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Supersymmetric Analysis of a Random Band Matrix at Criticality

Volume II

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AUTHOR'S NOTE

With the exception of this note, the entire document is machine-written. I have carefully supervised its preparation to ensure that the proofs are readable and that previous literature is appropriately cited. I welcome suggestions concerning mathematical corrections, omitted citations, and attribution of ideas.

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ABSTRACT

We study the $N \times N$ complex Hermitian Gaussian band matrix whose variance profile is the inverse of $\mathrm{Id} - W^2\Delta$, where Δ is the Neumann Laplacian on $\{1, \dots, N\}$, in the critical regime $W^2 \asymp N$. We prove that, for energies E in a neighborhood of zero, the unfolded eigenvalue process converges to a stationary point process, whose law depends only on the limit X of $N/((4 - E^2)W^2)$ and is given by the partition functions of a supersymmetric sigma model on an interval. We also determine the macroscopic law of the mass of an eigenvector selected near a given energy. Its density is the normalized exponential of a Brownian motion with drift, under the law tilted by its total mass; the same law describes the eigenvectors of the critical one-dimensional random Schrödinger operator. Further, we compute the limits of the two-energy correlations of the local densities of states at all pairs of interior sites, of the mean eigenvector profile at the scale of single sites, and of N times the mean inverse participation ratio, which is $2 + 2X/3$. As $X \to 0$ and $X \to \infty$, we recover the pair statistics of the GUE and of a Poisson process, and in the localized limit we obtain an explicit random shape for the eigenvectors. The number variance in a window of length L grows like $\sqrt{XL}$ as $L \to \infty$ at every fixed X, and we describe the crossover to Poisson statistics in windows of length of order X. Several of these results confirm predictions that physicists obtained from the sigma model. The proof starts from the supersymmetric representation of averages of ratios of characteristic polynomials, which for this profile is a transfer operator on a one-dimensional chain. We show that the transfer chain converges to the heat semigroup of the sigma model, whose noncompact direction produces the Brownian motion in the eigenvector law.

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PART III

From the sigma model to the eigenvalues and eigenvectors

SECTION 7

Inputs and the interface theorem

In Part III, we prove the results of Section 1 from the sigma model of Section 5. We use three kinds of input.

The first input is the sigma model itself. We use the definitions of Section 5, the weighted realization of the generator in Proposition 5.15, the multiplier bounds of Lemma 5.14, and the analyticity of the partition function in the sources in Lemma 6.14. We also use the supersymmetric localization Lemma 5.19 and its consequences Corollaries 5.20 and 5.21. Finally, we use the reduction of rank one in Section 6 (Lemmas 6.1, 6.3 and 6.5 to 6.7), with the rank-one calculus of Section 6.4 and the two source derivatives at rank one computed in Lemma 6.15.

The second input consists of facts about the band matrix H that do not involve supersymmetry: the lower bounds on its variance profile, the factorial moment bounds for eigenvalue counts, the asymptotics of the expected density, and the scalar Gaussian component of H. We prove them in Sections 7.1 to 7.4.

The third input is the interface theorem, Theorem 7.6, which collects the four statements about H that Part IV proves. We state it in terms of the matrix H and of the objects of Section 5 only, so that Part III can be read without Part IV.

The sections of Part III are organized as follows. Section 8 identifies the limiting spectral point process and its source law. Section 9 shows how moments of eigenvector entries are extracted from local resolvent moments at positive damping. Section 10 computes the limits of the sigma-model kernels as the damping tends to zero, in terms of the Brownian functionals of Definition 1.9. Sections 11 and 12 prove the results on local intensities, the inverse participation ratio and the coarse eigenvector law. Sections 13 and 14 analyze the function I_X and the spectral fluctuations on all scales.

Throughout this section, H is the Gaussian band matrix of Definition 1.1, with variance matrix $J = (\mathrm{Id} - W^2\Delta)^{-1}$, and we use the scaling of (1.3).

7.1 The variance profile

We first verify the image formula (1.2) for J, and we record two lower bounds on the entries of J that follow from it. Recall from Section 1.2 that $\theta_W = 2\,\mathrm{arsinh}(1/(2W))$ and $J_{\mathbb{Z}}(m) = (1+4W^2)^{-1/2}e^{-\theta_W|m|}$ for $m \in \mathbb{Z}$.

Lemma 7.1. *Let $N \geq 2$ and $W > 0$. Then the image formula* (1.2) *holds for all $i, j \in \{1, \dots, N\}$. In particular, $J_{ij} \geq J_{\mathbb{Z}}(i-j) > 0$ and $J_{ii} \geq (1+4W^2)^{-1/2}$. If $W \geq 1$, then*

$$J_{ij} \geq \frac{c_J}{W} \quad \text{for } |i-j| \leq W, \qquad c_J = \frac{e^{-1}}{\sqrt{5}}. \tag{7.1}$$

Proof. We first check that $J_{\mathbb{Z}}(i-j)$ is the kernel of $(\mathrm{Id} - W^2\Delta_{\mathbb{Z}})^{-1}$. By the definition of θ_W and the identities $\cosh\theta - 1 = 2\sinh^2(\theta/2)$ and $\sinh\theta = 2\sinh(\theta/2)\cosh(\theta/2)$, we have

$$\cosh\theta_W = 1 + \frac{1}{2W^2}, \qquad \sinh\theta_W = \frac{\sqrt{1+4W^2}}{2W^2}.$$

For $r \neq 0$, we therefore have

$$J_{\mathbb{Z}}(r) - W^2\big(J_{\mathbb{Z}}(r+1) + J_{\mathbb{Z}}(r-1) - 2J_{\mathbb{Z}}(r)\big) = J_{\mathbb{Z}}(r)\big(1 - 2W^2(\cosh\theta_W - 1)\big) = 0,$$

and for $r = 0$ the left side is

$$\begin{aligned} J_{\mathbb{Z}}(0)\big(1 + 2W^2(1 - e^{-\theta_W})\big) &= J_{\mathbb{Z}}(0)\big(1 + 2W^2(1 - \cosh\theta_W + \sinh\theta_W)\big) \\ &= J_{\mathbb{Z}}(0)\sqrt{1+4W^2} = 1. \end{aligned}$$

Next, let $\widetilde{J}_{ij}$ denote the right side of (1.2), extended to all $i \in \mathbb{Z}$ by the same formula; the series converges absolutely. Since $J_{\mathbb{Z}}$ is even, $\widetilde{J}_{ij}$ is invariant under the reflections $i \mapsto 1-i$ and $i \mapsto 2N+1-i$, that is,

$$\widetilde{J}_{1-i,j} = \widetilde{J}_{ij} = \widetilde{J}_{2N+1-i,j}, \qquad i \in \mathbb{Z}.$$

In particular, $\widetilde{J}_{0j} = \widetilde{J}_{1j}$ and $\widetilde{J}_{N+1,j} = \widetilde{J}_{Nj}$, so for $1 \leq i \leq N$ the Neumann Laplacian Δ acts on $\widetilde{J}_{\cdot j}$ as the Laplacian $\Delta_{\mathbb{Z}}$ of $\mathbb{Z}$. By the first step, we have, for $i \in \mathbb{Z}$,

$$\begin{aligned} (\mathrm{Id} - W^2\Delta_{\mathbb{Z}})\, J_{\mathbb{Z}}(\cdot - j + 2mN)(i) &= \mathbf{1}\{i = j - 2mN\}, \\ (\mathrm{Id} - W^2\Delta_{\mathbb{Z}})\, J_{\mathbb{Z}}(\cdot + j - 1 + 2mN)(i) &= \mathbf{1}\{i = 1 - j - 2mN\}. \end{aligned}$$

Only one of these points lies in $\{1, \dots, N\}$, namely $i = j$, from the first family with $m = 0$. We conclude that

$$\big((\mathrm{Id} - W^2\Delta)\widetilde{J}\big)_{ij} = \delta_{ij}, \qquad i, j \in \{1, \dots, N\},$$

which means that $\widetilde{J} = J$.

Keeping only the term $J_{\mathbb{Z}}(i-j)$ of (1.2), whose other terms are positive, we find

$$J_{ij} \geq J_{\mathbb{Z}}(i-j), \qquad J_{ii} \geq J_{\mathbb{Z}}(0) = (1+4W^2)^{-1/2}.$$

Now let $W \geq 1$ and $|i-j| \leq W$. Using $\operatorname{arsinh} y \leq y$ for $y \geq 0$, we have $\theta_W \leq 1/W$, and the first bound yields

$$J_{ij} \geq J_{\mathbb{Z}}(i-j) \geq \frac{e^{-\theta_W W}}{\sqrt{1+4W^2}} \geq \frac{e^{-1}}{\sqrt{5}\,W} = \frac{c_J}{W},$$

where we used $1 + 4W^2 \leq 5W^2$ in the last inequality. □

7.2 Factorial moments of eigenvalue counts

For an interval $I \subset \mathbb{R}$, let $\mathcal{N}_H(I)$ denote the number of eigenvalues of H in I, counted with multiplicity. For $a \in \mathbb{R}$ and an integer $m \geq 1$, let $(a)_m = a(a-1)\cdots(a-m+1)$ denote the falling factorial.

The following bound is due to Peled, Schenker, Shamis, and Sodin [104], who proved it for deformed block-Gaussian matrices. For $m = 1$ it is a Wegner estimate, and for $m \geq 2$ an estimate of Minami type. The bound depends on the blocks only through their total size, and this is what makes it uniform in W. We reduce our ensemble to theirs by splitting off a block-diagonal Gaussian matrix, as in their proof of [104, Corollary 1.6]. The only difference is that our variance profile is not translation invariant, and we use Lemma 7.1 in place of their assumption on the shape function.

Lemma 7.2. *There exists an absolute constant $C_{\mathrm{cnt}} > 0$ such that for all $N \geq W \geq 4$, all intervals $I \subset \mathbb{R}$ and all integers $m \geq 1$,*

$$\mathbb{E}\big(\mathcal{N}_H(I)\big)_m \leq \big(C_{\mathrm{cnt}} N |I|\big)^m. \tag{7.2}$$

In particular, for all $E \in (-2,2)$, all bounded intervals $A \subset \mathbb{R}$, and all $m \geq 1$, the unfolded process Ξ_N of Section 1.2 satisfies

$$\mathbb{E}\big(\Xi_N(A)\big)_m \leq \Big(\frac{C_{\mathrm{cnt}}|A|}{\rho(E)}\Big)^m. \tag{7.3}$$

Proof. We first recall the result of [104] in the form we use. In the normalization of [104, Section 1 and Remark 1.5], a GUE matrix of size n is a Hermitian matrix V with density proportional to $\exp(-\frac{n}{2}\operatorname{Tr} V^2)$. Its entries on and above the diagonal are independent, the diagonal entries are real centered Gaussians of variance $1/n$, and the off-diagonal entries have independent centered real and imaginary parts of variance $1/(2n)$ each. Let $V_1, \dots, V_\ell$ be independent GUE matrices of sizes $n_1, \dots, n_\ell$. Let A be a deterministic Hermitian matrix of size $n_1 + \cdots + n_\ell$. The unitary case of [104, Theorem 3] states that there exists an absolute constant C_1 such that for all intervals I and all $m \geq 1$,

$$\mathbb{E}\big(\mathcal{N}_{A+V_1\oplus\cdots\oplus V_\ell}(I)\big)_m \leq \Big(C_1|I|\sum_{j=1}^{\ell} n_j\Big)^m. \tag{7.4}$$

We now write H in this form. Let $\ell = \lceil N/\lfloor W\rfloor\rceil$, and partition $\{1,\dots,N\}$ into ℓ consecutive discrete intervals whose lengths $n_1,\dots,n_\ell$ satisfy $n_j \in \{\lfloor N/\ell\rfloor, \lceil N/\ell\rceil\}$. We claim that

$$\frac{W}{4} \leq n_j \leq W, \qquad 1 \leq j \leq \ell. \tag{7.5}$$

The upper bound follows from $N/\ell \leq \lfloor W\rfloor \leq W$. For the lower bound, the inequalities $\ell < N/\lfloor W\rfloor + 1$ and $N \geq \lfloor W\rfloor$ imply that

$$\frac{N}{\ell} > \frac{N\lfloor W\rfloor}{N+\lfloor W\rfloor} \geq \frac{\lfloor W\rfloor}{2}, \qquad \text{so that} \qquad n_j \geq \Big\lfloor \frac{N}{\ell}\Big\rfloor \geq \frac{\lfloor W\rfloor - 1}{2} \geq \frac{W-2}{2} \geq \frac{W}{4},$$

where we used $W \geq 4$ in the last step. Two sites i, i' in the jth interval satisfy

$$|i - i'| \leq n_j - 1 < W,$$

and using (7.1) and (7.5), we obtain

$$J_{ii'} \geq \frac{c_J}{W} \geq \frac{\sigma^2}{n_j}, \qquad \sigma^2 = \frac{c_J}{4}. \tag{7.6}$$

Let $V_1, \dots, V_\ell$ be independent GUE matrices of sizes $n_1, \dots, n_\ell$. We let H_0 be an independent Hermitian matrix with independent centered Gaussian entries on and above the diagonal, with real diagonal entries and with independent real and imaginary parts of equal variance off the diagonal, such that

$$\mathbb{E}|(H_0)_{ii'}|^2 = J_{ii'} - \frac{\sigma^2}{n_j}\mathbf{1}\{i, i' \text{ lie in the } j\text{th interval}\}.$$

These variances are nonnegative by (7.6). The matrix $H_0 + \sigma(V_1 \oplus \cdots \oplus V_\ell)$ has independent centered Gaussian entries on and above the diagonal, with the same variances and the same real structure as H. It therefore has the law of H.

Conditionally on H_0, the matrix $\sigma^{-1}H_0 + V_1 \oplus \cdots \oplus V_\ell$ has the form of (7.4), with

$$A = \sigma^{-1}H_0, \qquad \sum_{j=1}^{\ell} n_j = N.$$

We apply (7.4) to it with the interval $\sigma^{-1}I$. Since $\mathcal{N}_H(I) = \mathcal{N}_{\sigma^{-1}H}(\sigma^{-1}I)$, we find

$$\mathbb{E}\big[\big(\mathcal{N}_H(I)\big)_m \mid H_0\big] \leq \Big(\frac{C_1 N|I|}{\sigma}\Big)^m,$$

and taking the expectation over H_0 proves (7.2) with $C_{\mathrm{cnt}} = C_1/\sigma$. Finally, we have

$$\Xi_N(A) = \mathcal{N}_H\big(E + a_N^{-1}A\big), \qquad \big|E + a_N^{-1}A\big| = \frac{|A|}{N\rho(E)},$$

and (7.3) follows from (7.2). □

7.3 The expected density

The case $m = 1$ of (7.2) shows that the expected eigenvalue distribution $N^{-1}\mathbb{E}\sum_\alpha \delta_{\lambda_\alpha}$ is absolutely continuous, with density bounded by C_{cnt}. We let $\rho_{N,W}$ denote this density. We will use the following theorem of Disertori, Lohmann, and Sodin [29], which identifies $\rho_{N,W}$ with the semicircle density up to an error $O(N^{-1} + W^{-2})$ for our ensemble. Their ensemble [29, (1.4)] has the variance matrix $(-W^2\Delta_N + 1_N)^{-1}$, where Δ_N has diagonal

entries -1 at the two endpoints and -2 elsewhere, and off-diagonal entries 1 between neighbors. We have $\Delta_N = \Delta$, so that this variance matrix is our J, and their ensemble is ours.[1] Their finite-volume density of states is our $\rho_{N,W}$, and their $\rho_{\mathrm{s.c.}}$ is our ρ.

Lemma 7.3. *For every compact interval $I \subset (-2,2)$, there exists $C_I > 0$ such that for all $W \geq 2$ and $N \geq C_I W \log W$,*

$$\sup_{\lambda \in I} |\rho_{N,W}(\lambda) - \rho(\lambda)| \leq C_I\big(N^{-1} + W^{-2}\big). \tag{7.7}$$

Consequently, let $N, W \to \infty$ with N/W^2 bounded below, and let $E_0 \in (0,2)$ and $R > 0$. Then, uniformly for $|E| \leq E_0$, the density $\varrho_N(x) = \rho_{N,W}(E + x/a_N)/\rho(E)$ of $\mathbb{E}\Xi_N$ satisfies

$$\sup_{|x| \leq R} |\varrho_N(x) - 1| \to 0, \tag{7.8}$$

and for every deterministic sequence $\bar{r}_N \downarrow 0$,

$$\sup_{0 < r \leq \bar{r}_N} \Big| \frac{A_N(E,r)}{N\rho(E)} - 1 \Big| \to 0, \qquad A_N(E,r) = \mathbb{E} \sum_{\alpha} \phi_{E,r}(\lambda_\alpha), \tag{7.9}$$

where $A_N(E,r)$ is the denominator of the unrooted selection in Definition 1.8.

Proof. The bound (7.7) combines the two estimates of [29, Theorem 1],

$$|\rho_{N,W}(\lambda) - \rho_\infty(\lambda)| \leq \frac{C(\lambda)}{N}, \qquad |\rho_\infty(\lambda) - \rho(\lambda)| \leq \frac{C(\lambda)}{W^2},$$

where ρ_∞ is their infinite-volume density of states. These estimates hold for $|\lambda| < 2$ and $N \geq C(\lambda) W \log W$, and [29, Remark 1.3] states that the constants $C(\lambda)$ are bounded on compact subintervals of $(-2,2)$.

We now prove the consequences, and we set

$$I = \Big[-\frac{2+E_0}{2}, \frac{2+E_0}{2}\Big].$$

The condition $N \geq C_I W \log W$ holds for all sufficiently large W, since N/W^2 is bounded below. Further, the bound $a_N \geq N\rho(E_0)$ implies that $E + x/a_N \in I$ for $|x| \leq R$ and all sufficiently large N, and for such x and N we have

$$\begin{aligned} |\varrho_N(x) - 1| &\leq \frac{|\rho_{N,W}(E + x/a_N) - \rho(E + x/a_N)| + |\rho(E + x/a_N) - \rho(E)|}{\rho(E_0)} \\ &\leq C\Big(\frac{1}{N} + \frac{1}{W^2} + \frac{R}{N}\Big), \end{aligned}$$

[1] Disertori, Lohmann, and Sodin write the covariance in [29, (1.2) and (1.4)] as $\mathbb{E}H_{x,y}H_{x',y'} = \delta_{x,x'}\delta_{y,y'}J_{xy}$. Read literally, this is not the covariance of a complex Hermitian matrix. We read it as $\mathbb{E}H_{x,y}\overline{H_{x',y'}} = \delta_{x,x'}\delta_{y,y'}J_{xy}$, which is (1.1). Their supersymmetric representation [29, Proposition 2.1], with one complex boson and one pair of Grassmann variables per site, confirms this reading, since it is the representation for complex Hermitian matrices.

where we used (7.7) and the fact that ρ is Lipschitz on I, and (7.8) follows. Next, the change of variables $\lambda = E + rv$ gives

$$\frac{A_N(E,r)}{N} = \int \phi_{E,r}(\lambda)\rho_{N,W}(\lambda)\,\mathrm{d}\lambda = \int_{-1}^{1} \phi(v)\rho_{N,W}(E+rv)\,\mathrm{d}v.$$

For $r \le \bar{r}_N$ and $|v| \le 1$, the argument above shows that

$$|\rho_{N,W}(E+rv) - \rho(E)| \le C\big(N^{-1} + W^{-2} + \bar{r}_N\big).$$

Using $\int \phi = 1$ and $\rho(E) \ge \rho(E_0) > 0$, we obtain

$$\Big|\frac{A_N(E,r)}{N\rho(E)} - 1\Big| \le \frac{1}{\rho(E_0)} \int_{-1}^{1} \phi(v)\,\big|\rho_{N,W}(E+rv) - \rho(E)\big|\,\mathrm{d}v \le C\big(N^{-1} + W^{-2} + \bar{r}_N\big),$$

which proves (7.9). □

7.4 The scalar Gaussian component

The matrix H contains an independent Gaussian multiple of the identity, whose unfolded standard deviation is of order $W^{1/2}$. Translating it moves all unfolded eigenvalues by the same amount and leaves the eigenvectors unchanged. We use the resulting approximate translation invariance for stationarity in Section 8, and also in Section 9.

Lemma 7.4. *We set*

$$\sigma_N^2 = \Big(\sum_{i=1}^{N} J_{ii}^{-1}\Big)^{-1}, \qquad \gamma_N = \sigma_N^2 \sum_{i=1}^{N} \frac{H_{ii}}{J_{ii}}, \qquad \widetilde{H} = H - \gamma_N \mathrm{Id}. \tag{7.10}$$

(a) The variable γ_N is a real centered Gaussian of variance σ_N^2, it is independent of $\widetilde{H}$, and $\sigma_N^{-2} \le N(1+4W^2)^{1/2}$.

(b) For every bounded measurable function F on $N \times N$ Hermitian matrices, all $E \in (-2,2)$ and all $b \in \mathbb{R}$,

$$\big|\mathbb{E}F(H + ba_N^{-1}\mathrm{Id}) - \mathbb{E}F(H)\big| \le \sqrt{2/\pi}\,\|F\|_\infty \frac{|b|}{a_N\sigma_N}. \tag{7.11}$$

(c) For every nonnegative measurable function Φ on $N \times N$ Hermitian matrices and all $b \in \mathbb{R}$ and $K > 0$, with $\Phi_b(H) = \Phi(H + ba_N^{-1}\mathrm{Id})$,

$$\big|\mathbb{E}\Phi_b(H) - \mathbb{E}\Phi(H)\big| \le \sqrt{2/\pi}\,\frac{K|b|}{a_N\sigma_N} + \frac{\mathbb{E}\Phi(H)^2 + \mathbb{E}\Phi_b(H)^2}{K}. \tag{7.12}$$

If $N, W \to \infty$ with N/W^2 bounded below and $|E| \le E_0 < 2$, then $(a_N\sigma_N)^{-1} \le CW^{-1/2}$, where C depends only on E_0 and on the lower bound for N/W^2.

Proof. The diagonal entries H_{ii} are independent real centered Gaussians of variance J_{ii}, and they are independent of the off-diagonal entries. As a linear combination of them, γ_N is a real centered Gaussian. By (7.10), we have

$$\mathrm{Var}(\gamma_N) = \sigma_N^4 \sum_{i=1}^N \frac{J_{ii}}{J_{ii}^2} = \sigma_N^2, \qquad \mathbb{E}[\gamma_N H_{ii}] = \sigma_N^2 \frac{J_{ii}}{J_{ii}} = \sigma_N^2.$$

In particular, the diagonal entries $\widetilde{H}_{ii} = H_{ii} - \gamma_N$ satisfy

$$\mathbb{E}[\gamma_N \widetilde{H}_{ii}] = \mathbb{E}[\gamma_N H_{ii}] - \mathrm{Var}(\gamma_N) = 0, \qquad 1 \le i \le N,$$

and γ_N is uncorrelated with the off-diagonal entries of $\widetilde{H}$, which are those of H. Since the real coordinates of $(\gamma_N, \widetilde{H})$ are jointly Gaussian, γ_N and $\widetilde{H}$ are independent. Further, the bound $J_{ii} \ge (1+4W^2)^{-1/2}$ of Lemma 7.1 implies that

$$\sigma_N^{-2} = \sum_{i=1}^N J_{ii}^{-1} \le N(1+4W^2)^{1/2}.$$

This proves part (a).

We now prove part (b). Set $\beta = b/a_N$, and for $\sigma > 0$ let n_σ denote the centered Gaussian density on $\mathbb{R}$ of variance σ^2. Writing $H + \beta\mathrm{Id} = \widetilde{H} + (\gamma_N + \beta)\mathrm{Id}$ and using the independence of γ_N and $\widetilde{H}$ from part (a), we have

$$\begin{aligned} \big|\mathbb{E}F(H+\beta\mathrm{Id}) - \mathbb{E}F(H)\big| &= \Big|\mathbb{E}\int_{\mathbb{R}} F(\widetilde{H} + y\mathrm{Id})\big(\mathsf{n}_{\sigma_N}(y-\beta) - \mathsf{n}_{\sigma_N}(y)\big)\,\mathrm{d}y\Big| \\ &\le \|F\|_\infty\, |\beta|\, \|\mathsf{n}'_{\sigma_N}\|_{L^1}, \end{aligned}$$

and part (b) follows from

$$\|\mathsf{n}'_{\sigma_N}\|_{L^1} = 2\mathsf{n}_{\sigma_N}(0) = \sqrt{2/\pi}\,\sigma_N^{-1}.$$

For part (c), let $F = \min(\Phi, K)$, so that $\|F\|_\infty \le K$ and $0 \le \Phi - F = (\Phi - K)_+ \le \Phi^2/K$. The same holds for Φ_b and $F_b(H) = F(H + ba_N^{-1}\mathrm{Id}) = \min(\Phi_b(H), K)$. Using part (b) for F and these bounds, we obtain

$$\begin{aligned} \big|\mathbb{E}\Phi_b(H) - \mathbb{E}\Phi(H)\big| &\le \big|\mathbb{E}F_b(H) - \mathbb{E}F(H)\big| + \mathbb{E}\big(\Phi_b(H) - K\big)_+ + \mathbb{E}\big(\Phi(H) - K\big)_+ \\ &\le \sqrt{2/\pi}\,\frac{K|b|}{a_N\sigma_N} + \frac{\mathbb{E}\Phi(H)^2 + \mathbb{E}\Phi_b(H)^2}{K}, \end{aligned}$$

which is (7.12).

Finally, the bound on σ_N^{-2} in part (a) implies that

$$(a_N\sigma_N)^{-2} \le \frac{(1+4W^2)^{1/2}}{N\rho(E)^2} \le \frac{C}{W},$$

where we used $a_N = N\rho(E)$, $\rho(E) \ge \rho(E_0)$, and $N \ge cW^2$ in the last step. □

7.5 The interface theorem

We now state the interface theorem. Its four parts concern the source ratios $T_N^{(k)}$ of (1.4), their derivatives with respect to local sources at finitely many sites, the diagonal resolvent moments at all sites, and the one-point function. We begin with the local sources.

Throughout this section, $k \geq 1$ and $\varepsilon > 0$ are fixed, $\mathbf{u} = (u_1, \dots, u_k) \in \mathbb{R}^k$ satisfies $\sum_j u_j = 0$, and $\mathbf{b} = (b_1, \dots, b_k) \in (0, \infty)^k$. Recall from Definition 4.1 the source supermatrix $\Gamma = \Gamma(\mathbf{u}, \mathbf{b}) = \operatorname{diag}(\varepsilon \Lambda_{\mathrm{B}} \mid \Gamma_{\mathrm{F}})$, with

$$\Gamma_{\mathrm{F}} = \operatorname{diag}\big(b_1 - \mathrm{i}u_1, \dots, b_k - \mathrm{i}u_k, -b_1 - \mathrm{i}u_1, \dots, -b_k - \mathrm{i}u_k\big),$$

and $\eta_0 = a_N^{-1}$, where the $2k$ flavors are ordered as k retarded flavors followed by k advanced flavors. Then

$$\begin{gathered} E + \mathrm{i}\eta_0 \Gamma = \operatorname{diag}(Z_{\mathrm{B}} \mid Z_{\mathrm{F}}), \\ Z_{\mathrm{B}} = \operatorname{diag}\big(z_+^{\mathrm{B}} \mathrm{Id}_k, z_-^{\mathrm{B}} \mathrm{Id}_k\big), \quad Z_{\mathrm{F}} = \operatorname{diag}\big(z_{1,+}^{\mathrm{F}}, \dots, z_{k,+}^{\mathrm{F}}, z_{1,-}^{\mathrm{F}}, \dots, z_{k,-}^{\mathrm{F}}\big), \end{gathered} \tag{7.13}$$

where $z_\pm^{\mathrm{B}} = E \pm \mathrm{i}\varepsilon / a_N$ and $z_{j,\pm}^{\mathrm{F}} = E + (u_j \pm \mathrm{i}b_j)/a_N$. Let $e_1, \dots, e_N$ denote the standard basis of $\mathbb{C}^N$.

Definition 7.5. Let $m \geq 0$, let $1 \leq i_1 < \cdots < i_m \leq N$ be distinct sites, and for $1 \leq a \leq m$ let $Y_a = \operatorname{diag}(Y_a^{\mathrm{B}} \mid Y_a^{\mathrm{F}})$ be an even supermatrix with complex $2k \times 2k$ blocks Y_a^{B} and Y_a^{F}. We write $\mathbf{Y} = (Y_1, \dots, Y_m)$, and we define the *local source function*

$$T_N^{(k)}(\mathbf{u}, \mathbf{b}; \mathbf{Y}) = \mathbb{E} \frac{\det\big(\mathrm{Id}_N \otimes Z_{\mathrm{F}} - H \otimes \mathrm{Id}_{2k} + \sum_{a=1}^m e_{i_a} e_{i_a}^* \otimes Y_a^{\mathrm{F}}\big)}{\det\big(\mathrm{Id}_N \otimes Z_{\mathrm{B}} - H \otimes \mathrm{Id}_{2k} + \sum_{a=1}^m e_{i_a} e_{i_a}^* \otimes Y_a^{\mathrm{B}}\big)}. \tag{7.14}$$

In (7.14), the local source Y_a is added to $E + \mathrm{i}\eta_0\Gamma$ at the site i_a. In the notation of Section 4, we have

$$T_N^{(k)}(\mathbf{u}, \mathbf{b}; \mathbf{Y}) = \mathbb{E}\, \mathrm{SDet}\Big(\mathrm{Id}_N \otimes (E + \mathrm{i}\eta_0\Gamma) - H \otimes \mathrm{Id} + \sum_{a=1}^m e_{i_a} e_{i_a}^* \otimes Y_a\Big)^{-1},$$

and the right side is $T_N^{(k)}(\Gamma; \mathbf{Y})$ in Lemma 4.14. Since the eigenvalues $z_\pm^{\mathrm{B}} - \lambda_\alpha$ of $\mathrm{Id}_N \otimes Z_{\mathrm{B}} - H \otimes \mathrm{Id}_{2k}$ have modulus at least ε / a_N, the denominator in (7.14) is bounded away from zero, uniformly in H, when $\max_a \|Y_a^{\mathrm{B}}\| \leq \varepsilon/(2a_N)$. The numerator is a polynomial in the entries of H. The local source function is therefore finite and holomorphic in $\mathbf{Y}$ near zero, and we may differentiate it under the expectation. Only its derivatives at $\mathbf{Y} = 0$ are used.

The determinants at $\mathbf{Y} = 0$ factor over the eigenvalues of H. Using

$$(\lambda - z_{j,+}^{\mathrm{F}})(\lambda - z_{j,-}^{\mathrm{F}}) = \frac{(x - u_j)^2 + b_j^2}{a_N^2}, \qquad (\lambda - z_+^{\mathrm{B}})(\lambda - z_-^{\mathrm{B}}) = \frac{x^2 + \varepsilon^2}{a_N^2}$$

for $x = a_N(\lambda - E)$, we find $T_N^{(k)}(\mathbf{u}, \mathbf{b}; 0) = T_N^{(k)}(\mathbf{u}, \mathbf{b})$. The derivatives at $\mathbf{Y} = 0$ are expectations of the product in (1.4) times polynomials in the resolvent entries $G_{i_a i_b}(z)$ with $z \in \{z_\pm^{\mathrm{B}}, z_{j,\pm}^{\mathrm{F}}\}$. For instance, let $k = 1$, $\mathbf{u} = 0$, and $\mathbf{b} = \varepsilon$, so that $Z_{\mathrm{F}} = Z_{\mathrm{B}}$. Since

$$\det(z - H + te_i e_i^*) = \det(z - H)\big(1 - tG_{ii}(z)\big),$$

a fermionic source $Y^{\mathrm{F}} = \mathrm{diag}(t, 0)$ at the site i, with $Y^{\mathrm{B}} = 0$, gives the first identity in

$$-\partial_t T_N^{(1)}(0, \varepsilon; Y)\big|_{t=0} = \mathbb{E} G_{ii}(z_+^{\mathrm{B}}), \qquad \partial_t \partial_{t'} T_N^{(1)}(0, \varepsilon; Y_1, Y_2)\big|_{t=t'=0} = \mathbb{E} G_{ii}(z_+^{\mathrm{B}}) G_{jj}(z_-^{\mathrm{B}}), \tag{7.15}$$

and the fermionic sources $Y_1^{\mathrm{F}} = \mathrm{diag}(t, 0)$ at the site i and $Y_2^{\mathrm{F}} = \mathrm{diag}(0, t')$ at a site $j > i$, with $Y_1^{\mathrm{B}} = Y_2^{\mathrm{B}} = 0$, give the second.

The limits of these derivatives are described by the sigma model of Section 5. Recall the generator $\mathsf{G}_{X,\Gamma} = X\mathsf{C}_k + \pi\,\mathrm{STr}(Q\Gamma)$ of (5.23), realized as in Proposition 5.15, and the partition function

$$\mathcal{Z}_{X,\varepsilon}^{(k)}(\mathbf{u}, \mathbf{b}) = \mathcal{B}_k\big(e^{-s\mathsf{G}_{X,\Gamma}} 1, e^{-(1-s)\mathsf{G}_{X,\Gamma}} 1\big) = \mathcal{B}_k\big(1, e^{-\mathsf{G}_{X,\Gamma}} 1\big), \qquad 0 < s < 1,$$

of Definition 5.16, where $\Gamma = \Gamma(\mathbf{u}, \mathbf{b})$; the second equality and the independence of s are Proposition 5.15(d). In the sigma model, the local resolvent of the matrix at a site is replaced by the even supermatrix

$$\mathcal{G}(Q) = -\kappa_E \mathrm{Id} + \mathrm{i} s_E Q, \qquad \kappa_E = E/2. \tag{7.16}$$

Its retarded and advanced diagonal entries at $Q = \Lambda$ are the limits $-\kappa_E \pm \mathrm{i} s_E$ of $G_{ii}(E \pm \mathrm{i}0)$ for the semicircle law. Recall from Section 4.6 that the points of the balanced component of the saddle manifold are the supermatrices $S = \kappa_E - \mathrm{i} s_E Q$, where Q is the field on $\mathcal{M}_k$, and that $(E - S)^{-1} = \kappa_E - \mathrm{i} s_E Q = -\mathcal{G}(Q)$ at these points, by (4.24). For an even supermatrix Y as in Definition 7.5, we define the *orbit multiplier* $\mathrm{j}_Y(Q) = \mathrm{SDet}(\mathrm{Id} - \mathcal{G}(Q)Y)^{-1}$, as a formal power series in the entries of Y. By (4.30), a local source Y multiplies the onsite weight by $\mathrm{SDet}(\mathrm{Id} + (E - S)^{-1}Y)^{-1}$, and we have $\mathrm{SDet}(\mathrm{Id} + (E - S)^{-1}Y)^{-1} = \mathrm{j}_Y(Q)$ at the point $S = \kappa_E - \mathrm{i} s_E Q$. For a multi-index α in the entries of Y, we let

$$\mathrm{j}^{(\alpha)}(Q) = \partial_Y^\alpha\, \mathrm{SDet}\big(\mathrm{Id} - \mathcal{G}(Q)Y\big)^{-1}\big|_{Y=0} \tag{7.17}$$

denote its Taylor coefficients. Each $\mathrm{j}^{(\alpha)}$ is a polynomial in the entries of Q, hence a multiplier of finite degree on the weighted spaces L_p^2 of Section 5.2, by Lemma 5.14.

Theorem 7.6. *There exist energies $0 < E_* < E_{\mathrm{out}} \le 1/2$ such that the following holds for all $k \ge 1$ and $\varepsilon > 0$. In parts (a) and (b), $N, W \to \infty$ with $|E| \le E_{\mathrm{out}}$ and $X_N(E) \to X \in (0, \infty)$, where the energy E may depend on N.*

(a) Source limit. *There exists $r_{k,\varepsilon} > 0$ such that*

$$T_N^{(k)}(\mathbf{u},\mathbf{b}) \to \mathcal{Z}_{X,\varepsilon}^{(k)}(\mathbf{u},\mathbf{b}) \tag{7.18}$$

uniformly on compact subsets of the set of real $(\mathbf{u},\mathbf{b})$ with $\sum_j u_j = 0$ and $\|\mathbf{u}\| + \|\mathbf{b} - \varepsilon\mathbf{1}\| < r_{k,\varepsilon}$.

(b) Local jets. *Let $m \geq 1$ and $\delta \in (0,1/2)$, let $1 \leq i_1 < \cdots < i_m \leq N$ be sites with $\delta \leq i_a/N \leq 1-\delta$ and $i_a/N \to x_a$ for all a, and let $\alpha = (\alpha_1,\dots,\alpha_m)$, where α_a is a multi-index in the entries of Y_a. Then, uniformly on compact subsets of the set in part (a),*

$$\partial_{\mathbf{Y}}^{\alpha} T_N^{(k)}(\mathbf{u},\mathbf{b};\mathbf{Y})\big|_{\mathbf{Y}=0} \to \mathcal{B}_k\big(1,\, e^{-x_1\mathsf{G}_{X,\Gamma}}\mathfrak{j}^{(\alpha_1)}e^{-(x_2-x_1)\mathsf{G}_{X,\Gamma}}\mathfrak{j}^{(\alpha_2)}\cdots\mathfrak{j}^{(\alpha_m)}e^{-(1-x_m)\mathsf{G}_{X,\Gamma}}1\big), \tag{7.19}$$

where $\Gamma = \Gamma(\mathbf{u},\mathbf{b})$ and $\mathfrak{j}^{(\alpha_a)}$ acts by multiplication. No lower bound on $i_{a+1} - i_a$ is required. In particular, distinct sites whose scaled positions have the same limit $x_a = x_{a+1}$ give two separate multipliers, with the factor $e^{-0\cdot\mathsf{G}_{X,\Gamma}} = \mathrm{Id}$ between them, while all derivatives at one site enter through the single multiplier $\mathfrak{j}^{(\alpha_a)}$.

(c) All-site moments. *For all integers $p \geq 1$, all $u_0 > 0$, and all $0 < X_- \leq X_+ < \infty$, there exist C and N_0 such that*

$$\sup_{1\leq i\leq N}\ \sup_{|u|\leq u_0} \mathbb{E}\big|G_{ii}\big(E + (u+\mathrm{i}\varepsilon)/a_N\big)\big|^{2p} \leq C \tag{7.20}$$

for all $N \geq N_0$ and all (W,E) with $|E| \leq E_*$ and $X_- \leq X_N(E) \leq X_+$. For $u = 0$, the same holds with E_{out} in place of E_*.

(d) One-point normalization. *For all $\delta \in (0,1/2)$ and $u_0 > 0$,*

$$\mathbb{E}G_{ii}\big(E + (u+\mathrm{i}\varepsilon)/a_N\big) - \big(-\kappa_E + \mathrm{i}s_E\big) \to 0, \qquad \mathbb{E}\ell_{i,N}^{\varepsilon}(u) \to 1, \tag{7.21}$$

uniformly in the critical regime of Definition 1.2, and uniformly for $\delta \leq i/N \leq 1-\delta$ and $|u| \leq u_0$.

The energies E_* and E_{out} do not depend on k, on the number m of sites, or on the orders of the derivatives in part (b). These affect only the constants, the radius $r_{k,\varepsilon}$, and the rate of convergence.

The proof of Theorem 7.6 occupies Part IV and is completed in Section 20; its strategy is described in Section 2.

Remark 7.7. We call E_{out} the *outer energy bound* and E_* the *inner energy bound*. The outer bound is the minimum of $1/2$ and two energy thresholds up to which the estimates of Part IV on the chain hold, and it is fixed in (20.2). Parts (a) and (b) hold for $|E| \leq E_{\mathrm{out}}$, and so does part (c) at $u = 0$. In parts (c) and (d), the resolvent is evaluated at

$$E + \frac{u+\mathrm{i}\varepsilon}{a_N(E)} = E' + \frac{\mathrm{i}\varepsilon'}{a_N(E')}, \qquad E' = E + \frac{u}{a_N(E)}, \quad \varepsilon' = \varepsilon\,\frac{\rho(E')}{\rho(E)},$$

and Part IV deduces these parts from their case $u = 0$ at the energy E', using Lemma 20.1. For $|E| \le E_*$ and $|u| \le u_0$, we have $|E'| \le E_{\mathrm{out}}$ for all sufficiently large N, and the margin $E_{\mathrm{out}} - E_*$ absorbs this change of energy. The inner bound is fixed in Section 20.1, subject only to $0 < E_* < E_{\mathrm{out}}$. Their values are not explicit, since the thresholds of Part IV depend on constants that are not computed.

Throughout Part III, we use the inputs of Theorem 7.6 for $|E| \le E_{\mathrm{out}}$, and we state conclusions for $|E| \le E_*$, as in Section 1.2. We also set $E_{\mathrm{mid}} = (E_* + E_{\mathrm{out}})/2$, so that

$$0 < E_* < E_{\mathrm{mid}} < E_{\mathrm{out}} \le \frac{1}{2}. \tag{7.22}$$

Intermediate statements hold for $|E| \le E_{\mathrm{mid}}$, and the margin $E_{\mathrm{out}} - E_{\mathrm{mid}}$ is used to control the eigenvalues far from E.

Remark 7.8. Parts (a) and (b) are statements along sequences with $X_N(E) \to X$, while parts (c) and (d) are uniform over $|E| \le E_*$ (in (c) with $u = 0$, over $|E| \le E_{\mathrm{out}}$) and over $X_N(E)$ in compact subsets of $(0, \infty)$. Every sequence with $|E| \le E_{\mathrm{out}}$ and N/W^2 in a compact subset of $(0, \infty)$ has a subsequence along which $X_N(E)$ converges. By the argument of Remark 9.7, the uniform form of (c) and (d) is therefore equivalent to the same statements along all such sequences, and Part III uses them in either form, for instance in Lemma 9.2.

Note that part (a) identifies $T_N^{(k)}$ only near the base point $(\mathbf{u}, \mathbf{b}) = (0, \varepsilon\mathbf{1})$, at which both sides equal one. We extend the convergence to all real and complex fermionic source parameters, together with all derivatives, in Corollary 8.8, using only part (a), Lemma 7.2, and the analyticity of the partition function in Lemma 8.6. Parts (b)–(d) are used in Sections 9, 11 and 12. The first limit in (7.21) at $u = 0$ is a consequence of part (b) with $m = 1$, the first identity in (7.15), and the localization Lemma 5.19; we list it separately because it is used in this form. The second limit in (7.21) follows from the first, since

$$\ell_{i,N}^{\varepsilon}(u) = s_E^{-1} \operatorname{Im} G_{ii}\big(E + (u + \mathrm{i}\varepsilon)/a_N\big)$$

by Section 1.4.

Remark 7.9. The derivatives of the local source function at one site are polynomials in the resolvent entries at that site, and the multiplier $\mathrm{j}^{(\alpha)}$ is a Taylor coefficient of the complete local factor $\operatorname{SDet}(\mathrm{Id} - \mathcal{G}(Q)Y)^{-1}$, including the off-diagonal entries of $\mathcal{G}(Q)$. For this reason, one site and two adjacent sites give different limits. For instance, let $k = 1$, $\mathbf{u} = 0$, and $\mathbf{b} = \varepsilon$, and let $Y^{\mathrm{B}} = \operatorname{diag}(t, t')$ and $Y^{\mathrm{F}} = 0$ at the site i. Then

$$\partial_t \partial_{t'} T_N^{(1)}(0, \varepsilon; Y)\big|_{t=t'=0} = \mathbb{E} G_{ii}(z_+^{\mathrm{B}}) G_{ii}(z_-^{\mathrm{B}}),$$
$$\partial_t \partial_{t'} \det\big(\mathrm{Id} - \mathcal{G}^{\mathrm{BB}} Y^{\mathrm{B}}\big)^{-1}\big|_{t=t'=0} = \mathcal{G}^{\mathrm{BB}}_{\mathrm{RR}} \mathcal{G}^{\mathrm{BB}}_{\mathrm{AA}} + \mathcal{G}^{\mathrm{BB}}_{\mathrm{RA}} \mathcal{G}^{\mathrm{BB}}_{\mathrm{AR}},$$

where $\mathcal{G}^{\mathrm{BB}}_{\mathrm{RR}}, \mathcal{G}^{\mathrm{BB}}_{\mathrm{RA}}, \mathcal{G}^{\mathrm{BB}}_{\mathrm{AR}}, \mathcal{G}^{\mathrm{BB}}_{\mathrm{AA}}$ are the entries of the boson-boson block $\mathcal{G}^{\mathrm{BB}}$ of $\mathcal{G}(Q)$, in the notation of Section 5.2. When the two sources sit at two distinct sites instead, the limit contains the product $\mathcal{G}^{\mathrm{BB}}_{\mathrm{RR}}\mathcal{G}^{\mathrm{BB}}_{\mathrm{AA}}$ of two separate multipliers, with no term $\mathcal{G}^{\mathrm{BB}}_{\mathrm{RA}}\mathcal{G}^{\mathrm{BB}}_{\mathrm{AR}}$.

The energies of Part III. Recall the energies of (7.22) and their roles (Remark 7.7).

SECTION 8

The limiting point process

In this section, we prove Theorem 1.6. Parts (a) and (b) concern the unfolded spectral process Ξ_N: its convergence in law to a unique point process Ξ_X, the source law (1.9), and the stationarity, reflection invariance, simplicity, and unit intensity of Ξ_X. In part (c), we compute the first two derivatives of the rank-one source. The inputs are the matrix bounds of Section 7, part (a) of Theorem 7.6, and the partition function of Section 5, with its derivatives at rank one from Lemma 6.15.

The proof has three steps. First, the factorial moment bound of Lemma 7.2 implies tightness and shows that every subsequential limit is simple and of unit intensity. The scalar Gaussian component of H in Lemma 7.4 shows that every subsequential limit is stationary. Second, we show that the expectations of the products $\prod_{x\in\Xi} R(x)$ over the rational functions R of (1.9), which all have the denominator $(x^2+\varepsilon^2)^k$ for one fixed ε, determine the law of a point process. After the Cayley map $x = \varepsilon\tan(\vartheta/2)$, the differences $R-1$ become trigonometric polynomials, and we obtain the required approximation of Laplace functionals from Fejér's theorem. This is the main step specific to our setting. Third, Theorem 7.6(a) identifies $T_N^{(k)}$ only near the base point of the source parameters. Since $T_N^{(k)}$ is entire in the fermionic source parameters, with bounds uniform in N, a normal-families argument extends the convergence to all parameters.

In Section 8.6, we identify the first rank-one derivative with the unit intensity of Ξ_X, and we take the second from Lemma 6.15, which computes it in the sigma model by a Dyson expansion and supersymmetric localization. The function I_X comes from the two pairings in this expansion with one retarded and one advanced insertion, to which localization does not apply; the radial integration formula of Lemma 6.3 writes each of them as its corner term -1 minus the kernel $K_X(x,y;2\pi\varepsilon)$, and integrating this kernel over the positions $0<x<y<1$ of the two insertions gives $I_X(2\pi\varepsilon)$. Besides this lemma, only the analyticity of the partition function, which Lemma 8.6 takes from Lemma 6.14, uses the structure of the sigma model.

Throughout this section, (N,W,E) tends to the critical limit at a fixed $X\in(0,\infty)$, in the sense of Definition 1.2, and $\varepsilon>0$ is fixed. We let $\mathcal{N}(\mathbb{R})$ denote the space of locally finite counting measures on $\mathbb{R}$ with the vague topology of Section 1.2, which is a Polish space [23, Chapter 9]. A random point measure is a random element of $\mathcal{N}(\mathbb{R})$; multiple points are allowed. For $\zeta\in\mathcal{N}(\mathbb{R})$ and a function f, we write $\zeta(f)=\int f\,\mathrm{d}\zeta$. For a function $R\colon\mathbb{R}\to\mathbb{C}$ with $\sum_{x\in\zeta}|R(x)-1|<\infty$, we let $\prod_{x\in\zeta}R(x)$ denote the absolutely

convergent product over the points of ζ, counted with multiplicity. When $R > 0$, this product equals $\exp \zeta(\log R)$. Recall that $P_\varepsilon(x) = \varepsilon/(\pi(x^2+\varepsilon^2))$ denotes the Poisson kernel.

8.1 Ratios with poles at $\pm i\varepsilon$

For $k \geq 1$, $\mathbf{u} \in \mathbb{R}^k$ with $\sum_j u_j = 0$ and $\mathbf{b} \in (0,\infty)^k$, we define

$$R_{k,\mathbf{u},\mathbf{b}}(x) = \frac{\prod_{j=1}^k \big((x-u_j)^2 + b_j^2\big)}{(x^2+\varepsilon^2)^k}, \qquad x \in \mathbb{R}. \tag{8.1}$$

By (1.4), we have

$$T_N^{(k)}(\mathbf{u},\mathbf{b}) = \mathbb{E} \prod_{x \in \Xi_N} R_{k,\mathbf{u},\mathbf{b}}(x).$$

The centers u_j and the heights b_j of the numerator vary independently, subject to $\sum_j u_j = 0$. The denominator $(x^2+\varepsilon^2)^k$ depends on k, but it involves the one damping ε for all k, so that every ratio $R_{k,\mathbf{u},\mathbf{b}}$ has its poles at $\pm i\varepsilon$. We also define $\mathcal{V}$ to be the set of functions

$$x \mapsto a(x) P_\varepsilon(x)^k, \qquad k \geq 1, \quad a \in \mathbb{R}[x], \quad \deg a \leq 2k-2. \tag{8.2}$$

Lemma 8.1. *The following statements hold.*

(a) The set $\mathcal{V}$ is a real algebra of functions, and it is spanned by the functions P_ε^j with $j \geq 1$ and xP_ε^j with $j \geq 2$.

(b) Under the Cayley map $x = \varepsilon \tan(\vartheta/2)$, $\vartheta \in (-\pi,\pi)$, which is the stereographic parametrization of $\mathbb{R} \cup \{\infty\}$ by the circle, we have

$$P_\varepsilon(x) = \frac{1+\cos\vartheta}{2\pi\varepsilon}, \qquad xP_\varepsilon(x) = \frac{\sin\vartheta}{2\pi}, \tag{8.3}$$

and the set $P_\varepsilon^{-1}\mathcal{V} = \{f/P_\varepsilon : f \in \mathcal{V}\}$ is the algebra of real trigonometric polynomials in ϑ.

(c) For all k, $\mathbf{u}$, and $\mathbf{b}$ as in (8.1), we have $R_{k,\mathbf{u},\mathbf{b}} - 1 \in \mathcal{V}$. Moreover, $R_{k,\mathbf{u},\mathbf{b}}$ is bounded below by a positive constant, and there exists $C_R > 0$ such that $|\log R_{k,\mathbf{u},\mathbf{b}}(x)| \leq C_R P_\varepsilon(x)$ for all $x \in \mathbb{R}$.

Proof. We begin with part (a). Since $(x^2+\varepsilon^2)^{-1} = (\pi/\varepsilon)P_\varepsilon(x)$, we have, for $j \geq k$,

$$aP_\varepsilon^k = \Big(\frac{\pi}{\varepsilon}\Big)^{j-k} a\,(x^2+\varepsilon^2)^{j-k} P_\varepsilon^j,$$

and the polynomial $a(x^2+\varepsilon^2)^{j-k}$ has degree at most $2j-2$ if $\deg a \leq 2k-2$. We may thus write two elements of $\mathcal{V}$ with the same power of P_ε, and $\mathcal{V}$ is closed under sums.

The product of $a_1 P_\varepsilon^{k_1}$ and $a_2 P_\varepsilon^{k_2}$ has a numerator of degree at most $2(k_1+k_2)-4$, and $\mathcal{V}$ is closed under products.

For the spanning set, let $c \in \{0,1\}$ and $2i + c \le 2k-2$. Expanding $x^{2i} = (x^2+\varepsilon^2-\varepsilon^2)^i$ and using $(x^2+\varepsilon^2)P_\varepsilon = \varepsilon/\pi$, we find

$$x^{2i+c} P_\varepsilon^k = x^c \sum_{i'=0}^{i} \binom{i}{i'} (-\varepsilon^2)^{i-i'} \Big(\frac{\varepsilon}{\pi}\Big)^{i'} P_\varepsilon^{k-i'}.$$

The condition $2i + c \le 2k-2$ implies that $k - i' \ge k - i \ge 1$ for $c = 0$ and $k - i' \ge 2$ for $c = 1$. Conversely, the functions P_ε^j with $j \ge 1$ and xP_ε^j with $j \ge 2$ belong to $\mathcal{V}$.

For part (b), the identity $1 + \tan^2(\vartheta/2) = \cos^{-2}(\vartheta/2)$ gives

$$P_\varepsilon(x) = \frac{\cos^2(\vartheta/2)}{\pi\varepsilon} = \frac{1+\cos\vartheta}{2\pi\varepsilon}, \qquad xP_\varepsilon(x) = \frac{\sin(\vartheta/2)\cos(\vartheta/2)}{\pi} = \frac{\sin\vartheta}{2\pi}.$$

By part (a), $P_\varepsilon^{-1}\mathcal{V}$ is spanned by the functions P_ε^j with $j \ge 0$ and xP_ε^j with $j \ge 1$. Using $xP_\varepsilon^j = (xP_\varepsilon)P_\varepsilon^{j-1}$ and

$$(xP_\varepsilon)^2 = \frac{\varepsilon}{\pi} P_\varepsilon - \varepsilon^2 P_\varepsilon^2,$$

we see that this span is the unital algebra generated by P_ε and xP_ε. By (8.3), it is the unital algebra generated by $\cos\vartheta$ and $\sin\vartheta$, which is the algebra of real trigonometric polynomials.

We now prove part (c). The numerator of (8.1) satisfies

$$\prod_{j=1}^{k} \big((x-u_j)^2 + b_j^2\big) = x^{2k} - 2\Big(\sum_{j=1}^{k} u_j\Big) x^{2k-1} + O(x^{2k-2}) = x^{2k} + O(x^{2k-2}),$$

where $O(x^{2k-2})$ denotes a polynomial of degree at most $2k-2$, and the even polynomial $(x^2+\varepsilon^2)^k$ is also monic of degree $2k$. Their difference has degree at most $2k-2$, and

$$R_{k,\mathbf{u},\mathbf{b}} - 1 = \Big(\frac{\pi}{\varepsilon}\Big)^k \Big(\prod_{j=1}^{k} \big((x-u_j)^2 + b_j^2\big) - (x^2+\varepsilon^2)^k\Big) P_\varepsilon^k \in \mathcal{V}.$$

(The balance condition $\sum_j u_j = 0$ is exactly what is needed here.)

Next, the function $R_{k,\mathbf{u},\mathbf{b}}$ is continuous, positive since $b_j > 0$, and tends to one at $\pm\infty$. It is bounded below by some $c_R > 0$. Every element aP_ε^k of $\mathcal{V}$ satisfies $|aP_\varepsilon^k| \le CP_\varepsilon$, since $|a|P_\varepsilon^{k-1}$ is bounded when $\deg a \le 2k-2$. Using $|\log y| \le |y-1|/\min(y,1)$ for $y > 0$, we conclude that

$$|\log R_{k,\mathbf{u},\mathbf{b}}| \le c_R^{-1} |R_{k,\mathbf{u},\mathbf{b}} - 1| \le C_R P_\varepsilon. \qquad \square$$

8.2 Tightness and subsequential limits

We now restate the matrix inputs in the unfolded variables. By (7.3), there exists $C_{\exp} > 0$, depending only on E_*, such that for all sufficiently large N, all bounded intervals $I \subset \mathbb{R}$, and all $m \geq 1$,

$$\mathbb{E}\big(\Xi_N(I)\big)_m \leq (C_{\exp}|I|)^m. \tag{8.4}$$

By (7.8), the measures $\mathbb{E}\Xi_N$ converge vaguely to the Lebesgue measure. We say that a random point measure Ξ satisfies the *exponential bound with constant* $C_{\exp}$ if

$$\mathbb{E}e^{t\Xi(I)} \leq \exp\big\{C_{\exp}|I|(e^t-1)\big\} \qquad \text{for all } t \geq 0 \text{ and all bounded intervals } I \subset \mathbb{R}. \tag{8.5}$$

Proposition 8.2. *The laws of the processes Ξ_N are tight in $\mathcal{N}(\mathbb{R})$, and Ξ_N satisfies* (8.5) *for all sufficiently large N. Let Ξ be the limit in law of Ξ_N along a subsequence. Then Ξ satisfies* (8.5)*, it is stationary and simple, and $\mathbb{E}\Xi(I) = |I|$ for all bounded intervals I. Moreover, $\Xi_N(I) \to \Xi(I)$ in law along the subsequence for all bounded intervals I.*

Proof. We begin with the exponential bound for Ξ_N. For an integer $n \geq 0$ and $t \geq 0$, the binomial theorem gives

$$e^{tn} = \big(1 + (e^t - 1)\big)^n = \sum_{m=0}^{n} \frac{(n)_m}{m!}(e^t-1)^m.$$

Applying this with $n = \Xi_N(I)$ and using (8.4), we find

$$\mathbb{E}e^{t\Xi_N(I)} = \sum_{m\geq 0} \frac{(e^t-1)^m}{m!}\mathbb{E}\big(\Xi_N(I)\big)_m \leq \sum_{m\geq 0} \frac{(C_{\exp}|I|(e^t-1))^m}{m!} = \exp\big\{C_{\exp}|I|(e^t-1)\big\}.$$

In particular,

$$\sup_N \mathbb{E}\Xi_N(I) \leq C_{\exp}|I| \qquad \text{for all bounded intervals } I,$$

where the supremum is over all sufficiently large N. Markov's inequality then implies that the counts $\Xi_N(I)$ are tight, and a family of random point measures whose counts of bounded intervals are tight is tight in $\mathcal{N}(\mathbb{R})$ [23, Chapter 11].

Now let Ξ be a subsequential limit. Recall that for a bounded open interval I, the map $\zeta \mapsto \zeta(I)$ is lower semicontinuous on $\mathcal{N}(\mathbb{R})$. Since it takes integer values, the map $\zeta \mapsto F(\zeta(I))$ is lower semicontinuous for every nondecreasing $F\colon \{0,1,2,\dots\} \to [0,\infty)$. By the portmanteau theorem for nonnegative lower semicontinuous functions, we have

$$\mathbb{E}F\big(\Xi(I)\big) \leq \liminf_N \mathbb{E}F\big(\Xi_N(I)\big), \tag{8.6}$$

where the limit is taken along the subsequence. With $F(n) = e^{tn}$, we obtain (8.5) for Ξ and open intervals. Approximating a bounded interval from outside by open intervals,

we extend this bound to all bounded intervals. With $F(n) = (n)_2$, which is nondecreasing on the nonnegative integers, we find

$$\mathbb{E}\big(\Xi(I)\big)_2 \le (C_{\exp}|I|)^2 \qquad \text{for all bounded open intervals } I. \tag{8.7}$$

Next, the case $F(n) = n$ of (8.6) implies that, for all $a \in \mathbb{R}$ and $\eta > 0$,

$$\mathbb{E}\Xi(\{a\}) \le \mathbb{E}\Xi\big((a-\eta, a+\eta)\big) \le 2C_{\exp}\eta.$$

Letting $\eta \downarrow 0$, we see that $\Xi(\{a\}) = 0$ almost surely for every deterministic a. The boundary of a bounded interval I is thus almost surely not charged by Ξ, the map $\zeta \mapsto \zeta(I)$ is almost surely continuous at Ξ, and $\Xi_N(I) \to \Xi(I)$ in law along the subsequence. By (8.4) with $m = 2$, the variables $\Xi_N(I)$ are bounded in L^2, hence uniformly integrable, and (7.8) implies that $\mathbb{E}\Xi_N(I) \to |I|$. We conclude that $\mathbb{E}\Xi(I) = |I|$.

We now prove stationarity. For $\zeta \in \mathcal{N}(\mathbb{R})$ and $s \in \mathbb{R}$, we let $\zeta + s$ denote the image of ζ under the translation $x \mapsto x + s$, so that

$$(\zeta + s)(A) = \zeta(A - s)$$

for Borel sets A. The map $\zeta \mapsto \zeta + s$ is continuous on $\mathcal{N}(\mathbb{R})$. Adding $sa_N^{-1}\mathrm{Id}$ to H shifts every eigenvalue by s/a_N, and every unfolded eigenvalue by s.

Let Φ be a bounded continuous function on $\mathcal{N}(\mathbb{R})$. For a Hermitian matrix M with eigenvalues $\lambda_\alpha(M)$, we set

$$F(M) = \Phi\Big(\sum_\alpha \delta_{a_N(\lambda_\alpha(M) - E)}\Big).$$

Then $F(H) = \Phi(\Xi_N)$ and $F(H + sa_N^{-1}\mathrm{Id}) = \Phi(\Xi_N + s)$. Applying (7.11) with this F and with $b = s$, and using the last assertion of Lemma 7.4, we find

$$\big|\mathbb{E}\Phi(\Xi_N + s) - \mathbb{E}\Phi(\Xi_N)\big| \le \sqrt{2/\pi}\,\|\Phi\|_\infty \frac{|s|}{a_N\sigma_N} \le C\|\Phi\|_\infty |s| W^{-1/2}.$$

The function $\zeta \mapsto \Phi(\zeta + s)$ is bounded and continuous. Letting $N \to \infty$ along the subsequence, we find

$$\mathbb{E}\Phi(\Xi + s) = \mathbb{E}\Phi(\Xi).$$

The function Φ is arbitrary, and $\Xi + s$ and Ξ have the same law for every $s \in \mathbb{R}$.

Finally, we prove that Ξ is simple. Fix $A > 0$ and $\delta \in (0,1)$, and for $l \in \mathbb{Z}$ let

$$I_l = \big((l-1)\delta, (l+2)\delta\big).$$

Let $x \le y$ be points of $[-A, A]$ with $y - x \le \delta$, and let $l = \lfloor x/\delta \rfloor$. Then

$$(l-1)\delta < l\delta \le x \le y \le x + \delta < (l+2)\delta, \tag{8.8}$$

so that x and y both lie in I_l. Consequently, if Ξ has a multiple point in $[-A, A]$, then $\Xi(I_l) \geq 2$ for some l with $|l| \leq A/\delta + 1$. Using $\mathbf{1}\{n \geq 2\} \leq (n)_2/2$ for integers $n \geq 0$ and (8.7) for the intervals I_l, of length 3δ, we obtain

$$\mathbb{P}\big(\Xi \text{ has a multiple point in } [-A, A]\big) \leq \sum_{|l| \leq A/\delta + 1} \frac{\mathbb{E}(\Xi(I_l))_2}{2} \leq \Big(\frac{2A}{\delta} + 3\Big) \frac{(3C_{\exp}\delta)^2}{2},$$

which tends to zero as $\delta \downarrow 0$. The number A is arbitrary, and Ξ is simple. □

We will also use the following consequences of (8.5).

Lemma 8.3. *Let Ξ be a random point measure that satisfies* (8.5) *with constant $C_{\exp}$.*

(a) For every measurable $g \geq 0$ on $\mathbb{R}$ such that $A_g = \sum_{j \in \mathbb{Z}} \sup_{[j, j+1)} g < \infty$,

$$\mathbb{E}e^{\Xi(g)} \leq \exp\big\{C_{\exp}(e^{A_g} - 1)\big\}. \tag{8.9}$$

(b) Let $(\Xi^{(n)})_{n \geq 1}$ be random point measures that satisfy (8.5) *with the constant $C_{\exp}$ and converge in law to Ξ. Let $g_1, \dots, g_l$ be continuous functions on $\mathbb{R}$ such that $|g_i| \leq cP_\varepsilon$ for some $c > 0$, and write $\Xi(\mathbf{g}) = (\Xi(g_1), \dots, \Xi(g_l))$. Then $\Xi^{(n)}(\mathbf{g}) \to \Xi(\mathbf{g})$ in law. Moreover, $\mathbb{E}F(\Xi^{(n)}(\mathbf{g})) \to \mathbb{E}F(\Xi(\mathbf{g}))$ for every continuous $F\colon \mathbb{R}^l \to \mathbb{C}$ such that $|F(y)| \leq C_F e^{C_F|y|}$ for some $C_F > 0$.*

Proof. For part (a), let $I_j = [j, j+1)$ and $w_j = \sup_{I_j} g$. The weights w_j/A_g sum to one, and

$$\Xi(g) \leq \sum_{j \in \mathbb{Z}} \frac{w_j}{A_g} A_g \Xi(I_j).$$

By the convexity of the exponential and (8.5) with $t = A_g$, we have

$$\mathbb{E}e^{\Xi(g)} \leq \sum_{j \in \mathbb{Z}} \frac{w_j}{A_g} \mathbb{E}e^{A_g \Xi(I_j)} \leq \exp\big\{C_{\exp}(e^{A_g} - 1)\big\}.$$

For part (b), we first note that Jensen's inequality and (8.5) imply that

$$e^{t\mathbb{E}\Xi(I)} \leq \mathbb{E}e^{t\Xi(I)} \leq \exp\big\{C_{\exp}|I|(e^t - 1)\big\}, \qquad \text{so that} \qquad \mathbb{E}\Xi(I) \leq C_{\exp}|I| \, \frac{e^t - 1}{t} \to C_{\exp}|I|$$

as $t \downarrow 0$. For $A \geq 3$, let $\chi_A \in C_c(\mathbb{R})$ satisfy $0 \leq \chi_A \leq 1$ and $\chi_A = 1$ on $[-A, A]$. Then

$$\mathbb{E}\big|\Xi^{(n)}\big((1 - \chi_A)g_i\big)\big| \leq c \sum_{|j| \geq A-1} \mathbb{E}\Xi^{(n)}(I_j) \sup_{I_j} P_\varepsilon \leq \frac{C}{A} \tag{8.10}$$

uniformly in n, where we used the previous bound for the intervals I_j, of length one. The same bound holds for Ξ.

Now let F be bounded and Lipschitz. The map $\zeta \mapsto \zeta(\chi_A \mathbf{g})$ is continuous on $\mathcal{N}(\mathbb{R})$, since $\chi_A g_i \in C_c(\mathbb{R})$. Using this continuity and (8.10), we obtain

$$\begin{aligned}\limsup_{n\to\infty} &\big|\mathbb{E}F(\Xi^{(n)}(\mathbf{g})) - \mathbb{E}F(\Xi(\mathbf{g}))\big| \\ &\le \limsup_{n\to\infty} \big|\mathbb{E}F(\Xi^{(n)}(\chi_A\mathbf{g})) - \mathbb{E}F(\Xi(\chi_A\mathbf{g}))\big| + \frac{2Cl\,\mathrm{Lip}(F)}{A} = \frac{2Cl\,\mathrm{Lip}(F)}{A},\end{aligned}$$

and letting $A \to \infty$ proves the convergence in law.

For the second assertion, part (a) with $g = tclP_\varepsilon$ yields, for all $t \ge 0$,

$$\sup_n \mathbb{E}e^{t|\Xi^{(n)}(\mathbf{g})|} \le \sup_n \mathbb{E}e^{tcl\,\Xi^{(n)}(P_\varepsilon)} \le \exp\big\{C_{\exp}\big(e^{tclA_{P_\varepsilon}} - 1\big)\big\} < \infty,$$

where $|y|$ denotes the ℓ^1 norm of $y \in \mathbb{R}^l$ and $A_{P_\varepsilon} = \sum_j \sup_{I_j} P_\varepsilon$. Taking $t = 2C_F$, we obtain

$$\sup_n \mathbb{E}\big|F(\Xi^{(n)}(\mathbf{g}))\big|^2 \le C_F^2 \sup_n \mathbb{E}e^{2C_F|\Xi^{(n)}(\mathbf{g})|} < \infty.$$

The variables $F(\Xi^{(n)}(\mathbf{g}))$ are therefore uniformly integrable, and their expectations converge, by the convergence in law. □

8.3 A determining class

We now show that the products over the ratios (8.1) determine the law of a point process. The main step is the following approximation lemma, in which the rank k depends on f and η.

Lemma 8.4. *Let $f \in C_c(\mathbb{R})$ be nonnegative and let $\eta > 0$. Then there exist $k \ge 1$, $\mathbf{u} \in \mathbb{R}^k$ with $\sum_j u_j = 0$ and $\mathbf{b} \in (0,\infty)^k$ such that $R = R_{k,\mathbf{u},\mathbf{b}}$ satisfies*

$$c_f \le R \le 1 \quad \text{and} \quad |\log R(x) + f(x)| \le \eta P_\varepsilon(x) \quad \text{for all } x \in \mathbb{R}, \tag{8.11}$$

where $c_f = e^{-\|f\|_\infty}/2$.

Proof. We define

$$\psi_f = \frac{1 - e^{-f}}{P_\varepsilon},$$

which is nonnegative, continuous, and compactly supported. As a function of the Cayley angle ϑ of Lemma 8.1(b), the function $\sqrt{\psi_f}$ vanishes near $\vartheta = \pm\pi$ and extends to a continuous function on the circle. By Fejér's theorem [73, Chapter I], the Cesàro means of the Fourier series of $\sqrt{\psi_f}$ are real trigonometric polynomials q that converge to $\sqrt{\psi_f}$ uniformly. In particular, the number

$$\delta = \|q^2 - \psi_f\|_\infty \le \|q - \sqrt{\psi_f}\|_\infty\big(\|q\|_\infty + \|\sqrt{\psi_f}\|_\infty\big)$$

can be made arbitrarily small. By Lemma 8.1(b), each such q belongs to $P_\varepsilon^{-1}\mathcal{V}$. Every element of $\mathcal{V}$ has the form bP_ε^{M+1} with $M \geq 0$ and $\deg b \leq 2M$, and therefore there exist $M \geq 0$ and $a \in \mathbb{R}[x]$ with $\deg a \leq 2M$ such that $q = aP_\varepsilon^M$. We set

$$R = 1 - q^2 P_\varepsilon = 1 - a^2 P_\varepsilon^{2M+1}. \tag{8.12}$$

We first prove (8.11). By the definition of ψ_f, we have

$$R - e^{-f} = P_\varepsilon(\psi_f - q^2), \qquad \text{and therefore} \qquad |R - e^{-f}| \leq \delta P_\varepsilon.$$

We have $R \leq 1$, since $q^2 P_\varepsilon \geq 0$. Using $P_\varepsilon \leq (\pi\varepsilon)^{-1}$, we find

$$R \geq e^{-\|f\|_\infty} - \frac{\delta}{\pi\varepsilon} \geq c_f \qquad \text{if } \delta \leq \frac{\pi\varepsilon}{2} e^{-\|f\|_\infty}.$$

Since R and e^{-f} take values in $[c_f, 1]$, on which the logarithm has the Lipschitz constant c_f^{-1}, we obtain

$$|\log R + f| = |\log R - \log e^{-f}| \leq \frac{|R - e^{-f}|}{c_f} \leq \frac{\delta}{c_f} P_\varepsilon,$$

which proves (8.11) once $\delta \leq \eta c_f$.

It remains to show that R has the form (8.1). Let $k = 2M + 1$. Using $P_\varepsilon^k = (\varepsilon/\pi)^k (x^2 + \varepsilon^2)^{-k}$, we write $R = \mathsf{p}/(x^2 + \varepsilon^2)^k$ with

$$\mathsf{p}(x) = (x^2 + \varepsilon^2)^k - \Big(\frac{\varepsilon}{\pi}\Big)^k a(x)^2, \qquad \deg(a^2) \leq 4M = 2k - 2.$$

The polynomial p is monic of degree $2k$, with vanishing coefficient of x^{2k-1}. Further, p is positive on $\mathbb{R}$, since $R \geq c_f$. Having no real roots, it has k pairs of complex conjugate roots $u_j \pm \mathrm{i} b_j$ with $b_j > 0$, counted with multiplicity. This gives the factorization

$$\mathsf{p}(x) = \prod_{j=1}^k \big((x - u_j)^2 + b_j^2\big), \qquad \sum_{j=1}^k u_j = 0,$$

where the second identity follows from the vanishing coefficient of x^{2k-1}. This shows that $R = R_{k,\mathbf{u},\mathbf{b}}$. □

Proposition 8.5. *The following statements hold.*

(i) Let Ξ and Ξ' be random point measures such that $\mathbb{E}\Xi(P_\varepsilon) + \mathbb{E}\Xi'(P_\varepsilon) < \infty$. Suppose that $\mathbb{E}\prod_{x\in\Xi} R(x) = \mathbb{E}\prod_{x\in\Xi'} R(x)$ for all ratios $R = R_{k,\mathbf{u},\mathbf{b}}$ of the form (8.1) with $R \leq 1$. Then Ξ and Ξ' have the same law.

(ii) Let Ξ and $(\Xi^{(n)})_{n\ge1}$ be random point measures that satisfy (8.5) *with a common constant $C_{\exp}$, and suppose that $\Xi^{(n)} \to \Xi$ in law. Then $\mathbb{E}\prod_{x\in\Xi^{(n)}} R(x) \to \mathbb{E}\prod_{x\in\Xi} R(x)$ for all ratios R of the form* (8.1)*, including those that exceed one somewhere.*

Proof. Recall from Lemma 8.1(c) that each ratio R satisfies $|\log R| \le C_R P_\varepsilon$. Whenever $\Xi(P_\varepsilon) < \infty$, this bound shows that the product

$$\prod_{x\in\Xi} R(x) = \exp \Xi(\log R), \qquad |\Xi(\log R)| \le C_R\, \Xi(P_\varepsilon),$$

is well defined.

For part (i), let $f \in C_c(\mathbb{R})$ be nonnegative, let $\eta > 0$, and choose R by Lemma 8.4. The functions f and $-\log R$ are nonnegative, and $|e^{-s} - e^{-s'}| \le |s - s'|$ for $s, s' \ge 0$. Using (8.11), we have

$$\Big|\mathbb{E}e^{-\Xi(f)} - \mathbb{E}\prod_{x\in\Xi} R(x)\Big| \le \mathbb{E}\Xi\big(|f + \log R|\big) \le \eta\, \mathbb{E}\Xi(P_\varepsilon),$$

and the same bound holds for Ξ'. Letting $\eta \to 0$, we find

$$\mathbb{E}e^{-\Xi(f)} = \mathbb{E}e^{-\Xi'(f)}.$$

The Laplace functionals of the nonnegative functions in $C_c(\mathbb{R})$ determine the law of a random measure [23, Chapter 9], and therefore Ξ and Ξ' have the same law.

To prove part (ii), we apply Lemma 8.3(b) with $l = 1$, $g_1 = \log R$, and $F(y) = e^y$; its hypotheses hold, since $\log R$ is continuous and $|\log R| \le C_R P_\varepsilon$. □

8.4 Continuation to all source parameters

For $k \ge 1$, we parametrize the sources by

$$\mathbf{p} = (u_1, \dots, u_{k-1}, b_1, \dots, b_k) \in \mathbb{C}^{2k-1}, \qquad u_k = -\sum_{j<k} u_j, \tag{8.13}$$

and for complex $\mathbf{p}$ we define $R_{k,\mathbf{p}}$ by the formula (8.1), as a function of real x. The real parameters $\mathbf{p} \in \mathbb{R}^{k-1} \times (0,\infty)^k$ correspond to the sources of (8.1), and for them $R_{k,\mathbf{p}} = R_{k,\mathbf{u},\mathbf{b}}$. We write $T_N^{(k)}(\mathbf{p}) = \mathbb{E}\prod_{x\in\Xi_N} R_{k,\mathbf{p}}(x)$, and $\mathcal{Z}_{X,\varepsilon}^{(k)}(\mathbf{p}) = \mathcal{Z}_{X,\varepsilon}^{(k)}(\mathbf{u},\mathbf{b})$ for $\mathbf{p} \in \mathbb{R}^{k-1} \times (0,\infty)^k$.

We first deduce from Lemma 6.14 that the partition function is entire in these parameters.

Lemma 8.6. *For every $k \ge 1$, the function $\mathbf{p} \mapsto \mathcal{Z}_{X,\varepsilon}^{(k)}(\mathbf{p})$ on $\mathbb{R}^{k-1} \times (0,\infty)^k$ is the restriction of an entire function on $\mathbb{C}^{2k-1}$, which we also denote by $\mathcal{Z}_{X,\varepsilon}^{(k)}$.*

Proof. For $\mathbf{p} \in \mathbb{C}^{2k-1}$, with u_k as in (8.13), we let $\Gamma(\mathbf{p})$ denote the source supermatrix of Definition 4.1 with the fermion-fermion block

$$\Gamma_{\mathrm{F}}(\mathbf{p}) = \operatorname{diag}\bigl(b_1 - \mathrm{i}u_1, \dots, b_k - \mathrm{i}u_k, -b_1 - \mathrm{i}u_1, \dots, -b_k - \mathrm{i}u_k\bigr).$$

The diagonal entries of $\Gamma_{\mathrm{F}}(\mathbf{p})$ are affine functions of $\mathbf{p}$, and for $\mathbf{p} \in \mathbb{R}^{k-1} \times (0,\infty)^k$ we have $\Gamma(\mathbf{p}) = \Gamma(\mathbf{u}, \mathbf{b})$. We apply Lemma 6.14(c) with the present k, X, and ε. It shows that the partition function, as a function of the diagonal entries of Γ_{F}, is the restriction of an entire function on $\mathbb{C}^{2k}$. The composition of this entire function with the affine map $\mathbf{p} \mapsto \Gamma_{\mathrm{F}}(\mathbf{p})$ is entire on $\mathbb{C}^{2k-1}$, and it agrees with $\mathcal{Z}^{(k)}_{X,\varepsilon}(\mathbf{p})$ for $\mathbf{p} \in \mathbb{R}^{k-1} \times (0,\infty)^k$. □

The next lemma is a version of Vitali's theorem in several variables, for the expectations of products over point processes with uniform exponential bounds.

Lemma 8.7. *Let $(\Xi^{(n)})_{n\ge1}$ be random point measures that satisfy* (8.5) *with a common constant $C_{\exp}$. Fix $k \ge 1$, and set $T_n(\mathbf{p}) = \mathbb{E}\prod_{x\in\Xi^{(n)}} R_{k,\mathbf{p}}(x)$ for $\mathbf{p} \in \mathbb{C}^{2k-1}$. Then every T_n is entire, and*

$$\sup_n \sup_{\mathbf{p}\in K} |T_n(\mathbf{p})| < \infty \qquad \textit{for every compact } K \subset \mathbb{C}^{2k-1}.$$

If T_n converges pointwise on a nonempty open subset of $\mathbb{R}^{2k-1}$, then T_n converges locally uniformly on $\mathbb{C}^{2k-1}$, together with all its partial derivatives, to an entire function.

Proof. The numerator of $R_{k,\mathbf{p}}$ is a polynomial in $(x, \mathbf{p})$. It is monic of degree $2k$ in x, with vanishing coefficient of x^{2k-1}, since $\sum_j u_j = 0$. As in the proof of Lemma 8.1(c), its difference with $(x^2+\varepsilon^2)^k$ has degree at most $2k-2$ in x, with coefficients that are polynomials in $\mathbf{p}$. It follows that, for every compact $K \subset \mathbb{C}^{2k-1}$, there exists C_K such that

$$|R_{k,\mathbf{p}}(x) - 1| \le C_K P_\varepsilon(x), \qquad x \in \mathbb{R},\ \mathbf{p} \in K. \tag{8.14}$$

Let $\zeta \in \mathcal{N}(\mathbb{R})$ satisfy $\zeta(P_\varepsilon) < \infty$. By (8.14), the product $\prod_{x\in\zeta} R_{k,\mathbf{p}}(x)$ converges absolutely and uniformly for $\mathbf{p} \in K$. It is entire in $\mathbf{p}$, as a locally uniform limit of polynomials, and

$$\sup_{\mathbf{p}\in K}\Bigl|\prod_{x\in\zeta} R_{k,\mathbf{p}}(x)\Bigr| \le \exp\bigl\{C_K\zeta(P_\varepsilon)\bigr\}.$$

Using (8.9) with $g = C_K P_\varepsilon$, we find that the expectation of the right side is bounded uniformly in n,

$$\sup_n \mathbb{E}\exp\bigl\{C_K\Xi^{(n)}(P_\varepsilon)\bigr\} \le \exp\bigl\{C_{\exp}\bigl(e^{C_K A_{P_\varepsilon}} - 1\bigr)\bigr\},$$

where A_{P_ε} is as in the proof of Lemma 8.3. Dominated convergence shows that T_n is continuous, and Fubini's theorem and Morera's theorem in each variable show that T_n

is holomorphic in each variable separately. By Osgood's lemma, T_n is entire, and the uniform bound on compact sets holds.

Next, Cauchy's estimates show that the functions T_n are equicontinuous on compact sets. The Arzelà–Ascoli theorem then shows that every subsequence of (T_n) has a further subsequence that converges locally uniformly on $\mathbb{C}^{2k-1}$, and the limit is entire (this is Montel's theorem in several variables). Let T and T' be two such limits, which agree on a nonempty open subset U of $\mathbb{R}^{2k-1}$. The complex partial derivatives of $T - T'$ of every order at a point of U are equal to its real partial derivatives in the directions of $\mathbb{R}^{2k-1}$, which vanish. We have

$$\partial^\beta (T - T')(\mathbf{p}) = 0 \qquad \text{for all multi-indices } \beta \text{ and } \mathbf{p} \in U,$$

the Taylor series of $T - T'$ at a point of U vanishes, and since $\mathbb{C}^{2k-1}$ is connected, $T = T'$. We conclude that the whole sequence (T_n) converges locally uniformly. Cauchy's estimates then imply the locally uniform convergence of all partial derivatives. □

Corollary 8.8. *For every $k \geq 1$, $T_N^{(k)} \to \mathcal{Z}_{X,\varepsilon}^{(k)}$ locally uniformly on $\mathbb{C}^{2k-1}$, together with all partial derivatives. In particular, $T_N^{(k)}(\mathbf{u}, \mathbf{b}) \to \mathcal{Z}_{X,\varepsilon}^{(k)}(\mathbf{u}, \mathbf{b})$ for all $\mathbf{u} \in \mathbb{R}^k$ with $\sum_j u_j = 0$ and all $\mathbf{b} \in (0, \infty)^k$.*

Proof. Proposition 8.2 shows that the processes Ξ_N satisfy (8.5) with a constant independent of N, for all sufficiently large N. Let

$$U = \big\{ \mathbf{p} \in \mathbb{R}^{k-1} \times (0,\infty)^k : \|\mathbf{u}\| + \|\mathbf{b} - \varepsilon \mathbf{1}\| < r_{k,\varepsilon} \big\},$$

which is an open subset of $\mathbb{R}^{2k-1}$ containing $(0, \varepsilon\mathbf{1})$; on U, the partition function $\mathcal{Z}_{X,\varepsilon}^{(k)}(\mathbf{p})$ is defined by Definition 5.16. Theorem 7.6(a) shows that $T_N^{(k)}$ converges to $\mathcal{Z}_{X,\varepsilon}^{(k)}$ on U. We apply Lemma 8.7 to the processes Ξ_N in place of $\Xi^{(n)}$, and we obtain the locally uniform convergence of $T_N^{(k)}$ on $\mathbb{C}^{2k-1}$, with all partial derivatives, to an entire function T. By Lemma 8.6, the function $\mathcal{Z}_{X,\varepsilon}^{(k)}$ is entire. It agrees with T on U, and the argument in the proof of Lemma 8.7 shows that $T = \mathcal{Z}_{X,\varepsilon}^{(k)}$ everywhere. □

8.5 Proof of parts (a) and (b)

We now prove the statements of Theorem 1.6 about Ξ_N and its source law.

Proof of Theorem 1.6(a) and (b). By Proposition 8.2, the laws of Ξ_N are tight. Let Ξ be a subsequential limit. The same proposition shows that Ξ is stationary, simple, and of unit intensity, and Ξ and Ξ_N satisfy (8.5) with a common constant.

Fix $k \geq 1$, $\mathbf{u} \in \mathbb{R}^k$ with $\sum_j u_j = 0$, and $\mathbf{b} \in (0,\infty)^k$. By Corollary 8.8, we have $T_N^{(k)}(\mathbf{u},\mathbf{b}) \to \mathcal{Z}_{X,\varepsilon}^{(k)}(\mathbf{u},\mathbf{b})$, which is the first assertion of part (b). Next, we apply Proposition 8.5(ii) along the subsequence, with the processes Ξ_N in place of $\Xi^{(n)}$, and we obtain

$$\mathbb{E}\prod_{x\in\Xi}\prod_{j=1}^{k}\frac{(x-u_j)^2+b_j^2}{x^2+\varepsilon^2} = \mathcal{Z}_{X,\varepsilon}^{(k)}(\mathbf{u},\mathbf{b}). \tag{8.15}$$

Since Ξ has unit intensity, we have

$$\mathbb{E}\Xi(P_\varepsilon) = \int_{\mathbb{R}} P_\varepsilon(x)\,\mathrm{d}x = 1,$$

and Proposition 8.5(i) implies that the values (8.15), for all k, $\mathbf{u}$, and $\mathbf{b}$, determine the law of Ξ. These values are functions of X and ε alone, so all subsequential limits have the same law. We let Ξ_X denote a random point measure with this law.

The laws of Ξ_N are tight and all their subsequential limits agree, and we conclude that $\Xi_N \to \Xi_X$ in law. Further, the law of Ξ_X is the same for all sequences (N,W,E) that tend to the critical limit at X, and (8.15) at any one $\varepsilon > 0$ determines it. This proves part (b) and all statements of part (a) except reflection invariance.

It remains to prove that Ξ_X is invariant in law under the reflection $x \mapsto -x$. We may compute the law of Ξ_X along a sequence with $E = 0$, for instance with $W = (N/(4X))^{1/2}$, so that $X_N(0) = X$. The law of H is that of a centered Gaussian matrix, and $-H$ has the same law as H. Since $E = 0$, replacing H by $-H$ replaces each x_α by $-x_\alpha$. Therefore Ξ_N has the same law as its image under $x \mapsto -x$, and since this reflection is continuous on $\mathcal{N}(\mathbb{R})$, the reflection invariance of Ξ_X follows by passing to the limit. □

8.6 The rank-one derivatives

We now prove part (c) of Theorem 1.6. Throughout this section, $k = 1$, and we abbreviate

$$T_N(t) = T_N^{(1)}(0,\varepsilon+t), \qquad \mathcal{Z}(t) = \mathcal{Z}_{X,\varepsilon}^{(1)}(0,\varepsilon+t), \qquad t \in \mathbb{C}.$$

The only fermionic source parameter at rank one is $b_1 = \varepsilon + t$, and Corollary 8.8 shows that $T_N \to \mathcal{Z}$ locally uniformly on $\mathbb{C}$, together with all derivatives. Part (c) is therefore equivalent to

$$\mathcal{Z}'(0) = 2\pi, \qquad \mathcal{Z}''(0) = 4\pi^2\big(1 + I_X(2\pi\varepsilon)\big), \tag{8.16}$$

where we recall that I_X is the function defined in (1.8). We prove the first identity on the side of the matrices, where it expresses the unit intensity of Ξ_X, and we obtain the second from the computation of rank one in the sigma model in Lemma 6.15.

Proof of the first identity in (8.16). For real x, we have

$$\partial_t \frac{x^2+(\varepsilon+t)^2}{x^2+\varepsilon^2}\Big|_{t=0} = \frac{2\varepsilon}{x^2+\varepsilon^2} = 2\pi P_\varepsilon(x).$$

We differentiate the finite product in (1.4) under the expectation, which is justified by the bound (8.14) and (8.9), as in the proof of Lemma 8.7. This gives

$$T_N'(0) = 2\pi \mathbb{E} \sum_\alpha P_\varepsilon(x_\alpha) = 2\pi \mathbb{E}\Xi_N(P_\varepsilon).$$

We now show that $\mathbb{E}\Xi_N(P_\varepsilon) \to 1$. By parts (a) and (b), which we proved in Section 8.5, we have $\Xi_N \to \Xi_X$ in law. By Proposition 8.2, the processes Ξ_N, for all sufficiently large N, and Ξ_X satisfy (8.5) with a common constant $C_{\exp}$. We apply Lemma 8.3(b) with $l = 1$, $g_1 = P_\varepsilon$, $c = 1$, and $F(y) = y$, and we obtain

$$\lim_{N\to\infty} \mathbb{E}\Xi_N(P_\varepsilon) = \mathbb{E}\Xi_X(P_\varepsilon) = \int_{\mathbb{R}} P_\varepsilon(x)\,\mathrm{d}x = 1,$$

where the second equality holds since Ξ_X has unit intensity, as in the proof of parts (a) and (b). Finally, Corollary 8.8 shows that $T_N' \to \mathcal{Z}'$, and we conclude that

$$\mathcal{Z}'(0) = \lim_{N\to\infty} T_N'(0) = 2\pi. \qquad \square$$

Proof of the second identity in (8.16). We apply Lemma 6.15 with the present X and ε. The function $t \mapsto \mathcal{Z}^{(1)}_{X,\varepsilon}(0, \varepsilon + t)$ on $(-\varepsilon, \infty)$ of that lemma is the restriction of the entire function $\mathcal{Z}$ of this section, by Lemma 8.6, and the two have the same derivatives at $t = 0$. The second identity in (6.34) is therefore the second identity in (8.16). This completes the proof of Theorem 1.6. $\square$

The first identity in (6.34) is the first identity in (8.16), obtained within the sigma model from the one-insertion pairing in (6.33). The proof above identifies it, on the side of the matrices, with the unit intensity of Ξ_X.

Remark 8.9. The second derivative has the following meaning for Ξ_X. For real x, we have

$$\partial_t^2 \frac{x^2+(\varepsilon+t)^2}{x^2+\varepsilon^2}\Big|_{t=0} = \frac{2}{x^2+\varepsilon^2} = \frac{2\pi}{\varepsilon} P_\varepsilon(x),$$

and differentiating the product in (1.4) twice, we obtain

$$T_N''(0) = \frac{2\pi}{\varepsilon}\mathbb{E}\Xi_N(P_\varepsilon) + 4\pi^2 \mathbb{E}\big[\Xi_N(P_\varepsilon)^2 - \Xi_N(P_\varepsilon^2)\big].$$

We apply Lemma 8.3(b) with $g_1 = P_\varepsilon$, $g_2 = P_\varepsilon^2$, and $F(y_1, y_2) = y_1^2 - y_2$. It shows that the last expectation converges to $\mathbb{E}\sum_{x\neq y} P_\varepsilon(x)P_\varepsilon(y)$, where the sum runs over ordered pairs

of distinct points of the simple process Ξ_X. Using (8.16) and the limit $\mathbb{E}\Xi_N(P_\varepsilon) \to 1$ from its proof, we conclude that

$$\mathbb{E} \sum_{\substack{x,y \in \Xi_X \\ x \neq y}} P_\varepsilon(x) P_\varepsilon(y) = 1 + I_X(2\pi\varepsilon) - \frac{1}{2\pi\varepsilon}.$$

This is the case $v = 0$ of (1.12), with 2ε in place of ε, since $P_\varepsilon * P_\varepsilon = P_{2\varepsilon}$ and $P_{2\varepsilon}(0) = 1/(2\pi\varepsilon)$; the general case is proved in Section 14.

SECTION 9

Weighted moment measures and extraction of eigenvector moments

In this section, we show how resolvent averages at a fixed positive damping, which are the quantities controlled by Theorem 7.6, determine expected products of eigenvector weights over arbitrarily small energy windows. The main result is Theorem 9.6. In Section 11, we apply it to pairs of sites, which gives the local profile and the inverse participation ratio of Theorem 1.15. In Section 12, it is applied to products over k sites, which give the coarse eigenvector law.

We recall the mechanism from Section 2.7. For two sites i and j, the terms with $\alpha = \beta$ of the measure $N^2 \sum_{\alpha,\beta} w_\alpha(i) w_\beta(j) \delta_{(x_\alpha, x_\beta)}$ lie on the diagonal and form the eigenvector moment that we want. By (2.5), a diagonal part with constant density c contributes $c/(2\pi\varepsilon)$ to $\mathbb{E}\ell^\varepsilon_{i,N}(0)\ell^\varepsilon_{j,N}(0)$, while a measure that gives no mass to the diagonal contributes $o(\varepsilon^{-1})$ as $\varepsilon \downarrow 0$, so that the residue at $\varepsilon = 0$ should identify c. In physics, eigenvector moments are computed from such residues [94, Section 3.1].

Since the limits of Theorem 7.6 are taken at fixed $\varepsilon > 0$, and the residue only afterward, we pass to vague limits of the expected measures. Section 9.1 defines these measures and derives their moment bounds from Theorem 7.6(c). In Section 9.2, we show that the scalar Gaussian component of H makes the expected diagonal measures Gaussian mixtures, for which vague convergence implies locally uniform convergence of the densities. The main step is Lemma 9.5, which proves that every vague limit is stationary, that its part with distinct eigenvalue labels gives no mass to the diagonal, and that Poisson kernels may be integrated against it. For the second property, one might expect that products over k sites require estimates on k eigenvalues, but the bound on pairs of eigenvalues, which is the case $m = 2$ of Lemma 7.2, is enough for every k. We deduce the extraction theorem from these properties in Section 9.4, and in Section 9.5 we pass from densities to energy windows of arbitrarily small width.

Conventions. Recall the energies $E_* < E_{\mathrm{mid}} < E_{\mathrm{out}}$ of (7.22) and their roles (Remark 7.7). In this section, we consider sequences $N, W \to \infty$ with N/W^2 in a compact subset of $(0, \infty)$. The energy $E = E_N$, the sites and all other parameters may depend on N, and a_N and $x_\alpha = a_N(\lambda_\alpha - E)$ refer to the energy E, as in (1.3). Since our statements hold along every such sequence, they imply statements that are uniform over compact parameter sets; we spell this out in Remark 9.7.

9.1 Weighted moment measures

The local densities of Section 1 are Poisson smoothings of the following random measures.

Definition 9.1. For a site $i \in \{1, \dots, N\}$, the *unfolded local spectral measure* at i is the random measure

$$\ell_{i,N} = N \sum_{\alpha=1}^{N} w_\alpha(i) \delta_{x_\alpha} \tag{9.1}$$

on $\mathbb{R}$. Let $k \geq 1$. We call a random measure $\mathbf{\Pi}_N$ on $\mathbb{R}^k$ an *eigenvector-weighted moment measure of order k* if it has the form

$$\mathbf{\Pi}_N = \sum_{\alpha_1, \dots, \alpha_k = 1}^{N} c_N(\alpha_1, \dots, \alpha_k) \delta_{(x_{\alpha_1}, \dots, x_{\alpha_k})},$$

where each coefficient $c_N(\alpha_1, \dots, \alpha_k) \geq 0$ is a measurable function of the eigenprojections $u_{\alpha_1} u_{\alpha_1}^*, \dots, u_{\alpha_k} u_{\alpha_k}^*$. Let $\iota_k \colon \mathbb{R} \to \mathbb{R}^k$ denote the diagonal embedding $\iota_k(t) = (t, \dots, t)$, and let $r_k > 0$ be a normalization constant. We set

$$\mathbf{\Pi}_N^{\mathrm{d}} = \frac{1}{r_k} \sum_{\alpha=1}^{N} c_N(\alpha, \dots, \alpha) \delta_{x_\alpha}, \qquad \mathbf{\Pi}_N^{\neq} = \mathbf{\Pi}_N - r_k (\iota_k)_* \mathbf{\Pi}_N^{\mathrm{d}},$$

and we call $r_k (\iota_k)_* \mathbf{\Pi}_N^{\mathrm{d}}$ the *diagonal* (*coincident-eigenvalue*) *part* of $\mathbf{\Pi}_N$ and $\mathbf{\Pi}_N^{\neq}$ its *off-diagonal part*. We write $\Pi_N = \mathbb{E}\mathbf{\Pi}_N$, $\Pi_N^{\mathrm{d}} = \mathbb{E}\mathbf{\Pi}_N^{\mathrm{d}}$, and $\Pi_N^{\neq} = \mathbb{E}\mathbf{\Pi}_N^{\neq}$. Finally, let $i_1, \dots, i_k$ be sites and $C_0 > 0$. We say that $\mathbf{\Pi}_N$ is *dominated at the sites* $i_1, \dots, i_k$, with constant C_0, if for all nonnegative measurable functions $f_1, \dots, f_k$ on $\mathbb{R}$,

$$\mathbf{\Pi}_N(f_1 \otimes \cdots \otimes f_k) \leq C_0 \sum_{\pi \in S_k} \prod_{a=1}^{k} \ell_{i_a, N}(f_{\pi(a)}). \tag{9.2}$$

The eigenprojections in Definition 9.1 are almost surely well defined. Indeed, all entries of J are positive by Lemma 7.1, so that H has a density with respect to Lebesgue measure on Hermitian matrices, and its eigenvalues are almost surely simple. By the definition of $\ell_{i,N}^{\varepsilon}$ in Section 1, we have

$$\ell_{i,N}^{\varepsilon}(u) = \int_{\mathbb{R}} P_\varepsilon(x - u) \, \ell_{i,N}(\mathrm{d}x) = (P_\varepsilon * \ell_{i,N})(u), \tag{9.3}$$

and $\ell_{i,N}(\mathbb{R}) = N$, since u_α is a unit vector. With $f_a = 1$, the domination (9.2) therefore implies that $\mathbf{\Pi}_N(\mathbb{R}^k) \leq C_0 k! N^k$.

The basic example is the product $\mathbf{\Pi}_N = \ell_{i,N} \otimes \ell_{j,N}$, with coefficients $c_N(\alpha, \beta) = N^2 w_\alpha(i) w_\beta(j)$ and $r_2 = 1$. It is dominated at (i, j) with $C_0 = 1$, and

$$\mathbf{\Pi}_N^{\mathrm{d}} = N^2 \sum_{\alpha} w_\alpha(i) w_\alpha(j) \delta_{x_\alpha}.$$

In this case, Π_N is a mixed second moment measure of the random measures $\ell_{i,N}$ and $\ell_{j,N}$. Its off-diagonal part $\Pi_N^{\neq}$, which is carried by pairs of distinct labels, is an eigenvector-weighted second factorial-moment measure; see [23] for moment and factorial-moment measures of point processes. For $k \geq 3$, the off-diagonal part also contains label tuples with partial coincidences. In Section 12, the coefficients are squared moduli of permanents of eigenvector entries, which depend on the eigenvectors only through the eigenprojections, and $r_k = k!$.

Under $H \mapsto H + b\mathrm{Id}$ with $b \in \mathbb{R}$, the eigenprojections and the order of the eigenvalues do not change, and every λ_α increases by b. Therefore the coefficients of an eigenvector-weighted moment measure are invariant under scalar translations of the matrix, and the measure itself is translated by $a_N b(1, \dots, 1)$.

All estimates in this section rest on bounds for moments of local densities, which we now derive from part (c) of Theorem 7.6.

Lemma 9.2. *Let $p \geq 1$ and let $\mathcal{K} \subset (0, \infty)$ be compact. There exists C such that for all sufficiently large N, all sites i, all $\varepsilon \in \mathcal{K}$, all $|E| \leq E_{\mathrm{out}}$ and all $u \in \mathbb{R}$ with $|E + u/a_N| \leq E_{\mathrm{out}}$,*

$$\mathbb{E}\big[\ell_{i,N}^{\varepsilon}(u)^p\big] \leq C. \tag{9.4}$$

For every $m \geq 1$, the same bound holds for $\mathbb{E}\prod_{a=1}^{m} \ell_{i_a,N}^{\varepsilon_a}(u_a)$, uniformly over sites, dampings, and centers as above.

Proof. We recenter at $E' = E + u/a_N$. Let $a_N(E') = N\rho(E')$, and let

$$x'_\alpha = a_N(E')(\lambda_\alpha - E'), \qquad \text{so that} \qquad x_\alpha - u = a_N(\lambda_\alpha - E') = \frac{\rho(E)}{\rho(E')} x'_\alpha.$$

Since $P_\varepsilon(cy) = c^{-1} P_{\varepsilon/c}(y)$ for $c > 0$, we have

$$\ell_{i,N}^{\varepsilon}(u) = \frac{\rho(E')}{\rho(E)} N \sum_\alpha w_\alpha(i) P_{\varepsilon'}(x'_\alpha), \qquad \varepsilon' = \frac{\varepsilon \rho(E')}{\rho(E)}.$$

Note that

$$\frac{\rho(E_{\mathrm{out}})}{\rho(0)} \leq \frac{\rho(E')}{\rho(E)} \leq \frac{\rho(0)}{\rho(E_{\mathrm{out}})},$$

and in particular ε' ranges over a compact subset of $(0, \infty)$ when ε does.

By the elementary bound

$$P_{\varepsilon'} \leq \max\Big(\frac{\varepsilon'}{\varepsilon''}, \frac{\varepsilon''}{\varepsilon'}\Big) P_{\varepsilon''}, \qquad \varepsilon', \varepsilon'' > 0, \tag{9.5}$$

with $\varepsilon'' = 1$, we have

$$\ell_{i,N}^{\varepsilon}(u) \leq C \frac{1}{s_{E'}} \operatorname{Im} G_{ii}\big(E' + \mathrm{i}/a_N(E')\big) \leq C\big|G_{ii}\big(E' + \mathrm{i}/a_N(E')\big)\big|.$$

Further, $X_N(E') = N/((4-E'^2)W^2)$ ranges over a compact subset of $(0,\infty)$. We apply Theorem 7.6(c) with $u=0$, damping one and moment order $2p$, at the energy E'. Passing to subsequences along which $X_N(E')$ converges, we find

$$\sup_{1\le i\le N}\ \sup_{|E'|\le E_{\mathrm{out}}} \mathbb{E}\big|G_{ii}\big(E'+\mathrm{i}/a_N(E')\big)\big|^{2p} \le C.$$

Combining this with the previous display and Jensen's inequality, we obtain (9.4). The bound for products follows from Hölder's inequality. □

9.2 Gaussian mixtures

The matrix H contains a scalar Gaussian component. Recall from Lemma 7.4 that $H = \widetilde{H} + \gamma_N \mathrm{Id}$, where γ_N is a centered Gaussian variable of variance σ_N^2 that is independent of $\widetilde{H}$. We set $\tau_N = a_N\sigma_N$, the standard deviation of $a_N\gamma_N$. By Lemma 7.4, there exists C, depending only on E_{out} and the compact range of N/W^2, such that

$$\tau_N^{-1} \le CW^{-1/2}, \qquad |E| \le E_{\mathrm{out}}. \tag{9.6}$$

We let n_τ denote the centered Gaussian density on $\mathbb{R}$ with variance τ^2. A change of γ_N shifts every unfolded eigenvalue by one common amount and does not change the eigenvectors. As a consequence, expected eigenvector-weighted spectral measures are Gaussian mixtures.

Lemma 9.3. *Let $\boldsymbol{\mu}_N = \sum_\alpha c_N(\alpha)\delta_{x_\alpha}$, where $c_N(\alpha) \ge 0$ is a measurable function of $u_\alpha u_\alpha^*$, and suppose that $\mathbb{E}\boldsymbol{\mu}_N(\mathbb{R}) < \infty$. Then there exists a finite positive measure m_N on $\mathbb{R}$ with $m_N(\mathbb{R}) = \mathbb{E}\boldsymbol{\mu}_N(\mathbb{R})$ such that $\mathbb{E}\boldsymbol{\mu}_N = m_N * \mathsf{n}_{\tau_N}$. In particular, $\mathbb{E}\boldsymbol{\mu}_N$ has a smooth density.*

Proof. Let $\tilde\lambda_\alpha$ denote the eigenvalues of $\widetilde{H}$, in increasing order, and set $\tilde x_\alpha = a_N(\tilde\lambda_\alpha - E)$. The eigenprojections of H and $\widetilde{H}$ coincide, and $x_\alpha = \tilde x_\alpha + a_N\gamma_N$. Therefore $\boldsymbol{\mu}_N$ is the translate by $a_N\gamma_N$ of the measure

$$\widetilde{\boldsymbol{\mu}}_N = \sum_\alpha c_N(\alpha)\delta_{\tilde x_\alpha},$$

which is a function of $\widetilde{H}$. Since $a_N\gamma_N$ has density n_{τ_N} and is independent of $\widetilde{H}$, Fubini's theorem gives $\mathbb{E}\boldsymbol{\mu}_N = m_N * \mathsf{n}_{\tau_N}$ with $m_N = \mathbb{E}\widetilde{\boldsymbol{\mu}}_N$. □

By Lemma 9.3, the expected diagonal parts Π_N^{d} are Gaussian mixtures. We will need the following regularity property of such mixtures.

Lemma 9.4. *Let m be a finite positive measure on $\mathbb{R}$, let $\tau > 0$, and set $f = m * \mathsf{n}_\tau$. Then $u \mapsto \log f(u) + u^2/(2\tau^2)$ is convex, and for all $u_0 \in \mathbb{R}$,*

$$f(u_0) \le \frac{1}{2} e^{1/(2\tau^2)} \int_{u_0-1}^{u_0+1} f(u)\,\mathrm{d}u. \tag{9.7}$$

Further, let $f_N = m_N * \mathsf{n}_{\tau_N}$ for finite positive measures m_N and numbers τ_N with $\inf_N \tau_N > 0$. If $f_N(u)\,\mathrm{d}u \to c\,\mathrm{d}u$ vaguely for some $c \geq 0$, then $f_N \to c$ locally uniformly.

Proof. We may assume that $m \neq 0$. We have

$$f(u)e^{u^2/(2\tau^2)} = \frac{1}{\sqrt{2\pi}\,\tau} \int_{\mathbb{R}} e^{ut/\tau^2} e^{-t^2/(2\tau^2)}\, m(\,\mathrm{d}t).$$

The right side is the Laplace transform of the positive measure $e^{-t^2/(2\tau^2)} m(\,\mathrm{d}t)$, evaluated at u/τ^2. Laplace transforms of positive measures are log-convex by Hölder's inequality, and the first claim follows. Next, fix u_0 and set

$$\psi(u) = \log f(u) + \frac{(u-u_0)^2}{2\tau^2},$$

which differs from $\log f(u) + u^2/(2\tau^2)$ by an affine function and is therefore convex. For $0 \leq h \leq 1$, the inequality $2\psi(u_0) \leq \psi(u_0+h) + \psi(u_0-h)$ gives

$$f(u_0) \leq e^{h^2/(2\tau^2)} \sqrt{f(u_0+h)f(u_0-h)} \leq \frac{1}{2} e^{1/(2\tau^2)} \big(f(u_0+h) + f(u_0-h)\big).$$

Integrating over $h \in [0,1]$, we obtain (9.7).

We now prove the second claim. Fix $R > 0$ and set $\tau_0 = \inf_N \tau_N$. The indices with $m_N = 0$ satisfy $f_N = 0$, and we may discard them. Using (9.7) and the vague convergence, we see that f_N is bounded on $[-R-1, R+1]$, uniformly in N. Hence there exists M such that the convex functions

$$\psi_N(u) = \log f_N(u) + \frac{u^2}{2\tau_N^2}$$

satisfy $\psi_N \leq M$ on $[-R-1, R+1]$ for all N.

Let $|u| \leq R$. By convexity, the one-sided derivatives $\psi_N'(u\pm)$ satisfy

$$\psi_N(u) - M \leq \psi_N(u) - \psi_N(u-1) \leq \psi_N'(u\pm) \leq \psi_N(u+1) - \psi_N(u) \leq M - \psi_N(u).$$

Using $x(M - \log x) \leq e^{M-1}$ for $0 < x \leq e^M$, we find that e^{ψ_N} has Lipschitz constant at most e^{M-1} on $[-R, R]$. As $\tau_N \geq \tau_0$, the functions $f_N = e^{\psi_N} e^{-u^2/(2\tau_N^2)}$ are then uniformly bounded and equi-Lipschitz on $[-R, R]$. By the Arzelà–Ascoli theorem, every subsequence has a further subsequence that converges uniformly on $[-R, R]$, and the vague convergence identifies the limit as the constant c, also when $c = 0$. □

9.3 Stationary limits, collisions, and Poisson tails

We now prove the three properties of vague limits that were described at the beginning of the section. Recall that a positive Radon measure ν on $\mathbb{R}^d$ is translation bounded if $\sup_{s \in \mathbb{R}^d} \nu(s + [0,1]^d) < \infty$.

Lemma 9.5. *Let $k \geq 1$, and let $\mathbf{\Pi}_N$ be eigenvector-weighted moment measures of order k that are dominated at sites $i_1, \dots, i_k$, with a constant C_0 independent of N. Suppose that $|E| \leq E_{\mathrm{mid}}$.*

(a) For every compact set $V \subset \mathbb{R}$, we have $\sup_N \mathbb{E}[\mathbf{\Pi}_N(V^k)^2] < \infty$. In particular, the sequences Π_N, Π_N^{d}, and $\Pi_N^{\neq}$ are vaguely relatively compact.

(b) Suppose that $\Pi_N \to \Pi$, $\Pi_N^{\mathrm{d}} \to \Pi^{\mathrm{d}}$, and $\Pi_N^{\neq} \to \Pi^{\neq}$ vaguely along a subsequence. Then $\Pi = r_k(\iota_k)_* \Pi^{\mathrm{d}} + \Pi^{\neq}$, the measure $\Pi^{\neq}$ gives no mass to the diagonal $\iota_k(\mathbb{R})$, and $\Pi^{\mathrm{d}} = c\,\mathrm{d}t$ for a constant $c \geq 0$. Moreover, in the coordinates $(t_1, \mathbf{y})$ with $\mathbf{y} = (t_2 - t_1, \dots, t_k - t_1) \in \mathbb{R}^{k-1}$,

$$\Pi(\,\mathrm{d}\mathbf{t}) = \,\mathrm{d}t_1\, \nu(\,\mathrm{d}\mathbf{y}), \qquad \nu(\{0\}) = r_k c, \tag{9.8}$$

where ν is a translation-bounded positive Radon measure on $\mathbb{R}^{k-1}$. For $k = 1$, the space $\mathbb{R}^{k-1}$ is a point, and (9.8) reads $\Pi = r_1 c\,\mathrm{d}t$.

(c) Let $\varepsilon > 0$ and $u \in \mathbb{R}$, and let g_b and $g_{b,N}$, $1 \leq b \leq k$, be continuous functions on $\mathbb{R}$ such that $|g_b|, |g_{b,N}| \leq C P_\varepsilon(\cdot - u)$ for all N and $g_{b,N} \to g_b$ locally uniformly. Then, along the subsequence of (b),

$$\int_{\mathbb{R}^k} \prod_{b=1}^{k} g_{b,N}(t_b)\, \Pi_N(\,\mathrm{d}\mathbf{t}) \longrightarrow \int_{\mathbb{R}^k} \prod_{b=1}^{k} g_b(t_b)\, \Pi(\,\mathrm{d}\mathbf{t}), \tag{9.9}$$

and the same holds with (Π_N, Π) replaced by $(\Pi_N^{\neq}, \Pi^{\neq})$, or by $((\iota_k)_* \Pi_N^{\mathrm{d}}, (\iota_k)_* \Pi^{\mathrm{d}})$.

By (9.5), the functions in (c) include products of Poisson kernels $P_{\varepsilon_b}(\cdot - u_b)$ whose dampings and centers converge to a positive damping and a real center, respectively. They also include squared Cauchy kernels $(\cdot - u_b \pm \mathrm{i}\varepsilon_b)^{-2}$, since $|x - u_b \pm \mathrm{i}\varepsilon_b|^{-2} = \pi P_{\varepsilon_b}(x - u_b)/\varepsilon_b$, and derivatives of Poisson kernels in their parameters.

Proof. We begin with (a). Let $V \subset \mathbb{R}$ be compact, and let $m_V > 0$ be a lower bound for P_1 on V. By (9.3),

$$\ell_{i,N}(V) \leq \frac{1}{m_V} \int_{\mathbb{R}} P_1(x)\, \ell_{i,N}(\,\mathrm{d}x) = \frac{\ell^1_{i,N}(0)}{m_V}.$$

Inserting this bound into the domination (9.2) with $f_a = \mathbf{1}_V$, we obtain

$$\mathbf{\Pi}_N(V^k) \leq C_0\, k!\, m_V^{-k} \prod_{a=1}^{k} \ell^1_{i_a,N}(0).$$

Using Lemma 9.2 with $m = 2k$, we find that the right side has a second moment that is bounded uniformly in N. Further, $r_k(\iota_k)_* \mathbf{\Pi}_N^{\mathrm{d}} \leq \mathbf{\Pi}_N$ and $\mathbf{\Pi}_N^{\neq} \leq \mathbf{\Pi}_N$ as measures, which implies $\Pi_N^{\mathrm{d}}(V) \leq r_k^{-1} \Pi_N(V^k)$ and $\Pi_N^{\neq}(V^k) \leq \Pi_N(V^k)$. A sequence of positive Radon measures whose masses on compact sets are bounded is vaguely relatively compact, and (a) follows.

We now turn to (b). The relation $\Pi = r_k(\iota_k)_* \Pi^{\mathrm{d}} + \Pi^{\neq}$ follows from the corresponding relation for Π_N, since $\varphi \circ \iota_k$ is continuous with compact support for all $\varphi \in C_c(\mathbb{R}^k)$.

To show that $\Pi^{\neq}$ gives no mass to the diagonal, we use the pair count bound, which is the case $m = 2$ of Lemma 7.2 and an estimate of Minami type [92]. Fix $R \geq 1$ and $\gamma \in (0,1)$. Let $\mathcal{E}_\gamma$ be the event that there exist labels $\alpha \neq \beta$ such that $x_\alpha, x_\beta \in [-R, R]$ and $|x_\alpha - x_\beta| < \gamma$. By the covering (8.8) in the proof of Proposition 8.2, applied with $A = R$ and $\delta = \gamma$, two points $x \leq y$ of $[-R, R]$ with $y - x < \gamma$ lie in the interval I_l with $l = \lfloor x/\gamma \rfloor$, where

$$I_l = \big((l-1)\gamma, (l+2)\gamma\big), \qquad l \in \mathbb{Z},$$

as there, and this l satisfies $|l| \leq R/\gamma + 1$. Therefore, on $\mathcal{E}_\gamma$, one of the $2R/\gamma + 3 \leq 5R/\gamma$ intervals I_l with $|l| \leq R/\gamma + 1$ contains two eigenvalues, counted with multiplicity. Using $\mathbf{1}\{n \geq 2\} \leq (n)_2/2$ for integers $n \geq 0$, (7.3) with $m = 2$, and $\rho(E) \geq \rho(E_{\mathrm{out}})$, we find

$$\mathbb{P}(\mathcal{E}_\gamma) \leq \frac{1}{2} \sum_{|l| \leq R/\gamma + 1} \mathbb{E}(\Xi_N(I_l))_2 \leq \frac{5R}{2\gamma} \Big(\frac{3C_{\mathrm{cnt}}\gamma}{\rho(E_{\mathrm{out}})} \Big)^2 \leq C_R \gamma.$$

Next, we consider the band around the diagonal,

$$B_\gamma = \Big\{ \mathbf{t} \in [-R, R]^k : \max_a t_a - \min_a t_a < \gamma \Big\}.$$

An atom of $\mathbf{\Pi}_N^{\neq}$ in B_γ sits at a point $(x_{\alpha_1}, \dots, x_{\alpha_k})$ with $\alpha_a \neq \alpha_b$ for some a, b, and such a point can occur only on $\mathcal{E}_\gamma$. By the Cauchy–Schwarz inequality, (a), and the bound on $\mathbb{P}(\mathcal{E}_\gamma)$, we have

$$\Pi_N^{\neq}(B_\gamma) \leq \mathbb{E}\big[\mathbf{1}_{\mathcal{E}_\gamma} \mathbf{\Pi}_N([-R, R]^k)\big] \leq \mathbb{P}(\mathcal{E}_\gamma)^{1/2} \mathbb{E}\big[\mathbf{\Pi}_N([-R, R]^k)^2\big]^{1/2} \leq C_R \sqrt{\gamma}. \tag{9.10}$$

Let $\varphi \in C_c(\mathbb{R}^k)$ satisfy $0 \leq \varphi \leq \mathbf{1}_{B_\gamma}$ and $\varphi = 1$ on $\iota_k([-R+1, R-1])$. Then (9.10) implies

$$\Pi^{\neq}\big(\iota_k([-R+1, R-1])\big) \leq \int \varphi \, \mathrm{d}\Pi^{\neq} = \lim_N \int \varphi \, \mathrm{d}\Pi_N^{\neq} \leq C_R \sqrt{\gamma}.$$

Letting $\gamma \downarrow 0$, and then $R \to \infty$, we find that $\Pi^{\neq}(\iota_k(\mathbb{R})) = 0$.

Next, we show that the three limits are invariant under the translations $\mathbf{t} \mapsto \mathbf{t} + b\mathbf{1}$, where $\mathbf{1} = (1, \dots, 1)$, and $t \mapsto t + b$ for Π^{d}. Let $\boldsymbol{M}_N$ be one of $\mathbf{\Pi}_N$, $\mathbf{\Pi}_N^{\mathrm{d}}$, and $\mathbf{\Pi}_N^{\neq}$, let $\varphi \geq 0$ be continuous with compact support, and let $\Phi = \boldsymbol{M}_N[\varphi]$, viewed as a function of H. By the remarks after Definition 9.1, we have $\Phi_b = \boldsymbol{M}_N[\varphi(\cdot + b\mathbf{1})]$ in the notation of Lemma 7.4(c). We also have $\mathbb{E}\Phi_0^2 + \mathbb{E}\Phi_b^2 \leq C_{\varphi,b}$ uniformly in N, by (a). We apply (7.12) with $K = \tau_N^{1/2}$, recalling that $\tau_N = a_N \sigma_N$, and we find

$$\big|\mathbb{E}\boldsymbol{M}_N[\varphi(\cdot + b\mathbf{1})] - \mathbb{E}\boldsymbol{M}_N[\varphi]\big| \leq \sqrt{2/\pi} \, \frac{|b|}{\tau_N^{1/2}} + \frac{C_{\varphi,b}}{\tau_N^{1/2}} \longrightarrow 0, \qquad b \in \mathbb{R},$$

where we used $\tau_N \to \infty$, by (9.6). By linearity, the same holds for all real $\varphi \in C_c$, and the translation invariance of the limits follows.

We now derive (9.8). The linear map $\mathbf{t} \mapsto (t_1, \mathbf{y})$ has determinant one, and it carries the diagonal translations to translations in the first coordinate. A positive Radon measure on $\mathbb{R} \times \mathbb{R}^{k-1}$ that is invariant under translations in the first coordinate has the form $\mathrm{d}t_1\, \nu(\mathrm{d}\mathbf{y})$ for a unique positive Radon measure ν. (For $\psi \in C_c(\mathbb{R}^{k-1})$ with $\psi \geq 0$, the Radon measure on $\mathbb{R}$ that integrates $\mathbf{1}_A(t_1)\psi(\mathbf{y})$ is translation invariant and hence a multiple $\nu(\psi)$ of Lebesgue measure, and $\psi \mapsto \nu(\psi)$ is a positive linear functional.) The same argument with $k = 1$ shows that $\Pi^{\mathrm{d}} = c\,\mathrm{d}t$. Finally, we have $\nu(\{0\}) = r_k c$, since $(\iota_k)_*(c\,\mathrm{d}t) = c\,\mathrm{d}t_1\,\delta_0(\mathrm{d}\mathbf{y})$ and $\Pi^{\neq}$ gives no mass to $\{\mathbf{y} = 0\}$.

It remains to show that ν is translation bounded. Let $\mathbf{s} \in \mathbb{R}^{k-1}$, and let $A_1 = [0,1]$ and $A_b = [s_b, s_b + 2]$ for $2 \leq b \leq k$, so that $\nu(\mathbf{s} + [0,1]^{k-1}) \leq \Pi(A_1 \times \cdots \times A_k)$. The rectangle is compact, and $P_1 \geq P_1(1)$ on $[-1,1]$. Using (9.2) and Hölder's inequality, we have

$$\Pi(A_1 \times \cdots \times A_k) \leq \limsup_N \Pi_N(A_1 \times \cdots \times A_k) \leq C_0\, k!\, P_1(1)^{-k} \limsup_N \sup_{i,u} \bigl\| \ell^1_{i,N}(u) \bigr\|_{L^k}^k,$$

where the supremum is over all sites i and all centers u with $|E + u/a_N| \leq E_{\mathrm{out}}$. For fixed $\mathbf{s}$, the midpoints of the intervals A_b are such centers for all sufficiently large N (recall that $|E| \leq E_{\mathrm{mid}}$). The right side is bounded independently of $\mathbf{s}$, by Lemma 9.2.

Finally, we prove (c), and we begin with the tails of $\ell^{\varepsilon}_{i,N}(u)$. We set

$$c_1 = \frac{(E_{\mathrm{out}} - E_{\mathrm{mid}})\rho(E_{\mathrm{out}})}{2},$$

so that $|E + t/a_N| \leq E_{\mathrm{out}}$ for all $|t| \leq c_1 N + |u| + 2$ and all sufficiently large N. For $R \geq 3$, we define

$$T_R = \int_{|x-u|>R} P_\varepsilon(x - u)\, \ell_{i,N}(\mathrm{d}x).$$

The points with $|x - u| > c_1 N$ contribute at most

$$N \sum_\alpha w_\alpha(i) \frac{\varepsilon}{\pi c_1^2 N^2} = \frac{\varepsilon}{\pi c_1^2 N}$$

to T_R, since u_α is a unit vector.

On the remaining range, we bound T_R as in the tail bound (8.10), with $\ell_{i,N}$ in place of $\Xi^{(n)}$ and the L^{2k} norm in place of the expectation. We cover this range by the unit intervals $u + [m, m+1)$ with integers $R - 1 \leq |m| \leq c_1 N + 1$, on which $P_\varepsilon(x - u) \leq 4\varepsilon/(\pi m^2)$. In place of the bound on $\mathbb{E}\Xi^{(n)}(I_j)$ used there, we use

$$\ell_{i,N}\bigl(u + [m, m+1)\bigr) \leq P_1(1)^{-1} \ell^1_{i,N}(u + m + 1/2)$$

together with Lemma 9.2, which applies at the centers $u + m + 1/2$ by the choice of c_1. By Minkowski's inequality, we obtain

$$\|T_R\|_{L^{2k}} \leq \sum_{|m| \geq R-1} \frac{4\varepsilon}{\pi m^2 P_1(1)} \sup_{u'} \bigl\| \ell^1_{i,N}(u') \bigr\|_{L^{2k}} + \frac{\varepsilon}{\pi c_1^2 N} \leq C_\varepsilon \Bigl(\frac{1}{R} + \frac{1}{N} \Bigr), \tag{9.11}$$

where the supremum is over centers u' with $|E + u'/a_N| \le E_{\mathrm{out}}$.

Let $\chi_R \in C_c(\mathbb{R})$ satisfy $0 \le \chi_R \le 1$, $\chi_R = 1$ on $[u-R, u+R]$ and $\operatorname{supp}\chi_R \subset [u-R-1, u+R+1]$. We expand each factor as $g_{b,N} = g_{b,N}\chi_R + g_{b,N}(1-\chi_R)$. The product of the truncated factors is continuous, supported in a compact set that does not depend on N, and converges uniformly to $\prod_b g_b(t_b)\chi_R(t_b)$. Its integral against Π_N therefore converges, by the vague convergence and (a).

Every other term has at least one factor bounded by $CP_\varepsilon(\cdot - u)\mathbf{1}_{|\cdot-u|>R}$, while the remaining factors are bounded by $CP_\varepsilon(\cdot - u)$. By (9.2) and Hölder's inequality, the integral of its modulus against Π_N is at most

$$C^k C_0\, k!\, \max_a \|T_R(i_a)\|_{L^k} \max_a \big\|\ell^{\varepsilon}_{i_a,N}(u)\big\|_{L^k}^{k-1} \le C'\Big(\frac{1}{R} + \frac{1}{N}\Big),$$

where $T_R(i)$ denotes T_R at the site i, and we used (9.11) and Lemma 9.2. The same bound CR^{-1} holds for the limit Π. Indeed, for every nonnegative lower semicontinuous function f on $\mathbb{R}^k$, vague convergence implies

$$\int f \,\mathrm{d}\Pi \le \liminf_N \int f \,\mathrm{d}\Pi_N.$$

Letting $N \to \infty$, and then $R \to \infty$, we obtain (9.9). The measures $\Pi_N^{\neq}$ and $r_k(\iota_k)_*\Pi_N^{\mathrm{d}}$ are bounded by Π_N, so the same argument applies to them. □

9.4 The extraction theorem

We now identify the diagonal density from the residue of the source limits. For $\mathbf{y} = (y_2, \dots, y_k) \in \mathbb{R}^{k-1}$, we set $y_1 = 0$ and define

$$\mathsf{P}_k(\mathbf{y}) = \int_{\mathbb{R}} \prod_{b=1}^{k} P_1(t + y_b)\,\mathrm{d}t, \qquad \varkappa_{k,r} = r(2\pi)^{k-1}\mathsf{P}_k(0), \quad r > 0. \tag{9.12}$$

For $k = 1$, we set $\mathsf{P}_1 = 1$, so that $\varkappa_{1,r} = r$. The beta integral $\int_{\mathbb{R}}(1+t^2)^{-k}\,\mathrm{d}t = \sqrt{\pi}\,\Gamma(k - 1/2)/\Gamma(k)$ gives

$$\mathsf{P}_k(0) = \frac{(2k-3)!!}{2^{k-1}\pi^{k-1}(k-1)!}, \qquad \varkappa_{k,r} = \frac{r(2k-3)!!}{(k-1)!}, \tag{9.13}$$

with the convention $(-1)!! = 1$. For example, $\varkappa_{2,1} = 1$ and $\varkappa_{k,k!} = k(2k-3)!!$.

Theorem 9.6. *Let $k \ge 1$ and $r_k > 0$. For each N, let $|E| \le E_{\mathrm{mid}}$, let $i_1, \dots, i_k$ be sites, and let $\mathbf{\Pi}_N$ be an eigenvector-weighted moment measure of order k that is dominated at $i_1, \dots, i_k$ with a constant C_0 independent of N. Suppose that there exist a function $\mathfrak{f}\colon (0,\infty) \to \mathbb{R}$ and a constant $c_* \ge 0$ such that for all $\varepsilon > 0$,*

$$\int_{\mathbb{R}^k} \prod_{b=1}^{k} P_\varepsilon(t_b)\,\Pi_N(\,\mathrm{d}\mathbf{t}) \longrightarrow \mathfrak{f}(2\pi\varepsilon), \qquad \lim_{z\downarrow 0} z^{k-1}\mathfrak{f}(z) = \varkappa_{k,r_k} c_*. \tag{9.14}$$

Then $\Pi_N^{\mathrm{d}} \to c_*\,\mathrm{d}u$ *vaguely, and the densities of* Π_N^{d} *converge to* c_* *locally uniformly.*

The matrix limit in (9.14) is used only at fixed positive damping, which is the form in which Theorem 7.6 provides it. When $\mathbf{\Pi}_N$ is a tensor product of the measures $\ell_{i,N}$, the left side of (9.14) is an expected product of local densities, by (9.3).

Proof. By Lemma 9.5(a), it suffices to show that $\Pi^{\mathrm{d}} = c_*\,\mathrm{d}u$ for every vague limit point Π^{d} of Π_N^{d}. Passing to a further subsequence, we may assume that Π_N, Π_N^{d}, and $\Pi_N^{\neq}$ converge vaguely along it, and we let $\Pi^{\mathrm{d}} = c\,\mathrm{d}t$ and ν be as in Lemma 9.5(b). Applying Lemma 9.5(c) with $g_{b,N} = g_b = P_\varepsilon$ and using (9.8), we find that for all $\varepsilon > 0$,

$$\mathfrak{f}(2\pi\varepsilon) = \int_{\mathbb{R}^{k-1}} \int_{\mathbb{R}} \prod_{b=1}^{k} P_\varepsilon(t_1 + y_b)\,\mathrm{d}t_1\,\nu(\,\mathrm{d}\mathbf{y}) = \varepsilon^{1-k} \int_{\mathbb{R}^{k-1}} \mathsf{P}_k(\mathbf{y}/\varepsilon)\,\nu(\,\mathrm{d}\mathbf{y}),$$

where in the second step we used $P_\varepsilon(x) = \varepsilon^{-1}P_1(x/\varepsilon)$ and substituted $t_1 = \varepsilon t$. For $k = 1$, this reads $\mathfrak{f}(2\pi\varepsilon) = r_1 c$, and (9.14) implies that $c = c_*$. We assume $k \geq 2$ from now on.

To let $\varepsilon \downarrow 0$, we need three properties of P_k. First, we insert the representation

$$P_1(t) = \frac{1}{\pi} \int_0^\infty e^{-s(1+t^2)}\,\mathrm{d}s$$

into (9.12) and compute the Gaussian integral in t, which gives

$$\mathsf{P}_k(\mathbf{y}) = \pi^{1/2-k} \int_{(0,\infty)^k} \frac{e^{-|\mathbf{s}|}}{\sqrt{|\mathbf{s}|}} \exp\Bigl(-\frac{1}{|\mathbf{s}|} \sum_{1 \leq a < b \leq k} s_a s_b (y_a - y_b)^2\Bigr)\,\mathrm{d}\mathbf{s}, \qquad |\mathbf{s}| = \sum_{b=1}^{k} s_b.$$

It follows that $\mathsf{P}_k(v\mathbf{y}) \leq \mathsf{P}_k(\mathbf{y})$ for $v \geq 1$, and, by dominated convergence, that $\mathsf{P}_k(v\mathbf{y}) \to 0$ as $v \to \infty$ for $\mathbf{y} \neq 0$. Second, integrating first over $y_2, \dots, y_k$ and using that P_1 is a probability density, we find

$$\int_{\mathbb{R}^{k-1}} \mathsf{P}_k(\mathbf{y})\,\mathrm{d}\mathbf{y} = 1.$$

Third, we have $|P_1'| \leq P_1$, and hence $|\partial_{y_b}\mathsf{P}_k| \leq \mathsf{P}_k$ for $2 \leq b \leq k$. The third property implies that on every unit cube, the supremum of P_k is bounded by e^{k-1} times its infimum. Together with the second property, this shows that the suprema of P_k over the cubes $\mathbf{s} + [0,1)^{k-1}$, $\mathbf{s} \in \mathbb{Z}^{k-1}$, have sum at most e^{k-1}.

Since ν is translation bounded, the last statement implies that P_k is ν-integrable, with

$$\int_{\mathbb{R}^{k-1}} \mathsf{P}_k\,\mathrm{d}\nu \leq e^{k-1} \sup_{\mathbf{s} \in \mathbb{R}^{k-1}} \nu\bigl(\mathbf{s} + [0,1]^{k-1}\bigr).$$

The first property shows that P_k dominates $\mathsf{P}_k(\cdot/\varepsilon)$ for $\varepsilon \leq 1$. Using dominated convergence, we obtain

$$\lim_{z \downarrow 0} z^{k-1}\mathfrak{f}(z) = (2\pi)^{k-1} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^{k-1}} \mathsf{P}_k(\mathbf{y}/\varepsilon)\,\nu(\,\mathrm{d}\mathbf{y}) = (2\pi)^{k-1}\mathsf{P}_k(0)\nu(\{0\}) = \varkappa_{k,r_k} c,$$

where we used (9.8) in the last step. Comparing with (9.14), we find $c = c_*$. For $k = 2$, we have $\mathsf{P}_2 = P_2$, and this limit is Lemma A.4(c) with $a = 2\varepsilon$.

It remains to prove the convergence of densities. We have $\mathbb{E}\mathbf{\Pi}_N^{\mathrm{d}}(\mathbb{R}) \le r_k^{-1} C_0 k! N^k$, so Lemma 9.3 applies to $\mathbf{\Pi}_N^{\mathrm{d}}$, and the measure Π_N^{d} has the density $m_N * \mathsf{n}_{\tau_N}$ for a finite positive measure m_N. Since $\tau_N \to \infty$ by (9.6), and since $\Pi_N^{\mathrm{d}} \to c_*\,\mathrm{d}u$ vaguely, we conclude from Lemma 9.4 that the densities converge to c_* locally uniformly. □

Remark 9.7. Since all parameters in Theorem 9.6 may depend on N, the theorem implies uniform statements. Suppose that for each N the parameters (the energy, N/W^2, the sites, and any further parameters) range over a set Θ_N, and that $c_*(\theta)$ is a bounded function of the parameters. Suppose further that every sequence $\theta_N \in \Theta_N$ has a subsequence along which the hypotheses of Theorem 9.6 hold, for some $\mathfrak{f}$ and some c_* such that $c_*(\theta_N) \to c_*$. Let f_θ denote the density of Π_N^{d} at the parameter θ. Then for all $R > 0$,

$$\sup_{\theta\in\Theta_N} \sup_{|u|\le R} |f_\theta(u) - c_*(\theta)| \longrightarrow 0.$$

Indeed, otherwise there exist $\delta_0 > 0$ and parameters $\theta_N \in \Theta_N$ with $\sup_{|u|\le R} |f_{\theta_N}(u) - c_*(\theta_N)| \ge \delta_0$ for infinitely many N, and the theorem, applied along a suitable subsequence, yields a contradiction.

9.5 Densities and arbitrarily small windows

The case $k = 1$ of Theorem 9.6 normalizes the local spectral measures.

Corollary 9.8. *For each N, let $|E| \le E_{\mathrm{mid}}$ and let i be a site. If $\mathbb{E}\ell_{i,N}^{\varepsilon}(0) \to 1$ for all $\varepsilon > 0$, then the densities of $\mathbb{E}\ell_{i,N}$ converge to one locally uniformly.*

Proof. We apply Theorem 9.6 with $k = 1$, $\mathbf{\Pi}_N = \ell_{i,N}$, $r_1 = 1$, $\mathfrak{f} = 1$, and $c_* = 1$. Then $\mathbf{\Pi}_N^{\mathrm{d}} = \ell_{i,N}$ and $\mathbf{\Pi}_N^{\neq} = 0$, the domination holds with $C_0 = 1$, and the source limit is the hypothesis, by (9.3). □

We now pass from densities to energy windows. Let $\chi \ge 0$ be an integrable function with compact support and $\int \chi > 0$; the window ϕ of Definition 1.8 is an example. Let $k \ge 1$, and let $F_\alpha = F_{\alpha,N} \ge 0$ be bounded measurable functions of $u_\alpha u_\alpha^*$. For a physical energy e, we set $a_N(e) = N\rho(e)$, and we let $f_{N,e}$ denote the density of the measure

$$N^k \mathbb{E} \sum_\alpha F_\alpha\, \delta_{a_N(e)(\lambda_\alpha - e)}, \tag{9.15}$$

which exists by Lemma 9.3, applied with the energy e in place of E.

Corollary 9.9. *Let k, F_α, and $f_{N,e}$ be as above, and let $\bar{c}_N$ be functions on $[-E_{\mathrm{mid}}, E_{\mathrm{mid}}]$ that are bounded and equicontinuous, uniformly in N. Suppose that*

$$\sup_{|e|\le E_{\mathrm{mid}}} |f_{N,e}(0) - \bar{c}_N(e)| \longrightarrow 0.$$

Then for every sequence $\bar{r}_N \downarrow 0$, uniformly for $|E| \le E_*$ and $0 < r \le \bar{r}_N$,

$$N^k \frac{\mathbb{E}\sum_\alpha F_\alpha\, \chi((\lambda_\alpha - E)/r)}{\mathbb{E}\sum_\alpha \chi((\lambda_\alpha - E)/r)} - \bar{c}_N(E) \longrightarrow 0. \tag{9.16}$$

Let $i = i(N)$ be sites, and suppose in addition that the densities $g_{N,e}$ of the measures $N\mathbb{E}\sum_\alpha w_\alpha(i)\delta_{a_N(e)(\lambda_\alpha - e)}$ satisfy $\sup_{|e|\le E_{\mathrm{mid}}} |g_{N,e}(0) - 1| \to 0$. Then, uniformly in the same sense,

$$N^{k-1} \frac{\mathbb{E}\sum_\alpha F_\alpha\, \chi((\lambda_\alpha - E)/r)}{\mathbb{E}\sum_\alpha w_\alpha(i)\chi((\lambda_\alpha - E)/r)} - \bar{c}_N(E) \longrightarrow 0. \tag{9.17}$$

There is no lower bound on r, so the ratios in (9.16) and (9.17) are controlled also for windows that typically contain no eigenvalue.

Proof. The measure $N^k\mathbb{E}\sum_\alpha F_\alpha\delta_{\lambda_\alpha}$ has a density m_N in the physical variable, by Lemma 9.3. The change of variables $u = a_N(e)(\lambda - e)$ gives

$$m_N(e) = a_N(e) f_{N,e}(0) = N\rho(e) f_{N,e}(0)$$

for all e. Similarly, the measure $\mathbb{E}\sum_\alpha w_\alpha(i)\delta_{\lambda_\alpha}$ has the density $\rho(e)g_{N,e}(0)$. Recall from Lemma 7.3 that $\mathbb{E}\sum_\alpha \delta_{\lambda_\alpha}$ has the density $N\rho_{N,W}(e)$, and that $\rho_{N,W}(e) = \rho(e)(1 + o(1))$, uniformly for $|e| \le E_{\mathrm{mid}}$.

Let $\operatorname{supp}\chi \subset [-R_\chi, R_\chi]$. For $|E| \le E_*$ and $r \le \bar{r}_N$, all energies e with $\chi((e-E)/r) \ne 0$ satisfy $|e - E| \le R_\chi \bar{r}_N$, and in particular $|e| \le E_{\mathrm{mid}}$ for N large. For such e, the hypothesis and the equicontinuity of $\bar{c}_N$ imply that $f_{N,e}(0) = \bar{c}_N(E) + o(1)$, uniformly. We obtain

$$N^k \frac{\mathbb{E}\sum_\alpha F_\alpha \chi((\lambda_\alpha - E)/r)}{\mathbb{E}\sum_\alpha \chi((\lambda_\alpha - E)/r)} = \frac{\int \rho(e) f_{N,e}(0)\chi((e-E)/r)\,\mathrm{d}e}{\int \rho_{N,W}(e)\chi((e-E)/r)\,\mathrm{d}e} = \frac{\bar{c}_N(E) + o(1)}{1 + o(1)},$$

where we used that

$$\int \rho(e)\chi((e-E)/r)\,\mathrm{d}e \ge \rho(E_{\mathrm{mid}})\, r \int \chi > 0.$$

Since $\bar{c}_N$ is bounded, (9.16) follows. The proof of (9.17) is the same, with

$$N^{k-1}\mathbb{E}\sum_\alpha F_\alpha \chi\Big(\frac{\lambda_\alpha - E}{r}\Big) = \int \rho(e) f_{N,e}(0)\chi\Big(\frac{e-E}{r}\Big)\,\mathrm{d}e,$$
$$\mathbb{E}\sum_\alpha w_\alpha(i)\chi\Big(\frac{\lambda_\alpha - E}{r}\Big) = \int \rho(e) g_{N,e}(0)\chi\Big(\frac{e-E}{r}\Big)\,\mathrm{d}e$$

in place of the identities used above. □

In the applications, we verify the hypothesis of Corollary 9.9 with Theorem 9.6 and Remark 9.7. For each energy e, the measure (9.15) is the expected diagonal part, unfolded at e, of an eigenvector-weighted moment measure. Suppose that the hypotheses of Theorem 9.6 hold along all sequences of energies $e = e_N$ with $|e_N| \le E_{\mathrm{mid}}$, with targets c_* such that $\bar{c}_N(e_N) - c_* \to 0$. Then Remark 9.7 implies that $\sup_{|e| \le E_{\mathrm{mid}}} |f_{N,e}(0) - \bar{c}_N(e)| \to 0$.

SECTION 10

The noncompact boundary and Brownian functionals

In this section, we let the damping z of the sigma-model kernels tend to zero. There are two main results. Theorem 10.4 states that $zK_X(x, y; z)/2 \to \mathcal{C}_X(x, y)$, and Lemma 10.12 states that $z^{k-1}\mathfrak{p}_k^{X,z} \to k(2k-3)!!\, M_k^X$ for the permanent pairings $\mathfrak{p}_k^{X,z}$ of Definition 10.6. In Sections 11 and 12, these limits identify the eigenvector moments through the extraction theorem Theorem 9.6, which converts the coefficient of z^{1-k} at small damping into the density of the diagonal part of a moment measure.

The heuristic is explained in Section 2.7, around (2.6), and we recall it briefly. For small z, the killing rate $zd = z(B - F)$ is negligible unless B is of order $1/z$. On this scale, the radial part of the singlet diffusion is a geometric Brownian motion, killed at a rate proportional to its value, and the coordinate F and the angular and anticommuting variables decouple. The identity (2.7) then turns the Laplace transforms of the exponential functional of this geometric Brownian motion into the moment densities M_k^X of Definition 1.9.

In the physics literature, the same scaling appears in the computation of eigenfunction statistics from the one-dimensional sigma model. As the level broadening tends to zero, the noncompact radial variable is the one that survives, in the scaling form of [94, (3.9)], and the resulting transfer equation $\partial_\tau W = (z^2\partial_z^2 - z)W$ of [94, (3.22)–(3.23)] is the backward equation of the killed geometric Brownian motion above; see also [56]. Its stationary solution $2\sqrt{z}K_1(2\sqrt{z})$, which goes back to Efetov and Larkin [38, (55)], is the function Y of (10.5) below. In this section, we make the passage to this scaling limit rigorous, including the corner of D and, for the permanent pairings, all anticommuting variables. The main difficulty is the second point, and we treat it with a coupling argument in Section 10.7.

We now outline the section. In Section 10.1, we recall the radial and the singlet diffusions of Section 6.2. Section 10.2 introduces the scaled coordinates and the limiting semigroups, and Section 10.3 writes all moment densities M_k^X in terms of these semigroups. Section 10.4 proves a tail bound, and Section 10.5 proves Theorem 10.4. Finally, Sections 10.6 and 10.7 treat the permanent pairings, for which we need the complete rank-one sigma model in polar coordinates.

Conventions. Throughout this section, $z > 0$ is real unless stated otherwise, and we set

$$\hat{z} = \frac{z}{X}.$$

We fix $0 < X_- \le X_+ < \infty$ and $\delta \in (0, 1/2]$. Constants C may depend on $X_\pm$ and δ, and on the order k where it appears, and they may change from line to line. We let $D^\circ = (1,\infty) \times (-1,1)$ denote the interior of D.

10.1 The radial and the singlet diffusions

Recall from Definition 1.4 the domain D, the function $d = B - F$, the operator Δ_D, the diffusion with generator $X\Delta_D$, which Section 6.2 calls the radial diffusion, the semigroup $e^{t(X\Delta_D - zd)}$, and the singlet state $\Psi_t^{X,z}$. Recall from Section 6.2 that the coordinates of the radial diffusion are independent diffusions whose finite endpoints are entrance boundaries, so that the radial diffusion started in D° stays in D°, and that they solve the stochastic differential equations (6.7). Recall also the singlet diffusion of Lemma 6.5, which is obtained from the radial diffusion by the Doob transform with the function $1/d$ and is stopped at the corner at its corner time τ_{c}. It solves (6.9) before τ_{c}, and its law is given by the Doob identity (6.10). By (6.11), the singlet state is the expectation of its survival factor. All starting points below lie in D°.

As in Section 6.2, we let $(\tilde B_t, \tilde F_t)$ and (B_t, F_t) denote the radial and the singlet diffusions, driven by the Brownian motions β^1, β^2 and $\beta^{\mathrm{s},1}$, $\beta^{\mathrm{s},2}$ of (6.7) and (6.9); a pair (B, F) without a time index is a point of D. We will use the identity (6.8). The holomorphy and the absolute convergence of the kernels of Definition 1.4 are proved in Lemma 6.6(d), where the singlet state is estimated through Lemma 6.5(c).

10.2 The boundary at infinity

Scaled coordinates

The parameters X and z enter the objects of Definition 1.4 only through $\hat z = z/X$ and the time Xt. Indeed, $(\tilde B_{t/X}, \tilde F_{t/X})_{t\ge 0}$ is the radial diffusion with $X = 1$. Using the definition in Definition 1.4, we find

$$e^{t(X\Delta_D - zd)} = e^{Xt(\Delta_D - \hat z d)}, \qquad \Psi_t^{X,z} = \Psi_{Xt}^{1,\hat z},$$

where the second identity follows from the first and (1.5). From now on, we work with $X = 1$ and the damping $\hat z$, and we use the scaled radial coordinate and the scaled domain

$$q = \hat z B, \qquad D_{\hat z} = [\hat z, \infty) \times [-1, 1],$$

and a time s in the scaled variables corresponds to the time $t = s/X$ in Definition 1.4.

Let $(\tilde B_s, \tilde F_s)$ and (B_s, F_s) be the radial and the singlet diffusions with $X = 1$, started at $(q/\hat z, F)$ with $(q, F) \in (\hat z, \infty) \times (-1, 1)$, and let $\mathbb{E}$ denote the expectation with respect to their laws. We define the *scaled radial diffusion* and the *scaled singlet diffusion* by

$$(\tilde q_s, \tilde F_s) = (\hat z \tilde B_s, \tilde F_s), \qquad (q_s, F_s) = (\hat z B_s, F_s), \qquad s \ge 0.$$

Let $\tilde\beta = \beta^1$ and $\beta = \beta^{s,1}$ denote the Brownian motions that drive their first coordinates. We write τ_c for the corner time of the singlet. By (6.7) and (6.9),

$$\begin{aligned} \mathrm{d}\tilde q_s &= 2\tilde q_s\,\mathrm{d}s + \big(2(\tilde q_s^2 - \hat z^2)\big)^{1/2}\,\mathrm{d}\tilde\beta_s,\\ \mathrm{d}q_s &= 2\hat z\,\frac{\hat z - q_sF_s}{q_s - \hat zF_s}\,\mathrm{d}s + \big(2(q_s^2 - \hat z^2)\big)^{1/2}\,\mathrm{d}\beta_s, \qquad s < \tau_c, \end{aligned} \tag{10.1}$$

where we used $(1-BF)/(B-F) = (\hat z - qF)/(q - \hat zF)$. The killing integral becomes

$$\hat z\int_0^s d(B_u,F_u)\,\mathrm{d}u = \int_0^s (q_u - \hat zF_u)\,\mathrm{d}u,$$

and the corner corresponds to $(q,F) = (\hat z, 1)$, where $q - \hat zF = 0$. For $s \ge 0$, $(q,F) \in (\hat z,\infty)\times(-1,1)$, and bounded measurable functions f on $D_{\hat z}$, we define

$$\begin{aligned} \Psi_s^{(\hat z)}(q,F) &= \mathbb{E}\exp\Big(-\int_0^s (q_u - \hat zF_u)\,\mathrm{d}u\Big),\\ \big(\widetilde{\mathcal{T}}_s^{(\hat z)}f\big)(q,F) &= \mathbb{E}\Big[f(\tilde q_s,\tilde F_s)\exp\Big(-\int_0^s(\tilde q_u - \hat z\tilde F_u)\,\mathrm{d}u\Big)\Big]. \end{aligned} \tag{10.2}$$

Using Lemma 6.5(c) and the definition of the radial semigroup, we obtain the exact rescaling

$$\Psi_t^{X,z}(B,F) = \Psi_{Xt}^{(\hat z)}(\hat zB,F), \qquad \big(e^{t(X\Delta_D - zd)}f\big)(B,F) = \big(\widetilde{\mathcal{T}}_{Xt}^{(\hat z)}f_{\hat z}\big)(\hat zB,F), \tag{10.3}$$

where $f_{\hat z}(q,F) = f(q/\hat z,F)$. In particular, $0 \le \Psi_s^{(\hat z)} \le 1$.

The coefficients of (10.1) are close to those of geometric Brownian motion. Indeed, (6.8) implies that, for $q \ge \hat z$ and $F \in [-1,1]$,

$$\Big|2\hat z\,\frac{\hat z - qF}{q - \hat zF}\Big| \le 2\hat z, \qquad 0 \le \sqrt2\,q - \big(2(q^2-\hat z^2)\big)^{1/2} \le \sqrt2\,\hat z, \tag{10.4}$$

where for the second bound we used

$$q - (q^2-\hat z^2)^{1/2} = \frac{\hat z^2}{q + (q^2-\hat z^2)^{1/2}} \le \hat z.$$

Formally setting $\hat z = 0$ in (10.1), we obtain the geometric Brownian motions $\mathrm{d}q_s^0 = \sqrt2\,q_s^0\,\mathrm{d}\beta_s$ and $\mathrm{d}\tilde q_s^0 = 2\tilde q_s^0\,\mathrm{d}s + \sqrt2\,\tilde q_s^0\,\mathrm{d}\tilde\beta_s$, killed at the rates q_s^0 and $\tilde q_s^0$. We now introduce the corresponding semigroups.

