OS2 in clauses (v),(vi), the reflection positivity of exponentials $e^{\mathrm{i}\Phi(u)}$ with real u that gives hypothesis (H3) of the ergodic decomposition and of the clustering theorems (Definition 11.4), needs purely imaginary Wick coefficients only, that is amplitude 0 (Remark 10.9).

Remark 28.9 (the photon mass and the couplings). The photon mass $\mu_0 > 0$ is fixed throughout Theorem 28.2. No relation $\mu_0 \le m$ is used or needed: the lattice reflection positivity of King's tori holds for every $\mu_0 > 0$, its proof using no bound on μ_0 (Proposition 12.2, Lemma 29.3(d)). Reflection positivity of the cube limit needs the margin $R(\mu_0) = \pi\sqrt{3}/(2\mu_0)$ of (23) inside the upper half of the sandwich (Lemma 9.10) and nowhere in the conclusion. The constants of Leg C are absolute; those of Leg W depend on μ_0 and are not uniform in it (the limit of vanishing photon mass in the curvature sector is taken in Section 30). The couplings range over all of $e > 0$, $\lambda, m, \mu_0 > 0$: their quantifier is the sources' (Lemma 29.3(e), Remark 29.31), the stopping scale of the lattice constructions behind Theorems A and B and Proposition 12.6 being chosen after the point and the datum, with the mass cap of [2, Prop. 3.1] one more one-sided cap on that scale (Remark 29.5); so the theorem contains no sentence "e, λ sufficiently small", and the only coupling condition of the Wick sector is the one-sided sign clause (T3) (Lemma 29.3(b)), proved on the explicit regions of Proposition 29.29, among them a box uniform in the photon mass (Proposition 29.30).

29. The coupling range

In Theorem 28.2 the curvature sector is posed on all of King's range, and the Wick sector under the sign clause (T3), with the further restrictions stated there. This section proves the two statements about the couplings behind this: Lemma 29.3, by which the curvature sector needs no condition on the couplings, and Propositions 29.29 and 29.30, which give explicit regions on which the sign clause holds.

This section collects the conditions on the couplings in the literature behind Theorem 28.2 and shows that the only coupling condition anywhere in this paper is the one-sided sign clause (T3) of the Wick leg, which Propositions 29.29 and 29.30 prove on explicit regions. It uses the c-number statement (Lemma 12.7), the closed form of the limiting bare coefficient (Lemma 29.1) and the two-loop majorants of the lattice mass counterterm, proved below. Its results enter the Wick leg through Proposition 12.8 and the statement of Theorem 28.2, and nothing in the curvature sector depends on them.

King's range, units, two letters J. Let $e > 0$ and $\lambda, m, \mu_0 > 0$. Every constant of the cited constructions, including the number J_{K} of renormalisation steps after which the running couplings are small and the stopping scale $5^{-J_{\mathrm{K}}}$ (a member of the family $\{5^{-k}\}$), may depend on (e, λ, m, μ_0), on the large-block integer M and on the cube or torus. No numerical restriction is placed on the couplings themselves. That a stopping scale exists at every positive point is part of the cited constructions ([25, Thm. 3.1], [1, 2, 3, 4], and the existence and rotation parts of this paper), under the mass cap of [2, Prop. 3.1], a one-sided cap on the stopping scale (Remark 29.5). This is what "King's range" means here: all positive (e, λ, m, μ_0), with a stopping scale chosen after them. The cube index J of C_J is ours. King's stopping index, the "fixed integer J" of [25, Thm. 3.1], is written J_{K}. Couplings are measured in the units in which the coarsest spacing is 1. In $d = 3$ all four of e^2, λ, m, μ_0 have the dimension of a mass: rescaling to the unit lattice at spacing ε replaces m^2, λ, e by $m^2\varepsilon^2, \lambda\varepsilon, e\varepsilon^{1/2}$ [25, Sect. 0] and μ_0^2 like m^2.

The relative bare coefficient. Let $\alpha_k^\infty(\mathsf{M}) = \mathsf{M}^2 + \delta m_k^2(\mathsf{M})$, $s_k^\infty(M_1) = \frac{1}{2}(\alpha_k^\infty(m) - \alpha_k^\infty(M_1))$ and $\sigma_1(M_1) = \liminf_k s_k^\infty(M_1)$ be as in Definition 2.1. The theorem is stated through σ_1 alone. Describing *where* (T3) fails uses the continuum limit of this coefficient, which the following lemma, of the same kind as Lemma 12.7, gives in closed form. We write (CT_∞) for its first assertion: for every $\mathsf{M} > 0$ the limit $\alpha_\infty(\mathsf{M}) - \alpha_\infty(m) := \lim_{k\to\infty}[\alpha_k^\infty(\mathsf{M}) - \alpha_k^\infty(m)]$ exists and is a polynomial of degree ≤ 2 in (e^2, λ) with finite coefficients, differentiable in M^2.

Lemma 29.1 (the limiting bare coefficient)**.** *Let $N = 2$ (King's model), $e > 0$, $\lambda, m, \mu_0 > 0$, and write $\mu := \mu_0$. For every $M > 0$ the limit*

$$\alpha_\infty(M) - \alpha_\infty(m) := \lim_{k\to\infty} \big[\alpha_k^\infty(M) - \alpha_k^\infty(m)\big]$$

exists, and

$$\begin{aligned} \alpha_\infty(M) - \alpha_\infty(m) &= (M^2 - m^2) + \lambda D_{01} + e^2 D_{20} + e^4 D_{40} + \lambda e^2 D_{21} + \lambda^2 D_{02}, \\ D_{\alpha\beta} &:= F_{\alpha\beta}(M) - F_{\alpha\beta}(m), \end{aligned} \tag{75}$$

with

$$F_{01}(M) = \frac{4M}{\pi}, \qquad F_{20}(M) = -\frac{M^2}{4\pi(M+\mu)}, \qquad F_{02}(M) = -\frac{8}{\pi^2}\ln M, \tag{76}$$

$$F_{21}(M) = \frac{2}{\pi^2}\ln(2M+\mu) + \frac{\mu}{2\pi^2(M+\mu)}, \qquad F_{40}(M) = -\frac{1}{4\pi^2}\ln M + \Phi_{40}(\mu/M),$$

$$\begin{aligned} \Phi_{40}(x) = {} & \frac{-7 + 4x + 8x^2 - 3x^3 - 3x^4 + x^5}{32\pi^2(1+x)(1-x^2)^2} \\ & + \frac{(5 - 2x^2)\ln(2+x) - (9 - 10x^2 + 4x^4)\ln(1+2x)}{16\pi^2(1-x^2)^2}, \end{aligned} \tag{77}$$

where Φ_{40} extends analytically across $x = 1$. Consequently $\alpha_\infty(M) = M^2 + \lambda F_{01}(M) + e^2 F_{20}(M) + e^4 F_{40}(M) + \lambda e^2 F_{21}(M) + \lambda^2 F_{02}(M)$ up to an additive constant, a function of (e, λ, μ_0) and M

alone, real-analytic on $(0,\infty)$*; its derivatives in* $s=M^2$ *are*

$$
\begin{aligned}
y^{\infty}_{01}&=\frac{2}{\pi M},\quad y^{\infty}_{20}=-\frac{M+2\mu}{8\pi(M+\mu)^2},\quad y^{\infty}_{02}=-\frac{4}{\pi^2M^2},\\
y^{\infty}_{21}&=\frac{8M^2+14M\mu+7\mu^2}{4\pi^2M(M+\mu)^2(2M+\mu)}>0,\quad y^{\infty}_{40}=-\frac{1}{8\pi^2M^2}-\frac{\mu}{2M^3}\Phi'_{40}(\mu/M),
\end{aligned}
\tag{78}
$$

and $\alpha_\infty(M)-\alpha_\infty(m)=M^2+(4\lambda/\pi-e^2/4\pi)M-(e^4/4\pi^2-2\lambda e^2/\pi^2+8\lambda^2/\pi^2)\ln M+O(1)$ *as* $M\to\infty$*. In particular the limit is a polynomial of degree* ≤ 2 *in* (e^2,λ) *with finite coefficients, differentiable in* M^2 *at every* $M>0$*, and, for* $0<M_1<m$*,* $\sigma_1(M_1)=\lim_k s^{\infty}_k(M_1)=\frac12(\alpha_\infty(m)-\alpha_\infty(M_1))$ *exists and is finite. Moreover, with* $'=d/dM^2$*: if* m *is a global minimiser of* α_∞ *on* $(0,\infty)$*, then* $\alpha'_\infty(m)=0$*; and for every* (λ,m,μ_0) *the polynomial* $e^2\mapsto\alpha'_\infty(m)$*, of degree* ≤ 2*, is not identically zero, so it vanishes for at most two values of* $e^2>0$*.*

Proof. Existence. Each coefficient of the counterterm difference is

$$
\delta m^2_{(\alpha\beta)}(\varepsilon_k;M)-\delta m^2_{(\alpha\beta)}(\varepsilon_k;m)=\int_{m^2}^{M^2}y_{\alpha\beta}(s;\varepsilon_k)\,ds,
$$

where $y_{\alpha\beta}(s;\varepsilon)=\frac{d}{ds}\delta m^2_{(\alpha\beta)}(\varepsilon;\sqrt{s})$ is, by (84), the s-derivative of the zero-momentum one-particle-irreducible two-point function at that order without its own counterterm vertex, that is a sum over the scalar lines l of its graphs of the integrand with the line l doubled (a factor $-(\hat p_l^2+s)^{-1}$), the lower-order counterterm vertices and their s-derivatives inserted. Doubling a line lowers the lattice degree of divergence of every subgraph containing it by 2, so every subtracted derivative integrand has negative overall degree and negative degrees on all Zimmermann subspaces (the subtraction terms that thereby become over-subtractions integrate to zero by parity in the loop momentum of the subgraph, or are cancelled by the derivative of the counterterm vertex that the forest formula carries). Reisz's power counting theorem [34, Thm. 1], in its three-dimensional form (the dimension enters his proof only through the measure and the degrees) and with the absolute bound $\sup_{\varepsilon\le\varepsilon_0}\int|F|\le K$ that its proof gives at fixed masses, bounds $y_{\alpha\beta}(s;\varepsilon)$ uniformly in $\varepsilon\le\varepsilon_0$ and in s on compact intervals $[s_1,s_2]\subset(0,\infty)$: on such an interval the integrand is dominated by $\sum_i s_2^i|V_i|/C(s_1)$, with finitely many s-free trigonometric numerators V_i over the common denominator $C(s)=\prod_L(e_L+\mu_L^2)^{n_L}$, $C(s)\ge C(s_1)$, a finite family of fixed integrands of negative degrees. Reisz's renormalisation theorem [35], applied to the subtracted integrands of King's graph classes (his forest formula with periodic subtraction operators produces King's recursive counterterms, since the order-1 Taylor term of the scalar two-point subtraction vanishes by parity and the photon two-point function at zero momentum by the lattice Ward identity; the one-loop vertex subgraphs of degree 0 of the two crossed classes of order e^4, the crossed graph and the end seagull with both photons on the currents, subgraphs with two scalar legs and one photon leg, are subtracted at order 0 as his condition on the subtraction degrees requires, and their subtraction term, the vertex at zero external momenta, is odd in the loop momentum of the subgraph, a free integration variable at fixed outer momenta, so it integrates to zero over the Brillouin zone and adds no counterterm beyond King's), gives the convergence $y_{\alpha\beta}(s;\varepsilon)\to y^{\infty}_{\alpha\beta}(s)$ as $\varepsilon\to0$ for every $s>0$, the limit being the absolutely convergent continuum integral of the pointwise limit of King's subtracted integrand (the lattice-only structures, vertices with $n\ge3$ photon legs carrying ε^{n-2}, the cosine factors of the seagull and the phases, have pointwise limit 1 or 0); dominated convergence in s then gives $\lim_k D_{\alpha\beta,\varepsilon_k}=\int_{m^2}^{M^2}y^{\infty}_{\alpha\beta}(s)\,ds$. The limit is a polynomial of degree ≤2 in (e^2,λ) because the counterterm has the orders $\alpha+2\beta\le4$ with the odd powers of e vanishing (Lemma 29.14). The divergent parts of the counterterm, $1/\varepsilon$ at one loop and $\ln(1/\varepsilon)$ at two loops, do not depend on the scalar mass and cancel in the differences; this is what the negative degrees of the derivative integrands express.

Value. Continuum conventions: $\int_k := \int d^3k/(2\pi)^3$, lines $G(q) = (q^2+M^2)^{-1}$, $D(k) = (k^2+\mu^2)^{-1}\delta_{\mu\nu}$; for the complex field $\phi_c = (\phi_1 + \mathrm{i}\phi_2)/\sqrt{2}$ the vertices are the current $-e(p+p')_\mu$, the seagull $2e^2\delta_{\mu\nu}$, the quartic 16λ and the counterterm, with King's prescription $\delta m^2_{(\alpha\beta)} = \Sigma^{(\alpha\beta)}_{\mathrm{int}}(0)$ [25, (3.32)–(3.33)]. *One loop:* D_{01} and D_{20} are the limits of the Brillouin-zone integrals $16(M^2-m^2)\int_{\mathsf{B}} G_M G_m$ and $-(M^2-m^2)\int_{\mathsf{B}} \hat q^2 \hat D\, G_M G_m$, the differences between the masses M and m of Lemma 29.7(i), (ii), with $G_M(p) = (\hat p^2 + M^2)^{-1}$ and $\hat D$ the momentum-space propagators of that lemma, dominated by $(\pi^2/4)^2[(q^2+M^2)(q^2+m^2)]^{-1}$ since $\hat q^2 \geq \frac{4}{\pi^2}q^2$ and $\hat q^2\hat D \leq 1$; the continuum integrals $\int_k G_M G_m = \frac{1}{4\pi(M+m)}$ and $\int_k k^2 D G_M G_m$, by partial fractions, give F_{01} and F_{20}. *Order λ^2:* $y^\infty_{02}(s) = -384\iint G^2(q)G(k)G(q+k) = -384/(96\pi^2M^2) = -4/(\pi^2 s)$, from the two-loop vacuum integral $\iint G_a(q)^2 G_b(k) G_c(q+k) = \frac{1}{32\pi^2 a(a+b+c)}$ (in position space $(G^2_a)^\sim(x) = e^{-a|x|}/8\pi a$ and $\tilde G_a(x) = e^{-a|x|}/4\pi|x|$). *Order $e^2\lambda$:* the limit is $-16\int_k G^2\sigma$ up to an M-independent constant, where $\sigma(q) = 2q^2B(q) + (\mu^2-2M^2)(B(q)-B(0))$ is the renormalised one-loop self-energy subtracted at zero momentum and $B(q) = \arctan(|q|/(M+\mu))/(4\pi|q|)$ the bubble with lines of masses M,μ, the continuum limit of the exact lattice decomposition of the order-e^2 self-energy (Lemma 29.22(a)); partial fractions in q^2 with $q^2G^2 = G - M^2G^2$ reduce it to differences $V(a,b,c) - V(a',b',c') = -\frac{1}{16\pi^2}\ln\frac{a+b+c}{a'+b'+c'}$ of the vacuum integral with lines $q, k, q+k$ of masses a,b,c (with $\mathsf{s} := a+b+c$ and $\mathsf{s}' := a'+b'+c'$ the mass sums of the two triples, the Frullani integral $\frac{1}{16\pi^2}\int_0^\infty (e^{-\mathsf{s}r} - e^{-\mathsf{s}'r})\frac{dr}{r}$ in position space) and to the integrals above, which gives F_{21}. *Order e^4:* the twelve topologies of order e^4, with the order-e^2 counterterm inserted on the exchange line, reduce to five continuum classes, the rainbow with σ on its inner line, $\int Dq^2G^2\sigma$; the crossed graph, $V(M,\mu,\mu) - M^2I^2$ with $I = \int_k DG = \frac{1}{4\pi(M+\mu)}$; the seagull at an end with both photons to the currents; the two-photon seagull sunset $6V(M,\mu,\mu)$; and the seagull tadpole with the one-loop vacuum polarisation $\Pi_{\mu\mu}(k) = -e^2[k^2B_{MM}(k) + 4M^2(B_{MM}(k) - B_{MM}(0))]$, while the exchange with the vacuum-polarisation photon vanishes by the Ward identity $k_\mu\Pi_{\mu\nu} = 0$ and the middle seagull by parity. Each class is reduced to $V(M,\mu,\mu)$, $V(M,M,\mu)$ and elementary integrals by the exact identities $k_1\cdot(k_1+2k_2) = G(k_1+k_2)^{-1} - G(k_2)^{-1}$, $k_2\cdot(2k_1+k_2) = G(k_1+k_2)^{-1} - G(k_1)^{-1}$ and partial fractions; the two classes with a logarithmically divergent vertex subgraph are evaluated with the product cut-off $|k_1|, |k_2| < \Lambda$, under which the zeroth Taylor term of the subgraph is odd in its loop momentum and the one shift needed has a surface term whose s-derivative tends to 0. The coefficients of V add to $1+6-6+1+2 = 4$, which is the universal $\frac{1}{4\pi^2}\ln(1/\varepsilon)$ of the order e^4, cancelling in differences; the five class values add to F_{40} as displayed. *Analyticity and the derivatives.* $F_{01}, F_{20}, F_{02}, F_{21}$ are analytic on $(0,\infty)$; Φ_{40} is built from rational functions and $\ln(2+x)$, $\ln(1+2x)$, and at $x=1$, where its denominators vanish, the coefficients of $(x-1)^{-4}, \dots, (x-1)^{-1}$ of its Laurent expansion vanish identically, so the singularity is removable. The formulas (78) are the s-derivatives of (76)–(77), and the large-M form collects their terms. The identification $\sigma_1(M_1) = \lim_k s^\infty_k(M_1)$ for $0 < M_1 < m$ is the definition $s^\infty_k(M_1) = \frac12(\alpha^\infty_k(m) - \alpha^\infty_k(M_1))$ of Definition 2.1 together with the existence of the limit just proved, which turns the lower limit defining σ_1 into a limit.

The critical-point clause. A global minimum of a function differentiable at the interior point m is a critical point. By (78) at $M = m$,

$$\alpha'_\infty(m) = a_0(\lambda) + a_1(\lambda)e^2 + a_2e^4, \qquad a_0 = 1 + \frac{2\lambda}{\pi m} - \frac{4\lambda^2}{\pi^2m^2},$$
$$a_1 = -\frac{m+2\mu_0}{8\pi(m+\mu_0)^2} + \lambda\frac{8m^2+14m\mu_0+7\mu_0^2}{4\pi^2m(m+\mu_0)^2(2m+\mu_0)}, \qquad a_2 = y^\infty_{40}(m).$$

$a_0 = 0$ if and only if $\lambda = \lambda_0 := \pi m(1+\sqrt{5})/4$; at λ_0, with $x := \mu_0/m$, $8\pi(m+\mu_0)^2(2+x)\,a_1(\lambda_0) = m[(2+4\sqrt{5})+(2+7\sqrt{5})x+(\frac{3}{2}+\frac{7}{2}\sqrt{5})x^2] > 0$. So a_0 and a_1 never vanish together, the polynomial $e^2 \mapsto \alpha'_\infty(m)$ is never identically zero, and it has at most two zeros. □

Lemma 29.2 (the relative vacuum energy)**.** *Let $N = 2$, $e > 0$, $\lambda, m, \mu_0 > 0$ and $M_1 > 0$, $M_1 \neq m$, and let $b_k^\infty(M_1)$ be the quantity defined after Definition 12.5 with this M_1 (the limit defining it exists by Lemma 12.11, read on the interval between M_1 and m as in the proof of Proposition 12.13). Then $\sup_{k\geq 0} |b_k^\infty(M_1)| < \infty$. In particular $\beta_1(M_1) = \limsup_k b_k^\infty(M_1)$ is finite.*

Proof. Throughout, s denotes the squared scalar mass, ∂_s the derivative in s, and I the closed interval with endpoints M_1^2 and m^2. Fix k, write $\varepsilon = \varepsilon_k$ and $T = \mathbb{T}_J$. For the model on T at scalar mass $\sqrt{s}$ in King's torus convention let $c_T(s) = c_{k,T}(\sqrt{s})$ be the Wick constant, $\alpha_T(s) = \alpha^{\mathrm{tor}}_{k,T}(\sqrt{s}) = s + \delta m^2_{k,T}(\sqrt{s})$ the bare coefficient, $\delta m^2_{(\alpha\beta),T}(\varepsilon;\sqrt{s})$ the coefficients of $\delta m^2_{k,T}$ (those of (84), computed with the periodic propagators of T), and $E_T(s) = E_0 + E_1$ the field-independent constants of King's action, $E(\sqrt{s})$ in the notation of Section 12: E_0 is the logarithm of the Gaussian normalisation [25, (2.6)], and E_1 [25, (3.35)] is the sum of the connected vacuum graphs of King's action with the counterterm vertices $-\frac{1}{2}e^\alpha\lambda^\beta\delta m^2_{(\alpha\beta),T}\sum_x \varepsilon^3|\phi(x)|^2$ and the periodic propagators of T, over the orders $e^\alpha\lambda^\beta$ with $\alpha/2+\beta \leq 3$, the orders of the vacuum graphs of non-positive degree [25, (3.30)–(3.31)]. We write $\mathcal{O}$ for the set of these orders together with $(0,0)$; thus $E_T(s)$ is the expansion of $\ln\int e^{-S}$ in the couplings, counterterm vertices carrying their orders, truncated to $\mathcal{O}$. With $s_{k,T}(M_1) = \frac{1}{2}(\alpha_T(m^2) - \alpha_T(M_1^2))$, the definition of $B^z_{k,T}$ after Definition 12.5 says that

$$\Phi_T(s) := |T|^{-1}\big(E_T(s) - E_T(m^2)\big) - \big(c_T(m^2) + \tfrac{1}{2}z_k\big)\big(\alpha_T(m^2) - \alpha_T(s)\big)$$

satisfies $\Phi_T(M_1^2) = |T|^{-1}B^z_{k,T}(M_1)$ and $\Phi_T(m^2) = 0$.

The mass-insertion identity. The scalar mass enters King's action on T only through the term $\frac{1}{2}\alpha_T(s)\sum_x \varepsilon^3|\phi(x)|^2$. Each coefficient of the expansion of $\ln\int e^{-S}$ in the couplings is a finite sum of Gaussian moments of the torus field at mass $\sqrt{s}$, smooth in $s > 0$, and differentiation under the integral sign gives, order by order,

$$\partial_s \ln\int e^{-S} = -\tfrac{1}{2}\,\partial_s\alpha_T(s)\sum_x \varepsilon^3\langle|\phi(x)|^2\rangle_T = -\tfrac{1}{2}\,\partial_s\alpha_T(s)\,|T|\,\langle|\phi(0)|^2\rangle_T$$

by translation invariance. Since ∂_s preserves the order (the derivative of a graph doubles one of its scalar lines or differentiates one of its counterterm vertices), the identity holds for the truncations to $\mathcal{O}$:

$$\partial_s E_T(s) = -\tfrac{1}{2}|T|\sum_{o'} e^{\alpha'}\lambda^{\beta'}\,y_{o',T}(s)\sum_{o'':\,o'+o''\in\mathcal{O}} e^{\alpha''}\lambda^{\beta''}\,g_{o'',T}(s), \tag{79}$$

where $\partial_s\alpha_T(s) = \sum_{o'} e^{\alpha'}\lambda^{\beta'}y_{o',T}(s)$ with $o' = (\alpha',\beta')$ running over $(0,0)$ and the five orders of (84), $y_{(00),T} := 1$, $y_{(\alpha\beta),T}(s) := \partial_s\delta m^2_{(\alpha\beta),T}(\varepsilon;\sqrt{s})$, and $g_{o'',T}(s)$ is the coefficient of $e^{\alpha''}\lambda^{\beta''}$ in the expansion of $\langle|\phi(0)|^2\rangle_T$ (the connected graphs with the insertion $|\phi(0)|^2$, counterterm vertices included). Here $g_{(00),T} = 2c_T(s)$; $g_{(01),T} = 0$, the quartic tadpole being cancelled identically by $\delta m^2_{(01)}$ (Lemma 29.7(i)); and $e^2 g_{(20),T}$ is the torus form of the reference scheme $z_k(\sqrt{s}) = e^2 X_\varepsilon(\sqrt{s})$ at mass $\sqrt{s}$ (Definition 27.1(ii), Lemma 6.10(b)). Every o' has $\alpha'/2+\beta' \leq 2$, so $o'+(0,0)$ and $o'+(2,0)$ lie in $\mathcal{O}$, and the inner sum of (79) is $2c_T(s) + e^2 g_{(20),T}(s) + R_T^{o'}(s)$, where $R_T^{o'}$ collects the finitely many remaining orders o'', none of them $(0,0)$ or $(2,0)$. Hence

$$\partial_s\Phi_T(s) = \tfrac{1}{2}\sum_{o'} e^{\alpha'}\lambda^{\beta'}\,y_{o',T}(s)\Big[2\big(c_T(m^2) - c_T(s)\big) + \big(z_k - e^2 g_{(20),T}(s)\big) - R_T^{o'}(s)\Big]. \tag{80}$$

The volume limit at fixed spacing. Each term of $\Phi_T(s)$ and of (80) is a Riemann sum over the dual lattice of $\mathbb{T}_J$ of a smooth periodic function of the momenta and of $s \in I$, and so is its s-derivative; as in the proof of Lemma 12.11 these sums converge as $J \to \infty$, uniformly on I, to the corresponding Brillouin-zone integrals. Hence $\Phi_k := \lim_J \Phi_{\mathbb{T}_J}$ exists, is continuously differentiable on I, $\Phi_k(m^2) = 0$, $\Phi_k(M_1^2) = b_k^\infty(M_1)$, and

$$\partial_s \Phi_k(s) = \tfrac{1}{2} \sum_{o'} e^{\alpha'} \lambda^{\beta'} \, y_{\alpha'\beta'}(s;\varepsilon) \big[\Delta_k(\sqrt{s}) - R_k^{o'}(s) \big],$$

with $y_{\alpha\beta}(s;\varepsilon)$ the derivatives of the proof of Lemma 29.1, $\Delta_k(M) = 2(\mathsf{W}_\varepsilon(m) - \mathsf{W}_\varepsilon(M)) + z_k(m) - z_k(M)$ the quantity of Lemma 6.10(b) (as $\lim_J c_T(s) = \mathsf{W}_\varepsilon(\sqrt{s})$ and $\lim_J e^2 g_{(20),T}(s) = z_k(\sqrt{s})$), and $R_k^{o'}$ the same coefficients computed on Λ_k.

Bounds. Let $\varepsilon \le \varepsilon_0$ and $s \in I$. The $y_{\alpha\beta}(s;\varepsilon)$ are bounded uniformly in $\varepsilon \le \varepsilon_0$ and $s \in I$ (proof of Lemma 29.1). The coefficients in $R_k^{o'}$ are one-point coefficients of orders other than $e^2\lambda^0$; Reisz's power counting theorem, Theorem 19.14 applied as in Remark 19.15, bounds them uniformly in $\varepsilon \le \varepsilon_0$ and in the scalar mass on I, for the torus coefficients and their limits alike. Finally $\Delta_k(m) = 0$ and

$$\partial_s \Delta_k(\sqrt{s}) = 2 \int_{\mathsf{B}} \frac{1}{(\hat{p}^2 + s)^2} - e^2 \, \partial_s X_\varepsilon(\sqrt{s}),$$

the first term a convergent lattice integral, bounded uniformly in ε and in $s \in I$ ($\hat{p}^2 \ge 4|p|^2/\pi^2$ on the Brillouin zone), the second the order-e^2 one-point coefficient with one scalar line doubled, an integrand of negative degree on every Zimmermann subspace once its self-energy subgraph is subtracted at zero momentum by $\delta m^2_{(20)}$, bounded by the same theorem exactly as the $y_{\alpha\beta}$ are in the proof of Lemma 29.1. So $|\Delta_k(\sqrt{s})| \le |m^2 - M_1^2| \sup_I |\partial_s \Delta_k(\sqrt{\cdot})|$ on I, $\sup_I |\partial_s \Phi_k| \le \bar{\Phi}$ with $\bar{\Phi} < \infty$ independent of k, and

$$|b_k^\infty(M_1)| = \Big| \int_{m^2}^{M_1^2} \partial_s \Phi_k(s) \, ds \Big| \le |m^2 - M_1^2| \, \bar{\Phi} \qquad (\varepsilon_k \le \varepsilon_0).$$

For the finitely many k with $\varepsilon_k > \varepsilon_0$, $b_k^\infty(M_1)$ is a finite number by Lemma 12.11. Hence $\sup_k |b_k^\infty(M_1)| < \infty$, and $\beta_1(M_1)$ is finite. □

(CT_∞) enters Theorem 28.2 in two ways. The first is through the exceptional set E_W of the rotation part (Proposition 26.8 and Definition 27.2), defined in (81) below, off which the rotation theorem is proved first and whose points are reached by right-continuity in the scalar mass. The second is through the description of the failure set of (T3). By (CT_∞) put

$$s_\infty(M_1) := \tfrac{1}{2}\big(\alpha_\infty(m) - \alpha_\infty(M_1)\big), \qquad s_* := \sup_{\mathsf{M}>0} s_\infty(\mathsf{M}) \in [0,\infty],$$

$$\begin{aligned} E_W &:= \{s_* = 0\} = \{m \text{ is a global minimiser of } \mathsf{M} \mapsto \alpha_\infty(\mathsf{M}) \text{ on } (0,\infty)\}, \\ E_W^< &:= \{m \text{ minimises } \alpha_\infty \text{ on } (0,m]\} \supseteq E_W, \end{aligned} \tag{81}$$

subsets of the coupling space ($s_* \ge 0$ because $\mathsf{M} = m$ gives 0; the value $+\infty$ is harmless below). The two sign clauses are

(NE) m is not a global minimiser of α_∞ on $(0,\infty)$, i.e. $s_* > 0$, i.e. $(e,\lambda,m,\mu_0) \notin E_W$;

($\mathrm{NE}^<$) some $M_1 \in (0,m)$ has $\alpha_\infty(M_1) < \alpha_\infty(m)$, i.e. $(e,\lambda,m,\mu_0) \notin E_W^<$ (the one-sided form; it implies (NE)).

Throughout, $'$ denotes d/dM^2, N is the number of real components of the scalar field, and "one-loop order" means the orders e^2 and λ.

Lemma 29.3 traces every condition on the couplings e, λ, m, μ_0 that the results behind Theorem 28.2 impose. For this one has to know which results those are: for the curvature sector, the

existence theorem (Theorem A), which holds throughout King's range with no further condition, and the correlation inequalities Lemma 2.10, whose standing hypotheses (a finite single-site measure, positive hopping coefficients, a positive photon covariance) reduce to $\lambda > 0$ and $\mu_0 > 0$; for the Wick sector in Form (B), the torus theorems, the sign clause (T3) and the tube statements, whose conditions concern the Wick amplitudes of the sources and not the couplings. The conclusion is that the only condition on the couplings beyond their positivity, anywhere in the paper, is the sign clause (T3) of the Wick sector, with the explicit regions of Proposition 29.29; the existence and rotation theorems carry none, and no relation between the photon mass and the scalar mass is needed.

Throughout Lemma 29.3 and its proof, Theorem A is the existence clause of [26, Thm. 2.1] at zero scalar source, Lemma 2.10 is Theorem 6.2(1),(2) of [9], and *the results used in Form (B)* are, beyond Theorem B, Proposition 12.6, and Lemma 12.7 and (T3), the statements Proposition 12.2 and Lemmas 2.10, 12.4, 12.11 and 12.12 and the tube statements Theorems 10.6 and 13.10 and Lemma 13.11, whose conditions on the Wick amplitudes are not conditions on the couplings.

Lemma 29.3 (the coupling range)**.** *Let $e > 0$ and $\lambda, m, \mu_0 > 0$.*

(a) The curvature sector. *The existence theorem of the curvature sector, Theorem A, carries no condition on e, λ, m, μ_0. The correlation inequalities it uses, Lemma 2.10, have the standing hypotheses of a finite single-site measure, positive hopping coefficients and a positive photon covariance, which for King's action are $\lambda > 0$ and $\mu_0 > 0$; the margin $R(\mu_0) = \pi\sqrt{3}/(2\mu_0)$ of Lemma 9.10 needs $\mu_0 > 0$ only; and no lemma proved in this paper for the curvature sector carries a condition on the couplings beyond those of the results it uses. Hence the curvature sector of Theorem 28.2 holds on all of King's range, with no exceptional set. The one proof-internal proviso in the Bałaban antecedents of Theorem A, the mass clause of* [2, Prop. 3.1]*, is a cap on the dimensionless scalar mass at the stopping scales and no condition on the bare couplings (Remark 29.5). Of the bounds of the paper only the absolute bounds of Theorem 7.6 are uniform in μ_0.*

(b) The Wick sector, Form (B). *The results used in Form (B) state no condition on the bare couplings beyond positivity ($\lambda > 0$, $\mu_0 > 0$); Proposition 12.6 is used at the two masses m and M_1. The only coupling condition of Form (B) is therefore membership of the coupling region of Definition 2.1, the sign clause* (T3)*, which Proposition 29.29 proves on explicit regions. Its failure set is described in Lemma 29.4.*

(c) The existence and rotation parts. *The existence theorems (Theorems A and B) and the rotation theorem (Theorem C) are proved for $e > 0$, $\lambda, m, \mu_0 > 0$ in King's range, and none carries a condition on the bare couplings beyond positivity: every smallness condition of* [1, 2, 3, 4] *that they use (of the three kinds classified in the proof of (b)) is a one-sided cap on a stopping scale $\beta \le 1$, imposed last in the schedule of constants, and an admissible β exists at every positive point.*

(d) The photon mass. *No relation $\mu_0 \le m$ is needed: Proposition 12.2 holds for every $\mu_0 > 0$. The explicit regions of* (T3) *at $\mu_0 = m$ and at every $\mu_0 \in (0, m]$ are those of Proposition 29.29(iii),(iv).*

(e) The quantifier. *For all $e > 0$, $\lambda, m, \mu_0 > 0$ and each datum there are an integer J_{K} (*[25, Thm. 3.1]*) and a stopping scale $\beta \le 1$ such that the running couplings $(e\beta^{1/2}, \lambda\beta, m\beta, \mu_0\beta)$ satisfy finitely many one-sided caps on β and on three of the four of them ($\mu_0\beta$ is uncapped in the sources); no condition is placed on the bare couplings, and the theorems used from* [25, 26] *carry no coupling hypothesis. A sentence "e, λ sufficiently small, $m, \mu_0 > 0$ fixed", read as a hypothesis on the bare couplings, is strictly stronger and is stated in none of the sources (Remark 29.31).*

Proof. We prove the five clauses in turn; the classification of the smallness conditions into three kinds, made in (b), is used again in (c).

(a). Theorem 2.1 of [26] states, for g, h supported in C_J and all $z, w \in \mathbb{C}$, the existence of $\lim_K \langle \Theta(zg, wh) \rangle^{\varepsilon_K}_{N,D,C_J}$; no hypothesis on the couplings precedes it. Its proof rests on [25, Thm. 3.1] (the convergence of the effective action, referred there to [1, 2, 3, 4]), stated "for some fixed integer J", our J_{K}; neither it nor [25, Thms. 2.1, 3.4] carries a coupling hypothesis, the smallness in [25] being of the fields and of the stopping spacing. So the one proviso of [2, Prop. 3.1] is upstream of Theorem A. For King's action the single-site measure $e^{-\varepsilon^3[\frac{1}{2}\alpha r^2 + \lambda r^4]} r\, dr$ is finite iff $\lambda > 0$ (or $\lambda = 0$, $\alpha > 0$), the hopping weights are positive, and the Feynman-form photon covariance is positive iff $\mu_0 > 0$; the "$g > 0$" of Theorem 6.2 of [9] is the sign of the scalar source, and Lemma 2.10 covers $g = 0$. The margin of Lemma 9.10 concerns the photon Gaussians of the two boxes alone, so μ_0 is its only parameter (no fixed margin works at $\mu_0 = 0$). The lemmas of this paper see only the form of the action.

(b): the torus theorems. Proposition 12.6 is Bałaban's stability theorem [1, (1.14)], [2, (1.3)], at fixed scalar mass, read at m and at M_1 with block factor 5 and King's normalisation, with constants independent of ε and of the torus, under $\lambda > 0$, $\mu_0^2 > 0$, $m^2 > 0$. Every smallness condition stated in [1, 2, 3, 4] is of one of three kinds. *Running charge:* the bounds hold "for $e(L^k\varepsilon)$ sufficiently small", $e(L^k\varepsilon) = e\,(L^k\varepsilon)^{(4-d)/2}$ [1, (2.23)]; [4] proves them for a unit-lattice model whose charge is this running charge (its theorem is stated there as Proposition 2.1 of [1]), so "e sufficiently small" there is a cap on $e\beta^{1/2}$ at the stopping scale β. *Scale:* the stopping scale ε_0 with $L^K\varepsilon \le \varepsilon_0 < L^{K+1}\varepsilon$ is "some fixed sufficiently small number" [3], and "$L^{k+1}\varepsilon \le 1$", "$L^k\varepsilon$ sufficiently small" [1, 2] are caps on it; Bałaban's units have $\lambda = 1$ (in $d = 3$: spacing $\lambda\varepsilon$, mass squared m^2/λ^2, charge $e\lambda^{-1/2}$, the running charge invariant), so his "ε small" is "$\lambda\varepsilon$ small" in ours, and his constants may depend on $e\lambda^{-1/2}$, m/λ, μ_0/λ: a dependence, not a condition. *Mass:* Proposition 3.1 of [2] carries the sentence "we assume $m^2 \le O(1)$ also"; this is the mass cap of that proposition (Remark 29.5). Lemma 12.7 carries no coupling condition: the bound on the relative bare coefficient holds for every mass in a compact interval containing m, on either side of m, with a constant depending on the couplings (Lemma 12.7; the power-counting theorem of [34] that enters it has none). Lemma 12.12 needs a finite single-site integral ($\lambda > 0$) and a finite photon Gaussian on the torus ($\mu_0 > 0$; at $\mu_0 = 0$ the constant mode lies in the kernel); Lemma 12.11 needs massive propagators; Proposition 12.2 needs finite single-site moments and a positive Proca covariance and does not see the counterterm. The tube statements restrict Wick amplitudes only. So the only coupling condition of Form (B) is the sign clause of Proposition 12.8, i.e. (T3).