The limiting semigroups

We extend the definition of g_X in Definition 1.9 to all times. For a standard Brownian motion $\beta = (\beta_u)_{u\ge0}$, we set

$$g_1(u) = \sqrt2\,\beta_u - u, \qquad A_s = \int_0^s e^{g_1(u)}\,\mathrm{d}u, \qquad u,s \ge 0.$$

Then $q_s^0 = qe^{g_1(s)}$ and $\tilde{q}_s^0 = qe^{g_1(s)+2s}$ solve the equations above, with $\tilde{\beta}$ in place of β in the second case. For $s \geq 0$, $q > 0$, and measurable functions $f \geq 0$ on $(0,\infty)$, we define

$$\mathcal{T}_s f(q) = \mathbb{E}\big[f(qe^{g_1(s)})e^{-qA_s}\big], \qquad \mathrm{Y}_s = \mathcal{T}_s 1, \qquad \mathrm{Y}(q) = 2\sqrt{q}\,K_1(2\sqrt{q}), \tag{10.5}$$

where K_1 is the modified Bessel function. The function $\mathrm{Y}_s(q) = \mathbb{E}e^{-qA_s}$ is the Laplace transform of A_s, and $\mathcal{T}_s$ is the semigroup of the geometric Brownian motion q^0, killed at the rate q^0. We will see in Lemma 12.9 that Y is the limit of Y_s as $s \to \infty$. We also define the differential operator $\mathscr{L} = -\partial_q(q^2\partial_q) + q$ on $C_c^\infty((0,\infty))$, and the unitary map $\mathcal{U}\colon L^2((0,\infty), \mathrm{d}q) \to L^2(\mathbb{R}, \mathrm{d}y)$ by $(\mathcal{U}f)(y) = e^{y/2}f(e^y)$.

Lemma 10.1. *(a) For all $s \geq 0$ and all measurable $f, f' \geq 0$ on $(0,\infty)$,*

$$\int_0^\infty q^{-2} f'(q)\,\mathcal{T}_s f(q)\,\mathrm{d}q = \int_0^\infty q^{-2} f(q)\,\mathcal{T}_s f'(q)\,\mathrm{d}q.$$

(b) The operator $\mathscr{L}$ is essentially self-adjoint on $C_c^\infty((0,\infty))$ in $L^2((0,\infty), \mathrm{d}q)$. We also write $\mathscr{L}$ for its closure. It satisfies $\mathscr{L} \geq 1/4$, and $\mathcal{U}\mathscr{L}\mathcal{U}^{-1}$ is the closure of $-\partial_y^2 + e^y + 1/4$ on $C_c^\infty(\mathbb{R})$.

(c) For all $s \geq 0$ and $f \in L^2((0,\infty), \mathrm{d}q)$, and for almost every $q > 0$,

$$(e^{-s\mathscr{L}} f)(q) = \mathbb{E}\Big[f(qe^{g_1(s)+2s}) \exp\Big(-q\int_0^s e^{g_1(u)+2u}\,\mathrm{d}u\Big)\Big]. \tag{10.6}$$

For measurable $f \geq 0$, we use the right side of (10.6) *as the definition of $(e^{-s\mathscr{L}} f)(q)$ for every $q > 0$.*

(d) For all $s \geq 0$, $q > 0$, and measurable $f \geq 0$,

$$\mathcal{T}_s(qf)(q) = q\,(e^{-s\mathscr{L}} f)(q), \tag{10.7}$$

where qf denotes the function $q \mapsto qf(q)$.

The first statement is the symmetry of $\mathcal{T}_s$ with respect to the speed measure $q^{-2}\,\mathrm{d}q$ of q^0. The last is the Doob transform of $\mathcal{T}_s$ by the function q, which is harmonic for the generator $q^2\partial_q^2$ of q^0. Under this transform, the driftless geometric Brownian motion q^0 becomes the geometric Brownian motion $\tilde{q}^0$ with generator $q^2\partial_q^2 + 2q\partial_q$.

Proof. We begin with (a). Fix a path of β on $[0,s]$, and substitute $q = re^{-g_1(s)}$ in the left side, for which

$$q^{-2}\,\mathrm{d}q = r^{-2}e^{g_1(s)}\,\mathrm{d}r, \qquad qA_s = r\int_0^s e^{g_1(u)-g_1(s)}\,\mathrm{d}u.$$

Tonelli's theorem yields

$$\int_0^\infty q^{-2} f'(q)\,\mathcal{T}_s f(q)\,\mathrm{d}q = \mathbb{E}\int_0^\infty r^{-2} f(r)\,e^{g_1(s)} f'(re^{-g_1(s)}) \exp\Big(-r\int_0^s e^{g_1(u)-g_1(s)}\,\mathrm{d}u\Big)\,\mathrm{d}r.$$

We now reverse time. The processes

$$\hat\beta_u = \beta_s - \beta_{s-u}, \qquad \hat g(u) = \sqrt{2}\,\hat\beta_u - u = g_1(s) - g_1(s-u), \qquad u \in [0,s],$$

are a standard Brownian motion and a Brownian motion with drift -1. In particular,

$$g_1(s) = \hat g(s), \qquad \int_0^s e^{g_1(u)-g_1(s)}\,\mathrm{d}u = \int_0^s e^{-\hat g(v)}\,\mathrm{d}v.$$

Next, by the Cameron–Martin theorem [108], the variable

$$e^{\hat g(s)} = \exp\left(\sqrt{2}\,\hat\beta_s - s\right)$$

is the density of a measure under which $\hat\beta_u - \sqrt{2}\,u$, $u \le s$, is a standard Brownian motion. Under this measure,

$$-\hat g(u) = -\sqrt{2}\big(\hat\beta_u - \sqrt{2}\,u\big) - u$$

has the law of g_1, by the symmetry $\beta \mapsto -\beta$. We obtain

$$\int_0^\infty q^{-2} f'(q)\,\mathcal{T}_s f(q)\,\mathrm{d}q = \mathbb{E}\int_0^\infty r^{-2} f(r) f'(re^{g_1(s)}) e^{-rA_s}\,\mathrm{d}r,$$

which is the right side of (a).

For (d), we use the Cameron–Martin theorem directly. Under the measure with density $e^{g_1(s)}$, the process

$$\beta'_u = \beta_u - \sqrt{2}\,u, \qquad u \le s,$$

is a standard Brownian motion, and $g_1(u) = \sqrt{2}\,\beta'_u + u$. This yields

$$\mathcal{T}_s(qf)(q) = q\,\mathbb{E}\big[e^{g_1(s)} f(qe^{g_1(s)}) e^{-qA_s}\big] = q\,\mathbb{E}\Big[f(qe^{\sqrt{2}\beta'_s+s}) \exp\Big(-q\int_0^s e^{\sqrt{2}\beta'_u+u}\,\mathrm{d}u\Big)\Big],$$

and

$$\sqrt{2}\,\beta'_u + u = g_1(u) + 2u,$$

with g_1 built from β'. This is (10.7), with the right side of (10.6) as the definition of $e^{-s\mathscr{L}} f$.

We now prove (b). The map $\mathcal{U}$ is unitary, since the substitution $q = e^y$ gives

$$\int_{\mathbb{R}} |e^{y/2} f(e^y)|^2\,\mathrm{d}y = \int_0^\infty |f(q)|^2\,\mathrm{d}q,$$

and $\mathcal{U}$ maps $C_c^\infty((0,\infty))$ onto $C_c^\infty(\mathbb{R})$. Let $f \in C_c^\infty((0,\infty))$ and $v = \mathcal{U}f$. The operator $q\partial_q$ corresponds to ∂_y, and

$$f(e^y) = e^{-y/2} v(y), \qquad -\partial_q(q^2\partial_q) = -(q\partial_q)^2 - q\partial_q.$$

A direct computation gives

$$(\mathcal{U}\mathscr{L}f)(y) = -v''(y) + \big(e^y + \tfrac{1}{4}\big) v(y).$$

The operator $-\partial_y^2 + V$ with a nonnegative smooth potential V is essentially self-adjoint on $C_c^\infty(\mathbb{R})$ [107], and the lower bound follows from

$$\int_{\mathbb{R}} \big(|v'|^2 + (e^y + \tfrac{1}{4})|v|^2\big)\,\mathrm{d}y \geq \tfrac{1}{4}\|v\|^2.$$

Since $\mathcal{U}$ is unitary, (b) follows.

Finally, we prove (c). Let $\mathscr{H}$ denote the closure of $-\partial_y^2 + e^y + 1/4$. The potential is continuous and bounded below, and the Feynman–Kac formula [125] states that, for $v \in L^2(\mathbb{R})$ and almost every y,

$$(e^{-s\mathscr{H}}v)(y) = e^{-s/4}\,\mathbb{E}\Big[v(y + \sqrt{2}\,\beta_s)\exp\Big(-\int_0^s e^{y+\sqrt{2}\beta_u}\,\mathrm{d}u\Big)\Big],$$

where we used that $\sqrt{2}\,\beta$ has generator ∂_y^2. With $v = \mathcal{U}f$ and $y = \log q$, we find

$$(\mathcal{U}^{-1}e^{-s\mathscr{H}}\mathcal{U}f)(q) = \mathbb{E}\Big[e^{\beta_s/\sqrt{2}-s/4} f(qe^{\sqrt{2}\beta_s})\exp\Big(-q\int_0^s e^{\sqrt{2}\beta_u}\,\mathrm{d}u\Big)\Big].$$

Under the measure with density $e^{\beta_s/\sqrt{2}-s/4}$, the Cameron–Martin theorem shows that $\beta'_u = \beta_u - u/\sqrt{2}$ is a standard Brownian motion, and

$$\sqrt{2}\,\beta_u = \sqrt{2}\,\beta'_u + u = g_1(u) + 2u,$$

with g_1 built from β'. By (b), we have

$$e^{-s\mathscr{L}} = \mathcal{U}^{-1}e^{-s\mathscr{H}}\mathcal{U},$$

and we obtain (10.6). □

10.3 The moment densities as Brownian functionals

We now write all moment densities M_k^X of Definition 1.9 in terms of the semigroups of Lemma 10.1. For $k = 2$, this is the identity behind Theorem 10.4. For general k, it will be used in Lemma 10.12 and in the long-wire limit of Section 12.

Lemma 10.2. *Let $k \geq 2$, $X > 0$, and $0 \leq x_1 \leq \cdots \leq x_k \leq 1$. Set $s_0 = Xx_1$, $s_j = X(x_{j+1} - x_j)$ for $1 \leq j \leq k-1$, and $s_k = X(1 - x_k)$. Then*

$$\begin{aligned}\frac{(k-2)!}{X^{k-1}} M_k^X(x_1,\dots,x_k) &= \int_0^\infty q^{-2}\mathrm{Y}_{s_0}(q)\big[q\,\mathcal{T}_{s_1} q\,\mathcal{T}_{s_2}\cdots q\,\mathcal{T}_{s_{k-1}} q\,\mathrm{Y}_{s_k}\big](q)\,\mathrm{d}q \\ &= \int_0^\infty \mathrm{Y}_{s_0}(q)\big[e^{-s_1\mathscr{L}} q\, e^{-s_2\mathscr{L}}\cdots q\, e^{-s_{k-1}\mathscr{L}}\mathrm{Y}_{s_k}\big](q)\,\mathrm{d}q, \qquad (10.8)\end{aligned}$$

where q denotes multiplication by q, there are k such factors in the first line and $k-2$ in the second, and all operators act on nonnegative functions. All three expressions are finite. In particular, for $k = 2$ and $0 \leq x \leq y \leq 1$,

$$\frac{\mathcal{C}_X(x,y)}{X} = \int_0^\infty \mathrm{Y}_{Xx}(q)\big[e^{-X(y-x)\mathscr{L}}\mathrm{Y}_{X(1-y)}\big](q)\,\mathrm{d}q. \qquad (10.9)$$

Proof. By Brownian scaling, the process $(g_X(t))_{t\in[0,1]}$ of Definition 1.9 has the law of $(g_1(Xt))_{t\in[0,1]}$. Under this identification, $Z_X = X^{-1}A_X$. Inserting this into the definition of M_k^X, we find

$$M_k^X(x_1,\dots,x_k) = X^{k-1}\,\mathbb{E}\Big[A_X^{1-k}\prod_{a=1}^k e^{g_1(Xx_a)}\Big].$$

We insert the identity

$$A^{1-k} = \frac{1}{(k-2)!}\int_0^\infty q^{k-2}e^{-qA}\,\mathrm{d}q, \qquad A>0,$$

and use Tonelli's theorem. With $q_u^0 = qe^{g_1(u)}$, we have

$$qA_X = \int_0^X q_u^0\,\mathrm{d}u, \qquad \prod_{a=1}^k e^{g_1(Xx_a)} = q^{-k}\prod_{a=1}^k q^0_{Xx_a},$$

and hence

$$\frac{(k-2)!}{X^{k-1}}M_k^X(x_1,\dots,x_k) = \int_0^\infty q^{-2}\,\mathbb{E}\Big[\prod_{a=1}^k q^0_{Xx_a}\exp\Big(-\int_0^X q_u^0\,\mathrm{d}u\Big)\Big]\,\mathrm{d}q.$$

Recall that q^0 is a Markov process, killed at the rate q^0, with semigroup $\mathcal{T}_s$. The Markov property at the times $Xx_1 \le \cdots \le Xx_k$ writes the expectation as

$$\big[\mathcal{T}_{s_0} q\,\mathcal{T}_{s_1} q\cdots\mathcal{T}_{s_{k-1}} q\,\mathcal{T}_{s_k}1\big](q).$$

By Lemma 10.1(a), we may move $\mathcal{T}_{s_0}$ onto the constant function 1 in the integral against $q^{-2}\,\mathrm{d}q$, which turns it into Y_{s_0}. The result is the first line of (10.8).

For the second line, we apply (10.7) $k-1$ times, from the right. The first application gives

$$\mathcal{T}_{s_{k-1}}(q\mathrm{Y}_{s_k}) = q\,e^{-s_{k-1}\mathscr{L}}\mathrm{Y}_{s_k},$$

and each further application moves one factor q to the left through $\mathcal{T}_{s_j}$ and replaces $\mathcal{T}_{s_j}$ by $e^{-s_j\mathscr{L}}$. After $k-1$ steps, the expression in brackets in the first line becomes

$$q^2\big[e^{-s_1\mathscr{L}}q\cdots q\,e^{-s_{k-1}\mathscr{L}}\mathrm{Y}_{s_k}\big],$$

and the factor q^2 cancels q^{-2}.

The identities hold in $[0,\infty]$, as all expressions are integrals of nonnegative functions. To see that they are finite, we note that Jensen's inequality implies

$$Z_X \ge \exp\Big(\int_0^1 g_X(t)\,\mathrm{d}t\Big) \ge \exp\Big(-\sup_{t\le1}|g_X(t)|\Big).$$

This bounds the integrand in the definition of M_k^X by

$$Z_X^{1-k}\prod_{a=1}^k e^{g_X(x_a)} \le \exp\Big((2k-1)\sup_{t\le1}|g_X(t)|\Big),$$

which is integrable because $\sup_{t\le1}|\beta_t|$ has Gaussian tails. □

10.4 A tail envelope

The singlet state is close to one for small q and decays rapidly for large q. To integrate the limits of Section 10.5 over q, we need a bound for this decay that is uniform in $\hat{z}$.

Lemma 10.3. *Let $0 < t_0 \le T$ and $q_* = 4e^{6T}$. Define $\bar{\Upsilon}_{t_0,T}(q) = 1$ for $0 < q < q_*$ and*

$$\bar{\Upsilon}_{t_0,T}(q) = \exp\bigl(-t_0 e^{-3T}\sqrt{q}/2\bigr) + \exp\bigl(-(\log q)^2/(16T)\bigr), \qquad q \ge q_*.$$

Then for all $0 < \hat{z} \le 1$, all $(q,F) \in (\hat{z},\infty)\times(-1,1)$ and all $s \in [t_0,T]$,

$$0 \le \Psi_s^{(\hat{z})}(q,F) \le \bar{\Upsilon}_{t_0,T}(q), \qquad 0 \le \bigl(\widetilde{\mathcal{T}}_s^{(\hat{z})}1\bigr)(q,F) \le \bar{\Upsilon}_{t_0,T}(q), \qquad 0 \le \Upsilon_s(q) \le \bar{\Upsilon}_{t_0,T}(q). \tag{10.10}$$

Moreover, $\int_0^\infty (1+q)^r \bar{\Upsilon}_{t_0,T}(q)\,\mathrm{d}q < \infty$ for all $r \ge 0$. In particular, let $\bar{\Upsilon} = \bar{\Upsilon}_{X_-\delta,X_+}$. Then for all $X \in [X_-,X_+]$, $0 < z \le X_-$, $t \in [\delta,1]$, and $(B,F) \in D^\circ$,

$$0 \le \Psi_t^{X,z}(B,F) \le \bar{\Upsilon}(zB/X), \qquad 0 \le \bigl(e^{t(X\Delta_D - zd)}1\bigr)(B,F) \le \bar{\Upsilon}(zB/X). \tag{10.11}$$

Proof. The killing rates $q - \hat{z}F$, $\tilde{q} - \hat{z}\tilde{F}$, and q^0 are nonnegative, and all three functions take values in $[0,1]$. It suffices to consider $q \ge q_*$. We first treat $\Psi_s^{(\hat{z})}$. Itô's formula and (10.1) give, on $[0,\tau_{\mathrm{c}})$,

$$\log q_u = \log q + \int_0^u \gamma_v\,\mathrm{d}v + M_u, \qquad M_u = \int_0^u \Bigl(2 - \frac{2\hat{z}^2}{q_v^2}\Bigr)^{1/2}\mathrm{d}\beta_v,$$
$$\gamma_v = 2\hat{z}\,\frac{\hat{z} - q_v F_v}{q_v(q_v - \hat{z}F_v)} - 1 + \frac{\hat{z}^2}{q_v^2}.$$

Using (10.4) and $q_v \ge \hat{z}$, we see that

$$-3 \le \gamma_v \le 2, \qquad \langle M\rangle_u = \int_0^u \Bigl(2 - \frac{2\hat{z}^2}{q_v^2}\Bigr)\,\mathrm{d}v \le 2u.$$

We extend M to all times by stopping it at τ_{c}, and we define the event

$$\mathcal{A}_q = \Bigl\{\inf_{u\le T} M_u \ge -\frac{1}{2}\log q\Bigr\}.$$

We now bound the killing on $\mathcal{A}_q$. For $u < \tau_{\mathrm{c}} \wedge T$, the bound $\gamma_v \ge -3$ and $q \ge q_*$ give

$$\log q_u \ge \tfrac{1}{2}\log q - 3T, \qquad q_u \ge e^{-3T}\sqrt{q} \ge 2.$$

Therefore $\tau_{\mathrm{c}} > T$ on $\mathcal{A}_q$, since $q_u \to \hat{z} \le 1$ as $u \uparrow \tau_{\mathrm{c}}$. Recalling that $\hat{z} \le 1$, we obtain on $\mathcal{A}_q$ that

$$q_u - \hat{z}F_u \ge q_u - 1 \ge \frac{q_u}{2} \ge \frac{e^{-3T}\sqrt{q}}{2}, \qquad u \le T,$$

and the killing integral up to time $s \geq t_0$ is bounded below by $t_0 e^{-3T}\sqrt{q}/2$.

It remains to bound the probability of $\mathcal{A}_q^c$. The exponential martingale inequality [108] and $\langle M\rangle_T \leq 2T$ give

$$\mathbb{P}(\mathcal{A}_q^c) \leq \exp\Big(-\frac{(\frac{1}{2}\log q)^2}{2\cdot 2T}\Big) = \exp\Big(-\frac{(\log q)^2}{16T}\Big).$$

Splitting the expectation in (10.2) according to $\mathcal{A}_q$ proves the first bound in (10.10).

The same argument applies to the other two functions. For the radial diffusion, (10.1) and Itô's formula give

$$\mathrm{d}\log\tilde{q}_v = \Big(1+\frac{\hat{z}^2}{\tilde{q}_v^2}\Big)\,\mathrm{d}v + \Big(2-\frac{2\hat{z}^2}{\tilde{q}_v^2}\Big)^{1/2}\mathrm{d}\tilde{\beta}_v,$$

with drift in $[1,2]$, and $\tilde{q}$ never reaches $\hat{z}$. For Y_s, the logarithm of $q_u^0 = qe^{\mathcal{g}_1(u)}$ has drift -1 and quadratic variation $2u$.

The moments are finite because

$$\exp\Big(-\frac{(\log q)^2}{16T}\Big) = q^{-\log q/(16T)}$$

decays faster than every power of q. Finally, by (10.3), we have

$$\Psi_t^{X,z}(B,F) = \Psi_{Xt}^{(\hat{z})}(zB/X,F), \qquad \hat{z} = \frac{z}{X} \leq 1, \qquad Xt \in [X_-\delta, X_+],$$

and (10.10) implies the first bound in (10.11). The second follows in the same way. □

10.5 The scalar residue

We now prove the first main result of this section.

Theorem 10.4. *Let $0 < X_- \leq X_+ < \infty$ and $\delta \in (0,1/2]$. Then*

$$\lim_{z\downarrow 0}\frac{z}{2}K_X(x,y;z) = \mathcal{C}_X(x,y), \tag{10.12}$$

uniformly for $X \in [X_-,X_+]$ and $\delta \leq x \leq y \leq 1-\delta$, where the limit is taken along real z.

The proof compares the scaled diffusions with the geometric Brownian motions of Section 10.2, driven by the same Brownian motion. We first record this comparison.

Lemma 10.5. *For every $T > 0$, there exists C_T such that for all $0 < \hat{z} \leq 1$ and $(q,F) \in (\hat{z},\infty)\times(-1,1)$,*

$$\sup_{s\leq T}\big|\Psi_s^{(\hat{z})}(q,F) - \mathrm{Y}_s(q)\big| \leq C_T\hat{z}, \tag{10.13}$$

$$\sup_{s,s'\leq T}\big|\big(\widetilde{\mathcal{T}}_s^{(\hat{z})}\Psi_{s'}^{(\hat{z})}\big)(q,F) - \big(e^{-s\mathscr{L}}\mathrm{Y}_{s'}\big)(q)\big| \leq C_T\hat{z}. \tag{10.14}$$

Proof. We begin with (10.13). Let (q_u, F_u) be the scaled singlet diffusion started at (q, F), and recall that β drives q_u in (10.1). After the corner time τ_c, the process q_u is constant. We extend β beyond τ_c by an independent Brownian motion, on an enlarged probability space, and we compare q_u with the geometric Brownian motion q_u^0, where

$$q_u^0 = q e^{g_1(u)}, \qquad y_u = q_u - q_u^0,$$

and g_1 is built from β. Then $y_0 = 0$ and

$$\mathrm{d}y_u = a_u \,\mathrm{d}u + (\sqrt{2}\, y_u + e_u)\,\mathrm{d}\beta_u, \qquad |a_u| \le 2\hat{z}, \quad |e_u| \le \sqrt{2}\,\hat{z}.$$

Indeed, for $u < \tau_c$, the coefficients are

$$a_u = 2\hat{z}\,\frac{\hat{z} - q_u F_u}{q_u - \hat{z} F_u}, \qquad e_u = \bigl(2(q_u^2 - \hat{z}^2)\bigr)^{1/2} - \sqrt{2}\, q_u,$$

and the bounds are (10.4). For $u \ge \tau_c$, we have $q_u = \hat{z}$, $a_u = 0$, and $e_u = -\sqrt{2}\,\hat{z}$.

By the Burkholder–Davis–Gundy inequality, we have, for $u \le T$,

$$\mathbb{E} \sup_{v \le u} |y_v|^2 \le C\Bigl(T^2 \hat{z}^2 + T\hat{z}^2 + \int_0^u \mathbb{E} \sup_{w \le v} |y_w|^2 \,\mathrm{d}v \Bigr),$$

where the left side is finite by the moment bound of Lemma 6.5(a). By Gronwall's inequality,

$$\mathbb{E} \sup_{u \le T} |q_u - q_u^0|^2 \le C_T \hat{z}^2, \tag{10.15}$$

with a constant that is uniform in (q, F). Using $|e^{-a} - e^{-a'}| \le |a - a'|$ for $a, a' \ge 0$, we obtain

$$\bigl|\Psi_s^{(\hat{z})}(q, F) - \mathrm{Y}_s(q)\bigr| \le \mathbb{E}\Bigl| \int_0^s (q_u - \hat{z} F_u)\,\mathrm{d}u - \int_0^s q_u^0 \,\mathrm{d}u \Bigr| \le \int_0^s \mathbb{E}|q_u - q_u^0|\,\mathrm{d}u + \hat{z}s \le C_T \hat{z}.$$

We now turn to (10.14). The function $\mathrm{Y}_{s'}$ is Lipschitz with constant s', since

$$|\partial_q \mathrm{Y}_{s'}(q)| = \mathbb{E}\bigl[A_{s'} e^{-qA_{s'}}\bigr] \le \mathbb{E} A_{s'} = s'.$$

Let $(\tilde{q}_u, \tilde{F}_u)$ be the scaled radial diffusion started at (q, F). We compare it with

$$\tilde{q}_u^0 = q e^{g_1(u) + 2u}, \qquad \tilde{y}_u = \tilde{q}_u - \tilde{q}_u^0,$$

where g_1 is built from $\tilde{\beta}$. The difference satisfies

$$\mathrm{d}\tilde{y}_u = 2\tilde{y}_u \,\mathrm{d}u + (\sqrt{2}\,\tilde{y}_u + \tilde{e}_u)\,\mathrm{d}\tilde{\beta}_u, \qquad |\tilde{e}_u| \le \sqrt{2}\,\hat{z},$$

and the argument above shows that

$$\mathbb{E} \sup_{u \le T} |\tilde{y}_u|^2 \le C_T \hat{z}^2.$$

We set

$$\tilde\kappa = \int_0^s (\tilde q_u - \hat z \tilde F_u)\,\mathrm{d}u, \qquad \tilde\kappa^0 = \int_0^s \tilde q^0_u\,\mathrm{d}u.$$

By (10.2) and (10.6), the difference in (10.14) is

$$\mathbb{E}\big[\Psi^{(\hat z)}_{s'}(\tilde q_s, \tilde F_s) e^{-\tilde\kappa}\big] - \mathbb{E}\big[\Upsilon_{s'}(\tilde q^0_s) e^{-\tilde\kappa^0}\big].$$

All factors lie in $[0,1]$. Replacing the factors one at a time, we bound the modulus of this difference by

$$\begin{aligned}\mathbb{E}\big|\Psi^{(\hat z)}_{s'}(\tilde q_s, \tilde F_s) - \Upsilon_{s'}(\tilde q_s)\big| + \mathbb{E}\big|\Upsilon_{s'}(\tilde q_s) - \Upsilon_{s'}(\tilde q^0_s)\big| + \mathbb{E}|\tilde\kappa - \tilde\kappa^0| \\ \le C_T \hat z + T\,\mathbb{E}|\tilde y_s| + \int_0^s \mathbb{E}|\tilde y_u|\,\mathrm{d}u + T\hat z \le C_T \hat z,\end{aligned}$$

where we used (10.13) for the first term. □

Proof of Theorem 10.4. Let $s_0 = Xx$, $s_1 = X(y-x)$, and $s_2 = X(1-y)$. They lie in $[0, X_+]$, and $s_0, s_2 \ge X_-\delta$. In the definition (1.6) of K_X, we use (10.3) and the substitution $q = \hat z B$, for which $\mathrm{d}B = \mathrm{d}q/\hat z$ and $z/\hat z = X$. This gives

$$\frac{z}{2} K_X(x,y;z) = \frac{X}{2} \int_{\hat z}^\infty \int_{-1}^1 \Psi^{(\hat z)}_{s_0}(q,F) \big(\widetilde{\mathcal{T}}^{(\hat z)}_{s_1} \Psi^{(\hat z)}_{s_2}\big)(q,F)\,\mathrm{d}F\,\mathrm{d}q. \tag{10.16}$$

By (10.9), and since the integrand there is independent of F, the target is

$$\mathcal{C}_X(x,y) = \frac{X}{2} \int_0^\infty \int_{-1}^1 \Upsilon_{s_0}(q) \big(e^{-s_1\mathscr{L}} \Upsilon_{s_2}\big)(q)\,\mathrm{d}F\,\mathrm{d}q.$$

All factors of the two integrands take values in $[0,1]$.

Let $R \ge 1$, and split the integrals at $q = R$. For $\hat z < q \le R$, Lemma 10.5 with $T = X_+$ bounds the difference of the integrands by $C\hat z$, uniformly in F. The range $q \le \hat z$ contributes at most $X\hat z$ to the target, since its integrand is bounded by one.

For $q > R$, Lemma 10.3 with $t_0 = X_-\delta$ and $T = X_+$ bounds the first factors of the two integrands by $\bar\Upsilon(q)$. We obtain

$$\Big|\frac{z}{2} K_X(x,y;z) - \mathcal{C}_X(x,y)\Big| \le C\Big(R\hat z + \hat z + \int_R^\infty \bar\Upsilon(q)\,\mathrm{d}q\Big),$$

uniformly for $X \in [X_-, X_+]$ and $\delta \le x \le y \le 1-\delta$ (including $x = y$, for which $s_1 = 0$ and the middle factors are the identity). As $z \downarrow 0$, the right side tends to $C\int_R^\infty \bar\Upsilon$, which tends to zero as $R \to \infty$, and we obtain (10.12). □

10.6 The rank-one superspace in polar coordinates

The permanent pairings of Definition 10.6 below are built from semigroup products of the complete rank-one sigma model, with all its anticommuting variables, and not only from the radial diffusions of Section 10.1. To take their limit, we need the generator in polar coordinates near the noncompact boundary. In this section, we write it down, and we show that on the scale $q = \hat{z}B$ the terms acting on the hyperbolic angle θ_{B} and the anticommuting variables become negligible. We refer to this as freezing, since these variables do not move in the limit. The spherical diffusion persists and acts trivially on the limiting profiles, which are independent of the spherical variables.

The polar chart

We recall the rank-one objects of Section 5. The target space $\mathcal{M}_1$ has four odd generators, and its underlying manifold $\mathcal{M}_1^0 = H_1 \times S_1$, described in Section 5.2, is the product of a hyperbolic plane and a sphere. The field Q is a $(2|2)$ supermatrix with $Q^2 = \mathrm{Id}$, whose blocks Q^{BB} and Q^{FF} are indexed by the retarded and advanced flavors R and A, and $d_1(Q) = \frac{1}{2}\,\mathrm{STr}(\Lambda Q)$. The Casimir operator C_1 is defined in Definition 5.7, the invariant integral and the pairing $\mathcal{B}_1$ in Definition 5.6, and the semigroups $e^{-t\mathsf{G}_{X,z}}$ of $\mathsf{G}_{X,z} = X\mathsf{C}_1 + z d_1$ in Proposition 5.15. Since $\mathsf{G}_{X,z} = X\mathsf{G}_{1,\hat{z}}$, we have $e^{-t\mathsf{G}_{X,z}} = e^{-Xt\mathsf{G}_{1,\hat{z}}}$.

We use the polar coordinates $(B, F, \theta_{\mathrm{B}}, \theta_{\mathrm{F}}; \xi_1, \dots, \xi_4)$ of (6.2), in which $Q = e^{-\Theta_\xi} Q_0 e^{\Theta_\xi}$, and we index the rows and columns in the order $(\mathrm{RB}, \mathrm{RF}, \mathrm{AB}, \mathrm{AF})$. Recall that $Q_0 = Q_0(B, F, \theta_{\mathrm{B}}, \theta_{\mathrm{F}})$ denotes the point of $\mathcal{M}_1^0$ with the coordinates of Section 5.2, whose diagonal is $(B, F, -B, -F)$ and whose entries at $(\mathrm{RB}, \mathrm{AB})$ and $(\mathrm{RF}, \mathrm{AF})$ are $(B^2-1)^{1/2} e^{\mathrm{i}\theta_{\mathrm{B}}}$ and $\mathrm{i}(1-F^2)^{1/2} e^{\mathrm{i}\theta_{\mathrm{F}}}$. Recall also that $\Theta_\xi \in \mathfrak{k}_1$ has the entries ξ_1, ξ_2, ξ_3, ξ_4 at $(\mathrm{RB}, \mathrm{RF})$, $(\mathrm{RF}, \mathrm{RB})$, $(\mathrm{AB}, \mathrm{AF})$, $(\mathrm{AF}, \mathrm{AB})$; the blocks of $e^{\pm\Theta_\xi}$ are written out in the proof of Lemma 6.1. Thus Q^{BB} is, up to nilpotent terms, the point of the hyperboloid with polar coordinates (B, θ_{B}), and Q^{FF} is the point of the sphere with height F and longitude θ_{F}. We let $\mathbf{e} \in S^2$ denote the point with these spherical coordinates. Functions of (F, θ_{F}) that are smooth on S^2 are then smooth functions of $\mathbf{e}$, also at the poles $F = \pm 1$. As in Section 6.1, we set $n_{\mathrm{R}} = \xi_1\xi_2$ and $n_{\mathrm{A}} = \xi_3\xi_4$, and $[f]_{\mathrm{top}}$ is the coefficient of $\xi_1\xi_2\xi_3\xi_4 = n_{\mathrm{R}} n_{\mathrm{A}}$ in a superfunction f (Lemma 6.1(c)).

By Lemma 6.1(b), we have $Q^2 = \mathrm{Id}$ and

$$d_1(Q) = B - F, \tag{10.17}$$

without nilpotent terms. By Lemma 6.1(a), the polar chart is a coordinate system of $\mathcal{M}_1$ over the set of body points with $B > 1$ (near $F = \pm 1$, after (F, θ_{F}) is replaced by Cartesian coordinates of the sphere). We use two further facts from Section 6. First, let f be a rapidly decreasing superfunction supported in $\{B \geq 2\}$. Then $f_\varnothing(\Lambda) = 0$, and

Lemma 6.3 implies that

$$\int f = \int_D \frac{1}{(B-F)^2} \int_0^{2\pi}\int_0^{2\pi} [f]_{\mathrm{top}} \, \frac{\mathrm{d}\theta_{\mathrm{B}}\,\mathrm{d}\theta_{\mathrm{F}}}{4\pi^2} \, \mathrm{d}B\,\mathrm{d}F. \tag{10.18}$$

Second, by Lemma 6.7, invariant superfunctions are functions of (B, F) in the polar chart, and $e^{-t\mathsf{G}_{X,z}}1$ has the radial representative $\Psi_t^{X,z}$.

The source of the permanent pairings is the superfunction

$$\mathfrak{s}(Q) = \tfrac{1}{2}\big(Q^{\mathrm{BB}}_{\mathrm{RR}} + Q^{\mathrm{BB}}_{\mathrm{RA}} - Q^{\mathrm{BB}}_{\mathrm{AR}} - Q^{\mathrm{BB}}_{\mathrm{AA}}\big). \tag{10.19}$$

We will show in Proposition 12.3 that $s_E\mathfrak{s}$ is the orbit multiplier of Theorem 7.6(b) for the local source with $Y^{\mathrm{F}} = 0$ and $Y^{\mathrm{B}} = -\frac{\mathrm{i}}{2}\left(\begin{smallmatrix} 1 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ of Lemma 12.2(c), which produces permanents of $\operatorname{Im} G$. Multiplying out $Q = e^{-\Theta_\xi} Q_0 e^{\Theta_\xi}$, we find

$$\begin{aligned}\mathfrak{s} = {} & \big(B + (B^2-1)^{1/2}\cos\theta_{\mathrm{B}}\big)\Big(1 + \frac{n_{\mathrm{R}} + n_{\mathrm{A}}}{2}\Big) - \frac{F}{2}(n_{\mathrm{R}} + n_{\mathrm{A}}) + \frac{(B^2-1)^{1/2}\cos\theta_{\mathrm{B}}}{4} n_{\mathrm{R}} n_{\mathrm{A}} \\ & - \frac{\mathrm{i}}{2}(1-F^2)^{1/2} e^{\mathrm{i}\theta_{\mathrm{F}}} \xi_1 \xi_4 + \frac{\mathrm{i}}{2}(1-F^2)^{1/2} e^{-\mathrm{i}\theta_{\mathrm{F}}} \xi_2 \xi_3.\end{aligned} \tag{10.20}$$

In particular, $\mathfrak{s}$ is not invariant. Its body $B + (B^2-1)^{1/2}\cos\theta_{\mathrm{B}}$ is positive, since it is at least $B - (B^2-1)^{1/2} = (B + (B^2-1)^{1/2})^{-1}$.

Definition 10.6. Let $X > 0$, $z > 0$, and $k \geq 1$, and let $0 \leq x_1 \leq \cdots \leq x_k \leq 1$. We define the *permanent pairing*

$$\mathfrak{p}_k^{X,z}(x_1, \ldots, x_k) = \mathcal{B}_1\big(e^{-x_1\mathsf{G}_{X,z}}1,\, \mathfrak{s}\, e^{-(x_2-x_1)\mathsf{G}_{X,z}} \mathfrak{s} \cdots e^{-(x_k - x_{k-1})\mathsf{G}_{X,z}} \mathfrak{s}\, e^{-(1-x_k)\mathsf{G}_{X,z}} 1\big).$$

We extend $\mathfrak{p}_k^{X,z}$ to all $(x_1, \ldots, x_k) \in [0,1]^k$ by symmetry. More generally, for $\hat{z} > 0$, $n \geq 0$, and $t_0, \ldots, t_n \geq 0$, we call the superfunction $e^{-t_0\mathsf{G}_{1,\hat{z}}} \mathfrak{s}\, e^{-t_1\mathsf{G}_{1,\hat{z}}} \mathfrak{s} \cdots \mathfrak{s}\, e^{-t_n\mathsf{G}_{1,\hat{z}}} 1$ a *semigroup product with n sources*, and we let $\mathfrak{S}_k$ denote the set of finite linear combinations of semigroup products with at most k sources.

Since $\mathfrak{s}$ is a multiplier of degree one by Lemma 5.14, Proposition 5.15 shows that the pairing is well defined and finite. The name of $\mathfrak{p}_k^{X,z}$ refers to Section 12, where we show, using Theorem 7.6(b), that it is the limit of expected permanents of the matrix $\operatorname{Im} G$ at k sites, at the damping $\varepsilon = z/(2\pi)$.

Charges

Semigroup products are not invariant, but they transform in a simple way under phase rotations. For $\gamma = (\gamma_{\mathrm{B}}, \gamma_{\mathrm{F}}) \in \mathbb{R}^2$, let

$$\mathsf{k}_\gamma = \operatorname{diag}\big(e^{\mathrm{i}\gamma_{\mathrm{B}}/2}, e^{\mathrm{i}\gamma_{\mathrm{F}}/2}, e^{-\mathrm{i}\gamma_{\mathrm{B}}/2}, e^{-\mathrm{i}\gamma_{\mathrm{F}}/2}\big)$$

in the order of indices above. Recall from Section 5.2 that G_0 is the group of the block diagonal supermatrices $\mathrm{diag}(g_{\mathrm{B}}|g_{\mathrm{F}})$ with $g_{\mathrm{B}} \in U(1,1)$ and $g_{\mathrm{F}} \in U(2)$, which acts on $\mathcal{M}_1$ by conjugation, and that K_0 is the subgroup of the elements that commute with Λ. The matrix k_γ lies in K_0, since its blocks $\mathrm{diag}(e^{\mathrm{i}\gamma_{\mathrm{B}}/2}, e^{-\mathrm{i}\gamma_{\mathrm{B}}/2})$ and $\mathrm{diag}(e^{\mathrm{i}\gamma_{\mathrm{F}}/2}, e^{-\mathrm{i}\gamma_{\mathrm{F}}/2})$ are unitary and diagonal. Conjugating Q_0 and e^{Θ_ξ} by k_γ, we see that $\mathsf{k}_\gamma Q \mathsf{k}_\gamma^{-1}$ has the same B and F as Q, and the remaining polar coordinates

$$\theta_{\mathrm{B}} + \gamma_{\mathrm{B}}, \quad \theta_{\mathrm{F}} + \gamma_{\mathrm{F}}, \quad e^{\mathrm{i}(\gamma_{\mathrm{B}}-\gamma_{\mathrm{F}})/2}\xi_1, \quad e^{-\mathrm{i}(\gamma_{\mathrm{B}}-\gamma_{\mathrm{F}})/2}\xi_2, \quad e^{-\mathrm{i}(\gamma_{\mathrm{B}}-\gamma_{\mathrm{F}})/2}\xi_3, \quad e^{\mathrm{i}(\gamma_{\mathrm{B}}-\gamma_{\mathrm{F}})/2}\xi_4.$$

For a subset $I \subset \{1,2,3,4\}$, let ξ^I denote the monomial of Definition 3.1, and set $c_I = \frac{1}{2}(|I \cap \{1,4\}| - |I \cap \{2,3\}|)$. We say that a superfunction f has *charge* (m,m') if $f(\mathsf{k}_\gamma Q \mathsf{k}_\gamma^{-1}) = e^{\mathrm{i}(m\gamma_{\mathrm{B}} + m'\gamma_{\mathrm{F}})} f(Q)$ for all γ. In the polar chart, this means that

$$f = \sum_I e^{\mathrm{i}(m-c_I)\theta_{\mathrm{B}}} e^{\mathrm{i}(m'+c_I)\theta_{\mathrm{F}}} f_I(B,F)\, \xi^I \tag{10.21}$$

for functions f_I of (B,F). The operators C_1 and d_1 commute with the conjugation by k_γ, by Lemma 5.8(a) and the invariance of d_1, and so does the semigroup $e^{-t\mathsf{G}_{1,\hat{z}}}$. The entries $Q^{\mathrm{BB}}_{\mathrm{RR}}$ and $Q^{\mathrm{BB}}_{\mathrm{AA}}$ have charge $(0,0)$, while $Q^{\mathrm{BB}}_{\mathrm{RA}}$ and $Q^{\mathrm{BB}}_{\mathrm{AR}}$ have charges $(1,0)$ and $(-1,0)$. It follows that every semigroup product with n sources is a sum of even superfunctions of charges $(m,0)$ with $|m| \le n$. For even I, we have $c_I \in \{-1,0,1\}$.

The Casimir operator in the polar chart

Let ∂_{ξ_a} denote the left derivative with respect to ξ_a, and set $\partial_+ = \partial_{\theta_{\mathrm{B}}} + \partial_{\theta_{\mathrm{F}}}$. We define the odd differential operators

$$\mathcal{D}_1 = \partial_{\xi_1} + \tfrac{\mathrm{i}}{2}\xi_2\partial_+, \qquad \mathcal{D}_2 = \partial_{\xi_2} + \tfrac{\mathrm{i}}{2}\xi_1\partial_+, \qquad \mathcal{D}_3 = \partial_{\xi_3} - \tfrac{\mathrm{i}}{2}\xi_4\partial_+, \qquad \mathcal{D}_4 = \partial_{\xi_4} - \tfrac{\mathrm{i}}{2}\xi_3\partial_+,$$

and the antisymmetric matrix $(g^{ab})_{a,b=1}^4$ with $g^{13} = g^{24} = 0$ and

$$g^{12} = g^{34} = \frac{2(BF-1)}{(B-F)^2},$$

$$g^{14} = \frac{2\mathrm{i}\big((B^2-1)(1-F^2)\big)^{1/2} e^{\mathrm{i}(\theta_{\mathrm{B}}-\theta_{\mathrm{F}})}}{(B-F)^2}, \qquad g^{23} = -\frac{2\mathrm{i}\big((B^2-1)(1-F^2)\big)^{1/2} e^{-\mathrm{i}(\theta_{\mathrm{B}}-\theta_{\mathrm{F}})}}{(B-F)^2}.$$

In the coordinates (B,F), the singlet operator (6.1) reads

$$\Delta_D^{\mathrm{s}} = \partial_B\big((B^2-1)\partial_B\big) + \partial_F\big((1-F^2)\partial_F\big) - \frac{2(B^2-1)}{d}\partial_B + \frac{2(1-F^2)}{d}\partial_F.$$

Lemma 10.7. *For every smooth superfunction f on $\{B > 1\}$, in the polar chart,*

$$-\mathsf{C}_1 f = \Delta_D^{\mathrm{s}} f + \frac{1}{B^2-1}\partial^2_{\theta_{\mathrm{B}}} f + \frac{1}{1-F^2}\partial^2_{\theta_{\mathrm{F}}} f + \frac{1}{2}\sum_{a,b=1}^4 g^{ab}\mathcal{D}_b\mathcal{D}_a f. \tag{10.22}$$

We first describe the structure of (10.22). There are no mixed terms between the radial coordinates (B, F) and the other coordinates, and the radial part is the singlet operator Δ_D^{s}, also on noninvariant superfunctions. The phase terms are those of the polar coordinates of H_1 and S_1. The entries of the matrix (g^{ab}) of the odd part are of order $1/B$ for large B, uniformly in F, and its Pfaffian $g^{12}g^{34} + g^{14}g^{23} = 4(B-F)^{-2}$ degenerates only at the corner. The operators $\mathcal{D}_a$ involve the angles only through ∂_+, which shifts θ_{B} and θ_{F} together. Later, Lemma 10.11 uses three of these facts: the absence of mixed terms, the radial part Δ_D^{s}, and the decay of g^{ab}.

The proof reduces (10.22) to a few coefficients that depend only on (B, F), by the symmetry of C_1 under the stabilizer of Λ. It computes these coefficients from three sources: the Laplace–Beltrami operator of $\mathcal{M}_1^0$, the singlet operator of Lemma 6.8, and the action of C_1 on products of two entries of Q. We first state the two computations that it needs. We work over the set of body points with $B > 1$ and $|F| < 1$, where the polar coordinates are coordinates, by Lemma 6.1(a). The indices a, b run over $\{1, 2, 3, 4\}$. We call a' the partner of a, with $1' = 2$, $2' = 1$, $3' = 4$, and $4' = 3$, and we set $\sigma_a = 1$ for $a \in \{1, 2\}$ and $\sigma_a = -1$ for $a \in \{3, 4\}$, so that

$$\mathcal{D}_a = \partial_{\xi_a} + \tfrac{\mathrm{i}}{2}\sigma_a \xi_{a'} \partial_+.$$

We also write $c_a = c_{\{a\}}$ for the numbers of (10.21), so that $c_1 = c_4 = \frac{1}{2}$, $c_2 = c_3 = -\frac{1}{2}$, and $c_{a'} = -c_a$. At rank one, the even elements of $\mathfrak{k}$ are the diagonal matrices, and the odd elements are the linear combinations of the four matrix units $E_{\mathrm{RB,RF}}$, $E_{\mathrm{RF,RB}}$, $E_{\mathrm{AB,AF}}$, and $E_{\mathrm{AF,AB}}$ (Section 5.2).

Lemma 10.8. *In the polar chart, the fields $\widehat{Y}$ of Lemma 5.4 for $Y \in \mathfrak{k}$ are as follows. For $Y = \operatorname{diag}(y_1, y_2, y_3, y_4)$,*

$$\widehat{Y} = -\mathrm{i}(y_1 - y_3)\partial_{\theta_{\mathrm{B}}} - \mathrm{i}(y_2 - y_4)\partial_{\theta_{\mathrm{F}}} + (y_1 - y_2)(\xi_1\partial_{\xi_1} - \xi_2\partial_{\xi_2}) + (y_3 - y_4)(\xi_3\partial_{\xi_3} - \xi_4\partial_{\xi_4}), \tag{10.23}$$

and for the four odd matrix units of $\mathfrak{k}$,

$$\begin{aligned} \widehat{E_{\mathrm{RB,RF}}} &= -\partial_{\xi_1} + \tfrac{\mathrm{i}}{2}\xi_2\partial_+, \qquad & \widehat{E_{\mathrm{RF,RB}}} &= -\partial_{\xi_2} + \tfrac{\mathrm{i}}{2}\xi_1\partial_+, \\ \widehat{E_{\mathrm{AB,AF}}} &= -\partial_{\xi_3} - \tfrac{\mathrm{i}}{2}\xi_4\partial_+, \qquad & \widehat{E_{\mathrm{AF,AB}}} &= -\partial_{\xi_4} - \tfrac{\mathrm{i}}{2}\xi_3\partial_+. \end{aligned} \tag{10.24}$$

In particular, the four odd fields are $-\partial_{\xi_a} + \frac{\mathrm{i}}{2}\sigma_a\xi_{a'}\partial_+$, none of these fields differentiates in B or F, and their coefficients do not depend on $(B, F, \theta_{\mathrm{B}}, \theta_{\mathrm{F}})$.

Proof. Let V be a vector field in the polar chart such that $\vartheta VQ = [\vartheta Y, Q]$, where ϑ is the parameter of (5.6). Then $V = \widehat{Y}$. Indeed, by Lemma 6.1(b), the Jacobian of the eight entries $Q_{\mathrm{RR}}^{\mathrm{BB}}$, $Q_{\mathrm{RR}}^{\mathrm{FF}}$, $Q_{\mathrm{RA}}^{\mathrm{BB}}$, $Q_{\mathrm{RA}}^{\mathrm{FF}}$, $Q_{\mathrm{RR}}^{\mathrm{BF}} = d\xi_1$, $Q_{\mathrm{RR}}^{\mathrm{FB}} = -d\xi_2$, $Q_{\mathrm{AA}}^{\mathrm{BF}} = -d\xi_3$, and $Q_{\mathrm{AA}}^{\mathrm{FB}} = d\xi_4$ with respect to $(B, F, \theta_{\mathrm{B}}, \theta_{\mathrm{F}}, \xi_1, \dots, \xi_4)$ has a triangular body with nonzero diagonal entries, so a vector field is determined by its values on these entries.

We begin with $Y = \operatorname{diag}(y_1, y_2, y_3, y_4)$. Conjugation by e^{sY} multiplies the entries of Q_0 at $(\mathrm{RB}, \mathrm{AB})$ and $(\mathrm{RF}, \mathrm{AF})$ by $e^{s(y_1-y_3)}$ and $e^{s(y_2-y_4)}$, and the entries of $\Theta_{\tilde\xi}$ by the corresponding factors. In the polar coordinates, it therefore replaces θ_{B}, θ_{F}, ξ_1, ξ_2, ξ_3, and ξ_4 by

$$\theta_{\mathrm{B}} - \mathrm{i}s(y_1 - y_3), \quad \theta_{\mathrm{F}} - \mathrm{i}s(y_2 - y_4), \quad e^{s(y_1-y_2)}\xi_1, \quad e^{s(y_2-y_1)}\xi_2, \quad e^{s(y_3-y_4)}\xi_3, \quad e^{s(y_4-y_3)}\xi_4.$$

Since

$$e^{sY} Q e^{-sY} = e^{-e^{sY}\Theta_{\tilde\xi} e^{-sY}} \big(e^{sY} Q_0 e^{-sY}\big) e^{e^{sY}\Theta_{\tilde\xi} e^{-sY}},$$

the derivative at $s = 0$ gives (10.23).

Next, let $Y = \alpha E_{\mathrm{RB,RF}} + \beta E_{\mathrm{RF,RB}}$ with $\alpha, \beta \in \mathbb{C}$. Multiplying the 2×2 blocks in the flavor block RR, we find $[\vartheta Y, \Theta_{\tilde\xi}] = \vartheta(\alpha\xi_2 + \beta\xi_1)\mathrm{Id}_{\mathrm{R}}$, which commutes with ϑY and $\Theta_{\tilde\xi}$. Hence

$$\begin{gathered} e^{\vartheta Y} e^{-\Theta_{\tilde\xi}} = e^{-\Theta_{\tilde\xi'}} e^{c\mathrm{Id}_{\mathrm{R}}}, \qquad e^{\vartheta Y} Q e^{-\vartheta Y} = e^{-\Theta_{\tilde\xi'}} \big(e^{c\mathrm{Id}_{\mathrm{R}}} Q_0 e^{-c\mathrm{Id}_{\mathrm{R}}}\big) e^{\Theta_{\tilde\xi'}}, \\ \tilde\xi' = (\xi_1 - \vartheta\alpha,\, \xi_2 - \vartheta\beta,\, \xi_3,\, \xi_4), \qquad c = -\tfrac{1}{2}\vartheta(\alpha\xi_2 + \beta\xi_1). \end{gathered}$$

Conjugation by $e^{c\mathrm{Id}_{\mathrm{R}}}$ multiplies the entries of Q_0 at $(\mathrm{RB}, \mathrm{AB})$ and $(\mathrm{RF}, \mathrm{AF})$ by e^c, which amounts to replacing θ_{B} and θ_{F} by $\theta_{\mathrm{B}} - \mathrm{i}c$ and $\theta_{\mathrm{F}} - \mathrm{i}c$. Since $e^{\vartheta Y} Q e^{-\vartheta Y} = Q + [\vartheta Y, Q]$, the terms of first order in ϑ give the first line of (10.24). The same computation in the flavor block AA, where conjugation by $e^{c\mathrm{Id}_{\mathrm{A}}}$ replaces θ_{B} and θ_{F} by $\theta_{\mathrm{B}} + \mathrm{i}c$ and $\theta_{\mathrm{F}} + \mathrm{i}c$, gives the second line. □

Lemma 10.9. *We have $\mathsf{C}_1 Q_{\mu\nu} = 0$ for all entries $Q_{\mu\nu}$ of Q. Let $Q_{\mu\nu}$ and $Q_{\mu'\nu'}$ be odd entries, so that $|\mu| + |\nu| = |\mu'| + |\nu'| = 1$. Then*

$$\begin{aligned} -\mathsf{C}_1(Q_{\mu\nu} Q_{\mu'\nu'}) = {} & (-1)^{|\mu'|} Q_{\mu'\nu} Q_{\mu\nu'} + (-1)^{|\nu|} Q_{\mu\nu'} Q_{\mu'\nu} \\ & - \delta_{\mu\nu'} \sum_i (-1)^{|i|} Q_{i\nu} Q_{\mu' i} - (-1)^{|\nu|} \delta_{\nu\mu'} (Q^2)_{\mu\nu'}, \end{aligned} \tag{10.25}$$

and the body of this superfunction is

$$2(-1)^{|\nu|} \big(Q_{\mu\nu'} Q_{\mu'\nu} - \delta_{\mu\nu'}\delta_{\nu\mu'}\big)_{\varnothing}. \tag{10.26}$$

Proof. Recall that $|E_{ij}| = |i| + |j|$ is the parity of E_{ij}. By (5.6), for all indices i, j, μ, ν,

$$\widehat{E_{ij}}\, Q_{\mu\nu} = \delta_{\mu i} Q_{j\nu} - (-1)^{|E_{ij}|(|\mu|+|i|)} Q_{\mu i}\, \delta_{j\nu}, \tag{10.27}$$

where the sign comes from moving the odd parameter ϑ past $Q_{\mu i}$ when E_{ij} is odd. Recall from (5.14) that

$$\mathsf{C}_1 = -\frac{1}{2} \sum_{i,j} (-1)^{|j|}\, \widehat{E_{ji}}\, \widehat{E_{ij}}.$$

We apply $\widehat{E_{ji}}$ to (10.27), using (10.27) again, and we sum over i and j with the weights $(-1)^{|j|}$. Since $\sum_i (-1)^{|i|} = 0$ and $\operatorname{STr} Q = 0$ by (5.5), we find

$$-2\mathsf{C}_1 Q_{\mu\nu} = \sum_j (-1)^{|j|} Q_{\mu\nu} - 2\delta_{\mu\nu} \operatorname{STr} Q + \sum_i (-1)^{|i|} Q_{\mu\nu} = 0.$$

For homogeneous f and f', the Leibniz rule for the derivations $\widehat{E_{ij}}$, which are odd when E_{ij} is odd, and (5.14) give

$$\mathsf{C}_1(ff') = (\mathsf{C}_1 f) f' + f\, \mathsf{C}_1 f' - \sum_{i,j} (-1)^{|j|+|E_{ij}||f|}\, (\widehat{E_{ji}} f)(\widehat{E_{ij}} f').$$

For odd entries $f = Q_{\mu\nu}$ and $f' = Q_{\mu'\nu'}$, the first two terms vanish. Inserting (10.27) into the last sum and simplifying the signs with the relations $|\mu| + |\nu| = |\mu'| + |\nu'| = 1$, we obtain (10.25). Only products of two even entries have a body, and the diagonal blocks of the body Q_0 square to the identity. Using also $Q^2 = \mathrm{Id}$, we find (10.26). □

Proof of Lemma 10.7. Step 1 (a frame that commutes with the fields of $\mathfrak{k}$). We consider the two diagonal elements r_B and r_F of $\mathfrak{k}$ for which $\mathsf{k}_\gamma = e^{\gamma_\mathrm{B} \mathsf{r}_\mathrm{B} + \gamma_\mathrm{F} \mathsf{r}_\mathrm{F}}$, and the operators $\widetilde{\mathcal{D}}_a$, defined by

$$\mathsf{r}_\mathrm{B} = \tfrac{\mathrm{i}}{2} \operatorname{diag}(1,0,-1,0), \qquad \mathsf{r}_\mathrm{F} = \tfrac{\mathrm{i}}{2} \operatorname{diag}(0,1,0,-1), \qquad \widetilde{\mathcal{D}}_a = e^{\mathrm{i} c_a (\theta_\mathrm{B} - \theta_\mathrm{F})} \mathcal{D}_a.$$

By (10.23), we have

$$\widehat{\mathsf{r}_\mathrm{B}} = \partial_{\theta_\mathrm{B}} + \mathrm{i} \sum_a c_a \xi_a \partial_{\xi_a}, \qquad \widehat{\mathsf{r}_\mathrm{F}} = \partial_{\theta_\mathrm{F}} - \mathrm{i} \sum_a c_a \xi_a \partial_{\xi_a}.$$

We claim that each of the eight operators ∂_B, ∂_F, $\partial_{\theta_\mathrm{B}}$, $\partial_{\theta_\mathrm{F}}$, $\widetilde{\mathcal{D}}_1, \dots, \widetilde{\mathcal{D}}_4$ supercommutes with each of the six fields (10.24), $\widehat{\mathsf{r}_\mathrm{B}}$, and $\widehat{\mathsf{r}_\mathrm{F}}$. For the first four operators, this holds since the six fields have constant coefficients in $(B, F, \theta_\mathrm{B}, \theta_\mathrm{F})$. The phase of $\widetilde{\mathcal{D}}_b$ commutes with the odd fields, since ∂_+ annihilates $\theta_\mathrm{B} - \theta_\mathrm{F}$. Moreover,

$$\big\{ -\partial_{\xi_a} + \tfrac{\mathrm{i}}{2} \sigma_a \xi_{a'} \partial_+,\, \mathcal{D}_b \big\} = \tfrac{\mathrm{i}}{2} (\sigma_a - \sigma_b)\, \delta_{a'b}\, \partial_+ = 0, \qquad \Big[\mathrm{i} \sum_a c_a \xi_a \partial_{\xi_a},\, \mathcal{D}_b \Big] = -\mathrm{i} c_b \mathcal{D}_b,$$

where we used that $a' = b$ implies $\sigma_a = \sigma_b$, and that $c_{b'} = -c_b$. Since $\partial_{\theta_\mathrm{B}} e^{\mathrm{i} c_b (\theta_\mathrm{B} - \theta_\mathrm{F})} = \mathrm{i} c_b\, e^{\mathrm{i} c_b (\theta_\mathrm{B} - \theta_\mathrm{F})}$, the two contributions to $[\widehat{\mathsf{r}_\mathrm{B}}, \widetilde{\mathcal{D}}_b]$ cancel, and the same holds for $\widehat{\mathsf{r}_\mathrm{F}}$.

The eight operators form a frame: their coefficients with respect to the coordinate derivatives form a matrix with invertible body. By induction on the order, every differential operator P of order at most two in the polar chart can therefore be written uniquely as

$$P = \sum_w a_w W_w,$$

where W_w runs over the identity, the eight operators, and the products $W'W''$ of two of them with W' before W'' in the order $\partial_B, \partial_F, \partial_{\theta_\mathrm{B}}, \partial_{\theta_\mathrm{F}}, \widetilde{\mathcal{D}}_1, \dots, \widetilde{\mathcal{D}}_4$, and with $W' \neq W''$ if

both are odd. Here the a_w are superfunctions, and if P is even, then a_w has the parity of W_w.

Step 2 (reduction to functions of (B,F)). By the last assertion of Lemma 5.8(c), in the coordinates of a section, C_1 is a sum of finitely many terms, each of which is a superfunction, written on the left, times a product of at most two of the derivatives ∂_{x_μ} and ∂_{χ_a}. In the polar chart, each of these derivatives becomes a first-order operator in ∂_B, ∂_F, $\partial_{\theta_\mathrm{B}}$, $\partial_{\theta_\mathrm{F}}$, and $\partial_{\xi_1},\dots,\partial_{\xi_4}$. Hence $-\mathsf{C}_1$ is an even differential operator in the polar chart whose order, counting the odd derivatives ∂_{ξ_a} as well as the even ones, is at most two, which is the hypothesis of the expansion of Step 1. By Lemma 5.8(a), it commutes with all fields $\widehat{Y}$. We expand $-\mathsf{C}_1 = \sum_w a_w W_w$ as in Step 1. Let V be one of the six fields of Step 1. Since V supercommutes with each W_w, and a_w has the parity of W_w, we have

$$0 = [-\mathsf{C}_1, V] = -\sum_w (V a_w)\, W_w,$$

and the uniqueness of the expansion gives $Va_w = 0$ for all w. The first two fields in (10.24) have the anticommutator $-\mathrm{i}\partial_+$, and hence $\partial_+ a_w = 0$. The four odd fields then give $\partial_{\xi_a} a_w = 0$ for all a, and $\widehat{\mathsf{r}_\mathrm{B}}$ and $\widehat{\mathsf{r}_\mathrm{F}}$ show that a_w does not depend on θ_B and θ_F. Thus every a_w is a function of (B,F) without odd terms, and $a_w = 0$ whenever W_w is odd. We conclude that

$$-\mathsf{C}_1 = P_\mathrm{rad} + \sum_{x,\theta} a_{x\theta}\, \partial_x \partial_\theta + \sum_{\theta \le \theta'} a_{\theta\theta'}\, \partial_\theta \partial_{\theta'} + b_\mathrm{B}\, \partial_{\theta_\mathrm{B}} + b_\mathrm{F}\, \partial_{\theta_\mathrm{F}} + \sum_{a<b} a_{ab}\, \widetilde{\mathcal{D}}_a \widetilde{\mathcal{D}}_b, \tag{10.28}$$

where x runs over $\{B,F\}$ and θ,θ' over $\{\theta_\mathrm{B},\theta_\mathrm{F}\}$, P_rad is a differential operator in (B,F) of order two, and all coefficients are functions of (B,F). In particular, $-\mathsf{C}_1$ has no terms that mix a derivative in B or F with a derivative in an odd variable.

Step 3 (the radial and the angular parts). Let $\phi \in C_c^\infty(D^\circ)$, and let f be the superfunction that equals $\phi(B,F)$ in the polar chart and vanishes elsewhere. It is smooth by Lemma 6.1(a), and invariant under $\mathfrak{k}$ by Lemma 10.8, since none of the fields there differentiates in B or F. Every word in (10.28) outside P_rad ends with ∂_θ or with some $\widetilde{\mathcal{D}}_a$, which annihilate f. Hence $-\mathsf{C}_1 f = P_\mathrm{rad}\phi$, and Lemma 6.8 implies that $P_\mathrm{rad}\phi = \Delta_D^\mathrm{s}\phi$. As ϕ is arbitrary and the coefficients of P_rad are functions of (B,F), we conclude that $P_\mathrm{rad} = \Delta_D^\mathrm{s}$.

For the angular part, we use the principal part of C_1. By Lemma 5.8(c), in the coordinates of a section, the coefficients of the second derivatives of C_1 in the even coordinates are those of $-\Delta_{\mathcal{M}_1^0}$ plus superfunctions with nilpotent values. Under a change of coordinates, the bodies of these coefficients transform as for a differential operator on $\mathcal{M}_1^0$, since the derivatives of the even coordinates in the odd ones, and of the odd coordinates in the even ones, have no body. In the polar chart, the bodies of the

coefficients of $\partial_x\partial_y$ for $x, y \in \{B, F, \theta_{\mathrm{B}}, \theta_{\mathrm{F}}\}$ are therefore those of the Laplace–Beltrami operator of the metric of Section 5.2,

$$\Delta_{\mathcal{M}_1^0} = \partial_B\big((B^2-1)\partial_B\big) + \frac{1}{B^2-1}\partial_{\theta_{\mathrm{B}}}^2 + \partial_F\big((1-F^2)\partial_F\big) + \frac{1}{1-F^2}\partial_{\theta_{\mathrm{F}}}^2.$$

The products $\widetilde{\mathcal{D}}_a\widetilde{\mathcal{D}}_b$ contribute to these coefficients only multiples of $\xi_{a'}\xi_{b'}$, which have no body. Since the coefficients in (10.28) are functions, we conclude that

$$a_{x\theta} = a_{\theta_{\mathrm{B}}\theta_{\mathrm{F}}} = 0, \qquad a_{\theta_{\mathrm{B}}\theta_{\mathrm{B}}} = \frac{1}{B^2-1}, \qquad a_{\theta_{\mathrm{F}}\theta_{\mathrm{F}}} = \frac{1}{1-F^2}.$$

Step 4 (the first-order terms). We write each product $\widetilde{\mathcal{D}}_a\widetilde{\mathcal{D}}_b$ with $a < b$ as half the difference $\widetilde{\mathcal{D}}_a\widetilde{\mathcal{D}}_b - \widetilde{\mathcal{D}}_b\widetilde{\mathcal{D}}_a$ plus half the anticommutator. The phases commute with the operators $\mathcal{D}_a$, and

$$\widetilde{\mathcal{D}}_a\widetilde{\mathcal{D}}_b = \tfrac{1}{2}\big(\widetilde{\mathcal{D}}_a\widetilde{\mathcal{D}}_b - \widetilde{\mathcal{D}}_b\widetilde{\mathcal{D}}_a\big) + \tfrac{1}{2}e^{\mathrm{i}(c_a+c_b)(\theta_{\mathrm{B}}-\theta_{\mathrm{F}})}\{\mathcal{D}_a, \mathcal{D}_b\}, \qquad \{\mathcal{D}_a, \mathcal{D}_b\} = \mathrm{i}\sigma_a\delta_{a'b}\partial_+.$$

Since $c_a + c_{a'} = 0$, the anticommutators contribute multiples of ∂_+ with coefficients that are functions of (B, F). Combining this with Step 3, we find

$$-\mathsf{C}_1 = \Delta_D^{\mathrm{s}} + \frac{1}{B^2-1}\partial_{\theta_{\mathrm{B}}}^2 + \frac{1}{1-F^2}\partial_{\theta_{\mathrm{F}}}^2 + b'_{\mathrm{B}}\,\partial_{\theta_{\mathrm{B}}} + b'_{\mathrm{F}}\,\partial_{\theta_{\mathrm{F}}} + \frac{1}{2}\sum_{a,b}\tilde{g}^{ab}\mathcal{D}_b\mathcal{D}_a, \tag{10.29}$$

where b'_{B} and b'_{F} are functions of (B, F), and $(\tilde{g}^{ab})$ is an antisymmetric matrix such that $e^{-\mathrm{i}(c_a+c_b)(\theta_{\mathrm{B}}-\theta_{\mathrm{F}})}\tilde{g}^{ab}$ is a function of (B, F). We show in Step 5 that $\tilde{g}^{ab} = g^{ab}$.

We now show that $b'_{\mathrm{B}} = b'_{\mathrm{F}} = 0$. For superfunctions f and f' with compact support in the polar chart, we set

$$\langle f, f'\rangle = \int_D d^{-2}\int_0^{2\pi}\int_0^{2\pi}[ff']_{\mathrm{top}}\;\mathrm{d}\theta_{\mathrm{B}}\,\mathrm{d}\theta_{\mathrm{F}}\;\mathrm{d}B\,\mathrm{d}F.$$

By Lemma 5.8(b) and (6.3), the operator $-\mathsf{C}_1$ is symmetric for this form. The first three terms of (10.29) are symmetric, since

$$\Delta_D^{\mathrm{s}} = d^2\partial_B\big(d^{-2}(B^2-1)\partial_B\big) + d^2\partial_F\big(d^{-2}(1-F^2)\partial_F\big).$$

Integrating by parts in the odd variables and in the angles, we find that, for homogeneous f,

$$\langle\mathcal{D}_a f, f'\rangle = -(-1)^{|f|}\langle f, \mathcal{D}_a f'\rangle, \qquad \langle\mathcal{D}_b\mathcal{D}_a f, f'\rangle = -\langle f, \mathcal{D}_a\mathcal{D}_b f'\rangle.$$

Since $\mathcal{D}_a$ annihilates $\tilde{g}^{ab}$ and $\tilde{g}^{ab}$ is antisymmetric, the last term of (10.29) is symmetric. The remaining term $b'_{\mathrm{B}}\partial_{\theta_{\mathrm{B}}} + b'_{\mathrm{F}}\partial_{\theta_{\mathrm{F}}}$ is antisymmetric, by an integration by parts in the angles. Being also symmetric, it vanishes.

Step 5 (the odd coefficients). It remains to show that $\tilde{g}^{ab} = g^{ab}$. Since $e^{-\mathrm{i}(c_a+c_b)(\theta_\mathrm{B}-\theta_\mathrm{F})}\tilde{g}^{ab}$ is a function of (B,F), it suffices to compute bodies. Let $a<b$. We have

$$\mathcal{D}_{a_2}\mathcal{D}_{a_1}(\xi_a\xi_b) = \delta_{a_1a}\delta_{a_2b} - \delta_{a_1b}\delta_{a_2a}, \qquad \Delta_D^{\mathrm{s}}(d^2) = d\,\Delta_D d = 2d(B+F),$$

where the second identity follows from (6.1). Inserting these identities into (10.29), we obtain

$$-\mathsf{C}_1(d^2\xi_a\xi_b) = 2d(B+F)\,\xi_a\xi_b + d^2\tilde{g}^{ab}, \tag{10.30}$$

whose body is $d^2\tilde{g}^{ab}$. By Lemma 6.1(b), the superfunction $d^2\xi_a\xi_b$ is, up to sign, the product of two of the odd entries $Q^{\mathrm{BF}}_{\mathrm{RR}} = d\xi_1$, $Q^{\mathrm{FB}}_{\mathrm{RR}} = -d\xi_2$, $Q^{\mathrm{BF}}_{\mathrm{AA}} = -d\xi_3$, and $Q^{\mathrm{FB}}_{\mathrm{AA}} = d\xi_4$, and Lemma 10.9 computes the body of $-\mathsf{C}_1$ on such products.

We apply (10.26) to the six pairs $a<b$, with the bodies of the entries of Q_0. For $(a,b)=(1,2)$, we take $(\mu,\nu,\mu',\nu') = (\mathrm{RB},\mathrm{RF},\mathrm{RF},\mathrm{RB})$, and for $(a,b)=(1,4)$, we take $(\mu,\nu,\mu',\nu') = (\mathrm{RB},\mathrm{RF},\mathrm{AF},\mathrm{AB})$; the other pairs are analogous. This yields the bodies

$$\begin{aligned}
-\mathsf{C}_1\big(Q^{\mathrm{BF}}_{\mathrm{RR}}Q^{\mathrm{FB}}_{\mathrm{RR}}\big)\big|_\varnothing &= -\mathsf{C}_1\big(Q^{\mathrm{BF}}_{\mathrm{AA}}Q^{\mathrm{FB}}_{\mathrm{AA}}\big)\big|_\varnothing = -2(BF-1),\\
-\mathsf{C}_1\big(Q^{\mathrm{BF}}_{\mathrm{RR}}Q^{\mathrm{FB}}_{\mathrm{AA}}\big)\big|_\varnothing &= -2\,Q^{\mathrm{BB}}_{\mathrm{RA}}Q^{\mathrm{FF}}_{\mathrm{AR}}\big|_\varnothing = 2\mathrm{i}\big((B^2-1)(1-F^2)\big)^{1/2}e^{\mathrm{i}(\theta_\mathrm{B}-\theta_\mathrm{F})},\\
-\mathsf{C}_1\big(Q^{\mathrm{FB}}_{\mathrm{RR}}Q^{\mathrm{BF}}_{\mathrm{AA}}\big)\big|_\varnothing &= 2\,Q^{\mathrm{FF}}_{\mathrm{RA}}Q^{\mathrm{BB}}_{\mathrm{AR}}\big|_\varnothing = -2\mathrm{i}\big((B^2-1)(1-F^2)\big)^{1/2}e^{-\mathrm{i}(\theta_\mathrm{B}-\theta_\mathrm{F})},
\end{aligned}$$

for the pairs $(1,2)$, $(3,4)$, $(1,4)$, and $(2,3)$. For $(1,3)$ and $(2,4)$, the entry $Q_{\mu\nu'}$ in (10.26) is odd, and the body vanishes. We compare these bodies with the body of (10.30), using

$$Q^{\mathrm{BF}}_{\mathrm{RR}}Q^{\mathrm{FB}}_{\mathrm{RR}} = -d^2\xi_1\xi_2, \quad Q^{\mathrm{BF}}_{\mathrm{AA}}Q^{\mathrm{FB}}_{\mathrm{AA}} = -d^2\xi_3\xi_4, \quad Q^{\mathrm{BF}}_{\mathrm{RR}}Q^{\mathrm{FB}}_{\mathrm{AA}} = d^2\xi_1\xi_4, \quad Q^{\mathrm{FB}}_{\mathrm{RR}}Q^{\mathrm{BF}}_{\mathrm{AA}} = d^2\xi_2\xi_3,$$

and we find $\tilde{g}^{ab} = g^{ab}$ for all $a<b$. Together with (10.29), this proves (10.22). □

Remark 10.10. The formula (10.22) is also confirmed by exact computer algebra, with the supplementary scripts in the folder `polar`. The script `verify_tangency_exact.py` computes the sixteen fields $\widehat{E_{ij}}$ in the polar chart and checks (5.6) on all entries of Q, and `verify_op_exact.py` checks (10.22) on the superfunctions $\xi^I h$ with $h \in \{1, x_a, x_a x_b\}$, which determine a second-order operator (its output is in `verify_op_exact.log`).

Freezing at the noncompact boundary

We now pass to the scaled variables. Let $\hat{z} \in (0,1]$, and let f be an even superfunction on $\{B>1\}$ of charge $(m,0)$. By (10.21), we may write

$$f = \sum_{I\ \text{even}} e^{\mathrm{i}(m-c_I)\theta_\mathrm{B}}\xi^I f_I(q,\mathbf{e}), \qquad q = \hat{z}B, \tag{10.31}$$

where f_I is a function of q and of the point $\mathbf{e} \in S^2$, with charge c_I under the rotations of S^2 about its axis. The superfunction f is smooth if and only if the functions f_I are smooth on

$(\hat{z},\infty)\times S^2$, and we call $(f_I)_I$ the *components* of f. Let $\Delta_{S^2}=\partial_F((1-F^2)\partial_F)+(1-F^2)^{-1}\partial_{\theta_{\mathrm{F}}}^2$ denote the Laplace operator of S^2. We define

$$\Delta_{\hat{z}}=(q^2-\hat{z}^2)\partial_q^2+2\hat{z}\,\frac{\hat{z}-qF}{q-\hat{z}F}\,\partial_q+\Delta_{S^2}+\frac{2\hat{z}(1-F^2)}{q-\hat{z}F}\,\partial_F. \tag{10.32}$$

The first two terms form the generator of q_s in (10.1). The last two generate a Brownian motion on S^2 with a drift, since the vector field $(1-F^2)\partial_F$ is the gradient of F on S^2, which is smooth.

Lemma 10.11. *Let $k\geq 1$, $|m|\leq k$, and $\hat{z}\in(0,1]$. There exist a constant C_k and a matrix $\mathcal{V}_{\hat{z}}=\mathcal{V}_{\hat{z},m}$ indexed by the even subsets of $\{1,2,3,4\}$, with entries that are smooth functions of $(q,\mathbf{e})\in(\hat{z},\infty)\times S^2$, such that the following holds. For every smooth even superfunction f on $\{B>1\}$ of charge $(m,0)$, the components of $-\mathsf{G}_{1,\hat{z}}f$ are*

$$\big(-\mathsf{G}_{1,\hat{z}}f\big)_I=\big(\Delta_{\hat{z}}-(q-\hat{z}F)\big)f_I+\sum_J(\mathcal{V}_{\hat{z}})_{IJ}f_J,\qquad \|\mathcal{V}_{\hat{z}}(q,\mathbf{e})\|\leq C_k\frac{\hat{z}}{q}\quad \text{for } q\geq 2\hat{z}. \tag{10.33}$$

Moreover, the source satisfies

$$\hat{z}\,\mathfrak{s}=q\,\mathfrak{s}_0+\hat{z}\,\mathfrak{r}_{\hat{z}},\qquad \mathfrak{s}_0=(1+\cos\theta_{\mathrm{B}})\Big(1+\frac{n_{\mathrm{R}}+n_{\mathrm{A}}}{2}\Big)+\frac{\cos\theta_{\mathrm{B}}}{4}n_{\mathrm{R}}n_{\mathrm{A}}, \tag{10.34}$$

where the coefficients of $\mathfrak{r}_{\hat{z}}$ are bounded by an absolute constant on $\{B>1\}$, uniformly in $\hat{z}$.

In the scaled variables, the generator acts on the components as the scalar operator $\Delta_{\hat{z}}$ of the singlet diffusion with a spherical angle, plus a matrix potential that vanishes at the noncompact boundary. The source becomes multiplication by q times the element $\mathfrak{s}_0$, which depends only on θ_{B} and on the odd variables. In the limit, the generator and the source leave the variables θ_{B} and ξ fixed.

Proof. We use Lemma 10.7 and the substitution $B=q/\hat{z}$, for which $\partial_B=\hat{z}\partial_q$. Recall that

$$\Delta_D^{\mathrm{s}}=(B^2-1)\partial_B^2+\frac{2(1-BF)}{B-F}\partial_B+(1-F^2)\partial_F^2-2F\partial_F+\frac{2(1-F^2)}{B-F}\partial_F.$$

In the variable q, we have

$$(B^2-1)\partial_B^2=(q^2-\hat{z}^2)\partial_q^2,\qquad \frac{2(1-BF)}{B-F}\partial_B=2\hat{z}\,\frac{\hat{z}-qF}{q-\hat{z}F}\,\partial_q,\qquad \frac{2(1-F^2)}{B-F}=\frac{2\hat{z}(1-F^2)}{q-\hat{z}F},$$

Comparing with (10.32), we see that the operator

$$\Delta_D^{\mathrm{s}}+\frac{1}{1-F^2}\,\partial_{\theta_{\mathrm{F}}}^2$$

acts on each component as $\Delta_{\hat z}$. By (10.17), the multiplication by $\hat z d_1$ is the multiplication by $q - \hat z F$.

We next show that the remaining terms of (10.22) act on the components as a matrix. By (10.31), the operator $\partial_{\theta_{\mathrm B}}$ acts on the component of ξ^I as multiplication by $\mathrm i(m - c_I)$, and the term with $\partial^2_{\theta_{\mathrm B}}$ contributes the diagonal entries

$$-\frac{(m-c_I)^2}{B^2-1} = -\frac{(m-c_I)^2\hat z^2}{q^2-\hat z^2}.$$

For the odd part, note that ∂_+ acts on every superfunction of charge $(m,0)$ as multiplication by $\mathrm i m$, since the coefficient of ξ^I in (10.21) has the phase charges $m - c_I$ and c_I. Multiplication by ξ_a and the derivatives ∂_{ξ_a} preserve the sum of the two charges, so the operators $\mathcal D_a$ act on the coefficients of superfunctions of charge $(m,0)$ as finite matrices with entries in $\{0, \pm 1, \pm\frac{m}{2}\}$. The last term of (10.22) acts as a matrix whose entries are linear combinations of the g^{ab}, with coefficients bounded in terms of k. In the components, the phases $e^{\pm \mathrm i\theta_{\mathrm B}}$ of g^{14} and g^{23} are absorbed by the change of c_I, and the factors

$$(1-F^2)^{1/2} e^{\mp \mathrm i\theta_{\mathrm F}}$$

are smooth functions of $\mathbf e$. We let $\mathcal V_{\hat z}$ be the sum of these two matrices.

Let $B \ge 2$. The bounds $B - F \ge B - 1$ and $(B-1)^2 \ge B^2/4$ give

$$|g^{12}| = |g^{34}| \le \frac{2(B+1)}{(B-1)^2} \le \frac{12}{B}, \qquad |g^{14}| = |g^{23}| \le \frac{2B}{(B-1)^2} \le \frac{8}{B},$$
$$\frac{(m-c_I)^2\hat z^2}{q^2-\hat z^2} \le \frac{4(k+1)^2}{3B^2}.$$

Since $1/B = \hat z/q$, these bounds imply the bound on $\mathcal V_{\hat z}$ in (10.33).

Finally, (10.20) gives (10.34) with

$$\mathfrak r_{\hat z} = \frac{(q^2-\hat z^2)^{1/2} - q}{\hat z} \cos\theta_{\mathrm B}\Big(1 + \frac{n_{\mathrm R}+n_{\mathrm A}}{2} + \frac{n_{\mathrm R}n_{\mathrm A}}{4}\Big) - \frac{F}{2}(n_{\mathrm R}+n_{\mathrm A})$$
$$- \frac{\mathrm i}{2}(1-F^2)^{1/2}\big(e^{\mathrm i\theta_{\mathrm F}}\xi_1\xi_4 - e^{-\mathrm i\theta_{\mathrm F}}\xi_2\xi_3\big),$$

and (10.4) gives

$$\big|(q^2-\hat z^2)^{1/2} - q\big| \le \hat z. \qquad \square$$

10.7 The permanent residue

We now prove the second main result of this section. Its proof extends the coupling argument of Theorem 10.4 to the matrix potential of Lemma 10.11. This is the most technical step of the section.

Lemma 10.12. *Let $k \geq 2$, $0 < X_- \leq X_+ < \infty$, and $\delta \in (0, 1/2]$. Then*

$$\lim_{z \downarrow 0} z^{k-1} \mathfrak{p}_k^{X,z}(x_1, \dots, x_k) = k(2k-3)!!\, M_k^X(x_1, \dots, x_k), \tag{10.35}$$

uniformly for $X \in [X_-, X_+]$ and $(x_1, \dots, x_k)$ such that $\delta \leq x_1$, $x_k \leq 1 - \delta$, and $x_{j+1} - x_j \geq \delta$ for $1 \leq j \leq k-1$.

The constant $k(2k-3)!!$ is the constant $\varkappa_{k,k!}$ of (9.13), which the extraction theorem requires for the moment measures of Section 12 whose weights are squared permanents of eigenvector entries (Definition 12.1).

We fix the notation for the proof. We use the scaled times $s_0 = Xx_1$, $s_j = X(x_{j+1} - x_j)$ for $1 \leq j \leq k-1$, and $s_k = X(1 - x_k)$, which lie in $[s_*, T]$ with $s_* = X_-\delta$ and $T = X_+$. We let $0 < \hat{z} \leq \hat{z}_0 = \min(1, X_-)$. Since $e^{-t\mathsf{G}_{X,z}} = e^{-Xt\mathsf{G}_{1,\hat{z}}}$, we have

$$\mathfrak{p}_k^{X,z} = \hat{z}^{-k} \mathcal{B}_1\big(\Psi_{s_0}^{(\hat{z})}, \mathfrak{c}_1\big).$$

Here $\Psi_{s_0}^{(\hat{z})}$ stands for the invariant superfunction $e^{-s_0\mathsf{G}_{1,\hat{z}}}1$, whose radial representative, in the variables (q, F), is the function $\Psi_{s_0}^{(\hat{z})}$ of (10.3). The *partial products* $\mathfrak{c}_j$ are defined by

$$\mathfrak{c}_k = \hat{z}\,\mathfrak{s}\, e^{-s_k\mathsf{G}_{1,\hat{z}}}1, \qquad \mathfrak{c}_j = \hat{z}\,\mathfrak{s}\, e^{-s_j\mathsf{G}_{1,\hat{z}}}\mathfrak{c}_{j+1}, \quad 1 \leq j \leq k-1. \tag{10.36}$$

We define the *frozen partial products* by

$$\mathfrak{c}_j^0 = \mathfrak{s}_0^{k-j+1}\varphi_j, \qquad \varphi_k = q\,\Upsilon_{s_k}, \qquad \varphi_j = q\,\mathcal{T}_{s_j}\varphi_{j+1}, \quad 1 \leq j \leq k-1, \tag{10.37}$$

where $\mathfrak{s}_0$ is the element of (10.34) and φ_j is a function of q. The proof has four steps. We first prove a Feynman–Kac formula for semigroup products away from the corner (Step 1) and a polynomial bound (Step 2). We then show that $\mathfrak{c}_j \to \mathfrak{c}_j^0$ (Step 3), and we integrate over q (Step 4). In the first three steps, the norm $\|\cdot\|$ of a semigroup product at a point is the maximum of the moduli of its components (10.31), over the finitely many charges $|m| \leq k$ and even I.

We also use the diffusion of the scalar part of the generator. For $q > 2\hat{z}$ and $\mathbf{e} \in S^2$, let $(q_u, \mathbf{e}_u)$ denote the diffusion on $(\hat{z}, \infty) \times S^2$ with generator $\Delta_{\hat{z}}$, started at $(q, \mathbf{e})$. Its radial part q_u is driven by a Brownian motion β, and the motion on S^2 by an independent one. The coefficients of $\Delta_{\hat{z}}$ are smooth for $q > \hat{z}$, and the diffusion is well defined up to the hitting time τ_2 of $\{q = 2\hat{z}\}$, which corresponds to $B = 2$. By (10.32), the process (q_u, F_u), with F_u the height of $\mathbf{e}_u$, solves the equation of the scaled singlet diffusion in (10.1) up to τ_2. We need the following bounds for it.

Lemma 10.13. *Let $T > 0$ and $n \geq 1$. There exists C such that for all $0 < \hat{z} \leq 1$, $q > 2\hat{z}$, and $\mathbf{e} \in S^2$,*

$$\mathbb{E} \sup_{u \leq T} q_{u \wedge \tau_2}^n \leq C q^n, \qquad \mathbb{E} q_{u \wedge \tau_2}^{-2} \leq e^{10u} q^{-2}, \quad u \geq 0, \tag{10.38}$$

and

$$\mathbb{P}(\tau_2 \le T) \le \exp\Big(- \frac{(\log(q/(2\hat z)) - 3T)^2}{4T} \Big), \qquad q \ge 2\hat z e^{3T}. \tag{10.39}$$

Moreover, for every compact set $\mathcal{K} \subset (0,\infty)$,

$$\lim_{R\to\infty} \sup_{\hat z \le 1,\, q\in\mathcal{K},\, \mathbf{e}\in S^2} \mathbb{P}\big(q_u \notin [R^{-1}, R] \text{ for some } u \le T \wedge \tau_2\big) = 0. \tag{10.40}$$

Proof. For $u < \tau_2$, we have $q_u > 2\hat z$, and (10.4) bounds the drift of q_u by $2\hat z \le q_u$ and its diffusion coefficient by $\sqrt{2}\, q_u$. Itô's formula and the Burkholder–Davis–Gundy inequality give the first bound in (10.38). Further, the drift of q_u^{-2} satisfies

$$-2q_u^{-3} \cdot 2\hat z\, \frac{\hat z - q_u F_u}{q_u - \hat z F_u} + 6(q_u^2 - \hat z^2) q_u^{-4} \le 10 q_u^{-2},$$

and $q_u^{-2} \le (2\hat z)^{-2}$ before τ_2. Taking expectations of the stopped process and applying Gronwall's inequality, we obtain the second bound.

As in the proof of Lemma 10.3, the process $\log q_u - \log q + 3u$ is bounded below by a continuous martingale with quadratic variation at most $2u$, for $u < \tau_2$. The exponential martingale inequality implies (10.39) and the lower half of (10.40). The upper half follows from the first bound in (10.38) and Markov's inequality. □

The proof also needs the regularity of semigroup products for small values of the last time s_n in (5.26), since the Feynman–Kac formula below evaluates them there. This regularity is proved in Lemma 5.17. The semigroup products of Definition 10.6 are semigroup products (5.26) with the multipliers $\mathfrak{s}$ (Lemma 5.14), and we apply Lemma 5.17 with $X = 1$, $z_0 = 1$, $z = \hat z \in (0,1]$, and times in $[0,T]$. We use part (b) of Lemma 5.17, which is the last assertion of Proposition 5.15(e), to bound semigroup products on the compact set $\{B \le 3\}$ of $\mathcal{M}_1^0$, which contains the corner, in Step 2 below, uniformly in $\hat z$ and in the times, and its part (c) for the regularity up to time zero, at fixed $\hat z$. These parts bound coefficients in the odd generators χ_a. Recall that these are the coefficients in the coordinates of a section (the regular coordinates of Definition 5.2), and that the translated charts of Section 5.2 are the charts $x \mapsto g' x g'^{-1}$ obtained from one fixed chart at Λ by the action of an element $g' \in G_0$. In polar coordinates, part (c) implies the following corollary.

Corollary 10.14. *Let* $0 < \hat z \le 1$, *let* $f \in \mathfrak{S}_k$ *be a semigroup product of charge* $(m,0)$, *and let* $r \ge 0$. *The components* (10.31) *of* $e^{-v\mathsf{G}_{1,\hat z}} f$ *are smooth functions of* $(v,q,\mathbf{e}) \in [0,\infty) \times (\hat z,\infty) \times S^2$, *and they solve*

$$\partial_v u = \big(\Delta_{\hat z} - (q - \hat z F) + \mathcal{V}_{\hat z}\big) u \qquad \text{on } \{q > 2\hat z\}. \tag{10.41}$$

Moreover, there exist N *and* C *such that these components and their derivatives of order at most two are bounded by* $C(1+B)^N$ *on* $\{B \ge 2\}$, *for all* $v \in [0,r]$.

Proof. By Lemma 5.17(c), with $X = 1$, $z = \hat{z}$, and the multipliers $\mathfrak{s}$, the map

$$(v, Q) \mapsto \big(e^{-v\mathsf{G}_{1,\hat{z}}} f\big)(Q), \qquad (v, Q) \in [0, \infty) \times \mathcal{M}_1^0,$$

is smooth, also at $v = 0$. It solves $\partial_v u = -\mathsf{G}_{1,\hat{z}} u$, and Lemma 10.11 turns this equation into (10.41). The weighted bound in Lemma 5.17(c) and the Sobolev embedding theorem on the balls of the translated charts show that, for some N, the coefficients of $e^{-v\mathsf{G}_{1,\hat{z}}} f$ in the translated charts, and their derivatives of order at most two, are bounded by $C(1 + B)^N$, uniformly for $v \in [0, r]$.

It remains to pass to the polar chart on $\{B \geq 2\}$. We write $Q_0 = g \Lambda g^{-1}$ with a body element $g = \operatorname{diag}(g_{\mathrm{B}} | g_{\mathrm{F}})$. We may choose g_{B} as a smooth function of (B, θ_{B}) whose entries are bounded, with their derivatives, by $CB^{1/2}$ on $\{B \geq 2\}$ (a boost of rapidity $\operatorname{arccosh} B$), and g_{F} as a smooth function of $\mathbf{e}$ on each of two open sets that cover S^2. Then, by (6.2), we have

$$Q = g\, e^{-g^{-1}\Theta_{\tilde{\xi}} g} \Lambda e^{g^{-1}\Theta_{\tilde{\xi}} g}\, g^{-1}.$$

In the translated chart centered at the body point Q_0, the coordinates of Q are therefore a fixed smooth function of the odd matrix $g^{-1}\Theta_{\tilde{\xi}} g$. This matrix is linear in ξ, with coefficients bounded by CB. The transition between the two charts and its derivatives are thus bounded by powers of B, and the chain rule transfers the bounds to the components. □

Proof of Lemma 10.12. Step 1 (a Feynman–Kac formula). Let $(q_u, \mathbf{e}_u)$ and τ_2 be as in Lemma 10.13, and let $\mathcal{E}_u$ be the matrix solution of

$$\mathrm{d}\mathcal{E}_u = \mathcal{E}_u\, \mathcal{V}_{\hat{z}}(q_u, \mathbf{e}_u)\, \mathrm{d}u, \qquad \mathcal{E}_0 = \mathrm{Id}, \qquad \|\mathcal{V}_{\hat{z}}(q_u, \mathbf{e}_u)\| \leq \frac{C_k \hat{z}}{q_u} \leq C_k \quad (u < \tau_2),$$

where the bound is (10.33). Gronwall's inequality implies

$$\|\mathcal{E}_u\| \leq e^{C_k u}, \qquad \|\mathcal{E}_u - \mathrm{Id}\| \leq C_k e^{C_k u} \hat{z} \int_0^u q_v^{-1}\, \mathrm{d}v, \qquad u \leq \tau_2. \tag{10.42}$$

Let $f \in \mathfrak{S}_k$ be a semigroup product of charge $(m, 0)$ and $r \geq 0$. By Corollary 10.14, the components of $e^{-v\mathsf{G}_{1,\hat{z}}} f$ are smooth in $(v, q, \mathbf{e})$, also at $v = 0$, they solve (10.41), and they grow at most polynomially in B, uniformly for $v \in [0, r]$. We apply Itô's formula to the process

$$\mathcal{E}_v \exp\Big(- \int_0^v (q_w - \hat{z} F_w)\, \mathrm{d}w \Big) \big(e^{-(r-v)\mathsf{G}_{1,\hat{z}}} f\big)(q_v, \mathbf{e}_v), \qquad v \leq \tau_2 \wedge r.$$

The drift of this process vanishes. The polynomial bound on the components, (10.42), and (10.38) bound it by an integrable random variable. Hence its martingale part is a true martingale, and taking expectations, we obtain

$$\big(e^{-r\mathsf{G}_{1,\hat{z}}} f\big)(q, \mathbf{e}) = \mathbb{E}\Big[\mathcal{E}_{\tau_2 \wedge r} \exp\Big(- \int_0^{\tau_2 \wedge r} (q_v - \hat{z} F_v)\, \mathrm{d}v \Big) \big(e^{-(r - \tau_2 \wedge r)\mathsf{G}_{1,\hat{z}}} f\big)(q_{\tau_2 \wedge r}, \mathbf{e}_{\tau_2 \wedge r}) \Big], \tag{10.43}$$

as an identity between vectors of components, at points with $q > 2\hat{z}$.

Step 2 (a polynomial bound). We claim that for $0 \le n \le k$, there exists C such that

$$\left\| e^{-r\mathsf{G}_{1,\hat{z}}} \mathfrak{s}\, e^{-r_1\mathsf{G}_{1,\hat{z}}} \mathfrak{s} \cdots \mathfrak{s}\, e^{-r_n\mathsf{G}_{1,\hat{z}}} 1 \right\| \le C(1+B)^n \tag{10.44}$$

on $\{B > 1\}$ (at points with $B \le 3$, the norm is the maximum of the coefficients in the regular coordinates), for all $0 < \hat{z} \le \hat{z}_0$, $r \in [0,T]$, and $r_1, \dots, r_n \in [s_*, T]$. We argue by induction on n. Let $f = 1$ if $n = 0$, and

$$f = \mathfrak{s}\, e^{-r_1\mathsf{G}_{1,\hat{z}}} \cdots e^{-r_n\mathsf{G}_{1,\hat{z}}} 1$$

if $n \ge 1$. The entries of Q^{BB} are bounded by CB. Together with the induction hypothesis, this implies

$$\|f\| \le C(1+B)^n \qquad \text{on } \{B > 1\}.$$

We first bound $e^{-r\mathsf{G}_{1,\hat{z}}} f$ on the compact set $\{B \le 3\}$. This superfunction is a semigroup product (5.26) with the multipliers $\mathfrak{s}$ and times in $[0,T]$. We apply Lemma 5.17(b) with $X = 1$, $z_0 = 1$, $z = \hat{z}$, and the compact set $\{B \le 3\}$ of $\mathcal{M}_1^0$. It bounds the coefficients of $e^{-r\mathsf{G}_{1,\hat{z}}} f$ on $\{B \le 3\}$, uniformly in $r \in [0,T]$ and $0 < \hat{z} \le \hat{z}_0$; this includes times r close to zero.

We now insert these bounds on $\{B \le 3\}$ into (10.43), at points with $B > 2$. The last factor in (10.43) is bounded by a constant on the event $\{\tau_2 \le r\}$, where it is evaluated at $B = 2$. On the complementary event, it reduces to $f(q_r, \mathbf{e}_r)$, and

$$\|f(q_r, \mathbf{e}_r)\| \le C(1+B_r)^n = C\hat{z}^{-n}(\hat{z}+q_r)^n, \qquad B_r = \frac{q_r}{\hat{z}}.$$

Using (10.42), the nonnegativity of the killing and the moment bound

$$\mathbb{E}(\hat{z} + q_{r\wedge\tau_2})^n \le C(\hat{z}+q)^n = C\hat{z}^n(1+B)^n,$$

we obtain (10.44). Since $\mathfrak{c}_j$ is $\hat{z}^{k-j+1}$ times a semigroup product with $k-j+1$ sources, (10.44) with $r = 0$ gives

$$\|\mathfrak{c}_j\| \le C\hat{z}^{k-j+1}(1+B)^{k-j+1} = C(\hat{z}+q)^{k-j+1}, \qquad 1 \le j \le k. \tag{10.45}$$

For the frozen partial products, we differentiate

$$\mathcal{T}_s\varphi(q) = \mathbb{E}\big[\varphi(qe^{g_1(s)})e^{-qA_s}\big]$$

in q and argue by induction, using that $e^{g_1(s)}$ and A_s have moments of all orders. We find

$$|\varphi_j(q)| + |\varphi_j'(q)| \le C(1+q)^{k-j+1}, \qquad 1 \le j \le k.$$

Step 3 (the partial products freeze). We show by downward induction on j that $\mathfrak{c}_j \to \mathfrak{c}_j^0$ as $\hat{z} \to 0$, componentwise, uniformly for q in compact subsets of $(0,\infty)$, for $\mathbf{e} \in S^2$, and for the times in $[s_*, T]$. For $j = k$, (10.34) gives

$$\mathfrak{c}_k = (q\,\mathfrak{s}_0 + \hat{z}\,\mathfrak{r}_{\hat{z}})\Psi^{(\hat{z})}_{s_k}, \qquad \mathfrak{c}_k^0 = q\,\mathfrak{s}_0 \Upsilon_{s_k},$$

and (10.13) gives the claim.

For the induction step, let $j < k$, $s = s_j$, and $q > 0$, and take $\hat{z}$ so small that $q \geq 2\hat{z}e^{3T}$. By (10.34) and (10.45), it suffices to show that

$$\lim_{\hat{z}\to 0} e^{-s\mathsf{G}_{1,\hat{z}}}\mathfrak{c}_{j+1} = \mathfrak{s}_0^{k-j}\mathcal{T}_s\varphi_{j+1}.$$

Let $q_u^0 = qe^{g_1(u)}$, with g_1 built from the Brownian motion β of Step 1, so that

$$\mathfrak{s}_0^{k-j}\mathcal{T}_s\varphi_{j+1}(q) = \mathbb{E}\big[e^{-\kappa^0}\mathfrak{c}_{j+1}^0(q_s^0)\big], \qquad \kappa^0 = \int_0^s q_u^0\,\mathrm{d}u.$$

We compare this with (10.43) for $f = \mathfrak{c}_{j+1}$ and $r = s$, and we set

$$\kappa = \int_0^{\tau_2\wedge s} (q_u - \hat{z}F_u)\,\mathrm{d}u.$$

We first consider the event $\{\tau_2 \leq s\}$. There, the integrand of the frozen expectation is bounded by $C(1+q_s^0)^k$, and the integrand of (10.43) by a constant, by (10.44) at $B = 2$ and (10.42). The probability of this event tends to zero by (10.39), so the Cauchy–Schwarz inequality shows that its contribution tends to zero.

We now turn to the event $\{s < \tau_2\}$, on which the difference of the integrands is bounded by

$$\big(\|\mathcal{E}_s - \mathrm{Id}\| + |\kappa - \kappa^0|\big)\|\mathfrak{c}_{j+1}(q_s, \mathbf{e}_s)\| + \|\mathfrak{c}_{j+1}(q_s, \mathbf{e}_s) - \mathfrak{c}_{j+1}^0(q_s)\| + \|\mathfrak{c}_{j+1}^0(q_s) - \mathfrak{c}_{j+1}^0(q_s^0)\|.$$

We bound the three terms in turn. The proof of (10.15) rests on the equation of q_u in (10.1) and on the bounds (10.4). Both hold for the present process up to τ_2, since its pair (q_u, F_u) solves (10.1) there, and the same proof shows that

$$\mathbb{E}\sup_{u\leq s\wedge\tau_2}|q_u - q_u^0|^2 \leq C\hat{z}^2, \qquad \mathbb{E}\big[|\kappa - \kappa^0|^2; s < \tau_2\big] \leq C\hat{z}^2.$$

By (10.42), the Cauchy–Schwarz inequality and (10.38), we have

$$\mathbb{E}\big[\|\mathcal{E}_s - \mathrm{Id}\|^2; s < \tau_2\big] \leq C\hat{z}^2\int_0^s \mathbb{E}q_{u\wedge\tau_2}^{-2}\,\mathrm{d}u \leq C\hat{z}^2q^{-2}.$$

Together with (10.45), the moment bounds of Lemma 10.13 and the Cauchy–Schwarz inequality, these bounds show that the expectation of the first term tends to zero.

The expectation of the second term tends to zero by the induction hypothesis on the event $\{q_s \in [R^{-1}, R]\}$. On the complementary event, we use (10.45), the bound on φ_{j+1} and the tightness (10.40), and we let $R \to \infty$ after $\hat{z} \to 0$.

The third term is at most

$$\|\mathfrak{s}_0\|^{k-j} \sup |\varphi'_{j+1}| \, |q_s - q_s^0|,$$

with the supremum over the segment between q_s and q_s^0. Its expectation is $O(\hat{z})$, by the bound on φ'_{j+1}, the moment bounds, and the coupling. All these bounds are uniform in the stated parameters. We multiply by $\hat{z}\mathfrak{s}$ and use (10.34). This completes the induction step, since

$$q\mathfrak{s}_0 \cdot \mathfrak{s}_0^{k-j} \mathcal{T}_s \varphi_{j+1} = \mathfrak{s}_0^{k-j+1} \varphi_j = \mathfrak{c}_j^0.$$

Step 4 (integration). Let $\chi \in C^\infty(\mathbb{R})$ satisfy $\chi = 1$ on $(-\infty, 3]$ and $\chi = 0$ on $[4, \infty)$. The superfunction $\chi(d_1)$ is invariant, with radial representative $\chi(d)$, by (10.17). Since $B \geq d - 1$, the function $1 - \chi(d)$ vanishes on $\{B < 2\}$, and $\chi(d)$ vanishes outside the compact set $\{d \leq 4\}$. We split

$$\mathcal{B}_1\big(\Psi_{s_0}^{(\hat{z})}, \mathfrak{c}_1\big) = \int \chi(d_1) \Psi_{s_0}^{(\hat{z})} \mathfrak{c}_1 + \int \big(1 - \chi(d_1)\big) \Psi_{s_0}^{(\hat{z})} \mathfrak{c}_1.$$

Proposition 5.15(e) and (10.36) bound the coefficients of the first integrand in regular coordinates by $C\hat{z}^k$, on its compact support. Hence, by (5.11), the first integral is $O(\hat{z}^k)$, which is $o(\hat{z})$ since $k \geq 2$.

The second integrand is supported in $\{B \geq 2\}$, and we use (10.18). The functions $\Psi_{s_0}^{(\hat{z})}$ and $\chi(d)$ have no odd variables in the polar chart, and

$$\big[(1 - \chi(d)) \Psi_{s_0}^{(\hat{z})} \mathfrak{c}_1\big]_{\mathrm{top}} = (1 - \chi(d)) \Psi_{s_0}^{(\hat{z})} [\mathfrak{c}_1]_{\mathrm{top}}.$$

We substitute $q = \hat{z}B$, for which

$$(B - F)^{-2} \, \mathrm{d}B = \hat{z}(q - \hat{z}F)^{-2} \, \mathrm{d}q, \qquad d = \frac{q - \hat{z}F}{\hat{z}}.$$

We obtain

$$\hat{z}^{k-1} \mathfrak{p}_k^{X,z} + o(1) = \int_{2\hat{z}}^\infty \int_{-1}^1 \Big(1 - \chi\Big(\frac{q - \hat{z}F}{\hat{z}}\Big)\Big) \frac{\Psi_{s_0}^{(\hat{z})}(q, F)}{(q - \hat{z}F)^2} \big\langle [\mathfrak{c}_1]_{\mathrm{top}} \big\rangle (q, F) \, \mathrm{d}F \, \mathrm{d}q,$$

where $\langle \cdot \rangle$ denotes the average over θ_{B} and θ_{F}.

For $q \geq 2\hat{z}$ and $F \in [-1, 1]$, we have

$$q - \hat{z}F \geq q - \hat{z} \geq \frac{q}{2}.$$

Therefore Lemma 10.3 with $t_0 = s_*$ and (10.45) bound the integrand by

$$Cq^{-2}(\hat{z}+q)^k\,\bar{\Upsilon}_{s_*,T}(q) \le Cq^{k-2}\bar{\Upsilon}_{s_*,T}(q),$$

which is integrable since $k \ge 2$. By Step 3 and (10.13), the integrand converges to

$$q^{-2}\Upsilon_{s_0}(q)\big\langle[\mathfrak{s}_0^k]_{\mathrm{top}}\big\rangle\varphi_1(q),$$

uniformly for q in compact subsets of $(0,\infty)$, $F \in [-1,1]$, and the parameters. Splitting the integral over q at R^{-1} and R, as in the proof of Theorem 10.4, we conclude that, uniformly in the parameters,

$$\lim_{\hat{z}\downarrow 0}\hat{z}^{k-1}\mathfrak{p}_k^{X,z} = 2\,\big\langle[\mathfrak{s}_0^k]_{\mathrm{top}}\big\rangle\int_0^\infty q^{-2}\Upsilon_{s_0}(q)\varphi_1(q)\,\mathrm{d}q, \tag{10.46}$$

where the factor two is the integral over F.

It remains to compute the constant. Let $t = 1+\cos\theta_{\mathrm{B}}$. Since $n_{\mathrm{R}}^2 = n_{\mathrm{A}}^2 = 0$, the binomial expansion of

$$\mathfrak{s}_0^k = \Big(t + \frac{t}{2}(n_{\mathrm{R}}+n_{\mathrm{A}}) + \frac{t-1}{4}n_{\mathrm{R}}n_{\mathrm{A}}\Big)^k$$

gives

$$[\mathfrak{s}_0^k]_{\mathrm{top}} = kt^{k-1}\frac{t-1}{4} + \binom{k}{2}t^{k-2}\frac{t^2}{2} = \frac{k}{4}t^{k-1}(kt-1).$$

The averages of the powers of t are

$$\frac{1}{2\pi}\int_0^{2\pi}(1+\cos\theta)^n\,\mathrm{d}\theta = 2^{-n}\binom{2n}{n}, \qquad n \ge 0,$$

and we find

$$\big\langle[\mathfrak{s}_0^k]_{\mathrm{top}}\big\rangle = \frac{k}{4}\Big(k\,2^{-k}\binom{2k}{k} - 2^{1-k}\binom{2k-2}{k-1}\Big) = \frac{k(2k-3)!!}{2(k-2)!}.$$

In the last step, we used the identities

$$k\binom{2k}{k} = 2(2k-1)\binom{2k-2}{k-1}, \qquad \binom{2k-2}{k-1} = \frac{2^{k-1}(2k-3)!!}{(k-1)!}.$$

Finally, unwinding (10.37), we find

$$\varphi_1 = q\,\mathcal{T}_{s_1}q\cdots q\,\mathcal{T}_{s_{k-1}}q\,\Upsilon_{s_k},$$

and the first line of (10.8) identifies the integral in (10.46) with $(k-2)!\,X^{1-k}M_k^X$. Thus

$$\lim_{z\downarrow 0}z^{k-1}\mathfrak{p}_k^{X,z} = X^{k-1}\lim_{\hat{z}\downarrow 0}\hat{z}^{k-1}\mathfrak{p}_k^{X,z} = X^{k-1}\cdot 2\cdot\frac{k(2k-3)!!}{2(k-2)!}\cdot\frac{(k-2)!}{X^{k-1}}M_k^X = k(2k-3)!!\,M_k^X,$$

which is (10.35). □

SECTION 11

Local intensities and the inverse participation ratio

In this section, we prove Theorems 1.14 and 1.15. In Lemma 11.4, we also prove that $z\widetilde{K}_X(x;z) \to 2\mathcal{C}_X(x,x)$ as $z \downarrow 0$, which, together with Theorem 10.4, identifies the eigenvector correlations as the residues of the kernels of Definition 1.4.

The proof has three steps. First, Theorem 7.6(b) expresses the limits of the expected products of two resolvent entries at a fixed damping as pairings in the rank-one sigma model, with explicit multipliers at the sites. We evaluate these pairings with the supersymmetric localization of Lemma 5.19 and with the rank-one calculus of Section 6.4, which reduces retarded-covariant matrices of superfunctions to a value at the corner of D and a function that evolves by the radial diffusion of Section 6.2. As explained in Remark 7.9, one site and two distinct sites have different multipliers, and this is the origin of the contact kernel $\widetilde{K}_X$. For an infinite wire, the corresponding distinction between coincident and nearby points, with the weight $B/(B-F)$ of (1.7), was found by Skvortsov and Ostrovsky [126, (7)–(8)]. Second, the extraction theorem Theorem 9.6 converts the limits at fixed damping into the expected densities of $N^2 \sum_\alpha w_\alpha(i) w_\alpha(j) \delta_{x_\alpha}$, which give the profile. The full two-energy intensity requires the separation measure of the pair also away from its atom. We identify it from its values at coinciding energies, using that it is even, which we prove in Lemma 11.8. Third, the inverse participation ratio is a sum over all sites, and the sites near the ends of the chain are controlled by Theorem 7.6(c). Its limit is evaluated exactly in Proposition 11.12.

In the physics literature, eigenvector statistics of band matrices in this regime were derived from the one-dimensional supersymmetric sigma model by Fyodorov and Mirlin; see [94, Section 3.2] and the references there. In particular, the limit $2 + 2X/3$ of $N\langle \mathrm{IPR}(u_\alpha)\rangle_{E,r}$ agrees with their prediction [54, 56], and its expansion to second order in X appears in [57, (12), (17)].

Conventions. We keep the conventions of Section 9, and in particular the energies $E_* < E_{\mathrm{mid}} < E_{\mathrm{out}}$ of (7.22). We write $s = s_E$, so that $\kappa_E^2 + s^2 = 1$ with $\kappa_E = E/2$, and we set $z = 2\pi\varepsilon$ for a damping $\varepsilon > 0$. We fix $\delta \in (0, 1/2)$, and we call a site i *interior* if $\delta \le i/N \le 1 - \delta$. Statements are along sequences as in Section 9; uniform versions follow by Remark 9.7 or by the same argument.

11.1 Local sources

We use Theorem 7.6 at rank $k = 1$, with $\mathbf{u} = 0$ and $\mathbf{b} = \varepsilon$, and with local sources at one or two interior sites. We abbreviate $T(\mathbf{Y}) = T_N^{(1)}(0, \varepsilon; \mathbf{Y})$ for the local source function of Definition 7.5, and we write

$$z_\pm = E \pm \mathrm{i}\eta, \qquad \eta = \frac{\varepsilon}{a_N}, \qquad G^\pm = G(z_\pm).$$

Recall the supermatrix $\mathcal{G}(Q) = -\kappa_E \mathrm{Id} + \mathrm{i}sQ$ of (7.16) and the multipliers $\mathrm{j}^{(\alpha)}$ of (7.17). For a $(2|2)$ supermatrix M, whose indices are a flavor (R or A) and a type (B or F), we write $M_{\mathrm{RA}}^{\mathrm{BB}}$ for the entry with boson-boson type, retarded row, and advanced column, and similarly for the other entries. We let E_{RR} and E_{AA} denote the matrix units of the two flavors.

Lemma 11.1. *Let $i < j$ be sites, and let t, t' be complex parameters. The following derivatives at $\mathbf{Y} = 0$ hold, where the sources not listed vanish, and each line gives the multipliers at the sites, in the notation of* (7.19).

(a) For $Y^{\mathrm{F}} = tE_{\mathrm{RR}}$ at i, we have $\partial_t T = -\mathbb{E}G_{ii}^+$, with the multiplier $-\mathcal{G}_{\mathrm{RR}}^{\mathrm{FF}}$.

(b) For $Y^{\mathrm{F}} = tE_{\mathrm{RR}}$ at i and $Y^{\mathrm{F}} = t'E_{\mathrm{AA}}$ at j, we have $\partial_t \partial_{t'} T = \mathbb{E}G_{ii}^+ G_{jj}^-$, with the multipliers $-\mathcal{G}_{\mathrm{RR}}^{\mathrm{FF}}$ at i and $-\mathcal{G}_{\mathrm{AA}}^{\mathrm{FF}}$ at j, whose signs cancel in the product. Exchanging E_{RR} and E_{AA} gives $\mathbb{E}G_{ii}^- G_{jj}^+$, with the multipliers exchanged.

(c) For $Y^{\mathrm{F}} = tE_{\mathrm{RR}}$ at i and $Y^{\mathrm{F}} = t'E_{\mathrm{RR}}$ at j, we have $\partial_t \partial_{t'} T = \mathbb{E}(G_{ii}^+ G_{jj}^+ - G_{ij}^+ G_{ji}^+)$, with the multiplier $-\mathcal{G}_{\mathrm{RR}}^{\mathrm{FF}}$ at both sites. For $Y^{\mathrm{B}} = tE_{\mathrm{RR}}$ at i and $Y^{\mathrm{B}} = t'E_{\mathrm{RR}}$ at j, we have $\partial_t \partial_{t'} T = \mathbb{E}(G_{ii}^+ G_{jj}^+ + G_{ij}^+ G_{ji}^+)$, with the multiplier $\mathcal{G}_{\mathrm{RR}}^{\mathrm{BB}}$ at both sites.

(d) For $Y^{\mathrm{B}} = \mathrm{diag}(t, t')$ at i, we have $\partial_t \partial_{t'} T = \mathbb{E}G_{ii}^+ G_{ii}^-$, with the multiplier

$$\mathcal{G}_{\mathrm{RR}}^{\mathrm{BB}} \mathcal{G}_{\mathrm{AA}}^{\mathrm{BB}} + \mathcal{G}_{\mathrm{RA}}^{\mathrm{BB}} \mathcal{G}_{\mathrm{AR}}^{\mathrm{BB}}. \tag{11.1}$$

For $Y^{\mathrm{B}} = tE_{\mathrm{RR}}$ at i, we have $\frac{1}{2}\partial_t^2 T = \mathbb{E}(G_{ii}^+)^2$, with the multiplier $(\mathcal{G}_{\mathrm{RR}}^{\mathrm{BB}})^2$.

Proof. At $\mathbf{u} = 0$ and $\mathbf{b} = \varepsilon$, the energies and the matrices of (7.13) are

$$z_\pm^{\mathrm{B}} = z_{1,\pm}^{\mathrm{F}} = z_\pm, \qquad Z_{\mathrm{F}} = Z_{\mathrm{B}} = \mathrm{diag}(z_+, z_-).$$

With the flavors ordered as R, A, we have $tE_{\mathrm{RR}} = \mathrm{diag}(t, 0)$ and $t'E_{\mathrm{AA}} = \mathrm{diag}(0, t')$. Therefore (a) and the first identity in (b) are the two identities of (7.15), and the first identity in (d) is the first identity of Remark 7.9. The second identity in (b) is the computation of (7.15) with the two flavors exchanged.

For (c) and the last identity in (d), we compute in the same way. All sources listed are diagonal in the flavors, so that the numerator and the denominator in (7.14) factor over the two flavors, and the factors without sources cancel. The sources are added

to z, as in Definition 7.5. For a matrix $A = te_ie_i^* + t'e_je_j^*$ and $z \in \{z_+, z_-\}$, we have $\det(z - H + A) = \det(z - H)\det(\mathrm{Id} - G(z)A)$, since $G(z) = -(z-H)^{-1}$, and

$$\det\big(\mathrm{Id} - G(z)A\big) = 1 - tG_{ii} - t'G_{jj} + tt'(G_{ii}G_{jj} - G_{ij}G_{ji}), \tag{11.2}$$

with all entries at z. Fermionic sources appear in the numerator, and the first part of (c) follows from (11.2) directly. Bosonic sources appear in the denominator, and expanding the reciprocal of (11.2) to second order, we find

$$\partial_t\partial_{t'}\det\big(\mathrm{Id} - G(z)A\big)^{-1}\big|_{t=t'=0} = -(G_{ii}G_{jj} - G_{ij}G_{ji}) + 2G_{ii}G_{jj} = G_{ii}G_{jj} + G_{ij}G_{ji},$$

which is the second part of (c). When $Y^{\mathrm{B}} = tE_{\mathrm{RR}}$, the denominator contains the factor $1 - tG_{ii}^+$, and the last identity in (d) follows from $\frac{1}{2}\partial_t^2(1 - tG_{ii}^+)^{-1} = (G_{ii}^+)^2$ at $t = 0$.

The multipliers come from the block formula for the superdeterminant of Lemma 3.11. The supermatrix $\mathrm{Id} - \mathcal{G}Y$ of (7.17) is block upper triangular when $Y = \mathrm{diag}(0 \mid Y^{\mathrm{F}})$ and block lower triangular when $Y = \mathrm{diag}(Y^{\mathrm{B}} \mid 0)$, which gives

$$\mathrm{SDet}\big(\mathrm{Id} - \mathcal{G}\,\mathrm{diag}(0 \mid Y^{\mathrm{F}})\big)^{-1} = \det\big(\mathrm{Id} - \mathcal{G}^{\mathrm{FF}}Y^{\mathrm{F}}\big),$$
$$\mathrm{SDet}\big(\mathrm{Id} - \mathcal{G}\,\mathrm{diag}(Y^{\mathrm{B}} \mid 0)\big)^{-1} = \det\big(\mathrm{Id} - \mathcal{G}^{\mathrm{BB}}Y^{\mathrm{B}}\big)^{-1},$$

where $\mathcal{G}^{\mathrm{FF}}$ and $\mathcal{G}^{\mathrm{BB}}$ are the diagonal blocks of $\mathcal{G}$. Distinct sites carry separate multipliers, and the multipliers follow by differentiating these expressions. The multiplier (11.1) for $Y^{\mathrm{B}} = \mathrm{diag}(t, t')$ is the second identity of Remark 7.9. The fermionic factors have the first derivatives $-\mathcal{G}^{\mathrm{FF}}_{\mathrm{RR}}$ and $-\mathcal{G}^{\mathrm{FF}}_{\mathrm{AA}}$, and these signs cancel in pairs in (b) and (c). The same holds on the matrix side, where each fermionic first derivative carries the factor -1. □

11.2 Pairings at rank one

We now evaluate the pairings that Theorem 7.6(b) produces from the multipliers of Lemma 11.1, with the rank-one calculus of Section 6.4 at $z = 2\pi\varepsilon$. Recall from there the generator and the semigroup

$$\mathsf{G}_{X,z} = X\mathsf{C}_1 + z\,d_1(Q), \qquad d_1(Q) = \tfrac{1}{2}\,\mathrm{STr}(\Lambda Q), \qquad U_t = e^{-t\mathsf{G}_{X,z}}.$$

At $\mathbf{u} = 0$ and $\mathbf{b} = \varepsilon$, we have $\Gamma(0, \varepsilon\mathbf{1}) = \varepsilon\Lambda$, so that $\pi\,\mathrm{STr}(Q\Gamma) = z\,d_1(Q)$, and $\mathsf{G}_{X,z}$ is the generator $\mathsf{G}_{X,\Gamma}$ of (7.19). Using the symmetry of $\mathcal{B}_1$ under U_t of Proposition 5.15(c), we see that the limits in (7.19) with one site i and with two sites $i < j$, where $i/N \to x$ and $j/N \to y$, are

$$\mathcal{B}_1\big(U_x 1, \mathsf{j}\,U_{1-x}1\big), \qquad \mathcal{B}_1\big(U_x 1, \mathsf{j}\,U_{y-x}\,\mathsf{j}'\,U_{1-y}1\big), \tag{11.3}$$

where j and j' are the multipliers at the sites, which act by multiplication.

Recall that $U_t 1$ is an invariant superfunction with the radial representative Ψ_t and the value one at the corner, by fact (iii) of Section 6.4, and we denote it by Ψ_t, as in the proof of Lemma 6.15. We write $f_0 = f_\varnothing(\Lambda)$ for the value at the corner of a superfunction f. For the multipliers of Lemma 11.1(a)–(c), which are diagonal entries of $\mathcal{G}$, the pairings are those of Lemma 6.13. For the contact multiplier of Lemma 11.1(d), we use the radial integration formula Lemma 6.3, with the top coefficients $[A]_{\mathrm{top}}$ and their phase averages $\langle A\rangle_{\mathrm{top}}$ of Lemma 6.1(c), and the localization fact (i).

The contact multiplier contains the off-diagonal entries of the boson-boson block. By Lemma 6.1(c), the average over the phases of their top coefficient is

$$\big\langle Q^{\mathrm{BB}}_{\mathrm{RA}} Q^{\mathrm{BB}}_{\mathrm{AR}} \big\rangle_{\mathrm{top}} = -(B^2 - F^2) = -d(B+F). \tag{11.4}$$

Only this average enters the radial integration formula Lemma 6.3.

Lemma 11.2. *Let $\delta \le x \le 1-\delta$. Then*

$$\begin{aligned} \mathcal{B}_1\Big(\Psi_x, \big(\mathcal{G}^{\mathrm{BB}}_{\mathrm{RR}}\mathcal{G}^{\mathrm{BB}}_{\mathrm{AA}} + \mathcal{G}^{\mathrm{BB}}_{\mathrm{RA}}\mathcal{G}^{\mathrm{BB}}_{\mathrm{AR}}\big)\Psi_{1-x}\Big) &= 1 + 2s^2 \widetilde{K}_X(x;z), \\ \mathcal{B}_1\big(\Psi_x, (\mathcal{G}^{\mathrm{BB}}_{\mathrm{RR}})^2 \Psi_{1-x}\big) &= (-\kappa_E + \mathrm{i}s)^2. \end{aligned} \tag{11.5}$$

Proof. Let $\mathfrak{c}$ denote the multiplier (11.1). Inserting $\mathcal{G} = -\kappa_E \mathrm{Id} + \mathrm{i}sQ$, we have

$$\mathfrak{c} = \kappa_E^2 - \mathrm{i}s\kappa_E\big(Q^{\mathrm{BB}}_{\mathrm{RR}} + Q^{\mathrm{BB}}_{\mathrm{AA}}\big) - s^2\big(Q^{\mathrm{BB}}_{\mathrm{RR}} Q^{\mathrm{BB}}_{\mathrm{AA}} + Q^{\mathrm{BB}}_{\mathrm{RA}} Q^{\mathrm{BB}}_{\mathrm{AR}}\big).$$

At Λ, we have $Q^{\mathrm{BB}}_{\mathrm{RR}} = 1$, $Q^{\mathrm{BB}}_{\mathrm{AA}} = -1$, and $Q^{\mathrm{BB}}_{\mathrm{RA}} = 0$, so that

$$\mathfrak{c}_0 = \kappa_E^2 - \mathrm{i}s\kappa_E(1-1) + s^2 = 1.$$

By Lemma 6.1(b), the second term of $\mathfrak{c}$ is a sum of terms with n_{R} or n_{A} alone, and $[Q^{\mathrm{BB}}_{\mathrm{RR}} Q^{\mathrm{BB}}_{\mathrm{AA}}]_{\mathrm{top}} = -d^2$. Combining this with (11.4), we find

$$\langle \mathfrak{c} \rangle_{\mathrm{top}} = s^2\big(d^2 + d(B+F)\big) = 2s^2 Bd.$$

The functions Ψ_x and Ψ_{1-x} are invariant, so they depend only on (B, F) in the polar chart, and

$$\frac{\langle \Psi_x \mathfrak{c} \Psi_{1-x}\rangle_{\mathrm{top}}}{d^2} = 2s^2 \frac{B}{d}\, \Psi_x \Psi_{1-x}.$$

The right side is integrable on D. Indeed, B/d is integrable near the corner, and $\Psi_x \Psi_{1-x} \le C(1+B)^{-3}$ by Lemma 6.6(c), for every $z > 0$. Applying Lemma 6.3, with the corner term $\mathfrak{c}_0 = 1$, and using the definition (1.7) of $\widetilde{K}_X$, we obtain the first identity in (11.5). The second integrand is advanced-invariant, with the value $(-\kappa_E + \mathrm{i}s)^2$ at Λ, and fact (i) of Section 6.4 applies. □

Remark 11.3. The constant $2s^2$ and the weight B/d can be checked against the GUE. If we set $X = 0$ in Definition 1.4, then $\Psi_t = e^{-tzd}$, and a direct computation gives

$$K_0(x,y;z) = \frac{1-e^{-2z}}{z^2} = 2I_{\mathrm{GUE}}(z), \qquad \widetilde{K}_0(x;z) = \int_D \frac{B}{d} e^{-zd}\,\mathrm{d}B\,\mathrm{d}F = \frac{1}{z} + I_{\mathrm{GUE}}(z).$$

A single site carries the additional term $1/z$, whose residue doubles the coincident-eigenvalue contribution, in agreement with the exact identity $\mathbb{E}|u(i)|^4 = \frac{2N}{N+1}(\mathbb{E}|u(i)|^2)^2$ for Haar-distributed eigenvectors, whose factor tends to two as $N \to \infty$.

The contact kernel and the diagonal value of K_X have the same residue at $z = 0$, up to the factor two of Theorem 10.4.

Lemma 11.4. *Let $0 < X_- \le X_+ < \infty$. Uniformly for $X \in [X_-, X_+]$ and $\delta \le x \le 1-\delta$,*

$$\lim_{z \downarrow 0} z\widetilde{K}_X(x;z) = 2\mathcal{C}_X(x,x). \tag{11.6}$$

Proof. The definitions (1.6) at $y = x$ and (1.7) give

$$\widetilde{K}_X(x;z) - K_X(x,x;z) = \int_D \frac{F}{d}\Psi_x\Psi_{1-x}\,\mathrm{d}B\,\mathrm{d}F.$$

Let $0 < z \le X_-$. Recall that $\Psi_x\Psi_{1-x}$ takes values in $[0,1]$, and note that $1/d$ is integrable on $[1,2]\times[-1,1]$. For $B \ge 2$, we have

$$\frac{|F|}{d} \le \frac{1}{B-1} \le \frac{2}{B}, \qquad \Psi_x\Psi_{1-x} \le \min\bigl(1, \bar{\Upsilon}(zB/X)\bigr),$$

where the second bound is (10.11). Substituting $q = zB/X$, we obtain

$$\bigl|\widetilde{K}_X(x;z) - K_X(x,x;z)\bigr| \le C + 4\int_{2z/X}^{\infty} \min\bigl(1, \bar{\Upsilon}(q)\bigr)\frac{\mathrm{d}q}{q} \le C\Bigl(1 + \log\frac{X_+}{z}\Bigr),$$

where we used

$$\int_1^{\infty} \bar{\Upsilon}(q)\frac{\mathrm{d}q}{q} \le \int_0^{\infty} \bar{\Upsilon}(q)\,\mathrm{d}q < \infty,$$

which holds by Lemma 10.3. Multiplying by z and letting $z \downarrow 0$, we see that $z(\widetilde{K}_X - K_X(x,x;\cdot)) \to 0$ uniformly. Now Theorem 10.4 at $y = x$ implies (11.6). □

11.3 Local limits at positive damping

We now combine Theorem 7.6(b) with the pairings above. The result is the limit of the local densities at a fixed damping, at one or two interior sites, with no lower bound on the distance between the sites.

Proposition 11.5. *Let $|E| \le E_{\mathrm{out}}$, and let i, j be interior sites such that $i/N \to x$, $j/N \to y$, $X_N(E) \to X \in (0,\infty)$, and $\mathbf{1}_{i=j}$ is constant. Let $\varepsilon > 0$ and $z = 2\pi\varepsilon$. Then $\mathbb{E}G_{ii}^+ - (-\kappa_E + \mathrm{i}s) \to 0$ and $\mathbb{E}\ell_{i,N}^\varepsilon(0) \to 1$. If $i \neq j$, then*

$$\mathbb{E}G_{ii}^+ G_{jj}^- - 1 - s^2 K_X(x,y;z) \longrightarrow 0, \qquad \mathbb{E}G_{ii}^+ G_{jj}^+ - (-\kappa_E + \mathrm{i}s)^2 \longrightarrow 0, \tag{11.7}$$

$$\mathbb{E}\ell_{i,N}^\varepsilon(0)\ell_{j,N}^\varepsilon(0) \longrightarrow 1 + \tfrac{1}{2}K_X(x,y;z). \tag{11.8}$$

If $i = j$, then

$$\begin{aligned} \mathbb{E}G_{ii}^+ G_{ii}^- - 1 - 2s^2\widetilde{K}_X(x;z) &\to 0, \\ \mathbb{E}(G_{ii}^+)^2 - (-\kappa_E + \mathrm{i}s)^2 &\to 0, \\ \mathbb{E}\ell_{i,N}^\varepsilon(0)^2 &\to 1 + \widetilde{K}_X(x;z). \end{aligned} \tag{11.9}$$

Proof. Passing to subsequences, we may assume that E converges, so that κ_E and s converge. We first consider two distinct sites, and we assume $i < j$ and $x \le y$; the case $i > j$ is the same, with the second form of Lemma 11.1(b) and of (6.26), and with K_X extended symmetrically. By Theorem 7.6(b), Lemma 11.1(b), and (11.3), $\mathbb{E}G_{ii}^+ G_{jj}^-$ has the limit

$$\begin{aligned} &\mathcal{B}_1\big(\Psi_x, (-\kappa_E + \mathrm{i}sQ_{\mathrm{RR}}^{\mathrm{FF}})U_{y-x}(-\kappa_E + \mathrm{i}sQ_{\mathrm{AA}}^{\mathrm{FF}})\Psi_{1-y}\big) \\ &\qquad = \kappa_E^2 - \mathrm{i}s\kappa_E(1-1) + (\mathrm{i}s)^2\big(-1 - K_X(x,y;z)\big) = 1 + s^2 K_X(x,y;z), \end{aligned}$$

where we expanded the product, used Lemma 6.13, and used $\kappa_E^2 + s^2 = 1$ in the last step.

Next, let $c \in \{\mathrm{BB}, \mathrm{FF}\}$. Expanding in the same way and using (6.27) and (6.28), we find

$$\mathcal{B}_1\big(\Psi_x, (-\kappa_E + \mathrm{i}sQ_{\mathrm{RR}}^{c})U_{y-x}(-\kappa_E + \mathrm{i}sQ_{\mathrm{RR}}^{c})\Psi_{1-y}\big) = \kappa_E^2 - 2\mathrm{i}s\kappa_E - s^2 = (-\kappa_E + \mathrm{i}s)^2.$$

By Lemma 11.1(c), with $c = \mathrm{FF}$ and $c = \mathrm{BB}$, the expectations $\mathbb{E}(G_{ii}^+ G_{jj}^+ - G_{ij}^+ G_{ji}^+)$ and $\mathbb{E}(G_{ii}^+ G_{jj}^+ + G_{ij}^+ G_{ji}^+)$ converge to $(-\kappa_E + \mathrm{i}s)^2$. Taking their average, we find the second limit in (11.7). For one site, the first two limits in (11.9) follow from Lemma 11.1(d) and Lemma 11.2. Similarly, Lemma 11.1(a) and (6.27) imply that $\mathbb{E}G_{ii}^+ \to -\kappa_E + \mathrm{i}s$.

It remains to pass to the local densities. Recall that $\ell_{i,N}^\varepsilon(0) = s^{-1}\operatorname{Im} G_{ii}^+$, and that $\overline{G_{jj}(z_+)} = G_{jj}(z_-)$ since H is Hermitian. The identity $\operatorname{Im} a \operatorname{Im} b = \frac{1}{2}\operatorname{Re}(a\bar{b} - ab)$ gives

$$\mathbb{E}\ell_{i,N}^\varepsilon(0)\ell_{j,N}^\varepsilon(0) = \frac{1}{2s^2}\operatorname{Re}\big(\mathbb{E}G_{ii}^+ G_{jj}^- - \mathbb{E}G_{ii}^+ G_{jj}^+\big).$$

The kernels are real for real z, and

$$\operatorname{Re}(-\kappa_E + \mathrm{i}s)^2 = \kappa_E^2 - s^2 = 1 - 2s^2.$$

Inserting the limits above, we find for $i \neq j$ and $i = j$, respectively,

$$\frac{1 + s^2 K_X - (1 - 2s^2)}{2s^2} = 1 + \frac{1}{2}K_X(x,y;z), \qquad \frac{1 + 2s^2\widetilde{K}_X - (1 - 2s^2)}{2s^2} = 1 + \widetilde{K}_X(x;z),$$

which are (11.8) and the last limit in (11.9). Finally, $\mathbb{E}\ell^{\varepsilon}_{i,N}(0) = s^{-1}\operatorname{Im}\mathbb{E}G^{+}_{ii}$ tends to one. □

We also need these limits at shifted centers, where the damping of Theorem 7.6 is not fixed. The next lemma provides the necessary equicontinuity.

Lemma 11.6. *For sites i, j, an energy $|E| \le E_{\mathrm{mid}}$, $u \in \mathbb{R}$, and $\varepsilon > 0$, let*

$$F_N(u,\varepsilon) = \mathbb{E}G_{ii}\Big(E + \frac{u+\mathrm{i}\varepsilon}{a_N}\Big)G_{jj}\Big(E + \frac{u-\mathrm{i}\varepsilon}{a_N}\Big).$$

Then F_N is Lipschitz on compact subsets of $\mathbb{R}\times(0,\infty)$, uniformly in N, the sites, and E. If i, j, x, y, and X are as in Proposition 11.5 and $E = E_N \to E_0$, then, locally uniformly in (u,ε),

$$F_N(u,\varepsilon) \longrightarrow \begin{cases} 1 + s^2_{E_0} K_X(x,y;2\pi\varepsilon), & i \neq j, \\ 1 + 2s^2_{E_0}\widetilde{K}_X(x;2\pi\varepsilon), & i = j. \end{cases}$$

Proof. For $w = E + (u+\mathrm{i}\varepsilon)/a_N$, we have

$$\partial_u G_{ii}(w) = \frac{(G(w)^2)_{ii}}{a_N}, \qquad \partial_\varepsilon G_{ii}(w) = \frac{\mathrm{i}(G(w)^2)_{ii}}{a_N}.$$

By the Cauchy–Schwarz inequality and the spectral decomposition,

$$\big|(G(w)^2)_{ii}\big| \le \sum_k |G_{ik}(w)|^2 = \frac{\operatorname{Im} G_{ii}(w)}{\operatorname{Im} w} = \frac{a_N}{\varepsilon}\operatorname{Im} G_{ii}(w). \tag{11.10}$$

We first show that

$$\phi(\varepsilon) = \Big(\mathbb{E}\big|G_{jj}(E' + \mathrm{i}\varepsilon/a_N(E'))\big|^2\Big)^{1/2}$$

is bounded for ε in compact subsets of $(0,\infty)$, uniformly in N, j, and $|E'| \le E_{\mathrm{out}}$. Using (11.10) and the Cauchy–Schwarz inequality, we find

$$\big|\partial_\varepsilon \phi(\varepsilon)^2\big| \le \frac{2}{\varepsilon}\,\phi(\varepsilon)\,\Big(\mathbb{E}\big(\operatorname{Im} G_{jj}(E' + \mathrm{i}\varepsilon/a_N(E'))\big)^2\Big)^{1/2}.$$

The last factor is bounded by Lemma 9.2, since $s^{-1}_{E'}\operatorname{Im} G_{jj}(E' + \mathrm{i}\varepsilon/a_N(E')) = \ell^{\varepsilon}_{j,N}(0)$ at the energy E'. Therefore ϕ is Lipschitz on compact sets. At $\varepsilon = 1$, ϕ is bounded by Theorem 7.6(c). The recentering of the proof of Lemma 9.2 extends this bound to the arguments in F_N.

Let $w_\pm = E + (u \pm \mathrm{i}\varepsilon)/a_N$. Using (11.10) once more, at w_+ and at w_-, where $\operatorname{Im} w_- = -\varepsilon/a_N$, we find

$$|\partial_u F_N| + |\partial_\varepsilon F_N| \le \frac{2}{\varepsilon}\Big(\mathbb{E}\big[\operatorname{Im} G_{ii}(w_+)\,|G_{jj}(w_-)|\big] + \mathbb{E}\big[|G_{ii}(w_+)|\,\operatorname{Im} G_{jj}(w_+)\big]\Big),$$

where we used that $|\operatorname{Im} G_{jj}(w_-)| = \operatorname{Im} G_{jj}(w_+)$, since $w_- = \overline{w_+}$ and $G_{jj}(\bar{w}) = \overline{G_{jj}(w)}$. The right side is bounded on compact sets, by the Cauchy–Schwarz inequality and the bounds just proved, and the Lipschitz bound follows.

For the convergence, let $E' = E + u/a_N$ and $\varepsilon' = \varepsilon\rho(E')/\rho(E)$, so that

$$E + \frac{u \pm \mathrm{i}\varepsilon}{a_N} = E' \pm \frac{\mathrm{i}\varepsilon'}{a_N(E')}.$$

Then $|E'| \le E_{\mathrm{out}}$ for N large, and $\varepsilon' - \varepsilon$, $X_N(E') - X_N(E)$, and $s_{E'} - s_E$ tend to zero. The Lipschitz bound allows us to replace ε' by ε with an error $o(1)$. Since $E' \to E_0$, we also have $s_{E'} \to s_{E_0}$. Now Proposition 11.5 at the energy E' implies the stated convergence for each (u, ε), and the Lipschitz bound makes it locally uniform. □

11.4 The profile

We now apply the extraction theorem to the pairs of sites.

Proposition 11.7. *For sites i, j and an energy e, let $f^{ij}_{N,e}$ denote the density of $N^2\mathbb{E}\sum_\alpha w_\alpha(i)w_\alpha(j)\delta_{a_N(e)(\lambda_\alpha - e)}$. Then for every $R > 0$,*

$$\sup\left| f^{ij}_{N,e}(u) - (1 + \mathbf{1}_{i=j})\, \mathcal{C}_{X_N(e)}(i/N, j/N) \right| \longrightarrow 0, \tag{11.11}$$

where the supremum is over all interior sites i, j, all $|e| \le E_{\mathrm{mid}}$ and all $|u| \le R$. The convergence is uniform for N/W^2 in compact subsets of $(0, \infty)$.

Proof. By Remark 9.7, it suffices to consider sequences of sites i, j and energies e along which $i/N \to x$, $j/N \to y$, $X_N(e) \to X$, and $\mathbf{1}_{i=j}$ is constant, and to prove that $f^{ij}_{N,e}$ converges locally uniformly to $c_* = (1 + \mathbf{1}_{i=j})\mathcal{C}_X(x, y)$. Indeed, $\mathcal{C}_X(x, y)$ is continuous in (X, x, y), by dominated convergence in Definition 1.9, so that $c_* - (1 + \mathbf{1}_{i=j})\mathcal{C}_{X_N(e)}(i/N, j/N) \to 0$.

We apply Theorem 9.6 at the energy e with $k = 2$, $r_2 = 1$, and $\mathbf{\Pi}_N = \ell_{i,N} \otimes \ell_{j,N}$, which is dominated with $C_0 = 1$ and has the diagonal part

$$\mathbf{\Pi}^{\mathrm{d}}_N = N^2 \sum_\alpha w_\alpha(i)w_\alpha(j)\delta_{x_\alpha}.$$

Using (9.3), we write the left side of (9.14) as

$$\int_{\mathbb{R}^2} P_\varepsilon(t)P_\varepsilon(v)\, \Pi_N(\,\mathrm{d}t, \mathrm{d}v) = \mathbb{E}\ell^\varepsilon_{i,N}(0)\ell^\varepsilon_{j,N}(0),$$

and Proposition 11.5 shows that it converges to $\mathfrak{f}(2\pi\varepsilon)$, where

$$\mathfrak{f}(z) = 1 + \tfrac{1}{2}K_X(x, y; z) \quad (i \ne j), \qquad \mathfrak{f}(z) = 1 + \widetilde{K}_X(x; z) \quad (i = j).$$

By Theorem 10.4 and Lemma 11.4, we have $\lim_{z\downarrow 0} z\mathfrak{f}(z) = c_*$ in both cases. Since $\varkappa_{2,1} = 1$, Theorem 9.6 implies that the densities $f^{ij}_{N,e}$ of Π^{d}_N converge to c_* locally uniformly. □

Proof of Theorem 1.15, first part. Let $\bar{r}_N \downarrow 0$. We apply Corollary 9.9 with $k = 2$, $\chi = \phi$, $F_\alpha = w_\alpha(i)w_\alpha(j)$ and

$$\bar{c}_N(e) = (1 + \mathbf{1}_{i=j})\, \mathcal{C}_{X_N(e)}(i/N, j/N).$$

These functions are bounded and equicontinuous in $|e| \le E_{\mathrm{mid}}$, uniformly in N and in the sites, since $X_N(e) = N/((4-e^2)W^2)$ is Lipschitz in e uniformly for N/W^2 in a compact set, and $\mathcal{C}_X(x,y)$ is uniformly continuous on compact sets. The hypothesis of Corollary 9.9 is (11.11) at $u = 0$.

For the rooted ratio, the densities $g_{N,e}$ of $N\mathbb{E}\sum_\alpha w_\alpha(i)\delta_{a_N(e)(\lambda_\alpha - e)}$ converge to one, uniformly for $|e| \le E_{\mathrm{mid}}$ and interior sites i. This follows from Corollary 9.8 and Remark 9.7, since $\mathbb{E}\ell^\varepsilon_{i,N}(0) \to 1$ by Proposition 11.5. With these choices, (9.16) becomes the statement for $N^2\langle w_\alpha(i)w_\alpha(j)\rangle_{E,r}$, and (9.17) yields (1.24) for $P_N(E,r;i,j)$. These limits hold uniformly over interior pairs, since the hypotheses do. □

11.5 The two-energy intensity

We now turn to the full two-energy intensity, which requires the part of the separation measure of the pair away from zero. Let i, j be interior sites and $|E| \le E_{\mathrm{mid}}$, and set $\Pi_N = \mathbb{E}[\ell_{i,N} \otimes \ell_{j,N}]$. By Lemma 9.5, every vague limit of Π_N along a subsequence with $i/N \to x$, $j/N \to y$, $X_N(E) \to X$, and constant $\mathbf{1}_{i=j}$ has the form

$$\Pi(\,\mathrm{d}t, \mathrm{d}v) = \mathrm{d}t\, \nu_{x,y}(\,\mathrm{d}(v-t)), \tag{11.12}$$

where $\nu_{x,y}$ is a positive translation-bounded measure on $\mathbb{R}$. We call $\nu_{x,y}$ the *separation measure* of the limit. Exchanging the two sites reflects the separation measure, so that the limit for (j,i) has the separation measure $\nu_{x,y}(-\mathrm{d}r)$. Therefore evenness of $\nu_{x,y}$ is equivalent to its symmetry in the two sites. This symmetry does not follow from exchanging the eigenvalue labels, and we derive it from the limits at coinciding energies.

Lemma 11.8. *Every separation measure $\nu = \nu_{x,y}$ in* (11.12) *is even.*

Proof. Fix a subsequence along which $\Pi_N \to \Pi$ vaguely, with separation measure ν as in (11.12). Passing to a further subsequence, we may assume that $E = E_N \to E_0$; this preserves Π and ν. Let F_N be as in Lemma 11.6, which now shows that $F_N \to A$ locally uniformly on $\mathbb{R} \times (0,\infty)$, where

$$A(\varepsilon) = 1 + s^2_{E_0} K_X(x,y;2\pi\varepsilon) \quad (i \ne j), \qquad A(\varepsilon) = 1 + 2s^2_{E_0}\widetilde{K}_X(x;2\pi\varepsilon) \quad (i = j).$$

The function A is real and independent of u. With $w_\pm = u \pm \mathrm{i}\varepsilon$, we have $F_N(u,\varepsilon) = \Phi_N(w_+, w_-)$, where

$$\Phi_N(w,w') = \mathbb{E}\, G_{ii}(E + w/a_N)\, G_{jj}(E + w'/a_N)$$

is holomorphic in (w, w') for $\operatorname{Im} w > 0 > \operatorname{Im} w'$, and $\partial_u^2 + \partial_\varepsilon^2 = 4\partial_{w_+}\partial_{w_-}$. Using the spectral decomposition

$$G_{ii}(E + w/a_N) = \sum_\alpha \frac{a_N w_\alpha(i)}{x_\alpha - w},$$

we find

$$(\partial_u^2 + \partial_\varepsilon^2) F_N(u, \varepsilon) = 4\rho(E)^2 \int_{\mathbb{R}^2} \frac{\Pi_N(\,\mathrm{d}t,\,\mathrm{d}v)}{(t - u - \mathrm{i}\varepsilon)^2 (v - u + \mathrm{i}\varepsilon)^2}. \tag{11.13}$$

As $N \to \infty$, the left side converges to $A''(\varepsilon)$ in the sense of distributions on $\mathbb{R} \times (0, \infty)$, and A'' is real. By Lemma 9.5(c), applied to the squared Cauchy kernels, and $\rho(E) \to \rho(E_0) > 0$, the right side converges locally uniformly along the subsequence to $4\rho(E_0)^2$ times the corresponding integral against Π. This continuous limit equals A'' as a distribution. Since $4\rho(E_0)^2$ is positive and real, the integral against Π is real.

We compute this integral with (11.12). For $a = 2\varepsilon$, the residue theorem, applied in the upper half-plane at the double pole $t = \mathrm{i}\varepsilon$, gives

$$\int_{\mathbb{R}} \frac{\mathrm{d}t}{(t - \mathrm{i}\varepsilon)^2 (t + r + \mathrm{i}\varepsilon)^2} = -\frac{4\pi\mathrm{i}}{(r + \mathrm{i}a)^3}, \qquad r \in \mathbb{R}.$$

Substituting $t \mapsto t + u$, we conclude that

$$\operatorname{Re} \int_{\mathbb{R}} \frac{\nu(\,\mathrm{d}r)}{(r + \mathrm{i}a)^3} = 0, \qquad a > 0.$$

Let $\nu_\pm$ denote the even and the odd part of ν, so that $\nu_- = \frac{1}{2}(\nu - \check{\nu})$ with $\check{\nu}(\,\mathrm{d}r) = \nu(-\,\mathrm{d}r)$. Since

$$\overline{(r + \mathrm{i}a)^{-3}} = -(-r + \mathrm{i}a)^{-3}, \qquad r \in \mathbb{R},$$

the integral of $(r + \mathrm{i}a)^{-3}$ against the even measure ν_+ is purely imaginary, and against the odd measure ν_- it is real. Therefore the function

$$T(w) = \int_{\mathbb{R}} \frac{\nu_-(\,\mathrm{d}r)}{(r + w)^3}$$

vanishes on the positive imaginary axis. Since ν is translation bounded, T is holomorphic in the upper half-plane, and the identity theorem implies that $T = 0$ there. As ν_- is real, we have $T(\bar{w}) = \overline{T(w)}$, and $T = 0$ also in the lower half-plane.

Finally, write the Poisson convolution as

$$(P_a * \nu_-)(u) = \frac{1}{\pi} \int_{\mathbb{R}} \operatorname{Im} \frac{1}{r - u - \mathrm{i}a}\, \nu_-(\,\mathrm{d}r).$$

For fixed $a > 0$, this real kernel and its first two derivatives in u are bounded by $C(1 + r^2)^{-1}$, locally uniformly in u. The translation boundedness of $|\nu_-|$ therefore makes

these integrals absolutely convergent and justifies differentiating twice under the integral. Thus

$$\partial_u^2 (P_a * \nu_-)(u) = \frac{2}{\pi} \operatorname{Im} T(-u - \mathrm{i}a) = 0.$$

The function $P_a * \nu_-$ is bounded, since ν is translation bounded. Being affine and bounded, it is constant, and being odd, it vanishes. Therefore $P_a * \nu = P_a * \check{\nu}$ for all $a > 0$, and Lemma A.4(b) implies that $\nu = \check{\nu}$. □

We now identify the separation measure from its values at coinciding energies. For an even positive translation-bounded measure ν on $\mathbb{R}$, we define

$$\mathcal{F}_\nu(w) = \int_{\mathbb{R}} \frac{w}{\pi(w^2 + r^2)} \, \nu(\mathrm{d}r), \qquad \operatorname{Re} w > 0. \tag{11.14}$$

The integrand is holomorphic in w and bounded by $C(1 + r^2)^{-1}$ locally uniformly in $\operatorname{Re} w > 0$, so $\mathcal{F}_\nu$ is holomorphic. Using the evenness of ν and

$$\frac{w}{w^2 + r^2} = \frac{1}{2}\Big(\frac{1}{w - \mathrm{i}r} + \frac{1}{w + \mathrm{i}r}\Big),$$

we find for $a > 0$ and $d \in \mathbb{R}$

$$\mathcal{F}_\nu(a) = (P_a * \nu)(0), \qquad \operatorname{Re} \mathcal{F}_\nu(a - \mathrm{i}d) = (P_a * \nu)(d). \tag{11.15}$$

In particular, the values $(P_a * \nu)(0)$ for $a > 0$ determine $\mathcal{F}_\nu$ by the identity theorem, hence all functions $P_a * \nu$ by (11.15), hence ν by Lemma A.4(b).

The uniform statement of Theorem 1.14 also needs the continuity of the kernels in all their arguments.

Lemma 11.9. *The map $(X, x, y, z) \mapsto K_X(x, y; z)$ is continuous on $(0, \infty) \times \{0 < x \le y < 1\} \times \{\operatorname{Re} z > 0\}$, and the map $(X, x, z) \mapsto \widetilde{K}_X(x; z)$ is continuous on $(0, \infty) \times (0, 1) \times \{\operatorname{Re} z > 0\}$.*

Proof. We first remove X by a change of time. The diffusion with generator $X\Delta_D$ is the diffusion with generator Δ_D at time Xt. With $w = z/X$, the definitions in Definition 1.4 therefore imply that

$$e^{t(X\Delta_D - zd)} = e^{Xt(\Delta_D - wd)}, \qquad \Psi_t^{X,z} = \Psi_{Xt}^{1,w}. \tag{11.16}$$

Inserting (11.16) into the definitions (1.6) and (1.7), with $t_1 = Xx$, $t_2 = X(y - x)$, and $t_3 = X(1 - y)$, we have

$$K_X(x, y; z) = \int_D \Psi_{t_1}^{1,w} \, e^{t_2(\Delta_D - wd)} \Psi_{t_3}^{1,w} \, \mathrm{d}B \, \mathrm{d}F, \qquad \widetilde{K}_X(x; z) = \int_D \frac{B}{d} \, \Psi_{t_1}^{1,w} \Psi_{X(1-x)}^{1,w} \, \mathrm{d}B \, \mathrm{d}F.$$

Let

$$f(t,p,w) = \big(e^{t(\Delta_D - wd)}1\big)(p).$$

For fixed w with $\operatorname{Re} w > 0$, f is continuous in (t,p) on $[0,\infty)\times D$, as shown in the proof of Lemma 6.6(b). For fixed (t,p), f is holomorphic in w by dominated convergence in the Feynman–Kac formula, and $|f| \le 1$. Cauchy's estimate therefore shows that $|\partial_w f| \le 1/\alpha$ on $\{\operatorname{Re} w \ge 2\alpha\}$, uniformly in (t,p). Together with the continuity in (t,p), this implies that f is jointly continuous in (t,p,w). By (1.5),

$$\Psi_t^{1,w}(p) = 1 - w\,d(p)\int_0^t f(v,p,w)\,\mathrm{d}v$$

is then jointly continuous on $[0,\infty)\times D\times\{\operatorname{Re} w > 0\}$.

Next, let $(\tilde B_t, \tilde F_t)$ be the radial diffusion of Section 6.2 with $X = 1$, with generator Δ_D. By the Markov property,

$$\big(e^{t_2(\Delta_D - wd)}\Psi_{t_3}^{1,w}\big)(p) = \mathbb{E}_p\Big[\exp\Big(-w\int_0^{t_2} d(\tilde B_u,\tilde F_u)\,\mathrm{d}u\Big)\Psi_{t_3}^{1,w}(\tilde B_{t_2},\tilde F_{t_2})\Big].$$

When (t_2,t_3,w) converges, the integrand converges almost surely, by the previous paragraph and the continuity of the paths, and it is bounded by one. The bounded convergence theorem then shows that this function is continuous in (t_2,t_3,w) for every p.

Finally, fix a point in the domain of K and a compact neighborhood of it on which $t_1,t_3 \ge \sigma > 0$, $t_1,t_2,t_3 \le T$ and $\operatorname{Re} w \ge a > 0$. Using Lemma 6.6(b),(c), we bound the integrand of K there by

$$\big|\Psi_{t_1}^{1,w}\big| \le \Psi_{t_1}^{1,a} \le \Psi_{\sigma}^{1,a} \le C(1+B)^{-3},$$

which is integrable on D. The continuity of K follows by dominated convergence. For $\widetilde K$, we use the dominating function $(B/d)C(1+B)^{-3}$, which is integrable since B/d is integrable near the corner. □

Proof of Theorem 1.14. By the argument by contradiction of Remark 9.7, it suffices to consider sequences of interior sites i,j and of parameters $\varepsilon_1,\varepsilon_2 > 0$ and $u,v\in\mathbb{R}$, with $|E| \le E_*$, along which $i/N \to x$, $j/N \to y$, $X_N(E) \to X$, $\mathbf{1}_{i=j}$ is constant, and the parameters converge to positive dampings $\varepsilon_1,\varepsilon_2$ and real centers u,v. Passing to a further subsequence, we may assume that $\Pi_N = \mathbb{E}[\ell_{i,N}\otimes\ell_{j,N}]$ and its diagonal and off-diagonal parts converge vaguely, with the separation measure ν of (11.12). It suffices to identify the limits along this subsequence.

By (9.3) and Lemma 9.5(c), applied to the Poisson kernels with the converging parameters, $\mathbb{E}\ell_{i,N}^{\varepsilon_1}(u)\ell_{j,N}^{\varepsilon_2}(v)$ converges to

$$\int_{\mathbb{R}}\int_{\mathbb{R}} P_{\varepsilon_1}(t-u)P_{\varepsilon_2}(t+r-v)\,\mathrm{d}t\,\nu(\mathrm{d}r) = \int_{\mathbb{R}} P_{\varepsilon_1+\varepsilon_2}(v-u-r)\,\nu(\mathrm{d}r) = (P_{\varepsilon_1+\varepsilon_2}*\nu)(u-v),$$

where we used the semigroup property of the Poisson kernels and the evenness of ν from Lemma 11.8. For $\varepsilon_1 = \varepsilon_2 = a/2$ and $u = v = 0$, Proposition 11.5 identifies the same limit. Comparing the two limits and using (11.15), we find

$$\mathcal{F}_\nu(a) = 1 + \tfrac{1}{2}K_X(x, y; \pi a) \quad (i \neq j), \qquad \mathcal{F}_\nu(a) = 1 + \widetilde{K}_X(x; \pi a) \quad (i = j), \qquad a > 0.$$

The kernels are holomorphic in $\operatorname{Re} z > 0$ by Lemma 6.6(d), and the identity theorem implies that $\mathcal{F}_\nu(w) = 1 + \frac{1}{2}K_X(x, y; \pi w)$, or $1 + \widetilde{K}_X(x; \pi w)$, for $\operatorname{Re} w > 0$. Applying (11.15) with $a = \varepsilon_1 + \varepsilon_2$ and $d = u - v$, so that $\pi(a - \mathrm{i}d) = z$ in the notation of Theorem 1.14, we obtain (1.20) and (1.21).

For the same-eigenvalue part, the diagonal part Π_N^{d} converges vaguely to $(1 + \mathbf{1}_{i=j})\mathcal{C}_X(x, y)\,\mathrm{d}t$ by Proposition 11.7. We apply Lemma 9.5(c) to $(\iota_2)_*\Pi_N^{\mathrm{d}}$, and we use

$$\int_{\mathbb{R}} P_{\varepsilon_1}(t - u)P_{\varepsilon_2}(t - v)\,\mathrm{d}t = P_{\varepsilon_1+\varepsilon_2}(u - v).$$

This shows that the sum in (1.22) converges to $(1 + \mathbf{1}_{i=j})\mathcal{C}_X(x, y)P_{\varepsilon_1+\varepsilon_2}(u - v)$. Since the limits do not depend on the subsequence and, by Lemma 11.9 and the continuity of $\mathcal{C}_X(x, y)$ in (X, x, y), are continuous in all the parameters, the theorem follows. □

11.6 Boundary control and the inverse participation ratio

The inverse participation ratio also involves the sites within distance δN of the two ends, at which Theorem 7.6(b) is not available. We only need a uniform bound on the diagonal densities there.

Lemma 11.10. *For a site i and an energy e, let $f_{N,e}^{ii}$ be as in Proposition 11.7. Then*

$$\sup_N \sup_{1 \le i \le N} \sup_{|e| \le E_{\mathrm{mid}}} f_{N,e}^{ii}(0) < \infty. \tag{11.17}$$

Proof. Let Π_N^{d} denote the measure $N^2\mathbb{E}\sum_\alpha w_\alpha(i)^2\delta_{x_\alpha}$, unfolded at e. Using

$$\sum_\alpha w_\alpha(i)^2\mathbf{1}_{|x_\alpha|\le 1} \le \Big(\sum_\alpha w_\alpha(i)\mathbf{1}_{|x_\alpha|\le 1}\Big)^2$$

and the lower bound $P_1 \ge P_1(1)$ on $[-1, 1]$, we find

$$\Pi_N^{\mathrm{d}}([-1, 1]) \le \mathbb{E}\big[\ell_{i,N}([-1, 1])^2\big] \le P_1(1)^{-2}\,\mathbb{E}\big[\ell_{i,N}^1(0)^2\big] \le C,$$

uniformly in all sites and in $|e| \le E_{\mathrm{mid}}$, by Lemma 9.2. The density of Π_N^{d} is a Gaussian mixture with variance τ_N^2, by Lemma 9.3. Using (9.7) with $u_0 = 0$ and (9.6), we conclude that

$$f_{N,e}^{ii}(0) \le \frac{1}{2}e^{1/(2\tau_N^2)}\,\Pi_N^{\mathrm{d}}([-1, 1]) \le C.$$ □

Proof of Theorem 1.15, second part. For a physical energy e, the measure

$$\frac{1}{N}\sum_{i=1}^{N} N^2\mathbb{E}\sum_{\alpha} w_\alpha(i)^2\delta_{a_N(e)(\lambda_\alpha-e)} = N^2\mathbb{E}\sum_{\alpha} F_\alpha\,\delta_{a_N(e)(\lambda_\alpha-e)}, \qquad F_\alpha = \frac{1}{N}\,\mathrm{IPR}(u_\alpha),$$

has the density

$$f_{N,e} = \frac{1}{N}\sum_{i=1}^{N} f^{ii}_{N,e}.$$

By Proposition 11.7, $f^{ii}_{N,e}(0) - 2\mathcal{C}_{X_N(e)}(i/N, i/N) \to 0$ uniformly over interior sites and $|e| \le E_{\mathrm{mid}}$, and by Lemma 11.10, the remaining $2\delta N + O(1)$ sites contribute at most $C(\delta + N^{-1})$ to $f_{N,e}(0)$. The function $(X, x) \mapsto \mathcal{C}_X(x, x)$ is continuous and bounded on $[X_-, X_+] \times [0, 1]$ for $0 < X_- \le X_+$. Taking the Riemann sum over the interior sites, and then letting $\delta \downarrow 0$, we obtain

$$\sup_{|e|\le E_{\mathrm{mid}}} \left| f_{N,e}(0) - 2\int_0^1 \mathcal{C}_{X_N(e)}(x, x)\,\mathrm{d}x \right| \longrightarrow 0.$$

We set $\bar{c}_N(e) = 2 + 2X_N(e)/3$. By Proposition 11.12 below, we have

$$\bar{c}_N(e) = 2\int_0^1 \mathcal{C}_{X_N(e)}(x, x)\,\mathrm{d}x,$$

and these functions are bounded and equicontinuous in $|e| \le E_{\mathrm{mid}}$. We apply (9.16) with $k = 2$, $\chi = \phi$, and these F_α. Since $N^2 F_\alpha = N\,\mathrm{IPR}(u_\alpha)$, we conclude that $N\langle \mathrm{IPR}(u_\alpha)\rangle_{E,r} - 2 - 2X_N(E)/3 \to 0$, uniformly for $|E| \le E_*$ and $0 < r \le \bar{r}_N$, which is (1.25). □

11.7 The exact participation formula

We now evaluate the integrated diagonal of $\mathcal{C}_X$. We use the notation of Section 10, with

$$g_1(u) = \sqrt{2}\beta_u - u, \qquad A_t = \int_0^t e^{g_1(u)}\,\mathrm{d}u, \qquad \Upsilon_t(q) = \mathbb{E}e^{-qA_t}, \qquad t \ge 0,\ q > 0.$$

We also set

$$A_t^+ = \int_0^t e^{\sqrt{2}\beta_u + u}\,\mathrm{d}u, \qquad \Upsilon_t^+(q) = \mathbb{E}e^{-qA_t^+},$$

which correspond to the opposite drift, and we write $\langle f, g\rangle = \int_0^\infty f(q)g(q)\,\mathrm{d}q$ for nonnegative functions on $(0, \infty)$.

For $\lambda > 2$, we set

$$v_\lambda(q) = \int_0^\infty e^{-\lambda t}\Upsilon_t(q)\,\mathrm{d}t, \qquad v_\lambda^+(q) = \int_0^\infty e^{-\lambda t}\Upsilon_t^+(q)\,\mathrm{d}t,$$

which are the resolvents at λ, applied to the constant one, of the geometric Brownian motions with drifts -1 and $+1$, killed at rate q. We will need the following properties of v_λ.

Lemma 11.11. *Let $\lambda > 2$. The function v_λ is smooth on $(0,\infty)$ and satisfies*

$$q^2 v_\lambda'' - (q+\lambda)v_\lambda = -1. \tag{11.18}$$

Moreover, $v_\lambda(0+) = \lambda^{-1}$, $v_\lambda'(0+) = -\lambda^{-2}$, and for all $q > 0$,

$$0 \le v_\lambda(q) \le \min(\lambda^{-1}, q^{-1}), \qquad 0 \le -v_\lambda'(q) \le \frac{4}{eq^2}, \qquad v_\lambda^+(q) = \frac{1-\lambda v_\lambda(q)}{q}. \tag{11.19}$$

Proof. We have

$$\partial_q^m \mathrm{Y}_t(q) = \mathbb{E}\big[(-A_t)^m e^{-qA_t}\big], \qquad \big|q^m \partial_q^m \mathrm{Y}_t(q)\big| \le \sup_{a\ge 0} a^m e^{-a} = m^m e^{-m},$$

for all $m \ge 0$, uniformly in t. The process $q_u^0 = q e^{g_1(u)}$ satisfies $\mathrm{d}q_u^0 = \sqrt{2} q_u^0 \,\mathrm{d}\beta_u$, and A_t splits at $r \in (0,t)$ as $A_t = A_r + e^{g_1(r)}\tilde{A}_{t-r}$, where $\tilde{A}_{t-r}$ is independent of $(\beta_u)_{u\le r}$ and has the law of A_{t-r}. Conditioning on $(\beta_u)_{u\le r}$, we find

$$\mathrm{Y}_t(q) = \mathbb{E}\big[e^{-qA_r}\mathrm{Y}_{t-r}(q_r^0)\big].$$

Applying Itô's formula to $e^{-qA_u}\mathrm{Y}_{t-r}(q_u^0)$ on $[0,r]$, whose martingale part is a true martingale by the bound on $q\partial_q \mathrm{Y}_{t-r}$, and letting $r \downarrow 0$, we obtain the backward equation

$$\partial_t \mathrm{Y}_t = q^2 \partial_q^2 \mathrm{Y}_t - q\mathrm{Y}_t, \qquad \mathrm{Y}_0 = 1.$$

Multiplying by $e^{-\lambda t}$ and integrating over $t \in (0,\infty)$, with differentiation under the integral justified by the bounds above, we obtain

$$\lambda v_\lambda - 1 = \int_0^\infty e^{-\lambda t}\partial_t \mathrm{Y}_t \,\mathrm{d}t = q^2 v_\lambda'' - q v_\lambda,$$

which is (11.18). Since $\mathbb{E}A_t = t$, dominated differentiation gives $v_\lambda(0+) = \lambda^{-1}$ and

$$v_\lambda'(0+) = -\int_0^\infty e^{-\lambda t} t \,\mathrm{d}t = -\lambda^{-2},$$

and $0 \le \mathrm{Y}_t \le 1$ implies that $v_\lambda \le \lambda^{-1}$.

To prove $v_\lambda \le 1/q$, note that the generator $q^2\partial_q^2 - q$ of the killed process satisfies

$$\big(\lambda - q^2\partial_q^2 + q\big)\frac{1}{q} = 1 + \frac{\lambda - 2}{q} \ge 1.$$

We apply Dynkin's formula to $1/q$ and the killed process up to the exit time of $[n^{-1}, n]$, and we discard the nonnegative terminal term. Letting $n \to \infty$, which is allowed

since geometric Brownian motion reaches neither 0 nor ∞ in finite time, we find that $v_\lambda(q) \le 1/q$. For the derivative, we use $ae^{-qa/2} \le 2/(eq)$ for $a \ge 0$, which gives

$$-\partial_q \mathrm{Y}_t(q) = \mathbb{E}\big[A_t e^{-qA_t}\big] \le \frac{2}{eq}\mathrm{Y}_t(q/2).$$

Integrating against $e^{-\lambda t}\,\mathrm{d}t$ and using $v_\lambda(q/2) \le 2/q$, we obtain the bound on v'_λ.

Finally, dominated differentiation gives

$$\partial_t \mathrm{Y}_t(q) = -q\,\mathbb{E}\big[e^{g_1(t)}e^{-qA_t}\big].$$

The density $e^{g_1(t)}$ has mean one, and by the Cameron–Martin theorem, under the measure it defines, $(\beta_u - \sqrt{2}u)_{u\le t}$ is a Brownian motion, so that $g_1(u)$ becomes $\sqrt{2}\beta_u + u$ for $u \le t$. We conclude that $\partial_t \mathrm{Y}_t = -q\mathrm{Y}^+_t$. Integrating by parts in t, and using $0 \le \mathrm{Y}_t \le 1$, we obtain

$$q\,v^+_\lambda(q) = -\int_0^\infty e^{-\lambda t}\partial_t \mathrm{Y}_t(q)\,\mathrm{d}t = 1 - \lambda v_\lambda(q). \qquad \square$$

Proposition 11.12. *For all $X > 0$,*

$$\int_0^1 \mathcal{C}_X(x,x)\,\mathrm{d}x = 1 + \frac{X}{3}. \tag{11.20}$$

The proof rests on two identities, which come from one Markov decomposition of the exponential functional A_X, and on two integrated identities for v_λ, which we call the virial identities.

Proof. We set

$$\mathcal{A}(X) = \int_0^1 \mathcal{C}_X(x,x)\,\mathrm{d}x.$$

We first express $\mathcal{A}(X)$ and the number one as integrals of products of Y and Y^+. By (10.9) with $y = x$, we have

$$\mathcal{C}_X(x,x) = X\langle \mathrm{Y}_{Xx}, \mathrm{Y}_{X(1-x)}\rangle,$$

and the substitution $t = Xx$ gives

$$\mathcal{A}(X) = \int_0^X \langle \mathrm{Y}_t, \mathrm{Y}_{X-t}\rangle\,\mathrm{d}t. \tag{11.21}$$

For the second identity, we use Brownian scaling: $(g_X(x))_{x\le 1}$ has the law of $(g_1(Xx))_{x\le 1}$, under which $e^{g_X(x)}/Z_X$ becomes $Xe^{g_1(Xx)}/A_X$. Taking the expectation of the total mass of the probability measure μ_X and substituting $t = Xx$, we find

$$\int_0^X \mathbb{E}\Big[\frac{e^{g_1(t)}}{A_X}\Big]\,\mathrm{d}t = 1.$$

Next, for $0 \le t \le X$, we split the integral defining A_X at t and divide by $e^{g_1(t)}$, which gives

$$\frac{A_X}{e^{g_1(t)}} = \int_0^t e^{\sqrt{2}(\beta_{t-r}-\beta_t)+r}\,\mathrm{d}r + \int_0^{X-t} e^{\sqrt{2}(\beta_{t+r}-\beta_t)-r}\,\mathrm{d}r. \tag{11.22}$$

The increments of β before and after t are independent, and $r \mapsto \beta_{t-r} - \beta_t$ is a Brownian motion on $[0,t]$. The two integrals in (11.22) are therefore independent, with the laws of A_t^+ and A_{X-t}. Inserting the identity

$$\frac{1}{a} = \int_0^\infty e^{-qa}\,\mathrm{d}q, \qquad a = \frac{A_X}{e^{g_1(t)}},$$

and using the independence of the two integrals in (11.22), we find

$$1 = \int_0^X \langle \mathrm{Y}_t^+, \mathrm{Y}_{X-t}\rangle\,\mathrm{d}t. \tag{11.23}$$

Next, we take Laplace transforms in X. All integrands in (11.21) and (11.23) are nonnegative. Using Tonelli's theorem, we find, for $\lambda > 2$,

$$\int_0^\infty e^{-\lambda X}\mathcal{A}(X)\,\mathrm{d}X = \mathfrak{v}_1(\lambda), \qquad \frac{1}{\lambda} = \mathfrak{v}_2(\lambda), \tag{11.24}$$

where

$$\mathfrak{v}_1 = \langle v_\lambda, v_\lambda\rangle, \qquad \mathfrak{v}_2 = \langle v_\lambda, v_\lambda^+\rangle.$$

We also set

$$\mathfrak{v}_3 = \int_0^\infty q\, v_\lambda'(q)^2\,\mathrm{d}q.$$

The bounds (11.19) imply that the integrals $\mathfrak{v}_1$, $\mathfrak{v}_2$, and $\mathfrak{v}_3$ are finite, and that the functions $q^2(v_\lambda')^2$, qv_λ^2, and $qv_\lambda v_\lambda'$ tend to zero as $q \downarrow 0$ and as $q \to \infty$.

We now derive the virial identities, writing $v = v_\lambda$ for brevity. First, we multiply (11.18) by v' and integrate over $(0,\infty)$. Integrating by parts, and using $v(0+) = \lambda^{-1}$ from Lemma 11.11, $v(\infty) = 0$, and the vanishing boundary terms, we have

$$\int_0^\infty q^2 v'v''\,\mathrm{d}q = -\mathfrak{v}_3, \qquad -\int_0^\infty qvv'\,\mathrm{d}q = \frac{\mathfrak{v}_1}{2},$$
$$-\lambda\int_0^\infty vv'\,\mathrm{d}q = \frac{1}{2\lambda}, \qquad -\int_0^\infty v'\,\mathrm{d}q = \frac{1}{\lambda},$$

and therefore

$$-\mathfrak{v}_3 + \frac{\mathfrak{v}_1}{2} + \frac{1}{2\lambda} = \frac{1}{\lambda}.$$

Second, by (11.19), we may write (11.18) in the form

$$qv_\lambda'' - v_\lambda + v_\lambda^+ = 0.$$

We multiply by v and integrate. Here we keep $vv_\lambda^+ = v(1-\lambda v)/q$ together, since its two terms are not separately integrable at zero. Integrating by parts, we have

$$\int_0^\infty qvv''\,\mathrm{d}q = -\int_0^\infty \bigl(vv' + q(v')^2\bigr)\,\mathrm{d}q = \frac{1}{2\lambda^2} - \mathfrak{v}_3,$$

and we obtain $1/(2\lambda^2) - \mathfrak{v}_3 - \mathfrak{v}_1 + \mathfrak{v}_2 = 0$. Together, the two identities are

$$\mathfrak{v}_1 - 2\mathfrak{v}_3 = \frac{1}{\lambda}, \qquad \mathfrak{v}_2 = \mathfrak{v}_1 + \mathfrak{v}_3 - \frac{1}{2\lambda^2}. \tag{11.25}$$

Recall from (11.24) that $\mathfrak{v}_2 = \lambda^{-1}$. Eliminating $\mathfrak{v}_2$ and $\mathfrak{v}_3$ from (11.25), we obtain

$$\int_0^\infty e^{-\lambda X}\mathcal{A}(X)\,\mathrm{d}X = \mathfrak{v}_1(\lambda) = \frac{1}{\lambda} + \frac{1}{3\lambda^2} = \int_0^\infty e^{-\lambda X}\Bigl(1 + \frac{X}{3}\Bigr)\,\mathrm{d}X, \qquad \lambda > 2.$$

The function $\mathcal{A}$ is continuous on $(0,\infty)$, by dominated convergence in Definition 1.9. By the uniqueness of the Laplace transform, $\mathcal{A}(X) = 1 + X/3$ for all $X > 0$. □

Remark 11.13. Let G_{N} be the Neumann Green kernel of Proposition 1.17. Since $\operatorname{Tr} G_{\mathrm{N}} = \int_0^1 (x^2 - x + 1/3)\,\mathrm{d}x = 1/6$, the identity (11.20) reads $\int_0^1 \mathcal{C}_X(x,x)\,\mathrm{d}x = 1 + 2X \operatorname{Tr} G_{\mathrm{N}}$ for all $X > 0$. Compared with the expansion $\mathcal{C}_X = 1 + 2XG_{\mathrm{N}} + O(X^2)$ of Proposition 1.17, the expansion of the diagonal integrated over the chain terminates at first order. The first-order term is the physics prediction [94, (3.55)], and the vanishing of the second-order term for the inverse participation ratio is visible in [57, (12), (17)]. The limit 2 of $2 + 2X/3$ as $X \downarrow 0$ is the value for the GUE, and the slope $2/3$ as $X \to \infty$ agrees with $X^{-1}\mathcal{C}_X(x,x) \to 1/3$, which is proved in Section 12.

SECTION 12

The coarse eigenvector law

In this section, we prove Theorems 1.10 and 1.12 and Corollaries 1.11 and 1.13, and, in Proposition 12.14, the rows of Proposition 1.16 that concern the mass measure and the eigenvector correlation. The argument has four steps. First, we compute all fixed moments $N^k \langle \prod_a w_\alpha(i_a) \rangle_{E,r}$ at separated interior sites (Theorem 12.5). For this we use a positive source built from permanents of $\operatorname{Im} G$, whose limit at fixed damping is the permanent pairing of Definition 10.6, by Theorem 7.6(b); its residue at zero damping is given by Lemma 10.12, and Theorem 9.6 converts it into the moment. This is the main step, and the remaining ones are soft. Second, Scheffé's lemma upgrades the pointwise limits to L^1 limits of the step densities of the moments, including the collision and boundary strips that the first step excludes. Third, the Stone–Weierstrass theorem identifies the law of the mass measure. Finally, the rooted laws, the infinite-wire shape, and its mean follow from the Cameron–Martin theorem, Brownian scaling, Dufresne's identity, and the Kontorovich–Lebedev transform. In addition, Lemma 12.12 computes the second moment of the mean shape, which is used only in Remark 12.13, where we compare our results with the physics literature.

The coarse law was derived in the physics literature for a finite disordered wire in the unitary class. Mirlin [94, (3.32)–(3.34)], building on Fyodorov and Mirlin [56] and on Kolokolov's method for strictly one-dimensional systems [79], wrote the multipoint eigenfunction statistics as a Liouville path integral and gave the law of the envelope of an eigenfunction. In this description, an eigenfunction is the product of a fast factor with Gaussian statistics and a smooth envelope [94, (3.33)]. Positions along the wire are measured in units of the localization length, so that the wire is the interval $[0, X]$. Following [94, (3.34)], we let $\theta(t)$ denote the logarithm of the square of the envelope at the position $t \in [0, X]$. In particular, $\theta(0)$ and $\theta(X)$ are its values at the two ends of the wire. The law of the path θ in [94, (3.34)] has, formally, the density

$$e^{-X/4} e^{-(\theta(0)+\theta(X))/2} \exp\Big(-\frac{1}{4} \int_0^X \dot\theta(t)^2 \, \mathrm{d}t \Big) \tag{12.1}$$

with respect to a flat measure on paths, restricted by the normalization $X^{-1} \int_0^X e^{\theta(t)} \, \mathrm{d}t = 1$ of [94, (3.32)]. The Gaussian factor in (12.1) describes $\sqrt{2}$ times a Brownian motion, which is the diffusion with generator ∂_θ^2, started from a point $\theta(0) \in \mathbb{R}$ that carries Lebesgue measure.

We translate this law into our variables in two steps. First, we factor the weight in front of the Gaussian factor in (12.1) as

$$e^{-X/4}e^{-(\theta(0)+\theta(X))/2} = e^{-\theta(0)} \cdot e^{-X/4}e^{-(\theta(X)-\theta(0))/2}.$$

The second factor is the Girsanov density, on the time interval $[0, X]$, of the drift -1 for the diffusion with generator ∂_θ^2. After this change of measure, $(\theta(t) - \theta(0))_{t\in[0,X]}$ has the law of $(\sqrt{2}\beta_t - t)_{t\in[0,X]}$ for a standard Brownian motion β. By Brownian scaling, $(\theta(Xs) - \theta(0))_{s\in[0,1]}$ then has the law of the process g_X of Definition 1.9. Second, since $X^{-1}\int_0^X e^{\theta(t)}\,\mathrm{d}t = e^{\theta(0)}\int_0^1 e^{\theta(Xs)-\theta(0)}\,\mathrm{d}s$, the normalization constraint reads $e^{\theta(0)}Z_X = 1$, and the integration of $e^{-\theta(0)}$ over $\theta(0)$ against this constraint, after the substitution $u = e^{\theta(0)}$, produces the weight Z_X. On this constraint, the squared envelope at the position Xs satisfies $e^{\theta(Xs)} = e^{g_X(s)}/Z_X$, which is the density of μ_X, and the envelope law becomes the law of μ_X under $\mathbb{Q}_X$. The rooted shape at large X is implicit in the same formulas [94, (3.30)–(3.34)], and the tail of the mean shape, with the constant $\pi^{7/2}/32$, is due to Efetov and Larkin [38, (56)]. Our results give proofs of these predictions for the Gaussian band matrix.

The same law is known rigorously for the critical one-dimensional random Schrödinger operators recalled in Section 1.1, and Rifkind and Virág conjectured that their eigenvector shape is universal for critical one-dimensional random operators [109, Conjectures 1.2 and 1.3]. We show in Remark 12.7 that, with their parameter τ equal to $4X$, their limit [109, Theorem 1.1] is the law of μ_X under $\mathbb{Q}_X$, and at the start of Section 12.5 that p_∞ is the crossover shape of Dumaz and Labbé [35, Theorem 3], up to a linear change of the space variable and its centering; Theorems 1.10 and 1.12 therefore add Gaussian band matrices at the critical bandwidth to the operators that this shape describes.

Conventions. We keep the conventions of Section 9. In particular, $E_* < E_{\mathrm{mid}} < E_{\mathrm{out}}$ are the energies of (7.22), and statements hold along sequences $N, W \to \infty$ with N/W^2 in a compact subset of $(0, \infty)$. A damping $\varepsilon > 0$ corresponds to $z = 2\pi\varepsilon$, and we set $z_+ = E + \mathrm{i}\varepsilon/a_N$, which is the energy z_+^{B} of (7.13). For a matrix A, we let $\operatorname{Im} A = (A - A^*)/(2\mathrm{i})$; in particular, $\operatorname{Im} G(z_+)$ is positive definite. The cell of a site $j \in \{1, \dots, N\}$ is the interval $I_{j,N} = ((j-1)/N, j/N]$.

12.1 Permanent moment measures

The products $\prod_a w_\alpha(i_a)$ are the diagonal part of a positive moment measure, which we now define. Recall that the permanent of a $k \times k$ matrix A is

$$\operatorname{perm} A = \sum_{\pi\in S_k}\prod_{a=1}^{k} A_{a\pi(a)}.$$

Definition 12.1. Let $k \geq 1$, and let $i_1, \dots, i_k$ be distinct sites. We define the *permanent moment measure* at these sites to be the random measure on $\mathbb{R}^k$

$$\mathbf{\Pi}_N = \frac{N^k}{k!} \sum_{\alpha_1,\dots,\alpha_k=1}^{N} \big|\operatorname{perm}[u_{\alpha_b}(i_a)]_{a,b=1}^k\big|^2 \, \delta_{(x_{\alpha_1},\dots,x_{\alpha_k})}.$$

Lemma 12.2. *Let $k \geq 1$ and let $i_1 < \cdots < i_k$ be sites.*

(a) The measure $\mathbf{\Pi}_N$ is an eigenvector-weighted moment measure of order k in the sense of Definition 9.1, and it is dominated at $i_1, \dots, i_k$ with constant $C_0 = 1$. With $r_k = k!$, its diagonal part is given by

$$\mathbf{\Pi}_N^{\mathrm{d}} = N^k \sum_{\alpha=1}^{N} \prod_{a=1}^{k} w_\alpha(i_a) \, \delta_{x_\alpha}.$$

(b) For all $\varepsilon > 0$, with $\Pi_N = \mathbb{E}\mathbf{\Pi}_N$,

$$\int_{\mathbb{R}^k} \prod_{b=1}^{k} P_\varepsilon(t_b) \, \Pi_N(\,\mathrm{d}\mathbf{t}) = s_E^{-k} \, \mathbb{E} \operatorname{perm}\big[(\operatorname{Im} G(z_+))_{i_a i_b}\big]_{a,b=1}^k. \tag{12.2}$$

(c) Let $\varrho = (1,1)^{\mathrm{T}}$ and $v^{\mathrm{T}} = \frac{\mathrm{i}}{2}(1,-1)$, and let Y be the local source of Definition 7.5 with source rank one, that is, the even supermatrix with 2×2 blocks, with $Y^{\mathrm{F}} = 0$ and boson–boson block

$$Y^{\mathrm{B}} = -\varrho v^{\mathrm{T}} = -\frac{\mathrm{i}}{2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

in the order (R, A) of the flavors. (The block Y^{B} is also a matrix of rank one.) Then, with the local source function $T_N^{(1)}$ of Definition 7.5, of source rank one, at $\mathbf{u} = 0$ and $\mathbf{b} = \varepsilon$,

$$\mathbb{E} \operatorname{perm}\big[(\operatorname{Im} G(z_+))_{i_a i_b}\big]_{a,b=1}^k = \partial_{t_1} \cdots \partial_{t_k} T_N^{(1)}(0, \varepsilon; t_1 Y, \dots, t_k Y)\big|_{t=0}. \tag{12.3}$$

Proof. We begin with (a). Let

$$c_N(\alpha) = \frac{N^k}{k!} \big|\operatorname{perm}[u_{\alpha_b}(i_a)]\big|^2$$

denote the coefficient of the point mass at $(x_{\alpha_1}, \dots, x_{\alpha_k})$. Multiplying u_α by a phase multiplies a column of the matrix by this phase and leaves $c_N(\alpha)$ unchanged. Thus $c_N(\alpha)$ is a function of the eigenprojections. If all labels are equal to α, then all columns are equal, and

$$\operatorname{perm}[u_\alpha(i_a)]_{a,b=1}^k = k! \prod_{a=1}^{k} u_\alpha(i_a), \qquad c_N(\alpha, \dots, \alpha) = k! \, N^k \prod_{a=1}^{k} w_\alpha(i_a).$$

This is the diagonal part stated in (a).

For the domination, we apply the Cauchy–Schwarz inequality to the $k!$ terms of the permanent, and we find

$$\frac{1}{k!}\big|\operatorname{perm}[u_{\alpha_b}(i_a)]\big|^2 \le \sum_{\pi\in S_k}\prod_{a=1}^{k} w_{\alpha_{\pi(a)}}(i_a).$$

Let $f_1,\dots,f_k \ge 0$. We multiply by $N^k\prod_b f_b(x_{\alpha_b})$ and sum over $\alpha_1,\dots,\alpha_k$. For fixed π, the sum factorizes into $\prod_a \ell_{i_a,N}(f_{\pi(a)})$, by the definition (9.1) of $\ell_{i,N}$. This is (9.2) with $C_0 = 1$.

For (b), let $c_\alpha \ge 0$, and consider the matrix

$$K = \sum_\alpha c_\alpha u_\alpha u_\alpha^*.$$

Expanding the permanent of $[K_{i_a i_b}]$, we have

$$\operatorname{perm}[K_{i_a i_b}] = \sum_{\alpha_1,\dots,\alpha_k}\prod_a c_{\alpha_a}u_{\alpha_a}(i_a)\sum_{\pi\in S_k}\prod_a \overline{u_{\alpha_a}(i_{\pi(a)})}.$$

The inner sum is the complex conjugate of $\operatorname{perm}[u_{\alpha_b}(i_a)]$, which is invariant under permutations of the labels $\alpha_1,\dots,\alpha_k$. Averaging the factor $\prod_a u_{\alpha_a}(i_a)$ over these permutations turns it into $\operatorname{perm}[u_{\alpha_b}(i_a)]/k!$. We obtain the permanental Cauchy–Binet identity

$$\operatorname{perm}[K_{i_a i_b}] = \frac{1}{k!}\sum_{\alpha_1,\dots,\alpha_k}\prod_b c_{\alpha_b}\,\big|\operatorname{perm}[u_{\alpha_b}(i_a)]\big|^2.$$

The spectral theorem and (1.3) give

$$\operatorname{Im} G(z_+) = \sum_\alpha \frac{\varepsilon/a_N}{(\lambda_\alpha - E)^2 + (\varepsilon/a_N)^2}\,u_\alpha u_\alpha^* = s_E N\sum_\alpha P_\varepsilon(x_\alpha)\,u_\alpha u_\alpha^*,$$

where we used $\pi a_N = s_E N$. Applying the identity with $c_\alpha = s_E N P_\varepsilon(x_\alpha)$ and taking expectations, we obtain (12.2).

For (c), we first recall the matrices Z_{B} and Z_{F} of (7.13), which for source rank one are the diagonal 2×2 matrices

$$Z_{\mathrm{B}} = \operatorname{diag}(z_+^{\mathrm{B}}, z_-^{\mathrm{B}}), \qquad Z_{\mathrm{F}} = \operatorname{diag}(z_{1,+}^{\mathrm{F}}, z_{1,-}^{\mathrm{F}})$$

in the flavors (R,A), with $z_\pm^{\mathrm{B}} = E \pm \mathrm{i}\varepsilon/a_N$ and $z_{1,\pm}^{\mathrm{F}} = E + (u_1 \pm \mathrm{i}b_1)/a_N$. In particular, at $(\mathbf{u},\mathbf{b}) = (0,\varepsilon)$, we have

$$Z_{\mathrm{B}} = Z_{\mathrm{F}} = \operatorname{diag}(z_+, \bar z_+).$$

We set

$$A = \sum_{a=1}^{k} t_a e_{i_a} e_{i_a}^*, \qquad M = \mathrm{Id}_N \otimes Z_{\mathrm{B}} - H \otimes \mathrm{Id}_2,$$

where, as in (7.14), the first factor of each tensor product acts on the sites and the second on the flavors. With the matrix units E_{RR} and E_{AA} of the flavors R and A, we have

$$M = (z_+ - H) \otimes E_{\mathrm{RR}} + (\bar{z}_+ - H) \otimes E_{\mathrm{AA}}, \qquad M^{-1} = -G(z_+) \otimes E_{\mathrm{RR}} - G(\bar{z}_+) \otimes E_{\mathrm{AA}}.$$

The local source $t_a Y$ at the site i_a has the BB block $-t_a \varrho v^{\mathrm{T}}$, and therefore

$$\sum_{a=1}^{k} e_{i_a} e_{i_a}^* \otimes t_a Y^{\mathrm{B}} = -A \otimes \varrho v^{\mathrm{T}} = -(A \otimes \varrho)(\mathrm{Id}_N \otimes v^{\mathrm{T}}),$$

where $A \otimes \varrho$ is a $2N \times N$ matrix and $\mathrm{Id}_N \otimes v^{\mathrm{T}}$ is an $N \times 2N$ matrix.

Since $Y^{\mathrm{F}} = 0$ and $Z_{\mathrm{F}} = Z_{\mathrm{B}}$, the ratio in (7.14) reduces to

$$\frac{\det M}{\det(M - A \otimes \varrho v^{\mathrm{T}})}.$$

We now apply Sylvester's determinant identity $\det(\mathrm{Id}_{2N} - VV') = \det(\mathrm{Id}_N - V'V)$ with $V = M^{-1}(A \otimes \varrho)$ and $V' = \mathrm{Id}_N \otimes v^{\mathrm{T}}$. Writing $\varrho = (\varrho_{\mathrm{R}}, \varrho_{\mathrm{A}})^{\mathrm{T}} = (1,1)^{\mathrm{T}}$ and $v = (v_{\mathrm{R}}, v_{\mathrm{A}})^{\mathrm{T}} = (\mathrm{i}/2, -\mathrm{i}/2)^{\mathrm{T}}$, we find

$$\begin{aligned} \frac{\det(M - A \otimes \varrho v^{\mathrm{T}})}{\det M} &= \det\big(\mathrm{Id}_N - (\mathrm{Id}_N \otimes v^{\mathrm{T}}) M^{-1} (A \otimes \varrho)\big) \\ &= \det\big(\mathrm{Id}_N + v_{\mathrm{R}} \varrho_{\mathrm{R}}\, G(z_+) A + v_{\mathrm{A}} \varrho_{\mathrm{A}}\, G(\bar{z}_+) A\big) \\ &= \det\big(\mathrm{Id}_N + \tfrac{\mathrm{i}}{2}\big(G(z_+) - G(\bar{z}_+)\big) A\big) = \det\big(\mathrm{Id}_N - \mathrm{Im}\, G(z_+)\, A\big), \end{aligned}$$

where in the second step we used $(\mathrm{Id}_N \otimes v^{\mathrm{T}})(G \otimes E_{\mathrm{RR}})(A \otimes \varrho) = v_{\mathrm{R}} \varrho_{\mathrm{R}}\, GA$ and its analogue for E_{AA}, and in the last step $G(\bar{z}_+) = G(z_+)^*$. Since $\det(\mathrm{Id}_N - \mathrm{Im}\, G(z_+) A) = \det(\mathrm{Id}_N - A\, \mathrm{Im}\, G(z_+))$ by Sylvester's identity, the ratio in (7.14) reduces to $\det(\mathrm{Id}_N - A\, \mathrm{Im}\, G(z_+))^{-1}$.

Conditionally on H, let $\zeta = (\zeta_1, \dots, \zeta_N)$ be a centered complex Gaussian vector with

$$\mathbb{E}[\zeta \zeta^* \mid H] = \mathrm{Im}\, G(z_+), \qquad \mathbb{E}[\zeta \zeta^{\mathrm{T}} \mid H] = 0.$$

For small t, the Gaussian integral gives

$$\det\big(\mathrm{Id}_N - A\, \mathrm{Im}\, G(z_+)\big)^{-1} = \mathbb{E}\Big[\exp\Big(\sum_a t_a |\zeta_{i_a}|^2 \Big) \;\Big|\; H \Big].$$

We differentiate at $t = 0$ and use Wick's theorem for complex Gaussian vectors. We find that $\partial_{t_1} \cdots \partial_{t_k}$ of the ratio at $t = 0$ is

$$\mathbb{E}\Big[\prod_a |\zeta_{i_a}|^2 \;\Big|\; H \Big] = \mathrm{perm}\,\big[(\mathrm{Im}\, G(z_+))_{i_a i_b}\big].$$

As noted after Definition 7.5, the derivatives may be taken under the expectation, and (12.3) follows. □

We now identify the limit of (12.2), using Theorem 7.6(b). This is the only step of this section that works with the sigma model directly; the other steps use only its residues, computed in Section 10. We first recall the objects involved. The field Q of Definition 5.2, at rank one, is a $(2|2)$ supermatrix of superfunctions on $\mathcal{M}_1$ with $Q^2 = \mathrm{Id}$, and $Q^{\mathrm{BB}}_{\mathrm{RA}}$ denotes the entry of its boson–boson block in the retarded row and the advanced column. Recall the supermatrix $\Lambda = \mathrm{diag}(\Lambda_{\mathrm{B}} \mid \Lambda_{\mathrm{F}})$ of Definition 4.1, which at rank one has the blocks $\Lambda_{\mathrm{B}} = \Lambda_{\mathrm{F}} = \mathrm{diag}(1, -1)$ in the flavors (R, A). The superfunction $d_1(Q) = \frac{1}{2}\,\mathrm{STr}(\Lambda Q)$ is the damping term of the generator $\mathsf{G}_{X,z} = X\mathsf{C}_1 + z\,d_1$ of (5.23), and in the polar coordinates of Section 6.1, we have $d_1 = B - F$, by Lemma 6.1(b). Let $e^{-t\mathsf{G}_{X,z}}$ be the semigroup of Proposition 5.15, and let $\mathcal{B}_1$ be the Berezin pairing of Definition 5.6. For $0 \le x_1 \le \cdots \le x_k \le 1$, the permanent pairing of Definition 10.6 is

$$\mathfrak{p}_k^{X,z}(x_1, \dots, x_k) = \mathcal{B}_1\big(e^{-x_1\mathsf{G}_{X,z}}1,\, \mathfrak{s}\, e^{-(x_2-x_1)\mathsf{G}_{X,z}}\mathfrak{s} \cdots e^{-(x_k-x_{k-1})\mathsf{G}_{X,z}}\mathfrak{s}\, e^{-(1-x_k)\mathsf{G}_{X,z}}1\big), \quad (12.4)$$

where $\mathfrak{s}$ is the source (10.19),

$$\mathfrak{s}(Q) = \tfrac{1}{2}\big(Q^{\mathrm{BB}}_{\mathrm{RR}} + Q^{\mathrm{BB}}_{\mathrm{RA}} - Q^{\mathrm{BB}}_{\mathrm{AR}} - Q^{\mathrm{BB}}_{\mathrm{AA}}\big).$$

Finally, recall the orbit resolvent $\mathcal{G}(Q) = -\kappa_E\mathrm{Id} + \mathrm{i}s_E Q$ of (7.16), and the orbit multiplier $\mathrm{j}_Y(Q) = \mathrm{SDet}(\mathrm{Id} - \mathcal{G}(Q)Y)^{-1}$ of a local source Y, whose Taylor coefficients $\mathrm{j}^{(\alpha)}$ in (7.17) are the multipliers of Theorem 7.6(b). Here SDet is the superdeterminant of Definition 3.10. For an even supermatrix M with blocks A, B, C, D, this definition gives, for small t,

$$\begin{aligned} \mathrm{SDet}(\mathrm{Id} + tM) &= \frac{\det\big(\mathrm{Id} + tA - t^2B(\mathrm{Id} + tD)^{-1}C\big)}{\det(\mathrm{Id} + tD)}, \\ \partial_t\,\mathrm{SDet}(\mathrm{Id} + tM)^{-1}\big|_{t=0} &= -\,\mathrm{Tr}\,A + \mathrm{Tr}\,D = -\,\mathrm{STr}\,M. \end{aligned} \quad (12.5)$$

Proposition 12.3. *Let $k \ge 1$ and $\varepsilon > 0$, and let $|E| \le E_{\mathrm{out}}$, with E_{out} as in* (7.22). *Let $i_1 < \cdots < i_k$ be sites with $\delta \le i_a/N \le 1 - \delta$ and $i_a/N \to x_a$, and suppose that $X_N(E) \to X \in (0, \infty)$. Then*

$$\int_{\mathbb{R}^k} \prod_{b=1}^{k} P_\varepsilon(t_b)\,\Pi_N(\,\mathrm{d}\mathbf{t}) \longrightarrow \mathfrak{p}_k^{X,2\pi\varepsilon}(x_1, \dots, x_k).$$

Proof. Passing to subsequences, we may assume that E converges. Let Y be the local source of Lemma 12.2(c). We begin with its multiplier.

Since $Y^{\mathrm{B}} = -\varrho v^{\mathrm{T}}$ and $Y^{\mathrm{F}} = 0$, we have $\mathrm{Id} - t\mathcal{G}(Q)Y = \mathrm{Id} + tM$ with $M = \mathcal{G}(Q)\,\mathrm{diag}(\varrho v^{\mathrm{T}} \mid 0)$. The supermatrix M has the BB block $\mathcal{G}(Q)^{\mathrm{BB}}\varrho v^{\mathrm{T}}$ and a vanishing FF block. Using (12.5), we find

$$\partial_t \mathrm{j}_{tY}(Q)\big|_{t=0} = -\,\mathrm{Tr}\,\big(\mathcal{G}(Q)^{\mathrm{BB}}\varrho v^{\mathrm{T}}\big) = -v^{\mathrm{T}}\big(-\kappa_E\mathrm{Id} + \mathrm{i}s_E Q^{\mathrm{BB}}\big)\varrho = s_E\,\mathfrak{s}(Q), \quad (12.6)$$

where in the last step we used $v^{\mathrm{T}}\varrho = 0$ and

$$v^{\mathrm{T}}Q^{\mathrm{BB}}\varrho = \tfrac{\mathrm{i}}{2}\big(Q^{\mathrm{BB}}_{\mathrm{RR}} + Q^{\mathrm{BB}}_{\mathrm{RA}} - Q^{\mathrm{BB}}_{\mathrm{AR}} - Q^{\mathrm{BB}}_{\mathrm{AA}}\big) = \mathrm{i}\,\mathfrak{s}(Q).$$

We next apply Theorem 7.6(b) at source rank one, with the $m = k$ sites $i_1 < \cdots < i_k$ and at the base point $(\mathbf{u}, \mathbf{b}) = (0, \varepsilon)$. By the chain rule, ∂_{t_a} at $t = 0$ is the derivative $\sum_e Y_e \, \partial_{(Y_a)_e}$ in the direction Y of the local source at i_a, where e runs over the entries of the BB block. We expand $\partial_{t_1} \cdots \partial_{t_k}$ in this way, and we apply (7.19) to each term, with first-order multi-indices. The limits are pairings that are linear in each multiplier, and by (12.6), their sum has the multipliers

$$\sum_e Y_e \, \mathfrak{j}^{(e)} = \partial_t \mathfrak{j}_{tY}\big|_{t=0} = s_E \mathfrak{s}.$$

The base point corresponds to the source supermatrix $\Gamma = \varepsilon\Lambda$ of Definition 4.1, with $\Lambda = \operatorname{diag}(1, -1 \mid 1, -1)$ as recalled above. By (5.22), the generator of Theorem 7.6(b) satisfies

$$\mathsf{G}_{X,\Gamma} = X\mathsf{C}_1 + 2\pi\varepsilon \, d_1 = \mathsf{G}_{X,2\pi\varepsilon}.$$

Writing $\mathsf{G} = \mathsf{G}_{X,2\pi\varepsilon}$, we conclude that Theorem 7.6(b) yields

$$\partial_{t_1} \cdots \partial_{t_k} T_N^{(1)}(0, \varepsilon; t_1 Y, \ldots, t_k Y)\big|_{t=0} \longrightarrow s_E^k \, \mathcal{B}_1\big(1, \, e^{-x_1 \mathsf{G}} \mathfrak{s} \, e^{-(x_2 - x_1)\mathsf{G}} \mathfrak{s} \cdots \mathfrak{s} \, e^{-(1-x_k)\mathsf{G}} 1\big). \tag{12.7}$$

We now show that the pairing on the right is the permanent pairing. Let

$$f = \mathfrak{s} \, e^{-(x_2 - x_1)\mathsf{G}} \mathfrak{s} \cdots \mathfrak{s} \, e^{-(1-x_k)\mathsf{G}} 1.$$

Since $x_k \le 1 - \delta$, the superfunction $e^{-(1-x_k)\mathsf{G}} 1$ is rapidly decreasing, by Proposition 5.15(b). The source $\mathfrak{s}$ is a multiplier of degree one, by Lemma 5.14(a), and multiplication by $\mathfrak{s}$ maps L^2_p to L^2_{p-1} for all p, by Lemma 5.13(a) and the definition (5.20) of the norm of L^2_p. The semigroup maps each L^2_p to itself, and hence $f \in L^2_p$ for all p. By Proposition 5.15(d), we also have $1 \in L^2_{-q}$ for $q > q_1/2$, where q_1 is the exponent of the volume bound in Lemma 5.1(c). The transpose symmetry Proposition 5.15(c), applied with $p = -q$, the superfunction $1 \in L^2_{-q}$, and $f \in L^2_q$, gives

$$\mathcal{B}_1\big(1, e^{-x_1 \mathsf{G}} f\big) = \mathcal{B}_1\big(e^{-x_1 \mathsf{G}} 1, f\big) = \mathfrak{p}_k^{X, 2\pi\varepsilon}(x_1, \ldots, x_k),$$

where the last equality is (12.4) with $z = 2\pi\varepsilon$. Finally, combining this identity with (12.7), (12.2), and (12.3), we have

$$\begin{aligned} \int_{\mathbb{R}^k} \prod_{b=1}^k P_\varepsilon(t_b) \, \Pi_N(\, \mathrm{d}\mathbf{t}) &= s_E^{-k} \, \partial_{t_1} \cdots \partial_{t_k} T_N^{(1)}(0, \varepsilon; t_1 Y, \ldots, t_k Y)\big|_{t=0} \\ &\longrightarrow s_E^{-k} \cdot s_E^k \, \mathfrak{p}_k^{X, 2\pi\varepsilon}(x_1, \ldots, x_k) = \mathfrak{p}_k^{X, 2\pi\varepsilon}(x_1, \ldots, x_k). \end{aligned}$$

□

12.2 All fixed moments

We first record the properties of the limiting moment densities that we use.

Lemma 12.4. *Let $k \geq 1$. The function $(X, \mathbf{x}) \mapsto M_k^X(\mathbf{x})$ is positive and continuous on $(0,\infty) \times [0,1]^k$, and symmetric in $\mathbf{x}$. Moreover, $M_1^X = 1$, and for $k \geq 2$,*

$$\int_0^1 M_k^X(x_1,\dots,x_{k-1},y)\,\mathrm{d}y = M_{k-1}^X(x_1,\dots,x_{k-1}), \qquad \int_{[0,1]^k} M_k^X(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1. \tag{12.8}$$

Proof. All the assertions follow from the definition of M_k^X in Definition 1.9, by dominated convergence and Tonelli's theorem. We realize all g_X with one Brownian motion β (with X as a parameter in Definition 1.9), and the map $(X,t) \mapsto g_X(t)$ is continuous. For $X \leq X_+$, as in the proof of Lemma 10.2, the integrand in the definition of M_k^X is bounded by

$$\exp\Big((2k-1)\big(\sqrt{2X_+}\sup_{t\leq 1}|\beta_t| + X_+\big)\Big),$$

which is integrable. The continuity follows by dominated convergence. Positivity and symmetry are clear, and $\mathbb{E}e^{g_X(x)} = 1$ implies $M_1^X = 1$. Integrating $e^{g_X(y)}$ over $y \in [0,1]$ gives Z_X, which proves (12.8). □

Theorem 12.5. *Let $k \geq 1$, $\delta \in (0,1/2)$, and let $\bar{r}_N \to 0$ be a deterministic sequence. Then, uniformly in the critical regime, in $0 < r \leq \bar{r}_N$, and in sites $i_1,\dots,i_k$ with $\delta \leq i_a/N \leq 1-\delta$ and $|i_a - i_b| \geq \delta N$ for $a \neq b$,*

$$N^k\Big\langle \prod_{a=1}^k w_\alpha(i_a)\Big\rangle_{E,r} - M_k^{X_N(E)}(i_1/N,\dots,i_k/N) \longrightarrow 0. \tag{12.9}$$

Proof. We apply Corollary 9.9 with

$$\chi = \phi, \qquad F_\alpha = \prod_a w_\alpha(i_a), \qquad \bar{c}_N(e) = M_k^{X_N(e)}(i_1/N,\dots,i_k/N).$$

The parameter $X_N(e) = N/((4-e^2)W^2)$ ranges over a compact subset of $(0,\infty)$ and is Lipschitz in e, uniformly. Together with Lemma 12.4, this shows that the functions $\bar{c}_N$ are bounded and equicontinuous on $[-E_{\mathrm{mid}}, E_{\mathrm{mid}}]$, uniformly in N and in the sites.

It remains to verify the hypothesis of Corollary 9.9. Recall that $\chi = \phi$ is the window of Definition 1.8, and that $f_{N,e}$ is the density of the measure (9.15), which here is

$$N^k\mathbb{E}\sum_\alpha \prod_{a=1}^k w_\alpha(i_a)\,\delta_{a_N(e)(\lambda_\alpha - e)},$$

the expected diagonal part of the permanent moment measure of Definition 12.1 at the sites $i_1,\dots,i_k$, unfolded at the energy e (Lemma 12.2(a)). The hypothesis of Corollary 9.9 is

$$\sup_{|e|\leq E_{\mathrm{mid}}} |f_{N,e}(0) - \bar{c}_N(e)| \to 0,$$

uniformly in the sites.

We derive this hypothesis from Remark 9.7, with the energy and the sites as parameters. Thus we take a sequence of energies $|e_N| \le E_{\mathrm{mid}}$ and of admissible sites, and we show that, along a subsequence, the densities f_{N,e_N} converge locally uniformly to a constant c_*, and that $\bar{c}_N(e_N) \to c_*$. Both statements are to hold uniformly in the sites.

For $k = 1$, the measure in question is the local spectral measure $\ell_{i_1,N}$, unfolded at e_N. Here Corollary 9.8 applies with $c_* = 1 = M_1^X$, since $\mathbb{E}\ell^{\varepsilon}_{i_1,N}(0) \to 1$ at interior sites for all $|e_N| \le E_{\mathrm{out}}$, by Proposition 11.5.

Let $k \ge 2$. We pass to a subsequence along which $X_N(e_N)$ converges to some $X \in (0,\infty)$ and $i_a/N \to x_a$, and we order the sites. Then $\delta \le x_a \le 1-\delta$ and $|x_a - x_b| \ge \delta$. We apply Theorem 9.6 at the energy e_N to the permanent moment measure of Definition 12.1, with $r_k = k!$.

The hypotheses of Theorem 9.6 hold for the following reasons. Lemma 12.2(a) shows that the measure is dominated at the sites with $C_0 = 1$ and has the diagonal part above. The source limit (9.14) holds with $\mathfrak{f}(z) = \mathfrak{p}_k^{X,z}(x_1,\dots,x_k)$, by Proposition 12.3. Finally, Lemma 10.12 and (9.13) give

$$\lim_{z\downarrow 0} z^{k-1}\mathfrak{f}(z) = k(2k-3)!!\, M_k^X(x_1,\dots,x_k) = \varkappa_{k,k!}\, M_k^X(x_1,\dots,x_k).$$

Thus Theorem 9.6 applies with $c_* = M_k^X(x_1,\dots,x_k)$, and $\bar{c}_N(e_N) \to c_*$ by Lemma 12.4. Now (9.16) implies (12.9) along every sequence, uniformly in $0 < r \le \bar{r}_N$. As the hypotheses hold along every sequence in the critical regime, the convergence is uniform there. □

12.3 The coarse law

We now prove Theorem 1.10. The moments of Theorem 12.5 control the tensor powers of the mass measure away from the diagonals and the boundary. We first show that this suffices in L^1. For $k \ge 1$ and $\mathbf{y} \in (0,1]^k$, let $j_a = \lceil Ny_a \rceil$, and define the *step density*

$$M_{k,N}(\mathbf{y}) = N^k \Big\langle \prod_{a=1}^{k} w_\alpha(j_a) \Big\rangle_{E,r}.$$

It depends on E, r, and W, which we do not display.

Lemma 12.6. *Let $k \ge 1$, and let $\bar{r}_N \to 0$ be a deterministic sequence. Then, uniformly in the critical regime and in $0 < r \le \bar{r}_N$,*

$$\big\| M_{k,N} - M_k^{X_N(E)} \big\|_{L^1([0,1]^k)} \longrightarrow 0. \tag{12.10}$$

Proof. The weights of each eigenvector sum to one, and therefore

$$M_{k,N} \ge 0, \qquad \int_{[0,1]^k} M_{k,N}(\mathbf{y})\, \mathrm{d}\mathbf{y} = 1.$$

Suppose that (12.10) fails. Then there exist $\eta > 0$ and a sequence of parameters in the critical regime, with $r \le \bar{r}_N$, along which the L^1 distance is at least η. We pass to a subsequence along which $X_N(E)$ converges to some $X \in (0, \infty)$.

Let $\mathbf{y} \in (0,1)^k$ have distinct coordinates, and choose $\delta > 0$ with

$$2\delta < \min_a \min(y_a, 1 - y_a), \qquad 2\delta < \min_{a \neq b} |y_a - y_b|.$$

For large N, the sites $j_a = \lceil N y_a \rceil$ satisfy the hypotheses of Theorem 12.5 with this δ. That theorem and the continuity of M_k^X in Lemma 12.4 give $M_{k,N}(\mathbf{y}) \to M_k^X(\mathbf{y})$. The excluded set of $\mathbf{y}$ has Lebesgue measure zero.

We now apply Scheffé's lemma. The function M_k^X is a probability density by (12.8), and

$$\| M_{k,N} - M_k^X \|_{L^1} = 2 - 2 \int \min(M_{k,N}, M_k^X).$$

The minimum converges almost everywhere to M_k^X and is bounded by it. Dominated convergence implies $\| M_{k,N} - M_k^X \|_{L^1} \to 0$. Finally, $\| M_k^{X_N(E)} - M_k^X \|_{L^1} \to 0$ by the continuity of Lemma 12.4. This contradicts the choice of the sequence. □

Proof of Theorem 1.10. Let $f \in C([0,1]^k)$. The definitions of $M_{k,N}$ and of the cells give

$$\begin{aligned} \Big\langle \int f \, \mathrm{d}(\mu_\alpha^{(N)})^{\otimes k} \Big\rangle_{E,r} &= \sum_{j_1, \dots, j_k} f(j_1/N, \dots, j_k/N) \Big\langle \prod_a w_\alpha(j_a) \Big\rangle_{E,r} \\ &= \int_{[0,1]^k} f(\lceil N\mathbf{y} \rceil / N) \, M_{k,N}(\mathbf{y}) \, \mathrm{d}\mathbf{y}. \end{aligned}$$

The difference between the right side and $\int f M_{k,N}$ is bounded by the modulus of continuity of f at $\sqrt{k}/N$. Recall from Definition 1.9 that M_k^X is the density of the k-th moment measure of μ_X under $\mathbb{Q}_X$. Together with Lemma 12.6, we obtain, uniformly in the critical regime and in $0 < r \le \bar{r}_N$,

$$\Big\langle \int f \, \mathrm{d}(\mu_\alpha^{(N)})^{\otimes k} \Big\rangle_{E,r} - \mathbb{E}_{\mathbb{Q}_{X_N(E)}} \int f \, \mathrm{d}\mu_{X_N(E)}^{\otimes k} \longrightarrow 0. \tag{12.11}$$

Let $\mathcal{A}$ denote the linear span of the constant function and of the functions on $\mathcal{P}([0,1])$ given by

$$\mu \mapsto \prod_{a=1}^k \mu(f_a) = \int (f_1 \otimes \cdots \otimes f_k) \, \mathrm{d}\mu^{\otimes k},$$

for $k \ge 1$ and $f_a \in C([0,1])$. By (12.11), the convergence holds for every $\Phi \in \mathcal{A}$ in place of $\int f \, \mathrm{d}\mu^{\otimes k}$. The space $\mathcal{P}([0,1])$ is compact and metrizable, and $\mathcal{A}$ is an algebra of continuous functions that contains the constants. It separates points, since two probability measures that agree on all continuous functions are equal. By the Stone–Weierstrass theorem, $\mathcal{A}$ is dense in $C(\mathcal{P}([0,1]))$ for the supremum norm.

Given a continuous Φ and $\eta > 0$, we choose $\Phi_\eta \in \mathcal{A}$ with $\|\Phi - \Phi_\eta\|_\infty \le \eta$. Both selections in Theorem 1.10 are probability measures, and the difference in Theorem 1.10 is bounded by 2η plus the corresponding difference for Φ_η, which tends to zero. The uniform statement follows.

For the last statement, it suffices to show that the map

$$X \mapsto \mathbb{E}_{\mathbb{Q}_X} \Phi(\mu_X) = \mathbb{E}[Z_X \Phi(\mu_X)]$$

is continuous. We realize all g_X with one Brownian motion. As $X' \to X$, we have $g_{X'} \to g_X$ uniformly on $[0,1]$, and $Z_{X'} \to Z_X$ and $\mu_{X'} \to \mu_X$ in total variation, almost surely. For $X' \le X_+$, we have

$$Z_{X'} \le \exp\Big(\sqrt{2X_+} \sup_{t \le 1} |\beta_t| + X_+\Big),$$

and the continuity follows from dominated convergence, since Φ is bounded. □

Remark 12.7. The limit in Theorem 1.10 is the law found by Rifkind and Virág for critical random Schrödinger operators [109, Theorem 1.1]. For the operator on $\{1, \dots, n\}$ whose potential has variance σ^2/n, they show that the mass of an eigenvector chosen uniformly at random converges in law to a limit that depends on a parameter τ, which is determined by σ and the energy. Let $\tau > 0$, let β' be a two-sided standard Brownian motion and U an independent uniform variable on $[0,1]$, and set $S(t) = \exp(\beta'_t/\sqrt{2} - |t|/4)$. Their limit is the law of the random probability measure

$$\frac{S(\tau(t-U))}{\int_0^1 S(\tau(s-U))\,\mathrm{d}s}\,\mathrm{d}t, \qquad t \in [0,1].$$

For $\tau = 4X$, this is the law of μ_X under $\mathbb{Q}_X$. Indeed, by Brownian scaling, $(\beta'_{4Xr}/\sqrt{2})_{r\in\mathbb{R}}$ has the law of $(\sqrt{2X}\beta'_r)_{r\in\mathbb{R}}$. Hence, for each $u \in [0,1]$, the process $(S(4X(t-u)))_{t\in[0,1]}$ has the law of

$$\Big(\exp\left(\sqrt{2X}\beta'_{t-u} - X|t-u|\right)\Big)_{t\in[0,1]}.$$

For a standard Brownian motion β on $[0,1]$, the process $(\beta_t - \beta_u)_{t\in[0,1]}$ has the law of $(\beta'_{t-u})_{t\in[0,1]}$, since its two branches from u are independent Brownian motions. Comparing with (1.16), we find that the measure above, given $U = u$, has the law of $p_{X,u}(t)\,\mathrm{d}t$. By Lemma 12.8, the mixture of these laws over a uniform U coincides with the law of μ_X under $\overline{\mathbb{Q}}_X$, and $\overline{\mathbb{Q}}_X$ has first marginal $\mathbb{Q}_X$. The same comparison identifies the pair (shape, U) of [109, Theorem 1.1] with (μ_X, U) under $\overline{\mathbb{Q}}_X$, which is the limit in Theorem 1.12(a).

Proof of Corollary 1.11. The exact identities (1.15) are elementary. They follow from the definition $\mathrm{d}\mathbb{Q}_X = Z_X\,\mathrm{d}\mathbb{P}$ and the moment densities $M_1^X = 1$ and $M_2^X = \mathcal{C}_X$, by Tonelli's theorem, for bounded measurable f and g.

For the matrix limits, let f and g be bounded and Riemann integrable, and set

$$f_N(y) = f(\lceil Ny \rceil / N), \qquad g_N(y) = g(\lceil Ny \rceil / N), \qquad y \in (0,1].$$

As in the proof of Theorem 1.10,

$$\langle \mu_\alpha^{(N)}(f) \rangle_{E,r} = \int_0^1 f_N M_{1,N} \, \mathrm{d}y,$$
$$\langle \mu_\alpha^{(N)}(f) \, \mu_\alpha^{(N)}(g) \rangle_{E,r} = \int_{[0,1]^2} f_N(y_1) g_N(y_2) M_{2,N}(y_1, y_2) \, \mathrm{d}y_1 \, \mathrm{d}y_2.$$

The functions f_N and g_N are bounded by $\|f\|_\infty$ and $\|g\|_\infty$, and they converge to f and g at every point of continuity, hence almost everywhere. By Lemma 12.6 at $k = 1$ and $k = 2$, we may replace $M_{1,N}$ and $M_{2,N}$ by $M_1^X = 1$ and $M_2^X = \mathcal{C}_X$, with an error that tends to zero. Dominated convergence then gives the limits

$$\int_0^1 f(y) \, \mathrm{d}y, \qquad \iint f(y_1) g(y_2) \mathcal{C}_X(y_1, y_2) \, \mathrm{d}y_1 \, \mathrm{d}y_2,$$

and the covariance converges to the right side of (1.15). □

12.4 Rooted laws

We now prove parts (a) and (b) of Theorem 1.12. Recall the measure $\overline{\mathbb{Q}}_X$ and the rooted density $p_{X,x}$ of (1.16). We first describe the law of μ_X under the tilt by $e^{g_X(x)}$, which is the law of the mass measure seen from a root at x.

Lemma 12.8. *Let $X > 0$ and $x \in [0,1]$. Under the probability measure $e^{g_X(x)} \, \mathrm{d}\mathbb{P}$, the random measure μ_X has the law of $p_{X,x}(t) \, \mathrm{d}t$ under $\mathbb{P}$. Under $\overline{\mathbb{Q}}_X$, the root U is uniform on $[0,1]$, and given $U = u$, the measure μ_X has the law of $p_{X,u}(t) \, \mathrm{d}t$ under $\mathbb{P}$. The branches $(\beta_{x-t} - \beta_x)_{t \in [0,x]}$ and $(\beta_{x+t} - \beta_x)_{t \in [0,1-x]}$ that enter $p_{X,x}$ are independent.*

Proof. The density $e^{g_X(x)} = \exp(\sqrt{2X}\beta_x - Xx)$ is the Cameron–Martin density of the drift $\sqrt{2X}\mathbf{1}_{[0,x]}$. By the Cameron–Martin theorem [108], under $e^{g_X(x)} \, \mathrm{d}\mathbb{P}$, the process

$$\hat{\beta}_t = \beta_t - \sqrt{2X} \, (t \wedge x)$$

is a standard Brownian motion. In terms of $\hat{\beta}$, we have

$$g_X(t) - g_X(x) = \sqrt{2X}(\hat{\beta}_t - \hat{\beta}_x) + 2X(t \wedge x - x) - X(t - x) = \sqrt{2X}(\hat{\beta}_t - \hat{\beta}_x) - X|t - x|.$$

The measure μ_X has the density

$$\frac{e^{g_X(t) - g_X(x)}}{\int_0^1 e^{g_X(s) - g_X(x)} \, \mathrm{d}s},$$

and the first statement follows. The second follows from the definition of $\overline{\mathbb{Q}}_X$, whose second marginal is uniform, and the third from the independence of the increments of $\hat{\beta}$. □

Proof of Theorem 1.12(a) and (b). For (a), let Φ be continuous on $\mathcal{P}([0,1]) \times [0,1]$, and set

$$\widetilde{\Phi}(\mu) = \int_0^1 \Phi(\mu, t)\, \mu(\,\mathrm{d}t).$$

We first check that $\widetilde{\Phi}$ is continuous on $\mathcal{P}([0,1])$. Let $\mu_n \to \mu$. As Φ is uniformly continuous, we have

$$\int \Phi(\mu_n, t)\mu_n(\,\mathrm{d}t) - \int \Phi(\mu, t)\mu_n(\,\mathrm{d}t) \to 0, \qquad \int \Phi(\mu, t)\mu_n(\,\mathrm{d}t) \to \int \Phi(\mu, t)\mu(\,\mathrm{d}t).$$

Now Theorem 1.10 and the continuity of $X \mapsto \mathbb{E}_{\mathbb{Q}_X}\widetilde{\Phi}(\mu_X)$, shown in its proof, imply that the left side in (a) converges to

$$\mathbb{E}_{\mathbb{Q}_X}\widetilde{\Phi}(\mu_X) = \mathbb{E}\Big[Z_X \int_0^1 \Phi(\mu_X, t)\frac{e^{g_X(t)}}{Z_X}\,\mathrm{d}t\Big] = \mathbb{E}\int_0^1 \Phi(\mu_X, t)e^{g_X(t)}\,\mathrm{d}t = \mathbb{E}_{\overline{\mathbb{Q}}_X}\Phi(\mu_X, U),$$

uniformly in $0 < r \le \bar{r}_N$.

For (b), let $i = i_N$ be sites such that i/N converges to some $x \in (0,1)$. For $k \ge 1$, $\mathbf{y} \in (0,1]^k$, and $j_a = \lceil Ny_a \rceil$, we define the rooted step density as in Section 12.3. By Definition 1.8,

$$M^i_{k,N}(\mathbf{y}) = N^k \Big\langle \prod_a w_\alpha(j_a) \Big\rangle^i_{E,r} = \frac{N^{k+1}\langle w_\alpha(i)\prod_a w_\alpha(j_a)\rangle_{E,r}}{N\langle w_\alpha(i)\rangle_{E,r}}.$$

Let $\mathbf{y} \in (0,1)^k$ have distinct coordinates, all different from x. Applying Theorem 12.5 at the orders $k+1$ and 1, we see that the numerator converges to $M^X_{k+1}(x, \mathbf{y})$ and the denominator to one. The rooted step densities are nonnegative with integral one, and their limit $M^X_{k+1}(x, \cdot)$ has integral $M^X_1(x) = 1$, by (12.8). As in the proof of Lemma 12.6, Scheffé's lemma implies that

$$\big\| M^i_{k,N} - M^X_{k+1}(x, \cdot) \big\|_{L^1} \to 0,$$

uniformly in $0 < r \le \bar{r}_N$.

As in the proof of Theorem 1.10, we find

$$\Big\langle \int f\,\mathrm{d}(\mu^{(N)}_\alpha)^{\otimes k} \Big\rangle^i_{E,r} \longrightarrow \int_{[0,1]^k} f(\mathbf{y}) M^X_{k+1}(x, \mathbf{y})\,\mathrm{d}\mathbf{y} = \mathbb{E}\Big[e^{g_X(x)} \int f\,\mathrm{d}\mu_X^{\otimes k}\Big]$$

for all $f \in C([0,1]^k)$, where the last identity is the definition of M^X_{k+1}. By Lemma 12.8, the right side is

$$\mathbb{E}\int f\,\mathrm{d}\big(p_{X,x}(t)\,\mathrm{d}t\big)^{\otimes k}.$$

Part (b) now follows from the Stone–Weierstrass argument of the proof of Theorem 1.10. □

12.5 The infinite wire

As $X \to \infty$, the mass seen from the root lives on the scale $1/X$. In the notation of Remark 12.7, the limit $p_\infty(s)\,\mathrm{d}s$ of Theorem 1.12(c) is the normalized measure $S(4s)\,\mathrm{d}s$ on $\mathbb{R}$, by Brownian scaling. This is the shape of Dumaz and Labbé [35, Theorem 3], centered at the root instead of at its center of mass. We first identify the side states of the infinite wire. Recall g_1, A_s, and Y_s from Section 10.2, and set $A_\infty = \int_0^\infty e^{g_1(u)}\,\mathrm{d}u$.

Lemma 12.9. *(a) The variable A_∞ is finite almost surely, and it has the law of $1/\gamma$, where γ is a standard exponential variable.*

(b) For all $q > 0$, $\mathrm{Y}_s(q)$ decreases to $\mathrm{Y}(q) = \mathbb{E}e^{-qA_\infty} = 2\sqrt{q}\,K_1(2\sqrt{q})$ as $s \to \infty$, and $\mathrm{Y}_s \to \mathrm{Y}$ in $L^2((0,\infty),\,\mathrm{d}q)$.

(c) For all $s, p > 0$, we have $\mathbb{E}A_s^{-p} \le s^{-p}\exp(ps/2 + p^2 s/3)$.

Proof. The law of large numbers for β implies that $g_1(u)/u \to -1$ almost surely, and hence $A_\infty < \infty$ almost surely. For (a), the process $\tilde\beta_u = \beta_{2u}/\sqrt{2}$ is a standard Brownian motion, and

$$g_1(2u) = 2(\tilde\beta_u - u), \qquad A_\infty = 2\int_0^\infty \exp\big(2(\tilde\beta_u - u)\big)\,\mathrm{d}u.$$

Dufresne's identity [33], in the form of [88, Theorem 6.2], states that for $\mu > 0$,

$$\int_0^\infty \exp\big(2(\tilde\beta_u - \mu u)\big)\,\mathrm{d}u \quad \text{has the law of} \quad \frac{1}{2\gamma_\mu},$$

where γ_μ has the Gamma law with parameter μ. With $\mu = 1$, we obtain (a).

For (b), monotone convergence and (a) give

$$\mathrm{Y}_s(q) = \mathbb{E}e^{-qA_s} \downarrow \mathbb{E}e^{-qA_\infty} = \int_0^\infty e^{-q/x - x}\,\mathrm{d}x = 2\sqrt{q}\,K_1(2\sqrt{q}).$$

The last equality is [30, (10.32.10)] with $\nu = -1$ and $z = 2\sqrt{q}$, together with $K_{-1} = K_1$ [30, (10.27.3)]. Let $s \ge s_0 > 0$. Then $0 \le \mathrm{Y}_s \le \mathrm{Y}_{s_0} \le 1$, and

$$\int_0^\infty \mathrm{Y}_{s_0}(q)^2\,\mathrm{d}q \le \int_0^\infty \mathrm{Y}_{s_0}(q)\,\mathrm{d}q = \mathbb{E}A_{s_0}^{-1},$$

which is finite by (c) (with $p = 1$). The L^2 convergence follows by dominated convergence.

For (c), we note that Jensen's inequality implies

$$A_s \ge s\exp\Big(\frac{1}{s}\int_0^s g_1(u)\,\mathrm{d}u\Big).$$

The variable $s^{-1}\int_0^s g_1(u)\,\mathrm{d}u$ is Gaussian with mean $-s/2$ and variance

$$\frac{2}{s^2}\int_0^s\int_0^s (u \wedge v)\,\mathrm{d}u\,\mathrm{d}v = \frac{2s}{3}.$$

Its moment generating function gives

$$\mathbb{E}A_s^{-p} \le s^{-p}\,\mathbb{E}\exp\Big(-\frac{p}{s}\int_0^s g_1(u)\,\mathrm{d}u\Big) = s^{-p}\exp\Big(\frac{ps}{2}+\frac{p^2 s}{3}\Big). \qquad \square$$

Recall the two-sided Brownian motion $(\beta_s)_{s\in\mathbb{R}}$ of (1.17). We set

$$g_\infty(s) = \sqrt{2}\,\beta_s - |s|, \qquad Z_\infty = \int_{\mathbb{R}} e^{g_\infty(s)}\,\mathrm{d}s, \qquad Z_\pm = \int_0^\infty e^{g_\infty(\pm s)}\,\mathrm{d}s,$$

so that $p_\infty = e^{g_\infty}/Z_\infty$ and $Z_\infty = Z_+ + Z_-$. The variables Z_+ and Z_- are independent, and each has the law of A_∞.

Proof of Theorem 1.12(c). Let $X \ge 1$ and $u \in (0,1)$. Recall the map $\sigma_{X,u}(t) = X(t-u)$, defined before Theorem 1.12, and let

$$J_{X,u} = \sigma_{X,u}([0,1]) = [-Xu, X(1-u)].$$

Given $U = u$, Lemma 12.8 describes μ_X under $\overline{\mathbb{Q}}_X$ through the standard Brownian motion $\hat\beta$ defined in the proof of that lemma, with $x = u$. The pushforward of $p_{X,u}(t)\,\mathrm{d}t$ under $\sigma_{X,u}$ has the density

$$s \mapsto X^{-1} p_{X,u}(u + s/X), \qquad s \in J_{X,u}.$$

By Brownian scaling, the process

$$\check\beta_s = \sqrt{X}\,(\hat\beta_{u+s/X} - \hat\beta_u), \qquad s \in J_{X,u},$$

is the restriction of a two-sided standard Brownian motion to $J_{X,u}$ (its two branches are independent, by Lemma 12.8), and

$$\sqrt{2X}\,(\hat\beta_{u+s/X} - \hat\beta_u) - X|s/X| = \sqrt{2}\,\check\beta_s - |s|.$$

Consequently, the pushforward has the law of

$$p^{(X,u)}(s)\,\mathrm{d}s, \qquad p^{(X,u)}(s) = \frac{e^{g_\infty(s)}\mathbf{1}_{J_{X,u}}(s)}{Z_{X,u}}, \qquad Z_{X,u} = \int_{J_{X,u}} e^{g_\infty}.$$

We now realize all these laws on one probability space. Let $(\beta_s)_{s\in\mathbb{R}}$ be a two-sided standard Brownian motion, g_∞ as above, and U' an independent uniform variable on $[0,1]$. By the preceding paragraph, the law of $((\sigma_{X,U})_*\mu_X, U)$ under $\overline{\mathbb{Q}}_X$ is the law of $(p^{(X,U')}(s)\,\mathrm{d}s, U')$. As $X \to \infty$, the intervals $J_{X,U'}$ increase to $\mathbb{R}$ almost surely, and $Z_{X,U'} \uparrow Z_\infty$. We claim that

$$\big\|p^{(X,U')} - p_\infty\big\|_{L^1(\mathbb{R})} = 2\Big(1 - \frac{Z_{X,U'}}{Z_\infty}\Big) \longrightarrow 0 \qquad \text{almost surely.} \tag{12.12}$$

Indeed, on $J_{X,U'}$, the difference of the densities is

$$e^{g_\infty}\big(Z_{X,U'}^{-1} - Z_\infty^{-1}\big) \geq 0,$$

with integral $1 - Z_{X,U'}/Z_\infty$. This is also the mass of p_∞ outside $J_{X,U'}$, and the L^1 distance is twice this value.

Convergence in L^1 of densities implies weak convergence of the measures. Thus

$$\big(p^{(X,U')}(s)\,\mathrm{d}s, U'\big) \to \big(p_\infty(s)\,\mathrm{d}s, U'\big) \qquad \text{almost surely in } \mathcal{P}(\mathbb{R}) \times [0,1],$$

and the laws converge weakly. The same argument with $u = x$ fixed proves the second statement. □

12.6 The mean shape

We now prove Corollary 1.13, and we show that $X^{-1}\mathcal{C}_X(x,x) \to \bar{p}_\infty(0)$, as announced after Corollary 1.13. The first step writes the mean shape in terms of the operator $\mathscr{L}$ of Lemma 10.1.

Lemma 12.10. *For all $s \geq 0$,*

$$\bar{p}_\infty(s) = \int_0^\infty \mathrm{Y}(q)\big(e^{-s\mathscr{L}}\mathrm{Y}\big)(q)\,\mathrm{d}q = \big\langle \mathrm{Y}, e^{-s\mathscr{L}}\mathrm{Y}\big\rangle_{L^2((0,\infty),\,\mathrm{d}q)}. \tag{12.13}$$

In particular, $\bar{p}_\infty(0) = \int_0^\infty \mathrm{Y}(q)^2\,\mathrm{d}q$. Moreover, for $x \in (0,1)$, we have $X^{-1}\mathcal{C}_X(x,x) \downarrow \bar{p}_\infty(0)$ as $X \to \infty$.

Proof. By the symmetry $\beta \mapsto (\beta_{-s})_{s\in\mathbb{R}}$, it suffices to consider $s \geq 0$. We insert

$$\frac{1}{Z_\infty} = \int_0^\infty e^{-qZ_\infty}\,\mathrm{d}q,$$

and we use the independence of Z_+ and Z_- and Tonelli's theorem. We find

$$\bar{p}_\infty(s) = \int_0^\infty \mathbb{E}e^{-qZ_-}\,\mathbb{E}\big[e^{g_\infty(s)}e^{-qZ_+}\big]\,\mathrm{d}q.$$

Since Z_- has the law of A_∞, Lemma 12.9(b) gives $\mathbb{E}e^{-qZ_-} = \mathrm{Y}(q)$. For the second factor, we split

$$Z_+ = \int_0^s e^{g_\infty} + e^{g_\infty(s)}Z'_s, \qquad Z'_s = \int_0^\infty e^{g_\infty(s+v)-g_\infty(s)}\,\mathrm{d}v,$$

where Z'_s is independent of $(\beta_v)_{0\leq v\leq s}$ and has the law of A_∞ (by the Markov property of β at time s). We have $g_\infty = g_1$ on $[0,\infty)$, with g_1 built from β. Conditioning on $(\beta_v)_{0\leq v\leq s}$ and using Lemma 12.9(b), we obtain

$$\mathbb{E}\big[e^{g_\infty(s)}e^{-qZ_+}\big] = \mathbb{E}\big[e^{g_1(s)}e^{-qA_s}\mathrm{Y}(qe^{g_1(s)})\big] = q^{-1}\mathcal{T}_s(q\mathrm{Y})(q) = \big(e^{-s\mathscr{L}}\mathrm{Y}\big)(q),$$

where the last step is (10.7). The second identity in (12.13) follows from Lemma 10.1(c), since $Y \in L^2$ by Lemma 12.9(b).

For the last statement, Lemma 12.8 and the definition of $\mathcal{C}_X$ give

$$\mathcal{C}_X(x,x) = \mathbb{E}\Big[e^{g_X(x)}\frac{e^{g_X(x)}}{Z_X}\Big] = \mathbb{E}\, p_{X,x}(x) = X\,\mathbb{E}\big[Z_{X,x}^{-1}\big],$$

where $Z_{X,x}$ is the normalization of $p^{(X,x)}$ in the proof of Theorem 1.12(c), by the scaling used there. As X increases, $J_{X,x}$ increases to $\mathbb{R}$, and $Z_{X,x}^{-1}$ decreases to Z_∞^{-1}. For $X \geq 1/\min(x, 1-x)$, the interval $J_{X,x}$ contains $[-1,1]$, and

$$\mathbb{E}Z_{X,x}^{-1} \leq \mathbb{E}\Big(\int_{-1}^{1} e^{g_\infty}\Big)^{-1} < \infty,$$

by Jensen's inequality, as in Lemma 12.9(c). Monotone convergence yields

$$X^{-1}\mathcal{C}_X(x,x) = \mathbb{E}Z_{X,x}^{-1} \downarrow \mathbb{E}Z_\infty^{-1} = \bar{p}_\infty(0). \qquad \square$$

To compute the right side of (12.13), we diagonalize $\mathscr{L}$. By Lemma 10.1(b), the map $\mathcal{U}$ conjugates $\mathscr{L}$ to the closure $\mathscr{H}$ of $-\partial_y^2 + e^y + 1/4$ on $C_c^\infty(\mathbb{R})$. For $k > 0$ and $y \in \mathbb{R}$, we set

$$\psi_k(y) = \frac{2}{\pi}\big(k\sinh(2\pi k)\big)^{1/2} K_{2\mathrm{i}k}(2e^{y/2}).$$

The function $K_{\mathrm{i}\tau}(x)$ is real for $x, \tau > 0$, and ψ_k is real. It is bounded, since $K_{\mathrm{i}\tau}(x)$ is bounded as $x \downarrow 0$ and decays exponentially as $x \to \infty$.

Lemma 12.11. *There exists an isometry $\mathcal{W}\colon L^2(\mathbb{R}, \mathrm{d}y) \to L^2((0,\infty), \mathrm{d}k)$ such that $(\mathcal{W}v)(k) = \int_{\mathbb{R}} \psi_k(y)v(y)\,\mathrm{d}y$ for almost every k, for all $v \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and*

$$\big\langle v, e^{-s\mathscr{H}}v\big\rangle_{L^2(\mathbb{R})} = \int_0^\infty e^{-s(k^2+1/4)}|(\mathcal{W}v)(k)|^2\,\mathrm{d}k, \qquad v \in L^2(\mathbb{R}),\ s \geq 0. \tag{12.14}$$

Proof. The lemma is the Kontorovich–Lebedev inversion [30, Section 10.43(v)], written in the variables of $\mathscr{H}$. The normalization of ψ_k enters the constant of Corollary 1.13, and we first check it. Let ω be a function on $(0,\infty)$ such that $x^{-1}\omega(x)$ is continuously differentiable on $(0,\infty)$, and the functions $x\omega(x)$ and $x\,\mathrm{d}(x^{-1}\omega(x))/\,\mathrm{d}x$ are absolutely integrable on $(0,\infty)$; this is condition (a) of [30, Section 10.43(v)]. Then [30, (10.43.30)–(10.43.31)] state that

$$\omega(x) = \int_0^\infty \Omega(\tau)K_{\mathrm{i}\tau}(x)\,\mathrm{d}\tau, \qquad \Omega(\tau) = \frac{2\tau\sinh(\pi\tau)}{\pi^2}\int_0^\infty \omega(x)K_{\mathrm{i}\tau}(x)\frac{\mathrm{d}x}{x}. \tag{12.15}$$

Let $v \in C_c^\infty(\mathbb{R})$, and set

$$(\mathcal{W}v)(k) = \int_{\mathbb{R}} \psi_k(y)v(y)\,\mathrm{d}y, \qquad k > 0.$$

The function $\omega(x) = v(2\log(x/2))$ is smooth with compact support in $(0,\infty)$ and satisfies condition (a). We substitute $x = 2e^{y/2}$, for which $\mathrm{d}x/x = \mathrm{d}y/2$, and $\tau = 2k$. By the definition of ψ_k, the second identity in (12.15) becomes

$$\Omega(2k) = \frac{2k\sinh(2\pi k)}{\pi^2}\int_{\mathbb{R}} v(y)K_{2\mathrm{i}k}(2e^{y/2})\,\mathrm{d}y = \frac{(k\sinh(2\pi k))^{1/2}}{\pi}\,(\mathcal{W}v)(k),$$

and the first identity, at $x = 2e^{y/2}$, becomes

$$v(y) = 2\int_0^\infty \Omega(2k)K_{2\mathrm{i}k}(2e^{y/2})\,\mathrm{d}k = \int_0^\infty \psi_k(y)(\mathcal{W}v)(k)\,\mathrm{d}k. \tag{12.16}$$

Thus the factor $\frac{2}{\pi}(k\sinh(2\pi k))^{1/2}$ in ψ_k is the one for which the transform $\mathcal{W}$ and the expansion (12.16) are inverse to each other.

Next, we deduce the isometry. Since [30, Section 10.43(v)] does not state it, we include the short argument. We multiply (12.16) by $\overline{v(y)}$ and integrate over y, and for the interchange of the integrals we use two bounds.

First, by [30, (10.32.9)], we have

$$K_{\mathrm{i}\tau}(x) = \frac{1}{2}\int_{\mathbb{R}} e^{-x\cosh t + \mathrm{i}\tau t}\,\mathrm{d}t, \qquad x > 0,\ \tau \ge 0.$$

We shift the contour to $\mathbb{R} + \mathrm{i}\theta$ with $0 \le \theta < \pi/2$, which is allowed since $|e^{-x\cosh(t+\mathrm{i}\theta)}| = e^{-x\cos\theta\cosh t}$, and we arrive at

$$|K_{\mathrm{i}\tau}(x)| \le e^{-\theta\tau}K_0(x\cos\theta), \qquad 0 \le \theta < \pi/2. \tag{12.17}$$

With $\tau = 2k$ and $\theta = \pi/2 - (1+2k)^{-1}$, this implies that

$$|\psi_k(y)| \le C(1+k)^{1/2}\big(1+\log(1+k)\big),$$

uniformly for y in compact sets.

Second, the modified Bessel equation $(x\partial_x)^2 K_{\mathrm{i}\tau} = (x^2-\tau^2)K_{\mathrm{i}\tau}$ of [30, (10.25.1)] becomes $(-\partial_y^2 + e^y)\psi_k = k^2\psi_k$, since $x\partial_x = 2\partial_y$, and two integrations by parts give

$$\mathcal{W}(\mathscr{H}v) = (k^2 + 1/4)\,\mathcal{W}v, \qquad v \in C_c^\infty(\mathbb{R}). \tag{12.18}$$

Applying (12.18) n times and using the first bound, we find $|(\mathcal{W}v)(k)| \le C_n(1+k)^{-n}$ for all n. Hence the double integral converges absolutely, and since ψ_k is real, we obtain

$$\int_{\mathbb{R}} |v(y)|^2\,\mathrm{d}y = \int_0^\infty |(\mathcal{W}v)(k)|^2\,\mathrm{d}k, \qquad v \in C_c^\infty(\mathbb{R}).$$

Therefore $\mathcal{W}$ extends from $C_c^\infty(\mathbb{R})$ to an isometry. For $v \in L^1 \cap L^2$, we approximate v by $v_n \in C_c^\infty$ in L^1 and L^2. As ψ_k is bounded, the integrals of $\psi_k v_n$ converge to that of $\psi_k v$ for every k, and a subsequence of $\mathcal{W}v_n$ converges almost everywhere to $\mathcal{W}v$.

It remains to prove (12.14). Let M denote multiplication by $k^2 + 1/4$. By (12.18), we have

$$\mathcal{W}(\mathscr{H} \pm \mathrm{i})^{-1} w = (\mathsf{M} \pm \mathrm{i})^{-1} \mathcal{W} w, \qquad w \in (\mathscr{H} \pm \mathrm{i}) C_c^\infty(\mathbb{R}).$$

This set is dense, since $\mathscr{H}$ is essentially self-adjoint on $C_c^\infty(\mathbb{R})$ by Lemma 10.1(b). By continuity, the identity holds for all $w \in L^2$.

By the Stone–Weierstrass theorem, the polynomials in $(\lambda + \mathrm{i})^{-1}$ and $(\lambda - \mathrm{i})^{-1}$ are dense in the continuous functions on $[1/4, \infty]$. Functional calculus implies that $\mathcal{W} e^{-s\mathscr{H}} = e^{-s\mathsf{M}} \mathcal{W}$, and since $\mathcal{W}$ is an isometry, we conclude that

$$\bigl\langle v, e^{-s\mathscr{H}} v \bigr\rangle = \bigl\langle \mathcal{W} v, e^{-s\mathsf{M}} \mathcal{W} v \bigr\rangle,$$

which is (12.14). □

Proof of Corollary 1.13. The function $\bar{p}_\infty$ is even, by the symmetry $\beta \mapsto (\beta_{-s})_{s \in \mathbb{R}}$ of the two-sided Brownian motion, and Tonelli's theorem gives

$$\int_{\mathbb{R}} \bar{p}_\infty(s) \, \mathrm{d}s = \mathbb{E} \int_{\mathbb{R}} p_\infty(s) \, \mathrm{d}s = 1.$$

We next compute $\bar{p}_\infty(0)$, by Beta–Gamma algebra. By Lemma 12.9(a), the variables $\gamma_\pm = 1/Z_\pm$ are independent standard exponential variables, and

$$\bar{p}_\infty(0) = \mathbb{E} \frac{1}{Z_+ + Z_-} = \mathbb{E} \frac{\gamma_+ \gamma_-}{\gamma_+ + \gamma_-}.$$

The sum $\gamma_+ + \gamma_-$ has the Gamma law with parameter two, the ratio $\gamma_+ / (\gamma_+ + \gamma_-)$ is uniform on $[0, 1]$, and the two are independent. We write

$$\frac{\gamma_+ \gamma_-}{\gamma_+ + \gamma_-} = (\gamma_+ + \gamma_-) \cdot \frac{\gamma_+}{\gamma_+ + \gamma_-} \cdot \frac{\gamma_-}{\gamma_+ + \gamma_-},$$

and we obtain

$$\bar{p}_\infty(0) = \mathbb{E}[\gamma_+ + \gamma_-] \int_0^1 v(1 - v) \, \mathrm{d}v = 2 \cdot \frac{1}{6} = \frac{1}{3}.$$

For the tail, we apply Lemma 12.10, the identity

$$e^{-s\mathscr{L}} = \mathcal{U}^{-1} e^{-s\mathscr{H}} \mathcal{U},$$

and Lemma 12.11 with $v = \mathcal{U}\mathrm{Y}$. The function $\mathcal{U}\mathrm{Y}(y) = e^{y/2} \mathrm{Y}(e^y)$ is integrable and square integrable, since $\mathrm{Y} \le \mathrm{Y}_1 \le \bar{\mathrm{Y}}_{1,1}$ by Lemma 10.3. We obtain

$$\bar{p}_\infty(s) = e^{-s/4} \int_0^\infty e^{-sk^2} \varpi(k) \, \mathrm{d}k, \qquad \varpi(k) = \Bigl| \int_{\mathbb{R}} \psi_k(y) e^{y/2} \mathrm{Y}(e^y) \, \mathrm{d}y \Bigr|^2. \tag{12.19}$$

We now compute ϖ. With $x = 2e^{y/2}$, we have

$$e^{y/2} \mathrm{Y}(e^y) = 2e^y K_1(x) = \frac{x^2}{2} K_1(x), \qquad \mathrm{d}y = \frac{2 \, \mathrm{d}x}{x},$$

and therefore

$$\int_{\mathbb{R}} \psi_k(y) e^{y/2} \Upsilon(e^y)\, \mathrm{d}y = \frac{2}{\pi} \big(k \sinh(2\pi k)\big)^{1/2} \int_0^\infty x K_1(x) K_{2\mathrm{i}k}(x)\, \mathrm{d}x.$$

We evaluate the last integral with three identities. The product formula [30, (10.32.17)] gives

$$K_1(x) K_{2\mathrm{i}k}(x) = 2 \int_0^\infty K_{1+2\mathrm{i}k}(2x \cosh t) \cosh\big((1-2\mathrm{i}k)t\big)\, \mathrm{d}t.$$

Next, [30, (10.43.19)] with $\mu = 2$ and $\nu = 1 + 2\mathrm{i}k$, and the substitution $x' = 2x \cosh t$, give

$$\int_0^\infty x K_{1+2\mathrm{i}k}(2x \cosh t)\, \mathrm{d}x = (2 \cosh t)^{-2}\, \Gamma(\tfrac{1}{2} - \mathrm{i}k) \Gamma(\tfrac{3}{2} + \mathrm{i}k).$$

The remaining integral is a beta integral. With $w' = e^{2t}$,

$$\int_0^\infty \frac{\cosh(at)}{\cosh^2 t}\, \mathrm{d}t = \int_0^\infty \frac{(w')^{a/2}}{(1+w')^2}\, \mathrm{d}w' = \Gamma(1 + \tfrac{a}{2}) \Gamma(1 - \tfrac{a}{2}), \qquad |\operatorname{Re} a| < 2,$$

which we use with $a = 1 - 2\mathrm{i}k$. We combine these three identities with

$$\big|\Gamma(\tfrac{1}{2} + \mathrm{i}k)\big|^2 = \frac{\pi}{\cosh(\pi k)}, \qquad \Gamma(\tfrac{3}{2} + \mathrm{i}k) = (\tfrac{1}{2} + \mathrm{i}k) \Gamma(\tfrac{1}{2} + \mathrm{i}k),$$

where the first identity is [30, (5.4.4)]. We find

$$\int_0^\infty x K_1(x) K_{2\mathrm{i}k}(x)\, \mathrm{d}x = \frac{1}{2} \big|\Gamma(\tfrac{1}{2} + \mathrm{i}k)\big|^2 \big|\Gamma(\tfrac{3}{2} + \mathrm{i}k)\big|^2 = \frac{\pi^2 (k^2 + 1/4)}{2 \cosh^2(\pi k)}.$$

Inserting this into the definition of ϖ, we obtain

$$\varpi(k) = 2\pi^2 k \Big(k^2 + \frac{1}{4}\Big)^2 \frac{\tanh(\pi k)}{\cosh^2(\pi k)}. \tag{12.20}$$

Setting $s = 0$ in (12.19) and using $\bar{p}_\infty(0) = 1/3$, we find

$$\int_0^\infty \varpi(k)\, \mathrm{d}k = \frac{1}{3}.$$

A numerical evaluation of this integral of (12.20) gives the same value, which confirms the normalization of Lemma 12.11.

Finally, (12.20) implies that

$$\varpi(k) = \frac{\pi^3}{8} k^2 + O(k^4) \quad \text{as } k \downarrow 0, \qquad \varpi(k) \le C k^5 e^{-\pi k} \quad \text{for } k \ge 1.$$

We split the integral in (12.19) at $k = 1$ and use

$$\int_0^\infty k^2 e^{-sk^2}\, \mathrm{d}k = \frac{\sqrt{\pi}}{4} s^{-3/2}, \qquad \int_0^\infty k^4 e^{-sk^2}\, \mathrm{d}k = \frac{3\sqrt{\pi}}{8} s^{-5/2}.$$

We obtain

$$\int_0^\infty e^{-sk^2}\varpi(k)\,\mathrm{d}k = \frac{\pi^{7/2}}{32\,s^{3/2}}\big(1+O(s^{-1})\big), \qquad s\to\infty,$$

which proves (1.18). □

The second moment of the mean shape is also explicit.

Lemma 12.12. *We have* $\int_{\mathbb{R}} s^2\bar{p}_\infty(s)\,\mathrm{d}s = 8\zeta(3)$.

Proof. By the evenness of $\bar{p}_\infty$, (12.19) and Tonelli's theorem,

$$\int_{\mathbb{R}} s^2\bar{p}_\infty(s)\,\mathrm{d}s = 2\int_0^\infty \varpi(k)\int_0^\infty s^2 e^{-s(k^2+1/4)}\,\mathrm{d}s\,\mathrm{d}k = 4\int_0^\infty \frac{\varpi(k)}{(k^2+1/4)^3}\,\mathrm{d}k = 8\pi^2 J,$$

where we used (12.20) and set

$$J = \int_0^\infty \frac{k\tanh(\pi k)}{\cosh^2(\pi k)(k^2+1/4)}\,\mathrm{d}k.$$

It remains to show that $J = \zeta(3)/\pi^2$, and we do this by a residue computation. We use the identity

$$\tanh(\pi k)\operatorname{sech}^2(\pi k) = -\frac{1}{2\pi}\,\partial_k \operatorname{sech}^2(\pi k),$$

where $\operatorname{sech} = 1/\cosh$. An integration by parts without boundary terms gives

$$J = \frac{1}{2\pi}\int_0^\infty \operatorname{sech}^2(\pi k)\,\frac{1/4-k^2}{(k^2+1/4)^2}\,\mathrm{d}k = \frac{1}{2\pi}\operatorname{Re}\int_0^\infty f(k)\,\mathrm{d}k, \qquad f(k) = \frac{\operatorname{sech}^2(\pi k)}{(1/2+\mathrm{i}k)^2}.$$

For real k, we have $f(-k) = \overline{f(k)}$, and therefore

$$J = \frac{1}{4\pi}\int_{\mathbb{R}} f(k)\,\mathrm{d}k.$$

We now close the contour in the lower half-plane. Let $n \geq 1$ be an integer and $R > 0$, and consider the rectangle with vertices $\pm R$ and $\pm R - \mathrm{i}n$. On its bottom side, we have

$$|\cosh(\pi(x-\mathrm{i}n))| = \cosh(\pi x), \qquad |1/2+\mathrm{i}k| \geq n,$$

and this side contributes $O(n^{-2})$. On the vertical sides, $|\cosh(\pi k)| \geq \sinh(\pi R)$, and they contribute $O(ne^{-2\pi R})$.

Inside the rectangle, the poles of f are the zeros $k_m = -\mathrm{i}(m+1/2)$, $0 \leq m < n$, of $\cosh(\pi k)$. Near k_m, we have $\cosh(\pi k) = -\mathrm{i}(-1)^m \sinh(\pi(k-k_m))$, and hence

$$\operatorname{sech}^2(\pi k) = -\frac{1}{\pi^2(k-k_m)^2} + O(1),$$

with no term of order $(k-k_m)^{-1}$, since $\sinh^2$ is even. We write $(1/2+\mathrm{i}k)^2 = -(k-\mathrm{i}/2)^2$. The residue of f at k_m is therefore

$$\frac{1}{\pi^2}\,\partial_k (k-\mathrm{i}/2)^{-2}\big|_{k=k_m} = -\frac{2}{\pi^2}\big(-\mathrm{i}(m+1)\big)^{-3} = \frac{2\mathrm{i}}{\pi^2(m+1)^3}.$$

The rectangle is traversed clockwise, and we let $R\to\infty$ and then $n\to\infty$. We obtain

$$\int_{\mathbb{R}} f(k)\,\mathrm{d}k = -2\pi\mathrm{i}\sum_{m\ge 0}\frac{2\mathrm{i}}{\pi^2(m+1)^3} = \frac{4}{\pi}\,\zeta(3),$$

and we conclude that

$$J = \frac{1}{4\pi}\cdot\frac{4\zeta(3)}{\pi} = \frac{\zeta(3)}{\pi^2}. \qquad \square$$

Remark 12.13. The tail (1.18), with the same constant, is the asymptotic form of the eigenfunction correlation of an infinite wire obtained by Efetov and Larkin [38, (56)], in units in which their localization length L_c is one. Ivanov, Ostrovsky, and Skvortsov [71, (75)] give the zero-frequency limit $R(0,x)$ of the correlation of the local density of states of an infinite wire, which describes the decay of a single eigenfunction, in the form

$$R(0,x) = 4\pi^2\partial_t^2\int_0^\infty k\tanh(\pi k)\cosh^{-2}(\pi k)e^{-(k^2+1/4)t}\,\mathrm{d}k,$$

where $t = x/\xi$. By (12.19) and (12.20), we have $R(0,x) = 2\bar{p}_\infty(t)$, where the factor two is their normalization of the correlator. The expansion of Efetov and Larkin [38, (57)] corresponds to the second moment $\int_{\mathbb{R}} s^2\bar{p}_\infty(s)\,\mathrm{d}s = 8\zeta(3)$, which is Lemma 12.12. (The supplementary script `coarse_constants.py` confirms this value, and the tail constant, numerically.)

12.7 Endpoints of the coarse law

We now prove the third and fourth rows of the table in Proposition 1.16.

Proposition 12.14. *As $X\downarrow 0$, the law of μ_X under $\mathbb{Q}_X$ converges weakly to the point mass δ_{Leb}, and $C_X(x,y)\to 1$ uniformly in $x,y\in[0,1]$. As $X\to\infty$, the law of μ_X under $\mathbb{Q}_X$ converges weakly to the law of δ_U, with U uniform on $[0,1]$, and for $0<x<y<1$, there exists C such that*

$$C_X(x,y)\le CXe^{-X(y-x)/4}, \qquad X\ge 1/\min(x,1-y). \tag{12.21}$$

Proof. For $X\downarrow 0$, we realize all g_X with one Brownian motion. Then $g_X\to 0$ uniformly on $[0,1]$, and $Z_X\to 1$ and the density of μ_X converges uniformly to one, almost surely. For $X\le 1$, the variables Z_X and the integrands in the definition of M_2^X are bounded by

$$\exp\Big(3\big(\sqrt{2}\sup_{t\le 1}|\beta_t|+1\big)\Big),$$

which is integrable. Dominated convergence yields

$$\mathbb{E}[Z_X \Phi(\mu_X)] \to \Phi(\mathrm{Leb})$$

for every continuous Φ on $\mathcal{P}([0,1])$, and $\mathcal{C}_X \to 1$ uniformly.

For $X \to \infty$, we use the coupling in the proof of Theorem 1.12(c). Under $\overline{\mathbb{Q}}_X$, the pair (μ_X, U) has the law of $(\mu^{(X)}, U')$, where $\mu^{(X)}$ is the pushforward of $p^{(X,U')}(s)\,\mathrm{d}s$ under $s \mapsto U' + s/X$. Let $\eta > 0$. The Wasserstein distance satisfies

$$\begin{aligned} W_1(\mu^{(X)}, \delta_{U'}) = \int |t - U'|\, \mu^{(X)}(\,\mathrm{d}t) &\le \eta + \int_{|s|>X\eta} p^{(X,U')}(s)\,\mathrm{d}s \\ &\le \eta + \bigl\| p^{(X,U')} - p_\infty \bigr\|_{L^1} + \int_{|s|>X\eta} p_\infty(s)\,\mathrm{d}s. \end{aligned}$$

By (12.12), the right side tends to η almost surely, and $W_1(\mu^{(X)}, \delta_{U'}) \to 0$ almost surely. Since $\mathbb{Q}_X$ is the first marginal of $\overline{\mathbb{Q}}_X$, the law of μ_X under $\mathbb{Q}_X$ converges weakly to the law of $\delta_{U'}$.

For (12.21), Lemma 12.8 and the scaling of the proof of Theorem 1.12(c) give

$$\mathcal{C}_X(x,y) = \mathbb{E}\, p_{X,x}(y) = X\,\mathbb{E}\Bigl[\frac{e^{g_\infty(s)}}{Z_{X,x}}\Bigr], \qquad s = X(y-x).$$

As $X \ge 1/\min(x, 1-y)$, the interval $J_{X,x}$ contains $[-1,0]$ and $[s, s+1]$. Let

$$Z^{(0)} = \int_{-1}^{0} e^{g_\infty(v)}\,\mathrm{d}v, \qquad Z^{(s)} = \int_0^1 e^{g_\infty(s+v) - g_\infty(s)}\,\mathrm{d}v.$$

Then

$$Z_{X,x} \ge Z^{(0)} + e^{g_\infty(s)} Z^{(s)}, \qquad \frac{e^{g_\infty(s)}}{Z_{X,x}} \le \min\Bigl(\frac{e^{g_\infty(s)}}{Z^{(0)}}, \frac{1}{Z^{(s)}}\Bigr) \le \frac{e^{g_\infty(s)/2}}{(Z^{(0)} Z^{(s)})^{1/2}}.$$

The variables $g_\infty(s)$, $Z^{(0)}$, and $Z^{(s)}$ depend on the increments of β on $[0,s]$, $[-1,0]$, and $[s, s+1]$, and they are independent. The exponent $g_\infty(s)/2$ is Gaussian with mean $-s/2$ and variance $s/2$, and

$$\mathbb{E} e^{g_\infty(s)/2} = \mathbb{E} e^{\beta_s/\sqrt{2} - s/2} = e^{-s/4},$$

and the variables $Z^{(0)}$ and $Z^{(s)}$ have the same law, with $\mathbb{E}(Z^{(0)})^{-1/2} < \infty$ by Jensen's inequality, as in Lemma 12.9(c). Together, these bounds give (12.21). □

SECTION 13

The metallic expansion

In this section we prove Proposition 1.17, which gives the first corrections to the eigenvector correlation $\mathcal{C}_X$ and to the function I_X as $X \downarrow 0$. We also prove that the kernel $K_X(x, y; z)$ tends to $(1 - e^{-2z})/z^2 = 2I_{\mathrm{GUE}}(z)$ as $X \downarrow 0$, uniformly in $0 < x \leq y < 1$. This limit is the metallic endpoint of the last row of Proposition 1.16, which is proved in Section 14.

The two statements of Proposition 1.17 have different proofs. The correlation $\mathcal{C}_X$ is a Brownian functional, and we expand it by Taylor's theorem in the parameter $\sqrt{2X}$ of a Gaussian integral (Section 13.1). The function I_X is defined in Definition 1.4 through the diffusion $(\tilde{B}_t, \tilde{F}_t)$ with generator $X\Delta_D$. Heuristically, the semigroups in (1.6) act almost as multiplication operators for small X, since the diffusion moves only by $O(\sqrt{X})$ during the times x, $y - x$, and $1 - y$. In the limiting case $X = 0$, the semigroups would be the multiplications by e^{-tzd}, the singlet state would be e^{-tzd}, and the kernel would be $\int_D e^{-zd}\,\mathrm{d}B\,\mathrm{d}F$. We expand the semigroups in powers of X by Duhamel's formula, with remainders in closed form (Section 13.3). The main point is to bound these remainders uniformly over the closed time simplex $0 \leq x \leq y \leq 1$. The difficulty is that some of the three times vanish on the faces of the simplex, where the corresponding factors give no decay in B. We overcome this with an envelope for $\log(1 + \tilde{d}_t)$, where $\tilde{d}_t = \tilde{B}_t - \tilde{F}_t$ is the function $d = B - F$ of Definition 1.4 along the diffusion. This is the quantity $\log(1 + \gamma\tilde{d}_t)$ of Lemmas 13.2 and 13.3 with $\gamma = 1$, the value used in this section; Section 14 uses other values of γ. The envelope shows that the killing accumulated over all the time intervals decays faster than every power of B. The coefficients are then elementary integrals, which we evaluate in Section 13.4. In Section 13.5, we express them through the Neumann Laplacian and compare them, heuristically, with the corrections of Kravtsov and Mirlin [80].

13.1 The eigenvector correlation

Recall the Brownian motion β and the objects g_X, Z_X, and $\mathcal{C}_X$ of Definition 1.9, and the Neumann Green kernel G_{N} of Proposition 1.17.

Lemma 13.1. *Let $X_0 > 0$. There exists $C > 0$ such that for all $X \in (0, X_0]$,*

$$\sup_{x,y\in[0,1]} \left|\mathcal{C}_X(x, y) - 1 - 2XG_{\mathrm{N}}(x, y)\right| \leq CX^2. \tag{13.1}$$

Proof. For $\tau \in \mathbb{R}$ and $x, y \in [0,1]$, we set

$$g^\tau(u) = \tau\beta_u - \frac{\tau^2 u}{2}, \qquad Z^\tau = \int_0^1 e^{g^\tau(u)}\,\mathrm{d}u, \qquad R^\tau(x,y) = \frac{e^{g^\tau(x)+g^\tau(y)}}{Z^\tau}.$$

Since $g_X = g^{\sqrt{2X}}$, we have $\mathcal{C}_X(x,y) = \mathbb{E}R^{\sqrt{2X}}(x,y)$. We will show that $\tau \mapsto \mathbb{E}R^\tau(x,y)$ is an even function of class C^4 whose fourth derivative is bounded uniformly in x, y, and $|\tau| \le \sqrt{2X_0}$, and we will compute its second derivative at $\tau = 0$.

We begin by justifying differentiation under the expectation. Let $\tau_0 = \sqrt{2X_0}$, and set

$$\beta^* = \sup_{u\in[0,1]} |\beta_u|.$$

Differentiating $e^{g^\tau(u)}$ j times in τ produces $e^{g^\tau(u)}$ times a polynomial of degree at most j in $\beta_u - \tau u$ and u. For $|\tau| \le \tau_0$ and $u \in [0,1]$, we have

$$\left|\partial_\tau^j e^{g^\tau(u)}\right| \le C_j(1+\beta^*+\tau_0)^j e^{g^\tau(u)}, \qquad \left|\partial_\tau^j Z^\tau\right| \le C_j(1+\beta^*+\tau_0)^j Z^\tau,$$

and the same bound holds for $e^{g^\tau(x)+g^\tau(y)}$. Expanding the derivatives of $R^\tau = e^{g^\tau(x)+g^\tau(y)}(Z^\tau)^{-1}$ by the Leibniz rule, we find that for $0 \le j \le 4$

$$\sup_{x,y\in[0,1]} \left|\partial_\tau^j R^\tau(x,y)\right| \le C_j(1+\beta^*+\tau_0)^j R^\tau(x,y) \le C_j(1+\beta^*+\tau_0)^j e^{3\tau_0\beta^*+\tau_0^2/2}, \tag{13.2}$$

where we used $e^{g^\tau(x)+g^\tau(y)} \le e^{2\tau_0\beta^*}$ and $Z^\tau \ge e^{-\tau_0\beta^*-\tau_0^2/2}$ in the last step. The reflection principle implies that β^* has Gaussian tails, and the right side of (13.2) is therefore integrable.

Therefore $\tau \mapsto \mathbb{E}R^\tau(x,y)$ is of class C^4 on $[-\tau_0,\tau_0]$, with derivatives $\mathbb{E}\partial_\tau^j R^\tau(x,y)$. Further, its fourth derivative is bounded uniformly in x and y. Since $-\beta$ is a Brownian motion, $R^{-\tau}(x,y)$ has the same law as $R^\tau(x,y)$. It follows that $\mathbb{E}R^\tau(x,y)$ is an even function of τ, whose odd derivatives vanish at $\tau = 0$. Taylor's theorem and $R^0 = 1$ then yield

$$\mathbb{E}R^\tau(x,y) = 1 + \frac{\tau^2}{2}\mathbb{E}\partial_\tau^2 R^\tau(x,y)\big|_{\tau=0} + O(\tau^4), \tag{13.3}$$

uniformly in $x, y \in [0,1]$ and $|\tau| \le \tau_0$.

We now compute the second derivative. Let

$$\beta^{(1)} = \int_0^1 \beta_u\,\mathrm{d}u, \qquad \beta^{(2)} = \int_0^1 (\beta_u^2 - u)\,\mathrm{d}u.$$

At $\tau = 0$, we have

$$\partial_\tau e^{g^\tau(x)+g^\tau(y)} = \beta_x + \beta_y, \qquad \partial_\tau^2 e^{g^\tau(x)+g^\tau(y)} = (\beta_x+\beta_y)^2 - (x+y),$$
$$\partial_\tau Z^\tau = \beta^{(1)}, \qquad \partial_\tau^2 Z^\tau = \beta^{(2)}.$$

Differentiating the quotient twice, we find

$$\partial_\tau^2 R^\tau(x,y)\big|_{\tau=0} = (\beta_x+\beta_y)^2 - (x+y) - 2(\beta_x+\beta_y)\beta^{(1)} - \beta^{(2)} + 2\big(\beta^{(1)}\big)^2.$$

We have $\mathbb{E}\beta^{(2)} = 0$, $\mathbb{E}(\beta^{(1)})^2 = 1/3$, $\mathbb{E}\beta_x\beta_y = \min(x,y)$, and

$$\mathbb{E}\beta_x\beta^{(1)} = \int_0^1 \min(x,u)\,\mathrm{d}u = x - \frac{x^2}{2}.$$

Using these identities and $\min(x,y) - x - y = -\max(x,y)$, we conclude that

$$\mathbb{E}\partial_\tau^2 R^\tau(x,y)\big|_{\tau=0} = 2\min(x,y) - 2x - 2y + x^2 + y^2 + \frac{2}{3} = 2G_{\mathrm{N}}(x,y).$$

Inserting this into (13.3) with $\tau = \sqrt{2X}$ proves (13.1). □

13.2 Feynman–Kac preliminaries

We now collect the facts about the diffusions of Definition 1.4 and Remark 1.5 that the expansion of I_X uses. Since Section 14 uses them with a different time scale and a different size of the killing, we state them with general parameters.

We say that a function on D is smooth if it is the restriction to D of a smooth function on an open neighborhood of D in $\mathbb{R}^2$, and we let $C^\infty(D)$ denote the space of these functions. A smooth function of (B,F) is a smooth function on $\mathbb{H}^2 \times S^2$ (the hyperbolic cosine B and the cosine F of the two distances to the base points are smooth functions of the squares of these distances). For $f, g \in C^\infty(D)$, we set

$$\langle \nabla_D f, \nabla_D g\rangle = (B^2-1)\,\partial_B f\,\overline{\partial_B g} + (1-F^2)\,\partial_F f\,\overline{\partial_F g}, \qquad |\nabla_D f|^2 = \langle \nabla_D f, \nabla_D f\rangle,$$

which is the inner product of the Riemannian gradients on $\mathbb{H}^2 \times S^2$, linear in the first argument and conjugate linear in the second. We say that a function f on D has *polynomial growth* if there exist $C, q \geq 0$ such that $|f| \leq C(1+d)^q$ on D. Note that $1 \leq B \leq 1+d$, since $|F| \leq 1$. In particular, every polynomial in B and F has polynomial growth. We will often use the identities

$$\Delta_D d = 2(B+F), \qquad |\nabla_D d|^2 = B^2 - F^2 = d(B+F), \qquad 0 \leq B+F \leq d+2. \tag{13.4}$$

Recall that the radial diffusion $(\tilde{B}_t, \tilde{F}_t)$ of Definition 1.4 has generator $X\Delta_D$, and that the singlet diffusion (B_t, F_t) of Remark 1.5 has generator $X\Delta_D^{\mathrm{s}}$, where $\Delta_D^{\mathrm{s}} = d\Delta_D d^{-1}$, and is stopped at the corner $(1,1)$. We write $\tilde{d}_t = \tilde{B}_t - \tilde{F}_t$ and $d_t = B_t - F_t$. For a measurable function f of polynomial growth, $t \geq 0$, and $\operatorname{Re} z > 0$, we define $e^{t(X\Delta_D - zd)}f$ by the Feynman–Kac formula of Definition 1.4, whose expectation is finite by (13.6) below. By the Markov property of $(\tilde{B}_t, \tilde{F}_t)$, these operators form a semigroup on the functions of polynomial growth, and

$$\big|e^{t(X\Delta_D - zd)}f\big| \leq e^{t(X\Delta_D - (\operatorname{Re} z)d)}|f|. \tag{13.5}$$

Lemma 13.2. *Let X, γ, and T be positive, and let $\Theta \geq X(1+2\gamma)T$.*

(a) For $f \in C^\infty(D)$ and $p \in D$, the process

$$f(\tilde{B}_t, \tilde{F}_t) - f(p) - X\int_0^t \Delta_D f(\tilde{B}_s, \tilde{F}_s)\,\mathrm{d}s, \qquad t \geq 0,$$

is a continuous local martingale under $\mathbb{E}_p$, with quadratic variation

$$2X\int_0^t |\nabla_D f|^2(\tilde{B}_s, \tilde{F}_s)\,\mathrm{d}s.$$

For a complex martingale M, quadratic variation here means the positive process $[\operatorname{Re} M] + [\operatorname{Im} M]$.

(b) Let $(R_t)_{t\geq 0}$ denote either $\gamma\tilde{d}_t$ under $\mathbb{E}_p$ or γd_t under $\mathbb{E}_p^{\mathrm{s}}$, where in the second case p lies in the interior of D, and set $\ell_t = \log(1+R_t)$. There exists a continuous local martingale N with $N_0 = 0$ and $\langle N\rangle_T \leq 2\Theta$ such that $\ell_t - \ell_0 - N_t$ is a continuous process of finite variation with values in $[-\Theta, 2\Theta]$ for $t \leq T$.

(c) With R_t as in (b), for all $m \geq 0$,

$$\sup_{t\leq T} \mathbb{E}(1+R_t)^m \leq e^{m(m+2)\Theta}(1+R_0)^m. \tag{13.6}$$

(d) With R_t as in (b), and with $\mathbb{P}$ denoting the law under which R is defined,

$$\mathbb{P}\Big(\inf_{t\leq T}(1+R_t) < (1+R_0)^{1/2}\Big) \leq \exp\Big(-\frac{\big(\frac{1}{2}\log(1+R_0)-\Theta\big)_+^2}{4\Theta}\Big). \tag{13.7}$$

Proof. We begin with (a), and first let $p \in D^\circ$. By the boundary classification that precedes (6.7), the radial diffusion started at p stays in D°. Its coordinates solve the stochastic differential equations (6.7), driven by independent standard Brownian motions β^1 and β^2. In particular,

$$\mathrm{d}\langle\tilde{B}\rangle_t = 2X(\tilde{B}_t^2 - 1)\,\mathrm{d}t, \qquad \mathrm{d}\langle\tilde{F}\rangle_t = 2X(1-\tilde{F}_t^2)\,\mathrm{d}t, \qquad \langle\tilde{B},\tilde{F}\rangle_t = 0.$$

Since f is smooth on a neighborhood of D, Itô's formula applies to $f(\tilde{B}_t, \tilde{F}_t)$, and we find

$$\begin{aligned}\mathrm{d}f(\tilde{B}_t,\tilde{F}_t) = X\Delta_D f(\tilde{B}_t,\tilde{F}_t)\,\mathrm{d}t + \big(2X(\tilde{B}_t^2-1)\big)^{1/2}\partial_B f(\tilde{B}_t,\tilde{F}_t)\,\mathrm{d}\beta_t^1 \\ + \big(2X(1-\tilde{F}_t^2)\big)^{1/2}\partial_F f(\tilde{B}_t,\tilde{F}_t)\,\mathrm{d}\beta_t^2,\end{aligned}$$

where we used $\Delta_D = (B^2-1)\partial_B^2 + 2B\partial_B + (1-F^2)\partial_F^2 - 2F\partial_F$. The local martingale part has the quadratic variation $2X\int_0^t |\nabla_D f|^2(\tilde{B}_s,\tilde{F}_s)\,\mathrm{d}s$, which proves (a) for $p \in D^\circ$.

Now let p lie on the boundary of D. We write N_t for the process in (a), and we set

$$A_t = 2X\int_0^t |\nabla_D f|^2(\tilde{B}_s,\tilde{F}_s)\,\mathrm{d}s.$$

The diffusion does not reach the boundary of D at positive times (see the discussion after Definition 1.4). For $\varepsilon > 0$, the Markov property at time ε and the first case show that $N_t - N_\varepsilon$ and $|N_t - N_\varepsilon|^2 - (A_t - A_\varepsilon)$, $t \geq \varepsilon$, are continuous local martingales for the filtration of the diffusion. For $n \geq 1$, let T_n be the first time t with $|N_t| + A_t \geq n$. The stopped increments

$$N_{t\wedge T_n} - N_{\varepsilon\wedge T_n}, \qquad |N_{t\wedge T_n} - N_{\varepsilon\wedge T_n}|^2 - (A_{t\wedge T_n} - A_{\varepsilon\wedge T_n}), \qquad t \geq \varepsilon,$$

are bounded martingales: they vanish on $\{T_n \leq \varepsilon\}$, and on $\{T_n > \varepsilon\}$ the preceding local martingales are stopped at T_n. We let $\varepsilon \downarrow 0$, using $N_0 = A_0 = 0$ and bounded convergence, and find that $N_{t\wedge T_n}$ and $|N_{t\wedge T_n}|^2 - A_{t\wedge T_n}$ are martingales on $[0,\infty)$. Since $T_n \to \infty$, (a) follows for p.

We next prove (b) for the radial diffusion. Let $\ell(r) = \log(1+\gamma r)$ for $r \geq 0$. Applying (a) to $\ell(d)$, we see that $\ell_t - \ell_0 - X\int_0^t \Delta_D \ell(d)(\tilde B_s, \tilde F_s)\,\mathrm{d}s$ is a local martingale with quadratic variation $2X\int_0^t |\nabla_D \ell(d)|^2(\tilde B_s, \tilde F_s)\,\mathrm{d}s$. The identities (13.4) give

$$\Delta_D \ell(d) = \frac{2\gamma(B+F)}{1+\gamma d} - \frac{\gamma^2 d(B+F)}{(1+\gamma d)^2}, \qquad |\nabla_D \ell(d)|^2 = \frac{\gamma^2 d(B+F)}{(1+\gamma d)^2}.$$

We write $R = \gamma d$. The last inequality in (13.4) implies that $0 \leq \gamma(B+F) \leq R + 2\gamma$, and we find

$$0 \leq \frac{2\gamma(B+F)}{1+R} \leq \frac{2(R+2\gamma)}{1+R} \leq 2(1+2\gamma), \qquad 0 \leq \frac{\gamma^2 d(B+F)}{(1+R)^2} \leq \frac{R(R+2\gamma)}{(1+R)^2} \leq 1+2\gamma.$$

Multiplying by X, we obtain

$$-X(1+2\gamma) \leq X\Delta_D \ell(d) \leq 2X(1+2\gamma), \qquad 2X|\nabla_D \ell(d)|^2 \leq 2X(1+2\gamma).$$

These bounds, integrated over $[0,t]$ with $t \leq T$, imply (b).

For the singlet diffusion, the drift $-2Xd^{-1}\langle \nabla_D \cdot, \nabla_D d\rangle$ of $X\Delta_D^{\mathrm{s}}$ cancels the first-order term of $\Delta_D \ell(d)$. Indeed, by (13.4),

$$\Delta_D^{\mathrm{s}} \ell(d) = \Delta_D \ell(d) - \frac{2\ell'(d)|\nabla_D d|^2}{d} = \ell''(d)|\nabla_D d|^2 = -\frac{\gamma^2 d(B+F)}{(1+\gamma d)^2},$$

which is smooth on D and vanishes at the corner. By the stochastic differential equation for the singlet diffusion in Lemma 6.5, Itô's formula applies to $\ell(d)$ up to the time at which the corner is reached, and ℓ_t is constant after this time. The bounds above then imply (b), with $\ell_t - \ell_0 - N_t$ nonincreasing.

For (c), we apply Itô's formula to $e^{m\ell_t} = (1+R_t)^m$. By the bounds in the proof of (b), its drift is bounded by $m(2+m)X(1+2\gamma)e^{m\ell_t}$. We stop at the exit time of $\{R \leq n\}$, take expectations, and apply Gronwall's lemma. Letting $n \to \infty$ and using Fatou's lemma, we obtain (13.6).

We now prove (d). We may assume that $\delta = \frac{1}{2}\log(1+R_0) - \Theta$ is positive. Using (b), we see that on the event in (13.7),

$$\inf_{t\le T} N_t \le -\frac{1}{2}\log(1+R_0) + \Theta = -\delta.$$

For $\sigma > 0$, the process $\exp(-\sigma N_t - \sigma^2\langle N\rangle_t/2)$ is a nonnegative local martingale, hence a supermartingale, and $\langle N\rangle_T \le 2\Theta$. Doob's maximal inequality implies that

$$\mathbb{P}\Big(\inf_{t\le T} N_t \le -\delta\Big) \le \mathbb{P}\Big(\sup_{t\le T} e^{-\sigma N_t - \sigma^2\langle N\rangle_t/2} \ge e^{\sigma\delta - \sigma^2\Theta}\Big) \le e^{-\sigma\delta+\sigma^2\Theta},$$

and the choice $\sigma = \delta/(2\Theta)$ gives (13.7). □

We will use Lemma 13.2 through the following bound on the Feynman–Kac weight of a path along which polynomial factors and killing factors are collected at finitely many times. For $\lambda > 0$, $\Theta > 0$, and $\tau \ge 0$, we define

$$\mathcal{E}_\tau(R) = \exp\big(-\lambda\tau\big((1+R)^{1/2}-1\big)\big) + \exp\Big(-\frac{\big(\frac{1}{2}\log(1+R)-\Theta\big)_+^2}{8\Theta}\Big), \qquad R \ge 0, \tag{13.8}$$

and we suppress the dependence on λ and Θ in the notation. The function $\mathcal{E}_\tau$ is nonincreasing in τ and bounded by two. Observe that the second term in (13.8) decays faster than every power of $1+R$, since its logarithm behaves like $-(\log(1+R))^2/(32\Theta)$ as $R\to\infty$. In particular, $(1+R)^q\mathcal{E}_\tau(R)$ is integrable over $[0,\infty)$ for all $q \ge 0$ and $\tau > 0$.

Lemma 13.3. *Let X, γ, T, Θ, and R_t be as in Lemma 13.2, and let $\lambda > 0$, $n \ge 1$, and $q \ge 0$. Let $0 \le t_1 \le \cdots \le t_n \le T$ and $r_1, \dots, r_n \ge 0$, and set $\tau = t_n + r_1 + \cdots + r_n$. Then*

$$\mathbb{E}\Big[\prod_{i=1}^n (1+R_{t_i})^q e^{-\lambda r_i R_{t_i}} \exp\Big(-\lambda\int_0^{t_n} R_s\,ds\Big)\Big] \le e^{2nq(nq+1)\Theta}(1+R_0)^{nq}\,\mathcal{E}_\tau(R_0). \tag{13.9}$$

Proof. Let G denote the event that $1+R_t \ge (1+R_0)^{1/2}$ for all $t \le T$. The exponential factors in (13.9) satisfy, on G,

$$\prod_{i=1}^n e^{-\lambda r_i R_{t_i}} \exp\Big(-\lambda\int_0^{t_n} R_s\,ds\Big) \le \exp\big(-\lambda\tau\big((1+R_0)^{1/2}-1\big)\big).$$

Using Hölder's inequality and (13.6) with $m = nq$, we find

$$\mathbb{E}\prod_{i=1}^n (1+R_{t_i})^q \le \prod_{i=1}^n \Big(\mathbb{E}(1+R_{t_i})^{nq}\Big)^{1/n} \le e^{nq(nq+2)\Theta}(1+R_0)^{nq}.$$

We bound the exponential factors by one on the complement of G. By the Cauchy–Schwarz inequality, Hölder's inequality, (13.6) with $m = 2nq$, and (13.7), we have

$$\mathbb{E}\Big[\prod_{i=1}^n (1+R_{t_i})^q \mathbf{1}_{G^c}\Big] \le e^{nq(2nq+2)\Theta}(1+R_0)^{nq}\exp\Big(-\frac{\big(\frac{1}{2}\log(1+R_0)-\Theta\big)_+^2}{8\Theta}\Big).$$

Adding the two bounds yields (13.9). □

The expansions of the next section rest on the following form of Duhamel's formula.

Lemma 13.4. *Let $X > 0$, $\operatorname{Re} z > 0$, and $T > 0$. Let $u\colon [0,T] \times D \to \mathbb{C}$ be such that $u(t,\cdot) \in C^\infty(D)$ for all t, the functions u, $\partial_t u$, $\partial_B u$, $\partial_F u$, and $\Delta_D u$ are continuous on $[0,T] \times D$, and there exist $C, q \geq 0$ such that*

$$|u| + |\partial_t u| + |\nabla_D u| + |\Delta_D u| \leq C(1+d)^q \qquad \text{on } [0,T] \times D.$$

Let $g = \partial_t u - (X\Delta_D - zd)u$. Then

$$e^{T(X\Delta_D - zd)}u(0) = u(T) - \int_0^T e^{(T-r)(X\Delta_D - zd)} g(r)\,\mathrm{d}r. \tag{13.10}$$

Proof. Fix $p \in D$, and consider under $\mathbb{E}_p$ the process

$$N_r = \exp\Big(-z\int_0^r \tilde{d}_s\,\mathrm{d}s\Big) u(T-r, \tilde{B}_r, \tilde{F}_r), \qquad 0 \leq r \leq T.$$

By Lemma 13.2(a) and Itô's formula for its product with the factor of finite variation, the process

$$N_r - N_0 + \int_0^r e^{-z\int_0^s \tilde{d}_v\,\mathrm{d}v} g(T-s, \tilde{B}_s, \tilde{F}_s)\,\mathrm{d}s, \qquad 0 \leq r \leq T,$$

is a local martingale, whose quadratic variation at time T has expectation at most $2X\int_0^T \mathbb{E}_p|\nabla_D u|^2(T-s,\tilde{B}_s,\tilde{F}_s)\,\mathrm{d}s$. This expectation is finite by the growth assumption and (13.6) with $\gamma = 1$, and a local martingale with this property is a martingale with mean zero.

Taking expectations at $r = T$, and using Fubini's theorem, which applies since g has polynomial growth uniformly in t, we obtain

$$\mathbb{E}_p N_T - u(T,p) = -\int_0^T \Big(e^{s(X\Delta_D - zd)} g(T-s)\Big)(p)\,\mathrm{d}s.$$

Since $\mathbb{E}_p N_T = (e^{T(X\Delta_D - zd)}u(0))(p)$, the change of variables $r = T - s$ gives (13.10). □

13.3 Expansion of the singlet state

We now expand the singlet state and the right branch $e^{b(X\Delta_D - zd)}\Psi_c^{X,z}$ of the kernel (1.6) in powers of X. Throughout this section, $\operatorname{Re} z > 0$. Recall the singlet operator $\Delta_D^{\mathrm{s}} = d\Delta_D d^{-1}$, and let $\mathbb{C} + d\,C^\infty(D)$ denote the space of functions $c + dg$ with $c \in \mathbb{C}$ and $g \in C^\infty(D)$. Since $1/d$ is harmonic for Δ_D away from the corner, we have

$$\Delta_D^{\mathrm{s}}(c + dg) = d\,\Delta_D g, \qquad c \in \mathbb{C},\ g \in C^\infty(D), \tag{13.11}$$

away from the corner, and we use the right side to define Δ_D^{s} on $\mathbb{C} + d\,C^\infty(D)$. For $k \geq 0$ and $t \geq 0$, we define functions $\Psi_t^{[k]}$ on D by

$$\Psi_t^{[0]} = e^{-tzd}, \qquad \Psi_t^{[k]} = \int_0^t e^{-(t-r)zd}\,\Delta_D^{\mathrm{s}}\Psi_r^{[k-1]}\,\mathrm{d}r \quad (k \geq 1). \tag{13.12}$$

We will see that $\Psi_t^{[k]}$ is the coefficient of X^k in the singlet state $\Psi_t^{X,z}$. For $\tau \geq 0$, we call a function on D an *exponential polynomial of order* τ if it is the product of $e^{-\tau zd}$ and a polynomial in B and F.

Lemma 13.5. *The following statements hold.*

(a) For all $k \geq 0$ and $t \geq 0$, we have $\Psi_t^{[k]} \in \mathbb{C} + d\,C^\infty(D)$, and $\Psi_t^{[k]}$ is an exponential polynomial of order t, whose polynomial factor has coefficients that are polynomials in t and z and is divisible by d if $k \geq 1$. Moreover, $d^{-1}\Delta_D^{\mathrm{s}}\Psi_t^{[k]}$ is an exponential polynomial of order t, and

$$\Delta_D^{\mathrm{s}}\Psi_t^{[0]} = z^2t^2d(B+F)e^{-tzd}, \qquad \Psi_t^{[1]} = \frac{z^2t^3}{3}\,d(B+F)e^{-tzd}. \tag{13.13}$$

(b) Let $X > 0$ and $M \geq 0$. For all $t \geq 0$,

$$\Psi_t^{X,z} = \sum_{k=0}^{M} X^k\Psi_t^{[k]} + X^{M+1}d\,\mathcal{R}_M(t), \qquad \mathcal{R}_M(t) = \int_0^t e^{(t-r)(X\Delta_D - zd)}\big[d^{-1}\Delta_D^{\mathrm{s}}\Psi_r^{[M]}\big]\,\mathrm{d}r. \tag{13.14}$$

Proof. We begin with (a), and first treat $k = 0$. We have $\Psi_t^{[0]} = 1 + dg_t$, where $g_t = (e^{-tzd} - 1)/d$ is an entire function of d. Let ψ be a smooth function on $[0, \infty)$. By (13.4), we have

$$\Delta_D\psi(d) = (B+F)\big(2\psi'(d) + d\psi''(d)\big) = (B+F)\,\frac{\mathrm{d}^2}{\mathrm{d}r^2}\big(r\psi(r)\big)\Big|_{r=d}.$$

With $rg_t(r) = e^{-tzr} - 1$, this identity yields $\Delta_D g_t = z^2t^2(B+F)e^{-tzd}$, and the first identity in (13.13) follows from (13.11). Inserting it into (13.12) gives the second.

Now let $k \geq 2$, and suppose that $\Psi_r^{[k-1]} = de^{-rzd}\rho_r$, where ρ_r is a polynomial in B and F whose coefficients are polynomials in r and z. The product rule and (13.4) show that $e^{rzd}\Delta_D(e^{-rzd}\rho_r)$ is a polynomial of the same kind. By (13.11), we have $\Delta_D^{\mathrm{s}}\Psi_r^{[k-1]} = d\Delta_D(e^{-rzd}\rho_r)$, and inserting this into (13.12), we see that $\Psi_t^{[k]}$ has the same form, with t in place of r and another polynomial of the same kind. By induction, (a) follows.

For (b), we compare the difference quotients by d. By (1.5), we have

$$\frac{\Psi_t^{X,z} - 1}{d} = -z\int_0^t e^{v(X\Delta_D - zd)}1\,\mathrm{d}v.$$

We set

$$u(t) = \frac{1}{d}\Big(\sum_{k=0}^{M} X^k \Psi_t^{[k]} - 1 \Big),$$

which belongs to $C^\infty(D)$ by (a), and which satisfies $u(0) = 0$, since $\Psi_0^{[0]} = 1$ and $\Psi_0^{[k]} = 0$ for $k \geq 1$. Differentiating (13.12) in t, we find $\partial_t \Psi_t^{[0]} = -zd\Psi_t^{[0]}$, and $\partial_t \Psi_t^{[k]} = -zd\Psi_t^{[k]} + \Delta_D^{\mathrm{s}} \Psi_t^{[k-1]}$ for $k \geq 1$. We sum these identities with the weights X^k and use $\Delta_D^{\mathrm{s}}(1 + du(t)) = d\Delta_D u(t)$, which follows from (13.11). This gives

$$d\,\partial_t u(t) = -zd\big(1 + du(t)\big) + Xd\Delta_D u(t) - X^{M+1}\Delta_D^{\mathrm{s}} \Psi_t^{[M]},$$

and after division by d,

$$\partial_t u(t) - (X\Delta_D - zd)u(t) = -z - X^{M+1} d^{-1} \Delta_D^{\mathrm{s}} \Psi_t^{[M]}.$$

The function u satisfies the assumptions of Lemma 13.4 on $[0, t]$, by (a). We apply Lemma 13.4 with $T = t$. Using (13.10) and $u(0) = 0$, we find

$$u(t) = -z \int_0^t e^{(t-r)(X\Delta_D - zd)} 1 \,\mathrm{d}r - X^{M+1} \mathcal{R}_M(t) = \frac{\Psi_t^{X,z} - 1}{d} - X^{M+1} \mathcal{R}_M(t).$$

Multiplying by d proves (13.14). □

We next treat the right branch. For $n \geq 0$ and $b, c \geq 0$, we define

$$\Psi_{b,c}^{[0]} = e^{-(b+c)zd}, \qquad \Psi_{b,c}^{[n]} = e^{-bzd} \Psi_c^{[n]} + \int_0^b e^{-(b-r)zd} \Delta_D \Psi_{r,c}^{[n-1]} \,\mathrm{d}r \quad (n \geq 1), \tag{13.15}$$

so that $\Psi_{0,c}^{[n]} = \Psi_c^{[n]}$. By Lemma 13.5(a) and the product rule, $\Psi_{b,c}^{[n]}$ is an exponential polynomial of order $b + c$, whose polynomial factor has coefficients that are polynomials in b, c, and z.

Lemma 13.6. *Let $X > 0$, $M \geq 0$, and $b, c \geq 0$. Then*

$$e^{b(X\Delta_D - zd)} \Psi_c^{X,z} = \sum_{n=0}^{M} X^n \Psi_{b,c}^{[n]} + X^{M+1} \mathcal{R}_M(b, c), \tag{13.16}$$

where

$$\mathcal{R}_M(b, c) = e^{b(X\Delta_D - zd)} \big[d\,\mathcal{R}_M(c) \big] + \int_0^b e^{(b-r)(X\Delta_D - zd)} \Delta_D \Psi_{r,c}^{[M]} \,\mathrm{d}r.$$

Proof. We set

$$u(b) = \sum_{n=0}^{M} X^n \Psi_{b,c}^{[n]}.$$

Differentiating (13.15) in b, we find

$$\partial_b u(b) = (X\Delta_D - zd)u(b) - X^{M+1}\Delta_D \Psi_{b,c}^{[M]},$$

and by (13.14), we have $u(0) = \Psi_c^{X,z} - X^{M+1} d\mathcal{R}_M(c)$. The function u satisfies the assumptions of Lemma 13.4, and (13.10) implies that

$$e^{b(X\Delta_D - zd)} u(0) = u(b) + X^{M+1} \int_0^b e^{(b-r)(X\Delta_D - zd)} \Delta_D \Psi_{r,c}^{[M]} \, \mathrm{d}r.$$

The function $d\mathcal{R}_M(c)$ has polynomial growth by (13.5), Lemma 13.5(a), and (13.6), and the semigroup may be applied to it. Inserting the expression for $u(0)$ into this identity gives (13.16). □

13.4 The kernel and the function I_X

We now insert the expansions of Section 13.3 into the definition (1.6) of K_X. For $0 < x \le y < 1$, we write $a = x$, $b = y - x$, and $c = 1 - y$, so that (a, b, c) ranges over the time simplex $a, b, c \ge 0$, $a + b + c = 1$, with $a, c > 0$. For $m \ge 0$, we define

$$K^{[m]}(x, y; z) = \int_D \sum_{i+n=m} \Psi_a^{[i]} \Psi_{b,c}^{[n]} \, \mathrm{d}B \, \mathrm{d}F, \qquad I^{[m]}(z) = \int_0^1 \int_x^1 K^{[m]}(x, y; z) \, \mathrm{d}y \, \mathrm{d}x. \quad (13.17)$$

The integrand of $K^{[m]}$ is an exponential polynomial of order $a + b + c = 1$, and these integrals converge absolutely (see (13.21) below), and $K^{[m]}$ extends continuously to the closed simplex.

Proposition 13.7. *Let $X_0 > 0$ and $M \ge 0$, and let $\mathcal{K} \subset \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ be compact. There exists $C > 0$ such that for all $X \in (0, X_0]$, $z \in \mathcal{K}$, and $0 < x \le y < 1$,*

$$\left| K_X(x, y; z) - \sum_{m=0}^{M} X^m K^{[m]}(x, y; z) \right| \le C X^{M+1}. \quad (13.18)$$

Consequently,

$$\left| I_X(z) - \sum_{m=0}^{M} X^m I^{[m]}(z) \right| \le C X^{M+1}. \quad (13.19)$$

Proof. Let $\alpha = \min_{\mathcal{K}} \operatorname{Re} z$, which is positive. By (13.14) with $t = a$ and (13.16), we have

$$\Psi_a^{X,z} \, e^{b(X\Delta_D - zd)} \Psi_c^{X,z} = \sum_{m=0}^{M} X^m \sum_{i+n=m} \Psi_a^{[i]} \Psi_{b,c}^{[n]} + X^{M+1} \mathcal{R},$$

where

$$\mathcal{R} = d\,\mathcal{R}_M(a)\,e^{b(X\Delta_D - zd)}\Psi_c^{X,z} + \Big(\sum_{i=0}^{M} X^i \Psi_a^{[i]}\Big)\mathcal{R}_M(b,c) + \sum_{\substack{0\le i,n\le M\\ i+n\ge M+1}} X^{i+n-M-1}\Psi_a^{[i]}\Psi_{b,c}^{[n]}. \tag{13.20}$$

It suffices to show that $\int_D |\mathcal{R}|\,\mathrm{d}B\,\mathrm{d}F$ is bounded uniformly in X, z, and the simplex. We bound each factor in (13.20) pointwise with Lemma 13.3, applied to the radial diffusion with $\gamma = 1$, $T = 1$, $\Theta = 3X_0$, and $\lambda = \alpha$. Below, C and q denote constants depending only on X_0, M, and $\mathcal{K}$, whose values may change from line to line.

We first describe the form in which the semigroups appear. Let $f_1, \dots, f_n$ be exponential polynomials of orders $r_1, \dots, r_n$ whose polynomial factors have coefficients bounded for $z \in \mathcal{K}$, and let $s_1, \dots, s_n \ge 0$ with $s_1 + \cdots + s_n \le 1$. Consider the function

$$e^{s_1(X\Delta_D - zd)}\Big[f_1\, e^{s_2(X\Delta_D - zd)}\big[f_2 \cdots e^{s_n(X\Delta_D - zd)} f_n\big]\Big].$$

The Markov property and (13.5) bound its absolute value at p by a constant times the left side of (13.9), with $t_i = s_1 + \cdots + s_i$. Lemma 13.3 therefore bounds it by $C(1 + d(p))^q \mathcal{E}_\tau(d(p))$, where

$$\tau = s_1 + \cdots + s_n + r_1 + \cdots + r_n.$$

We call τ the total time of such an expression.

We now bound the factors of the first two terms of (13.20). Recall from Lemma 13.5(a) that $\Psi_a^{[i]}$ is an exponential polynomial of order a. Using $(1+d)^{1/2} - 1 \le d$, we obtain

$$|\Psi_a^{[i]}| \le C(1+d)^q e^{-\alpha a d} \le C(1+d)^q \mathcal{E}_a(d).$$

The function $\mathcal{R}_M(a)$ in (13.14) is an integral over $r \in [0,a]$ of expressions of total time $(a-r) + r = a$. It follows that

$$|d\mathcal{R}_M(a)| \le C(1+d)^q \mathcal{E}_a(d).$$

For the right branch, we recall from (13.14) with $M = 0$ that $\Psi_c^{X,z} = e^{-czd} + Xd\mathcal{R}_0(c)$, and therefore

$$e^{b(X\Delta_D - zd)}\Psi_c^{X,z} = e^{b(X\Delta_D - zd)}e^{-czd} + X\int_0^c e^{b(X\Delta_D - zd)}\Big[d\,e^{(c-r)(X\Delta_D - zd)}\big[d^{-1}\Delta_D^{\mathrm{s}}\Psi_r^{[0]}\big]\Big]\,\mathrm{d}r.$$

Both terms are expressions, or integrals of expressions, of total time $b + c$, so that

$$\big|e^{b(X\Delta_D - zd)}\Psi_c^{X,z}\big| \le C(1+d)^q \mathcal{E}_{b+c}(d).$$

The same argument applies to the two terms of $\mathcal{R}_M(b,c)$ in Lemma 13.6. In the first term, the times add up to $b + (c-r) + r = b + c$. In the second, $\Delta_D \Psi_{r,c}^{[M]}$ is an exponential

polynomial of order $r + c$, and the times add up to $(b - r) + (r + c) = b + c$. We conclude that

$$|\mathcal{R}_M(b, c)| \le C(1 + d)^q \mathcal{E}_{b+c}(d).$$

Finally, each term of the last sum in (13.20) is an exponential polynomial of order $a + b + c = 1$. Combining these bounds, we find

$$|\mathcal{R}| \le C(1 + d)^{2q} \big(\mathcal{E}_a(d)\, \mathcal{E}_{b+c}(d) + \mathcal{E}_1(d)\big) \le 3C(1 + d)^{2q} \mathcal{E}_{1/2}(d),$$

where we used that $\max(a, b + c) \ge 1/2$, that $\mathcal{E}_\tau$ is nonincreasing in τ, and that $\mathcal{E}_\tau \le 2$. (When a is small, the left factor decays slowly, but then $b + c \ge 1/2$, and the right factor provides the decay.)