(c). The smallness conditions are of the three kinds listed in the proof of (b) (running charge, scale, mass); the order in which the constants of the existence and rotation parts are fixed, and the proof that an admissible stopping scale exists at every positive point, are as follows: the running couplings $(e\beta^{1/2}, \lambda\beta, m\beta, \mu\beta)$ vanish as $\beta \to 0$, so every smallness condition of the existence and rotation parts holds below a stopping scale $\beta_0(e, \lambda, m, \mu) > 0$, the constants being fixed first; every smallness condition used is a one-sided cap, monotone in the stopping scale β, on β or on the running couplings at β, so with the other constants fixed first a $\beta = 5^{-J}$ below the finitely many positive caps exists at every positive point; and the constants are fixed in an acyclic order ($L = 5$; the decay and coercivity constants before M; $M \in 50\mathbb{N}$; the operator and class constants; the localisation radius R; b_0; the stopping scale β last; $J \in 5^{k_*}\mathbb{N}$, $\eta_0 \in 5^{-\mathbb{N}}$), each step imposing finitely many one-sided or divisibility conditions on one entry, with right sides depending on the couplings and on the earlier entries only, so an admissible choice exists at every positive point.

(d). The proof of Proposition 12.2 uses site-independent coefficients, $\lambda > 0$, $\mu_0 > 0$ and even sides, and no bound on μ_0. The statements at $\mu_0 = m$ and for $\mu_0 \in (0, m]$ are Proposition 29.29(iii),(iv), whose proof uses Lemma 29.4(i) and neither Lemma 29.4(iii) nor (d).

(e). The quantifier is read off [25, Thm. 3.1] ("for some fixed integer J, for all $K \geq J$, and all $k \leq K - J$") and the stopping-scale caps listed above; the last sentence is Remark 29.31. □

The next lemma describes where the sign clause fails: exactly on $E_W^<$, and in closed form to one-loop order.

Lemma 29.4 (the failure set of the sign clause)**.** *Let $e > 0$ and $\lambda, m, \mu_0 > 0$. By* (CT_∞) *(Lemma 29.1):*

(i) $\sigma_1(M_1) = \lim_k s_k^\infty(M_1) = s_\infty(M_1)$ *for every $M_1 > 0$. Hence* (T3) *holds if and only if* ($\mathrm{NE}^<$)*: it fails exactly on $E_W^<$. Since Proposition 12.6 and Lemma 12.7 hold also at auxiliary masses $M_1 > m$ (Bałaban's theorem at a fixed mass, and the c-number bound on a compact mass interval on either side of m; see the proof of Lemma 29.3), by Proposition 12.13 the two-sided clause* (NE) *replaces* (T3)*, with failure set exactly E_W; the branch $M_1 > m$ is needed exactly on $E_W^< \setminus E_W$, for instance when $\alpha'_\infty(m) < 0$ and m minimises α_∞ on $(0, m]$.*

(ii) *To one-loop order, for every $M_1 > 0$,*

$$s_\infty(M_1) = \tfrac{1}{2}(m^2 - M_1^2) + \frac{\lambda(N+2)(m - M_1)}{2\pi} - e^2\,\frac{(m - M_1)\,(m\mu_0 + mM_1 + M_1\mu_0)}{8\pi(\mu_0 + M_1)(m + \mu_0)},$$

so that

$$\alpha'_\infty(m) = 1 + \frac{\lambda(N+2)}{2\pi m} - e^2\,\frac{m + 2\mu_0}{8\pi(m+\mu_0)^2}, \qquad \frac{m+2\mu_0}{8\pi(m+\mu_0)^2} \leq \frac{1}{4\pi(m+\mu_0)}, \tag{82}$$

and to one-loop order (T3) *holds if and only if*

$$e^2 < e_c^2(\lambda, m, \mu_0) := \max\Big\{\frac{8\pi(m+\mu_0)^2}{m+2\mu_0}\Big(1 + \frac{\lambda(N+2)}{2\pi m}\Big),\ 4\pi(m+\mu_0)\Big(1 + \frac{\lambda(N+2)}{\pi m}\Big)\Big\}, \tag{83}$$

the first entry being the larger if and only if $\lambda(N+2)\mu_0 \leq \pi m^2$. So the one-loop failure set of (T3) *is the half-line $e^2 \geq e_c^2$. The first entry is the zero $e_*^2 \geq 4\pi(m+\mu_0)$ of (82); at $m = \mu_0 = 1$, $\lambda = 0$ it equals $32\pi/3 \approx 33.5$ (for orientation only). Nothing finer than one-loop order is claimed about $E_W^<$.*

(iii) *For the two-sided set, $E_W \subset \{\alpha'_\infty(m) = 0\}$, and for each (λ, m, μ_0) the polynomial $e^2 \mapsto \alpha'_\infty(m)$, of degree ≤ 2, is not identically zero (Lemma 29.1), so E_W contains at most two values of e^2. At $\mu_0 = m$ the box $0 < \lambda, e^2 \leq 0.239\,m$ lies outside $E_W^<$, by (i) and Proposition 29.29(iii).*

Proof. We first identify $\sigma_1(M_1)$ with the limit of the relative bare coefficient (i), then evaluate that limit in closed form to one-loop order and optimise it over M_1 (ii), and finally locate the exceptional set (iii).

(i). By Lemma 12.11, $s_k^\infty(M_1)$ is the fixed-k volume limit of the torus coefficient, the relative bare coefficient on Λ_k in the free convention; by (CT_∞) (Lemma 29.1) its limit $s_\infty(M_1)$ exists, hence $\sigma_1(M_1) = \liminf_k = \lim_k = s_\infty(M_1)$. Thus some $M_1 \in (0, m)$ has $\sigma_1(M_1) > 0$ iff ($\mathrm{NE}^<$), and some $M_1 > 0$ has $\sigma_1(M_1) > 0$ iff (NE). Proposition 12.13 gives (I2) from Proposition 12.6 and Lemma 12.7 at an auxiliary mass on either side of m, so with those statements (NE) suffices; on $E_W^< \setminus E_W$ every $M_1 < m$ fails and some $M_1 > m$ works.

(ii). The closed form is the one-loop evaluation of King's counterterm [25, (3.32)–(3.33)] (tadpole and zero-momentum photon-exchange graphs; the $1/\varepsilon$ divergence cancels in the difference). It equals $\frac{1}{2}\int_{M_1^2}^{m^2} \alpha'_\infty(\mu)\,d\mu^2$ with $\alpha'_\infty(\mu) = 1 + \frac{\lambda(N+2)}{2\pi\mu} - e^2\frac{\mu+2\mu_0}{8\pi(\mu+\mu_0)^2}$, an orientation-free definite

integral, valid on either side of m; at $\mu = m$ this is (82), whose inequality is $m \geq 0$. For $M_1 \in (0, m)$ write $s_\infty(M_1) = \frac{m-M_1}{2}\beta(M_1)$ with

$$\beta(M_1) = (m + M_1) + c - e^2\,\frac{m\mu_0 + (m+\mu_0)M_1}{4\pi(m+\mu_0)(\mu_0+M_1)}, \qquad c := \frac{\lambda(N+2)}{\pi};$$

$s_\infty(M_1) > 0$ iff $\beta(M_1) > 0$ iff $e^2 < e_c^2(M_1) := \big[(m+M_1)+c\big]\frac{4\pi(m+\mu_0)(\mu_0+M_1)}{m\mu_0+(m+\mu_0)M_1}$. With $\mathcal{N}(M_1) := (m+M_1+c)(\mu_0+M_1)$ and $\mathcal{D}(M_1) := m\mu_0 + (m+\mu_0)M_1$, the derivative of $e_c^2(M_1)$ has the sign of $\mathcal{N}'\mathcal{D} - \mathcal{N}\mathcal{D}' = 2m\mu_0 M_1 + (m+\mu_0)M_1^2 - c\mu_0^2$, positive on $(0,m)$ when $c = 0$ and, when $c > 0$, negative then positive with one zero. So $e_c^2(\cdot)$ is increasing, or decreasing then increasing, on $(0,m)$, and its supremum over the open interval is the larger of the two end values, not attained: $M_1 \to m^-$ gives $\frac{8\pi(m+\mu_0)^2}{m+2\mu_0}(1+\frac{c}{2m})$, the zero e_*^2 of (82), and $M_1 \to 0^+$ gives $4\pi(m+\mu_0)(1+\frac{c}{m})$. Hence to one-loop order (T3) holds iff $e^2 < e_c^2(\lambda, m, \mu_0)$ as in (83), and fails for every $e^2 \geq e_c^2$ (then $\beta \leq 0$ for every M_1); the first entry is the larger iff $(m+\mu_0)(2m+c) \geq (m+c)(m+2\mu_0)$, i.e. $c\mu_0 \leq m^2$.

(iii). By Lemma 29.1, $\mathsf{M}^2 \mapsto \alpha_\infty(\mathsf{M})$ is differentiable at m, a global minimum at the interior point m is a critical point, so $E_W \subset \{\alpha_\infty'(m) = 0\}$, and the polynomial $e^2 \mapsto \alpha_\infty'(m)$ is never identically zero, so it has at most two zeros. For the clause at $\mu_0 = m$: on that box (T3) holds by Proposition 29.29(iii), and under (CT$_\infty$) (T3) is (NE$^<$) by (i), so the box lies outside $E_W^<$. □

The next remark returns to the mass clause of Bałaban's upper bound, the third of the three kinds of smallness condition classified in the proof of Lemma 29.3(b), and records the reading under which this paper cites it.

Remark 29.5 (the mass clause of Bałaban's upper bound). Proposition 3.1 of [2] states an inequality (3.26), summed over the scales $k = 0, \dots, K$, with a constant $\gamma_0 = \gamma_0(d,a)$ and mass terms $(L^k\varepsilon)^d m^2|\phi_k|^2$; rescaled to the unit lattice, its k-th term is the per-scale inequality (3.29) with the mass term $(L^k\varepsilon)^2 m^2|\phi_k'|^2$, so $(L^k\varepsilon)^2 m^2$ is the dimensionless mass at scale k. The proposition carries the sentence "we assume $m^2 \leq O(1)$ also", m^2 being the bare scalar mass squared in Bałaban's units ($\lambda = 1$), i.e. m^2/λ^2 in ours. The clause under which this paper cites the proposition concerns the scales at which it is applied:

> *Inequality (3.29) of* [2] *holds, with $\gamma_0 = \gamma_0(d,a)$, at every scale $k \leq K$ at which $(L^k\varepsilon)^2 m^2 \leq c_m(d,a)$; hence (3.26) holds whenever this is so for every $k \leq K$.*

Here $c_m(d,a)$ is a constant depending on d and a only. The clause belongs to the statement of the proposition: the mass that enters its proof as given there, referred to [4, Sect. 4], is the unit-lattice mass at scale k, the text of [2] makes no further use of the sentence, and the proposition is cited here under the clause. It is then one more one-sided cap on the stopping scale $\beta = \varepsilon_0$ ($(L^k\varepsilon)^2 m^2 \leq \beta^2 m^2$ at every scale used), of the kind every other smallness condition of [1, 2, 3, 4] has; a further monotone cap on β leaves the existence of an admissible stopping scale unchanged, and nothing is imposed on the bare mass.

Read instead as an absolute bound on m^2/λ^2, the sentence would exclude small λ at fixed m, including every $\lambda < 0.1m$ ($m^2/\lambda^2 > 100$), which meets the explicit regions of Proposition 29.29, and would be a hypothesis of the theorem (1.3) of [2] that is not stated there, the only stated condition on m being $m^2 > 0$; the proposition is not proved under that interpretation, and it is used nowhere. The clause sits upstream of Proposition 12.6 at both masses, of Theorems A and B (through [25, Thm. 3.1]) and of the steps of the rotation part that rest on [1, 2], as every cap on the stopping scale does; it is not a condition of this paper.

The explicit coupling region. The sign clause (T3) is proved below on an explicit region of the (λ, e^2)-plane at every photon mass $\mu_0 \in (0, m]$, with $N = 2$ and the auxiliary mass $M_1 = 0.7m$.

The argument has three parts. The first is the three low orders of King's counterterm, whose contributions to $s_k^\infty(M_1)$ are lattice integrals with exact limits (Lemmas 29.6, 29.7 and 29.9 to 29.11). The second is the two orders $e^2\lambda$ and e^4, whose contributions are bounded from below, uniformly in k, through the two-loop majorants of King's counterterm on periodic tori (the facts (G1)–(G5) stated in the subsection on the two-loop orders, before Lemma 29.25, and proved there); these majorants are read at the photon mass μ_0, with the torus floor removed (Lemmas 29.25 and 29.26), made uniform on intervals of photon masses (Lemma 29.27) and, in a weaker closed form, uniform in the photon mass (Lemma 29.28). The resulting lower bound σ_{low} on $\sigma_1(0.7m)$, (89), is explicit through Table 2. The third part, Proposition 29.30, replaces the majorants of the two-loop orders by the exact limit coefficients of Lemma 29.1 and obtains a region uniform in the photon mass.

King's counterterm order by order. Throughout this subsection and the next, $\varepsilon = \varepsilon_k$, $\Lambda = \Lambda_k$, $\mathsf{B} = \mathsf{B}_\varepsilon := [-\pi/\varepsilon, \pi/\varepsilon)^3$ is the Brillouin zone with $\int_{\mathsf{B}} := \int_{\mathsf{B}} \frac{d^3p}{(2\pi)^3}$, $\hat p_\mu := \frac{2}{\varepsilon}\sin\frac{\varepsilon p_\mu}{2}$ and $\hat p^2 := \sum_\mu \hat p_\mu^2$ (periodic in p), and for a mass $M > 0$

$$G_M(p) := \frac{1}{\hat p^2 + M^2}, \qquad \hat D(p) := \frac{1}{\hat p^2 + \mu_0^2},$$
$$C_M(x) := \int_{\mathsf{B}} e^{\mathrm{i}p\cdot x} G_M(p), \qquad D(x) := \int_{\mathsf{B}} e^{\mathrm{i}p\cdot x}\hat D(p) \quad (x \in \Lambda)$$

are the free scalar and photon propagators of King's action on Λ (per real scalar component and per photon component; $C_M(0)$ is the free Wick constant at mass M, $C_m(0) = \mathsf{W}_\varepsilon$). King's free-boundary counterterm [25, (3.32)–(3.33)], [26, p. 330] at scalar mass M is the polynomial

$$\delta m_\varepsilon^2(M) = \sum_{(\alpha,\beta)} e^\alpha \lambda^\beta\, \delta m^2_{(\alpha\beta)}(\varepsilon; M), \qquad (\alpha,\beta) \in \{(0,1),(2,0),(0,2),(2,1),(4,0)\}, \tag{84}$$

over the superficially divergent orders $2 \le \alpha + 2\beta \le 4$ with the odd powers of e dropped (they vanish: an odd power of e comes with an odd number of photon fields, whose centred Gaussian expectation is zero); the coefficient $\delta m^2_{(\alpha\beta)}(\varepsilon; M)$ is fixed recursively by the condition that the order-$e^\alpha\lambda^\beta$ part of the amputated one-particle-irreducible two-point function of ϕ at zero external momentum, computed on Λ with the propagators above and with all counterterm vertices of lower order inserted, be cancelled by the counterterm vertex of that order. In the sign convention in which a term of the expansion of e^{-V} that reduces to $+\frac12\sum_{x,y}\varepsilon^6\phi_a(x)\mathsf{K}(x-y)\phi_a(y)$ contributes $+\hat{\mathsf{K}}(p)$ to the self-energy $\Sigma(p)$, so that the inverse propagator per component is $\hat p^2 + M^2 - \Sigma(p)$ and the counterterm vertex $-\frac12\delta m^2\sum\varepsilon^3|\phi|^2$ contributes $-\delta m^2$, the condition reads $\delta m^2_{(\alpha\beta)}(\varepsilon; M) = \Sigma^{(\alpha\beta)}(0; M)$, $\Sigma^{(\alpha\beta)}$ the order-$e^\alpha\lambda^\beta$ part of the sum of the one-particle-irreducible two-point graphs built from the vertices of King's action and from the counterterm vertices of lower order. Every such graph is a finite sum of absolutely convergent integrals over B^L, $L \le 2$ (all masses are $\ge \min(M,\mu_0) > 0$ and B is compact), so each coefficient is a finite real number, smooth in M (a rational function of M^2 without poles on $(0,\infty)$ under each integral). With $\alpha_k^\infty(M) = M^2 + \delta m^2_{\varepsilon_k}(M)$ as in Definition 2.1 and $M_1 \in (0, m)$:

Lemma 29.6 (structure)**.** *Put* $S_{\alpha\beta,k} := \frac12\big[\delta m^2_{(\alpha\beta)}(\varepsilon_k; m) - \delta m^2_{(\alpha\beta)}(\varepsilon_k; M_1)\big]$*, numbers free of the couplings. Then*

$$s_k^\infty(M_1) = \tfrac12(m^2 - M_1^2) + \lambda S_{01,k} + e^2 S_{20,k} + \lambda^2 S_{02,k} + e^2\lambda S_{21,k} + e^4 S_{40,k}. \tag{85}$$

In $d = 3$, $S_{01,k}$ and $S_{20,k}$ have the dimension of a mass and $S_{02,k}$, $S_{21,k}$, $S_{40,k}$ are dimensionless.

Proof. $s_k^\infty(M_1) = \frac12(\alpha_k^\infty(m) - \alpha_k^\infty(M_1))$ and (84). □

Lemma 29.7 (the three low orders, exact)**.** *For every ε and $M>0$:*
(i) $\delta m^2_{(01)}(\varepsilon;M) = -4(N+2)\,C_M(0)$*;*
(ii) $\delta m^2_{(20)}(\varepsilon;M) = \int_{\mathsf{B}} \hat{k}^2 \hat{D}(k) G_M(k) - 3D(0) = -2D(0) - M^2 \int_{\mathsf{B}} \hat{D} G_M$*;*
(iii) $\delta m^2_{(02)}(\varepsilon;M) = 32(N+2)\sum_{x\in\Lambda} \varepsilon^3 C_M(x)^3$.
Consequently, with

$$J_\varepsilon := \int_{\mathsf{B}} G_m G_{M_1}, \qquad I_\varepsilon := \int_{\mathsf{B}} \hat{k}^2 \hat{D}(k) G_m(k) G_{M_1}(k)$$
$$\text{and} \qquad F_\varepsilon := \sum_{x\in\Lambda} \varepsilon^3 [C_{M_1}(x)^3 - C_m(x)^3],$$

we have

$$\begin{aligned} S_{01,k} &= 2(N+2)\big(C_{M_1}(0) - C_m(0)\big) = 2(N+2)(m^2 - M_1^2) J_{\varepsilon_k}, \\ S_{20,k} &= -\tfrac{1}{2}(m^2 - M_1^2) I_{\varepsilon_k}, \qquad S_{02,k} = -16(N+2) F_{\varepsilon_k}. \end{aligned} \tag{86}$$

Proof. Write $\phi = (\phi_1,\dots,\phi_N)$ with $\phi\cdot\psi = \sum_a \phi_a\psi_a$ and, for $N=2$, $\phi\wedge\psi = \phi_1\psi_2 - \phi_2\psi_1 = J_{ab}\phi_a\psi_b$ with $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $J^2 = -1$. With $y = x + \varepsilon e_\mu$ and $\theta = e\varepsilon A_\mu(x)$ the hopping term of King's action is $\frac{1}{2}\varepsilon|e^{\mathrm{i}\theta}\phi(y) - \phi(x)|^2 = \frac{1}{2}\varepsilon(|\phi(x)|^2 + |\phi(y)|^2) - \varepsilon\cos\theta\,\phi(x)\cdot\phi(y) + \varepsilon\sin\theta\,\phi(x)\wedge\phi(y)$; expanding $\cos\theta$ and $\sin\theta$ gives the free action $S_0 = \frac{1}{2}\sum_x \varepsilon^3[\sum_\mu |\nabla_\mu\phi|^2 + M^2|\phi|^2] + \frac{1}{2}\sum_x \varepsilon^3[\sum_{\mu,\nu}(\nabla_\nu A_\mu)^2 + \mu_0^2|A|^2]$, $\nabla_\mu f(x) = (f(x+\varepsilon e_\mu) - f(x))/\varepsilon$, whose covariances are $\langle\phi_a(x)\phi_b(y)\rangle_0 = \delta_{ab} C_M(x-y)$ and $\langle A_\mu(x) A_\nu(y)\rangle_0 = \delta_{\mu\nu} D(x-y)$, and the vertices

$$V_1 = e\sum_{x,\mu} \varepsilon^2 A_\mu(x)\,\phi(x)\wedge\phi(y), \qquad V_2 = \tfrac{e^2}{2}\sum_{x,\mu} \varepsilon^3 A_\mu(x)^2\,\phi(x)\cdot\phi(y),$$
$$V_\lambda = \lambda\sum_x \varepsilon^3(\phi\cdot\phi)^2, \qquad V_{\mathrm{ct}} = \tfrac{1}{2}\delta m^2 \sum_x \varepsilon^3 \phi\cdot\phi,$$

together with vertices of order e^3, e^4 and higher, which do not enter the three orders below. The graphs are those of $-V + \frac{1}{2}V^2 - \cdots$ with the normalisation fixed above; by $O(N)$ symmetry they are proportional to δ_{ab}.

(i) At order λ the only one-particle-irreducible two-point graph is the tadpole of $-V_\lambda$. Contracting one pair of fields in $(\phi\cdot\phi)^2 = \phi_a\phi_a\phi_b\phi_b$ gives $2(N+2)C_M(0)\,\phi\cdot\phi$. So $-V_\lambda$ contributes $-2(N+2)\lambda C_M(0)\sum_x \varepsilon^3\phi\cdot\phi = \frac{1}{2}\sum \phi_a \mathsf{K}\phi_a$ with $\hat{\mathsf{K}} = -4(N+2)\lambda C_M(0)$, and $\delta m^2_{(01)} = \Sigma^{(01)}(0)/\lambda = -4(N+2)C_M(0)$.

(ii) At order e^2 there are two graphs. *Seagull:* contracting $A_\mu(x)^2$ to $D(0)$ in $-V_2$ gives $-\frac{e^2}{2}D(0)\sum_{x,\mu}\varepsilon^3\phi(x)\cdot\phi(x+\varepsilon e_\mu)$, i.e. $\Sigma_{\mathrm{seag}}(p) = -e^2 D(0)\sum_\mu \cos(\varepsilon p_\mu)$ after symmetrisation, and $\Sigma_{\mathrm{seag}}(0) = -3e^2 D(0)$. *Exchange:* in momentum space the vertex V_1 carries the symmetrised factor $W^\mu_{ij}(p,p') = \mathrm{i}e\,J_{ij}\,e^{-\mathrm{i}\varepsilon k_\mu/2}\frac{1}{\varepsilon}\sin\frac{\varepsilon(p'-p)_\mu}{2}$ at momenta $k+p+p'=0$; in $\frac{1}{2}V_1^2$ contract the photon pair and one scalar pair, so that with external pair $\hat\phi_i(p)\hat\phi_l(-p)$ the internal line carries $p+k$ and the two vertex factors multiply to $e^2\delta_{il}\frac{1}{\varepsilon^2}\sin^2\frac{\varepsilon(2p+k)_\mu}{2}$ ($J^2=-1$; the phases cancel). Hence $\Sigma_{\mathrm{exch}}(p) = e^2\int_{\mathsf{B}}\hat{D}(k)G_M(p+k)\sum_\mu\big(\frac{2}{\varepsilon}\sin\frac{\varepsilon(2p+k)_\mu}{2}\big)^2$ and $\Sigma_{\mathrm{exch}}(0) = e^2\int_{\mathsf{B}}\hat{D}(k)\hat{k}^2 G_M(k)$. So $\delta m^2_{(20)} = \int_{\mathsf{B}}\hat{k}^2\hat{D}G_M - 3D(0)$; the second form follows from $\hat{k}^2\hat{D}G_M = \hat{D}(1 - M^2 G_M)$ and $\int_{\mathsf{B}}\hat{D} = D(0)$.

(iii) At order λ^2 (and $e=0$) the one-particle-irreducible two-point graphs with two quartic vertices u,v are the sunset (external legs at u and at v, three lines u–v) and the tadpole with

a tadpole inserted on its loop line (external legs at u, two lines u–v, one loop at v); the configurations with a single bridge are not one-particle irreducible and the rest are disconnected. The insertion graph is cancelled exactly by the graph with the counterterm vertex $-V_{\mathrm{ct}}^{(01)}$ on the loop line, because the order-λ self-energy with its counterterm, $\Sigma^{(01)}(q) - \lambda\delta m^2_{(01)}$, vanishes for every q (the tadpole does not depend on the momentum). For the sunset, the contractions of $\frac{1}{2}V_\lambda^2 = \frac{1}{2}\lambda^2 \sum_{u,v} \varepsilon^6 (\phi\cdot\phi)_u^2 (\phi\cdot\phi)_v^2$ with one external field at u and one at v carry the factor $16\cdot 2(N+2)\delta_{ab}$; so $\frac{1}{2}V_\lambda^2$ contributes $\frac{1}{2}\sum_{u,v}\varepsilon^6 \phi_a(u)\mathsf{K}(u-v)\phi_a(v)$ with $\mathsf{K}(x) = 32(N+2)\lambda^2 C_M(x)^3$, and $\delta m^2_{(02)} = \hat{\mathsf{K}}(0)/\lambda^2 = 32(N+2)\sum_x \varepsilon^3 C_M(x)^3$.

The formulas (86) follow from (i)–(iii) and the resolvent identity

$$G_{M_1} - G_m = (m^2 - M_1^2) G_{M_1} G_m;$$

the M-independent tadpole $3D(0)$ and the term $-2D(0)$ cancel in $S_{20,k}$. □

At $N = 2$ these coefficients are the terms $-16\lambda C_T(M)$, $-2e^2 D_T - e^2 J_T(M)$ and $128\lambda^2 S_T(M)$ of the exact two-loop form of the torus coefficient, namely (G1) of the list before Lemma 29.25 and Lemma 29.14, read on Λ in place of the torus; the orders $e^2\lambda$ and e^4 are the terms $16e^2\lambda F_T(M)$ and $e^4 Q_T(M)$ there. Every graph of these two orders is an absolutely convergent integral over B^2 at fixed ε (massive propagators, bounded trigonometric vertex factors), so $S_{21,k}$ and $S_{40,k}$ are finite real numbers for every k; their bounds, uniform in k, are the subject of the next subsection.

Lemma 29.8 (four lattice facts)**.** (L1) *For $p \in \mathsf{B}$: $c\,p^2 \le \hat{p}^2 \le p^2$ with $c = 4/\pi^2$. Hence, for a mass $a > 0$, $\bar{G}_a(p) := (p^2+a^2)^{-1} \le G_a(p) \le \bar{G}^c_a(p) := (cp^2+a^2)^{-1}$ on B, and for $a, b > 0$: $\int_{\mathsf{B}} \bar{G}^c_a \bar{G}^c_b \le \int_{\mathbb{R}^3} \bar{G}^c_a \bar{G}^c_b \frac{d^3p}{(2\pi)^3} = \frac{\pi^2}{32(a+b)}$, $\int_{\mathsf{B}} (\bar{G}^c_a)^2 \le \frac{\pi^2}{64a}$.* (L2) *$C_M(x) = \varepsilon^{-3}\int_0^\infty e^{-tM^2}(e^{t\Delta_\varepsilon})_{x,0}\,dt$ with $(e^{t\Delta_\varepsilon})_{x,y} = e^{-6t/\varepsilon^2}\sum_{n\ge 0}\frac{(t/\varepsilon^2)^n}{n!}\#\{n\text{-step nearest-neighbour paths } y \to x\} > 0$. Hence $C_M(x) > 0$ for every $x \in \Lambda$, $M \mapsto C_M(x)$ is strictly decreasing, $\sum_x \varepsilon^3 C_M(x) = M^{-2}$, and the resolvent identity $C_{M_1} - C_m = (m^2 - M_1^2)\, C_{M_1} * C_m$ holds pointwise with a positive right side, $(f*g)(x) := \sum_y \varepsilon^3 f(x-y)g(y)$.* (L3) *With $\int := \int_{\mathbb{R}^3}\frac{d^3p}{(2\pi)^3}$ and $a, b, c > 0$: $\int \frac{1}{(p^2+a^2)(p^2+b^2)} = \frac{1}{4\pi(a+b)}$, $\int \frac{1}{(p^2+a^2)^2} = \frac{1}{8\pi a}$, $\int \frac{1}{(p^2+a^2)(p^2+b^2)(p^2+c^2)} = \frac{1}{4\pi(a+b)(b+c)(c+a)}$.* (L4) *If h_ε on B_ε satisfies $|h_\varepsilon| \le \bar{h} \in L^1(\mathbb{R}^3)$ and $h_\varepsilon(p) \to h(p)$ for every p as $\varepsilon \to 0$, then $\int_{\mathsf{B}_\varepsilon} h_\varepsilon \to \int_{\mathbb{R}^3} h$. As $\varepsilon \to 0$, $\hat{p}^2 \to p^2$ at every fixed p, so $G_M(p) \to \bar{G}_M(p)$ and $\hat{D}(p) \to \bar{G}_{\mu_0}(p)$.*

Proof. (L1) $|\sin x| \le |x|$ and $\sin x \ge 2x/\pi$ on $[0, \pi/2]$, at $x = \varepsilon|p_\mu|/2 \le \pi/2$; the integrals follow from (L3) by the scaling $p \mapsto c^{-1/2}p$, $c^{-3/2} = \pi^3/8$. (L2) $(-\Delta_\varepsilon + M^2)^{-1} = \int_0^\infty e^{-tM^2} e^{t\Delta_\varepsilon} dt$ as matrices on Λ, with kernel $\varepsilon^3 C_M(x-y)$; the path expansion is $e^{t\Delta_\varepsilon} = e^{-6t/\varepsilon^2} e^{(t/\varepsilon^2)\mathcal{A}}$ with $\mathcal{A}$ the adjacency matrix; $\sum_x (e^{t\Delta_\varepsilon})_{x,0} = 1$; the identity is $G_{M_1} - G_m = (m^2 - M_1^2) G_{M_1} G_m$ in Fourier space. (L3) Radially, $\frac{1}{2\pi^2}\int_0^\infty \frac{p^2 dp}{(p^2+a^2)(p^2+b^2)} = \frac{1}{2\pi^2}\cdot\frac{\pi}{2(a+b)}$, the second integral is the limit $b \to a$, and the third follows by partial fractions from the first. (L4) Dominated convergence after extension by 0. □

Lemma 29.9 (order λ)**.** *For every k:* (i) $S_{01,k} > 0$*;* (ii) $J_\varepsilon \le \frac{\pi^2}{32(m+M_1)}$*, hence $S_{01,k} \le \frac{(N+2)\pi^2(m-M_1)}{16}$;* (iii) $J_{\varepsilon_k} \to \frac{1}{4\pi(m+M_1)}$*, hence $S_{01,k} \to \frac{(N+2)(m-M_1)}{2\pi}$ as $k \to \infty$.*

Proof. (i) (86) and $G_m G_{M_1} > 0$. (ii) (L1). (iii) (L4) with $\bar{h} = \bar{G}^c_m \bar{G}^c_{M_1}$, and (L3). □

Lemma 29.10 (order e^2)**.** *For every k:* (i) $S_{20,k} < 0$*;* (ii) $0 < I_\varepsilon \le \bar{I} := \frac{\pi^2}{32(m+M_1)}$*, a bound free of μ_0 and k;* (iii) $I_{\varepsilon_k} \to I_\infty(\mu_0) := \frac{m\mu_0 + mM_1 + M_1\mu_0}{4\pi(m+M_1)(\mu_0+M_1)(m+\mu_0)}$*, hence $S_{20,k} \to -\frac{1}{2}(m^2 - M_1^2) I_\infty(\mu_0) =$*

$-\frac{(m-M_1)(m\mu_0+mM_1+M_1\mu_0)}{8\pi(\mu_0+M_1)(m+\mu_0)}$; (iv) $\mu_0 \mapsto I_\infty(\mu_0)$ *is strictly decreasing on* $(0,\infty)$, *with* $I_\infty(\mu_0) \uparrow I^* := \frac{1}{4\pi(m+M_1)}$ *as* $\mu_0 \downarrow 0$.

Proof. (i) (86). (ii) $\hat{k}^2\hat{D}(k) \in [0,1)$ and (L1). (iii) (L4) with $\bar{h} = \bar{G}^c_m\bar{G}^c_{M_1}$; the limit is $\int \frac{k^2}{(k^2+\mu_0^2)(k^2+m^2)(k^2+M_1^2)} = \frac{1}{4\pi(m+M_1)} - \frac{\mu_0^2}{4\pi(\mu_0+m)(m+M_1)(M_1+\mu_0)}$ by (L3), and $(\mu_0+M_1)(m+\mu_0) - \mu_0^2 = m\mu_0 + mM_1 + M_1\mu_0$. (iv) With $\mathcal{N}(\mu) = (m+M_1)\mu + mM_1$ and $\mathcal{D}(\mu) = (\mu+M_1)(m+\mu)$, $\mathcal{N}'\mathcal{D} - \mathcal{N}\mathcal{D}' = -(m+M_1)\mu^2 - 2mM_1\mu < 0$. □

Lemma 29.11 (order λ^2)**.** *For every* k: (i) $F_\varepsilon > 0$, *so* $S_{02,k} < 0$; (ii) $F_\varepsilon \le 3(m^2-M_1^2)\big(\int_{\mathsf{B}} G^2_{M_1}\big)^2$ *and so* $F_\varepsilon \le \frac{3\pi^4(m^2-M_1^2)}{4096M_1^2}$, *hence* $|S_{02,k}| \le \bar{L} := \frac{3(N+2)\pi^4(m^2-M_1^2)}{256M_1^2}$; (iii) $F_{\varepsilon_k} \to \frac{\ln(m/M_1)}{16\pi^2}$, *hence* $S_{02,k} \to -L(M_1) := -\frac{(N+2)\ln(m/M_1)}{\pi^2}$. *The same holds with any* $M > 0$ *in place of* M_1*:*

$$\lim_k [\delta m^2_{(02)}(\varepsilon_k; m) - \delta m^2_{(02)}(\varepsilon_k; M)] = -\frac{N+2}{\pi^2}\ln\frac{m^2}{M^2}$$

(each $\delta m^2_{(02)}(\varepsilon_k; M)$ *alone diverges like* $\ln(1/\varepsilon_k)$*).*

Proof. (i) By (L2), $C_{M_1}(x) > C_m(x) > 0$ pointwise, and the sum converges absolutely ($C_M \in \ell^1 \cap \ell^\infty$).

(ii) Pointwise $C^3_{M_1} - C^3_m = (C_{M_1} - C_m)(C^2_{M_1} + C_{M_1}C_m + C^2_m) \le 3(C_{M_1} - C_m)C^2_{M_1}$, and by (L2) $C_{M_1} - C_m = (m^2 - M_1^2)C_{M_1} * C_m \le (m^2 - M_1^2)C_{M_1} * C_{M_1}$. For $f, g, h \in \ell^1(\Lambda)$, $\sum_x \varepsilon^3 f(x)g(x)h(x) = \int_{\mathsf{B}}\int_{\mathsf{B}} \hat{f}(p)\hat{g}(q)\hat{h}(-p-q)$ (Fourier inversion and Fubini, $\hat{h}$ periodic); here $\hat{f} = G^2_{M_1}$, $\hat{g} = \hat{h} = G_{M_1}$, all even. By $G(q)G(p+q) \le \frac{1}{2}[G(q)^2 + G(p+q)^2]$ and the measure-preserving shift $q \mapsto p+q$ of B,

$$\int_{\mathsf{B}}\int_{\mathsf{B}} G_{M_1}(p)^2 G_{M_1}(q)G_{M_1}(p+q) \le \Big(\int_{\mathsf{B}} G^2_{M_1}\Big)^2 \le \Big(\frac{\pi^2}{64M_1}\Big)^2$$

by (L1). Multiply by $3(m^2 - M_1^2)$, and by $16(N+2)$ for $\bar{L}$.

(iii) Write $G_1 := G_{M_1}$, $G_2 := G_m$ (at spacing ε) and $\Delta := G_1 - G_2 = (m^2 - M_1^2)G_1G_2 \in [0, (m^2-M_1^2)G_1^2]$, so that, by the formula of (ii), $F_\varepsilon = \int_{\mathsf{B}}\int_{\mathsf{B}}[G_1G_1G_1 - G_2G_2G_2](p,q,p+q) = T_1 + T_2 + T_3$ with $T_1 = \iint \Delta(p)G_1(q)G_1(p+q)$, $T_2 = \iint G_2(p)\Delta(q)G_1(p+q)$, $T_3 = \iint G_2(p)G_2(q)\Delta(p+q)$, all ≥ 0 (telescoping). Consider T_3; the other two are the same argument after the measure-preserving change of variables $(p,q) \mapsto (q, -p-q \bmod \mathsf{B})$ on $\mathsf{B}\times\mathsf{B}$. Split $\mathsf{B}\times\mathsf{B} = N_\varepsilon \cup U_\varepsilon$ with $U_\varepsilon := \{(p,q) : |p_\mu + q_\mu| > \pi/\varepsilon \text{ for some } \mu\}$, the set where $p+q$ is folded back into B. On U_ε, $\max(|p|,|q|) \ge \pi/(2\varepsilon)$, so $\min(G_2(p), G_2(q)) \le \varepsilon^2$ by (L1), and with $\Delta \le (m^2 - M_1^2)G_1G_2$ the contribution of U_ε is at most $2(m^2-M_1^2)\,\varepsilon^2 \int_{\mathsf{B}} G_2 \int_{\mathsf{B}} G_1G_2 = O(\varepsilon)$ by (L1). On N_ε the integrand, extended by 0, converges pointwise by (L4) to $(m^2-M_1^2)\bar{G}_m(p)\bar{G}_m(q)\bar{G}_{M_1}(p+q)\bar{G}_m(p+q)$ and is dominated by $(m^2 - M_1^2)\bar{G}^c_{M_1}(p)\bar{G}^c_{M_1}(q)\bar{G}^c_{M_1}(p+q)^2 \in L^1(\mathbb{R}^6)$ (substitute $r = p+q$ and use $\bar{G}^c(q)\bar{G}^c(r-q) \le \frac{1}{2}[\bar{G}^c(q)^2 + \bar{G}^c(r-q)^2]$). So $T_3 \to T_3^\infty := (m^2 - M_1^2)\iint_{\mathbb{R}^6} \bar{G}_m(p)\bar{G}_m(q)\bar{G}_{M_1}(p+q)\bar{G}_m(p+q)$, and likewise T_1, T_2. *Back to position space:* with the continuum Green's function $\bar{C}_M(x) = \frac{e^{-M|x|}}{4\pi|x|}$ (Fourier transform $\bar{G}_M$) and $\hat{h} := \bar{G}_{M_1}\bar{G}_m \in L^1 \cap L^2$, $h = \bar{C}_{M_1} * \bar{C}_m = (\bar{C}_{M_1} - \bar{C}_m)/(m^2 - M_1^2)$, a bounded continuous function in L^2: for fixed p, $\int \bar{G}_m(q)\hat{h}(p+q)\,dq = \widehat{(\bar{C}_m h)}(p)$ as a Plancherel identity for the L^2 functions $\bar{G}_m$ and $\hat{h}$ ($\bar{C}_m h \in L^1 \cap L^2$, being $O(|x|^{-1})$ near 0 and exponentially small at infinity), and $\int \bar{G}_m(p)\widehat{(\bar{C}_m h)}(p)\,dp = \int d^3x\, \bar{C}_m(x)(\bar{C}_m h)(x)$ by Plancherel again ($\bar{C}_m \in L^2(\mathbb{R}^3)$). Hence $T_3^\infty = \int d^3x\, \bar{C}^2_m(\bar{C}_{M_1} - \bar{C}_m)$, $T_1^\infty = \int \bar{C}^2_{M_1}(\bar{C}_{M_1} - \bar{C}_m)$,

$T_2^\infty = \int \bar C_m \bar C_{M_1}(\bar C_{M_1} - \bar C_m)$, each absolutely convergent (the integrands are $O(|x|^{-2})$ at 0), and

$$\lim_{\varepsilon\to 0} F_\varepsilon = \int_{\mathbb{R}^3} d^3x \left[\bar C_{M_1}(x)^3 - \bar C_m(x)^3\right] = \frac{4\pi}{(4\pi)^3}\int_0^\infty \frac{e^{-3M_1 x} - e^{-3mx}}{x}\,dx = \frac{\ln(m/M_1)}{16\pi^2}$$

by Frullani's integral $\int_0^\infty (e^{-ax} - e^{-bx})\frac{dx}{x} = \ln\frac{b}{a}$. Multiplying by $-16(N+2)$ gives the limit of $S_{02,k}$. For an arbitrary $M > 0$ in place of M_1 the same proof applies (for $M > m$ with the roles of the two masses exchanged); the divergence of a single $\delta m^2_{(02)}(\varepsilon; M)$ is that of $\sum_x \varepsilon^3 C_M(x)^3$, whose terms near $x = 0$ are of size ε^{-3} on $O(1)$ sites, with the logarithm from the region $\varepsilon \ll |x| \ll 1/M$, where $C_M(x) \approx (4\pi|x|)^{-1}$. □

Remark 29.12 (the one-loop form and the $e = 0$ limit). Inserted into (85), the limits of Lemmas 29.9 to 29.11 are the one-loop closed form of Lemma 29.4(ii), and at $e = 0$ the limit $\lim_k s_k^\infty(M_1)$ of Proposition 29.29(vi) is exact; of the three low orders only I_∞ depends on μ_0, and $I_\infty(\mu_0) \le I^*$ for every $\mu_0 > 0$ (Lemma 29.10(iv)).

The two-loop orders at every photon mass. From here to the end of the proof of Proposition 29.29 we use the units $m = 1$ (λ and e^2 in units of m, σ in units of m^2; Remark 29.13), $N = 2$ and $M_1 = 0.7$; "$\varepsilon_k \le 1$" means $\varepsilon_k \le 1/m$. The two-loop coefficients $S_{21,k}$, $S_{40,k}$ are bounded from below through the facts (G1)–(G5) below, proved after the list ((G1) in Lemma 29.14 with Lemma 29.7 and Lemmas 29.21, 29.23; (G2) in Lemma 29.18; (G3) in Lemma 29.24; (G4) and (G5) where they are stated), which are written on periodic tori T of side $\ell_T \ge \ell_0$ with $\ell_T/\varepsilon \in 2\mathbb{N}$, in units $m = 1$, and carry the photon mass μ of the photon propagator $\hat D = (\hat k^2 + \mu^2)^{-1}$ as a parameter; below we write it μ_γ (a parameter of (G1)–(G5) below; it is not the fixed reference photon mass μ_* of the scheme of Definition 27.7) and read it at $\mu_\gamma = \mu_0$. Its objects are the following.

(G1) *The torus coefficient* $a_{K,T}(M)$: King's recursion (84) computed on T with the periodic propagators of T at scalar mass M and photon mass μ_γ. Read at $\mu_\gamma = \mu_0$, this is the bare coefficient $\alpha^{\mathrm{tor}}_{k,T}(M) = M^2 + \delta m^2_{k,T}(M)$ of King's torus convention (Definition 12.1(ii)): both are the same recursion with the same propagators. Its exact two-loop form (Lemma 29.14: the odd orders vanish, and the orders $e^2\lambda$ and e^4 are the two-loop graphs) is, at $N = 2$,

$$a_{K,T}(M) = M^2 - 16\lambda C_T(M) - 2e^2 D_T - e^2 J_T(M) + 128\lambda^2 S_T(M) + 16e^2\lambda F_T(M) + e^4 Q_T(M),$$

with $C_T(M) = \int_T \hat G_M$, $D_T = \int_T \hat D$, $J_T(M) = M^2\int_T \hat D\hat G_M$, $S_T(M) = \sum_{x\in T}\varepsilon^3 G_M(x)^3$, F_T the order-$e^2\lambda$ and Q_T the order-e^4 coefficient, the term $\frac34\varepsilon^2 D_T^2$ of Q_T being independent of M; $M \mapsto a_{K,T}(M)$ is smooth. (The three low orders are Lemma 29.7 on the torus.)

(G2) *The one-loop majorants* (Lemma 29.18): for $\alpha, \beta \ge 0$ with $g := \alpha + \beta > \frac32$, $a, b > 0$, $\nu := \min(a, b)$ and $\ell \in (0, \infty]$,

$$\mathcal{I}_{\alpha,\beta}(a, b; \ell) := c_N C_3 + \frac{3\pi}{16\ell}C_2 + \frac{3}{4\ell^2}C_1 + \frac{a^{-2\alpha}b^{-2\beta}}{\ell^3}, \qquad c_N = \frac{\pi^3}{8}, \tag{87}$$

each $C_d = C_d(\alpha, \beta; a, b)$ the minimum of its admissible closed forms: for C_3 the ν-form $\Gamma(g-\frac32)\nu^{3-2g}/((4\pi)^{3/2}\Gamma(g))$, the Beta form in a, $\frac{1}{4\pi^2}B(c, \alpha-c)a^{2c-2\alpha}$ with $c = \frac{3-2\beta}{2}$ if $\beta < \frac32$, the same with $(a, \alpha) \leftrightarrow (b, \beta)$ if $\alpha < \frac32$, the exact values $\frac{1}{4\pi(a+b)}$ at $(\alpha, \beta) = (1, 1)$ and $\frac{1}{8\pi a(a+b)^2}$ at $(2, 1)$, and the Hölder form $\gamma_3(g)a^{(3-2g)\alpha/g}b^{(3-2g)\beta/g}$, $\gamma_3(g)a^{3-2g}$ the closed form of $\int(u+a^2)^{-g}$; for C_2 the ν-form $\nu^{2-2g}/(g-1)$, the Beta form $a^{2(1-g)}B(1-\beta, g-1)$ if $0 < \beta < 1$ and its symmetric form, and the Hölder form; for C_1 the ν-form $\sqrt{\pi}\Gamma(g-\frac12)\nu^{1-2g}/\Gamma(g)$, the Beta form $a^{1-2g}B(\frac12-\beta, g-\frac12)$ if $0 < \beta < \frac12$ and its symmetric form, and the Hölder form. For every torus of side ℓ and every ε, $\int_T(\hat k^2 + a^2)^{-\alpha}(\hat k^2 + b^2)^{-\beta} \le \mathcal{I}_{\alpha,\beta}(a, b; \ell)$, on $\varepsilon\mathbb{Z}^3$ with $\ell = \infty$; $\mathcal{I}_{\alpha,\beta}$ is decreasing in ℓ, and at $(a, b) = (M, \mu)$ every admissible form is

non-increasing in M. The majorants, all at $(M,\mu;\ell_0)$ with $\mu=\mu_\gamma$, are $\bar\tau=\mathcal{I}_{2,0}$, $\bar\psi=\mathcal{I}_{1,1}$, $\bar\delta=\mathcal{I}_{0,2}$, $\bar\Theta=\mathcal{I}_{1,3/4}+\mathcal{I}_{3/4,1}$, $\bar J_s=\mathcal{I}_{3-s/2,0}$ and $\bar J'_s=\mathcal{I}_{2-s/2,1}$ for $s\in\{0,1,\frac32\}$, $\bar\Lambda_0=\mathcal{I}_{3/4,1}$, $\bar\Lambda_1=\mathcal{I}_{5/4,1}+\mathcal{I}_{2,1/4}$, $\bar\Lambda_2=\mathcal{I}_{1/4,2}+\mathcal{I}_{1,5/4}$: fourteen distinct pairs (α,β), all with $g\ge\frac74>\frac32$; and $\bar\omega:=w+\hat E(\mu,\ell_0)$ with $w=0.252731$ (used as $w\le0.25273101$), $\hat E:=\min\{\bar E,\tilde E\}$, where $\bar E(\mu,\ell)=\frac{7.37}{\ell}+\frac{3\pi}{4\ell}E_1(\frac{\mu^2\ell^2}{8})+\frac{12\pi\,\mathrm{erfc}(\mu\ell/(2\sqrt2))}{\mu\ell^2}+\frac{64e^{-\mu^2\ell^2/8}}{\mu^2\ell^3}$ (Lemma 29.15(a)) and $\tilde E\ge0$ the image bound of Lemma 29.15(b), defined for $\ell>\frac12$ (for $\ell\le\frac12$, $\hat E:=\bar E$); for $\varepsilon\le1$ and $\ell_T\ge\ell_0$, $\varepsilon D_T\le\bar\omega$ (Lemma 29.15(c)), and on $\varepsilon\mathbb{Z}^3$ ($\ell_0=\infty$) one takes $\bar\omega=w$, since $\varepsilon D_f\le w$ there (Lemma 29.15(c)).

(G3) *The slope bounds* (Lemma 29.24): for $\varepsilon\le1$, $\ell_T\ge\ell_0$, $M>0$, with the majorants at $(M,\mu_\gamma;\ell_0)$, $\bar\beta:=2M[2\bar\tau+8\bar\psi]$ and

$$j:=2M\min\{\bar\tau,\bar\psi\},\qquad \varphi_3:=4M\big[2\sqrt2\,\bar\Theta\bar J_{3/2}+|\mu_\gamma^2-2M^2|\bar\psi\bar J_0+3\sqrt3\,\bar\omega\bar J_1\big]+\bar\beta\bar\tau,$$
$$b_4:=\varphi_A+28M\bar\delta\bar\tau+\varphi_{D2},\quad \varphi_A:=6M\big[2\sqrt2\,\bar\Theta\bar J'_{3/2}+|\mu_\gamma^2-2M^2|\bar\psi\bar J'_0+3\sqrt3\,\bar\omega\bar J'_1\big]+\bar\beta\bar\psi,$$
$$\varphi_{D2}:=2M\bar\psi(2\bar\psi+4\bar\delta)+\min\{8M\bar\psi(6\bar\psi+2\bar\tau+4\bar\delta),\ 2^{5/2}M(2\bar\Lambda_1+4\bar\Lambda_2)\bar\Lambda_0\},$$

one has $0\le\partial_M J_T\le j(M)$, $0\le-128\,\partial_M S_T\le768M\bar\tau(M)^2$, $|\partial_M F_T|\le\varphi_3(M)$ and $\partial_M Q_T\ge-b_4(M)$.

(G4) *The two-loop gap*: for $0<M_1<M_2$, $\mu_\gamma\in(0,1]$, $\ell_0>0$, every $\varepsilon\le1$ and every torus of side $\ell_T\ge\ell_0$, $a_{K,T}(M_2)-a_{K,T}(M_1)\ge(M_2^2-M_1^2)+16\lambda h_T-e^2\bar j-\lambda^2\bar s-e^2\lambda\bar B_3-e^4\bar B_4^-$ with $h_T:=C_T(M_1)-C_T(M_2)\ge0$, $\bar j=\int_{M_1}^{M_2}j$, $\bar s=\int768M\bar\tau^2$, $\bar B_3=16\int\varphi_3$, $\bar B_4^-=\int b_4$; its proof writes each order as $\int_{M_1}^{M_2}\partial_M(\cdot)$ and applies (G3) *order by order*: $-[J_T(M_2)-J_T(M_1)]\ge-\bar j$, $128[S_T(M_2)-S_T(M_1)]\ge-\bar s$, $16[F_T(M_2)-F_T(M_1)]\ge-\bar B_3$, $Q_T(M_2)-Q_T(M_1)\ge-\bar B_4^-$.

(G5) *The cell rule*: for a density $\phi\in\{j,768M\bar\tau^2,16\varphi_3,b_4\}$ and a cell $[a,b]\subset(0,\infty)$ let $\phi^{\mathrm{cell}}[a,b](\mu,\ell_0)$ be the number obtained from the formula for ϕ by evaluating every majorant at the left endpoint a, the prefactor M at the right endpoint b, the prefactor $|\mu^2-2M^2|$ at its maximum over $\{a,b\}$ and $\bar\omega$ as it is; since the majorants are non-negative and non-increasing in M and the formulas are sums, products and minima with non-negative coefficients, $\phi\le\phi^{\mathrm{cell}}[a,b]$ on $[a,b]$, and for a partition $M_1=M^{(0)}<\cdots<M^{(n)}=M_2$

$$\int_{M_1}^{M_2}\phi\,dM\ \le\ \Phi_n(\mu,\ell_0):=\sum_{i=0}^{n-1}\big(M^{(i+1)}-M^{(i)}\big)\,\phi^{\mathrm{cell}}[M^{(i)},M^{(i+1)}](\mu,\ell_0).\tag{88}$$

We write $\bar j_n,\bar s_n,\bar B_{3,n},\bar B_{4,n}^-$ for the four upper sums on the uniform partition of $[0.7,1]$ into n cells, and $\bar X_n$ for any of them; $n=400$ unless said otherwise.

Remark 29.13 (units). Every coefficient of King's counterterm on a torus is a finite sum of lattice graphs, and each is homogeneous under the change of units $(\varepsilon,\ell,x)\mapsto(\varepsilon,\ell,x)/s$, $(M,\mu)\mapsto s(M,\mu)$, $(e^2,\lambda)\mapsto s(e^2,\lambda)$, bare coefficients $\mapsto s^2\cdot$, for $s>0$: with $\phi'(x')=s^{1/2}\phi(sx')$ and $A'(x')=s^{1/2}A(sx')$, King's action [25, (2.4), p. 652] at spacing ε/s, on the torus of side ℓ/s, with masses sM, $s\mu$, couplings se^2, $s\lambda$ and bare coefficient s^2a, evaluated at (ϕ',A'), equals the action at the original parameters evaluated at (ϕ,A), since each sum $\sum_{x'}(\varepsilon/s)^3$ contributes s^{-3}, each lattice derivative s, each field $s^{1/2}$, and the phase $e^{\mathrm{i}e'\varepsilon'A'}=e^{\mathrm{i}e\varepsilon A}$. The renormalisation condition defining the counterterm, the vanishing of a zero-momentum two-point function, is covariant under this change, so $\delta m^2_{(\alpha\beta)}$ is multiplied by $s^{2-\alpha/2-\beta}$ and $a_{K,T}(M)$ by s^2. Hence it suffices to prove every statement in units in which a fixed reference mass m equals 1; in these units "$\varepsilon\le1$" means $\varepsilon\le1/m$, a torus floor ℓ_0 means ℓ_0/m, and λ,e^2 stand for $\lambda/m,e^2/m$.

Lemma 29.14 (odd orders vanish)**.** *Let $\delta m^2_{(\alpha\beta),T}(\varepsilon;M)$, $2\le\alpha+2\beta\le 4$, be the coefficients of King's counterterm* [25, (3.32)–(3.33)] *computed on a torus T with the periodic propagators (the recursion* (84)*: each coefficient is fixed, in the order in which the recursion determines it, by the vanishing of the order-$e^\alpha\lambda^\beta$ part of the amputated one-particle-irreducible two-point function of ϕ at zero momentum, all counterterm vertices of lower order inserted). Then $\delta m^2_{(\alpha\beta),T}=0$ for odd α; hence*

$$\delta m^2_T = e^2\delta m^2_{(20),T}+\lambda\delta m^2_{(01),T}+e^4\delta m^2_{(40),T}+e^2\lambda\delta m^2_{(21),T}+\lambda^2\delta m^2_{(02),T},$$

and $a_{K,T}(M)=M^2+\delta m^2_T(\varepsilon;M)$ is a polynomial in (e^2,λ), which is the form (G1).

Proof. Vertices. Expanding the hopping term of King's action in e, the part of order e^n is homogeneous of degree n in A (a sum of terms $A^n_{x,\mu}$ times a quadratic form in ϕ); the quartic and counterterm vertices contain no A. So a hopping vertex of order e^n has exactly n photon legs, every other vertex has none, and the photon covariance does not depend on e. *Contractions.* Under the free measure A and ϕ are independent and the centred Gaussian law of A is invariant under $A\mapsto -A$; so every polynomial of odd degree in A has expectation 0, and a graph with an odd number of photon legs vanishes (the external legs are scalar). *Induction.* Order the pairs (α,β) as the recursion determines them and assume $\delta m^2_{(\alpha'\beta')}=0$ for the earlier pairs with α' odd. A graph of order $e^\alpha\lambda^\beta$ other than the single counterterm vertex, with hopping vertices of orders $n_1,\dots,n_r$ and counterterm vertices of orders (α_i,β_i), has $\alpha=\sum n_j+\sum\alpha_i$; it vanishes unless every α_i is even (the earlier pairs), and then unless $\sum n_j$, the number of photon legs, is even. So the order-$e^\alpha\lambda^\beta$ part of the two-point function vanishes for odd α, and the recursion gives $\delta m^2_{(\alpha\beta)}=0$. Of the pairs $(2,0),(0,1),(3,0),(1,1),(4,0),(2,1),(0,2)$ the five with even α remain. □

The two-loop form (G1) collects the five orders: the three low orders are Lemma 29.7 on the torus (its proof is a computation graph by graph with the free propagators and holds with $\int_T$ in place of the Brillouin-zone integral and the sites of T in place of Λ), and the coefficients F_T and Q_T of the orders $e^2\lambda$ and e^4 are the finite sums of two-loop lattice graphs with the periodic propagators of Lemmas 29.21 and 29.23 below; at fixed ε and T each term is a finite sum of products of factors $\hat G_M(k)$ and of M-independent factors, a rational function of M^2 without poles on $(0,\infty)$, which is the smoothness of (G1). The one-loop majorants (G2) are Lemma 29.18 and the slope bounds (G3) are Lemma 29.24, both proved below from the lattice tools (Lemmas 29.16–29.20), the structure of the subtracted self-energy (Lemma 29.22) and the e^4 decomposition (Lemma 29.23); (G4) and (G5) follow as stated there. For a torus T of side ℓ and $M>0$ let $G_{M,f}$ be the massive propagator $(-\Delta_\varepsilon+M^2)^{-1}$ on $\varepsilon\mathbb{Z}^3$ and $G_{M,T}$ the periodic one on T, and put $E_T(M):=G_{M,T}(0)-G_{M,f}(0)=\sum_{z\in\ell\mathbb{Z}^3\setminus\{0\}}G_{M,f}(z)$, the image excess at coincident points (every term is positive by the random-walk representation of $G_{M,f}$); $c_N=\pi^3/8$ and $w=\varepsilon\int_f\hat k^{-2}=0.2527310098\dots$ are the constants of (G2) ($\int_f:=\int_{\mathsf B_\varepsilon}\frac{d^3k}{(2\pi)^3}$ is the Brillouin-zone integral, as in the next subsection).

Lemma 29.15 (torus images; the constant $\bar\omega$)**.** *Let T be a torus of side $\ell=\ell_T$ and $M>0$.*

(a) $E_T(M)\le\bar E(M,\ell):=\dfrac{7.37}{\ell}+\dfrac{3\pi}{4\ell}E_1\Big(\dfrac{M^2\ell^2}{8}\Big)+\dfrac{12\pi\,\mathrm{erfc}(M\ell/(2\sqrt2))}{M\ell^2}+\dfrac{64e^{-M^2\ell^2/8}}{M^2\ell^3}$, *$E_1$ the exponential integral; $\bar E$ is decreasing in ℓ and does not depend on ε.*

(b) *For $z \in \varepsilon\mathbb{Z}^3 \setminus \{0\}$ and $r := |z|_\infty$:* $G_{M,f}(z) \le \dfrac{c_N}{\pi(2r-\varepsilon)}\Big(1+\dfrac{\varepsilon M}{2\sqrt{2}}\Big)^{1-2r/\varepsilon}$. *Hence, for $\varepsilon \le 1$ and $\ell > \frac12$, with $\kappa(x) := \ln\big(1+\frac{x}{2\sqrt{2}}\big)\big/\frac{x}{2\sqrt{2}}$,*

$$E_T(M) \le \tilde E(M,\ell) := \sum_{j\ge1}(24j^2+2)\,\frac{c_N(1+M/(2\sqrt{2}))}{\pi(2j\ell-1)}\,e^{-\kappa(M)Mj\ell/\sqrt{2}},$$

which is decreasing in $\ell > \frac12$.

(c) *Put $\hat E := \min\{\bar E, \tilde E\}$ for $\ell > \frac12$ and $\hat E := \bar E$ for $\ell \le \frac12$; $\hat E$ is non-increasing in ℓ. For $\varepsilon \le 1$ and $\ell_T \ge \ell_0$: $\varepsilon D_T \le \bar\omega := w + \hat E(\mu,\ell_0)$, where $D_T = \int_T \hat D$ is the periodic photon propagator at coincident points. On $\varepsilon\mathbb{Z}^3$, $\varepsilon D_f \le w$ for every ε.*

Proof. (a) By the random-walk representation, $G_{M,f}(x) = \int_0^\infty e^{-M^2t}\prod_\mu p_t(x_\mu)\,dt$, where $p_t(x) = \varepsilon^{-1}\mathbb{P}(X_t = x)$ is the kernel of the continuous-time random walk X_t on $\varepsilon\mathbb{Z}$ with jumps $\pm\varepsilon$ at rate ε^{-2} each, started at 0, so that $\operatorname{Var} X_t = 2t$; p_t is non-increasing in $|x|$ (for $P_n := \mathbb{P}(X_t = n\varepsilon)$ and $q_n := P_n - P_{n+1}$, $n \ge 0$, the forward equation gives $\dot q = Lq$ with $L + 3\varepsilon^{-2}$ entrywise non-negative, so $q(t) \ge 0$ follows from $q(0) = (1,0,0,\dots)$), and $p_t(0) \le a_* t^{-1/2}$ with $a_* := \sqrt{\pi}/4$ (from $\mathbb{P}(X_t = 0) = \int_{-\pi}^{\pi}\frac{d\theta}{2\pi}e^{-v(1-\cos\theta)}$, $v = 2t/\varepsilon^2$, and $1-\cos\theta \ge 2\theta^2/\pi^2$ on $[0,\pi]$). Let $p^0 := p_t(0)$ and $\delta(t) := \sum_{n\ne0} p_t(n\ell)$. For $n \ge 1$ the interval $(n\ell - \ell/2, n\ell]$ contains at least $\ell/(4\varepsilon)$ sites, at each of which $p_t \ge p_t(n\ell)$, so $p_t(n\ell) \le \frac{4}{\ell}\mathbb{P}(X_t \in (n\ell - \frac{\ell}{2}, n\ell])$, and symmetrically for $n \le -1$; these intervals are disjoint, so by Chebyshev's inequality $\delta(t) \le \frac{4}{\ell}\mathbb{P}(|X_t| \ge \frac{\ell}{2}) \le \frac{4}{\ell}\min(1, 8t/\ell^2)$. By the image formula (Lemma 29.19(b) below) and Tonelli,

$$E_T(M) = \int_0^\infty e^{-M^2t}\big[(p^0+\delta)^3 - (p^0)^3\big]dt = \int_0^\infty e^{-M^2t}\big[3(p^0)^2\delta + 3p^0\delta^2 + \delta^3\big]dt.$$

Split at $t_0 := \ell^2/8$, use $\min(1, 8t/\ell^2) = t/t_0$ and $e^{-M^2t} \le 1$ on $t \le t_0$, and $\min = 1$ on $t \ge t_0$. The first term gives $\frac{3\pi}{4\ell t_0}\int_0^{t_0}dt + \frac{3\pi}{4\ell}\int_{t_0}^\infty \frac{e^{-M^2t}}{t}dt = \frac{3\pi}{4\ell} + \frac{3\pi}{4\ell}E_1(\frac{M^2\ell^2}{8})$; the second gives $48a_*\ell^{-2}t_0^{-2}\int_0^{t_0}t^{3/2}dt + 48a_*\ell^{-2}\int_{t_0}^\infty t^{-1/2}e^{-M^2t}dt = \frac{24\sqrt{\pi}}{5\sqrt{8}\,\ell} + \frac{12\pi\operatorname{erfc}(M\ell/(2\sqrt{2}))}{M\ell^2}$; the third gives $\frac{64}{\ell^3}\big[\frac{t_0}{4} + \frac{e^{-M^2t_0}}{M^2}\big] = \frac{2}{\ell} + \frac{64e^{-M^2\ell^2/8}}{M^2\ell^3}$. The constants add to $\frac{3\pi}{4} + \frac{24\sqrt{\pi}}{5\sqrt{8}} + 2 = 7.3641\ldots \le 7.37$. Each term of $\bar E$ is a product of positive decreasing functions of ℓ. (b) By Lemma 29.19(a) below, $G_M = \int_{M^2}^\infty (G_{\sqrt\sigma} * G_{\sqrt\sigma})\,d\sigma$. By the lattice symmetries we may take $z_1 = r > 0$. In $(G_{\sqrt\sigma} * G_{\sqrt\sigma})(z) = \int_f e^{\mathrm{i}k\cdot z}\hat G_{\sqrt\sigma}(k)^2$ shift $k_1 \to k_1 + \mathrm{i}\eta$; the integrand is periodic, so the vertical sides cancel, provided the denominator does not vanish. Since $\operatorname{Re}\frac{4}{\varepsilon^2}\sin^2\frac{\varepsilon(k_1+\mathrm{i}\eta)}{2} \ge \hat k_1^2 - \frac{4}{\varepsilon^2}\sinh^2\frac{\varepsilon\eta}{2}$, the choice $\frac{4}{\varepsilon^2}\sinh^2\frac{\varepsilon\eta_\sigma}{2} = \frac{\sigma}{2}$ gives $|\hat G_{\sqrt\sigma}| \le (\hat k^2 + \sigma/2)^{-1}$ on the shifted line, and the density-of-states bound (Lemma 29.16 below) gives $|(G_{\sqrt\sigma} * G_{\sqrt\sigma})(z)| \le e^{-\eta_\sigma r}c_N/(8\pi(\sigma/2)^{1/2})$, with $\eta_\sigma = \frac{2}{\varepsilon}\operatorname{arsinh}\frac{\varepsilon\sqrt\sigma}{2\sqrt2} \ge \frac{2}{\varepsilon}\ln(1+\frac{\varepsilon\sqrt\sigma}{2\sqrt2})$. With $\sigma = u^2$, $\int_M^\infty \frac{c_N\sqrt2}{4\pi}(1+\frac{\varepsilon u}{2\sqrt2})^{-2r/\varepsilon}du$ is the first bound. For $\tilde E$ use the image formula, $\#\{n \in \mathbb{Z}^3 : |n|_\infty = j\} = 24j^2+2$ and $r = j\ell$; each factor of the first bound is non-decreasing in ε while $2j\ell - \varepsilon > 0$ ($\ln(1+y)/y$ is decreasing), and for $\ell > \frac12$ this holds for every $\varepsilon \le 1$, so $\varepsilon \le 1$ gives the value at $\varepsilon = 1$. (c) $D_T = D_f + E_T(\mu)$ by the image formula, $\varepsilon D_f \le \varepsilon\int_f \hat k^{-2} = w$, and $\varepsilon E_T(\mu) \le E_T(\mu) \le \hat E(\mu,\ell_T) \le \hat E(\mu,\ell_0)$ for $\varepsilon \le 1$, by (a), (b) and the monotonicity of $\hat E$ in ℓ. □

The one-loop majorants and the slope bounds. In this subsection $\mathsf{X} \in \{T, f\}$ is a torus T of side ℓ_T with $\ell_T/\varepsilon \in 2\mathbb{N}$ or the lattice $\varepsilon\mathbb{Z}^3$ (written f); $\int_T := |T|^{-1}\sum_{k\in T^*}$ over the dual lattice of T, $\int_f := \int_{\mathsf{B}_\varepsilon}\frac{d^3k}{(2\pi)^3}$ over the Brillouin zone $\mathsf{B}_\varepsilon = [-\pi/\varepsilon, \pi/\varepsilon)^3$, $\hat k_\mu = \frac{2}{\varepsilon}\sin\frac{\varepsilon k_\mu}{2}$, $\hat k^2 = \sum_\mu \hat k_\mu^2$,

$\hat G=\hat G_M(k)=(\hat k^2+M^2)^{-1}$, $\hat D(k)=(\hat k^2+\mu^2)^{-1}$ with the photon mass $\mu=\mu_\gamma$, and $G_{M,\mathsf X}$, $D_{\mathsf X}$ are the position kernels (periodic on T), so that $C_{\mathsf X}=\int_{\mathsf X}\hat G_M=G_{M,\mathsf X}(0)$ and $D_{\mathsf X}=\int_{\mathsf X}\hat D$. The one-loop sums are $\tau_{\mathsf X}:=\int\hat G^2$, $\Xi_{\mathsf X}(p):=\int_{\mathsf X,p'}\hat D(p-p')\hat G_M(p')$ (so $\Xi_{\mathsf X}(0)=\int\hat D\hat G$), $\delta_{\mathsf X}:=\int\hat D^2$ and $\Theta_{\mathsf X}:=\int\hat D^{3/4}\hat G+\int\hat D\hat G^{3/4}$. With $\psi:=(\phi_1+\mathrm i\phi_2)/\sqrt2$, so that $|\phi|^2=2|\psi|^2$, the order-e^2 part of the shift Σ of the coefficient of $|\psi|^2$ (the inverse of the full propagator being $\hat p^2+M^2+\delta m^2+\Sigma(p)$, so that Σ is the negative of the self-energy in the convention of (84) and the counterterm condition reads $\delta m^2_{(\alpha\beta)}e^\alpha\lambda^\beta=-\Sigma^{(\alpha\beta)}(0)$) is $e^2\Sigma_{2,\mathsf X}$ with

$$\Sigma_{2,\mathsf X}(p;M):=D_{\mathsf X}\sum_\mu\cos(\varepsilon p_\mu)-\int_{\mathsf X,p'}\hat D(p-p')\,\widehat{(p+p')}^{\,2}\,\hat G_M(p'),$$

the seagull and the exchange graph, so that

$$\delta m^2_{(20),T}=-\Sigma_{2,T}(0;M)=-2D_T-J_T(M)$$

(Lemma 29.7(ii) on T); the *subtracted self-energy* is

$$\pi_{\mathsf X}(p;M):=\Sigma_{2,\mathsf X}(p;M)-\Sigma_{2,\mathsf X}(0;M),$$

explicitly

$$\pi_T(p;M)=-D_T\sum_\mu(1-\cos\varepsilon p_\mu)-\int_{T,p'}\big[\hat D(p-p')\widehat{(p+p')}^{\,2}-\hat D(p')\hat p'^2\big]\hat G_M(p').$$

Lemma 29.16 (density of states)**.** *For every $\varepsilon>0$ and every non-increasing $F:[0,\infty)\to[0,\infty)$, $\int_{\mathsf B_\varepsilon}F(\hat k^2)\,d^3k\le c_N\int_{\mathbb R^3}F(|k|^2)\,d^3k$, $c_N=\pi^3/8$.*

Proof. On $[-\pi,\pi]$, $|\sin(y/2)|\ge|y|/\pi$, so $\hat k^2\ge(4/\pi^2)|k|^2$ on $\mathsf B_\varepsilon$ and $\{k\in\mathsf B_\varepsilon:\hat k^2\le E\}\subset\{|k|\le\frac\pi2E^{1/2}\}$, a set of volume $c_N\frac{4\pi}{3}E^{3/2}$. By the layer-cake formula $\int_{\mathsf B_\varepsilon}F(\hat k^2)=\int_0^\infty|\{k\in\mathsf B_\varepsilon:F(\hat k^2)>t\}|\,dt$; since F is non-increasing, $\{u:F(u)>t\}$ is an interval $[0,E_t)$ or $[0,E_t]$, so the set in the integrand has measure at most $c_N\frac{4\pi}{3}E_t^{3/2}\le c_N|\{k\in\mathbb R^3:F(|k|^2)>t\}|$. Integrate in t. □

Lemma 29.17 (torus Riemann sums)**.** *Let F be as in Lemma 29.16 and T a torus of side ℓ. Then*

$$\int_T F(\hat k^2)\le c_N\int_{\mathbb R^3}F(|k|^2)\tfrac{d^3k}{(2\pi)^3}+\tfrac{3\pi}{16\ell}\int_0^\infty F(u)\,du+\tfrac{3}{4\ell^2}\int_{\mathbb R}F(q^2)\,dq+\tfrac{F(0)}{\ell^3}.$$

Proof. One coordinate. Let g be even and non-increasing in $|k|$ on $[-\pi/\varepsilon,\pi/\varepsilon]$, $h=2\pi/\ell$, $N'=\ell/\varepsilon$. The points of T^* in one coordinate are jh, $-N'/2<j\le N'/2$. For $1\le j\le N'/2$, $h\,g(jh)\le\int_{(j-1)h}^{jh}g$, and symmetrically for $j<0$; summing, $\ell^{-1}\sum_j g(jh)\le\int_{-\pi/\varepsilon}^{\pi/\varepsilon}g\,\frac{dk}{2\pi}+\frac{g(0)}{\ell}$. *Three coordinates.* $\hat k^2$ is increasing in each $|k_\mu|$ on $\mathsf B_\varepsilon$, so $F(\hat k^2)$, and every partial sum or integral of it over some coordinates, is even and non-increasing in each remaining coordinate; iterating the one-coordinate bound gives $\int_T F(\hat k^2)\le\sum_{S\subset\{1,2,3\}}\ell^{-|S|}\int F(\hat k^2)|_{k_S=0}$, the integral over the remaining $3-|S|$ coordinates on $[-\pi/\varepsilon,\pi/\varepsilon)$ with $\frac{dk}{2\pi}$ per coordinate. *The pieces.* The term $S=\emptyset$ is bounded by Lemma 29.16; in the others use $\hat k_\mu^2\ge(4/\pi^2)k_\mu^2$ and extend to $\mathbb R^d$: $\int_{\mathbb R^2}F(\frac{4}{\pi^2}|k|^2)\frac{d^2k}{(2\pi)^2}=\frac{\pi}{16}\int_0^\infty F$ and $\int_{\mathbb R}F(\frac{4}{\pi^2}k^2)\frac{dk}{2\pi}=\frac14\int_{\mathbb R}F(q^2)\,dq$; there are three sets S of each size $1,2$. □

Lemma 29.18 (one-loop majorants: the bounds (G2))**.** *Let $\alpha,\beta\ge0$ with $g:=\alpha+\beta>\frac32$, let $a,b>0$, $\nu:=\min(a,b)$, $F(u):=(u+a^2)^{-\alpha}(u+b^2)^{-\beta}$, and let $\mathcal I_{\alpha,\beta}(a,b;\ell)$, $\ell\in(0,\infty]$, be as in (87), with $\gamma_d(g)x^{d-2g}$ the closed form of $\int_{\mathbb R^3}(|k|^2+x^2)^{-g}\frac{d^3k}{(2\pi)^3}$, $\int_0^\infty(u+x^2)^{-g}du$, $\int_{\mathbb R}(q^2+$*

$x^2)^{-g}dq$ *for* $d = 3, 2, 1$ *in the Hölder forms. Then:* (a) *for every* ε *and every torus of side* ℓ, $\int_T F(\hat{k}^2) \le \mathcal{I}_{\alpha,\beta}(a, b; \ell)$, *and* $\int_f F(\hat{k}^2) \le \mathcal{I}_{\alpha,\beta}(a, b; \infty) = c_N C_3$; (b) $\mathcal{I}_{\alpha,\beta}$ *is decreasing in* ℓ; (c) *at* $(a, b) = (M, \mu)$ *every admissible form is non-increasing in* M, *hence so is* $\mathcal{I}_{\alpha,\beta}(M, \mu; \ell)$. *A factor* $\hat{p}^s$ *with* $0 \le s < 2\alpha$ *is absorbed by* $\hat{p}^s \le (\hat{p}^2 + a^2)^{s/2}$, *i.e.* $\alpha \to \alpha - \frac{s}{2}$. *The one-loop majorants of* (G2), *all at* $(M, \mu; \ell_0)$, *bound the following sums* ($s \in \{0, 1, \frac{3}{2}\}$): $\bar{\tau} = \mathcal{I}_{2,0}$ *bounds* τ_{X}; $\bar{\psi} = \mathcal{I}_{1,1}$ *bounds* $\Xi_{\mathsf{X}}(0)$; $\bar{\delta} = \mathcal{I}_{0,2}$ *bounds* δ_{X}; $\bar{\Theta} = \mathcal{I}_{1,3/4} + \mathcal{I}_{3/4,1}$ *bounds* Θ_{X}; $\bar{J}_s = \mathcal{I}_{3-s/2,0}$ *bounds* $\int \hat{G}^3 \hat{p}^s$; $\bar{J}'_s = \mathcal{I}_{2-s/2,1}$ *bounds* $\int \hat{D}\hat{G}^2 \hat{p}^s$; $\bar{\Lambda}_0 = \mathcal{I}_{3/4,1}$ *bounds* $\int \hat{k}^{1/2}\hat{G}\hat{D}$; $\bar{\Lambda}_1 = \mathcal{I}_{5/4,1} + \mathcal{I}_{2,1/4}$ *bounds* $\int \hat{G}^{5/4}\hat{D} + \int \hat{G}^2\hat{D}^{1/4}$; $\bar{\Lambda}_2 = \mathcal{I}_{1/4,2} + \mathcal{I}_{1,5/4}$ *bounds* $\int \hat{G}^{1/4}\hat{D}^2 + \int \hat{G}\hat{D}^{5/4}$. *Every instance has* $g \ge \frac{7}{4}$, *and by* (a), (b) *each one-loop sum on a torus of side* $\ell_T \ge \ell_0$, *and on* $\varepsilon\mathbb{Z}^3$ *with* $\ell_0 = \infty$, *is at most its majorant.*

Proof. (a) F is non-increasing, so Lemma 29.17 applies (on $\varepsilon\mathbb{Z}^3$, Lemma 29.16). Each closed form is the integral of a pointwise larger function: $F \le (u + \nu^2)^{-g}$ (the ν-forms); or $(u + b^2)^{-\beta} \le u^{-\beta}$ (the Beta forms, and symmetrically); or Hölder's inequality with the exponents g/α, g/β (the Hölder forms). They are evaluated by $\int_0^\infty t^{c-1}(t + a^2)^{-x} dt = a^{2(c-x)} B(c, x - c)$ ($0 < c < x$), after the substitution $t = |k|^2$ in polar coordinates ($\int_{\mathbb{R}^3} h(|k|^2) \frac{d^3k}{(2\pi)^3} = \frac{1}{4\pi^2} \int_0^\infty t^{1/2} h(t)\, dt$) or $t = q^2$; the exact values are $\int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} \frac{1}{(k^2+a^2)(k^2+b^2)} = \frac{1}{4\pi(a+b)}$ and its derivative $-\partial_{a^2}$. The last term of (87) is $F(0)/\ell^3$. (b) The ℓ-terms are positive multiples of $\ell^{-1}, \ell^{-2}, \ell^{-3}$. (c) The ν-forms: $\nu = \min(M, \mu)$ is non-decreasing in M and the exponent $d - 2g$ is negative; the Beta forms carry $M^{2c-2\alpha}$ with $c < \alpha$ (resp. $M^{2(1-g)}$, M^{1-2g}), or do not depend on M; the exact forms are decreasing; the Hölder forms are multiples of $M^{(d-2g)\alpha/g}$; the zero-mode term is $M^{-2\alpha}\mu^{-2\beta}\ell^{-3}$. A minimum of non-increasing functions is non-increasing. The sums listed are of the form $\int F(\hat{k}^2)$ with the stated (α, β) after the absorption of the factors $\hat{p}^s$, $\hat{k}^{1/2}$. □

Lemma 29.19 (positivity and images)**.** *Let* $M, \mu > 0$ *and* $\mathsf{X} \in \{T, f\}$.