It remains to integrate over D. For a nonnegative measurable function ψ on $[0, \infty)$, the change of variables to $d = B - F$ and $B + F$ gives

$$\int_D \psi(d)\, \mathrm{d}B\, \mathrm{d}F = \int_0^\infty \psi(d) \min(d, 2)\, \mathrm{d}d. \tag{13.21}$$

Applying (13.21) with $\psi = (1 + d)^{2q} \mathcal{E}_{1/2}(d)$, which is integrable, we find that $\int_D |\mathcal{R}|\, \mathrm{d}B\, \mathrm{d}F \le C$. We have proved (13.18). Integrating it over the simplex, we find (13.19). □

We now evaluate the coefficients, using the variables $d = B - F$ and $B + F$ on D. For fixed $d \ge 0$, the variable $B + F$ ranges over $[|d - 2|, d + 2]$, and $\mathrm{d}B\, \mathrm{d}F = \frac{1}{2}\, \mathrm{d}d\ \mathrm{d}(B + F)$. It follows that for $j \ge 0$ and a measurable function ψ on $[0, \infty)$ such that $|\psi(d)|(1 + d)^{j+1}$ is integrable,

$$\int_D \psi(d)(B + F)^j\, \mathrm{d}B\, \mathrm{d}F = \int_0^\infty \psi(d)\, \frac{(d + 2)^{j+1} - |d - 2|^{j+1}}{2(j + 1)}\, \mathrm{d}d. \tag{13.22}$$

For $j = 0$, 1, and 2, the fraction in (13.22) takes the values

$$\min(d, 2), \qquad 2d, \qquad \begin{cases} d^3/3 + 4d, & d \le 2, \\ 2d^2 + 8/3, & d \ge 2, \end{cases}$$

respectively. For $m \ge 0$, we also define

$$K_\triangle^{[m]}(B, F; z) = e^{zd} \int_0^1 \int_0^{1-a} \sum_{i+n=m} \Psi_a^{[i]} \Psi_{b,1-a-b}^{[n]}\, \mathrm{d}b\, \mathrm{d}a,$$

which is a polynomial in B and F, since the integrand is an exponential polynomial of order one. By Fubini's theorem, we have

$$I^{[m]}(z) = \int_D e^{-zd} K_\triangle^{[m]}\, \mathrm{d}B\, \mathrm{d}F. \tag{13.23}$$

Lemma 13.8. *For* $\operatorname{Re} z > 0$ *and* $0 < x \le y < 1$,

$$K^{[0]}(x,y;z) = \frac{1-e^{-2z}}{z^2}, \qquad I^{[0]}(z) = I_{\mathrm{GUE}}(z), \qquad I^{[1]}(z) = 0, \qquad I^{[2]}(z) = \frac{1-e^{-2z}}{45}. \tag{13.24}$$

Proof. By the definitions (13.12) and (13.15), we have

$$\Psi_a^{[0]}\Psi_{b,c}^{[0]} = e^{-azd}e^{-(b+c)zd} = e^{-zd},$$

and (13.21) gives

$$K^{[0]}(x,y;z) = \int_0^\infty e^{-zd}\min(d,2)\,\mathrm{d}d = \frac{1-e^{-2z}}{z^2}.$$

Since the simplex has area $1/2$, this identity implies that $I^{[0]} = I_{\mathrm{GUE}}$.

We next compute $K_\triangle^{[1]}$. The sum in its definition has the three terms $\Psi_a^{[1]}\Psi_{b,c}^{[0]}$, $\Psi_a^{[0]}e^{-bzd}\Psi_c^{[1]}$, and $\Psi_a^{[0]}$ times the integral in (13.15) with $n = 1$. By (13.13), the first two terms are given by

$$\Psi_a^{[1]}\Psi_{b,c}^{[0]} = \frac{z^2a^3}{3}\,d(B+F)e^{-zd}, \qquad \Psi_a^{[0]}e^{-bzd}\Psi_c^{[1]} = \frac{z^2c^3}{3}\,d(B+F)e^{-zd}.$$

For the third term, we use the identity

$$\Delta_D e^{-\tau zd} = \big(\tau^2z^2d(B+F) - 2\tau z(B+F)\big)e^{-\tau zd}, \qquad \tau \ge 0,$$

which follows from (13.4), and we find

$$\int_0^b e^{-(b-r)zd}\Delta_D e^{-(r+c)zd}\,\mathrm{d}r = e^{-(b+c)zd}(B+F)\Big(\frac{z^2d}{3}\big((b+c)^3-c^3\big) - z\big((b+c)^2-c^2\big)\Big).$$

Adding the three terms and using $b+c = 1-a$, we find

$$K_\triangle^{[1]} = (B+F)\int_0^1\int_0^{1-a}\Big(\frac{z^2d}{3}\big(a^3+(1-a)^3\big) - z\big((1-a)^2-c^2\big)\Big)\,\mathrm{d}b\,\mathrm{d}a, \qquad c = 1-a-b.$$

Integrating componentwise over the simplex, we have

$$\int_0^1\int_0^{1-a}\big(a^3,(1-a)^3,(1-a)^2,c^2\big)\,\mathrm{d}b\,\mathrm{d}a = \Big(\frac{1}{20},\frac{1}{5},\frac{1}{4},\frac{1}{12}\Big),$$

where for c^2 we used that a, b, and c play symmetric roles. Inserting these values, we find

$$K_\triangle^{[1]} = (B+F)\Big(\frac{dz^2}{12} - \frac{z}{6}\Big).$$

Inserting this into (13.23), and using (13.22) with $j = 1$ and the integrals

$$\int_0^\infty e^{-zd} d^n \,\mathrm{d}d = \frac{n!}{z^{n+1}}, \qquad n \geq 0, \tag{13.25}$$

we obtain

$$I^{[1]}(z) = \int_0^\infty e^{-zd} \Big(\frac{dz^2}{12} - \frac{z}{6} \Big) 2d \,\mathrm{d}d = \frac{z^2}{6} \cdot \frac{2}{z^3} - \frac{z}{3} \cdot \frac{1}{z^2} = 0.$$

The second-order term is computed in the same way, and we give the steps. The sum in the definition of $K_{\triangle}^{[2]}$ has the three terms $\Psi_a^{[2]} \Psi_{b,c}^{[0]}$, $\Psi_a^{[1]} \Psi_{b,c}^{[1]}$, and $\Psi_a^{[0]} \Psi_{b,c}^{[2]}$. Besides (13.4), we use the identities

$$\begin{aligned} \Delta_D(B+F) = 2d, \qquad |\nabla_D(B+F)|^2 = d(B+F), \\ 2\langle \nabla_D d, \nabla_D(B+F) \rangle = d^2 + (B+F)^2 - 4, \end{aligned}$$

which follow from $\Delta_D B = 2B$, $\Delta_D F = -2F$, and the definition of $\langle \nabla_D \cdot, \nabla_D \cdot \rangle$. Inserting them into the complex product rule $\Delta_D(fg) = f\Delta_D g + g\Delta_D f + 2\langle \nabla_D f, \nabla_D \overline{g} \rangle$, we find, for $w \in \mathbb{C}$,

$$\begin{aligned} e^{wd} \Delta_D \big((B+F) e^{-wd} \big) &= (w^2 d - 3w)(B+F)^2 - wd^2 + 2d + 4w, \\ e^{wd} \Delta_D \big(d(B+F) e^{-wd} \big) &= (w^2 d^2 - 5wd + 3)(B+F)^2 + 3d^2 - wd^3 + 4wd - 4. \end{aligned} \tag{13.26}$$

Below, all integrals over the simplex are over $a \in [0,1]$ and $b \in [0, 1-a]$, with $c = 1-a-b$, and we will use the identities

$$\int_0^1 \int_0^{1-a} a^n \,\mathrm{d}b \,\mathrm{d}a = \int_0^1 \tau^n (1-\tau) \,\mathrm{d}\tau = \frac{1}{(n+1)(n+2)}, \qquad n \geq 0. \tag{13.27}$$

We begin with the first term. Using (13.13) and (13.11), we find

$$\Delta_D^{\mathrm{s}} \Psi_r^{[1]} = \frac{z^2 r^3}{3} \, d \, \Delta_D \big((B+F) e^{-rzd} \big).$$

We insert the first identity in (13.26) with $w = rz$ into (13.12) with $k = 2$, and we integrate over $r \in [0,t]$, which gives

$$e^{tzd} \Psi_t^{[2]} = \Big(\frac{z^4 t^6 d^2}{18} - \frac{z^3 t^5 d}{5} \Big) (B+F)^2 + \frac{z^2 t^4 d^2}{6} + \frac{z^3 t^5 (4d - d^3)}{15}.$$

We have $e^{zd} \Psi_{b,c}^{[0]} = e^{azd}$, and integrating this function with $t = a$ over the simplex with (13.27) gives the first line of (13.29) below.

For the second term, we add the two terms of $\Psi_{b,c}^{[1]}$ computed above, which gives

$$e^{(b+c)zd} \Psi_{b,c}^{[1]} = (B+F) \Big(\frac{z^2 d}{3} (b+c)^3 - z \big((b+c)^2 - c^2 \big) \Big). \tag{13.28}$$

We multiply by $e^{azd}\Psi_a^{[1]} = (z^2a^3/3)\,d(B+F)$, use $b+c = 1-a$, and integrate over the simplex with the values

$$\int_0^1 \int_0^{1-a} \big(a^3(1-a)^3, a^3(1-a)^2, a^3c^2\big)\,\mathrm{d}b\,\mathrm{d}a = \Big(\frac{1}{280}, \frac{1}{140}, \frac{1}{420}\Big),$$

which follow from the Beta integral. This gives the second line of (13.29).

For the third term, we recall from (13.15) that

$$\Psi_{b,c}^{[2]} = e^{-bzd}\Psi_c^{[2]} + \int_0^b e^{-(b-r)zd}\Delta_D\Psi_{r,c}^{[1]}\,\mathrm{d}r.$$

The part $\Psi_a^{[0]}e^{-bzd}\Psi_c^{[2]} = e^{-zd}e^{czd}\Psi_c^{[2]}$ has the same simplex integral as the first term, since a and c play symmetric roles.

In the second part, we substitute $\tau = r + c$, and we integrate over (b, c) instead of (a, b), which preserves the Lebesgue measure. For $0 \le c \le \tau \le 1$, the variable b then ranges over $[\tau - c, 1 - c]$, an interval of length $1 - \tau$. Multiplied by e^{zd}, the second part therefore has the simplex integral

$$\int_0^1 (1-\tau)\int_0^\tau G(\tau,c)\,\mathrm{d}c\,\mathrm{d}\tau, \qquad G(\tau,c) = e^{\tau zd}\Delta_D\Big((B+F)e^{-\tau zd}\Big(\frac{z^2d}{3}\tau^3 - z(\tau^2-c^2)\Big)\Big),$$

where we used (13.28) with (r, c) in place of (b, c). Inserting (13.26) with $w = \tau z$ and using $\int_0^\tau(\tau^2 - c^2)\,\mathrm{d}c = 2\tau^3/3$, we obtain

$$\begin{aligned}\int_0^\tau G(\tau,c)\,\mathrm{d}c = {}& \frac{z^2\tau^4}{3}\big(z^2\tau^2d^2 - 7z\tau d + 9\big)(B+F)^2 \\ & + \frac{z^2\tau^4}{3}(5d^2-12) + \frac{z^3\tau^5}{3}(4d-d^3) - \frac{4z\tau^3d}{3}.\end{aligned}$$

Integrating against $1 - \tau$ with (13.27) and adding the first part, we obtain the third line of

$$\begin{aligned}
&\int_0^1\int_0^{1-a} e^{zd}\Psi_a^{[2]}\Psi_{b,c}^{[0]}\,\mathrm{d}b\,\mathrm{d}a = \frac{z^2d^2}{180} + \frac{z^3(4d-d^3)}{630} + (B+F)^2\Big(\frac{z^4d^2}{1008} - \frac{z^3d}{210}\Big),\\
&\int_0^1\int_0^{1-a} e^{zd}\Psi_a^{[1]}\Psi_{b,c}^{[1]}\,\mathrm{d}b\,\mathrm{d}a = (B+F)^2\Big(\frac{z^4d^2}{2520} - \frac{z^3d}{630}\Big),\\
&\int_0^1\int_0^{1-a} e^{zd}\Psi_a^{[0]}\Psi_{b,c}^{[2]}\,\mathrm{d}b\,\mathrm{d}a = -\frac{zd}{15} - \frac{2z^2}{15} + \frac{11z^2d^2}{180} + \frac{z^3(4d-d^3)}{105}\\
&\qquad\qquad + (B+F)^2\Big(\frac{z^2}{10} - \frac{19z^3d}{315} + \frac{z^4d^2}{144}\Big).
\end{aligned} \tag{13.29}$$

Adding the three lines, we find

$$K_\triangle^{[2]} = -\frac{dz}{15} + \frac{(d^2-2)z^2}{15} + \frac{(4d-d^3)z^3}{90} + (B+F)^2\Big(\frac{z^2}{10} - \frac{dz^3}{15} + \frac{d^2z^4}{120}\Big). \tag{13.30}$$

It remains to integrate over D. We write the right side of (13.30) as $\psi_0(d) + \psi_2(d)(B+F)^2$, and we insert it into (13.23) with $m = 2$. For $j = 0$ and $j = 2$, we write the fractions in (13.22) as

$$\min(d,2) = d - (d-2)_+, \qquad \frac{(d+2)^3 - |d-2|^3}{6} = \frac{d^3}{3} + 4d - \frac{(d-2)_+^3}{3}.$$

Substituting $d = 2+u$ in the integrals over $\{d > 2\}$, we find

$$\begin{aligned} I^{[2]}(z) = {} & \int_0^\infty e^{-zd}\Big(\psi_0(d)\,d + \psi_2(d)\Big(\frac{d^3}{3} + 4d\Big)\Big)\,\mathrm{d}d \\ & - e^{-2z}\int_0^\infty e^{-zu}\Big(\psi_0(2+u)\,u + \psi_2(2+u)\frac{u^3}{3}\Big)\,\mathrm{d}u. \end{aligned}$$

Expanding the polynomial factors of the two integrands, we find

$$\begin{aligned} \psi_0(d)\,d + \psi_2(d)\Big(\frac{d^3}{3} + 4d\Big) &= \frac{4z^2}{15}d - \Big(\frac{z}{15} + \frac{2z^3}{9}\Big)d^2 + \Big(\frac{z^2}{10} + \frac{z^4}{30}\Big)d^3 - \frac{z^3}{30}d^4 + \frac{z^4}{360}d^5, \\ \psi_0(2+u)\,u + \psi_2(2+u)\frac{u^3}{3} &= \frac{2(z^2-z)}{15}u + \Big(\frac{4z^2}{15} - \frac{z}{15} - \frac{4z^3}{45}\Big)u^2 + \Big(\frac{z^2}{10} - \frac{z^3}{9} + \frac{z^4}{90}\Big)u^3 \\ &\quad + \Big(\frac{z^4}{90} - \frac{z^3}{30}\Big)u^4 + \frac{z^4}{360}u^5. \end{aligned}$$

By (13.25), the terms of order z^{-2} and z^{-1} cancel in both integrals, and the constant terms add up to $1/45$. We conclude that $I^{[2]}(z) = (1 - e^{-2z})/45$. □

Remark 13.9. The supplementary script `metallic_coefficients.py` confirms the computations in the proof of Lemma 13.8 in exact arithmetic. It implements (13.12), (13.15), the integrals over the simplex, and the integrals over D, the latter both through (13.22) and directly in the coordinates (B, F). It confirms (13.30) and the four values in (13.24). (All the functions involved are holomorphic in z, and the script evaluates them for real $z > 0$.)

Proof of Proposition 1.17. The expansion (1.26) is Lemma 13.1. We apply Proposition 13.7 with $M = 2$ and $X_0 = 1$. Using Lemma 13.8, we obtain

$$I_X(z) = I_{\mathrm{GUE}}(z) + \frac{X^2}{45}(1 - e^{-2z}) + O(X^3),$$

uniformly for z in compact subsets of $\{\mathrm{Re}\, z > 0\}$, which is (1.27). □

The case $M = 0$ of Proposition 13.7 will be used in Section 14.

Corollary 13.10. *Let $\mathcal{K} \subset \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$ be compact. There exists $C > 0$ such that for all $X \in (0,1]$, $z \in \mathcal{K}$, and $0 < x \le y < 1$,*

$$\Big|K_X(x,y;z) - \frac{1 - e^{-2z}}{z^2}\Big| \le CX.$$

Proof. This is (13.18) with $M = 0$ and $X_0 = 1$, together with the first identity in (13.24). □

13.5 The coefficients and the diffusion modes

We now express the coefficients through the Neumann Laplacian on $[0,1]$, with eigenvalues 0 and $(\pi n)^2$ for $n \geq 1$ and normalized eigenfunctions 1 and $\sqrt{2}\cos(\pi n t)$. Recall from Section 1.2 that

$$G_{\mathrm{N}}(x,y) = \sum_{n \geq 1} \frac{2\cos(\pi n x)\cos(\pi n y)}{(\pi n)^2},$$

so that the integral operator with kernel G_{N} satisfies $\operatorname{Tr} G_{\mathrm{N}}^p = \sum_{n\geq 1}(\pi n)^{-2p}$ for $p \geq 1$.

Remark 13.11. The coefficients in Proposition 1.17 are spectral quantities of G_{N}. The first correction to $\mathcal{C}_X$ is the kernel of $2XG_{\mathrm{N}}$. In I_X, the coefficient $I^{[1]}$ of X vanishes, and the coefficient of X^2 satisfies $I^{[2]}(z) = (1-e^{-2z})/45$ by Lemma 13.8, with

$$\frac{1}{45} = 2\sum_{n\geq 1}\frac{1}{(\pi n)^4} = 2\operatorname{Tr} G_{\mathrm{N}}^2.$$

Carrying the recursion one order further, the supplementary script of Remark 13.9 gives $I^{[3]}(z) = -8z/945 = -8z\operatorname{Tr} G_{\mathrm{N}}^3$.

In the metallic perturbation theory of the physics literature, $2XG_{\mathrm{N}}$ has the form of the diffusion propagator of a wire with hard-wall ends. In the normalization of Mirlin's review [94], in which the dimensionless conductance is g_{M}, the propagator on its diagonal is $(2/g_{\mathrm{M}})G_{\mathrm{N}}(x,x)$ [94, (3.55)], and the correction to the correlation of one eigenfunction at two distinct points is the propagator itself [94, (3.74)]. This agrees with (1.26) under the identification $g_{\mathrm{M}} = 1/X$ of the conductance, which we make in Remark 13.12.

The quadratic term has the following meaning for the pair measure ν_X of Corollary 1.7. For $z = \pi\varepsilon - \mathrm{i}\pi v$, we have

$$\operatorname{Re}\left(1-e^{-2z}\right) = 1 - e^{-2\pi\varepsilon}\cos(2\pi v) = \int_{\mathbb{R}} P_\varepsilon(v-u)\,2\sin^2(\pi u)\,\mathrm{d}u,$$

where we used $\int_{\mathbb{R}} P_\varepsilon(v-u)\cos(2\pi u)\,\mathrm{d}u = e^{-2\pi\varepsilon}\cos(2\pi v)$. The measure $\nu_{\mathrm{GUE}}(\mathrm{d}s) = (1-(\sin\pi s/(\pi s))^2)\,\mathrm{d}s$ satisfies

$$(P_\varepsilon * \nu_{\mathrm{GUE}})(v) = 1 + \operatorname{Re} I_{\mathrm{GUE}}(\pi\varepsilon - \mathrm{i}\pi v) - P_\varepsilon(v)$$

by Lemma 14.3, which Section 14 proves by a direct computation. Together with (1.12) and Proposition 1.17, this implies that the Poisson transform $(P_\varepsilon * \nu_X)(v)$ is the sum of the Poisson transforms of the densities $1-(\sin\pi u/(\pi u))^2$ and $4X^2\operatorname{Tr}(G_{\mathrm{N}}^2)\sin^2(\pi u)$, up to an error $O(X^3)$, uniformly for ε and v in compact subsets of $(0,\infty)$ and $\mathbb{R}$. This is a statement about the Poisson transforms at positive ε, and we do not prove a corresponding expansion of ν_X itself.

Remark 13.12. The quadratic term in Remark 13.11 has the form of the correction of order $1/g_{\mathrm{KM}}^2$ that Kravtsov and Mirlin [80, (15)] obtained for the pair correlation of a metallic sample in the unitary class, whose dimensionless conductance g_{KM} is the ratio of the Thouless energy to the mean level spacing. In dimension one, their correction has the coefficient $a_1/(\pi^2 g_{\mathrm{KM}}^2)$ with $a_1 = \operatorname{Tr} G_{\mathrm{N}}^2$ [80, (13)]. In their notation, the diffusion modes appear through the denominators $-\mathrm{i}s + \pi^2 g_{\mathrm{KM}} n^2$ [80, (16)], so that their correction is $\pi^{-2} \sum_{n \geq 1} (\pi^2 g_{\mathrm{KM}} n^2)^{-2} \sin^2(\pi s)$. The two corrections agree if $g_{\mathrm{KM}} = 1/(2\pi X)$. We use this heuristic identification of the conductance in our comparisons with the physics literature. In the normalization of [94, (2.27)], where the conductance is $g_{\mathrm{M}} = 2\pi g_{\mathrm{KM}}$, it reads $g_{\mathrm{M}} = 1/X$.

This identification can also be read off from our generator by the following heuristic computation, which we do not justify. Let $\varrho \in \mathbb{R}^4$ denote geodesic normal coordinates on $\mathbb{H}^2 \times S^2$ at the point that corresponds to the corner, so that $d = |\varrho|^2/2 + O(|\varrho|^4)$, and Δ_D is the Euclidean Laplacian to leading order. In the Gaussian approximation of $X\Delta_D - zd$, a path $\varrho(t)$, $0 \leq t \leq 1$, with coefficients ϱ_n in the Neumann basis $1, \sqrt{2}\cos(\pi n t)$, has the Feynman–Kac action

$$\sum_{n \geq 0} \Big(z + \frac{(\pi n)^2}{2X} \Big) \frac{|\varrho_n|^2}{2}.$$

The spectral variable corresponds to $z = -\mathrm{i}\pi s$, so that $\pi^{-1}(z + (\pi n)^2/(2X)) = -\mathrm{i}s + (\pi n)^2/(2\pi X)$, and the comparison with the denominators $-\mathrm{i}s + \pi^2 g_{\mathrm{KM}} n^2$ gives $g_{\mathrm{KM}} = 1/(2\pi X)$. Then $\pi^{-2} \sum_n (\pi^2 g_{\mathrm{KM}} n^2)^{-2} = 4X^2 \sum_n (\pi n)^{-4}$, in agreement with the coefficient $2X^2/45$ of $\sin^2(\pi s)$ in Remark 13.11. This agreement is a consistency check, not part of our proofs.

SECTION 14

Spectral fluctuations across scales

In this section we prove the results of Section 1.2 on the second-order statistics of the limiting process Ξ_X, namely Corollary 1.7 on the pair measure ν_X and the count covariances, the rows of Proposition 1.16 other than the law of μ_X, the large-scale covariance of Theorem 1.18, and the crossover of Theorem 1.19. The law of μ_X in Proposition 1.16 is treated in Section 12.

We follow the plan of Section 2.9, which explains how these statistics are read off from the function I_X of Definition 1.4. The main estimate of this section is Lemma 14.10, which shows that $XI_X(Xa)$ is close to a stationary pairing $\mathcal{I}(a)$ when the wire is long compared with the relaxation time of the diffusion, and we compute $\mathcal{I}(a)$ in closed form in Lemma 14.12.

Throughout this section, we use the notation of Section 13.2. For functions f and g on D such that fg is integrable, we write $\langle f, g\rangle_D = \int_D fg \, \mathrm{d}B \, \mathrm{d}F$, and we write $\|f\|_{L^p(D)}$ for the norms with respect to $\mathrm{d}B \, \mathrm{d}F$. For functions on $\mathbb{R}$, the Fourier transform $\hat{f}(k) = \int_{\mathbb{R}} e^{-2\pi i k s} f(s) \, \mathrm{d}s$ is that of Section 1.4.

14.1 The pair measure

We begin with the existence of ν_X. Following [22, Section 8.1], where it is written $\check{M}_{[2]}$, we call the measure ν_X of (1.11) the *reduced second factorial moment measure* of Ξ_X. We also keep the shorter name used in Section 1.2, the pair measure. Recall from Proposition 8.2 that Ξ_X is stationary and simple, with $\mathbb{E}\Xi_X(I) = |I|$ for all bounded intervals I, that it satisfies the exponential bound (8.5) with the constant C_{exp}, which does not depend on X, and that it satisfies (8.7). The existence of ν_X is a standard fact about stationary point processes with locally finite second moments [22, Section 8.1], and the bound (14.1), which is uniform in X, is the part that we need beyond it.

Lemma 14.1. *Let $X > 0$. There exists an even positive Radon measure ν_X on $\mathbb{R}$ such that* (1.11) *holds for all nonnegative measurable functions f on $\mathbb{R}^2$. Further, there exists C_ν, depending only on C_{exp}, such that for all $X > 0$,*

$$\sup_{s \in \mathbb{R}} \nu_X([s, s+1]) \le C_\nu. \tag{14.1}$$

Proof. For a Borel set $A \subset \mathbb{R}^2$, we let

$$\mu^{[2]}(A) = \mathbb{E} \sum_{\substack{x,y \in \Xi_X \\ x \neq y}} \mathbf{1}_A(x,y)$$

denote the second factorial moment measure of Ξ_X. By (8.7), we have $\mu^{[2]}(I \times I) \leq (C_{\exp}|I|)^2$ for all bounded open intervals I, and Ξ_X has locally finite second moments. For Borel sets $A \subset \mathbb{R}^2$, we also let

$$M_2(A) = \mathbb{E} \sum_{x,y \in \Xi_X} \mathbf{1}_A(x,y)$$

denote the second moment measure of Ξ_X. Since Ξ_X is stationary, [22, Proposition 8.1.I(b)] implies that there exists a positive Radon measure $\check{M}_2$ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}^2} f \, \mathrm{d}M_2 = \int_{\mathbb{R}} \int_{\mathbb{R}} f(v, v+s) \, \check{M}_2(\, \mathrm{d}s) \, \mathrm{d}v$$

for all bounded measurable functions f with bounded support. We call $\check{M}_2$ the reduced second moment measure of Ξ_X, as in that reference. It is symmetric by [22, Proposition 8.1.II(i)].

Since Ξ_X is simple and has intensity one, M_2 is the sum of $\mu^{[2]}$ and the image of the Lebesgue measure under $v \mapsto (v,v)$. Therefore (1.11) holds with $\nu_X = \check{M}_2 - \delta_0$ for bounded measurable f with bounded support, and monotone convergence extends it to all nonnegative measurable f. Taking $f(x,y) = \mathbf{1}_{[0,1)}(x)\mathbf{1}_S(y-x)$ for bounded Borel sets S, we see that ν_X is positive and that (1.11) determines it. It is even, since $\check{M}_2$ and δ_0 are.

It remains to prove (14.1). Let $s \in \mathbb{R}$ and $\eta \in (0,1)$, and let $I_1 = (-\eta, 1+\eta)$ and $I_2 = (s-\eta, s+2+\eta)$. Every pair $(v, v+r)$ with $v \in [0,1)$ and $r \in [s, s+1]$ belongs to $I_1 \times I_2$. Now (1.11) and the Cauchy–Schwarz inequality give

$$\nu_X([s,s+1]) \leq \mathbb{E}\big[\Xi_X(I_1)\Xi_X(I_2)\big] \leq \Big(\mathbb{E}\Xi_X(I_1)^2 \, \mathbb{E}\Xi_X(I_2)^2\Big)^{1/2}.$$

For a bounded open interval I, (8.7) and the unit intensity of Ξ_X give

$$\mathbb{E}\Xi_X(I)^2 = \mathbb{E}\big(\Xi_X(I)\big)_2 + \mathbb{E}\Xi_X(I) \leq (C_{\exp}|I|)^2 + |I|.$$

Letting $\eta \downarrow 0$ yields (14.1) with $C_\nu = ((C_{\exp}^2 + 1)(4C_{\exp}^2 + 2))^{1/2}$. □

We now identify the Poisson transform of ν_X, using the rank-one derivatives of Theorem 1.6(c).

Lemma 14.2. *Let $X > 0$. The measure ν_X satisfies* (1.12)*, and it is the unique positive Radon measure ν with $\sup_s \nu([s,s+1]) < \infty$ that satisfies* (1.12) *for all $\varepsilon > 0$ and $v \in \mathbb{R}$.*

Proof. Fix $\varepsilon > 0$, and set

$$c_\varepsilon(x) = \frac{2\varepsilon}{x^2+\varepsilon^2} = 2\pi P_\varepsilon(x), \qquad x \in \mathbb{R}.$$

For finite N, the product in (1.4) with $k = 1$, $u = 0$, and $b = \varepsilon + t$ is

$$T_N^{(1)}(0, \varepsilon + t) = \mathbb{E} \prod_{\alpha=1}^{N} \Big(1 + c_\varepsilon(x_\alpha)t + \frac{t^2}{x_\alpha^2 + \varepsilon^2}\Big),$$

a polynomial in t with bounded coefficients. Differentiating twice at $t = 0$, we find

$$\partial_t^2 T_N^{(1)}(0, \varepsilon + t)\big|_{t=0} = \frac{1}{\varepsilon}\mathbb{E}\Xi_N(c_\varepsilon) + \mathbb{E}\big[\Xi_N(c_\varepsilon)^2 - \Xi_N(c_\varepsilon^2)\big]. \tag{14.2}$$

The functions c_ε and c_ε^2 are continuous and bounded by multiples of P_ε. Applying Lemma 8.3(b) to $\Xi_N \to \Xi_X$, we see that the expectations on the right side of (14.2) converge to the same expressions for Ξ_X, and $\mathbb{E}\Xi_X(c_\varepsilon) = 2\pi$. Comparing with (1.10), we obtain

$$4\pi^2\mathbb{E} \sum_{\substack{x,y\in\Xi_X \\ x\neq y}} P_\varepsilon(x)P_\varepsilon(y) = \mathbb{E}\big[\Xi_X(c_\varepsilon)^2 - \Xi_X(c_\varepsilon^2)\big] = 4\pi^2\big(1 + I_X(2\pi\varepsilon)\big) - \frac{2\pi}{\varepsilon}.$$

We now apply (1.11) to the left side, and use the identity

$$\int_{\mathbb{R}} P_\varepsilon(v)P_\varepsilon(v+s)\,\mathrm{d}v = P_{2\varepsilon}(s), \qquad s \in \mathbb{R},$$

which follows from $P_\varepsilon * P_\varepsilon = P_{2\varepsilon}$ and the evenness of P_ε. With $a = 2\varepsilon$, we obtain

$$\int_{\mathbb{R}} P_a(s)\,\nu_X(\,\mathrm{d}s) = 1 + I_X(\pi a) - P_a(0), \qquad a > 0. \tag{14.3}$$

We now remove the restriction to the center $v = 0$. For $\operatorname{Re} w > 0$, we set

$$\mathcal{Y}_X(w) = \int_{\mathbb{R}} \frac{w}{\pi(w^2+s^2)}\,\nu_X(\,\mathrm{d}s).$$

For w in a compact subset of the half-plane, $w^2 + s^2$ does not vanish, and $|w/(w^2+s^2)| \le C(1+s^2)^{-1}$. Using (14.1), dominated convergence, and Morera's theorem, we see that $\mathcal{Y}_X$ is holomorphic. The function I_X is holomorphic in the half-plane as well, and (14.3) shows that $\mathcal{Y}_X(a) = 1 + I_X(\pi a) - (\pi a)^{-1}$ for $a > 0$. The identity theorem now implies that

$$\mathcal{Y}_X(w) = 1 + I_X(\pi w) - \frac{1}{\pi w}, \qquad \operatorname{Re} w > 0.$$

For $w = \varepsilon - \mathrm{i}v$, we have

$$\operatorname{Re}\frac{w}{\pi(w^2+s^2)} = \frac{1}{2\pi}\operatorname{Re}\Big(\frac{1}{w+\mathrm{i}s} + \frac{1}{w-\mathrm{i}s}\Big) = \frac{1}{2}\big(P_\varepsilon(v-s) + P_\varepsilon(v+s)\big).$$

Integrating against the even measure ν_X, we obtain $\operatorname{Re} \mathcal{Y}_X(\varepsilon - \mathrm{i}v) = (P_\varepsilon * \nu_X)(v)$. Taking real parts in the formula for $\mathcal{Y}_X$ and using $\operatorname{Re}(\pi(\varepsilon - \mathrm{i}v))^{-1} = P_\varepsilon(v)$, we arrive at (1.12).

Finally, let ν be a translation-bounded positive Radon measure that satisfies (1.12). Then $P_\varepsilon * \nu = P_\varepsilon * \nu_X$ for all $\varepsilon > 0$, and Lemma A.4(b) implies that $\nu = \nu_X$. □

Proof of Corollary 1.7. The existence of ν_X, its translation bound, and (1.11) are Lemma 14.1, and (1.12) and the uniqueness are Lemma 14.2.

Let A_1 and A_2 be bounded intervals. Applying (1.11) with $f(x, y) = \mathbf{1}_{A_1}(x)\mathbf{1}_{A_2}(y)$, and using that Ξ_X has intensity one, we find

$$\mathbb{E}\big[\Xi_X(A_1)\Xi_X(A_2)\big] = |A_1 \cap A_2| + \int_{\mathbb{R}} |A_1 \cap (A_2 - s)|\, \nu_X(\mathrm{d}s),$$

Subtracting $\mathbb{E}\Xi_X(A_1)\,\mathbb{E}\Xi_X(A_2) = |A_1||A_2|$ from this identity gives the formula for the covariance of $\Xi_X(A_1)$ and $\Xi_X(A_2)$.

We now prove the convergence in (1.13). Almost surely, Ξ_X does not charge the endpoints of A_1 and A_2, since it has intensity one. As in the proof of Proposition 8.2, the map $\Pi \mapsto (\Pi(A_1), \Pi(A_2))$ on $\mathcal{N}(\mathbb{R})$ is continuous at every point measure that does not charge these endpoints, and therefore $(\Xi_N(A_1), \Xi_N(A_2))$ converges in law to $(\Xi_X(A_1), \Xi_X(A_2))$. The bound (8.4) with $m = 4$ shows that the variables $\Xi_N(A_1)\Xi_N(A_2)$ are bounded in L^2, hence uniformly integrable, and the same holds for $\Xi_N(A_i)$. The means and the covariances converge, which proves (1.13). □

14.2 The Bartlett spectrum

We next record the Poisson transform of the pair measure of the sine process, which identifies the metallic endpoint of ν_X.

Lemma 14.3. *The measure* $\nu_{\mathrm{GUE}}(\mathrm{d}s) = \big(1 - (\sin \pi s/(\pi s))^2\big)\,\mathrm{d}s$ *satisfies, for all* $\varepsilon > 0$ *and* $v \in \mathbb{R}$,

$$(P_\varepsilon * \nu_{\mathrm{GUE}})(v) = 1 + \operatorname{Re} I_{\mathrm{GUE}}(\pi\varepsilon - \mathrm{i}\pi v) - P_\varepsilon(v). \tag{14.4}$$

Proof. We set

$$\sigma(s) = \Big(\frac{\sin \pi s}{\pi s}\Big)^2, \qquad s \in \mathbb{R}.$$

The function $\sin(\pi s)/(\pi s)$ is the inverse Fourier transform of the indicator of $[-\frac{1}{2}, \frac{1}{2}]$, and σ is the inverse Fourier transform of the convolution of this indicator with itself. The Fourier transforms of σ and P_ε are

$$\hat{\sigma}(k) = (1 - |k|)_+, \qquad \hat{P}_\varepsilon(k) = e^{-2\pi\varepsilon|k|}.$$

Both functions are integrable, and with $z = \pi\varepsilon - \mathrm{i}\pi v$, the Fourier inversion formula yields

$$(P_\varepsilon * \sigma)(v) = \int_{\mathbb{R}} e^{-2\pi\varepsilon|k|+2\pi\mathrm{i}kv}(1-|k|)_+ \,\mathrm{d}k = 2\operatorname{Re}\int_0^1 e^{-2zk}(1-k)\,\mathrm{d}k.$$

Integrating by parts, we find

$$\int_0^1 e^{-2zk}(1-k)\,\mathrm{d}k = \frac{1}{2z} - \frac{1-e^{-2z}}{4z^2},$$

and therefore $(P_\varepsilon * \sigma)(v) = \operatorname{Re} z^{-1} - \operatorname{Re} I_{\mathrm{GUE}}(z)$. Inserting $\operatorname{Re} z^{-1} = P_\varepsilon(v)$ and $P_\varepsilon * 1 = 1$, we obtain (14.4). □

We now recall the spectral theory of second-order stationary point processes [22, Section 8.2], in the normalization of our Fourier transform. We let ν_X^{c} denote the signed measure $\delta_0 + \nu_X - \mathrm{d}s$, and we call it the *reduced covariance measure* of Ξ_X. By the proof of Lemma 14.1, $\delta_0 + \nu_X$ is the reduced second moment measure of Ξ_X, and since Ξ_X has intensity one, ν_X^{c} is the reduced covariance measure $\check{C}_2$ of [22, (8.1.6)]. Recall from [22, Proposition 8.2.I and Definition 8.2.II] that the Bartlett spectrum of Ξ_X is defined as the positive measure on $\mathbb{R}$ with Fourier transform ν_X^{c}, in the sense of the Parseval relation. That reference uses the Fourier transform $\int_{\mathbb{R}} e^{\mathrm{i}\omega u} f(u)\,\mathrm{d}u = \hat{f}(-\omega/(2\pi))$, and we let m_X denote the image of the Bartlett spectrum of Ξ_X under $\omega \mapsto \omega/(2\pi)$. We also call m_X the *Bartlett spectrum* of Ξ_X. In physics, m_X is called the structure factor of Ξ_X (see the discussion after Theorem 1.19).

The following lemma states the properties of m_X that we use. Its first assertion and part (b) are standard, and the Laplace identity (14.6) of part (a), which links m_X to the function I_X, is the part specific to Ξ_X.

Lemma 14.4. *Let $X > 0$. The Bartlett spectrum m_X is an even, positive, and translation-bounded Radon measure on $\mathbb{R}$, and*

$$\int_{\mathbb{R}} f\,\mathrm{d}\nu_X^{\mathrm{c}} = \int_{\mathbb{R}} \hat{f}\,\mathrm{d}m_X \tag{14.5}$$

for all Schwartz functions f on $\mathbb{R}$. Moreover, the following statements hold.

(a) For $\operatorname{Re} z > 0$, the function $e^{-2z|k|}$ is integrable with respect to m_X, and

$$I_X(z) = \int_{\mathbb{R}} e^{-2z|k|} m_X(\,\mathrm{d}k). \tag{14.6}$$

(b) Let f and g be functions on $\mathbb{R}$, each of which is the sum of a bounded measurable function with bounded support and a Schwartz function. Then

$$\operatorname{Cov}\big(\Xi_X(f), \Xi_X(g)\big) = \int_{\mathbb{R}} \hat{f}(k)\overline{\hat{g}(k)}\, m_X(\,\mathrm{d}k). \tag{14.7}$$

Proof. Recall from Proposition 8.2 and (8.7) that Ξ_X is stationary and has locally finite second moments, so that it is second-order stationary in the sense of [22, Section 8.1]. By [22, Proposition 8.2.I(a)], its Bartlett spectrum is a symmetric and translation-bounded measure that satisfies the Parseval relation [22, (8.2.1)] for all Schwartz functions. It is positive; see the remarks after [22, Definition 8.2.II]. Its image m_X under $\omega \mapsto \omega/(2\pi)$ is therefore even, positive, and translation-bounded. After the change of variables $\omega = 2\pi k$, the Parseval relation becomes

$$\int_{\mathbb{R}} f \, \mathrm{d}\nu_X^{\mathrm{c}} = \int_{\mathbb{R}} \hat{f}(-k) \, m_X(\, \mathrm{d}k),$$

which is (14.5), since m_X is even. In the same way, [22, Proposition 8.2.I(c)] implies that

$$\operatorname{Var} \Xi_X(f) = \int_{\mathbb{R}} |\hat{f}|^2 \, \mathrm{d}m_X \tag{14.8}$$

for all bounded measurable functions f with bounded support and for all Schwartz functions f.

We now prove (a). Let $a > 0$. For $\delta \in (0, 1]$, we set

$$\psi_\delta(s) = \delta^{-1} e^{-\pi s^2/\delta^2}, \qquad s \in \mathbb{R},$$

so that $\hat{\psi}_\delta(k) = e^{-\pi\delta^2 k^2}$. For $r \in \mathbb{R}$, we apply (14.5) to the Schwartz function $\psi_\delta(\cdot - r)$, with Fourier transform $k \mapsto e^{-2\pi i k r}\hat{\psi}_\delta(k)$. We multiply by $P_a(r)$ and integrate over r. Since ψ_δ is even and $\hat{P}_a(k) = e^{-2\pi a|k|}$, Fubini's theorem gives

$$\int_{\mathbb{R}} (P_a * \psi_\delta) \, \mathrm{d}\nu_X^{\mathrm{c}} = \int_{\mathbb{R}} e^{-2\pi a|k|} e^{-\pi\delta^2 k^2} \, m_X(\, \mathrm{d}k). \tag{14.9}$$

To justify the use of Fubini's theorem, we note that

$$(P_a * \psi_\delta)(s) = \int_{\mathbb{R}} P_a(s - \delta t) e^{-\pi t^2} \, \mathrm{d}t \le 4P_a(s) + \frac{e^{-\pi s^2/4}}{\pi a}, \qquad s \in \mathbb{R},$$

where we split the integral at $|\delta t| = |s|/2$ and used $P_a(s/2) \le 4P_a(s)$ and $P_a \le 1/(\pi a)$. The total variation of ν_X^{c} is at most $\delta_0 + \nu_X + \mathrm{d}s$, which is translation-bounded by (14.1), and m_X is translation-bounded. Therefore the integrals in (14.9) converge absolutely.

As $\delta \downarrow 0$, we have $P_a * \psi_\delta \to P_a$ pointwise, and the bound above does not depend on δ. Dominated convergence on the left side of (14.9) and monotone convergence on the right side give

$$\int_{\mathbb{R}} e^{-2\pi a|k|} m_X(\, \mathrm{d}k) = \int_{\mathbb{R}} P_a \, \mathrm{d}\nu_X^{\mathrm{c}} = P_a(0) + (P_a * \nu_X)(0) - 1 = I_X(\pi a),$$

where we used that P_a is even with integral one, (1.12) with $\varepsilon = a$ and $v = 0$, and that I_X is real on the positive axis. In particular, $e^{-2\pi a|k|}$ is integrable with respect to m_X for

all $a > 0$. By dominated convergence and Morera's theorem, the right side of (14.6) is holomorphic in $\operatorname{Re} z > 0$, and so is I_X. The identity theorem now implies (14.6).

Finally, we prove (b). The functions in (b) form a vector space, and the left and right sides of (14.7) are sesquilinear forms on it. By the polarization identity, it suffices to prove (14.8) for all f in this space. Let $f = f_1 + f_2$, where f_1 is a bounded measurable function with bounded support and f_2 is a Schwartz function. Let χ be a smooth function with compact support such that $0 \le \chi \le 1$ and $\chi = 1$ on $[-1, 1]$, and set

$$g_n(s) = \chi(s/n) f_2(s), \qquad s \in \mathbb{R},\ n \ge 1.$$

Then $f_1 + g_n$ has bounded support, and $f_2 - g_n$ is a Schwartz function, so that (14.8) holds for $f_1 + g_n$ and for $f_2 - g_n$. Since $f_2 - g_n \to 0$ in the Schwartz space, there exist $\varepsilon_n \to 0$ such that $|\widehat{f_2 - g_n}(k)| \le \varepsilon_n (1 + k^2)^{-1}$ for all k, and the translation bound of m_X implies that

$$\operatorname{Var} \Xi_X(f_2 - g_n) = \int_{\mathbb{R}} |\widehat{f_2 - g_n}|^2 \, \mathrm{d}m_X \le \varepsilon_n^2 \int_{\mathbb{R}} (1 + k^2)^{-2} \, m_X(\,\mathrm{d}k) \longrightarrow 0.$$

It follows that $\operatorname{Var} \Xi_X(f_1 + g_n) \to \operatorname{Var} \Xi_X(f)$ and that $\hat{f}_1 + \hat{g}_n \to \hat{f}$ in $L^2(m_X)$, which proves (14.8) for f. □

By (14.7) with $f = g = \mathbf{1}_{[0,L]}$, we have

$$\Sigma_X^2(L) = \int_{\mathbb{R}} \Big(\frac{\sin \pi L k}{\pi k} \Big)^2 m_X(\,\mathrm{d}k), \qquad L \ge 0. \tag{14.10}$$

At the two endpoints of Proposition 1.16, the corresponding measures are explicit. By the proof of Lemma 14.3, the reduced covariance measure $\delta_0 + \nu_{\mathrm{GUE}} - \mathrm{d}s$ of the sine process has the Bartlett spectrum $\min(|k|, 1)\,\mathrm{d}k$, and

$$\int_{\mathbb{R}} e^{-2z|k|} \min(|k|, 1)\,\mathrm{d}k = I_{\mathrm{GUE}}(z), \qquad \operatorname{Re} z > 0.$$

For a Poisson process of intensity one, the reduced covariance measure reduces to the atom δ_0, and the Bartlett spectrum to the Lebesgue measure [22, Example 8.2(a)], with Laplace transform $1/z$.

14.3 Laplace transforms of positive measures

The following lemma passes from the convergence of Laplace transforms of even positive measures to their convergence against test functions that are not compactly supported. Its first part is a form of the continuity theorem for Laplace transforms [50, Section XIII.1]. We give the short proof, since we need the bound (14.11) for the second part.

Lemma 14.5. *Let $(\mu_n)_{n\ge 1}$ be even positive Radon measures on $\mathbb{R}$, and set $L_n(a) = \int e^{-2a|u|}\mu_n(\mathrm{d}u)$ for $a > 0$. Suppose that there exist $C > 0$ and $\beta \in (0,2)$ such that $L_n(a) \le Ca^{-\beta}$ for all $n \ge 1$ and $a > 0$, and that $L_n(a)$ converges to a limit $L(a)$ for all $a > 0$.*

(a) There exists a unique even positive Radon measure μ on $\mathbb{R}$ such that $\int e^{-2a|u|}\mu(\mathrm{d}u) = L(a)$ for all $a > 0$, and μ_n converges vaguely to μ.

(b) Let $\Phi\colon \mathbb{R} \to \mathbb{C}$ be continuous, and suppose that there exists $A > 0$ such that $|\Phi(u)| \le A\min(1, u^{-2})$ for all $u \ne 0$. Then $\int \Phi\,\mathrm{d}\mu_n \to \int \Phi\,\mathrm{d}\mu$.

(c) Let $\mathcal{T}$ be a compact metric space, and let $(u,t) \mapsto \Phi_t(u)$ be a continuous function on $\mathbb{R}\times\mathcal{T}$ such that each Φ_t satisfies the assumption of (b) with the same constant A. Then the convergence in (b) is uniform in $t \in \mathcal{T}$.

Proof. We begin with (a). Bounding $e^{-2|u|/R}$ from below by e^{-2} on $[-R,R]$, we find

$$\mu_n([-R,R]) \le e^2 L_n(1/R) \le Ce^2R^{\beta}, \qquad R > 0. \tag{14.11}$$

The measures μ_n are bounded on compact sets uniformly in n, and every subsequence has a further subsequence that converges vaguely. Let μ be such a limit. It is even and positive, and it satisfies (14.11), since $\mu([-R,R]) \le \liminf_n \mu_n((-R-\eta, R+\eta))$ for $\eta > 0$. For $a > 0$ and $R > 0$, the tails satisfy

$$\int_{|u|>R} e^{-2a|u|}\mu_n(\mathrm{d}u) \le e^{-aR}L_n(a/2) \le Ce^{-aR}(a/2)^{-\beta},$$

uniformly in n. Approximating $e^{-2a|u|}$ by continuous functions with compact support, we obtain

$$\int_{\mathbb{R}} e^{-2a|u|}\mu(\mathrm{d}u) = L(a), \qquad a > 0.$$

It remains to show that an even positive Radon measure μ is determined by its Laplace transform L, when L is finite on $(0,\infty)$; then all subsequential limits agree, and the whole sequence converges. Let ρ denote the image of $e^{-2|u|}\mu(\mathrm{d}u)$ under $u \mapsto |u|$, which is a finite measure on $[0,\infty)$ with

$$\int_{[0,\infty)} e^{-2\lambda x}\rho(\mathrm{d}x) = L(1+\lambda), \qquad \lambda \ge 0.$$

The linear span of the functions $x \mapsto e^{-2\lambda x}$, $\lambda \ge 0$, is an algebra that contains the constants and separates the points of the one-point compactification $[0,\infty]$, to which these functions extend continuously. By the Stone–Weierstrass theorem, it is dense in $C([0,\infty])$, and ρ is determined by L. The even measure μ is in turn determined by ρ.

For (b), let $R \ge 1$. Covering $\{|u| > R\}$ by the shells $2^jR < |u| \le 2^{j+1}R$, $j \ge 0$, and using (14.11), we have

$$\int_{|u|>R} u^{-2}\mu_n(\mathrm{d}u) \le \sum_{j\ge 0}(2^jR)^{-2}\mu_n([-2^{j+1}R, 2^{j+1}R]) \le C_\beta R^{\beta-2}, \tag{14.12}$$

where

$$C_\beta = Ce^2 2^\beta \sum_{j\ge 0} 2^{(\beta-2)j} < \infty,$$

since $\beta < 2$. The same bound holds for μ. Let η_R be continuous, with $0 \le \eta_R \le 1$, $\eta_R = 1$ on $[-R, R]$, and $\eta_R = 0$ outside $[-2R, 2R]$. Part (a) implies that

$$\int_{\mathbb{R}} \Phi\eta_R \,\mathrm{d}\mu_n \longrightarrow \int_{\mathbb{R}} \Phi\eta_R \,\mathrm{d}\mu.$$

Using (14.12), we bound the integrals of $\Phi(1-\eta_R)$ against μ_n and μ by $AC_\beta R^{\beta-2}$, and letting $R \to \infty$ proves (b).

For (c), we note that the tail bound in the proof of (b) is uniform in t. For fixed R, the functions $\Phi_t\eta_R$ form a compact set in the space of continuous functions supported in $[-2R, 2R]$ with the uniform norm, since $(u,t) \mapsto \Phi_t(u)\eta_R(u)$ is uniformly continuous on $[-2R, 2R] \times \mathcal{T}$. We cover this set by finitely many balls of radius δ and use the uniform bound (14.11) on $[-2R, 2R]$. We conclude that the convergence of $\int \Phi_t\eta_R \,\mathrm{d}\mu_n$ to $\int \Phi_t\eta_R \,\mathrm{d}\mu$ is uniform in t, and (c) follows. □

14.4 The stationary approximation

In this section and the next, $a > 0$, and we write $\bar{a} = \max(1, a)$. We consider the semigroup $e^{t(\Delta_D - ad)}$ of Definition 1.4 with $X = 1$ and $z = a$, and the singlet states $\Psi_t^{1,a}$. By the scaling of time in the Feynman–Kac formula, we have

$$e^{t(X\Delta_D - zd)} = e^{Xt(\Delta_D - (z/X)d)}, \qquad \Psi_t^{X,z} = \Psi_{Xt}^{1,z/X}, \qquad X, t > 0,\ \mathrm{Re}\, z > 0. \tag{14.13}$$

Inserting (14.13) with $z = Xa$ into (1.6) and (1.8), we obtain

$$XI_X(Xa) = X \int_0^1 \int_x^1 \left\langle \Psi_{Xx}^{1,a}, e^{X(y-x)(\Delta_D - ad)} \Psi_{X(1-y)}^{1,a} \right\rangle_D \mathrm{d}y \,\mathrm{d}x. \tag{14.14}$$

Heuristically, the diffusion relaxes on the time scale $1/\sqrt{\bar{a}}$, and when $X\sqrt{\bar{a}}$ is large, the two outer singlet states in (14.14) are close to their limit as $t \to \infty$, except near the faces of the simplex. We make this precise in Lemma 14.10, after four preliminary lemmas.

We next introduce the limit of the singlet states $\Psi_t^{1,a}$ as $t \to \infty$, which is identified in Lemma 14.8 below. For $a > 0$, we define functions on D by

$$p_a = 2\sqrt{a(B+1)}, \qquad q_a = 2\sqrt{a(F+1)}, \tag{14.15}$$

and

$$\Psi_\infty^{1,a} = p_a K_1(p_a) I_0(q_a) + q_a I_1(q_a) K_0(p_a), \tag{14.16}$$

where I_0, I_1, K_0, and K_1 denote the modified Bessel functions of orders zero and one [30, Section 10.25]. Since $B \ge 1$ and $F \le 1$, we have $p_a \ge \sqrt{8a} \ge q_a$, with equality at the

corner. The function $\Psi_\infty^{1,a}$ was found by Skvortsov and Ostrovsky [126, (14)] as the exact zero mode of the transfer-matrix Hamiltonian of a disordered wire in the unitary class; in their notation, $B = \lambda_1$, $F = \lambda$, and $4a = -2\mathrm{i}\omega/\Delta_\xi$. We verify the properties that we need directly in Lemma 14.6 and Lemma 14.8.

Lemma 14.6. *Let $a > 0$. The following statements hold.*

(a) Let $\mathfrak{u}_a = 2K_0(p_a)I_0(q_a)$. The functions $\Psi_\infty^{1,a}$ and $\mathfrak{u}_a$ are smooth, positive, and bounded on D, and

$$(ad - \Delta_D)\mathfrak{u}_a = \Psi_\infty^{1,a}, \qquad \partial_a \Psi_\infty^{1,a} = -d\mathfrak{u}_a. \tag{14.17}$$

(b) Let $\mathfrak{w}_a = \int_0^a \mathfrak{u}_{a'}\,\mathrm{d}a'$. Then $\mathfrak{w}_a$ is smooth, nonnegative, and bounded on D, with $\nabla_D \mathfrak{w}_a$ bounded, and

$$\Psi_\infty^{1,a} = 1 - d\mathfrak{w}_a, \qquad (ad - \Delta_D)\mathfrak{w}_a = a. \tag{14.18}$$

In particular, $\Psi_\infty^{1,a} \le 1$, with equality at the corner.

Proof. We begin with (a). By the series [30, (10.25.2)], $I_0(q)$ and $qI_1(q)$ are entire functions of $q^2 = 4a(F+1)$, and in particular smooth functions of F. The functions $K_0(p)$ and $pK_1(p)$ are analytic for $p > 0$, and $p_a \ge \sqrt{8a} > 0$; they are therefore smooth functions of B. All four functions are nonnegative; only $qI_1(q)$ vanishes, and only at $q = 0$. By [30, (10.29.3), (10.29.4)], we have $K_0' = -K_1$, $(pK_1(p))' = -pK_0(p)$, $I_0' = I_1$, and $(qI_1(q))' = qI_0(q)$. It follows that $K_0(p)$ and $pK_1(p)$ are decreasing in p, and $I_0(q)$ and $qI_1(q)$ are increasing in q. The bounds $p_a \ge \sqrt{8a} \ge q_a$ show that all four functions are bounded by their values at $\sqrt{8a}$, and $\Psi_\infty^{1,a}$ and $\mathfrak{u}_a$ are bounded.

We now prove the first identity in (14.17). Let $\zeta_a = \sqrt{8a}$. In the variables $(p, q) = (p_a, q_a)$, we have $B + 1 = p^2/(4a)$, $B - 1 = (p^2 - \zeta_a^2)/(4a)$, $F + 1 = q^2/(4a)$, $1 - F = (\zeta_a^2 - q^2)/(4a)$, and $d = (p^2 - q^2)/(4a)$. Since $\partial_B = (2a/p)\partial_p$ and $\partial_F = (2a/q)\partial_q$, we find

$$\Delta_D = \frac{1}{4p}\partial_p\big(p(p^2 - \zeta_a^2)\partial_p\big) + \frac{1}{4q}\partial_q\big(q(\zeta_a^2 - q^2)\partial_q\big).$$

By the modified Bessel equation [30, (10.25.1)] with order zero, $K_0'' = K_0 - K_0'/p$ and $I_0'' = I_0 - I_0'/q$. Inserting these identities, we find

$$\frac{1}{4p}\partial_p\big(p(p^2 - \zeta_a^2)K_0'(p)\big) = \frac{(p^2 - \zeta_a^2)K_0(p) + 2pK_0'(p)}{4},$$
$$\frac{1}{4q}\partial_q\big(q(\zeta_a^2 - q^2)I_0'(q)\big) = \frac{(\zeta_a^2 - q^2)I_0(q) - 2qI_0'(q)}{4}.$$

Multiplying the first identity by $I_0(q)$ and the second by $K_0(p)$, adding, and using $K_0' = -K_1$ and $I_0' = I_1$, we find

$$\begin{aligned}\Delta_D\big[K_0(p)I_0(q)\big] &= \frac{p^2-q^2}{4}K_0(p)I_0(q) - \frac{1}{2}\big(pK_1(p)I_0(q) + qI_1(q)K_0(p)\big)\\ &= ad\,K_0(p)I_0(q) - \frac{1}{2}\Psi_\infty^{1,a},\end{aligned}$$

which is the first identity. For the second, we use $\partial_a p_a = p_a/(2a)$ and $\partial_a q_a = q_a/(2a)$ and the derivatives above, which give

$$\begin{aligned}\partial_a\Psi_\infty^{1,a} &= \frac{1}{2a}\Big(-p^2K_0(p)I_0(q) + pqK_1(p)I_1(q) + q^2I_0(q)K_0(p) - pqI_1(q)K_1(p)\Big)\\ &= -\frac{p^2-q^2}{2a}K_0(p)I_0(q),\end{aligned}$$

and $(p^2-q^2)/(2a) = 2d$.

For (b), we note that $K_0(p_{a'}) \le C(1+|\log a'|)$ for $0 < a' \le a$ by [30, (10.30.3)]. The integral defining $\mathfrak{w}_a$ therefore converges, and $\mathfrak{w}_a$ is bounded. Its derivatives are given by differentiating under the integral sign.

To justify this, we bound the derivatives of $\mathfrak{u}_{a'}$ for $0 < a' \le a$. Let $p = p_{a'}$ and $q = q_{a'}$, so that

$$\partial_B p = \frac{2a'}{p} = \frac{p}{2(B+1)}, \qquad \partial_F q = \frac{2a'}{q}.$$

By [30, (10.29.2), (10.29.4)], we have

$$K_0' = -K_1, \qquad \Big(\frac{K_1(p)}{p}\Big)' = -\frac{pK_0(p)+2K_1(p)}{p^2}, \qquad I_0' = I_1, \qquad \Big(\frac{I_1(q)}{q}\Big)' = \frac{I_2(q)}{q}.$$

These identities give

$$|\partial_B K_0(p)| = \frac{pK_1(p)}{2(B+1)} \le \frac{1}{2(B+1)}, \qquad \partial_B^2 K_0(p) = \frac{p^2K_0(p)+2pK_1(p)}{4(B+1)^2},$$

where we used $pK_1(p) \le 1$, a consequence of [30, (10.30.2)] and the monotonicity of $pK_1(p)$, and

$$\partial_F I_0(q) = 2a'\,\frac{I_1(q)}{q}, \qquad \partial_F^2 I_0(q) = 4a'^2\,\frac{I_2(q)}{q^2}.$$

The functions $p^2K_0(p)$ and $pK_1(p)$ are bounded on $(0,\infty)$, and $I_1(q)/q$ and $I_2(q)/q^2$ are bounded on $[0,\sqrt{8a}]$, since they are entire functions of q^2 by [30, (10.25.2)]. The mixed derivative of $K_0(p)I_0(q)$ is the product $\partial_B K_0(p)\,\partial_F I_0(q)$. Together with $K_0(p) \le C(1+|\log a'|)$ and $I_0(q) \le I_0(\sqrt{8a})$, these bounds show that all derivatives of $\mathfrak{u}_{a'}$ of order at most two are bounded on D by an integrable function of $a' \in (0,a]$. This justifies the

differentiation under the integral sign. Since $(B^2-1)/(B+1)^2 \le 1$, they also show that $\nabla_D \mathfrak{w}_a$ is bounded.

As $a \downarrow 0$, we have $p_a K_1(p_a) \to 1$, $I_0(q_a) \to 1$, and $q_a I_1(q_a) K_0(p_a) \to 0$, by [30, (10.30.1)–(10.30.3)] and $q_a \le p_a$. Integrating the second identity in (14.17) over a yields the first identity in (14.18). For the second, we note that (a) and the first identity give $(a'd - \Delta_D)\mathfrak{u}_{a'} = 1 - d\mathfrak{w}_{a'}$ for $a' > 0$. We write $ad = a'd + (a-a')d$ and integrate over $a' \in (0,a)$, which gives

$$\begin{aligned}(ad - \Delta_D)\mathfrak{w}_a &= \int_0^a \left((a'd - \Delta_D)\mathfrak{u}_{a'} + (a-a')d\mathfrak{u}_{a'}\right) \mathrm{d}a' \\ &= a - d\int_0^a \mathfrak{w}_{a'}\,\mathrm{d}a' + d\int_0^a (a-a')\mathfrak{u}_{a'}\,\mathrm{d}a' = a,\end{aligned}$$

where the last step uses the identity

$$\int_0^a \mathfrak{w}_{a'}\,\mathrm{d}a' = \int_0^a \int_0^{a'} \mathfrak{u}_{a''}\,\mathrm{d}a''\,\mathrm{d}a' = \int_0^a (a-a'')\mathfrak{u}_{a''}\,\mathrm{d}a'',$$

which follows from Fubini's theorem. □

Lemma 14.7. *Let $a > 0$. The following statements hold.*

(a) For nonnegative measurable functions f and g on D and $t \ge 0$, we have $\langle f, e^{t(\Delta_D - ad)} g\rangle_D = \langle e^{t(\Delta_D - ad)} f, g\rangle_D$.

(b) The operators $e^{t(\Delta_D - ad)}$, $t \ge 0$, restrict to a semigroup of self-adjoint contractions on $L^2(D)$, and for all $f \in L^2(D)$ and $t \ge 0$,

$$\left\| e^{t(\Delta_D - ad)} f \right\|_{L^2(D)} \le e^{-t\sqrt{a}/4} \|f\|_{L^2(D)}. \tag{14.19}$$

(c) For $t > 0$,

$$\left\| e^{t(\Delta_D - ad)} 1 \right\|_{L^2(D)}^2 \le \frac{1 - e^{-4at}}{4a^2 t^2}. \tag{14.20}$$

In particular, $\|e^{t(\Delta_D - ad)}1\|_{L^2(D)}^2 \le 2/a$ for $t \ge 1/(2\sqrt{a})$.

(d) There exists an absolute constant C such that for all $g \in L^2(D)$,

$$\left\| d\, e^{(\Delta_D - ad)/\sqrt{a}} g \right\|_{L^2(D)} \le \frac{C\sqrt{a}}{a} \|g\|_{L^2(D)}. \tag{14.21}$$

Proof. We write $S_t = e^{t(\Delta_D - ad)}$ in this proof. We begin with (a). By Lemma 6.6(a) with $X = 1$ and $z = a$, the identity holds for bounded integrable functions f and g. For nonnegative measurable f and g, we apply it to $\min(f,n)\mathbf{1}_{\{B \le n\}}$ and $\min(g,n)\mathbf{1}_{\{B\le n\}}$, and (a) follows by monotone convergence as $n \to \infty$.

For (b), the Cauchy–Schwarz inequality for the positive kernel of S_t, the bound $S_t 1 \le 1$, and (a) give

$$\|S_t f\|_{L^2(D)}^2 \le \int_D S_t(|f|^2)\, S_t 1 \le \int_D |f|^2\, S_t 1 \le \|f\|_{L^2(D)}^2.$$

In particular, S_t is a contraction on $L^2(D)$, and (a) shows that it is self-adjoint.

To prove (14.19), we use the following form of Schur's test. Let S be a self-adjoint operator with a nonnegative kernel, and suppose that $S\vartheta \le \theta\vartheta$ for a positive function ϑ and a constant $\theta > 0$. Then $\|S\|_{L^2\to L^2} \le \theta$. Indeed, by the Cauchy–Schwarz inequality, we have $(S|f|)^2 \le S(|f|^2/\vartheta)\, S\vartheta$, and integrating with the symmetry of S, we find

$$\|Sf\|^2 \le \theta \int \vartheta\, S\big(|f|^2/\vartheta\big) = \theta \int |f|^2 \frac{S\vartheta}{\vartheta} \le \theta^2 \|f\|^2.$$

We apply Schur's test with two functions. First, let $\vartheta = (B+1)^{-1/2}$. A computation gives

$$\Delta_D \vartheta = -\frac{B+3}{4(B+1)}\vartheta \le -\frac{1}{4}\vartheta.$$

By Lemma 13.2(a) with $X = 1$, the process $e^{t/4 - a\int_0^t \tilde d_s\,\mathrm{d}s}\vartheta(\tilde B_t, \tilde F_t)$ is a positive local supermartingale, hence a supermartingale, which implies that $S_t\vartheta \le e^{-t/4}\vartheta$. Second, we use the function $\mathfrak{u}_a$ of Lemma 14.6(a), which satisfies $(ad - \Delta_D)\mathfrak{u}_a = \Psi_\infty^{1,a}$. Using (14.16), we find

$$\frac{\Psi_\infty^{1,a}}{\mathfrak{u}_a} = \frac{p_a K_1(p_a)}{2K_0(p_a)} + \frac{q_a I_1(q_a)}{2I_0(q_a)} \ge \frac{p_a}{2} \ge \sqrt{2a},$$

where we used $K_1 \ge K_0$, which follows from [30, (10.32.9)], and $p_a \ge \sqrt{8a}$. The same supermartingale argument shows that $S_t\mathfrak{u}_a \le e^{-\sqrt{2a}t}\mathfrak{u}_a$. Schur's test now implies that $\|S_t\| \le \exp(-t\max(\frac14, \sqrt{2a}))$, and $\max(\frac14, \sqrt{2a}) \ge \sqrt{\bar a}/4$, which proves (14.19).

For (c), we note that (a) and the semigroup property give

$$\|S_t 1\|^2_{L^2(D)} = \langle S_t 1, S_t 1\rangle_D = \langle 1, S_{2t} 1\rangle_D.$$

By Jensen's inequality for the time average of $\tilde d_s$ on $[0, 2t]$,

$$\exp\Big(-a\int_0^{2t} \tilde d_s\,\mathrm{d}s\Big) \le \frac{1}{2t}\int_0^{2t} e^{-2ta\tilde d_s}\,\mathrm{d}s.$$

The radial diffusion without killing preserves $\mathrm{d}B\,\mathrm{d}F$, since its mass is conserved and its transition kernel is symmetric, by (a) and monotone convergence as $a \downarrow 0$. Taking expectations and integrating over the starting point, we find

$$\int_D S_{2t}1 \le \int_D e^{-2tad}\,\mathrm{d}B\,\mathrm{d}F = \int_0^\infty e^{-2tad}\min(d,2)\,\mathrm{d}d = \frac{1-e^{-4at}}{4a^2t^2},$$

where we used (13.21). The last assertion of (c) follows from the bound

$$\frac{1-e^{-4at}}{4a^2t^2} \le \min\Big(\frac{1}{at}, \frac{1}{4a^2t^2}\Big),$$

applied with the first term if $a \le 1$ and with the second if $a \ge 1$.

For (d), let $r = 1/\sqrt{\bar{a}}$ and $\gamma = a/\sqrt{\bar{a}}$. Positivity of the kernel, the Cauchy–Schwarz inequality, and (a) give, also for complex-valued g,

$$\|d\, S_r g\|^2 \le \int_D d^2\, S_r(|g|^2)\, S_r 1 \le \langle S_r(d^2), |g|^2 \rangle_D \le \sup_D S_r(d^2)\, \|g\|^2.$$

To bound $S_r(d^2)$, we apply Lemma 13.3 to the radial diffusion with $X = 1$, this γ, $T = r$, $\lambda = \sqrt{\bar{a}}$, $n = 1$, $q = 2$, $t_1 = r$, and $r_1 = 0$, so that $\lambda\gamma d = ad$ and $\lambda\tau = 1$. We may take $\Theta = 3$, since $(1 + 2\gamma)r \le 3$ for all $a > 0$. With $R = \gamma d$, the lemma yields

$$S_r\big((1+R)^2\big) \le e^{36}(1+R)^2 \mathcal{E}_r(R) \le C,$$

where C is absolute, since $(1+R)^2\mathcal{E}_r(R)$ is bounded on $[0,\infty)$. With $d^2 \le \gamma^{-2}(1+R)^2$, it follows that $S_r(d^2) \le C\gamma^{-2}$. Inserting this into the previous bound and using $\gamma^{-2} = \bar{a}/a^2$, we obtain (14.21). □

Lemma 14.8. *Let $a > 0$. For all $t \ge 0$,*

$$\Psi_t^{1,a} = \Psi_\infty^{1,a} + d\, e^{t(\Delta_D - ad)} \mathfrak{w}_a. \tag{14.22}$$

Consequently, $0 < \Psi_\infty^{1,a} \le \Psi_t^{1,a} \le 1$, the function $\Psi_t^{1,a}$ is nonincreasing in t and converges to $\Psi_\infty^{1,a}$ at every point of D as $t \to \infty$, and

$$\Psi_t^{1,a} - \Psi_\infty^{1,a} = ad \int_t^\infty e^{s(\Delta_D - ad)} 1 \, \mathrm{d}s. \tag{14.23}$$

Proof. We apply Lemma 13.4 with $X = 1$, $z = a$, and the time-independent function $u = \mathfrak{w}_a$, which satisfies its assumptions by Lemma 14.6(b). Using $(\Delta_D - ad)\mathfrak{w}_a = -a$ in (13.10), we find

$$e^{t(\Delta_D - ad)} \mathfrak{w}_a = \mathfrak{w}_a - a \int_0^t e^{s(\Delta_D - ad)} 1 \, \mathrm{d}s.$$

Multiplying by d and using (1.5) and (14.18), we obtain

$$d\, e^{t(\Delta_D - ad)} \mathfrak{w}_a = (1 - \Psi_\infty^{1,a}) - (1 - \Psi_t^{1,a}),$$

which is (14.22). Since $\mathfrak{w}_a \ge 0$, this identity implies that $\Psi_t^{1,a} \ge \Psi_\infty^{1,a} > 0$, and (1.5) shows that $\Psi_t^{1,a} \le 1$ and that $\Psi_t^{1,a}$ is nonincreasing in t.

We now identify the limit, and we first show that $e^{t(\Delta_D - ad)} 1 \to 0$ at every point $p \in D$. The radial diffusion escapes to infinity in the coordinate B. Indeed, we have $\Delta_D \log B = 1 + B^{-2}$ and $|\nabla_D \log B|^2 = 1 - B^{-2}$. Applying Lemma 13.2(a) with $X = 1$ and $f = \log B$, which is smooth on a neighborhood of D, we find

$$\log \tilde{B}_t = \log \tilde{B}_0 + \int_0^t \big(1 + \tilde{B}_s^{-2}\big)\, \mathrm{d}s + \tilde{M}_t \ge t + \tilde{M}_t,$$

where $\tilde{M}$ is a continuous local martingale with $\langle \tilde{M} \rangle_t \le 2t$. For $\theta > 0$, the process $\exp(-\theta \tilde{M}_t - \theta^2 \langle \tilde{M} \rangle_t / 2)$ is a supermartingale, as in the proof of Lemma 13.2(d), and Doob's maximal inequality with $\theta = 1/8$ implies that

$$\mathbb{P}_p\Big(\inf_{s \le t} \tilde{M}_s \le -\frac{t}{4} \Big) \le e^{-t/64}.$$

On the complementary event, we have for all $s \in [t/2, t]$

$$\log \tilde{B}_s \ge \frac{s}{2} \ge \frac{t}{4}, \qquad \tilde{d}_s \ge \tilde{B}_s - 1 \ge e^{t/4} - 1.$$

It follows that

$$\big(e^{t(\Delta_D - ad)} 1\big)(p) \le \exp\Big(-\frac{at}{2}\big(e^{t/4} - 1\big)\Big) + e^{-t/64},$$

which tends to zero as $t \to \infty$.

Next, by (14.22), we have

$$0 \le \Psi^{1,a}_t - \Psi^{1,a}_\infty = d\, e^{t(\Delta_D - ad)} \mathfrak{w}_a \le d\, \|\mathfrak{w}_a\|_\infty\, e^{t(\Delta_D - ad)} 1,$$

and the right side tends to zero at every point of D. Finally, for $t < t'$, (1.5) implies that

$$\Psi^{1,a}_t - \Psi^{1,a}_{t'} = ad \int_t^{t'} e^{s(\Delta_D - ad)} 1 \, \mathrm{d}s,$$

and letting $t' \to \infty$ with monotone convergence gives (14.23). □

Lemma 14.9. *There exists an absolute constant C such that for all $a > 0$ the following statements hold.*

(a) For $t \ge 1/\sqrt{\bar{a}}$, we have $\|\Psi^{1,a}_t\|_{L^1(D)} \le C/a$. Moreover, $\|\Psi^{1,a}_\infty\|^2_{L^2(D)} \le \|\Psi^{1,a}_\infty\|_{L^1(D)} \le C/a$.

(b) For $t \ge 2/\sqrt{\bar{a}}$,

$$\big\|\Psi^{1,a}_t - \Psi^{1,a}_\infty\big\|_{L^2(D)} \le \frac{C}{\sqrt{a}} e^{-t\sqrt{\bar{a}}/4}. \tag{14.24}$$

(c) For $s, v, t \ge 0$ with $s + v + t \ge 3/\sqrt{\bar{a}}$,

$$0 \le \big\langle \Psi^{1,a}_s, e^{v(\Delta_D - ad)} \Psi^{1,a}_t \big\rangle_D \le \frac{C}{a} e^{-v\sqrt{\bar{a}}/4}. \tag{14.25}$$

(d) For $X \ge 3$, we have $X I_X(Xa) \le C/a$.

Proof. Below, C denotes an absolute constant whose value may change from line to line, and we write $S_t = e^{t(\Delta_D - ad)}$. We begin with (a). As $\Psi^{1,a}_t$ is nonincreasing in t (Lemma 14.8), we may assume that $t = 1/\sqrt{\bar{a}}$. By Lemma 6.5(c) with $X = 1$ and $z = a$, we have

$$\Psi^{1,a}_t(p) = \mathbb{E}^{\mathrm{s}}_p \exp\Big(-a \int_0^t d_s \, \mathrm{d}s \Big), \qquad p \in D^\circ,$$

where the singlet diffusion has generator Δ_D^{s}. We apply Lemma 13.3 to the singlet diffusion with $X = 1$, $\gamma = a/\sqrt{\bar{a}}$, $T = t$, $\lambda = \sqrt{\bar{a}}$, $n = 1$, $q = 0$, $t_1 = t$, and $r_1 = 0$, so that $\lambda\gamma d = ad$ and $\lambda\tau = 1$. We may take $\Theta = 3$, since $(1 + 2\gamma)t \le 3$. With $R = \gamma d$, we obtain

$$\Psi_t^{1,a} \le \mathcal{E}_t(R) = e^{-((1+R)^{1/2}-1)} + e^{-(\frac{1}{2}\log(1+R)-3)_+^2/24} \qquad \text{on } D^\circ.$$

The boundary of D is a null set, and this bound suffices for the L^1 norm.

Using (13.21), we find

$$\|\Psi_t^{1,a}\|_{L^1(D)} \le \int_0^\infty \mathcal{E}_t(\gamma d)\min(d,2)\,\mathrm{d}d.$$

If $a \le 1$, then $\gamma = a$, and we bound $\min(d,2) \le 2$. If $a \ge 1$, then $\gamma = \sqrt{a}$, and we bound $\min(d,2) \le d$. In both cases, the change of variables $R = \gamma d$ gives

$$\|\Psi_t^{1,a}\|_{L^1(D)} \le \frac{2}{a}\int_0^\infty (1+R)\,\mathcal{E}_t(R)\,\mathrm{d}R \le \frac{C}{a},$$

since $\mathcal{E}_t(R)$ decays faster than every power of $1 + R$. The second assertion of (a) follows from $0 < \Psi_\infty^{1,a} \le \Psi_t^{1,a} \le 1$.

We next prove (b). Let $r = 1/\sqrt{\bar{a}}$ and $r' = 1/(2\sqrt{\bar{a}})$, and let $t \ge 2/\sqrt{\bar{a}}$. The identity (14.23), the semigroup property, and Tonelli's theorem give

$$\Psi_t^{1,a} - \Psi_\infty^{1,a} = ad\,S_r u, \qquad u = \int_{t-r}^\infty S_s 1\,\mathrm{d}s.$$

For $s \ge r'$, (14.19) and Lemma 14.7(c) give

$$\|S_s 1\|_{L^2(D)} \le e^{-(s-r')\sqrt{\bar{a}}/4}\|S_{r'}1\|_{L^2(D)} \le e^{-(s-r')\sqrt{\bar{a}}/4}\Big(\frac{2}{a}\Big)^{1/2}.$$

Integrating over $s \ge t - r$ and using $t - r \ge r'$, we find

$$\|u\|_{L^2(D)} \le \Big(\frac{2}{a}\Big)^{1/2}\frac{4}{\sqrt{\bar{a}}}e^{-(t-r-r')\sqrt{\bar{a}}/4} \le \frac{C}{\sqrt{a\bar{a}}}e^{-t\sqrt{\bar{a}}/4}.$$

Finally, by (14.21), we have $\|ad\,S_r u\| \le C\sqrt{\bar{a}}\|u\|$, which proves (14.24).

For (c), the lower bound follows from $\Psi_t^{1,a} \ge 0$ and the positivity of S_v. Suppose first that $v \ge 1/\sqrt{\bar{a}}$. The bound $0 \le \Psi_t^{1,a} \le 1$ for all t, Lemma 14.7(a), (14.19), and Lemma 14.7(c) give

$$\big\langle \Psi_s^{1,a}, S_v\Psi_t^{1,a}\big\rangle_D \le \langle 1, S_v 1\rangle_D = \|S_{v/2}1\|^2 \le e^{-(v-2r')\sqrt{\bar{a}}/4}\|S_{r'}1\|^2 \le \frac{2e^{1/4}}{a}e^{-v\sqrt{\bar{a}}/4}.$$

Suppose next that $v < 1/\sqrt{\bar{a}}$. Then $s + t > 2/\sqrt{\bar{a}}$, and $s \ge 1/\sqrt{\bar{a}}$ or $t \ge 1/\sqrt{\bar{a}}$. In the first case, we use $S_v\Psi_t^{1,a} \le S_v 1 \le 1$ and (a), which give

$$\big\langle \Psi_s^{1,a}, S_v\Psi_t^{1,a}\big\rangle_D \le \|\Psi_s^{1,a}\|_{L^1(D)} \le \frac{C}{a} \le \frac{Ce^{1/4}}{a}e^{-v\sqrt{\bar{a}}/4}.$$

In the second case, we use Lemma 14.7(a) to move S_v to the other side, and conclude in the same way. This proves (14.25).

For (d), we use (14.14) and (c) with $s = Xx$, $v = X(y-x)$, and $t = X(1-y)$, whose sum X satisfies $X \geq 3 \geq 3/\sqrt{\bar{a}}$. Integrating, we obtain

$$XI_X(Xa) \leq X \int_0^1 \int_x^1 \frac{C}{a} e^{-X(y-x)\sqrt{\bar{a}}/4} \, \mathrm{d}y \, \mathrm{d}x \leq \frac{4C}{a\sqrt{\bar{a}}},$$

which proves (d). □

We define the *stationary pairing* by

$$\mathcal{I}(a) = \int_0^\infty \left\langle \Psi_\infty^{1,a}, e^{v(\Delta_D - ad)} \Psi_\infty^{1,a} \right\rangle_D \mathrm{d}v, \qquad a > 0, \tag{14.26}$$

which is finite by (14.19) and Lemma 14.9(a).

The stationary pairing and its integrand have a meaning in physics, which we explain in Remark 14.13.

The next lemma is the main estimate of this section. The difficulty is that the bounds must be uniform in $a \in (0, \infty)$: the relaxation time $1/\sqrt{\bar{a}}$ and the size $1/a$ of the stationary state both vary with a, and the three regimes of Proposition 14.14 below use small a, large a, and fixed a.

Lemma 14.10. *There exists an absolute constant C such that for all $a > 0$ and $X > 0$ with $X\sqrt{\bar{a}} \geq 3$,*

$$\left| XI_X(Xa) - \mathcal{I}(a) \right| \leq \frac{C}{a\bar{a}X}. \tag{14.27}$$

Proof. Below, C denotes an absolute constant whose value may change from line to line, and we write $S_v = e^{v(\Delta_D - ad)}$ and $\Psi_t = \Psi_t^{1,a}$ for $t \in [0, \infty]$. In (14.14), we change variables to $b = x$ and $v = X(y-x)$, which gives

$$XI_X(Xa) = \int_0^X \int_0^{1-v/X} \left\langle \Psi_{Xb}, S_v \Psi_{X(1-b)-v} \right\rangle_D \mathrm{d}b \, \mathrm{d}v. \tag{14.28}$$

The three times Xb, v, and $X(1-b)-v$ add up to $X \geq 3/\sqrt{\bar{a}}$, and (14.25) bounds the integrand by $(C/a)e^{-v\sqrt{\bar{a}}/4}$.

We first remove the region where one of the times Xb and $X(1-b)-v$ of the two end states is less than $2/\sqrt{\bar{a}}$. For fixed v, the set of b in this region has measure at most $4/(X\sqrt{\bar{a}})$. Its contribution to (14.28) is therefore at most

$$\int_0^X \frac{4}{X\sqrt{\bar{a}}} \frac{C}{a} e^{-v\sqrt{\bar{a}}/4} \, \mathrm{d}v \leq \frac{C}{a\bar{a}X}.$$

On the remaining region, both of these times are at least $2/\sqrt{\bar{a}}$, and we replace the two end states by Ψ_∞. Writing $s = Xb$ and $t = X(1-b) - v$, we have

$$\big|\langle \Psi_s, S_v \Psi_t\rangle_D - \langle \Psi_\infty, S_v \Psi_\infty\rangle_D\big| \le \|\Psi_s - \Psi_\infty\|\,\|S_v\Psi_t\| + \|\Psi_\infty\|\,\|S_v(\Psi_t - \Psi_\infty)\|.$$

We bound the right side with (14.19), (14.24), and the bounds $\|\Psi_t\|^2 \le \|\Psi_t\|_{L^1(D)} \le C/a$ of Lemma 14.9(a). The right side is then at most

$$\frac{C}{a} e^{-v\sqrt{\bar{a}}/4}\big(e^{-s\sqrt{\bar{a}}/4} + e^{-t\sqrt{\bar{a}}/4}\big).$$

The integral of $e^{-s\sqrt{\bar{a}}/4} + e^{-t\sqrt{\bar{a}}/4}$ over b is bounded by $8/(X\sqrt{\bar{a}})$, and the integral of $e^{-v\sqrt{\bar{a}}/4}$ over v by $4/\sqrt{\bar{a}}$. The error of the replacement is therefore bounded by $C/(a\bar{a}X)$.

The stationary integrand $\langle \Psi_\infty, S_v\Psi_\infty\rangle_D$ is also bounded by $(C/a)e^{-v\sqrt{\bar{a}}/4}$, by (14.19) and Lemma 14.9(a). Adding it back on the removed region costs at most $C/(a\bar{a}X)$, as in the first step. We have shown that

$$\Big| X I_X(Xa) - \int_0^X \Big(1 - \frac{v}{X}\Big)\langle \Psi_\infty, S_v\Psi_\infty\rangle_D \,\mathrm{d}v\Big| \le \frac{C}{a\bar{a}X}.$$

Finally, we compare this integral with $\mathcal{I}(a)$. Their difference is bounded by

$$\frac{C}{a}\int_0^X \frac{v}{X} e^{-v\sqrt{\bar{a}}/4}\,\mathrm{d}v + \frac{C}{a}\int_X^\infty e^{-v\sqrt{\bar{a}}/4}\,\mathrm{d}v \le \frac{16C}{a\bar{a}X} + \frac{4C}{a\sqrt{\bar{a}}} e^{-X\sqrt{\bar{a}}/4} \le \frac{C}{a\bar{a}X},$$

where we used $e^{-y} \le 1/y$ for $y > 0$ in the last step. This completes the proof. □

14.5 The stationary pairing

We now compute $\mathcal{I}(a)$. The computation is elementary once the stationary state is explicit, and we will need a Laplace transform of the product $I_0 K_0$.

Lemma 14.11. *For $a > 0$,*

$$I_0(\sqrt{8a})K_0(\sqrt{8a}) = \int_0^\infty e^{-at}\frac{e^{-4/t} I_0(4/t)}{2t}\,\mathrm{d}t. \tag{14.29}$$

Proof. For $r, \rho \ge 0$, let

$$\mathrm{Av}(r,\rho) = \frac{1}{2\pi}\int_0^{2\pi} K_0\big(\sqrt{a}\,|re^{\mathrm{i}\theta} - \rho|\big)\,\mathrm{d}\theta,$$

which is the average of $K_0(\sqrt{a}|x-y|)$ over x on the circle of radius r centered at zero, for $|y| = \rho$. We first show that

$$\mathrm{Av}(r,\rho) = I_0\big(\sqrt{a}\min(r,\rho)\big)K_0\big(\sqrt{a}\max(r,\rho)\big), \qquad r,\rho > 0. \tag{14.30}$$

Fix $\rho > 0$. For $y \in \mathbb{R}^2$, the function $x \mapsto K_0(\sqrt{a}|x-y|)$ is smooth on $\mathbb{R}^2 \setminus \{y\}$ and satisfies $\Delta u = au$ there, by the modified Bessel equation [30, (10.25.1)] in polar coordinates centered at y. Averaging over the rotations of y about zero, we see that $\mathrm{Av}(\cdot, \rho)$ is smooth on $[0, \rho)$ and satisfies the radial equation

$$\partial_r^2 \mathrm{Av}(r,\rho) + \frac{1}{r}\partial_r \mathrm{Av}(r,\rho) = a\,\mathrm{Av}(r,\rho), \qquad 0 < r < \rho.$$

The solutions of this equation are the combinations of $I_0(\sqrt{a}r)$ and $K_0(\sqrt{a}r)$, and $K_0(\sqrt{a}r)$ is unbounded as $r \downarrow 0$ by [30, (10.30.3)]. The function $\mathrm{Av}(\cdot, \rho)$ is continuous at $r = 0$, with $\mathrm{Av}(0,\rho) = K_0(\sqrt{a}\rho)$. With $I_0(0) = 1$, this gives (14.30) for $r < \rho$.

The identity $|re^{\mathrm{i}\theta} - \rho| = |\rho e^{\mathrm{i}\theta} - r|$ shows that Av is symmetric, which implies (14.30) for $r > \rho$. It remains to treat $r = \rho$. For $r \geq \rho/2$, we have

$$|re^{\mathrm{i}\theta} - \rho|^2 = (r-\rho)^2 + 2r\rho(1-\cos\theta) \geq \rho^2(1-\cos\theta),$$

and $K_0(\sqrt{a}\rho(1-\cos\theta)^{1/2})$ is integrable in θ, since K_0 is decreasing and has a logarithmic singularity at zero. Dominated convergence shows that $\mathrm{Av}(\cdot, \rho)$ is continuous at ρ, and (14.30) follows for $r = \rho$.

We now take $r = \rho = \sqrt{8}$. By [30, (10.32.1)], we have

$$I_0(4/t) = \frac{1}{2\pi}\int_0^{2\pi} e^{(4/t)\cos\theta}\,\mathrm{d}\theta, \qquad t > 0,$$

and since $|\sqrt{8}e^{\mathrm{i}\theta} - \sqrt{8}|^2 = 16(1-\cos\theta)$, we obtain

$$\frac{e^{-4/t}I_0(4/t)}{2t} = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2t}\exp\Big(-\frac{|\sqrt{8}e^{\mathrm{i}\theta} - \sqrt{8}|^2}{4t}\Big)\,\mathrm{d}\theta.$$

For $R > 0$, the substitution $s = at$ and [30, (10.32.10)] with order zero give

$$\int_0^\infty e^{-at}\frac{1}{2t}e^{-R^2/(4t)}\,\mathrm{d}t = \frac{1}{2}\int_0^\infty \exp\Big(-s - \frac{aR^2}{4s}\Big)\frac{\mathrm{d}s}{s} = K_0(\sqrt{a}R).$$

Integrating over t, using Tonelli's theorem and (14.30), we obtain (14.29). □

Recall the function $m_\infty(u) = e^{-2/|u|}I_0(2/|u|)$ of Theorem 1.19 (with $m_\infty(0) = 0$). By [30, (10.32.1)],

$$m_\infty(u) = \frac{1}{\pi}\int_0^\pi \exp\Big(-\frac{2(1-\cos\theta)}{|u|}\Big)\,\mathrm{d}\theta, \qquad u \neq 0. \tag{14.31}$$

Lemma 14.12. *The following statements hold.*

(a) The function m_∞ is even and continuous, $0 < m_\infty(u) < 1$ for $u \neq 0$, and $m_\infty(u)$ is strictly increasing in $|u|$. Moreover, $m_\infty(u) \to 1$ as $|u| \to \infty$,

$$m_\infty(u) \sim \frac{|u|^{1/2}}{2\sqrt{\pi}} \quad (u \to 0), \qquad m_\infty(u) \leq \min\Big(1, \frac{\sqrt{\pi|u|}}{4}\Big) \quad (u \in \mathbb{R}). \tag{14.32}$$

(b) For $a > 0$, with $\zeta_a = \sqrt{8a}$,

$$\begin{aligned} \mathcal{I}(a) &= \frac{8}{\zeta_a}\big(I_0(\zeta_a)K_1(\zeta_a) - I_1(\zeta_a)K_0(\zeta_a)\big) \\ &= -2\frac{\mathrm{d}}{\mathrm{d}a}\Big(I_0(\sqrt{8a})K_0(\sqrt{8a})\Big) = \int_{\mathbb{R}} e^{-2a|u|} m_\infty(u)\,\mathrm{d}u. \end{aligned} \tag{14.33}$$

(c) As $a \downarrow 0$, we have $a\mathcal{I}(a) \to 1$, and as $a \to \infty$, we have $a^{3/2}\mathcal{I}(a) \to 1/(4\sqrt{2})$.

Proof. We begin with (a). The integrand in (14.31) lies in $(0,1)$ for $\theta \in (0,\pi]$ and is strictly increasing in $|u|$, which implies the first assertions, and the limit as $|u| \to \infty$ follows by dominated convergence. For the behavior at zero, we substitute $\theta = |u|^{1/2}v$, which gives

$$m_\infty(u) = \frac{|u|^{1/2}}{\pi}\int_0^{\pi|u|^{-1/2}} \exp\Big(-\frac{2(1-\cos(|u|^{1/2}v))}{|u|}\Big)\,\mathrm{d}v.$$

The inequality $1 - \cos\theta \geq 2\theta^2/\pi^2$ for $\theta \in [0,\pi]$ bounds the integrand by e^{-4v^2/π^2}, and the integrand tends to e^{-v^2} as $u \to 0$. Dominated convergence and the integral $\int_0^\infty e^{-v^2}\,\mathrm{d}v = \sqrt{\pi}/2$ give the asymptotic in (14.32). The same bound implies that

$$m_\infty(u) \leq \frac{|u|^{1/2}}{\pi}\int_0^\infty e^{-4v^2/\pi^2}\,\mathrm{d}v = \frac{\sqrt{\pi|u|}}{4},$$

which completes the proof of (a).

We now prove (b). By Lemma 14.6(a), the function $\mathfrak{u}_a$ is smooth and bounded, $(\Delta_D - ad)\mathfrak{u}_a = -\Psi_\infty^{1,a}$, and $\nabla_D\mathfrak{u}_a$ is bounded, by the argument in the proof of Lemma 14.6(b). Applying Lemma 13.4 with $X = 1$, $z = a$, and the time-independent function $u = \mathfrak{u}_a$, we obtain

$$\mathfrak{u}_a = e^{t(\Delta_D - ad)}\mathfrak{u}_a + \int_0^t e^{v(\Delta_D - ad)}\Psi_\infty^{1,a}\,\mathrm{d}v, \qquad t \geq 0.$$

The function $\mathfrak{u}_a$ belongs to $L^2(D)$, since K_0 decays exponentially by [30, (10.40.2)] and $p_a \geq 2\sqrt{aB}$. We pair this identity with $\Psi_\infty^{1,a}$ and let $t \to \infty$. The contribution of the first term on the right tends to zero by (14.19), and that of the second tends to $\mathcal{I}(a)$ by monotone convergence. We conclude that

$$\mathcal{I}(a) = \langle \Psi_\infty^{1,a}, \mathfrak{u}_a\rangle_D.$$

We change variables to $(p,q) = (p_a, q_a)$, which range over $p \geq \zeta_a$ and $0 \leq q \leq \zeta_a$, with $\mathrm{d}B\,\mathrm{d}F = pq\,\mathrm{d}p\,\mathrm{d}q/(4a^2)$. This gives

$$\mathcal{I}(a) = \frac{1}{2a^2}\bigg(\int_{\zeta_a}^\infty p^2 K_0K_1\,\mathrm{d}p\int_0^{\zeta_a} qI_0^2\,\mathrm{d}q + \int_{\zeta_a}^\infty pK_0^2\,\mathrm{d}p\int_0^{\zeta_a} q^2 I_0I_1\,\mathrm{d}q\bigg),$$

where the arguments of K_0, K_1 are p and those of I_0, I_1 are q. Using $K_0' = -K_1$ and $I_0' = I_1$, we integrate by parts in the mixed integrals, which gives

$$\int_{\zeta_a}^{\infty} p^2 K_0 K_1 \,\mathrm{d}p = \frac{\zeta_a^2}{2} K_0(\zeta_a)^2 + \int_{\zeta_a}^{\infty} p K_0^2 \,\mathrm{d}p, \qquad \int_0^{\zeta_a} q^2 I_0 I_1 \,\mathrm{d}q = \frac{\zeta_a^2}{2} I_0(\zeta_a)^2 - \int_0^{\zeta_a} q I_0^2 \,\mathrm{d}q.$$

We also use

$$\int_{\zeta_a}^{\infty} p K_0(p)^2 \,\mathrm{d}p = \frac{\zeta_a^2}{2}\big(K_1(\zeta_a)^2 - K_0(\zeta_a)^2\big), \qquad \int_0^{\zeta_a} q I_0(q)^2 \,\mathrm{d}q = \frac{\zeta_a^2}{2}\big(I_0(\zeta_a)^2 - I_1(\zeta_a)^2\big),$$

which follow by differentiation in ζ_a from the derivative formulas [30, (10.29.2), (10.29.3)]. Inserting these four identities and using $\zeta_a^4 = 64a^2$, we obtain

$$\mathcal{I}(a) = \frac{\zeta_a^4}{8a^2}\Big(K_0^2\big(I_0^2 - I_1^2\big) + I_0^2\big(K_1^2 - K_0^2\big)\Big)(\zeta_a) = 8\Big((I_0K_1)^2 - (I_1K_0)^2\Big)(\zeta_a).$$

By the Wronskian [30, (10.28.2)], $I_0K_1 + I_1K_0 = 1/\zeta_a$, and the first identity in (14.33) follows. The second follows from $\mathrm{d}\zeta_a/\,\mathrm{d}a = 4/\zeta_a$ and $(I_0K_0)' = I_1K_0 - I_0K_1$.

For the third identity, we differentiate the identity of Lemma 14.11 in a. This is justified by dominated convergence, since the integrand of (14.29) can be written as $e^{-at} m_\infty(t/2)/(2t)$ and $m_\infty \le 1$. We obtain

$$-2\frac{\mathrm{d}}{\mathrm{d}a}\Big(I_0(\sqrt{8a})K_0(\sqrt{8a})\Big) = \int_0^{\infty} e^{-at} m_\infty(t/2)\,\mathrm{d}t = \int_{\mathbb{R}} e^{-2a|u|} m_\infty(u)\,\mathrm{d}u,$$

where we substituted $t = 2|u|$ and used that m_∞ is even.

For (c), we substitute $u = v/a$ in the last expression in (14.33), which gives

$$a\mathcal{I}(a) = \int_{\mathbb{R}} e^{-2|v|} m_\infty(v/a)\,\mathrm{d}v, \qquad a^{3/2}\mathcal{I}(a) = \int_{\mathbb{R}} e^{-2|v|}\sqrt{a}\, m_\infty(v/a)\,\mathrm{d}v.$$

As $a \downarrow 0$, we find the limit $\int e^{-2|v|}\,\mathrm{d}v = 1$ by dominated convergence with $m_\infty \le 1$. As $a \to \infty$, dominated convergence with the bound $\sqrt{a}\, m_\infty(v/a) \le \sqrt{\pi|v|}/4$ of (14.32) gives the limit

$$\frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} e^{-2|v|} |v|^{1/2}\,\mathrm{d}v = \frac{1}{2\sqrt{\pi}} \cdot \frac{2\Gamma(3/2)}{2^{3/2}} = \frac{1}{4\sqrt{2}}.$$

This completes the proof. □

Remark 14.13. The stationary pairing and its inverse Laplace transform m_∞ appear in the physics literature on long wires in the unitary class. We begin with the integrand of (14.26). In the notation of Ivanov, Ostrovsky, and Skvortsov [71], where $2\tilde{H} = ad - \Delta_D$ with $a = 2\Omega$, the integrand in (14.26) at $v = t$ is the bilinear expression $\langle \Psi_0 | e^{-2\tilde{H}t} | \Psi_0 \rangle$ of [71, (16)], which enters their formula for the local density of states correlation at two

points of an infinite wire at distance t, in units of the localization length. In [90, (16)–(22)], Micklitz computed the correlation of levels in a long wire from the zero mode and from the solution of the transfer-matrix equation with the zero mode as its source. The stationary pairing corresponds to the integral of the bilinear expression over the distance, and he evaluated it in closed form.

We now compare the closed forms precisely. Micklitz [90, (8)–(9)] and Marinho and Micklitz [87, (12)–(13)] computed the connected correlation function K of the density of states at a separation ω of energies, for a wire whose length L is large compared with its localization length ξ. We use the localization length ξ and the level spacing Δ_ξ of a localization volume as defined in [90, after (2)], we let Δ denote the mean level spacing, and we set

$$\frac{L}{\xi} = X, \qquad \Delta_\xi = \frac{X\Delta}{2\pi}, \qquad s = \frac{\omega}{\Delta}.$$

This is the identification $g_{\mathrm{KM}} = 1/(2\pi X)$ of Remark 13.12, since in these conventions the conductance g_{KM} of [80], the ratio of the Thouless energy to Δ, equals $\xi/(2\pi L)$. The localization length and the level spacing in [87] are those of the orthogonal class, and they correspond to $\xi/2$ and $4\Delta_\xi$ in our notation [87, notes 31 and 43]. With $a = -\mathrm{i}\pi s/X$, the arguments of the Bessel functions in both formulas are $\sqrt{8a}$, and the first identity in (14.33) shows that the formula of [87, (12)–(13)] for the unitary class is

$$K = \frac{1}{X}\operatorname{Re}\mathcal{I}\Big(-\frac{\mathrm{i}\pi s}{X}\Big), \tag{14.34}$$

where $\mathcal{I}$ is continued to $\{\operatorname{Re} a \geq 0,\ a \neq 0\}$ by its closed form. The formula of [90, (8)–(9)] is twice the right side of (14.34), and [87, note 34] records the correction of a factor 4 in [90], in the relation between the zero mode and the generating function.

In our setting, (14.34) arises as follows. Formally, (1.12) at $\varepsilon = 0$ identifies the density of $\nu_X - \mathrm{d}s$ at $s \neq 0$ with $\operatorname{Re} I_X(-\mathrm{i}\pi s)$, and Proposition 14.14(c) gives $I_X(z) \approx X^{-1}\mathcal{I}(z/X)$ for large X. The right side of (14.34) is therefore the pair function of Ξ_X on the scale X that our results suggest, although we do not prove a pointwise statement of this kind. As $s/X \to \infty$, the asymptotic expansions [30, (10.40.1), (10.40.2)] give $\mathcal{I}(a) \sim a^{-3/2}/(4\sqrt{2})$ also for a on the imaginary axis, and the right side of (14.34) behaves like $-X^{1/2}s^{-3/2}/(8\pi^{3/2})$. This is the regime of Theorem 1.18, whose constant agrees with the law of Altshuler and Shklovskii (see the discussion after Theorem 1.18).

By Lemma 14.12(b), the function m_∞ is the inverse Laplace transform of $\mathcal{I}$. In the physics literature, m_∞ is computed in [91, (12)–(13)], as the forward-scattering contrast discussed after Theorem 1.19, and in [87, (19)], as a form factor, where u is the time in units of the Heisenberg time $1/\Delta_\xi$ of a localization volume. At finite X, the corresponding object is the correlation function of the density of states of a wire of finite length. Altland and Fuchs [4] computed it in the supersymmetric σ-model for practically all lengths and energy separations, from numerical solutions of the transfer-matrix equations according

to [90, Section V], and Micklitz [90, (25)–(26)] gives a closed formula for separations large compared with Δ_ξ at every ratio L/ξ.

We now collect the consequences of Lemma 14.10 and Lemma 14.12 for the function I_X.

Proposition 14.14. *The following statements hold.*

(a) As $X \to \infty$, $I_X(z) \to 1/z$ locally uniformly in $\operatorname{Re} z > 0$.

(b) For fixed $X > 0$, as $z \to \infty$ along the positive axis,

$$I_X(z) = \frac{\sqrt{X}}{4\sqrt{2}\, z^{3/2}}\big(1 + o(1)\big). \tag{14.35}$$

(c) For fixed $a > 0$, $XI_X(Xa) \to \mathcal{I}(a)$ as $X \to \infty$. Moreover, $XI_X(Xa) \le C/a$ for all $X \ge 3$ and $a > 0$, with an absolute constant C.

Proof. We begin with (a), first for z in a compact interval $[z_0, z_1] \subset (0, \infty)$. For $X \ge \max(z_1, 3)$, we apply Lemma 14.10 with $a = z/X \le 1$, for which $\bar{a} = 1$, and we find

$$I_X(z) = \frac{1}{X}\mathcal{I}\Big(\frac{z}{X}\Big) + O\Big(\frac{1}{zX}\Big) = \frac{1}{z}\Big(\frac{z}{X}\,\mathcal{I}\Big(\frac{z}{X}\Big)\Big) + O\Big(\frac{1}{zX}\Big).$$

The right side tends to $1/z$ uniformly for $z \in [z_0, z_1]$, by Lemma 14.12(c).

We now treat complex z. For $\operatorname{Re} z \ge \alpha > 0$, (14.6) implies that

$$|I_X(z)| \le I_X(\operatorname{Re} z) \le I_X(\alpha),$$

and the right side is bounded for large X by what we just proved. The functions I_X are therefore locally bounded on the half-plane, uniformly for large X. By Montel's theorem, every sequence $X_n \to \infty$ has a subsequence along which I_{X_n} converges locally uniformly to a holomorphic function, which equals $1/z$ on the positive axis. The identity theorem shows that this limit coincides with $1/z$ on the whole half-plane. This proves (a).

For (b), let $a = z/X$. When $z \ge X$, we have $\bar{a} = a$ and $X\sqrt{\bar{a}} = \sqrt{Xz}$, which is at least three for large z. Applying Lemma 14.10, we find

$$I_X(z) = \frac{1}{X}\mathcal{I}\Big(\frac{z}{X}\Big) + O\Big(\frac{1}{a^2X^2}\Big) = \frac{1}{X}\mathcal{I}\Big(\frac{z}{X}\Big) + O(z^{-2}),$$

and Lemma 14.12(c) gives $X^{-1}\mathcal{I}(z/X) \sim X^{-1}(4\sqrt{2})^{-1}(X/z)^{3/2}$, which is (14.35).

The first assertion of (c) is Lemma 14.10 with a fixed, and the second is Lemma 14.9(d). □

14.6 The endpoints

We now prove the rows of Proposition 1.16 for the pair measure, the number variance, and the kernel. The rows for the law of μ_X and for the eigenvector correlation are Proposition 12.14.

Proof of the rows of Proposition 1.16 for ν_X, Σ^2_X, and K_X. We begin with the pair measure. By (14.1), the measures ν_X have unit-interval masses bounded by C_ν for all $X > 0$. As $X \downarrow 0$, Proposition 1.17 implies that $I_X \to I_{\mathrm{GUE}}$ pointwise in $\operatorname{Re} z > 0$. Using (1.12) and (14.4), we find

$$(P_\varepsilon * \nu_X)(v) \longrightarrow 1 + \operatorname{Re} I_{\mathrm{GUE}}(\pi\varepsilon - \mathrm{i}\pi v) - P_\varepsilon(v) = (P_\varepsilon * \nu_{\mathrm{GUE}})(v), \qquad \varepsilon > 0,\ v \in \mathbb{R}.$$

By Lemma A.4(d), applied along every sequence $X_n \downarrow 0$, the measures ν_{X_n} converge vaguely to a translation-bounded measure ν with $P_\varepsilon * \nu = P_\varepsilon * \nu_{\mathrm{GUE}}$ for all $\varepsilon > 0$, and Lemma A.4(b) gives $\nu = \nu_{\mathrm{GUE}}$.

As $X \to \infty$, Proposition 14.14(a) implies that $I_X(z) \to 1/z$. Using $\operatorname{Re}(\pi\varepsilon - \mathrm{i}\pi v)^{-1} = P_\varepsilon(v)$, we see that the right side of (1.12) tends to one (the Poisson transform of the Lebesgue measure), and the same argument gives $\nu_X \to \mathrm{d}s$ vaguely.

For the number variance, we note that $s \mapsto (L - |s|)_+$ is continuous with compact support, and $\int (L - |s|)_+ \,\mathrm{d}s = L^2$. By (1.14) and the vague convergence just proved, $\Sigma^2_X(L)$ tends to

$$L + L^2 - 2\int_0^L (L - s)\Big(\frac{\sin \pi s}{\pi s}\Big)^2 \mathrm{d}s - L^2 = \Sigma^2_{\mathrm{GUE}}(L)$$

as $X \downarrow 0$, and to $L + L^2 - L^2 = L$ as $X \to \infty$.

For the kernel, Corollary 13.10 implies that

$$\frac{1}{2} K_X(x, y; z) \longrightarrow \frac{1 - e^{-2z}}{2z^2} = I_{\mathrm{GUE}}(z) \qquad (X \downarrow 0),$$

uniformly for z in compact subsets of $\{\operatorname{Re} z > 0\}$.

We now consider the limit $X \to \infty$. Let $\mathcal{K}$ be such a compact set, and let $\alpha > 0$ and $\alpha' < \infty$ be the minimum and the maximum of $\operatorname{Re} z$ over $\mathcal{K}$. By Remark 1.5 and (13.5), we have $|K_X(x, y; z)| \le K_X(x, y; \operatorname{Re} z)$. Using (13.13) with $a = \operatorname{Re} z / X$ and (14.25) with $s = Xx$, $v = X(y - x)$, and $t = X(1 - y)$, we obtain for $X \ge \max(\alpha', 3)$, so that $a \le 1$,

$$|K_X(x, y; z)| \le \big\langle \Psi^{1,a}_{Xx}, e^{X(y-x)(\Delta_D - ad)} \Psi^{1,a}_{X(1-y)} \big\rangle_D \le \frac{CX}{\alpha} e^{-X(y-x)/4},$$

which tends to zero uniformly for $z \in \mathcal{K}$. □

The decay of the bound on K_X is not uniform as $y - x \downarrow 0$; this is why the last two rows of Proposition 1.16 concern fixed separated sites.

14.7 Large-scale covariance

We now prove Theorem 1.18. For $X > 0$ and $L > 0$, we define $m_{X,L}$ as the image of $L^{3/2} m_X$ under $k \mapsto Lk$.

Proof of Theorem 1.18. Fix $X > 0$, and let f and g be as in the statement. With $\hat{f}_L(k) = L\hat{f}(Lk)$, (14.7) gives

$$\operatorname{Cov}\big(\Xi_X(f_L), \Xi_X(g_L)\big) = L^2 \int_{\mathbb{R}} \hat{f}(Lk)\overline{\hat{g}(Lk)}\, m_X(\mathrm{d}k) = \sqrt{L} \int_{\mathbb{R}} \hat{f}(u)\overline{\hat{g}(u)}\, m_{X,L}(\mathrm{d}u). \tag{14.36}$$

We apply Lemma 14.5 to the measures $m_{X,L}$ along an arbitrary sequence $L \to \infty$, with $\beta = 3/2$. By (14.6), their Laplace transforms are

$$\int_{\mathbb{R}} e^{-2a|u|} m_{X,L}(\mathrm{d}u) = L^{3/2} I_X(aL), \qquad a > 0.$$

Proposition 14.14(b) shows that these converge to $\sqrt{X}/(4\sqrt{2}a^{3/2})$.

We next verify the growth bound of Lemma 14.5. Summing the integrand of (14.6) over unit intervals and using the translation bound of m_X in Lemma 14.4, we find $I_X(s) \le C_X s^{-1}$ for $0 < s \le 1$. By (14.35) and the continuity of I_X on $[1, \infty)$, we have $I_X(s) \le C_X s^{-3/2}$ for $s \ge 1$. For $L \ge 1$ and $a > 0$, these two bounds give

$$L^{3/2} I_X(aL) \le \begin{cases} C_X L^{1/2} a^{-1} \le C_X a^{-3/2} & \text{if } aL \le 1, \\ C_X a^{-3/2} & \text{if } aL \ge 1. \end{cases}$$

The even measure $\sqrt{X}|u|^{1/2}\,\mathrm{d}u/(2\sqrt{\pi})$ has the Laplace transform $\sqrt{X}/(4\sqrt{2}a^{3/2})$, since

$$\int_{\mathbb{R}} e^{-2a|u|} \frac{\sqrt{X}|u|^{1/2}}{2\sqrt{\pi}}\,\mathrm{d}u = \frac{\sqrt{X}}{\sqrt{\pi}} \cdot \frac{\Gamma(3/2)}{(2a)^{3/2}} = \frac{\sqrt{X}}{4\sqrt{2}\,a^{3/2}}.$$

The uniqueness in Lemma 14.5(a) shows that $m_{X,L}$ converges vaguely to this measure.

If f is of bounded variation with compact support, then f is bounded and integrable, and an integration by parts in the Stieltjes sense gives

$$|\hat{f}(u)| \le \min\Big(\|f\|_{L^1}, \frac{\mathrm{TV}(f)}{2\pi|u|}\Big), \qquad u \ne 0, \tag{14.37}$$

where $\mathrm{TV}(f)$ is the total variation of f on $\mathbb{R}$. Schwartz functions satisfy $|\hat{f}(u)| \le C(1+u^2)^{-1}$. Therefore the function $\Phi = \hat{f}\overline{\hat{g}}$ is continuous, and there exists $A > 0$ such that $|\Phi(u)| \le A \min(1, u^{-2})$ for all $u \ne 0$. Lemma 14.5(b) and (14.36) give

$$\frac{\operatorname{Cov}\big(\Xi_X(f_L), \Xi_X(g_L)\big)}{\sqrt{XL}} \longrightarrow \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} |u|^{1/2} \hat{f}(u)\overline{\hat{g}(u)}\,\mathrm{d}u = \frac{1}{2\sqrt{\pi}} \langle f, g \rangle_{\dot{H}^{1/4}},$$

which is the first limit in (1.28), and the bound on Φ shows that this integral converges absolutely.

We next prove the second expression in (1.28). For $c > 0$, an integration by parts and the integral $\int_0^\infty x^{-1/2} \sin x \, \mathrm{d}x = \sqrt{\pi/2}$ give

$$\int_0^\infty \frac{1 - \cos(cy)}{y^{3/2}} \, \mathrm{d}y = \sqrt{c} \int_0^\infty \frac{1 - \cos x}{x^{3/2}} \, \mathrm{d}x = 2\sqrt{c} \int_0^\infty \frac{\sin x}{x^{1/2}} \, \mathrm{d}x = \sqrt{2\pi c}.$$

Taking $c = 2\pi|u|$, we find

$$\int_{\mathbb{R}} \left| e^{2\pi i u y} - 1 \right|^2 \frac{\mathrm{d}y}{|y|^{3/2}} = 2 \int_{\mathbb{R}} \big(1 - \cos(2\pi u y)\big) \frac{\mathrm{d}y}{|y|^{3/2}} = 8\pi |u|^{1/2}, \qquad u \in \mathbb{R}. \tag{14.38}$$

By the Plancherel theorem and Tonelli's theorem,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(s) - f(s')|^2}{|s - s'|^{3/2}} \, \mathrm{d}s \, \mathrm{d}s' = \int_{\mathbb{R}} \frac{\|f(\cdot + y) - f\|_{L^2}^2}{|y|^{3/2}} \, \mathrm{d}y = 8\pi \int_{\mathbb{R}} |u|^{1/2} |\hat{f}(u)|^2 \, \mathrm{d}u,$$

which is finite. Both sides are Hermitian forms in f, and polarization gives the identity for $f \neq g$, and the Cauchy–Schwarz inequality shows that the double integral in (1.28) converges absolutely. Dividing by $16\pi^{3/2}$, we arrive at the second expression in (1.28).

Finally, $\Sigma_X^2(L) = \operatorname{Var} \Xi_X(f_L)$ for $f = \mathbf{1}_{[0,1]}$, and for this function the double integral equals

$$\begin{aligned} 2 \int_0^1 \Big(\int_{-\infty}^0 (x - y)^{-3/2} \, \mathrm{d}y + \int_1^\infty (y - x)^{-3/2} \, \mathrm{d}y \Big) \, \mathrm{d}x &= 2 \int_0^1 \big(2x^{-1/2} + 2(1 - x)^{-1/2} \big) \, \mathrm{d}x \\ &= 16. \end{aligned}$$

Inserting this value into the second expression in (1.28), we find

$$\frac{\Sigma_X^2(L)}{\sqrt{XL}} \longrightarrow \frac{16}{16\pi^{3/2}} = \frac{1}{\pi^{3/2}}.$$

□

14.8 The crossover

We now prove Theorem 1.19. For $X > 0$, we define $\bar{m}_X$ as the image of $X m_X$ under $k \mapsto Xk$. The next proposition describes the covariance of linear statistics on the scale X; it shows that m_∞ is the limit of the Bartlett spectra m_X in the scaling of Theorem 1.19.

Proposition 14.15. *As $X \to \infty$, the measures $\bar{m}_X$ converge vaguely to $m_\infty(u) \, \mathrm{d}u$. Moreover, for compactly supported functions f and g of bounded variation, or Schwartz functions,*

$$\frac{1}{X} \operatorname{Cov} \big(\Xi_X(f_X), \Xi_X(g_X) \big) \longrightarrow \int_{\mathbb{R}} m_\infty(u) \hat{f}(u) \overline{\hat{g}(u)} \, \mathrm{d}u. \tag{14.39}$$

Proof. By (14.6), the Laplace transforms of $\bar m_X$ are

$$\int_{\mathbb{R}} e^{-2a|u|} \bar m_X(\,\mathrm{d}u) = X I_X(Xa), \qquad a > 0.$$

Proposition 14.14(c) shows that they converge to $\mathcal{I}(a)$, and they are bounded by C/a for $X \geq 3$. Lemma 14.12(b) shows that $\mathcal{I}$ is the Laplace transform of the even measure $m_\infty(u)\,\mathrm{d}u$. Applying Lemma 14.5(a) with $\beta = 1$ along every sequence $X \to \infty$, we obtain the vague convergence. As in (14.36), (14.7) gives

$$\frac{1}{X}\operatorname{Cov}\big(\Xi_X(f_X), \Xi_X(g_X)\big) = X \int_{\mathbb{R}} \hat f(Xk)\overline{\hat g(Xk)}\, m_X(\,\mathrm{d}k) = \int_{\mathbb{R}} \hat f(u)\overline{\hat g(u)}\, \bar m_X(\,\mathrm{d}u),$$

and Lemma 14.5(b), applied with $\Phi = \hat f\overline{\hat g}$ and the bound $|\Phi(u)| \leq A\min(1, u^{-2})$ from the proof of Theorem 1.18, gives (14.39). □

Proof of Theorem 1.19. For $\ell \geq 0$, we let

$$\Phi_\ell(u) = \Big(\frac{\sin \pi \ell u}{\pi u}\Big)^2, \quad u \neq 0, \qquad \Phi_\ell(0) = \ell^2,$$

so that $\Phi_\ell = |\hat f|^2$ for $f = \mathbf{1}_{[0,\ell]}$. By (14.10) with $L = X\ell$, we have

$$\frac{\Sigma_X^2(X\ell)}{X} = \int_{\mathbb{R}} \Phi_\ell \,\mathrm{d}\bar m_X.$$

Let $\ell_0 > 0$. The function $(u, \ell) \mapsto \Phi_\ell(u)$ is continuous on $\mathbb{R} \times [0, \ell_0]$, and $\Phi_\ell(u) \leq \max(\ell_0^2, \pi^{-2})\min(1, u^{-2})$. Proposition 14.15 and Lemma 14.5(c) give $\Sigma_X^2(X\ell)/X \to V(\ell)$ uniformly for $\ell \in [0, \ell_0]$, and the same lemma shows that V is continuous, with $V(0) = 0$.

We now prove the properties of V. The substitution $v = \ell u$ gives

$$\frac{V(\ell)}{\ell} = \int_{\mathbb{R}} m_\infty(v/\ell)\Big(\frac{\sin \pi v}{\pi v}\Big)^2 \,\mathrm{d}v, \qquad \ell > 0. \tag{14.40}$$

By Lemma 14.12(a), the function $\ell \mapsto m_\infty(v/\ell)$ is strictly decreasing for $v \neq 0$, with values in $(0, 1)$. It follows from (14.40) that $V(\ell)/\ell$ is strictly decreasing in $\ell > 0$, and that

$$0 < V(\ell) < \ell \int_{\mathbb{R}} \Big(\frac{\sin \pi v}{\pi v}\Big)^2 \,\mathrm{d}v = \ell.$$

As $\ell \downarrow 0$, we have $m_\infty(v/\ell) \to 1$ for $v \neq 0$, and dominated convergence gives $V(\ell)/\ell \to 1$. As $\ell \to \infty$, the bound $\sqrt{\ell}\, m_\infty(v/\ell) \leq \sqrt{\pi|v|}/4$ of (14.32) and the asymptotic there give, by dominated convergence,

$$\frac{V(\ell)}{\sqrt{\ell}} = \int_{\mathbb{R}} \sqrt{\ell}\, m_\infty(v/\ell)\Big(\frac{\sin \pi v}{\pi v}\Big)^2 \,\mathrm{d}v \longrightarrow \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} |v|^{1/2}\Big(\frac{\sin \pi v}{\pi v}\Big)^2 \,\mathrm{d}v = \frac{1}{\pi^{3/2}}.$$

In the last step, we used $(\sin \pi v)^2 = (1 - \cos 2\pi v)/2$ and (14.38) with $u = 1$, which give

$$\int_{\mathbb{R}} |v|^{1/2} \Big(\frac{\sin \pi v}{\pi v} \Big)^2 \, \mathrm{d}v = \frac{1}{2\pi^2} \int_{\mathbb{R}} \frac{1 - \cos 2\pi v}{|v|^{3/2}} \, \mathrm{d}v = \frac{2}{\pi}.$$

This completes the proof. □

PART IV

From the matrices to the sigma model

SECTION 15

A priori bounds on transfer chains

Part IV proves Theorem 7.6, which is the only result of Part IV that Part III uses. The proof compares the exact transfer chain of Section 4 with the continuum semigroup of the sigma model of Section 5. Recall from Proposition 4.11 that the source ratio $T_N^{(k)}$ is, up to an explicit prefactor, the pairing of two endpoints with a power of the transfer operator T_Γ, which acts on superfunctions on the Schäfer–Wegner cycle. In this section we prove a priori bounds on the partial chains $T_\Gamma^j 1$. Section 16 introduces the normal tube around the saddle manifold and the normal oscillator, and Section 17 proves that the transfer kernel is stable on this tube. In Section 18 we prove that the transfer operator agrees with the continuum generator to first order on the invariant normal line, and in Section 19 we attach the endpoints of the chain to the tube. Finally, Section 20 combines these results into the proof of Theorem 7.6. The heuristics behind this strategy are explained in Section 2.

The main result of this section is Theorem 15.14. It states that the partial chains $T_\Gamma^j 1$, for $1 \le j \le N$, and their versions with finitely many local source derivatives, have norm $e^{o(W)}$ in a space with a negative polynomial weight. We now explain why a bound of this order is needed. In Section 20, the physical chain is compared with a chain on the normal tube. Every local error in this comparison is of order $e^{-\gamma W}$ for some $\gamma > 0$, and the telescoping identity of Lemma 20.4 pairs it with a partial chain. The errors are therefore harmless only if the partial chains are of size $e^{o(W)}$. One step satisfies the trivial bound $\|T_\Gamma\| \le e^{C/W}$, by Lemmas 15.7 and 15.9. Over the $N \asymp W^2$ steps of the chain, this bound grows to e^{CW}, which is of the same exponential order as the error.

We obtain the bound $e^{o(W)}$ from two estimates of a different nature. Away from the saddle manifold, one step of the chain contracts the norm by a factor $e^{-c(\log W)^2/W}$, by the onsite bound Lemma 15.9. Near the saddle manifold, on cores of radius $h \log W$, we use the anchored identity of Proposition 4.13. It writes a partial chain at a point of the cycle as the expectation, over a Gaussian band matrix, of a ratio of a fermionic and a bosonic determinant. We compare the two determinants by Hua's determinant inequality, and we take the expectation by spectral averaging over the diagonal entries of the matrix, in the form of Minami's estimate due to Graf and Vaghi. These tools yield a bound $e^{C\sqrt{W}(\log W)^2}$ on the cores, and a recurrence combines the two estimates. Since the fermionic determinants can vanish, we only bound them from above and never divide by them.

In Section 15.1 we fix the coordinates on the cycle and the weighted spaces, and in Section 15.2 we prove the determinant inequalities. Section 15.3 contains the two bounds for one step on the cycle, the kinetic bound Lemma 15.7 and the onsite bound Lemma 15.9. Sections 15.4 and 15.5 contain the spectral averaging and the coefficient bounds that we use on the cores, and Section 15.6 proves Theorem 15.14.

15.1 Setting

Throughout Part IV, the rank $k \geq 1$ is fixed, and we write

$$h = W^{-1/2}, \qquad \delta_W = W^{-2} = h^4, \qquad \tau_{\mathrm{phys}} = (2W^2)^{-1}, \qquad r_W = h \log W. \tag{15.1}$$

In this section we assume that $|E| \leq 1/2$, so that $|\kappa_E| \leq 1/4$ and $s_E \geq \sqrt{15}/4$, and we abbreviate $\kappa = \kappa_E$ and $s = s_E$. Let $\|\cdot\|$ denote the operator norm of a matrix and $\|\cdot\|_{\mathrm{HS}}$ its Hilbert–Schmidt norm. In Section 15.3, and only there, we use instead the convention of Section 4, in which $\|\cdot\|$ denotes the Hilbert–Schmidt norm and $\|\cdot\|_{\mathrm{op}}$ the operator norm. Constants $C, c > 0$ may depend on k and on the parameters fixed in the statements, but not on N, W, E, η, the point of the cycle, or the positions of the marks defined below. Their values may change from line to line.

The cycle and its coordinates. Recall the Schäfer–Wegner cycle $\mathcal{S}$ and its coordinates (P, X, M) from Definition 4.6. We set $P_0 = P - \kappa\mathrm{Id}$, so that the BB and FF blocks of a point $S \in \mathcal{S}$ are

$$S_{BB} = \kappa\mathrm{Id} + P_0 - \mathrm{i}s\Lambda_B \mathsf{g}^2, \qquad S_{FF} = \kappa\mathrm{Id} + \mathrm{i}M, \tag{15.2}$$

where $\mathsf{g} = \mathsf{g}_X$ is the positive definite matrix of (4.11), which we call the *boost* of S. The odd blocks S_{BF} and S_{FB} of S have free odd generators as entries. Recall from Lemma 4.7(a) that $\mathsf{g}\Lambda_B\mathsf{g} = \Lambda_B$ and $\det \mathsf{g} = 1$. We let $v_{\mathcal{S}} = 1 + \operatorname{Tr} f_0(X)$ denote the *orbit height* of S. It is the counterpart on $\mathcal{S}$ of the weight v of (5.3): at the point of $\mathcal{S}$ with $P = \kappa\mathrm{Id}$, Lemma B.5 gives $v_{\mathcal{S}} = 1 + 2s(v(Q_X) - 1)$, with Q_X as in Lemma 4.7(d). Since $\|X\| \leq \|f_0(X)\| \leq \operatorname{Tr} f_0(X)$ and $\Lambda_B \mathsf{g}^2 \Lambda_B = \mathsf{g}^{-2}$, we have

$$\|\mathsf{g}^2\| = \|\mathsf{g}^{-2}\| \leq 2s^{-1} v_{\mathcal{S}} \leq 3v_{\mathcal{S}}, \qquad \operatorname{Tr} \mathsf{g}^2 = s^{-1} \operatorname{Tr} f_0(X) = s^{-1}(v_{\mathcal{S}} - 1), \tag{15.3}$$

where we used $\operatorname{Tr} X = 0$ in the second identity.

We define the *transported odd coordinates* of S to be the entries of the $2k \times 2k$ matrices

$$O_1 = \mathsf{g}^{-1}\Lambda_B S_{BF}, \qquad O_2 = S_{FB}\,\mathsf{g}^{-1}. \tag{15.4}$$

At each point of the underlying manifold of $\mathcal{S}$, these $8k^2$ entries are odd generators, obtained from the entries of S_{BF} and S_{FB} by an invertible linear map, and we call them the *transported odd frame*. For a superfunction f on $\mathcal{S}$, let $|f|_{1;h}$ and $|f|_{2;h}$ denote the coefficient norms of Definition 5.12 with the weight h, taken at each point of the body in the monomials of the transported odd coordinates. For a matrix with such entries, $\|\cdot\|_{1;h}$ is the matrix norm of Definition 5.12.

Weighted spaces. We parametrize the underlying manifold of $\mathcal{S}$ by (P_0, X, M), and we let $\mathrm{d}P_0\,\mathrm{d}X\,\mathrm{d}M$ denote Lebesgue measure in these coordinates, with the Hilbert–Schmidt metric. For $p \in \mathbb{R}$, we define $\mathcal{H}_p(\mathcal{S})$ to be the space of superfunctions f on $\mathcal{S}$ with measurable coefficients such that

$$\|f\|^2_{\mathcal{H}_p(\mathcal{S})} = \int v_{\mathcal{S}}^{2p}\, |f|^2_{2;h}\,\mathrm{d}P_0\,\mathrm{d}X\,\mathrm{d}M < \infty. \tag{15.5}$$

By Lemma 5.13(a), we have $|af|_{2;h} \le |a|_{1;h}|f|_{2;h}$, and therefore multiplication by a superfunction a has norm at most $\sup |a|_{1;h}$ on $\mathcal{H}_p(\mathcal{S})$. We will also need the following bound for the pairing $\mathcal{B}_{\mathcal{S}}$ of Definition 4.9, which is used in Section 20.

Lemma 15.1. *For all $p \in \mathbb{R}$, $f \in \mathcal{H}_{-p}(\mathcal{S})$, and $u \in \mathcal{H}_p(\mathcal{S})$, we have*

$$|\mathcal{B}_{\mathcal{S}}(f,u)| \le C_k W^{4k^2} \|f\|_{\mathcal{H}_{-p}(\mathcal{S})} \|u\|_{\mathcal{H}_p(\mathcal{S})},$$

where C_k depends only on k.

Proof. By Definition 4.6 and the normalization (4.6), $\mathcal{B}_{\mathcal{S}}(f,u)$ is a constant c_k times the integral over the body, with respect to $\mathrm{d}P_0\,\mathrm{d}X\,\mathrm{d}M$, of the coefficient of the top monomial of fu in the entries of S_{BF} and S_{FB}. The map (15.4) from these entries to the transported odd coordinates has determinant

$$\det(\mathrm{g}^{-1}\Lambda_B)^{2k} \det(\mathrm{g}^{-1})^{2k} = 1,$$

so by Lemma 3.6(c) the top coefficient is the same in both frames. In the h-scaled monomials, the top coefficient of fu is the product of $h^{-8k^2} = W^{4k^2}$ with a signed sum, over complementary monomials, of products of a coefficient of f and a coefficient of u. The Cauchy–Schwarz inequality bounds this sum by $|f|_{2;h}|u|_{2;h}$, and we obtain

$$|\mathcal{B}_{\mathcal{S}}(f,u)| \le |c_k| W^{4k^2} \int v_{\mathcal{S}}^{-p}|f|_{2;h}\, v_{\mathcal{S}}^{p}|u|_{2;h}\,\mathrm{d}P_0\,\mathrm{d}X\,\mathrm{d}M \le |c_k| W^{4k^2} \|f\|_{\mathcal{H}_{-p}(\mathcal{S})}\|u\|_{\mathcal{H}_p(\mathcal{S})}. \quad \square$$

Sources and marks. Recall from Definitions 4.1 and 4.9 the source supermatrix $\Gamma = \mathrm{diag}(\varepsilon\Lambda_B \,|\, \Gamma_F)$, the onsite weight w_Γ, the marked onsite weight $w_{\Gamma,Y}$, and the transfer operator $T_\Gamma = M_{w_\Gamma} K_{\tau_{\mathrm{phys}}}$, and recall that $\eta = \varepsilon\eta_0$. These objects depend on η_0 and Γ only through $\eta_0\Gamma$. We define the *normalized source* $\hat\Gamma = \varepsilon^{-1}\Gamma$, so that

$$\eta_0\Gamma = \eta\hat\Gamma, \qquad \hat\Gamma = \mathrm{diag}(\Lambda_B \,|\, \hat\Gamma_F), \qquad w_\Gamma(S) = e^{-\mathrm{STr}(S^2)/2 - \mathrm{i}\eta\,\mathrm{STr}(S\hat\Gamma)}\,\mathrm{SDet}(E-S)^{-1}. \tag{15.6}$$

In this section, $\eta > 0$ is arbitrary, and $\hat\Gamma_F$ is a complex $2k \times 2k$ matrix with $\mathrm{Tr}\,\hat\Gamma_F = 0$ and $\|\hat\Gamma_F\| \le C_\Gamma$. The sources $\Gamma(\mathbf{u},\mathbf{b})$ of Definition 4.1 are of this form, also after complex continuation of $(\mathbf{u},\mathbf{b})$, since their FF block is traceless. We write the FF spectral matrix as

$$\mathrm{i}\hat\Gamma_F = \gamma_1 + \mathrm{i}\gamma_2, \qquad \gamma_1 = \gamma_1^*, \quad \gamma_2 = \gamma_2^*, \qquad \mathrm{Tr}\,\gamma_1 = \mathrm{Tr}\,\gamma_2 = 0, \qquad \|\gamma_1\|, \|\gamma_2\| \le C_\Gamma. \tag{15.7}$$

Recall that a local source at one site is a supermatrix $Y = \operatorname{diag}(Y^B \mid Y^F)$ with zero odd blocks, and that $w_{\Gamma,Y}(S)$ is obtained from $w_\Gamma(S)$ by replacing $\operatorname{SDet}(E-S)^{-1}$ with $\operatorname{SDet}(E-S+Y)^{-1}$. A jet of local source derivatives at one site gives the following onsite weights.

Definition 15.2. Let $Y_1, \dots, Y_m$ be local sources with $\|Y_l^B\|, \|Y_l^F\| \le 1$. The *onsite jet* of order m with these directions, a mixed derivative of the marked onsite weight, is

$$w_\Gamma[Y_1, \dots, Y_m](S) = \frac{\partial^m}{\partial t_1 \cdots \partial t_m}\, w_{\Gamma, t_1 Y_1 + \cdots + t_m Y_m}(S)\Big|_{t=0},$$

and for $m = 0$ it is w_Γ. A *marked partial chain* of length $j \ge 1$ and total order d is a superfunction

$$F_j = M_{w_j} K_{\tau_{\text{phys}}} M_{w_{j-1}} K_{\tau_{\text{phys}}} \cdots M_{w_2} K_{\tau_{\text{phys}}} w_1, \tag{15.8}$$

where each w_i is an onsite jet of some order m_i, and $m_1 + \cdots + m_j = d$. We call the steps i with $m_i \ge 1$ the *marked steps*, and we set $F_0 = 1$.

The derivatives at zero of $w_{\Gamma,Y}$ are defined at every point of $\mathcal{S}$, as noted after Definition 4.9, and every derivative of finite order in the entries of Y is a linear combination of onsite jets. Since $K_{\tau_{\text{phys}}} 1 = 1$, the unmarked partial chain of length j satisfies $F_j = T_\Gamma^j 1$. By Proposition 4.11(c), the derivatives of the source ratio in local sources are chain integrals of onsite jets, and they are pairings of marked partial chains. In the analytic transpose order of Definition 4.9, the marked partial chains are the functions $K_{\tau_{\text{phys}}} F_j$, just as $\ell_m = K_{\tau_{\text{phys}}} f_m$ in Proposition 4.11(b).

The anchored identity. For $n \ge 1$, recall the anchored covariance J_n, the boundary profile $\varsigma \in (0,1)^n$, the number σ_n, and the Gaussian matrix $H^{(n)}$ of (4.25). We also write ς for the diagonal matrix $\operatorname{diag}(\varsigma_1, \dots, \varsigma_n)$. For $S \in \mathcal{S}$, we define the supermatrix $\mathbb{M}_n(S)$ on $\mathbb{C}^n \otimes \mathbb{C}^{2k|2k}$ and the Gaussian factor $\mathcal{G}_n(S)$ by

$$\mathbb{M}_n(S) = \operatorname{Id}_n \otimes E - H^{(n)} \otimes \operatorname{Id} - \varsigma \otimes S + \mathrm{i}\eta(\operatorname{Id}_n - \varsigma) \otimes \hat{\Gamma}, \tag{15.9}$$

$$\mathcal{G}_n(S) = \exp\Big[-\frac{\sigma_n}{2} \operatorname{STr}(S^2) - \mathrm{i}\eta\sigma_n \operatorname{STr}(S\hat{\Gamma}) - \frac{\eta^2(n-\sigma_n)}{2} \operatorname{STr}(\hat{\Gamma}^2)\Big]. \tag{15.10}$$

By (15.6), Proposition 4.13 states that

$$(K_{\tau_{\text{phys}}} T_\Gamma^n 1)(S) = \mathcal{G}_n(S)\, \mathbb{E} \operatorname{SDet}\big[\mathbb{M}_n(S)\big]^{-1}. \tag{15.11}$$

Let $\mathbf{B}$, $\mathbf{F}$, $\mathbf{O}_{BF}$, and $\mathbf{O}_{FB}$ denote the blocks of $\mathbb{M}_n(S)$. As in Definition 4.9, and by Lemma 3.11(a), the inverse superdeterminant is

$$\operatorname{SDet}\big[\mathbb{M}_n(S)\big]^{-1} = \frac{\det(\mathbf{F} - \mathbf{O}_{FB}\mathbf{B}^{-1}\mathbf{O}_{BF})}{\det \mathbf{B}},$$

which only requires $\mathbf{B}$ to be invertible; we check this in Lemma 15.6. In the anchored chain, the site i is linked to S through $i-1$ other sites, so it carries the step $n+1-i$ of the chain.

15.2 Determinant inequalities

We now prove the determinant inequalities that compare the bosonic and fermionic blocks of the anchored identity. We begin with elementary bounds for logarithms of determinants, which we will use several times.

Lemma 15.3. *Let A be a Hermitian and Y a positive definite matrix of the same size, and set $R = (A + \mathrm{i}Y)^{-1}$ and $\operatorname{Im} R = (R - R^*)/(2\mathrm{i})$. Then $A + \mathrm{i}Y$ is invertible, and*

$$\|R\| \le \|Y^{-1}\|, \qquad 0 \le -\operatorname{Im} R = RYR^* \le Y^{-1}. \tag{15.12}$$

For all positive semidefinite Z of the same size,

$$\frac{\mathrm{d}}{\mathrm{d}t} \log|\det(A + \mathrm{i}(Y + tZ))|\Big|_{t=0} = \operatorname{Tr}\big(Z(-\operatorname{Im} R)\big) \in \big[0, \operatorname{Tr}(ZY^{-1})\big]. \tag{15.13}$$

In particular, $Y \mapsto |\det(A + \mathrm{i}Y)|$ is nondecreasing for the order of positive definite matrices. Moreover, for all Hermitian Z of the same size, the function $\ell(u) = \log|\det(A + uZ + \mathrm{i}Y)|$ of $u \in \mathbb{R}$ satisfies

$$|\ell'(0)| \le \|Z\| \operatorname{Tr}(Y^{-1}), \qquad |\ell''(0)| \le \|Z\|^2 \|Y^{-1}\| \operatorname{Tr}(-\operatorname{Im} R). \tag{15.14}$$

Proof. Set $A' = Y^{-1/2} A Y^{-1/2}$. Then

$$A + \mathrm{i}Y = Y^{1/2}(A' + \mathrm{i}\mathrm{Id})Y^{1/2}, \qquad \|(A' + \mathrm{i}\mathrm{Id})^{-1}\| \le 1,$$

where the second bound holds since A' is Hermitian. The invertibility and the bound $\|R\| \le \|Y^{-1}\|$ follow. Further,

$$R - R^* = R\big((A - \mathrm{i}Y) - (A + \mathrm{i}Y)\big)R^* = -2\mathrm{i}RYR^*,$$
$$Y^{1/2} R Y R^* Y^{1/2} = (A'^2 + \mathrm{Id})^{-1} \le \mathrm{Id},$$

and (15.12) follows.

The derivative in (15.13) is

$$\operatorname{Re}\operatorname{Tr}(\mathrm{i}RZ) = -\operatorname{Tr}(Z \operatorname{Im} R) = \operatorname{Tr}\big(Z^{1/2}(-\operatorname{Im} R)Z^{1/2}\big),$$

and (15.12) places it in the stated interval. The monotonicity follows by integrating (15.13) along the segment between two ordered positive definite matrices.

We now prove (15.14). Set $Z' = Y^{-1/2} Z Y^{-1/2}$. We have

$$\ell'(0) = \operatorname{Re}\operatorname{Tr}(RZ), \qquad \ell''(0) = -\operatorname{Re}\operatorname{Tr}(RZRZ), \qquad -\|Z\|Y^{-1} \le Z' \le \|Z\|Y^{-1},$$

where the last bound follows by conjugating $-\|Z\|\mathrm{Id} \le Z \le \|Z\|\mathrm{Id}$ with $Y^{-1/2}$. Let $\Pi_\pm$ be the spectral projections of Z' onto its positive and negative eigenspaces. Then

$$\begin{aligned} |\operatorname{Tr}(RZ)| = \big|\operatorname{Tr}\big((A' + \mathrm{i}\mathrm{Id})^{-1} Z'\big)\big| &\le \operatorname{Tr}|Z'| = \operatorname{Tr}(\Pi_+ Z' \Pi_+) - \operatorname{Tr}(\Pi_- Z' \Pi_-) \\ &\le \|Z\| \operatorname{Tr}(Y^{-1}). \end{aligned}$$

For the second derivative, the Cauchy–Schwarz inequality for the Hilbert–Schmidt inner product gives

$$|\operatorname{Tr}(RZRZ)| \le \operatorname{Tr}(ZR^*RZ) = \operatorname{Tr}(RZ^2R^*) \le \|Z\|^2 \operatorname{Tr}(RR^*) \le \|Z\|^2 \|Y^{-1}\| \operatorname{Tr}(-\operatorname{Im} R),$$

where in the last step we used $RR^* \le \|Y^{-1}\| RYR^*$, which follows from $Y \ge \|Y^{-1}\|^{-1}\mathrm{Id}$. □

The next lemma is the main determinant inequality. Let Λ' be a Hermitian involution (a Hermitian matrix with $\Lambda'^2 = \mathrm{Id}$, such as Λ_B). The first part of the lemma says that, if L is Hermitian and commutes with Λ', and if Y is positive definite with $\Lambda' Y \Lambda' = Y^{-1}$, then replacing Y by the identity does not increase the modulus of $\det(L + \mathrm{i}Y)$. The proof rests on Hua's determinant inequality (15.15) below, which Hua derived from a matrix identity [68]. Paige, Styan, Wang, and Zhang state the identity and the inequality as [102, (1), (4)], and they locate both in the proof of Theorem 1 of [68]. Since Hua's paper is not easily available, we include the short derivation, following [102].

Lemma 15.4. *The following statements hold.*

(a) Let Λ' be a Hermitian involution, let Y be positive definite with $\Lambda' Y \Lambda' = Y^{-1}$, and let L be Hermitian with $L\Lambda' = \Lambda' L$, all of the same size. Then

$$|\det(L + \mathrm{i}Y)| \ge |\det(L + \mathrm{i}\mathrm{Id})|.$$

(b) Let g be a positive definite $2k \times 2k$ matrix with $\mathrm{g}\Lambda_B\mathrm{g} = \Lambda_B$, let Y_0 be a positive semidefinite $n \times n$ matrix, let $s_0 > 0$, and let L be a Hermitian matrix on $\mathbb{C}^n \otimes \mathbb{C}^{2k}$ that commutes with $\mathrm{Id}_n \otimes \Lambda_B$. Then

$$\bigl|\det\bigl(L + \mathrm{i}(Y_0 \otimes \mathrm{Id} + s_0 \mathrm{Id}_n \otimes \mathrm{g}^2)\bigr)\bigr| \ge \bigl|\det\bigl(L + \mathrm{i}(Y_0 + s_0\mathrm{Id}_n) \otimes \mathrm{Id}\bigr)\bigr|.$$

Proof. We first recall Hua's inequality. Let A and B be complex $d_1 \times d_2$ matrices with $\|A\|, \|B\| < 1$. Hua's matrix identity [102, (1)], with A and B replaced by B^* and A^*, states that

$$(\mathrm{Id} - AB^*)(\mathrm{Id} - BB^*)^{-1}(\mathrm{Id} - BA^*) - (\mathrm{Id} - AA^*) = (A - B)(\mathrm{Id} - B^*B)^{-1}(A - B)^*.$$

One checks it by multiplying out, using

$$(\mathrm{Id} - BB^*)^{-1} = \mathrm{Id} + B(\mathrm{Id} - B^*B)^{-1}B^*.$$

Since the right side is positive semidefinite, the first term on the left is bounded below by $\mathrm{Id} - AA^* > 0$. Taking determinants, we find

$$|\det(\mathrm{Id} - AB^*)|^2 \ge \det(\mathrm{Id} - AA^*) \det(\mathrm{Id} - BB^*), \tag{15.15}$$

which is [102, (4)] with A and B replaced by A^* and B^*.

We now prove part (a) by Cayley transforms. Set

$$Y_c = (Y - \mathrm{Id})(Y + \mathrm{Id})^{-1}, \qquad U = (\mathrm{Id} + \mathrm{i}L)^{-1}(\mathrm{Id} - \mathrm{i}L).$$

The matrix Y_c is Hermitian with $\|Y_c\| < 1$, and

$$\Lambda' Y_c \Lambda' = (Y^{-1} - \mathrm{Id})(Y^{-1} + \mathrm{Id})^{-1} = -Y_c.$$

The matrix U is unitary and commutes with Λ'. In an orthonormal basis in which $\Lambda' = \mathrm{diag}(\mathrm{Id}, -\mathrm{Id})$, we therefore have

$$Y_c = \begin{pmatrix} 0 & y \\ y^* & 0 \end{pmatrix}, \qquad U = \mathrm{diag}(U_+, U_-),$$

with $\|y\| < 1$.

We now compute the determinant. Since $Y = (\mathrm{Id} + Y_c)(\mathrm{Id} - Y_c)^{-1}$ and $(\mathrm{Id} + \mathrm{i}L)U = \mathrm{Id} - \mathrm{i}L$, we have

$$Y + \mathrm{i}L = (\mathrm{Id} + \mathrm{i}L)(\mathrm{Id} + UY_c)(\mathrm{Id} - Y_c)^{-1}.$$

Taking determinants and computing $\det(\mathrm{Id} + UY_c)$ by a Schur complement, we find

$$\frac{\det(Y + \mathrm{i}L)}{\det(\mathrm{Id} + \mathrm{i}L)} = \frac{\det(\mathrm{Id} + UY_c)}{\det(\mathrm{Id} - Y_c)} = \frac{\det(\mathrm{Id} - U_+ y U_- y^*)}{\det(\mathrm{Id} - yy^*)}.$$

We apply (15.15) with $A = U_+ y$ and $B = yU_-^*$, for which

$$\det(\mathrm{Id} - AB^*) = \det(\mathrm{Id} - U_+ y U_- y^*),$$
$$\det(\mathrm{Id} - AA^*) = \det(\mathrm{Id} - BB^*) = \det(\mathrm{Id} - yy^*),$$

and we conclude that the right side has modulus at least one. Taking moduli in the identities

$$L + \mathrm{i}Y = \mathrm{i}(Y + \mathrm{i}L)^*, \qquad L + \mathrm{i}\mathrm{Id} = \mathrm{i}(\mathrm{Id} + \mathrm{i}L)^*,$$

we obtain part (a).

We now prove part (b). Set $\Lambda' = \mathrm{Id}_n \otimes \Lambda_B$ and

$$Y_1 = Y_0 \otimes \mathrm{Id} + s_0 \mathrm{Id}_n \otimes \mathrm{g}^2, \qquad \Lambda' Y_1 \Lambda' = Y_0 \otimes \mathrm{Id} + s_0 \mathrm{Id}_n \otimes \mathrm{g}^{-2},$$

where we used $\Lambda_B \mathrm{g}^2 \Lambda_B = \mathrm{g}^{-2}$. The matrices Y_1 and $\Lambda' Y_1 \Lambda'$ commute. Let G_1 be the positive square root of $Y_1\, \Lambda' Y_1 \Lambda'$. Then

$$G_1^2 = Y_0^2 \otimes \mathrm{Id} + s_0 Y_0 \otimes (\mathrm{g}^2 + \mathrm{g}^{-2}) + s_0^2 \mathrm{Id} \geq (Y_0 + s_0 \mathrm{Id}_n)^2 \otimes \mathrm{Id}, \tag{15.16}$$

where we used $\mathrm{g}^2 + \mathrm{g}^{-2} \geq 2\mathrm{Id}$ and $Y_0 \geq 0$.

We have

$$\Lambda' G_1^2 \Lambda' = \Lambda' Y_1 \Lambda' Y_1 = G_1^2,$$

so the matrix G_1 commutes with Λ'; it also commutes with Y_1. Consider the positive definite matrix

$$Y = G_1^{-1/2} Y_1 G_1^{-1/2} = Y_1 G_1^{-1}, \qquad \text{which satisfies} \qquad Y\,\Lambda' Y \Lambda' = Y_1\,\Lambda' Y_1 \Lambda'\, G_1^{-2} = \mathrm{Id},$$

and $L' = G_1^{-1/2} L G_1^{-1/2}$ is Hermitian and commutes with Λ'. Using part (a) for L' and Y, we find

$$|\det(L + \mathrm{i}Y_1)| = \det G_1\, |\det(L' + \mathrm{i}Y)| \geq \det G_1\, |\det(L' + \mathrm{i}\mathrm{Id})| = |\det(L + \mathrm{i}G_1)|.$$

Since G_1 commutes with $(Y_0 + s_0 \mathrm{Id}_n) \otimes \mathrm{Id}$, (15.16) implies $G_1 \geq (Y_0 + s_0 \mathrm{Id}_n) \otimes \mathrm{Id}$. The monotonicity in Lemma 15.3 completes the proof. □

We next bound the modulus of the bosonic part of the onsite weight from above. For $-1 < t < 1$, the energy $E = 2t$ has $\kappa_E = t$ and $s_E = (1 - t^2)^{1/2}$. The following lemma compares the BB part of the onsite weight at a point with BB coordinate P_0 with its value at $P_0 = 0$, for every boost and every rank. Its proof shows that the deficit $V_t(P_0)$ defined below, which is minus the logarithm of the ratio of the two moduli, is a concave function of t, and it bounds $V_{\pm 1/2}(P_0)$ from below by Hua's inequality.

Lemma 15.5. *Let* g *be a positive definite* $2k \times 2k$ *matrix with* $\mathrm{g}\Lambda_B \mathrm{g} = \Lambda_B$*, let* P_0 *be a Hermitian* $2k \times 2k$ *matrix commuting with* Λ_B*, and set* $L = \mathrm{g}^{-1} \Lambda_B P_0 \mathrm{g}^{-1}$*. For* $-1 < t < 1$*, set* $s_t = (1 - t^2)^{1/2}$ *and* $R_t(P_0) = t\mathrm{Id} + P_0 - \mathrm{i}s_t \Lambda_B \mathrm{g}^2$*, and define the* BB onsite deficit

$$V_t(P_0) = \Big[\tfrac{1}{2} \operatorname{Re} \operatorname{Tr} R_t(P_0)^2 + \log |\det(2t\mathrm{Id} - R_t(P_0))|\Big] - \Big[\tfrac{1}{2} \operatorname{Re} \operatorname{Tr} R_t(0)^2 + \log |\det(2t\mathrm{Id} - R_t(0))|\Big]. \quad (15.17)$$

Then L *is Hermitian,*

$$\mathrm{g}^{-1} \Lambda_B \big(2t\mathrm{Id} - R_t(P_0)\big) \mathrm{g}^{-1} = t\Lambda_B - L + \mathrm{i}s_t \mathrm{Id}, \quad (15.18)$$

and

$$V_t(P_0) = \tfrac{1}{2} \operatorname{Tr} P_0^2 + t \operatorname{Tr} P_0 + \tfrac{1}{2} \operatorname{Tr} \log\big[\mathrm{Id} + L^2 - t(\Lambda_B L + L \Lambda_B)\big]. \quad (15.19)$$

Moreover, for $|t| \leq 1/2$*,*

$$V_t(P_0) \geq (1 - 2|t|) V_0(P_0), \quad (15.20)$$

and in particular $V_t(P_0) \geq \frac{1}{2} V_0(P_0) \geq \frac{1}{4} \operatorname{Tr} P_0^2$ *for* $|t| \leq 1/4$*.*

When $t = \kappa_E$, the matrix $R_t(P_0)$ is the BB block (15.2) of the point of $\mathcal{S}$ with coordinates $(\kappa \mathrm{Id} + P_0, X, M)$, and $e^{-V_t(P_0)}$ is the modulus of the BB part $e^{-\operatorname{Tr} S_{BB}^2/2} \det(E - S_{BB})^{-1}$ of the onsite weight, divided by its value at $P_0 = 0$.