(a) $G_{M,\mathsf{X}}(x) > 0$ *for every* x, *and* $-\partial_{M^2} G_{M,\mathsf{X}} = G_{M,\mathsf{X}} * G_{M,\mathsf{X}} \ge 0$ *pointwise.*

(b) $G_{M,T}(x) = \sum_{n\in\mathbb{Z}^3} G_{M,f}(x + n\ell_T)$, *the series converging absolutely, and likewise for* D *and for* $G * G$. *In particular* $E_T(M) = C_T(M) - C_f(M) = \sum_{n\neq 0} G_{M,f}(n\ell_T) \ge 0$ *and* $D_T - D_f = E_T(\mu)$.

(c) *For* $s, t > 0$ *the position kernels of* $\hat{G}_M^s$ *and* $\hat{D}^t$ *are pointwise positive. Hence, for every* q ($q \in T^*$ *on a torus*),

$$\int_{\mathsf{X},P} \hat{G}(P)^s \hat{D}(P \pm q)^t \le \int_{\mathsf{X}} \hat{G}^s \hat{D}^t, \qquad \int_{\mathsf{X},P} \hat{G}(P)\hat{G}(P + q) \le \int_{\mathsf{X}} \hat{G}^2,$$
$$\int_{\mathsf{X},P} \hat{D}(P - q)\hat{D}(P + q) \le \int_{\mathsf{X}} \hat{D}^2,$$

the first also with $s = 0$. *Also* $|(K * K')(x)| \le (K * K')(0)$ *whenever* $\hat{K}, \hat{K}' \ge 0$.

Proof. (a) Let P be the nearest-neighbour adjacency matrix of X (entries ≥ 0, row sums 6). Then $-\Delta_\varepsilon + M^2 = (6\varepsilon^{-2} + M^2)(1 - P/(6 + \varepsilon^2 M^2))$ and $\|P\| \le 6 < 6 + \varepsilon^2 M^2$, so the Neumann series of the inverse has non-negative terms, its zeroth term is positive on the diagonal and its n-th term reaches graph distance n; as X is connected, the kernel is positive everywhere. Moreover $-\partial_{M^2}(-\Delta_\varepsilon + M^2)^{-1} = (-\Delta_\varepsilon + M^2)^{-2}$, whose kernel is $G * G$. (b) $\hat{G}_M$, $\hat{D}$ and $\hat{G}_M^2$ are C^∞ and $(2\pi/\varepsilon)$-periodic in each coordinate, so their position kernels on $\varepsilon\mathbb{Z}^3$ decay faster than every power and the image series converge absolutely; for $k \in T^*$, $\sum_{x\in T} \varepsilon^3 e^{-\mathrm{i}k\cdot x} \sum_n G_{M,f}(x + n\ell_T) = \sum_{y\in\varepsilon\mathbb{Z}^3} \varepsilon^3 e^{-\mathrm{i}k\cdot y} G_{M,f}(y) = \hat{G}_M(k)$, so the image sum has the Fourier coefficients of $G_{M,T}$ on T. Positivity of E_T follows from (a). (c) By subordination, $(-\Delta_\varepsilon + M^2)^{-s} = \Gamma(s)^{-1} \int_0^\infty \vartheta^{s-1} e^{-\vartheta M^2} e^{\vartheta\Delta_\varepsilon} d\vartheta$, and $e^{\vartheta\Delta_\varepsilon} = e^{-6\vartheta/\varepsilon^2} e^{\vartheta P/\varepsilon^2}$ has a positive

kernel; the same for $\hat D^t$. For even kernels $K, K' \ge 0$, Parseval gives $\int_P \hat K(P)\hat K'(P-q) = \sum_x \varepsilon^3 K(x)K'(x)e^{\mathrm{i}q\cdot x}$, whose modulus is at most its value at $q=0$; for $s=0$ the left side does not depend on q. The last claim: $(K * K')(x) = \int e^{\mathrm{i}k\cdot x}\hat K\hat K'$. □

Lemma 29.20 (splitting and lattice identities)**.** (a) *For $q \neq 0$ and $0 < \theta \le \min(s,t)$ (with $\hat G^0 := 1$, $\hat D^0 := 1$): $\int_{\mathsf{X},P} \hat G(P)^s \hat D(P-q)^t \le (2/\hat q)^{2\theta}\big[\int_{\mathsf{X}} \hat G^{s-\theta}\hat D^t + \int_{\mathsf{X}} \hat G^s \hat D^{t-\theta}\big]$.*

(b) $|\widehat{(a+b)}_\nu| \le |\hat a_\nu| + |\hat b_\nu|$, *hence* $\widehat{(a+b)}^2 \le 2\hat a^2 + 2\hat b^2$; *and* $\widehat{(p+p')}^2 \le 2\widehat{(p-p')}^2 + 8\hat p'^2$.

(c) $|\hat D(P-q) - \hat D(P+q)| \le 4\hat P\hat q\, \hat D(P-q)\hat D(P+q)$.

(d) *Componentwise,* $\widehat{(a+b)}_\mu^2 + \widehat{(a-b)}_\mu^2 = 2\hat a_\mu^2 + 2\hat b_\mu^2 - \varepsilon^2 \hat a_\mu^2 \hat b_\mu^2$.

Proof. (a) By $|\sin(x+y)| \le |\sin x| + |\sin y|$ componentwise and Minkowski, $\hat q \le \hat P + \widehat{(q-P)}$, so either $\hat P \ge \hat q/2$, and then $\hat G(P)^\theta \le (2/\hat q)^{2\theta}$, or $\widehat{(P-q)} \ge \hat q/2$, and then $\hat D(P-q)^\theta \le (2/\hat q)^{2\theta}$; bound the integrand by the sum of the two cases and apply Lemma 29.19(c). (b) The first claim is the same subadditivity; the second uses $|\sin 2x| \le 2|\sin x|$ with $p+p' = (p-p') + 2p'$ and $(x+y)^2 \le 2x^2 + 2y^2$. (c) $\hat D(P-q) - \hat D(P+q) = \hat D(P-q)\hat D(P+q)[\widehat{(P+q)}^2 - \widehat{(P-q)}^2]$ and $\sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)$ give $|\widehat{(P+q)}^2 - \widehat{(P-q)}^2| = \sum_\mu \frac{4}{\varepsilon^2}|\sin(\varepsilon P_\mu)\sin(\varepsilon q_\mu)| \le 4\sum_\mu |\hat P_\mu||\hat q_\mu| \le 4\hat P\hat q$. (d) $4 - 4\cos a\cos b = 2(2-2\cos a) + 2(2-2\cos b) - (2-2\cos a)(2-2\cos b)$, divided by ε^2. □

Lemma 29.21 (the order $e^2\lambda$)**.** *On a torus T, $\delta m^2_{(21),T} = 16 F_T(M)$ with*

$$F_T(M) := \int_T \hat G_M^2\, \pi_T(\cdot; M).$$

Proof. Write the hopping term of King's action as $\sum_x \varepsilon^3 \bar\psi(-\Delta_A\psi)$ and let V_n be the part of $-\Delta_A + \Delta_\varepsilon$ of degree n in A: $(V_n\psi)(x) = c_n \sum_\mu [A_\mu(x)^n \psi(x+\varepsilon e_\mu) + (-1)^n A_\mu(x-\varepsilon e_\mu)^n \psi(x-\varepsilon e_\mu)]$, $c_n = -\varepsilon^{-2}(\mathrm{i}\varepsilon e)^n/n!$; a vertex λ carries $4\lambda|\psi|^4$ and a counterterm vertex $c_{\alpha,\beta}$ the factor $e^\alpha \lambda^\beta \delta m^2_{(\alpha\beta)}$. The coefficient $\delta m^2_{(21),T} e^2\lambda$ is minus the zero-momentum value of the sum of the amputated one-particle-irreducible two-point graphs of order $e^2\lambda$ other than the single vertex $c_{2,1}$ (the chain of scalar-irreducible parts sums to a geometric series, and a scalar-irreducible graph that is not one-particle irreducible contains a one-photon part, whose value vanishes by the inversion $x \mapsto -x$ of T, which multiplies each vertex of type V_n by $(-1)^n$). Apart from $c_{2,1}$, the vertex sets of order $e^2\lambda$ with non-vanishing factors are $\{\lambda, V_2\}$, $\{\lambda, V_1, V_1'\}$, $\{\lambda, c_{2,0}\}$, $\{V_2, c_{0,1}\}$, $\{V_1, V_1', c_{0,1}\}$, $\{c_{2,0}, c_{0,1}\}$ (the odd counterterms vanish, Lemma 29.14, and V_n carries e^n), and the one-particle-irreducible graphs on them are exactly: (a_1)–(a_3) the order-λ tadpole $16\lambda\int_T \hat G_M$ with the seagull, the exchange graph or the counterterm $c_{2,0}$ inserted on its loop line; (b_1) the exchange graph with the tadpole $16\lambda C_T$ inserted on its internal scalar line and (b_2) the same graph with the insertion $\lambda\delta m^2_{(01),T} = -16\lambda C_T$; (c) the graphs with external legs at λ and at V_1, one scalar line λ–V_1, two scalar lines λ–V_1' and the photon line V_1–V_1' (and the same with V_1, V_1' exchanged); all other scalar pairings are disconnected or have a bridge. The sum of (a_1)–(a_3) is the tadpole with $\hat G$ replaced by $-\hat G(e^2\Sigma_{2,T} + e^2\delta m^2_{(20),T})\hat G = -e^2\hat G^2\pi_T$, that is $-16e^2\lambda\int_T \hat G_M^2\pi_T$; ($b_1$) and ($b_2$) cancel; each graph (c) vanishes before the sum over the position w of its λ-vertex, since for V_1' on the bond $(z, z+\varepsilon e_\mu)$ both its scalar lines end at w and the sum over its two orientations is $c_1[G_{M,T}(z+\varepsilon e_\mu - w)G_{M,T}(z-w) - G_{M,T}(z-w)G_{M,T}(z+\varepsilon e_\mu - w)] = 0$. Hence $\delta m^2_{(21),T} = 16\int_T \hat G_M^2\pi_T$. □

Lemma 29.22 (structure of the subtracted self-energy)**.** *Put $Y_\mu(p) := \int_{\mathsf{X},p'} \hat D(p-p')\hat p_\mu'^2 \hat G_M(p')$. Then $\Xi_{\mathsf{X}}(p) \in [0, \Xi_{\mathsf{X}}(0)]$, $Y_\mu(p) \in [0, D_{\mathsf{X}}]$, and:*

(a) $\pi_{\mathsf X}(p;M) = -\big\{2\hat p^2\Xi_{\mathsf X}(p) + (\mu^2-2M^2)[\Xi_{\mathsf X}(p)-\Xi_{\mathsf X}(0)] - \varepsilon^2\sum_\mu \hat p_\mu^2 Y_\mu(p) + D_{\mathsf X}\tfrac{\varepsilon^2}{2}\hat p^2\big\}$;
(b) $|\pi_{\mathsf X}(p;M)| \le 2\sqrt2\,\Theta_{\mathsf X}(M)\hat p^{3/2} + |\mu^2-2M^2|\,\Xi_{\mathsf X}(0) + 3\sqrt3\,(\varepsilon D_{\mathsf X})\hat p$;
(c) $0 \le \partial_M\Sigma_{2,\mathsf X}(p;M) \le \beta_{\mathsf X}(M) := 2M[2\tau_{\mathsf X} + 8\Xi_{\mathsf X}(0)]$, *and* $|\partial_M\pi_{\mathsf X}(p;M)| \le \beta_{\mathsf X}(M)$ *for all* p.

Proof. The ranges of $\Xi_{\mathsf X}$, Y_μ follow from Lemma 29.19(c) and $\hat p_\mu'^2\hat G \le 1$. (a) By Lemma 29.20(d) with $a=p$, $b=p'$, summed over μ: $\widehat{(p+p')}^{\,2} = 2\hat p^2 + 2\hat p'^2 - \varepsilon^2\sum_\mu \hat p_\mu^2\hat p_\mu'^2 - \hat k^2$ with $k = p-p'$. With $\hat p'^2\hat G = 1 - M^2\hat G$ and $\hat k^2\hat D = 1-\mu^2\hat D$, the exchange integral is $2\hat p^2\Xi(p) + (\mu^2-2M^2)\Xi(p) - \varepsilon^2\sum_\mu\hat p_\mu^2 Y_\mu(p) + 2D_{\mathsf X} - C_{\mathsf X}$; the constant $2D_{\mathsf X}-C_{\mathsf X}$ cancels in $\pi = \Sigma_2(p)-\Sigma_2(0)$, and the seagull part of π is $-D_{\mathsf X}\sum_\mu(1-\cos\varepsilon p_\mu) = -D_{\mathsf X}\frac{\varepsilon^2}{2}\hat p^2$. (b) Lemma 29.20(a) with $s=t=1$, $\theta=\frac14$ gives $\Xi(p) \le \sqrt2\,\hat p^{-1/2}\Theta$, which bounds $2\hat p^2\Xi(p)$; $|\Xi(p)-\Xi(0)| \le \Xi(0)$; the last two terms of (a) are at most $\frac32\varepsilon^2 D_{\mathsf X}\hat p^2$, and $\varepsilon\hat p \le 2\sqrt3$. (c) The seagull does not depend on M, and the derivative of the exchange part of $\Sigma_2(p)$ is $2M\int \hat D(p-p')\widehat{(p+p')}^{\,2}\hat G(p')^2 \ge 0$; by Lemma 29.20(b), $\widehat{(p+p')}^{\,2} \le 2\hat k^2 + 8\hat p'^2$, and $\hat D\hat k^2 \le 1$, $\hat p'^2\hat G \le 1$, Lemma 29.19(c) bound it by $2M[2\tau + 8\Xi(0)]$. So $\partial_M\Sigma_2(p)$ and $\partial_M\Sigma_2(0)$ lie in $[0,\beta_{\mathsf X}]$, and their difference $\partial_M\pi(p)$ is at most $\beta_{\mathsf X}$ in modulus. □

Lemma 29.23 (the e^4 coefficient)**.** $\delta m^2_{(40),T} = Q_T(M) = \frac34\varepsilon^2 D_T^2 + (A) + (C) + (D1) + (D2) + (D3)$, *where, with* $s_\mu(k) := \hat k_\mu$ *and* $\hat G = \hat G_M$,

$$(A) = M^2\int_T \hat D\hat G^2\pi_T, \qquad (C) = -\tfrac{\varepsilon^2 D_T}{2}\big(D_T - J_T(M)\big),$$

$$(D1) = \tfrac12\sum_\mu\sum_z \varepsilon^3\big[D_T(z)^2 + D_T(z+\varepsilon e_\mu)^2\big]\big[G_{M,T}(z) + G_{M,T}(z+\varepsilon e_\mu)\big],$$

$$(D2) = -\sum_\mu\int_{T,k}\int_{T,P} s_\mu(k)\big[s_\mu(k) + s_\mu(2P+k)\big]\hat G(k)\hat G(P)\hat D(k)\hat D(P-k),$$

$$(D3) = \int_T \hat D(k)^2\sum_\mu \Pi_{\mu\mu}(k;M), \qquad \Pi_{\mu\mu}(k;M) := 2G_{M,T}(\varepsilon e_\mu) - \int_{T,q}\widehat{(2q+k)}_\mu^{\,2}\hat G(q)\hat G(q+k).$$

Proof. The mechanism is a Ward identity. Let $\lambda = 0$, $a = M^2 + \delta m^2$, $\mathcal G_{A,a} := (-\Delta_A + a)^{-1}$, and let $\langle\cdot\rangle$ be the photon expectation of the model after the scalar field is integrated out (the Gaussian of covariance $\delta_{\mu\nu}D_T$ times $\det(-\Delta_A+a)^{-1}$, normalised), so that the full two-point function is $G_{\rm full}(x-y) = \langle\mathcal G_{A,a}(x,y)\rangle$; all identities are identities of formal power series in e. King's condition is equivalent to $\hat G_{\rm full}(0) = M^{-2}$ through order e^4. With $W_A := -\Delta_A\mathbf 1$ one has $(-\Delta_A+a)\mathbf 1 = a + W_A$, hence $a\,\hat G_{\rm full}(0) = 1 - \langle(\mathcal G_{A,a}W_A)(x)\rangle$ and $\delta m^2 = -M^2\langle(\mathcal G_{A,a}W_A)(x)\rangle$ through order e^4. Gauge covariance, $-\Delta_{A+\nabla^+\chi} = e^{-\mathrm{i}e\chi}(-\Delta_A)e^{\mathrm{i}e\chi}$, and Gaussian integration by parts in the direction $\nabla^+\chi_y$, $\chi_y = -D_T(y-\cdot)$, give $\langle\mathcal G_{A,a}(x,y)\,\mathrm{div}A(y)\rangle = -\mathrm{i}e(D_T(0) - D_T(x-y))G_{\rm full}(x-y)$, whence the *Ward form* $\delta m^2 = e^2D_T - e^2M^2\int_T\hat D\hat G_{\rm full} - M^2\langle(\mathcal G_{A,a}R_A)(x)\rangle$, $R_A := W_A + \mathrm{i}e\,\mathrm{div}A = R^{(2)} + R^{(3)} + R^{(4)} + O(e^5)$ with $R^{(2)} = \frac{e^2}{2}\sum_\mu[A_\mu(x)^2 + A_\mu(x-\varepsilon e_\mu)^2]$, $R^{(3)} = \frac{\mathrm{i}\varepsilon^2e^3}{6}\sum_\mu\nabla_\mu^-(A_\mu^3)(x)$, $R^{(4)} = -\frac{\varepsilon^2e^4}{24}\sum_\mu[A_\mu(x)^4 + A_\mu(x-\varepsilon e_\mu)^4]$ (the expansion of $1-\cos\vartheta$ and $\sin\vartheta-\vartheta$, $\vartheta = \varepsilon eA$). Expanding $\mathcal G_{A,a} = G - GV_1G - G(V_2 + e^2\delta m^2_{(20)})G + GV_1GV_1G + O(e^3)$ and the determinant to order e^2 ($U_2 = \frac12\langle A, \Pi A\rangle$, whose diagonal symbol is $e^2\Pi_{\mu\mu}(k;M)$), and using $\sum_y\varepsilon^3G(x-y) = M^{-2}$, the order-$e^4$ part of the Ward form is the sum of five terms: the order-e^4 part of $-e^2M^2\int\hat D\hat G_{\rm full}$, which is (A) since the order-e^2 part of $\hat G_{\rm full}$ is $-e^2\hat G^2\pi_T$; $-\mathbb E_0R^{(4)}(0) = \frac34\varepsilon^2e^4D_T^2$ ($\mathbb E_0A_\mu^4 = 3D_T^2$); the term with $R^{(3)}$, which by $\mathbb E_0[L\,A_\mu^3] = 3D_T\mathbb E_0[L\,A_\mu]$ for linear L and the order-e Ward identity is (C); the terms with $R^{(2)}$ contracted with V_2 and with V_1V_1, which are $(D1)$ and $(D2)$ (the vertex factors multiply to $e^4s_\mu(k)[s_\mu(k)+s_\mu(2P+k)]$, from

$e^{\mathrm{i}\varepsilon k_\mu}-1=\mathrm{i}\varepsilon e^{\mathrm{i}\varepsilon k_\mu/2}s_\mu(k)$), the pairing of $R^{(2)}$ with itself contributing $-3e^4D_T\hat G(0)^2\pi_T(0)=0$; and the covariance of $R^{(2)}(0)$ with U_2, which is $(D3)$. The identity holds for King's counterterm through order e^4, and the coefficient of $e^4\lambda^0$ comes from graphs without a λ-vertex, so it is the same at $\lambda\neq0$. □

Lemma 29.24 (slope bounds for each coupling order: the bounds (G3))**.** *Let $\varepsilon\le1$, let T have side $\ell_T\ge\ell_0>0$, and let $M>0$. With the one-loop majorants of Lemma 29.18, all at $(M,\mu;\ell_0)$, the constant $\bar\omega=w+\hat E(\mu,\ell_0)$ of Lemma 29.15(c), and $\bar\beta$, j, φ_3, b_4, φ_A, φ_{D2} as defined in* (G3)*:* (o-λ) $\partial_M C_T\le0$; (o-e^2) $0\le\partial_M J_T\le j(M)$; (o-$\lambda^2$) $0\le-128\,\partial_M S_T\le768M\bar\tau(M)^2$; (o-$e^2\lambda$) $|\partial_M F_T|\le\varphi_3(M)$; (o-$e^4$) $\partial_M Q_T\ge-b_4(M)$. *All the majorants and $\bar\omega$ are non-negative and independent of the couplings and of ε; the one-loop majorants of Lemma 29.18 and $\bar\omega$ are non-increasing in M and in ℓ_0, and $\bar\omega$ does not depend on M.*

Proof. Every one-loop sum on T ($\ell_T\ge\ell_0$) is at most its majorant (Lemma 29.18(a),(b)); write $\tau=\tau_T$, $\Xi=\Xi_T$, $\delta=\delta_T$, $\Theta=\Theta_T$. The bound $\varepsilon D_T\le\bar\omega$ (Lemma 29.15(c)) is the only place where $\varepsilon\le1$ enters.

(o-λ). $\partial_M\hat G_M=-2M\hat G_M^2$, so $\partial_M C_T=-2M\tau\le0$.

(o-e^2). $\partial_M J_T=2M\int\hat D\hat G(1-M^2\hat G)=2M\int\hat D\hat k^2\hat G^2\ge0$. With $\hat D\hat k^2\le1$ it is at most $2M\tau\le2M\bar\tau$; with $\hat k^2\hat G\le1$ at most $2M\Xi(0)\le2M\bar\psi$.

(o-λ^2). By Lemma 29.19(a), $-\partial_M S_T=3\sum_x\varepsilon^3G_M^2(-\partial_M G_M)=6M\sum_x\varepsilon^3G_M(x)^2(G_M*G_M)(x)\ge0$. Next $(G*G)(x)\le(G*G)(0)=\tau$ (Lemma 29.19(c)) and $\sum_x\varepsilon^3G_M^2=\tau$ (Parseval), so $-128\,\partial_M S_T\le768M\tau^2\le768M\bar\tau^2$.

(o-$e^2\lambda$). By Lemma 29.21, $\partial_M F_T=-4M\int\hat G^3\pi_T+\int\hat G^2\partial_M\pi_T$. Lemma 29.22(b) bounds the first term by $4M[2\sqrt2\Theta J_{3/2}+|\mu^2-2M^2|\Xi(0)J_0+3\sqrt3(\varepsilon D_T)J_1]$, $J_s=\int\hat G^3\hat p^s$, and Lemma 29.22(c) bounds the second by $\beta_T\tau$. Replacing each factor by its majorant and εD_T by $\bar\omega$ gives φ_3.

(o-e^4). By Lemma 29.23, $\partial_M Q_T$ is the sum of the following five derivatives.

(C) $\partial_M(C)=\frac{\varepsilon^2D_T}{2}\partial_M J_T\ge0$ by (o-e^2); it is dropped.

(A) $(A)=M^2H$ with $H:=\int\hat D\hat G^2\pi_T$, so $\partial_M(A)=2MH-4M^3\int\hat D\hat G^3\pi_T+M^2\int\hat D\hat G^2\partial_M\pi_T$. By $M^2\hat G\le1$ and Lemma 29.22(b),(c), $|\partial_M(A)|\le6M\int\hat D\hat G^2|\pi_T|+\beta_T\Xi(0)\le\varphi_A$, with $\int\hat D\hat G^2\hat p^s\le\bar J'_s$.

(D1) $\partial_M G_{M,T}(z)=-2M(G*G)(z)\in[-2M\tau,0]$ (Lemma 29.19(a),(c)) and

$$\sum_z\varepsilon^3D_T(z)^2=\sum_z\varepsilon^3D_T(z+\varepsilon e_\mu)^2=\delta;$$

over three directions, $-12M\bar\delta\bar\tau\le\partial_M(D1)\le0$.

(D3) $\sum_\mu\partial_M[2G_{M,T}(\varepsilon e_\mu)]\in[-12M\tau,0]$. With $\hat G':=\hat G(q+k)$, the derivative of the bubble part of $\sum_\mu\Pi_{\mu\mu}$ is $2M\int\widehat{(2q+k)}^2(\hat G^2\hat G'+\hat G\hat G'^2)\ge0$; by Lemma 29.20(b) with $a=q$, $b=q+k$, $\hat q^2\hat G\le1$ and Lemma 29.19(c) it is at most $2M[4\int\hat G\hat G'+4\tau]\le16M\tau$. Since $\int\hat D^2=\delta$, $|\partial_M(D3)|\le16M\bar\delta\bar\tau$.

(D2) Write $(D2)=-\sum_\mu\int_k s_\mu(k)\hat G(k)\hat D(k)S_\mu(k)$ with $S_\mu(k):=\int_P[s_\mu(k)+s_\mu(2P+k)]\hat G(P)\hat D(P-k)$, and decompose $s_\mu(2P+k)=\frac{2}{\varepsilon}[\sin(\varepsilon P_\mu)\cos\frac{\varepsilon k_\mu}{2}+\cos(\varepsilon P_\mu)\sin\frac{\varepsilon k_\mu}{2}]$. The first piece is odd in P except through $\hat D(P-k)$; symmetrising $P\to-P$, $|\frac{2}{\varepsilon}\sin\varepsilon P_\mu|\le2|\hat P_\mu|$ and Lemma 29.20(c) bound its contribution to S_μ by $\int4|\hat P_\mu|\hat P\hat k\,\hat G\hat D(P-k)\hat D(P+k)$. The other two pieces are each at most $|s_\mu(k)|\int\hat G\hat D(\cdot-k)\le|s_\mu(k)|\Xi(0)$. With $\sum_\mu|\hat k_\mu||\hat P_\mu|\le\hat k\hat P$, $\hat P^2\hat G\le1$ and Lemma 29.19(c), $\sum_\mu|s_\mu||S_\mu|\le\hat k^2(2\Xi(0)+4\delta)$. Likewise, with $\hat D\hat D'\le\frac12(\hat D^2+\hat D'^2)$ and Lemma 29.20(a) for $\theta\in(\frac12,1]$ (at $k=0$ the factor $s_\mu(k)$ vanishes), $\sum_\mu|s_\mu||\partial_M S_\mu|\le2M\hat k^2(2/\hat k)^{2\theta}\big[2(\int\hat G^{2-\theta}\hat D+\int\hat G^2\hat D^{1-\theta})+4(\int\hat G^{1-\theta}\hat D^2+\int\hat G\hat D^{2-\theta})\big]$. When ∂_M hits $\hat G(k)$,

$\hat{k}^2\hat{G}^2\hat{D} \le \hat{G}\hat{D}$ bounds the contribution by $2M\Xi(0)(2\Xi(0)+4\delta)$. When it hits $\hat{G}(P)$, the contribution is at most $8M\Xi(0)(6\Xi(0)+2\tau+4\delta)$ at $\theta=1$ and at most $2^{5/2}M(2\Lambda_1+4\Lambda_2)\Lambda_0$ at $\theta=\frac{3}{4}$, with Λ_i the sums bounded by $\bar{\Lambda}_i$. So $|\partial_M(D2)| \le \varphi_{D2}$.

Summing, $\partial_M Q_T \ge -[\varphi_A + (12+16)M\bar{\delta}\bar{\tau} + \varphi_{D2}] = -b_4$.

Properties of the majorants. They are non-negative and coupling-free; the $\mathcal{I}$'s and $\bar{\omega}$ do not depend on ε, and they are non-increasing in M and in ℓ_0 (Lemma 29.18(b),(c), Lemma 29.15); $\bar{\omega}$ does not depend on M. □

Lemma 29.25 (the four orders on Λ_k at every photon mass)**.** *Let $\mu_0 \in (0,1]$, $\ell_0 > 0$, $n \ge 1$. For every k with $\varepsilon_k \le 1$:*

$$2S_{20,k} \ge -\bar{j}_n(\mu_0,\ell_0), \qquad 2S_{02,k} \ge -\bar{s}_n, \qquad 2S_{21,k} \ge -\bar{B}_{3,n}(\mu_0,\ell_0), \qquad 2S_{40,k} \ge -\bar{B}^{-}_{4,n}(\mu_0,\ell_0).$$

Proof. Fix such a k, and let $T = T_J$ be King's tori (Definition 12.1; side $2MJ$, $2MJ\cdot 5^k$ sites per side, an even number), J so large that $2MJ \ge \ell_0$. By (G1) at $\mu_\gamma = \mu_0$, the coefficients of the torus-convention bare coefficient $\alpha^{\mathrm{tor}}_{k,T}(M)$ in the five orders are the terms of the exact two-loop form, so that for $S^T_{\alpha\beta,k} := \frac{1}{2}[\delta m^2_{(\alpha\beta),T}(\varepsilon_k;1) - \delta m^2_{(\alpha\beta),T}(\varepsilon_k;0.7)]$,

$$2S^T_{20,k} = -[J_T(1) - J_T(0.7)], \quad 2S^T_{02,k} = 128[S_T(1) - S_T(0.7)],$$
$$2S^T_{21,k} = 16[F_T(1) - F_T(0.7)], \quad 2S^T_{40,k} = Q_T(1) - Q_T(0.7),$$

the M-independent terms $-2D_T$ and $\frac{3}{4}\varepsilon^2 D_T^2$ cancelling in the differences. The conditions of (G3) and (G4) are met: $\varepsilon_k \le 1$, $\ell_T \ge \ell_0$, $\mu_\gamma = \mu_0 \in (0,1]$, $M \in [0.7,1]$, and Lemmas 29.18, 29.15 and 29.24, which prove (G2) and (G3), carry μ_γ as a parameter in their statements (Lemma 29.18: $a,b>0$ and $g > \frac{3}{2}$ for every instance; Lemma 29.15: $\varepsilon \le 1$ and $\ell_T \ge \ell_0$ for $\bar{\omega}$, and $\ell_T/\varepsilon \in \mathbb{N}$, $\ell_T/\varepsilon \ge 2$ for $\bar{E}$). Hence, by the order-by-order bounds of (G4) with the majorants at (μ_0,ℓ_0) and by the cell rule (88), $2S^T_{20,k} \ge -\bar{j}_n(\mu_0,\ell_0)$, $2S^T_{02,k} \ge -\bar{s}_n$, $2S^T_{21,k} \ge -\bar{B}_{3,n}(\mu_0,\ell_0)$, $2S^T_{40,k} \ge -\bar{B}^{-}_{4,n}(\mu_0,\ell_0)$, uniformly in J ($\bar{s}$ does not depend on μ: $\bar{\tau} = \mathcal{I}_{2,0}$ has $\beta = 0$). Now let $J \to \infty$ at fixed k. By Lemma 12.11, $\alpha^{\mathrm{tor}}_{k,T_J}(M) \to \alpha^{\infty}_k(M)$ for $M \in \{0.7,1\}$, and the same argument (the volume limit at fixed ε is the convergence of torus Riemann sums of smooth periodic functions, faster than any power, by Poisson summation) gives the convergence of the recursion (84) order by order: $\delta m^2_{(\alpha\beta),T_J}(\varepsilon_k;M) \to \delta m^2_{(\alpha\beta)}(\varepsilon_k;M)$ for each of the five orders (alternatively: a polynomial of bounded degree in (e^2,λ) that converges at every point of an open set converges coefficient by coefficient, and Lemma 12.11 holds at every point of King's range). So $S^{T_J}_{\alpha\beta,k} \to S_{\alpha\beta,k}$, and closed inequalities pass to the limit. □

Lemma 29.26 (the torus floor can be removed)**.** *For every $\mu_0 \in (0,1]$, $n \ge 1$ and every k with $\varepsilon_k \le 1$, the four inequalities of Lemma 29.25 hold with $\bar{X}_n(\mu_0,\infty)$, the upper sums formed with the $\varepsilon\mathbb{Z}^3$ majorants $\mathcal{I}_{\alpha,\beta}(M,\mu_0;\infty) = c_N C_3(M,\mu_0)$ and with $\bar{\omega} = w$.*

Proof. Fix k, μ_0, n. By Lemma 29.25, $2S_{\alpha\beta,k} \ge -\bar{X}_n(\mu_0,\ell_0)$ for *every* $\ell_0 > 0$: the torus floor is a free parameter of (G4), and the tori T_J of the volume limit eventually exceed any ℓ_0. It therefore suffices that $\bar{X}_n(\mu_0,\ell_0) \to \bar{X}_n(\mu_0,\infty)$ as $\ell_0 \to \infty$. Each $\bar{X}_n$ is a finite sum of cell values, and each cell value is a fixed finite expression (sums, products and minima, with the fixed prefactors) in the majorants $\mathcal{I}_{\alpha,\beta}(a,\mu_0;\ell_0)$ at the left endpoint a of the cell and in $\bar{\omega}(\mu_0,\ell_0)$, hence continuous in these. As $\ell_0 \to \infty$, $\mathcal{I}_{\alpha,\beta}(a,\mu_0;\ell_0) \to c_N C_3(a,\mu_0)$ by (87), since C_2 and C_1 are finite numbers that do not depend on ℓ_0; and $0 \le \hat{E}(\mu_0,\ell_0) \le \bar{E}(\mu_0,\ell_0) \to 0$, each of the four terms of $\bar{E}$ being a positive constant times ℓ_0^{-1}, $\ell_0^{-1}E_1(\mu_0^2\ell_0^2/8)$, $\ell_0^{-2}\mathrm{erfc}(\mu_0\ell_0/(2\sqrt{2}))$ or $\ell_0^{-3}e^{-\mu_0^2\ell_0^2/8}$, so $\bar{\omega} \to w$. □

Lemma 29.27 (monotonicity in the photon mass; interval bounds)**.** *Let $0 < \mu_{\mathrm{lo}} \le \mu_{\mathrm{hi}} \le 1$. Define the* interval upper sum $\bar{X}^{\mathrm{int}}_n([\mu_{\mathrm{lo}},\mu_{\mathrm{hi}}])$ *as $\bar{X}_n(\cdot,\infty)$ with every majorant evaluated at*

$\mu = \mu_{\rm lo}$ and, on the cell $[a,b]$, the prefactor $|\mu^2 - 2M^2|$ replaced by $\max\{|\mu^2 - 2M^2| : \mu \in \{\mu_{\rm lo}, \mu_{\rm hi}\},\ M \in \{a,b\}\}$. Then $\bar{X}_n(\mu,\infty) \le \bar{X}_n^{\rm int}([\mu_{\rm lo},\mu_{\rm hi}])$ for every $\mu \in [\mu_{\rm lo},\mu_{\rm hi}]$, so the inequalities of Lemma 29.26 hold with $\bar{X}_n^{\rm int}([\mu_{\rm lo},\mu_{\rm hi}])$ for all $\mu_0 \in [\mu_{\rm lo},\mu_{\rm hi}]$ at once. Moreover, if $\mu_{\rm hi} \le 0.989949$, the interval upper sum equals the point upper sum $\bar{X}_n(\mu_{\rm lo},\infty)$.