Proof. The matrix $\Lambda_B P_0$ is Hermitian, since P_0 commutes with Λ_B, and therefore $L = \mathsf{g}^{-1}(\Lambda_B P_0)\mathsf{g}^{-1}$ is Hermitian.

We begin with the Gaussian part. The diagonal blocks of $\Lambda_B \mathsf{g}^2$ are Hermitian, since g^2 is, and $t\mathrm{Id} + P_0$ is block diagonal and Hermitian. Therefore $\mathrm{Tr}((t\mathrm{Id} + P_0)\Lambda_B \mathsf{g}^2)$ is real. Using $(\Lambda_B \mathsf{g}^2)^2 = \mathrm{Id}$, which follows from $\Lambda_B \mathsf{g}^2 \Lambda_B = \mathsf{g}^{-2}$, we find

$$\operatorname{Re}\operatorname{Tr} R_t(P_0)^2 = \operatorname{Tr}(t\mathrm{Id} + P_0)^2 - 2ks_t^2 = \operatorname{Tr} P_0^2 + 2t \operatorname{Tr} P_0 + 2k(t^2 - s_t^2).$$

We now turn to the determinant. The identity $\mathsf{g}\Lambda_B \mathsf{g} = \Lambda_B$ implies that

$$\mathsf{g}^{-1}\Lambda_B \mathsf{g}^{-1} = \Lambda_B, \qquad \mathsf{g}^{-1}\Lambda_B(\Lambda_B \mathsf{g}^2)\mathsf{g}^{-1} = \mathrm{Id}.$$

Multiplying the identity

$$2t\mathrm{Id} - R_t(P_0) = t\mathrm{Id} - P_0 + \mathrm{i}s_t \Lambda_B \mathsf{g}^2$$

by $\mathsf{g}^{-1}\Lambda_B$ on the left and by g^{-1} on the right, we obtain (15.18). The matrix $t\Lambda_B - L$ is Hermitian, and $\det \mathsf{g} = 1$ and $t^2 + s_t^2 = 1$. Using (15.18), we find

$$\begin{aligned} \log|\det(2t\mathrm{Id} - R_t(P_0))| &= \tfrac{1}{2}\operatorname{Tr}\log\bigl[(t\Lambda_B - L)^2 + s_t^2\mathrm{Id}\bigr] \\ &= \tfrac{1}{2}\operatorname{Tr}\log\bigl[\mathrm{Id} + L^2 - t(\Lambda_B L + L\Lambda_B)\bigr]. \end{aligned}$$

The two parts equal $k(t^2 - s_t^2)$ and 0 at $P_0 = 0$, and (15.19) follows.

We now prove (15.20) by concavity in t, at fixed g and P_0. Set

$$A_L = \mathrm{Id} + L^2, \qquad M_L = A_L^{-1/2}(\Lambda_B L + L\Lambda_B)A_L^{-1/2},$$

so that

$$\mathrm{Id} + L^2 - t(\Lambda_B L + L\Lambda_B) = A_L^{1/2}(\mathrm{Id} - tM_L)A_L^{1/2}.$$

Then (15.19) becomes

$$V_t(P_0) = V_0(P_0) + t \operatorname{Tr} P_0 + \tfrac{1}{2}\operatorname{Tr}\log(\mathrm{Id} - tM_L), \qquad |t| < 1, \tag{15.21}$$

where M_L does not depend on t.

The matrix M_L is Hermitian with $\|M_L\| \le 1$. Indeed, the Hermitian matrices $Z_1 = LA_L^{-1/2}$ and $Z_2 = A_L^{-1/2}$ commute with L, and they satisfy

$$Z_1^2 + Z_2^2 = \mathrm{Id}, \qquad M_L = Z_1 \Lambda_B Z_2 + Z_2 \Lambda_B Z_1.$$

Using these identities, we find

$$|\langle x, M_L x\rangle| = 2|\operatorname{Re}\langle Z_1 x, \Lambda_B Z_2 x\rangle| \le \|Z_1 x\|^2 + \|Z_2 x\|^2 = \|x\|^2.$$

Differentiating (15.21) twice, we find

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} V_t(P_0) = -\tfrac{1}{2}\operatorname{Tr}\big[\big((\mathrm{Id} - tM_L)^{-1} M_L\big)^2\big] \le 0,$$

since $(\mathrm{Id} - tM_L)^{-1} M_L$ is Hermitian. Thus $t \mapsto V_t(P_0)$ is concave on $(-1, 1)$.

It remains to show that $V_{\pm 1/2}(P_0) \ge 0$, and we show that $V_t(P_0) \ge 0$ for $|t| \le 1/2$. Note that

$$\Lambda_B\big(2t\mathrm{Id} - R_t(P_0)\big) = t\Lambda_B - \Lambda_B P_0 + \mathrm{i} s_t \mathsf{g}^2.$$

We apply Lemma 15.4(a) with $\Lambda' = \Lambda_B$, $Y = \mathsf{g}^2$, and L replaced by $s_t^{-1}(t\Lambda_B - \Lambda_B P_0)$, which is Hermitian and commutes with Λ_B. It implies that

$$|\det(2t\mathrm{Id} - R_t(P_0))| \ge |\det(t\Lambda_B - \Lambda_B P_0 + \mathrm{i} s_t \mathrm{Id})| = \prod_{\nu} (1 - 2t\nu + \nu^2)^{1/2},$$

where the product is over the eigenvalues ν of P_0, counted with multiplicity. Here we diagonalized P_0 by a unitary that commutes with Λ_B, and we used $(t-\nu)^2 + s_t^2 = 1 - 2t\nu + \nu^2$.

Combining this bound with the formula for the Gaussian part, we obtain

$$V_t(P_0) \ge \sum_{\nu} \phi_t(\nu), \qquad \phi_t(\nu) = \tfrac{1}{2}\nu^2 + t\nu + \tfrac{1}{2}\log(1 - 2t\nu + \nu^2).$$

We have $\phi_t(0) = 0$ and

$$\phi_t'(\nu) = \frac{\nu\big[\nu^2 - t\nu + 2(1-t^2)\big]}{1 - 2t\nu + \nu^2}.$$

The quadratic polynomial in brackets has discriminant $9t^2 - 8 < 0$. Therefore $\phi_t'(\nu)$ has the sign of ν, so that $\phi_t \ge 0$ and $V_t(P_0) \ge 0$.

Now let $|t| \le 1/2$, and let $\pm$ be the sign of t. Since t is the convex combination of 0 and $\pm 1/2$ with weights $1 - 2|t|$ and $2|t|$, concavity implies that

$$V_t(P_0) \ge (1 - 2|t|) V_0(P_0) + 2|t|\, V_{\pm 1/2}(P_0) \ge (1 - 2|t|) V_0(P_0).$$

Finally, at $t = 0$, the identity (15.19) reads

$$V_0(P_0) = \tfrac{1}{2}\operatorname{Tr} P_0^2 + \tfrac{1}{2}\operatorname{Tr}\log(\mathrm{Id} + L^2) \ge \tfrac{1}{2}\operatorname{Tr} P_0^2. \qquad \square$$

We now apply Hua's inequality to the BB block of the anchored identity. In the following lemma, ς is an arbitrary diagonal matrix with entries in $(0, 1]$; in our application it is the boundary profile.

Lemma 15.6. *Let A be a Hermitian $n \times n$ matrix, let $\varsigma = \operatorname{diag}(\varsigma_1, \dots, \varsigma_n)$ with $0 < \varsigma_i \le 1$, and let $\eta, s_0 > 0$. Let g be a positive definite $2k \times 2k$ matrix with $\mathsf{g}\Lambda_B\mathsf{g} = \Lambda_B$, and let Z be a Hermitian $2k \times 2k$ matrix that commutes with Λ_B and satisfies $\|Z\| \le r$. Set*

$$\mathbf{B} = A \otimes \mathrm{Id} - \varsigma \otimes Z + \mathrm{i}\eta(\mathrm{Id}_n - \varsigma) \otimes \Lambda_B + \mathrm{i} s_0 \varsigma \otimes \Lambda_B \mathsf{g}^2, \qquad \Pi = \sum_{i=1}^{n} \frac{\varsigma_i}{\eta(1-\varsigma_i) + s_0 \varsigma_i}. \tag{15.22}$$

Then $\mathbf{B}$ *is invertible, and*

$$|\det \mathbf{B}| \geq e^{-2kr\Pi} \left|\det\bigl(A + \mathrm{i}\eta(\mathrm{Id}_n - \varsigma) + \mathrm{i}s_0\varsigma\bigr)\right|^{2k}, \tag{15.23}$$

$$\|\mathbf{B}^{-1}\| \leq \max\bigl\{\eta^{-1}, s_0^{-1}\|\mathsf{g}^2\|\bigr\}. \tag{15.24}$$

Proof. We first prove (15.24). We write

$$(\mathrm{Id}_n \otimes \Lambda_B)\mathbf{B} = X_1 + \mathrm{i}Y_1,$$
$$X_1 = A \otimes \Lambda_B - \varsigma \otimes \Lambda_B Z, \qquad Y_1 = \eta(\mathrm{Id}_n - \varsigma) \otimes \mathrm{Id} + s_0\varsigma \otimes \mathsf{g}^2.$$

The matrices X_1 and Y_1 are Hermitian, since $\Lambda_B Z$ is Hermitian. Since $\Lambda_B \mathsf{g}^2 \Lambda_B = \mathsf{g}^{-2}$, we have $\|\mathsf{g}^{-2}\| = \|\mathsf{g}^2\|$, and therefore

$$\mathsf{g}^2 \geq \|\mathsf{g}^{-2}\|^{-1}\mathrm{Id} = \|\mathsf{g}^2\|^{-1}\mathrm{Id}, \qquad Y_1 \geq \min\bigl\{\eta, s_0\|\mathsf{g}^2\|^{-1}\bigr\}\mathrm{Id}.$$

The matrix $\mathrm{Id}_n \otimes \Lambda_B$ is unitary. Therefore $\|\mathbf{B}^{-1}\| = \|(X_1 + \mathrm{i}Y_1)^{-1}\|$, and the first bound in (15.12) implies (15.24).

For (15.23), we set

$$A' = \varsigma^{-1/2} A \varsigma^{-1/2}, \qquad Y_0 = \eta\varsigma^{-1}(\mathrm{Id}_n - \varsigma) \geq 0.$$

Then

$$(\varsigma^{-1/2} \otimes \mathrm{Id})(X_1 + \mathrm{i}Y_1)(\varsigma^{-1/2} \otimes \mathrm{Id}) = L + \mathrm{i}(Y_0 \otimes \mathrm{Id} + s_0\mathrm{Id}_n \otimes \mathsf{g}^2),$$
$$L = A' \otimes \Lambda_B - \mathrm{Id}_n \otimes \Lambda_B Z,$$

and L is Hermitian and commutes with $\mathrm{Id}_n \otimes \Lambda_B$. Since $|\det(\mathrm{Id}_n \otimes \Lambda_B)| = 1$, Lemma 15.4(b) implies that

$$|\det \mathbf{B}| \geq (\det\varsigma)^{2k} \bigl|\det\bigl(L + \mathrm{i}(Y_0 + s_0\mathrm{Id}_n) \otimes \mathrm{Id}\bigr)\bigr|.$$

The last matrix is block diagonal in the flavor space. We diagonalize Z by a unitary that commutes with Λ_B, with eigenvalues $\nu_1, \dots, \nu_{2k}$, and use $|\det(-Y' + \mathrm{i}Y'')| = |\det(Y' + \mathrm{i}Y'')|$ for Hermitian Y' and Y''. This yields

$$\bigl|\det\bigl(L + \mathrm{i}(Y_0 + s_0\mathrm{Id}_n) \otimes \mathrm{Id}\bigr)\bigr| = \prod_{a=1}^{2k} \bigl|\det\bigl(A' - \nu_a\mathrm{Id}_n + \mathrm{i}(Y_0 + s_0\mathrm{Id}_n)\bigr)\bigr|.$$

By (15.14) with $Z = \mathrm{Id}_n$, we have

$$\Bigl|\frac{\mathrm{d}}{\mathrm{d}u} \log\bigl|\det\bigl(A' - u\mathrm{Id}_n + \mathrm{i}(Y_0 + s_0\mathrm{Id}_n)\bigr)\bigr|\Bigr| \leq \mathrm{Tr}(Y_0 + s_0\mathrm{Id}_n)^{-1} = \Pi \qquad \text{for all } u \in \mathbb{R}.$$

Using $|\nu_a| \leq r$ and

$$\det\varsigma\,|\det(A' + \mathrm{i}(Y_0 + s_0\mathrm{Id}_n))| = |\det(A + \mathrm{i}\eta(\mathrm{Id}_n - \varsigma) + \mathrm{i}s_0\varsigma)|,$$

we obtain (15.23). □

15.3 One step on the cycle

One step of the chain is the heat link followed by the onsite weight, $T_\Gamma = M_{w_\Gamma} K_{\tau_{\mathrm{phys}}}$, and in this section we bound the two factors separately on the weighted spaces $\mathcal{H}_p(\mathcal{S})$. The kinetic bound Lemma 15.7 shows that $K_{\tau_{\mathrm{phys}}}$ has norm at most $e^{C/W}$. The onsite bound Lemma 15.9 shows that $|w_\Gamma|_{1;h} \le e^{C/W} e^{-c\vartheta}$, where the escape function ϑ of (15.32) vanishes on the saddle manifold and grows with the distance from it. The proof of Theorem 15.14 needs both bounds. Together they give the trivial bound $\|T_\Gamma\| \le e^{C/W}$, and away from the cores, where $\vartheta \ge c r_W^2$, they give the contraction by $e^{-c(\log W)^2/W}$. For this reason we prove them here, before Theorem 15.14; Section 19 uses them as well. The proof of the onsite bound rests on the concavity bound Lemma 15.5 and the congruence (15.18).

We use the following objects of Section 4. Recall from Definition 4.6 that the Schäfer–Wegner cycle $\mathcal{S}$ has the coordinates (P, X, M), where P is Hermitian and block diagonal, X is Hermitian and block off-diagonal, and M is Hermitian, and that the odd entries of $S \in \mathcal{S}$ are free generators. By (15.2), the blocks of S are

$$S_{BB} = \kappa \mathrm{Id} + P_0 - \mathrm{i} s \Lambda_B \mathsf{g}^2, \qquad S_{FF} = \kappa \mathrm{Id} + \mathrm{i} M, \qquad P_0 = P - \kappa \mathrm{Id},$$

where the boost $\mathsf{g} = \mathsf{g}_X$ of (4.11) satisfies $s\mathsf{g}^2 = f_0(X) + X$, with $f_0(X) = (s^2 \mathrm{Id} + X^2)^{1/2}$. Recall also from Definition 4.9 the onsite weight w_Γ, the heat operators K_t with the kernels $k_t(S, S') = e^{-\mathrm{STr}(S-S')^2/(4t)}$, and the transfer operator T_Γ. Finally, recall from Section 4.6 the saddle manifold, the union of the $2k+1$ orbits of the diagonal supermatrices $R_*^{(p)}$, $p \in \{0, \dots, 2k\}$, under conjugation by even invertible supermatrices, where $R_*^{(p)}$ has the BB block $\kappa \mathrm{Id} - \mathrm{i} s \Lambda_B$ and an FF block that depends on the signature p.

From here to the end of this section, $\|\cdot\|$ denotes the Hilbert–Schmidt norm of a matrix and $\|\cdot\|_{\mathrm{op}}$ its operator norm, as in Section 4. This replaces the convention of Section 15.1, since the gradients, the Fourier variable, and the defect below are taken for the Hilbert–Schmidt inner product. Recall also that $|E| \le 1/2$, so that $s^{-1} \le 4/\sqrt{15}$.

The cycle near the saddle manifold

We first write the BB block of the cycle in coordinates adapted to the saddle manifold, and we prove the kinetic bound.

The boost. By Lemma 4.7(a), the boost satisfies

$$s\mathsf{g}^2 = f_0(X) + X, \qquad \mathsf{g} \Lambda_B \mathsf{g} = \Lambda_B, \qquad \det \mathsf{g}^2 = 1. \tag{15.25}$$

Let $\log \mathsf{g}$ be its Hermitian logarithm, which is off-diagonal, and define the operators $\mathrm{ch}_\mathsf{g} = \cosh(\mathrm{ad}_{\log \mathsf{g}})$ and $\mathrm{th}_\mathsf{g} = \tanh(\mathrm{ad}_{\log \mathsf{g}})$ on $2k \times 2k$ matrices. Since $\mathrm{ad}_{\log \mathsf{g}}$ is selfadjoint for the Hilbert–Schmidt inner product and exchanges block-diagonal and off-diagonal

matrices, ch_{g} preserves both classes and satisfies $\mathrm{ch}_{\mathrm{g}} \geq \mathrm{Id}$, while th_{g} exchanges them and has norm at most one. A direct computation gives

$$\mathrm{g} S_{BB} \mathrm{g}^{-1} = R_{*,B} + \widetilde{P} + \mathrm{th}_{\mathrm{g}}\, \widetilde{P}, \qquad R_{*,B} = \kappa \mathrm{Id} - \mathrm{i} s \Lambda_B, \quad \widetilde{P} = \mathrm{ch}_{\mathrm{g}}\, P_0, \tag{15.26}$$

where $R_{*,B}$ is the BB block of $R_*^{(p)}$, which is the same for all p, and we used $S_{BB} = \kappa \mathrm{Id} + P_0 - \mathrm{i} s \Lambda_B \mathrm{g}^2$, $\mathrm{g} \Lambda_B \mathrm{g} = \Lambda_B$, and $\mathrm{Ad}_{\mathrm{g}} = \mathrm{ch}_{\mathrm{g}} + \mathrm{th}_{\mathrm{g}}\, \mathrm{ch}_{\mathrm{g}}$. We call the block-diagonal matrix $\widetilde{P}$ the *physical normal* of the point. Since $\mathrm{ch}_{\mathrm{g}} \geq \mathrm{Id}$, we have $\|\widetilde{P}\| \geq \|P_0\|$.

Divided differences. We diagonalize X at a given point, with eigenvalues $x_a = s \sinh r_a$. Differentiating $f_0(X)^2 = s^2 \mathrm{Id} + X^2$ in the direction V gives the Sylvester equation $f_0\, Df_0(V) + Df_0(V) f_0 = XV + VX$, and therefore

$$\big(Df_0(X)V\big)_{ab} = c_{ab} V_{ab}, \qquad c_{ab} = \frac{x_a + x_b}{(s^2 + x_a^2)^{1/2} + (s^2 + x_b^2)^{1/2}} = \tanh \frac{r_a + r_b}{2}, \tag{15.27}$$

where repeated eigenvalues use the continuous value. For a block-diagonal Hermitian Ξ, we set $\phi_\Xi(X) = \langle \Xi, \Lambda_B f_0(X) \rangle$ and define the *defect*

$$d_X(\Xi) = \|\Xi\|^2 - \|\nabla \phi_\Xi(X)\|^2 = \sum_{a,b} (1 - c_{ab}^2) |(\Lambda_B \Xi)_{ab}|^2 \geq 0. \tag{15.28}$$

Since $|c_{ab}| < 1$, the map $X \mapsto -\Lambda_B f_0(X)$ is Lipschitz with constant one, and by Definition 4.6, the BB block of S differs from that of the flat cycle $\mathcal{S}_\flat$ by i times this map. In the proof of Lemma 15.7, we take the Fourier transform of the heat kernel in P_0, with Fourier variable Ξ, and the bound $d_X(\Xi) \geq 0$ then shows that the transformed generator is dissipative. For the logarithmic derivative of the boost, with $\bar{r} = (r_a + r_b)/2$ and $\delta r = (r_a - r_b)/2$, the same calculation gives

$$\big(\mathrm{g}^{-1} D\mathrm{g}(V)\big)_{ab} = \frac{e^{-\delta r/2}}{2s \cosh \bar{r} \cosh(\delta r/2)} V_{ab}, \tag{15.29}$$

and the right logarithmic derivative has $e^{\delta r/2}$ in place of $e^{-\delta r/2}$. It follows that

$$\begin{aligned} \|\mathrm{g}^{-1} D\mathrm{g}(V)\| + \|D\mathrm{g}(V)\mathrm{g}^{-1}\| &\leq C s^{-1} \|V\|, \\ \|\mathrm{g}^{-1} D\mathrm{g}(\nabla \phi_\Xi)\| + \|D\mathrm{g}(\nabla \phi_\Xi)\mathrm{g}^{-1}\| &\leq C s^{-1} d_X(\Xi)^{1/2}, \end{aligned} \tag{15.30}$$

where the second bound uses $\cosh^{-2} \bar{r} = 1 - \tanh^2 \bar{r}$ and (15.28). The constant C in (15.30) is universal; in particular, it does not depend on k.

Norms on the cycle. Recall the transported odd frame of (15.4), the orbit height $v_{\mathcal{S}} = 1 + \mathrm{Tr}\, f_0(X)$, and the weighted spaces $\mathcal{H}_p(\mathcal{S})$ of (15.5), whose norms are built from the coefficient norm $|\cdot|_{2;h}$ of Definition 5.12, taken in the transported odd frame. We let $\rho(\mathrm{g})$ denote the linear map (15.4) from the odd entries of S to the transported odd frame.

Recall from Lemma 5.13(b) that multiplication by an h-scaled monomial is a partial isometry for $|\cdot|_{2;h}$, and from Section 15.1 that multiplication by a coefficient function a has norm at most $\sup |a|_{1;h}$ on $\mathcal{H}_p(\mathcal{S})$.

Lemma 15.7. *Let K_t be the heat operators of Definition 4.9(b), acting on superfunctions on $\mathcal{S}$ whose coefficients are taken in the transported odd frame. For all $p \in \mathbb{R}$,*

$$\|K_t\|_{\mathcal{H}_p(\mathcal{S})\to\mathcal{H}_p(\mathcal{S})} \le \exp\big[(C_k W + C_{k,p})t\big], \qquad 0 \le t \le \tau_{\mathrm{phys}},$$

and the same bound holds for the adjoint of K_t in the Hilbert space $\mathcal{H}_p(\mathcal{S})$. Moreover, the operators K_t satisfy the semigroup law and are strongly continuous at $t = 0$ on $\mathcal{H}_p(\mathcal{S})$.

Proof. We first treat the BB block without weight and without odd variables. In the coordinates (P_0, X), we have

$$S_{BB} - S'_{BB} = \big[(P_0 - P'_0) - \mathrm{i}\Lambda_B(f_0(X) - f_0(X'))\big] - \mathrm{i}\Lambda_B(X - X'),$$

where the first term is block diagonal and the second is off-diagonal, so that the cross term of the two parts has zero trace. Since $\Lambda_B Y \Lambda_B = -Y$ for off-diagonal Y, the off-diagonal part contributes $\mathrm{Tr}(X - X')^2$ to $\mathrm{Tr}(S_{BB} - S'_{BB})^2$. The block-diagonal part is a Gaussian in $P_0 - P'_0$ shifted by the imaginary amount $\mathrm{i}\Lambda_B(f_0(X) - f_0(X'))$, and partial Fourier transformation in P_0, with the Fourier variable Ξ, turns the BB kernel into

$$T_{t,\Xi} = e^{-t\|\Xi\|^2}\, e^{\phi_\Xi}\, e^{t\Delta_X}\, e^{-\phi_\Xi},$$

where Δ_X is the flat Laplacian in X. Its generator is

$$\mathcal{L}_\Xi = \sum_j (\partial_j - \partial_j\phi_\Xi)^2 - \|\Xi\|^2.$$

Since ϕ_Ξ is real, the operator $\sum_j(\partial_j\phi_\Xi\,\partial_j + \partial_j\,\partial_j\phi_\Xi)$ is skew-adjoint. Integration by parts on a bounded Dirichlet domain, together with (15.28), gives

$$\mathrm{Re}\langle u, \mathcal{L}_\Xi u\rangle = -\|\nabla u\|^2 + \int \big(\|\nabla\phi_\Xi\|^2 - \|\Xi\|^2\big)|u|^2 = -\|\nabla u\|^2 - \int d_X(\Xi)|u|^2 \le 0.$$

Next, we add the odd variables. In the transported frame the fibre generator is conjugated by $\rho(\mathrm{g})$. On a bounded Dirichlet domain we set

$$C_j = \rho(\mathrm{g})^{-1}\partial_j\rho(\mathrm{g}) = A_j + S_j,$$

where A_j and S_j are the skew-adjoint and selfadjoint parts of C_j for the inner product of $|\cdot|_{2;h}$. With the skew-adjoint operators $D_j^0 = \partial_j + A_j$, the fibre generator becomes

$$\mathcal{L}_\Xi = \sum_j \big(D_j^0 + S_j - \partial_j\phi_\Xi\big)^2 - \|\Xi\|^2.$$

Since the anticommutator of a skew-adjoint operator with a selfadjoint multiplication is skew-adjoint, integration by parts yields the identity

$$\operatorname{Re}\langle u, \mathcal{L}_\Xi u\rangle = -\sum_j \|D_j^0 u\|^2 + \sum_j \|(S_j - \partial_j \phi_\Xi)u\|^2 - \|\Xi\|^2\|u\|^2.$$

The operators C_j act on the finite-dimensional Grassmann algebra of the transported odd coordinates through the logarithmic derivatives of the boost, and (15.30) therefore implies that

$$\Big\|\sum_j (\partial_j \phi_\Xi) S_j\Big\| \le C_k d_X(\Xi)^{1/2}, \qquad \sum_j \|S_j\|^2 \le C_k.$$

Expanding $\|(S_j - \partial_j\phi_\Xi)u\|^2$ and using (15.28), we see that the last two terms in the identity for $\operatorname{Re}\langle u, \mathcal{L}_\Xi u\rangle$ are bounded by

$$\int \big(-d_X(\Xi) + C_k d_X(\Xi)^{1/2} + C_k\big)|u|^2 \le C_k'\|u\|^2.$$

The odd part of the Laplacian of the heat kernel is preserved by the congruence actions, since $\mathrm{g}\Lambda_B\mathrm{g} = \Lambda_B$ preserves the odd bilinear form, and by Lemma 5.13(c), two odd derivatives have norm at most $h^{-2} = W$ for $|\cdot|_{2;h}$. Therefore the odd part of the heat has norm at most $e^{C_k W t}$, and the energy inequality yields the bound $e^{(C_k W + C_k)t}$ for the fibre operator.

We now remove the Dirichlet domain. For compactly supported input, the Dirichlet heat kernels on an exhaustion of the Cartesian domain converge pointwise to the free Gaussian kernel. The frame factors are polynomially bounded and ϕ_Ξ grows at most linearly. Fatou's lemma then implies the same norm bound for the actual fibre kernel, and by density the bound extends to all inputs. Plancherel's theorem in P_0 yields the BB bound. The FF block of the cycle is the flat Cartesian space of Hermitian matrices M, on which the FF part of the heat kernel defines the flat heat $e^{t\Delta_M}$, which is a contraction. In a product of two kernels, the frame changes cancel at the middle point, so that the semigroup law follows from Lemma 4.10. Strong continuity holds on compactly supported smooth inputs, by local heat approximation, and it extends to all inputs by the uniform bound.

Finally, we add the weight $v_{\mathcal{S}}^p$. Conjugation by $v_{\mathcal{S}}^p$ adds first- and zeroth-order terms with coefficients given by the derivatives of $p \log v_{\mathcal{S}}$. The only potentially unbounded cross term in the Fourier variable is aligned with the defect. Indeed, in the eigenbasis of X we have

$$|\langle \nabla\phi_\Xi, \nabla \log v_{\mathcal{S}}\rangle| \le d_X(\Xi)^{1/2}\, \frac{(\sum_a \sinh^4 r_a/\cosh^2 r_a)^{1/2}}{v_{\mathcal{S}}} \le s^{-1} d_X(\Xi)^{1/2}.$$

The other derivatives of $\log v_{\mathcal{S}}$ and their frame terms are bounded. Retaining part of the negative defect, we obtain the weighted bound by the same exhaustion. The adjoint of K_t in $\mathcal{H}_p(\mathcal{S})$ has the same norm. □

The escape function and the onsite weight

In this subsection we define the escape function, which measures the distance from the saddle manifold, and we prove that the onsite weight gains a universal fraction of it, in the form of an ℓ^1 bound on its coefficients.

Fix a nondecreasing smooth function χ_0 on $[0,\infty)$ with $\chi_0(0) = 0$ and $\chi_0'(0) > 0$, which is linear near zero and constant on $[1,\infty)$, and fix a radius $0 < r_{\mathrm{esc}} \le 1$. Then $\chi_0(x)/\min(x,1)$ is continuous and positive on $(0,\infty)$, with the positive limit $\chi_0'(0)$ at zero and the constant value $\chi_0(1) > 0$ on $[1,\infty)$, so there exist $c, C > 0$ such that

$$c\min(x,1) \le \chi_0(x) \le C\min(x,1), \qquad x \ge 0. \tag{15.31}$$

We define the *escape function*

$$\vartheta = \vartheta_B + \vartheta_F, \qquad \vartheta_B = r_{\mathrm{esc}}^2\,\chi_0\big(\|\widetilde{P}\|^2/r_{\mathrm{esc}}^2\big), \qquad \vartheta_F = \mathrm{Tr}\,\frac{(M^2 - s^2\mathrm{Id})^2}{\mathrm{Id} + M^2}. \tag{15.32}$$

By (15.31), we have

$$c\min(\|\widetilde{P}\|^2, r_{\mathrm{esc}}^2) \le \vartheta_B \le C\min(\|\widetilde{P}\|^2, r_{\mathrm{esc}}^2).$$

Moreover, with $x = \|\widetilde{P}\|^2/r_{\mathrm{esc}}^2$,

$$|\nabla_{\widetilde{P}}\vartheta_B|^2 = 4r_{\mathrm{esc}}^2\chi_0'(x)^2 x \le C\vartheta_B,$$

since $\chi_0'(x)^2 x \le C\chi_0(x)$ near zero and χ_0' vanishes on $[1,\infty)$. All BB derivatives of ϑ_B vanish where $\|\widetilde{P}\| \ge r_{\mathrm{esc}}$. Writing $\vartheta_F = \sum_i \theta(\mu_i)$ over the eigenvalues μ_i of M, with $\theta(\mu) = (\mu^2 - s^2)^2/(1+\mu^2)$, we have

$$|\nabla_M \vartheta_F|^2 = \sum_i \theta'(\mu_i)^2 \le C_F\vartheta_F, \qquad \frac{\theta'(\mu)^2}{\theta(\mu)} = \frac{4\mu^2(\mu^2+2+s^2)^2}{(1+\mu^2)^3} \le C_F, \tag{15.33}$$

with a universal constant C_F. Since $\mathrm{ch}_{\mathrm{g}} \ge \mathrm{Id}$, the escape function vanishes exactly at the points where $P_0 = 0$ and M has only the eigenvalues $\pm s$. By Lemma 4.7(d), these are the points of $\mathcal{S}$ with BB block $\kappa\mathrm{Id} - \mathrm{i}sQ_X$ and with FF block of eigenvalues $\kappa \pm \mathrm{i}s$, and they lie on the saddle manifold, in the $2k+1$ signatures.

We first bound the FF part of the onsite weight. For $D = 0$ and $d = 2k$, the function of the following lemma satisfies

$$\mathcal{W}_\kappa(M,0) = e^{k(s^2-\kappa^2)}\big|e^{\mathrm{Tr}\,S_{FF}^2/2}\det(E - S_{FF})\big|,$$

where $e^{\mathrm{Tr}\,S_{FF}^2/2}\det(E - S_{FF})$ is the FF part of the onsite weight, and $\mathcal{W}_\kappa(M,0) = 1$ when M has only the eigenvalues $\pm s$. The lemma also treats a shift by a Hermitian matrix D, which Section 19.5 needs; here we use only the case $D = 0$.

Lemma 15.8. *Let $|\kappa| \le \kappa_0 < 1$, $s^2 = 1 - \kappa^2$, and let M and D be Hermitian $d \times d$ matrices with $\|D\|_{\mathrm{op}} \le d_0$. Define*

$$\mathcal{W}_\kappa(M, D) = e^{(ds^2 - \operatorname{Tr} M^2)/2 + \kappa \operatorname{Tr} D + \operatorname{Tr} D^2/2} \, |\det(\kappa \mathrm{Id} - D + \mathrm{i}M)|.$$

There exist $c, C > 0$, depending only on κ_0 and d_0 and not on d, such that

$$\mathcal{W}_\kappa(M, D) \le \exp\Big\{ C\|D\|^2 - c \operatorname{Tr} \frac{(M^2 - s^2 \mathrm{Id})^2}{\mathrm{Id} + M^2} \Big\}.$$

Proof. We diagonalize M but not D. Let λ_i be the eigenvalues of M, and in the eigenbasis set

$$d_i = D_{ii} \in \mathbb{R}, \qquad t_i = \sum_j |D_{ji}|^2.$$

By Hadamard's inequality for the columns, we have

$$|\det(\kappa \mathrm{Id} - D + \mathrm{i}M)| \le \prod_i y_i^{1/2}, \qquad y_i = \lambda_i^2 + \kappa^2 - 2\kappa d_i + t_i \ge 0.$$

We may assume that every y_i is nonzero, since otherwise the corresponding column vanishes and the conclusion is immediate. Set $x_i = \lambda_i^2 + \kappa^2$. Using $s^2 = 1 - \kappa^2$, we find that the logarithm of the resulting bound for $\mathcal{W}_\kappa(M, D)$ is

$$\sum_i \Big\{ -\frac{x_i - 1}{2} + \kappa d_i + \frac{t_i}{2} + \frac{1}{2} \log y_i \Big\} = \sum_i \Big\{ t_i - \frac{1}{2}(y_i - 1 - \log y_i) \Big\}.$$

The terms linear in D cancel in this identity, because of the factor $e^{\kappa \operatorname{Tr} D}$ in the definition of $\mathcal{W}_\kappa$.

For $y > 0$ we have $y - 1 - \log y \ge (y-1)^2/(2(1+y))$, as one checks by differentiating the difference on $(0,1)$ and on $(1,\infty)$. The function $x \mapsto (x-1)(1+x)^{-1/2}$ has bounded derivative on $[0,\infty)$, and

$$|x_i - y_i|^2 = |2\kappa d_i - t_i|^2 \le 8\kappa_0^2 t_i + 2t_i^2 \le C_{\kappa_0, d_0} t_i,$$

since $t_i \le \|D\|_{\mathrm{op}}^2 \le d_0^2$. It follows that

$$\frac{(x_i - 1)^2}{1 + x_i} \le C(y_i - 1 - \log y_i) + C t_i.$$

The quantities $1 + x_i$ and $1 + \lambda_i^2$ are comparable with constants independent of d, and $x_i - 1 = \lambda_i^2 - s^2$. Substituting the last inequality into the logarithmic bound, we obtain

$$\log \mathcal{W}_\kappa(M, D) \le C \sum_i t_i - c \sum_i \frac{(\lambda_i^2 - s^2)^2}{1 + \lambda_i^2} = C\|D\|^2 - c \operatorname{Tr} \frac{(M^2 - s^2 \mathrm{Id})^2}{\mathrm{Id} + M^2},$$

where we used $\sum_i t_i = \|D\|^2$. □

We now state the onsite bound. We write the damping of the onsite weight as $e^{-\mathrm{i}\eta\,\mathrm{STr}(S\Gamma)}$, with a matrix Γ whose BB block is close to Λ_B, as in (15.6), and we allow complex values of the damping, which Sections 17 and 19 use.

Lemma 15.9. *Let $|E| \le 1/2$ and $0 < \eta \le C_X W^{-2}$. Let Γ be a constant even $(2k|2k)$ supermatrix with zero odd blocks such that $\|\Gamma_{BB} - \Lambda_B\|_{\mathrm{op}} \le 1/4$ and $\|\Gamma_{FF}\|_{\mathrm{op}} \le C_\Gamma$, and set*

$$w_\Gamma(S) = e^{-\mathrm{STr}(S^2)/2 - \mathrm{i}\eta\,\mathrm{STr}(S\Gamma)}\ \mathrm{SDet}(E-S)^{-1}.$$

There exists a universal $c > 0$ such that for all sufficiently large W and all $S \in \mathcal{S}$,

$$|w_\Gamma|_{1;h} \le e^{C_k/W + C_k\eta}\, e^{-c\vartheta - c\eta v_{\mathcal{S}}}. \tag{15.34}$$

More precisely, the body of w_Γ has modulus at most $e^{C_k\eta - c\vartheta - c\eta v_{\mathcal{S}}}$, and the absolute values of its other coefficients sum to at most $C_k h^2 e^{C_k\eta - c\vartheta - c\eta v_{\mathcal{S}}}$. The constants C_k and the threshold on W depend on k, C_X, and C_Γ.

The hypothesis on Γ covers the normalized sources $\hat\Gamma$ of (15.6), for which $\hat\Gamma_{BB} = \Lambda_B$ and $\|\hat\Gamma_F\|_{\mathrm{op}} \le C_\Gamma$, and the parameter balls of (15.36) below. For $\Gamma = \hat\Gamma$, the function defined in the lemma is the onsite weight of Definition 4.9 for the source supermatrix $\varepsilon\hat\Gamma$, by (15.6). The body bound in (15.34) carries no constant factor, and the factor $e^{C_k/W}$ comes only from the nonbody coefficients. This is the form that the contraction away from the saddle manifold in the proof of Theorem 15.14 needs, since a constant factor larger than one would be raised to the power of the number of steps.

Proof. We first bound the body. Its modulus is the product of the BB factor $e^{-V_\kappa(P_0)}$, where V_κ is the BB onsite deficit (15.17) of Lemma 15.5 (the real part of $\frac12 \operatorname{Tr} S_{BB}^2 + \log\det(E - S_{BB})$ minus its value at $P_0 = 0$), of the FF factor $\mathcal{W}_\kappa(M,0)$ of Lemma 15.8, and of the damping $|e^{-\mathrm{i}\eta\,\mathrm{STr}(S\Gamma)}|$. Indeed, the BB part of $\log|w_0|$ is $-V_\kappa(P_0) - k(\kappa^2 - s^2)$, since its value at $P_0 = 0$ is $k(\kappa^2 - s^2)$, and the FF part is

$$\tfrac12 \operatorname{Re}\operatorname{Tr}(\kappa\mathrm{Id} + \mathrm{i}M)^2 + \log|\det(\kappa\mathrm{Id} - \mathrm{i}M)| = \log\mathcal{W}_\kappa(M,0) + k(\kappa^2 - s^2).$$

The two constants cancel, so that the product of the BB and FF parts equals one at the saddle points.

By the concavity bound (15.20), for $|E| \le 1/2$ we have

$$V_\kappa \ge (1 - 2|\kappa|)V_0 \ge V_0/2,$$

where

$$V_0 = \tfrac12 \operatorname{Tr} P_0^2 + \tfrac12 \operatorname{Tr}\log(\mathrm{Id} + L^2), \qquad L = \mathrm{g}^{-1}\Lambda_B P_0 \mathrm{g}^{-1} = \Lambda_B\, \mathrm{g} P_0 \mathrm{g}^{-1},$$

and we used $\mathrm{g}^{-1}\Lambda_B = \Lambda_B \mathrm{g}$. Since $\mathrm{g}P_0\mathrm{g}^{-1} = \widetilde{P} + \mathrm{th}_{\mathrm{g}}\,\widetilde{P}$ is an orthogonal sum, we have $\|L\| \geq \|\widetilde{P}\|$. Let k_j be the eigenvalues of the Hermitian matrix L. Then

$$\begin{aligned}\tfrac{1}{2}\operatorname{Tr}\log(\mathrm{Id}+L^2) &= \tfrac{1}{2}\log\prod_j(1+k_j^2) \geq \tfrac{1}{2}\log\Big(1+\sum_j k_j^2\Big)\\ &= \tfrac{1}{2}\log\big(1+\|L\|^2\big) \geq \tfrac{1}{2}\log\big(1+\|\widetilde{P}\|^2\big).\end{aligned}$$

Using also $\log(1+x) \geq (\log 2)\min(x,1)$ and $\vartheta_B \leq C\min(\|\widetilde{P}\|^2, r_{\mathrm{esc}}^2)$ with $r_{\mathrm{esc}} \leq 1$, we obtain

$$V_\kappa \geq \tfrac{1}{4}\operatorname{Tr} P_0^2 + \tfrac{1}{4}\log\big(1+\|\widetilde{P}\|^2\big) \geq \tfrac{1}{4}\operatorname{Tr} P_0^2 + c_B\vartheta_B \tag{15.35}$$

for a universal $c_B > 0$. By Lemma 15.8 with $D = 0$, we also have $\mathcal{W}_\kappa(M,0) \leq e^{-c_F\vartheta_F}$.

Next, we bound the damping. Write $\Gamma_{BB} = \Lambda_B + \gamma$. Since X is off-diagonal, $\operatorname{Tr} X = 0$, and therefore

$$\begin{aligned}\operatorname{Tr}(S_{BB}\Lambda_B) &= \operatorname{Tr}(P_0\Lambda_B) - \mathrm{i}\operatorname{Tr} f_0(X),\\ \operatorname{Re}\big(-\mathrm{i}\eta\operatorname{Tr}(S_{BB}\Lambda_B)\big) &= -\eta\operatorname{Tr} f_0(X) = -\eta(v_{\mathcal{S}}-1).\end{aligned}$$

The positive matrix $f_0(X) + X = s\mathrm{g}^2$ has trace norm $\operatorname{Tr} f_0(X)$, and therefore

$$|\eta\operatorname{Tr}(S_{BB}\gamma)| \leq \frac{\eta}{4}\big(2k|\kappa| + \|P_0\|_1 + \operatorname{Tr} f_0(X)\big) \leq C_k\eta + \tfrac{1}{8}\operatorname{Tr} P_0^2 + \tfrac{1}{4}\eta\operatorname{Tr} f_0(X),$$

where we used $\|P_0\|_1 \leq 2k + \operatorname{Tr} P_0^2$ and $\eta \leq 1/2$. For the FF block, the bounds $\|\kappa\mathrm{Id} + \mathrm{i}M\|_1 \leq 2k + \|M\|_1$ and $|\mu| \leq 2 + 2\theta(\mu)$ for real μ give

$$|\eta\operatorname{Tr}(S_{FF}\Gamma_{FF})| \leq C_k\eta + 2C_\Gamma\eta\vartheta_F,$$

and we take W so large that $2C_\Gamma\eta \leq c_F/2$. Combining these bounds with (15.35), we find that the modulus of the body is at most

$$\exp\big\{C_k\eta - c_B\vartheta_B - c_F\vartheta_F/2 - 3\eta(v_{\mathcal{S}}-1)/4\big\}.$$

We now bound the other coefficients. By (15.18), congruence by g^{-1} turns the BB block of $\Lambda_B(E-S)$ into $\kappa\Lambda_B - L + \mathrm{i}s\mathrm{Id}$, whose inverse has operator norm at most s^{-1}, and the congruence changes the superdeterminant only by a sign, since $\det\mathrm{g}^2 = 1$. In the transported frame, the odd blocks of $E-S$ are the transported odd coordinates, and the Schur complement of the BB block yields

$$\operatorname{SDet}(E-S)^{-1} = \pm\frac{\det(E - S_{FF} - \mathcal{N})}{\det(\kappa\Lambda_B - L + \mathrm{i}s\mathrm{Id})},$$

where the entries of $\mathcal{N}$ are quadratic in the transported odd coordinates, with coefficients bounded by C/s. The congruence preserves the odd part of $\operatorname{STr}(S^2)$, since $\mathrm{g}\Lambda_B\mathrm{g} = \Lambda_B$, and that part has bounded coefficients. Every nonbody coefficient therefore contains at

least two h-scaled odd coordinates, and hence a factor h^2. We expand $\det(E - S_{FF} - \mathcal{N})$ by complementary minors, without inverting $E - S_{FF} = \kappa\mathrm{Id} - \mathrm{i}M$. Let $\mu_1, \dots, \mu_{2k}$ be the eigenvalues of M. After a unitary diagonalization of M, whose minors have modulus at most one, each minor of $E - S_{FF}$ is bounded by C_k times a product $\prod_{i\in I} |\kappa + \mathrm{i}\mu_i|$ over a set $I \subset \{1, \dots, 2k\}$. The proof of Lemma 15.8, with $D = 0$, shows that

$$|\kappa + \mathrm{i}\mu_i| e^{(s^2-\mu_i^2)/2} \le e^{-c\theta(\mu_i)} \quad (i \in I), \qquad e^{(s^2-\mu_i^2)/2} \le Ce^{-c\theta(\mu_i)} \quad (i \notin I).$$

Each minor, times the FF Gaussian, is thus bounded by $C_k e^{-c\vartheta_F}$. Multiplying by the bounded coefficients of $\mathcal{N}$, the BB body and the damping, and summing over the finitely many monomials, we obtain the bound on the nonbody coefficients. Adding the body bound and using $1 + C_k h^2 \le e^{C_k/W}$, we obtain (15.34). □

Parameter balls. In Sections 17 and 19, the damping of the onsite weight takes complex values. We call a constant even $(2k|2k)$ supermatrix Γ with zero odd blocks, used in the damping $e^{-\mathrm{i}\eta\,\mathrm{STr}(S\Gamma)}$ as in Lemma 15.9, a *linear damping direction*. For the sources of Definition 4.1, it is the normalized source $\hat{\Gamma}$ of (15.6), whose BB and FF blocks are the normalized damping and source matrices. We fix nested complex balls $\mathcal{U}_0 \Subset \mathcal{U}_1 \Subset \mathcal{U}_2$ of linear damping directions around the normalized base point $\Lambda = \varepsilon^{-1}\Gamma_0$ of Definition 4.1, with uniformly bounded FF blocks, such that

$$\mathrm{Re}(\Gamma_{BB}\Lambda_B) \ge c_\varepsilon \mathrm{Id}, \qquad \Gamma \in \mathcal{U}_2, \tag{15.36}$$

for some $c_\varepsilon > 0$, and such that $\mathcal{U}_2$ lies inside the ball $\|\Gamma_{BB} - \Lambda_B\|_{\mathrm{op}} \le 1/4$ of Lemma 15.9. The order in which these balls are chosen, relative to the other parameters of Part IV, is given in Table 19.1.

15.4 Spectral averaging

The determinant inequalities reduce the anchored identity to exponential moments of traces of imaginary parts of resolvents of the Gaussian matrix $H^{(n)}$. We bound these by averaging over the diagonal entries. The following lemma is a consequence of Minami's estimate [92], in the form proved by Graf and Vaghi [63, Lemma 2], which applies to determinants of all orders and does not need the symmetry of the matrix. (We use the numbering of the arXiv version of [63].)

Lemma 15.10. *Let A be a random Hermitian $n \times n$ matrix whose diagonal entries are, conditionally on the off-diagonal entries, independent real random variables with densities bounded by $\mathsf{d}_1, \dots, \mathsf{d}_n$. Let $\eta, c > 0$, let Y be a deterministic positive definite matrix with $Y \ge c\eta\mathrm{Id}$, let Z_0 be a deterministic real diagonal matrix, and set $R = (A + Z_0 + \mathrm{i}Y)^{-1}$. Then for all $t \ge 0$,*

$$\mathbb{E}e^{t\eta\,\mathrm{Tr}(-\operatorname{Im} R)} \le \exp\Big(\pi c\big(e^{t/c} - 1\big)\eta \sum_{i=1}^{n} \mathsf{d}_i\Big). \tag{15.37}$$

Proof. Since $A + Z_0$ satisfies the same hypothesis as A, with the same bounds d_i, we may assume that $Z_0 = 0$. Then (15.12) implies that

$$0 \le c\eta(-\operatorname{Im} R) \le c\eta Y^{-1} \le \mathrm{Id}.$$

For $0 \le x \le 1$ and $\lambda \ge 0$, we have $e^{\lambda x} \le 1 + (e^\lambda - 1)x$, by convexity. We apply this bound to the eigenvalues of $c\eta(-\operatorname{Im} R)$, with $\lambda = t/c$, and we expand the resulting determinant into principal minors. We obtain

$$e^{t\eta \operatorname{Tr}(-\operatorname{Im} R)} \le \det\bigl[\mathrm{Id} + (e^{t/c} - 1)c\eta(-\operatorname{Im} R)\bigr] = \sum_{I \subset [n]} \bigl((e^{t/c} - 1)c\eta\bigr)^{|I|} \det\bigl[(-\operatorname{Im} R)_{II}\bigr],$$

where $(-\operatorname{Im} R)_{II}$ is the principal submatrix with rows and columns in I, and the term with $I = \emptyset$ is one. We will show that

$$\mathbb{E} \det\bigl[(-\operatorname{Im} R)_{II}\bigr] \le \pi^{|I|} \prod_{i \in I} \mathsf{d}_i \qquad \text{for all } I \subset [n]. \tag{15.38}$$

Inserting (15.38) into the sum, we then find

$$\mathbb{E} e^{t\eta \operatorname{Tr}(-\operatorname{Im} R)} \le \prod_{i=1}^{n} \bigl[1 + \pi(e^{t/c} - 1)c\eta\mathsf{d}_i\bigr] \le \exp\Bigl(\pi c(e^{t/c} - 1)\eta \sum_{i=1}^{n} \mathsf{d}_i\Bigr).$$

To prove (15.38), let $I \ne \emptyset$, and condition on all entries of A except the diagonal entries a_i with $i \in I$; let $V_I = \operatorname{diag}(a_i)_{i \in I}$. Recall from (15.12) that $\operatorname{Im} R^* = -\operatorname{Im} R = RYR^*$, where $R^* = (A - \mathrm{i}Y)^{-1}$, and this matrix is positive definite. It follows that the principal submatrix $C_I = (R^*)_{II}$ has positive definite imaginary part as well. By the Schur complement formula, $C_I = (V_I - Z_I)^{-1}$, where the matrix Z_I is a function of the entries we conditioned on. Moreover,

$$\operatorname{Im} Z_I = -\operatorname{Im}(C_I^{-1}) = C_I^{-1}(\operatorname{Im} C_I)(C_I^{-1})^*$$

is positive definite.

By [63, Lemma 2], we have

$$\int_{\mathbb{R}^{|I|}} \det\bigl[\operatorname{Im}(V_I - Z_I)^{-1}\bigr] \prod_{i \in I} \mathrm{d}a_i \le \pi^{|I|}$$

for every square matrix Z_I with positive definite imaginary part $\operatorname{Im} Z_I = (Z_I - Z_I^*)/(2\mathrm{i})$. Recall that $\operatorname{Im} C_I = (-\operatorname{Im} R)_{II}$, so that the integrand is

$$\det \operatorname{Im}(V_I - Z_I)^{-1} = \det \operatorname{Im} C_I = \det(-\operatorname{Im} R)_{II} \ge 0.$$

The conditional density of V_I is bounded by $\prod_{i \in I} \mathsf{d}_i$, and (15.38) follows. □

We now record the properties of the anchored covariance and of the boundary profile that the averaging uses.

Lemma 15.11. *Let $n, W \geq 1$, and define $\theta_W > 0$ by $\cosh \theta_W = 1 + (2W^2)^{-1}$, so that $\theta_W = 2\,\mathrm{arsinh}(1/(2W))$ as in Section 1. Then the following statements hold.*

(a) For $1 \leq i \leq n$, we have $\varsigma_i = \cosh(\theta_W(n + \frac{1}{2} - i))/\cosh(\theta_W(n + \frac{1}{2}))$, and in particular $0 < \varsigma_i \leq \min(1, 2e^{-\theta_W i})$. Moreover, $1/(2W) \leq \theta_W \leq 1/W$ and $\sigma_n \leq 4W$.

(b) For $1 \leq i \leq n$, we have $(J_n)_{ii} \geq \min(i, W)/(8W^2)$. Consequently, the diagonal entries of $H^{(n)}$ are independent of the other entries and have densities bounded by $\mathsf{d}_i = (2\pi (J_n)_{ii})^{-1/2}$, with

$$\sum_{i=1}^{n} \mathsf{d}_i \leq 3\big(W^{3/2} + n\sqrt{W}\big).$$

(c) For all $\eta > 0$ and $s_0 \in (0, 1]$,

$$\Pi_n = \sum_{i=1}^{n} \frac{\varsigma_i}{\eta(1 - \varsigma_i) + s_0 \varsigma_i} \leq \frac{6W}{s_0}\big(1 + \log(1 + \eta^{-1})\big).$$

Proof. We begin with (a). Set $\varsigma_0 = 1$ and $\varsigma_{n+1} = \varsigma_n$. Recall that

$$\varsigma = W^2 J_n e_1, \qquad J_n^{-1} = \mathrm{Id}_n - W^2 \Delta_n + W^2 e_1 e_1^*.$$

The equation $J_n^{-1}\varsigma = W^2 e_1$ then says that

$$\varsigma_{i+1} + \varsigma_{i-1} = (2 + W^{-2})\varsigma_i = 2\cosh(\theta_W)\varsigma_i, \qquad 1 \leq i \leq n.$$

The solutions of this recurrence are linear combinations of $e^{\pm \theta_W i}$, and the conditions $\varsigma_0 = 1$ and $\varsigma_{n+1} = \varsigma_n$ determine the stated formula. This formula implies that $0 < \varsigma_i \leq 1$ and

$$\varsigma_i \leq \frac{e^{\theta_W(n+1/2-i)}}{e^{\theta_W(n+1/2)}/2} = 2e^{-\theta_W i}.$$

Comparing the Taylor series term by term, we have

$$\frac{\theta_W^2}{2} \leq \cosh \theta_W - 1 \leq \frac{1}{2}\theta_W^2 \cosh \theta_W.$$

The first inequality implies that $\theta_W \leq 1/W \leq 1$. Using $\cosh 1 \leq 8/5$, we deduce from the second inequality that

$$\frac{1}{2W^2} = \cosh \theta_W - 1 \leq \frac{4}{5}\theta_W^2,$$

so that $\theta_W \geq 1/(2W)$. Finally,

$$\sigma_n = \sum_{i=1}^{n} \varsigma_i \leq \sum_{i \geq 1} 2e^{-\theta_W i} \leq \frac{2}{\theta_W} \leq 4W.$$

For (b), let $\ell = \min(i, W)$, and define $f \in \mathbb{R}^n$ by

$$f_j = \max\bigl(0, 1 - |j - i|/\ell\bigr), \qquad 1 \le j \le n.$$

By the variational characterization of the diagonal entries of the inverse of a positive definite matrix, we have

$$(J_n)_{ii} \ge \frac{f_i^2}{\langle f, J_n^{-1} f\rangle}, \qquad \langle f, J_n^{-1} f\rangle = \sum_{j=1}^{n} f_j^2 + W^2 \sum_{j=1}^{n-1} (f_{j+1} - f_j)^2 + W^2 f_1^2.$$

We have $f_i = 1$, and f vanishes outside an interval of length 2ℓ about i. In particular, the first sum is bounded by $2\ell + 1 \le 3\ell$, and the second sum has at most $2\ell + 2 \le 4\ell$ nonzero terms, each bounded by ℓ^{-2}.

Moreover, $f_1 \le 1/\ell$. Indeed, if $\ell = i$, then $f_1 = 1/i$, and if $\ell = W < i$, then $(i-1)/W > 1 - 1/W$ and $f_1 < 1/W$. Using $1 \le \ell \le W$, we find

$$\langle f, J_n^{-1} f\rangle \le 3\ell + \frac{4W^2}{\ell} + \frac{W^2}{\ell^2} \le \frac{8W^2}{\ell},$$

which implies the lower bound on $(J_n)_{ii}$. The matrix $H^{(n)}$ has independent entries up to the Hermitian symmetry, and $H_{ii}^{(n)}$ is a real centered Gaussian variable with variance $(J_n)_{ii}$, whose density is bounded by d_i. We conclude that

$$\sum_{i=1}^{n} \mathsf{d}_i \le \frac{2W}{\sqrt{\pi}} \sum_{i=1}^{n} \min(i, W)^{-1/2} \le \frac{2W}{\sqrt{\pi}} \bigl(2\sqrt{W} + nW^{-1/2}\bigr) \le 3\bigl(W^{3/2} + n\sqrt{W}\bigr).$$

For (c), set

$$i_0 = \bigl\lceil \theta_W^{-1} \log(4(1 + \eta^{-1}))\bigr\rceil.$$

Each term of Π_n is bounded by s_0^{-1}. Therefore the terms with $i \le i_0$ contribute at most

$$\frac{i_0}{s_0} \le \frac{2W \log(4(1+\eta^{-1})) + 1}{s_0} \le \frac{W}{s_0}\bigl(4 + 2\log(1+\eta^{-1})\bigr).$$

For $i > i_0$, part (a) gives

$$\varsigma_i \le 2e^{-\theta_W i_0} \le \frac{\eta}{2(1+\eta)} \le \frac{1}{2},$$

so that $1 - \varsigma_i \ge 1/2$ and

$$\frac{\varsigma_i}{\eta(1-\varsigma_i) + s_0\varsigma_i} \le \frac{2\varsigma_i}{\eta} \le \frac{4e^{-\theta_W i}}{\eta}.$$

These terms therefore contribute at most

$$\frac{4}{\eta} \sum_{i > i_0} e^{-\theta_W i} \le \frac{4}{\eta} \cdot \frac{e^{-\theta_W i_0}}{e^{\theta_W} - 1} \le \frac{4}{\eta} \cdot \frac{\eta}{4\theta_W} \le 2W \le \frac{2W}{s_0}.$$

Adding the two contributions proves (c). □

15.5 Coefficient bounds

The anchored identity contains the odd coordinates of the free site S and, after differentiation, the local sources. The next lemma bounds the coefficients of determinants and inverses whose body is controlled, relative to a determinant that may vanish. We state it for a finite commutative algebra with a monomial basis, which covers the even part of a Grassmann algebra tensored with truncated polynomials in the source variables.

Let $\mathcal{A}$ be a finite-dimensional commutative algebra over $\mathbb{C}$ with a basis of *monomials* that contains 1, such that the product of two monomials is a monomial, minus a monomial, or zero, and such that every product of $m_{\mathcal{A}}+1$ monomials different from 1 vanishes. For $a \in \mathcal{A}$, let $|a|_1$ be the sum of the absolute values of the coefficients of a in this basis. For a matrix $\mathcal{N} = \sum_\mu \mu\, \mathcal{N}_\mu$ with entries in $\mathcal{A}$, where μ runs over the monomials and the $\mathcal{N}_\mu$ are complex matrices, let $\|\mathcal{N}\|_1 = \sum_\mu \|\mathcal{N}_\mu\|$. These norms are submultiplicative, by the argument of Lemma 5.13(a). We say that $\mathcal{N}$ has no constant term if $\mathcal{N}_1 = 0$.

Lemma 15.12. *In this setting, let $\mathcal{N}$ be a $d \times d$ matrix with entries in $\mathcal{A}$ and no constant term.*

(a) Let A and Y be Hermitian $d \times d$ matrices, and let Y_+ be a positive definite $d \times d$ matrix with $-Y_+ \le Y \le Y_+$. Then

$$|\det(A + \mathrm{i}Y + \mathcal{N})|_1 \le |\det(A + \mathrm{i}Y_+)| \big(1 + d\|\mathcal{N}\|_1 \|Y_+^{-1}\|\big)^{m_{\mathcal{A}}}. \tag{15.39}$$

(b) Let $\mathbf{B}$ be an invertible complex $d \times d$ matrix, and set $\alpha = \|\mathbf{B}^{-1}\|\, \|\mathcal{N}\|_1$. Then $\mathbf{B} - \mathcal{N}$ is invertible over $\mathcal{A}$, and

$$\|(\mathbf{B} - \mathcal{N})^{-1}\|_1 \le \|\mathbf{B}^{-1}\| (1 + \alpha)^{m_{\mathcal{A}}}. \tag{15.40}$$

If, moreover, every $\mathcal{N}_\mu$ has rank at most m_0, then

$$\Big| \frac{\det \mathbf{B}}{\det(\mathbf{B} - \mathcal{N})} \Big|_1 \le \big(1 + (m_0 + m_{\mathcal{A}})\alpha\big)^{m_{\mathcal{A}}}. \tag{15.41}$$

The case $\mathcal{N} = 0$ of (15.39) is the bound $|\det(A + \mathrm{i}Y)| \le |\det(A + \mathrm{i}Y_+)|$, which holds also when $\det(A + \mathrm{i}Y) = 0$.

Proof. Each of the three quantities is obtained by substituting $t_\mu = \mu$ into a formal power series in commuting variables t_μ, one for each monomial $\mu \neq 1$, whose constant term is the value at $\mathcal{N} = 0$. This substitution is a ring homomorphism from formal power series to $\mathcal{A}$, since only the terms of total degree at most $m_{\mathcal{A}}$ survive it. Since every product of monomials has norm at most one, the norm $|\cdot|_1$ of the result is bounded by the sum of the absolute values of the coefficients of total degree at most $m_{\mathcal{A}}$.

We begin with (a). Set

$$A' = Y_+^{-1/2} A Y_+^{-1/2},$$
$$Y' = Y_+^{-1/2} Y Y_+^{-1/2},$$
$$Q_+ = \tfrac{1}{2}(\mathrm{Id} + Y'),$$
$$U = (A' + \mathrm{i}\mathrm{Id})^{-1}(A' - \mathrm{i}\mathrm{Id}).$$

Then $0 \le Q_+ \le \mathrm{Id}$, U is unitary, and

$$(A' + \mathrm{i}\mathrm{Id})^{-1}(A' + \mathrm{i}Y') = Q_+ + U(\mathrm{Id} - Q_+).$$

Let (e_a) be an orthonormal eigenbasis of Q_+, with eigenvalues $q_a \in [0,1]$. In this basis, the ath column of this matrix is

$$q_a e_a + (1 - q_a)Ue_a,$$

which has norm at most one.

The matrices

$$\mathcal{N}'_\mu = (A' + \mathrm{i}\mathrm{Id})^{-1} Y_+^{-1/2} \mathcal{N}_\mu Y_+^{-1/2}$$

have norm at most $\|\mathcal{N}_\mu\| \|Y_+^{-1}\|$, and we have

$$\det\Big(A + \mathrm{i}Y + \sum_\mu t_\mu \mathcal{N}_\mu\Big) = \det(A + \mathrm{i}Y_+) \det\Big(Q_+ + U(\mathrm{Id} - Q_+) + \sum_\mu t_\mu \mathcal{N}'_\mu\Big),$$

where we used the identity

$$\det Y_+ \det(A' + \mathrm{i}\mathrm{Id}) = \det\big(Y_+^{1/2}(A' + \mathrm{i}\mathrm{Id})Y_+^{1/2}\big) = \det(A + \mathrm{i}Y_+).$$

We expand the last determinant by columns in the basis (e_a). Let $\nu = (\nu_\mu)$ be a multi-index of total degree r. The coefficient of $\prod_\mu t_\mu^{\nu_\mu}$ is a sum of $d!/((d-r)!\prod_\mu \nu_\mu!)$ determinants, in which ν_μ of the columns are columns of $\mathcal{N}'_\mu$ and the others are columns of $Q_+ + U(\mathrm{Id} - Q_+)$. By Hadamard's inequality, each of them is bounded in modulus by $\prod_\mu (\|\mathcal{N}_\mu\| \|Y_+^{-1}\|)^{\nu_\mu}$. Summing over ν with $|\nu| = r$, and using the multinomial theorem and $d!/(d-r)! \le d^r$, we find that the coefficients of degree r contribute at most

$$|\det(A + \mathrm{i}Y_+)| \, \frac{(d\|\mathcal{N}\|_1 \|Y_+^{-1}\|)^r}{r!}.$$

Summing over $r \le m_{\mathcal{A}}$ and using $1/r! \le \binom{m_{\mathcal{A}}}{r}$, we obtain (15.39).

We now prove (b). Set $\widetilde{\mathcal{N}}_\mu = \mathbf{B}^{-1}\mathcal{N}_\mu$, so that

$$\sum_\mu \|\widetilde{\mathcal{N}}_\mu\| \le \|\mathbf{B}^{-1}\| \sum_\mu \|\mathcal{N}_\mu\| = \alpha.$$

Every entry of $(\mathbf{B}^{-1}\mathcal{N})^{m_{\mathcal{A}}+1}$ is a sum of products of $m_{\mathcal{A}}+1$ monomials different from 1, and therefore

$$(\mathbf{B}^{-1}\mathcal{N})^{m_{\mathcal{A}}+1} = 0.$$

It follows that the finite geometric series

$$(\mathbf{B} - \mathcal{N})^{-1} = \sum_{p=0}^{m_{\mathcal{A}}} (\mathbf{B}^{-1}\mathcal{N})^p \mathbf{B}^{-1}$$

is the inverse. Its norm satisfies

$$\|(\mathbf{B}-\mathcal{N})^{-1}\|_1 \le \|\mathbf{B}^{-1}\| \sum_{p=0}^{m_{\mathcal{A}}} \alpha^p \le \|\mathbf{B}^{-1}\|(1+\alpha)^{m_{\mathcal{A}}},$$

which is (15.40).

Next, we prove (15.41). As formal power series,

$$\frac{\det \mathbf{B}}{\det\big(\mathbf{B}-\sum_\mu t_\mu \mathcal{N}_\mu\big)} = \det\Big(\mathrm{Id} - \sum_\mu t_\mu \widetilde{\mathcal{N}}_\mu\Big)^{-1} = \exp\Big(\sum_{l\ge 1} \frac{1}{l}\,\mathrm{Tr}\Big(\sum_\mu t_\mu \widetilde{\mathcal{N}}_\mu\Big)^l\Big).$$

A product $\widetilde{\mathcal{N}}_{\mu_1}\cdots\widetilde{\mathcal{N}}_{\mu_l}$ has rank at most m_0, and therefore its trace has modulus at most $m_0\prod_i \|\widetilde{\mathcal{N}}_{\mu_i}\|$. The coefficients of the exponent are therefore dominated in modulus by those of the series

$$-m_0 \log\Big(1-\sum_\mu \|\widetilde{\mathcal{N}}_\mu\| t_\mu\Big) = m_0 \sum_{l\ge 1}\frac{1}{l}\Big(\sum_\mu \|\widetilde{\mathcal{N}}_\mu\| t_\mu\Big)^l.$$

Since the exponential has nonnegative Taylor coefficients, the coefficients of the left side are dominated by those of $(1-\sum_\mu \|\widetilde{\mathcal{N}}_\mu\| t_\mu)^{-m_0}$. For $p \le m_{\mathcal{A}}$, the coefficients of total degree p of the latter series sum to

$$\binom{m_0+p-1}{p}\Big(\sum_\mu \|\widetilde{\mathcal{N}}_\mu\|\Big)^p \le \frac{((m_0+m_{\mathcal{A}})\alpha)^p}{p!}.$$

Summing over $p \le m_{\mathcal{A}}$ as in part (a), we obtain (15.41). □

We now apply this lemma to the onsite jets. The following bound is far from optimal near the saddle manifold, where Lemma 15.9 is sharper. However, it holds on the whole cycle. We use it to bound the first step $F_1 = w_1$ of a chain, whose norm is an integral over all of $\mathcal{S}$.

Lemma 15.13. *Let $m \ge 0$ and $\eta \in (0,1]$, and let $w = w_\Gamma[Y_1,\dots,Y_m]$ be an onsite jet. Set $A_1 = 2(4k^2+m)^2$. Then at every point of $\mathcal{S}$,*

$$|w|_{1;h} \le C v_{\mathcal{S}}^{A_1} \exp\Big[-\frac{1}{4}\|P_0\|_{\mathrm{HS}}^2 - \frac{1}{8}\|M\|_{\mathrm{HS}}^2 - \eta v_{\mathcal{S}}\Big], \tag{15.42}$$

where C depends only on k, m, and C_Γ. In particular,

$$\sup_{\mathcal{S}} |w|_{1;h} \le C\eta^{-A_1}, \qquad \|w\|_{\mathcal{H}_{-p}(\mathcal{S})} \le C\eta^{-A_1-k^2} \quad \text{for all } p \ge 0.$$

Proof. Let $S \in \mathcal{S}$. We work in the algebra $\mathcal{A}$ obtained from the even part of the Grassmann algebra generated by the transported odd coordinates (15.4) by adjoining commuting

variables $t_1, \dots, t_m$ with $t_l^2 = 0$. Its monomials are the products of h-scaled monomials of even degree with monomials in the t_l, so that $|\cdot|_1 = |\cdot|_{1;h}$ on the even Grassmann part, and $m_{\mathcal{A}} = 4k^2 + m$. The onsite jet w is the coefficient of $t_1 \cdots t_m$ in

$$\tilde{w} = e^{-\operatorname{STr}(S^2)/2 - \mathrm{i}\eta \operatorname{STr}(S\hat{\Gamma})} \operatorname{SDet}\Big(E - S + \sum_l t_l Y_l\Big)^{-1},$$

so that $|w|_{1;h} \le |\tilde{w}|_1$. We will bound the superdeterminant and the exponential factor in turn.

The BB block. Recall from (15.2) that

$$E - S_{BB} = \kappa \mathrm{Id} - P_0 + \mathrm{i}s\Lambda_B \mathrm{g}^2.$$

By (15.18) with $t = \kappa$, we have

$$E\mathrm{Id} - S_{BB} = \Lambda_B \mathrm{g} B_0 \mathrm{g}, \qquad B_0 = \kappa\Lambda_B - L + \mathrm{i}s\mathrm{Id}, \qquad L = \mathrm{g}^{-1}\Lambda_B P_0 \mathrm{g}^{-1}.$$

Therefore the BB block of $E - S + \sum_l t_l Y_l$ is

$$\mathbf{B}_t = \Lambda_B \mathrm{g}(B_0 - \mathcal{N}_B)\mathrm{g}, \qquad \mathcal{N}_B = -\sum_{l=1}^{m} t_l\, \mathrm{g}^{-1}\Lambda_B Y_l^B \mathrm{g}^{-1}.$$

Since $\kappa\Lambda_B - L$ is Hermitian, the first bound in (15.12) implies that $\|B_0^{-1}\| \le s^{-1} \le 2$. By (15.3), $\|\mathcal{N}_B\|_1 \le 3mv_{\mathcal{S}}$, and every coefficient of $\mathcal{N}_B$ has rank at most $2k$. We apply Lemma 15.12(b) with $\mathbf{B} = B_0$ and $\mathcal{N} = \mathcal{N}_B$, for which $\alpha \le 6mv_{\mathcal{S}}$, and the lemma yields

$$\Big|\frac{\det B_0}{\det(B_0 - \mathcal{N}_B)}\Big|_1 \le (1 + Cv_{\mathcal{S}})^{m_{\mathcal{A}}}, \qquad \|(B_0 - \mathcal{N}_B)^{-1}\|_1 \le 2(1 + 6mv_{\mathcal{S}})^{m_{\mathcal{A}}}. \tag{15.43}$$

The Schur complement. By (15.4), we have $S_{BF} = \Lambda_B \mathrm{g} O_1$ and $S_{FB} = O_2 \mathrm{g}$. By Lemma 3.11(a),

$$\operatorname{SDet}\Big(E - S + \sum_l t_l Y_l\Big)^{-1} = \frac{\det\big(\mathbf{F}_t - O_2 \mathrm{g} \mathbf{B}_t^{-1} \Lambda_B \mathrm{g} O_1\big)}{\det \mathbf{B}_t}, \qquad \mathbf{F}_t = \kappa\mathrm{Id} - \mathrm{i}M + \sum_l t_l Y_l^F.$$

Here

$$\mathrm{g}\mathbf{B}_t^{-1}\Lambda_B \mathrm{g} = (B_0 - \mathcal{N}_B)^{-1}, \qquad \det \mathbf{B}_t = \det(\Lambda_B)\det(B_0 - \mathcal{N}_B),$$

where we used $\det \mathrm{g} = 1$. Thus the numerator is the determinant of $\kappa\mathrm{Id} - \mathrm{i}M + \mathcal{N}_F$, where

$$\mathcal{N}_F = \sum_l t_l Y_l^F - O_2(B_0 - \mathcal{N}_B)^{-1} O_1$$

has no constant term.

Each odd term of $\mathcal{N}_F$ is the product of two transported odd coordinates, each of norm h, with a matrix of norm at most $\|(B_0 - \mathcal{N}_B)^{-1}\|_1$. By (15.43), we have

$$\|\mathcal{N}_F\|_1 \le m + (2k)^4 h^2 \|(B_0 - \mathcal{N}_B)^{-1}\|_1 \le C(1 + v_{\mathcal{S}})^{m_{\mathcal{A}}}.$$

We apply Lemma 15.12(a) to the matrix

$$M + \mathrm{i}\kappa\mathrm{Id} + \mathrm{i}\mathcal{N}_F = \mathrm{i}(\kappa\mathrm{Id} - \mathrm{i}M + \mathcal{N}_F),$$

with $A = M$, $Y = \kappa\mathrm{Id}$, and $Y_+ = \mathrm{Id}$. Writing $\mu_1, \dots, \mu_{2k}$ for the eigenvalues of M, we find

$$\big|\det(\kappa\mathrm{Id} - \mathrm{i}M + \mathcal{N}_F)\big|_1 \le |\det(M + \mathrm{i}\mathrm{Id})|\big(1 + 2k\|\mathcal{N}_F\|_1\big)^{m_{\mathcal{A}}} \le C v_{\mathcal{S}}^{m_{\mathcal{A}}^2} \prod_{j=1}^{2k} (1 + \mu_j^2)^{1/2}.$$

Combining this with (15.43) and $|\det B_0| = |\det(E - S_{BB})|$, we obtain

$$\Big|\mathrm{SDet}\Big(E - S + \sum_l t_l Y_l\Big)^{-1}\Big|_1 \le \frac{C v_{\mathcal{S}}^{A_1}}{|\det(E - S_{BB})|} \prod_{j=1}^{2k} (1 + \mu_j^2)^{1/2}. \tag{15.44}$$

The exponential factor. We have

$$\begin{aligned} \mathrm{STr}(S^2) &= \mathrm{Tr}\, S_{BB}^2 - \mathrm{Tr}\, S_{FF}^2 + 2\,\mathrm{Tr}(S_{BF} S_{FB}), \\ \mathrm{Tr}(S_{BF} S_{FB}) &= \mathrm{Tr}(\Lambda_B \mathrm{g} O_1 O_2 \mathrm{g}) = \mathrm{Tr}(\Lambda_B O_1 O_2), \end{aligned} \tag{15.45}$$

where we used $\mathrm{g}\Lambda_B\mathrm{g} = \Lambda_B$ in the last identity. The odd trace has norm at most $4k^2h^2 \le 4k^2$, and Lemma 5.13(e) shows that the odd factor $e^{-\mathrm{Tr}(S_{BF}S_{FB})}$ has norm at most e^{4k^2}. The damping involves only the even blocks of S, since $\hat{\Gamma}$ has zero odd blocks.

We now bound the body. The computation in the proof of Lemma 15.5 gives

$$\frac{|e^{-\mathrm{Tr}\, S_{BB}^2/2}|}{|\det(E - S_{BB})|} = e^{-V_\kappa(P_0) - k(\kappa^2 - s^2)}, \qquad |e^{\mathrm{Tr}\, S_{FF}^2/2}| = e^{k\kappa^2 - \mathrm{Tr}\, M^2/2}.$$

For the damping, we have

$$\mathrm{STr}(S\hat{\Gamma}) = \mathrm{Tr}(S_{BB}\Lambda_B) - \mathrm{Tr}(S_{FF}\hat{\Gamma}_F), \qquad \mathrm{Tr}(\Lambda_B \mathrm{g}^2 \Lambda_B) = \mathrm{Tr}\,\mathrm{g}^2 = s^{-1}(v_{\mathcal{S}} - 1),$$

where we used (15.3). Since $\mathrm{Tr}(P_0\Lambda_B)$ is real, we find

$$\mathrm{Re}\big(-\mathrm{i}\eta\,\mathrm{Tr}(S_{BB}\Lambda_B)\big) = -\eta(v_{\mathcal{S}} - 1), \qquad |\eta\,\mathrm{Tr}(S_{FF}\hat{\Gamma}_F)| \le \eta C_\Gamma(2k + \|M\|_1).$$

Using these bounds, (15.44), $\eta \le 1$, and Lemma 15.5 in the form $V_\kappa(P_0) \ge \frac{1}{4}\|P_0\|_{\mathrm{HS}}^2$, we conclude that

$$|\tilde{w}|_1 \le C v_{\mathcal{S}}^{A_1} \exp\Big[-\frac{1}{4}\|P_0\|_{\mathrm{HS}}^2 - \eta v_{\mathcal{S}} + C_\Gamma \|M\|_1\Big] \prod_{j=1}^{2k} e^{(s^2 - \mu_j^2)/2} (1 + \mu_j^2)^{1/2}.$$

Finally, for real μ we have

$$e^{-\mu^2/2+C_\Gamma|\mu|}(1+\mu^2)^{1/2} \le Ce^{-\mu^2/8},$$

and (15.42) follows.

The supremum bound follows from (15.42) and

$$\sup_{v_{\mathcal{S}}\ge 1} v_{\mathcal{S}}^{A_1} e^{-\eta v_{\mathcal{S}}} \le \Big(\frac{A_1}{e\eta}\Big)^{A_1}.$$

To bound the norm, we use $|w|_{2;h} \le |w|_{1;h}$ and $v_{\mathcal{S}}^{-2p} \le 1$, we integrate the Gaussian factors in P_0 and M, and we use $v_{\mathcal{S}} \ge 1 + \|X\|_{\mathrm{HS}}$, which follows from Lemma 4.7(a). Since X ranges over a real vector space of dimension $2k^2$, we obtain

$$\|w\|^2_{\mathcal{H}_{-p}(\mathcal{S})} \le C\int v_{\mathcal{S}}^{2A_1} e^{-2\eta v_{\mathcal{S}}}\,\mathrm{d}X \le C\eta^{-2A_1-2k^2}. \qquad \square$$

15.6 The a priori bound on partial chains

We now state the main result of this section. Recall that $r_W = h\log W$, as in (15.1). For a constant $C_1 > 0$, we define

$$\mathcal{E}_W(\eta) = C_1\Big(\sqrt{W} + r_W W\big(1+\log(1+\eta^{-1})\big) + \eta W + N\eta^2\Big). \tag{15.46}$$

Theorem 15.14. *Fix integers $k \ge 1$ and $d \ge 0$, and fix $C_\Gamma, C_X \ge 1$. There exist p, A, C_1 and W_0, depending only on k, d, C_Γ, and C_X, such that the following holds for all $W \ge W_0$. Let $N \le C_X W^2$, $|E| \le 1/2$, and $0 < \eta \le C_X W^{-2}$, and let $\hat\Gamma$ be a normalized source as in* (15.6)*, with $\operatorname{Tr}\hat\Gamma_F = 0$ and $\|\hat\Gamma_F\| \le C_\Gamma$. Then every marked partial chain F_j of length $1 \le j \le N$ and total order at most d satisfies*

$$\|F_j\|_{\mathcal{H}_{-p}(\mathcal{S})} + \|K_{\tau_{\mathrm{phys}}}F_j\|_{\mathcal{H}_{-p}(\mathcal{S})} \le W^A(1+\eta^{-1})^A e^{\mathcal{E}_W(\eta)}, \tag{15.47}$$

uniformly in the positions and the directions of the marks.

We have $r_W W = \sqrt{W}\log W$ and $N\eta^2 \le C_X^3 W^{-2}$. If moreover $\eta \ge W^{-C}$ for some constant C, then $\log(1+\eta^{-1}) \le (C+1)\log W$, and the right side of (15.47) is at most $\exp(C\sqrt{W}(\log W)^2) = e^{o(W)}$. This is the case in the applications of Section 20, where $\eta = \varepsilon/a_N \asymp W^{-2}$, so that $(1+\eta^{-1})^A$ is a power of W. Every marked chain in the analytic transpose order of Definition 4.9 is of the form $K_{\tau_{\mathrm{phys}}}F_j$, so the theorem also covers that order.

The proof combines two estimates. The first is a bound on the coefficients of the anchored marked chains on the *core*

$$\mathcal{K}_W = \big\{S \in \mathcal{S} : \|P_0\|_{\mathrm{HS}} \le r_W,\ \big||\mu| - s\big| \le r_W \text{ for every eigenvalue } \mu \text{ of } M\big\}, \tag{15.48}$$

which contains a neighborhood of the BB saddle orbit in every FF signature of the saddle manifold. The second is the contraction of one step away from the core. In the proofs below we assume that W is so large that $\eta \le C_X W^{-2} \le s/2$ and $r_W \le 1/2$.

Local sources in the anchored identity

To bound marked chains on the core, we need the anchored identity (15.11) with local sources inserted, which is the second part of Proposition 4.13. We restate it in the form used below. Let F_n be a marked partial chain whose marked step j_a carries the directions $Y_{a,1}, \dots, Y_{a,m_a}$, and let $i_a = n + 1 - j_a$ be the corresponding site of the anchored chain. We introduce commuting variables $t_{a,l}$ with $t_{a,l}^2 = 0$, and we set

$$\mathbb{M}_n^t(S) = \mathbb{M}_n(S) + \sum_{a,l} t_{a,l}\, e_{i_a} e_{i_a}^* \otimes Y_{a,l}, \tag{15.49}$$

where e_i is the ith standard basis vector of $\mathbb{C}^n$.

Lemma 15.15. *Let $n \ge 1$, let F_n be a marked partial chain, and let $S \in \mathcal{S}$. Then $(K_{\tau_{\mathrm{phys}}} F_n)(S)$ is the coefficient of $\prod_{a,l} t_{a,l}$ in*

$$\mathcal{G}_n(S)\, \mathbb{E}\, \mathrm{SDet}\big[\mathbb{M}_n^t(S)\big]^{-1}, \tag{15.50}$$

where the expectation is taken coefficientwise.

Proof. For a complex vector $\zeta = (\zeta_{a,l})$, let $\mathbb{M}_n^\zeta(S)$ be defined by (15.49) with ζ in place of t. The function $K_{\tau_{\mathrm{phys}}} F_n$ is the function $\ell_n^{(\alpha)}$ of Proposition 4.13, for the derivatives $\partial_{\zeta_{a,1}} \cdots \partial_{\zeta_{a,m_a}}$ at $\zeta = 0$ in the directions $Y_{a,1}, \dots, Y_{a,m_a}$ at the site i_a, which are linear combinations of the derivatives in the entries of Y_{i_a}. By Proposition 4.13, we therefore have

$$(K_{\tau_{\mathrm{phys}}} F_n)(S) = \partial_\zeta^{\mathbf{1}} \Big[\mathcal{G}_n(S)\, \mathbb{E}\, \mathrm{SDet}\big[\mathbb{M}_n^\zeta(S)\big]^{-1} \Big]\Big|_{\zeta = 0},$$

where $\partial_\zeta^{\mathbf{1}}$ is the mixed derivative of first order in each $\zeta_{a,l}$.

It remains to identify this derivative with the coefficient of $\prod t_{a,l}$ in (15.50). After multiplication by $\mathrm{Id}_n \otimes \Lambda_B$, the BB block of $\mathbb{M}_n(S)$ has imaginary part at least $\min_i \eta(1 - \varsigma_i) > 0$, uniformly in $H^{(n)}$, by the remark after Proposition 4.13. Consider the polydisk

$$\mathcal{D}_\zeta = \Big\{ \zeta : \sum_l |\zeta_{a,l}| \le \tfrac{1}{2} \min_i \eta(1 - \varsigma_i) \text{ for all } a \Big\}.$$

The local BB sources keep half of this margin on $\mathcal{D}_\zeta$. There, $\mathrm{SDet}[\mathbb{M}_n^\zeta(S)]^{-1}$ is holomorphic in ζ for every $H^{(n)}$, the inverse of its BB block is bounded uniformly in $H^{(n)}$, and its coefficients are bounded by a polynomial in $\|H^{(n)}\|$. Dominated convergence therefore allows us to differentiate under the expectation. Finally, for a function holomorphic near $\zeta = 0$, the mixed derivative $\partial_\zeta^{\mathbf{1}}$ at zero is the coefficient of $\prod t_{a,l}$ in its Taylor expansion evaluated at $\zeta = t$, since $t_{a,l}^2 = 0$. This completes the proof. □

The bound on the core

Lemma 15.16. *In the setting of Theorem 15.14, there exist A, p_0, and C_1, depending only on k, d, C_Γ, and C_X, such that for all marked partial chains F_n with $0 \le n \le N-1$ and total order at most d, and all $S \in \mathcal{K}_W$,*

$$|(K_{\tau_{\mathrm{phys}}} F_n)(S)|_{1;h} \le W^A (1+\eta^{-1})^A v_{\mathcal{S}}^{p_0} e^{\mathcal{E}_W(\eta)}, \tag{15.51}$$

with $\mathcal{E}_W(\eta)$ given by (15.46).

Proof. For $n = 0$ we have $K_{\tau_{\mathrm{phys}}} F_0 = 1$, and there is nothing to prove. Let $n \ge 1$ and $S \in \mathcal{K}_W$. Let $\mathcal{A}$ be the algebra of the proof of Lemma 15.13, with the variables $t_{a,l}$ of (15.49) in place of the t_l, so that $m_{\mathcal{A}} \le 4k^2 + d$. By Lemma 15.15, it suffices to bound the norm $|\cdot|_1$ in $\mathcal{A}$ of (15.50). Since the expectation is taken coefficientwise, this norm is at most

$$|\mathcal{G}_n(S)|_{1;h}\, \mathbb{E}\big|\mathrm{SDet}[\mathbb{M}_n^t(S)]^{-1}\big|_1.$$

We bound the second factor in four steps, and the first factor at the end. Throughout the proof, we write $D_\varsigma = \eta(\mathrm{Id}_n - \varsigma)$, and we recall that $\|D_\varsigma\| \le \eta$.

The blocks. Using (15.2) and $\hat{\Gamma}_{BB} = \Lambda_B$, we see that the BB block of $\mathbb{M}_n(S)$ is the matrix $\mathbf{B}$ of (15.22), with

$$A = E\mathrm{Id}_n - \kappa\varsigma - H^{(n)}, \qquad Z = P_0, \qquad s_0 = s, \qquad r = r_W. \tag{15.52}$$

We write the FF block of $\mathbb{M}_n(S)$ as $\mathbf{F} = X_F + \mathrm{i}Y_F$, with the Hermitian matrices

$$X_F = A \otimes \mathrm{Id} + D_\varsigma \otimes \gamma_1, \qquad Y_F = D_\varsigma \otimes \gamma_2 - \varsigma \otimes M,$$

where γ_1 and γ_2 are as in (15.7), and the odd blocks of $\mathbb{M}_n(S)$ are given by $-\varsigma \otimes S_{BF}$ and $-\varsigma \otimes S_{FB}$. In $\mathbb{M}_n^t(S)$, the blocks $\mathbf{B}$ and $\mathbf{F}$ are replaced by

$$\mathbf{B}_t = \mathbf{B} + \sum_{a,l} t_{a,l}\, e_{i_a} e_{i_a}^* \otimes Y_{a,l}^B, \qquad \mathbf{F}_t = \mathbf{F} + \sum_{a,l} t_{a,l}\, e_{i_a} e_{i_a}^* \otimes Y_{a,l}^F.$$

By Lemma 15.6 and (15.3), $\mathbf{B}$ is invertible, uniformly in $H^{(n)}$, with

$$\|\mathbf{B}^{-1}\| \le \max\{\eta^{-1}, 3s^{-1}v_{\mathcal{S}}\} \le 4v_{\mathcal{S}}/\eta. \tag{15.53}$$

Odd coefficients and marks. We apply Lemma 15.12(b) to $\mathbf{B}_t = \mathbf{B} - \mathcal{N}_B$, where $\|\mathcal{N}_B\|_1 \le d$ and every coefficient of $\mathcal{N}_B$ has rank at most $2k$. By (15.53), we have $\alpha \le 4dv_{\mathcal{S}}/\eta$, and the lemma yields

$$\Big|\frac{\det \mathbf{B}}{\det \mathbf{B}_t}\Big|_1 \le \Big(1 + \frac{Cv_{\mathcal{S}}}{\eta}\Big)^{m_{\mathcal{A}}}, \qquad \|\mathbf{B}_t^{-1}\|_1 \le \frac{4v_{\mathcal{S}}}{\eta}\Big(1 + \frac{4dv_{\mathcal{S}}}{\eta}\Big)^{m_{\mathcal{A}}}. \tag{15.54}$$

By (15.4), we may write the odd blocks as

$$-\varsigma \otimes S_{BF} = -(\mathrm{Id}_n \otimes \Lambda_B \mathfrak{g})(\varsigma \otimes O_1), \qquad -\varsigma \otimes S_{FB} = -(\varsigma \otimes O_2)(\mathrm{Id}_n \otimes \mathfrak{g}).$$

Then Lemma 3.11(a) yields

$$\mathrm{SDet}\big[\mathbb{M}_n^t(S)\big]^{-1} = \frac{\det(\mathbf{F} + \mathcal{N}_F)}{\det \mathbf{B}_t},$$

where

$$\mathcal{N}_F = \mathbf{F}_t - \mathbf{F} - (\varsigma \otimes O_2)(\mathrm{Id}_n \otimes \mathfrak{g})\mathbf{B}_t^{-1}(\mathrm{Id}_n \otimes \Lambda_B \mathfrak{g})(\varsigma \otimes O_1)$$

has no constant term. Its odd part is a sum of $(2k)^4$ terms, each the product of two transported odd coordinates with a matrix of norm at most $\|\mathfrak{g}\|^2 \|\mathbf{B}_t^{-1}\|_1$. The two coordinates contribute $h^2 \le 1$, and (15.3) implies that

$$\|\mathfrak{g}\|^2 = \|\mathfrak{g}^2\| \le 3v_{\mathcal{S}}.$$

Using (15.54), we find

$$\|\mathcal{N}_F\|_1 \le d + (2k)^4 \cdot 3v_{\mathcal{S}} \cdot \frac{4v_{\mathcal{S}}}{\eta}\Big(1 + \frac{4dv_{\mathcal{S}}}{\eta}\Big)^{m_{\mathcal{A}}} \le C\Big(\frac{v_{\mathcal{S}}}{\eta}\Big)^{m_{\mathcal{A}}+2}. \tag{15.55}$$

The fermionic determinant. Let $c_+ = \max(1, \|\gamma_2\|)$, and set

$$Y_+ = c_+ D_\varsigma + (s + r_W)\varsigma.$$

Every eigenvalue of M has modulus at most $s + r_W$ on $\mathcal{K}_W$. Using also $\eta \le s$, we find

$$-Y_+ \otimes \mathrm{Id} \le Y_F \le Y_+ \otimes \mathrm{Id}, \qquad Y_+ \ge D_\varsigma + s\varsigma \ge \min(\eta, s)\mathrm{Id} = \eta \mathrm{Id}.$$

We apply Lemma 15.12(a) to $X_F + \mathrm{i}Y_F + \mathcal{N}_F$, with the envelope $Y_+ \otimes \mathrm{Id}$ and the matrix size $2kn \le 2kC_X W^2$. Using (15.55), we find

$$|\det(\mathbf{F} + \mathcal{N}_F)|_1 \le \big|\det(X_F + \mathrm{i}Y_+ \otimes \mathrm{Id})\big|\Big(1 + CW^2\Big(\frac{v_{\mathcal{S}}}{\eta}\Big)^{m_{\mathcal{A}}+3}\Big)^{m_{\mathcal{A}}}.$$

We diagonalize γ_1 by a unitary in the flavor space, with eigenvalues $u_1, \dots, u_{2k}$, which satisfy

$$|u_a| \le C_\Gamma, \qquad \sum_{a=1}^{2k} u_a = \mathrm{Tr}\,\gamma_1 = 0.$$

In this basis, we have

$$\big|\det(X_F + \mathrm{i}Y_+ \otimes \mathrm{Id})\big| = \prod_{a=1}^{2k} |\det(A + u_a D_\varsigma + \mathrm{i}Y_+)|. \tag{15.56}$$

Comparison of the determinants. We compare each factor in (15.56) with $|\det(A+\mathrm{i}(D_\varsigma + s\varsigma))|$, which appears in (15.23), in three steps. First, for $u \in \mathbb{R}$, set

$$\ell(u) = \log|\det(A + uD_\varsigma + \mathrm{i}Y_+)|, \qquad R_u = (A + uD_\varsigma + \mathrm{i}Y_+)^{-1}.$$

Since $\|D_\varsigma\| \le \eta$ and $\|Y_+^{-1}\| \le \eta^{-1}$, the second bound in (15.14) implies that

$$|\ell''(u)| \le \|D_\varsigma\|^2 \|Y_+^{-1}\| \operatorname{Tr}(-\operatorname{Im} R_u) \le \eta \operatorname{Tr}(-\operatorname{Im} R_u), \qquad u \in \mathbb{R}.$$

Using Taylor's formula and the tracelessness of γ_1, we find

$$\sum_{a=1}^{2k} \ell(u_a) - 2k\ell(0) = \sum_{a=1}^{2k} u_a^2 \int_0^1 (1-x)\ell''(xu_a)\,\mathrm{d}x \le C_\Gamma^2 \eta \sum_{a=1}^{2k} \int_0^1 \operatorname{Tr}(-\operatorname{Im} R_{xu_a})\,\mathrm{d}x.$$

This is the step that uses the tracelessness of the FF source. Without it, the left side would contain the linear term $\ell'(0)\sum_a u_a$, which (15.14) bounds only by $C\eta \operatorname{Tr}(Y_+^{-1}) \le Cn$, and the resulting factor e^{Cn} could be as large as e^{CW^2}.

Second, for $1 \le \tau \le c_+$, set

$$R'_\tau = \big(A + \mathrm{i}(\tau D_\varsigma + (s + r_W)\varsigma)\big)^{-1}.$$

By (15.13), we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \log\big|\det\big(A + \mathrm{i}(\tau D_\varsigma + (s+r_W)\varsigma)\big)\big| = \operatorname{Tr}\big(D_\varsigma(-\operatorname{Im} R'_\tau)\big) \le \eta \operatorname{Tr}(-\operatorname{Im} R'_\tau).$$

Integrating from 1 to c_+, we obtain

$$\ell(0) \le \log|\det(A + \mathrm{i}(D_\varsigma + (s+r_W)\varsigma))| + \eta \int_1^{c_+} \operatorname{Tr}(-\operatorname{Im} R'_\tau)\,\mathrm{d}\tau.$$

Third, the upper bound in (15.13) shows that the derivative in $\sigma \ge s$ of $\log|\det(A + \mathrm{i}(D_\varsigma + \sigma\varsigma))|$ is bounded by

$$\operatorname{Tr}\big(\varsigma(D_\varsigma + \sigma\varsigma)^{-1}\big) \le \operatorname{Tr}\big(\varsigma(D_\varsigma + s\varsigma)^{-1}\big) = \Pi_n,$$

where Π_n is as in Lemma 15.11(c) with $s_0 = s$. Integrating from s to $s + r_W$, we find

$$\log|\det(A + \mathrm{i}(D_\varsigma + (s+r_W)\varsigma))| \le \log|\det(A + \mathrm{i}(D_\varsigma + s\varsigma))| + r_W\Pi_n.$$

Combining the three steps with (15.56) and (15.23), we find

$$\begin{aligned} \frac{|\det(X_F + \mathrm{i}Y_+ \otimes \mathrm{Id})|}{|\det \mathbf{B}|} &\le e^{4kr_W\Pi_n} e^{\eta \mathcal{T}_W}, \\ \mathcal{T}_W &= 2k \int_1^{c_+} \operatorname{Tr}(-\operatorname{Im} R'_\tau)\,\mathrm{d}\tau + C_\Gamma^2 \sum_{a=1}^{2k} \int_0^1 \operatorname{Tr}(-\operatorname{Im} R_{xu_a})\,\mathrm{d}x. \end{aligned} \tag{15.57}$$

Expectation. The quantity $\mathcal{T}_W$ is a sum of $2k+1$ integrals, over parameter intervals of length at most c_+, of $\lambda \operatorname{Tr}(-\operatorname{Im} R)$ with $\lambda \le 2k + C_\Gamma^2$. Each resolvent R here is of the form of Lemma 15.10, with a deterministic real diagonal shift $Z_0 = uD_\varsigma$ or $Z_0 = 0$ and a deterministic diagonal $Y \ge \eta\mathrm{Id}$. We use Hölder's inequality for the $2k+1$ terms, Jensen's inequality for the normalized parameter integrals, and Fubini's theorem. They show that $\mathbb{E}e^{\eta\mathcal{T}_W}$ is bounded by the supremum over the parameter of $\mathbb{E}e^{t_*\eta \operatorname{Tr}(-\operatorname{Im} R)}$, where

$$t_* = (2k+1)(2k + C_\Gamma^2)c_+.$$

The diagonal entries of $A = E\mathrm{Id}_n - \kappa\varsigma - H^{(n)}$ have densities bounded by the d_i of Lemma 15.11(b). Applying Lemma 15.10 with $c = 1$ and $t = t_*$, and using Lemma 15.11(b), $n \le N \le C_X W^2$, and $\eta \le C_X W^{-2}$, we obtain

$$\mathbb{E}e^{\eta\mathcal{T}_W} \le \exp\Big(\pi(e^{t_*}-1)\eta \sum_{i=1}^n \mathsf{d}_i\Big) \le \exp\big(C\eta(W^{3/2} + N\sqrt{W})\big) \le e^{C\sqrt{W}}.$$

By Lemma 15.11(c), we also have

$$4kr_W\Pi_n \le Cr_W W\big(1 + \log(1+\eta^{-1})\big).$$

Combining these bounds with (15.54) and the bound on the fermionic determinant, we conclude that

$$\mathbb{E}\big|\operatorname{SDet}\big[\mathbb{M}_n^t(S)\big]^{-1}\big|_1 \le CW^{2m_{\mathcal{A}}}\Big(\frac{v_{\mathcal{S}}}{\eta}\Big)^{Cm_{\mathcal{A}}^2} \exp\Big(C\sqrt{W} + Cr_W W\big(1+\log(1+\eta^{-1})\big)\Big). \tag{15.58}$$

The Gaussian factor. By (15.45), the odd part of the exponent of (15.10) is the product of $-\sigma_n$ with $\operatorname{Tr}(\Lambda_B O_1 O_2)$. Using Lemma 15.11(a), we bound its norm by

$$\sigma_n(2k)^2h^2 \le 4W \cdot 4k^2W^{-1} = 16k^2.$$

For the body, Lemma 4.7(c) with $P = \kappa\mathrm{Id} + P_0$ shows that

$$\operatorname{Re}\operatorname{STr}(S^2) = \operatorname{Tr}P_0^2 + 2\kappa\operatorname{Tr}P_0 + \operatorname{Tr}M^2 - 2ks^2 \qquad \text{on the body of } \mathcal{S}.$$

On $\mathcal{K}_W$, every eigenvalue μ of M and the trace of P_0 satisfy

$$s^2 - \mu^2 = (s-|\mu|)(s+|\mu|) \le 3r_W, \qquad |\operatorname{Tr}P_0| \le \sqrt{2k}\|P_0\|_{\mathrm{HS}} \le \sqrt{2k}\,r_W.$$

Using also $\sigma_n \le 4W$, we find

$$-\tfrac{1}{2}\sigma_n \operatorname{Re}\operatorname{STr}(S^2) \le C\sigma_n r_W \le CWr_W \qquad \text{on } \mathcal{K}_W.$$

The computation of the damping in the proof of Lemma 15.13 and the bound $|\operatorname{STr}(\hat\Gamma^2)| \le C$ give

$$\operatorname{Re}\big(-\mathrm{i}\eta\sigma_n\operatorname{STr}(S\hat\Gamma)\big) \le \eta\sigma_n C_\Gamma(2k + \|M\|_1) \le C\eta W, \qquad \frac{\eta^2(n-\sigma_n)}{2}|\operatorname{STr}(\hat\Gamma^2)| \le CN\eta^2$$

on $\mathcal{K}_W$. By Lemma 5.13(e), we conclude that

$$|\mathcal{G}_n(S)|_{1;h} \le \exp\bigl(16k^2 + CWr_W + C\eta W + CN\eta^2\bigr) \qquad \text{for } S \in \mathcal{K}_W.$$

Combining this bound with (15.58), we conclude that (15.51) holds, where C_1 is the largest of the constants in the exponents. □

Proof of the theorem

Proof of Theorem 15.14. Let $\|\cdot\|$ denote the norm of $\mathcal{H}_{-p}(\mathcal{S})$, where p will be chosen below, and let $\mathbf{1}_{\mathcal{K}_W}$ and $\mathbf{1}_{\mathcal{K}_W^c}$ denote the indicator functions of $\mathcal{K}_W$ and of its complement. Set

$$a_j = \|\mathbf{1}_{\mathcal{K}_W^c} F_j\|, \qquad b_j = \|\mathbf{1}_{\mathcal{K}_W} F_j\|, \qquad 1 \le j \le N.$$

Let w_j be the onsite weight of the step j in (15.8), so that

$$F_j = w_j K_{\tau_{\text{phys}}} F_{j-1}, \qquad 1 \le j \le N.$$

Let A_1 be the largest of the constants $A_1 = 2(4k^2 + m)^2$ of Lemma 15.13 for $m \le d$. By that lemma, there exists a constant C_2 such that

$$\sup_{\mathcal{S}} |w|_{1;h} \le Q_m, \qquad Q_m = C_2(1 + \eta^{-1})^{A_1}, \tag{15.59}$$

for all onsite jets w of order at most d, including the unmarked weight w_Γ.

The core. By Lemma 5.13(a), we have the pointwise bound

$$|F_j|_{1;h} \le |w_j|_{1;h}\, |K_{\tau_{\text{phys}}} F_{j-1}|_{1;h}.$$

Using (15.59) and Lemma 15.16, we find

$$|F_j|_{1;h} \le Q_m W^A (1 + \eta^{-1})^A v_{\mathcal{S}}^{p_0} e^{\mathcal{E}_W(\eta)} \qquad \text{on } \mathcal{K}_W.$$

The coordinates P_0 and M range over bounded sets on $\mathcal{K}_W$, while X ranges over all off-diagonal Hermitian matrices. We choose $p = p_0 + k^2 + 1$. Using $|F_j|_{2;h} \le |F_j|_{1;h}$ and $v_{\mathcal{S}} \ge 1 + \|X\|_{\text{HS}}$, we find

$$b_j \le Q_m W^A (1 + \eta^{-1})^A e^{\mathcal{E}_W(\eta)} \Bigl(C \int v_{\mathcal{S}}^{2p_0 - 2p}\, \mathrm{d}X\Bigr)^{1/2} \le C Q_m W^A (1 + \eta^{-1})^A e^{\mathcal{E}_W(\eta)}. \tag{15.60}$$

The far region. Let $S \notin \mathcal{K}_W$. Then either $\|P_0\|_{\text{HS}} > r_W$, or M has an eigenvalue μ with $||\mu| - s| > r_W$. We claim that the escape function $\vartheta = \vartheta_B + \vartheta_F$ of (15.32) satisfies $\vartheta(S) \ge c_\vartheta r_W^2$ in both cases, once $r_W \le r_{\text{esc}}$, where $c_\vartheta > 0$ depends only on the choice of χ_0 in (15.32). Both ϑ_B and ϑ_F are nonnegative.

In the first case, recall that the physical normal $\widetilde{P} = \mathrm{ch}_{\mathrm{g}}\, P_0$ of (15.26) satisfies $\|\widetilde{P}\|_{\text{HS}} \ge \|P_0\|_{\text{HS}}$. Recall also the lower bound in (15.31), which holds since $\chi_0(0) = 0$ and $\chi_0'(0) > 0$.

We let $c_0 > 0$ denote the constant c of that bound, so that $\chi_0(x) \geq c_0 \min(x, 1)$ for all $x \geq 0$. Using this bound and the definition of ϑ_B in (15.32), we find

$$\vartheta(S) \geq \vartheta_B(S) = r_{\mathrm{esc}}^2 \chi_0\big(\|\widetilde{P}\|_{\mathrm{HS}}^2 / r_{\mathrm{esc}}^2\big) \geq c_0 \min\big(\|\widetilde{P}\|_{\mathrm{HS}}^2, r_{\mathrm{esc}}^2\big) \geq c_0 r_W^2.$$

In the second case, ϑ_F is the sum of the nonnegative terms $(\mu'^2 - s^2)^2/(1+\mu'^2)$ over the eigenvalues μ' of M, and for the given μ,

$$\frac{(\mu^2 - s^2)^2}{1+\mu^2} = (|\mu| - s)^2 \, \frac{(|\mu| + s)^2}{1+\mu^2} \geq s^2 r_W^2 \geq \tfrac{1}{2} r_W^2,$$

where we used $(x+s)^2 \geq s^2(1+x^2)$ for $x \geq 0$ and $s \leq 1$. This proves the claim, with $c_\vartheta = \min(c_0, 1/2)$.

We now bound one unmarked step on the far region. We apply Lemma 15.9 with $\Gamma = \hat{\Gamma}$, for which $\hat{\Gamma}_{BB} = \Lambda_B$ and $\|\hat{\Gamma}_{FF}\| \leq C_\Gamma$. Since $\vartheta \geq c_\vartheta r_W^2$ off the core, the lemma implies that

$$|w_\Gamma|_{1;h} \leq e^{C/W + C\eta - c c_\vartheta r_W^2} \qquad \text{on the complement of } \mathcal{K}_W.$$

By Lemma 15.7 with the weight $-p$, we have $\|K_{\tau_{\mathrm{phys}}}\| \leq e^{C/W}$ on $\mathcal{H}_{-p}(\mathcal{S})$.

Recall that $r_W^2 = (\log W)^2/W$ and $\eta \leq C_X W^{-2}$. Therefore, for all sufficiently large W,

$$\|\mathbf{1}_{\mathcal{K}_W^c} M_{w_\Gamma} K_{\tau_{\mathrm{phys}}}\| \leq q_W, \qquad q_W = \exp\big[-c_2 (\log W)^2 / W\big], \tag{15.61}$$

with $c_2 = c c_\vartheta / 2$. Note that the indicator acts on the output of the step, after the heat.

At a marked step j, we use (15.59) and Lemma 15.7 instead, and we find

$$\|\mathbf{1}_{\mathcal{K}_W^c} M_{w_j} K_{\tau_{\mathrm{phys}}}\| \leq 2Q_m.$$

The recurrence. Set $q_j = q_W$ if the step j is unmarked and $q_j = 2Q_m$ if it is marked; at most d steps are marked. Using $\|F_j\| \leq a_j + b_j$ and

$$\mathbf{1}_{\mathcal{K}_W^c} F_{j+1} = \mathbf{1}_{\mathcal{K}_W^c} M_{w_{j+1}} K_{\tau_{\mathrm{phys}}} F_j,$$

we obtain

$$a_{j+1} \leq q_{j+1}(a_j + b_j), \qquad 1 \leq j \leq N-1.$$

Iterating this inequality, we find

$$a_j \leq \Big(\prod_{l=2}^{j} q_l\Big) a_1 + \sum_{i=1}^{j-1} \Big(\prod_{l=i+1}^{j} q_l\Big) b_i.$$

Each product contains at most d marked factors, and all its other factors equal q_W. Using $q_W^{-d} \leq 2$ for large W and summing the geometric series, we find

$$a_j \leq 2(2Q_m)^d \Big[a_1 + (1-q_W)^{-1} \max_{i<j} b_i\Big], \qquad (1-q_W)^{-1} \leq CW. \tag{15.62}$$

The recurrence starts at $j = 1$, since $F_0 = 1$ has no finite norm in $\mathcal{H}_{-p}(\mathcal{S})$. Instead, $F_1 = w_1$ is an onsite jet, and Lemma 15.13 implies that

$$a_1 \leq \|w_1\| \leq C\eta^{-A_1-k^2}.$$

Conclusion. Combining (15.60) and (15.62), and recalling (15.59), we obtain (15.47) for $\|F_j\| \leq a_j + b_j$, with a larger constant A. By Lemma 15.7, the final heat in $K_{\tau_{\text{phys}}} F_j$ multiplies the norm by at most $e^{C/W} \leq 2$. All the estimates above are uniform in the positions and the directions of the marks, and so is the bound. □

SECTION 16

The normal tube and the normal oscillator

This section and the next prove that the transfer operator of the supersymmetric representation is stable over the $N \asymp W^2$ steps of the chain, uniformly in W, when it is written in coordinates adapted to its saddle manifold. In this section we set up these coordinates and study the frozen normal transfer, the normal part of the transfer operator, which is a supersymmetric harmonic oscillator. In Section 17 we compare the full transfer operator with a product of this oscillator and a heat semigroup on the saddle manifold. The results are used in three places. The stability theorem of Section 17, the fixed state of the normal oscillator (Proposition 16.22), and the identification of the heat generator on the saddle manifold with a multiple of the Casimir operator in Lemma 16.5, which fixes the time normalization of one link, are the inputs of the averaging argument of Section 18, and the endpoint and assembly arguments of Sections 19 and 20 use the coordinates and the gauge of Section 16.5. The transfer-operator approach to one-dimensional band matrices in the supersymmetric representation goes back to M. Shcherbina and T. Shcherbina (see [118] and the introduction).

Throughout Part IV we write

$$h = W^{-1/2}, \qquad \delta_W = W^{-2} = h^4,$$

as in Section 15. The reader who knows Ornstein–Uhlenbeck semigroups but not superanalysis may read this section as follows. The normal variables are Gaussian at scale h, and in standardized coordinates the frozen normal transfer is an Ornstein–Uhlenbeck operator at a small complex time of order h^2, tensored with a finite fermionic analogue. Every estimate below is an estimate for this operator and for its perturbations by multipliers, cutoffs, and first- and second-order differential operators. The fermionic variables enter only through finite-dimensional linear algebra, which we carry out before taking any norm. Grassmann variables, Berezin integrals, supermatrices, STr, and SDet are as in Section 3, and Section 16.2 collects the further superalgebra that we use, from super vector spaces to invariant Berezin integration on homogeneous superspaces.

16.1 Two scales

We first explain heuristically why there are two scales. Recall from Proposition 4.11 that the source ratio is given by powers of the transfer operator T_Γ, which multiplies by the

onsite weight w_Γ after applying the ambient heat factor $K_{\tau_{\text{phys}}}$. The heat factor has the Gaussian kernel

$$K_{\tau_{\text{phys}}}(R, R') = \exp\Big\{-\frac{\mathrm{STr}(R-R')^2}{2h^4}\Big\}, \tag{16.1}$$

where R and R' are the supermatrix variables of two neighboring sites. The kernel carries no normalizing constant. Indeed, the Berezin form $\mathrm{d}R$ of Definition 4.5 is not rescaled on the cycle, its order of the odd coordinates makes the Gaussian integral (4.6) equal to $+1$ with $c_k = (2\pi)^{-4k^2}$, and Section 4.5 shows that $K_t 1 = 1$ without a constant factor. Recall from Section 2.3 that the chain stays near the saddle manifold of Section 4.6, the union of $2k+1$ orbits of critical points of the onsite weight, whose balanced component is a copy of the target superspace $\mathcal{M}_k$ of Section 5, and that one link moves the supermatrix by $h^2 = W^{-1}$. Normal to the saddle manifold, the chain behaves as a discretized Ornstein–Uhlenbeck process that relaxes over about h^{-2} links to a size of order h, and along it as a random walk with variance $\delta_W = h^4$ per link, which runs for a time of order one over the $N \asymp h^{-4}$ links (Figure 16.1). We make the first statement precise in Lemma 16.10: in the balanced signature the exact quadratic normal transfer is the Mehler operator $\mu^{\mathbf{N}}$ with $-\log\mu \asymp h^2$. In the other signatures it also carries the scalar root phase and the zero-point factor computed there.

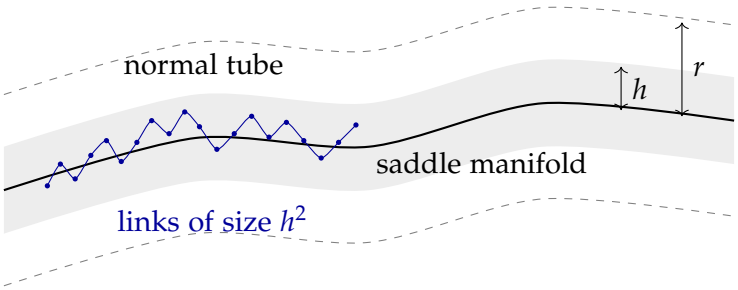


Figure 16.1. The saddle manifold (solid), the region within normal distance of order h of it where the normal variables of the chain are concentrated (shaded), and the normal cutoff at a fixed radius r (dashed). Each link moves the supermatrix by h^2. The normal coordinate relaxes after about h^{-2} links, during which the tangent coordinate moves by about h; over the h^{-4} links of the chain the tangent coordinate moves by a distance of order one.

The heuristic suggests the method. We write the transfer operator in coordinates (x, U), where x is a point of the saddle manifold and U is a normal displacement, and we restrict U to a ball of fixed radius r by a cutoff. In the standardized normal coordinate $Z = U/h$, the frozen normal transfer is, to leading order, an Ornstein–Uhlenbeck operator at time of order h^2, and the normal relaxation comes from its spectral gap of order h^2 off the vacuum. Recall from Section 2.4 that an error of size h^3 per step accumulates to h^{-1} over the chain, so that every error term must be compared with this gap or with the size δ_W of the tangent step. We measure errors relative to the defect operators of

a reference contraction, in the form required by Lemma A.1. The cutoff and Mehler estimates of Sections 16.7 and 16.8 provide these relative bounds for the frozen normal transfer, and Section 16.10 proves the Ward identity of Section 2.4, the exact preservation of the Gaussian mean on closed vectors, which no perturbative estimate could give.

16.2 Super vector spaces and homogeneous superspaces

The tubular coordinates of Section 16.3 use some superalgebra beyond that of Section 3 and Section 5.2: super vector spaces with the supertraces and superdeterminants of operators on them, the stabilizer of a point of the saddle manifold, and the invariant Berezin volume of the quotient by this stabilizer. We use them to prove the identity (16.15) of Lemma 16.2, which is the supersymmetric analogue of the Weyl integration formula. For a Hermitian $n \times n$ matrix $H = g\,\mathrm{diag}(\lambda)g^{-1}$ with $g \in U(n)$, this formula reads $\mathrm{d}H = \prod_{i<j}(\lambda_j - \lambda_i)^2\,\mathrm{d}\lambda\,\mathrm{d}g$ up to a constant factor, where $\mathrm{d}g$ is the invariant measure on the orbit of $\mathrm{diag}(\lambda)$. In (16.15), a supermatrix near the saddle manifold is written as a conjugate $T(R_* + U)T^{-1}$ of a block-diagonal supermatrix $R_* + U$, the invariant Berezin volume ν of this section plays the role of the orbit measure $\mathrm{d}g$, and a superdeterminant ϖ, defined in (16.11), plays the role of the Vandermonde factor. In the ordinary case, in which all entries commute, ϖ is the product of the differences between the eigenvalues of the two diagonal blocks of $R_* + U$, a cross-Vandermonde factor. The differences within a block do not appear, since U ranges over block-diagonal rather than diagonal matrices. Remark 16.4 treats this ordinary case. We collect these notions here, in the concrete form in which we use them. We follow [132, Chapter 3] for super linear algebra, [139, Section II.F] for symmetric superspaces, and [2] for invariant Berezin integration on homogeneous superspaces. Recall the supermatrices, their supertrace STr and their superdeterminant SDet from Definition 3.10, and recall from Section 5.2 that $U(k,k|2k)$ denotes the pair of the group $G_0 = \{\mathrm{diag}(g_\mathrm{B}|g_\mathrm{F}) : g_\mathrm{B} \in U(k,k),\ g_\mathrm{F} \in U(2k)\}$ and the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(2k|2k)$ of all $(2k|2k)$ supermatrices, with the bracket $[Y, Y'] = YY' - (-1)^{|Y||Y'|}Y'Y$.

Super vector spaces

We define a *super vector space* to be a finite-dimensional complex vector space V with a decomposition $V = V_0 \oplus V_1$ into an even part V_0 and an odd part V_1. If $\dim V_0 = m$ and $\dim V_1 = n$, we write $\dim V = m|n$, and we call $\mathrm{sdim}\, V = m - n$ the *superdimension* of V. A homogeneous basis of V is a basis of V_0 followed by a basis of V_1, and we write $|i| \in \{0,1\}$ for the parity of its ith vector. For instance, the columns of size $(2k|2k)$, with even bosonic and odd fermionic coordinates, form a super vector space of dimension $2k|2k$, and the $(2k|2k)$ supermatrices are the matrices of its linear maps in the standard basis. For super vector spaces V and W, the space $\mathrm{Hom}(W, V)$ of linear maps is a super

vector space, whose even part consists of the maps that preserve the parity and whose odd part of the maps that reverse it. In homogeneous bases, the matrix unit $E_{i\alpha}$ has the parity $|i| + |\alpha|$, as in Section 5.2. If $\dim V = m|n$ and $\dim W = m'|n'$, then $\mathrm{Hom}(W, V)$ has dimension $(mm' + nn')|(mn' + nm')$, and therefore

$$\operatorname{sdim} \mathrm{Hom}(W, V) = \operatorname{sdim} V \operatorname{sdim} W. \tag{16.2}$$

We write $\mathrm{End}(V) = \mathrm{Hom}(V, V)$.

Let $\mathcal{R}$ be a Grassmann algebra, possibly with auxiliary generators (Section 3.2). An element of V with coefficients in $\mathcal{R}$ is a sum $X = \sum_I X_I e_I$ over a homogeneous basis (e_I), with coordinates $X_I \in \mathcal{R}$. It is even if every X_I has the parity $|I|$. For instance, an even supermatrix over $\mathcal{R}$ is an even element of the endomorphisms of the columns, with its entries as coordinates. We call a map L of the even elements of V with coefficients in $\mathcal{R}$ into themselves an *even operator* on V if there exists an even supermatrix $M = (M_{IJ})$ over $\mathcal{R}$, indexed by the basis, such that $(LX)_I = \sum_J M_{IJ} X_J$ for all X, and we call M the matrix of L. We define

$$\mathrm{STr}_V L = \mathrm{STr}\, M, \qquad \mathrm{SDet}_V L = \mathrm{SDet}\, M,$$

the second when M is invertible. These do not depend on the homogeneous basis. Indeed, a change of homogeneous basis replaces M by $B^{-1}MB$ for an even invertible supermatrix B with complex entries, and by Lemma 3.11(c) and (b), we have $\mathrm{STr}(B^{-1}MB) = \mathrm{STr}\, M$ and $\mathrm{SDet}(B^{-1}MB) = \mathrm{SDet}\, M$. For example, $\mathrm{SDet}_V(c\mathrm{Id}) = c^{\operatorname{sdim} V}$ for every even $c \in \mathcal{R}$ with nonzero body, and Lemma 3.11(d) implies that $\mathrm{SDet}_V e^L = e^{\mathrm{STr}_V L}$.

For even supermatrices A on V and B on W with entries in $\mathcal{R}$, we let λ_A and ρ_B denote the even operators $X \mapsto AX$ and $X \mapsto XB$ on $\mathrm{Hom}(W, V)$. They commute, and

$$\mathrm{STr}_{\mathrm{Hom}(W,V)}(\lambda_A \rho_B) = \mathrm{STr}_V(A)\, \mathrm{STr}_W(B). \tag{16.3}$$

Indeed, in homogeneous bases of V and W, we have

$$(AXB)_{i\alpha} = \sum_{j,\beta} (-1)^{(|j|+|\beta|)(|\beta|+|\alpha|)} A_{ij} B_{\beta\alpha} X_{j\beta}.$$

The matrix of $\lambda_A \rho_B$ has the diagonal entries $A_{ii} B_{\alpha\alpha}$, at $(j, \beta) = (i, \alpha)$, and the coordinate $X_{i\alpha}$ has the parity $|i| + |\alpha|$. Summing the diagonal entries with the signs $(-1)^{|i|+|\alpha|}$, we obtain (16.3). For $W = V$, the identity says that the operator $Y \mapsto AYA'$ of $\mathrm{End}(V)$ has the supertrace $\mathrm{STr}(A)\, \mathrm{STr}(A')$. In particular, the operator $\mathrm{ad}_A = \lambda_A - \rho_A$ of $\mathrm{End}(V)$ satisfies

$$\mathrm{STr}_{\mathrm{End}(V)}(\mathrm{ad}_A) = \mathrm{STr}(A) \operatorname{sdim} V - \operatorname{sdim} V\, \mathrm{STr}(A) = 0. \tag{16.4}$$

The stabilizer of a root point

Fix a signature $p \in \{0, \dots, 2k\}$, and let $\Sigma = \mathrm{diag}\big(\Lambda_{\mathrm{B}} \mid \mathrm{diag}(\mathrm{Id}_p, -\mathrm{Id}_{2k-p})\big)$, so that $R_* = \kappa_E \mathrm{Id} - \mathrm{i}s_E \Sigma$ is the root point of signature p of Section 4.6. We let V_+ and V_- denote

the eigenspaces of Σ in $\mathbb{C}^{2k|2k}$ for the eigenvalues 1 and -1. Then $\dim V_+ = k|p$ and $\dim V_- = k|(2k-p)$, so that

$$\operatorname{sdim} V_+ = k - p, \qquad \operatorname{sdim} V_- = p - k.$$

We call the signature $p = k$ balanced. In the balanced signature $\Sigma = \Lambda$, and V_+ and V_- are spanned by the retarded and by the advanced basis vectors of Section 5.2. We set

$$\mathfrak{k} = \{Y \in \mathfrak{g} : \Sigma Y = Y\Sigma\}, \qquad \mathfrak{m} = \{Y \in \mathfrak{g} : \Sigma Y = -Y\Sigma\},$$

which are the block-diagonal and the block off-diagonal supermatrices relative to the splitting $V_+ \oplus V_-$, so that $\mathfrak{k} = \operatorname{End}(V_+) \oplus \operatorname{End}(V_-)$ and $\mathfrak{m} = \operatorname{Hom}(V_-, V_+) \oplus \operatorname{Hom}(V_+, V_-)$, and for $p = k$ these are the spaces of (5.1). The map $Y \mapsto \Sigma Y \Sigma$ is an automorphism of order two of $\mathfrak{g}$, with the eigenspaces $\mathfrak{k}$ and $\mathfrak{m}$ for the eigenvalues 1 and -1, and therefore

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \qquad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}. \tag{16.5}$$

Since the supertrace of a block off-diagonal supermatrix vanishes, the spaces $\mathfrak{k}$ and $\mathfrak{m}$ are orthogonal for the form $\operatorname{STr}(YY')$, and this form is nondegenerate on each of them. We let $K_0 = \{g \in G_0 : g\Sigma = \Sigma g\}$, which is isomorphic to $U(k) \times U(k) \times U(p) \times U(2k-p)$ and connected, and we call the pair $K = (K_0, \mathfrak{k})$ the *stabilizer* of R_* in $U(k,k|2k)$. For $p = k$, the group K_0 is the group of Section 5.2. We let $\mathfrak{m}_{\mathbb{R}} = \mathfrak{m}_0 \cap \operatorname{Lie}(G_0)$ denote the real form of the even part of $\mathfrak{m}$ that is tangent to G_0.

The odd elements of $\mathfrak{k}$ do not change volumes on $\mathfrak{m}$. Precisely, let $C = \operatorname{diag}(C_+, C_-)$ be an even supermatrix that commutes with Σ, with entries in a Grassmann algebra. Then ad_C preserves $\mathfrak{m}$, it acts as $\lambda_{C_+} - \rho_{C_-}$ on $\operatorname{Hom}(V_-, V_+)$ and as $\lambda_{C_-} - \rho_{C_+}$ on $\operatorname{Hom}(V_+, V_-)$. By (16.3), we have

$$\begin{aligned} \operatorname{STr}_{\mathfrak{m}}(\operatorname{ad}_C) = {} & \big(\operatorname{STr} C_+ \operatorname{sdim} V_- - \operatorname{sdim} V_+ \operatorname{STr} C_-\big) \\ & + \big(\operatorname{STr} C_- \operatorname{sdim} V_+ - \operatorname{sdim} V_- \operatorname{STr} C_+\big) = 0. \end{aligned} \tag{16.6}$$

By Lemma 3.11(d), this implies that $\operatorname{SDet}_{\mathfrak{m}}(e^{\operatorname{ad}_C}) = 1$. Since K_0 is connected, the conjugation $\operatorname{Ad}_k \colon Y \mapsto kYk^{-1}$ also satisfies $\operatorname{SDet}_{\mathfrak{m}} \operatorname{Ad}_k = 1$ for all $k \in K_0$, and therefore for all products ke^C of such elements.

The homogeneous superspace and its invariant volume

We define the superfunctions on the superspace $U(k,k|2k)/K$ as in Definition 5.2, with Λ replaced by Σ. We fix a basis (e_a) of the odd part $\mathfrak{m}_1$ of $\mathfrak{m}$ that is orthonormal for the Hilbert–Schmidt inner product $\langle Y, Y' \rangle = \operatorname{Tr}(Y^*Y')$, as in Section 5.2. The space $\mathfrak{m}_1$ has dimension $4k^2$ for every signature, and we set $\Theta = \sum_a \chi_a e_a$. A superfunction is a smooth map f from G_0 into the Grassmann algebra of the χ_a such that $f(g\mathsf{k}) = \alpha_{\mathsf{k}}(f(g))$ for all

$\mathsf{k} \in K_0$, where α_k replaces Θ by $\mathsf{k}\Theta\mathsf{k}^{-1}$. We identify the underlying manifold G_0/K_0 with the set of matrices $g\Sigma g^{-1}$, and we call the matrix of superfunctions $Q = ge^{\Theta}\Sigma e^{-\Theta}g^{-1}$ the field. For $p = k$, this is the target space $\mathcal{M}_k$ of Definition 5.2. Let $\mathcal{O}$ be an open subset of G_0/K_0 with coordinates x, and let s be a section over $\mathcal{O}$, that is, a smooth map into G_0 with $\mathsf{s}(x)\Sigma\mathsf{s}(x)^{-1} = x$. As in Section 5.2, the superfunctions on $\mathcal{O}$ are then the superfunctions of the coordinates $z = (x, \chi)$. We call an even invertible supermatrix T of superfunctions on $\mathcal{O}$ with $T\Sigma T^{-1} = Q$ a *local section* of $U(k,k|2k)$ over $\mathcal{O}$. For instance, $T = \mathsf{s}(x)e^{\Theta}$ is a local section. Two local sections T and T' over $\mathcal{O}$ differ by $T' = Tk$, where k commutes with Σ.

The quotient of $\mathrm{GL}(2k|2k)$ by the stabilizer $\mathrm{GL}(k|p) \times \mathrm{GL}(k|2k-p)$ of Σ, together with the real manifold G_0/K_0, is a Riemannian symmetric superspace in the sense of [139, Definition 2.5], of the type AIII|AIII of Table 2 there, and we call $U(k,k|2k)/K$ a symmetric superspace. We use this only through the automorphism $Y \mapsto \Sigma Y\Sigma$, which fixes K and acts as $-\mathrm{Id}$ on $\mathfrak{m}$, and through the form $\mathrm{STr}(YY')$, for which $\mathfrak{k}$ and $\mathfrak{m}$ are orthogonal and which is positive definite on $\mathfrak{m}_\mathbb{R}$.

Let T be a local section over $\mathcal{O}$. For an even derivation $\mathcal{V}$ of the superfunctions on $\mathcal{O}$, that is, a vector field in the coordinates z or the product of an odd vector field with an auxiliary odd parameter as in Section 5.2, we set

$$\mathfrak{a}(\mathcal{V}) = T^{-1}(\mathcal{V}T), \tag{16.7}$$

where $\mathcal{V}$ acts on each entry of T, and we write $\mathfrak{a}(\mathcal{V}) = \mathfrak{a}_\mathfrak{k}(\mathcal{V}) + \mathfrak{a}_\mathfrak{m}(\mathcal{V})$ for its components in $\mathfrak{k}$ and $\mathfrak{m}$. We call $\mathfrak{a}$ the *Maurer–Cartan form* of T. Its value is an even element of $\mathfrak{g}$ with coefficients in the superfunctions. The Maurer–Cartan form does not change if T is replaced by gT with $g \in G_0$. If T is replaced by $T' = Tk$, then

$$\mathfrak{a}'(\mathcal{V}) = k^{-1}\mathfrak{a}(\mathcal{V})k + k^{-1}\mathcal{V}k, \qquad \mathfrak{a}'_\mathfrak{m}(\mathcal{V}) = k^{-1}\mathfrak{a}_\mathfrak{m}(\mathcal{V})k, \tag{16.8}$$

since $k^{-1}\mathcal{V}k$ commutes with Σ.

We call a local section T together with a homogeneous basis (Y_b) of $\mathfrak{m}$, whose even elements form a basis of $\mathfrak{m}_\mathbb{R}$, a *local orbit frame*. In a local orbit frame we let $\mathcal{J}_\mathfrak{a}$ denote the even supermatrix, with rows indexed by the coordinates z_I and columns by the basis (Y_b), such that

$$\mathfrak{a}_\mathfrak{m}(\vartheta^{|I|}\partial_{z_I}) = \vartheta^{|I|}\sum_b (\mathcal{J}_\mathfrak{a})_{Ib}Y_b,$$

where ϑ is an auxiliary odd parameter and $|I|$ is the parity of z_I. It is the analogue of the Jacobian supermatrix (3.15), with the components of $\mathfrak{a}_\mathfrak{m}$ in place of the differentials of new coordinates. Its body is invertible: in the even rows, the body of $\mathfrak{a}_\mathfrak{m}$ maps the tangent space of G_0/K_0 isomorphically onto $\mathfrak{m}_\mathbb{R}$, and in the odd rows, for $T = \mathsf{s}(x)e^{\Theta}$, we have $\mathfrak{a}_\mathfrak{m}(\vartheta\partial_{\chi_a}) = \vartheta e_a$ up to terms of positive degree in the χ. We define the invariant Berezin volume ν by

$$\int f \,\mathrm{d}\nu = \int f \,\mathrm{SDet}(\mathcal{J}_\mathfrak{a})\,\mathrm{d}x\,\mathrm{d}\chi \tag{16.9}$$

for superfunctions f whose coefficients have compact support in $\mathcal{O}$.

The volume ν does not depend on the coordinates, among those that induce a fixed orientation of G_0/K_0, which we choose before Lemma 16.2: by the chain rule, another system of coordinates replaces $\mathcal{J}_\mathfrak{a}$ by the product of the Jacobian supermatrix (3.15) of the coordinate change with the new $\mathcal{J}_\mathfrak{a}$, and Lemma 3.11(b) and Theorem 3.15, which applies to changes of coordinates that preserve orientation, show that (16.9) is unchanged. It does not depend on the local section: by (16.8), a second section multiplies $\mathcal{J}_\mathfrak{a}$ by the matrix of Ad_k^{-1} on $\mathfrak{m}$, whose superdeterminant is one by (16.6). The integrals over overlapping coordinate patches therefore agree, and (16.9) defines $\int f \,\mathrm{d}\nu$ for all superfunctions with compactly supported integrable coefficients. Since $\mathfrak{a}$ is unchanged under $T \mapsto gT$, the volume ν is invariant under G_0. It is also invariant under the odd symmetries, by [2, Theorem 4.13], which we apply below. In the terminology of [2], ν is the Berezinian form whose value in the frame $\mathfrak{a}_\mathfrak{m}$ is constant. By [2, Corollary 4.12], the Berezinian bundle of $U(k,k|2k)/K$ is the bundle associated with the Berezinian of $(\mathfrak{g}/\mathfrak{k})^*$, so that an invariant Berezinian form is a constant element of this Berezinian that is invariant under the stabilizer. By [2, Theorem 4.13], applied with the condition that the Berezinian of $(\mathfrak{g}/\mathfrak{k})^*$ be a trivial module of the stabilizer, which is $\mathrm{SDet}_\mathfrak{m} \mathrm{Ad}_k = 1$ and $\mathrm{STr}_\mathfrak{m}(\mathrm{ad}_C) = 0$ by (16.6), the form ν is invariant under the whole of $U(k,k|2k)$; such a form is unique up to a constant factor (see also [2, Remark 4.15] for symmetric pairs). In particular, for $p = k$ the volume ν is a constant multiple of the invariant Berezin integral of Definition 5.6, by the uniqueness proved in Section 5.2.

Near the body point Σ we use the exponential chart, in which $T = e^Y$ with $Y = \sum_b y_b Y_b \in \mathfrak{m}$ and the coordinates y_b (real for even b). It is a local section, since the change from the coordinates y to the coordinates of a section satisfies the hypotheses of Lemma 3.14: its body is the exponential map of G_0/K_0 at Σ, and its linear part in the odd coordinates is the identity at Σ. For an even derivation $\mathcal{V}$, the differential of the exponential map gives

$$\mathfrak{a}(\mathcal{V}) = e^{-Y}\mathcal{V}e^Y = \frac{1-e^{-\mathrm{ad}_Y}}{\mathrm{ad}_Y}(\mathcal{V}Y) = \frac{\sinh \mathrm{ad}_Y}{\mathrm{ad}_Y}(\mathcal{V}Y) + \frac{1-\cosh \mathrm{ad}_Y}{\mathrm{ad}_Y}(\mathcal{V}Y),$$

where the first term lies in $\mathfrak{m}$ and the second in $\mathfrak{k}$, since ad_Y exchanges $\mathfrak{k}$ and $\mathfrak{m}$ by (16.5). It follows that $\mathcal{J}_\mathfrak{a}$ is the matrix of the operator $\sinh(\mathrm{ad}_Y)/\mathrm{ad}_Y$ of $\mathfrak{m}$, and

$$\mathrm{d}\nu = j(Y)\,\mathrm{d}y, \qquad j(Y) = \mathrm{SDet}_\mathfrak{m}\Big(\frac{\sinh \mathrm{ad}_Y}{\mathrm{ad}_Y}\Big), \tag{16.10}$$

which is the formula of [139, Section II.F]. In particular $j(0) = 1$, and $j(-Y) = j(Y)$, since $\sinh(u)/u$ is even; in the language of symmetric spaces, the geodesic symmetry $Y \mapsto -Y$ at Σ preserves ν. We normalize the basis (Y_b) by requiring that $\mathrm{d}y$ be the flat Berezin form $\mathrm{d}Y$ of $\mathfrak{m}$ defined as follows. We let $\mathrm{d}U$ denote the flat Berezin form of $\mathfrak{k}$, normalized by $\int e^{-\mathrm{STr}(U^2)/2}\,\mathrm{d}U = 1$ over the real form of $\mathfrak{k}$ with Hermitian boson–boson block, i

times a Hermitian fermion–fermion block, and free odd entries, and we let $\mathrm{d}Y$ denote the flat Berezin form of $\mathfrak{m}$ for which the flat Berezin form $\mathrm{d}R$ of Definition 4.5 splits as $\mathrm{d}R = \mathrm{d}U\,\mathrm{d}Y$ along $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. With this choice we say that ν is normalized at the root.

16.3 Tubular coordinates around the saddle manifold

We compute the volume of the tubular coordinates, including every odd coefficient, prove that the square root of the Berezinian of the tubular map is harmonic in the normal variables, and derive the resulting decomposition of the ambient Laplacian.

Root spaces and the tubular map

Fix a signature p, and recall from Section 16.2 the root point $R_* = \kappa_E \mathrm{Id} - \mathrm{i}s_E\Sigma$, the eigenspaces $V_\pm$ of Σ, and the spaces $\mathfrak{k}$ and $\mathfrak{m}$, where $\kappa_E = E/2$ and $s_E = (1 - E^2/4)^{1/2}$ as in Equation (1.3). The points R_*, for the $2k+1$ signatures, are the saddle points of the onsite weight of Section 4 up to conjugation by $\mathrm{GL}(2k|2k)$. All matrices below are even supermatrices with entries in a Grassmann algebra that contains all odd coordinates. We call the elements of $\mathfrak{k}$ normal and those of $\mathfrak{m}$ tangent. For a normal $U = \mathrm{diag}(U_+, U_-)$ we set $S = R_* + U$, with blocks $S_\pm$, and we define the operator L_S on $\mathcal{H}_{+-} = \mathrm{Hom}(V_-, V_+)$ and the function ϖ by

$$L_S A = S_+ A - A S_-, \qquad \varpi(S) = \mathrm{SDet}_{\mathcal{H}_{+-}} L_S. \tag{16.11}$$

Note that a scalar shift of S does not change L_S. At the root we have $L_{R_*} = -2\mathrm{i}s_E\mathrm{Id}$, and by (16.2) we have $\mathrm{sdim}\,\mathcal{H}_{+-} = (k-p)(p-k)$, so that $\varpi(R_*) = (-2\mathrm{i}s_E)^{-(p-k)^2}$. In particular, $\varpi(R_*) = 1$ in the balanced signature.

We fix $r_* \in (0, s_E/8]$ so small that the body of L_S is invertible, with $\|L_S^{-1}\| \le 1/s_E$, whenever the body of U has norm at most $2r_*$. Then ϖ and ϖ^{-1} are analytic in the entries of U, with all derivatives bounded in terms of s_E, k, and r_*. For $0 < r \le r_*$ we call the set of normal U whose body has norm less than $2r$ the normal tube of radius r. All statements of this section and the next are local to a normal tube with $r \le r_*$. Throughout, the energy E ranges over a compact subset of $(-2, 2)$ that does not depend on the rank k, and constants may depend on it.

Recall the stabilizer $K = (K_0, \mathfrak{k})$ of R_* from Section 16.2. The symmetric superspace $U(k,k|2k)/K$, with the underlying manifold G_0/K_0, is the real form of the component of the saddle manifold through R_*; in the balanced signature, it is the balanced component of Section 4.6. Let $T(x)$ be a local section of $U(k,k|2k)$ over a coordinate patch of $U(k,k|2k)/K$, as in Section 16.2, where x stands for all coordinates of the patch, even and odd. We call the map

$$(x, U) \longmapsto R(x, U) = T(x)(R_* + U)T(x)^{-1} \tag{16.12}$$

the tubular map, and (x, U) the tubular coordinates. We let ν denote the invariant Berezin volume (16.9) on $U(k,k|2k)/K$, normalized at the root, and $\mathrm{d}U$ the flat Berezin form of $\mathfrak{k}$ of Section 16.2. In a second local section Tk, the same points R have the normal coordinate $k^{-1}Uk$. We call a function of the normal coordinate U alone that is invariant under the conjugations $U \mapsto kUk^{-1}$, for all even invertible supermatrices k with Grassmann entries that commute with Σ, a *normal class function*. Such functions do not depend on the choice of the local section, and they are annihilated by the normal charges of Section 16.3.

The normal integration in this section and in Section 17 runs over a rotated real form of the normal space, which we now fix. For $\sigma \in \{+,-\}$ let $r_\sigma = \kappa_E + \sigma \mathrm{i} s_E$ be the two roots, and set

$$m_\sigma = 1 - r_\sigma^2, \qquad c_\sigma = \big(W^2 m_\sigma + \tfrac{1}{4} m_\sigma^2\big)^{1/2}, \qquad \theta_\sigma = \tfrac{1}{2} \arg c_\sigma, \tag{16.13}$$

with the branch of positive real part. Then $|\theta_\sigma| \le C|E|$.

Definition 16.1. For each root σ, let $\mathfrak{k}_\sigma^{\mathbb{R}}$ denote the real form of the normal block of root σ: Hermitian boson–boson block, i times a Hermitian fermion–fermion block, and free odd entries. We call $\bigoplus_\sigma e^{-\mathrm{i}\theta_\sigma} \mathfrak{k}_\sigma^{\mathbb{R}}$ the adapted real normal space. We call the set of points $T(x)(R_* + U)T(x)^{-1}$ with $x \in G_0/K_0$ and U in the adapted real normal space whose body has norm less than $2r$ the real tube.

All operators and Hilbert spaces of this section and Section 17 are defined on the real tube, which is their integration cycle. It differs from the Schäfer–Wegner cycle $\mathcal{S}$ of Section 4, and the two are connected in Sections 19 and 20. On the unrotated real normal space the frozen quadratic kernel, whose coefficient m_σ is complex, is a complex harmonic oscillator whose Hermite eigenfunctions are not orthogonal; on the rotated space its Gaussian ground state is real, as we will see in Lemma 16.10.

Integrals over the real tube depend on two orientations, which we fix here. First, the volume ν depends on the orientation of G_0/K_0 induced by the coordinates x in (16.9), and reversing this orientation changes the sign of ν. In the balanced signature, ν is a constant multiple of the integral of Definition 5.6 by Section 16.2, and we show in Section 16.5 that the constant is ± 1. We orient G_0/K_0 so that it is $+1$. In an unbalanced signature, either orientation of G_0/K_0 may be used, and Section 20.4 fixes one. Second, the flat form $\mathrm{d}R$ is holomorphic in the entries of R, and its integral over the real tube, a real cycle in the complex supermatrices, depends on an orientation of the real tube. We orient the real tube so that the identity (16.15) of the next lemma holds; in its proof, this choice absorbs a sign $(-1)^{(p-k)^2}$. All later sections use these orientations. In particular, Section 16.5 concludes that, in the balanced signature, $\mathrm{d}\nu$ is the invariant measure of $\mathcal{M}_k$ with the orientation of the pairing $\mathcal{B}_k$, and Section 20.4 uses this orientation of the real tube when it deforms the Schäfer–Wegner cycle into the real tube.

Lemma 16.2. *Let* $\mathfrak{a}$ *be the Maurer–Cartan form* (16.7) *of* T*, applied to even derivations* $\mathcal{V}$ *of the superfunctions of the tubular coordinates* (x, U)*. Then*

$$T^{-1}(\mathcal{V}R)T = \mathcal{V}U + [\mathfrak{a}_{\mathfrak{k}}(\mathcal{V}), U] + [\mathfrak{a}_{\mathfrak{m}}(\mathcal{V}), S]. \tag{16.14}$$

The first two terms are normal and the last is tangent, and the flat Berezin form $\mathrm{d}R$ *of Definition 4.5 satisfies*

$$\mathrm{d}R = \varpi(S)^2 \,\mathrm{d}U \,\mathrm{d}\nu(x), \tag{16.15}$$

including all nilpotent terms, with the orientations fixed above.

Proof. Since T depends only on x, the derivation $\mathcal{V}$ acts on T through its tangent part. Applying $\mathcal{V}$ to $R = TST^{-1}$ and using $\mathcal{V}(T^{-1}) = -T^{-1}(\mathcal{V}T)T^{-1}$, we have

$$T^{-1}(\mathcal{V}R)T = \mathfrak{a}(\mathcal{V})S + \mathcal{V}U - S\mathfrak{a}(\mathcal{V}) = \mathcal{V}U + [\mathfrak{a}_{\mathfrak{k}}(\mathcal{V}), S] + [\mathfrak{a}_{\mathfrak{m}}(\mathcal{V}), S].$$

Since $\mathfrak{a}_{\mathfrak{k}}(\mathcal{V})$ is block diagonal, it commutes with R_*, and therefore $[\mathfrak{a}_{\mathfrak{k}}(\mathcal{V}), S] = [\mathfrak{a}_{\mathfrak{k}}(\mathcal{V}), U]$, and (16.14) follows. By (16.5), the first two terms lie in $\mathfrak{k}$ and the last in $\mathfrak{m}$.

To compute the volume, we use linear coordinates of the ambient supermatrices adapted to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, the coordinates of the normal part and those of the tangent part in the basis (Y_b) of Section 16.2, in which $\mathrm{d}R = \mathrm{d}U \,\mathrm{d}Y$. Let $\mathcal{J}$ be the Jacobian supermatrix (3.15) of the tubular coordinates relative to these coordinates, whose row indexed by a tubular coordinate w consists of the coordinates of $\mathcal{V}_w R$, where $\mathcal{V}_w = \vartheta^{|w|}\partial_w$. By Theorem 3.15, we have $\mathrm{d}R = \mathrm{SDet}(\mathcal{J})\,\mathrm{d}x\,\mathrm{d}U$.

We multiply $\mathcal{J}$ on the right by the matrix of Ad_T^{-1} on $\mathfrak{g}$. This does not change the superdeterminant: T is a product of exponentials of even supermatrices A, and ad_A has supertrace zero on $\mathfrak{g} = \mathrm{End}(\mathbb{C}^{2k|2k})$ by (16.4). The rows of the new matrix $\mathcal{J}'$ consist of the coordinates of $T^{-1}(\mathcal{V}_w R)T$, which are given by (16.14). For a normal coordinate w, we have $\mathfrak{a}(\mathcal{V}_w) = 0$, and the row is the unit vector of w in $\mathfrak{k}$. For a coordinate z_I of x, we have $\mathcal{V}_{z_I} U = 0$, and the tangent part of the row is the row of $\mathcal{J}_{\mathfrak{a}}$ multiplied by the matrix of the operator $Y \mapsto [Y, S]$ of $\mathfrak{m}$. With the rows ordered as (x, U) and the columns as $(\mathfrak{m}, \mathfrak{k})$, the matrix $\mathcal{J}'$ is therefore block upper triangular, with the diagonal blocks $\mathcal{J}_{\mathfrak{a}} \cdot [Y \mapsto [Y, S]]$ and Id. Then Lemma 3.11(e) and (b), together with (16.9), imply that

$$\mathrm{d}R = \mathrm{SDet}_{\mathfrak{m}}\big(Y \mapsto [Y, S]\big)\,\mathrm{d}U\,\mathrm{d}\nu(x).$$

It remains to compute this superdeterminant. The operator $Y \mapsto [Y, S]$ preserves the two summands of $\mathfrak{m}$. It acts as $-L_S$ on $\mathcal{H}_{+-} = \mathrm{Hom}(V_-, V_+)$, and on $\mathrm{Hom}(V_+, V_-)$ as

$$B \longmapsto BS_+ - S_- B = (\rho_{S_+} - \lambda_{S_-})B.$$

Using (16.3) on the two spaces, we find that for every polynomial f we have

$$\mathrm{STr}_{\mathrm{Hom}(V_+, V_-)} f(\rho_{S_+}, \lambda_{S_-}) = \mathrm{STr}_{\mathcal{H}_{+-}} f(\lambda_{S_+}, \rho_{S_-}),$$

since the two sides are the same combination of $\operatorname{STr} S_+^i \operatorname{STr} S_-^j$. Since $\operatorname{SDet} = \exp \operatorname{STr} \log$, we obtain

$$\operatorname{SDet}(\rho_{S_+} - \lambda_{S_-}) = \operatorname{SDet}_{\mathcal{H}_{+-}}(\lambda_{S_+} - \rho_{S_-}) = \varpi(S),$$
$$\operatorname{SDet}_{\mathcal{H}_{+-}}(-L_S) = (-1)^{\operatorname{sdim} \mathcal{H}_{+-}} \varpi(S) = (-1)^{(p-k)^2} \varpi(S),$$

and our choice of orientation for the real tube absorbs the sign. Inserting the two superdeterminants into the last display, we conclude that (16.15) holds. □

The harmonic half-density

We let $\Delta_{\mathrm{nor}} = \Delta_+ + \Delta_-$ denote the flat Laplacian of the normal variables for the metric $\operatorname{STr}(\mathrm{d}U_+)^2 + \operatorname{STr}(\mathrm{d}U_-)^2$, which is the restriction of the ambient metric of (16.1). Precisely, we define Δ_+ to be the second-order operator with constant coefficients in the entries of U_+ such that

$$\Delta_+ e^{\operatorname{STr}(PU_+)} = \operatorname{STr}(P^2) e^{\operatorname{STr}(PU_+)} \tag{16.16}$$

for every even supermatrix P on V_+ whose entries lie in an auxiliary Grassmann algebra, and we define Δ_- in the same way. For a function ϕ of U_+ we let $\nabla_+ \phi$ denote the supermatrix on V_+ such that $\frac{\mathrm{d}}{\mathrm{d}t} \phi(U_+ + tX)|_{t=0} = \operatorname{STr}(X \nabla_+ \phi)$ for all even X, and we set $|\nabla_+ \phi|^2 = \operatorname{STr}(\nabla_+ \phi \, \nabla_+ \phi)$. We use the same notation for U_-. Applying (16.16) to products of exponentials, we find that

$$\Delta_\pm(\phi\psi) = \phi \, \Delta_\pm \psi + \psi \, \Delta_\pm \phi + 2 \operatorname{STr}(\nabla_\pm \phi \, \nabla_\pm \psi) \tag{16.17}$$

for an even function ϕ and an arbitrary function ψ (with odd auxiliary parameters in the exponentials when ψ has odd parts), and therefore

$$e^{-g} \Delta_\pm e^g = \Delta_\pm g + |\nabla_\pm g|^2 \tag{16.18}$$

for every even function g. Moreover, the value of $\Delta_+ \phi$ at a point is determined by the Hessian of ϕ there. Precisely, suppose that $\mathcal{B}$ is a linear map of $\operatorname{End}(V_+)$ such that

$$\frac{\partial^2}{\partial s \partial t} \phi(U_+ + sX + tY)\Big|_{s=t=0} = \operatorname{STr}(X \mathcal{B} Y)$$

for all even X and Y. Then $\Delta_+ \phi = \operatorname{STr}_{\operatorname{End}(V_+)} \mathcal{B}$, the supertrace of $\mathcal{B}$ as an endomorphism of the super vector space $\operatorname{End}(V_+)$. To see this, note that both sides are linear in the Hessian and that the Hessians of the exponentials in (16.16) span all Hessians. It suffices to check the claim for $\phi = e^{\operatorname{STr}(PU_+)}$. Then $\mathcal{B}Y = P \operatorname{STr}(PY) \phi$, and $\operatorname{STr}_{\operatorname{End}(V_+)} \mathcal{B} = \operatorname{STr}(P^2)\phi$.

Recall the operators λ_A and ρ_B of left and right multiplication, here on $\mathcal{H}_{+-}$, with even supermatrices A on V_+ and B on V_-, and the identity (16.3) $\operatorname{STr}_{\mathcal{H}_{+-}}(\lambda_A \rho_B) = \operatorname{STr}(A) \operatorname{STr}(B)$, together with its form $\operatorname{STr}_{\operatorname{End}(V_+)}(Y \mapsto AYA') = \operatorname{STr}(A) \operatorname{STr}(A')$ for $W = V$.

Lemma 16.3. *On the normal tube of radius r_*, we have*

$$\Delta_+ \log \varpi = -|\nabla_- \log \varpi|^2, \qquad \Delta_- \log \varpi = -|\nabla_+ \log \varpi|^2, \tag{16.19}$$

and consequently

$$\Delta_{\mathrm{nor}} \varpi = 0. \tag{16.20}$$

These are identities of all Grassmann coefficients. For every normal superfunction f, with $\varpi^{-2} \operatorname{div}_{\mathrm{nor}}(\varpi^2 \nabla_{\mathrm{nor}} f)$ defined as $\Delta_{\mathrm{nor}} f + 2 \operatorname{STr}(\nabla_{\mathrm{nor}} \log \varpi \, \nabla_{\mathrm{nor}} f)$,

$$\varpi \big[\varpi^{-2} \operatorname{div}_{\mathrm{nor}}(\varpi^2 \nabla_{\mathrm{nor}} f) \big] = \Delta_{\mathrm{nor}}(\varpi f). \tag{16.21}$$

The identity (16.21) says that the half-density change $f \mapsto \varpi f$, which turns the tubular measure $\varpi^2 \, \mathrm{d}U$ into the flat measure $\mathrm{d}U$, conjugates the normal part of the tubular Laplacian to the flat normal Laplacian with no potential term. Without harmonicity, a potential $\varpi^{-1} \Delta_{\mathrm{nor}} \varpi$ would appear, of order one on the tube.

Proof. Let $G = L_S^{-1}$, an even invertible operator on $\mathcal{H}_{+-}$. We define the even supermatrices Γ_+ on V_+ and Γ_- on V_- by

$$\operatorname{STr}(X\Gamma_+) = \operatorname{STr}_{\mathcal{H}_{+-}}(G\lambda_X), \qquad \operatorname{STr}(Y\Gamma_-) = \operatorname{STr}_{\mathcal{H}_{+-}}(G\rho_Y) \tag{16.22}$$

for all even X and Y. They exist and are unique, since STr is a nondegenerate pairing. Since $\log \operatorname{SDet} = \operatorname{STr} \log$ and $\mathrm{d}L_S = \lambda_{\mathrm{d}U_+} - \rho_{\mathrm{d}U_-}$, we have $\mathrm{d} \log \varpi = \operatorname{STr}(G\lambda_{\mathrm{d}U_+}) - \operatorname{STr}(G\rho_{\mathrm{d}U_-})$, and therefore

$$\nabla_+ \log \varpi = \Gamma_+, \qquad \nabla_- \log \varpi = -\Gamma_-.$$

We now compute $\Delta_+ \log \varpi$. On the tube, the body of $\mathrm{i}L_S$ has spectrum in the half-plane $\{\operatorname{Re} w \geq s_E\}$, since $\mathrm{i}L_{R_*} = 2s_E$ and $\|L_U\| \leq 4r_* \leq s_E/2$. Using that λ_{S_+} and ρ_{S_-} commute, we obtain

$$G = \mathrm{i} \int_0^\infty e^{-\mathrm{i}tL_S} \, \mathrm{d}t = \int_0^\infty \lambda_{A(t)} \rho_{B(t)} \, \mathrm{d}t, \qquad A(t) = \mathrm{i}e^{-\mathrm{i}tS_+}, \quad B(t) = e^{\mathrm{i}tS_-}, \tag{16.23}$$

where the integral converges absolutely, since the nilpotent parts of $S_\pm$ contribute only polynomial factors in t.

Differentiating $\log \varpi$ twice and using $\mathrm{d}G = -G(\mathrm{d}L_S)G$, we find that the Hessian of $\log \varpi$ in U_+ is the bilinear form $(X, Y) \mapsto -\operatorname{STr}_{\mathcal{H}_{+-}}(G\lambda_X G\lambda_Y)$. Since λ and ρ commute and $\rho_B \rho_{B'} = \rho_{B'B}$, inserting (16.23) twice and using (16.3), we find that

$$\operatorname{STr}_{\mathcal{H}_{+-}}(G\lambda_X G\lambda_Y) = \int_0^\infty \int_0^\infty \operatorname{STr}\big(A(t)XA(t')Y\big) \operatorname{STr}\big(B(t')B(t)\big) \, \mathrm{d}t \, \mathrm{d}t'.$$

The Hessian is therefore $\operatorname{STr}(X\mathcal{B}Y)$, where

$$\mathcal{B}Y = -\iint \operatorname{STr}\big(B(t')B(t)\big) \, A(t')YA(t) \, \mathrm{d}t \, \mathrm{d}t',$$

and the characterization of Δ_+ by the Hessian, together with the identity $\mathrm{STr}_{\mathrm{End}(V_+)}(Y \mapsto AYA') = \mathrm{STr}(A)\,\mathrm{STr}(A')$, implies that

$$\Delta_+ \log \varpi = -\iint \mathrm{STr}\, A(t)\ \mathrm{STr}\, A(t')\ \mathrm{STr}\big(B(t')B(t)\big)\,\mathrm{d}t\,\mathrm{d}t'.$$

On the other hand, inserting (16.23) into (16.22) and using (16.3), we have

$$\Gamma_- = \int_0^\infty \mathrm{STr}\, A(t)\, B(t)\,\mathrm{d}t,$$

and hence

$$|\nabla_- \log \varpi|^2 = \mathrm{STr}(\Gamma_-^2) = \iint \mathrm{STr}\, A(t)\ \mathrm{STr}\, A(t')\ \mathrm{STr}\big(B(t)B(t')\big)\,\mathrm{d}t\,\mathrm{d}t'.$$

Since $\mathrm{STr}(B(t)B(t')) = \mathrm{STr}(B(t')B(t))$, this proves the first identity in (16.19). The second identity follows in the same way. The Hessian of $\log \varpi$ in U_- is

$$(X, Y) \mapsto -\mathrm{STr}_{\mathcal{H}_{+-}}(G\rho_X G\rho_Y) = -\iint \mathrm{STr}\big(A(t)A(t')\big)\,\mathrm{STr}\big(XB(t)YB(t')\big)\,\mathrm{d}t\,\mathrm{d}t',$$

and

$$\Gamma_+ = \int_0^\infty \mathrm{STr}\, B(t)\, A(t)\,\mathrm{d}t.$$

Adding the two identities in (16.19) and using that the two blocks are orthogonal, we obtain $\Delta_{\mathrm{nor}} \log \varpi + |\nabla_{\mathrm{nor}} \log \varpi|^2 = 0$, which is (16.20) by (16.18). Finally, by the product rule (16.17), we have

$$\Delta_{\mathrm{nor}}(\varpi f) = \varpi\, \Delta_{\mathrm{nor}} f + 2\varpi\, \mathrm{STr}(\nabla_{\mathrm{nor}} \log \varpi\, \nabla_{\mathrm{nor}} f) + f\, \Delta_{\mathrm{nor}} \varpi,$$

where the first two terms on the right add up to $\varpi[\varpi^{-2}\,\mathrm{div}_{\mathrm{nor}}(\varpi^2 \nabla_{\mathrm{nor}} f)]$ by the definition in the statement. Together with (16.20), this proves (16.21). □

Remark 16.4. In the purely even analogue, where V_+ and V_- are ordinary vector spaces, the proof is the classical computation for $\varpi = \prod_{i,a}(x_i - y_a)$ in terms of the eigenvalues x_i of S_+ and y_a of S_-. The two sides of the first identity in (16.19) equal $-\sum_a (\sum_i (x_i - y_a)^{-1})^2$. The identities (16.19) and (16.20) were checked numerically, coefficient by coefficient in a finite Grassmann algebra, at selected complex base points in six signatures with $\dim V_\pm \leq 3$, and the scripts are in the supplementary material (`tube_cross_identity.py`, `tube_super_check.py` and `tube_ordinary_check.py`). The second script also confirms the normalization of the odd part of Δ_{nor} through the identity $\Delta_+ \,\mathrm{STr}(U_+^2) = 2(\mathrm{sdim}\, V_+)^2$, which follows from (16.16) and (16.3).

The decomposition of the ambient Laplacian

We first define the normal charges. For an even element $\mathfrak{e}$ of $\mathfrak{k}$, possibly with Grassmann coefficients, we let $\mathcal{N}_\mathfrak{e}$ denote the first-order differential operator $f \mapsto \frac{\mathrm{d}}{\mathrm{d}t} f(e^{-t\mathfrak{e}} U e^{t\mathfrak{e}})|_{t=0}$ in the normal variable, which is the derivative in the direction $[U, \mathfrak{e}]$. We call the operators $\mathcal{N}_\mathfrak{e}$ with even $\mathfrak{e} \in \mathfrak{k}$, and the operators $\mathcal{N}_{\vartheta\mathfrak{e}}$ with odd $\mathfrak{e} \in \mathfrak{k}$ and an auxiliary odd parameter ϑ, divided by ϑ, the compact and the odd normal charges. On the normal oscillator of Section 16.4 they are the operators $\mathfrak{j}_a$ and $\mathfrak{q}_a$ of Section 16.7. The normal charges are divergence-free for the flat normal form $\mathrm{d}U$, since the supertrace of $\mathrm{ad}_\mathfrak{e}$ on $\mathfrak{k} = \mathrm{End}(V_+) \oplus \mathrm{End}(V_-)$ vanishes by (16.3). They annihilate every normal class function, in particular ϖ, which is invariant under conjugation by block-diagonal supermatrices, since conjugating S conjugates L_S by $\lambda_{k_+}\rho_{k_-^{-1}}$. They also preserve the normal metric $\mathrm{STr}(\mathrm{d}U)^2$.

We now define the horizontal covariant fields. Let T be a local section. For an even element Y of $\mathfrak{m}$ whose coefficients are superfunctions of x, we let X_Y denote the even derivation of the superfunctions of x with $\mathfrak{a}_\mathfrak{m}(X_Y) = Y$, which exists and is unique since $\mathcal{J}_\mathfrak{a}$ is invertible, and we set

$$\mathbb{D}_Y = X_Y + \mathcal{N}_{\mathfrak{a}_\mathfrak{k}(X_Y)},$$

an even derivation of the superfunctions of (x, U). By (16.14), $\mathbb{D}_Y$ is characterized by

$$\mathfrak{a}_\mathfrak{m}(\mathbb{D}_Y) = Y, \qquad \mathbb{D}_Y U + [\mathfrak{a}_\mathfrak{k}(\mathbb{D}_Y), U] = 0, \qquad T^{-1}(\mathbb{D}_Y R)T = [Y, S]. \tag{16.24}$$

We call $\mathbb{D}_Y$ horizontal because it moves the point R only along its orbit under conjugation, and covariant because its normal part compensates the rotation $\mathfrak{a}_\mathfrak{k}$ of the frame. In a local orbit frame $(T, (Y_b))$, we let $\mathbb{D}_b = \mathbb{D}_{Y_b}$, defined for odd Y_b by $\mathbb{D}_{\vartheta Y_b} = \vartheta \mathbb{D}_b$, and we call the $\mathbb{D}_b$ the horizontal covariant fields of the frame. The normal part of $\mathbb{D}_b$ is the normal charge of $\mathfrak{e}_b = \mathfrak{a}_\mathfrak{k}(X_{Y_b})$.

Every even derivation $\mathcal{V}$ of the superfunctions of (x, U) splits into a normal and a horizontal part. Indeed, $\mathcal{V} - \mathbb{D}_{\mathfrak{a}_\mathfrak{m}(\mathcal{V})}$ has vanishing Maurer–Cartan form, so that it has no tangent part, and by (16.24) it is the derivative in U in the direction $\mathcal{V}U + [\mathfrak{a}_\mathfrak{k}(\mathcal{V}), U]$. For a superfunction ϕ on the tube, we let $\nabla_{\mathrm{nor}}\phi \in \mathfrak{k}$ denote its gradient in U, as in Section 16.3 for the two blocks, and we let $\nabla_H \phi \in \mathfrak{m}$ denote the horizontal gradient, defined by $\mathrm{STr}(Y \nabla_H \phi) = \mathbb{D}_Y \phi$ for all even $Y \in \mathfrak{m}$. Then

$$\mathcal{V}\phi = \mathrm{STr}\big((\mathcal{V}U + [\mathfrak{a}_\mathfrak{k}(\mathcal{V}), U])\nabla_{\mathrm{nor}}\phi\big) + \mathrm{STr}\big(\mathfrak{a}_\mathfrak{m}(\mathcal{V})\nabla_H \phi\big). \tag{16.25}$$

We now derive the decomposition of the ambient flat Laplacian Δ_{amb} of (16.1), which is the operator with constant coefficients in the linear ambient coordinates such that $\Delta_{\mathrm{amb}} e^{\mathrm{STr}(PR)} = \mathrm{STr}(P^2) e^{\mathrm{STr}(PR)}$, as in (16.16). Let ∇F denote the ambient gradient of a function F of R, defined by $\frac{\mathrm{d}}{\mathrm{d}t} F(R + tX)|_{t=0} = \mathrm{STr}(X \nabla F)$. For an even derivation $\mathcal{V}$, we obtain from (16.14) and the orthogonality of $\mathfrak{k}$ and $\mathfrak{m}$ that

$$\mathcal{V}F = \mathrm{STr}\big(T^{-1}(\mathcal{V}R)T \cdot T^{-1}(\nabla F)T\big) = \mathrm{STr}\big((\mathcal{V}U + [\mathfrak{a}_\mathfrak{k}(\mathcal{V}), U])G_\mathfrak{k}\big) + \mathrm{STr}\big(\mathfrak{a}_\mathfrak{m}(\mathcal{V})[S, G_\mathfrak{m}]\big),$$

where $G = T^{-1}(\nabla F)T = G_{\mathfrak{k}} + G_{\mathfrak{m}}$. The operator $\mathrm{ad}_S = [S, \cdot\,]$ is invertible on $\mathfrak{m}$ on the tube, by the proof of Lemma 16.2. Comparing with (16.25) for all $\mathcal{V}$, we conclude that

$$T^{-1}(\nabla F)T = \nabla_{\mathrm{nor}} F + \mathrm{ad}_S^{-1}(\nabla_H F). \tag{16.26}$$

Applying the definition of Δ_{amb} to products of exponentials, as for (16.17), we find that

$$\Delta_{\mathrm{amb}}(\phi\psi) = \phi\,\Delta_{\mathrm{amb}}\psi + \psi\,\Delta_{\mathrm{amb}}\phi + 2\mathcal{Q}_{\mathrm{amb}}(\phi,\psi), \qquad \mathcal{Q}_{\mathrm{amb}}(\phi,\psi) = \mathrm{STr}(\nabla\phi\,\nabla\psi),$$

for even ϕ, and we call $\mathcal{Q}_{\mathrm{amb}}$ the carré du champ of Δ_{amb}. Inserting (16.26), we find that

$$\mathcal{Q}_{\mathrm{amb}}(\phi,\psi) = \mathrm{STr}(\nabla_{\mathrm{nor}}\phi\,\nabla_{\mathrm{nor}}\psi) + \mathrm{STr}\big(\mathrm{ad}_S^{-1}\,\nabla_H\phi\;\mathrm{ad}_S^{-1}\,\nabla_H\psi\big) = \mathcal{Q}_{\mathrm{nor}}(\phi,\psi) + \mathcal{Q}_H(\phi,\psi). \tag{16.27}$$

The second term is the inverse of the horizontal metric

$$g_S(Y,Z) = \mathrm{STr}([Y,S][Z,S]), \qquad Y, Z \in \mathfrak{m},$$

applied to the horizontal gradients: since ad_S is skew for the form $\mathrm{STr}(YZ)$, the metric g_S corresponds to the operator $-\mathrm{ad}_S^2$ of $\mathfrak{m}$, and $\mathcal{Q}_H(\phi,\psi) = -\mathrm{STr}(\nabla_H\phi\;\mathrm{ad}_S^{-2}\,\nabla_H\psi)$. In this sense, the ambient metric is the orthogonal sum of the constant normal metric and the horizontal metric g_S.

Let ϕ and ψ be superfunctions on the tube, with ϕ even and with compactly supported coefficients. We integrate by parts in the linear ambient coordinates, by Lemma 3.6(a) and (b), and we write the result in tubular coordinates by (16.15) and (16.27). This yields

$$\int \phi\,\Delta_{\mathrm{amb}}\psi\,\varpi^2\,\mathrm{d}U\,\mathrm{d}\nu = -\int \mathcal{Q}_{\mathrm{nor}}(\phi,\psi)\,\varpi^2\,\mathrm{d}U\,\mathrm{d}\nu - \int \mathcal{Q}_H(\phi,\psi)\,\varpi^2\,\mathrm{d}U\,\mathrm{d}\nu.$$

For the first term on the right, we integrate by parts in U and use the product rule (16.17) for Δ_{nor}, and we obtain

$$\begin{aligned} -\int \mathcal{Q}_{\mathrm{nor}}(\phi,\psi)\,\varpi^2\,\mathrm{d}U &= \int \phi\,\big(\Delta_{\mathrm{nor}}\psi + 2\,\mathrm{STr}(\nabla_{\mathrm{nor}}\log\varpi\,\nabla_{\mathrm{nor}}\psi)\big)\varpi^2\,\mathrm{d}U \\ &= \int \phi\,\varpi^{-2}\,\mathrm{div}_{\mathrm{nor}}(\varpi^2\nabla_{\mathrm{nor}}\psi)\,\varpi^2\,\mathrm{d}U, \end{aligned}$$

with the notation of Lemma 16.3. Let $L_H(U)$ denote the operator $\Delta_{\mathrm{amb}} - \varpi^{-2}\,\mathrm{div}_{\mathrm{nor}}(\varpi^2\nabla_{\mathrm{nor}})$ in tubular coordinates. The last two displays show that

$$\int \phi\,L_H(U)\psi\,\varpi^2\,\mathrm{d}U\,\mathrm{d}\nu = -\int \mathcal{Q}_H(\phi,\psi)\varpi^2\,\mathrm{d}U\,\mathrm{d}\nu.$$

Since $\mathbb{D}_Y\varpi = 0$, we have $\mathcal{Q}_H(\varpi^{-2}\phi,\psi) = \varpi^{-2}\mathcal{Q}_H(\phi,\psi)$, and replacing ϕ by $\varpi^{-2}\phi$ yields

$$\int \phi\,L_H(U)\psi\,\mathrm{d}U\,\mathrm{d}\nu = -\int \mathcal{Q}_H(\phi,\psi)\,\mathrm{d}U\,\mathrm{d}\nu. \tag{16.28}$$

For the same reason, $\mathcal{Q}_H(\phi, \varpi\psi) = \mathcal{Q}_H(\varpi\phi, \psi)$, so that $L_H(U)$ commutes with multiplication by ϖ. Together with (16.21), applied to $\varpi^{-1}\psi$, this implies the exact identity

$$\varpi\, \Delta_{\mathrm{amb}}\, \varpi^{-1} = \Delta_{\mathrm{nor}} + L_H(U), \qquad L_H(U) = \mathrm{div}_{\nu,\mathbb{D}}(g_S^{-1}\mathbb{D}), \tag{16.29}$$

for all Grassmann coefficients, where we write $\mathrm{div}_{\nu,\mathbb{D}}(g_S^{-1}\mathbb{D})$ for the operator characterized by (16.28), the divergence-form operator of the horizontal metric with respect to $\mathrm{d}U\, \mathrm{d}\nu$. We call $L_H(U)$ the horizontal operator.

In a local orbit frame, integrating by parts along the fields $\mathbb{D}_b$ in (16.28), whose normal parts are divergence-free, shows that $L_H(U)$ is a finite sum of products of at most two of the fields $\mathbb{D}_b$, with coefficients that are superfunctions of (x, U), analytic in U. Its ordinary second-order part has the form $\sum_{b,c} g_S^{bc} X_{Y_b} X_{Y_c}$, summed over the even b and c, where (g_S^{bc}) is the inverse of the matrix $(g_S(Y_b, Y_c))$. In particular, all normal derivatives in $L_H(U)$ come from the normal charges, and $L_H(U)1 = 0$. The inverse horizontal metric is analytic in U on the tube. It satisfies $g_S^{-1} - g_{R_*}^{-1} = O(U)$, and its derivatives of every fixed order are bounded. At $U = 0$, we have $g_{R_*}(Y, Z) = 4s_E^2\, \mathrm{STr}(YZ)$.

At $U = 0$, the horizontal operator is a constant multiple of the Casimir operator of the sigma model. In the balanced signature, $U(k, k|2k)/K$ is the target space $\mathcal{M}_k$, and we identify the functions of the tangent variables x alone with the superfunctions on $\mathcal{M}_k$ through the field $Q = T\Lambda T^{-1}$, so that $R(x, 0) = \kappa_E - \mathrm{i}s_E Q$. For such a function ψ, we write $L_H(0)\psi$ for the restriction of $L_H(U)\psi$ to $U = 0$.

Lemma 16.5. *In the balanced signature, on superfunctions on $\mathcal{M}_k$,*

$$L_H(0) = -\frac{1}{2s_E^2}\, \mathsf{C}_k, \tag{16.30}$$

where C_k is the Casimir operator of Definition 5.7.

In words, at $U = 0$ the horizontal operator is the Laplace–Beltrami operator of the orbit metric

$$g_{R_*}(Y, Z) = 4s_E^2\, \mathrm{STr}(YZ) = -s_E^2\, \mathrm{STr}([Y, \Lambda][Z, \Lambda]),$$

that is, of $2s_E^2$ times the metric (5.2) on the vectors $[Y, \Lambda]$ (see the proof of Lemma 5.8(c)), and C_k is minus the Laplace–Beltrami operator of the metric (5.2), extended to the odd directions as the supermetric $-\frac{1}{2}\, \mathrm{STr}(\, \mathrm{d}Q\, \mathrm{d}Q)$.

Proof. We first express $L_H(0)$ through its horizontal carré du champ. Let ϕ and ψ be functions of x alone, with ϕ even and compactly supported, and let ζ be a normal function with compactly supported coefficients. Then $\mathbb{D}_Y\phi = X_Y\phi$, while $\mathbb{D}_Y\zeta = \mathcal{N}_{\mathfrak{a}_\natural(X_Y)}\zeta$ is a derivative of ζ along a vector field in U that is divergence-free and vanishes at $U = 0$. We apply (16.28) to $\zeta\phi$ and ψ, and we move these normal charges from ζ to the coefficients of $\mathcal{Q}_H$ by parts in U; the factors ϕ and $\nabla_H\psi$ do not depend on U. The resulting terms

vanish at $U = 0$. Since ζ is arbitrary, the integral against $\mathrm{d}\nu$ of $\phi\, L_H(U)\psi + \mathcal{Q}_H(\phi,\psi)$ plus these terms vanishes identically in U. At $U = 0$ only the first two terms remain, and we conclude that the restriction to $U = 0$ is the operator on $\mathcal{M}_k$ with the carré du champ $\mathcal{Q}_H$ at $U = 0$:

$$\int \phi\, L_H(0)\psi \,\mathrm{d}\nu = -\int \mathcal{Q}_H(\phi,\psi)|_{U=0}\,\mathrm{d}\nu = -\frac{1}{4s_E^2}\int \mathrm{STr}(\nabla_H\phi\,\nabla_H\psi)\,\mathrm{d}\nu.$$

In the second step we used that $\mathrm{ad}_{R_*} A = -2\mathrm{i}s_E\Lambda A$ for $A \in \mathfrak{m}$, so that $\mathrm{ad}_{R_*}^{-1} A = (\mathrm{i}/2s_E)\Lambda A$ and, since $\Lambda A\Lambda = -A$,

$$\mathrm{STr}(\mathrm{ad}_{R_*}^{-1} A\ \mathrm{ad}_{R_*}^{-1} B) = -\frac{1}{4s_E^2}\,\mathrm{STr}(\Lambda A\Lambda B) = \frac{1}{4s_E^2}\,\mathrm{STr}(AB).$$

Next, we express ∇_H through the symmetries $\widehat{Y}$ of Lemma 5.4. Let $Y \in \mathfrak{g}$ be homogeneous, with the parameter ϑ_Y of Section 5.2. Applying the even derivation $\vartheta_Y\widehat{Y}$ to $Q = T\Lambda T^{-1}$ and using (5.6), we find $[(\vartheta_Y\widehat{Y}T)T^{-1} - \vartheta_Y Y, Q] = 0$. Hence $\mathfrak{a}(\vartheta_Y\widehat{Y}) - T^{-1}\vartheta_Y YT$ commutes with Λ. Its component in $\mathfrak{m}$ vanishes, and $\mathfrak{a}_\mathfrak{m}(\vartheta_Y\widehat{Y}) = (T^{-1}\vartheta_Y YT)_\mathfrak{m}$. For an even function f of x, (16.25) with $\nabla_{\mathrm{nor}} f = 0$ and the orthogonality of $\mathfrak{k}$ and $\mathfrak{m}$ then imply that

$$\vartheta_Y\widehat{Y}f = \mathrm{STr}\big(\mathfrak{a}_\mathfrak{m}(\vartheta_Y\widehat{Y})\nabla_H f\big) = \mathrm{STr}\big(\vartheta_Y Y\,\widehat{\nabla} f\big),$$

where we write $\widehat{\nabla} f = T(\nabla_H f)T^{-1}$. With $Y = E_{ij}$, this gives $(\widehat{\nabla} f)_{ji} = (-1)^{|i|}\widehat{E_{ij}}f$, and therefore

$$\mathrm{STr}(\nabla_H\phi\,\nabla_H\psi) = \mathrm{STr}(\widehat{\nabla}\phi\,\widehat{\nabla}\psi) = \sum_{i,j}(-1)^{|j|}(\widehat{E_{ji}}\phi)(\widehat{E_{ij}}\psi).$$

Finally, by Section 16.2 the volume ν is a constant multiple of the integral of Definition 5.6. Using the integration by parts formula (5.13) with the even function ϕ, and then the definition (5.14), we obtain

$$\begin{aligned}\int \phi\, L_H(0)\psi\,\mathrm{d}\nu &= -\frac{1}{4s_E^2}\sum_{i,j}(-1)^{|j|}\int(\widehat{E_{ji}}\phi)(\widehat{E_{ij}}\psi)\,\mathrm{d}\nu \\ &= \frac{1}{4s_E^2}\sum_{i,j}(-1)^{|j|}\int \phi\,\widehat{E_{ji}}\widehat{E_{ij}}\psi\,\mathrm{d}\nu = -\frac{1}{2s_E^2}\int \phi\,\mathsf{C}_k\psi\,\mathrm{d}\nu.\end{aligned}$$

Since ϕ is arbitrary, this proves (16.30) for even ψ, and the case of odd ψ follows by applying it to $\vartheta\psi$. □

Together with the time normalization of one link, this identity determines the generator of the tangent motion of the chain; this is done in Section 18.4.

The tangent Gaussian normalization is fixed by the same geometry. By the proof of Lemma 16.2, the linear map $Y \mapsto [Y, S]$ of $\mathfrak{m}$ has Berezinian $\pm\varpi(S)^2$. The normalized

Gaussian integral of $e^{-\operatorname{STr}[Y,S]^2/2}$ over $\mathfrak{m}$ is therefore $\varpi(S)^{-2}$ times its value for the flat metric, which cancels the tubular density. Recall from (16.10) that the density j of ν in the exponential chart is even, with $j(0) = 1$; by the invariance of ν under G_0, the same holds in the exponential chart $g_0 e^Y$ around every point $g_0 \Sigma g_0^{-1}$. Consequently $j(h^2 Y) = 1 + O(h^4 |Y|^2)$ on a chart of fixed size.

16.4 The normal oscillator

We now realize the normal variables on a Hilbert space, the positive Wick realization, on which the frozen normal transfer of Section 16.5 will be a contraction.

We first consider one normal matrix coordinate, with real variables x, y, which we treat as independent standard Gaussian variables, and Grassmann variables α, β. We let $\mathbb{E}_{\mathrm{odd}}$ denote the Gaussian Berezin functional of the odd variables, the Berezin integral against $e^{-\alpha\beta}$ normalized by $\mathbb{E}_{\mathrm{odd}} 1 = 1$. For $f = f_\varnothing + f_\alpha \alpha + f_\beta \beta + f_{\alpha\beta} \alpha\beta$, we have

$$\mathbb{E}_{\mathrm{odd}}(f) = f_\varnothing - f_{\alpha\beta}.$$

Since $\mathbb{E}_{\mathrm{odd}}(\alpha\beta) = -1$, it is not a positive functional, and we use instead the following positive metric.

Definition 16.6. Let $\mathcal{H}_{\mathrm{odd}}$ denote the four-dimensional space spanned by 1, α, β, and $\alpha\beta + 1$, with these four vectors declared orthonormal, and give the functions of (x, y) the Gaussian L^2 norm. We define the normal Hilbert space $\mathcal{H}_{\mathrm{nor}}$ to be the tensor product of these spaces over the finitely many coordinates of an orthonormal basis of normal Hermitian matrices and over the two roots, and the Wick basis to be the product of the Hermite basis in the even variables and the basis above in the odd variables. We call the inner product $\langle \cdot, \cdot \rangle$ of $\mathcal{H}_{\mathrm{nor}}$, in which the Wick basis is orthonormal, the positive Wick metric, and $\mathcal{H}_{\mathrm{nor}}$ with this metric the positive Wick realization of the normal variables.

The degree of the Wick basis defines the normal number operator $\mathbf{N} \geq 0$, which counts all excitations, even and odd. We let $\mathbf{N}_\sigma$ denote the part of root σ and $\mathbf{N}_F$ the odd part. The vacuum $\Omega_0 = 1$ is the only Wick monomial of degree zero.

We let ℓ_0 denote the tensor product, over the coordinates, of the Gaussian expectation of the even variables and of the functionals $\mathbb{E}_{\mathrm{odd}}$, and we call it the Gaussian mean. It is the normalized Gaussian Berezin integral against $e^{-\frac{1}{2}\sum(x^2+y^2+2\alpha\beta)}$. We define the *analytic Berezin pairing* of the normal variables by

$$\mathcal{B}_{\mathrm{nor}}(f, g) = \ell_0(fg), \qquad f, g \in \mathcal{H}_{\mathrm{nor}}. \tag{16.31}$$

It is bilinear and indefinite, whereas the Wick metric is sesquilinear and positive. For one coordinate, for instance, $\mathcal{B}_{\mathrm{nor}}(1, 1) = 1$, $\mathcal{B}_{\mathrm{nor}}(\alpha, \beta) = -1$, and $\mathcal{B}_{\mathrm{nor}}(\alpha\beta + 1, \alpha\beta + 1) = -1$,

while all three vectors have norm one. In general, $\mathcal{B}_{\mathrm{nor}}$ pairs each Wick monomial with exactly one Wick monomial of the same degree, with a sign, since the Hermite functions are real and orthonormal and $\mathbb{E}_{\mathrm{odd}}$ pairs 1 with 1, α with β, and $\alpha\beta + 1$ with itself. In this way, $\mathcal{B}_{\mathrm{nor}}(f, g)$ becomes the inner product of g with the vector obtained from f by conjugating its coefficients in the Wick basis and permuting them with these signs, an antilinear bijective isometry of $\mathcal{H}_{\mathrm{nor}}$. In particular, $|\mathcal{B}_{\mathrm{nor}}(f, g)| \le \|f\|\|g\|$.

We use the two forms for different purposes, and we fix the corresponding words. Let X be a bounded operator on $\mathcal{H}_{\mathrm{nor}}$, or on a space of which $\mathcal{H}_{\mathrm{nor}}$ is a tensor factor, with the corresponding pairing and metric. We call the operator X^* with $\langle X^* f, g\rangle = \langle f, Xg\rangle$ the (Hilbert) adjoint of X, and the operator X^{tr} with $\mathcal{B}(X^{\mathrm{tr}} f, g) = \mathcal{B}(f, Xg)$ its (analytic) transpose. If a linear form on the vectors is written as $\mathcal{B}(g, \cdot)$, we call g its dual vector. For an operator on $\mathcal{H}_{\mathrm{nor}}$, the transpose is the conjugate of the adjoint by the antilinear isometry above, so that the two have the same norm. The transpose is defined without complex conjugation, so that it depends analytically on analytic parameters of X, while the adjoint appears in all norm estimates. We let $\mathbf{P}_0$ denote the orthogonal projection onto $\mathbb{C}\Omega_0$ and set $\mathbf{P}_0^\perp = \mathrm{Id} - \mathbf{P}_0$. Since every Wick monomial of positive degree has Gaussian mean zero, we have $\mathbf{P}_0 f = \ell_0(f)\Omega_0$ for all $f \in \mathcal{H}_{\mathrm{nor}}$. For an operator X on a Hilbert space of which $\mathcal{H}_{\mathrm{nor}}$ is a tensor factor, with $\mathbf{P}_0$ acting on this factor, we call $\mathbf{P}_0^\perp X \mathbf{P}_0^\perp$ its *fast block*, $\mathbf{P}_0 X \mathbf{P}_0^\perp$ and $\mathbf{P}_0^\perp X \mathbf{P}_0$ its *vacuum row* and *vacuum column*, and $\mathbf{P}_0 X \mathbf{P}_0$ its *vacuum scalar*.

For the analytic computations it is convenient to also use flat coordinates. For a single species of d real even coordinates, we call $L^2(\mathbb{R}^d) \otimes \mathcal{H}_{\mathrm{odd}}$ the flat realization, with

$$\mathbf{N} = \tfrac{1}{2}(-\Delta + |x|^2 - d) + \mathbf{N}_F,$$

and the two realizations are related by the ground state $\Omega_B(x) = \pi^{-d/4} e^{-|x|^2/2}$. Since $|\Omega_B|^2$ has variance $1/2$ and the Wick variables have variance one, the flat coordinate is the Wick coordinate divided by $\sqrt{2}$. Mehler's formula [89] gives the kernel of $e^{-z\mathbf{N}}$ on the even variables,

$$k_z(x, y) = \frac{e^{dz/2}}{(2\pi \sinh z)^{d/2}} \exp\Big\{ -\frac{1}{4} \coth(z/2)|x - y|^2 - \frac{1}{4} \tanh(z/2)|x + y|^2 \Big\}, \qquad (16.32)$$

for $\mathrm{Re}\, z > 0$. The ground-state transform $\Omega_B^{-1} e^{-z\mathbf{N}} \Omega_B$ on the even variables is the Ornstein–Uhlenbeck semigroup at the complex time z. For real z its kernel is the Gaussian density in y with mean $e^{-z}x$ and covariance $\frac{1}{2}(1 - e^{-2z})\mathrm{Id}$, and by analytic continuation the same formula holds for $\mathrm{Re}\, z > 0$, with the bilinear square in the exponent.

16.5 The adapted transfer operator and the normal gauge

We fix the normalization of the transfer operator in tubular coordinates. Let w_Γ be the onsite weight of Section 4 with the source supermatrix Γ, and let w_0 be its value at $\Gamma = 0$.

We set $s_\Gamma = w_\Gamma / w_0$ and call it the source factor. Recall from (16.15) that $\mathrm{d}R = \varpi^2 \, \mathrm{d}U \, \mathrm{d}\nu$ in tubular coordinates.

In the balanced signature, the tangent variable x runs over the balanced component, with the points $\kappa_E - \mathrm{i}s_E Q$ for $Q \in \mathcal{M}_k$, and by Section 16.2 the volume ν agrees with the integral of Definition 5.6 up to a constant factor. The normalization of ν at the root assigns unit Gaussian mass to the ambient metric, which on the tangent space at the root is a positive multiple of $\mathrm{STr}\, Y^2$, $Y \in \mathfrak{m}$. Since $d_k(\Lambda e^{-2Y}) = \mathrm{STr}\, Y^2 + O(|Y|^4)$, the normalization (5.12) also gives unit Gaussian mass at the root to a positive multiple of $\mathrm{STr}\, Y^2$. The Gaussian mass of $e^{-c\,\mathrm{STr}\, Y^2}$ does not depend on $c > 0$, since $\mathfrak{m}$ has superdimension zero, and the constant factor is therefore ± 1. With the orientation of G_0/K_0 fixed before Lemma 16.2, it is $+1$. Hence $\mathrm{d}\nu$ is the invariant measure of $\mathcal{M}_k$ with the orientation of $\mathcal{B}_k$; Section 20 uses this in the proofs of Propositions 20.9 and 20.10.

We let K^ϖ denote the operator $\varpi K_{\tau_{\mathrm{phys}}} \varpi^{-1}$, acting on functions of (x, U) with the flat measure $\mathrm{d}U \, \mathrm{d}\nu$. Its kernel is

$$K^\varpi(x, U; y, U') = \varpi(U)\varpi(U') K_{\tau_{\mathrm{phys}}}\big(R(x, U), R(y, U')\big), \tag{16.33}$$

where we write $\varpi(U) = \varpi(R_* + U)$. The two factors ϖ come from the half-density change. Indeed, multiplying functions by ϖ turns the pulled-back measure $\varpi(U')^2 \, \mathrm{d}U' \, \mathrm{d}\nu(y)$ into the flat measure. We let K^ϖ_{nor} denote the flat normal heat kernel, the operator in the normal variable U with the kernel $c_{\mathrm{nor}} \exp\{-\mathrm{STr}(U - U'')^2/(2h^4)\}$, where c_{nor} is the constant for which this operator maps the constant function 1 to itself; in the balanced signature $c_{\mathrm{nor}} = 1$, since the normal space has superdimension zero. We call it the frozen normal heat kernel of K^ϖ. Setting $x = y$ in K^ϖ and integrating out the tangent increment against its normalized Gaussian, we obtain K^ϖ_{nor} up to the factor $\varpi(U)\varpi(U'')\varpi(S)^{-2} = 1 + O(\delta_W |V|^2)$, with $S = R_* + (U + U'')/2$ and $V = (U - U'')/h^2$; this factor is treated in Section 17.6. We compute the spectral decomposition of K^ϖ_{nor} in Lemma 16.10.

Definition 16.7. We call the realization of the chain on functions of the tubular coordinates (x, U), with the flat measure $\mathrm{d}U \, \mathrm{d}\nu$, in which one link acts by K^ϖ and the constant function 1 is represented by ϖ, the *raw realization*.

The gauged realization below is obtained from the raw realization by conjugation with $w_0^{1/2}\gamma_h$, with the Gaussian gauge γ_h of (16.35).

We call the square root $w_0^{1/2}$ of the source-free weight the half-weight. On the normal tube, we write it as

$$w_0^{1/2} = w_{\mathrm{quad}}^{1/2} \, w_{\mathrm{anh}}^{1/2}, \tag{16.34}$$

where $w_{\mathrm{quad}}^{1/2}$ contains the constant and quadratic Taylor terms of $\log w_0^{1/2}$ in the normal variables and $w_{\mathrm{anh}}^{1/2}$ is the remaining anharmonic part. We retain the physical weight w_0

in every signature and set

$$\zeta_p = w_0(R_*) = \left(e^{2\mathrm{i}\kappa_E s_E} \frac{\kappa_E - \mathrm{i}s_E}{\kappa_E + \mathrm{i}s_E} \right)^{k-p}, \qquad |\zeta_p| = 1, \qquad \zeta_k = 1.$$

This follows by evaluating $e^{-\operatorname{STr}(R_*^2)/2} \operatorname{SDet}(E - R_*)^{-1}$ at the root of signature p. Choose a continuous branch of $\zeta_p^{1/2}$ on the energy interval, with $\zeta_k^{1/2} = 1$ in the balanced signature, and fix $w_0^{1/2}(R_*) = w_{\mathrm{quad}}^{1/2}(0) = \zeta_p^{1/2}$ and $w_{\mathrm{anh}}^{1/2}(0) = 1$. Thus the two quadratic half-weights retain one factor ζ_p per step. The endpoint factors below use the same physical half-weight, with its inverse at the right endpoint. For operator norms, we extend $\log w_0^{1/2}$ and $\log w_{\mathrm{anh}}^{1/2}$ from the tube by a fixed smooth invariant cutoff on a slightly larger tube. The exponentials of the extended functions are bounded invertible multipliers that agree with the analytic ones on the tube.

We define the *Gaussian gauge* to be the function

$$\gamma_h(U) = \exp\Big\{ -\frac{1}{2} \sum_\sigma c_\sigma \operatorname{STr}(U_\sigma^2) \Big\}, \tag{16.35}$$

where U_σ is the normal block of root σ and c_σ is the number of (16.13). On the adapted real normal space of Definition 16.1, we have $U_\sigma = e^{-\mathrm{i}\theta_\sigma} X_\sigma$ with $X_\sigma \in \mathfrak{k}_\sigma^{\mathbb{R}}$, and $c_\sigma \operatorname{STr}(U_\sigma^2) = |c_\sigma| \operatorname{STr}(X_\sigma^2)$, since $e^{-2\mathrm{i}\theta_\sigma} c_\sigma = |c_\sigma|$. In particular, γ_h is real on the real tube, and its body is a Gaussian whose precision is of order W in each even coordinate. We show in Lemma 16.10 that conjugation by γ_h maps the flat realization of the exact quadratic normal transfer to the positive Wick realization of Section 16.4, in the standardized coordinates $Z_\sigma = (2c_\sigma)^{1/2} U_\sigma$, which are real on the adapted real normal space. In these coordinates $\gamma_h^2 = e^{-\operatorname{STr}(Z^2)/2}$, the Gaussian of the Gaussian mean ℓ_0. We define the analytic Berezin pairing on the adapted tube by

$$\mathcal{B}_{\mathrm{ad}}(f, g) = \int f g \, \gamma_h^2 \, \mathrm{d}U \, \mathrm{d}\nu. \tag{16.36}$$

In the balanced signature, $\int \gamma_h^2 \, \mathrm{d}U = 1$, since the change of variables to Z has Berezinian one on a space of superdimension zero and the Gaussian $e^{-\operatorname{STr}(U^2)/2}$ has the integral one for $\mathrm{d}U$. The normal part of $\mathcal{B}_{\mathrm{ad}}$ is then the pairing $\mathcal{B}_{\mathrm{nor}}$ of (16.31): for normal vectors u and u', with u' even, and functions f and f' of the tangent variables,

$$\mathcal{B}_{\mathrm{ad}}(uf, u'f') = \mathcal{B}_{\mathrm{nor}}(u, u') \int f f' \, \mathrm{d}\nu. \tag{16.37}$$

In an unbalanced signature, $\int \gamma_h^2 \, \mathrm{d}U$ is a nonzero constant, a fixed power of the c_σ.

We define the adapted transfer operator A_h, the exact quadratic normal transfer $\mathbf{M}_h^{\mathrm{q}}$, the frozen normal transfer $\mathbf{F}_h$, and the right and left endpoint factors g_R, g_L by

$$\begin{aligned} A_h &= \gamma_h^{-1} w_0^{-1/2} \, T_\Gamma \, w_0^{1/2} \gamma_h = s_\Gamma \, \gamma_h^{-1} w_0^{1/2} K^\omega w_0^{1/2} \gamma_h, \\ \mathbf{M}_h^{\mathrm{q}} &= \gamma_h^{-1} w_{\mathrm{quad}}^{1/2} K_{\mathrm{nor}}^\omega w_{\mathrm{quad}}^{1/2} \gamma_h, \qquad \mathbf{F}_h = w_{\mathrm{anh}}^{1/2} \, \mathbf{M}_h^{\mathrm{q}} \, w_{\mathrm{anh}}^{1/2}, \\ g_R &= \omega \, w_0^{-1/2} \gamma_h^{-1}, \qquad g_L = \omega \, w_0^{1/2} \gamma_h^{-1}, \end{aligned} \tag{16.38}$$

where T_Γ is written in tubular coordinates with the half-density change, that is, $T_\Gamma = w_\Gamma K^\varpi$. The second expression for A_h uses $w_\Gamma w_0^{-1/2} = s_\Gamma w_0^{1/2}$. The source factor s_Γ stands on the left. We call the normalization (16.38) the symmetric normal gauge.

The two endpoint factors represent the two ends of the chain. In the raw realization, both ends are the half-density form ϖ of the constant function 1, a vector at the right end and the linear form $F \mapsto \int \varpi F \, \mathrm{d}U \, \mathrm{d}\nu$ at the left end. In the gauged realization, vectors F become $\gamma_h^{-1} w_0^{-1/2} F$, so that $g_R = \gamma_h^{-1} w_0^{-1/2} \varpi$, and

$$\int \varpi F \, \mathrm{d}U \, \mathrm{d}\nu = \mathcal{B}_{\mathrm{ad}}\big(g_L, \gamma_h^{-1} w_0^{-1/2} F\big) \tag{16.39}$$

for all raw vectors F. This identity says that g_L is the dual vector, for the analytic Berezin pairing, of the linear form that ends the chain, and we call it the analytic dual of g_R. It is not the Hilbert adjoint of g_R. The frozen normal transfer $\mathbf{F}_h$ is symmetric for the analytic Berezin pairing, $\mathbf{F}_h^{\mathrm{tr}} = \mathbf{F}_h$, since

$$\gamma_h \mathbf{F}_h \gamma_h^{-1} = w_{\mathrm{anh}}^{1/2} w_{\mathrm{quad}}^{1/2} K_{\mathrm{nor}}^{\varpi} w_{\mathrm{quad}}^{1/2} w_{\mathrm{anh}}^{1/2}$$

and $K_{\mathrm{nor}}^{\varpi}$ has a symmetric kernel. All transposes in Part IV are transposes for the analytic Berezin pairing, in the sense of Section 16.4.

Lemma 16.8. *The functions $\varpi^{\pm 1}$, $w_0^{\pm 1/2}$, $w_{\mathrm{quad}}^{\pm 1/2}$, $w_{\mathrm{anh}}^{\pm 1/2}$ and $\gamma_h^{\pm 1}$ on the normal tube are normal class functions. In particular, the endpoint factors g_R and g_L are normal class functions, and they are annihilated by the normal charges.*

Proof. Let k be an even invertible supermatrix with Grassmann entries that commutes with Σ. Then $k = \mathrm{diag}(k_+, k_-)$ and $k(R_* + U)k^{-1} = R_* + kUk^{-1}$. For ϖ, conjugation of S by k conjugates L_S by $\lambda_{k_+}\rho_{k_-^{-1}}$, and Lemma 3.11(b) shows that ϖ is unchanged. The weight w_0 is invariant under conjugation by Remark 4.12, and the same holds for $w_0^{1/2}$, whose branch is fixed at $U = 0$. Since $U \mapsto kUk^{-1}$ is linear, it maps the Taylor part of each degree of $\log w_0^{1/2}$ at $U = 0$ to the Taylor part of the same degree. Hence the constant and quadratic terms $\log w_{\mathrm{quad}}^{1/2}$ are invariant, and therefore also $w_{\mathrm{anh}}^{1/2} = w_0^{1/2} / w_{\mathrm{quad}}^{1/2}$. Finally,

$$\mathrm{STr}\big((kUk^{-1})_\sigma^2\big) = \mathrm{STr}(k_\sigma U_\sigma^2 k_\sigma^{-1}) = \mathrm{STr}(U_\sigma^2)$$

by Lemma 3.11(c), so that γ_h is invariant. The products g_R and g_L of (16.38) are therefore invariant, and the normal charges, which are derivatives along conjugations by elements of this kind, annihilate them. □

We keep the name $\mathbf{F}_h$ for the frozen normal transfer with the normal cutoffs of Section 16.7, and Sections 16.9 and 16.10 study it.

Remark 16.9. On the tube, $w_0^{\pm 1/2}$ and $w_{\mathrm{anh}}^{\pm 1/2}$ are bounded analytic multipliers, and the identities (16.38) are exact for all odd coefficients and all source jets. Indeed, every normal determinant in w_0 has nonzero body on the tube. Its square root, its inverse and its finite nilpotent completion are therefore analytic and bounded there. In the standardized coordinates $U = hZ$, each derivative of a coefficient carries a factor h, and the product rule gives bounds in the graph norms $\|(\mathbf{N}+1)^m f\|$ of the number operator of Section 16.4, which we use from Section 16.7 on. Setting $B_h = \gamma_h^{-1} T_\Gamma \gamma_h$, we have

$$A_h = w_0^{-1/2} B_h w_0^{1/2}, \qquad B_h^n = w_0^{1/2} A_h^n w_0^{-1/2}.$$

The condition numbers of the half-weight therefore appear once in each power. All source and cutoff multipliers are even functions, and they commute with these scalar normal multipliers. In the balanced signature the equal even and odd dimensions cancel the standardizing Berezinians and fix the Gaussian vacuum pairing at one. In an unbalanced signature the constant $\varpi(R_*)$ is kept separately until it is suppressed in Proposition 16.22.

The transfer operator of Proposition 4.11 applies the onsite weight after the heat factor. Its analytic transpose ends with a heat factor, and the identities (16.38) transport this endpoint without changing the order of the factors.

16.6 The quadratic normal transfer and the supercharge

We now compute the exact quadratic normal transfer of (16.38) in the positive Wick realization of Section 16.4, and we introduce the supercharge of the normal variables.

The exact quadratic normal transfer

Recall the roots r_σ and the numbers m_σ, c_σ and θ_σ of (16.13), and set

$$A_\sigma = W^2 + \tfrac{1}{2} m_\sigma, \qquad \mu_\sigma = \frac{W^2}{A_\sigma + c_\sigma}, \qquad z_\sigma = -\log \mu_\sigma, \tag{16.40}$$

with the branches of positive real part, which are complex conjugate for the two roots. We have $z_\sigma = 2\,\mathrm{arsinh}(m_\sigma^{1/2}/(2W))$, so that $\mathrm{Re}\, z_\sigma \asymp W^{-1} = h^2$ and $|\mathrm{Im}\, z_\sigma| \le C|E|\,\mathrm{Re}\, z_\sigma$. Recall the standardized coordinates $Z_\sigma = (2c_\sigma)^{1/2} U_\sigma$ of Section 16.5, which scale the even and odd coordinates of each root block by the same factor. Since $2|c_\sigma|h^2 = 2|m_\sigma|^{1/2}(1 + O(h^4))$ lies between two positive constants, Z is comparable to U/h. We write $U = hZ$ for this relation, absorbing the bounded factors $(2c_\sigma h^2)^{1/2}$ into constants and cutoff radii, and we call U the unscaled normal coordinate.

We also fix coordinates on a normal block. In the balanced signature, we write a root block U_σ through its even blocks u_{11} and u_{22} and its odd blocks ξ_1 and ξ_2, all of size $k \times k$, and we set

$$\begin{aligned} U_\sigma &= \begin{pmatrix} u_{11} & \xi_1 \\ \xi_2 & u_{22} \end{pmatrix}, \qquad \operatorname{STr} U_\sigma^2 = \sum_e (x_e^2 + y_e^2 + 2\alpha_e\beta_e), \\ u_{11} &= \sum_e x_e e, \quad u_{22} = \mathrm{i}\sum_e y_e e, \quad \xi_1 = \sum_e \alpha_e e, \quad \xi_2 = \sum_e \beta_e e, \end{aligned} \tag{16.41}$$

where $\{e\}$ is a basis of the Hermitian $k \times k$ matrices that is orthonormal for the form $\operatorname{Tr}(ee')$, the coordinates x_e and y_e are even, and α_e and β_e are odd. The identity for $\operatorname{STr} U_\sigma^2$ follows from $\operatorname{STr} U_\sigma^2 = \operatorname{Tr}(u_{11}^2) - \operatorname{Tr}(u_{22}^2) + 2\operatorname{Tr}(\xi_1\xi_2)$. On the adapted real normal space of Definition 16.1, the coordinates x_e and y_e are $e^{-\mathrm{i}\theta_\sigma}$ times real numbers. In an unbalanced signature, the two even blocks have different sizes. We then expand u_{11} and $-\mathrm{i}u_{22}$ in orthonormal bases of the Hermitian matrices of their sizes, and we take the pairs of odd coordinates $\alpha = (\xi_1)_{ij}$ and $\beta = (\xi_2)_{ji}$, so that $\operatorname{STr} U_\sigma^2$ has the same form.

Lemma 16.10. *In the balanced signature, the exact quadratic normal transfer of* (16.38) *is*

$$\mathbf{M}_h^{\mathrm{q}} = \mu_+^{\mathbf{N}_+}\mu_-^{\mathbf{N}_-}$$

on the positive Wick realization. In the signature $p \neq k$, it is $\lambda\, \mu_+^{\mathbf{N}_+}\mu_-^{\mathbf{N}_-}$ on the normal space of that signature, where

$$\lambda = \zeta_p(\mu_+\mu_-)^{(p-k)^2/2}, \qquad |\lambda| = e^{-(p-k)^2 \operatorname{Re} z_+}.$$

We use the same notation $\lambda = 1$ in the balanced signature.

Proof. The two constant factors $\zeta_p^{1/2}$ in the quadratic half-weights contribute ζ_p once. We first compute the quadratic transfer with this scalar removed. We treat one even coordinate of root σ, and write it as $U = e^{-\mathrm{i}\theta_\sigma}x$ with x real, as in Definition 16.1. In this coordinate, the kernel of one step of the exact quadratic normal transfer, with the quadratic half-weights, is the symmetric Gaussian kernel

$$\frac{We^{-\mathrm{i}\theta_\sigma}}{\sqrt{2\pi}} \exp\{-\tfrac{1}{2}W^2e^{-2\mathrm{i}\theta_\sigma}(x-y)^2 - \tfrac{1}{4}m_\sigma e^{-2\mathrm{i}\theta_\sigma}(x^2+y^2)\}$$

with respect to $\mathrm{d}y$. We set

$$A'_\sigma = e^{-2\mathrm{i}\theta_\sigma}A_\sigma, \qquad c'_\sigma = e^{-2\mathrm{i}\theta_\sigma}c_\sigma = |c_\sigma| > 0,$$

where the last identity is the definition of θ_σ in (16.13). Completing the square in y, we find that

$$\begin{aligned} &\frac{We^{-\mathrm{i}\theta_\sigma}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}W^2e^{-2\mathrm{i}\theta_\sigma}(x-y)^2 - \frac{1}{4}m_\sigma e^{-2\mathrm{i}\theta_\sigma}(x^2+y^2) - \frac{1}{2}c'_\sigma y^2}\,\mathrm{d}y \\ &\qquad = \Big(\frac{W^2}{A_\sigma + c_\sigma}\Big)^{1/2} e^{-c'_\sigma x^2/2} = \mu_\sigma^{1/2}e^{-|c_\sigma|x^2/2}, \end{aligned}$$

where we used the identity

$$\frac{W^4 e^{-4\mathrm{i}\theta_\sigma}}{A'_\sigma + c'_\sigma} = A'_\sigma - c'_\sigma,$$

which follows from $A_\sigma^2 - c_\sigma^2 = W^4$.

The same Gaussian integral with an additional linear term e^{ty}, expanded in t, shows that the kernel acts on the nth Hermite function of the real Gaussian $e^{-|c_\sigma|x^2/2}$ by the factor $\mu_\sigma^{n+1/2}$. These Hermite functions form an orthonormal basis, so the kernel is diagonal in a real orthonormal basis, and the corresponding ground state, the Gaussian gauge γ_h, is real.

We now add the odd coordinates, which appear in $\mathrm{STr}\, U_\sigma^2$ in pairs (α, β), through $2\alpha\beta$, as in (16.41). This step is the supersymmetric cancellation of the zero-point factors.

For one pair, the odd parts of the heat kernel, of the two quadratic half-weights and of the Gaussian gauge (16.35) are

$$-W^{-2}\exp\{-W^2(\alpha-\alpha')(\beta-\beta')\}, \qquad e^{-m_\sigma\alpha\beta/2}, \quad e^{-m_\sigma\alpha'\beta'/2}, \qquad e^{-c_\sigma\alpha\beta},$$

where the factor $-W^{-2}$ is the one for which the odd heat kernel maps 1 to 1, with the Berezin integral of Definition 3.5 in (α', β'). Collecting the exponents, we find that the odd factor of $\mathbf{M}_h^{\mathrm{q}}$ acts on functions of (α', β') by

$$f \longmapsto -W^{-2} e^{(c_\sigma - A_\sigma)\alpha\beta} \int \exp\big\{-(A_\sigma + c_\sigma)\alpha'\beta' + W^2(\alpha\beta' + \alpha'\beta)\big\} f(\alpha', \beta')\, \mathrm{d}\alpha'\, \mathrm{d}\beta'.$$

Since the three terms in the last exponent are even and have square zero, we have

$$\exp\big\{-(A_\sigma + c_\sigma)\alpha'\beta' + W^2(\alpha\beta' + \alpha'\beta)\big\} = 1 - (A_\sigma + c_\sigma)\alpha'\beta' + W^2\alpha\beta' + W^2\alpha'\beta - W^4\alpha'\beta'\alpha\beta,$$

and we obtain

$$\begin{aligned} 1 &\longmapsto W^{-2}(A_\sigma + c_\sigma) = \mu_\sigma^{-1}, \qquad \alpha \longmapsto \alpha, \qquad \beta \longmapsto \beta, \\ \alpha\beta + (2c_\sigma)^{-1} &\longmapsto W^{-2}(A_\sigma - c_\sigma)\big(\alpha\beta + (2c_\sigma)^{-1}\big) = \mu_\sigma\big(\alpha\beta + (2c_\sigma)^{-1}\big), \end{aligned} \tag{16.42}$$

where we used $A_\sigma^2 - c_\sigma^2 = W^4$, which cancels the term with $\alpha\beta$ in the image of 1 and gives $W^{-2}(A_\sigma - c_\sigma) = \mu_\sigma$. The supplementary script `tube_odd_wick.py` confirms the four images in (16.42) in exact arithmetic, with all their Grassmann components.

In the standardized odd coordinates $\tilde\alpha = (2c_\sigma)^{1/2}\alpha$ and $\tilde\beta = (2c_\sigma)^{1/2}\beta$, these four vectors become 1, $\tilde\alpha$, $\tilde\beta$ and $(2c_\sigma)^{-1}(\tilde\alpha\tilde\beta + 1)$, multiples of the odd Wick basis of Section 16.4. Hence the odd factor of one pair acts as $\mu_\sigma^{-1}\mu_\sigma^{\mathbf{N}_F}$ on the Wick realization. This is the Wick identity for the odd coordinates: each odd coordinate carries the zero-point factor $\mu_\sigma^{-1/2}$, which cancels the zero-point factor $\mu_\sigma^{1/2}$ of an even coordinate.

Tensoring over an orthonormal basis of normal Hermitian matrices, we obtain $\mu_\sigma^{\mathbf{N}_\sigma}$ times the zero-point factor $\mu_\sigma^{\mathrm{sdim}\,/2}$, where sdim is the superdimension of the normal

space of root σ. In the balanced signature these superdimensions vanish. In the signature p, the normal blocks $\mathrm{End}(V_+) \cong \mathfrak{gl}(k|p)$ and $\mathrm{End}(V_-) \cong \mathfrak{gl}(k|2k-p)$ both have the superdimension $(p-k)^2$, by (16.2), and the zero-point factors are $\mu_\sigma^{(p-k)^2/2}$. Their product $(\mu_+\mu_-)^{(p-k)^2/2}$ has modulus $e^{-(p-k)^2 \operatorname{Re} z_+}$, since $z_- = \bar{z}_+$. Restoring the scalar ζ_p gives the stated λ and leaves its modulus unchanged. □

Remark 16.11. We use two descriptions of one step of the chain, and only one of them leads to stability. The differential description (16.29) identifies the generator without an expansion around the marginal, and Section 18 turns it into consistency on smooth vectors. The kernel description, through the midpoint formula of Section 17, bounds the marginals of the kernels by $C\delta_W$ and thereby verifies, for each step, the hypothesis of Lemma A.1, which can be iterated over the h^{-4} steps of the chain on arbitrary inputs.

It is natural to try to prove stability from an energy bound for the heat generator in the positive Wick metric. Lemma 18.7 transports this metric through the full Gaussian gauge to a flat positive coefficient metric and estimates the normal charges and their Hilbert adjoints there. The ordinary normal Laplacian supplies the gradient energy needed to absorb the charge terms on a sufficiently small fixed tube. This gives the bound $\exp(Cth^{-2})$ at time t, which is $1 + O(h^2)$ over one link of duration $h^4/2$, but grows like $\exp(Ch^{-2})$ over a fixed macroscopic time. It therefore does not give the iterable bound $1 + O(h^4)$ per link. Invariance of the Gaussian Berezin density is not used to assert skew-adjointness of odd charges in the positive metric.

The obstruction is in the odd normal variables, whose heat has a nilpotent generator, which is dissipative in no metric. By (16.42), the odd factor of one step has the eigenvalues μ_σ^{-1}, 1, 1, and μ_σ, and $|\mu_\sigma|^{-1} = e^{\operatorname{Re} z_\sigma} \geq 1 + ch^2$. The eigenvalue μ_σ^{-1} is compensated by the even zero-point factor μ_σ, which appears only in the product of the heat with the half-weight. Since a norm is at least the spectral radius, in every metric of tensor-product form the odd heat and the odd part of the half-weight cannot both have norm $1 + O(\delta_W)$. In rank one there are even invariant vectors $u = \varphi(\mathrm{STr}\, U_\sigma^2)$, annihilated by the horizontal operator, on which the real part of the quadratic form of the gauged generator is at least $ch^{-2}\|u\|^2$; a numerical model in the supplementary material, `fock_dissipativity.py`, finds about $0.2h^{-2}$. For this reason we integrate the odd normal variables against the complete Mehler factor, which contains the half-weight, before taking any norm.

The supercharge and localization

Lemma 16.12. *On the span of the Wick polynomials, let* $\mathfrak{q}_{\mathrm{nor}}$ *be the sum over all balanced normal coordinates of the odd differential operators*

$$\beta(\partial_x - \mathrm{i}\partial_y) - (x - \mathrm{i}y)\partial_\alpha.$$

Then $\mathfrak{q}_{\mathrm{nor}}^2 = 0$ *and* $\mathfrak{q}_{\mathrm{nor}}\mathfrak{q}_{\mathrm{nor}}^* + \mathfrak{q}_{\mathrm{nor}}^*\mathfrak{q}_{\mathrm{nor}} = 2\mathbf{N}$, *and* $\mathfrak{q}_{\mathrm{nor}}$ *preserves the number of each coordinate. Its direct sum over the number spaces is a closed operator,* $\ker \mathfrak{q}_{\mathrm{nor}}$ *is closed, the spectral projections of* $\mathbf{N}$ *commute with* $\mathfrak{q}_{\mathrm{nor}}$, *and the Wick polynomials in* $\ker \mathfrak{q}_{\mathrm{nor}}$ *are dense in* $\ker \mathfrak{q}_{\mathrm{nor}}$.

The operator $\mathfrak{q}_{\mathrm{nor}}$ is a supercharge of the supersymmetric harmonic oscillator with number operator $\mathbf{N}$, in the sense of these two identities. It is one of the odd normal charges of Section 16.3, and therefore every invariant function of U is annihilated by it. To verify this, we write a balanced root block U through its blocks u_{11}, ξ_1, ξ_2, and u_{22} as in (16.41), and we take $\mathfrak{e} = \zeta \begin{pmatrix} 0 & \mathrm{Id} \\ 0 & 0 \end{pmatrix} \in \mathfrak{k}$ with an odd parameter ζ. For this $\mathfrak{e}$, we have

$$[\mathfrak{e}, U] = \zeta \begin{pmatrix} \xi_2 & u_{22} - u_{11} \\ 0 & \xi_2 \end{pmatrix},$$

so the variations of u_{11}, u_{22}, ξ_1, and ξ_2 are $\zeta\xi_2$, $\zeta\xi_2$, $\zeta(u_{22} - u_{11})$, and 0. In the coordinates of (16.41), the induced derivation on functions of U therefore takes the form

$$\zeta \sum_e \big[\beta_e(\partial_{x_e} - \mathrm{i}\partial_{y_e}) - (x_e - \mathrm{i}y_e)\partial_{\alpha_e}\big] = \zeta\mathfrak{q}_{\mathrm{nor}}.$$

The rotation and the standardization of Definition 16.1 preserve this form, since the field is homogeneous of degree zero. In the coordinates of the remark after Corollary 3.20, $\mathfrak{q}_{\mathrm{nor}}$ is the half $\mathfrak{q}_+$ of the supercharge $\mathfrak{q}$ of Definition 3.16; unlike $\mathfrak{q}$, whose square is a rotation generator, it squares to zero.

Proof. For one coordinate, let $a = (\partial_x - \mathrm{i}\partial_y)/\sqrt{2}$ and $b = (\partial_x + \mathrm{i}\partial_y)/\sqrt{2}$. Together with their Gaussian adjoints, they form two independent pairs of bosonic creation and annihilation operators. In the odd basis above, set

$$f_\alpha = \partial_\alpha, \quad f_\alpha^* = \alpha + \partial_\beta, \qquad f_\beta = \partial_\beta, \quad f_\beta^* = \beta - \partial_\alpha.$$

Their action on the four basis vectors verifies the canonical anticommutation relations, and that f_α^* and f_β^* are the adjoints of f_α and f_β in the positive metric. Since $x - \mathrm{i}y = \sqrt{2}(a + b^*)$ on Gaussian functions, the terms $\beta \cdot \sqrt{2}a$ and $-\sqrt{2}a\,\partial_\alpha$ combine into $\sqrt{2}f_\beta^* a$, and for one coordinate

$$\mathfrak{q}_{\mathrm{nor}} = \sqrt{2}(f_\beta^* a - f_\alpha b^*).$$

Each term exchanges one even and one odd excitation of the same coordinate. From the canonical relations we obtain, for one coordinate,

$$\mathfrak{q}_{\mathrm{nor}}^2 = 0, \qquad \mathfrak{q}_{\mathrm{nor}}\mathfrak{q}_{\mathrm{nor}}^* + \mathfrak{q}_{\mathrm{nor}}^*\mathfrak{q}_{\mathrm{nor}} = 2(a^*a + b^*b + f_\alpha^* f_\alpha + f_\beta^* f_\beta),$$

and the cross terms between different coordinates cancel by graded anticommutation. Summing over the coordinates proves the two identities.

Let $\mathcal{H}_{\mathrm{nor}} = \bigoplus_{n\geq 0} \mathcal{E}_n$ be the decomposition into the finite-dimensional eigenspaces of $\mathbf{N}$, and let $\mathfrak{q}_{\mathrm{nor},n}$ be the restriction of $\mathfrak{q}_{\mathrm{nor}}$ to $\mathcal{E}_n$. We define $\mathfrak{q}_{\mathrm{nor}}$ by

$$\mathfrak{q}_{\mathrm{nor}} f = \sum_{n\geq 0} \mathfrak{q}_{\mathrm{nor},n} f_n \quad \text{on the domain} \quad \Big\{ f = \sum_{n\geq 0} f_n : \sum_{n\geq 0} \|\mathfrak{q}_{\mathrm{nor},n} f_n\|^2 < \infty \Big\}.$$

Componentwise limits show that this operator is closed, $\ker \mathfrak{q}_{\mathrm{nor}} = \bigoplus_n \ker \mathfrak{q}_{\mathrm{nor},n}$ is closed, and the truncations to finitely many n commute with $\mathfrak{q}_{\mathrm{nor}}$ and converge strongly to the identity. On the domain of $\mathbf{N}$, this operator agrees with the differential operator, since the two operators are continuous for the graph norm of $\mathbf{N}$ and agree on Wick polynomials. □

Lemma 16.13. *Let U be a balanced normal supermatrix, and let $\mathrm{STr}\, U^2 = \sum_a (x_a^2 + y_a^2 + 2\alpha_a\beta_a)$ in the coordinates of* (16.41)*. Let f be a smooth function of U, with Grassmann coefficients, such that $\mathfrak{q}_{\mathrm{nor}} f = 0$ and such that f and its derivatives are bounded by $Ce^{a|U|^2}$ for some $a < 1/2$; for instance, f may be a Wick polynomial. Then the normalized Gaussian Berezin integral of f is the value $f_{\mathrm{body}}(0)$ of its body at the origin. The same holds for a uniformly accretive complex Gaussian precision, that is, for the Gaussian $\exp\{-\frac{1}{2}\sum_a \lambda_a (x_a^2 + y_a^2 + 2\alpha_a\beta_a)\}$ with one complex number λ_a per normal coordinate and $\mathrm{Re}\,\lambda_a \geq c > 0$, provided that $a < c/2$; each term of this exponent is annihilated by $\mathfrak{q}_{\mathrm{nor}}$.*

Proof. This is the localization of an integral under a $\mathfrak{q}_{\mathrm{nor}}$-exact deformation [103, 113], in the form of Corollary 3.20. We use the coordinates $\zeta_a = x_a + \mathrm{i}y_a$, $\bar{\chi}_a = \alpha_a$, and $\chi_a = 2\beta_a$ of the remark after that corollary, in which $\zeta^*\zeta + \chi^*\chi = \mathrm{STr}\, U^2$ and $\mathfrak{q}_+ = \mathfrak{q}_{\mathrm{nor}}$. Linear changes of coordinates multiply the Berezin integral by a constant, by Lemma 3.6(c) for the odd coordinates, so normalized Gaussian integrals are the same in both coordinate systems. We apply Corollary 3.20 with $\mathfrak{q}' = \mathfrak{q}_+$, $A = \frac{1}{2}\mathrm{Id}$, and $\gamma = a$, coefficientwise in the Grassmann coefficients of f. Its hypothesis $\gamma < \lambda_A = 1/2$ is our growth bound, and the corollary implies the first claim. For the complex precision, we take the diagonal matrix A with entries $\lambda_a/2$, for which $\lambda_A \geq c/2 > a$. □

In particular, every function of $\mathrm{STr}\, V^2$, such as the invariant cutoff of Section 16.7, is annihilated by $\mathfrak{q}_{\mathrm{nor}}$, including all its nilpotent terms.

16.7 Cutoffs and Mehler defects

We now prove the estimates for the frozen normal transfer that the comparison of Section 17 needs. They concern the Mehler operator with complex times of order h^2, a normal cutoff at a radius r that does not depend on h, and the normal charges.

Setting

Let $\mathcal{H}_{\mathrm{nor}} = L^2(\mathbb{R}^d) \otimes \mathcal{H}_{\mathrm{odd}}$ in the flat realization, with $\mathcal{H}_{\mathrm{odd}}$ a fixed finite-dimensional fermionic Fock space, and let the coordinates and odd modes be grouped into finitely many species σ, with $\mathbf{N} = \sum_\sigma \mathbf{N}_\sigma$. We consider

$$\mathbf{M} = \lambda \prod_\sigma e^{-z_\sigma \mathbf{N}_\sigma}, \qquad c_0 h^2 \le \operatorname{Re} z_\sigma \le C_0 h^2, \qquad |\operatorname{Im} z_\sigma| \le C_0 \operatorname{Re} z_\sigma,$$

where $0 < |\lambda| \le 1$ and $1 - |\lambda|^2 \le C_0 h^2$. We set $\underline{z} = \min_\sigma \operatorname{Re} z_\sigma$ and $D_{\mathrm{nor}} = \mathrm{Id} - \mathbf{M}^*\mathbf{M}$. We call $\mathbf{M}$ the Mehler operator and D_{nor} the normal defect. Since the $\mathbf{N}_\sigma$ commute, $\mathbf{M}$ is a normal contraction, $\mathbf{M}^* = \bar\lambda \prod_\sigma e^{-\bar z_\sigma \mathbf{N}_\sigma}$, and

$$D_{\mathrm{nor}} = \mathrm{Id} - \mathbf{M}\mathbf{M}^* = \mathrm{Id} - |\lambda|^2 \prod_\sigma e^{-2 \operatorname{Re} z_\sigma \mathbf{N}_\sigma}.$$

Recall from Lemma 16.10 that the exact quadratic transfer $\mathbf{M}_h^{\mathrm{q}}$ is of this form, with z_σ as in (16.40) and the full scalar λ, including ζ_p, of that lemma. In particular, $\mathbf{M}\Omega_0 = \lambda\Omega_0$. On $\mathbf{P}_0^\perp$, its defect is comparable to D_+ below: the phase-free defect is bounded below by $ch^2\mathbf{P}_0^\perp$, while $1 - |\lambda|^2 = O(h^2)$ adds at most a fixed multiple of that defect. The scalar λ therefore preserves these comparisons. The proofs below use only the moduli $|e^{-z_\sigma n}| = e^{-n \operatorname{Re} z_\sigma}$ and the positive real parts of the Mehler precisions, and for this reason every statement also holds for $\mathbf{M}^*$. The zero eigenspace of $\mathbf{N}$ is spanned by Ω_0. We fix $0 < h \le r \le r_*$. Constants may depend on the dimension, on $\mathcal{H}_{\mathrm{odd}}$, on c_0 and C_0, and on the indicated finite data, but not on h, and we indicate dependence on r explicitly. For an integer $m \ge 0$ we let $\mathcal{G}^m$ denote the domain of $(\mathbf{N}+1)^m$ with the graph norm $\|(\mathbf{N}+1)^m f\|$, and we say that an operator is bounded in the graph norm of order m if it is bounded on $\mathcal{G}^m$.

Let η be a smooth function on $[0,\infty)$, with $0 \le \eta \le 1$, equal to one on $[0,1]$ and to zero on $[4,\infty)$, and set $\chi(x) = \eta(|x|^2)$ for $x \in \mathbb{R}^d$. Thus χ is radial, equals one on the unit ball, and is supported in the ball of radius two. We set $\chi_{r/h}(x) = \chi(hx/r) = \eta(h^2|x|^2/r^2)$ and

$$\begin{aligned} \mathbf{M}^\chi &= \chi_{r/h}\mathbf{M}\chi_{r/h}, \\ \bar D_R &= \mathrm{Id} - (\mathbf{M}^\chi)^*\mathbf{M}^\chi, \\ \bar D_L &= \mathrm{Id} - \mathbf{M}^\chi(\mathbf{M}^\chi)^*, \\ D_+ &= \mathrm{Id} - e^{-2\underline{z}(\mathbf{N}+1)}. \end{aligned}$$

We call $\mathbf{M}^\chi$ the cutoff Mehler operator; it is a contraction, since $0 \le \chi_{r/h} \le 1$ and $\|\mathbf{M}\| \le 1$. We define the invariant cutoff by

$$\chi_{\mathrm{inv}}(Z) = \eta\Big(\frac{h^2 \operatorname{STr}(Z^2)}{2r^2}\Big).$$

Here x is the even flat coordinate: the even coordinates of the standardized normal supermatrix Z are $\sqrt{2}x$, by Section 16.4. Thus $\frac{1}{2}\operatorname{STr}(Z^2)$ has body $|x|^2$, and χ_{inv} has

exactly the body $\chi_{r/h}$. The invariant cutoff differs from $\chi_{r/h}$ by finitely many nilpotent terms, which we estimate in Lemma 16.17.

Lemma 16.14. *Let b be a bounded scalar or matrix-valued multiplier with bounded first derivatives. Then*

$$\|D_+^{-1/2} b D_+^{1/2}\| + \|D_+^{1/2} b D_+^{-1/2}\| \le C(\|b\|_\infty + \|\nabla b\|_\infty).$$

If a is a multiplier bounded by one and supported in $|x| \ge cr/h$, then

$$\|a D_+^{-1/2}\| + \|D_+^{-1/2} a^*\| \le C_c(1 + r^{-1}).$$

Finally, $D_{\mathrm{nor}} \le 2\bar{D}_R$ and $D_{\mathrm{nor}} \le 2\bar{D}_L$ as quadratic forms.

Proof. The function $u \mapsto (1 - e^{-2u})^{-1}$ is comparable, with absolute constants, to $1 + u^{-1}$ on $u > 0$. By spectral calculus, it follows that

$$\|D_+^{-1/2} f\|^2 \asymp \|f\|^2 + \underline{z}^{-1} \|(\mathbf{N}+1)^{-1/2} f\|^2.$$

The form norm $\|(\mathbf{N}+1)^{1/2} f\|$ is equivalent to $\|\nabla f\| + \||x| f\| + \|f\|$, and the product rule bounds multiplication by b on the form domain by $C(\|b\|_\infty + \|\nabla b\|_\infty)$. By duality the same bound holds on the dual space, with the norm $\|(\mathbf{N}+1)^{-1/2} f\|$, and the first estimate follows from the displayed equivalence.

The form inequality $|x|^2 \le C_d(\mathbf{N}+1)$ implies that $\|a(\mathbf{N}+1)^{-1/2}\| \le C_c h/r$. Combining this with $D_+^{-1/2} \le C(\mathrm{Id} + \underline{z}^{-1/2}(\mathbf{N}+1)^{-1/2})$, which follows from the scalar inequality above by spectral calculus, we obtain the second estimate.

For the last assertion we use the identity

$$\bar{D}_R = (1 - \chi_{r/h}^2) + \chi_{r/h} \mathbf{M}^* (1 - \chi_{r/h}^2) \mathbf{M} \chi_{r/h} + \chi_{r/h} D_{\mathrm{nor}} \chi_{r/h},$$

and the same identity for $\bar{D}_L$ with $\mathbf{M}$ and $\mathbf{M}^*$ exchanged, which holds since $\mathbf{M}\mathbf{M}^* = \mathbf{M}^*\mathbf{M}$. Since $0 \le D_{\mathrm{nor}} \le \mathrm{Id}$, we have

$$\|D_{\mathrm{nor}}^{1/2} f\| \le \|D_{\mathrm{nor}}^{1/2} \chi_{r/h} f\| + \|(1 - \chi_{r/h}) f\|.$$

Squaring and using $(1 - \chi_{r/h})^2 \le 1 - \chi_{r/h}^2$, we obtain $D_{\mathrm{nor}} \le 2\bar{D}_R$, and likewise $D_{\mathrm{nor}} \le 2\bar{D}_L$. □

We call the operators x_j and ∂_j letters. An oscillator polynomial of degree s is a finite sum of products of at most s letters with fixed finite matrices on $\mathcal{H}_{\mathrm{odd}}$ as coefficients, and we let O_s denote such a polynomial. All bounds below are uniform over a fixed bounded set of coefficients.

Lemma 16.15. *For every oscillator polynomial O_s of degree at most s,*

$$\|D_+^{-1/2} O_s \mathbf{M} D_+^{-1/2}\| \le C_s \underline{z}^{-\max(1, s/2)},$$

and the same bound holds with the factors in the opposite order.

Proof. In creation and annihilation coordinates, O_s changes the number by one of finitely many integers, bounded in terms of s and $\mathcal{H}_{\mathrm{odd}}$, and its norm on the eigenspace of $\mathbf{N}$ with eigenvalue n is at most $C_s(n+1)^{s/2}$, since $\|a_j P_n\| \le \sqrt{n}$ and $\|a_j^* P_n\| \le \sqrt{n+1}$ for the spectral projections P_n of $\mathbf{N}$. The finitely many shifted values $n'+1$ are comparable to $n+1$, and so are the scalar weights $1-e^{-2\underline{z}(n+1)}$. On the eigenspace with eigenvalue n, $\mathbf{M}$ has norm at most $e^{-\underline{z}n}$. Each component of fixed number shift is bounded by a constant times

$$\sup_{n\ge 0} \frac{(n+1)^{s/2} e^{-\underline{z}n}}{1-e^{-c_s \underline{z}(n+1)}}.$$

For $\underline{z}(n+1) \le 1$ this supremum is bounded by $C_s \underline{z}^{-1}(n+1)^{s/2-1}$, and for the complementary range the exponential gives $C_s \underline{z}^{-s/2}$. Taking the maximum proves the bound, and taking adjoints proves the opposite order. □

We will also use the following interpolation inequality. If D is a bounded positive operator with bounded inverse and X is bounded, then

$$\|D^{-1/2} X D^{-1/2}\| \le \max\bigl(\|D^{-1}X\|, \|XD^{-1}\|\bigr). \tag{16.43}$$

Indeed, the function $\xi \mapsto D^{-\xi} X D^{\xi-1}$ is bounded and analytic on the strip $0 \le \operatorname{Re}\xi \le 1$, and its norm is at most $\|XD^{-1}\|$ on $\operatorname{Re}\xi = 0$ and at most $\|D^{-1}X\|$ on $\operatorname{Re}\xi = 1$; the three-lines theorem gives (16.43) at $\xi = 1/2$. It applies to $D = D_+$, since $D_+ \ge 1 - e^{-2\underline{z}}$.

Normal charges and words

We call an oscillator polynomial $\mathfrak{q}_a$ of degree at most one with finite fermionic coefficients an odd charge, and an operator

$$\mathfrak{j}_a = \sum_{i,j} a_{ij} x_i \partial_j + \mathfrak{j}_{a,F}, \qquad a_{ij} = -a_{ji},$$

with a fixed finite matrix $\mathfrak{j}_{a,F}$, a compact charge. We assume that the charges are the normal charges of Section 16.3, that is, the actions of the stabilizer generators. Under this assumption they preserve every $\mathbf{N}_\sigma$. Hence they commute with $\mathbf{M}$ and with $\mathbf{P}_0$, they annihilate Ω_0 and its dual, and they commute with the invariant cutoff χ_{inv}, which is a function of the invariant $\mathrm{STr}(hZ)^2$. We call a finite product $\mathbf{w}$ of charges a word, and its number of factors its degree. Since $\mathfrak{j}_a$ is quadratic in the letters, its norm on the eigenspace with eigenvalue n grows like n, and the cutoff is needed to bound its coordinate factor.

We now count, heuristically, the powers of h in a word; the next lemma makes the count precise. In Section 17, each charge carries the factor h^2, which is the size of the endpoint increment of one link. On the number levels $n \lesssim h^{-2}$, where the Mehler factor is not small, an odd charge costs h^{-1}, and a compact charge costs $(r/h)h^{-1} = rh^{-2}$, since the cutoff bounds its coordinate factor by r/h and its derivative costs h^{-1}. A Mehler

factor between two inverse half-defects $D_+^{-1/2}$ absorbs a polynomial of degree s at the cost $h^{-\max(2,s)}$, by Lemma 16.15. It thus supplies the two inverse half-defects, each of size h^{-1}, at no extra cost once $s \geq 2$. For a word with a odd and b compact charges and $a + b \geq 2$, the count is

$$h^{2(a+b)} h^{-a} (rh^{-2})^b = h^a r^b.$$

Lemma 16.16. *Let* $\mathbf{w}$ *be a fixed word with a odd charges and b compact charges, and let* $n = a + b \geq 2$. *Then*

$$\| D_+^{-1/2} h^{2n} \chi_{\mathrm{inv}} \, \mathbf{w} \, \mathbf{M} \chi_{\mathrm{inv}} D_+^{-1/2} \| \leq C_{\mathbf{w}} h^a r^b + C_{\mathbf{w}}(r) h,$$

and the adjoint satisfies the same bound. On the range of $\mathbf{P}_0^\perp$, *the same estimate holds with* $D_{\mathrm{nor}}^{-1/2} \mathbf{P}_0^\perp$ *in place of* $D_+^{-1/2}$. *The vacuum row and column vanish:*

$$\mathbf{P}_0 \chi_{\mathrm{inv}} \, \mathbf{w} \, \mathbf{M} \chi_{\mathrm{inv}} = \chi_{\mathrm{inv}} \, \mathbf{w} \, \mathbf{M} \chi_{\mathrm{inv}} \mathbf{P}_0 = 0. \tag{16.44}$$

For a single odd charge with the factor h^4, $\| D_+^{-1/2} h^4 \chi_{\mathrm{inv}} \mathfrak{q}_a \mathbf{M} \chi_{\mathrm{inv}} D_+^{-1/2} \| \leq C h^2$, *and for a single compact charge the corresponding bound is* $C(rh + h^2)$.

Proof. We move the coordinate factor of each compact charge to the left, using $[\partial_i, x_j] = \delta_{ij}$, and we leave the coordinate factors of the odd charges in place. Every resulting term is of the form $x^\alpha O_s$ with $|\alpha| = j \leq b$ and $s \leq a + j$, where the finite matrix terms of the compact charges only lower these degrees. Commuting a coordinate past a derivative removes one coordinate and one derivative. It therefore preserves this relation, and induction on the degree of the word proves the expansion.

We now define the multiplier b_α by

$$\chi_{\mathrm{inv}} x^\alpha = (r/h)^j \, b_\alpha.$$

The matrix-valued multiplier b_α and its first derivatives are bounded, since $r/h \geq 1$ and each nilpotent term of χ_{inv} is a multiplier of the same kind times a power of $(h/r)^2$. Using Lemma 16.14 at both ends and Lemma 16.15, we bound the contribution of this term by

$$C_{\mathbf{w}} h^{2n} (r/h)^j h^{-\max(2, a+j)}.$$

For $j = b$, we have $\max(2, a + j) = n$, since $a + b = n \geq 2$, and the bound reduces to $C_{\mathbf{w}} h^a r^b$. For $j < b$, the bound carries the power $h^{a+2(b-j)}$ with $a + 2(b - j) \geq 2$ if $a + j \geq 2$, and the power h^{2n-j-2} with $2n - j - 2 \geq 1$ if $a + j < 2$; thus every such term is at most $C_{\mathbf{w}}(r)h$. The opposite order follows from the same proof, or by taking adjoints. On the range of $\mathbf{P}_0^\perp$, the scalar weights of D_{nor} and D_+ are comparable, which gives the assertion with $D_{\mathrm{nor}}^{-1/2} \mathbf{P}_0^\perp$.

Since every charge commutes with χ_{inv} and $\mathbf{M}$, we have $\mathbf{w}\mathbf{M}\chi_{\mathrm{inv}} = \chi_{\mathrm{inv}}\mathbf{M}\mathbf{w}$ and $\chi_{\mathrm{inv}}\mathbf{w} = \mathbf{w}\chi_{\mathrm{inv}}$. Together with $\mathbf{w}\mathbf{P}_0 = \mathbf{P}_0\mathbf{w} = 0$, these identities give (16.44). For a single

odd charge, the same argument with $s = 1$ and $j = 0$ gives $h^4 \underline{z}^{-1} \le Ch^2$. A compact charge gives a term with $j = s = 1$ and a bounded finite matrix term, which are bounded by $h^4(r/h)\underline{z}^{-1} \le Crh$ and by Ch^2. □

The reference $\mathbf{M}^\chi$ uses the scalar cutoff $\chi_{r/h}$, since χ_{inv} is not a contraction in the positive metric. The next lemma bounds the difference between the two cutoffs relative to the normal defect.

Lemma 16.17. *There is a decomposition*

$$[\chi_{r/h}, \mathbf{M}] = \mathbf{P}_0^\perp \mathbf{K}_h \mathbf{P}_0^\perp + \mathbf{R}_h,$$
$$\|D_{\mathrm{nor}}^{-1/2} \mathbf{P}_0^\perp \mathbf{K}_h \mathbf{P}_0^\perp D_{\mathrm{nor}}^{-1/2}\| \le C(r)h^2,$$
$$\|\mathbf{R}_h\| \le C(r) h^{-L} e^{-cr^2/h^2},$$

for some L. The same holds for the operators $b_h \mathbf{M} \chi_{r/h}$ and $\chi_{r/h} \mathbf{M} b_h$, where b_h is a fixed matrix-valued multiplier supported in a fixed enlargement of the transition annulus of $\chi_{r/h}$ such that $\|b_h\|_\infty \le C(r)h^2$. In particular, it holds for the nilpotent terms of χ_{inv}.

Proof. For one even coordinate of species σ, the kernel of $e^{-z_\sigma \mathbf{N}_\sigma}$ is (16.32), and the even kernel of $\mathbf{M}$ is the product of these factors. In the sector we have $\operatorname{Re}\coth(z/2) \ge c/\underline{z}$, $\operatorname{Re}\tanh(z/2) \ge c\underline{z}$ and $|z| \le C\underline{z}$. Completing the squares, we find that the absolute row and column integrals of the kernel are bounded, and that with an additional factor $|x - y|$ they are bounded by $C\underline{z}^{1/2}$. The finite odd Mehler factor is bounded.

Let a be an annular multiplier, equal to one on a fixed neighborhood of the transition annulus of $\chi_{r/h}$. If $\chi_{r/h}(x) \ne \chi_{r/h}(y)$ and x or y lies outside the support of a, then $|x - y| \ge cr/h$. The part of the kernel with $|x - y| \ge cr/h$ has Schur norm at most $C(r)h^{-L} e^{-cr^2/(h^2 \underline{z})}$, by the Gaussian bound. For the remaining part, $|\chi_{r/h}(x) - \chi_{r/h}(y)| \le C(h/r)|x - y|$. We therefore have

$$[\chi_{r/h}, \mathbf{M}] = a S_h a + \mathbf{R}'_h, \qquad \|S_h\| \le C(h/r)\underline{z}^{1/2} \le Ch^2/r,$$

with a remainder $\mathbf{R}'_h$ of the stated Gaussian size. Applying the annulus estimate of Lemma 16.14 on both sides, we obtain

$$\|D_+^{-1/2} a S_h a D_+^{-1/2}\| \le C(h^2/r)(1 + r^{-1})^2.$$

On the range of $\mathbf{P}_0^\perp$ the defects D_{nor} and D_+ are comparable, which implies the bound on $D_{\mathrm{nor}}^{-1/2} \mathbf{P}_0^\perp \mathbf{K}_h \mathbf{P}_0^\perp D_{\mathrm{nor}}^{-1/2}$. The vacuum row and column are Gaussian-small, because $[\chi_{r/h}, \mathbf{M}]\mathbf{P}_0 = (\lambda \chi_{r/h} - \mathbf{M}\chi_{r/h})\mathbf{P}_0$, $\mathbf{P}_0[\chi_{r/h}, \mathbf{M}] = \mathbf{P}_0(\chi_{r/h}\mathbf{M} - \lambda \chi_{r/h})$, and $(1 - \chi_{r/h})\Omega_0$ is a Gaussian tail. We put them and $\mathbf{R}'_h$ into $\mathbf{R}_h$.

For $b_h \mathbf{M} \chi_{r/h}$, we localize the input of the Mehler kernel to an enlarged annulus. The discarded part again has separation cr/h and the same Gaussian bound. The retained

kernel has operator norm at most $C(r)h^2$ by Schur's test. Its output and its input are both supported in annuli, so the annulus estimate of Lemma 16.14 on each side gives the bound with inverse half-defects on both sides, as for the commutator, and the opposite order follows by taking adjoints. For the invariant cutoff, finite Taylor expansion in the nilpotent part gives terms $(h/r)^{2j}\eta^{(j)}(h^2|x|^2/r^2)\,F_j$ with $j \geq 1$, fixed finite matrices F_j and the same annular support. These terms satisfy the hypothesis, since $(h/r)^{2j} \leq C(r)h^2$. □

16.8 Mehler kernels with amplitudes

The transfer kernel of Section 17 inserts into the Mehler kernel amplitudes that need not preserve the normal number. We treat them through exact identities for their kernels. For simplicity we take one species and first remove the constant scalar λ, so that $\lambda^{-1}\mathbf{M} = e^{-z\mathbf{N}}$. The scalar is restored by multiplying the amplitude by λ; all bounds below are unchanged since $|\lambda| \leq 1$. For several species the Gaussian below is replaced by the product of the Gaussians of the species. In the midpoint and increment variables

$$U = \frac{h(x+y)}{2}, \qquad V = \frac{x-y}{h},$$

the even kernel (16.32) is

$$k_z(x,y) = \frac{e^{dz/2}}{(2\pi \sinh z)^{d/2}} \exp\{-\tfrac{1}{2}c_z|V|^2 - t_z|U|^2\},$$
$$c_z = \tfrac{1}{2}h^2 \coth(z/2), \quad t_z = h^{-2}\tanh(z/2).$$

In the sector we have $\operatorname{Re} c_z, \operatorname{Re} t_z \geq c$ and $|c_z| + |t_z| \leq C$. Here U is the unscaled normal midpoint and V the normalized increment, and both are of order one under the kernel.

For integers $k, L \geq 0$ and numbers $\varkappa_0, \varkappa_1 \geq 0$, we let $\mathcal{A}_{k,L}(\varkappa_0, \varkappa_1)$ denote the class of C^k maps $a\colon \mathbb{R}^d \times \mathbb{R}^d \to \operatorname{End}(\mathcal{H}_{\mathrm{odd}})$ such that for all $|\alpha| + |\beta| \leq k$,

$$\|\partial_U^\alpha \partial_V^\beta a(U,V)\| \leq \varkappa_1 \langle U\rangle^L \langle V\rangle^L, \qquad \|\partial_V^\beta a(U,V)\| \leq \varkappa_0 \langle U\rangle^L \langle V\rangle^L,$$

where $\langle u\rangle = (1+|u|^2)^{1/2}$. We call such maps amplitudes. Since the second bound includes $\beta = 0$, the parameter $\varkappa_0$ bounds the amplitude itself, while derivatives in the midpoint may cost $\varkappa_1$. For an amplitude a we define the operator

$$(\mathbf{M}[a]f)(x) = \int k_z(x,y)\, a\Big(\frac{h(x+y)}{2}, \frac{x-y}{h}\Big) f(y)\, \mathrm{d}y.$$

The odd Mehler factor $e^{-z\mathbf{N}_F}$ and fixed finite odd coefficients are absorbed into a. For instance, $\mathbf{M} = \mathbf{M}[a]$ with $a = \lambda e^{-z\mathbf{N}_F}$. Finite sums of such operators are treated in the same way.

Lemma 16.18. *Let $a \in \mathcal{A}_{k,L}(\varkappa_0, \varkappa_1)$, where k is large enough depending on the orders m and s below. Constants depend on k, L, m, s, the dimension, $\mathcal{H}_{\mathrm{odd}}$, and the sector.*

(i) We have $\|\mathbf{M}[a]\| \le C \sup_{U,V} \langle U\rangle^{-L}\langle V\rangle^{-L}\|a(U,V)\| \le C\varkappa_0$, and $\mathbf{M}[a]^* = \mathbf{M}[a^\sharp]$ with z replaced by $\bar z$ and $a^\sharp(U,V) = a(U,-V)^*$.

(ii) For all j, the following identities hold:

$$[x_j, \mathbf{M}[a]] = h\,\mathbf{M}[V_j a], \qquad [\partial_j, \mathbf{M}[a]] = \mathbf{M}[b_j],$$
$$x_j \mathbf{M}[a] = \mathbf{M}[(h^{-1}U_j + hV_j/2)a], \qquad \partial_j \mathbf{M}[a] = \mathbf{M}[a^{(j)}],$$

where $b_j = h\partial_{U_j} a - 2ht_z U_j a$ and

$$a^{(j)} = h^{-1}(\partial_{V_j} a - c_z V_j a) + h(\tfrac{1}{2}\partial_{U_j} a - t_z U_j a),$$

and $[\mathbf{N}_F, \mathbf{M}[a]] = \mathbf{M}[[\mathbf{N}_F, a]]$.

(iii) There exists $a_0 \in \mathrm{End}(\mathcal{H}_{\mathrm{odd}})$ such that

$$\Big\| a_0 - \int \mathfrak{g}_z(V)\, a(0,V)\, \mathrm{d}V \Big\| \le Ch^2\varkappa_1, \qquad \|(\mathbf{N}+1)^m(\mathbf{M}[a]\Omega_0 - a_0\Omega_0)\| \le Ch(\varkappa_0+\varkappa_1)$$

for every $m \ge 0$, where $\mathfrak{g}_z$ is the normalized complex Gaussian (16.45) *below and $a_0\Omega_0$ is the even vacuum times a_0 applied to the odd vacuum. In particular, if $a(0,\cdot)$ has vanishing Gaussian moment, then $\|(\mathbf{N}+1)^m\mathbf{M}[a]\Omega_0\| \le Ch(\varkappa_0+\varkappa_1)$.*

(iv) For all integers m and oscillator polynomials O_s of degree at most s,

$$\|(\mathbf{N}+1)^m O_s \mathbf{M}[a](\mathbf{N}+1)^{-m}\| \le Ch^{-s}(\varkappa_0 + h\varkappa_1),$$
$$\|D_+^{-1/2} O_s \mathbf{M}[a]\| + \|O_s \mathbf{M}[a] D_+^{-1/2}\| \le Ch^{-\max(1,s)}(\varkappa_0 + h\varkappa_1),$$
$$\|D_+^{-1/2} O_s \mathbf{M}[a] D_+^{-1/2}\| \le Ch^{-\max(2,s)}(\varkappa_0 + h\varkappa_1).$$

In particular, the estimate of Lemma 16.16 on the range of $\mathbf{P}_0^\perp$ remains true with $\mathbf{M}$ replaced by $\mathbf{M}[a]$ and multiplied by $C(\varkappa_0 + h\varkappa_1)$, and its vacuum row and column are given by part (iii).

Proof. We begin with (i). The row and column integrals of $|k_z(x,y)|\langle U\rangle^L\langle V\rangle^L$ are bounded, since the increment Gaussian has width h in $x-y$, the midpoint Gaussian $e^{-c|U|^2}$ absorbs $\langle U\rangle^L$, and $|e^{dz/2}(2\pi\sinh z)^{-d/2}| \le Ch^{-d}$. Schur's test gives the bound. The kernel of $\mathbf{M}[a]^*$ is

$$\overline{k_z(y,x)}\, a(U,-V)^* = k_{\bar z}(x,y)\, a^\sharp(U,V),$$

where we used that k_z is symmetric and $\overline{k_z} = k_{\bar z}$.

We now prove (ii). The kernels of $x_j\mathbf{M}[a]$ and $\mathbf{M}[a]x_j$ are the kernel of $\mathbf{M}[a]$ multiplied by x_j and by y_j, and $x_j = h^{-1}U_j + hV_j/2$, $y_j = h^{-1}U_j - hV_j/2$. The kernel of $\partial_j\mathbf{M}[a]$ is the

x_j-derivative of the kernel, and integration by parts shows that the kernel of $\mathbf{M}[a]\partial_j$ is minus its y_j-derivative. We have

$$\partial_{x_j} k_z = -(h^{-1}c_z V_j + h t_z U_j)\, k_z, \qquad (\partial_{x_j} + \partial_{y_j}) k_z = -2h t_z U_j\, k_z,$$

and $\partial_{x_j} = \frac{h}{2}\partial_{U_j} + h^{-1}\partial_{V_j}$, $\partial_{x_j} + \partial_{y_j} = h\partial_{U_j}$ on functions of (U, V). The identities follow from these relations. All of them are algebraic in z, so they hold in the complex sector.

Next, we prove (iii). We write $\Omega_0 = \Omega_B \otimes \Omega_F$. By the description of the ground-state transform after (16.32), substituting $y = x - hV$, we obtain

$$\frac{(\mathbf{M}[a]\Omega_0)(x)}{\Omega_B(x)} = \beta_a(x)\Omega_F, \qquad \beta_a(x) = \int \mathfrak{g}_z(V - \lambda_z x)\, a\big(hx - \tfrac{1}{2}h^2 V, V\big)\, \mathrm{d}V,$$

where

$$\mathfrak{g}_z(w) = (\pi s_z)^{-d/2} \exp\{-w\cdot w/s_z\}, \qquad s_z = (1 - e^{-2z})/h^2, \qquad \lambda_z = (1 - e^{-z})/h = O(h). \tag{16.45}$$

In the sector $\mathrm{Re}(1/s_z) \geq c$, so $|\mathfrak{g}_z(V - \lambda_z x)| \leq Ce^{-c|V|^2 + Ch^2|x|^2}$. We set $a_0 = \beta_a(0)$. The first bound in (iii) follows from $|U| \leq h^2|V|$ at $x = 0$. Each derivative of β_a brings out a factor λ_z or $h\partial_U$, which gives, for $|\nu| \geq 1$,

$$|\partial_x^\nu \beta_a(x)| \leq Ch(\varkappa_0 + \varkappa_1)\langle x\rangle^L e^{Ch^2|x|^2}.$$

Writing

$$\beta_a(x) - \beta_a(0) = \int_0^1 x\cdot\nabla\beta_a(tx)\, \mathrm{d}t,$$

we find the same bound for $\beta_a - \beta_a(0)$, with $L + 1$ in place of L. Since $(\mathbf{N}+1)^m$ is a differential operator of order $2m$ with polynomial coefficients and $e^{-|x|^2/2 + Ch^2|x|^2} \leq e^{-|x|^2/4}$ for small h, the second bound in (iii) follows.

It remains to prove (iv). We first record how the identities of (ii) change the class. Let $b \in \mathcal{A}_{k,L}(\varkappa_0', \varkappa_1')$ and let ℓ be a letter. A commutator gains a factor h:

$$[\ell, \mathbf{M}[b]] = \mathbf{M}[b'], \qquad b' \in \mathcal{A}_{k-1,L+1}\big(Ch(\varkappa_0' + \varkappa_1'), Ch\varkappa_1'\big),$$

and $[\mathbf{N}_F, \mathbf{M}[b]] = \mathbf{M}[[\mathbf{N}_F, b]]$ with $[\mathbf{N}_F, b] \in \mathcal{A}_{k,L}(C\varkappa_0', C\varkappa_1')$. A letter absorbed into the kernel on the left costs a factor h^{-1}:

$$\ell\,\mathbf{M}[b] = \mathbf{M}[b''], \qquad b'' \in \mathcal{A}_{k-1,L+1}\big(C(h^{-1}\varkappa_0' + h\varkappa_1'), Ch^{-1}\varkappa_1'\big),$$

and the same holds on the right, since $\mathbf{M}[b]\ell = \ell\mathbf{M}[b] - [\ell, \mathbf{M}[b]]$.

We first bound $(\mathbf{N}+1)^m \mathbf{M}[b](\mathbf{N}+1)^{-m}$ for an integer $m \geq 1$. Writing $\mathrm{ad}_X(Y) = [X, Y]$, we have

$$(\mathbf{N}+1)^m \mathbf{M}[b](\mathbf{N}+1)^{-m} = \sum_{i \leq m} \binom{m}{i} \mathrm{ad}^i_{\mathbf{N}+1}(\mathbf{M}[b])(\mathbf{N}+1)^{-i}.$$

Since $[\mathbf{N}, x_j] = -\partial_j$ and $[\mathbf{N}, \partial_j] = -x_j$, the commutator rule shows by induction that $\mathrm{ad}^i_{\mathbf{N}+1}(\mathbf{M}[b])$ is a finite sum of terms $\Lambda_1 \mathbf{M}[b_\Lambda]\Lambda_2$, where Λ_1 and Λ_2 are products of letters of total degree at most i and the coefficients of b_Λ satisfy $\varkappa_0(b_\Lambda) + h\varkappa_1(b_\Lambda) \le C(\varkappa_0' + h\varkappa_1')$. Moving the letters of Λ_1 to the right by the commutator rule preserves this form with $\Lambda_1 = \mathrm{Id}$. Since a product of at most $2i$ letters is bounded relative to $(\mathbf{N}+1)^i$, part (i) implies that

$$\|(\mathbf{N}+1)^m \mathbf{M}[b](\mathbf{N}+1)^{-m}\| \le C(\varkappa_0' + h\varkappa_1').$$

For negative m we apply this to $\mathbf{M}[b]^* = \mathbf{M}[b^\sharp]$ and take adjoints.

Now let O_s be given. Absorbing its letters on the left, we have $O_s\mathbf{M}[a] = \mathbf{M}[a_s]$ with $\varkappa_0(a_s) \le Ch^{-s}(\varkappa_0 + h^2\varkappa_1)$ and $\varkappa_1(a_s) \le Ch^{-s}\varkappa_1$. The first bound in (iv) then follows from the previous paragraph. For the third, we use (16.43) with $X = O_s\mathbf{M}[a]$, the bound $\|D_+^{-1}Y\| \le C(\|Y\| + h^{-2}\|(\mathbf{N}+1)^{-1}Y\|)$, which holds by spectral calculus, and its mirror image on the right. We have

$$\|O_s\mathbf{M}[a]\| \le Ch^{-s}(\varkappa_0 + h^2\varkappa_1), \qquad \|(\mathbf{N}+1)^{-1}O_s\mathbf{M}[a]\| \le Ch^{-\max(s-2,0)}(\varkappa_0 + h^2\varkappa_1),$$

where the second bound uses that $(\mathbf{N}+1)^{-1}$ times a product of two letters is bounded.

We now bound $\|O_s\mathbf{M}[a](\mathbf{N}+1)^{-1}\|$. We move the letters of O_s to the right through $\mathbf{M}[a]$, one at a time, by $\ell\mathbf{M}[b] = \mathbf{M}[b]\ell + [\ell, \mathbf{M}[b]]$. This writes $O_s\mathbf{M}[a]$ as a sum of terms $\mathbf{M}[b]\Lambda$, where Λ is a product of $p \le s$ letters and b has gone through $s-p$ commutators, so that $\varkappa_0(b) + h^2\varkappa_1(b) \le Ch^{s-p}(\varkappa_0 + \varkappa_1)$ if $p < s$, while $b = a$ if $p = s$. In each term $(\mathbf{N}+1)^{-1}$ absorbs the last two letters of Λ, and the other letters are absorbed into $\mathbf{M}[b]$ on the right, at the cost $h^{-\max(p-2,0)}$. Since $s - p - \max(p-2,0) \ge 1 - \max(s-2,0)$ for $p < s$, we obtain

$$\|O_s\mathbf{M}[a](\mathbf{N}+1)^{-1}\| \le Ch^{-\max(s-2,0)}(\varkappa_0 + h\varkappa_1),$$

and the third bound in (iv) follows. The second bound follows in the same way from

$$\|D_+^{-1/2}Y\| \le C\big(\|Y\| + h^{-1}\|(\mathbf{N}+1)^{-1/2}Y\|\big),$$

its mirror image, and the boundedness of $(\mathbf{N}+1)^{-1/2}$ times one letter. The last assertion of (iv) follows from the proof of Lemma 16.16, with this bound in place of Lemma 16.15. □

For a word with interleaved amplitudes, part (ii) gives explicit formulas for the commutators. For a compact charge, the Gaussian factor of k_z is invariant under simultaneous rotations of x and y, and we have

$$\Big[\sum_{i,j} a_{ij}x_i\partial_j, \mathbf{M}[a]\Big] = \frac{h}{2}\sum_{i,j} a_{ij}\big(x_i\mathbf{M}[\partial_{U_j}a] + \mathbf{M}[\partial_{U_j}a]x_i + \partial_j\mathbf{M}[V_ia] + \mathbf{M}[V_ia]\partial_j\big). \tag{16.46}$$

To prove this, note that by (ii) the left side equals $\mathbf{M}[b]$ with $b = \sum_{i,j} a_{ij}(U_i\partial_{U_j} + V_i\partial_{V_j})a$, since the terms $t_z U_i U_j a$, $c_z V_i V_j a$, and $\delta_{ij} a$ cancel by antisymmetry. Also by (ii), we have

$$\mathbf{M}[U_i b'] = \tfrac{h}{2}(x_i \mathbf{M}[b'] + \mathbf{M}[b']x_i), \qquad \mathbf{M}[\partial_{V_j} b' - c_z V_j b'] = \tfrac{h}{2}(\partial_j \mathbf{M}[b'] + \mathbf{M}[b']\partial_j).$$

Applying these with $b' = \partial_{U_j} a$ and $b' = V_i a$, and using antisymmetry once more to remove the terms with c_z, we obtain (16.46). For an odd charge $\mathfrak{q}_a = \sum_j (F_j x_j + G_j \partial_j)$ with finite matrices F_j, G_j, part (ii) gives

$$[\mathfrak{q}_a, \mathbf{M}[a]] = \sum_j \big(hF_j \mathbf{M}[V_j a] + \mathbf{M}[[F_j, a]]x_j + G_j \mathbf{M}[b_j] + \mathbf{M}[[G_j, a]]\partial_j \big), \tag{16.47}$$

where $[\cdot,\cdot]$ is the graded commutator. In both cases the commutator is h times operators of the same kind, with at most one letter on either side, except for the matrix commutators $[F_j, a]$ and $[G_j, a]$, which Lemma 17.4 controls in Section 17 for the amplitudes that occur there.

Finite odd Gaussian increments

The odd increments in these kernels are handled by one finite Gaussian computation, which we carry out before taking any norm. Multiplying the Berezin kernel of an operator X of parity $|X|$ by $\theta - \theta'$, where θ and θ' are the copies of an odd coordinate at the two endpoints, gives the kernel of the graded commutator $[\theta, X] = \theta X - (-1)^{|X|} X\theta$. A product of normalized odd increments $(\theta_a - \theta'_a)/\kappa_{\mathrm{o}}$ corresponds to an iterated graded commutator divided by a power of κ_{o}. We will use $\kappa_{\mathrm{o}} = h$ for the normal odd variables, whose Mehler factor runs for a time of order h^2, and $\kappa_{\mathrm{o}} = h^2$ for the tangent odd variables, whose Gaussian is the odd heat over the time δ_W.

Lemma 16.19. *Let $\mathcal{H}_0$ be a finite-dimensional fermionic Fock space, let $0 < \kappa_{\mathrm{o}} \le 1$, and let $\mathbf{M}_0 = e^{-\mathfrak{h}_0}$, where $\mathfrak{h}_0$ is an even element of the algebra of canonical anticommutation relations of $\mathcal{H}_0$, quadratic in the creation and annihilation operators, with coefficients bounded by $C_0\kappa_{\mathrm{o}}^2$. Let $\theta_1, \dots, \theta_n$ be pairwise anticommuting linear forms in these operators, with coefficients bounded by C_0. For a sequence $\mathbf{a} = (a_1, \dots, a_l)$ of indices we set*

$$\Delta_{\mathbf{a}} \mathbf{M}_0 = \kappa_{\mathrm{o}}^{-l} \operatorname{ad}_{\theta_{a_1}} \cdots \operatorname{ad}_{\theta_{a_l}} \mathbf{M}_0,$$

with graded commutators. Then $\Delta_{\mathbf{a}}\mathbf{M}_0$ is a finite sum, over the partial pairings π of $\{1, \dots, l\}$, of terms

$$\pm \prod_{\{b,b'\} \in \pi} c_{a_b a_{b'}} \cdot \prod_{d \notin \cup \pi} \lambda_{a_d} \cdot \mathbf{M}_0,$$

where the c_{ab} are numbers with $|c_{ab}| \le C$, the λ_a are linear forms with coefficients bounded by $C\kappa_{\mathrm{o}}$, the unpaired factors are ordered as in $\mathbf{a}$, and C depends only on C_0, n and $\dim \mathcal{H}_0$. Moreover $\|\mathbf{M}_0 - \mathrm{Id}\| \le C\kappa_{\mathrm{o}}^2$. In particular, $\|\Delta_{\mathbf{a}}\mathbf{M}_0\| \le C_l$, $\|\Delta_{\mathbf{a}}\mathbf{M}_0\| \le C_l\kappa_{\mathrm{o}}$ if l is odd, and $\Delta_{\mathbf{a}}\mathbf{M}_0 = c_{\mathbf{a}}\mathrm{Id} + O(\kappa_{\mathrm{o}})$ with a number $c_{\mathbf{a}}$.

Thus a pair of normalized odd increments has a bounded covariance, and every unpaired normalized increment carries a factor κ_{o}.

Proof. We introduce Grassmann sources $\xi_1, \dots, \xi_n$ that anticommute with the θ_a, and we set

$$L_\xi = \sum_a \xi_a \theta_a,$$

which is even. Then

$$\operatorname{ad}_{L_\xi} X = \sum_a \xi_a [\theta_a, X]$$

for all X. Hence the coefficient of $\xi_{a_1} \cdots \xi_{a_l}$ in

$$e^{L_\xi/\kappa_{\mathrm{o}}} \mathbf{M}_0 e^{-L_\xi/\kappa_{\mathrm{o}}} = \sum_{l \ge 0} \frac{1}{l!} \kappa_{\mathrm{o}}^{-l} \operatorname{ad}_{L_\xi}^l \mathbf{M}_0$$

is $\pm \Delta_{\mathbf{a}} \mathbf{M}_0$, where we used that the iterated commutators are antisymmetric in their indices, since the θ_a anticommute.

We now compute this conjugation. Since $\mathfrak{h}_0$ is quadratic, the operator

$$L'_\xi = \mathbf{M}_0 L_\xi \mathbf{M}_0^{-1} = e^{-\operatorname{ad}_{\mathfrak{h}_0}} L_\xi$$

is again linear in the creation and annihilation operators, and the coefficients of $L_\xi - L'_\xi$ are $O(\kappa_{\mathrm{o}}^2)$. The anticommutator of two linear forms is a number. Therefore $[L_\xi, L'_\xi]$ is a quadratic polynomial in the sources with numerical coefficients, it is central, and $[L_\xi, L'_\xi] = [L_\xi, L'_\xi - L_\xi]$ has coefficients $O(\kappa_{\mathrm{o}}^2)$. The Baker–Campbell–Hausdorff formula with a central commutator gives

$$e^{L_\xi/\kappa_{\mathrm{o}}} \mathbf{M}_0 e^{-L_\xi/\kappa_{\mathrm{o}}} = e^{L_\xi/\kappa_{\mathrm{o}}} e^{-L'_\xi/\kappa_{\mathrm{o}}} \mathbf{M}_0 = \exp\Big\{ \frac{L_\xi - L'_\xi}{\kappa_{\mathrm{o}}} - \frac{[L_\xi, L'_\xi]}{2\kappa_{\mathrm{o}}^2} \Big\} \mathbf{M}_0.$$

We write the first term in the exponent as $\sum_a \xi_a \lambda_a$, with $\lambda_a = O(\kappa_{\mathrm{o}})$, and the second as $\frac{1}{2} \sum_{a,b} c_{ab} \xi_a \xi_b$, with numbers $c_{ab} = O(1)$. Expanding the exponential, which is a polynomial in the finitely many sources, and extracting the coefficient of $\xi_{a_1} \cdots \xi_{a_l}$, we obtain the sum over partial pairings. Finally, $\|\mathbf{M}_0 - \mathrm{Id}\| \le C\kappa_{\mathrm{o}}^2$ since $\mathfrak{h}_0 = O(\kappa_{\mathrm{o}}^2)$ on the finite-dimensional space $\mathcal{H}_0$. □

16.9 Graph powers of the frozen normal transfer

Four-term bounds

All perturbation estimates in this section and the next have the same form. Let C be a contraction on a Hilbert space with the defect operators $D_R = \mathrm{Id} - C^*C$ and

$D_L = \mathrm{Id} - CC^*$, as in Lemma A.1, and let $\mathbf{T}$ be a bounded operator. We say that $\mathbf{T}$ satisfies a four-term bound relative to C, with the parameters $a, A_1 \geq 0$ and $0 < \delta_0 \leq 1$, if for all vectors f and g,

$$|\langle g, (\mathbf{T} - C)f\rangle| \leq a\|D_L^{1/2} g\| \|D_R^{1/2} f\| + A_1\sqrt{\delta_0}\big(\|g\|\|D_R^{1/2} f\| + \|D_L^{1/2} g\|\|f\|\big) + A_1\delta_0\|g\|\|f\|. \tag{16.48}$$

We call the first term the bilateral term, the second the one-sided terms, and the third the scalar term.

Remark 16.20. The bound (16.48) is the hypothesis (A.10) of Corollary A.2 with $\mathbf{T}$ in place of T and $A_0 = A_1$, and we use that corollary in this form. Its constant A of (A.11) is then $B = \max\{1, 2304A_1^2\}$. If $a \leq 1/16$ and $B\delta_0 \leq 1$, the corollary shows that $\mathbf{T}$ satisfies (A.4) with $B/2$ in place of A_1, that is,

$$\mathbf{T}^*\mathbf{T} \leq (1 + B\delta_0/2)\mathrm{Id} - \tfrac{1}{2}D_R, \qquad \mathbf{T}\mathbf{T}^* \leq (1 + B\delta_0/2)\mathrm{Id} - \tfrac{1}{2}D_L,$$

and Lemma A.1(b) then gives $\|\mathbf{T}^n\| \leq e^{Bt/4}$ whenever $n\delta_0 \leq t$.

The frozen normal transfer with cutoffs

In the normal gauge (16.38), the frozen normal transfer with cutoffs is

$$\mathbf{F}_h = w_{\mathrm{anh}}^{1/2}\chi_{\mathrm{inv}}\mathbf{M}_h^{\mathrm{q}}\chi_{\mathrm{inv}}w_{\mathrm{anh}}^{1/2}.$$

We call the left factor χ_{inv} the *output cutoff* and the right factor the *input cutoff*; the same names are used for the cutoffs of A_h in later sections. We write $\mathbf{F}_h$ as $(\mathrm{Id} + \mathfrak{b}_{+,h})\chi_{\mathrm{inv}}\mathbf{M}\chi_{\mathrm{inv}}(\mathrm{Id} + \mathfrak{b}_{-,h})$, with $\mathbf{M} = \mathbf{M}_h^{\mathrm{q}}$, where $\mathfrak{b}_{\pm,h} = w_{\mathrm{anh}}^{1/2} - 1$, extended as in Section 16.5, are smooth matrix-valued multipliers supported in $|hx| \leq 3r$.

The structure of these multipliers comes from the Taylor expansion at $U = 0$ of the phase $\phi_{\mathrm{anh}} = \log w_{\mathrm{anh}}^{1/2}$. Recall from Section 4.6 that $w_0 = e^{-\mathcal{F}}$ with $\mathcal{F}(S) = \frac{1}{2}\,\mathrm{STr}\,S^2 + \mathrm{STr}\log(E - S)$, so that

$$\log w_0^{1/2}(R_* + U) = \log w_{\mathrm{quad}}^{1/2}(U) + \phi_{\mathrm{anh}}(U).$$

We claim that ϕ_{anh} vanishes to third order at $U = 0$, in all normal coordinates, even and odd. The term of degree zero vanishes because both physical half-weights $w_0^{1/2}$ and $w_{\mathrm{quad}}^{1/2}$ have the same value $\zeta_p^{1/2}$ at the root. The first variation of $\mathcal{F}$ at S in the direction U satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(S + tU)\Big|_{t=0} = \mathrm{STr}\big((S - (E - S)^{-1})U\big),$$

which vanishes at $S = R_*$, since $(E - R_*)^{-1} = R_*$ by (4.24); hence the term of degree one vanishes. By definition, $\log w_{\mathrm{quad}}^{1/2}$ contains the constant term and the term of degree two. Since w_0 is even, ϕ_{anh} contains only monomials of even degree in the odd normal

coordinates, and we write $\phi_{\mathrm{anh}} = \sum_{j\ge 0}\phi_{\mathrm{anh},2j}$, where $\phi_{\mathrm{anh},2j}$ has degree $2j$ in them. Each normal coordinate carries a factor h in the standardized coordinates $U = hZ$. Measuring the odd coefficients in the monomials of the standardized odd coordinates, as in Definition 5.12 with the weight h, we therefore obtain

$$|\phi_{\mathrm{anh},0}| \le C|hx|^3, \qquad |\phi_{\mathrm{anh},2}| \le Ch^2|hx|, \qquad |\phi_{\mathrm{anh},2j}| \le Ch^4 \quad (j \ge 2), \tag{16.49}$$

where hx is the unscaled even normal coordinate. In words, the remaining body phase $\phi_{\mathrm{anh},0}$ starts in degree three, its coefficient of degree two in the normal odd variables starts with h^2 times one body coordinate, and its terms of higher odd degree are of order h^4. Since $\mathfrak{b}_{\pm,h} = e^{\phi_{\mathrm{anh}}} - 1$ and ϕ_{anh} is bounded on the tube, Lemma 5.13(a) gives the same bounds for the coefficients of $\mathfrak{b}_{\pm,h}$. The next lemma states the consequences that we need as hypotheses, and the paragraph after its proof verifies them.

Lemma 16.21. *Let* $\mathbf{F}_h = (\mathrm{Id} + \mathfrak{b}_{+,h})\chi_{\mathrm{inv}}\mathbf{M}\chi_{\mathrm{inv}}(\mathrm{Id} + \mathfrak{b}_{-,h})$, *and suppose that for each integer* $m \ge 0$ *the multipliers* $\mathfrak{b}_h \in \{\mathfrak{b}_{+,h}, \mathfrak{b}_{-,h}, \mathfrak{b}^*_{+,h}, \mathfrak{b}^*_{-,h}\}$ *satisfy*

$$\begin{aligned} \|(\mathbf{N}+1)^m \mathfrak{b}_h (\mathbf{N}+1)^{-m}\| &\le Cr^3 + C_m(r)h, \\ \|(\mathbf{N}+1)^m \mathfrak{b}_h (\mathbf{N}+1)^{-m-1}\| &\le Crh^2 + C_m(r)h^3, \\ \|(\mathbf{N}+1)^m \mathfrak{b}_h \Omega_0\| \le C_m h^3, \qquad |\langle \Omega_0, \mathfrak{b}_h \Omega_0\rangle| &\le Ch^4. \end{aligned}$$

Fix a finite set of graph orders. Then for all sufficiently small r, *and then all sufficiently small* h, *the operators* $\mathbf{F}_{m,h} = (\mathbf{N}+1)^m \mathbf{F}_h (\mathbf{N}+1)^{-m}$ *satisfy*

$$\|\mathbf{F}_{m,h} f\|^2 \le (1 + C_m(r)h^4)\|f\|^2 - c_m \|\bar{D}_R^{1/2} f\|^2, \tag{16.50}$$

where $\bar{D}_R$ *is the right defect of the cutoff Mehler operator and* $D_{\mathrm{nor}} \le 2\bar{D}_R$. *Consequently*

$$\|(\mathbf{N}+1)^m \mathbf{F}_h^j (\mathbf{N}+1)^{-m}\| \le e^{C_m(r) j h^4} \qquad (jh^4 \le T),$$

with a constant depending also on T, *and*

$$\|(\mathbf{N}+1)^m (\mathbf{F}_h \Omega_0 - \lambda \Omega_0)\| \le C_m h^3. \tag{16.51}$$

Moreover, $\mathbf{F}_{m,h}$ *satisfies* (16.48) *relative to* $\mathbf{M}^\chi$ *with* $\delta_0 = h^4$, $a = Cr + C_m(r)h$, *and* $A_1 = C_m(r)$. *The opposite graph order and the Hilbert adjoint satisfy the corresponding bounds with the left defect* $\bar{D}_L$.

Proof. We first show that conjugation by $(\mathbf{N}+1)^m$ changes the cutoff only by a small bilateral perturbation. For an integer m, $(\mathbf{N}+1)^m \chi_{r/h} (\mathbf{N}+1)^{-m} - \chi_{r/h}$ is a finite sum of the terms $\mathrm{ad}^l_{\mathbf{N}}(\chi_{r/h})(\mathbf{N}+1)^{-l}$ with $1 \le l \le m$. Every such term contains a derivative of $\chi_{r/h}$, has its output supported in the transition annulus, and has norm at most $C_m h/r$. This follows from $[\mathbf{N}, \chi_{r/h}] = -(\nabla\chi_{r/h})\cdot\nabla - \frac{1}{2}\Delta\chi_{r/h}$ and the bound of oscillator

polynomials by powers of $\mathbf{N}+1$. The further commutators with $|x|^2$ insert coordinates, which the factor $(\mathbf{N}+1)^{-l}$ pays, and they do not remove the first factor h/r.

After composition with $\mathbf{M}\chi_{r/h}$, these terms can be localized on both sides to enlarged transition annuli, up to an error $C_m(r)h^{-L_m}e^{-c_m r^2/h^2}$. To see this, we use the representation

$$(\mathbf{N}+1)^{-l} = \frac{1}{\Gamma(l)}\int_0^\infty t^{l-1}e^{-t(\mathbf{N}+1)}\,\mathrm{d}t,$$

whose integrand has Mehler kernels at positive times. If an annular endpoint at distance of order r/h from the origin is separated by cr/h from the other endpoint, the Mehler Gaussian has exponent at most $-cr^2/h^2$ after integration over t. (For $t \le 1$ we use the increment exponent, and for $t \ge 1$ the confining exponent $-c(|x|^2+|y|^2)$.) Polynomial factors are absorbed by reducing c, and Schur's test gives the operator bound. Applying Lemma 16.14 at both annular ends, we find that the graph-conjugated cutoff differs from the cutoff Mehler operator by a bilateral perturbation of size $C_m(r)h$ on the range of $\mathbf{P}_0^\perp$, plus Gaussian-small vacuum rows. The invariant cutoff differs from its body by the nilpotent terms of Lemma 16.17. Every graph commutator of such a term keeps its first factor h^2. The corresponding estimate follows from the product rule and the same localization, and the vacuum rows of these terms are Gaussian-small.

Next, we bound the multipliers. We set

$$\mathfrak{b}_m = (\mathbf{N}+1)^m \mathfrak{b}_h (\mathbf{N}+1)^{-m}.$$

By spectral calculus, we have $\|D_+^{-1}Y\| \le C(\|Y\| + h^{-2}\|(\mathbf{N}+1)^{-1}Y\|)$ and the mirror bound on the right. Together with the assumed bounds and their adjoint versions, this implies

$$\|\mathfrak{b}_m D_+^{-1}\| + \|D_+^{-1}\mathfrak{b}_m\| \le Cr + C_m(r)h, \qquad \|D_+^{-1/2}\mathfrak{b}_m D_+^{-1/2}\| \le Cr + C_m(r)h,$$

where the second bound follows from the first and (16.43). The operator $D_+^{1/2}\mathbf{M}^\chi D_+^{-1/2}$ is bounded by Lemma 16.14, since $\mathbf{M}$ commutes with D_+. Expanding $\mathfrak{b}_{+,m}\mathbf{M}^\chi + \mathbf{M}^\chi\mathfrak{b}_{-,m} + \mathfrak{b}_{+,m}\mathbf{M}^\chi\mathfrak{b}_{-,m}$ and adding the cutoff error of the previous paragraph, we find that the part of $\mathbf{F}_{m,h} - \mathbf{M}^\chi$ between two vectors in the range of $\mathbf{P}_0^\perp$ satisfies

$$|\langle \mathbf{P}_0^\perp g, (\mathbf{F}_{m,h} - \mathbf{M}^\chi)\mathbf{P}_0^\perp f\rangle| \le (Cr + C_m(r)h)\|D_+^{1/2}\mathbf{P}_0^\perp g\|\,\|D_+^{1/2}\mathbf{P}_0^\perp f\|.$$

On the range of $\mathbf{P}_0^\perp$, D_+ is comparable to D_{nor}, and $D_{\mathrm{nor}} \le 2\bar{D}_L$, $D_{\mathrm{nor}} \le 2\bar{D}_R$ by Lemma 16.14.

The parts $\mathbf{P}_0(\cdot)\mathbf{P}_0^\perp$ and $\mathbf{P}_0^\perp(\cdot)\mathbf{P}_0$ of $\mathbf{F}_{m,h} - \mathbf{M}^\chi$ have norm $C_m h^3$ by the assumed vacuum bounds and the Gaussian-small cutoff rows, and its vacuum scalar $\mathbf{P}_0(\cdot)\mathbf{P}_0$ has size $C_m h^4$. Since $D_{\mathrm{nor}} \ge ch^2\mathbf{P}_0^\perp$, we have $\|\mathbf{P}_0^\perp f\| \le Ch^{-1}\|\bar{D}_R^{1/2} f\|$, and similarly on the left. Therefore $\mathbf{F}_{m,h}$ satisfies the four-term bound (16.48) relative to $\mathbf{M}^\chi$ with

$$\delta_0 = h^4, \qquad a = Cr + C_m(r)h, \qquad A_1 = C_m(r),$$

up to a Gaussian-small error, which we add to the scalar term. We choose r and then h so small that $a \le 1/16$ and $Bh^4 \le 1$, with the constant $B = \max\{1, 2304A_1^2\}$ of Remark 16.20. We then apply Remark 16.20 with $\mathbf{T} = \mathbf{F}_{m,h}$, $C = \mathbf{M}^\chi$, $\delta_0 = h^4$, and $A_1 = C_m(r)$. Since the right defect of $\mathbf{M}^\chi$ is $\bar{D}_R$, its first inequality is (16.50), with $c_m = 1/2$ and the constant $B/2$ in the first term, and its power bound with $t = jh^4$ implies the power bound of the lemma. The inequality $D_{\mathrm{nor}} \le 2\bar{D}_R$ is Lemma 16.14.

For (16.51), we expand $\mathbf{F}_h\Omega_0 - \lambda\Omega_0$ into the two terms with $\mathfrak{b}_{\pm,h}$ and the term $\chi_{\mathrm{inv}}\mathbf{M}\chi_{\mathrm{inv}}\Omega_0 - \lambda\Omega_0$. The last is a Gaussian cutoff tail, since $\mathbf{M}\Omega_0 = \lambda\Omega_0$. By the assumed bounds, each multiplier term has the size $O_m(h^3)$, and the boundedness of the other factors in the graph norm of order m preserves this order. The opposite graph order follows by the same argument for the adjoint, since

$$\big((\mathbf{N}+1)^m\mathbf{F}_h^*(\mathbf{N}+1)^{-m}\big)^* = (\mathbf{N}+1)^{-m}\mathbf{F}_h(\mathbf{N}+1)^m.$$

□

Verification of the hypotheses. For the analytic half-weight of the onsite weight, the hypotheses of Lemma 16.21 reduce to a finite computation at each graph order. By (16.49), the coefficients satisfy

$$|\mathfrak{b}_h(x)| \le C\big(|hx|^3 + h^2|hx| + h^4\big).$$

On the tube we have $|hx|^3 \le Crh^2|x|^2$, and the bound of oscillator polynomials by $\mathbf{N}+1$ implies the second hypothesis at $m = 0$. In a commutator with $\mathbf{N}$, the first derivative of the cubic remainder carries the factor h times a quadratic unscaled coordinate, its second derivative the factor h^2 times a linear one, and its higher derivatives carry at least three powers of h. The additional factor $(\mathbf{N}+1)^{-1}$ in each graph commutator pays its differential order. We obtain the leading bound Crh^2, and the bound $C_m(r)h^3$ for all differentiated terms. The commutators with fixed odd matrices have the same order, since every nonconstant normal odd coefficient of the remainder starts with $h^2|hx|$. The smooth extension off the tube and the derivatives of its cutoff satisfy the same bounds with constants that depend on r. Gaussian vacuum moments give $O_m(h^3)$ for the vacuum column. Since the Gaussian expectation of the term of degree three vanishes by parity, the vacuum scalar starts with h^4.

16.10 The normal contraction and the Ward identity

We now prove that the frozen normal transfer contracts the complement of the Gaussian mean at the rate h^2, and that it preserves the Gaussian mean exactly on the kernel of the supercharge.

Recall from Section 2.4 why the exact preservation, the Ward identity, is needed. By Lemma 16.21, $\mathbf{F}_h$ differs from the Mehler operator by vacuum rows of order h^3 and a

vacuum scalar of order h^4, so that a perturbative argument places its top eigenvalue only at $1 + O(h^4)$, an error of order one over the h^{-4} steps of the chain. The identity $\mathbf{P}_0\mathbf{F}_h f = \mathbf{P}_0 f$ for $f \in \ker \mathfrak{q}_{\mathrm{nor}}$, proved below, removes this error. As for the transfer operator of [29] (see Section 1.3), supersymmetry fixes an eigenvalue at one; here the supercharge acts only on the normal variables. It is used only to make the closed subspace $\ker \mathfrak{q}_{\mathrm{nor}} \cap \ker \ell_0$ invariant, since the contraction holds on the whole kernel of the Gaussian mean.

Proposition 16.22. *Let $\mathbf{F}_h$ be the frozen normal transfer of Section 16.9, including all nilpotent terms of the invariant cutoff and the analytic normal half-weight. Let $\mathbf{P}_0$, ℓ_0 and Ω_0 be as in Section 16.4, and fix a finite set of graph orders $m \geq 0$. For a sufficiently small tube and then all sufficiently small h, the following holds in the balanced signature. The operator $\mathbf{F}_h$ maps* $\ker \mathfrak{q}_{\mathrm{nor}}$ *into itself, and*

$$\begin{aligned} \mathbf{P}_0\mathbf{F}_h f &= \mathbf{P}_0 f \quad (f \in \ker \mathfrak{q}_{\mathrm{nor}}), \\ \mathbf{F}_h(\ker \mathfrak{q}_{\mathrm{nor}} \cap \ker \ell_0) &\subset \ker \mathfrak{q}_{\mathrm{nor}} \cap \ker \ell_0, \\ \|\mathbf{F}_h^j f\| &\leq e^{-cjh^2}\|f\| \quad (f \in \ker \mathfrak{q}_{\mathrm{nor}} \cap \ker \ell_0). \end{aligned} \tag{16.52}$$

For every prescribed graph order m,

$$\|(\mathbf{N}+1)^m \mathbf{F}_h^j f\| \leq e^{-c_m jh^2}\|(\mathbf{N}+1)^m f\|, \qquad f \in \ker \mathfrak{q}_{\mathrm{nor}} \cap \ker \ell_0 \cap \mathcal{G}^m, \tag{16.53}$$

and $\|(\mathbf{N}+1)^m \mathbf{F}_h f\| \leq e^{-c_m h^2}\|(\mathbf{N}+1)^m f\|$ for all $f \in \mathcal{G}^m$ with $\ell_0(f) = 0$. In every signature $p \neq k$, the exact quadratic transfer is $\lambda \mu_+^{\mathbf{N}_+}\mu_-^{\mathbf{N}_-}$ as in Lemma 16.10, its defect satisfies

$$\mathrm{Id} - (\mathbf{M}_h^{\mathfrak{q}})^* \mathbf{M}_h^{\mathfrak{q}} \geq (1 - |\lambda|^2)\mathrm{Id} \geq ch^2 \mathrm{Id} \tag{16.54}$$

on the whole normal space of that signature, and $\|\mathbf{F}_h^j\| \leq Ce^{-cjh^2}$ there. All constants are uniform for E in a compact subset of $(-2, 2)$, and the tube and the threshold for h may depend on k and on the chosen graph orders.

In Section 18, we obtain from Proposition 16.22 and Banach's fixed point theorem the fixed state Ω_h of $\mathbf{F}_h$ in $\ker \mathfrak{q}_{\mathrm{nor}}$ with $\ell_0(\Omega_h) = 1$.

Proof. We first show that $\mathbf{F}_h$ preserves $\ker \mathfrak{q}_{\mathrm{nor}}$. Every factor of $\mathbf{F}_h$ commutes with $\mathfrak{q}_{\mathrm{nor}}$ on smooth functions with Gaussian decay. Indeed, the half-weight and the invariant cutoff are invariant even multipliers, and $\mathfrak{q}_{\mathrm{nor}}$ is an odd derivation that annihilates invariant functions. The Mehler operator $\mathbf{M}_h^{\mathfrak{q}} = \mu_+^{\mathbf{N}_+}\mu_-^{\mathbf{N}_-}$ commutes with $\mathfrak{q}_{\mathrm{nor}}$, since $\mathfrak{q}_{\mathrm{nor}}$ preserves the number of each coordinate by Lemma 16.12. Now let g be a Wick polynomial in $\ker \mathfrak{q}_{\mathrm{nor}}$. By the Gaussian kernel and the graph bounds of Lemma 16.21, we have $\mathbf{F}_h g \in \mathcal{G}^1$, and Lemma 16.12 implies that

$$\mathfrak{q}_{\mathrm{nor}}\mathbf{F}_h g = \mathbf{F}_h \mathfrak{q}_{\mathrm{nor}} g = 0.$$

Since $\ker \mathfrak{q}_{\mathrm{nor}}$ is closed, such polynomials are dense in it, and $\mathbf{F}_h$ is bounded, $\mathbf{F}_h$ preserves $\ker \mathfrak{q}_{\mathrm{nor}}$.

Next, we show that $\mathbf{F}_h$ preserves the Gaussian mean on $\ker \mathfrak{q}_{\mathrm{nor}}$. Let f be a Wick polynomial in $\ker \mathfrak{q}_{\mathrm{nor}}$. Then $w_{\mathrm{anh}}^{1/2}\chi_{\mathrm{inv}} f$ is in $\ker \mathfrak{q}_{\mathrm{nor}}$ and has Gaussian decay, and Lemma 16.13 identifies its Gaussian mean with its value at the origin, which equals $f_{\mathrm{body}}(0) = \ell_0(f)$, since the invariant multipliers equal one at the origin. The Mehler operator preserves the Gaussian mean, since $\mathbf{P}_0\mathbf{M}_h^{\mathfrak{q}} = \mathbf{P}_0$. Applying the same argument to the output multipliers, we obtain $\mathbf{P}_0\mathbf{F}_h f = \mathbf{P}_0 f$. By density and the boundedness of $\mathbf{P}_0\mathbf{F}_h$, this identity extends to $\ker \mathfrak{q}_{\mathrm{nor}}$, and in particular $\ker \mathfrak{q}_{\mathrm{nor}} \cap \ker \ell_0$ is invariant.

We now prove the contraction, using the graph defect estimate. Recall from (16.50) that

$$\|\mathbf{F}_{m,h} f\|^2 \le (1 + C_m h^4)\|f\|^2 - c_m \|\bar{D}_R^{1/2} f\|^2,$$

and that $D_{\mathrm{nor}} \le 2\bar{D}_R$. Using $\operatorname{Re} z_\sigma \ge ch^2$, we also find that

$$D_{\mathrm{nor}} = \mathrm{Id} - \mathbf{M}^*\mathbf{M} \ge c_0 h^2 \mathbf{P}_0^{\perp}.$$

Together these give

$$\|\mathbf{F}_{m,h} f\|^2 \le (1 + C_m h^4 - c'_m h^2)\|f\|^2 \le e^{-2c_m h^2}\|f\|^2, \qquad \mathbf{P}_0 f = 0, \tag{16.55}$$

for small h, after renaming c_m. Since $\mathbf{N}$ commutes with $\mathbf{P}_0$, the operator $(\mathbf{N}+1)^m$ preserves $\ker \ell_0$, and (16.55) implies the one-step bound for all $f \in \mathcal{G}^m$ with $\ell_0(f) = 0$. For $f \in \ker \mathfrak{q}_{\mathrm{nor}} \cap \ker \ell_0 \cap \mathcal{G}^m$, the iterates $\mathbf{F}_h^i f$ remain in $\ker \mathfrak{q}_{\mathrm{nor}} \cap \ker \ell_0$ by the invariance proved above, and in $\mathcal{G}^m$ since $\mathbf{F}_{m,h}$ is bounded. Iterating the one-step bound proves (16.53), and the case $m = 0$ completes (16.52).

It remains to treat the unbalanced signatures. By Lemma 16.10, the scalar of $\mathbf{M}_h^{\mathfrak{q}}$ satisfies $|\lambda| = e^{-(p-k)^2 \operatorname{Re} z_+}$. Its Mehler factor is a contraction, and $(p-k)^2 \operatorname{Re} z_+ \ge ch^2$, which proves (16.54). Thus the normal defect of the physical quadratic transfer satisfies $D_{\mathrm{nor}} \ge ch^2 \mathrm{Id}$. Applying (16.50) with $m = 0$ and using $D_{\mathrm{nor}} \le 2\bar{D}_R$, we obtain

$$\|\mathbf{F}_h^j\| \le (1 + Ch^4 - c'h^2)^{j/2} \le e^{-cjh^2}$$

for small h, as desired. □

SECTION 17

Stability of the transfer kernel

In this section we prove that the powers of the adapted transfer operator A_h of Section 16.5 stay bounded over h^{-4} steps, on arbitrary inputs and uniformly in h. This is the stability hypothesis of the approximation theorem used in Section 18, and Sections 19 and 20 use the underlying four-term bound for one step. The main result is Theorem 17.1.

We compare A_h with the product C_h of the cutoff Mehler operator of Section 16.7 and the heat semigroup of the saddle manifold over the time of one link. Since $A_h - C_h$ is not small in norm, we measure it relative to the defects of C_h, in the form (16.48), which Corollary A.2 turns into the hypothesis of Lemma A.1. The comparison is made through the kernel of A_h, written in coordinates centered at the midpoint of the two endpoints of a link, and it rests on three structural facts. First, the chordal Gaussian of a link is exactly even under the reversal of the tangent increment (Lemma 17.2). Second, once the action of the stabilizer on the normal variables is factored out, the remaining kernel is a scalar superfunction of the unscaled normal coordinates. The odd normal variables, integrated against the odd Mehler factor, therefore produce a multiple of the identity up to $O(h)$ (Lemmas 17.3 and 17.4). Third, we control differences across a link by the heat semigroup of the saddle manifold, through a difference energy along geodesics (Lemma 17.7), and the marginals of kernels that are odd under the reversal gain a factor h^2 (Lemma 17.8). Recall that $h^2 = W^{-1}$ is the size of the increment of one link and $\delta_W = h^4$ its variance. By Remark 16.11, an energy estimate in a metric of product form cannot control the odd normal variables, and the proof below integrates them out against the full Mehler factor of the step, half-weight included, before any norm is taken.

17.1 The reference contraction and the main theorem

The underlying manifold, the coefficient bundle, and the body heat

We let $|\mathcal{M}|$ denote the underlying manifold of the saddle manifold through R_*, with the invariant Riemannian metric that $g_{R_*}(Y, Z) = 4s_E^2 \operatorname{STr}(YZ)$ induces on the real form of the even tangent directions. (On the fermion–fermion block, the sign of the supertrace is compensated by the real form, as in Section 5.) We let μ denote its Riemannian volume and n_b its dimension. In the balanced signature, $|\mathcal{M}|$ is the product of the hyperbolic and the compact symmetric spaces that form the underlying manifold of $\mathcal{M}_k$ (see Section 5). Recall from Section 16.2 that $U(k, k|2k)$ denotes the pair of the group

$G_0 = \{\mathrm{diag}(g_{\mathrm{B}} \,|\, g_{\mathrm{F}}) : g_{\mathrm{B}} \in U(k,k),\ g_{\mathrm{F}} \in U(2k)\}$ and the Lie superalgebra $\mathfrak{gl}(2k|2k)$, so that G_0 is the underlying manifold of this supergroup. In every signature, $|\mathcal{M}|$ is a complete Riemannian symmetric space on which G_0 acts transitively by isometries, with compact stabilizer K_0. We fix $\varrho_0 > 0$ smaller than half the injectivity radius of $|\mathcal{M}|$. For $d(x,y) < 2\varrho_0$ there exists a unique minimal geodesic from y to x, and we let $m(x,y)$ denote its midpoint.

We let Λ_T denote the exterior algebra of the odd tangent coordinates of the saddle manifold. We measure its elements in the coefficient norm $|\cdot|_{2;\omega}$ of Definition 5.12, taken in an orthonormal basis of the odd tangent coordinates, with a scalar weight $0 < \omega \le 1$:

$$|c|_{2;\omega}^2 = \sum_{j \ge 0} \omega^{2j} |c_j|^2,$$

where c_j is the component of c of odd degree j and $|c_j|$ its invariant norm. This is the norm of the continuum model in Section 5.2. By Lemma 5.13, multiplication by a nilpotent even element has norm $O(\omega^2)$ in this norm, and an odd derivative has norm $O(\omega^{-1})$. The compact group K_0 acts unitarily on $\mathcal{H}_{\mathrm{nor}} \otimes \Lambda_T$, by rotations of the normal and odd tangent coordinates, and this action preserves $\mathbf{N}$. We let $\mathcal{E}$ denote the associated Hilbert bundle over $|\mathcal{M}|$ with fiber $\mathcal{H}_{\mathrm{nor}} \otimes \Lambda_T$, and ∇ its canonical connection, induced by the $\mathfrak{k}_0$-component of the Maurer–Cartan form. For $d(x,y) < 2\varrho_0$ we let τ_{xy} denote the parallel transport of $\mathcal{E}$ along the minimal geodesic from y to x. This transport is unitary, it commutes with $\mathbf{N}$, and

$$\tau_{yx} = \tau_{xy}^{-1}.$$

We let Δ_∇ denote the Friedrichs extension of $-\nabla^*\nabla$ from the smooth compactly supported sections. On each summand of $\mathcal{E}$ of finite rank (an eigenspace of $\mathbf{N}$ tensored with Λ_T), Δ_∇ is essentially self-adjoint on these sections, since $|\mathcal{M}|$ is complete [18]. For $q \in \mathbb{R}$ we set

$$\mathcal{H}_q^{\mathrm{tube}} = L^2\big(|\mathcal{M}|, v^{2q}\,\mathrm{d}\mu; \mathcal{E}\big), \qquad \mathbf{H}_h = e^{\delta_W \Delta_\nabla / 2},$$

where v is the weight (5.3). Since v is a function of the BB block of the point, and the BB factor of $|\mathcal{M}|$ is the hyperbolic space H_k of Section 5 in every signature, v is defined on $|\mathcal{M}|$. By Lemma 5.1(b), $\log v$ is Lipschitz with bounded second derivatives. We call $\mathbf{H}_h$ the body heat. In terms of a Brownian motion on $|\mathcal{M}|$ with generator $\frac{1}{2}\Delta$, it is the semigroup at time δ_W. With the generator Δ it is the semigroup at time $\tau_{\mathrm{phys}} = \delta_W/2$, which is the time of one link: by (16.1) and the discussion after (16.29), the tangent part of one link is the heat semigroup of $L_H(0)$ at time τ_{phys}.

The reference contraction

We now define the frozen source kick. In the normalization of Section 15, (15.6) gives the bosonic part $e^{-\mathrm{i}\eta\,\mathrm{Tr}(S^{\mathrm{BB}}\Gamma_{BB})}$ of the source factor s_Γ at a point S, where $\eta = \varepsilon\eta_0$ is of

order $W^{-2} = \delta_W$. Since the BB factor of $|\mathcal{M}|$ is the hyperbolic space H_k, the body of the BB block of the point $R(x,0)$ of the saddle manifold has the form $\kappa_E \mathrm{Id} - \mathrm{i} s_E Q_B$ with $Q_B \in H_k$, and we write $Q_B = Q_B(x)$. The body of the bosonic part of s_Γ at this point is then

$$e^{-\mathrm{i}\eta \operatorname{Tr}((\kappa_E \mathrm{Id} - \mathrm{i} s_E Q_B)\Gamma_{BB})} = c_\Gamma \, e^{-\eta s_E \operatorname{Tr}(Q_B \Gamma_{BB})}, \qquad c_\Gamma = e^{-\mathrm{i}\eta \kappa_E \operatorname{Tr} \Gamma_{BB}}.$$

We define the *frozen source kick* $\mathbf{S}_h$ to be the multiplication by $e^{-\eta s_E \operatorname{Tr}(Q_B \Gamma_{BB})}$. By the last display, it is the multiplication by the body of the bosonic part of $c_\Gamma^{-1} s_\Gamma$ at $U = 0$. We keep the constant c_Γ out of the kick, since $\operatorname{Tr} \Gamma_{BB}$ is not real in general on the parameter ball $\mathcal{U}_2$ of (15.36), and $|c_\Gamma|$ may then exceed one. Since $\operatorname{Tr} \Lambda = 0$ and $\|\Gamma_{BB} - \Lambda\|_{\mathrm{op}} \le 1/4$ on $\mathcal{U}_2$, we have $|\operatorname{Tr} \Gamma_{BB}| \le k/2$. Since η is of order δ_W, this implies

$$|c_\Gamma - 1| \le C \delta_W. \tag{17.1}$$

We account for c_Γ in the bound for the source, Lemma 17.19.