Proof. A cell value is non-decreasing in each majorant and in the prefactor (sums, products and minima of non-negative quantities with non-negative coefficients). (i) Each admissible closed form of C_3 is non-increasing in $b = \mu$: the ν-form, since $\nu = \min(a,b)$ is non-decreasing in b and $3 - 2g < 0$; the Beta form in a is b-free; the Beta form in b carries $b^{2c'-2\beta}$ with $c' = \frac{3-2\alpha}{2} < \beta$, as $g > \frac{3}{2}$; the Hölder form carries $b^{(3-2g)\beta/g}$; the exact forms $\frac{1}{4\pi(a+b)}$ and $\frac{1}{8\pi a(a+b)^2}$ decrease in b. A minimum of non-increasing functions is non-increasing, so every $\mathcal{I}_{\alpha,\beta}(a,\mu;\infty)$ is non-increasing in μ, and $\bar{\omega} = w$ does not depend on μ. (ii) $\mu \mapsto |\mu^2 - 2M^2|$ is convex in μ^2 and $M \mapsto |\mu^2 - 2M^2|$ convex in M^2, so the maximum over the rectangle $[\mu_{\rm lo},\mu_{\rm hi}] \times [a,b]$ is attained at a corner. Hence every ingredient of the cell value at μ is at most the corresponding ingredient of the interval cell value. For the last claim: on $[0.7,1]$ one has $2M^2 \ge 0.98 > 0.989949^2$, so $|\mu^2 - 2M^2| = 2M^2 - \mu^2$ on the whole rectangle, maximal at $(\mu_{\rm lo}, b)$, which is also the point prefactor at $\mu_{\rm lo}$. (For the interval $[0.98, 0.99]$ the corner $(0.99, 0.7)$ of the first cell gives $|0.9801 - 0.98| = 10^{-4}$, below the value $2 \cdot 0.70075^2 - 0.98^2 = 0.0217$ at $(0.98, 0.70075)$, so the equality holds there too; only the interval $[0.99,1]$ differs, by 8.5786210 against the point value 8.5785880 of $\bar{B}_3$.) □

Lemma 29.28 (a closed form, uniform in the photon mass)**.** *For $M \in [0.7,1]$ put $\bar{\tau}^*(M) := \mathcal{I}_{2,0}(M;\infty) = \frac{c_N}{8\pi M}$, $\bar{\psi}^*(M) := \frac{c_N}{4\pi M}$, and for $X \in \{\Theta, J_s, J_s'\}$ let $X^*(M)$ be c_N times the minimum over the μ-free admissible forms of C_3 only (the Beta forms in $a = M$ and, for $\beta = 0$, the Hölder form $\gamma_3(g) M^{3-2g}$; the ν-forms, the Beta forms in b, the exact form at $(2,1)$ and the Hölder forms with $\beta > 0$ are dropped, and the exact form at $(1,1)$ is replaced by its bound $\frac{1}{4\pi a}$, which is $\bar{\psi}^*$). Put $\bar{\beta}^* := 2M[2\bar{\tau}^* + 8\bar{\psi}^*]$,*

$$\varphi_3^*(M) := 4M\big[2\sqrt{2}\,\bar{\Theta}^* \bar{J}_{3/2}^* + 2M^2 \bar{\psi}^* \bar{J}_0^* + 3\sqrt{3}\,w\,\bar{J}_1^*\big] + \bar{\beta}^* \bar{\tau}^*,$$
$$\varphi_A^*(M) := 6M\big[2\sqrt{2}\,\bar{\Theta}^* \bar{J}_{3/2}'^* + 2M^2 \bar{\psi}^* \bar{J}_0'^* + 3\sqrt{3}\,w\,\bar{J}_1'^*\big] + \bar{\beta}^* \bar{\psi}^*,$$
$$P(M) := \varphi_A^* + 52M\bar{\psi}^{*2} + 16M\bar{\psi}^*\bar{\tau}^*, \qquad Q(M) := \frac{c_N}{8\pi} M\big(28\bar{\tau}^* + 40\bar{\psi}^*\big).$$

Then $16\varphi_3(M;\mu,\infty) \le 16\varphi_3^*(M)$ and $b_4(M;\mu,\infty) \le P(M) + Q(M)/\mu$ for all $M \in [0.7,1]$ and $\mu \in (0,1]$. Consequently, with the 400-cell upper sums $B_3^* := \sum h\,(16\varphi_3^*)^{\rm cell}$, $P^* := \sum h P^{\rm cell}$, $Q^* := \sum h Q^{\rm cell}$ (majorants at the left endpoint, the prefactors M and $2M^2$ at the right endpoint), for every $\mu_0 \in (0,1]$ and every k with $\varepsilon_k \le 1$:

$$2S_{21,k} \ge -B_3^*, \qquad 2S_{40,k} \ge -\big(P^* + Q^*/\mu_0\big),$$
$$B_3^* = 13.187549, \quad P^* = 4.7629000, \quad Q^* = 0.77120851.$$

Proof. Each starred majorant is an upper bound of the corresponding majorant at $(M,\mu;\infty)$ for every $\mu > 0$: a minimum over a subset of the admissible forms is at least the minimum over all of them, and $\mathcal{I}_{1,1}(M,\mu;\infty) \le c_N \frac{1}{4\pi(M+\mu)} \le \frac{c_N}{4\pi M}$. Further $\bar{\delta}(\mu) = \mathcal{I}_{0,2}(M,\mu;\infty) \le c_N \frac{1}{4\pi^2} B(\frac{3}{2},\frac{1}{2}) \mu^{-1} = \frac{c_N}{8\pi\mu}$ (the Beta form in b, admissible since $\alpha = 0 < \frac{3}{2}$ and $\beta = 2 > \frac{3}{2}$); $|\mu^2 - 2M^2| \le 2M^2$, since $\mu^2 \le 1 \le 4M^2$; $\bar{\omega} = w$ (Lemma 29.26); and in φ_{D2} the minimum is at most its first branch. Substituting these bounds into the formulas (G3), which are non-decreasing in every majorant, gives $16\varphi_3 \le 16\varphi_3^*$ and

$$b_4 \le \varphi_A^* + 28M\bar{\tau}^* \frac{c_N}{8\pi\mu} + 2M\bar{\psi}^*\Big(2\bar{\psi}^* + \frac{4c_N}{8\pi\mu}\Big) + 8M\bar{\psi}^*\Big(6\bar{\psi}^* + 2\bar{\tau}^* + \frac{4c_N}{8\pi\mu}\Big) = P(M) + \frac{Q(M)}{\mu}.$$

The starred majorants are non-increasing in M (powers $M^{2c-2\alpha}$ with $c<\alpha$, and $1/M$), so the cell rule (88) applies to $16\varphi_3^*$, P and Q with the prefactors M, $2M^2$ at the right endpoint. The two inequalities follow *cellwise* from Lemma 29.26: for a cell $[a,b]$ the cell value of $16\varphi_3$ (resp. b_4) at (μ_0,∞) is a non-decreasing expression in the majorants at $(a,\mu_0;\infty)$, in $\bar\omega$ and in the prefactor, each of which is at most its starred counterpart at a (with $\bar\delta\le c_N/(8\pi\mu_0)$, $\max_{M\in\{a,b\}}|\mu_0^2-2M^2|\le 2b^2$, $\bar\omega=w$); so the cell value is at most the cell value of $16\varphi_3^*$ (resp. of $P+Q/\mu_0$). Summing over the 400 cells, $\bar B_{3,400}(\mu_0,\infty)\le B_3^*$ and $\bar B^-_{4,400}(\mu_0,\infty)\le P^*+Q^*/\mu_0$, and Lemma 29.26 gives the two inequalities. The three numbers are the 400-cell sums, computed as described after Table 2. (The slope bounds (G3) are not invoked with the starred majorants in place of the one-loop majorants; the domination above is all that is used.) □

The numbers. Let $\hat\mu=\mu_0/m\in(0,1]$. The grid is $\{1, 0.99, 0.98, 0.95, 0.9, 0.85, \dots, 0.15, 0.1, 0.08, 0.06, 0.05, 0.04, 0.03, 0.02, 0.015, 0.01, 0.005, 0.002, 0.001, 0.0005, 0.0002, 0.0001\}$ (35 points), and the *rows* of Table 2 are the 34 intervals $[\mu_{\rm lo},\mu_{\rm hi}]$ between consecutive grid points. For $\hat\mu\in[10^{-4},1]$ the *row of* $\hat\mu$ is the row containing $\hat\mu$ (at a grid point, where two rows contain $\hat\mu$, the row with $\mu_{\rm lo}=\hat\mu$), and $\bar B_3(\hat\mu)$, $\bar B_4^-(\hat\mu)$, $\bar j(\hat\mu)$ denote the interval upper sums $\bar B^{\rm int}_{3,400}$, $\bar B^{-,\rm int}_{4,400}$, $\bar j^{\rm int}_{400}$ of that row (Lemma 29.27), and $\bar s:=\bar s_{400}=6.5231964$; for $\hat\mu<10^{-4}$, below the table, we read $\bar B_3(\hat\mu):=B_3^*$, $\bar B_4^-(\hat\mu):=P^*+Q^*/\hat\mu$ and $\bar j(\hat\mu):=0.092610085$ (Lemma 29.28 for the first two; the last is the μ-free branch $\int 2M\bar\tau$ of $\bar j$, see the proof of Proposition 29.29(i)). With $A:=\frac12(1-M_1^2)=0.255$, $b:=\frac{4(1-M_1)}{2\pi}=0.1909859$, $L:=\frac{4\ln(1/M_1)}{\pi^2}=0.1445549$ (Lemmas 29.9 to 29.11 at $N=2$, $M_1=0.7$), $I_\infty(\hat\mu)=\frac{0.7+1.7\hat\mu}{4\pi\cdot1.7\,(\hat\mu+0.7)(1+\hat\mu)}$ (so $I_\infty(1)=0.0330425$) and $I^*=\frac{1}{4\pi\cdot1.7}=0.0468103$, put

$$\sigma_{\rm low}(\hat\lambda,\hat e^2;\hat\mu):=A\big(1-\hat e^2 I_\infty(\mu_{\rm lo})\big)+b\hat\lambda-L\hat\lambda^2-\tfrac12\bar B_3(\hat\mu)\,\hat e^2\hat\lambda-\tfrac12\bar B_4^-(\hat\mu)\,\hat e^4\qquad(\hat\mu\in[10^{-4},1]),\tag{89}$$

$$\sigma^*(\hat\lambda,\hat e^2;\hat\mu):=A\big(1-\hat e^2 I^*\big)+b\hat\lambda-L\hat\lambda^2-\tfrac12 B_3^*\,\hat e^2\hat\lambda-\tfrac12\big(P^*+Q^*/\hat\mu\big)\hat e^4\qquad(\hat\mu\in(0,1]),\tag{90}$$

and $\sigma_{\rm low}(\cdot;\hat\mu):=\sigma^*(\cdot;\hat\mu)$ for $\hat\mu<10^{-4}$. The derived columns of the table are: $e^2_{\max}$, the positive root of $\sigma_{\rm low}(0,\cdot;\hat\mu)$, $e^2_{\max}=\big(-AI_\infty+\sqrt{A^2I_\infty^2+2A\bar B_4^-}\big)/\bar B_4^-$; the side c of the largest box $[0,c]^2$ on which $\sigma_{\rm low}\ge0$; and the value $\sigma_{\rm low}(0.1,0.1;\hat\mu)$, where a negative entry means only that the bound does not place $[0,0.1]^2$ in the region.

How the entries were computed, and what is exact. The upper sums are the cell sums (88) with the interval rule of Lemma 29.27. At $\ell_0=\infty$ only the c_NC_3 term of (87) and $\bar\omega=w\le0.25273101$ enter, so no series and no special function beyond Γ and B is involved. The evaluation was carried out in floating-point arithmetic with 50 significant digits; every final upper sum was multiplied by $1+10^{-30}$ and rounded upward to eight significant digits, and the derived region numbers were computed from the rounded upper sums with coefficients rounded up and sizes rounded down. The rounding of the elementary and Gamma functions at 50 digits is many orders below the safety factor and the final rounding; the entries are 50-digit floating-point evaluations with the safety factor $1+10^{-30}$, not a certified (interval-arithmetic) computation; their validity as upper bounds rests on Lemma 29.27 and on these finitely many evaluations. Internal consistency of the table: $\bar s_{400}=6.5231964$ at every $\hat\mu$ ($\bar s$ is μ-free); $\bar j=0.092610085$ for all $\hat\mu\le0.7$, the μ-free branch $\int 2M\bar\tau$ of $\bar j$ (analytically $\frac{c_N}{4\pi}(1-M_1)=0.09253$ before the cell rounding); the interval values equal the point values at $\mu_{\rm lo}$ except on $[0.99,1]$ (Lemma 29.27); $\bar B^-_{4,400}(\hat\mu,\infty)\le P^*+Q^*/\hat\mu$ at every grid point ($7716.17\le7716.85$ at $\hat\mu=10^{-4}$, $1.5854\le5.534$ at $\hat\mu=1$); and $\hat\mu\,\bar B_4^-(\hat\mu)\to0.7716$ at $\hat\mu=10^{-4}$ against $Q^*=0.7712$. The closed form of Lemma 29.28 is weaker than the table near $\hat\mu=1$ (it uses $\bar\psi^*$, $|\mu^2-2M^2|\le2M^2$ and the first branch of the minimum) and is within one per cent of it for $\hat\mu\le10^{-2}$.

$[\mu_{\mathrm{lo}},\mu_{\mathrm{hi}}]$	$\bar{j}$	$\bar{B}_3$	$\bar{B}_4^-$	$AI_\infty(\mu_{\mathrm{lo}})$	$e^2_{\max}(\lambda=0)$	c	$\sigma_{\mathrm{low}}(0.1,0.1)$
$[0.99,1]$	0.085313936	8.5786210	1.5854190	0.008458	0.5618	0.2390	0.2209
$[0.98,0.99]$	0.085757165	8.6036471	1.6023646	0.008491	0.5588	0.2385	0.2207
$[0.95,0.98]$	0.087020288	8.6804035	1.6554211	0.008589	0.5498	0.2369	0.2201
$[0.9,0.95]$	0.088887935	8.8139269	1.7519811	0.008757	0.5345	0.2341	0.2189
$[0.85,0.9]$	0.090427040	8.9551127	1.8603008	0.008930	0.5188	0.2313	0.2176
$[0.8,0.85]$	0.091595352	9.1048479	1.9826438	0.009108	0.5026	0.2283	0.2163
$[0.75,0.8]$	0.092342924	9.2641791	2.1218953	0.009291	0.4858	0.2251	0.2147
$[0.7,0.75]$	0.092610085	9.4343534	2.2812536	0.009480	0.4686	0.2217	0.2131
$[0.65,0.7]$	0.092610085	9.6168755	2.4625429	0.009673	0.4511	0.2182	0.2112
$[0.6,0.65]$	0.092610085	9.8135862	2.6698136	0.009871	0.4333	0.2145	0.2092
$[0.55,0.6]$	0.092610085	10.026776	2.9084862	0.010073	0.4152	0.2105	0.2069
$[0.5,0.55]$	0.092610085	10.259350	3.1856310	0.010279	0.3969	0.2063	0.2044
$[0.45,0.5]$	0.092610085	10.515087	3.5155645	0.010488	0.3779	0.2017	0.2014
$[0.4,0.45]$	0.092610085	10.799045	3.9183359	0.010697	0.3580	0.1967	0.1979
$[0.35,0.4]$	0.092610085	11.118258	4.4224644	0.010906	0.3371	0.1912	0.1938
$[0.3,0.35]$	0.092610085	11.477171	5.0733664	0.011111	0.3148	0.1850	0.1887
$[0.25,0.3]$	0.092610085	11.851059	5.9437544	0.011309	0.2910	0.1781	0.1825
$[0.2,0.25]$	0.092610085	12.193509	7.0632067	0.011495	0.2670	0.1709	0.1752
$[0.15,0.2]$	0.092610085	12.433452	8.5440078	0.011662	0.2429	0.1635	0.1665
$[0.1,0.15]$	0.092610085	12.659419	11.303445	0.011801	0.2113	0.1525	0.1516
$[0.08,0.1]$	0.092610085	12.756284	13.316343	0.011846	0.1948	0.1460	0.1411
$[0.06,0.08]$	0.092610085	12.857230	16.620716	0.011884	0.1744	0.1370	0.1240
$[0.05,0.06]$	0.092610085	12.909321	19.239396	0.011899	0.1621	0.1310	0.1107
$[0.04,0.05]$	0.092610085	12.962542	23.145186	0.011912	0.1479	0.1233	0.0909
$[0.03,0.04]$	0.092610085	13.016932	29.623519	0.011923	0.1308	0.1132	0.0582
$[0.02,0.03]$	0.092610085	13.072534	42.530531	0.011931	0.1092	0.0987	-0.0066
$[0.015,0.02]$	0.092610085	13.100802	55.411533	0.011933	0.0957	0.0887	-0.0712
$[0.01,0.015]$	0.092610085	13.129390	81.146532	0.011935	0.0791	0.0753	-0.2000
$[0.005,0.01]$	0.092610085	13.158304	158.29597	0.011937	0.0566	0.0555	-0.5859
$[0.002,0.005]$	0.092610085	13.175811	389.67594	0.011937	0.0361	0.0360	-1.7428
$[0.001,0.002]$	0.092610085	13.181673	775.28604	0.011937	0.0256	0.0256	-3.6709
$[0.0005,0.001]$	0.092610085	13.184609	1546.4975	0.011937	0.0181	0.0181	-7.5270
$[0.0002,0.0005]$	0.092610085	13.186373	3860.1248	0.011937	0.0114	0.0114	-19.096
$[0.0001,0.0002]$	0.092610085	13.186961	7716.1679	0.011937	0.0081	0.0081	-38.376

TABLE 2. The interval upper sums $\bar{j}$, $\bar{B}_3$, $\bar{B}_4^-$ of Lemma 29.27 on the 400-cell partition of $[0.7,1]$ at $\ell_0=\infty$ (Lemma 29.26), valid for every $\hat{\mu}$ in the row; $\bar{s}=6.5231964$ on every row. The three upper sums are rounded upward to eight significant digits; the column $AI_\infty(\mu_{\mathrm{lo}})$ is rounded up, and the derived columns $e^2_{\max}$, c and $\sigma_{\mathrm{low}}(0.1,0.1)$ are rounded down (towards $-\infty$). Units $m=1$, $N=2$, $M_1=0.7$.

Throughout Proposition 29.29 and its proof, let $N=2$, $M_1=0.7m$ and $\mu_0\in(0,m]$; write $\hat{\lambda}=\lambda/m$, $\hat{e}^2=e^2/m$, $\hat{\mu}=\mu_0/m$. Let $\delta m_k^2(\mathsf{M})$ be King's counterterm (84) on Λ_k at the photon mass μ_0, $S_{\alpha\beta,k}$ the coefficients of Lemma 29.6, $\sigma_1(M_1)=\liminf_k s_k^\infty(M_1)$ as in Definition 2.1, and let σ_{low}, σ^*, $\bar{j}(\hat{\mu})$, $\bar{s}$, $\bar{B}_3(\hat{\mu})$, $\bar{B}_4^-(\hat{\mu})$, A, b, L, I_∞, I^* be as defined before Table 2, B_3^*, P^*, Q^* as in Lemma 29.28, and $\lambda_c:=\frac{b+\sqrt{b^2+4AL}}{2L}=2.14398\ldots$.

Proposition 29.29 (the explicit coupling region of (T3) at every photon mass)**.**

(i) (The five orders.) *For every k with $\varepsilon_k\le 1/m$, in units $m=1$: $S_{01,k}>0$; $2S_{20,k}\ge -\bar{j}(\hat{\mu})\ge -0.092611$; $2S_{02,k}\ge -\bar{s}=-6.5231964$; $2S_{21,k}\ge -\bar{B}_3(\hat{\mu})$ and $2S_{21,k}\ge -B_3^*$; $2S_{40,k}\ge -\bar{B}_4^-(\hat{\mu})$ and $2S_{40,k}\ge -(P^*+Q^*/\hat{\mu})$.*

(ii) (The lower bound.) $\sigma_1(0.7m)\ge m^2\max\{\sigma_{\mathrm{low}}(\hat{\lambda},\hat{e}^2;\hat{\mu}),\ \sigma^*(\hat{\lambda},\hat{e}^2;\hat{\mu})\}$.

(iii) (The region.) (T3) *holds on* $R(\mu_0) := \{(\lambda, e^2) : \sigma_{\rm low}(\hat\lambda, \hat e^2; \hat\mu) > 0\}$, *with* $\sigma_1(0.7m) \geq m^2\sigma_{\rm low} > 0$ *there.* $R(\mu_0)$ *contains the box* $[0, c(\hat\mu)m]^2$ *and the segment* $\{\lambda = 0,\ e^2 < e^2_{\max}(\hat\mu)m\}$, *with* c *and* $e^2_{\max}$ *from Table 2 for* $\hat\mu \in [10^{-4}, 1]$, *and the segment* $\{e = 0,\ \lambda < \lambda_c m\}$ *for every* $\hat\mu$. *The map* $\hat\mu \mapsto \sigma_{\rm low}(\hat\lambda, \hat e^2; \hat\mu)$ *is non-decreasing (on* $[10^{-4}, 1]$ *a step function of* $\hat\mu$, *constant on each row of the table; below* 10^{-4} *the closed form* σ^**), so* $R(\mu_0) \supseteq R(\mu_0')$ *for* $\mu_0 \geq \mu_0'$. *In particular* $[0, 0.2m]^2 \subseteq R(\mu_0)$ *for* $\mu_0 \in [0.5m, m]$; $[0, 0.15m]^2 \subseteq R(\mu_0)$ *for* $\mu_0 \in [0.1m, m]$; $[0, 0.1m]^2 \subseteq R(\mu_0)$ *for* $\mu_0 \in [0.03m, m]$, *with* $\sigma_1(0.7m) \geq 0.0582\, m^2$ *at* $(\lambda, e^2) = (0.1m, 0.1m)$; $[0, 0.075m]^2 \subseteq R(\mu_0)$ *for* $\mu_0 \in [0.01m, m]$; *and at* $\mu_0 = m$, $R(m)$ *contains the box* $[0, 0.239m]^2$ *and the segment* $\{\lambda = 0,\ e^2 < 0.5618\, m\}$, *and* $\sigma_1(0.7m) \geq 0.2209\, m^2$ *on* $[0, 0.1m]^2$.

(iv) (Uniform in the photon mass; closed form.) (T3) *holds whenever* $\sigma^*(\hat\lambda, \hat e^2; \hat\mu) > 0$, *with* $\sigma_1(0.7m) \geq m^2\sigma^*$; *in particular, for every* $\mu_0 \in (0, m]$, *on each of the boxes* $\{\lambda \leq \lambda_1 m,\ e^2 \leq c_1\sqrt{\mu_0 m}\}$ *with* $(\lambda_1, c_1) \in \{(0, 0.3014), (0.05, 0.2533), (0.1, 0.2152), (0.2, 0.1616), (0.5, 0.0884), (1, 0.0447)\}$.

(v) (Every k.) *For every* k *with* $\varepsilon_k \leq 1/m$: $s_k^\infty(0.7m) \geq m^2\big[A - \frac12\bar j(\hat\mu)\hat e^2 - \frac12\bar s\hat\lambda^2 - \frac12\bar B_3(\hat\mu)\hat e^2\hat\lambda - \frac12\bar B_4^-(\hat\mu)\hat e^4\big]$.

(vi) ($e = 0$; exact.) *At* $e = 0$, *for every* $M_1 \in (0, m)$, *every* $\mu_0 > 0$ *and every* $N \geq 1$: $\lim_k s_k^\infty(M_1) = \frac12(m^2 - M_1^2) + \frac{(N+2)\lambda(m - M_1)}{2\pi} - \frac{(N+2)\lambda^2}{\pi^2}\ln\frac{m}{M_1}$ *exactly, and* (T3) *holds if and only if* $\lambda < \frac{\pi}{4}\big(1 + \sqrt{1 + \frac{16}{N+2}}\big)m$, *which is* $\frac{\pi}{4}(1 + \sqrt5)\, m = 2.5416\ldots\, m$ *for* $N = 2$; *at* $M_1 = 0.7m$, $N = 2$, *the exact window is* $\lambda < \lambda_c m$, $\lambda_c = 2.14398\ldots$, *in particular* $\lambda \leq 2.1439\, m$.

(vii) (Two-sided.) *On each region of (iii), (iv) and (vi) the two-sided clause* (NE) *holds as well, with no use of Proposition 12.13.*

Outside these regions and those of Proposition 29.30 below the sign clause (T3) is not decided here (it fails on $E_W^<$, Lemma 29.4(i), so the regions are not the whole coupling space). The bound $\sigma_{\rm low}$ is explicit at every photon mass but not uniform in it: by (90), $e^2_{\max}(\hat\mu) \sim \sqrt{2A/Q^*}\sqrt{\hat\mu} = 0.81\sqrt{\hat\mu}$ as $\hat\mu \to 0$ (Table 2: $\hat\mu\,\bar B_4^-(\hat\mu) \to 0.77$), and $\bigcap_{\mu_0 \in (0,m]} R(\mu_0) = \{e = 0,\ \lambda < \lambda_c m\}$. The $1/\hat\mu$ is a feature of the order-e^4 majorant alone, not of the coupling region: the exact limit coefficient of order e^4 is bounded below by $-\frac{\ln(10/7)}{8\pi^2}$ at every photon mass, and Proposition 29.30 below proves (T3) on the box $(0, \lambda_c m) \times (0, 6.30\, m)$ for every $\mu_0 \in (0, m]$. (The sign clause is a condition of the Wick sector alone, Lemma 29.3; Section 30 takes the limit of vanishing photon mass in the curvature sector, which carries no coupling condition, and the Wick sector in that limit is left open, Section 32.) The proof below treats the seven clauses in turn; in (iii) and (iv) the concavity of $\sigma_{\rm low}$ and of σ^* in each variable separately reduces each box to its corners.

Proof of Proposition 29.29. Units $m = 1$, $N = 2$, $M_1 = 0.7$; $\hat\mu = \mu_0 \in (0, 1]$.

(i). $S_{01,k} > 0$ is Lemma 29.9(i). The four bounds with interval upper sums are Lemma 29.26 with Lemma 29.27, applied to the row of $\hat\mu$ (for $\hat\mu \geq 10^{-4}$); the bounds with B_3^* and $P^* + Q^*/\hat\mu$ are Lemma 29.28; for $\hat\mu < 10^{-4}$ the first bound is the μ-free branch of $\bar j$: $j \leq 2M\bar\tau$ gives $\bar j_{400}(\hat\mu, \infty) \leq \sum h\,(2M\bar\tau^*)^{\rm cell} = 0.092610085$ for every $\hat\mu$, by the cell rule (88). The numbers $\bar j \leq 0.092610085 < 0.092611$ and $\bar s = 6.5231964$ are read off Table 2. The interval value $\bar B_4^-(\hat\mu)$, taken at $\mu_{\rm lo}$, and the closed form $P^* + Q^*/\hat\mu$, taken at $\hat\mu$, are two separate bounds, neither of which dominates the other on a whole row.

(ii). By (85) and Lemma 29.9(iii), Lemma 29.10(iii), Lemma 29.11(iii), the first four coefficients converge, so

$$\sigma_1(0.7) = \liminf_k s_k^\infty(0.7) = A + b\hat\lambda - \hat e^2 A I_\infty(\hat\mu) - L\hat\lambda^2 + \liminf_k \big(\hat e^2\hat\lambda S_{21,k} + \hat e^4 S_{40,k}\big),$$

and by (i), for all but finitely many k (those with $\varepsilon_k \le 1$), $\hat{e}^2\hat{\lambda}S_{21,k} + \hat{e}^4 S_{40,k} \ge -\frac{1}{2}\bar{B}_3(\hat{\mu})\hat{e}^2\hat{\lambda} - \frac{1}{2}\bar{B}_4^-(\hat{\mu})\hat{e}^4$; with $I_\infty(\hat{\mu}) \le I_\infty(\mu_{\rm lo})$ (Lemma 29.10(iv)) this is $\sigma_1(0.7) \ge \sigma_{\rm low}$. The same argument with the closed-form bounds of (i) and $I_\infty(\hat{\mu}) \le I^*$ gives $\sigma_1(0.7) \ge \sigma^*$. (Cellwise, the interval value of $16\varphi_3$ at $\mu_{\rm lo}$ is at most the starred cell value, so $\bar{B}_3(\hat{\mu}) \le B_3^*$ always holds; for b_4 the interval value carries $\bar{\delta}(\mu_{\rm lo}) \approx c_N/(8\pi\mu_{\rm lo})$ while the closed form carries $Q^*/\hat{\mu}$, so neither bound dominates the other on a whole row; hence the maximum in (ii).)

(iii). The first sentence is (ii) and the definition of (T3) (Definition 2.1). $\sigma_{\rm low}$ is concave in $\hat{\lambda}$ at fixed $\hat{e}^2$ (the term $-L\hat{\lambda}^2$) and concave in $\hat{e}^2$ at fixed $\hat{\lambda}$ (the term $-\frac{1}{2}\bar{B}_4^-\hat{e}^4$), so its minimum over a box $[0,c]^2$ is attained at a corner: the minimum over $\hat{\lambda} \in [0,c]$ of a concave function is attained at an endpoint, the two resulting functions of $\hat{e}^2$ are concave and so is their minimum, whose minimum over $[0,c]$ is again at an endpoint. The side $c(\hat{\mu})$ of Table 2 is the largest c for which the four corner values are ≥ 0, rounded down; $e^2_{\max}$ is the positive root of $\sigma_{\rm low}(0,\cdot) = A(1-\hat{e}^2 I_\infty) - \frac{1}{2}\bar{B}_4^-\hat{e}^4$, rounded down; and at $e=0$, $\sigma_{\rm low} = A + b\hat{\lambda} - L\hat{\lambda}^2 > 0$ if and only if $\hat{\lambda} < \lambda_c = \frac{b+\sqrt{b^2+4AL}}{2L} = 2.14398\ldots$ (as in (vi)). *Monotonicity in $\hat{\mu}$:* on $[10^{-4},1]$, $\sigma_{\rm low}(\cdot;\hat{\mu})$ is a step function of $\hat{\mu}$, built row by row from $I_\infty(\mu_{\rm lo})$ and the interval values $\bar{B}_3$, $\bar{B}_4^-$ of the row, which enter with negative coefficients; so it suffices that these three row values do not decrease from each row to the row below it. For every row but the top one the interval values are the point values at $\mu_{\rm lo}$ (Lemma 29.27, last clause, and the remark at the end of its proof for the row $[0.98, 0.99]$), the point values are non-increasing in μ (Lemma 29.27(i), with the point prefactor $\max_{M\in\{a,b\}}|\mu^2 - 2M^2| = 2b^2 - \mu^2$ non-increasing in μ for $\mu \le 0.989949$) and so is I_∞ (Lemma 29.10(iv)); hence the values of a row, taken at its $\mu_{\rm lo}$, are at most those of the row below, whose left end is smaller. For the top row $[0.99, 1]$ this is read off the table ($8.5786210 \le 8.6036471$, $1.5854190 \le 1.6023646$, $0.008458 \le 0.008491$); by inspection each of the three columns is non-decreasing down the table through all 34 rows. Across $\hat{\mu} = 10^{-4}$: σ^* is non-decreasing in $\hat{\mu}$, and $\sigma^*(\cdot;\hat{\mu}) \le \sigma_{\rm low}(\cdot;10^{-4})$ for $\hat{\mu} < 10^{-4}$ because $I^* \ge I_\infty(10^{-4})$, $B_3^* \ge \bar{B}_3(10^{-4}) = 13.186961$ and $P^* + Q^*/\hat{\mu} > 7716.84 > \bar{B}_4^-(10^{-4}) = 7716.1679$. At a grid point both rows containing it give valid bounds, and the convention selects the row with $\mu_{\rm lo} = \hat{\mu}$. Hence $\hat{\mu} \mapsto \sigma_{\rm low}(\hat{\lambda}, \hat{e}^2; \hat{\mu})$ is non-decreasing and $R(\mu_0) \supseteq R(\mu_0')$ for $\mu_0 \ge \mu_0'$. *The listed boxes:* $c \ge 0.2063$ on the row $[0.5, 0.55]$, $c \ge 0.1525$ on $[0.1, 0.15]$, $c \ge 0.1132$ and $\sigma_{\rm low}(0.1, 0.1) \ge 0.0582$ on $[0.03, 0.04]$, $c \ge 0.0753$ on $[0.01, 0.015]$ (Table 2), together with the monotonicity; at $\hat{\mu} = 1$ the row $[0.99, 1]$ gives $c \ge 0.2390$ and $e^2_{\max} \ge 0.5618$.

(iv). σ^* is concave in each variable separately, so again the corners decide. Let $(\hat{\lambda}, \hat{e}^2)$ lie in the box $\{\hat{\lambda} \le \lambda_1,\ \hat{e}^2 \le c_1\sqrt{\hat{\mu}}\}$ with $\hat{\mu} \le 1$. At a corner $(\hat{\lambda}, c_1\sqrt{\hat{\mu}})$, $\hat{\lambda} \in \{0, \lambda_1\}$,

$$\begin{aligned}\sigma^* &= A - AI^* c_1\sqrt{\hat{\mu}} + b\hat{\lambda} - L\hat{\lambda}^2 - \tfrac{1}{2}B_3^* c_1\sqrt{\hat{\mu}}\,\hat{\lambda} - \tfrac{1}{2}\big(P^*\hat{\mu} + Q^*\big)c_1^2 \\ &\ge\ A - AI^* c_1 + b\hat{\lambda} - L\hat{\lambda}^2 - \tfrac{1}{2}B_3^* c_1\hat{\lambda} - \tfrac{1}{2}(P^* + Q^*)c_1^2 = \sigma^*(\hat{\lambda}, c_1; 1),\end{aligned}$$

using $\sqrt{\hat{\mu}} \le 1$ and $\hat{\mu} \le 1$ in the negative terms; the corners with $e = 0$ give $A > 0$ and $A + b\lambda_1 - L\lambda_1^2 > 0$ ($\lambda_1 \le 1 < \lambda_c$). So it suffices that $\sigma^*(0, c_1; 1) \ge 0$ and $\sigma^*(\lambda_1, c_1; 1) \ge 0$ for each listed pair, which is how the c_1 were determined (bisection on c_1, rounded down); explicitly, $\sigma^*(0, c_1; 1) = A(1 - I^* c_1) - \frac{1}{2}(P^* + Q^*)c_1^2 = 3.8 \cdot 10^{-5}$ at $c_1 = 0.3014$, and $\sigma^*(\lambda_1, c_1; 1)$ equals $1.2 \cdot 10^{-4}$, $4.1 \cdot 10^{-5}$, $1.1 \cdot 10^{-4}$, $2.3 \cdot 10^{-4}$, $6.3 \cdot 10^{-4}$ at the five pairs with $\lambda_1 > 0$, each positive.

(v). (85), $\hat{\lambda}S_{01,k} \ge 0$ (Lemma 29.9(i)) and (i).

(vi). At $e = 0$ only the orders $(0,1)$ and $(0,2)$ of (84) are present, so by Lemma 29.9(iii), Lemma 29.11(iii) (for an arbitrary $M_1 \in (0, m)$, and now in general units) $\lim_k s_k^\infty(M_1) = g_\lambda(M_1) := (m - M_1)\big[\frac{m+M_1}{2} + \frac{(N+2)\lambda}{2\pi}\big] - \frac{(N+2)\lambda^2}{\pi^2}\ln\frac{m}{M_1}$ exactly, and $\sigma_1(M_1) = g_\lambda(M_1)$. As a function of $M_1 \in (0, m]$: $g_\lambda(m) = 0$, $g_\lambda'(M_1) = -M_1 - \frac{(N+2)\lambda}{2\pi} + \frac{(N+2)\lambda^2}{\pi^2 M_1}$, $g_\lambda''(M_1) = -1 - \frac{(N+2)\lambda^2}{\pi^2 M_1^2} <$

0, so g_λ is strictly concave. If $g'_\lambda(m) \geq 0$, concavity gives $g_\lambda(M_1) < g_\lambda(m) + g'_\lambda(m)(M_1 - m) \leq 0$ for every $M_1 < m$, and (T3) fails; if $g'_\lambda(m) < 0$, then $g_\lambda > 0$ on a left neighbourhood of m, and (T3) holds. Now $g'_\lambda(m) < 0$ if and only if $x = \lambda/m$ lies below the positive root of $\frac{N+2}{\pi^2}x^2 - \frac{N+2}{2\pi}x - 1$, which is $x_* = \frac{\pi}{4}\big(1 + \sqrt{1 + \frac{16}{N+2}}\big)$, equal to $\frac{\pi}{4}(1+\sqrt{5}) = 2.5416\ldots$ for $N = 2$. At a fixed M_1, $g_\lambda(M_1) > 0$ if and only if $L\lambda^2 - b\lambda - A < 0$ with A, b, L at M_1, i.e. $\lambda < \lambda_c(M_1)m := \frac{b+\sqrt{b^2+4AL}}{2L}m$; at $M_1 = 0.7m$, $\lambda_c = 2.14398\ldots$.

(vii). By (CT_∞) (Lemma 29.1) the relative limit $\alpha_\infty(M) - \alpha_\infty(m)$ exists for every $M > 0$ and $\sigma_1(M_1) = \lim_k s_k^\infty(M_1) = \frac{1}{2}(\alpha_\infty(m) - \alpha_\infty(M_1))$. On each region of (iii), (iv), (vi), $\sigma_1(0.7m) > 0$ (resp. $\sigma_1(M_1) > 0$), so $\alpha_\infty(0.7m) < \alpha_\infty(m)$: m is not a minimiser of α_∞ on $(0, m]$, which is ($\mathrm{NE}^<$), and ($\mathrm{NE}^<$) implies (NE) (Lemma 29.4(i)). Proposition 12.13 is needed only on $E_W^< \setminus E_W$, which these regions avoid. □

The exact limit coefficients of Lemma 29.1 give a second region, uniform in the photon mass. Throughout Proposition 29.30 and its proof, $N = 2$ and $M_1 = 0.7m$; $\hat\lambda = \lambda/m$, $\hat e^2 = e^2/m$ and $\hat\mu = \mu_0/m$ as before, the photon mass $\mu_0 > 0$ is arbitrary unless restricted, $A = 0.255$, b, L, I_∞, I^* are the numbers defined before Table 2 (units $m = 1$), and $\lambda_c = 2.14398\ldots$ is as in Proposition 29.29(vi).

Proposition 29.30 (a coupling region uniform in the photon mass)**.** *Put* $\bar c := \frac{\ln(10/7)}{8\pi^2} = 0.0045173\ldots$.

(i) (The exact coefficients.) *For every* $\mu_0 > 0$ *and every* (α, β) *in* (84) *the limit* $S_{\alpha\beta,\infty} := \lim_k S_{\alpha\beta,k}$ *exists and equals* $\frac{1}{2}\big[F_{\alpha\beta}(m) - F_{\alpha\beta}(M_1)\big]$ *with the closed forms of Lemma 29.1; in units* $m = 1$, $S_{01,\infty} = b$, $S_{02,\infty} = -L$, $S_{20,\infty} = -AI_\infty(\hat\mu)$,

$$S_{21,\infty}(\hat\mu) = \frac{1}{\pi^2}\ln\frac{2+\hat\mu}{1.4+\hat\mu} - \frac{\hat\mu}{4\pi^2}\Big(\frac{1}{0.7+\hat\mu} - \frac{1}{1+\hat\mu}\Big),$$

$$S_{40,\infty}(\hat\mu) = \frac{\ln 0.7}{8\pi^2} + \frac{1}{2}\big[\Phi_{40}(\hat\mu) - \Phi_{40}(\hat\mu/0.7)\big]$$

with Φ_{40} *of* (77)*, and*

$$\sigma_1(0.7m) = m^2\big[A + b\hat\lambda - L\hat\lambda^2 - AI_\infty(\hat\mu)\hat e^2 + S_{21,\infty}(\hat\mu)\hat\lambda\hat e^2 + S_{40,\infty}(\hat\mu)\hat e^4\big].$$

(ii) (Floors uniform in the photon mass.) *For every* $\mu_0 > 0$*:* $-AI_\infty(\hat\mu) > -AI^*$, $S_{21,\infty}(\hat\mu) > 0$ *and* $S_{40,\infty}(\hat\mu) > -\bar c$*; moreover* $S_{21,\infty}(\hat\mu) \geq 0.020355$ *for* $\mu_0 \in (0, m]$.

(iii) (The uniform region.) *Put* $\sigma_{\mathrm{unif}}(\hat\lambda, \hat e^2) := A + b\hat\lambda - L\hat\lambda^2 - AI^*\hat e^2 - \bar c\,\hat e^4$. *Then* $\sigma_1(0.7m) \geq m^2\sigma_{\mathrm{unif}}(\hat\lambda, \hat e^2)$ *for every* $\mu_0 > 0$*, so* (T3) *holds at every photon mass on* $\{\sigma_{\mathrm{unif}} > 0\}$*, which contains the segment* $\{\lambda = 0,\ e^2 < 6.3073\,m\}$*, the box* $[0, 2.0397\,m]^2$ *and the segment* $\{e = 0,\ \lambda < \lambda_c m\}$*, and at* $\hat e^2 = 0.5,\ 1,\ 2,\ 4$ *the windows* $\hat\lambda < 2.1273,\ 2.1051,\ 2.0427,\ 1.8311$.

(iv) (The product box.) *For every* $\mu_0 \in (0, m]$, (T3) *holds, with* $\sigma_1(0.7m) > 0$*, at every* $(\lambda, e^2) \in (0, \lambda_c m) \times (0, 6.30\,m)$.

Proof of Proposition 29.30. (i). By Lemma 29.6, $s_k^\infty(0.7m)$ is a polynomial of degree ≤ 2 in (e^2, λ) with the coefficients $S_{\alpha\beta,k}$ of (85); by Lemma 29.1 it converges at every $e, \lambda > 0$ to $\frac{1}{2}\big[\alpha_\infty(m) - \alpha_\infty(0.7m)\big]$, a polynomial of the same degree in (e^2, λ) whose coefficient of $e^\alpha\lambda^\beta$ is $-\frac{1}{2}D_{\alpha\beta} = \frac{1}{2}\big[F_{\alpha\beta}(m) - F_{\alpha\beta}(0.7m)\big]$ by (75). Polynomials of bounded degree converging on an open set converge coefficient by coefficient, whence the formula for $S_{\alpha\beta,\infty}$; at the orders $(0,1)$, $(2,0)$, $(0,2)$ it agrees with Lemmas 29.9 to 29.11(iii), and at $(2,1)$, $(4,0)$ the closed forms are read off (76)–(77). The formula for $\sigma_1(0.7m)$ is the limit of (85), so that here $\sigma_1(0.7m) = \lim_k s_k^\infty(0.7m)$, not only the lower limit.

(ii). I_∞ is strictly decreasing with $I_\infty \uparrow I^*$ as $\mu_0 \downarrow 0$ (Lemma 29.10(iv)). $S_{21,\infty} > 0$ is $y_{21}^\infty > 0$ in (78): F_{21} is increasing in M. On $\hat\mu \in (0,1]$ the two bounds $\ln\frac{2+\hat\mu}{1.4+\hat\mu} \geq \ln\frac{3}{2.4}$ and $\frac{0.3\,\hat\mu}{(0.7+\hat\mu)(1+\hat\mu)} \leq \frac{0.3}{(1+\sqrt{0.7})^2} < 0.08894$ (the maximum over $\hat\mu > 0$ is at $\hat\mu = \sqrt{0.7}$) give $S_{21,\infty} \geq \frac{1}{\pi^2}\big(\ln 1.25 - \frac{1}{4}\cdot 0.08894\big) > 0.020355$. $S_{40,\infty} > -\bar c$ because Φ_{40} is strictly decreasing on $[0,\infty)$ (Lemma 6.12) and $\hat\mu/0.7 > \hat\mu$; the infimum $-\bar c = \frac{\ln 0.7}{8\pi^2}$ is the limit $\hat\mu \to 0$, by continuity of Φ_{40} at 0, and is not attained.

(iii). In the formula of (i) drop $S_{21,\infty}\hat\lambda\hat e^2 \geq 0$ and insert the floors of (ii). σ_{unif} is concave in $\hat\lambda$ and in $\hat e^2$ separately, so on a box its minimum sits at a corner, as in the proof of Proposition 29.29(iii): $\sigma_{\text{unif}}(0,x) = 0.255 - 0.0119366\,x - \bar c\,x^2$ has the positive root 6.3073; $\sigma_{\text{unif}}(2.0397, 2.0397) = 1.0\cdot 10^{-5} > 0$ and the other three corners of the box are larger; the $e = 0$ segment is Proposition 29.29(vi); the windows are the positive roots in $\hat\lambda$.

(iv). Keep the floor of $S_{21,\infty}$: $h(\hat\lambda,\hat e^2) := A + b\hat\lambda - L\hat\lambda^2 - AI^*\hat e^2 + 0.020355\,\hat\lambda\hat e^2 - \bar c\,\hat e^4 \leq \sigma_1(0.7m)/m^2$ for $\mu_0 \in (0,m]$. h is strictly concave in $\hat\lambda$, $h(0,x) = \sigma_{\text{unif}}(0,x) > 0$ for $0 \leq x < 6.3073$ by (iii), $h(\lambda_c,0) = 0$ by the definition of λ_c, and $h(\lambda_c,x) = x\,(0.020355\,\lambda_c - AI^*) - \bar c\,x^2 = x\,(0.03170 - 0.0045173\,x) > 0$ for $0 < x < 7.01$; so on each segment $\{\hat e^2 = x\}$, $0 < x < 6.30$, h is positive at $\hat\lambda = 0$ and at $\hat\lambda = \lambda_c$ and strictly concave between, hence positive on $(0,\lambda_c)$. At $(0,6.30)$ and $(\lambda_c,6.30)$ its values are $5.1\cdot 10^{-4}$ and $2.0\cdot 10^{-2}$. □

The two-sided clause (NE) of Proposition 29.29(vii) is not claimed on the regions of Proposition 29.30, whose statement concerns the limit σ_1 alone. The proposition concerns the coupling region alone: the constants of the Wick leg of Theorem 28.2, among them C_U and κ of the extensive bound and the spacing threshold k_0 of Theorem 6.25, are not shown uniform in the photon mass (Remark 28.9).

Remark 29.31 (the quantifier over the couplings). The theorems of this paper carry the sources' quantifier of Lemma 29.3(e), in the units of the coarsest spacing, with the cube family C_J not rescaled and $\beta = 5^{-J_{\text{K}}}$. A sentence "$e,\lambda$ sufficiently small, $m,\mu_0 > 0$ fixed" describes *where* the smallness lives (the running couplings, through Bałaban's ε_0), but as a hypothesis on the bare couplings it is strictly stronger than what the sources need. (i) All four couplings have mass dimension one, so its dimensionless content is a bound on e^2/m and λ/m, while the sources' hypothesis on the bare couplings is empty beyond positivity: the point $(e^2,\lambda,m,\mu_0) = (100,100,1,1)$ is admitted by every source (with a large J_{K}) and excluded by the phrase in every system of units. (ii) In the units in which the stopping scale is 1, m and μ_0 are not fixed but rescaled to $m\beta$ and $\mu_0\beta$, of which $m\beta$ is capped (by the mass cap of [2, Prop. 3.1] and by a further cap of the schedule) and $\mu_0\beta$ is not: in those units the theorem reads "three couplings small, the fourth rescaled". (iii) Even at a fixed stopping scale β_0 the admissible set is not a product neighbourhood of $(0,0)$ in (e,λ): one cap of the schedule of constants of the rotation part has the form $e \leq \lambda^{1/4}c(\beta_0)$, a cusp that fails as $\lambda \to 0$ at fixed e. So the phrase names no set exactly, and the sources' form survives because β can be taken to 0.

The photon mass μ_0 has been fixed throughout. The next section lets it go to zero in the curvature sector, whose constants are absolute (Remark 7.5) and which carries no coupling condition (Lemma 29.3(a)); the Wick sector, whose coupling region contains a box uniform in the photon mass (Proposition 29.30) but whose constants (Remark 28.9) are not uniform in it, stays at the fixed photon mass, and its limit is among the questions of Section 32.

30. The limit of vanishing photon mass in the curvature sector

Every statement of the preceding sections is made at the fixed photon mass $\mu_0 > 0$, the infrared regulator of King's construction [25, 26], which keeps it fixed (Section 1). This section lets the photon mass go to zero in the curvature sector, where the constants are absolute (Remark 7.5, Theorem 28.2 A).

The limit is taken along King's own states at the photon masses $\mu_n \downarrow 0$: every such sequence has a subsequence along which they converge to a state with full Euclidean invariance, reflection positivity about every plane and the Euclidean cluster clauses (Theorem 30.2). Along King's own states the bare coefficient depends on the photon mass, and the limit is extracted by compactness. It is taken along King's states at the fixed scalar mass m; whether it exists as a full limit is the question of Open Problem 31.11.

Not proved here: anything about the Wick sector, whose constants are not uniform in the photon mass (Remark 28.9), and the convergence of the measures. Masslessness proper, that is a lower bound showing that the photon of the limit theory is massless, and the non-degeneracy of the limit are outside the scope of this paper. A second limit, along a family of states with fixed counterterm, and the conditions under which it would be unique and Euclidean invariant are taken up as further directions in Section 31, where the open problems of both sections are posed (Section 31.5); nothing there is used in the proof of Theorem 30.2. Beyond Theorem 30.2 and the statements of Section 31 the limit of vanishing photon mass remains among the open problems of Section 32.

Throughout this section and the next $e > 0$ and $\lambda, m, \mu_0 > 0$ are the couplings of Section 2.1, with no further condition (the curvature sector carries none, Lemma 29.3(a), Remark 28.9), the block factor is 5 and M is King's large-block integer. Nothing below depends on M (Remark 2.4). No scalar source occurs, so no scheme enters (Definition 2.5). Families are finite families $f = (f_1, \dots, f_N)$ of real 2-forms in $C_c^\infty(\mathbb{R}^3; \Lambda^2)$, with $J_0(f)$ as in Theorem A and $g(z) = \sum_j (\operatorname{Re} z_j) f_j$ as in Definition 2.8. The word *family* alone means such a family of test forms (the normal families of entire functions in two proofs excepted). The fixed-counterterm family of Definition 31.1 and the comparison family of Definition 2.2 are always named in full.

Definition 30.1 (King's state at a point)**.** For a scalar mass $\mathsf{M} > 0$ and a photon mass $\mu > 0$ let $\delta m_k^2(\mathsf{M}; \mu)$ be the counterterm (84) at the scalar mass M computed with the photon propagator $\hat{D}(p) = (\hat{p}^2 + \mu^2)^{-1}$ at the mass μ in place of μ_0, and $\alpha_k^\infty(\mathsf{M}; \mu) := \mathsf{M}^2 + \delta m_k^2(\mathsf{M}; \mu)$, so that $\alpha_k^\infty(\mathsf{M}) = \alpha_k^\infty(\mathsf{M}; \mu_0)$. *King's state at the point* $(e, \lambda, \mathsf{M}, \mu)$ is the state of Theorem 28.2 A for the couplings $(e, \lambda, \mathsf{M}, \mu)$, that is the model of Section 2 with the photon at the mass μ and the bare coefficient $\alpha_k^\infty(\mathsf{M}; \mu)$ on King's convention, written $Z_\infty^K(\,\cdot\,; \mathsf{M}, \mu)$; it exists at every point of King's range, $e > 0$, $\lambda, \mathsf{M}, \mu > 0$, by Theorem A (Theorem 5.47), and $Z_\infty^K = Z_\infty^K(\,\cdot\,; m, \mu_0)$.

In Theorem 30.2 a limit is written $Z^{(0)}$; its Schwinger functions $S_n^{(0)}$, which replace the S_n^K in the clauses of Theorem 28.2 B, are its derivatives at $z = 0$, as in Section 14.

Theorem 30.2 (subsequential limits of King's states as the photon mass vanishes)**.** *Let $\mu_n \downarrow 0$. There is a subsequence $(\mu_{n_l})_l$ such that for every family f King's states $Z_\infty^K(\,\cdot\,; m, \mu_{n_l})$ at the points $(e, \lambda, m, \mu_{n_l})$ converge locally uniformly on $\mathbb{C}^N$ to an entire $Z^{(0)}(\sum_j z_j f_j, 0)$, and $Z^{(0)}$ satisfies clauses (i)–(vi) of Theorem 28.2 B with the first* OS0 *bound, the lattice sentences excepted: not inherited are the sentence of B(i) on the lattice functionals (the two bounds at every k and J) and the second sentence of B(v) (the laws of $\chi\Phi_{k,J}^K$ under the cube states converging along the full sequence k, then J), there being no lattice states at $\mu = 0$; every other clause holds in full, in particular $Z^{(0)}(\sum_j z_j \gamma \cdot f_j, 0) = Z^{(0)}(\sum_j z_j f_j, 0)$ for every Euclidean motion γ and all complex z, reflection positivity about every plane $\{x \cdot n = c\}$, and (vi) with its Euclidean sentence: ν-almost*

every component is $\mathbb{R}^3 \rtimes O(3)$-invariant, reflection positive about every plane, clusters as $|a| \to \infty$ in any manner and has a unique vacuum in every direction. Whether the limit depends on the subsequence is not decided (Open Problem 31.11).

Proof. The members. Each $Z^K_\infty(\,\cdot\,; m, \mu_n)$ satisfies Theorem 28.2 A and B(i)–(vi): Theorem A holds at every point of King's range (Theorem 5.47), and so does the rotation theorem (Theorem 15.1, at every point of King's range; Theorem 8.8).

Extraction. Let $\mathcal{P}$ be the countable core of Step 1 of the proof of Lemma 14.7: finite families from the countable $\|\cdot\|_\infty$-dense sets $S_0(B)$, B a finite union of rational boxes. For each core family the sequence $(Z^K_\infty(\,\cdot\,; m, \mu_n))_n$ is locally bounded on $\mathbb{C}^N$ by $e^{\|g(z)\|_2^2}$, which carries no dependence on the photon mass (Theorem 28.2 A), hence is a normal family; enumerate $\mathcal{P}$, extract successively and take the diagonal subsequence (μ_{n_l}). For an arbitrary family the Lipschitz bound $|Z^K_\infty(\mathrm{i}u; m, \mu) - Z^K_\infty(\mathrm{i}u'; m, \mu)| \le \sqrt{2}\|u - u'\|_2$ of Corollary 8.4, valid at every point with the absolute constant $\sqrt{2}$ (it is the second-order bound $\langle F(u)^2\rangle \le 2\|u\|^2_{2,k}$ of Lemma 7.1 and Theorem 7.6(a), passed to the limit), gives convergence at every imaginary point by the density argument of Step 3 of the proof of Lemma 14.7, and then Montel–Vitali on $\mathbb{C}^N$ as in its Step 4: $Z^K_\infty(\,\cdot\,; m, \mu_{n_l}) \to Z^{(0)}$ locally uniformly, $Z^{(0)}$ entire.

The axioms. The clause numbers below are those of Theorem 28.2 B, and from here on a *member* is one of King's states $Z^K_\infty(\,\cdot\,; m, \mu_{n_l})$ of the subsequence. Each member depends on the family only through the total source (Theorem 28.2 A), hence so does the pointwise limit $Z^{(0)}$; in particular $Z^{(0)}(\mathrm{i}u, 0)$ is a function of the real 2-form u, and the $S^{(0)}_n$ are symmetric and multilinear over the reals. The Schwinger functions of the members, derivatives at 0 of locally uniformly convergent entire functions, converge to the $S^{(0)}_n$ by Weierstrass' theorem. Below, n in $S^{(0)}_n$ is the order, n in $\{x \cdot n = c\}$, H_n and Ω_n is a unit vector, and the photon masses of the subsequence are always written μ_{n_l}.

(i) The bound $e^{\|g(z)\|_2^2}$ of the members, the first OS0 bound, passes to the limit. The Schwinger bounds, temperedness and E0″ are Cauchy's estimates and the kernel theorem applied to the limit function with its bound, the argument of Theorem 7.6(c) run on $Z^{(0)}$ as Theorem 14.9 runs it on the LIM states (replacement (γ)); the constants are those of B(i). The second OS0 bound of a member carries the factor $\mu_{n_l}^{-2}$ and degenerates along the subsequence; it is not claimed.

(ii) Each member is $\mathbb{R}^3 \rtimes O(3)$-invariant at all complex coefficients (Theorem 28.2 B(ii), Theorem 8.8), the transformed family is a family, and pointwise limits of invariant functionals are invariant. The translations and the cubic group O_h about every point are contained in $\mathbb{R}^3 \rtimes O(3)$. Differentiation at $z = 0$ gives the invariance of the $S^{(0)}_n$ under translations and O_h about every point and their Euclidean covariance.

(iii) Fix a plane $\{x \cdot n = c\}$, n a unit vector, with its reflection θ, and sources $\zeta_1, \dots, \zeta_N$ supported in $\{x \cdot n > c\}$ with complex coefficients. For every member the matrix $(Z^K_\infty(\theta\zeta_i^* \sqcup \zeta_j; m, \mu_{n_l}))_{ij}$ is positive semidefinite (B(iii) for the member, about every plane); its entries converge along the subsequence, and the positive semidefinite matrices form a closed set, so the limit matrix is positive semidefinite: as the remark after Lemma 9.10 says, a limit of vanishing photon mass inherits reflection positivity, if it exists, as a pointwise limit of positive semidefinite matrices. In the polynomial form $S(\theta(P)P)$ is a finite linear combination of Schwinger functions, each converging, and $[0, \infty)$ is closed.

(iv) Realness at real points, symmetry (the Schwinger functions are Taylor coefficients of a function of the total source) and the evenness of the members in z, which is the vanishing of their Schwinger functions of odd order (B(iv)), pass to the limit.

(v) The characteristic functional $\mathcal{Z}^{(0)}(u) := Z^{(0)}(\mathrm{i}u, 0)$, $u \in C^\infty_c(\mathbb{R}^3; \Lambda^2)$ real, is normalised, $\mathcal{Z}^{(0)}(0) = 1$; positive definite, because for every member $Z^K_\infty(\mathrm{i}\,\cdot\,; m, \mu_{n_l})$ is the characteristic

functional of a probability measure (B(v) for the member), so that the matrices $(Z^K_\infty(\mathrm{i}(u_b - u_a); m, \mu_{n_l}))_{a,b\le p}$, $u_1, \dots, u_p \in C^\infty_c$ real, are positive semidefinite, and limits of positive semidefinite matrices are positive semidefinite; and Lipschitz for $\sqrt{2}\|\cdot\|_2$, because $|Z^K_\infty(\mathrm{i}u; m, \mu_{n_l}) - Z^K_\infty(\mathrm{i}u'; m, \mu_{n_l})| \le \sqrt{2}\|u - u'\|_2$ for every member (Corollary 8.4, as in the extraction) and this bound is closed under pointwise limits; hence $\mathcal{Z}^{(0)}$ extends uniquely to a continuous, normalised, positive definite functional on the real Schwartz forms $\mathcal{S}(\mathbb{R}^3; \Lambda^2)$. With the moment bounds of (i) these are exactly what the construction of the measure from a limit functional uses (Steps 1 and 2 of the proof of Theorem 11.5, as applied to the LIM states in the proof of Theorem 14.9(v)): the Bochner–Minlos theorem on $\mathcal{S}$, the placement on $\mathcal{H}^{-r}$ by the moment bounds, the identification of the moments and the uniqueness; the image on H^{-r}_{loc} likewise. The moments are identified from the limit object alone: for real $u \in C^\infty_c$ the map $t \mapsto \mathcal{Z}^{(0)}(tu) = Z^{(0)}(\mathrm{i}tu, 0)$ extends to an entire function of t, so by Lukacs' theorem [28] (a characteristic function analytic near 0 has moments of every order, equal to its derivatives at 0) $\int \Phi(u)^n \, d\mu^{(0)}_F = (-\mathrm{i})^n \partial^n_t Z^{(0)}(\mathrm{i}tu, 0)|_{t=0} = S^{(0)}_n(u, \dots, u)$, where $\mu^{(0)}_F$ denotes the measure so obtained (the analogue of $\mu^{K,H}$; it is carried by $\mathcal{H}^{-r}$, as shown next); polarisation, $S^{(0)}_n$ being symmetric, gives the mixed moments on C^∞_c; the Schwinger bounds of (i) then place the measure on the Hermite–Sobolev carrier $\mathcal{H}^{-r}$, $r \ge 14$ even, and give the uniqueness by the moments, on $\mathcal{H}^{-r}$ and for the image on H^{-r}_{loc}; the uniqueness of the image by its characteristic functional needs no bound (Theorem 11.5(a); [32, Lemma A.8], [32, Proposition A.9], [32, Lemma A.5]). No lattice statement is used.

(vi) The image measure on $X = H^{-r}_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^2)$, with the translations and the coordinate reflections of (ii) and the reflection positivity of (iii) about the coordinate planes extended to L^2 as in Steps 3 and 4 of the proof of Theorem 11.5, is a reflection-positive stationary system in the sense of Definition 11.4: (H1) holds because by (ii) $\mathcal{Z}^{(0)}$ is invariant under τ_a and θ_j, both being Euclidean motions, and hence so is the image measure, a Borel probability measure on X being determined by its characteristic functional on the real forms in C^∞_c ([32, Proposition A.9], the uniqueness of (v) for the image; for θ_j this is Lemma 11.11(ii)); (H2) concerns the local σ-algebras of X, a property of the space and not of the measure, as is the space part of (H1); (H3) is (iii) about the coordinate planes. The general results of [32, Appendix A] then give every clause of Theorem 11.5(b) about μ^K_S, with the image measure in its place, as the proof of Theorem 14.9(vi) applies them to the LIM states: the decomposition $\int \mu_\xi \, d\nu(\xi)$ into translation-ergodic, θ_j-invariant, reflection-positive components, clustering in the mean and under diagonal separation, the unique joint vacuum in each coordinate direction, weak OS4, the growth bound for the components' moments (from the moment bounds of (i) for the measure of (v)) and the O_h-invariant decomposition (from the O_h-invariance of the image measure about the origin, which is (ii) with Lemma 11.11(ii), and about every point by (H1)). $\mathbb{R}^3 \rtimes O_h$-invariance of the image measure follows from (ii) by the uniqueness used for (H1). The Euclidean sentence: by Theorem 11.5(c) its clauses use only the $O(3)$-invariance of the characteristic functional, which the limit has by (ii). Explicitly, $\mathcal{Z}^{(0)}$ is $O(3)$-invariant (the members are, by B(ii), and the invariance passes to the limit), so the image measure on X is $O(3)$-invariant by Lemma 11.11(ii); with (H1)–(H3) as above and the admissible action of Lemma 11.11(i), Corollary 11.16 gives, for ν-almost every component, $\mathbb{R}^3 \rtimes O(3)$-invariance, reflection positivity about every plane, clustering in any manner, the forms (E_n), (C_n) and (OS_n) of OS4 (Definition 11.15) and $\ker H_n = \mathbb{C}\Omega_n$ for every unit vector n; the moments of ν-almost every component exist by the growth bound and are real and symmetric, being integrals of products of the real random variables $\Phi(u_i)$. This is the Euclidean sentence of B(vi); the vanishing of a component's odd moments is not claimed.

The lattice sentences of B(i) and B(v) refer to lattice states at the photon mass of the member and have no counterpart at $\mu = 0$; they are not claimed. □

31. Further directions: the limit along the family with fixed counterterm

Nothing in this section is used in the proof of Theorem 30.2 or in the preceding sections. The section takes the limit of vanishing photon mass along a second family of states. For the limit states of a common continuum subsequence the limit exists along a countable set of photon masses, as a full limit along that set, with invariance under translations and the lattice point group and reflection positivity about the coordinate planes. Its identification with the limit of King's states along a curve of shifted scalar masses, which would make it unique and Euclidean invariant, is proved under two conditions that are left open.

The limit is along the *family with fixed counterterm*: the cube states of Definitions 2.2 and 2.3 with the photon at a mass $\mu \in (0, \mu_0]$ and the bare scalar coefficient held at its value at the reference photon mass μ_0 (the cube states at one such mass μ, on both conventions, form the *fixed-counterterm member* at μ, Definition 31.1). Along it the limit states of any common continuum subsequence are monotone in μ by the correlation inequalities and converge as $\mu \downarrow 0$ for every family of curvature sources (Theorem 31.4). The limit satisfies the regularity and symmetry clauses, invariance under translations and the lattice point group and reflection positivity about the coordinate planes, and carries the measure with its ergodic decomposition and the weak cluster clauses, with the same absolute constants (Theorem 31.4; the second OS0 bound and the Euclidean clause are not claimed for it, Remark 31.5).

The coefficient is held fixed so that two members of the family differ only in the photon mass term, and the correlation inequalities then order their states in the photon mass (Lemma 2.10(a), and (d) at real coefficients; Theorem 31.4(iii)). Along King's own states the bare coefficient depends on the photon mass, and the limit of Theorem 30.2 is extracted by compactness instead.

A further result is conditional: under two conditions on the counterterm and on King's state in the scalar mass, the limit state of the fixed-counterterm member at the mass μ is King's state at the same photon mass and a shifted scalar mass (Proposition 31.6). The identification would make the limit along the fixed-counterterm family independent of the subsequence and of the set of photon masses, and equal to the limit of King's own states along a shifted curve of scalar masses. Under the identification the limit along the fixed-counterterm family runs along the shifted curve $\mu \mapsto \mathsf{M}(\mu)$ of scalar masses and the limit of Theorem 30.2 along the fixed scalar mass m; whether the two limits coincide is not decided here.

The conditions, and what else is left open, are posed as open problems in Section 31.5. Not proved here: anything about the Wick sector; the identification itself; and the convergence of the measures. The limit along the fixed-counterterm family is taken along a countable set of photon masses and may depend on the subsequence and on the set; the identification would make it unique (Proposition 31.6). The conventions of Section 30 are in force.

31.1. **The family with fixed counterterm.**

Definition 31.1 (the reference photon mass and the fixed-counterterm members)**.** The photon mass μ_0 of the sections preceding Section 30 is the *reference photon mass*. For $\mu \in (0, \mu_0]$ the *fixed-counterterm member at the photon mass* μ consists of the Neumann–Dirichlet cube states on the cubes C_J at the spacings $\varepsilon_k = 5^{-k}$, on the comparison convention of Definition 2.2 and on King's convention of Definition 2.3, written $\langle\cdot\rangle_{k,J}^{(a),\mu}$ and $\langle\cdot\rangle_{k,J}^{K,\mu}$, in which the photon is in Feynman form at the mass μ, with the mass term $\frac{\mu^2}{2}\sum_{x\in X_J}\sum_\alpha \varepsilon^3 A_\alpha(x)^2$ on X_J and the Neumann condition, and the single-site weight at every alive site is $\exp(-\varepsilon^3[\frac{1}{2}b_k r^2 + \lambda r^4])$ with the bare scalar coefficient *held at the reference photon mass*,

$$b_k := \alpha_k^\infty(m) = m^2 + \delta m_k^2(m),$$

King's free-boundary counterterm (84) computed with the photon propagator at the mass μ_0, as everywhere before Section 30. The member at $\mu = \mu_0$ is the pair of cube states of the sections preceding Section 30. For a family f, $J \geq J_0(f)$ and $z \in \mathbb{C}^N$ put

$$a^{(a),\mu}_{k,J}(z) := \big\langle e^{\sum_j z_j F_{\Lambda_k}(f_j)} \big\rangle^{(a),\mu}_{k,J}, \qquad a^{K,\mu}_{k,J}(z) := \big\langle e^{\sum_j z_j F_{\Lambda_k}(f_j)} \big\rangle^{K,\mu}_{k,J},$$

so that $a^{(a),\mu_0}_{k,J} = a^{(a)}_{k,J}$ and $a^{K,\mu_0}_{k,J} = a^{K}_{k,J}$ in the notation of Corollary 4.4.

Two members differ only in the photon mass term. The single-site measures, the hopping terms and the Dirichlet and Neumann conditions are the same. Every fixed-counterterm member is a BFS measure in the sense of Section 2.2: $c_{xy} = \varepsilon$ on the bonds present, single-site measures $e^{-\varepsilon^3[\frac{1}{2}b_k r^2 + \lambda r^4]}$ times the diagonal bond parts, with finite exponential moments because $\lambda > 0$, and the Neumann photon Gaussian at the mass μ, non-degenerate because $\mu > 0$. It is also a lattice model of King's type in the sense of the clause "several models at once" of Lemma 14.7(a): a Neumann–Dirichlet cube state with a site-independent real bare scalar coefficient, photon mass $\mu > 0$ and $\lambda > 0$ (a model on either convention qualifies; this is wider than "of King's type" in the Terminology paragraph of Section 2, whose box-model sense is King's convention (it records the present wider sense too)). That clause was formulated for the fixed-counterterm family. One remark on the kill steps is needed. The killing lemma, Lemma 2.12, is stated for BFS measures *of King type* in the sense of Section 2.2, and it is the lemma used in the kill steps of Lemma 4.2. Whether a fixed-counterterm member, or a member of the class $\mathcal{M}$ of Lemma 31.3, is of King type is not used here. Those steps are available for every BFS measure all the same, in the form given after the proof of Lemma 4.2: since the observable does not depend on the fields at the killed sites, one sets $c_{xy} = 0$ on every bond at the set to be killed, one bond at a time (Lemma 2.10(b),(d)). The measure is then a product (Lemma 2.10(e)) whose factor over the killed set cancels. Wherever Lemma 4.2, Corollary 4.3 or Lemma 2.13 is applied to a fixed-counterterm member below, its kill steps are read in this form.

Definition 31.2 (continuum subsequences and limit states of a member)**.** For a fixed-counterterm member at the mass μ, continuum subsequences $\mathbf{k} = (k_m)$, the fixed-cube limits $Z^{\mathbf{k},\mu}_J(\sum_j z_j f_j, 0) := \lim_m a^{(a),\mu}_{k_m,J}(z)$ and the LIM state $Z^{\mathbf{k},\mu}_\infty(\cdot, 0) := \lim_J Z^{\mathbf{k},\mu}_J$ are defined by Definitions 14.1 and 14.2 with $a^{(a),\mu}_{k,J}$ in place of $a^{(a)}_{k,J}$; on King's convention, $Z^{K,\mathbf{k},\mu}_J$ and $Z^{K,\mathbf{k},\mu}_\infty$ are defined with $a^{K,\mu}_{k,J}$, as in Corollary 14.15. The Schwinger functions of a limit state are its derivatives at $z = 0$, as in Section 14. In the notation of Definition 30.1, $b_k = \alpha^\infty_k(m; \mu_0)$.

31.2. **One continuum subsequence for countably many models.** A member at $\mu < \mu_0$ is not King's model at the point (e, λ, m, μ): its bare coefficient $b_k = \alpha^\infty_k(m; \mu_0)$ is computed with the photon propagator at the reference photon mass, not at μ (Theorem 31.4(ii)). The existence of its fixed-cube limits along the full sequence of spacings is neither used nor claimed (see the end of Section 31.5). Its limit states are taken along continuum subsequences, as in Section 14. Members at different photon masses are compared along the same spacings (Theorem 31.4(iii)), so one subsequence must serve all of them. The diagonal extraction of Lemma 14.7 provides one for countably many members at once, and accordingly Theorem 31.4 takes the limit along a countable set of photon masses. Both conventions are carried: the LIM state $Z^{\mathbf{k},\mu}_\infty$ of a member is taken on the comparison convention (Definition 31.2), while King's states, with which Proposition 31.6 compares the members, are on King's convention. Clause (b) of Lemma 31.3 below lets the two be interchanged.

Lemma 31.3 (Section 14 for countably many models at once)**.** *Let $\mathcal{M}$ be a countable set of lattice models of King's type in the sense of Lemma 14.7(a): Neumann–Dirichlet cube states*

on the cubes C_J at the spacings ε_k, each with a site-independent real bare scalar coefficient, the photon in Feynman form at some mass $\mu > 0$, and $\lambda > 0$; for instance the fixed-counterterm members at the photon masses of a countable set.

(a) *There is one increasing sequence $\mathbf{k}$ of spacings which is a continuum subsequence for every member of $\mathcal{M}$ on both conventions, and along it every member satisfies the curvature-sector clauses of Lemma 14.8 on both conventions and clauses (i)–(vi) of Theorem 14.9 on the comparison convention, each at the member's own photon mass: in the second bound of clause (i) the member's photon mass replaces μ_0, and all other constants are the absolute constants of Remark 7.5.*

(b) *For every member and every family the LIM states of the two conventions coincide: $Z_\infty^{\mathbf{k},\mu}(\cdot,0) = Z_\infty^{K,\mathbf{k},\mu}(\cdot,0)$ on $\mathbb{C}^N$.*

Clause (vii) of Theorem 14.9, which rests on the rotation theorem Theorem C for the model in question, is not claimed for a member.

Proof. Clause (a) occupies the next three paragraphs (the common subsequence; the cube limit, that is the curvature-sector clauses of Lemma 14.8; the axioms, that is clauses (i)–(vi) of Theorem 14.9), clause (b) the last.

The common subsequence. This is the clause "several models at once" of Lemma 14.7(a), proved in Step 5 of its proof: the constant of Lemma 7.3, the bound $e^{\|g\|_{2,k}^2}$ and the Lipschitz constant $\sqrt{2}$ are model-free, because Theorem 7.4 is Lemma 2.11(a) for a measure of the class of Lemma 2.10 together with Lemma 7.1 for the Neumann–Dirichlet photon covariance, valid for every real site-independent bare coefficient and every photon mass $\mu > 0$; for countable $\mathcal{M}$ Step 2 of that proof is run over $\mathcal{M}\times\mathcal{P}$ and Steps 3–4 member by member. On King's convention the same two bounds hold with the same constants (step (2) of the proof of Corollary 4.4(ii)(α), which sees of the matter only that the state is a BFS measure; Corollary 14.15(a)), so Step 2 is run over $\mathcal{M}\times\{(a),K\}\times\mathcal{P}$ and the diagonal sequence $\mathbf{k}$ serves every member on both conventions.