When the damping is real, we have $\Gamma_{BB} = \Lambda$ and $c_\Gamma = 1$, and Lemma B.5 gives $0 < \mathbf{S}_h \le 1$. For complex $\Gamma \in \mathcal{U}_2$, the kick is complex, and we have

$$|\mathbf{S}_h| \le e^{-c\delta_W v} \le 1, \qquad |\nabla \mathbf{S}_h| + |\nabla^2 \mathbf{S}_h| \le C \delta_W v \, e^{-c\delta_W v}. \tag{17.2}$$

Indeed, by (5.3) and Lemma 5.1(a), we have $\Lambda Q_B \ge 0$ and $\operatorname{Tr}(\Lambda Q_B) = 2(v-1) \ge 2k$. Together with these relations, (15.36) implies that

$$\operatorname{Re} \operatorname{Tr}(Q_B \Gamma_{BB}) = \operatorname{Re} \operatorname{Tr}\big(\Lambda Q_B \, \Gamma_{BB} \Lambda\big) \ge c_\varepsilon \operatorname{Tr}(\Lambda Q_B) = 2 c_\varepsilon (v-1) \ge c_\varepsilon v.$$

The derivatives of $\operatorname{Tr}(Q_B \Gamma_{BB})$ along the orbit are bounded by Cv, since a unit tangent generator acts on Q_B by a commutator of size $O(v)$ and $\|\Gamma\|$ is bounded, and the second bound in (17.2) follows as in the proof of Lemma B.5. We define the reference step

$$C_h = \mathbf{M}^\chi \otimes \mathbf{S}_h \mathbf{H}_h,$$

where $\mathbf{M}^\chi$ is the cutoff Mehler operator of Section 16.7 with $\mathbf{M} = \mathbf{M}_h^{\mathrm{q}}$. Thus $\mathbf{M}^\chi$ already contains the full scalar λ of Lemma 16.10, including the physical root phase ζ_p, exactly once. Since the scalar cutoff $\chi_{r/h}$ is invariant under K_0, it commutes with the parallel transport, with $\mathbf{S}_h$, and with $\mathbf{H}_h$. Since $\|\mathbf{M}^\chi\| \le 1$, $|\mathbf{S}_h| \le 1$ and $\|\mathbf{H}_h\| \le 1$ on $\mathcal{H}_0^{\mathrm{tube}}$, the operator C_h is a contraction of $\mathcal{H}_0^{\mathrm{tube}}$, and we let

$$D_R = \mathrm{Id} - C_h^* C_h, \qquad D_L = \mathrm{Id} - C_h C_h^*$$

denote its defects. We also set

$$D_{\mathrm{body,R}} = \mathrm{Id} - \mathbf{H}_h^2 + \mathbf{H}_h(\mathrm{Id} - |\mathbf{S}_h|^2)\mathbf{H}_h, \qquad D_{\mathrm{body,L}} = \mathrm{Id} - |\mathbf{S}_h|^2 + \mathbf{S}_h(\mathrm{Id} - \mathbf{H}_h^2)\mathbf{S}_h^*,$$

so that $D_{\mathrm{body,R}} = \mathrm{Id} - \mathbf{H}_h |\mathbf{S}_h|^2 \mathbf{H}_h$ and $D_{\mathrm{body,L}} = \mathrm{Id} - \mathbf{S}_h \mathbf{H}_h^2 \mathbf{S}_h^*$.

We now compare these defects. The normal factor $(\mathbf{M}^{\chi})^{*}\mathbf{M}^{\chi}$ and the body factor $\mathbf{H}_h|\mathbf{S}_h|^2\mathbf{H}_h$ act on different tensor factors, so they commute. For commuting operators a, b with $0 \le a, b \le \mathrm{Id}$ we have

$$\mathrm{Id} - ab - \tfrac{1}{2}(\mathrm{Id} - a) - \tfrac{1}{2}(\mathrm{Id} - b) = \tfrac{1}{2}\big(a(\mathrm{Id} - b) + b(\mathrm{Id} - a)\big) \ge 0.$$

Combining this inequality with $D_{\mathrm{nor}} \le 2\bar{D}_R$ from Lemma 16.14, and arguing in the same way on the left, we obtain

$$D_{\mathrm{nor}} + D_{\mathrm{body,R}} \le 4D_R, \qquad D_{\mathrm{nor}} + D_{\mathrm{body,L}} \le 4D_L. \tag{17.3}$$

The weighted spaces need a conjugated reference. Multiplication by v^q is a unitary map from $\mathcal{H}^{\mathrm{tube}}_q$ onto $\mathcal{H}^{\mathrm{tube}}_0$. We set

$$C_{h,q} = v^{-q} C_h v^q.$$

Since C_h is a contraction of $\mathcal{H}^{\mathrm{tube}}_0$, the operator $C_{h,q}$ is a contraction of $\mathcal{H}^{\mathrm{tube}}_q$, with the defects $v^{-q}D_R v^q$ and $v^{-q}D_L v^q$. Note that C_h itself is in general not a contraction of $\mathcal{H}^{\mathrm{tube}}_q$ for $q \neq 0$, since the conjugate $v^q \mathbf{H}_h v^{-q}$ has norm $1 + O(q^2\delta_W)$, as the model case $e^{qx}e^{t\partial_x^2}e^{-qx} = e^{t(\partial_x - q)^2}$ on $\mathbb{R}$ shows.

The main theorem

We consider the adapted transfer operator A_h of (16.38), with the invariant cutoff χ_{inv} at radius r on both sides and all odd coefficients retained. It acts on functions on the real tube of Definition 16.1. The Schäfer–Wegner cycle of Section 4 is different, and the two are connected in Sections 19 and 20.

Theorem 17.1. *Let $T > 0$ and $q \in \mathbb{R}$. There exist $\omega \in (0,1]$, then $r \in (0, r_*]$, then $h_0 > 0$ and constants A_1 and B, depending on k, q, T, the compact sets of energies, of ε and of X, and the damping ball $\mathcal{U}_2$, with the following property. For $0 < h \le h_0$, in every signature, and uniformly for the damping Γ in the parameter ball $\mathcal{U}_2$ of* (15.36)*, the operator A_h on $\mathcal{H}^{\mathrm{tube}}_q$ satisfies the four-term bound*

$$\begin{aligned} |\langle g, (A_h - C_{h,q})f\rangle| \le a\|D^{1/2}_{L,q} g\|\|D^{1/2}_{R,q} f\| \\ + A_1\sqrt{\delta_W}\big(\|g\|\|D^{1/2}_{R,q} f\| + \|D^{1/2}_{L,q} g\|\|f\|\big) + A_1\delta_W\|g\|\|f\| \end{aligned} \tag{17.4}$$

relative to the contraction $C_{h,q} = v^{-q}C_h v^q$, with its defects $D_{R,q} = v^{-q}D_R v^q$ and $D_{L,q} = v^{-q}D_L v^q$ and with $a \le 1/16$. Consequently

$$\begin{aligned} A_h^* A_h &\le (1 + B\delta_W/2)\mathrm{Id} - \tfrac{1}{2}D_{R,q}, \\ A_h A_h^* &\le (1 + B\delta_W/2)\mathrm{Id} - \tfrac{1}{2}D_{L,q}, \\ \sup_{n\delta_W \le T} &\|A_h^n\|_{\mathcal{H}^{\mathrm{tube}}_q \to \mathcal{H}^{\mathrm{tube}}_q} \le e^{BT/4}, \end{aligned}$$

with adjoints in $\mathcal{H}_q^{\mathrm{tube}}$. The same holds for every step of the same form with a smaller damping in $\mathcal{U}_2$. The conclusion remains true, with the same r, if ω is replaced by any $\omega' \in (0, \omega]$, with h_0, A_1 and B depending also on ω'.

The second part of the theorem follows from the first by the reduction to Corollary A.2 in Remark 16.20, applied on $\mathcal{H}_q^{\mathrm{tube}}$ with $\mathbf{T} = A_h$, $C = C_{h,q}$, $\delta_0 = \delta_W$, and $t = T$. Recall that this reduction takes $B = \max\{1, 2304 A_1^2\}$ and requires $a \le 1/16$ and $B\delta_0 \le 1$. We therefore choose h_0 so small that $B h_0^4 \le 1$.

Since multiplication by v^q is unitary from $\mathcal{H}_q^{\mathrm{tube}}$ onto $\mathcal{H}_0^{\mathrm{tube}}$, (17.4) is equivalent to the four-term bound for $A_{h,q} = v^q A_h v^{-q}$ on $\mathcal{H}_0^{\mathrm{tube}}$ relative to C_h. This is the form in which we prove it. The constants ω and r are fixed before $h \to 0$. We write $A_h - C_h$ as a sum of terms in Sections 17.2 and 17.3, prove the three tools of Sections 17.4 and 17.5, bound the terms in Sections 17.6 and 17.7, and collect the bounds, together with the weights, in Section 17.8.

17.2 The kernel in midpoint coordinates

By (16.38) and (16.33), the kernel of A_h with respect to $\mathrm{d}U'\,\mathrm{d}\nu(y)$ is

$$A_h(x, U; y, U') = s_\Gamma\big(R(x,U)\big)\, w_{\mathrm{out}}(U)\, K_{\tau_{\mathrm{phys}}}\big(R(x,U), R(y,U')\big)\, w_{\mathrm{in}}(U'), \tag{17.5}$$

where we set

$$w_{\mathrm{out}} = \chi_{\mathrm{inv}} \varpi w_0^{1/2} \gamma_h^{-1}, \qquad w_{\mathrm{in}} = \chi_{\mathrm{inv}} \varpi w_0^{1/2} \gamma_h.$$

Both normal multipliers are invariant under conjugation by K, since ϖ, w_0, γ_h, and χ_{inv} are. This is an identity of all Grassmann coefficients. The kernel decays like a Gaussian at scale h^2 in the distance between the endpoints, so that up to an error of size e^{-c/δ_W} in every norm used below we may restrict to $d(x,y) < \varrho_0$.

Midpoint frames

We now introduce coordinates in which the reversal of a link is explicit. For $d(x,y) < \varrho_0$, with all odd coordinates retained by finite nilpotent expansion around the body, let $m = m(x,y)$ be the midpoint and let $Y = Y(x,y) \in \mathfrak{m}$ be such that

$$x = T(m) e^{h^2 Y/2} \cdot R_*, \qquad y = T(m) e^{-h^2 Y/2} \cdot R_*,$$

where $g \cdot R_*$ denotes the point $g R_* g^{-1}$ of the saddle manifold. Here Y is the full tangent increment, with its even part Y_0 and its odd part. These coordinates exist and are unique near the diagonal by the inverse function theorem for supermanifolds [84, 132], applied to the map $(m, Y) \mapsto (x, y)$. The body of this map is a diffeomorphism near the diagonal,

by the geodesic midpoint of the symmetric space for $d(x,y) < 2\varrho_0$. Its odd differential at $Y = 0$ is the invertible map

$$(\delta m, \delta Y) \mapsto (\delta m + h^2 \delta Y/2, \delta m - h^2 \delta Y/2),$$

and the differential stays invertible for $|h^2 Y|$ small. We define the stabilizer elements

$$k_+(x,y) = e^{-h^2 Y/2} T(m)^{-1} T(x), \qquad k_-(x,y) = e^{h^2 Y/2} T(m)^{-1} T(y), \tag{17.6}$$

which lie in K by the choice of Y. Exchanging x and y fixes m and replaces Y by $-Y$. By (17.6), this implies

$$k_+(y,x) = k_-(x,y), \qquad k_-(y,x) = k_+(x,y), \tag{17.7}$$

and $k_\pm(x,x) = \mathrm{Id}$. We call (m, Y) the midpoint coordinates of the pair (x,y). By homogeneity, all bounds below on these coordinates hold on uniformly sized charts.

Lemma 17.2. *Let U and U'' be normal, and write $U = U_m + h^2 V/2$ and $U'' = U_m - h^2 V/2$, with $S = R_* + U_m$. Then*

$$\begin{aligned} &e^{h^2 \operatorname{ad}_Y/2}(R_* + U) - e^{-h^2 \operatorname{ad}_Y/2}(R_* + U'') \\ &\quad = 2\sinh(h^2 \operatorname{ad}_Y/2) S + h^2 \cosh(h^2 \operatorname{ad}_Y/2) V \\ &\quad = h^2 (V + [Y,S]) + h^6 \big([Y,[Y,V]]/8 + [Y,[Y,[Y,S]]]/24 \big) + O(h^{10}) \end{aligned} \tag{17.8}$$

as a convergent series, where $O(h^{10})$ is bounded by $C h^{10} (|V| + |Y|)^5$ on the tube. The function

$$\zeta(U, U''; Y) = \mathrm{STr}\big(e^{h^2 \operatorname{ad}_Y/2}(R_* + U) - e^{-h^2 \operatorname{ad}_Y/2}(R_* + U'') \big)^2$$

satisfies $\zeta(U, U''; -Y) = \zeta(U, U''; Y)$ exactly, and

$$\zeta = h^4 \big(\mathrm{STr}\, V^2 + \mathrm{STr}[Y,S]^2 \big) + O\big(h^8 (|V| + |Y|)^4 \big).$$

Proof. The identity (17.8) follows from $e^{\pm h^2 \operatorname{ad}_Y/2}(S \pm h^2 V/2)$ and the definitions of sinh and cosh, and the expansion is the Taylor series. Since Σ commutes with the block-diagonal S and V, and $\Sigma Y \Sigma^{-1} = -Y$ for off-diagonal Y, we have

$$\Sigma (\operatorname{ad}_Y^j X) \Sigma^{-1} = (-1)^j \operatorname{ad}_Y^j X$$

for block-diagonal X. Conjugating the left side of (17.8) by Σ replaces Y by $-Y$, and ζ is invariant under this conjugation, since STr is. Finally, V is block diagonal and $[Y,S]$ is off diagonal, and the cross term in $\mathrm{STr}(V + [Y,S])^2$ vanishes. □

The leading Gaussian of a link is thus the flat normal increment Gaussian times the conditional tangent Gaussian with the quadratic form $\mathrm{STr}[Y,S]^2$, whose body is uniformly elliptic on the tube by root separation. By the end of Section 16.3, its normalized integral over $\mathfrak{m}$ is $\varpi(S)^{-2}$ times the flat one.

17.3 The frame factorization and scalar amplitudes

We now factor out the way in which the stabilizer acts on the normal variables. For $k \in K$ and a function f of U, we set

$$(\pi(k)f)(U) = f(k^{-1}Uk).$$

For the body k_0 of k, the operator $\pi(k_0)$ is unitary on $\mathcal{H}_{\mathrm{nor}}$ and preserves $\mathbf{N}$. We write $k = k_0 k_{\mathrm{rem}}$, where k_{rem} has positive Grassmann degree. Then $\pi(k_{\mathrm{rem}}) = \exp(\mathrm{d}\pi(\log k_{\mathrm{rem}}))$ is a finite sum, since $\log k_{\mathrm{rem}}$ is nilpotent, and for $\mathfrak{e} \in \mathfrak{k}$ the operator $\mathrm{d}\pi(\mathfrak{e})$ is the odd or compact charge of $\mathfrak{e}$ of Section 16.3.

Lemma 17.3. *For $d(x,y) < \varrho_0$, let (m, Y) be the midpoint coordinates and $k_\pm = k_\pm(x,y)$. Let $\widetilde{\Pi}(Y)$ be the operator on functions of the normal variable with the kernel*

$$\Pi(U, U''; Y) = w_{\mathrm{out}}(U) \exp\Big\{-\frac{\zeta(U, U''; Y)}{2h^4}\Big\} w_{\mathrm{in}}(U'')$$

with respect to $\mathrm{d}U''$. Then the kernel (17.5)*, as an operator from functions of U' at y to functions of U at x, is*

$$s_\Gamma\big(R(x,\cdot)\big)\, \pi(k_+)^{-1}\, \widetilde{\Pi}(Y)\, \pi(k_-). \tag{17.9}$$

The kernel Π is a scalar superfunction, in the following sense. It is a function of the normal coordinates U and U'' and of Y with values in the Grassmann algebra generated by their odd coordinates, and its coefficients are smooth in the even coordinates on the tube and analytic on regions where the cutoffs are locally constant. It does not depend on m, and $\widetilde{\Pi}(-Y) = \widetilde{\Pi}(Y)$.

Proof. By (17.6), we have $T(x) = T(m)e^{h^2 Y/2}k_+$ and $T(y) = T(m)e^{-h^2 Y/2}k_-$. The elements $k_\pm$ stabilize R_*, and hence

$$R(x,U) = T(m)e^{h^2 Y/2}(R_* + k_+ U k_+^{-1})e^{-h^2 Y/2}T(m)^{-1},$$

and similarly for $R(y,U')$ with k_- and $e^{-h^2 Y/2}$. Using the invariance of the supertrace under conjugation by $T(m)$, we obtain

$$\mathrm{STr}\big(R(x,U) - R(y,U')\big)^2 = \zeta(k_+ U k_+^{-1}, k_- U' k_-^{-1}; Y).$$

We now substitute $U'' = k_- U' k_-^{-1}$ in the integral over U'. The multipliers w_{out} and w_{in} are invariant under conjugation by K. On each block of $\mathfrak{k}$, we have $k_- U' k_-^{-1} = \lambda_{k_-}\rho_{k_-^{-1}}U'$, with the left and right multiplications λ and ρ. Using $\mathrm{SDet} = \exp \mathrm{STr} \log$, (16.3), and the fact that k_- is close to the identity, we find

$$\mathrm{SDet}\,\lambda_{k_-} = \mathrm{SDet}(k_-)^{\mathrm{sdim}}, \qquad \mathrm{SDet}\,\rho_{k_-^{-1}} = \mathrm{SDet}(k_-)^{-\mathrm{sdim}},$$

where sdim is the superdimension of the block. Hence the Berezinian of the substitution is one, and we obtain (17.9).

The geometric factors of Π are exponentials, Berezinians, and analytic functions of the supermatrices U, U'', and Y. The invariant cutoffs are smooth, so their finite nilpotent expansions give smooth coefficients, which are analytic where the cutoffs are locally constant. None of these factors involves m. This proves the second assertion. The evenness of $\widetilde{\Pi}$ in Y follows from Lemma 17.2. □

The next lemma rests on Lemma 17.3. All non-scalar structure of the kernel in the odd normal variables comes from the frame actions $\pi(k_\pm)$, which are finite sums of words in the normal charges. Without the factorization, a constant matrix on $\mathcal{H}_{\mathrm{odd}}$, such as an odd creation operator, could not be excluded from the amplitude, and the conclusion of Lemma 17.4 would fail for it.

Lemma 17.4. *In the standardized coordinates $U = hZ$ and $U'' = hZ''$ and the positive Wick realization, let $V = (Z - Z'')/h$ and $U_m = h(Z + Z'')/2$, and let $\Pi(U, U''; Y) = k_z(Z, Z'')\, a(U_m, V; Y)$ with the phase-free even Mehler kernel k_z of* (16.32)*; the scalar λ of $\mathbf{M}_h^{\mathrm{q}}$ is included in a. Let b be a polynomial in the normalized normal increments, with coefficients that are scalar superfunctions of the same kind as Π. After the finite integration over the odd normal variables, the operator with the kernel $b\,\Pi$ is $\mathbf{M}[c]$, in the notation of Lemma 16.18, with*

$$c(U_m, V; Y) = c^{\mathrm{sc}}(U_m, V; Y)\,\mathrm{Id}_{\mathcal{H}_{\mathrm{odd}}} + h\, c^{\mathrm{rem}}(U_m, V; Y),$$

where c^{sc} is scalar, and c^{sc} and c^{rem} are amplitudes with bounded seminorms, times a polynomial envelope in Y. Consequently its vacuum column and the vacuum column of its adjoint are scalar multiples of Ω_0 up to $O(h)$ in every graph norm.

Proof. We first express the odd normal dependence. Let θ and θ'' be the copies of a standardized odd normal coordinate at the two endpoints. The corresponding odd coordinates of the unscaled midpoint and of the unscaled increment, normalized by h^2, are

$$\tfrac{h}{2}(\theta + \theta''), \qquad h^{-1}(\theta - \theta'').$$

Since Π and b are scalar superfunctions of the unscaled coordinates, a and b are polynomials in $h(\theta + \theta'')/2$ and $(\theta - \theta'')/h$, with coefficients that are smooth functions of the even variables with the bounds of the amplitude class. For a, this follows from Lemma 17.2 after division by the quadratic exponent of k_z, restricted to $|V| + |Y| \le h^{-\alpha}$ for a small fixed $\alpha > 0$, with a Gaussian-small complement.

Next, we integrate the odd variables. Multiplying the Berezin kernel of an operator X of parity $|X|$ by $(\theta + \theta'')/2$ corresponds to the bounded map $X \mapsto \frac{1}{2}(\theta X + (-1)^{|X|} X\theta)$, and multiplying by $\theta - \theta''$ corresponds to the graded commutator with θ. Every non-constant monomial in the midpoint odd variables carries an explicit factor h. By

Lemma 16.19 with $\kappa_{\mathrm{o}} = h$, applied to the odd Mehler factor $e^{-z\mathbf{N}_F}$, every product of normalized increments acting on this factor is a number times the identity plus $O(h)$, and $e^{-z\mathbf{N}_F} = \mathrm{Id} + O(h^2)$. We conclude that at every point (U_m, V) the coefficient is a scalar times the identity plus $O(h)$. The same holds for every derivative in (U_m, V), since the odd computation does not involve these variables. This proves the decomposition.

Finally, applying part (iii) of Lemma 16.18 to c^{sc} and to c^{rem}, we obtain $\mathbf{M}[c]\Omega_0 = c_0^{\mathrm{sc}}\Omega_0 + O(h)$ in every graph norm, with a scalar c_0^{sc}. The same argument applies to $\mathbf{M}[c]^* = \mathbf{M}[c^\sharp]$. □

In particular, for an odd charge $\mathfrak{q}_a = \sum_j (F_j x_j + G_j \partial_j)$, the matrix commutators in (16.47) satisfy $[F_j, c] = h[F_j, c^{\mathrm{rem}}]$ and $[G_j, c] = h[G_j, c^{\mathrm{rem}}]$. Together with (16.47), this implies that $[\mathfrak{q}_a, \mathbf{M}[c]]$ is h times a sum of operators $\mathbf{M}[c']$ with at most one letter on either side. By (16.46), the same holds for the even part of a compact charge, and its finite matrix part gives $[\mathrm{j}_{a,F}, \mathbf{M}[c]] = h\mathbf{M}[[\mathrm{j}_{a,F}, c^{\mathrm{rem}}]]$.

Words in the charges

Lemma 17.5. *For $d(x,y) < \varrho_0$, write $k_\pm = k_{\pm,0} k_{\pm,\mathrm{rem}}$. Then*

$$\log k_{\pm,\mathrm{rem}} = h^2 \sum_\alpha c_{\pm,\alpha}\, \mathfrak{e}_\alpha,$$

where the $\mathfrak{e}_\alpha$ run over a basis of the even and odd generators of $\mathfrak{k}$, and the $c_{\pm,\alpha}$ are analytic superfunctions of the midpoint coordinates and of the odd coordinates of m, with every fixed covariant derivative bounded on uniformly sized charts, times a polynomial in Y. By (17.7)*, $c_{+,\alpha}(y,x) = c_{-,\alpha}(x,y)$. The expansion*

$$\pi(k_{\pm,\mathrm{rem}}) = \sum_{n=0}^{D} \frac{h^{2n}}{n!} \Big(\sum_\alpha c_{\pm,\alpha}\, \mathrm{d}\pi(\mathfrak{e}_\alpha) \Big)^n$$

terminates at an order D depending only on k. It is a finite sum of words in the odd and compact charges in which a word of degree n carries the factor h^{2n}, and after the integration over the odd tangent increments, normalized by h^2, no negative power of h arises.

Proof. By the construction of the midpoint coordinates above, $k_\pm$ are analytic functions of the pair (x,y), with all derivatives of fixed order bounded on uniformly sized translated charts, and $k_\pm = \mathrm{Id}$ at coincident endpoints. The factor $k_{\pm,\mathrm{rem}} = k_{\pm,0}^{-1} k_\pm$ has positive Grassmann degree, and its logarithm is a finite Grassmann polynomial. Since every nonzero coefficient of this polynomial vanishes at coincident endpoints, it contains at least one even or odd endpoint difference. Such a difference is of size h^2, and this gives the factor h^2. The odd coordinates of m are not small; they occur only in the bounded coefficients.

The induced action of $\mathfrak{e}_\alpha$ on the normal variable is linear with Berezinian one, and its differential $\mathrm{d}\pi(\mathfrak{e}_\alpha)$ is an odd charge $\mathfrak{q}_a$ or a compact charge $\mathfrak{j}_a$. The exponential series terminates since each coefficient has positive degree in a finite Grassmann algebra, and the noncommuting products of charges are kept in every term. Finally, the odd tangent endpoint differences, normalized by h^2, are integrated against the odd tangent Gaussian, which is the odd heat over the time δ_W. By Lemma 16.19 with $\kappa_{\mathrm{o}} = h^2$, pairs of them have bounded covariance and an unpaired one gains h^2, so that the explicit factor of a word survives the integration. □

The decomposition of the step

We define, for $d(x,y) < \varrho_0$,

$$\tau'_{xy} = \pi(k_{+,0})^{-1}\pi(k_{-,0}), \qquad \Pi_x(x,y) = \pi(k_{+,0})^{-1}\widetilde{\Pi}(Y)\pi(k_{+,0}), \qquad L_\pm = \mathrm{d}\pi(\log k_{\pm,\mathrm{rem}}),$$

where L_+ acts on the fiber at x and L_- on the fiber at y. By (17.6), the transvection $T(m)e^{h^2Y}T(m)^{-1}$ (the isometry of the saddle manifold that translates along the geodesic through m and transports frames in parallel along it) carries the frame $T(y)$ to the frame $T(x)k_+^{-1}k_-$ at x. The body of τ'_{xy} is therefore the parallel transport along the minimal geodesic in the frames T, and we identify it with τ_{xy} on the normal factor. Note also that, by the conjugation invariance of STr and of the multipliers, $\pi(k)^{-1}\widetilde{\Pi}(Y)\pi(k) = \widetilde{\Pi}(k^{-1}Yk)$ for $k \in K_0$. Thus

$$\Pi_x(x,y) = \widetilde{\Pi}(Y_x), \tag{17.10}$$

where $Y_x = k_{+,0}^{-1}Yk_{+,0}$ is the increment written in the frame $T(x)$. By Lemma 17.3, the kernel of A_h at (x,y) is

$$s_\Gamma\, \pi(k_{+,\mathrm{rem}})^{-1}\Pi_x(x,y)\tau_{xy}\pi(k_{-,\mathrm{rem}}) = s_\Gamma\Big(\sum_{n\le D}\frac{(-L_+)^n}{n!}\Big)\Pi_x(x,y)\tau_{xy}\Big(\sum_{n\le D}\frac{L_-^n}{n!}\Big).$$

We call the term with $n = 0$ on both sides the uncharged part, the two terms with one charge factor the degree-one part, and the rest the higher words. We set $L_-^{\mathrm{out}} = \tau_{xy}L_-\tau_{xy}^{-1}$, which acts on the fiber at x. The degree-one part is then

$$\Pi_x\tau_{xy}L_- - L_+\Pi_x\tau_{xy} = \tfrac{1}{2}\{\Pi_x, L_-^{\mathrm{out}} - L_+\}\tau_{xy} + \tfrac{1}{2}[\Pi_x, L_-^{\mathrm{out}} + L_+]\tau_{xy}, \tag{17.11}$$

with the anticommutator $\{X,X'\} = XX' + X'X$ and the commutator.

The reference C_h has the kernel $\mathbf{S}_h(x)\,\mathbf{M}^\chi\,p_h^\nabla(x,y)$, where $p_h^\nabla(x,y)$ is the kernel of $\mathbf{H}_h$, an operator from the fiber at y to the fiber at x. We let $\mathbf{F}_h$ be the frozen normal transfer with cutoffs of Section 16.9, and we let p_h be the scalar heat kernel of $e^{\delta_W\Delta/2}$ on $|\mathcal{M}|$. We write

$$A_h - C_h = \mathbf{E}_{\mathrm{src}} + \mathbf{E}_{\mathrm{nor}} + \mathbf{E}_0 + \mathbf{E}_{\mathrm{hol}} + \mathbf{E}_1 + \mathbf{E}_{\ge 2},$$

where, with kernels on $d(x,y) < \varrho_0$ and with the Gaussian-small remainder included in $\mathbf{E}_0$, we set

$$\mathbf{E}_{\mathrm{src}} = (s_\Gamma - \mathbf{S}_h)\,\mathcal{K}_h, \qquad \mathbf{E}_{\mathrm{nor}} = (\mathbf{F}_h - \mathbf{M}^\chi) \otimes \mathbf{S}_h\mathbf{H}_h,$$
$$\mathbf{E}_0(x,y) = \mathbf{S}_h(x)\big(\Pi_x(x,y) - \mathbf{F}_h p_h(x,y)\big)\tau_{xy},$$
$$\mathbf{E}_{\mathrm{hol}}(x,y) = \mathbf{S}_h(x)\mathbf{F}_h\big(p_h(x,y)\tau_{xy} - p_h^\nabla(x,y)\big),$$

where $\mathcal{K}_h$ denotes A_h without the source factor, and where $\mathbf{E}_1$ and $\mathbf{E}_{\geq 2}$ are $\mathbf{S}_h$ times the degree-one part and the higher words. The sum telescopes, since $\mathbf{F}_h$ commutes with the transports and with $\mathbf{S}_h$. Each of $\mathbf{E}_{\mathrm{nor}}$, $\mathbf{E}_0$, $\mathbf{E}_{\mathrm{hol}}$, $\mathbf{E}_1$, and $\mathbf{E}_{\geq 2}$ has the form $\mathbf{S}_h X$ with an operator X that does not depend on the damping, and these five operators X add up to $\mathcal{K}_h - \mathbf{M}^\chi \otimes \mathbf{H}_h$.

We bound each term in the following form. For an operator X on $\mathcal{H}_0^{\mathrm{tube}}$ and numbers $a', A' \geq 0$, we say that X *satisfies the four-term bound with parameters* a' and A' if $C_h + X$ satisfies (16.48) relative to C_h with $\delta_0 = \delta_W$, $a = a'$, and $A_1 = A'$, that is, if (17.4) holds with $q = 0$, with X in place of $A_h - C_h$, a' in place of a, and A' in place of A_1. If two operators satisfy such bounds, then their sum satisfies the four-term bound whose parameters are the sums of theirs. For an operator X on $\mathcal{H}_0^{\mathrm{tube}}$, recall from Section 16.4 its fast block $\mathbf{P}_0^\perp X \mathbf{P}_0^\perp$, its vacuum row $\mathbf{P}_0 X \mathbf{P}_0^\perp$ and vacuum column $\mathbf{P}_0^\perp X \mathbf{P}_0$, and its vacuum scalar $\mathbf{P}_0 X \mathbf{P}_0$, where $\mathbf{P}_0$ acts on the normal factor. We will need the following block criterion for $\mathbf{E}_1$ and $\mathbf{E}_{\geq 2}$.

Lemma 17.6. *Let X be an operator on $\mathcal{H}_0^{\mathrm{tube}}$, and let $b_0, b, b_1, b_2 \geq 0$. Suppose that the fast block of X satisfies* (16.48)*, with $\mathbf{P}_0^\perp X \mathbf{P}_0^\perp$ in place of $\mathbf{T} - C$, with $D_{\mathrm{nor}} + D_{\mathrm{body,L}}$ and $D_{\mathrm{nor}} + D_{\mathrm{body,R}}$ in place of D_L and D_R, with $\delta_0 = \delta_W$, $a = b_0$, and $A_1 = b$, and that*

$$\|\mathbf{P}_0 X \mathbf{P}_0^\perp\| + \|\mathbf{P}_0^\perp X \mathbf{P}_0\| \leq b_1 h^3, \qquad \|\mathbf{P}_0 X \mathbf{P}_0\| \leq b_2 h^4. \tag{17.12}$$

Then X satisfies the four-term bound with parameters $4b_0$ and $C(b + b_1 + b_2)$.

Proof. We write X as the sum of its fast block, its vacuum row and column, and its vacuum scalar. By (17.3), we have $\|(D_{\mathrm{nor}} + D_{\mathrm{body,R}})^{1/2} f\| \leq 2\|D_R^{1/2} f\|$, and similarly on the left, so that the fast block satisfies the four-term bound with parameters $4b_0$ and $2b$. Since $\operatorname{Re} z_\sigma \geq c_0 h^2$ for all σ, we have $D_{\mathrm{nor}} \geq c_0 h^2 \mathbf{P}_0^\perp$ for h small. This implies

$$\|\mathbf{P}_0^\perp f\| \leq C h^{-1} \|D_{\mathrm{nor}}^{1/2} f\| \leq 2C h^{-1} \|D_R^{1/2} f\|.$$

Using (17.12) and $h^2 = \sqrt{\delta_W}$, we obtain

$$|\langle g, \mathbf{P}_0 X \mathbf{P}_0^\perp f\rangle| \leq b_1 h^3 \|g\| \|\mathbf{P}_0^\perp f\| \leq 2C b_1 \sqrt{\delta_W} \|g\| \|D_R^{1/2} f\|,$$

the same bound with the roles of f and g exchanged for the vacuum column, and $|\langle g, \mathbf{P}_0 X \mathbf{P}_0 f\rangle| \leq b_2 \delta_W \|g\| \|f\|$. Adding these bounds proves the lemma. □

17.4 The geodesic difference energy

We measure the difference across one link of a section of $\mathcal{E}$ by the following energy. For $c > 0$ and $0 < t \le 1$, we set

$$\mathfrak{D}_{c,t}(f) = \iint_{d(x,y)<\varrho_0} t^{-n_b/2} e^{-c\,d(x,y)^2/t} \|f(x) - \tau_{xy} f(y)\|^2 \,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y),$$

and we write $\mathfrak{D}_c = \mathfrak{D}_{c,\delta_W}$. The next lemma bounds this energy by the defect of the body heat. It holds for every fiber, uniformly, and in particular for the normal fiber, on which the connection acts through the compact charges, which are unbounded.

Lemma 17.7. *Let $(\mathcal{Y}, g)$ be a Riemannian manifold of dimension n on which a group of isometries acts transitively, let ϱ_0 be smaller than its injectivity radius, and let $\mathcal{E} = \bigoplus_j \mathcal{E}_j$ be an orthogonal sum of Hermitian vector bundles of finite rank over $\mathcal{Y}$ with a metric connection ∇ that preserves the summands. Let τ_{xy} be the parallel transport along the minimal geodesic from y to x, let Δ_∇ be the Friedrichs extension of $-\nabla^*\nabla$, and define $\mathfrak{D}_{c,t}$ as above with n in place of n_b. Then for every $c > 0$ there exists C, depending only on c, n, ϱ_0, and the geometry of $\mathcal{Y}$, such that for all $0 < t \le 1$ and all $f \in L^2(\mathcal{Y}; \mathcal{E})$,*

$$\mathfrak{D}_{c,t}(f) \le C \langle f, (\mathrm{Id} - e^{t\Delta_\nabla}) f \rangle. \tag{17.13}$$

With $\mathcal{Y} = |\mathcal{M}|$, the bundle of Section 17.1 decomposed into the eigenspaces of $\mathbf{N}$ and $t = \delta_W$, we have $e^{t\Delta_\nabla} = \mathbf{H}_h^2$, so that the right side of (17.13) becomes $C\langle f, (\mathrm{Id} - \mathbf{H}_h^2) f\rangle$. In the proof, we bound the energy of a smooth section by its gradient along geodesics, extend this bound to the form domain, and pass to all of L^2 by the spectral theorem.

Proof. We first prove that for every smooth compactly supported section f,

$$\mathfrak{D}_{c,t}(f) \le Ct \|\nabla f\|^2. \tag{17.14}$$

Let $d(x,y) < \varrho_0$, let $u = \exp_x^{-1}(y)$, so that $|u| = d(x,y)$, and let $\gamma(s) = \exp_x(su)$ for $s \in [0,1]$. The section $\sigma(s) = \tau_{x\gamma(s)} f(\gamma(s))$ of the fiber at x satisfies $\sigma(0) = f(x)$ and $\sigma(1) = \tau_{xy} f(y)$, and $\sigma'(s) = \tau_{x\gamma(s)}(\nabla_{\gamma'(s)} f)(\gamma(s))$, since τ is the parallel transport of ∇ along γ. By the Cauchy–Schwarz inequality,

$$\|f(x) - \tau_{xy} f(y)\|^2 \le d(x,y)^2 \int_0^1 \|\nabla f(\gamma(s))\|^2 \,\mathrm{d}s. \tag{17.15}$$

We now insert (17.15) into $\mathfrak{D}_{c,t}(f)$, and we set $G = \|\nabla f\|^2$. Let $J_x(u)$ be the Jacobian of $\exp_x$, so that $\mathrm{d}\mu(\exp_x u) = J_x(u)\,\mathrm{d}u$. The isometries act transitively, so J_x is the same function for all x in isometric frames; it is continuous and positive on $|u| \le \varrho_0$, and hence bounded above and below by positive constants there.

For $s \in [1/2, 1]$, we fix x and substitute $z = \exp_x(su)$ for $y = \exp_x(u)$. Then

$$\mathrm{d}\mu(y) = \frac{J_x(u)}{J_x(su)s^n}\,\mathrm{d}\mu(z) \le C\,\mathrm{d}\mu(z), \qquad d(x,z) \le d(x,y) = d(x,z)/s \le 2d(x,z).$$

These bounds imply

$$\int_{d(x,y)<\varrho_0} t^{-n/2} e^{-c\,d(x,y)^2/t} d(x,y)^2 G(\exp_x(su))\,\mathrm{d}\mu(y) \\ \le C \int_{d(x,z)<\varrho_0} t^{-n/2} e^{-c\,d(x,z)^2/t} d(x,z)^2 G(z)\,\mathrm{d}\mu(z).$$

Integrating over x and exchanging the integrals, we bound the contribution of $s \in [1/2, 1]$ by

$$C \int G(z) \int_{d(x,z)<\varrho_0} t^{-n/2} e^{-c\,d(x,z)^2/t} d(x,z)^2\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(z) \le Ct \int G\,\mathrm{d}\mu,$$

where we used the bound on J and the identity

$$\int_{\mathbb{R}^n} t^{-n/2} e^{-c|u|^2/t} |u|^2\,\mathrm{d}u = C_c t.$$

For $s \in [0, 1/2]$ we use $\gamma(s) = \exp_y((1-s)\exp_y^{-1} x)$ and exchange the roles of x and y, which is allowed since the weight is symmetric. This proves (17.14).

Next, we extend (17.14) to the form domain. Using $\|a-b\|^2 \le 2\|a\|^2 + 2\|b\|^2$ and the bound on J, we have $\mathfrak{D}_{c,t}(f) \le C_0 \|f\|^2$, with C_0 depending on c, n, ϱ_0, and the geometry. By the definition of the Friedrichs extension, the smooth compactly supported sections are a core for its form $\|\nabla f\|^2$. Approximating in the form norm and using the continuity of $\mathfrak{D}_{c,t}$ on L^2, we obtain (17.14) for all f in the form domain of $-\Delta_\nabla$, with $\|\nabla f\|^2 = \langle f, -\Delta_\nabla f\rangle$.

Finally, let $f \in L^2$, and set

$$f_1 = \mathbf{1}_{[0,1/t]}(-\Delta_\nabla) f, \qquad f_2 = f - f_1.$$

Since $\lambda \mapsto 1 - e^{-\lambda}$ is concave and vanishes at zero, we have $\lambda \le (1-e^{-1})^{-1}(1-e^{-\lambda})$ for $0 \le \lambda \le 1$, and $1 \le (1-e^{-1})^{-1}(1-e^{-\lambda})$ for $\lambda \ge 1$. By the spectral theorem,

$$t\|\nabla f_1\|^2 \le (1-e^{-1})^{-1}\langle f_1, (\mathrm{Id} - e^{t\Delta_\nabla}) f_1\rangle, \qquad \|f_2\|^2 \le (1-e^{-1})^{-1}\langle f_2, (\mathrm{Id} - e^{t\Delta_\nabla}) f_2\rangle.$$

We use $\mathfrak{D}_{c,t}(f) \le 2\mathfrak{D}_{c,t}(f_1) + 2\mathfrak{D}_{c,t}(f_2)$, (17.14) for f_1, the bound $\mathfrak{D}_{c,t}(f_2) \le C_0\|f_2\|^2$, and the orthogonality of f_1 and f_2 for the spectral measure of Δ_∇, and we obtain (17.13). The constants do not depend on the summands $\mathcal{E}_j$, and both sides of (17.13) are sums over j, since τ and Δ_∇ preserve the decomposition. □

We use Lemma 17.7 through the symmetrization identity in the proof of Lemma A.3(b). Let $R(x,y)$, for $d(x,y) < \varrho_0$, be operators on the fiber at x such that

$$R(y,x) = \tau_{yx} R(x,y) \tau_{xy}, \qquad \|R(x,y)\| \le b\, \delta_W^{-n_b/2} e^{-c\, d(x,y)^2/\delta_W},$$

let T have the kernel $R(x,y)\tau_{xy}$, and let $R_{\mathrm{marg}}(x) = \int R(x,y)\, \mathrm{d}\mu(y)$ be its marginal. Exchanging x and y and averaging, as in that proof, we have

$$\langle g, (T - R_{\mathrm{marg}}) f \rangle = -\frac{1}{2} \iint \big\langle g(x) - \tau_{xy} g(y), R(x,y)\big(f(x) - \tau_{xy} f(y)\big)\big\rangle\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y).$$

The Cauchy–Schwarz inequality and (17.13) now imply

$$|\langle g, (T - R_{\mathrm{marg}}) f \rangle| \le C b\, \|(\mathrm{Id} - \mathbf{H}_h^2)^{1/2} g\|\, \|(\mathrm{Id} - \mathbf{H}_h^2)^{1/2} f\|. \tag{17.16}$$

This identity uses the symmetry of the kernel, and (17.13) takes the place of the positivity hypothesis of Lemma A.3(b). Without the symmetry, we write

$$\langle g, (T - R_{\mathrm{marg}}) f \rangle = \iint \big\langle R(x,y)^* g(x), \tau_{xy} f(y) - f(x)\big\rangle\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y),$$

and applying the Cauchy–Schwarz inequality with the weight $\delta_W^{\pm n_b/2} e^{\mp c' d(x,y)^2/\delta_W}$, where $c' = c/2$, we obtain the one-sided version

$$\begin{aligned} &|\langle g, (T - R_{\mathrm{marg}}) f \rangle| \\ &\qquad \le \Big(\iint \delta_W^{n_b/2} e^{c' d(x,y)^2/\delta_W} \|R(x,y)^* g(x)\|^2\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y) \Big)^{1/2} \mathfrak{D}_{c'}(f)^{1/2}. \end{aligned} \tag{17.17}$$

By (17.13), its last factor satisfies $\mathfrak{D}_{c'}(f)^{1/2} \le C \|(\mathrm{Id} - \mathbf{H}_h^2)^{1/2} f\|$.

17.5 Marginals and midpoint parity

The marginal of a kernel of size one is of size one. The next lemma shows that the marginal of a kernel that is odd under the reversal of the link gains a factor h^2. This is the source of the factor δ_W on the marginals of the degree-one words.

Lemma 17.8. *Let $\mathcal{X}$ be a Banach space, let $x \in |\mathcal{M}|$, and let $F(w,u) \in \mathcal{X}$ be defined for w, u in the tangent space at x with $|w| < \varrho_0$. Suppose that $s \mapsto F(su,u)$ is continuously differentiable and that*

$$F(0,-u) = -F(0,u), \qquad \|F(su,u)\| + \Big\| \frac{\mathrm{d}}{\mathrm{d}s} F(su,u) \Big\| \le A e^{-c|u|^2}.$$

Then for $0 < t \le 1$,

$$\Big\| \int_{|u| < \varrho_0/t} F(tu/2, u) J_x(tu)\, \mathrm{d}u \Big\| \le C A t,$$

where J_x is the Jacobian of $\exp_x$, and C depends on c, n_b, and the geometry.

Proof. Since the geodesic symmetry at x is an isometry with differential $-\mathrm{Id}$ at x, we have $J_x(-w) = J_x(w)$. Substituting $u \mapsto -u$ in half of the integral, we can replace $F(tu/2, u)$ by $\frac{1}{2}(F(tu/2, u) + F(-tu/2, -u))$. Since $F(0, u) + F(0, -u) = 0$,

$$F(tu/2, u) + F(-tu/2, -u) = \int_0^{t/2} \Big(\frac{\mathrm{d}}{\mathrm{d}s} F(su, u) + \frac{\mathrm{d}}{\mathrm{d}s} F(-su, -u) \Big) \, \mathrm{d}s,$$

and by the hypothesis, the norm of the right side is bounded by $tAe^{-c|u|^2}$. Since J_x is bounded, integrating over u proves the lemma. □

Corollary 17.9. *Let $\mathcal{X}_x$ be a Banach space of operators on the fiber at x whose norm is invariant under conjugation by the transports, as is the case for the operator norm and for $X \mapsto \|D_+^{-1/2} X D_+^{-1/2}\|$. Let $R(x, y)$, for $d(x, y) < \varrho_0$, be of the form $R(x, y) = h^{-2n_b} \tau_{xm} \widetilde{R}(m, Y) \tau_{mx}$ in the midpoint coordinates (m, Y), with $\widetilde{R}(m, -Y) = -\widetilde{R}(m, Y)$ and*

$$\|\widetilde{R}(m, Y)\| + \big\|(\nabla_Y \widetilde{R})(m, Y)\big\| \leq Ae^{-c|Y|^2},$$

where $(\nabla_Y \widetilde{R})(m, Y)$ is the covariant derivative of $\widetilde{R}$ in m along the geodesic through m in the direction Y, with Y parallel. Then $\| \int R(x, y) \, \mathrm{d}\mu(y)\| \leq CAh^2$ for all x.

Proof. We write $y = \exp_x(h^2 u)$, so that

$$\mathrm{d}\mu(y) = h^{2n_b} J_x(h^2 u) \, \mathrm{d}u, \qquad m = \exp_x(h^2 u/2),$$

and $Y = -P_{x \to m} u$, where $P_{x \to m}$ is the parallel transport of tangent vectors along the radial geodesic. Indeed, $x = \exp_m(h^2 Y/2)$, and the geodesic from x to y passes through m with direction $P_{x \to m} u$. Thus $h^{2n_b} R(x, y) = F(h^2 u/2, u)$ with

$$F(w, u) = \tau_{x, \exp_x w} \widetilde{R}\big(\exp_x w, -P_{x \to \exp_x w} u\big) \tau_{\exp_x w, x}.$$

Since $\widetilde{R}$ is odd in Y, the function F satisfies

$$F(0, -u) = \widetilde{R}(x, u) = -F(0, u).$$

Along $w = su$, set $z(s) = \exp_x(su)$ and $Y(s) = -P_{x \to z(s)} u$. Then $z'(s) = -Y(s)$, while $Y(s)$ and the fiber transports are parallel along this geodesic. Hence $\frac{\mathrm{d}}{\mathrm{d}s} F(su, u)$ is the negative of the transported derivative $(\nabla_Y \widetilde{R})(z(s), Y(s))$. Its norm has the bound required in Lemma 17.8. Applying Lemma 17.8 with $t = h^2$, we obtain the claim. □

We now apply Corollary 17.9 to the degree-one part of the step. We let W^{a} denote the anticommutator part of (17.11), the operator with the kernel $W^{\mathrm{a}}(x, y) \tau_{xy}$, where

$$W^{\mathrm{a}}(x, y) = \tfrac{1}{2} \big\{ \Pi_x(x, y), (L_-^{\mathrm{out}} - L_+)(x, y) \big\},$$

and we let $W^{\mathrm{a}}_{\mathrm{marg}}(x) = \int W^{\mathrm{a}}(x,y)\,\mathrm{d}\mu(y)$ denote its marginal. The next lemma is the source of the factor δ_W on the marginal of the degree-one part. It measures operators on the fiber at x in the norm

$$\|X\|_{\mathrm{f}} = \big\|D_{\mathrm{nor}}^{-1/2}\mathbf{P}_0^{\perp}X\mathbf{P}_0^{\perp}D_{\mathrm{nor}}^{-1/2}\big\|,$$

which bounds the fast block of X bilaterally in the normal defect, and in which Lemma 16.16 measures words.

Lemma 17.10. *Before the odd tangent integration, the full superkernel satisfies*

$$W^{\mathrm{a}}(y,x) = -\tau_{yx}W^{\mathrm{a}}(x,y)\tau_{xy}, \tag{17.18}$$

where the reversal changes the signs of both the even and the odd tangent increments. For the resulting operator kernel on the body, for all x,

$$\|W^{\mathrm{a}}_{\mathrm{marg}}(x)\|_{\mathrm{f}} \le C(rh + h^2). \tag{17.19}$$

Proof. We begin with (17.18). By (17.7), we have $L_-(y,x) = L_+(x,y)$ and $L_+(y,x) = L_-(x,y)$, as operators on the fibers at x and y, respectively. Recall that $L_-^{\mathrm{out}} = \tau_{xy}L_-\tau_{xy}^{-1}$ and $\tau_{yx} = \tau_{xy}^{-1}$. Therefore

$$(L_-^{\mathrm{out}} - L_+)(y,x) = \tau_{yx}L_+(x,y)\tau_{xy} - L_-(x,y) = -\tau_{yx}(L_-^{\mathrm{out}} - L_+)(x,y)\tau_{xy}.$$

On the other hand, before the odd tangent integration, $\Pi_x(y,x) = \tau_{yx}\Pi_x(x,y)\tau_{xy}$. Indeed, (17.10) gives $\Pi_x(x,y) = \widetilde{\Pi}(Y_x)$, with the increment Y_x in the frame $T(x)$. The transport changes this frame to $T(y)$, the reversal changes the sign of the increment, and $\widetilde{\Pi}(-Y) = \widetilde{\Pi}(Y)$ by Lemma 17.3. Inserting the two superkernel identities into the definition of W^{a}, we obtain (17.18). This is not a reversal identity for the operator-valued body kernel: an odd tangent increment becomes an odd derivative after integration, and that derivative does not change sign when only the body endpoints are exchanged.

For (17.19), we write the kernel in the midpoint coordinates of (x,y), as in Corollary 17.9:

$$W^{\mathrm{a}}(x,y) = h^{-2n_b}\tau_{xm}\widetilde{W}(m,Y)\tau_{mx}.$$

The pair (y,x) has the midpoint m and the increment $-Y$, and the transports along one geodesic compose. By (17.18), we have $\widetilde{W}(m,-Y) = -\widetilde{W}(m,Y)$, where Y is the full tangent increment, and the reversal changes the sign of its even part Y_0 and of its odd part. We expand $\widetilde{W}$ in the normalized odd tangent increments. A term of even degree in these increments is then odd in Y_0, and a term of odd degree is even in Y_0.

We next bound the size of these terms. The norm $\|\cdot\|_{\mathrm{f}}$ is invariant under conjugation by the transports, since they commute with $\mathbf{N}$. By Lemma 17.5, $h^{-2}L_{\pm}$ is a sum of odd and compact charges, with coefficients that are bounded, with bounded covariant

derivatives, times a polynomial in Y. By Lemma 17.4, Π_x is an operator $\mathbf{M}[c]$ whose amplitude carries the tangent Gaussian of the link, which absorbs these polynomials. The single-charge bounds at the end of Lemma 16.16, which part (iv) of Lemma 16.18 extends from $\mathbf{M}$ to $\mathbf{M}[c]$, give $\|h^4 \mathfrak{q}_a \mathbf{M}[c]\|_{\mathrm{f}} \le Ch^2$ and $\|h^4 \mathfrak{j}_a \mathbf{M}[c]\|_{\mathrm{f}} \le C(rh + h^2)$, and the adjoint version bounds the products with the charge on the right. Since $\widetilde{\Pi}$ does not depend on m, by Lemma 17.3, the same bounds hold for the covariant derivative along the geodesic. Since $\widetilde{W}$ is a sum of such products of a charge with Π_x, with the factor h^2 of $L_\pm$, every coefficient of its expansion, and its covariant derivative, has norm $\|\cdot\|_{\mathrm{f}}$ at most $Ae^{-c|Y_0|^2}$, with

$$A = Ch^{-2}(rh + h^2) = C(r/h + 1).$$

Finally, we integrate. For a term of even degree, which is odd in Y_0, the full-pairing part of the finite odd Gaussian computation is a scalar on Λ_T, and its coefficient remains odd in Y_0. Corollary 17.9, applied in the norm $\|\cdot\|_{\mathrm{f}}$, bounds its marginal by CAh^2. The remaining even-degree terms are $O(Ah^4)$: every non-full pairing leaves at least two factors of size h^2, and the odd heat differs from the identity by $O(h^4)$. A term of odd degree contains an odd number of normalized odd tangent increments. As in the proof of Lemma 17.5, its integration over the odd tangent coordinates is the computation of Lemma 16.19 with $\kappa_{\mathrm{o}} = h^2$, and it gives an extra factor h^2. The marginal of such a term is again bounded by CAh^2. Since $Ah^2 = C(rh + h^2)$, we obtain (17.19). Odd midpoint coordinates are retained as bounded midpoint multiplications throughout this computation; the precise finite operator rules, including their covariance and derivative bounds, are given in Lemma 17.12 below. □

The proof of (17.19) also shows that the marginal of the first variation of a weight or of the source factor across a link, which is antisymmetric under the reversal and carries the factor h^2, is of order δ_W.

17.6 The uncharged part

Lemma 17.11. *Let $a(U_m, V; Y)$ be a normal amplitude in the unscaled midpoint and the normalized increment, as in Lemma 17.4, with $\|a(U_m, V; Y)\| \le e_0(1 + |V| + |Y|)^l$ on the tube. After the integration over the odd normal variables, the operator with the kernel $k_z\, a$ has norm at most $Ce_0(1 + |Y|)^l$ on $\mathcal{H}_{\mathrm{nor}}$, and so does its adjoint. With the tangent Gaussian included, the bound is $Ce_0(1 + |Y|)^l e^{-c|Y|^2}$.*

Proof. By Lemma 16.19 with $\kappa_{\mathrm{o}} = h$, every word of fixed length in the normalized odd increments acting on the odd Mehler factor is a bounded matrix, and the odd midpoint variables carry their factor h. The odd integration thus costs no power of h, and we obtain an operator $\mathbf{M}[c]$ with $\|c(U_m, V; Y)\| \le Ce_0(1 + |V| + |Y|)^l$ at fixed Y. By part (i) of Lemma 16.18, the norm of this operator and that of its adjoint are bounded by

$Ce_0(1+|Y|)^l$. The anharmonic normal multipliers in w_{out} and w_{in} are bounded on the tube. □

Lemma 17.12. *Consider a scalar midpoint superkernel of the kind in Lemma 17.3, with its complete normal Mehler factor, and factor out its body Gaussian at scale h^2. Suppose that its amplitude and the finitely many derivatives used below, including two midpoint covariant derivatives, have polynomial Gaussian envelopes bounded by A. The same hypothesis may be imposed in the fast-block norm $\|\cdot\|_{\mathrm{f}}$ when normal charges are present and their bounds have already been applied. Suppose that full link reversal has sign $\sigma \in \{1,-1\}$. After the normal and odd tangent integrations, the operator-valued body kernel has a decomposition*

$$\begin{aligned} R &= R^{(0)} + R^{(1)} + R^{(2)}, \\ R^{(j)}(y,x) &= \sigma(-1)^j \tau_{yx} R^{(j)}(x,y) \tau_{xy} \quad (j=0,1), \end{aligned} \tag{17.20}$$

where, for a slightly widened body Gaussian $w_h(x,y) = \delta_W^{-n_b/2} e^{-cd(x,y)^2/\delta_W}$,

$$\|R^{(j)}(x,y)\| \le CAh^{2j} w_h(x,y) \quad (j=0,1,2). \tag{17.21}$$

The corresponding midpoint amplitudes have the same bounds after up to two covariant derivatives in m in the direction Y_0, with Y_0 parallel. Here and below the constants may depend on the fixed degree metric ω. In particular, when $\sigma = 1$,

$$\sup_x \Big\| \int R^{(1)}(x,y)\, \mathrm{d}\mu(y) \Big\| \le CAh^4. \tag{17.22}$$

Proof. Use body midpoint frames, transporting the two endpoint odd tangent coordinates to m. Denote their odd midpoint by ϑ and their difference divided by h^2 by η. The change from the full supermidpoint coordinates to these endpoint coordinates has bounded scaled coefficients. Indeed, if $\Phi(m,\Delta) = (x,y)$ is the analytic midpoint map, then $x - y = 0$ at $\Delta = 0$, and

$$\frac{x-y}{h^2} = \int_0^1 D_\Delta (x-y)(m, th^2 Y)[Y]\, \mathrm{d}t.$$

Together with the endpoint average, this map has an invertible differential at $h = 0$. Its analytic inverse, expanded in the finitely many odd coordinates, therefore has bounded coefficients with polynomial increment envelopes. Retain the full finite Berezinian of the endpoint measure in the kernel coefficients. After removing the Gaussian scale factors, it too has bounded analytic coefficients and inherits the full reversal covariance, with the fixed Berezin orientation retained. For the kernels here this uses the invariant measure ν and the even exponential density j of (16.10); geodesic symmetry preserves the endpoint measure. Nilpotent even coordinate shifts are Taylor expanded in the kernel coefficients in the fixed endpoint body coordinates, before applying the kernel

to a vector. Equivalently, each unscaled endpoint difference supplies h^2, cancelling the h^{-2} of a derivative of the normalized body Gaussian. No derivative of the input, and no negative power of h, is introduced by this change. The change intertwines endpoint interchange with $(Y_0, \eta) \mapsto (-Y_0, -\eta)$ and fixes ϑ. Normal frame actions remain in the charged factors of (17.11) and (17.29); no normal change of variables or charge is inserted in this coefficient conversion.

Factor out the frozen odd tangent heat H_F, whose quadratic generator and its required coefficient derivatives are $O(h^4)$. Expand the remaining amplitude in ϑ and η. The coefficient of $\vartheta^J \eta^I$ has reversal parity $\sigma(-1)^{|I|}$ in Y_0. First integrate the full normal Mehler factor, retaining the order of all coefficient operators. By Lemma 17.11, this costs no power of h: an odd normal midpoint carries h, and normalized odd normal increments are treated with $\kappa_{\mathrm{o}} = h$ in Lemma 16.19. The same applies to normal-amplitude derivatives, which differentiate bounded unscaled midpoint coefficients, and to the stated fast-block bounds. Thus the coefficient operators and their midpoint covariant derivatives have the assumed envelopes.

For the tangent integration, write $\Delta_I H_F = h^{-2|I|} \operatorname{ad}_{\theta_{i_1}} \cdots \operatorname{ad}_{\theta_{i_{|I|}}} H_F$, with graded commutators. The partial-pairing formula of Lemma 16.19, now with $\kappa_{\mathrm{o}} = h^2$, gives the sharper even-degree assertion

$$\Delta_I H_F = \begin{cases} c_I \mathrm{Id}_{\Lambda_T} + O(h^4), & |I| \text{ even}, \\ O(h^2), & |I| \text{ odd}. \end{cases}$$

For even $|I|$, every non-full pairing leaves at least two linear factors of size h^2, and $H_F - \mathrm{Id} = O(h^4)$; this also covers $I = \emptyset$. For odd $|I|$, at least one such factor remains. Differentiating the finite formula preserves these orders. Tangent midpoint coordinates act by the bounded maps

$$J_\theta(X) = \tfrac{1}{2}\big(\theta X + (-1)^{|X|} X\theta\big).$$

In particular, applying the midpoint maps to $c_I \mathrm{Id}$ gives multiplication by the corresponding midpoint monomial. These maps preserve both error orders. They act in the graded tensor product with the normal coefficient operators in their original order; only fixed signs arise, and none depends on Y_0.

Let $R^{(0)}$ consist of the full-pairing terms of even $|I|$, let $R^{(1)}$ consist of all odd-$|I|$ terms, and put the even-degree $O(h^4)$ terms in $R^{(2)}$. The moment and midpoint maps are independent of the sign of Y_0 and covariant under K_0. The coefficient parities therefore give the genuine body-kernel identities (17.20), while the Gaussian envelope bounds give (17.21), including the covariant derivatives. For $\sigma = 1$, the midpoint amplitude of $R^{(1)}$ is odd in Y_0 and has size and derivative CAh^2. Applying Corollary 17.9 proves (17.22). □

We now bound $\mathbf{E}_0$, which has the kernel $\mathbf{S}_h(x) R_0(x, y) \tau_{xy}$ with $R_0 = \Pi_x - \mathbf{F}_h p_h$. Recall that w_{out} and w_{in} contain the same half-weights and cutoffs as $\mathbf{F}_h$, so that R_0 is the

difference of two normal operators with the same outer multipliers. We compare their exponents. By (17.10), $\Pi_x(x,y) = \widetilde{\Pi}(Y_x)$, and by Lemma 17.2,

$$\begin{aligned} \frac{\zeta}{2h^4} &= \frac{1}{2}\operatorname{STr} V^2 + \frac{1}{2}\operatorname{STr}[Y, R_*]^2 + \mathcal{Q}_\zeta(U_m; Y) + h^4 \mathcal{R}_h(V, Y; U_m), \\ \mathcal{Q}_\zeta &= \operatorname{STr}\big([Y, R_*][Y, U_m]\big) + \frac{1}{2}\operatorname{STr}[Y, U_m]^2, \end{aligned} \tag{17.23}$$

where $\mathcal{R}_h$ retains the full analytic remainder obtained by squaring the convergent series in (17.8); its leading term is quartic in (V, Y). Write V_0 and Y_0 for the even increments. On the local increment domain $|h^2 Y_0| < \varrho_0$, with the normal endpoints in the tube, every Grassmann coefficient of $\mathcal{R}_h$, and every fixed derivative of that coefficient in the even variables, is bounded by a fixed polynomial envelope in (V_0, Y_0), uniformly for small h. Indeed, after the leading quadratic terms are removed, the convergent series has an overall factor h^4; the remaining analytic functions of $h^2 Y_0$ and their derivatives are bounded on this domain. Finite expansion in the odd variables preserves polynomial envelopes. The cosh term in (17.8) is block diagonal and the sinh term is off diagonal, so their cross term has zero supertrace. The body remainder therefore satisfies the sharper bound $|\mathcal{R}_{h,\varnothing}| \le C|Y_0|^2(|V_0|^2 + |Y_0|^2)$.

(a) The body part of $\mathcal{Q}_\zeta$ is $O(r|Y|^2)$ on the tube, where the body of U_m has norm less than $2r$. Its odd parts come from the odd normal midpoint coordinates in $[Y_0, U_m]$, which carry the factor h, and from $[Y_{\mathrm{odd}}, U_{m,\mathrm{body}}]$, which carries r. Differentiating the normalized Gaussian in $\mathcal{Q}_\zeta$ and using Lemma 17.11, we find that replacing $\mathcal{Q}_\zeta$ by zero changes the normal operator by at most $C(r+h)$ times a slightly widened Gaussian in Y. The normalized Gaussians with and without $\mathcal{Q}_\zeta$ have the same integral in Y.

(b) The odd tangent Gaussian $e^{-\frac{1}{2}\operatorname{STr}[Y_{\mathrm{odd}}, R_*]^2}$ is the odd heat over the time δ_W. In the norm $|\cdot|_{2;\omega}$ of Λ_T, which is the degree metric of Section 5.2, odd derivatives cost ω^{-1}, and by Lemma 16.19 with $\kappa_{\mathrm{o}} = h^2$ this factor differs from the identity of Λ_T by $O(\delta_W \omega^{-2})$, which gives a scalar term with constant $C(\omega)$.

(c) The factor $\varpi(U_m + h^2 V/2)\varpi(U_m - h^2 V/2)\varpi(S)^{-2}$, which combines the half-density with the normalization of the tangent Gaussian of Section 16.3, differs from one by $O(\delta_W |V|^2)$, since it is even in V and equals one at $V = 0$.

(d) The term $h^4 \mathcal{R}_h$, the density $j(h^2 Y) = 1 + O(\delta_W |Y|^2)$ of the measure in midpoint coordinates, and the difference between p_h and the Gaussian $(2\pi\delta_W)^{-n_b/2} e^{-d(x,y)^2/(2\delta_W)}$ with $d(x,y) = h^2|Y|_{g_{R_*}}$ change the kernel by δ_W times a polynomial envelope against a slightly widened Gaussian in (V_0, Y_0). For the remainder in the exponent, decrease ϱ_0 if necessary: its body bound gives $h^4|\mathcal{R}_{h,\varnothing}| \le C\varrho_0^2(|V_0|^2 + |Y_0|^2)$, so the interpolated exponents remain uniformly coercive. The identity $e^{-A-B} - e^{-A} = -B\int_0^1 e^{-A-tB}\,\mathrm{d}t$ and the finite nilpotent expansion then give this bound, with the coefficient and derivative envelopes just established.

Apply Lemma 17.12 with $\sigma = 1$ to the uncharged superkernel, and subtract the symmetric body kernel $\mathbf{F}_h p_h$ from its leading part. This gives body operator kernels

$$\begin{aligned} R_0 &= R_{\mathrm{ev}} + R_{\mathrm{odd}} + R_{\mathrm{rem}}, \\ R_{\mathrm{ev}}(y,x) &= \tau_{yx} R_{\mathrm{ev}}(x,y) \tau_{xy}, \\ R_{\mathrm{odd}}(y,x) &= -\tau_{yx} R_{\mathrm{odd}}(x,y) \tau_{xy}, \\ \|R_{\mathrm{ev}}(x,y)\| &\le C(r+\omega^2+h) w_h(x,y), \\ \|R_{\mathrm{odd}}(x,y)\| &\le C(r,\omega) h^2 w_h(x,y), \\ \|R_{\mathrm{rem}}(x,y)\| &\le C(r,\omega) \delta_W w_h(x,y), \\ \sup_x \Big\| \int R_{\mathrm{odd}}(x,y)\, \mathrm{d}\mu(y) \Big\| &\le C(r,\omega)\delta_W. \end{aligned} \tag{17.24}$$

The midpoint covariant derivatives through order two have the same Gaussian envelopes, by the bounded fixed-order derivatives of the geometric factors and the smooth cutoffs in the unscaled normal variables, and by the homogeneous midpoint frames. To see the small coefficient in the leading even part, items (a), (c), and (d) give $C(r+h)$ before the endpoint coefficient conversion. Its body is unchanged by that conversion. In the leading even normal contractions, a new tangent midpoint soul has even degree at least two, so its multiplication norm is $O(\omega^2)$. A term with an unscaled odd normal midpoint factor, or with odd normal parity after contraction, is $O(h)$ by Lemma 17.4. Paired normalized odd normal increments remain in the bounded even coefficient; no small factor is attributed to such a pair. Terms with an unpaired odd tangent increment belong to R_{odd}, and the even-degree moment remainders belong to R_{rem}. Fixed positive powers of h^2 from the analytic coefficient conversion are absorbed in Ch by decreasing h_0 after r and ω are fixed. Thus C in the leading bound may be chosen independently of these small parameters; the remaining constants are allowed to depend on r and ω.

We also need the marginal as an operator on all of Λ_T. Freeze the tangent midpoint when taking its leading even contractions. They act by multiplication in the odd midpoint, and the normalized tangent super-Gaussian has integral $\varpi(S)^{-2}$ times the flat one, independently of that midpoint. Its leading mass is therefore $\varpi(S)^{-2}\mathrm{Id}_{\Lambda_T}$, not merely its value on the constant tangent coefficient. The half-density cancellation in item (c) then gives $\mathbf{F}_h$. The variation of an even midpoint amplitude has first term $h^2 \nabla_{Y_0}$, which is odd in Y_0 and integrates to zero when the midpoint is frozen; its second-order Taylor remainder is $O(h^4)$ with a polynomial Gaussian envelope. For the odd-degree terms, (17.22) supplies the same order. The even-degree moment remainders, the odd heat minus the identity, the density $j-1$, and item (d) are already $O(\delta_W)$. Applying Lemma 17.11 to these errors and using $\int p_h(x,y)\, \mathrm{d}\mu(y) = 1$, we obtain

$$\Big\| \int R_0(x,y)\, \mathrm{d}\mu(y) \Big\| \le C(r,\omega)\delta_W. \tag{17.25}$$

Subtracting the last two parts of (17.24) shows that the marginal of R_{ev} is also $O(\delta_W)$. The full body operator kernel R_0 need not be symmetric: full superkernel reversal and ordinary body-kernel reversal differ on the odd-degree moment terms.

Lemma 17.13. *Let h be small enough, depending on r and ω. Then for all f and g in $\mathcal{H}_0^{\mathrm{tube}}$,*

$$|\langle g, \mathbf{E}_0 f\rangle| \le C(r+\omega^2+h)\|D^{1/2}_{\mathrm{body,L}} g\|\,\|D^{1/2}_{\mathrm{body,R}} f\| + C(r,\omega)\sqrt{\delta_W}\|g\|\,\|D^{1/2}_{\mathrm{body,R}} f\| + C(r,\omega)\delta_W\|g\|\,\|f\|.$$

Proof. We have $\langle g, \mathbf{E}_0 f\rangle = \langle \mathbf{S}_h^* g, T_0 f\rangle$, where T_0 has the kernel $R_0(x,y)\tau_{xy}$. Decompose it by (17.24). The kernel R_{ev} is genuinely symmetric on the body, so (17.16), applied after subtracting its marginal, gives

$$|\langle \mathbf{S}_h^* g, (T_{\mathrm{ev}} - R_{\mathrm{ev,marg}}) f\rangle| \le C(r+\omega^2+h)\|(\mathrm{Id}-\mathbf{H}_h^2)^{1/2}\mathbf{S}_h^* g\|\,\|(\mathrm{Id}-\mathbf{H}_h^2)^{1/2} f\|.$$

For R_{odd} use (17.17) instead. Its Gaussian bound in (17.24) makes the first factor there at most $C(r,\omega)h^2\|\mathbf{S}_h^* g\|$, while Lemma 17.7 bounds the second by $C\|(\mathrm{Id}-\mathbf{H}_h^2)^{1/2} f\|$. Its marginal is $O(\delta_W)$ by (17.24). The remainder has operator norm $C(r,\omega)\delta_W$ by Schur's test, and the even marginal has the same bound by (17.25) and the paragraph following it. Finally, $\mathbf{S}_h(\mathrm{Id}-\mathbf{H}_h^2)\mathbf{S}_h^* \le D_{\mathrm{body,L}}$ and $\mathrm{Id}-\mathbf{H}_h^2 \le D_{\mathrm{body,R}}$, while $\|\mathbf{S}_h^* g\| \le \|g\|$ and $h^2 = \sqrt{\delta_W}$. These observations give the stated three terms. □

17.7 Holonomy, charged words, the frozen normal transfer and the source

Holonomy

We now compare the body heat $\mathbf{H}_h$ with the operator with kernel $p_h(x,y)\tau_{xy}$, which uses the geodesic transport. Recall that p_h is the scalar heat kernel of $e^{\delta_W\Delta/2}$ on $|\mathcal{M}|$. Let (X_t) be a Brownian motion on $|\mathcal{M}|$ with generator $\frac{1}{2}\Delta$, started at x, and let $//_t$ be the stochastic parallel transport of $\mathcal{E}$ along X, from the fiber at x to the fiber at X_t. On each summand of $\mathcal{E}$ of finite rank, Δ_∇ is essentially self-adjoint, as noted in Section 17.1, and the Feynman–Kac–Itô formula for the Bochner Laplacian [65] gives

$$(\mathbf{H}_h f)(x) = \mathbb{E}_x\big[//^{-1}_{\delta_W} f(X_{\delta_W})\big]. \tag{17.26}$$

The law of X_{δ_W} has the density $p_h(x,\cdot)$, and the operator with kernel $p_h(x,y)\tau_{xy}$ acts by $f \mapsto \mathbb{E}_x[\tau_{xX_{\delta_W}} f(X_{\delta_W})]$, up to the Gaussian-small part of the kernel with $d(x,y) \ge \varrho_0$. We define the holonomy

$$\gamma_x = //^{-1}_{\delta_W}\tau_{X_{\delta_W}x} \in K_0,$$

of the loop that goes from x to X_{δ_W} along the geodesic and returns along the Brownian path. Then

$$(\mathbf{E}_{\mathrm{hol}} f)(x) = -\mathbf{S}_h(x)\mathbf{F}_h\, \mathbb{E}_x\big[(\pi(\gamma_x) - \mathrm{Id})\tau_{x X_{\delta_W}} f(X_{\delta_W})\big] \tag{17.27}$$

up to a Gaussian-small error.

Lemma 17.14. *Let τ_x be the exit time of X from the ball $B(x, \varrho_0/2)$, and let $|\cdot|$ be the norm in a fixed faithful unitary representation of K_0. For every $j \geq 1$ there exists C_j, and there exist $C, c > 0$, such that for all x and $0 < t \leq 1$,*

$$\mathbb{E}_x\big[|\gamma_x(t) - \mathrm{Id}|^j; \tau_x > t\big] \leq C_j t^j, \qquad \mathbb{P}_x(\tau_x \leq t) \leq C e^{-c/t},$$

where $\gamma_x(t)$ is the holonomy defined above with δ_W replaced by t.

Proof. On $B(x, \varrho_0)$ we use the radial gauge: the frame of the principal K_0-bundle obtained by parallel transport along the geodesics from x. By homogeneity, it is uniform in x. The connection form A of this frame, with values in $\mathfrak{k}_0$, satisfies $A_x = 0$ and $|A(z)| \leq C d(x,z)$, with all derivatives bounded. Since the transport along radial geodesics is trivial in this gauge, $\gamma_x(t) = g_t^{-1}$, where g_t solves the Stratonovich equation $\mathrm{d}g = -A(X) \circ \mathrm{d}X\, g$ with $g_0 = \mathrm{Id}$.

In normal coordinates at x, we have $\mathrm{d}X^i = \sigma_a^i(X)\,\mathrm{d}\beta^a + \frac{1}{2} b^i(X)\,\mathrm{d}s$, with σ and b bounded uniformly in x, where β is a Brownian motion in $\mathbb{R}^{n_b}$. In Itô form,

$$g_t - \mathrm{Id} = -\int_0^t A_i \sigma_a^i(X_s) g_s\,\mathrm{d}\beta_s^a + \int_0^t \widetilde{B}(X_s) g_s\,\mathrm{d}s,$$

with $|\widetilde{B}| \leq C$ on the ball, since A, ∂A, and σ are bounded there and $|g_s| = 1$. By the Burkholder–Davis–Gundy inequality,

$$\mathbb{E}_x \sup_{s \leq t\wedge\tau_x} \Big|\int_0^s A\sigma g\,\mathrm{d}\beta\Big|^j \leq C_j \mathbb{E}_x\Big(\int_0^{t\wedge\tau_x} |A(X_s)|^2\,\mathrm{d}s\Big)^{j/2} \leq C_j t^{j/2} \mathbb{E}_x \sup_{s\leq t\wedge\tau_x} d(x, X_s)^j \leq C_j t^j,$$

where the last step is the same inequality for the coordinate martingale. The drift term is bounded by Ct. The exit bound follows from the exponential martingale inequality for the coordinate martingale, whose quadratic variation grows at a bounded rate. □

We now bound $\mathbf{E}_{\mathrm{hol}}$ through (17.27).

Lemma 17.15. *For all f and g in $\mathcal{H}_0^{\mathrm{tube}}$,*

$$|\langle g, \mathbf{E}_{\mathrm{hol}} f\rangle| \leq C(r+h)\sqrt{\delta_W}\,\|g\|\|D_{\mathrm{nor}}^{1/2} f\| + C\delta_W \|g\|\|f\|. \tag{17.28}$$

In particular, $\mathbf{E}_{\mathrm{hol}}$ satisfies the four-term bound with parameters 0 and C.

Proof. We first consider the event

$$\mathcal{A}_x = \{\tau_x > \delta_W,\ |\gamma_x - \mathrm{Id}| \le 1/2\}.$$

On $\mathcal{A}_x$, we set $\mathfrak{e} = \log \gamma_x \in \mathfrak{k}_0$, so that $|\mathfrak{e}| \le 2|\gamma_x - \mathrm{Id}|$ and

$$\pi(\gamma_x) - \mathrm{Id} = \int_0^1 \pi(e^{s\mathfrak{e}})\, \mathrm{d}\pi(\mathfrak{e})\, \mathrm{d}s, \qquad \mathrm{d}\pi(\mathfrak{e}) = \sum_a \mathfrak{e}^a (\mathsf{j}_a \otimes \mathrm{Id} + \mathrm{Id} \otimes m_a),$$

where the m_a are bounded matrices on Λ_T. The operator $\pi(e^{s\mathfrak{e}})$ is unitary and commutes with $\mathbf{F}_h$ and with $\mathbf{N}$, and $\mathsf{j}_a \mathbf{P}_0 = 0$. By part (iv) of Lemma 16.18, in which the cutoff bounds the coordinate factor of a compact charge by r/h, we have

$$\|\mathbf{F}_h \mathsf{j}_a D_{\mathrm{nor}}^{-1/2} \mathbf{P}_0^{\perp}\| \le C(r/h + 1)h^{-1}.$$

Therefore, on $\mathcal{A}_x$ and for every vector u in the fiber at x,

$$\big\|\mathbf{F}_h(\pi(\gamma_x) - \mathrm{Id})u\big\| \le C|\gamma_x - \mathrm{Id}|\big((r/h+1)h^{-1}\|D_{\mathrm{nor}}^{1/2}u\| + \|u\|\big),$$

where the second term comes from the matrices m_a on Λ_T. On the complementary event, we use $\|\pi(\gamma_x) - \mathrm{Id}\| \le 2$. By Lemma 17.14 with $j = 2$ and Chebyshev's inequality, this event has probability at most $C\delta_W^2 + Ce^{-c/\delta_W}$, and Lemma 17.14 with $j = 2$ also gives

$$\mathbb{E}_x\big[|\gamma_x - \mathrm{Id}|^2; \tau_x > \delta_W\big] \le C\delta_W^2.$$

We insert these bounds into (17.27) with $u = \tau_{x X_{\delta_W}} f(X_{\delta_W})$, and we apply the Cauchy–Schwarz inequality in $\mathbb{E}_x \otimes \mathrm{d}\mu(x)$. The invariance of μ for X,

$$\int \mathbb{E}_x \varphi(X_{\delta_W})\, \mathrm{d}\mu(x) = \int \varphi\, \mathrm{d}\mu \qquad \text{for all measurable } \varphi \ge 0,$$

and the fact that the transports commute with D_{nor} then give

$$|\langle g, \mathbf{E}_{\mathrm{hol}} f\rangle| \le C\delta_W(r/h+1)h^{-1}\|g\|\|D_{\mathrm{nor}}^{1/2}f\| + C\delta_W\|g\|\|f\| + C\delta_W\|g\|\|f\|,$$

where the last term comes from the complementary event, on which we take the square root of its probability in the Cauchy–Schwarz inequality. Since $\delta_W(r/h+1)h^{-1} = \sqrt{\delta_W}(r+h)$, this proves (17.28). The last assertion follows from $D_{\mathrm{nor}} \le 4D_R$, which is (17.3), and from $r + h \le 2$. □

The degree-one part

Lemma 17.16. *The operator* $\mathbf{E}_1$ *satisfies the four-term bound with parameters* $C(r+h)$ *and* C.

Proof. We have $\mathbf{E}_1 = \mathbf{S}_h X$, where X is the degree-one part, the operator with the kernel (17.11) on $d(x,y) < \varrho_0$. Since $\mathbf{S}_h$ is a function on $|\mathcal{M}|$ with $|\mathbf{S}_h| \le 1$, it commutes with D_{nor} and $\mathbf{P}_0$, and $\|D_{\mathrm{nor}}^{1/2}\mathbf{S}_h^* g\| \le \|D_{\mathrm{nor}}^{1/2} g\|$. We verify the hypotheses of Lemma 17.6 for $\mathbf{E}_1$, and to this end we bound $\langle g', Xf\rangle$ for a vector g' in the place of $\mathbf{S}_h^* g$. Throughout, $w(x,y) = C\delta_{\mathrm{W}}^{-n_b/2} e^{-c\,d(x,y)^2/\delta_{\mathrm{W}}}$ denotes a slightly widened body Gaussian. It absorbs the polynomials in Y, and $\int w(x,y)\,\mathrm{d}\mu(y)$ and $\int w(x,y)\,\mathrm{d}\mu(x)$ are bounded.

We begin with the fast block of the commutator part $X^{\mathrm{c}} = \frac{1}{2}[\Pi_x, L_-^{\mathrm{out}} + L_+]\tau_{xy}$ of (17.11). Lemma 17.5 shows that $h^{-2}(L_-^{\mathrm{out}} + L_+)$ is a sum of charges, with coefficients bounded by a polynomial in Y. The remark after Lemma 17.4 shows that the commutator of a charge with Π_x is h times a sum of operators $\mathbf{M}[c']$ with at most one letter on either side. By part (iv) of Lemma 16.18 with $s \le 1$, and by its adjoint for a letter on the right, each of these operators has norm $\|\cdot\|_{\mathrm{f}}$ at most Ch^{-2}, where $\|\cdot\|_{\mathrm{f}}$ is the norm of Section 17.5; here we used that D_+ and D_{nor} are comparable on the range of $\mathbf{P}_0^\perp$. The kernel of the fast block of X^{c} has norm $\|\cdot\|_{\mathrm{f}}$ at most $Ch^2 h \cdot h^{-2} w(x,y) = Ch\, w(x,y)$. Since the transports commute with D_{nor}, the Cauchy–Schwarz inequality implies

$$|\langle g', \mathbf{P}_0^\perp X^{\mathrm{c}} \mathbf{P}_0^\perp f\rangle| \le Ch \|D_{\mathrm{nor}}^{1/2} g'\| \|D_{\mathrm{nor}}^{1/2} f\|.$$

Next, we treat the fast block of the anticommutator part W^{a}, with the one-sided form (17.17) of the energy, applied to $R(x,y) = \mathbf{P}_0^\perp W^{\mathrm{a}}(x,y)\mathbf{P}_0^\perp$, whose marginal is $\mathbf{P}_0^\perp W_{\mathrm{marg}}^{\mathrm{a}} \mathbf{P}_0^\perp$. Using part (iv) of Lemma 16.18 with one inverse half-defect, we find

$$\|W^{\mathrm{a}}(x,y)^* \mathbf{P}_0^\perp u\| \le C(r+h) w(x,y) \|D_{\mathrm{nor}}^{1/2} u\|$$

for vectors u in the fiber at x. Indeed, for an odd charge, part (iv) with $s = 1$ gives the factor $h^2 h^{-1} = h$, and for a compact charge the coordinate factor, cut off at r/h, and one derivative give $h^2 (r/h) h^{-1} = r$.

With $c' = c/2$ in (17.17), where c is the constant in w, the first factor there is therefore at most $C(r+h)\|D_{\mathrm{nor}}^{1/2} g'\|$. Since $\mathbf{P}_0^\perp$ commutes with $\mathbf{H}_h$, (17.13) implies that the second factor satisfies $\mathfrak{D}_{c'}(\mathbf{P}_0^\perp f)^{1/2} \le C\|(\mathrm{Id} - \mathbf{H}_h^2)^{1/2} f\|$. We obtain

$$|\langle g', \mathbf{P}_0^\perp (W^{\mathrm{a}} - W_{\mathrm{marg}}^{\mathrm{a}}) \mathbf{P}_0^\perp f\rangle| \le C(r+h) \|D_{\mathrm{nor}}^{1/2} g'\| \|(\mathrm{Id} - \mathbf{H}_h^2)^{1/2} f\|.$$

By (17.19) of Lemma 17.10, the marginal satisfies

$$|\langle g', \mathbf{P}_0^\perp W_{\mathrm{marg}}^{\mathrm{a}} \mathbf{P}_0^\perp f\rangle| \le C(rh + h^2) \|D_{\mathrm{nor}}^{1/2} g'\| \|D_{\mathrm{nor}}^{1/2} f\|.$$

We add the three bounds, take $g' = \mathbf{S}_h^* g$, and use $\mathrm{Id} - \mathbf{H}_h^2 \le D_{\mathrm{body,R}}$. The fast block of $\mathbf{E}_1$ then satisfies the hypothesis of Lemma 17.6 with $b_0 = C(r+h)$ and $b = 0$.

It remains to bound the vacuum blocks. The degree-one part has the kernel $\Pi_x \tau_{xy} L_- - L_+ \Pi_x \tau_{xy}$, the left side of (17.11). The charges commute with $\mathbf{P}_0$ and annihilate Ω_0, so that $\mathbf{P}_0 L_\pm = L_\pm \mathbf{P}_0 = 0$. Hence the vacuum scalar of X vanishes, and its vacuum row

has the kernel $\mathbf{P}_0(\Pi_x - c_\Pi)\tau_{xy}L_-$, where c_Π is the scalar of the vacuum row of Π_x given by Lemma 17.4. That lemma also shows that $\mathbf{P}_0(\Pi_x - c_\Pi)$ is $O(h)$ in every graph norm. Since $h^{-2}L_-$ is a sum of charges, which are bounded relative to $\mathbf{N}+1$, the norm of this kernel is bounded by $Ch^3 w(x,y)$. The vacuum column is bounded in the same way, and Schur's test gives $\|\mathbf{P}_0 X \mathbf{P}_0^\perp\| + \|\mathbf{P}_0^\perp X \mathbf{P}_0\| \le Ch^3$. The same bounds hold for $\mathbf{E}_1 = \mathbf{S}_h X$, and Lemma 17.6 with $b_1 = C$ and $b_2 = 0$ proves the lemma. □

Higher words

Lemma 17.17. *The operator $\mathbf{E}_{\ge 2}$ satisfies the four-term bound with parameters $Cr^2 + C(r)h$ and C.*

Proof. We have $\mathbf{E}_{\ge 2} = \mathbf{S}_h X$, where X is the sum of the higher words. As in the proof of Lemma 17.16, we bound $\langle g', Xf\rangle$ for a vector g' in the place of $\mathbf{S}_h^* g$, and $w(x,y)$ denotes the widened body Gaussian of that proof.

We first write out the terms. By the expansion of the kernel of A_h after (17.10), the kernel of X on $d(x,y) < \varrho_0$ is the sum, over $0 \le n_1, n_2 \le D$ with $n = n_1 + n_2 \ge 2$, of

$$\frac{(-1)^{n_1}}{n_1!\, n_2!} L_+^{n_1}\, \Pi_x(x,y)\, \tau_{xy} L_-^{n_2} = \frac{(-1)^{n_1}}{n_1!\, n_2!} L_+^{n_1}\, \Pi_x(x,y)\, (L_-^{\mathrm{out}})^{n_2} \tau_{xy}. \tag{17.29}$$

The transported charges are again charges, since the transport acts on the normal variables through K_0. Then Lemma 17.5 shows that $h^{-2n} L_+^{n_1} (L_-^{\mathrm{out}})^{n_2}$ is a finite sum of products $\mathbf{w}_+\mathbf{w}_-$, with coefficients bounded by polynomials in Y, where $\mathbf{w}_+$ is a word of degree n_1 and $\mathbf{w}_-$ a word of degree n_2. The number of terms depends only on k, and we fix one of them. We let a and b denote the numbers of odd and compact charges in $\mathbf{w}_+\mathbf{w}_-$, so that $a + b = n$.

Next, we bound the vacuum blocks. The charges commute with $\mathbf{P}_0$ and annihilate Ω_0, so that $\mathbf{P}_0\mathbf{w} = \mathbf{w}\mathbf{P}_0 = 0$ for every word $\mathbf{w}$ of positive degree. All vacuum blocks of the term thus vanish if $n_1 \ge 1$ and $n_2 \ge 1$. If $n_2 = 0$, its vacuum row and vacuum scalar vanish, and its vacuum column has the kernel $h^{2n}\mathbf{w}_+(\Pi_x - c'_\Pi)\mathbf{P}_0\tau_{xy}$ times a coefficient, where c'_Π is the scalar of the vacuum column of Π_x given by Lemma 17.4. That lemma also shows that $(\Pi_x - c'_\Pi)\Omega_0$ is $O(h)$ in every graph norm, and a word of degree n is bounded relative to $(\mathbf{N}+1)^n$. The norm of this kernel is therefore bounded by $Ch^{2n+1}w(x,y) \le Ch^5 w(x,y)$. The case $n_1 = 0$ is symmetric. Schur's test gives (17.12) with $b_1 = Ch^2$ and $b_2 = 0$.

We now turn to the fast block, in the norm $\|\cdot\|_{\mathrm{f}}$ of Section 17.5. We move the charges of $\mathbf{w}_-$ to the left of Π_x. Writing $\mathbf{w}_- = \mathfrak{c}_1 \cdots \mathfrak{c}_{n_2}$ with charges $\mathfrak{c}_i$, we have

$$\mathbf{w}_+\Pi_x\mathbf{w}_- = \mathbf{w}_+\mathbf{w}_-\Pi_x + \sum_{i=1}^{n_2} \pm\, \mathbf{w}_+\mathfrak{c}_1 \cdots \mathfrak{c}_{i-1}\, [\Pi_x, \mathfrak{c}_i]\, \mathfrak{c}_{i+1} \cdots \mathfrak{c}_{n_2}, \tag{17.30}$$

with graded commutators. Recall from Lemma 17.4 that Π_x is an operator $\mathbf{M}[c]$ between the invariant cutoffs contained in w_{out} and w_{in}, which commute with the charges, and the tangent Gaussian in its amplitude absorbs the polynomials in Y. For the first term of (17.30), Lemma 16.16, which part (iv) of Lemma 16.18 extends to $\mathbf{M}[c]$ on the range of $\mathbf{P}_0^\perp$, gives

$$\|h^{2n}\mathbf{w}_+\mathbf{w}_-\Pi_x\|_{\mathrm{f}} \le \big(Ch^a r^b + C(r)h\big)w(x,y) \le \big(Cr^2 + C(r)h\big)w(x,y),$$

where we used $h \le r \le 1$ and $a + b \ge 2$ in the second inequality.

For the other terms of (17.30), we count the letters. The word $\mathbf{w}_+\mathbf{w}_-$ is a sum of products of at most $a + 2b$ letters, with fixed matrices as coefficients. The commutator $[\Pi_x, \mathfrak{c}_i]$ is h times a sum of operators $\mathbf{M}[c']$ with at most one letter. For the terms of $\mathfrak{c}_i$ that contain letters, this follows from the remark after Lemma 17.4 and from (16.46). For a finite matrix F, it follows from $[F, \mathbf{M}[c]] = h\mathbf{M}[[F, c^{\mathrm{rem}}]]$, which holds by Lemma 17.4. Replacing $\mathfrak{c}_i$ by its commutator with Π_x therefore produces the factor h and does not increase the number of letters.

We then move the remaining charges and letters to the left of the amplitude operator, one at a time. By part (ii) of Lemma 16.18 and by (16.47) and (16.46), each such move produces commutators that either carry a factor h and remove a letter or keep the number of letters. Every resulting term is h^{2n+1} times an oscillator polynomial of degree $s \le a + 2b$ applied to an operator $\mathbf{M}[c'']$, and part (iv) of Lemma 16.18 bounds it in the norm $\|\cdot\|_{\mathrm{f}}$ by

$$Ch^{2n+1}h^{-\max(2,s)}w(x,y) \le Ch^{a+1}w(x,y) \le Ch\,w(x,y),$$

since $2n - (a + 2b) = a$.

Combining the bounds for the fast block and using the Cauchy–Schwarz inequality as in the proof of Lemma 17.16, we obtain

$$|\langle g', \mathbf{P}_0^\perp X \mathbf{P}_0^\perp f\rangle| \le \big(Cr^2 + C(r)h\big)\|D_{\mathrm{nor}}^{1/2} g'\|\|D_{\mathrm{nor}}^{1/2} f\|.$$

With $g' = \mathbf{S}_h^* g$, this is the hypothesis of Lemma 17.6 for $\mathbf{E}_{\ge 2}$ with $b_0 = Cr^2 + C(r)h$ and $b = 0$, and the lemma follows from Lemma 17.6. □

The frozen normal transfer

Lemma 17.18. *The operator* $\mathbf{E}_{\mathrm{nor}}$ *satisfies the four-term bound with parameters* $Cr + C(r)h$ *and* $C(r)$.

Proof. By Lemma 16.21 with $m = 0$, $\mathbf{F}_h - \mathbf{M}^\chi$ satisfies (16.48) relative to $\mathbf{M}^\chi$, with the defects $\bar{D}_L$ and $\bar{D}_R$, the parameter $\delta_0 = \delta_W$, the bilateral coefficient $Cr + C(r)h$ and $A_1 = C(r)$, including the nilpotent terms of the invariant cutoff. This is a bound on the normal factor. It extends to the operator $(\mathbf{F}_h - \mathbf{M}^\chi) \otimes \mathrm{Id}$, which acts on each fiber, with

$\bar D_L \otimes \mathrm{Id}$ and $\bar D_R \otimes \mathrm{Id}$ in place of $\bar D_L$ and $\bar D_R$. To see this, we apply the bound at each point of $|\mathcal{M}|$ to the components of f and g in an orthonormal basis of Λ_T, and we sum with the Cauchy–Schwarz inequality. Let $C_{\mathrm{body}} = \mathbf{S}_h \mathbf{H}_h$ denote the body factor of C_h, a contraction that commutes with the normal operators. Then

$$\langle g, \mathbf{E}_{\mathrm{nor}} f\rangle = \big\langle (\mathrm{Id}\otimes C_{\mathrm{body}}^*)g, \big((\mathbf{F}_h - \mathbf{M}^\chi)\otimes \mathrm{Id}\big) f\big\rangle,$$
$$\|(\bar D_L \otimes \mathrm{Id})^{1/2}(\mathrm{Id}\otimes C_{\mathrm{body}}^*)g\| \le \|(\bar D_L\otimes \mathrm{Id})^{1/2} g\|.$$

Moreover, $(\mathbf{M}^\chi)^*\mathbf{M}^\chi \otimes \mathbf{H}_h|\mathbf{S}_h|^2\mathbf{H}_h \le (\mathbf{M}^\chi)^*\mathbf{M}^\chi \otimes \mathrm{Id}$ gives $\bar D_R \otimes \mathrm{Id} \le D_R$, since these positive operators commute, and likewise $\bar D_L \otimes \mathrm{Id} \le D_L$. Inserting these bounds into the four-term bound for $(\mathbf{F}_h - \mathbf{M}^\chi)\otimes \mathrm{Id}$ proves the lemma. □

The source

Let s_Γ^{BB} and s_Γ^{FF} denote the bosonic and the fermionic part of the source factor. Since Γ is block diagonal, (15.6) gives, at a point S of the tube,

$$s_\Gamma = s_\Gamma^{\mathrm{BB}} s_\Gamma^{\mathrm{FF}}, \qquad s_\Gamma^{\mathrm{BB}}(S) = e^{-\mathrm{i}\eta\,\mathrm{Tr}(S^{\mathrm{BB}}\Gamma_{BB})}, \qquad s_\Gamma^{\mathrm{FF}}(S) = e^{\mathrm{i}\eta\,\mathrm{Tr}(S^{\mathrm{FF}}\Gamma_{FF})}. \tag{17.31}$$

We let $D_S = \mathrm{Id} - |\mathbf{S}_h|^2$ denote the defect of the frozen source kick. It is the multiplication by a function on $|\mathcal{M}|$, and (17.2) gives $D_S \ge 1 - e^{-2c\delta_W v}$, where c is the constant of (17.2). The soul of the BB block in the odd tangent coordinates contributes the term ω^2 to the bilateral parameter in the next lemma. For this reason ω must be small in Theorem 17.1, as in the continuum model, where the soul of the potential is bounded in (C.3).

Lemma 17.19. *There exists C_1 such that the following holds if $C_1(r+\omega^2) \le c/4$, where c is the constant of* (17.2)*. For h small enough, depending on r and ω, the operator $\mathbf{E}_{\mathrm{src}}$ satisfies the four-term bound with parameters $C(r+\omega^2) + C(r,\omega)h$ and $C(r,\omega)$.*

Proof. We first prove that, on the tube and in the operator norm on the fiber,

$$\big\| c_\Gamma^{-1} s_\Gamma^{\mathrm{BB}} - \mathbf{S}_h \big\| \le C_1(r+\omega^2+h)\,\delta_W v\, e^{-c\delta_W v/2}. \tag{17.32}$$

Let $S = R(x,U)$ be a point of the tube. As in Definition 5.2 and the proof of Lemma 5.14, we write the frame as $T(x) = g e^{\Theta}$, where $g = \mathrm{diag}(g_{\mathrm{B}} \,|\, g_{\mathrm{F}})$ is a body element with g_{B} positive definite and $\|g_{\mathrm{B}}\|\|g_{\mathrm{B}}^{-1}\| \le 2v$, and where Θ is an odd tangent supermatrix whose entries are the odd tangent coordinates. Then $S^{\mathrm{BB}} = g_{\mathrm{B}}\big(e^{\Theta}(R_* + U)e^{-\Theta}\big)^{\mathrm{BB}} g_{\mathrm{B}}^{-1}$, and we split

$$S^{\mathrm{BB}} = \kappa_E \mathrm{Id} - \mathrm{i}s_E Q_B + \mathcal{N}_T + \mathcal{N}_U, \qquad \mathcal{N}_U = g_{\mathrm{B}}\big(e^{\Theta} U e^{-\Theta}\big)^{\mathrm{BB}} g_{\mathrm{B}}^{-1}, \tag{17.33}$$

where $\kappa_E\mathrm{Id} - \mathrm{i}s_E Q_B$ is the body of the BB block of $R(x,0)$, and $\mathcal{N}_T$ is its soul.

We bound the two parts in turn. Since g is block diagonal and the odd powers of Θ have vanishing diagonal blocks, the entries of $\mathcal{N}_T$ are polynomials of even degree at

least two in the odd tangent coordinates, with coefficients bounded by $C\|g_{\mathrm{B}}\|\|g_{\mathrm{B}}^{-1}\| \le Cv$. Multiplication by $\mathcal{N}_T$ therefore has norm at most $C\omega^2 v$, by Lemma 5.13(a) and (d). The body of $\mathcal{N}_U$ is $g_{\mathrm{B}} U^{\mathrm{BB}} g_{\mathrm{B}}^{-1}$, and on the tube

$$\|g_{\mathrm{B}} U^{\mathrm{BB}} g_{\mathrm{B}}^{-1}\| \le 2v|U^{\mathrm{BB}}| \le 4rv.$$

Every term of its soul contains two odd tangent coordinates and an even entry of U, or one odd tangent coordinate and one odd entry of U. In the standardized coordinates $U = hZ$, an odd entry of U is h times a bounded operator on $\mathcal{H}_{\mathrm{odd}}$, and the soul of $\mathcal{N}_U$ therefore has norm at most $C(\omega^2 r + \omega h)v \le C(\omega^2 + h)v$.

By the definitions of c_Γ and $\mathbf{S}_h$ in Section 17.1, the body of the BB block of $R(x,0)$ contributes $c_\Gamma \mathbf{S}_h$ to s_Γ^{BB}. All these terms are even and commute, and we obtain

$$c_\Gamma^{-1} s_\Gamma^{\mathrm{BB}} = \mathbf{S}_h e^{\psi}, \qquad \psi = -\mathrm{i}\eta \operatorname{Tr}\big((\mathcal{N}_T + \mathcal{N}_U)\Gamma_{BB}\big).$$

Using $\eta \le C\delta_W$, $|\operatorname{Tr} A| \le 2k\|A\|$, and $\|\Gamma_{BB}\| \le 2$ on $\mathcal{U}_2$, we find from the bounds above that $\|\psi\| \le C_1(r + \omega^2 + h)\delta_W v$. Together with $\|e^{\psi} - \mathrm{Id}\| \le \|\psi\| e^{\|\psi\|}$ and (17.2), this gives

$$\big\|c_\Gamma^{-1} s_\Gamma^{\mathrm{BB}} - \mathbf{S}_h\big\| = \|\mathbf{S}_h(e^{\psi} - \mathrm{Id})\| \le C_1(r + \omega^2 + h)\, \delta_W v\, e^{(C_1(r+\omega^2+h) - c)\delta_W v}.$$

By the hypothesis of the lemma, and for h so small that $C_1 h \le c/4$, the exponent is at most $-c\delta_W v/2$, which proves (17.32).

Next, we factor the difference through D_S. For $y \ge 0$, we have $ye^{-cy/2} \le C(1 - e^{-2cy})$, since the quotient of the two sides is continuous on $(0,\infty)$ and tends to $(2c)^{-1}$ as $y \to 0$ and to zero as $y \to \infty$. Since $D_S \ge 1 - e^{-2c\delta_W v}$, the bound (17.32) implies that

$$c_\Gamma^{-1} s_\Gamma^{\mathrm{BB}} - \mathbf{S}_h = D_S^{1/2} b_S D_S^{1/2}, \qquad b_S = D_S^{-1}\big(c_\Gamma^{-1} s_\Gamma^{\mathrm{BB}} - \mathbf{S}_h\big), \qquad \|b_S\| \le C(r + \omega^2 + h), \tag{17.34}$$

and b_S commutes with D_S. The FF block of S is bounded on the tube, with all its coefficients, since g_{F} is unitary, and therefore $\|s_\Gamma^{\mathrm{FF}} - \mathrm{Id}\| \le C\delta_W$. Using (17.31), (17.34), and (17.1), we obtain

$$s_\Gamma - \mathbf{S}_h = D_S^{1/2} b'_S D_S^{1/2} + \mathbf{S}_h\big(c_\Gamma s_\Gamma^{\mathrm{FF}} - \mathrm{Id}\big), \qquad b'_S = c_\Gamma s_\Gamma^{\mathrm{FF}} b_S, \tag{17.35}$$

with $\|b'_S\| \le 2\|b_S\|$ and $\|c_\Gamma s_\Gamma^{\mathrm{FF}} - \mathrm{Id}\| \le C\delta_W$ for h small.

We also need a bound for the step without the frozen kick. Recall that $\mathcal{K}_h - \mathbf{M}^{\chi} \otimes \mathbf{H}_h$ is the sum of the five operators X with $\mathbf{S}_h X \in \{\mathbf{E}_{\mathrm{nor}}, \mathbf{E}_0, \mathbf{E}_{\mathrm{hol}}, \mathbf{E}_1, \mathbf{E}_{\ge 2}\}$. In the proofs of Lemma 17.13, Lemma 17.15, Lemma 17.16, Lemma 17.17, and Lemma 17.18, the vector g enters only through $\mathbf{S}_h^* g$, or through $(\mathrm{Id} \otimes C_{\mathrm{body}}^*)g$ for $\mathbf{E}_{\mathrm{nor}}$, and only through the norm of this vector and its norms with D_{nor}, $\mathrm{Id} - \mathbf{H}_h^2$, and $\bar{D}_L \otimes \mathrm{Id}$, which are at most its norm. These proofs therefore apply with this vector replaced by an arbitrary vector of norm

one. Adding the resulting bounds, and using (17.3) on the right, we find that for h small enough, depending on r and ω,

$$\big\|(\mathcal{K}_h - \mathbf{M}^{\chi} \otimes \mathbf{H}_h) f\big\| \le C\|D_R^{1/2} f\| + C(r,\omega)\sqrt{\delta_W}\|f\|. \tag{17.36}$$

In particular, $\|\mathcal{K}_h\| \le C$.

We now bound $\mathbf{E}_{\mathrm{src}} = (s_\Gamma - \mathbf{S}_h)\mathcal{K}_h$. The second term of (17.35) has norm at most $C\delta_W$, and together with $\|\mathcal{K}_h\| \le C$ it contributes a scalar term $C\delta_W\|g\|\|f\|$. For the first term, we have

$$|\langle g, D_S^{1/2} b_S' D_S^{1/2} \mathcal{K}_h f\rangle| \le \|b_S'\|\, \|D_S^{1/2} g\|\, \|D_S^{1/2}\mathcal{K}_h f\|.$$

We have $\|D_S^{1/2} g\| \le 2\|D_L^{1/2} g\|$, since $D_S \le D_{\mathrm{body,L}}$ and by (17.3). The multiplier $D_S^{1/2}$ commutes with $\mathbf{M}^{\chi}$, and $\mathbf{H}_h D_S \mathbf{H}_h \le D_{\mathrm{body,R}}$. Therefore

$$\|D_S^{1/2}(\mathbf{M}^{\chi} \otimes \mathbf{H}_h) f\| \le \langle f, \mathbf{H}_h D_S \mathbf{H}_h f\rangle^{1/2} \le \|D_{\mathrm{body,R}}^{1/2} f\| \le 2\|D_R^{1/2} f\|,$$

and (17.36) bounds the remaining part of $\|D_S^{1/2}\mathcal{K}_h f\|$. Combining these bounds with (17.34), we obtain

$$|\langle g, \mathbf{E}_{\mathrm{src}} f\rangle| \le C(r + \omega^2 + h)\|D_L^{1/2} g\|\big(\|D_R^{1/2} f\| + C(r,\omega)\sqrt{\delta_W}\|f\|\big) + C\delta_W\|g\|\|f\|,$$

which is the four-term bound with the stated parameters. □

17.8 Proof of the main theorem

We first treat the weights. Let $q \in \mathbb{R}$, and set

$$\Delta_v(x,y) = \log v(x) - \log v(y),$$

which is exactly antisymmetric under the reversal and satisfies $|\Delta_v(x,y)| \le C d(x,y)$, since $\log v$ is Lipschitz with bounded second derivatives. The kernel of $A_{h,q} = v^q A_h v^{-q}$ is $e^{q\Delta_v(x,y)}$ times the kernel of A_h, and we write

$$A_{h,q} - C_h = v^q (A_h - C_h) v^{-q} + \big(v^q C_h v^{-q} - C_h\big). \tag{17.37}$$

The second term comes from the conjugation of the reference:

$$v^q C_h v^{-q} - C_h = \mathbf{M}^{\chi} \otimes \mathbf{S}_h\big(v^q \mathbf{H}_h v^{-q} - \mathbf{H}_h\big),$$

with the kernel $\mathbf{S}_h(x)\mathbf{M}^{\chi}(e^{q\Delta_v(x,y)} - 1)p_h^{\nabla}(x,y)$. We expand

$$e^{q\Delta_v} - 1 = q\Delta_v + O\big(q^2 d(x,y)^2 e^{C|q|d(x,y)}\big),$$

and we write $p_h^\nabla = p_h\tau + (p_h^\nabla - p_h\tau)$. The term $q\Delta_v p_h\tau$ is antisymmetric under the reversal and of size $C|q|\sqrt{\delta_W}$ times a widened Gaussian. By (17.17), its part without its marginal gives a one-sided term with coefficient $C|q|$, and its marginal is $O(\delta_W)$ by Corollary 17.9. The remainder is $O(q^2\delta_W)$ by Schur's test, and the part with $p_h^\nabla - p_h\tau$ is bounded as $\mathbf{E}_{\mathrm{hol}}$, with the factor $\mathbb{E}_x[e^{2q\Delta_v(x,X_{\delta_W})}] \le C(q)$. For the first term of (17.37), we conjugate each of the terms $\mathbf{E}_{\mathrm{src}}$, $\mathbf{E}_{\mathrm{nor}}$, $\mathbf{E}_0$, $\mathbf{E}_{\mathrm{hol}}$, $\mathbf{E}_1$, $\mathbf{E}_{\ge 2}$. The factor $e^{q\Delta_v}$ is absorbed by widening the Gaussians, and its first variation $q\Delta_v$, which carries h^2 and is antisymmetric, produces one-sided terms through (17.17) and marginals of order δ_W through Corollary 17.9. In $\mathbf{E}_{\mathrm{nor}}$, the operator $v^q C_{\mathrm{body}} v^{-q}$, with $C_{\mathrm{body}} = \mathbf{S}_h\mathbf{H}_h$ as in the proof of Lemma 17.18, has norm at most $1 + C(q)\delta_W$ by Schur's test, since $e^{q\Delta_v} > 0$ and its marginal is $1 + O(\delta_W)$ by the same parity argument. The argument for $\mathbf{E}_{\mathrm{nor}}$ then applies with this constant.

Proof of Theorem 17.1. It suffices, by the reduction after the theorem, to prove the four-term bound for $A_{h,q}$ on $\mathcal{H}_0^{\mathrm{tube}}$ relative to C_h. The reference C_h is a contraction for every $\Gamma \in \mathcal{U}_2$, by (17.2), as Corollary A.2 requires. We first collect the bounds for the terms of $A_h - C_h$. Using Lemma 17.13, for h small enough depending on r and ω, and (17.3), we see that the term $\mathbf{E}_0$, which contains the Gaussian-small remainder from $d(x,y) \ge \varrho_0$, satisfies the four-term bound with parameters $C(r + \omega^2 + h)$ and $C(r,\omega)$. The operators $\mathbf{E}_{\mathrm{hol}}$, $\mathbf{E}_1$, $\mathbf{E}_{\ge 2}$, $\mathbf{E}_{\mathrm{nor}}$, and $\mathbf{E}_{\mathrm{src}}$ satisfy the four-term bound with the parameters stated in Lemma 17.15, Lemma 17.16, Lemma 17.17, Lemma 17.18, and Lemma 17.19, for h small enough, depending on r and ω, if $C_1(r + \omega^2) \le c/8$. For $q \ne 0$, the paragraph above adds the weight terms of (17.37), with constants that depend also on q. All of these bounds hold uniformly for $\Gamma \in \mathcal{U}_2$, since they use the damping only through (17.2) and (17.1). Adding them, we find that $A_{h,q}$ satisfies (17.4) relative to C_h with

$$a = C_k(r + \omega^2) + C_k(r,\omega)h, \qquad A_1 = C_k(r,\omega,q).$$

We choose ω so small that $C_k\omega^2 \le 1/64$ and $C_1\omega^2 \le c/16$, and then r so small that $C_k r \le 1/64$ and $C_1 r \le c/16$. We let h_0 be so small that $C_k(r,\omega)h_0 \le 1/32$, that $Bh_0^4 \le 1$ for the constant B recalled after the theorem, and that the lemmas above apply for $h \le h_0$. Then $a \le 1/16$ and $B\delta_W \le 1$ for $h \le h_0$. By Remark 16.20, applied on $\mathcal{H}_0^{\mathrm{tube}}$ with $\mathbf{T} = A_{h,q}$, $C = C_h$, $\delta_0 = \delta_W$, and $t = T$, we obtain the two operator inequalities of the theorem for $A_{h,q}$, with the defects of C_h, and the power bound $\|A_{h,q}^n\| \le e^{BT/4}$ on $\mathcal{H}_0^{\mathrm{tube}}$ for $n\delta_W \le T$. Since multiplication by v^q is unitary, these statements are equivalent to those on $\mathcal{H}_q^{\mathrm{tube}}$. The constant B depends on k, q, r, and ω, which are fixed before $h \to 0$. The constant c of (17.2), which enters the choices of ω and r, can be taken to be $s_E c_\varepsilon \eta/\delta_W$, by the derivation of (17.2), where $\eta/\delta_W = \varepsilon/((4 - E^2)X_N(E)\rho(E))$ and c_ε is the constant of (15.36) on $\mathcal{U}_2$. It therefore depends on the compact sets of energies, of ε and of X, and on $\mathcal{U}_2$, as stated.

The two conditions on ω above hold for every $\omega' \in (0,\omega]$ as well, and the conditions on r do not involve ω. The argument above with ω' in place of ω, and with h_0 chosen depending on ω', therefore proves the theorem for ω' with the same r.

In a signature $p \neq k$, the reference contains the full factor $\lambda = \zeta_p(\mu_+\mu_-)^{(p-k)^2/2}$ through $\mathbf{M}^\chi$, as defined in Lemma 16.10. The bounds above are unchanged, and the defect of the reference contains $(1-|\lambda|^2)\mathrm{Id}$, which by (16.54) is bounded below by a positive multiple of $h^2\mathrm{Id}$. A smaller damping changes only $\mathbf{S}_h$ and c_Γ. It preserves (17.1), and it preserves (17.2), possibly with $c/2$ in place of c. Since we chose $C_1(r+\omega^2) \leq c/8$, the hypothesis of Lemma 17.19 holds with $c/2$ in place of c, and the proof applies without change. □

17.9 Gain of body weights

We now deduce from Theorem 17.1 that the adapted step improves polynomial body weights; Sections 19 and 20 use this. Recall from Section 17.1 that the weighted space $\mathcal{H}_p^{\mathrm{tube}}$ carries the weight v^{2p}, where v is the weight (5.3), so that

$$\|v^{1/2}f\|_{\mathcal{H}_p^{\mathrm{tube}}} = \|f\|_{\mathcal{H}_{p+1/2}^{\mathrm{tube}}}. \tag{17.38}$$

The continuum semigroup improves polynomial body weights by Proposition 5.15(b), because its potential contains a positive BB damping term, which grows like the weight v of (5.3). The following lemma is the discrete analogue for the adapted chain. Its proof is an energy estimate followed by the pigeonhole principle. The idea is to set aside part of the BB damping of the source factor, and the defect sums of Lemma A.1 then show that this part dissipates the weight $\delta_W v$ on average over the steps.

Lemma 17.20. *Let $p \in \mathbb{R}$, let $m \geq 1$ be an integer and let $T > 0$. There exists $C = C_{p,m,T}$ such that for all integers $1 \leq n \leq T/\delta_W$,*

$$\|A_h^n\|_{\mathcal{H}_p^{\mathrm{tube}} \to \mathcal{H}_{p+m/2}^{\mathrm{tube}}} \leq C(n\delta_W)^{-m/2}. \tag{17.39}$$

The constant may depend on the tube.

Proof. We begin by setting aside part of the BB damping. Let c_1 denote the constant c of (17.2), so that the frozen kick $\mathbf{S}_h$ of A_h has modulus at most $e^{-c_1\delta_W v}$. Recall that $\mathbf{S}_h$ is the frozen kick of $c_\Gamma^{-1}s_\Gamma$, with the constant c_Γ of (17.1). Let $0 < c \leq c_1/2$, and set

$$\mathbf{S}_\flat = e^{-c\delta_W v/2}, \qquad A_h^\flat = \mathbf{S}_\flat^{-2}A_h, \qquad A_h^\sharp = \mathbf{S}_\flat A_h^\flat \mathbf{S}_\flat.$$

Then $A_h^\flat$ differs from A_h only through its frozen source kick. Indeed, $\mathbf{S}_\flat$ depends only on the tangent variables, so that the normal part of the source factor of $A_h^\flat$ is that of A_h, and

its frozen kick is $\mathbf{S}_\flat^{-2}\mathbf{S}_h$, where $\mathbf{S}_h$ is the frozen kick of A_h in (17.2). Since $c \le c_1/2$, the kick $\mathbf{S}_\flat^{-2}\mathbf{S}_h$ has modulus at most $e^{-(c_1-c)\delta_W v} \le e^{-c_1\delta_W v/2}$, and it satisfies (17.2) with $c_1/2$ in place of c_1. Moreover, $A_h^n = \mathbf{S}_\flat (A_h^\sharp)^{n-1}\mathbf{S}_\flat A_h^\flat$ for all $n \ge 1$.

We apply the stability theorem Theorem 17.1 in $\mathcal{H}_p^{\mathrm{tube}}$ to the step $A_h^\flat$. This is allowed by the last paragraph of its proof, since $A_h^\flat$ differs from A_h only through its frozen kick, which satisfies (17.2) with half the constant. By the theorem, $A_h^\flat$ satisfies the four-term bound (17.4) relative to a contraction

$$C_\flat = v^{-p} C_{h,\flat} v^p$$

of $\mathcal{H}_p^{\mathrm{tube}}$, where $C_{h,\flat}$ is the reference of $A_h^\flat$ on $\mathcal{H}_0^{\mathrm{tube}}$, with $a \le 1/16$. Since multiplication by v^p is unitary from $\mathcal{H}_p^{\mathrm{tube}}$ onto $\mathcal{H}_0^{\mathrm{tube}}$, the operator $C_\flat$ is a contraction of $\mathcal{H}_p^{\mathrm{tube}}$. The first part of the proof of Corollary A.2, applied with A_0 equal to the constant A_1 of (17.4), shows that the four-term bound implies (A.2) relative to $C_\flat$, with $\delta_0 = \delta_W$, $\theta = 1/8$, and $A = A_\flat = \max\{1, 2304 A_1^2\}$, since $A_\flat \delta_W \le 1$ for $h \le h_0$. Let $D_{R,\flat}$ and $D_{L,\flat}$ denote the defects of $C_\flat$ in $\mathcal{H}_p^{\mathrm{tube}}$.

We first show that for all $f \in \mathcal{H}_p^{\mathrm{tube}}$ and $n \le T/\delta_W$,

$$\sum_{j=0}^{n-1} \|(\mathrm{Id} - \mathbf{S}_\flat^2)^{1/2} (A_h^\sharp)^j f\|_{\mathcal{H}_p^{\mathrm{tube}}}^2 \le C_{p,T} \|f\|_{\mathcal{H}_p^{\mathrm{tube}}}^2. \tag{17.40}$$

We use $C' = \mathbf{S}_\flat C_\flat \mathbf{S}_\flat$, which is a contraction since $0 < \mathbf{S}_\flat \le 1$, as the reference for $A_h^\sharp$. Its defects satisfy

$$\begin{aligned} D'_R &= \mathrm{Id} - C'^* C' = (\mathrm{Id} - \mathbf{S}_\flat^2) + \mathbf{S}_\flat D_{R,\flat} \mathbf{S}_\flat + \mathbf{S}_\flat C_\flat^* (\mathrm{Id} - \mathbf{S}_\flat^2) C_\flat \mathbf{S}_\flat \ge (\mathrm{Id} - \mathbf{S}_\flat^2) + \mathbf{S}_\flat D_{R,\flat} \mathbf{S}_\flat, \\ D'_L &= \mathrm{Id} - C' C'^* = (\mathrm{Id} - \mathbf{S}_\flat^2) + \mathbf{S}_\flat D_{L,\flat} \mathbf{S}_\flat + \mathbf{S}_\flat C_\flat (\mathrm{Id} - \mathbf{S}_\flat^2) C_\flat^* \mathbf{S}_\flat \ge (\mathrm{Id} - \mathbf{S}_\flat^2) + \mathbf{S}_\flat D_{L,\flat} \mathbf{S}_\flat. \end{aligned} \tag{17.41}$$

Observe that $A_h^\sharp - C' = \mathbf{S}_\flat (A_h^\flat - C_\flat) \mathbf{S}_\flat$. Since $\mathbf{S}_\flat$ is self-adjoint, the form bound for $A_h^\flat$ implies

$$|\langle g, (A_h^\sharp - C') f \rangle| \le \tfrac{1}{8} \, \|(D_{L,\flat} + A_\flat \delta_W)^{1/2} \mathbf{S}_\flat g\| \, \|(D_{R,\flat} + A_\flat \delta_W)^{1/2} \mathbf{S}_\flat f\|.$$

Using (17.41) and $\mathbf{S}_\flat^2 \le \mathrm{Id}$, we find

$$\|(D_{R,\flat} + A_\flat \delta_W)^{1/2} \mathbf{S}_\flat f\|^2 = \langle f, (\mathbf{S}_\flat D_{R,\flat} \mathbf{S}_\flat + A_\flat \delta_W \mathbf{S}_\flat^2) f \rangle \le \|(D'_R + A_\flat \delta_W)^{1/2} f\|^2,$$

and similarly on the left. Therefore $A_h^\sharp$ satisfies (A.2) relative to C', with the same constants. Since $D'_R \ge \mathrm{Id} - \mathbf{S}_\flat^2$, the defect sums (A.5) of Lemma A.1 imply (17.40).

We now convert (17.40) into a gain of weight. By the elementary inequality $x e^{-cx} \le c^{-1}(1 - e^{-cx})$ for $x \ge 0$, we have $\delta_W v \mathbf{S}_\flat^2 \le c^{-1}(\mathrm{Id} - \mathbf{S}_\flat^2)$ pointwise. Using $A_h^{j+1} = \mathbf{S}_\flat (A_h^\sharp)^j \mathbf{S}_\flat A_h^\flat$, (17.38), and the fact that $\mathbf{S}_\flat$ commutes with v, we obtain

$$\sum_{j=1}^{n} \delta_W \|A_h^j f\|_{\mathcal{H}_{p+1/2}^{\mathrm{tube}}}^2 \le c^{-1} \sum_{j=0}^{n-1} \|(\mathrm{Id} - \mathbf{S}_\flat^2)^{1/2} (A_h^\sharp)^j \mathbf{S}_\flat A_h^\flat f\|_{\mathcal{H}_p^{\mathrm{tube}}}^2 \le C'_{p,T} \|f\|_{\mathcal{H}_p^{\mathrm{tube}}}^2,$$

where in the last inequality we used (17.40) and the boundedness of $\mathbf{S}_\flat A_h^\flat$ on $\mathcal{H}_p^{\mathrm{tube}}$.

The interval $[\lceil n/2\rceil, n]$ contains at least $n/2$ integers. By the pigeonhole principle, there exists j in this interval such that

$$\|A_h^j f\|^2_{\mathcal{H}^{\mathrm{tube}}_{p+1/2}} \le 2C'_{p,T}(n\delta_W)^{-1}\|f\|^2_{\mathcal{H}^{\mathrm{tube}}_p}.$$

Since $(n-j)\delta_W \le T$, the power bound of Theorem 17.1 in $\mathcal{H}^{\mathrm{tube}}_{p+1/2}$ bounds A_h^{n-j} on that space, and we conclude that

$$\|A_h^n f\|_{\mathcal{H}^{\mathrm{tube}}_{p+1/2}} \le C(n\delta_W)^{-1/2}\|f\|_{\mathcal{H}^{\mathrm{tube}}_p}.$$

This is (17.39) for $m = 1$.

Let $m \ge 2$, and suppose first that $n \ge m$. We split the n steps into m consecutive blocks of lengths at least $n/(2m)$, and we apply the case $m = 1$ on the i-th block, from $\mathcal{H}^{\mathrm{tube}}_{p+(i-1)/2}$ to $\mathcal{H}^{\mathrm{tube}}_{p+i/2}$. The product of the m bounds is bounded by $C(n\delta_W)^{-m/2}$.

Now suppose that $1 \le n < m$. We write $A_h = \mathbf{S}_\flat^2 A_h^\flat$, where $\mathbf{S}_\flat^2 = e^{-c\delta_W v}$ and $A_h^\flat$ is bounded on $\mathcal{H}_p^{\mathrm{tube}}$ by Theorem 17.1. Then

$$\begin{aligned}\|A_h^n f\|_{\mathcal{H}^{\mathrm{tube}}_{p+m/2}} &= \|v^{m/2}\mathbf{S}_\flat^2 A_h^\flat A_h^{n-1} f\|_{\mathcal{H}_p^{\mathrm{tube}}} \\ &\le \Big(\sup_{x>0} x^{m/2}e^{-c\delta_W x}\Big)\|A_h^\flat A_h^{n-1} f\|_{\mathcal{H}_p^{\mathrm{tube}}} \le C_m\delta_W^{-m/2}\|f\|_{\mathcal{H}_p^{\mathrm{tube}}},\end{aligned}$$

and $\delta_W^{-m/2} \le m^{m/2}(n\delta_W)^{-m/2}$ since $n < m$. □

The last step of the proof applies to all $n \ge 1$ and gives

$$\|A_h^n\|_{\mathcal{H}_p^{\mathrm{tube}} \to \mathcal{H}^{\mathrm{tube}}_{p+m/2}} \le C_{p,m,T} W^m, \qquad 1 \le n \le T/\delta_W. \tag{17.42}$$

In Section 20, this bound is used for the first steps after the right block of the chain, where a cost polynomial in W is acceptable.

SECTION 18

Consistency on the invariant normal line

In this section we prove that, in the balanced root signature $p = k$, the adapted chain of Section 16 converges to the continuum semigroup on vectors of the form $\Omega_h \otimes f$, where Ω_h is the invariant state of the frozen normal transfer and f is a function on the saddle manifold. The precise statement is Theorem 18.14. We call a set of consecutive steps of the chain a *leg*. The result will be used in Section 20 for the legs of macroscopic length, after the initial layers in which the endpoints relax to Ω_h. The endpoints themselves are treated in Section 19.

Fast and slow variables. We first recall the heuristic of Section 2.4, which leads to the consistency relation (2.4). By Proposition 16.22, the frozen normal transfer $\mathbf{F}_h$, which is the adapted step A_h with the tangent motion and the source removed, contracts closed normal vectors of zero mean by the factor e^{-ch^2} per step, while one step moves the tangent variables, which parametrize the saddle manifold, by a heat flow of duration $\tau_{\mathrm{phys}} = \delta_W/2$. The normal variables therefore equilibrate after about h^{-2} steps, long before the tangent variables move over the h^{-4} steps of the chain, and to first order the tangent variables should see them only through their equilibrium state. This is first-order averaging, or adiabatic elimination of fast variables, for operator semigroups, in the sense of Kurtz's limit theorem for perturbed semigroups [83]; see also [46, Chapter 1].

One also expects first order to suffice. Let Π be the projection onto the line of vectors $\Omega_h \otimes f$, and let L_H be the tangent generator. The fast resolvent on the complement of the line has norm of order h^{-2} in units of steps. The second-order averaging correction to the generator is then of order $\delta_W h^{-2} \|(\mathrm{Id} - \Pi) L_H \Pi\|^2$ per unit of tangent time. We will see in (18.34) that the tangent generator leaves the line only at order h, which makes this correction of order $\delta_W = h^4$.

The argument. This picture is only a heuristic, and we do not use an averaging theorem in the proof. Instead, we verify the two hypotheses of the Trotter–Kurtz approximation theorem, which we state in the form we need as Lemma 18.1. The first is stability of the powers of A_h on all inputs, which is the main result of Section 17, Theorem 17.1. The second is consistency on the line $\iota_h f = \Omega_h \otimes f$ for f in an operator core, Proposition 18.12.

We construct the normal state Ω_h in Section 18.2 by Banach's fixed point theorem, from the contraction of Proposition 16.22. The consistency calculation of Section 18.5 is a Duhamel expansion of one heat link around its frozen normal heat kernel. It needs

only a bound for the heat semigroup on the tube over the duration of one link. This bound is of order one and cannot be iterated over the chain; the iterable bound is the stability estimate of Section 17, which is proved from the kernel of the step. The one-link bound rests on two facts that Section 19 also uses: the horizontal operator is sectorial in group charts, and the actual Gaussian kernel agrees with the Dirichlet heat on the tube, also when the normal coordinates are moved to complex graphs that change in time. We prove both in Section 18.3. In Section 18.4, we combine Lemma 16.5, which identifies the horizontal operator on the saddle manifold as $-(2s_E^2)^{-1}\mathsf{C}_k$, with the duration of one link, and we identify the generator that the chain produces on the tangent variables with the generator of the sigma model, which fixes the limit semigroup of Theorem 18.14 and its time scale.

Inputs from Sections 16 and 17. For the normal fixed state and the normal-line consistency statements, we fix the balanced root signature $p = k$ and work on the real tube of Definition 16.1, of radius r, where $r \le r_*$ is fixed before $h \to 0$. The abstract approximation results of Section 18.1 do not depend on a root signature. The analytic heat estimates of Section 18.3 apply in every signature, as specified there. Recall that the points of the tube are

$$R(x, U) = T(x)(R_* + U)T(x)^{-1},$$

where x runs over the saddle manifold $U(k, k|2k)/K$, $T(x)$ is a local section, $R_* = \kappa_E \mathrm{Id} - \mathrm{i}s_E\Sigma$ is the root matrix, and U is a normal coordinate in the adapted real normal space of Definition 16.1, with standardized form $U = hZ$. Recall also the normal Wick space $\mathcal{H}_{\mathrm{nor}}$ of Section 16.4, with vacuum $\Omega_0 = 1$, number operator $\mathbf{N}$, and Gaussian mean ℓ_0, so that the orthogonal projection onto $\mathbb{C}\Omega_0$ is $\mathbf{P}_0 f = \ell_0(f)\Omega_0$. We let $\mathcal{G}^m$ denote the domain of $(\mathbf{N}+1)^m$, with the norm $\|f\|_{\mathcal{G}^m} = \|(\mathbf{N}+1)^m f\|$. By Lemma 16.12, the supercharge $\mathfrak{q}_{\mathrm{nor}}$ is a closed operator. We let $\mathcal{K} = \ker \mathfrak{q}_{\mathrm{nor}}$ denote the closed subspace of closed normal vectors, and we set $\mathcal{K}_0 = \mathcal{K} \cap \ker \ell_0$. Let $\mathbf{F}_h$ be the frozen normal transfer of (16.38) and Section 16.9. Inputs (N1)–(N3) below concern the balanced signature; the stability input (N4) also holds in every other signature.

(N1) By Proposition 16.22, $\mathbf{F}_h$ maps $\mathcal{K}$ and $\mathcal{K}_0$ into themselves, satisfies $\ell_0(\mathbf{F}_h f) = \ell_0(f)$ for $f \in \mathcal{K}$, and contracts $\mathcal{K}_0 \cap \mathcal{G}^m$ by the factor $e^{-c_m h^2}$ per step in the norm of $\mathcal{G}^m$, for each of the finitely many prescribed graph orders m, as in (16.53). The exact preservation of the mean is the Ward identity of Section 2.4, and Section 16.10 explains why a perturbative argument would not suffice.

(N2) By Lemma 16.21, $\mathbf{F}_h$ is bounded on each $\mathcal{G}^m$, and its vacuum residual satisfies $\|\mathbf{F}_h\Omega_0 - \Omega_0\|_{\mathcal{G}^m} \le C_m h^3$, which is (16.51).

(N3) Localization, Lemma 16.13: for a closed normal function g with compactly supported body, the normalized Gaussian Berezin integral of g is the value of its body at the origin.

(N4) Stability, Theorem 17.1: for all $T > 0$ and $q \in \mathbb{R}$, $\sup_{0 \le n\delta_W \le T} \|A_h^n\| \le e^{BT/4}$ on the weighted space $\mathcal{H}_q^{\mathrm{tube}}$ of Section 17, on all inputs, uniformly for the damping in $\mathcal{U}_2$ and on compact energy sets.

By (16.38), we have $A_h = s_\Gamma T_h$, where s_Γ is the multiplication by the physical source factor and T_h is the unsourced step in the symmetric normal gauge. We define the *source potential* V_h on the tube by $s_\Gamma = e^{-\delta_W V_h}$. The frozen normal transfer $\mathbf{F}_h$ is the normal part of T_h; below (18.24) we compare these Gaussian steps with their Dirichlet versions.

18.1 The Trotter–Kurtz approximation theorem

The following lemma is the discrete-time approximation theorem of Trotter and Kurtz [82, 131], in the core form of [46, Chapter 1, Section 6]. That theorem assumes contractions. Trotter's Theorem 5.3 [131] already allows powers that are bounded but not contractive, but it uses hypotheses on the limit of the generators and proves convergence for each fixed time. Since we need the core form of the consistency hypothesis and uniformity in time, we deduce the lemma from the contractive theorem by an equivalent renorming.

Lemma 18.1. *Let $\mathcal{H}$ be a Banach space, and let $(U_t)_{t\ge 0}$ be a strongly continuous semigroup on $\mathcal{H}$ with generator L. Let $\mathcal{D} \subset D(L)$ be a linear subspace that is a core for L. For $0 < h \le h_0$, let $\mathcal{H}_h$ be Banach spaces, let $\iota_h \colon \mathcal{H} \to \mathcal{H}_h$ be bounded linear maps with $\sup_h \|\iota_h\| < \infty$, let A_h be bounded operators on $\mathcal{H}_h$, and let $\delta_h > 0$ with $\delta_h \to 0$ as $h \to 0$. Suppose that for all $T > 0$,*

$$\sup_{0<h\le h_0} \sup_{0\le n\delta_h \le T} \|A_h^n\| \le C_T < \infty, \tag{18.1}$$

and that for all $f \in \mathcal{D}$,

$$\lim_{h\to 0} \big\| \delta_h^{-1}(A_h \iota_h f - \iota_h f) - \iota_h L f \big\| = 0. \tag{18.2}$$

Then for all $f \in \mathcal{H}$ and $T > 0$,

$$\lim_{h\to 0} \sup_{0\le n\delta_h \le T} \|A_h^n \iota_h f - \iota_h U_{n\delta_h} f\| = 0. \tag{18.3}$$

We first recall the cited theorem. Let L_n and L be Banach spaces and $\pi_n \colon L \to L_n$ bounded linear maps with $\sup_n \|\pi_n\| < \infty$. Let T_n be linear contractions on L_n, let $\varepsilon_n > 0$ with $\varepsilon_n \to 0$, and let $T(t)$ be a strongly continuous contraction semigroup on L whose generator A has a core D. Suppose that for each $f \in D$ there exist $f_n \in L_n$ such that $\|f_n - \pi_n f\| \to 0$ and $\|\varepsilon_n^{-1}(T_n - \mathrm{Id}) f_n - \pi_n A f\| \to 0$. Then the theorem of [46, Chapter 1, Section 6] states that for all $f \in L$ and $T > 0$,

$$\sup_{0\le t\le T} \|T_n^{\lfloor t/\varepsilon_n \rfloor} \pi_n f - \pi_n T(t) f\| \longrightarrow 0.$$

Proof of Lemma 18.1. It suffices to prove (18.3) along an arbitrary sequence $h_n \downarrow 0$. We write $A = A_{h_n}$, $\delta = \delta_{h_n}$, and $\iota = \iota_{h_n}$ when the index is clear.

We begin by renorming. Let $C_1 \geq 1$ be the constant of (18.1) with $T = 1$, suppose that $\delta \leq 1/2$, and let $n_1 = \lfloor 1/\delta \rfloor$, so that $n_1\delta \geq 1/2$. Writing $n = jn_1 + i$ with $0 \leq i < n_1$, we have $j \leq 2n\delta$, and therefore

$$\|A^n\| \leq \|A^{n_1}\|^j \|A^i\| \leq C_1^{1+2n\delta} \qquad \text{for all } n \geq 0.$$

Let $M_0 \geq 1$ and $\alpha_0 \in \mathbb{R}$ be such that $\|U_t\| \leq M_0 e^{\alpha_0 t}$ for all $t \geq 0$, and fix $\alpha \geq \max(2\log C_1, \alpha_0)$. We define the norms

$$\|g\|_\sharp = \sup_{n\geq 0} e^{-\alpha n\delta}\|A^n g\|, \qquad \|f\|_* = \sup_{t\geq 0} e^{-\alpha t}\|U_t f\|.$$

By the two growth bounds,

$$\|g\| \leq \|g\|_\sharp \leq C_1\|g\|, \qquad \|f\| \leq \|f\|_* \leq M_0\|f\|,$$

uniformly along the sequence. In these norms, $e^{-\alpha\delta}A$ is a contraction, and $e^{-\alpha t}U_t$ is a strongly continuous contraction semigroup. Its generator is $L - \alpha$, with the same domain, and $\mathcal{D}$ is a core for it. The maps ι remain uniformly bounded.

Next, we verify the hypothesis of the cited theorem with $L_n = \mathcal{H}_{h_n}$, $\pi_n = \iota_{h_n}$, $\varepsilon_n = \delta_{h_n}$, $T_n = e^{-\alpha\delta}A$, the generator $L - \alpha$, the core $\mathcal{D}$, and $f_n = \iota f$. For $f \in \mathcal{D}$, a direct computation gives

$$\begin{aligned}\delta^{-1}(e^{-\alpha\delta}A - \mathrm{Id})\iota f - \iota(L-\alpha)f = e^{-\alpha\delta}\big[\delta^{-1}(A\iota f - \iota f) - \iota L f\big] \\ + \big[\delta^{-1}(e^{-\alpha\delta} - 1) + \alpha\big]\iota f + (e^{-\alpha\delta} - 1)\iota L f.\end{aligned}$$

The first term tends to zero by (18.2), and the other two tend to zero since ι is uniformly bounded. The cited theorem therefore applies. Since $\lfloor t/\delta \rfloor = n$ at $t = n\delta$, its conclusion implies that

$$\sup_{0\leq n\delta\leq T} e^{-\alpha n\delta}\|A^n\iota f - \iota U_{n\delta} f\|_\sharp \longrightarrow 0.$$

Multiplying by $e^{\alpha n\delta} \leq e^{\alpha T}$ and using $\|\cdot\| \leq \|\cdot\|_\sharp$, we obtain (18.3). □

We also need the conclusion uniformly in parameters. Let $\mathcal{X}$ be a compact metric space of parameters (in our application, energies, sources, and aspect ratios). In the application the spaces depend on the parameters, but they can be identified with one another: we say that a family of Banach spaces $\mathcal{H}_{h,\mathfrak{p}}$, $\mathfrak{p} \in \mathcal{X}$, is *uniformly identified* with a Banach space $\mathcal{H}_h$ if there exist linear isomorphisms $J_{h,\mathfrak{p}} \colon \mathcal{H}_{h,\mathfrak{p}} \to \mathcal{H}_h$ with $\sup_{h,\mathfrak{p}}(\|J_{h,\mathfrak{p}}\| + \|J_{h,\mathfrak{p}}^{-1}\|) < \infty$.

Corollary 18.2. *For $\mathfrak{p} \in \mathcal{X}$, let $(U_t^{\mathfrak{p}})$, $L_{\mathfrak{p}}$, $\iota_{h,\mathfrak{p}}$, and $A_{h,\mathfrak{p}}$ be as in Lemma 18.1, on a fixed space $\mathcal{H}$ and on spaces $\mathcal{H}_{h,\mathfrak{p}}$ that are uniformly identified with spaces $\mathcal{H}_h$. Suppose that $\|U_t^{\mathfrak{p}}\| \leq M_0 e^{\alpha_0 t}$*

for all $\mathfrak{p}$ *and* t*, that* (18.1) *holds uniformly in* $\mathfrak{p}$*, that* $\sup_{h,\mathfrak{p}} \|\iota_{h,\mathfrak{p}}\| < \infty$*, that* $\mathcal{D}$ *is a common core on which* $\mathfrak{p} \mapsto L_{\mathfrak{p}} f$ *is continuous, and that* (18.2) *holds uniformly in* $\mathfrak{p}$ *for each* $f \in \mathcal{D}$*. Then* (18.3) *holds uniformly in* $\mathfrak{p} \in \mathcal{X}$*.*

Proof. Transporting by $J_{h,\mathfrak{p}}$ changes all the norms by bounded factors, and we may assume that $\mathcal{H}_{h,\mathfrak{p}} = \mathcal{H}_h$. Suppose that the conclusion fails. Then there exist $f \in \mathcal{H}$, $T > 0$, $\gamma > 0$, $h_n \downarrow 0$, and $\mathfrak{p}_n \in \mathcal{X}$ such that the supremum in (18.3) at $(h_n, \mathfrak{p}_n)$ is at least γ. By compactness we may assume that $\mathfrak{p}_n \to \mathfrak{p}$. Along this sequence, the consistency (18.2) holds with the fixed generator $L_{\mathfrak{p}}$, since

$$\|\iota_{h_n,\mathfrak{p}_n}(L_{\mathfrak{p}_n} f - L_{\mathfrak{p}} f)\| \le C\|L_{\mathfrak{p}_n} f - L_{\mathfrak{p}} f\| \longrightarrow 0, \qquad f \in \mathcal{D}.$$

Applying the proof of Lemma 18.1 to the sequence $(A_{h_n,\mathfrak{p}_n}, \iota_{h_n,\mathfrak{p}_n})$ and the semigroup $U^{\mathfrak{p}}$, we obtain

$$\sup_{0 \le j\delta_{h_n} \le T} \|A^j_{h_n,\mathfrak{p}_n} \iota_{h_n,\mathfrak{p}_n} f - \iota_{h_n,\mathfrak{p}_n} U^{\mathfrak{p}}_{j\delta_{h_n}} f\| \longrightarrow 0.$$

The generators $L_{\mathfrak{p}_n}$ converge to $L_{\mathfrak{p}}$ on the common core $\mathcal{D}$, and the semigroups satisfy a common exponential bound. By the Trotter–Kato theorem for semigroups with a common exponential bound [72, 131], whose conclusion is uniform for t in bounded intervals, we have $\sup_{t \le T} \|U^{\mathfrak{p}_n}_t f - U^{\mathfrak{p}}_t f\| \to 0$. Since the maps $\iota_{h_n,\mathfrak{p}_n}$ are uniformly bounded, the supremum in (18.3) at $(h_n, \mathfrak{p}_n)$ tends to zero, which is a contradiction. □

18.2 The normal fixed state

In this section we construct the invariant state Ω_h of the frozen normal transfer. It is the vector in the embedding $\iota_h f = \Omega_h \otimes f$, and the initial layer of Section 20 moves both endpoints of the chain to it. The idea is simple. Since $\mathbf{F}_h$ preserves the mean on closed vectors and contracts closed vectors of zero mean, its restriction to the closed vectors of mean one is a contraction of a complete metric space, and Ω_h is the fixed point of this restriction.

Lemma 18.3. *In the balanced signature* $p = k$*, there exists a unique* $\Omega_h \in \mathcal{K}$ *such that* $\ell_0(\Omega_h) = 1$ *and* $\mathbf{F}_h \Omega_h = \Omega_h$*. It is annihilated by every normal charge. For each prescribed graph order* m*,*

$$\mathbf{P}_0 \Omega_h = \Omega_0, \qquad \|\Omega_h - \Omega_0\|_{\mathcal{G}^m} \le C_m h. \tag{18.4}$$

For all $f \in \mathcal{K} \cap \mathcal{G}^m$ *and* $j \ge 0$*,*

$$\|\mathbf{F}_h^j f - \ell_0(f)\Omega_h\|_{\mathcal{G}^m} \le e^{-c_m j h^2} \|f - \ell_0(f)\Omega_h\|_{\mathcal{G}^m}, \tag{18.5}$$

where c_m *is the rate in* (16.53)*. The same bound holds for* $\mathbf{F}_h \otimes \mathrm{Id}$ *on vectors with values in a fixed auxiliary Hilbert space. For each* h*, the vector* Ω_h *is smooth and supported in the support*

of the output cutoff of $\mathbf{F}_h$*, and its normalized analytic Berezin pairing with itself in the normal variables satisfies*

$$\mathcal{B}_{\mathrm{nor}}(\Omega_h, \Omega_h) = 1. \tag{18.6}$$

Proof. We first construct Ω_h. By (N1), $\mathbf{F}_h$ maps $\mathcal{K}$ into itself and preserves ℓ_0 there. Since $\mathfrak{q}_{\mathrm{nor}}\Omega_0 = 0$ and $\ell_0(\Omega_0) = 1$, it maps the closed affine subset

$$\Omega_0 + \mathcal{K}_0 = \{f \in \mathcal{K} : \ell_0(f) = 1\}$$

into itself. Fix a prescribed graph order m. Note that $(\Omega_0 + \mathcal{K}_0) \cap \mathcal{G}^m$ is closed in $\mathcal{G}^m$, since $\mathfrak{q}_{\mathrm{nor}}$ is closed and ℓ_0 is bounded, and that $\mathbf{F}_h$ maps it into itself by (N2). The difference of two of its points lies in $\mathcal{K}_0 \cap \mathcal{G}^m$. Using (16.53) with $j = 1$, we obtain

$$\|\mathbf{F}_h f - \mathbf{F}_h g\|_{\mathcal{G}^m} \leq e^{-c_m h^2} \|f - g\|_{\mathcal{G}^m}, \qquad f, g \in (\Omega_0 + \mathcal{K}_0) \cap \mathcal{G}^m.$$

By Banach's fixed point theorem, $\mathbf{F}_h$ has a unique fixed point Ω_h in this set, and

$$\|\Omega_h - \Omega_0\|_{\mathcal{G}^m} \leq \frac{\|\mathbf{F}_h\Omega_0 - \Omega_0\|_{\mathcal{G}^m}}{1 - e^{-c_m h^2}} \leq \frac{C_m h^3}{c_m h^2/2},$$

where we used (N2) and $1 - e^{-x} \geq x/2$ for $0 \leq x \leq 1$. The fixed point is unique already in $\Omega_0 + \mathcal{K}_0$, by the case $m = 0$, and it does not depend on m. Finally, we have $\mathbf{P}_0\Omega_h = \ell_0(\Omega_h)\Omega_0 = \Omega_0$, which completes the proof of (18.4).

Next, we prove that the normal charges annihilate Ω_h. Fix a prescribed order $m \geq 1$. The normal charges are first-order differential operators in the normal variables with linear coefficients, hence bounded from $\mathcal{G}^1$ to $\mathcal{H}_{\mathrm{nor}}$. The vectors of $(\Omega_0 + \mathcal{K}_0) \cap \mathcal{G}^m$ that they all annihilate therefore form a closed subset of $\mathcal{G}^m$. This subset contains Ω_0, and $\mathbf{F}_h$ maps it into itself, since the normal Gaussian, the half-weight, and the invariant cutoff commute with the normal charges. By uniqueness, the fixed point of $\mathbf{F}_h$ in this subset is Ω_h.

We now prove the relaxation bound. For $f \in \mathcal{K} \cap \mathcal{G}^m$, we have $f - \ell_0(f)\Omega_h \in \mathcal{K}_0 \cap \mathcal{G}^m$ and

$$\mathbf{F}_h^j f - \ell_0(f)\Omega_h = \mathbf{F}_h^j\big(f - \ell_0(f)\Omega_h\big),$$

The bound (18.5) therefore follows from (16.53), applied to $f - \ell_0(f)\Omega_h$. For vectors with values in an auxiliary Hilbert space, the same argument applies to $(\mathbf{F}_h|_{\mathcal{K}_0})^j \otimes \mathrm{Id}$, which has the same norm as $(\mathbf{F}_h|_{\mathcal{K}_0})^j$.

Finally, we prove (18.6). For fixed $h > 0$, the Gaussian kernel of $\mathbf{F}_h$ maps $\mathcal{H}_{\mathrm{nor}}$ into smooth functions, and its output cutoff is smooth. The identity $\Omega_h = \mathbf{F}_h\Omega_h$ therefore shows that Ω_h is smooth and supported in the support of that cutoff. Then Ω_h^2 is smooth, compactly supported and closed, since $\mathfrak{q}_{\mathrm{nor}}$ is an odd derivation. Applying (N3) to Ω_h and to Ω_h^2, we find that $\ell_0(\Omega_h)$ and $\mathcal{B}_{\mathrm{nor}}(\Omega_h, \Omega_h)$ are the values at the origin of the bodies of Ω_h and Ω_h^2. Since the body of Ω_h^2 is the square of the body of Ω_h, we obtain

$$\mathcal{B}_{\mathrm{nor}}(\Omega_h, \Omega_h) = \ell_0(\Omega_h)^2 = 1. \qquad \square$$

Only finitely many graph orders are used. They supply the normal moments needed in the consistency calculation, and they make the contributions of the annular parts of cutoffs smaller than every prescribed power of h. Indeed, let a be a smooth function of the unscaled normal coordinate with bounded derivatives, supported where $|U| \geq r_1$ for some $r_1 > 0$. On its support we have $|Z| \geq W^{1/2} r_1 = r_1/h$, and every commutator of multiplication by a with $\mathbf{N}$ is again supported there. For every $L \geq 1$ such that $m + L$ is among the prescribed orders, it follows that

$$\|a\,\Omega_h\|_{\mathcal{G}^m} \leq C_{m,L}(r_1)\,h^{2L}\|\Omega_h\|_{\mathcal{G}^{m+L}} \leq C_{m,L}(r_1)\,h^{2L}. \tag{18.7}$$

We include these orders before the tube is chosen. The same bound holds after finitely many derivatives of a, and for the nilpotent coefficients of a smooth invariant cutoff.

18.3 The heat semigroup on the tube

The analytic estimates in this section hold in every root signature $p \in \{0, \dots, 2k\}$.

In this section we prove three facts about one heat link on the adapted tube. First, Lemma 18.4 shows that the coefficients of the horizontal operator are uniform in group charts and that its form is sectorial. Second, Lemma 18.6 shows that the Gaussian kernel of the link agrees, on vectors supported inside the tube, with the evolution of the Dirichlet problem on a slightly larger tube, also on moving complex normal graphs. Third, Lemma 18.7 bounds this evolution uniformly over the duration of one link. The first two facts are also the analytic inputs of the weighted energy estimate of Section 19. The identification uses a lower bound for the real part of the Gaussian exponent between points of the tube, Lemma 18.5.

The generator

Recall from Section 4 that the ambient heat semigroup K_t has the kernel $\exp(-\operatorname{STr}(S - S')^2/(4t))$. Its generator is the ambient flat Laplacian Δ_{amb} of the bilinear form $\operatorname{STr}(\mathrm{d}S^2)$, in the sense that the kernel satisfies $\partial_t k_t = \Delta_{\mathrm{amb}} k_t$, and one link is $K_{\tau_{\mathrm{phys}}}$. Recall from Lemma 16.2 that the tubular map has Berezinian ϖ^2, where the half-density ϖ is analytic and normal-harmonic on the tube by Lemma 16.3. By (16.29), the half-density conjugation of Δ_{amb} is the exact decomposition

$$\varpi \Delta_{\mathrm{amb}} \varpi^{-1} = \Delta_{\mathrm{nor}} + L_H(U). \tag{18.8}$$

Here Δ_{nor} is the flat Laplacian of the normal variables, ordinary and odd, and $L_H(U) = \operatorname{div}_{\nu,\mathbb{D}}(g_S^{-1}\mathbb{D})$ is the horizontal operator of (16.29), where g_S denotes the orbit metric

$$g_S(Y, Y') = \operatorname{STr}\big([Y, S][Y', S]\big), \qquad S = R_* + U, \tag{18.9}$$

on the tangent directions $Y \in \mathfrak{m}$. By Lemma 18.4(a) below, the ordinary second-order part of $L_H(U)$ is given by the inverse of g_S. All normal derivatives in $L_H(U)$ come from the normal charges of Section 16.3, which form the infinitesimal action of the stabilizer K on the normal variables. Since ϖ is normal-harmonic, the decomposition has no additional scalar potential.