The cube limit. The curvature-sector clauses of Lemma 14.8 for a member along $\mathbf{k}$: its proof uses the first inequality of Lemma 2.13, valid for every Neumann–Dirichlet box state, the cut-out factor being the Neumann–Dirichlet state on C_J because the bare coefficient is site-independent, together with the model-free bound and Montel's theorem. On King's convention the monotonicity in J is Corollary 4.3, from Lemma 4.2 with its kill steps read as explained after Definition 31.1.

The axioms. Clauses (i)–(vi) of Theorem 14.9 for a member along $\mathbf{k}$: the proof of that theorem uses, for (i), Lemmas 14.7 and 14.8 and Cauchy's estimates on the limit function from its bound (the argument of Theorem 7.6(c), replacement (γ)), the second bound coming from Theorem 7.4 at the model's own photon mass; for (ii), the exact lattice covariance under $5^{-k_0}\mathbb{Z}^3$ with Lemma 2.13, and the frame flips of Lemma 2.15 with Lemma 2.10(a),(b); for (iii), Lemmas 9.3, 9.9 and 9.10, which concern the photon form and the inequalities of Lemma 2.10 only, Lemma 9.10 with the margin $N\varepsilon \geq \pi\sqrt{3}/(2\mu)$ of the model's own photon mass, finite for every $\mu > 0$, and Theorem 7.4; for (iv), charge conjugation and closedness under limits; for (v) and (vi), the moment bounds of Theorem 7.4, the Gram positivity and the Lipschitz bound passed along $\mathbf{k}$ and J, the construction of the measure from the limit functional (replacement (γ)), and the abstract setting of Definition 11.4, (H1)–(H3), with the general results of [32, Appendix A] as in Theorem 11.5(b), applied with $\mu_F^{\mathbf{k}}$ in place of μ_S^K. Each of these statements is made for BFS measures or for either convention with no hypothesis beyond the definition of the cube state (Lemma 7.3; Theorem 7.4 for every k, J and every photon mass; Lemma 2.10 for every BFS measure; Lemma 9.10 for the photon Gaussian alone), and none refers to the value of the bare coefficient or to the counterterm, as Remark 14.14 records: the proofs of Section 14 use of the lattice functionals only the k-uniform bounds, the cone monotonicity at fixed k and the

convergence along $\mathbf{k}$. Clause (vii) uses Theorem C, a statement about the model at the reference photon mass, and is not claimed.

(b), the two conventions. For a member at the mass μ, Lemma 4.2 at fixed k and $J \geq J_0(f)$, whose proof uses the correlation inequalities of Lemma 2.10 only, with its kill steps in the form explained after Definition 31.1, gives $a^{K,\mu}_{k,J}(\mathrm{i}t) \leq a^{(a),\mu}_{k,J}(\mathrm{i}t) \leq a^{K,\mu}_{k,J+1}(\mathrm{i}t)$ for $t \in \mathbb{R}^N$. Along $\mathbf{k}$ all three terms converge (part (a) on both conventions), and then $J \to \infty$ (Lemma 14.8 on both conventions): the two LIM states agree on $(\mathrm{i}\mathbb{R})^N$, a maximal totally real subspace of $\mathbb{C}^N$ and a set of uniqueness for entire functions (Step 4 of the proof of Lemma 14.7), hence on $\mathbb{C}^N$. □

31.3. **The limit along the fixed-counterterm family.** The limit along the fixed-counterterm family is written $Z^{\mathbf{k},0}$ (Theorem 31.4) and, in its form free of $\mathbf{k}$ and Q, Z^0 (Proposition 31.6(a)), to keep it apart from the limit $Z^{(0)}$ along King's states (Theorem 30.2).

Theorem 31.4 (the limit of vanishing photon mass along the fixed-counterterm family)**.** *Let $Q \subset (0, \mu_0]$ be countable with $\mu_0 \in Q$ and $\inf Q = 0$.*

(i) (A common subsequence.) *There is an increasing sequence $\mathbf{k}$ of spacings which is a continuum subsequence for every fixed-counterterm member at a mass $\mu \in Q$, on both conventions, and for every $\mu \in Q$ and every family the LIM states of the two conventions coincide: $Z^{\mathbf{k},\mu}_\infty(\cdot, 0) = Z^{K,\mathbf{k},\mu}_\infty(\cdot, 0)$.*
In (ii)–(v), $\mathbf{k}$ is any increasing sequence of spacings with the property of (i); one exists by Lemma 31.3.

(ii) (The members.) *For every $\mu \in Q$ and every family, $Z^{\mathbf{k},\mu}_\infty(\sum_j z_j f_j, 0)$ exists locally uniformly in $z \in \mathbb{C}^N$, is entire, and satisfies clauses (i)–(vi) of Theorem 14.9 for the fixed-counterterm member at the mass μ (bare coefficient $\alpha^\infty_k(m; \mu_0)$, photon Gaussian at the mass μ; it is not the model of Section 2 at the couplings (e, λ, m, μ)): OS0 with $|Z^{\mathbf{k},\mu}_\infty(\sum_j z_j f_j, 0)| \leq e^{\|g(z)\|_2^2}$, the second bound of clause (i) of Theorem 14.9 with μ in place of μ_0, the Schwinger bounds $(\sqrt{2}\, e)^n (n!)^{1/2} \prod_j \|f_j\|_2$ and E0″ with $s = 4$; invariance under translations and under O_h about every point; reflection positivity about every coordinate plane in matrix and polynomial form; realness, symmetry and vanishing at odd order; the unique measure $\mu^{\mathbf{k},\mu}_F$ on $\mathcal{H}^{-r}(\mathbb{R}^3; \Lambda^2)$, every even $r \geq 14$, with its image on H^{-r}_{loc} and the convergence of the laws of the localised lattice fields along $\mathbf{k}$ and then in J; and $\mathbb{R}^3 \rtimes O_h$-invariance with the ergodic decomposition and the weak cluster clauses of Theorem 11.5(b). The constants are the absolute ones of Remark 7.5.*

(iii) (Monotonicity in the photon mass.) *For $\mu \leq \mu'$ in Q, every family, $t \in \mathbb{R}^N$ and $x \in \mathbb{R}^N$:*

$$Z^{\mathbf{k},\mu}_\infty(\mathrm{i}t, 0) \leq Z^{\mathbf{k},\mu'}_\infty(\mathrm{i}t, 0), \qquad Z^{\mathbf{k},\mu}_\infty(x, 0) \geq Z^{\mathbf{k},\mu'}_\infty(x, 0).$$

(iv) (Convergence.) *For every family the limit*

$$Z^{\mathbf{k},0}\Big(\sum_j z_j f_j, 0\Big) := \lim_{\mu \downarrow 0,\ \mu \in Q} Z^{\mathbf{k},\mu}_\infty\Big(\sum_j z_j f_j, 0\Big)$$

exists locally uniformly on $\mathbb{C}^N$, as a full limit along Q, equal to $\inf_{\mu \in Q}$ at the points of $(\mathrm{i}\mathbb{R})^N$ and to $\sup_{\mu \in Q}$ at the points of $\mathbb{R}^N$; $Z^{\mathbf{k},0}$ is entire, real and ≥ 1 at real points, in $[0, 1]$ at imaginary points, and $|Z^{\mathbf{k},0}(\sum_j z_j f_j, 0)| \leq e^{\|g(z)\|_2^2}$; every Schwinger function converges, $S^{\mathbf{k},\mu}_n(f_1, \ldots, f_n) \to S^{\mathbf{k},0}_n(f_1, \ldots, f_n)$ as $\mu \downarrow 0$ in Q.

(v) (The axioms of the limit.) *$Z^{\mathbf{k},0}$ satisfies clauses (i)–(vi) of (ii) with the same absolute constants, the second bound of clause (i) of Theorem 14.9 excepted (it degenerates as $\mu \downarrow 0$) and clause (v) in its first sentence only, the sentence about the measure: the second sentence of Theorem 14.9(v), the convergence of the laws of the localised lattice fields $\chi F_{\Lambda_{k_m}}$ under*

the cube states along $\mathbf{k}$ *and then in* J*, has no analogue at* $\mu = 0$*, where there is no member (the Neumann Feynman form is degenerate at zero photon mass), and is not inherited. Its replacements are the sentence for each member, which holds for every* $\mu \in Q$ *by (ii), and the convergence of the limit laws along* μ*, which is Open Problem 31.12.*

Remark 31.5 (what is not claimed in Theorem 31.4). The limit is written $Z^{\mathbf{k},0}$, without the subscript ∞ that marks the LIM states $Z_\infty^{\mathbf{k},\mu}$ of the members: there is no fixed-counterterm member at photon mass zero (the Neumann Feynman form is degenerate there, clause (v)), so $Z^{\mathbf{k},0}$ is not the LIM state of a member but the limit, along Q, of the LIM states of the members at $\mu \in Q$, as defined in clause (iv). Not claimed for $Z^{\mathbf{k},0}$: $\mathbb{R}^3 \rtimes O(3)$-invariance and reflection positivity about non-coordinate planes; the Euclidean clause (vii) of Theorem 14.9; independence of $Z^{\mathbf{k},0}$ from the subsequence $\mathbf{k}$ and from the set Q; the convergence of the lattice laws or of the limit laws along μ (Open Problem 31.12); and the second bound of OS0, $\exp(\frac{1}{2}\mu^{-2}\|\sum_j (\mathrm{Re}\, z_j) d^* f_j\|_2^2)$, which degenerates as $\mu \downarrow 0$. The constants called uniform in the photon mass here are exactly those the paper declares absolute: the bound $e^{\|g\|_2^2}$, the Schwinger constants $(\sqrt{2}\,e)^n (n!)^{1/2}$ and $(e\pi/2)^n (n!)^{1/2}$ and the constants of E0″ (Remark 7.5; Theorem 14.9(i); Theorem 28.2 A), and the Lipschitz constant $\sqrt{2}$ of the characteristic functional (Step 5 of the proof of Lemma 14.7; Corollary 8.4). The non-uniform ones, the second OS0 bound, the margin $\pi\sqrt{3}/(2\mu)$ of Lemma 9.10 and every constant of the Wick leg (Remark 28.9), are either excluded or used for one member at a time.

Proof of Theorem 31.4. (i), (ii). Lemma 31.3 with $\mathcal{M}$ the set of fixed-counterterm members at the masses $\mu \in Q$ (the remarks after Definition 31.1): (a) gives the common continuum subsequence and the clauses of (ii), (b) the coincidence of the two conventions.

(iii). Fix k, $J \geq J_0(f)$, either convention and $\mu < \mu'$ in Q. The two members have the same single-site measures (the coefficient is held fixed) and the same hopping terms; the action of the member at μ' is that of the member at μ plus $\frac{1}{2}(\mu'^2 - \mu^2) \sum_{x \in X_J} \sum_\alpha \varepsilon^3 A_\alpha(x)^2$, the photon mass term on X_J of Definition 2.2, a finite sum of terms $s|\ell(A)|^2$ with $s = \frac{1}{2}(\mu'^2 - \mu^2)\varepsilon^3 \geq 0$ and $\ell(A) = A_\alpha(x)$ a real linear functional. Add them one at a time; each intermediate measure is a BFS measure, a positive quadratic form keeping the Gaussian non-degenerate. Lemma 2.10(a) (adding $s|\ell(A)|^2$ to the action, $s \geq 0$, does not decrease $\langle\Theta\rangle$) with $\Theta = \Theta_{F,0} = e^{\mathrm{i}F(u)}$, the case $g = 0$ being included, gives $\langle e^{\mathrm{i}F(u)}\rangle_{k,J}^{\mu} \leq \langle e^{\mathrm{i}F(u)}\rangle_{k,J}^{\mu'}$ for every real 2-form u with plaquettes in X_J, and Lemma 2.10(d) with $\Theta^- = \Theta_{F,0}^- = e^{F(u)}$ gives $\langle e^{F(u)}\rangle_{k,J}^{\mu} \geq \langle e^{F(u)}\rangle_{k,J}^{\mu'}$. With $u = \sum_j t_j f_j$, respectively $u = \sum_j x_j f_j$: $a_{k,J}^{\mu}(\mathrm{i}t) \leq a_{k,J}^{\mu'}(\mathrm{i}t)$ and $a_{k,J}^{\mu}(x) \geq a_{k,J}^{\mu'}(x)$ on either convention. Pass along $\mathbf{k}$, both sides converging by (i), and then $J \to \infty$ (Lemma 14.8).

(iv). Fix $\mathbf{k}$ and the family. On $(\mathrm{i}\mathbb{R})^N$ the map $\mu \mapsto Z_\infty^{\mathbf{k},\mu}(\mathrm{i}t, 0)$ is non-decreasing on Q with values in $[0,1]$ (Lemma 14.8), so its limit as $\mu \downarrow 0$ along Q exists and equals the infimum; on $\mathbb{R}^N$ it is non-increasing with values in $[1, e^{\|g(x)\|_2^2}]$, so the limit exists and equals the supremum. The family $\{Z_\infty^{\mathbf{k},\mu}(\cdot, 0)\}_{\mu \in Q}$ is locally bounded on $\mathbb{C}^N$ by the bound $e^{\|g(z)\|_2^2}$, which is free of μ (Lemma 14.8; Remark 7.5). The Montel–Vitali argument of Step 4 of the proof of Lemma 14.7 and of Corollary 4.4(ii)(β) applies: $(\mathrm{i}\mathbb{R})^N$ is a maximal totally real subspace of $\mathbb{C}^N$, a set of uniqueness for entire functions; every sequence $\mu_n \downarrow 0$ in Q has locally uniformly convergent subsequences, whose limits all agree on $(\mathrm{i}\mathbb{R})^N$ with the monotone limit, hence agree; so $Z_\infty^{\mathbf{k},\mu} \to Z^{\mathbf{k},0}$ locally uniformly as $\mu \downarrow 0$ in Q. The bound, realness and the ranges $[1, \infty)$ and $[0, 1]$ are closed under pointwise limits; the Schwinger functions, derivatives at 0 of locally uniformly convergent entire functions, converge by Weierstrass' theorem. (The clauses of (iii) and (iv) at

real points use Lemma 2.10(d); they are correct but not needed for the rest, since the Montel–Vitali argument from the imaginary axes suffices, as in the main line of the paper, which avoids (d), Remark 7.5.)

(v). The items (i)–(vi) below are the clauses of Theorem 14.9, in the form listed in Theorem 31.4(ii), taken one at a time; an unqualified clause number refers to them, and the clauses of Theorem 31.4 used here are cited as such. (i) The bounds pass to the limit; the Schwinger bounds, temperedness and E0′′ are Cauchy's estimates and the kernel theorem applied to the limit function with its bound, the argument of Theorem 7.6(c) run on $Z^{\mathbf{k},0}$ as Theorem 14.9 runs it on the LIM states (replacement (γ)); the constants are those of Theorem 31.4(ii). (ii) For $b \in \mathbb{R}^3$ and $R \in O_h$ about any point, $Z^{\mathbf{k},\mu}_\infty(\sum_j z_j \tau_b R \cdot f_j, 0) = Z^{\mathbf{k},\mu}_\infty(\sum_j z_j f_j, 0)$ for every $\mu \in Q$ by Theorem 31.4(ii); the transformed family is a family, so both sides converge as $\mu \downarrow 0$ by Theorem 31.4(iv), and the limit is invariant. (iii) Fix a coordinate plane $\{x_\nu = c\}$ and sources $\zeta_1, \dots, \zeta_N$ supported in $\{x_\nu > c\}$ with complex coefficients. For every μ the matrix $(Z^{\mathbf{k},\mu}_\infty(\theta\zeta_i^* \sqcup \zeta_j, 0))_{ij}$ is positive semidefinite by Theorem 31.4(ii); its entries converge by Theorem 31.4(iv), and the positive semidefinite matrices form a closed set, so the limit matrix is positive semidefinite: as the remark after Lemma 9.10 says, a limit of vanishing photon mass inherits reflection positivity, if it exists, as a pointwise limit of positive semidefinite matrices. In the polynomial form $S(\theta_c(P)P)$ is a finite linear combination of Schwinger functions, each converging, and $[0, \infty)$ is closed. (iv) Realness at real points, symmetry (the Schwinger functions are Taylor coefficients of a function of the total source) and $Z^{\mathbf{k},\mu}_\infty(-z, 0) = Z^{\mathbf{k},\mu}_\infty(z, 0)$ pass to the limit. (v) The characteristic functional $\mathcal{Z}^0(u) := Z^{\mathbf{k},0}(\mathrm{i}u, 0)$, $u \in C^\infty_c(\mathbb{R}^3; \Lambda^2)$ real, is normalised, $\mathcal{Z}^0(0) = 1$; positive definite, because for every μ the matrix $(Z^{\mathbf{k},\mu}_\infty(\mathrm{i}(u_m - u_l), 0))_{lm}$ is positive semidefinite (the Gram argument in the proof of Theorem 14.9(v)) and limits of positive semidefinite matrices are positive semidefinite; and Lipschitz for $\sqrt{2}\|\cdot\|_2$, because $|Z^{\mathbf{k},\mu}_\infty(\mathrm{i}u, 0) - Z^{\mathbf{k},\mu}_\infty(\mathrm{i}u', 0)| \le \sqrt{2}\|u - u'\|_2$ for every μ (Step 3 of the proof of Lemma 14.7, passed to the limits as in the proof of Theorem 14.9(ii)) and this bound is closed under pointwise limits. With the moment bounds of (i) these are exactly what the construction of the measure from a limit functional uses (Steps 1 and 2 of Theorem 11.5, as applied to the LIM states in the proof of Theorem 14.9(v)): the Bochner–Minlos theorem on $\mathcal{S}$, the placement on $\mathcal{H}^{-r}$ by the moment bounds, the identification of the moments and the uniqueness; the image on H^{-r}_{loc} likewise. The moments are identified from the limit object alone: for real $u \in C^\infty_c$ the map $t \mapsto \mathcal{Z}^0(tu) = Z^{\mathbf{k},0}(\mathrm{i}tu, 0)$ extends to an entire function of t, so by Lukacs' theorem [28] (a characteristic function analytic near 0 has moments of every order, equal to its derivatives at 0) $\int \Phi(u)^n \, d\mu^{\mathbf{k},0}_F = (-\mathrm{i})^n \partial_t^n Z^{\mathbf{k},0}(\mathrm{i}tu, 0)|_{t=0} = S^{\mathbf{k},0}_n(u, \dots, u)$, and polarisation, $S^{\mathbf{k},0}_n$ being symmetric, gives the mixed moments on C^∞_c; the Schwinger bounds of (i) then place the measure on the Hermite–Sobolev carrier $\mathcal{H}^{-r}$, $r \ge 14$ even, and give both uniqueness statements (Theorem 11.5(a); [32, Lemma A.8], [32, Proposition A.9], [32, Lemma A.5]), with no lattice statement used. (vi) The image measure on $X = H^{-r}_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^2)$, with the translations and the coordinate reflections of (ii) and the coordinate reflection positivity of (iii) extended to L^2 as in Steps 3 and 4 of the proof of Theorem 11.5, is a reflection-positive stationary system in the sense of Definition 11.4: (H1) is the invariance under τ_a and θ_j from (ii), $\theta_j \in O_h$; (H2) concerns the local σ-algebras of X, a property of the space and not of the measure; (H3) is (iii). The general results of [32, Appendix A] then give the ergodic decomposition and the weak cluster clauses of Theorem 11.5(b), as the proof of Theorem 14.9(vi) applies them to the LIM states. $\mathbb{R}^3 \rtimes O_h$-invariance is (ii). □

31.4. **Identification with King's state at a shifted scalar mass.** The fixed-counterterm member at the mass $\mu < \mu_0$ carries the bare coefficient $b_k = \alpha^\infty_k(m; \mu_0)$, King's model at the point $(e, \lambda, \mathsf{M}, \mu)$ the coefficient $\alpha^\infty_k(\mathsf{M}; \mu)$; both live on the same cube with the same photon

Gaussian. If the two coefficients become equal in the limit at a scalar mass $\mathsf{M}(\mu)$, and if King's state is continuous in the scalar mass there, the fixed-counterterm member's limit state is King's state at $(e,\lambda,\mathsf{M}(\mu),\mu)$. Neither condition is proved in this paper; they are stated in full in the proposition and posed as Open Problems 31.8 to 31.10 (the two conditions are (ID$_1$) and (ID$_2$); the further problem (ID$_1'$), a monotonicity of the bare coefficient in the scalar mass, enters only parts (b) and (c) of the proposition, where continuity is not assumed). Throughout, $a^K_{k,J}(\,\cdot\,;\mathsf{M},\mu)$ and $Z^K_J(\,\cdot\,;\mathsf{M},\mu)$ denote the functional of King's convention and its fixed-cube limit (Theorem A at the point) for the model at $(e,\lambda,\mathsf{M},\mu)$, so that $Z^K_\infty(\,\cdot\,;\mathsf{M},\mu)=\lim_J Z^K_J(\,\cdot\,;\mathsf{M},\mu)$.

Proposition 31.6 (the fixed-counterterm member as King's state at a shifted scalar mass)**.** *Let $\mu\in(0,\mu_0]$.*

(a) *Assume* (ID$_1$) *at μ: the relative limit*

$$\Psi_\mu(\mathsf{M}):=\lim_{k\to\infty}\big[\alpha_k^\infty(\mathsf{M};\mu)-\alpha_k^\infty(m;\mu_0)\big]$$

exists for every $\mathsf{M}>0$, $\mathsf{M}\mapsto\Psi_\mu(\mathsf{M})$ is continuous and strictly increasing on $(0,\infty)$, $\Psi_\mu(\mathsf{M})\to+\infty$ as $\mathsf{M}\to\infty$ and $\inf_{\mathsf{M}>0}\Psi_\mu(\mathsf{M})<0$; let $\mathsf{M}(\mu)$ be the unique zero of Ψ_μ. Assume (ID$_2$) *at μ: for every real $u\in C_c^\infty(\mathbb{R}^3;\Lambda^2)$ the map $\mathsf{M}\mapsto Z^K_\infty(\mathrm{i}u;\mathsf{M},\mu)$ is continuous at $\mathsf{M}=\mathsf{M}(\mu)$. Then for every continuum subsequence $\mathbf{k}$ of the fixed-counterterm member at μ on the comparison convention and every family,*

$$Z^{\mathbf{k},\mu}_\infty(\,\cdot\,,0)=Z^K_\infty(\,\cdot\,;\mathsf{M}(\mu),\mu)\,:$$

the LIM state of the member does not depend on $\mathbf{k}$, is $\mathbb{R}^3\rtimes O(3)$-invariant for all complex coefficients, and satisfies clauses (i)–(vi) of Theorem 28.2 B in full, the lattice sentences of B(i) and B(v) referring to King's lattice states at the point $(e,\lambda,\mathsf{M}(\mu),\mu)$. If the hypotheses hold at every $\mu\in Q$, the limit $Z^{\mathbf{k},0}$ of Theorem 31.4 does not depend on $\mathbf{k}$. If they hold at every $\mu\in(0,\mu_0]$, then $Z^{\mathbf{k},0}=:Z^0$ depends neither on $\mathbf{k}$ nor on Q, equals the full monotone limit $\lim_{\mu\downarrow0}Z^K_\infty(\,\cdot\,;\mathsf{M}(\mu),\mu)$ of King's own states along the shifted curve $\mu\mapsto(e,\lambda,\mathsf{M}(\mu),\mu)$, and satisfies clauses (i)–(vi) of Theorem 28.2 B ($\mathbb{R}^3\rtimes O(3)$-invariance, reflection positivity about every plane, the Euclidean sentence of (vi)), the lattice sentences of B(i) and B(v) excepted as in Theorem 30.2.

(b) (The squeeze without continuity.) *Assume* (ID$_1$) *at μ and* (ID$_1'$) *at $\mathsf{M}_0=\mathsf{M}(\mu)$: there are $\delta_0>0$ and k_0 such that for $k\ge k_0$ the map $\mathsf{M}\mapsto\alpha_k^\infty(\mathsf{M};\mu)$ is non-decreasing on $[\mathsf{M}(\mu)-\delta_0,\mathsf{M}(\mu)+\delta_0]$. Then $\mathsf{M}\mapsto Z^K_\infty(\mathrm{i}t;\mathsf{M},\mu)$ is non-increasing on that interval, its one-sided limits $Z^K_\infty(\mathrm{i}t;\mathsf{M}(\mu)\pm,\mu)$ at $\mathsf{M}(\mu)$ exist, and for every continuum subsequence $\mathbf{k}$ of the member, every family and $t\in\mathbb{R}^N$,*

$$Z^K_\infty(\mathrm{i}t;\mathsf{M}(\mu)+,\mu)\;\le\;Z^{\mathbf{k},\mu}_\infty(\mathrm{i}t,0)\;\le\;Z^K_\infty(\mathrm{i}t;\mathsf{M}(\mu)-,\mu).$$

(c) (Countably many exceptional reference scalar masses.) *Assume* (ID$_1$) *for every reference scalar mass $m'>0$ in place of m (with $\Psi_\mu(\mathsf{M})=\Phi_\mu(\mathsf{M})-\Phi_{\mu_0}(m')$ for a continuous, strictly increasing Φ_μ and the normalisation $\Phi_{\mu_0}(m)=0$, as in Open Problem 31.8) and* (ID$_1'$) *at every $\mathsf{M}_0>0$, for every $\mu\in Q$, Q countable. Then there is a countable set $N(Q)\subset(0,\infty)$ such that for every reference scalar mass $m'\notin N(Q)$ the condition* (ID$_2$) *holds at every $\mu\in Q$ for the fixed-counterterm family with reference scalar mass m' and reference photon mass μ_0, so that (a) applies to that family at every $\mu\in Q$. The set $N(Q)$ depends on Q; nothing is claimed for all $\mu\in(0,\mu_0]$ at once.*

Proof. (a), the squeeze. King's model at $(e,\lambda,\mathsf{M},\mu)$ and the fixed-counterterm member at μ live on the same cube with the same photon Gaussian, and their single-site weights differ by the factor $\exp(-\frac12\varepsilon^3(\alpha_k^\infty(\mathsf{M};\mu)-b_k)r^2)$. Lemma 2.10(c) (multiplying ρ_x by e^{-sr^2}, $s\ge0$, does not increase $\langle\Theta\rangle$), applied as in Lemma 26.5(a) by interpolation within the class of BFS measures, gives: if

$\alpha_k^\infty(\mathsf{M};\mu) \geq b_k$ then $\langle e^{\mathrm{i}F(u)}\rangle_{k,J}^K(\mathsf{M},\mu) \leq \langle e^{\mathrm{i}F(u)}\rangle_{k,J}^{K,\mu}$ for every real 2-form u with plaquettes in X_J, and the reverse inequality if $\alpha_k^\infty(\mathsf{M};\mu) \leq b_k$. By (ID$_1$), for every $\delta \in (0,\delta_0)$, $\delta_0 := \mathsf{M}(\mu)/2$, there is $k_0(\delta)$ with $\alpha_k^\infty(\mathsf{M}(\mu)-\delta;\mu) \leq b_k \leq \alpha_k^\infty(\mathsf{M}(\mu)+\delta;\mu)$ for $k \geq k_0$: Ψ_μ is continuous and strictly increasing with $\Psi_\mu(\mathsf{M}(\mu)) = 0$, so $\Psi_\mu(\mathsf{M}(\mu)-\delta) < 0 < \Psi_\mu(\mathsf{M}(\mu)+\delta)$, and $\alpha_k^\infty(\mathsf{M}(\mu)\pm\delta;\mu) - b_k \to \Psi_\mu(\mathsf{M}(\mu)\pm\delta)$. Hence, on King's convention, for $k \geq k_0$, $J \geq J_0(f)$ and $t \in \mathbb{R}^N$,

$$a_{k,J}^K(\mathrm{i}t;\mathsf{M}(\mu)+\delta,\mu) \;\leq\; a_{k,J}^{K,\mu}(\mathrm{i}t) \;\leq\; a_{k,J}^K(\mathrm{i}t;\mathsf{M}(\mu)-\delta,\mu).$$

Let $\mathbf{k}$ be a continuum subsequence of the member on the comparison convention and refine it to a subsequence $\mathbf{k}' \subset \mathbf{k}$ which is a continuum subsequence of the member on King's convention (Corollary 14.15(a) applied to the sequence $\mathbf{k}$, its bounds being model-free as in the proof of Lemma 31.3); $\mathbf{k}'$ is a continuum subsequence of the comparison convention with the same LIM state (Definition 14.2), and the two conventions have the same LIM state along $\mathbf{k}'$ (Lemma 31.3(b)). Let $k \to \infty$ along $\mathbf{k}'$: the outer terms converge along the full sequence by Theorem A at the points $(e,\lambda,\mathsf{M}(\mu)\pm\delta,\mu)$ of King's range (Corollary 4.4(ii)(α)), so $Z_J^K(\mathrm{i}t;\mathsf{M}(\mu)+\delta,\mu) \leq Z_J^{K,\mathbf{k}',\mu}(\mathrm{i}t) \leq Z_J^K(\mathrm{i}t;\mathsf{M}(\mu)-\delta,\mu)$. Let $J \to \infty$: all three terms are non-decreasing in J (Corollary 4.3 for King's states; Lemma 14.8 on King's convention for the member, Lemma 31.3(a)), so

$$Z_\infty^K(\mathrm{i}t;\mathsf{M}(\mu)+\delta,\mu) \;\leq\; Z_\infty^{\mathbf{k},\mu}(\mathrm{i}t,0) \;\leq\; Z_\infty^K(\mathrm{i}t;\mathsf{M}(\mu)-\delta,\mu),$$

the middle term by the coincidence of the two conventions along $\mathbf{k}'$. Under (ID$_2$) both ends tend to $Z_\infty^K(\mathrm{i}t;\mathsf{M}(\mu),\mu)$ as $\delta \downarrow 0$, so $Z_\infty^{\mathbf{k},\mu}(\,\cdot\,,0) = Z_\infty^K(\,\cdot\,;\mathsf{M}(\mu),\mu)$ on $(\mathrm{i}\mathbb{R})^N$, hence on $\mathbb{C}^N$ by the identity theorem. The remaining clauses of the first statement are Theorem 28.2 at the point $(e,\lambda,\mathsf{M}(\mu),\mu)$. If the hypotheses hold at every $\mu \in Q$, then on $(\mathrm{i}\mathbb{R})^N$ $Z^{\mathbf{k},0} = \inf_{\mu\in Q} Z_\infty^K(\,\cdot\,;\mathsf{M}(\mu),\mu)$ (Theorem 31.4(iv)), which does not involve $\mathbf{k}$; an entire function is determined on $(\mathrm{i}\mathbb{R})^N$. If they hold at every $\mu \in (0,\mu_0]$, then $\mu \mapsto Z_\infty^K(\mathrm{i}t;\mathsf{M}(\mu),\mu)$ is non-decreasing on $(0,\mu_0]$: for $\mu < \mu'$ apply Theorem 31.4(iii) to a countable $Q' \ni \mu,\mu'$ with $\mu_0 \in Q'$ and $\inf Q' = 0$, the LIM states along its subsequence being the two King states by the first statement. Hence $\inf_Q = \inf_{(0,\mu_0]}$ for every Q accumulating at 0: the limit is free of Q and equals the full monotone limit along the shifted curve on $(\mathrm{i}\mathbb{R})^N$, and by the μ-free bound and Montel–Vitali the full limit exists locally uniformly on $\mathbb{C}^N$; $\mathbb{R}^3 \rtimes O(3)$-invariance, reflection positivity about every plane and the Euclidean sentence of (vi) pass to the limit as in the proof of Theorem 30.2.

(b). Under (ID$_1'$) at $\mathsf{M}_0 = \mathsf{M}(\mu)$, for $k \geq k_0$ the map $\mathsf{M} \mapsto a_{k,J}^K(\mathrm{i}t;\mathsf{M},\mu)$ is non-increasing on $[\mathsf{M}(\mu)-\delta_0,\mathsf{M}(\mu)+\delta_0]$ (Lemma 2.10(c) as above), hence so is $\mathsf{M} \mapsto Z_\infty^K(\mathrm{i}t;\mathsf{M},\mu)$, a pointwise limit (in k, then J) of non-increasing functions; its one-sided limits at $\mathsf{M}(\mu)$ exist, and the displayed inequalities of (a), valid for every $\delta \in (0,\min(\delta_0,\mathsf{M}(\mu)/2))$, give the claim as $\delta \downarrow 0$.

(c). Fix $\mu \in Q$. By (ID$_1'$) at every $\mathsf{M}_0 > 0$ and Lemma 2.10(c), $\mathsf{M} \mapsto Z_\infty^K(\mathrm{i}u;\mathsf{M},\mu)$ is non-increasing on $[\mathsf{M}_0-\delta_0(\mathsf{M}_0),\mathsf{M}_0+\delta_0(\mathsf{M}_0)]$ for every $\mathsf{M}_0 > 0$ and every real u; countably many such intervals cover $(0,\infty)$. Let $D(\mu)$ be the set of $\mathsf{M} > 0$ at which $\mathsf{M} \mapsto Z_\infty^K(\mathrm{i}u;\mathsf{M},\mu)$ is discontinuous for some u in the countable core $\mathcal{P}$ of Step 1 of the proof of Lemma 14.7. A monotone function has countably many discontinuities, so $D(\mu)$, a countable union of countable sets, is countable. Continuity on the core gives continuity for every real $u \in C_c^\infty$: for u supported in the interior of a rational box B and $u' \in S_0(B)$ with $\|u-u'\|_\infty \leq \eta$, the Lipschitz bound of Corollary 8.4 at the points $(e,\lambda,\mathsf{M},\mu)$, with its absolute constant, gives $|Z_\infty^K(\mathrm{i}u;\mathsf{M},\mu) - Z_\infty^K(\mathrm{i}u;\mathsf{M}',\mu)| \leq 2\sqrt{2}\,(3\,\mathrm{vol}\,B)^{1/2}\eta + |Z_\infty^K(\mathrm{i}u';\mathsf{M},\mu) - Z_\infty^K(\mathrm{i}u';\mathsf{M}',\mu)|$, the density argument of Step 3 of that proof. Under (ID$_1$) for every reference scalar mass m', the zero of $\mathsf{M} \mapsto \Phi_\mu(\mathsf{M}) - \Phi_{\mu_0}(m')$ is $\mathsf{M}(m',\mu) = \Phi_\mu^{-1}(\Phi_{\mu_0}(m'))$, a continuous injection of m', Φ_μ and Φ_{μ_0} being continuous and strictly increasing. Hence $N(Q) := \bigcup_{\mu\in Q}\{m' > 0 :\ \mathsf{M}(m',\mu) \in D(\mu)\}$ is countable, and for every $m' \notin N(Q)$ the

map $\mathsf{M} \mapsto Z^K_\infty(iu;\mathsf{M},\mu)$ is continuous at $\mathsf{M}(m',\mu)$ for every $\mu \in Q$ and every real u, which is (ID$_2$) at every $\mu \in Q$ for the reference (m',μ_0). □

Remark 31.7 (no exceptional set of photon masses). The set $D(\mu)$ of Proposition 31.6(c) depends on μ. Monotonicity in the scalar mass therefore yields no statement of the form "(ID$_2$) holds for all photon masses outside a countable set" at a fixed reference scalar mass m; the exceptional set obtained is a set of reference scalar masses, and it depends on the countable set Q of photon masses.

31.5. **What is left open.** The following statements are not proved in this paper. The first three are the hypotheses of Proposition 31.6; the last two concern the uniqueness of the limit along King's states and the convergence of the measures.

(ID$_1$) provides the shifted scalar mass $\mathsf{M}(\mu)$ of Proposition 31.6; with (ID$_2$) it identifies the limit state of the fixed-counterterm member at μ as King's state at $(e,\lambda,\mathsf{M}(\mu),\mu)$, which, if it holds at every $\mu \in (0,\mu_0]$, makes the limit of Theorem 31.4 independent of the subsequence and of Q (Proposition 31.6(a)).

Open Problem 31.8 ((ID$_1$): the counterterm at two photon masses)**.** Let $e,\lambda > 0$ and let $m' > 0$ be a reference scalar mass. For every $\mu \in (0,\mu_0]$ (the natural statement has every photon mass $\mu > 0$ and every reference pair), is it true that the relative limit $\lim_{k\to\infty}[\alpha^\infty_k(\mathsf{M};\mu) - \alpha^\infty_k(m';\mu_0)] =: \Phi_\mu(\mathsf{M}) - \Phi_{\mu_0}(m')$ exists for every $\mathsf{M} > 0$, with Φ_μ defined through these differences and normalised by $\Phi_{\mu_0}(m) = 0$, that $\mathsf{M} \mapsto \Phi_\mu(\mathsf{M})$ is continuous and strictly increasing on $(0,\infty)$ with $\Phi_\mu(\mathsf{M}) \to +\infty$ as $\mathsf{M} \to \infty$, and that $\inf_{\mathsf{M}} \Phi_\mu(\mathsf{M}) < \Phi_{\mu_0}(m')$, so that the shifted scalar mass $\mathsf{M}(m',\mu)$ exists and is unique? Two further clauses complete the natural statement but are not used in Proposition 31.6: that the limit is locally uniform in M, and that $\Phi_\mu \to \Phi_0$ as $\mu \downarrow 0$, a finite shift of the scalar mass in the limit of vanishing photon mass.

Of (ID$_1$), Proposition 31.6(a) uses the convergence at $\mathsf{M}(\mu) \pm \delta$ and the continuity and strict monotonicity of Φ_μ, and (c) also those of Φ_{μ_0}. What this paper proves is the same statement at two scalar masses and one photon mass: the relative limit of Lemma 12.7 and its closed form in Lemma 29.1 (the $1/\varepsilon$ and $\ln(1/\varepsilon)$ parts of the counterterm do not depend on the scalar mass and cancel in the difference), with the structure of Lemma 29.6; Section 29 also gives the orders of the counterterm at every photon mass (Lemmas 29.25 to 29.28). What is missing is the independence of the divergent parts from the photon mass (the $1/\varepsilon$ part of the one-loop photon tadpole and the $\ln(1/\varepsilon)$ coefficient at two loops), and the sign and size of the finite shift.

(ID$_1'$) stands in for (ID$_2$) in parts (b) and (c) of Proposition 31.6: with (ID$_1$) it gives the two-sided squeeze of the member's limit state between the one-sided limits of King's state (b), and the countable exceptional set of reference scalar masses (c).

Open Problem 31.9 ((ID$_1'$): monotonicity of the bare coefficient in the scalar mass)**.** For every $\mu \in (0,\mu_0]$ and every $\mathsf{M}_0 > 0$, are there $\delta_0(\mathsf{M}_0,\mu) > 0$ and $k_0(\mathsf{M}_0,\mu)$ such that for $k \geq k_0$ the map $\mathsf{M} \mapsto \alpha^\infty_k(\mathsf{M};\mu)$ is non-decreasing on $[\mathsf{M}_0 - \delta_0, \mathsf{M}_0 + \delta_0]$?

Proposition 31.6(b) uses it at $\mathsf{M}_0 = \mathsf{M}(\mu)$ only, and (c) at every $\mathsf{M}_0 > 0$. It follows if $\sup_k |\partial_{\mathsf{M}} \delta m^2_k(\mathsf{M};\mu)| < 2\mathsf{M}$ locally uniformly in M.

(ID$_2$) is the continuity that closes the squeeze in the proof of Proposition 31.6(a), where the member's limit state is caught between King's states at the scalar masses $\mathsf{M}(\mu) \pm \delta$.

Open Problem 31.10 ((ID$_2$): continuity of King's state in the scalar mass at the shifted point)**.** For every $\mu \in (0,\mu_0]$ and every real $u \in C^\infty_c(\mathbb{R}^3;\Lambda^2)$, is $\mathsf{M} \mapsto Z^K_\infty(iu;\mathsf{M},\mu)$ continuous at $\mathsf{M} = \mathsf{M}(\mu)$ (two-sided; equivalently, by the density argument in the proof of Proposition 31.6(c), for u in the countable core)?

What this paper proves is the monotonicity in the bare coefficient (Lemma 26.5(a)), the right-continuity in the scalar mass at the points of E_W (Theorem 26.7(b), Remark 28.3), and the continuity in a counterterm offset uniformly in the spacing within the dead-shell geometry (Theorem 26.6). Two-sided continuity at a general point is not proved.

(U_K) would make the subsequential limit of Theorem 30.2 a full limit, independent of the subsequence.

Open Problem 31.11 ((U_K): uniqueness of the limit along King's states)**.** Does $\lim_{\mu\downarrow 0} Z^K_\infty(\,\cdot\,; m, \mu)$ exist locally uniformly on $\mathbb{C}^N$, as a full limit, for every family? Under (ID_1) and (ID_2) at every $\mu \in (0, \mu_0]$ this is a question about the dependence of the limit Z^0 of Proposition 31.6(a) on the reference scalar mass: King's state at (e, λ, m, μ) is the fixed-counterterm member of the reference scalar mass $m'(\mu) := \Phi^{-1}_{\mu_0}(\Phi_\mu(m))$, and $m'(\mu) \to \Phi^{-1}_{\mu_0}(\Phi_0(m))$ under the last clause of Open Problem 31.8.

The limit along the shifted curve $\mu \mapsto (e, \lambda, \mathsf{M}(\mu), \mu)$ of Proposition 31.6(a) does not need it.

(W) asks for the convergence of the limit laws along μ, the second of the two replacements named in Theorem 31.4(v) for the sentence of Theorem 14.9(v) that the limit does not inherit.

Open Problem 31.12 ((W): convergence of the measures)**.** In the setting of Theorem 31.4, do the measures $\mu_F^{\mathbf{k},\mu}$ of (ii) converge weakly on $H^{-r}_{\mathrm{loc}}(\mathbb{R}^3; \Lambda^2)$, as $\mu \downarrow 0$ in Q, to the measure $\mu_F^{\mathbf{k},0}$ of (v)?

Theorem 31.4 gives the pointwise convergence of the characteristic functionals and uniform moment bounds, and the Lipschitz constant $\sqrt{2}$ of the characteristic functionals is uniform in the photon mass (equicontinuity at 0; Remark 31.5); a continuity theorem of Lévy's type for measures on the dual of a nuclear space, together with tightness on $\mathcal{H}^{-r}$, would turn these into the convergence of the measures. We do not pursue this here. Finally, the fixed-cube limits $\lim_k a^{(a),\mu}_{k,J}$ of a fixed-counterterm member at $\mu < \mu_0$ along the full sequence of spacings are neither needed for Theorems 30.2 and 31.4 and Proposition 31.6 nor claimed; under (ID_1) they would follow at every J from the squeeze of Proposition 31.6(a) at fixed J, by the pattern of Corollary 4.4(ii)(γ), given the continuity of King's fixed-cube limits in the scalar mass.

32. Open problems

The questions left open are stated here in precise form. Four of the questions are posed as formal open problems (Open Problems 32.2–32.5); open problems of the limit of vanishing photon mass, among them the two conditions of Proposition 31.6, are posed in Section 31.5.

This section collects what is not proved in this paper: the relation of King's state to the periodic-torus states, its uniqueness and ergodicity, non-triviality, the limit of vanishing photon mass beyond the curvature-sector statements of Sections 30 and 31, the Wick sector at arbitrary amplitudes with its Euclidean clause and the stability of the shifted torus model behind it, the rotation invariance with scalar sources at the points of E_W and above the threshold, and the coupling condition. Each is posed as a question about the objects of Theorem 28.2 and of the rotation part; none is used anywhere in the paper (the conditions under which Proposition 31.6 identifies the fixed-counterterm members with King's states at a shifted scalar mass are among the open problems of Section 31 (Section 31.5) and enter the paper only as hypotheses of that conditional proposition).

Throughout, K denotes King's state of Theorem 28.2, the infinite-volume state of King's construction [25, 26]: the Neumann–Dirichlet cube limit on King's RG cube at block factor 5, fixed photon mass $\mu_0 > 0$, with an admissible scheme a in the Wick-square sector; S denotes a periodic-torus limit in the sense of Definition 32.1(iii) below (dyadic spacings, periodic tori, the

mass counterterm computed with the photon propagator at the mass m), at couplings at which one exists. No identification between the two is proved, and the existence of S is not proved here. "Symmetric phase" is, for both, the expectation that the states lie above the surface predicted by one-loop perturbation theory; it is neither a hypothesis nor a conclusion of any theorem here, and the curvature leg of Theorem 28.2 holds on all of King's coupling range (Lemma 29.3(a)), the surface included, the Wick leg on the coupling region of Definition 2.1, where the sign clause (T3) holds (Lemma 29.3(b), Section 32.7). The problems below are extensions of the main theorem, not conditions on it.

32.1. **Relation to the periodic-torus states.** For which couplings do the periodic-torus states of Definition 32.1 have continuum limits S, and does such a limit coincide with K, possibly at a matched bare mass? Three statements bear on this, none of them proved; we call them (FAM$_L$), King's cube construction exists with block factor 2 along the dyadic spacings ε'_K of Definition 32.1, with the counterterm defined there; (KI$_L$), the fixed-torus continuum limit exists in that family; (CE), a convergent unit-scale cluster expansion with sources on the cubic tori, their tilted copies and their cuts along King's cubes, uniform in the lattice spacing. Whether (CE) holds anywhere at $e > 0$ is open. Two obstacles are structural. The counterterm of Definition 32.1 is computed with the photon propagator at the mass m, King's with the model's own μ_0; already at order e^2 the two relative bare coefficients, $s^{\mathrm{per}}_{j,K}(M_1)$ and King's $s^{\infty}_{k}(M_1)$, have different limits unless $\mu_0 = m$ (the first as $j \to \infty$ and then $K \to \infty$), the difference being at most of order $e^2|m - \mu_0|$, of exactly that order for μ_0 in compact subsets of $(0, \infty)$, and not small when $|m - \mu_0|$ is comparable to m (the one-loop comparison stated after Definition 32.1); and King's construction needs block factor 5 for the rotation by $\arctan(3/4)$, while no lattice family admits both a dyadic structure and a rotation of infinite order. On the one-loop surface $\lambda m^2 = e^6/(72\pi^2)(1 + O(e^2/m))$ the two constructions are expected to select different phases (periodic tori the broken one, Dirichlet faces the symmetric one); this is a heuristic, not a theorem, and no statement of this paper uses the phase.

Definition 32.1 (periodic-torus states with the counterterm at photon mass m)**.** Let $e > 0$ and $\lambda, m, \mu_0 > 0$. Fix an integer $n_0 \geq 0$ and an even integer $\ell_0 \geq 2$; for $K \geq 0$ and $j \geq 1$ put

$$\varepsilon'_K := 2^{-K-n_0}, \qquad \ell_j := j!\,\ell_0, \qquad T_j := \mathbb{R}^3/(\ell_j\mathbb{Z})^3, \qquad T_{j,K} := T_j \cap \varepsilon'_K\mathbb{Z}^3.$$

Since $\ell_j/\varepsilon'_K = j!\,\ell_0\,2^{K+n_0}$ is an even integer, $\varepsilon'_K\mathbb{Z}^3$ descends to T_j with an even number of sites per side; $\ell_j \mid \ell_{j+1}$ and $\ell_j \to \infty$.

(i) *The counterterm.* For $M > 0$ let $\delta m^2_{\mathrm{per};j,K}(M)$ be the polynomial (84) over the orders $(\alpha, \beta) \in \{(0,1),(2,0),(0,2),(2,1),(4,0)\}$, with every coefficient fixed by the recursive zero-momentum condition stated there, computed on the finite periodic graph $T_{j,K}$ with the periodic scalar propagator $(-\Delta_{\varepsilon'_K} + M^2)^{-1}$ and the periodic photon propagator $(-\Delta_{\varepsilon'_K} + m^2)^{-1}$, both without block averaging. The photon propagator of these graphs has the mass m, whatever the value of μ_0. Both propagators are massive, so every graph is a finite sum on the dual group of $T_{j,K}$ and each coefficient is a finite real number.

(ii) *The state.* $\langle\cdot\rangle^{\mathrm{per}}_{K,j}$ is King's action on $T_{j,K}$ with periodic conditions for both fields, exactly as the periodic state of Definition 12.1 with the spacing ε'_K in place of ε_k: Feynman-form gradient terms for every pair of parallel neighbouring bonds, the photon mass μ_0^2, hopping terms on every bond, and the single-site weights $e^{-\varepsilon'^3_K[\frac{1}{2}br^2+\lambda r^4]}$ with the site-independent bare coefficient

$$b = b^{\mathrm{per}}_{j,K} := m^2 + \delta m^2_{\mathrm{per};j,K}(m).$$

The normalisation is finite and positive because $\lambda > 0$ and $\mu_0 > 0$. The observables are the gauge-invariant functions of the fields, generated by the curvature $F(f)$ and the Wick

square $\sum_x \varepsilon_K'^3 h(x)\big(r_x^2 - 2\mathsf{W}_{\varepsilon_K'}\big)$, with the free Wick constant $\mathsf{W}_\varepsilon = (-\Delta_{\varepsilon\mathbb{Z}^3} + m^2)^{-1}(0,0)$ of the free convention.

(iii) *Continuum limits.* A *periodic-torus limit* is a family S of Schwinger functions of the curvature and the Wick square such that, along some sequence (K_l, j_l) with $K_l \to \infty$ and $j_l \to \infty$, the lattice Schwinger functions of $\langle\cdot\rangle^{\mathrm{per}}_{K_l,j_l}$, at test functions in C_c^∞ sampled on the lattice, converge to those of S for all orders.

The bare coefficient of King's state is computed on the infinite lattice with the photon propagator at μ_0 ((84)); that of Definition 32.1 on the tori with the photon propagator at m, for every μ_0. The two differ already at order e^2: for $M_1 \in (0, m)$ the relative bare coefficient $s^{\mathrm{per}}_{j,K}(M_1) := \frac{1}{2}[b^{\mathrm{per}}_{j,K} - (M_1^2 + \delta m^2_{\mathrm{per};j,K}(M_1))]$, the analogue of King's $s^\infty_k(M_1)$ of Lemma 29.6, has at order e^2 the limit $-e^2(m - M_1)R_m/8\pi$ as $j \to \infty$ and then $K \to \infty$, while King's has $-e^2(m - M_1)R_{\mu_0}/8\pi$, where $R_\nu := \frac{m\nu + mM_1 + M_1\nu}{(\nu+M_1)(m+\nu)} = 1 - \frac{\nu^2}{(\nu+M_1)(\nu+m)}$ is strictly decreasing in ν, tends to 1 as $\nu \downarrow 0$ and is Lipschitz with constant $2/m$ (a one-loop computation: the orders λ, λ^2 contain no photon line and agree, the order e^2 is $-\frac{1}{2}(m^2 - M_1^2)\int \hat{k}^2 \hat{D}_\nu G_m G_{M_1}$ on the lattice, whose continuum limit is $R_\nu/(4\pi(m + M_1))$, and the torus sums are Riemann sums of this integral); so the two limits differ unless $\mu_0 = m$, by at most $e^2(m - M_1)|m - \mu_0|/(4\pi m)$ and by a positive multiple of $e^2(m - M_1)|m - \mu_0|$ for μ_0 in compact subsets of $(0, \infty)$.

32.2. **Uniqueness and ergodicity of K.** Uniqueness of King's cube limit is part of the construction (monotonicity in the cube; Theorem A, proved in Theorem 5.47), but uniqueness of the infinite-volume state, in the sense of King's expected identity between cube limits and periodic-box limits, is not proved, and neither is ergodicity of K under translations: Theorem 28.2 gives clustering only on the ergodic components. The usual route, a cluster expansion with a mass gap, is (CE). This is distinct from the independence of a LIM state (Section 14) from its continuum subsequence, (UNIQ), which holds (Proposition 14.6, from Theorem C and from Theorem A; Proposition 14.13, Corollary 4.4). Cubic symmetry of K is a theorem (Theorem 8.6): it needs only the uniqueness of the monotone cube limit and the reflection sandwich (Remark 8.7).

32.3. **Non-triviality.** For a state with Schwinger functions of the curvature and the Wick square let $M(f;h) := S^T_{2,1}(f, f; h)$ be the truncated function with two curvature arguments and one Wick-square argument. If the curvature and the Wick square are independent under the law of a state, all their mixed truncated functions vanish; if their joint law is Gaussian, all truncated functions of order three vanish. In either case $M \equiv 0$; so $M(f;h) \neq 0$ for one real 2-form f and one $h \geq 0$ would show that under K the curvature and the Wick square are neither independent ("decoupled") nor jointly Gaussian. On every BFS measure, in particular on every lattice approximant of K, $\big\langle \sum_x \varepsilon^3 h(x) r_x^2; F(f)^2 \big\rangle \leq 0$ for $h \geq 0$: this is Lemma 2.10(a) for $\Theta = e^{t\sum_x \varepsilon^3 h(x) r_x^2}$ and $\ell = F(f)$, divided by $t > 0$ and taken as $t \downarrow 0$ (dominated convergence, $\lambda > 0$). The sign is not strict. A route to $M(f;h) < 0$ at small couplings has two steps, both open: (C) at $e = \lambda = 0$ the coefficient of e^2 in that lattice covariance has a strictly negative continuum limit $c^{\mathrm{free}}(f,h)$ for $d^*f \not\equiv 0$ and $h \geq 0$, $h \not\equiv 0$ (a one-loop statement); (R) e^{-2} times the covariance differs from that coefficient by at most $\rho((e^2 + \lambda)/m)\,\kappa(f,h)$, with $\rho(t) \to 0$ as $t \downarrow 0$, uniformly along the approximants of K. (C) and (R) together would give $M(f;h) < 0$ at small couplings; both are open. The bounds of the existence theory are not uniform in the couplings, and the cited constructions state no asymptotics in (e, λ), so neither yields (R). A non-vanishing curvature two-point function, that is, the non-degeneracy of K in the curvature sector, is also left open. For K nothing is proved here beyond the non-strict lattice sign above. Whether K is non-Gaussian, or even not decoupled, is open.

32.4. **The limit of vanishing photon mass.** Every constant of the Wick leg of Theorem 28.2 depends on μ_0 and is not uniform in it (Remark 28.9), though the coupling region contains a box uniform in the photon mass (Proposition 29.30); the absolute bounds of the curvature leg are uniform in it. In the curvature sector the limit is taken in Section 30: King's own states at photon masses $\mu_n \downarrow 0$ converge along a subsequence to a limit with full Euclidean invariance and reflection positivity about every plane (Theorem 30.2). Section 31 takes a second limit: along the family with the bare scalar coefficient held at the reference photon mass μ_0 (whose members are monotone in the photon mass on the cone by the correlation inequalities, Lemma 2.10(a); Theorem 31.4(iii)), the LIM states of a common continuum subsequence converge as $\mu \downarrow 0$ along a countable set of photon masses, and the limit satisfies clauses (i)–(vi) of Theorem 14.9 with the same absolute constants, in the form stated in Theorem 31.4(v), which excepts the second OS0 bound (it degenerates as $\mu \downarrow 0$) and the second sentence of Theorem 14.9(v), on the laws of the lattice fields. Left open there (Section 31.5): the identification of the fixed-counterterm member at the mass μ with King's state at a shifted scalar mass, an identification which would make the limit independent of the subsequence and of the set of photon masses; Proposition 31.6 proves it only under the conditions Open Problems 31.8 and 31.10 on the counterterm at two photon masses and on the continuity of King's state in the scalar mass (Open Problem 31.9 for its parts (b) and (c), the variants without continuity), and these conditions are themselves open. Also left open there: the uniqueness of the limit along King's states (Open Problem 31.11); the coincidence of that limit with the limit along the fixed-counterterm family; the convergence of the measures (Open Problem 31.12). Masslessness proper (the vanishing of the photon mass in the limit theory), the rate of decay of the curvature two-point function and the non-degeneracy of the limit are outside the scope of this paper. In the Wick sector the limit is not taken: it would need a cell bound uniform in the photon mass (the analogue, along the family, of the cell bound ($\mathrm{B}_{\mathrm{cell}}$) of Lemma 14.4, which is proved at a fixed mass), together with the torus theorems at the reference photon mass and the transfer of Theorem 28.5 for the members on King's convention. Of the three statements proved for an admissible scheme at a fixed photon mass, the cell bound ($\mathrm{B}_{\mathrm{cell}}$), the scheme bound (SCH) and the independence (UNIQ) of the continuum-subsequence statements from the subsequence (Lemmas 14.4 and 14.5 and Proposition 14.6, Proposition 14.13), the first two would have to be re-established for a LIM state along a family of photon masses, as for a LIM state of a scheme that is not admissible. The pieces of the Wick leg that are already uniform in the photon mass are the lattice reflection positivity of King's tori for every $\mu_0 > 0$ (Proposition 12.2, Remark 28.9) and the sign clause (T3) on the box $0 < \lambda < 2.14398\,m$, $0 < e^2 < 6.30\,m$ at every $\mu_0 \in (0, m]$ (Proposition 29.30; the explicit regions at each photon mass are those of Proposition 29.29, monotone in the photon mass by Lemma 29.27).

32.5. **The Wick sector above the amplitude threshold, and its Euclidean clause.** Form (B) of Theorem 28.2 lives on the tube $\mathcal{D}_{\sigma_+,H}$: the only positive-source bound it uses is the one proved on the cone $\sup q \le s_1 < \sigma_1(M_1)$ (Corollary 13.3), and the only existence statement is Theorem B, whose coefficient vectors have positive scalar part below the same threshold. Above the threshold the existence of the fixed-cube limits and a uniform positive-source bound are open (Open Problem 32.2), and so are a Euclidean clause for the Wick sector (Open Problem 32.3) and the corresponding statements on King's convention (Open Problem 32.4). Below the threshold and outside E_W the Euclidean clause is proved: Theorem 27.13 gives the $E(3)$ invariance of King's state with curvature and neutral scalar sources on the tube $\Omega_*(h)$, hence of every Schwinger function with curvature and Wick-square insertions (Theorem 27.13(iii)); what remains open is the clause above the threshold and at the points of E_W (Section 32.6).

Open Problem 32.2 (W2: the Wick sector at arbitrary amplitudes)**.** Let a be an admissible scheme, and let $Z^a_{k,J}$ denote the scheme functional of Definition 2.5 on King's convention or on the comparison convention. It is not known whether (a) for every finite family of real 2-forms and real functions, every $J \geq J_0(f,h)$ and *every* real coefficient vector (z,w) the limit $\lim_k Z^a_{k,J}(z,w)$ exists and lies in $(0,\infty)$, nor whether (b) for every $q \in C^\infty_c(\mathbb{R}^3)$ with $q \geq 0$

$$\sup_{J \geq J_q} \limsup_{k\to\infty} Z^a_{k,J}(0,q) < \infty,$$

where J_q is the least J with $\operatorname{supp} q$ at ℓ^∞-distance greater than 1 from $\mathbb{R}^3 \setminus C_J$.

What is proved is the part below the amplitude threshold, which is all that Form (B) uses: (a) on King's convention for every real coefficient vector whose positive scalar part satisfies $2\|k_+\|_\infty < s_W$, the negative part being unrestricted (Theorem B), and (b) in L^1 form on the cone of amplitudes $\sup q \leq s_1 < \sigma_1(M_1)$, on both conventions (Corollary 13.3, Remark 4.9). Above the threshold both clauses are open, for either convention, and the fixed-cube limits on one convention do not give those on the other by the results of this paper (Remark 4.5). The exact obstruction is identified in Open Problem 32.5 below: the free-energy bound $(\mathrm{B}^-)[u]$ at an amplitude u above the threshold, the ε-uniform stability of King's torus model with its bare coefficient shifted by $-2u$. For the same reason the existence of the full object $Z^a_\infty = \lim_J \lim_k Z^a_{k,J}$ on all of $\mathbb{C}^{n+p}$, its reflection positivity at arbitrary complex Wick coefficients and the measure of the joint field off the tube are open. No clause of Theorem 28.2 depends on the answer: the Wick result of the main theorem is Form (B) on the tube $\mathcal{D}_{\sigma_+,H}$, where only the proved cone instance enters, through Theorem 28.5.

Open Problem 32.3 (R_W: Euclidean invariance of the Wick sector)**.** Let a be an admissible scheme. It is not known whether the tube object G^K_∞ of King's state (Theorem 28.5), or the full object Z^a_∞ of Open Problem 32.2 where it exists, is invariant under every Euclidean motion for all complex coefficients, F transforming as a 2-form and W^a as a scalar.

Theorem 28.2 gives invariance under $\mathbb{R}^3 \rtimes O(3)$ for the curvature sector; for the Wick sector it gives invariance under $\mathbb{R}^3 \rtimes O_h$ on the tube (Proposition 28.6(b)) and no Euclidean clause. On the tube $\Omega_*(h)$ of Definition 27.2, in the scheme $\Theta^\sharp$, the Euclidean invariance of the joint field is Theorem 27.13; whether the heavy-shell comparison of the rotation part extends to the Wick square at arbitrary amplitudes is open. Two statements about the walled tori of the rotation part, both open, are what its method would call for in the Wick sector. The first is the torus comparison with scalar sources: for every rotation R of the generating set $\mathcal{R}_0$, every King torus D_J, every finite family of real smooth D_J-periodic 2-forms G_j and functions h_j, and every real coefficient vector (z,w), the limits $\lim_k Z^a_{\Lambda_k,D_J}(\sum_j z_j G_j, \sum_j w_j h_j)$ and $\lim_k Z^a_{R^{-1}\Lambda_k,D_J}(\sum_j z_j G_j, \sum_j w_j h_j)$ exist in $(0,\infty)$ and are equal. The second is the closing property in the Wick sector: on the cone $\mathcal{K}$ the lower envelope of the walled-torus functionals with scalar sources is bounded above by the full object Z^a_∞, the counterpart with scalar sources of the envelope statement Theorem 21.4. Below the threshold, on $\mathcal{K}_{<s_*}$, both hold in the walled form (Theorem 27.9; Proposition 27.12), and Theorem 27.13 rests on exactly those; open are every real coefficient and all of $\mathcal{K}$. In the curvature sector their counterparts are theorems of the rotation part off the set E_W (Theorems 18.1 and 21.4, Proposition 26.8 and Definition 27.2), and Theorem 15.1 holds at every point of King's range.

Open Problem 32.4 (FA^K: the Wick sector on King's convention at arbitrary amplitudes)**.** On King's convention ND^K, at amplitudes above the threshold of Theorem B, it is not known whether the limit functional $Z^{a,K}_\infty$ exists and is entire in all coefficients, whether the lattice symmetries O_h about every point hold for the joint field, whether reflection positivity holds at

arbitrary complex coefficients, whether the measure of the joint field exists off the tube, and whether the Euclidean clause of Open Problem 32.3 holds for the King-family object.

Behind these open statements lies a single stability question for the torus model at a constant scalar source.

Open Problem 32.5 ($\mathrm{B}^-[u]$: stability of the shifted torus model)**.** Let $u > 0$. It is not known whether, in the free convention and the scheme a,

$$\sup_{k\ge 0}\ \limsup_{J\to\infty}\ \alpha^a_{k,T_J}(u) < \infty, \qquad \alpha^a_{k,T}(u) = |T|^{-1}\ln N^a_{k,T}(0, u\mathbf{1}),$$

the free-energy density of the torus functional at the constant scalar source u; that is, whether (I2) of Definition 12.5 holds at the amplitude $s_1 = u$.

For u below the threshold s_* defined before (81), $(\mathrm{B}^-)[u]$ is proved: it is Proposition 12.8 (or Proposition 12.13) at an auxiliary mass M_1 with $\sigma_1(M_1) > u$, so Form (B), whose constants are built from (I2) at one such amplitude, already contains every amplitude at which the positive-source bound is known. For $u \ge s_*$ the statement is open, and it is the exact obstruction. The scheme enters only through the exact relation $\alpha^a_{k,T}(u) = \alpha^z_{k,T}(u) + (z_k - a_k)u$ (proof of Proposition 12.8); the discussion below is in the reference scheme z. By Lemma 12.9, exact at fixed (k, T), $\alpha^z_{k,T}(u)$ is, up to explicit c-numbers, $|T|^{-1}$ times the difference of the free energies of King's torus model with bare scalar coefficient $\alpha^{\mathrm{tor}}_{k,T}(m) - 2u$ and with $\alpha^{\mathrm{tor}}_{k,T}(m)$, both with the counterterms of the mass m: the second is Proposition 12.6 at m, the first the ultraviolet stability of the shifted model. For $u < s_*$ the shifted coefficient is that of King's model at a positive mass M_1; for $u \ge s_*$ it lies, asymptotically in k, below $\inf_{\mathsf{M}>0}\alpha^\infty_k(\mathsf{M})$, and no positive scalar mass realises it. The proofs of [1, 2] expand about the Gaussian at the positive scalar mass m and use its mass term in the small-field bounds; no published theorem covers a bare coefficient below every positive-mass value. The two open statements are related by Proposition 13.2, whose steps do not depend on the amplitude: run at $s_1 = u$ it turns $(\mathrm{B}^-)[u]$ into the bound $\ln N^a_{k,X}(0, q) \le (C_U/u)\|q\|_{1,k}$ for $0 \le q \le u$ on every Neumann–Dirichlet box, that is into clause (b) of Open Problem 32.2 in L^1 form on $\sup q \le u$. The converse direction is not known: the inequalities available bound a constant-source density from below by local sources, never from above. Also open: a form of the tube bound (37) in which the negative part k_- of the Wick source enters through its L^1 norm in place of the weighted norm $|k_-|_w$.

32.6. **Rotation invariance with scalar sources: the points of E_W and the sources above the threshold.** Theorem 15.1 concerns curvature sources, and Theorem 27.13 (Section 27, whose notation we use) curvature sources together with neutral scalar sources on the tube $\Omega_*(h)$, outside the set E_W. The following two problems are not solved in this paper; neither is a hypothesis of Theorem 15.1 or of Theorem 27.13.

1. The points of E_W. At a point of E_W (Proposition 26.8), $s_*(e) = 0$ and the cone, the tube and the split domain of Definitions 27.2 and 27.3 are empty; Theorem 15.1 covers $h = 0$ there by right-continuity in the scalar mass (Theorem 26.7), and nothing with a scalar source is proved. The statement that would extend Theorem 27.13 to such a point is: $(\mathrm{W}^\sharp 2)$ at that point with some positive threshold s' in place of $s_*(e)$ and $(\mathrm{W}^\sharp 1)$ on the split domain of s' there (Theorem 27.6(i) uses it), together with the two walled-torus statements of the Wick sector in the scheme $a^\sharp$ on the cone of s' at that point: the walled torus comparison on the cone (Theorem 27.9, Section 27.3) and the closing statement on the cone (Section 27.4); both rest on the walled large-field envelope, statement (3) of Section 27.3, which is proved where $s_*(e) > 0$. With these, every statement of Section 27 holds at that point with s' in place of $s_*(e)$, by the

same proofs. Right-continuity in the scalar mass does not provide them: the tubes $\Omega_*(h)$ at the nearby masses shrink to the empty set as the mass decreases to m.

2. The generating functional above the threshold. ($\mathrm{W}^\sharp 1$) at the points outside $D(h)$, where the positive part of the split source exceeds half the threshold, and ($\mathrm{W}^\sharp 2$) at every amplitude are not proved (the former would need the large-field half of the cube bound above the threshold, the latter the ultraviolet stability bound at a negative mass); they would extend the domain of $Z^\sharp_\infty$ beyond $\Omega_*(h)$. The invariance there would need, in addition, the walled torus comparison and the closing statement of the scheme $a^\sharp$ above the cone (Sections 27.3 and 27.4); the plain-torus comparison of King's type with scalar sources, the equality of the two lattice limits of $Z^a_{\Lambda_k,D_J}$ for every smooth periodic family at every real coefficient, which would give the walled form on all of $\mathcal{K}$, is likewise not proved. None of this restricts a Schwinger function: every Schwinger function with curvature and Wick-square insertions is a derivative at the interior point 0 of $\Omega_*(h)$ (Theorem 27.13(iii)).

32.7. **The coupling region.** The sign clause (T3) is proved on the explicit regions of Propositions 29.29 and 29.30 (for every $\mu_0 \in (0, m]$ on the box $0 < \lambda < 2.14398\, m$, $0 < e^2 < 6.30\, m$, uniform in the photon mass; exactly for $\lambda < \frac{\pi}{4}(1+\sqrt{5})\, m$ at $e = 0$), and the Wick leg is stated on the coupling region of Definition 2.1, which contains them. Outside those regions the exact position of the failure set $E^<_W$ is not determined; the two-sided clause that m is not a global minimiser of α_∞, with the auxiliary mass on either side of m (Proposition 12.13, Lemma 29.4(i)), would enlarge the region.

Index of notation

The index lists the principal symbols and tags, with the place where each is introduced.

Couplings, lattices and cubes.

e, λ, m, μ_0	the charge, the quartic coupling, the scalar mass and the photon mass; King's range is the set of all positive values	Section 2, Section 29
ε_k, Λ_k	the spacing 5^{-k} and the lattice $\varepsilon_k\mathbb{Z}^3$	Section 2
M	King's large-block integer	Section 2, Remark 2.4
C_J, X_J	King's cube $[-MJ, MJ)^3$ and its site set $C_J \cap \Lambda_k$	Section 2
J	the index of the cube C_J	Section 2
T_J	the three top face layers: the sites of X_J with $x_\mu = MJ - \varepsilon_k$ for some μ	Definition 2.3
J_{K}	King's stopping index	Section 29

States and functionals.

$\langle\cdot\rangle^{\mathrm{ND}}_{k,X}$	the Neumann–Dirichlet state on a finite set $X \subset \Lambda_k$	Definition 2.2
$\mathrm{ND}^{(a)}_{k,J}$, $\langle\cdot\rangle^{(a)}_{k,J}$	the comparison cube: the state on the comparison convention (the scalar alive on all of X_J), and its expectation	Definition 2.2
$\mathrm{ND}^{K}_{k,J}$, $\langle\cdot\rangle^{K}_{k,J}$	King's RG cube: the state on King's convention (the scalar killed on T_J), and its expectation	Definition 2.3

$Z_{k,J}$, $Z^K_{k,J}$	the generating functionals of the curvature on the comparison convention and on King's convention	Definition 2.5
$Z^a_{k,J}$, $Z^{a,K}_{k,J}$	the same with scalar sources, in the scheme a	Definition 2.5
Z^K_∞	King's state: the limit along King's cubes, $k \to \infty$ first and then $J \to \infty$	Section 1
G^K_∞	King's tube object: the limit on the tube of the joint functional of (F, W^a) on King's cubes	Section 1.2, item (6); Theorem 28.5
S_n, $S_{n,p}$	the Schwinger functions, defined from the derivatives of the limit functionals at 0	Section 1.2
μ_F	the measure of the curvature sector	Section 1.2, item (5); Theorem 11.5 (there $\mu^{K,H}$)

Sources, norms and the tube.

f, h	a curvature source, a real 2-form in C^∞_c; a scalar source, a real function in C^∞_c	Definition 2.5
$F(f)$, $W^a(h)$	the curvature paired with f; the renormalised Wick square in the scheme a, paired with h	Definition 2.5
$a = (a_k)$, $a^\sharp$	a scheme; the reference scheme	Definition 2.5, Definition 2.7
ζ, $\Theta(\zeta)$	a finite family of sources with complex coefficients, $(z_j, f_j; w_j, h_j)_j$; its exponential observable	Definition 2.5
$\mathcal{K}$, $\mathcal{I}$, $\mathcal{D}$	the cone and its two classes of observables	Definition 2.5
θ, τ_b	the reflection about a coordinate plane; the translation by b	Definition 2.5
$\mathsf{a}(\zeta)$	the amplitude of a family	Definition 2.5
$g(\zeta)$, $k(\zeta) = k_+ - k_-$	the real parts of the curvature source and of the scalar source of a family; $k_\pm$ are the pointwise positive and negative parts of $k(\zeta)$	Definition 2.8
$\|\cdot\|_2$, $\|\cdot\|_w$, $\|\cdot\|_4$	the norms of the regularity bounds	Definition 2.8
$\mathcal{D}_{\sigma,H}$, $\mathcal{D}_{\sigma_+,H}$	the tube with parameters σ, H; the tube of the Wick leg, with $\sigma = \sigma_+$	Definition 2.8

Groups and axioms.

O_h, $\mathbb{R}^3 \rtimes O(3)$	the cubic group; the Euclidean group	Section 1.2
OS0, OS1, OS2, OS3, OS4	the axioms: regularity, Euclidean invariance, reflection positivity, symmetry, clustering	Section 1
E0″	the form of the regularity axiom that is proved	Theorem 7.6
weak form	the form of the cluster axiom proved in the Wick sector and in the limit of vanishing photon mass (ergodicity of the components under translations)	Definition 11.2

Tags.

(T3)	the sign clause: $\sigma_1(M_1) > 0$ for some auxiliary scalar mass $M_1 \in (0, m)$, where $\sigma_1(M_1)$ is the lower limit of half the difference of King's free-boundary bare coefficients at the scalar masses m and M_1	Section 2, Definition 2.1
(I2)	the extensive bound for constant positive scalar sources	Definition 12.5
(1)–(7), (R0)–(R4)	the numbered rows of the table of results, with (1′) and sub-rows such as (3a), (4b) and (6a)	Table 1
(N1)–(N6)	the two-flow comparison bound; decay and regularity of the block-spin operators; King's upper bound; the hybrid lower bound; the polynomial lower bound; the walled axis-lattice limit	Section 19
(SK), $(\mathrm{BAL})_5$, (CT)	the skew-torus instances; Bałaban's estimates at blocking factor five; the convention bridge	Section 19
(F), (F_0)	one-sided bounds at one spacing	Definition 18.4
$(\mathrm{SW1}^\circ)_{\mathrm{int}}$, $(\mathrm{SW2}^{\mathrm{min}})$, $(\mathrm{SW3}^\circ)_{\mathrm{int}}$, (SWT)	the four shell bounds: the two-sided bound, the disc bound, the κ-Cauchy property, the threshold clause	Section 23.4; Theorem 24.5, Lemma 25.13, Theorem 25.11, Corollary 25.12
$(\mathrm{W}^\sharp)$	the scheme condition on the cone	Definition 27.4

The rotation part and the limits.

θ_0	the rotation angle: $\cos\theta_0 = \frac{4}{5}$, $\sin\theta_0 = \frac{3}{5}$	Definition 16.1
King's tori	the periodic tori $\mathbb{R}^3/(\ell\mathbb{Z})^3$, $\ell \in 2M\mathbb{N}$; in the rotation part $D_J = \mathbb{R}^3/(50MJ\,\mathbb{Z})^3$	Section 2, Definition 12.1, Definition 16.1
walled tori	periodic tori with a mass wall	Definition 16.4, Definition 21.1
$\mathfrak{c}$	a configuration (J, ψ, n) of the class $\mathcal{C}$	Definition 16.6
E_W	the exceptional set of couplings: a finite set of scalar masses when the other couplings are fixed	Proposition 26.8
modified box state	King's modified box state, reflection positive (Lemma 9.3); the sandwich carries the positivity to the cube	Definition 9.2
continuum subsequence, LIM state	subsequences of spacings and the limit states that they define	Definitions 14.1 and 14.2
fixed-counterterm members	the family with the bare scalar coefficient held at the reference photon mass, that is, computed with the photon propagator at the mass μ_0	Definition 31.1
$\mathsf{M}(\mu)$	the shifted scalar mass at the photon mass μ: the unique zero of Ψ_μ, defined under the first of the two open conditions of Proposition 31.6	Proposition 31.6

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