We now fix the chart in which the coefficients are uniform along the noncompact directions of the saddle manifold. Recall that the group G_0 of Section 5.2 is the body of the symmetry supergroup $U(k,k|2k)$; it is a real reductive group and hence unimodular. Let K_0 be the body of the stabilizer K of Section 16.3, and let $\mathfrak{m}_0$ and $\mathfrak{m}_1$ be the even and odd parts of the tangent space $\mathfrak{m}$. We call the coordinates

$$(g_0, \xi, U) \longmapsto R\big(g_0 \exp(\xi) K, U\big), \qquad g_0 \in G_0,\ \xi \in \mathfrak{m}_1,\ U \text{ normal}, \tag{18.10}$$

the *group chart* of the tube; it is the analogue for the tube of the translated charts of Section 5.2. Functions on the tube are functions of (g_0, ξ, U) that are invariant under the right action of K_0, which acts on g_0 by right multiplication and on (ξ, U) by conjugation. Let (Y_a) be a basis of $\mathfrak{m}_0$ that is orthonormal for the positive form $4s_E^2\,\mathrm{STr}(YY')$ on the real tangent space, and let X_a be the corresponding left-invariant vector fields on G_0, so that $X_a f(g_0)$ is the derivative of $t \mapsto f(g_0 e^{tY_a})$ at $t = 0$.

Lemma 18.4. *At fixed rank k, there exist $r_k > 0$, a degree-metric parameter $\varkappa_k \in (0,1]$ and a constant C_k such that the following holds for every block-diagonal normal coordinate U, real or complex, with $|U| \le r_k$, and for all sufficiently small h. Give the coefficient space the positive metric in which the standardized odd normal monomials and the monomials $\varkappa_k^{-|I|}\xi^I$ in the odd tangent coordinates are orthonormal. Then*

(a) in the group chart, the coefficients of $L_H(U)$ do not depend on $g_0 \in G_0$, and its ordinary second-order part is $\sum_{a,b} X_a\, g^{ab}(U,\xi)\, X_b$, where the matrix g^{ab} is the inverse of the orbit metric (18.9) *in the basis (Y_a) and its body at $U = 0$ is the identity;*

(b) for all u in the form domain of the Dirichlet problem on the tube of radius r_k, writing $\mathcal{E}_{\tan}(u) = \sum_a \|X_a u\|^2$ and $\mathcal{E}_{\mathrm{nor}}(u) = \|\nabla_U u\|^2$ for the ordinary tangent and normal energies in physical units,

$$\begin{aligned} \mathrm{Re}\langle u, L_H(U)u\rangle &\le -\tfrac{1}{2}\mathcal{E}_{\tan}(u) + C_k r_k\, \mathcal{E}_{\mathrm{nor}}(u) + C_k h^{-2}\|u\|^2, \\ |\,\mathrm{Im}\langle u, L_H(U)u\rangle| &\le C_k(r_k + \varkappa_k + h)\, \mathcal{E}_{\tan}(u) + C_k r_k\, \mathcal{E}_{\mathrm{nor}}(u) + C_k h^{-2}\|u\|^2. \end{aligned} \tag{18.11}$$

The constants do not depend on h or on the point of the saddle manifold. The same bounds hold in every fixed polynomial body weight v^p, with C_k replaced by $C_{k,p}$. They also hold if the squared norm of each standardized odd normal monomial of degree j is multiplied by a prescribed positive number d_j, with the constants and the choices of r_k and $\varkappa_k$ allowed to depend on these fixed degree weights. A common choice may be made for any prescribed finite collection of such metrics.

The term $C_k h^{-2}\|u\|^2$ comes from the odd normal variables, which also give the odd part of Δ_{nor} its norm $C_k h^{-2}$. In every application this lower-order term is integrated over at most $O(W)$ links of duration τ_{phys}, where it contributes a factor e^{C_k}. Only the normal block of the generator needs a margin that does not depend on the rank, and we treat that block explicitly where it is used.

Proof. We begin with (a). The operator Δ_{amb} commutes with conjugation by G, since the bilinear form $\mathrm{STr}(\,\mathrm{d}S^2)$ and the measure $\mathrm{d}S$ are invariant under it. The tubular map is equivariant, $gR(x,U)g^{-1} = R(gx,U)$, and the half-density ϖ depends only on U. Therefore $\varpi\Delta_{\mathrm{amb}}\varpi^{-1}$, and with it $L_H(U) = \varpi\Delta_{\mathrm{amb}}\varpi^{-1} - \Delta_{\mathrm{nor}}$, commutes with the left action of G_0, which is $g_0 \mapsto g_1 g_0$ in the group chart. In the frame of left-invariant fields, an operator that commutes with all left translations has coefficients that do not depend on g_0. In this frame, the horizontal part of the tubular differential in the direction Y_a is $[Y_a, S]$, and the ordinary principal part is the inverse of the matrix $g_S(Y_a, Y_b)$.

The first-order part consists of the fields X_a, the odd tangent derivatives, and the infinitesimal stabilizer action on U, which is given by the even normal charges $[\mathfrak{e}, U]\cdot\nabla_U$ with even $\mathfrak{e} \in \mathfrak{k}$, and by odd ones that exchange ordinary and odd normal coordinates. All coefficients are polynomial in the odd variables, since there are finitely many of them. They are analytic in U, since (18.9) is analytic in S and root separation keeps it invertible on the tube.

It remains to compute the principal part at $U = 0$. The scalar part $\kappa_E\mathrm{Id}$ of R_* commutes with Y, and for off-diagonal Y we have $\Sigma Y = -Y\Sigma$. Hence

$$[Y, R_*] = 2\mathrm{i}s_E\Sigma Y, \qquad [Y, R_*][Y', R_*] = -4s_E^2\Sigma Y\Sigma Y' = 4s_E^2 YY',$$

and on the body, the metric (18.9) at $U = 0$ satisfies $g_{R_*}(Y, Y') = 4s_E^2\,\mathrm{STr}(YY')$. On the real tangent space of the saddle manifold, the body of $\mathrm{STr}(YY')$ is positive definite. Indeed, for $Y_{BB} = \left(\begin{smallmatrix} 0 & B \\ B^* & 0 \end{smallmatrix}\right)$ in the BB block and $Y_{FF} = \left(\begin{smallmatrix} 0 & B \\ -B^* & 0 \end{smallmatrix}\right)$ in the FF block, we have

$$\mathrm{Tr}(Y_{BB}^2) = 2\,\mathrm{Tr}(BB^*), \qquad -\mathrm{Tr}(Y_{FF}^2) = 2\,\mathrm{Tr}(BB^*).$$

In our basis the body of $g^{ab}(0,0)$ is therefore the identity matrix, which proves (a).

Next, we bound the coefficients on a fixed analytic tube supplied by root separation, before decreasing its radius. By analyticity, the body of $g^{ab}(U,0)$ differs from the identity by at most $C_k|U|$; this covers its imaginary part as well, since U may be complex. The nonbody part of g^{ab} is a polynomial without constant term in the odd variables. Multiplication by an odd tangent coordinate has norm $\varkappa_k$, while multiplication by an unscaled odd normal coordinate has norm $C_d h$ in either of the stated normal metrics. Thus the nonbody principal coefficient has norm at most $C_{k,d}(\varkappa_k + h)$. An odd tangent derivative has norm $\varkappa_k^{-1}$, which is fixed before the final tube radius is chosen.

We estimate normal charges using their positive Hilbert adjoints. Writing U for the ordinary normal coordinates in the following formulas, an even charge and an odd charge have, respectively, the forms

$$R_{\mathrm{even}} = \sum_{i,j} U_i C_{ij} \partial_{U_j} + D, \qquad R_{\mathrm{odd}} = h \sum_j A_j \partial_{U_j} + h^{-1} \sum_j U_j B_j,$$

with fixed finite coefficient matrices; the matrix D includes the action on the odd normal coordinates. Their formal Hilbert adjoints in the flat positive metric satisfy the same estimates. If a is a coefficient matrix with bounded ordinary derivatives, then, on the tube of radius r,

$$\|R(au)\| + \|R^*(au)\| \le C\big((r+h)\|\nabla_U u\| + (1 + r/h)\|u\|\big). \tag{18.12}$$

The constants are uniform as r decreases inside the initial analytic tube. In particular, on compactly supported smooth vectors,

$$\langle u, aR_1R_2u\rangle = \langle R_1^*(a^*u), R_2u\rangle,$$

so a product of two charges contributes at most $C(r+h)^2\mathcal{E}_{\mathrm{nor}}(u) + Ch^{-2}\|u\|^2$ for $r, h \le 1$. Mixed terms with one X_a and one charge are estimated by the same inequality and Young's inequality. No skew-adjointness of odd charges is used.

Each X_a is skew-adjoint on $L^2(G_0)$, since G_0 is unimodular. Expand the products of horizontal fields in (16.28) before integrating by parts in the ordinary variables. We obtain

$$\langle u, L_H(U)u\rangle = -\sum_{a,b} \langle X_a u, g^{ab} X_b u\rangle + \mathcal{R}(u),$$

where $\mathcal{R}(u)$ contains the terms involving charges, odd tangent derivatives, and derivatives of coefficients. By the principal coefficient bounds,

$$\begin{aligned} \mathrm{Re} \sum_{a,b} \langle X_a u, g^{ab} X_b u\rangle &\ge \big(1 - C_k r_k - C_k(\varkappa_k + h)\big)\mathcal{E}_{\tan}(u), \\ \Big|\mathrm{Im} \sum_{a,b} \langle X_a u, g^{ab} X_b u\rangle\Big| &\le C_k(r_k + \varkappa_k + h)\mathcal{E}_{\tan}(u), \end{aligned}$$

and, for every fixed $\varepsilon > 0$, (18.12) and Young's inequality give

$$|\mathcal{R}(u)| \le \varepsilon \mathcal{E}_{\tan}(u) + C_{\varepsilon,k,d,\varkappa_k}(r+h)^2\mathcal{E}_{\mathrm{nor}}(u) + C_{\varepsilon,k,d,\varkappa_k} h^{-2}\|u\|^2.$$

Fix $\varkappa_k$ sufficiently small for the ordinary tangent principal part. Use $\varepsilon = 1/4$ for the real part and $\varepsilon = \varkappa_k$ for the imaginary part, then decrease r_k and take $h \le r_k$ sufficiently small. Since $(r_k + h)^2 \le 4r_k^2 \le 4r_k$, this proves (18.11), with the constants chosen after the coefficient metric and uniformly as the tube is decreased. In particular, its normal-energy

loss can be absorbed by the flat normal Laplacian after a further decrease of r_k. A body weight v^p adds first- and zeroth-order terms with bounded coefficients, since $\log v$ has bounded derivatives along the saddle manifold; the same estimates apply. Fixed degree weights d_j change only the constants in odd multiplication, odd differentiation, and their adjoints, which proves the assertion about those metrics. □

Gaussian separation

To identify the actual heat with the Dirichlet heat, we need a lower bound for the real part of the Gaussian exponent between a point near the core of the tube and a point near its boundary. The bound must hold uniformly along the noncompact directions, and it must allow complex normal coordinates of small height, since Section 19 moves the normal coordinates to complex graphs. Recall from Definition 16.1 that the adapted real normal space of root σ is $e^{-\mathrm{i}\theta_\sigma}$ times the real form, which consists of the Hermitian matrices in the BB block, of i times the Hermitian matrices in the FF block, and of free odd entries. Here $\theta_\sigma = \frac{1}{2}\arg c_\sigma$ is the phase of (16.13), and $|\theta_\sigma| \le C|E|$. For the body of a normal coordinate U we write

$$U = e^{-\mathrm{i}\theta_\sigma}(u + \mathrm{i}w)$$

on each root block, with u and w in the real form, and we call u the real part and w the height of U.

Lemma 18.5. *There exist universal constants $r_{\mathrm{sep}}, E_{\mathrm{sep}}, c > 0$ such that the following holds for $|E| \le E_{\mathrm{sep}}$. Let x, x' be points of the real saddle manifold, and let U, U' be normal coordinates with zero odd part, with real parts u, u' and heights w, w', such that $|U|, |U'| \le r_{\mathrm{sep}}$. Then*

$$\operatorname{Re}\operatorname{STr}\big(R(x,U) - R(x',U')\big)^2 \ge \tfrac{1}{4}\big(|u| - |u'|\big)^2 - 4\big(|w|^2 + |w'|^2\big) + c\, s_E^2\, \mathfrak{d}(x,x'), \quad (18.13)$$

where $\mathfrak{d}(x,x') \ge 0$ is a function that vanishes only for $x = x'$ and grows exponentially in the distance between the BB parts of x and x'.

Proof. The scalar part $\kappa_E \mathrm{Id}$ of R_* cancels in the difference, and we may replace R_* by $-\mathrm{i}s_E\Sigma$. It suffices to treat the two blocks separately, since the body of STr is the trace of the BB block minus the trace of the FF block.

We begin with the BB block, which carries the noncompact directions. Let $g, g' \in U(k,k)$ be representatives of the BB parts of x and x', and write $S = R_* + U$ and $S' = R_* + U'$ for the BB blocks, so that $R(x,U)^{\mathrm{BB}} = gSg^{-1}$. Conjugation by g^{-1} does not change the trace, and we have

$$\operatorname{Tr}\big(gSg^{-1} - g'S'g'^{-1}\big)^2 = \operatorname{Tr}\big(S - g''S'g''^{-1}\big)^2, \qquad g'' = g^{-1}g'.$$

We use the Cartan decomposition $g'' = k_1 e^A k_2$, with k_1, k_2 in the stabilizer $U(k) \times U(k)$ and A Hermitian and off-diagonal. Replacing S by $k_1^{-1}Sk_1$ and S' by $k_2 S' k_2^{-1}$ preserves $|u|$,

$|u'|$, $|w|$, $|w'|$ and the block structure, and we may assume that $g'' = e^A$. Since $\mathrm{Tr}(S'^2) = \mathrm{Tr}((e^A S' e^{-A})^2)$, and since in $\mathrm{Tr}(S\, e^{\mathrm{ad}_A} S')$ only the even powers of ad_A contribute (odd powers map block-diagonal matrices to off-diagonal ones), we have

$$\mathrm{Tr}\big(S - e^A S' e^{-A}\big)^2 = \mathrm{Tr}(S - S')^2 - 2\,\mathrm{Tr}\big(S(\cosh(\mathrm{ad}_A) - 1)S'\big). \tag{18.14}$$

We now compute the main term. On the BB block we have $R_* = -\mathrm{i}s_E\Lambda$. Since $A\Lambda = -\Lambda A$, we have $\mathrm{ad}_A^2\,\Lambda = 4\Lambda A^2$, and by induction $(\cosh(\mathrm{ad}_A) - 1)\Lambda = \Lambda(\cosh(2A) - \mathrm{Id})$. Therefore

$$-2\,\mathrm{Tr}\big(R_*(\cosh(\mathrm{ad}_A) - 1)R_*\big) = 2s_E^2\,\mathrm{Tr}\big(\cosh(2A) - \mathrm{Id}\big) = 2s_E^2\,\mathfrak{d}_B(A),$$

where we define

$$\mathfrak{d}_B(A) = \mathrm{Tr}\big(\cosh(2A) - \mathrm{Id}\big).$$

The quantity $\mathfrak{d}_B(A)$ is real and nonnegative, since A is Hermitian, and it vanishes only for $A = 0$ and grows exponentially in $\|A\|$.

Next, we bound the cross terms in (18.14). Using that the operator ad_A^2 is symmetric for the bilinear form $\mathrm{Tr}(YY')$, we find that the two terms that contain one factor R_* are $2\mathrm{i}s_E\,\mathrm{Tr}(\Lambda(\cosh(2A) - \mathrm{Id})U')$ and the same expression with U. The matrix $\Lambda(\cosh(2A) - \mathrm{Id})$ is a positive matrix times a unitary one, with trace norm $\mathfrak{d}_B(A)$. It follows that

$$\begin{aligned}&\big|2\mathrm{i}s_E\,\mathrm{Tr}\big(\Lambda(\cosh(2A) - \mathrm{Id})U'\big)\big| + \big|2\mathrm{i}s_E\,\mathrm{Tr}\big(\Lambda(\cosh(2A) - \mathrm{Id})U\big)\big| \\ &\qquad \le 2s_E\big(\|U\| + \|U'\|\big)\mathfrak{d}_B(A).\end{aligned}$$

For the last term, we use that as an operator on Hilbert–Schmidt matrices,

$$\|\cosh(\mathrm{ad}_A) - 1\| \le \cosh(2\|A\|) - 1 \le \mathfrak{d}_B(A),$$

which bounds that term by $2\|U\|\|U'\|\mathfrak{d}_B(A)$. For $r_{\mathrm{sep}} \le s_0/8$, where $s_0 = \inf_{|E|\le 1/2} s_E$, the cross terms are at most $s_E^2\mathfrak{d}_B(A)$ in absolute value.

It remains to bound $\mathrm{Re}\,\mathrm{Tr}(S - S')^2 = \mathrm{Re}\,\mathrm{Tr}(U - U')^2$. On a root block of type σ, we have

$$\begin{aligned}\mathrm{Re}\,\mathrm{Tr}\big(e^{-\mathrm{i}\theta_\sigma}((u - u') + \mathrm{i}(w - w'))\big)^2 &= \cos(2\theta_\sigma)\big(\|u - u'\|^2 - \|w - w'\|^2\big) \\ &\quad + 2\sin(2\theta_\sigma)\,\mathrm{Tr}\big((u - u')(w - w')\big) \\ &\ge \tfrac{1}{2}\|u - u'\|^2 - 2\|w - w'\|^2,\end{aligned}$$

where the inequality holds once $|\theta_\sigma| \le 1/10$, which follows from $|\theta_\sigma| \le C|E|$ for E_{sep} small. Since $\|u - u'\| \ge \big|\,|u| - |u'|\,\big|$, the BB block satisfies (18.13) with $\mathfrak{d} = \mathfrak{d}_B$, with the BB parts of u, u', w, and w'.

The FF block is compact, and the argument is the same. The points of the FF block are i times Hermitian matrices up to the shift by κ_E, and on the FF block $-\mathrm{Tr}$ is the

positive part of STr. In the Cartan decomposition of $U(2k)$ relative to the FF stabilizer, A is anti-Hermitian, $A = \mathrm{i}B$ with B Hermitian, and $\cosh(2A) = \cos(2B)$. The main term becomes $2s_E^2 \operatorname{Tr}(\mathrm{Id} - \cos(2B)) \geq 0$. The Cartan decomposition can be chosen with $\|B\| \leq \pi/2$, and the cross terms are then bounded by a small multiple of the main term, as before. Adding the BB and FF blocks, and using $(x+y)^2 \leq 2x^2 + 2y^2$ to combine the norms of the two blocks, we conclude that (18.13) holds. □

Identification with the Dirichlet heat

We now compare one link of the actual kernel with the Dirichlet evolution of the same generator. We state the comparison on moving normal graphs, which is the form used in Section 19; the static case is the case of zero velocity.

Let ψ be a smooth nonincreasing function on $[0,\infty)$, equal to one on $[0,1]$ and to zero on $[4,\infty)$, let $\bar{r} > 0$, and set $\psi_{\bar{r}}(U) = \psi(\|U\|_{\mathrm{HS}}^2/\bar{r}^2)$. For a real normal direction b and $\lambda \in [0,1]$, we define the *compact normal graph*

$$U_\lambda = U + \mathrm{i}\lambda\psi_{\bar{r}}(U)b, \qquad \rho_\lambda = \det \frac{\partial U_\lambda}{\partial U}. \tag{18.15}$$

It moves only the body of the normal coordinate and is the identity outside a compact set; we evaluate functions of the normal variable at the complex points U_λ instead of U. This is an analytic distortion of the kind used by Hunziker [69]. The derivative $D\psi_{\bar{r}}$ has Hilbert–Schmidt operator norm at most $C/\bar{r}$, with C independent of the matrix dimension, and $\partial U_\lambda/\partial U - \mathrm{Id}$ has rank one and norm at most $C|b|/\bar{r}$.

Fix a link $[t_0, t_0 + \tau_{\mathrm{phys}}]$ and numbers $\lambda_0, \lambda_1 \in [0,1]$, and let $\lambda(t)$ interpolate linearly between them over the link. Let $\mathcal{T}_{\mathrm{in}} \subset \mathcal{T}_{\mathrm{out}}$ be the tubes of normal radii $r_{\mathrm{in}} < r_{\mathrm{out}}$, and suppose that $r_{\mathrm{in}} \leq 3r_{\mathrm{out}}/4$ and $16\bar{r} \leq r_{\mathrm{out}}$, $4\bar{r} \leq r_{\mathrm{in}}$, so that the graphs are the identity outside $\mathcal{T}_{\mathrm{in}}$. Let $y = (g_0, \xi, U)$ denote a point of the group chart and $\mathrm{d}y$ the chart measure, and set

$$R_\lambda(y) = R\big(g_0 \exp(\xi) K, U_\lambda\big), \qquad (\varpi\rho^{1/2})_\lambda(y) = \varpi(U_\lambda)\,\rho_\lambda(U)^{1/2}$$

for the corresponding point of the complexified tube on the graph of parameter λ and for the half-density of that graph. For a coefficient function f supported in $\mathcal{T}_{\mathrm{in}}$, we define the *pulled Gaussian evolution*

$$u(t,y) = \int k_{t-t_0}\big(R_{\lambda(t)}(y), R_{\lambda_0}(y')\big)\,(\varpi\rho^{1/2})_{\lambda(t)}(y)\,(\varpi\rho^{1/2})_{\lambda_0}(y')\,f(y')\,\mathrm{d}y', \qquad t > t_0. \tag{18.16}$$

To specify the half-density convention, write $\Phi_\lambda(U) = U_\lambda$ and define the graph pull on raw half densities by

$$(\mathcal{P}_\lambda F)(g_0, \xi, U) = \rho_\lambda(U)^{1/2} F(g_0, \xi, \Phi_\lambda(U)).$$

For $\lambda_0 > 0$, the input in (18.16) has the form $f = \mathcal{P}_{\lambda_0} F$, where F is holomorphic in the normal variables wherever the graph is deformed. Thus $f/\rho_{\lambda_0}^{1/2}$ is the raw graph

value; the second half-Jacobian needed for the graph measure is supplied by f. For $\lambda_0 = 0$ there is no holomorphy requirement on the input. This is the convention used in Lemma 19.8. The factor ϖ has already been incorporated into the raw half density and into the generator $L_{\mathrm{raw}} = \Delta_{\mathrm{nor}} + L_H$ of (18.8).

We let $v(t)$ solve the Dirichlet problem on $\mathcal{T}_{\mathrm{out}}$ for the pulled operator,

$$\begin{aligned} &\partial_t v = (L_t + V_t)v, \qquad v(t_0) = f, \\ &L_t = \mathcal{P}_{\lambda(t)} L_{\mathrm{raw}} \mathcal{P}_{\lambda(t)}^{-1}, \\ &V_t = \mathsf{v}_t \cdot \nabla_U + \tfrac{1}{2} \operatorname{div}_U \mathsf{v}_t, \qquad \mathsf{v}_t = (D\Phi_{\lambda(t)})^{-1} \partial_t \Phi_{\lambda(t)}. \end{aligned} \tag{18.17}$$

Here the expression for L_t means the differential operator obtained by the chain rule, including conjugation by $\rho_{\lambda(t)}^{1/2}$, and is extended coefficientwise to the real graph coordinate U.

Lemma 18.6. *At fixed rank k, let $2r_{\mathrm{out}} \le \min(r_k, r_{\mathrm{sep}})$ and $|E| \le E_{\mathrm{sep}}$. There exist universal constants $c_1, c > 0$ such that the following holds for $|b| \le c_1 \bar{r}$. The forms of $L_t + V_t$ are sectorial on $\mathcal{T}_{\mathrm{out}}$, uniformly in t, and the Dirichlet problem* (18.17) *has a unique weak solution. For all f as above and $t_0 \le t \le t_0 + \tau_{\mathrm{phys}}$,*

$$\| \mathbf{1}_{\mathcal{T}_{\mathrm{in}}} (u(t) - v(t)) \| \le C_k W^{C_k} e^{-c\, r_{\mathrm{out}}^2 W^2} \| f \|. \tag{18.18}$$

The same holds in every fixed polynomial body weight, with C_k depending on the weight.

Proof. We first derive the transport of a half density. Set $q(U) = b \cdot \nabla_U \psi_{\bar{r}}(U)$, which is real. The rank-one determinant and inverse identities give

$$\rho_\lambda = 1 + \mathrm{i}\lambda q, \qquad \mathsf{v}_t = \frac{\mathrm{i}\lambda'(t) \psi_{\bar{r}} b}{1 + \mathrm{i}\lambda(t) q}, \qquad \| \mathsf{v}_t \|_\infty \le |\lambda'(t) b|. \tag{18.19}$$

In particular $\operatorname{Re} \rho_\lambda = 1$, so its square root is the branch continuous from $\rho_0^{1/2} = 1$. The Jacobian identity $\partial_t \rho_\lambda = \operatorname{div}_U(\rho_\lambda \mathsf{v}_t)$ implies

$$\dot{\mathcal{P}}_\lambda \mathcal{P}_\lambda^{-1} = \mathsf{v}_t \cdot \nabla_U + \tfrac{1}{2} \big(\partial_t \log \rho_\lambda - \mathsf{v}_t \cdot \nabla_U \log \rho_\lambda \big) = V_t.$$

The second term in parentheses is required because the transport field also differentiates the half-Jacobian. The full ambient half-density pull is $\mathcal{P}_\lambda M_\varpi$; M_ϖ is independent of t, so it adds no term to $\dot{\mathcal{P}}_\lambda \mathcal{P}_\lambda^{-1}$.

For $s > 0$, the integral of $k_s(S, \cdot)$ against the fixed input graph in (18.16) is coefficientwise holomorphic in a sufficiently small complex neighborhood of each output point $S_0 = R_{\lambda(t)}(y)$ of the tube. To see this, let x be the saddle coordinate of S_0. By Lemma 18.5 and the smallness of the graph height, the real part of $\mathrm{STr}(S_0 - R_{\lambda_0}(y'))^2$ is bounded below by $c\mathfrak{d}(x, x') - C$. A sufficiently small complex perturbation of S_0 preserves this

bound with a smaller positive coefficient of $\mathfrak{d}$: its additional terms are bounded by $C|S - S_0|(1 + \mathfrak{d}(x, x'))$. Every fixed derivative in S or s, including the finite odd expansion, introduces only a polynomial in $\mathfrak{d}$ and is absorbed by this orbit decay. Since the input has bounded normal support and the tangent measure grows at most exponentially in distance, the resulting kernels are locally uniformly in L^2 in the input variable, also in each fixed polynomial body weight. Cauchy–Schwarz therefore permits differentiation under the integral for the allowed L^2 inputs, uniformly for s in compact subsets of $(0, \infty)$. The integral solves $\partial_s = \Delta_S$ on these neighborhoods. Apply the preceding chain rule to this integral, first absorbing ϖ into its raw half density. We obtain $\partial_t u = (L_t + V_t)u$ classically on $\mathcal{T}_{\mathrm{out}}$.

We will also use the exact velocity form. In an orthonormal coefficient frame, ordinary integration by parts gives, for compactly supported f, g,

$$\int \sum_I \overline{f_I}(V_t g)_I \, \mathrm{d}y = \tfrac{1}{2} \int \mathsf{v}_t \cdot \sum_I (\overline{f_I} \nabla_U g_I - (\nabla_U \overline{f_I}) g_I) \, \mathrm{d}y. \tag{18.20}$$

This identity extends to H^1_0 coefficient vectors. It is valid for the complex field v_t and any of our fixed positive coefficient metrics. A body weight commutes with V_t. By (18.19), for every $\varepsilon > 0$,

$$|\langle u, V_t u \rangle| \le \varepsilon \mathcal{E}_{\mathrm{nor}}(u) + \frac{|\lambda' b|^2}{4\varepsilon} \|u\|^2. \tag{18.21}$$

Thus differentiating the Jacobian introduces no separate loss in this energy estimate.

We now bound u near the boundary of $\mathcal{T}_{\mathrm{out}}$. We set $d = r_{\mathrm{out}} - r_{\mathrm{in}}$, and we note that $d \ge r_{\mathrm{out}}/4$, since $r_{\mathrm{in}} \le 3r_{\mathrm{out}}/4$. Let $\tilde{\chi}$ be a smooth real function of $|U|$, equal to one where $|U| \le r_{\mathrm{in}} + d/3$ and to zero where $|U| \ge r_{\mathrm{in}} + 2d/3$, and let $\mathcal{A}$ be the collar where $\nabla \tilde{\chi} \ne 0$. For an output point in $\mathcal{A}$ and an input point in the support of f, the real parts of the normal coordinates satisfy $|u| - |u'| \ge d/3$. The output point has height zero, since the graph is the identity there, and the input point has height at most $|b| \le c_1 \bar{r} \le c_1 r_{\mathrm{out}}/16$. By Lemma 18.5, the real part of the exponent $\mathrm{STr}(\cdot)^2/(4s)$ of k_s is at least

$$\frac{1}{4s}\Big(\frac{d^2}{36} - \frac{c_1^2 r_{\mathrm{out}}^2}{64} + c s_E^2 \mathfrak{d}(x, x')\Big) \ge \frac{1}{4s}\Big(\frac{r_{\mathrm{out}}^2}{1200} + c s_E^2 \mathfrak{d}(x, x')\Big)$$

for $c_1 \le 1/10$.

The nilpotent part of the exponent is a polynomial in the odd variables whose coefficients are bounded by $C_k(1 + \mathfrak{d})$. Its exponential is a finite sum, bounded by $C_k s^{-C_k}$ times a polynomial in $\mathfrak{d}$, and a fraction of $e^{-c s_E^2 \mathfrak{d}/(4s)}$ absorbs the polynomial. The same holds for one derivative of the kernel in the output variable. Once W is large, the function $s \mapsto s^{-C_k} e^{-r_{\mathrm{out}}^2/(9600 s)}$ is increasing on $(0, \tau_{\mathrm{phys}}]$, and $\mathfrak{d}$ grows exponentially in the distance on the saddle manifold while the invariant measure grows at most exponentially. Combining these bounds with Schur's test, we obtain

$$\sup_{t_0 < t \le t_0 + \tau_{\mathrm{phys}}} \big(\|\mathbf{1}_{\mathcal{A}} u(t)\| + \|\mathbf{1}_{\mathcal{A}} \nabla u(t)\|\big) \le C_k W^{C_k} e^{-r_{\mathrm{out}}^2 W^2/4800} \|f\|, \tag{18.22}$$

where we used $\tau_{\mathrm{phys}} = (2W^2)^{-1}$. A body weight v^p changes only the constant, since v grows at most exponentially in the distance.

Next, we show that the forms are sectorial. The ordinary part of Δ_{nor} has constant coefficient matrix $D_E = \bigoplus_\sigma e^{2i\theta_\sigma}\mathrm{Id}$. If $J_\lambda = D\Phi_\lambda$, its principal coefficient on the graph is $J_\lambda^{-1} D_E J_\lambda^{-T}$; this formula also allows the graph to mix root blocks. Its distance from D_E is at most $C|b|/\bar{r} \le Cc_1$, so its form has real part at most $-\frac{1}{2}\mathcal{E}_{\mathrm{nor}}(u)$ for c_1 and $|E|$ small. Differentiating J_λ^{-1} and the half-Jacobian also gives first-order and zeroth-order coefficients. All derivatives required here are bounded at fixed $\bar{r}$, uniformly for $|b| \le c_1\bar{r}$, and their lower-order form costs are included in $C_k(\bar{r})$ below.

The horizontal part satisfies (18.11), since the normal coordinates of the graph have norm at most $r_{\mathrm{out}} + |b| \le 2r_{\mathrm{out}} \le r_k$, and its normal energy term $C_k r_k \mathcal{E}_{\mathrm{nor}}$ is at most $\frac{1}{4}\mathcal{E}_{\mathrm{nor}}$ once r_k is small. The odd normal part of Δ_{nor} is a finite matrix of norm $C_k h^{-2}$ in the standardized metric. Since $|\lambda'| \le \tau_{\mathrm{phys}}^{-1}$, (18.21) with $\varepsilon = 1/8$ gives

$$|\langle u, V_t u\rangle| \le \tfrac{1}{8}\mathcal{E}_{\mathrm{nor}}(u) + C\tau_{\mathrm{phys}}^{-2}|b|^2\|u\|^2.$$

The forms are therefore bounded on the form domain of the Dirichlet problem, which is $H_0^1(\mathcal{T}_{\mathrm{out}})$ with coefficients in the finite exterior space, and satisfy a Gårding inequality with constant

$$C_{\mathrm{G}} \le C_k(\bar{r})h^{-2} + C\tau_{\mathrm{phys}}^{-2}|b|^2, \qquad C_{\mathrm{G}}\tau_{\mathrm{phys}} \le C_k(\bar{r}) + CW^2|b|^2,$$

where the static lower-order graph terms are absorbed into $C_k(\bar{r})$. The form identity (18.20) also bounds the sesquilinear velocity form on H_0^1, without a derivative-of-Jacobian term. By the theorem of Lions on evolution equations for time-dependent forms [85], the Dirichlet problem (18.17) has a unique weak solution, and its evolution operators $\mathcal{D}(t,s)$ satisfy the energy inequality $\|\mathcal{D}(t,s)\| \le e^{C_{\mathrm{G}}(t-s)}$.

We now compare, first for smooth f. The function $w = \tilde{\chi}u$ lies in the form domain and satisfies

$$\partial_t w = (L_t + V_t)w - [L_t + V_t, \tilde{\chi}]u,$$

where the commutator is a first-order operator with coefficients supported in $\mathcal{A}$. The graph and its velocity are the identity and zero, respectively, on this outer collar, so $[V_t, \tilde{\chi}] = 0$ there.

We next check that $w(t) \to f$ as $t \downarrow t_0$. Let $\mathbf{Q}_\lambda = DR_\lambda^T DR_\lambda$ denote the pullback of the bilinear form STr by the parametrization $y \mapsto R_\lambda(y)$ of the graph of the tube, and let $s = t - t_0$. In the integral (18.16), the exponent is $-(y-y')^T\mathbf{Q}_\lambda(y)(y-y')/(4s)$ up to cubic terms in $y - y'$. The real part of $\mathbf{Q}_\lambda$ is positive definite near the real tube, and the Laplace method shows that the integral tends to $f(y)\,(\varpi^2\rho)(y)\,\mathrm{Ber}(\mathbf{Q}_\lambda(y))^{-1/2}$, where the branch is fixed by continuity from the real tube and the constants of the Gaussian are

the normalized ambient ones, since the flat Gaussian has integral one. By Lemma 16.2, the holomorphic Jacobian of the parametrization in the chart satisfies

$$\mathrm{Ber}(\mathbf{Q}_\lambda)^{1/2} = \mathrm{Ber}(DR_\lambda) = \varpi^2 \rho_\lambda,$$

and the limit is $f(y)$, in L^2 for smooth f.

By uniqueness of weak solutions, Duhamel's formula holds,

$$w(t) - v(t) = -\int_{t_0}^{t} \mathcal{D}(t,s)[L_s + V_s, \tilde{\chi}]u(s)\,\mathrm{d}s.$$

Using (18.22) and the energy inequality, we obtain

$$\|w(t) - v(t)\| \le C_k W^{C_k} e^{CW^2|b|^2 - r_{\mathrm{out}}^2 W^2/4800} \|f\|.$$

The exponent is at most $-r_{\mathrm{out}}^2 W^2/9600$ once c_1 is small enough, since $|b| \le c_1 r_{\mathrm{out}}/16$. The last display implies (18.18) for smooth f, since $w = u$ on $\mathcal{T}_{\mathrm{in}}$.

For general f, we approximate by smooth functions f_n. The evolutions v depend continuously on f, by the energy inequality. Applied to f_n, the bound (18.18) implies that

$$\|\mathbf{1}_{\mathcal{T}_{\mathrm{in}}} u_n(t)\| \le \|v_n(t)\| + C_k W^{C_k} e^{-c r_{\mathrm{out}}^2 W^2} \|f_n\|,$$

and the operators $f \mapsto \mathbf{1}_{\mathcal{T}_{\mathrm{in}}} u(t)$ are therefore uniformly bounded on smooth functions and extend by continuity. This completes the proof of (18.18) for all f. □

The heat over one link

We now prove the one-link bound for the gauged generator used in the consistency calculation. In the symmetric normal gauge of Section 16, vectors carry the Wick metric of Section 16.4, and the Gaussian gauge γ_h relates it to the flat metric. We define the gauged generator

$$\mathcal{L}_h = \gamma_h^{-1}\big(\Delta_{\mathrm{nor}} + L_H(U)\big)\gamma_h = L_{N,h} + L_{H,h}, \qquad L_{N,h} = \gamma_h^{-1}\Delta_{\mathrm{nor}}\gamma_h, \quad L_{H,h} = L_H(U). \tag{18.23}$$

The second identity holds because all normal derivatives of $L_H(U)$ come from the normal charges, which annihilate the normal class function γ_h of Lemma 16.8. We let $L_{H,0}$ denote the operator $L_H(0)$ acting on tangent functions. On vectors that are invariant under the normal charges, all normal derivatives of $L_H(0)$ vanish, and by Lemma 18.4(a), the ordinary second-order part of $L_{H,0}$ is given by the inverse of the orbit metric $4s_E^2\,\mathrm{STr}(YY')$ at the saddle manifold. In the balanced signature, Lemma 16.5 shows that $L_{H,0} = -(2s_E^2)^{-1}\mathsf{C}_k$ as operators on superfunctions on $\mathcal{M}_k$, odd coefficients included.

With the half-weight $w_0^{1/2}$ and the invariant cutoff $\chi = \chi_{\mathrm{inv}}$ of Section 16.7, define the Dirichlet versions of the unsourced and frozen symmetric steps by

$$T_h^{\mathrm{D}} = \chi w_0^{1/2}\, e^{\tau_{\mathrm{phys}}\mathcal{L}_h}\, w_0^{1/2}\chi, \qquad \mathbf{F}_h^{\mathrm{D}} = \chi w_0^{1/2}\, e^{\tau_{\mathrm{phys}}L_{N,h}}\, w_0^{1/2}\chi. \tag{18.24}$$

Here $e^{t\mathcal{L}_h}$ and $e^{tL_{N,h}}$ denote the Dirichlet evolutions on a fixed tube slightly larger than the support of χ. We retain T_h and $\mathbf{F}_h$ for the actual Gaussian steps of (16.38). Write $B_h(t)$ for the free frozen normal heat on the whole normal space, with the same differential expression $L_{N,h}$. Thus

$$\mathbf{F}_h = \chi w_0^{1/2} B_h(\tau_{\mathrm{phys}}) w_0^{1/2} \chi.$$

The fixed gap between $\operatorname{supp}\chi$ and the Dirichlet boundary gives

$$\|\mathbf{F}_h - \mathbf{F}_h^{\mathrm{D}}\| + \|T_h - T_h^{\mathrm{D}}\| \le C_k W^{C_k} e^{-cW^2+CW},$$

in every fixed body weight. For the normal step this is the Gaussian boundary estimate; for the full step it is Lemma 18.6 with the static graph $\lambda_0 = \lambda_1 = 0$, applied between the cutoffs. The factor e^{CW} bounds the Gaussian gauges on the fixed tube. These errors are smaller than every power of h. We keep them in the consistency calculation so that the exact fixed-state identity for the original $\mathbf{F}_h$ remains available.

Lemma 18.7. *At fixed rank and fixed cutoff radius $r \le r_k$, for every body weight p there exists C_p such that, in the Wick metric with body weight v^p,*

$$\sup_{0\le t\le \tau_{\mathrm{phys}}} \|e^{t\mathcal{L}_h}\| \le \exp\bigl(C_p \tau_{\mathrm{phys}} h^{-2}\bigr) \le 2 \tag{18.25}$$

for all sufficiently small h. The same base-norm bound holds for the Dirichlet frozen normal heat $e^{tL_{N,h}}$ and for the free frozen normal heat $B_h(t)$. The free heat also satisfies

$$\|B_h(t)\|_{\mathcal{G}^m \to \mathcal{G}^m} \le e^{C_m t h^{-2}}, \qquad 0 \le t \le \tau_{\mathrm{phys}}, \tag{18.26}$$

for each fixed prescribed graph order m. Consequently, for every vector g in the domain of $\mathcal{L}_h$ and $0 \le t \le \tau_{\mathrm{phys}}$,

$$\|(e^{t\mathcal{L}_h} - \mathrm{Id})g\| \le 2t\,\|\mathcal{L}_h g\|. \tag{18.27}$$

Proof. We transport the positive Wick metric through the full Gaussian gauge. For one standardized odd pair, write $F = a + b\alpha + c\beta + d\alpha\beta$. By Definition 16.6,

$$\|F\|_{\mathrm{W}}^2 = |a-d|^2 + |b|^2 + |c|^2 + |d|^2.$$

The odd Gaussian factor is $\gamma_F = 1 - \alpha\beta/2$, so

$$\|\gamma_F^{-1} F\|_{\mathrm{W}}^2 = \tfrac{1}{2}|a|^2 + |b|^2 + |c|^2 + 2|d|^2.$$

For M odd pairs the tensor product of these metrics is the flat positive coefficient metric

$$|F|_*^2 = \sum_I 2^{|I|-M} |F_I|^2 \tag{18.28}$$

in standardized odd normal monomials; the tangent degree metric is unchanged. Multiplication by γ_h is therefore unitary from the positive Wick space to flat ordinary L^2 with metric (18.28), after a common scalar normalization of the ordinary measure. It also preserves the Dirichlet boundary condition and commutes with the body weight v^p. Thus $\mathcal{L}_h$ is unitarily equivalent to $\Delta_{\mathrm{nor}} + L_H(U)$ in this flat positive metric.

We apply the degree-weighted version of Lemma 18.4; this uses its coefficient and adjoint estimates directly in (18.28), rather than transferring a numerical-range bound by equivalence of norms. The ordinary part of Δ_{nor} has form bounded above by $-c_0\mathcal{E}_{\mathrm{nor}}$, since its root coefficients are $e^{2\mathrm{i}\theta_\sigma}$ with $\cos 2\theta_\sigma$ bounded below. Its odd part is a finite matrix of norm $C_k h^{-2}$. Fix a common coefficient metric and then decrease the fixed tube radius using the unweighted estimates, so that the normal-energy loss in Lemma 18.4 is at most $c_0\mathcal{E}_{\mathrm{nor}}/2$. Conjugating by v^p adds only first-order and zeroth-order terms, since $\log v$ has bounded tangent derivatives and is independent of U. By (18.12) and Young's inequality, these terms are bounded by any fixed small fraction of the two gradient energies plus $C_p h^{-2}\|F\|_*^2$. Thus the tube radius remains independent of p, and in every fixed body weight we obtain

$$\mathrm{Re}\langle F, (\Delta_{\mathrm{nor}} + L_H)F\rangle_* \le -c\big(\mathcal{E}_{\mathrm{nor}}(F) + \mathcal{E}_{\tan}(F)\big) + C_p h^{-2}\|F\|_*^2.$$

Unitary transport and the Dirichlet energy inequality give $\|e^{t\mathcal{L}_h}\| \le e^{C_p t h^{-2}}$, with no norm-equivalence prefactor. Omitting L_H gives the same base estimate for both frozen normal evolutions, on the tube and on the whole normal space. Since $\tau_{\mathrm{phys}} = h^4/2$, this proves (18.25).

For the free normal heat on $\mathcal{G}^m$, work first on the whole-space Schwartz core. The generator is h^{-2} times a quadratic oscillator polynomial. Each monomial changes the number n by at most two with amplitude $O(n+1)$. Consequently

$$\big\|[(\mathbf{N}+1)^m, L_{N,h}](\mathbf{N}+1)^{-m}\big\| \le C_m h^{-2}.$$

Conjugating by $(\mathbf{N}+1)^m$ adds this bounded operator to the base generator. The base energy estimate, followed by closure, proves (18.26). This graph-space argument concerns the free heat and uses no Dirichlet boundary integration. Finally, (18.27) follows from

$$(e^{t\mathcal{L}_h} - \mathrm{Id})g = \int_0^t e^{\sigma\mathcal{L}_h}\mathcal{L}_h g \,\mathrm{d}\sigma$$

and (18.25). □

The bound (18.25) is of order one for a single link and cannot be iterated over the h^{-4} links of the chain. The reason is that the generator of the heat in the finitely many odd normal coordinates is nilpotent; Remark 16.11 explains why no metric makes the gauged generator dissipative. The iterable bound (N4) comes instead from the kernel of the step.

18.4 The tangent generator

We return to the balanced signature $p = k$ for the remainder of this chapter.

In this section we identify the generator that the adapted chain produces on the tangent variables. The product limit of Theorem 18.14 needs this identification. The consistency estimate of Section 18.5 expresses one adapted step on the invariant normal line through the horizontal operator $L_{H,0}$ and the source potential, combined into the operator $L_h^{\tan}$ below. Proposition 5.15 constructs the limit semigroup once $L_h^{\tan}$ is known to be a multiple of the generator of the sigma model. The identification rests on Lemma 16.5, which shows that $L_{H,0} = -(2s_E^2)^{-1}\mathsf{C}_k$. Lemma 18.8 below combines this identity with the time τ_{phys} of one link and with the source factor. As a consequence, j steps of the chain run the semigroup of the sigma model for the time j/N.

Recall the generator of the sigma model from (5.23),

$$\mathsf{G}_{X,\Gamma} = X\mathsf{C}_k + \pi\,\mathrm{STr}(Q\Gamma), \tag{18.29}$$

where C_k is the Casimir operator of Definition 5.7, Q is the field of Definition 5.2, Γ is the source supermatrix, and the second term acts by multiplication. Recall also the operator $L_{H,0}$, defined after (18.23) as $L_H(0)$ acting on tangent functions, and the source potential V_h, defined by $s_\Gamma = e^{-\delta_W V_h}$ on the tube. We let V_0 denote the value of V_h at $U = 0$, a function of the tangent variables that acts on tangent functions by multiplication, and we set

$$L_h^{\tan} = \frac{\tau_{\mathrm{phys}}}{\delta_W} L_{H,0} - V_0 = \tfrac{1}{2} L_{H,0} - V_0.$$

We will show in Proposition 18.12 that, for compactly supported smooth tangent functions f,

$$A_h(\Omega_h \otimes f) = \Omega_h \otimes f + \delta_W\, \Omega_h \otimes L_h^{\tan} f + O(\delta_W h)\|f\|_{p',m},$$

where $\|f\|_{p',m}$ is the tangent norm of Section 18.5. In this expansion, the term $\tau_{\mathrm{phys}} L_{H,0}$ comes from one heat link, which acts on $\Omega_h \otimes f$ as $\Omega_h \otimes e^{\tau_{\mathrm{phys}} L_{H,0}} f$ up to the error, and the term $\delta_W V_0$ comes from the source factor $e^{-\delta_W V_0}$ of one onsite weight on the saddle manifold.

Lemma 18.8. *In the balanced signature $p = k$, we have*

$$N\delta_W L_h^{\tan} = -\mathsf{G}_{X_N(E),\Gamma}. \tag{18.30}$$

Consequently, j steps of the adapted chain correspond to the macroscopic time j/N of the semigroup $e^{-t\mathsf{G}_{X_N(E),\Gamma}}$.

Proof. We begin with the tangent part. As in Section 16.3, we identify the tangent functions with the superfunctions on $\mathcal{M}_k$ through Q, so that $R(x,0) = \kappa_E \mathrm{Id} - \mathrm{i}s_E Q$

on the balanced saddle manifold. By Lemma 16.5, applied to $L_{H,0} = L_H(0)$ on these functions, and since $\tau_{\mathrm{phys}} = (2W^2)^{-1}$, we have

$$L_{H,0} = -\frac{1}{2s_E^2}\,\mathsf{C}_k, \qquad \tau_{\mathrm{phys}} L_{H,0} = -\frac{1}{4s_E^2 W^2}\,\mathsf{C}_k. \tag{18.31}$$

Using $4s_E^2 = 4 - E^2$, we obtain

$$N\tau_{\mathrm{phys}} L_{H,0} = -\frac{N}{(4-E^2)W^2}\,\mathsf{C}_k = -X_N(E)\mathsf{C}_k,$$

where in the last step we used the definition of $X_N(E)$ in (1.3).

We next compute the source term. Recall from (4.15) that one onsite weight carries the source factor $e^{-\mathrm{i}\eta_0 \operatorname{STr}(S\Gamma)}$, where $\eta_0 = a_N^{-1}$ by Definition 4.1. On the saddle manifold we have $S = \kappa_E \mathrm{Id} - \mathrm{i}s_E Q$, and $\operatorname{STr}\Gamma = 0$, since $\sum_j u_j = 0$ for the source supermatrix $\Gamma = \Gamma(\mathbf{u}, \mathbf{b})$. Therefore

$$\delta_W V_0 = \mathrm{i}\eta_0 \operatorname{STr}\big((\kappa_E \mathrm{Id} - \mathrm{i}s_E Q)\Gamma\big) = \frac{s_E}{a_N}\operatorname{STr}(Q\Gamma),$$
$$N\delta_W V_0 = \frac{N s_E}{a_N}\operatorname{STr}(Q\Gamma) = \pi \operatorname{STr}(Q\Gamma),$$

where we used $a_N = N\rho(E)$ and $\pi\rho(E) = s_E$. Combining the two computations with $N\delta_W L_h^{\tan} = N\tau_{\mathrm{phys}} L_{H,0} - N\delta_W V_0$ and (18.29), we obtain (18.30). □

Remark 18.9. The constant in (18.31) can also be read off, heuristically, from the variance of a single link, which explains the time $(4s_E^2 W^2)^{-1}$ of Section 5.1. In one bosonic retarded–advanced plane, write $Q = e^{2Y}\Lambda$ with $Y = \left(\begin{smallmatrix} 0 & z \\ \bar z & 0 \end{smallmatrix}\right)$ and $z \in \mathbb{C}$. By (5.2), the real and imaginary parts of $2z$ are orthonormal coordinates at Λ. To first order, the saddle point $\kappa_E \mathrm{Id} - \mathrm{i}s_E Q$ moves by $-2\mathrm{i}s_E Y\Lambda$, and since $\Lambda Y \Lambda = -Y$, the link factor of Definition 4.9 restricted to this displacement is

$$\exp\Big(-\frac{W^2}{2}\operatorname{STr}\big((2\mathrm{i}s_E Y\Lambda)^2\big)\Big) = \exp\big(-4s_E^2 W^2 |z|^2\big).$$

Each real component of $2z$ therefore has variance $(2s_E^2 W^2)^{-1}$ under one link. This is the variance $2t$ of the heat semigroup of $-\mathsf{C}_k$, whose body is $\Delta_{\mathcal{M}_k^0}$, at the time $t = (4s_E^2 W^2)^{-1}$ of (18.31).

18.5 Consistency

We now prove consistency on the line of vectors $\iota_h f = \Omega_h \otimes f$. Since Ω_h is invariant under the normal charges by Lemma 18.3, the product $\Omega_h(U) f(g_0, \xi)$ is invariant under

the body stabilizer in the group chart, and $\Omega_h \otimes f$ is a well-defined function on the tube. For $p' \in \mathbb{R}$ and an integer $m \geq 0$, we define the tangent norm

$$\|f\|_{p',m} = \sum_{j \leq m} \sum_{a_1,\dots,a_j} \|v^{p'} \mathbf{D}_{a_1} \cdots \mathbf{D}_{a_j} f\|,$$

where the $\mathbf{D}_a$ run over the left-invariant fields X_a and the odd tangent derivatives. It is finite for compactly supported smooth tangent functions.

We will need the following bounds for the normal heat applied to the fixed state. Choose a second invariant normal cutoff ρ equal to one on a neighborhood of $\operatorname{supp} \chi$, with support strictly inside the enlarged Dirichlet tube. It includes its nilpotent terms, as do the cutoffs of Section 16.7. We localize the free heat with ρ so that normal-charge invariance and oscillator graph norms are retained, while every vector used below has compact Dirichlet support.

Lemma 18.10. *For $0 \leq s \leq \tau_{\mathrm{phys}}$, set $g_h = w_0^{1/2} \chi \Omega_h$ and $\Omega_{h,s} = \rho B_h(s) g_h$. Then $\Omega_{h,s}$ is invariant under the normal charges, has compact support in the enlarged tube, and for each fixed prescribed graph order m,*

$$\|U \Omega_{h,s}\|_{\mathcal{G}^m} \leq Ch, \qquad \|\Omega_{h,s} - \Omega_h\|_{\mathcal{G}^m} \leq Ch^2. \tag{18.32}$$

Here U denotes an ordinary or odd unscaled normal coordinate. Moreover, with $L_{N,h}$ acting by its differential expression on these compactly supported vectors,

$$(\partial_s - L_{N,h})\Omega_{h,s} = R_{h,s}, \qquad R_{h,s} = -[L_{N,h}, \rho] B_h(s) g_h, \qquad \sup_{0 \leq s \leq \tau_{\mathrm{phys}}} \|R_{h,s}\|_{\mathcal{G}^m} = O(h^L) \tag{18.33}$$

for every prescribed m and L.

Proof. The free normal heat, the half-weight $w_0^{1/2}$, and the invariant cutoffs commute with the normal charges. Since Ω_h is invariant, so is $\Omega_{h,s}$. For the second bound, recall that $w_0^{1/2}(0) = 1$ and that $w_0^{1/2}$ has a critical point at the origin, since the onsite weight w_0 has one at the saddle. Hence $w_0^{1/2} - 1 = O(|U|^2)$ on the tube, and since $U = O(h|Z|)$,

$$\|(w_0^{1/2}\chi - 1)\Omega_h\|_{\mathcal{G}^m} \leq C \big\| |U|^2 \Omega_h \big\|_{\mathcal{G}^m} + \|(\chi - 1)\Omega_h\|_{\mathcal{G}^m} \leq Ch^2 \|\Omega_h\|_{\mathcal{G}^{m+1}} + Ch^2,$$

where in the last step we used (18.7) for the cutoff. The whole-space generator $L_{N,h}$ is bounded from $\mathcal{G}^{m+1}$ to $\mathcal{G}^m$ by $C_m h^{-2}$. By (18.26), for sufficiently small h,

$$\|(B_h(s) - \mathrm{Id})g\|_{\mathcal{G}^m} \leq 2s \|L_{N,h} g\|_{\mathcal{G}^m} \leq C_m s h^{-2} \|g\|_{\mathcal{G}^{m+1}}.$$

The fixed physical cutoff ρ is uniformly bounded on each prescribed $\mathcal{G}^m$: its standardized derivatives gain powers of h, and its nilpotent coefficients are finite cutoff derivatives. Since $\rho g_h = g_h$, we have $\Omega_{h,s} - \Omega_h = \rho(B_h(s) - \mathrm{Id})g_h + (g_h - \Omega_h)$. The preceding

bounds and (18.4) therefore give the second estimate. For the first, each unscaled normal coordinate is h times a standardized one, which is bounded from $\mathcal{G}^{m+1}$ to $\mathcal{G}^m$. Using the second estimate with $m+1$ in place of m, we obtain

$$\|U\Omega_{h,s}\|_{\mathcal{G}^m} \le Ch\|\Omega_{h,s}\|_{\mathcal{G}^{m+1}} \le Ch.$$

Differentiating $\rho B_h(s)g_h$ gives the defect identity. Its commutator is supported where ρ varies, at a fixed positive normal distance from $\operatorname{supp} g_h \subset \operatorname{supp}\chi$. The differentiated free Gaussian kernel there is bounded by a fixed power of h^{-1} times e^{-c/h^4}, uniformly for $0 < s \le h^4/2$. Gaussian gauge factors on the fixed tube cost at most e^{C/h^2}. The same bounds hold after the finitely many derivatives and polynomial factors that define a prescribed graph norm. Thus the defect is $O(h^L)$ for every prescribed m, L; at $s = 0$ it vanishes by the separation of supports. This proves (18.33). □

The horizontal operator leaves the invariant normal line only at order h, as the following lemma shows.

Lemma 18.11. *Let Ω be a normal vector that is invariant under the normal charges, with $\|U\Omega\|_{\mathcal{G}^m} \le Ch$ and $\|\Omega\|_{\mathcal{G}^m} \le C$, for a sufficiently large prescribed order m. Then for all compactly supported smooth tangent functions f,*

$$L_{H,h}(\Omega \otimes f) = \Omega \otimes L_{H,0}f + O(h)\|f\|_{p',2}, \tag{18.34}$$

in every body weight p with $p' = p + C$. In particular this holds for $\Omega = \Omega_h$ and for $\Omega = \Omega_{h,s}$.

Proof. We use the structure of $L_H(U)$ described in the proof of Lemma 18.4(a). Its normal derivatives are normal charges, which annihilate Ω. When a normal charge falls instead on a coefficient of $L_H(U)$, the result is the derivative of that coefficient along $[\mathfrak{e}, U]$ for some $\mathfrak{e} \in \mathfrak{k}$, which carries one factor of U. By analyticity in U, each remaining coefficient differs from its value at $U = 0$ by a function that carries one factor of an ordinary or odd normal coordinate. Every term of $L_{H,h}(\Omega \otimes f) - \Omega \otimes L_{H,0}f$ is therefore a sum of products

$$(\phi(U)U_j\Omega) \otimes (\mathbf{D}f),$$

where ϕ is bounded with bounded derivatives on the tube, U_j is a normal coordinate and $\mathbf{D}$ is a product of at most two tangent derivatives. Each term is $O(h)\|f\|_{p',2}$, since multiplication by ϕ is bounded on $\mathcal{G}^m$ and $\|U_j\Omega\|_{\mathcal{G}^m} \le Ch$. The body weight accounts for the coefficients of the odd tangent part, which are bounded in the group chart. For $\Omega = \Omega_h$, the hypotheses follow from Lemma 18.3, since $U = O(h|Z|)$, and for $\Omega = \Omega_{h,s}$ they follow from Lemma 18.10. □

Proposition 18.12. *In the balanced signature, for every body weight p there exist p', an integer m and C such that for all compactly supported smooth tangent functions f,*

$$A_h\iota_h f = \iota_h f + \delta_W\,\iota_h\Big(\frac{\tau_{\mathrm{phys}}}{\delta_W}L_{H,0} - V_0\Big)f + O(\delta_W h)\|f\|_{p',m}, \tag{18.35}$$

where V_0 is the source potential at $U = 0$ and the error is measured in the adapted space with body weight p. The constant is uniform on compact sets of energies, sources, and aspect ratios.

Proof. We first expand the unsourced step around the exact frozen normal transfer. Use the Dirichlet full heat of (18.24) and the localized free normal path $\Omega_{h,s}$ of Lemma 18.10. This path has compact Dirichlet support and starts at g_h. Differentiating $e^{(\tau_{\mathrm{phys}}-s)\mathcal{L}_h}(\Omega_{h,s}\otimes f)$ and using (18.33) gives

$$(T_h^{\mathrm{D}} - \mathbf{F}_h)\iota_h f = \chi w_0^{1/2}\int_0^{\tau_{\mathrm{phys}}} e^{(\tau_{\mathrm{phys}}-s)\mathcal{L}_h}\big(L_{H,h}(\Omega_{h,s}\otimes f) - R_{h,s}\otimes f\big)\,\mathrm{d}s.$$

At the right endpoint we used $\chi\rho = \chi$ and the free-heat expression for the original $\mathbf{F}_h$. The full heat has norm at most two by Lemma 18.7, so the defect term is $O(h^L)\|f\|_{p',0}$ for every prescribed L. The same is true of $T_h - T_h^{\mathrm{D}}$. Consequently,

$$(T_h - \mathbf{F}_h)\iota_h f = \chi w_0^{1/2}\int_0^{\tau_{\mathrm{phys}}} e^{(\tau_{\mathrm{phys}}-s)\mathcal{L}_h} L_{H,h}(\Omega_{h,s}\otimes f)\,\mathrm{d}s + O(h^L)\|f\|_{p',0}. \tag{18.36}$$

All vectors $\Omega_{h,s}\otimes f$ in this formula have compact support in the enlarged tube. Choose L sufficiently large to absorb the finitely many derivatives and factors h^{-2} below into the final $O(\delta_W h)$ remainder.

Next, we evaluate the integrand. By Lemma 18.11 with $\Omega = \Omega_{h,s}$, followed by (18.32),

$$L_{H,h}(\Omega_{h,s}\otimes f) = \Omega_{h,s}\otimes L_{H,0}f + O(h)\|f\|_{p',2} = \Omega_h\otimes L_{H,0}f + O(h)\|f\|_{p',2}.$$

The outer evolution has norm at most 2 by Lemma 18.7, and it maps the error to an error of the same order. Applying (18.27) to the main term, we obtain

$$\begin{aligned}\big\|\big(e^{(\tau_{\mathrm{phys}}-s)\mathcal{L}_h} - \mathrm{Id}\big)(\Omega_h\otimes L_{H,0}f)\big\| &\le 2\tau_{\mathrm{phys}}\|\mathcal{L}_h(\Omega_h\otimes L_{H,0}f)\| \\ &\le C\tau_{\mathrm{phys}}h^{-2}\|f\|_{p',4} = O(h^2)\|f\|_{p',4},\end{aligned}$$

where we used that $L_{N,h}$ is h^{-2} times a second-order polynomial in the standardized normal variables, which is bounded from $\mathcal{G}^1$ to $\mathcal{H}_{\mathrm{nor}}$. Finally, $\|(\chi w_0^{1/2} - \mathrm{Id})\Omega_h\| = O(h^2)$, as in the proof of Lemma 18.10. The integrand in (18.36) is therefore $\iota_h L_{H,0}f + O(h)\|f\|_{p',4}$, uniformly in s. Integrating over an interval of length $\tau_{\mathrm{phys}} = \delta_W/2$, we obtain

$$T_h\iota_h f = \mathbf{F}_h\iota_h f + \tau_{\mathrm{phys}}\,\iota_h L_{H,0}f + O(\delta_W h)\|f\|_{p',4} = \iota_h f + \tau_{\mathrm{phys}}\,\iota_h L_{H,0}f + O(\delta_W h)\|f\|_{p',4}, \tag{18.37}$$

where in the second step we used $\mathbf{F}_h\iota_h f = (\mathbf{F}_h\Omega_h)\otimes f = \iota_h f$.

We now add the source. Recall that $s_\Gamma = e^{-\delta_W V_h}$, where V_h is analytic in U and $V_h - V_0$ carries one factor of a normal coordinate. Using this factor and (18.4), we obtain

$$V_h\iota_h f = \iota_h V_0 f + O(h)\|f\|_{p',0}.$$

The real part of the BB part of $\delta_W V_h$ is a positive damping, and its FF part is bounded, so that $\mathrm{Re}(\delta_W V_h) \geq -C\delta_W$ on the tube. Therefore $|s_\Gamma| \leq e^{C\delta_W}$, and the Euler remainder satisfies

$$\|(s_\Gamma - \mathrm{Id} + \delta_W V_h)g\| \leq e^{C\delta_W}\delta_W^2 \|V_h^2 g\|.$$

For $g = T_h \iota_h f$, the right side is $O(\delta_W^2)\|f\|_{p',0}$, after finitely many stronger body weights, since V_h grows at most linearly in v. Combining these bounds with (18.37), and using $\delta_W \tau_{\mathrm{phys}} = O(\delta_W^2)$, we conclude that (18.35) holds with $m = 4$. The bound is uniform on compact parameter sets, since every constant above depends continuously on the parameters. □

Remark 18.13. The same consistency holds for the analytic transpose $A_h^{\mathrm{tr}} = T_h s_\Gamma$ of the adapted step, with the same limit generator. Indeed, applying s_Γ before T_h changes the proof above only in the order of the two expansions, and the errors are of the same order. The powers of the transpose are also stable, since the identity $(A_h^{\mathrm{tr}})^n = T_h A_h^{n-1} s_\Gamma$ implies that $\|(A_h^{\mathrm{tr}})^n\| \leq \|T_h\|\|A_h^{n-1}\|\|s_\Gamma\|$. Theorem 18.14 below therefore also holds for A_h^{tr}.

We now state the product limit. Let $\mathcal{H}_p^{\tan}$ be the space L_p^2 of Section 5.2, with the degree-metric parameter ω of Theorem 17.1, chosen with $\omega \leq \min(\varkappa_k, \omega_0)$, where ω_0 is the parameter of Proposition 5.15. This choice is possible since Theorem 17.1 remains true for smaller ω, and the spaces $\mathcal{H}_p^{\mathrm{tube}}$ below carry the same ω. The odd frames of Section 5.2 and of the group chart (18.10) differ by a linear change that is equivariant under K_0 and bounded in the translated charts, and the two coefficient norms are equivalent. Recall from Section 18.4 that $L_h^{\tan} = (\tau_{\mathrm{phys}}/\delta_W)L_{H,0} - V_0$ is the operator in (18.35). By Lemma 18.8, we have $L_h^{\tan} = -(N\delta_W)^{-1}\mathsf{G}_{X_N(E),\Gamma}$, with the generator $\mathsf{G}_{X,\Gamma}$ of (18.29). By Proposition 5.15, the closure of $L_h^{\tan}$ on compactly supported smooth superfunctions generates a strongly continuous semigroup (U_t) on $\mathcal{H}_p^{\tan}$, and these superfunctions form a core for it in every body weight.

Theorem 18.14. *In the balanced signature, for every body weight p, all $f \in \mathcal{H}_p^{\tan}$ and all $T > 0$,*

$$\lim_{h\to 0} \sup_{0 \leq n\delta_W \leq T} \|A_h^n \iota_h f - \iota_h U_{n\delta_W} f\| = 0,$$

uniformly on compact sets of energies, sources, and aspect ratios.

Proof. We apply Lemma 18.1 with $\mathcal{H} = \mathcal{H}_p^{\tan}$, the weighted space $\mathcal{H}_p^{\mathrm{tube}}$ of Theorem 17.1 as $\mathcal{H}_h$, $\delta_h = \delta_W$, the maps $\iota_h f = \Omega_h \otimes f$, and the core $\mathcal{D}$ of compactly supported smooth superfunctions. The maps ι_h are uniformly bounded, since $\|\Omega_h\| \leq 1 + Ch$ by (18.4) and the coefficient norms of the odd tangent coordinates in the two frames are equivalent, with constants that do not depend on h. The power bound (18.1) is the stability estimate (N4). For $f \in \mathcal{D}$, the norm $\|f\|_{p',m}$ is finite, and (18.35) implies (18.2), since its error

term divided by δ_W is $O(h)\|f\|_{p',m}$. The conclusion of Lemma 18.1 is the limit at fixed parameters.

For the uniformity, we apply Corollary 18.2. The adapted spaces for different parameters are identified with one space through the chart (g_0, ξ, Z), and their norms are uniformly equivalent on compact parameter sets, since c_σ, θ_σ, and γ_h depend continuously on E. The generators depend continuously on the parameters on the common core, and the semigroups obey a common exponential bound on compact parameter sets, by Proposition 5.15 with constants that are uniform for X, ε, and Γ in compact sets. Finally, the stability and consistency estimates are uniform there. □

18.6 The initial layer

Theorem 18.14 applies to vectors on the invariant normal line, but the endpoints of the chain are not of this form. In Section 20, Lemma 20.3 moves both endpoints onto the line in an initial layer of order $h^{-2}\log(1/h)$ steps, whose tangent time tends to zero. Its proof relaxes the frozen path with the bound (18.5) of Lemma 18.3. We record here a second property of $\mathbf{F}_h$, which concerns the left endpoint.

The original frozen normal transfer $\mathbf{F}_h$ of (16.38), with its cutoffs, is its own analytic transpose for the analytic Berezin pairing in the normal variables (a transpose for this indefinite pairing, not a Hilbert adjoint for the positive Wick metric). Indeed, the Wick density γ_h^2 of the pairing cancels the two Gaussian gauges in its free-heat kernel. This leaves the kernel $w_0^{1/2}(U)K^{\varpi}_{\mathrm{nor}}(U,U')w_0^{1/2}(U')$ between identical cutoffs, which is symmetric under the interchange of its endpoints, with the Berezin orientation retained. The transposed chain, which carries the left endpoint g_L, therefore has the same frozen normal transfer as the chain itself, and g_L relaxes under the same contraction (18.5) as g_R.

SECTION 19

The endpoints of the chain

In this section we attach the endpoints of the chain to the adapted tube. In the one-sided form of Section 20.2, the chain integral is the pairing of the constant function 1 with $T_\Gamma^{N-m} f_m$, where $f_m = T_\Gamma^m 1$ and $m = \lceil C_0 W \rceil$. We prove that the right block f_m is, up to an error $C_k W^{A_k} e^{-\gamma_k W}$, a sum over the root signatures of local reference vectors on the tube (Theorem 19.21), and that these references, and every family obtained from them by adapted steps, can be compared with the physical chain on the cycle with an exponentially small error (Proposition 19.20). The comparison itself is carried out in Section 20. As explained in Section 2.5, both errors must be of order $e^{-\gamma W}$ for some $\gamma > 0$, since every change of integration cycle at an endpoint is paired in Section 20 with a physical partial chain, whose norm we only know to be $e^{o(W)}$ by Theorem 15.14.

The two steps. We proceed in two steps. In the first, the block f_m is replaced by local references on the tube with an error $e^{-\gamma_k W}$. Its main ingredients are an Agmon estimate with a time-dependent weight for the physical chain on the cycle, which concentrates f_m near the saddle manifold (Proposition 19.4), and a Davies–Gaffney estimate on the tube, which compares the physical chain with the references there (Theorem 19.21). In the second step, a block of adapted steps with Gaussian weights that increase from $e^{-a|Z|^2}$ to $e^{a|Z|^2}$ turns the negative Gaussian weight of the references into a positive one (Corollary 19.11). Both steps are over a block of $m = \lceil C_0 W \rceil$ steps, whose physical heat time $m\tau_{\mathrm{phys}} \approx C_0/(2W)$ is short. A term of order $C_k W$ in the generator changes norms by at most a factor $e^{C_k C_0}$ over the block, and plain weighted energy estimates in the raw realization suffice (Lemma 19.8). The joint normal–tangent analysis of the kernel in Section 17 is needed only for the legs of macroscopic length. The point that needs care is the coefficient of $|b|^2$ per step on a complex normal graph of height b, which must not depend on the rank; Remark 19.10 explains why and checks it.

Conventions. Three conventions hold throughout this section. First, we assert analytic continuation only for the families generated from the endpoint (Definition 19.1) and their finite jets, while the operator bounds hold on arbitrary inputs. Second, real cutoffs are evaluated only at real points. We call the set where a real cutoff takes values strictly between zero and one its *collar*. Every complex deformation and every submean ball lies where the relevant real cutoffs equal one, and the collar of each cutoff is estimated separately on the real cycle. Third, adjoints are positive Hilbert adjoints for the fixed

positive coefficient metric. They differ from the analytic Berezin transpose, which defines the left endpoint. Throughout, $h = W^{-1/2}$, $\tau = \tau_{\mathrm{phys}} = (2W^2)^{-1}$, and $\eta = \varepsilon\eta_0 \asymp W^{-2}$ is the physical imaginary part. Constants with a subscript k may depend on the rank, on finitely many prescribed jets and polynomial weights, and on every quantity chosen before them in Table 19.1. The table and the next paragraphs also use three terms, which we define precisely before Corollary 19.11. Recall that Z is the standardized normal coordinate, defined after (16.40) by $U = hZ$. Roughly, a function f on the tube has Gaussian margin a if $e^{a|Z|^2}f$ has finite norm, a signed ramp is a sequence of such Gaussian weights that increases from $e^{-a|Z|^2}$ to $e^{a|Z|^2}$ in steps of size $O(h^2)$, and a prepared family is one with a positive Gaussian margin.

The order of the constants. Table 19.1 lists the constants and radii of this section in the order in which they are chosen. The constants in the first group are universal and are chosen before the rank k. At fixed k, we then choose the coefficient metric of Lemma 18.4, the radii in the second group, and finally the lower threshold on W. The radii R_k, d_{col}, ρ_{pd}, r_{phys}, and r_1 are chosen in the order listed, while r_{esc} is subject only to the constraints shown.

Table 19.1. The constants of Section 19, in the order in which they are chosen. Here $s_0 = \inf_{|E|\le 1/2} s_E$.

Quantity	Role and constraint	Introduced in
Universal, chosen before k		
a_0, v_0	largest amplitude of the signed weights $w_a = e^{a\|Z\|^2}$, and largest change per step in units of h^2	Lemma 19.8
c_1, C_1	largest relative graph height $\|b\|/R_k$, and coefficient of $\|\beta_j\|^2 + W^2\|\beta_j - \beta_{j-1}\|^2$ per step	Lemma 19.8
a	positive Gaussian margin of prepared families, $0 < a < a_0$	Corollary 19.11
ϱ	the shortest signed ramp has ϱW steps, $\varrho > 2a/v_0$	Corollary 19.11
C_0	block length $m = \lceil C_0 W \rceil$, $C_0 \ge \max(1, \varrho)$	this table
C_n	coefficient of $C_0 W\|b\|^2$ in the exponent of the bound (19.36) for a block of adapted steps on a graph of height b, $C_n = 3C_1$	Proposition 19.13
C_*	coefficient of $\|D\|^2 + \|D_F\|^2$ in the shifted onsite bound	(19.42)
C_{geo}	canonical normal height	Lemma 19.15
a_{esc}	amplitude of the escape ramp, chosen after C_0	Proposition 19.4
c_{D}	rate of the Davies–Gaffney estimate	Lemma 19.9
C_{gau}	coefficient of $W\|b\|^2$ from conversion to the physical norm on a normal graph, $C_{\mathrm{gau}} = C_Z/4$	(19.49)
c_{sh}	rate of the Gaussian decay $e^{-c_{\mathrm{sh}} W r_1^2}$ on the support of $\nabla\chi_{\mathrm{phys}}$, $c_{\mathrm{sh}} = c_{\mathrm{ann}}/4$, proportional to a	Lemma 19.18
E_{geom}	energy threshold	(19.50)
E_{sep}, r_{sep}	energy and radius of the Gaussian separation	Lemma 18.5

Continued on next page

Table 19.1 (continued)

Quantity	Role and constraint	Introduced in
Chosen at fixed rank k		
r_k, $\varkappa_k$	radius and degree metric of the sectorial bound	Lemma 18.4
ω	degree-metric parameter of the stability theorem, $\omega \le \min(\varkappa_k, \omega_0)$ with ω_0 of Proposition 5.15	Theorem 17.1
r	cutoff radius of the stability theorem, $r \le r_k$	Theorem 17.1
graph orders	the finitely many prescribed orders m	Section 18.2
$\mathcal{U}_0 \Subset \mathcal{U}_1 \Subset \mathcal{U}_2$	complex damping balls about Λ; the closure of $\mathcal{U}_0$ lies in the open ball of radius $\rho/2$ about Λ, where ρ is the radius of a ball about Λ inside $\mathcal{U}_2$; the tube is then chosen so that conjugated dampings stay in $\mathcal{U}_1$	(15.36)
ρ, ρ_{out}	Hardy radii of the escape ramp, with ρ as in the previous row and $\rho_{\mathrm{out}} = \rho/2$	Proposition 19.4
R_k	outer root tube, which carries the normal cutoffs, the normal graphs, and the Taylor remainders; $C_k R_k$ below fixed universal fractions; $R_k \le R_{\mathrm{s}}$; $CR_k \le c_0/2$ for the rate c_0 of the BB damping and the constant C of the proof of Lemma 19.7; the identification lemma is applied with $16R_k \le r_{\mathrm{out}}$, $4R_k \le r_{\mathrm{in}} \le 3r_{\mathrm{out}}/4$, and $2r_{\mathrm{out}} \le \min(r_k, r_{\mathrm{sep}})$	Lemmas 18.6, 19.7 and 19.8 and Propositions 19.13 and 19.17
$d_{\mathrm{col}} = R_k/8$	distance between the collar of the barrier cutoff and the inner region	Lemma 19.24
$\rho_{\mathrm{pd}} \asymp R_k$	radius of the polydisk that carries the continued error	Lemma 19.26
r_{phys}	support radius of the physical cutoffs, r_{phys}/R_k small at fixed k	Theorem 19.21
r_1	inner radius of the annulus $r_1 \le \lvert U\rvert \le r_{\mathrm{phys}}$ on which the physical cutoff varies, $r_1 < r_{\mathrm{phys}}$	Lemma 19.18
A_{sh}	exponent of W in the residual bound (19.58), $A_{\mathrm{sh}} = A_{\mathrm{ann}} + A_1$, depending on k, the body weight, the orders of the shell jets, and the prescribed local jets of the family	(19.52)
r_{esc}	saturation radius of the escape function ϑ_B; $r_{\mathrm{esc}} < s_0/2$ and $C_k a_{\mathrm{esc}} C_0 r_{\mathrm{esc}}$ small	Section 15.3

The block length C_0 is chosen once and has three roles. In the signed preparation, Corollary 19.11 and Proposition 19.13 need $C_0 \ge \max(1, \varrho)$, so that the signed ramp fits into one block. In the escape ramp of Proposition 19.4, C_0 fixes the number of steps over which the escape weight is raised, and the amplitude a_{esc} is chosen afterward with $3a_{\mathrm{esc}}/C_0 < c/4$. In the barrier of Lemma 19.24, the physical duration $C_0/(2W)$ of the block gives the decay factor $e^{-c_{\mathrm{D}} W d_{\mathrm{col}}^2/C_0}$ of the Davies–Gaffney estimate across the collar, which must dominate the growth factor $e^{16C_{\mathrm{geo}}^2 C_* C_0 W(E^2R_k^2+R_k^4)}$ that comes from the height of the canonical graph over the collar; this is the second condition in (19.50). Only the first role constrains C_0 itself, and the other two constrain constants chosen after it.

We use the coordinates on the cycle of Section 15.3: the boost g, the physical normal $\widetilde{P} = \mathrm{ch}_{\mathrm{g}}\, P_0$ of (15.26), the escape function ϑ of (15.32), and the parameter balls of (15.36), together with the kinetic and onsite bounds Lemmas 15.7 and 15.9. Recall from (15.4) that the transported odd frame consists of the entries of $O_1 = \mathrm{g}^{-1}\Lambda_B S_{BF}$ and $O_2 = S_{FB}\mathrm{g}^{-1}$, and that the norms of $\mathcal{H}_p(\mathcal{S})$ are taken in its monomials. We abbreviate $s = s_E$ and $\kappa = \kappa_E$, and in this section all norms of matrices are Hilbert–Schmidt norms unless stated otherwise. The first step of the chain starts from the constant function, which has no finite norm in $\mathcal{H}_p(\mathcal{S})$. Since $K_\tau 1 = 1$, we have $f_1 = w_\Gamma$, and we integrate this first factor directly with a fixed fraction of its BB and FF Gaussians and its damping $e^{-c\eta v}$. Since $v^q e^{-c\eta v} \le C_q \eta^{-q}$ for $q \ge 0$, we obtain $\|w_\Gamma\|_{\mathcal{H}_p(\mathcal{S})} \le C_k \eta^{-A_k}$, also in the parameter Hardy norm used below.

19.1 Families generated from the endpoint and their equivariance

The comparison of Section 20 continues certain functions on the tube from real to complex points. Such a continuation is legitimate only for functions with a known holomorphic structure, and we now describe the class of functions for which we assert it, together with their equivariance under conjugation, which controls their continuation along the complexified orbit.

The realizations on the tube. Recall from Definition 16.7 that the raw realization is the realization of the chain on functions of the tubular coordinates (x, U) of (16.12) with the flat measure $\mathrm{d}U\,\mathrm{d}\nu$, in which one link acts by the operator K^{ω} with the kernel (16.33). The symmetric normal gauge (16.38) is obtained from it by conjugation with $w_0^{1/2}\gamma_h$. Recall that w_0 denotes the source-free onsite weight, that the half-weight $w_0^{1/2}$ factors as in (16.34) into its exact Gaussian and anharmonic parts, and that the Gaussian gauge γ_h is the Gaussian function (16.35) of U, which is real on the real tube. By Lemma 16.10, conjugation by γ_h turns the exact quadratic normal transfer, written in the flat realization, into its positive Wick realization. In the symmetric normal gauge we use the positive Wick metric of Definition 16.6, the norm $\|\cdot\|$ in which the Wick basis is orthonormal, together with the orbit measure.

Parameters. Recall the linear damping direction Γ and the nested complex balls $\mathcal{U}_0 \Subset \mathcal{U}_1 \Subset \mathcal{U}_2$ of (15.36). Local inverse-resolvent sources are treated differently. Their directions $\mathbf{V} = (V_1, \dots, V_d)$ are used only through finite derivatives at zero. We choose a basis of the corresponding finite jet tensor space, and for a family $f_{\Gamma,\mathbf{V}}$ we write $f_{\Gamma,\ell}$ for its components in this basis. We define the *family norm*

$$\|f\|^2_{\mathcal{U};p,a} = \int_{\mathcal{U}} \sum_{\ell} \left\| e^{a|Z|^2} v^p f_{\Gamma,\ell} \right\|^2 \mathrm{d}\Gamma, \tag{19.1}$$

where $\mathrm{d}\Gamma$ is the complex volume on the parameter space and all finite odd coefficients are retained in the norm. A family norm on a larger ball controls every fixed parameter derivative on a smaller ball, by Cauchy's formula.

Definition 19.1. We say that a family $f_{\Gamma,\mathbf{V}}$ of coefficient functions on the tube is *generated from the endpoint* if it is obtained from the right endpoint g_R of (16.38), or from its representative in the raw realization ϖ, by applying finitely many of the following operations in chronological order: adapted steps $A_{\Gamma,h}$; the two factors s_Γ and T_h of such a step; the half-weights $w_0^{\pm 1/2}$; invariant normal input and output cutoffs; partial heats of duration at most τ_{phys}; and a prescribed finite number of complete local jets.

The comparison of the physical and adapted chains in Section 20 uses exactly this class. By (19.20) below, the families obtained from g_L by the transposed steps are also generated from the endpoint.

The next lemma states the equivariance of these families under simultaneous conjugation of the matrix point and of the parameters, in its finite and infinitesimal forms, together with its standard consequence: holomorphic continuation along complexified conjugation orbits is the same as continuation in the parameters.

Lemma 19.2. *Let $f_{\Gamma,\mathbf{V}}$ be a family generated from the endpoint, written in pushed-forward normal and tangent frames, or the physical family $f_{j,\Gamma} = T_\Gamma^j 1$ on the cycle, for which $\mathbf{V}$ is absent.*

(a) For all real even matrices S near the identity,

$$f_{\Gamma,\mathbf{V}}(SRS^{-1}) = f_{S^{-1}\Gamma S,\, S^{-1}\mathbf{V}S}(R). \tag{19.2}$$

Equivalently, for all constant even matrices $\mathfrak{e}$, the generator $\mathcal{W}_\mathfrak{e}$ of the conjugation $R \mapsto e^{t\mathfrak{e}} R e^{-t\mathfrak{e}}$ satisfies

$$\mathcal{W}_\mathfrak{e} f_{\Gamma,\mathbf{V}} = \langle [\Gamma, \mathfrak{e}], \partial_\Gamma \rangle f_{\Gamma,\mathbf{V}} + \langle [\mathbf{V}, \mathfrak{e}], \partial_\mathbf{V} \rangle f_{\Gamma,\mathbf{V}}. \tag{19.3}$$

(b) For complex S, as long as $S^{-1}\Gamma S \in \mathcal{U}_1$ and the normal point stays in the tube, the right side of (19.2) *is holomorphic in S. At every real output in the interior of the set where the output cutoffs of the family equal one, it is the holomorphic continuation of $R \mapsto f_{\Gamma,\mathbf{V}}(R)$ along the complexified orbit $\{SRS^{-1}\}$.*

(c) Let f be generated from the endpoint, let $\Gamma \in \mathcal{U}_0$, and let $R \mapsto S(R)$ be measurable with $\|S(R)^{\pm 1} - \mathrm{Id}\| \le c_k$ for a sufficiently small $c_k > 0$. Write $f_\ell^S(R)$ for the components of the selected values $f_{S(R)^{-1}\Gamma S(R),\, S(R)^{-1}\mathbf{V}S(R)}(R)$. Then

$$\int \sum_\ell \left\| e^{a|Z|^2} v^p f_\ell^S(R) \right\|^2 \mathrm{d}R \le C_k \|f\|^2_{\mathcal{U}_2;p,a}. \tag{19.4}$$

Proof. We first prove (19.2) for a family generated from the endpoint. We change the body integration variables and use the pushed-forward frame $T \mapsto ST$. The ambient

Gaussian and the tubular measure are invariant. The normal coordinates, the half-density, the Gaussian gauge, and the invariant cutoffs, including their nilpotent terms, are unchanged, and g_R and $w_0^{1/2}$ are normal class functions by Lemma 16.8. The identities $\mathrm{STr}(SRS^{-1}\Gamma) = \mathrm{STr}(RS^{-1}\Gamma S)$ and $\det(E - SRS^{-1} - tV) = \det(E - R - tS^{-1}VS)$ imply (19.2) for one step and one complete jet, and by induction it holds for their ordered products and for partial heats. The positive source and a fixed part of the weight margin justify the change of variables over the noncompact body.

For the physical family the substitution $R \mapsto SRS^{-1}$ moves the cycle, and we argue infinitesimally, by induction on j. The idea is that simultaneous conjugation of the two arguments leaves the heat kernel invariant, and that the conjugation field $\mathcal{W}_\mathfrak{e}$ can be moved from one argument to the other by an integration by parts over the cycle, since it is holomorphic and has divergence zero. For $j = 0$, the identity (19.3) is trivial. Let $j \geq 1$, and write $f_{j,\Gamma} = w_\Gamma\, g$ with $g = K_\tau f_{j-1,\Gamma}$. The factors w_Γ and g are holomorphic near $\mathcal{S}$, the first since $E - R$ is invertible on $\mathcal{S}$ by Lemma 4.7(b), and the second by the Gaussian bound of the link kernel from complex output points in Proposition 19.17(c). That proposition, and the formula (19.45) for $d(X, X')$ used below, are proved in Section 19.5 without the present lemma. By the identities used above for one step, we have $w_\Gamma(SRS^{-1}) = w_{S^{-1}\Gamma S}(R)$, and differentiating at $S = e^{t\mathfrak{e}}$, we obtain (19.3) for w_Γ. By the Leibniz rule for the even derivation $\mathcal{W}_\mathfrak{e}$, it suffices to prove (19.3) for g. The case $j = 1$ is trivial, since $g = K_\tau 1 = 1$, and we let $j \geq 2$.

The kernel $k_\tau(R, R')$ is invariant under the simultaneous conjugation $(R, R') \mapsto (SRS^{-1}, SR'S^{-1})$, since $\mathrm{STr}(SRS^{-1} - SR'S^{-1})^2 = \mathrm{STr}(R - R')^2$. Let $\mathcal{W}_\mathfrak{e}^{(1)}$ and $\mathcal{W}_\mathfrak{e}^{(2)}$ denote the field acting on R and on R'. Then $\mathcal{W}_\mathfrak{e}^{(1)} k_\tau = -\mathcal{W}_\mathfrak{e}^{(2)} k_\tau$, and differentiating under the integral, we find

$$\mathcal{W}_\mathfrak{e} g(R) = -\int_\mathcal{S} \big(\mathcal{W}_\mathfrak{e}^{(2)} k_\tau(R, R')\big) f_{j-1,\Gamma}(R')\, \mathrm{d}R'. \tag{19.5}$$

We now integrate by parts on the cycle. We use linear coordinates (z, ζ) of the ambient space, given by the even and the odd entries, and we write

$$\mathcal{W}_\mathfrak{e} = \sum_\mu a^\mu \partial_{z_\mu} + \sum_b a^b \partial_{\zeta_b},$$

where the coefficients a^μ and a^b are those of the linear map $Z \mapsto [\mathfrak{e}, Z]$. This map has divergence zero, since

$$[\mathfrak{e}, Z]_{il} = (\mathfrak{e}_{ii} - \mathfrak{e}_{ll}) Z_{il} + \sum_{m \neq i} \mathfrak{e}_{im} Z_{ml} - \sum_{m \neq l} Z_{im} \mathfrak{e}_{ml},$$

and the supertrace of its diagonal part vanishes by the antisymmetry of $\mathfrak{e}_{ii} - \mathfrak{e}_{ll}$ in (i, l). It follows that every superfunction F that is holomorphic near $\mathcal{S}$ satisfies

$$\mathcal{W}_\mathfrak{e} F = \sum_\mu \partial_{z_\mu}(a^\mu F) - \sum_b \partial_{\zeta_b}(a^b F).$$

The Berezin integral of the second sum vanishes. Coefficientwise in ζ, the first sum times $\mathrm{d}z$ is the exterior derivative of the holomorphic form $\sum_\mu \pm a^\mu F\, \iota_{\partial_{z_\mu}} \mathrm{d}z$. Let ρ be a smooth function on $[0,\infty)$ that equals one on $[0,1]$ and zero on $[2,\infty)$, and set

$$\rho_n(R') = \rho(|y'|/n),$$

where $y' = (P_0', X', M')$ are the body coordinates of $R' \in \mathcal{S}$. The coefficients a^μ are linear in R', and the parametrization of $\mathcal{S}$ has bounded derivatives by (15.27). Stokes' theorem on the body of $\mathcal{S}$, as in the proof of Lemma B.1, gives

$$\Big| \int_{\mathcal{S}} \rho_n\, \mathcal{W}_{\mathfrak{e}} F \,\mathrm{d}R' \Big| \le C \int_{n \le |y'| \le 2n} |F|_1 \,\mathrm{d}y', \tag{19.6}$$

where $|F|_1$ is the sum of the moduli of the coefficients of F in ζ.

We apply (19.6) with $F = k_\tau(R,\cdot)\, f_{j-1,\Gamma}$, for fixed $R \in \mathcal{S}$. As in the proof of Lemma 15.7, the body of the kernel satisfies

$$\big|k_\tau(R,R')_\varnothing\big| = \exp\Big\{ -\frac{W^2}{2} \big(\|P_0 - P_0'\|^2 + d(X,X') + \|M - M'\|^2 \big) \Big\},$$

and its other coefficients carry polynomial factors. The formula (19.45) implies that $d(X,X') \ge c_X \|X'\| - C_X$, and therefore $k_\tau(R,\cdot)$, times every polynomial in $|y'|$, is square integrable. The function $f_{j-1,\Gamma}$ lies in $\mathcal{H}_0(\mathcal{S})$, since $j \ge 2$, and the transported odd frame changes coefficient norms only by polynomial factors in $v_{\mathcal{S}}$. By the Cauchy–Schwarz inequality, F and $\mathcal{W}_{\mathfrak{e}} F$ are integrable; for $\mathcal{W}_{\mathfrak{e}} f_{j-1,\Gamma}$ we use the induction hypothesis and Cauchy's formula in Γ. Letting $n \to \infty$ in (19.6), we find

$$\int_{\mathcal{S}} \big(\mathcal{W}_{\mathfrak{e}}^{(2)} k_\tau(R,R') \big) f_{j-1,\Gamma}(R') \,\mathrm{d}R' = - \int_{\mathcal{S}} k_\tau(R,R')\, \mathcal{W}_{\mathfrak{e}} f_{j-1,\Gamma}(R') \,\mathrm{d}R'.$$

Inserting this into (19.5), and using the induction hypothesis and the fact that k_τ does not depend on Γ, we obtain $\mathcal{W}_{\mathfrak{e}} g = \langle [\Gamma, \mathfrak{e}], \partial_\Gamma \rangle g$. This is (19.3) on the cycle. Each side of (19.3) is holomorphic near $\mathcal{S}$, which is totally real of maximal dimension, and the identity extends there.

The two forms in (a) are equivalent. Differentiating (19.2) at $S = e^{t\mathfrak{e}}$, $t = 0$, we obtain (19.3). Conversely, (19.3) at every $(\Gamma, \mathbf{V}, R)$ implies that the function

$$t \mapsto f_{e^{t\mathfrak{e}}\Gamma e^{-t\mathfrak{e}},\, e^{t\mathfrak{e}}\mathbf{V}e^{-t\mathfrak{e}}}(e^{t\mathfrak{e}} R e^{-t\mathfrak{e}})$$

has zero derivative, which is (19.2) for $S = e^{t\mathfrak{e}}$, and products of such S cover a neighborhood of the identity.

For (b), the right side of (19.2) is holomorphic in complex S by Lemma 19.14 below, whose proof does not use the present lemma, since the dependence on $\mathbf{V}$ is a finite multilinear tensor. Fix a real output R in the interior of the set where the output cutoffs

equal one. There the output of the last heat is real analytic, and its germ at R extends holomorphically to SRS^{-1} for S in a small complex ball $\mathcal{B}$ about the identity. The two sides of (19.2) are holomorphic on the connected set $\mathcal{B}$ and agree for real S, which form a totally real subspace of maximal dimension. By the identity theorem they agree on $\mathcal{B}$.

For (c), a small ambient S keeps the damping in $\mathcal{U}_1$ and transforms each jet into a bounded combination of basis jets. We apply Lemma A.5 with $\zeta = \Gamma$, with the real body point R as the variable x, with the measure m given by the coefficient measure with weight $e^{2a|Z|^2}v^{2p}$, with the selection $\zeta_*(R) = S(R)^{-1}\Gamma S(R)$, and with ρ the distance from $\mathcal{U}_1$ to the complement of $\mathcal{U}_2$. The right side of that lemma is at most $|B_\rho|^{-1}\|f\|^2_{\mathcal{U}_2;p,a}$, which proves (19.4). Conjugation by a full supermatrix may create formal BF and FB entries of Γ and odd entries of the jet directions. We keep their finite coefficient expansion, using the same finite Duhamel and cofactor identities as for complete source jets; this changes the bound only by fixed powers of W and requires fixed stronger Gaussian and body weights. Only the ordinary BB and FF entries belong to the parameter balls. □

19.2 The escape ramp

In this section we show that, after one block of $m = \lceil C_0 W \rceil$ steps, the partial chain $f_m = T_\Gamma^m 1$ is concentrated near the saddle manifold, with a Gaussian tail e^{-cWr^2} at distance r. The tool is an Agmon estimate with a time-dependent weight [1]. We conjugate the heat by $e^{A\vartheta}$, where ϑ is the escape function of (15.32) and A is a real coefficient of size $O(a_{\mathrm{esc}}W)$. The coefficient decreases during each heat interval and is raised over the block using the onsite factors. The time derivative of the weight supplies a negative term, which dominates the positive term that the conjugation creates. Since the heat is not selfadjoint on the cycle, the first-order term of the conjugated generator in the BB block is not skew, and we control it with the equivariance of Lemma 19.2. In the FF block the cycle is flat and this difficulty does not arise (Lemma 19.3).

The conjugated generator. Let L be the generator of the ambient heat pulled to the cycle, as in the proof of Lemma 15.7. The cycle has unit holomorphic Jacobian, so L has divergence form. We first compute its principal part in the BB block. Write $B = DF_X$ for the derivative of $X \mapsto -\Lambda f_0(X)$. Expanding the body of $\mathrm{STr}(\,\mathrm{d}S_B^2)$ in the coordinates (P_0, X), we find

$$\|\,\mathrm{d}P_0\|^2 + \|\,\mathrm{d}X\|^2 - \|B\,\mathrm{d}X\|^2 + 2\mathrm{i}\langle\,\mathrm{d}P_0, B\,\mathrm{d}X\rangle.$$

The inverse of this form gives the principal coefficient of the BB body of $L = \nabla \cdot a\nabla$, namely

$$a = \begin{pmatrix} \mathrm{Id} - BB^* & -\mathrm{i}B \\ -\mathrm{i}B^* & \mathrm{Id} \end{pmatrix}, \qquad \mathrm{Re}\, a = \begin{pmatrix} D_N & 0 \\ 0 & \mathrm{Id} \end{pmatrix}, \qquad \mathrm{Im}\, a = -\begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}, \tag{19.7}$$

where $D_N = \mathrm{Id} - BB^* = \mathrm{ch}_{\mathrm{g}}^{-2}$ is the normal real diffusion metric; one checks $a \cdot (\text{form}) = \mathrm{Id}$ by multiplying the block matrices. For a real function ϑ and a real constant A, we have $e^{A\vartheta} \nabla e^{-A\vartheta} = \nabla - A\nabla\vartheta$, and therefore

$$e^{A\vartheta} L e^{-A\vartheta} = L - A\big(\nabla\vartheta \cdot a\nabla + \nabla \cdot (a\nabla\vartheta\,\cdot)\big) + A^2\, \nabla\vartheta \cdot a\nabla\vartheta. \tag{19.8}$$

In the first-order part, the term with $\mathrm{Re}\, a$ is $Y \cdot \nabla + \nabla \cdot (Y\,\cdot)$ with the real field $Y = \mathrm{Re}\, a\nabla\vartheta$, which is exactly skew-adjoint for the flat measure, with no zeroth-order remainder. The term with $\mathrm{Im}\, a$ is $\mathrm{i}(2\mathcal{Z} \cdot \nabla + \mathrm{div}\, \mathcal{Z})$ with the real field

$$\mathcal{Z} = \mathrm{Im}\, a\, \nabla\vartheta = -(T_P, V), \qquad T_P = B\nabla_X\vartheta, \quad V = B^*\nabla_{P_0}\vartheta.$$

Taking real parts, we obtain

$$\mathrm{Re}\big\langle u, e^{A\vartheta} L e^{-A\vartheta} u\big\rangle = \mathrm{Re}\langle u, Lu\rangle + 2A\, \mathrm{Im}\langle u, \mathcal{Z} \cdot \nabla u\rangle + A^2 \int \nabla\vartheta \cdot \mathrm{Re}\, a\nabla\vartheta\, |u|^2. \tag{19.9}$$

The last term is the squared gradient of ϑ in the real diffusion metric. We will see that it is at most $C\vartheta$, and that the current term $\mathcal{Z} \cdot \nabla$ is the delicate one.

The FF part of the current. On the FF block of the cycle, the Hermitian matrices M form a flat Cartesian space, and the FF part of the generator is a real symmetric diffusion. The next lemma computes the FF part of the conjugated form exactly, so that the equivariance is needed only in the BB block.

Lemma 19.3. *In the coordinates (P_0, X, M) of the cycle and the transported odd frame, the generator decomposes as $L = L_B + \Delta_M + L_{\mathrm{odd}}$, where Δ_M is the flat Laplacian of the Hilbert–Schmidt metric on Hermitian $2k \times 2k$ matrices, L_B acts only in (P_0, X) and in the odd coordinates through the frame, and L_{odd} acts only in the odd coordinates, with coefficients depending only on X. Consequently, for every real constant A and every u in the form domain,*

$$\mathrm{Re}\big\langle u, e^{A\vartheta_F} \Delta_M e^{-A\vartheta_F} u\big\rangle = -\|\nabla_M u\|^2 + A^2 \int |\nabla_M \vartheta_F|^2 |u|^2 \le -\|\nabla_M u\|^2 + C_F A^2 \int \vartheta_F |u|^2, \tag{19.10}$$

where the inner product is that of $\mathcal{H}_p(\mathcal{S})$, of the parameter Hardy space $\mathcal{A}_\rho(\mathcal{H}_p(\mathcal{S}))$ introduced below, or of their weighted versions with any weight that does not depend on M, and C_F is the universal constant of (15.33)*. Moreover, $e^{A\vartheta_F}$ commutes with L_B, L_{odd}, the body weights v^p, the parameter structure, and the BB escape weight.*

Proof. We first prove the decomposition. The FF block of a point S of the cycle satisfies $S_{FF} = \kappa\mathrm{Id} + \mathrm{i}M$, and its holomorphic derivatives are $-\mathrm{i}$ times the derivatives in M. The FF part of the ambient Laplacian of STr carries the sign -1 of the supertrace, which cancels the factor $(-\mathrm{i})^2$. The FF part of the generator is therefore $+\Delta_M$, where Δ_M is the Laplacian of the metric $-\mathrm{Tr}(\mathrm{i}\,\mathrm{d}M)^2 = \|\,\mathrm{d}M\|^2$, that is, the flat Laplacian for the Hilbert–Schmidt metric in an orthonormal basis of Hermitian matrices. The measure of the cycle

is Lebesgue measure in M. The ambient Laplacian has no mixed second derivatives between different entries, and the transported odd frame is defined by congruence with the BB boost $\mathrm{g} = \mathrm{g}(X)$, which does not depend on M. In particular, no derivative in M acts on the frame, and Δ_M commutes with L_B and L_{odd}.

Next, we compute the conjugation. For real A and the real function ϑ_F of M, we have

$$e^{A\vartheta_F}\Delta_M e^{-A\vartheta_F}u = \Delta_M u - 2A\nabla_M\vartheta_F\cdot\nabla_M u - A(\Delta_M\vartheta_F)u + A^2|\nabla_M\vartheta_F|^2 u.$$

Here Δ_M acts on coefficient vectors componentwise, since the frame does not depend on M, and the coefficient metric is positive. Integrating by parts in M, we find that $\mathrm{Re}\langle u, \Delta_M u\rangle = -\|\nabla_M u\|^2$ and

$$\mathrm{Re}\big\langle u, -2A\nabla_M\vartheta_F\cdot\nabla_M u\big\rangle = -A\int \nabla_M\vartheta_F\cdot\nabla_M|u|^2 = A\int(\Delta_M\vartheta_F)|u|^2,$$

which cancels the contribution of $-A(\Delta_M\vartheta_F)u$. This is the identity in (19.10), and the inequality follows from (15.33). The weights of the norms are independent of M and do not change the integration by parts. The commutation statements hold since ϑ_F depends only on M, while L_B, L_{odd}, v, ϑ_B, and the parameter Γ do not involve M. □

The identity (19.10) is the Davies–Agmon identity for a real weight and a real symmetric diffusion. The divergence term of the current cancels exactly, which is why the constant is universal. For the rigorous argument below we replace ϑ_F by $\zeta_n(\vartheta_F)$, where ζ_n is bounded, smooth, increasing, and concave with $\zeta_n(0) = 0$, $\zeta_n' \le 1$, and $\zeta_n(x) \uparrow x$. Since $\zeta_n' \le 1$ and $x\zeta_n'(x) \le \zeta_n(x)$ by concavity, (15.33) implies that

$$|\nabla_M\zeta_n(\vartheta_F)|^2 = \zeta_n'(\vartheta_F)^2|\nabla_M\vartheta_F|^2 \le C_F\zeta_n'(\vartheta_F)^2\vartheta_F \le C_F\zeta_n(\vartheta_F).$$

Hence (19.10) holds for the saturated weights with the same constant.

The BB part of the current. In the BB block the field $\mathcal{Z}$ is not small, since $\|V\|$ grows like the boost. We now show that, up to a remainder bounded by the real gradient energy, it is the generator of a conjugation, to which the equivariance (19.3) applies. Write $G = \nabla_{\widetilde{P}}\vartheta_B$ and $\Xi = \mathrm{ch}_{\mathrm{g}}\, G$. Since ϑ_B depends on P_0 through $\widetilde{P} = \mathrm{ch}_{\mathrm{g}}\, P_0$, we have $\nabla_{P_0}\vartheta_B = \Xi$ and $V = B^*\Xi$, and

$$\nabla\vartheta_B\cdot\mathrm{Re}\, a\nabla\vartheta_B = \langle\Xi, D_N\Xi\rangle + \|\nabla_X\vartheta_B\|^2 = \|G\|^2 + \|\nabla_X\vartheta_B\|^2. \tag{19.11}$$

Let $\mathcal{M} = \lambda_f + \rho_f$, where λ_f and ρ_f are left and right multiplication by $f_0(X)$, so that $\mathcal{M} = 2s\cosh(\lambda_{\log\mathrm{g}} + \rho_{\log\mathrm{g}})\,\mathrm{ch}_{\mathrm{g}}$. Substituting into (15.27) in an eigenbasis of $\log\mathrm{g}$, we obtain

$$\|\,\mathrm{ch}_{\mathrm{g}}\,\mathcal{M}^{-1}V\| \le \|G\|/(2s), \qquad \mathrm{ch}_{\mathrm{g}}\,\mathcal{M}^{-1}\Lambda = \Lambda\,\mathrm{ch}_{\mathrm{g}}^{-1}/(2s). \tag{19.12}$$

Let $\mathfrak{e}$ be the off-diagonal solution of the dissipative Sylvester equation

$$-\mathcal{M}\mathfrak{e} + \mathrm{i}\Lambda[\mathfrak{e}, P_0] = V, \tag{19.13}$$

whose unweighted inverse has norm at most $(2s)^{-1}$.

We first identify the conjugation field of $\mathfrak{e}$. On restrictions to $\mathcal{S}$ of holomorphic functions, the generator $\mathcal{W}_{\mathfrak{e}}$ of the conjugation $S \mapsto e^{t\mathfrak{e}} S e^{-t\mathfrak{e}}$ acts as $\delta P \cdot \nabla_{P_0} + \delta X \cdot \nabla_X$ plus terms in the odd variables, which we denote by $\mathcal{W}_{\mathfrak{e}}^{\mathrm{odd}}$, where $(\delta P, \delta X)$ is the tangent vector of $\mathcal{S}$ whose image under the complexified tangent map $(\delta P, \delta X) \mapsto \delta P - \mathrm{i}\Lambda(Df_0\,\delta X + \delta X)$ is $[\mathfrak{e}, S_B]$. Comparing off-diagonal and block-diagonal parts, we have

$$\delta X = \mathrm{i}\Lambda[\mathfrak{e}, S_B]_{\mathrm{off}}, \qquad \delta P = [\mathfrak{e}, S_B]_{\mathrm{diag}} + \mathrm{i}\Lambda Df_0(\delta X).$$

Since $\mathfrak{e}$ is off-diagonal, $[\mathfrak{e}, S_B]_{\mathrm{off}} = [\mathfrak{e}, P_0] - \mathrm{i}[\mathfrak{e}, \Lambda f_0]$ and $[\mathfrak{e}, S_B]_{\mathrm{diag}} = -\mathrm{i}[\mathfrak{e}, \Lambda X] = \mathrm{i}\Lambda\{\mathfrak{e}, X\}$, and since $\Lambda\mathfrak{e}\Lambda = -\mathfrak{e}$, we have $\Lambda[\mathfrak{e}, \Lambda f_0] = -\{\mathfrak{e}, f_0\} = -\mathcal{M}\mathfrak{e}$. The Sylvester equation (19.13) implies that $\delta X = V$. Next, differentiating $f_0^2 = s^2 + X^2$ shows that $Df_0(\{\mathfrak{e}, f_0\}) = \{\mathfrak{e}, X\}$, so the anticommutator terms cancel, and

$$\delta P = -\Lambda Df_0\big(\Lambda[\mathfrak{e}, P_0]\big) = \mathrm{th}_{\mathrm{g}}[\mathfrak{e}, P_0].$$

For the last identity, note that in an eigenbasis of X we have $(Df_0 Y)_{ab} = \tanh((r_a + r_b)/2) Y_{ab}$ by (15.27), and that Λ maps the eigenvector of r_a to that of $-r_a$, since $\Lambda X \Lambda = -X$; hence $-\Lambda Df_0(\Lambda Y) = \mathrm{th}_{\mathrm{g}} Y$ for block-off-diagonal Y, in particular for $Y = [\mathfrak{e}, P_0]$. Since $\mathcal{Z} = -(T_P, V)$, we conclude that

$$\mathcal{Z} \cdot \nabla = -\mathcal{W}_{\mathfrak{e}} + \mathcal{W}_{\mathfrak{e}}^{\mathrm{odd}} - (T_P - \delta P) \cdot \nabla_{P_0}. \tag{19.14}$$

We now bound the pieces of (19.14); all constants are dimension-free. Assume $\|\widetilde{P}\| \le s/2$, which holds on the support of $\nabla\vartheta_B$ when $r_{\mathrm{esc}} < s_0/2$. Multiplying (19.13) by $\mathrm{ch}_{\mathrm{g}}\,\mathcal{M}^{-1}$, and using (19.12) and $\|[\mathfrak{e}, P_0]\| \le 2\|\mathrm{ch}_{\mathrm{g}}\,\mathfrak{e}\|\|\widetilde{P}\|$, we find

$$\|\mathrm{ch}_{\mathrm{g}}\,\mathfrak{e}\| \le s^{-1}\|G\| \le C\|\widetilde{P}\|. \tag{19.15}$$

The inequality $\cosh(a-c) \le 2\cosh(a-b)\cosh(b-c)$ implies $\|\mathrm{ch}_{\mathrm{g}}(A_1 A_2)\| \le 2\|\mathrm{ch}_{\mathrm{g}}\,A_1\|\|\mathrm{ch}_{\mathrm{g}}\,A_2\|$, and therefore

$$\|\mathrm{ch}_{\mathrm{g}}\,\delta P\| \le 4\|\mathrm{ch}_{\mathrm{g}}\,\mathfrak{e}\|\|\widetilde{P}\| \le 4s^{-1}\|G\|\|\widetilde{P}\|.$$

For T_P we use duality. For a unit block-diagonal B', the direction $V' = DF_X^*\,\mathrm{ch}_{\mathrm{g}}\,B'$ has defect $d_X(\mathrm{ch}_{\mathrm{g}}\,B') = \|B'\|^2 = 1$, and $\nabla_X\vartheta_B$ involves the derivative of $\|\widetilde{P}\|^2 = (\|P_0\|^2 + \|\mathrm{g}P_0\mathrm{g}^{-1}\|^2)/2$. By the second line of (15.30), we have

$$\|\mathrm{ch}_{\mathrm{g}}\,T_P\| = \sup_{\|B'\|=1} \langle V', \nabla_X\vartheta_B\rangle \le 8\sup|\chi_0'|\,\|\widetilde{P}\|^2 \sup_{\|B'\|=1} \|D\mathrm{g}(V')\mathrm{g}^{-1}\|_{\mathrm{op}} \le C\|\widetilde{P}\|^2.$$

Combining the last two bounds with $D_N^{-1/2} = \mathrm{ch}_{\mathrm{g}}$ and $\|G\| \le C\|\widetilde{P}\|$, we obtain

$$\|D_N^{-1/2}(T_P - \delta P)\| \le C\|\widetilde{P}\|^2, \qquad \langle \Xi, D_N \Xi\rangle = \|G\|^2. \tag{19.16}$$

Since $\nabla\vartheta_B$ vanishes where $\|\widetilde{P}\| \geq r_{\mathrm{esc}}$, the squared remainder satisfies $\|\widetilde{P}\|^4 \leq C r_{\mathrm{esc}}^2 \vartheta_B$, and by (19.11) the squared gradient of the weight in the real diffusion metric satisfies $\nabla\vartheta_B \cdot \operatorname{Re} a \nabla\vartheta_B \leq C(\|G\|^2 + \|\widetilde{P}\|^4) \leq C\vartheta_B$. The same argument gives the bound on the commutator of the conjugation field with ϑ_B,

$$\begin{aligned} |\mathcal{W}_{\mathfrak{e}}\vartheta_B| &\leq |\langle \delta P, \Xi\rangle| + |\langle V, \nabla_X \vartheta_B\rangle| \\ &\leq \|\operatorname{ch}_{\mathsf{g}} \delta P\| \|G\| + 8 \sup|\chi_0'| \|\widetilde{P}\|^2 \|D\mathsf{g}(V)\mathsf{g}^{-1}\| \leq C r_{\mathrm{esc}} \vartheta_B, \end{aligned}$$

where for the middle term we used the second line of (15.30) for $V = DF_X^* \Xi$, whose defect is $\|G\|^2$. Finally, in the transported frame, $\mathcal{W}_{\mathfrak{e}}^{\mathrm{odd}}$ acts on the odd coordinates by $O_1 \mapsto -\mathsf{g}^{-1}\mathfrak{e}\mathsf{g}\, O_1$ plus the frame term $\mathsf{g}^{-1}D\mathsf{g}(V)$. Since $\|\mathsf{g}^{-1}\mathfrak{e}\mathsf{g}\|^2 = 2\|\operatorname{ch}_{\mathsf{g}} \mathfrak{e}\|^2 - \|\mathfrak{e}\|^2$, this action has norm at most $C\|G\|$ by (19.15) and (15.30).

The parameter Hardy space. Let $\mathcal{A}_\rho$ be the Hardy space of holomorphic functions on the parameter ball of radius ρ about the normalized base point Λ, with values in $\mathcal{H}_p(\mathcal{S})$, and let $\mathbf{N}_\Gamma$ count the homogeneous degree in the parameter. A homogeneous polynomial p of degree n satisfies $\|p\|_{\mathcal{A}_\rho}^2 = \rho^{2n}\|p\|_{\mathcal{A}_1}^2$, so for fixed u,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\mathcal{A}_{\rho(t)}}^2 = \frac{2\rho'(t)}{\rho(t)} \langle u, \mathbf{N}_\Gamma u\rangle. \tag{19.17}$$

Decreasing the radius ρ during the evolution therefore supplies a negative number form. This is the device of a scale of Banach spaces of Ovsyannikov and Nirenberg–Nishida [98, 99, 101]. A bounded affine parameter field moves the degree by at most one, and Cauchy–Schwarz between neighboring degrees bounds its quadratic form by $C_k\langle u, (\mathbf{N}_\Gamma + 1)u\rangle$.

The weighted heat form. Let $u = e^{A\vartheta} f_\Gamma$, where f_Γ is the physical family and $|A| \leq 2a_{\mathrm{esc}} W + C$. We insert (19.14) into the current term of (19.9). The field $\mathfrak{e} = \mathfrak{e}(S)$ depends on the point, but $\mathcal{W}_{\mathfrak{e}(S)} f(S)$ involves only the value $\mathfrak{e}(S)$, so the equivariance (19.3) applies pointwise and gives

$$\mathcal{W}_{\mathfrak{e}} u = \big\langle [\Gamma, \mathfrak{e}(S)], \partial_\Gamma \big\rangle u + A(\mathcal{W}_{\mathfrak{e}}\vartheta) u,$$

since ϑ does not depend on Γ. By (19.15), $\|\mathfrak{e}\| \leq \|\operatorname{ch}_{\mathsf{g}} \mathfrak{e}\| \leq C r_{\mathrm{esc}}$, so the parameter field has form at most $C r_{\mathrm{esc}}\langle u, (\mathbf{N}_\Gamma + 1)u\rangle$ on $\mathcal{A}_\rho$, and the scalar commutator contributes at most $CA^2 r_{\mathrm{esc}} \int \vartheta_B |u|^2$. By (19.16) and Young's inequality in the metric D_N, the contribution of the remainder $(T_P - \delta P)\cdot\nabla_{P_0}$ is at most a fraction of the real gradient energy plus $CA^2 r_{\mathrm{esc}}^2 \int \vartheta_B |u|^2$, and $\mathcal{W}_{\mathfrak{e}}^{\mathrm{odd}}$ has norm $C\|G\|$. Collecting these bounds with the real-form calculation of Lemma 15.7 and with Lemma 19.3 for the FF part, we obtain

$$\operatorname{Re}\big\langle u, e^{A\vartheta} L e^{-A\vartheta} u\big\rangle_{\mathcal{A}_\rho} \leq -c\, \mathcal{E}_{\mathrm{real}}(u) + C_k W \|u\|^2 + CA^2 \int \vartheta |u|^2 + C_k |A| r_{\mathrm{esc}} \langle u, (\mathbf{N}_\Gamma + 1)u\rangle, \tag{19.18}$$

where $\mathcal{E}_{\mathrm{real}}(u)$ is the real gradient energy of the proof of Lemma 15.7 and C is universal. The last term comes from the parameter field. By (19.17), it is absorbed when ρ decreases at the logarithmic speed $C_k|A|r_{\mathrm{esc}}$.

Proposition 19.4. *There exist universal constants $c, a_{\mathrm{esc}} > 0$ with $3a_{\mathrm{esc}}/C_0 < c/4$ such that the following holds at fixed rank k after choosing r_{esc} so small that $C_k a_{\mathrm{esc}} C_0 r_{\mathrm{esc}} \le \log 2$, where C_k is the constant of the last term of* (19.18). *Let ρ be the radius of a parameter ball about Λ inside $\mathcal{U}_2$, and set $\rho_{\mathrm{out}} = \rho/2$. For $m = \lceil C_0 W \rceil$, every body weight p and all large W,*

$$\left\| e^{2a_{\mathrm{esc}} W\vartheta} f_{m,\Gamma} \right\|_{\mathcal{A}_{\rho_{\mathrm{out}}}(\mathcal{H}_p(\mathcal{S}))} \le C_k \eta^{-A_k}. \tag{19.19}$$

Proof. Let c be the onsite gain of Lemma 15.9, and choose a_{esc} so small that $C(2a_{\mathrm{esc}}W + C)^2 \le \frac{c}{8}W^2$ for the constant C of (19.18), and $3a_{\mathrm{esc}}/C_0 < c/4$. We raise the escape coefficient along the block. We set

$$a_1 = -a_{\mathrm{esc}}, \qquad a_m = 2a_{\mathrm{esc}}, \qquad a_j - a_{j-1} = \frac{3a_{\mathrm{esc}}}{m-1} \quad (2 \le j \le m).$$

Since $K_\tau 1 = 1$, we have $f_1 = w_\Gamma$, and since $a_1 < 0$, the bound on w_Γ at the start of this section gives $\|e^{Wa_1\vartheta} f_1\|_{\mathcal{A}_\rho(\mathcal{H}_p(\mathcal{S}))} \le \|w_\Gamma\|_{\mathcal{A}_\rho(\mathcal{H}_p(\mathcal{S}))} \le C_k\eta^{-A_k}$; we start the induction at $j = 1$, since the raw constant function has no coefficient Hilbert norm.

For $2 \le j \le m$, during the heat interval of step j we interpolate the coefficient $A(t)$ linearly from Wa_{j-1} to $Wa_j - c/2$. Since $m - 1 = C_0W + O(1)$ and $3a_{\mathrm{esc}}/C_0 < c/4$, we have $W(a_j - a_{j-1}) = (3a_{\mathrm{esc}}/C_0)(1 + O(W^{-1})) \le c/4$ for all sufficiently large W. Its derivative therefore satisfies

$$A'(t) = \frac{W(a_j - a_{j-1}) - c/2}{\tau} \le \Big(\frac{c}{4} - \frac{c}{2}\Big)2W^2 = -\frac{c}{2}W^2.$$

In the energy identity for $\|e^{A(t)\vartheta}u(t)\|^2_{\mathcal{A}_{\rho(t)}}$, the term $A'(t)\int \vartheta|u|^2$ is therefore at most $-\frac{c}{2}W^2\int\vartheta|u|^2$. By our choice of a_{esc}, we have $CA^2 \le \frac{c}{8}W^2$, and therefore this term dominates the term $CA^2\int\vartheta|u|^2$ of (19.18).

The other terms of (19.18) are lower order. The term $C_kW\|u\|^2$ integrates over the heat interval to the factor $e^{C_kW\tau} = e^{C_k/W}$, and the number form is absorbed by the decrease of ρ. After the heat, the function is bounded in the weight $e^{(Wa_j - c/2)\vartheta}$. By (15.34), the whole onsite factor satisfies $|w_\Gamma|_{1;h} \le e^{C_k/W + C_k\eta}e^{-c\vartheta}$, and therefore multiplication by it restores the weight $e^{Wa_j\vartheta}$, up to the factor $e^{C_k/W + C_k\eta}$. The order of the factors is the actual onsite-after-heat order of T_Γ.

Iterating over the $m - 1 \le 2C_0W$ steps, these factors multiply to at most $e^{C_kC_0}$, and the total decrease of $\log\rho$ is at most $C_ka_{\mathrm{esc}}C_0r_{\mathrm{esc}} \le \log 2$, so the final radius is at least ρ_{out}. The universal a_{esc} is not shrunk with the rank, and this completes the proof of (19.19).

The calculation above is formal in two respects, which we now remove. First, we saturate ϑ_F by the bounded concave functions ζ_n described after Lemma 19.3, whose

gradient and current bounds are uniform in n. Second, we carry out the calculation for the physical family with compact spatial cutoffs and on a larger parameter ball, retaining the boundary terms. Fixed unscaled derivatives and polynomial moments of the family are finite after the first step, because of the positive damping, and they dominate each cutoff error. We pass to the limit by spatial exhaustion and then by Fatou's lemma in the saturation. The equivariance (19.3) is used only for the actual family, and the truncations enter only through these boundary terms. □

The zero set of ϑ is the union of the $2k+1$ FF signatures of the saddle manifold. Since $\vartheta \geq cr^2$ outside the union of the normal cores of radius r, (19.19) implies the global tail bound $C_k W^{A_k} e^{-cWr^2}$ for f_m outside any fixed sufficiently small union of such cores.

19.3 The endpoint factors on the tube

We now turn to the adapted tube, with the tubular coordinates (x, U) of (16.12), in one root signature. Recall the right endpoint g_R and its analytic dual g_L from (16.38), and the rotation phases $e^{-\mathrm{i}\theta_\sigma}$ of the root supermatrices, with $\theta_\sigma = \frac{1}{2}\arg c_\sigma$ as in (16.13) and $|\theta_\sigma| \leq C|E|$. We write U for the coordinate on the adapted real normal space of Definition 16.1 and Z for its standardized version, so that $c_Z W|U|^2 \leq |Z|^2 \leq C_Z W|U|^2$ with universal constants c_Z, C_Z, and we set $w_a = e^{a|Z|^2}$ for $a \in \mathbb{R}$. Let $\widetilde{\chi}$ be a real invariant normal cutoff with $0 \leq \widetilde{\chi} \leq 1$, supported in the tube, such as χ_{inv}.

Lemma 19.5. *In the balanced signature, for every sufficiently small fixed $a > 0$,*

$$\|e^{-a|Z|^2}\widetilde{\chi} g_R\| + \|e^{-a|Z|^2}\widetilde{\chi} g_L\| \leq C_{k,a}.$$

At every other signature the same norms are at most $C_{k,a}W^{A_k}$. The tangent part of the endpoint is the constant function, and for some sufficiently negative $p_0 = p_0(k)$ the same bounds hold in the body weight v^{p_0}.

Proof. We compute the quadratic parts first; the full endpoints differ from them by the bounded factors $w_{\mathrm{anh}}^{\mp 1/2}$ on the tube (Remark 16.9). In one even coordinate of root σ, write $U = e^{-\mathrm{i}\theta_\sigma} x$ with x real. The proof of Lemma 16.10 gives the quadratic half-weight $w_{\mathrm{quad}}^{1/2} = e^{-m_\sigma e^{-2\mathrm{i}\theta_\sigma} x^2/4}$, while the Gaussian gauge maps the flat ground state $e^{-|c_\sigma| x^2/2}$ to the Wick vacuum $\Omega_0 = 1$. Here $Z = \sqrt{2|c_\sigma|}\, x$, so that $\gamma_h^2 = e^{-|Z|^2/2}$. Against this Gaussian density of the positive Wick metric of Definition 16.6, the squared moduli of the quadratic parts of g_R and g_L therefore carry the residual real exponents

$$\pm \frac{\mathrm{Re}\big(m_\sigma e^{-2\mathrm{i}\theta_\sigma}\big)}{4|c_\sigma|}|Z|^2 = O(W^{-1})|Z|^2,$$

since $|c_\sigma| \asymp W$ by (16.40). The odd coefficients contain the factors $(c_\sigma \pm m_\sigma/2)e^{-2\mathrm{i}\theta_\sigma}/(2|c_\sigma|)$, which are bounded. In the balanced signature each

normal root has equal even and odd dimensions, and the standardizing change of variables has canceling Berezinians. The norm is therefore bounded by a constant depending on k times $\int e^{-(2a-C/W)|Z|^2}\,\mathrm{d}Z$, uniformly for large W. The half-density ϖ and the factors $w_{\mathrm{anh}}^{\pm 1/2}$ have bounded coefficients on the tube and preserve this estimate. In an unbalanced signature the same scaling calculation leaves a fixed power of $|c_\sigma|$, hence a fixed power of W, and the Gaussian estimate is unchanged; these powers of W are kept until the unbalanced signature is suppressed in Section 20. The tangent part is the constant function, whose norm is finite in the body weight v^{p_0} for p_0 sufficiently negative. □

The left endpoint. The analytic left endpoint reduces to the right one. By (16.38), we have $g_L = w_0 g_R$ and $A_h = s_\Gamma T_h$, where T_h and s_Γ are symmetric for the analytic Berezin pairing. Thus $A_h^{\mathrm{tr}} = T_h s_\Gamma$, and

$$(A_h^{\mathrm{tr}})^n g_L = T_h A_h^{n-1}(s_\Gamma w_0 g_R), \qquad \ell_m = (T_\Gamma^{\mathrm{tr}})^m 1 = K_{\tau_{\mathrm{phys}}} f_m. \tag{19.20}$$

The second identity is Proposition 4.11. Every family obtained from the left endpoint is therefore obtained from the right one, with the input $s_\Gamma w_0 g_R$ in place of g_R, followed by one extra unsourced step. Since $w_0^{\pm 1/2}$ are bounded multipliers on the tube and s_Γ is the source factor of an adapted step, all the estimates below for families obtained from the right endpoint apply to those obtained from the left endpoint, and we state only the former.

19.4 A weighted energy estimate for short blocks

A block of $\lceil C_0 W \rceil$ steps has physical heat time of order C_0/W, which is why, as explained at the start of this section, we can work with energy estimates in the raw realization. We prove one such estimate, Lemma 19.8, for signed Gaussian weights and compact normal graphs, and transport it to the symmetric normal gauge by the similarity of (16.38), where it gives the signed ramp Corollary 19.11 and the bound Proposition 19.13 for a block on a normal graph. Its input on the onsite factor is Lemma 19.7. In the raw realization, the coefficient of $|b|^2$ per step on a graph of height b comes only from the flat normal metric, the signed weight, the quadratic part of the onsite factor, and the velocity of the graph, and Remark 19.10 checks that it does not depend on the rank. We also prove the Davies–Gaffney estimate Lemma 19.9, which Section 19.7 uses.

The raw steps. We work on the tube of one signature, in the raw realization of Definition 16.7, in which one link acts by the operator K^ω of (16.33), and in the coefficient metric of Lemma 18.4, in which the odd normal coordinates carry their standardized

scaling and the odd tangent coordinates a small degree metric. The adapted step and the frozen normal transfer, with their cutoffs, in the raw realization are

$$S_\Gamma = \chi_{\mathrm{inv}}\, s_\Gamma w_0 K^{\omega} \chi_{\mathrm{inv}}, \qquad S_N = \chi_{\mathrm{inv}}\, w_0 K^{\omega}_{\mathrm{nor}} \chi_{\mathrm{inv}}. \tag{19.21}$$

Here w_0 is the physical source-free weight, including its saddle value $\zeta_p = w_0(R_*)$ in signature p. Each raw step retains this scalar phase; since $|\zeta_p| = 1$, it does not change the norm estimates below. All the multipliers involved commute, and by (16.38) we have, for all $n \geq 0$,

$$A_h^n = \gamma_h^{-1} w_0^{-1/2}\, S_\Gamma^n\, w_0^{1/2} \gamma_h, \qquad \mathbf{F}_h^n = \gamma_h^{-1} w_0^{-1/2}\, S_N^n\, w_0^{1/2} \gamma_h, \tag{19.22}$$

where A_h and $\mathbf{F}_h$ carry the same cutoffs. Let $\|\cdot\|_{\mathrm{raw}}$ be the norm of L^2 with respect to $\mathrm{d}Z$ and the orbit measure, in the coefficient metric above. Recall from (16.35) that the body of γ_h is a Gaussian in U; multiplication by the body of γ_h^{-1} is unitary from the raw metric onto the Wick metric up to a constant factor, and $w_0^{\pm 1/2}$ and the odd part of $\gamma_h^{\pm 1}$ are bounded coefficient multipliers on the tube. It follows that for all $|a| \leq 1$,

$$C_k^{-1} \|w_a g\|_{\mathrm{raw}} \leq \|w_a \gamma_h^{-1} w_0^{-1/2} g\| \leq C_k \|w_a g\|_{\mathrm{raw}}, \tag{19.23}$$

where the constant factor of γ_h cancels in (19.22). The upper bound also holds for values on a compact normal graph $U \mapsto U_\lambda$ as in (18.15), supported where the cutoffs equal one. Indeed, $w_0^{-1/2}$ is analytic and bounded on the complex tube, and if Z_β denotes the standardized version of the height $\beta = \lambda \psi_{R_k}(U) b$, then the Gaussian body of γ_h^{-1} satisfies

$$|\gamma_h^{-1}(Z + \mathrm{i} Z_\beta)|^2 e^{-|Z|^2/2} = e^{-|Z_\beta|^2/2} \leq 1.$$

Graph norms. We use the compact normal graphs (18.15) with radius $\bar{r} = R_k$, with ψ_{R_k} supported where $\chi_{\mathrm{inv}} = 1$. In the applications the heights are canonical heights at radius at most r_{phys}, and Lemma 19.15 below gives $|b|/R_k \leq C_{\mathrm{geo}}(|E| + r_{\mathrm{phys}}) r_{\mathrm{phys}}/R_k$, which is small once r_{phys}/R_k is small. For f holomorphic in the normal variables where $\psi_{R_k} \neq 0$, we define

$$\|f\|_{\lambda;p,a} = \big\| w_a(U_\lambda)\, v^p \rho_\lambda^{1/2}\, f(U_\lambda) \big\|_{\mathrm{raw}}, \tag{19.24}$$

where $w_a(U_\lambda) = e^{a Z_\lambda^T Z_\lambda}$ is the signed weight evaluated holomorphically, Z_λ being the standardized version of U_λ. For $\lambda = 0$, the norm (19.24) is $\|w_a v^p f\|_{\mathrm{raw}}$ for all f.

Onsite factors and jet steps. Recall from Definition 5.12 and Lemma 5.13 that the ℓ^1 coefficient norm $|w|_1$, in the normalized monomials of the coefficient metric, bounds the norm of multiplication by w. Besides S_Γ and S_N, we use the jet steps

$$S^{(\alpha)} = \chi_{\mathrm{inv}}\, s_\Gamma w_0 J_\alpha K^{\omega} \chi_{\mathrm{inv}}, \tag{19.25}$$

where $J_\alpha = w_\Gamma^{-1} \partial_Y^\alpha w_{\Gamma,Y}|_{Y=0}$ is the ratio of a complete local jet of degree d_α, a derivative at $Y = 0$ of the marked onsite weight $w_{\Gamma,Y}$ of (4.15), to the onsite weight.

We first record the damping that the source factor s_Γ provides on the parameter ball $\mathcal{U}_2$. It needs only the coercivity (15.36) of the parameter balls. Recall from Section 17.1 that, at the point $R(x,0)$ of the saddle manifold, the BB block has the body $\kappa_E \mathrm{Id} - \mathrm{i}s_E Q_B$, where $Q_B = Q_B(x)$ lies in the BB factor H_k of the underlying manifold, and recall the body weight v of (5.3).

Lemma 19.6. *Let $\Gamma \in \mathcal{U}_2$. For every $Q_B \in H_k$, with body weight $v = 1 + \operatorname{Tr}(\Lambda_B Q_B)/2$,*

$$\operatorname{Re}\operatorname{Tr}(Q_B \Gamma_{BB}) \geq c_\varepsilon \operatorname{Tr}(\Lambda_B Q_B) \geq c_\varepsilon v, \tag{19.26}$$

where c_ε is the constant of (15.36)*. Moreover, there exists $c_0 > 0$, depending on $\mathcal{U}_2$, such that the source factor satisfies $|s_\Gamma|_1 \leq e^{C_k\eta - c_0\eta v}$ on the real tube.*

Proof. We first prove (19.26). Write $Q_B = g\Lambda_B g^{-1}$ with $g \in U(k,k)$, as in the proof of Lemma 5.1. Since $g^{-1} = \Lambda_B g^* \Lambda_B$, the matrix $\Lambda_B Q_B = (\Lambda_B g)(\Lambda_B g)^*$ is positive definite. Using $\Lambda_B^2 = \mathrm{Id}$, we write $\operatorname{Tr}(Q_B\Gamma_{BB}) = \operatorname{Tr}(\Lambda_B Q_B \Gamma_{BB}\Lambda_B)$, and the real part of this trace is the trace of $(\Lambda_B Q_B)^{1/2} \operatorname{Re}(\Gamma_{BB}\Lambda_B)(\Lambda_B Q_B)^{1/2}$, where $\operatorname{Re}(\Gamma_{BB}\Lambda_B)$ is the Hermitian part of $\Gamma_{BB}\Lambda_B$. Using (15.36), we obtain

$$\operatorname{Re}\operatorname{Tr}(Q_B\Gamma_{BB}) \geq c_\varepsilon \operatorname{Tr}(\Lambda_B Q_B) = 2c_\varepsilon(v-1).$$

By Lemma 5.1(a), we have $v - 1 \geq k \geq 1$, and therefore $2(v-1) \geq v$. This proves (19.26).

We now bound the source factor. By (15.6), the body of the bosonic part of s_Γ at $R(x,0)$ is

$$e^{-\mathrm{i}\eta \operatorname{Tr}((\kappa_E \mathrm{Id} - \mathrm{i}s_E Q_B)\Gamma_{BB})} = c_\Gamma\, e^{-\eta s_E \operatorname{Tr}(Q_B\Gamma_{BB})}, \qquad c_\Gamma = e^{-\mathrm{i}\eta\kappa_E \operatorname{Tr}\Gamma_{BB}}.$$

The imaginary part of $\operatorname{Tr}(Q_B\Gamma_{BB})$ gives a factor of modulus one, and $|c_\Gamma| \leq e^{C_k\eta}$, since $|\operatorname{Tr}\Gamma_{BB}| \leq k/2$ on $\mathcal{U}_2$. By (19.26), the modulus of this body is at most $e^{C_k\eta - c_\varepsilon s_0 \eta v}$. The normal remainder of the source has size $\delta_W O_k(|U|)v$, which is bounded by a fixed fraction of the same damping on the tube. The nilpotent source terms carry the factors of the degree metric and damped envelopes of the form $\delta_W v$, and the compact FF terms contribute only a scalar error $O_k(\delta_W)$. These bounds give $|s_\Gamma|_1 \leq e^{C_k\eta - c_0\eta v}$ for some $c_0 > 0$ that depends on $\mathcal{U}_2$. □

The following lemma bounds the onsite factors $s_\Gamma w_0$, w_0 and $s_\Gamma w_0 J_\alpha$ of S_Γ, S_N and $S^{(\alpha)}$ on the compact normal graphs.

Lemma 19.7. *There exist universal constants $c_{\mathrm{kick}}, C_2 > 0$ such that the following holds at fixed rank k, after choosing R_k sufficiently small and then W sufficiently large. Let $\Gamma \in \mathcal{U}_2$, let b be a real normal direction with $|b| \leq R_k$, let $\lambda \in [0,1]$, and let w be one of the onsite factors $s_\Gamma w_0$ and w_0. Then at every point (x,U) of the tube with U real and in the support of χ_{inv},*

$$|w(U_\lambda)|_1 \leq \exp\bigl\{C_k/W - c_{\mathrm{kick}}|U|^2 + C_2|\beta|^2 + C_k R_k|\beta|^2\bigr\}, \qquad \beta = \lambda\psi_{R_k}(U)b. \tag{19.27}$$

Moreover, the ratio J_α of (19.25) *satisfies $|J_\alpha|_1 \le C_k v^{d_\alpha}$ at the points $T(x)(R_* + U)T(x)^{-1}$ with U complex in the tube and $T(x)$ a complex frame near a real one, and the onsite factor $s_\Gamma w_0 J_\alpha$ satisfies* (19.27) *with the additional factor $C_k v^{d_\alpha}$.*

Proof. We first bound the body of w_0. Recall that the weight w_0 is invariant under conjugation, so that on the tube it is a function of $R_* + U$ alone. By the proof of Lemma 16.10, its normal Hessian at R_* is $m_\sigma \operatorname{STr}(U^2)$ on a root block of type σ, and on the adapted real normal space of Definition 16.1 this form is $m_\sigma e^{-2\mathrm{i}\theta_\sigma}|U|^2$. Since $|m_\sigma| \ge 2s_0^2$ and $|\arg(m_\sigma e^{-2\mathrm{i}\theta_\sigma})| \le C|E|$, its real part is at least $4c_{\mathrm{kick}}|U|^2$ for a universal $c_{\mathrm{kick}} > 0$.

Let q_2 be this quadratic form, and write the body of $\log w_0(R_* + U) - \log w_0(R_*)$ as $-q_2(U)/2 + r_{\mathrm{T}}(U)$. The remainder r_{T} is analytic on the tube, and

$$r_{\mathrm{T}}(0) = Dr_{\mathrm{T}}(0) = D^2 r_{\mathrm{T}}(0) = 0, \qquad \|D^3 r_{\mathrm{T}}\| \le C_k.$$

Taylor's formula gives

$$|r_{\mathrm{T}}(U + \mathrm{i}\beta) - r_{\mathrm{T}}(U) - \mathrm{i}Dr_{\mathrm{T}}(U)\beta| \le C_k(R_k + |\beta|)|\beta|^2, \qquad |Dr_{\mathrm{T}}(U)| \le C_k|U|^2. \quad (19.28)$$

Let U and β be real, with $|\beta| \le R_k$ and $C_k R_k \le c_{\mathrm{kick}}/4$. Using Young's inequality on the cross term $\operatorname{Im}(m_\sigma e^{-2\mathrm{i}\theta_\sigma})\langle U, \beta\rangle$, together with (19.28) and $|r_{\mathrm{T}}(U)| \le C_k R_k |U|^2$, we obtain

$$\operatorname{Re}\Big(-\frac{q_2(U + \mathrm{i}\beta)}{2}\Big) \le -\frac{3c_{\mathrm{kick}}}{2}|U|^2 + C|\beta|^2,$$
$$\operatorname{Re}\Big(-\frac{q_2(U + \mathrm{i}\beta)}{2} + r_{\mathrm{T}}(U + \mathrm{i}\beta)\Big) \le -c_{\mathrm{kick}}|U|^2 + C_2|\beta|^2 + 2C_k R_k|\beta|^2.$$

Since $|w_0(R_*)| = 1$ by the saddle normalization, the second bound controls the body of w_0 on the graph.

We now bound the remaining factors in (19.27). The nonbody coefficients of w_0, divided by its body, are analytic and bounded on the tube, since every normal determinant has nonzero body there (Remark 16.9). Each of them contains at least two standardized odd normal coordinates, and hence a factor h^2. Next, by Lemma 19.6, the source factor satisfies $|s_\Gamma|_1 \le e^{C_k\eta - c_0\eta v}$ on the real tube, with $c_0 > 0$ depending on the damping ball $\mathcal{U}_2$. Moving to complex normal points changes the exponent of s_Γ by at most $C\eta|\beta|v \le CR_k\eta v$. We choose R_k so small that $CR_k \le c_0/2$. Then half of the damping absorbs this change, and $|s_\Gamma(U_\lambda)|_1 \le e^{C_k\eta - c_0\eta v/2}$. Since $\eta \le CW^{-2}$ and $1 + C_k h^2 \le e^{C_k/W}$, these bounds give (19.27). Since $|e^{c'\eta v} s_\Gamma(U_\lambda)|_1 \le e^{C_k\eta}$ for $0 \le c' \le c_0/2$, the same argument shows that (19.27) also holds with s_Γ replaced by $e^{c'\eta v}s_\Gamma$, for every $0 \le c' \le c_0/2$.

Finally, we treat a jet step. The ratio J_α is a polynomial of degree d_α in the entries of

$$(E - S)^{-1} = T(x)(E - R_* - U)^{-1}T(x)^{-1}.$$

At the points of the statement, $E - R_* - U$ has a bounded inverse and $\|T(x)\|\|T(x)^{-1}\| \le Cv$ by (15.3), and therefore these entries are bounded by Cv. We conclude that $|J_\alpha|_1 \le C_k v^{d_\alpha}$, with no growth in $\langle Z\rangle$, and $s_\Gamma w_0 J_\alpha$ satisfies (19.27) with the additional factor $C_k v^{d_\alpha}$. □

Lemma 19.8. *There exist universal constants $a_0, v_0, c_1, C_1 > 0$ such that the following holds at fixed rank k, after choosing R_k sufficiently small and then W sufficiently large. Let $1 \le n \le C_0 W$, let each of $S_1, \dots, S_n$ be S_Γ or S_N, with $\Gamma \in \mathcal{U}_2$, and let p be fixed. Let $a'_0, \dots, a'_n$ satisfy $|a'_j| \le a_0$ and $|a'_j - a'_{j-1}| \le v_0 h^2$, let b be a real normal direction with $|b| \le c_1 R_k$, let $\lambda_0, \dots, \lambda_n \in [0,1]$, and set $\beta_j = \lambda_j b$. Then*

$$\|S_n \cdots S_1 f\|_{\lambda_n;p,a'_n} \le C_k \exp\Big\{ \sum_{j=1}^{n} \big[C_1\big(|\beta_j|^2 + W^2|\beta_j - \beta_{j-1}|^2\big) + C_k R_k |\beta_j|^2 \big] \Big\} \|f\|_{\lambda_0;p,a'_0} \tag{19.29}$$

for all f if $\lambda_0 = 0$, and for all f that are holomorphic in the normal variables where $\psi_{R_k} \neq 0$ if $\lambda_0 > 0$. The constant C_k may also depend on C_0. Moreover, let a fixed number J of the S_j be replaced by jet steps $S^{(\alpha_1)}, \dots, S^{(\alpha_J)}$ of (19.25)*, and let $d = d_{\alpha_1} + \cdots + d_{\alpha_J}$ be their total degree. Then, for the same f,*

$$\|S_n \cdots S_1 f\|_{\lambda_n;p,a'_n} \le C_k^{1+J} \exp\Big\{ \sum_{j=1}^{n} \big[C_1\big(|\beta_j|^2 + W^2|\beta_j - \beta_{j-1}|^2\big) + C_k R_k |\beta_j|^2 \big] \Big\} \|f\|_{\lambda_0;p+d,a'_0}. \tag{19.30}$$

Proof. We first represent the block by evolutions on moving graphs. The kernel of $S_n \cdots S_1$ is an iterated integral over the integration variables between consecutive steps, which we call the *nodes*. Where the cutoffs equal one, its integrand is holomorphic in the normal variables of each node, since the onsite factors are analytic on the tube and the ambient Gaussian is entire in the matrix point, which is affine in U at each point of the saddle manifold. By Cauchy's theorem, we may move the normal integration at node j to the graph of parameter λ_j, for $1 \le j < n$, without changing the kernel. The output is holomorphic where its cutoff equals one, and we evaluate it on the graph of parameter λ_n.

Fix $1 \le j \le n$, let $\lambda(t)$ interpolate linearly from λ_{j-1} to λ_j over $0 \le t \le \tau = \tau_{\mathrm{phys}}$, and set $\beta(t) = \lambda(t) b$. We use the graph pull $\mathcal{P}_\lambda$ defined before (18.17): the initial vector for the pulled heat is $\mathcal{P}_{\lambda_{j-1}} F$, including its half-Jacobian, where F is the raw input of this step. Evaluating the output with $\mathcal{P}_{\lambda(t)}$ gives (18.16). By Lemma 18.6, it agrees on the support of the cutoffs with the Dirichlet evolution of (18.17) on a slightly larger tube, up to an error $C_k W^{C_k} e^{-c R_k^2 W^2}$ times the input norm. The weights below are bounded by e^{CW} on the tube. We may therefore neglect this error, and we work with the Dirichlet evolution $u(t)$ of $\partial_t u = (L_t + V_t)u$, which satisfies the energy identity [85]. For S_N the operator L_t is replaced by its normal part; the bounds below hold for both.

We now bound the forms. The ordinary part of Δ_{nor} has constant coefficients: on a root block of type σ it is $e^{2\mathrm{i}\theta_\sigma}$ times the flat Laplacian of the real coordinate, and on the graph its coefficient matrix differs from this by at most $C|b|/R_k$. The horizontal part satisfies (18.11) by Lemma 18.4, and its normal-energy term is absorbed by the normal part for R_k small. The horizontal coefficients at $U + \mathrm{i}\beta$ differ from those at U by $O_k(|\beta|)$. The odd normal part of Δ_{nor} has norm $C_k h^{-2}$. The first- and zeroth-order coefficients from the half-densities, the graph Jacobian, and the body weight v^p are bounded by $C_k(R_k)$.

Next, we conjugate by the real norm weights ψ_t specified below, which satisfy $|\nabla_U \psi_t| \le CW|U|$ with universal C. Outside the deformed region they are linear combinations of the rootwise Gaussian quadratics $|Z_\sigma|^2$ and $|U|^2$. The holomorphic Gaussian factor in (19.24) has modulus $e^{a\,\mathrm{Re}(Z_\lambda^T Z_\lambda)}$. We therefore apply this real multiplier directly to the pulled solution; its evolution contains $\partial_t \psi_t + e^{\psi_t}(L_t + V_t)e^{-\psi_t}$, without any need to conjugate by the complex phase.

The body weight v^p commutes with the normal velocity. Writing $z = e^{\psi_t} v^p u$, we have

$$e^{\psi_t} V_t e^{-\psi_t} = V_t - \mathsf{v}_t \cdot \nabla_U \psi_t.$$

By (18.21), (18.19), and Young's inequality, for all $\varepsilon, \mu > 0$,

$$\begin{aligned} |\langle z, e^{\psi_t} V_t e^{-\psi_t} z\rangle| \le{}& \varepsilon \mathcal{E}_{\mathrm{nor}}(z) + \mu W^2 \||U|z\|^2 \\ &+ C_{\varepsilon,\mu} |\lambda' b|^2 \|z\|^2. \end{aligned} \tag{19.31}$$

All constants in this velocity estimate are independent of rank, dimension, and the graph radius. In particular, no term proportional to $|\lambda' b|/R_k$ is introduced.

The normal charges require separate weight estimates. Outside $\{|U| \le 2R_k\}$ the graph is the identity, its half-Jacobian is one, and every compact even normal charge annihilates the rootwise quadratics in ψ_t. Thus the even-charge weight commutators are supported where $|U| \le 2R_k$. There their ordinary derivative coefficients are bounded by $C_k(|U| + |\beta|) \le C_k R_k$, giving a commutator bound $C_k R_k |\nabla_U \psi_t|$. An odd charge has ordinary derivative coefficients $O_k(h)$, also after the uniformly bounded graph change, so its weight commutator is $O_k(h|\nabla_U \psi_t|) = O_k(h^{-1}|U|)$ throughout the fixed enlarged tube. Estimate these commutators and their products by the adjoint bounds (18.12). Their forms are bounded by a small fraction of the gradient energies, $C_k R_k^2 W^2 \||U|z\|^2$, and $C_k h^{-2}\|z\|^2$. Fix the universal value of μ used below before decreasing R_k at fixed rank, and absorb the middle term into a prescribed fraction of $\mu W^2 \||U|z\|^2$. The enlarged Dirichlet tube need not have radius comparable to R_k. This uses direct odd-charge bounds, rather than invariance of an ordinary radial weight under odd charges.

Combining these estimates with the normal diffusion form, and choosing c_1 and R_k sufficiently small, gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|e^{\psi_t}v^p u\|^2 \le 2\big\langle e^{\psi_t}v^p u,\, \big(\partial_t\psi_t + C_5|\nabla_U\psi_t|^2 + \mu W^2|U|^2\big)e^{\psi_t}v^p u\big\rangle \\ + C_\mu\big(|\lambda' b|^2 + W^2|\beta(t)|^2 + C_k h^{-2}\big)\|e^{\psi_t}v^p u\|^2,\end{aligned} \tag{19.32}$$

where C_5 is universal and C_μ depends only on μ. The constant C_5 is universal because the normal metric is flat. Indeed, for the operator $a\Delta$ with a complex constant a of positive real part and a real weight, the divergence terms in $e^{\psi}a\Delta e^{-\psi}$ cancel exactly, and

$$\operatorname{Re}\langle u, e^{\psi}a\Delta e^{-\psi}u\rangle \le -(\operatorname{Re}a - |\operatorname{Im}a|)\|\nabla u\|^2 + (\operatorname{Re}a + |\operatorname{Im}a|)\|(\nabla\psi)u\|^2.$$

Here $a = e^{2\mathrm{i}\theta_\sigma}$, and the graph changes the coefficient of $|\nabla_U\psi|^2$ by a factor $1 + C(|E| + |b|/R_k)$. We may thus take $C_5 = 3$, after reserving the small gradient fractions in the preceding estimates. The velocity contribution is (19.31); the rank-dependent weight commutators enter only the allotted $\mu W^2|U|^2$ term and $C_k h^{-2}$ term. Thus the coefficient of $|\lambda' b|^2$ remains universal.

We next choose the weight on the heat interval of step j. Let $\alpha(t)$ interpolate linearly from a'_{j-1} to a'_j, and define

$$\psi_t = \alpha(t)\,\operatorname{Re}\big(Z^T_{\lambda(t)}Z_{\lambda(t)}\big) - \frac{c_{\mathrm{kick}}t}{2\tau}|U|^2,$$

where $c_{\mathrm{kick}} > 0$ is the universal constant of Lemma 19.7. Since $\operatorname{Re}(Z_\lambda^T Z_\lambda) = |Z|^2 - \lambda^2\psi^2_{R_k}|Z_b|^2$, $|b| \le c_1 R_k$ and $|\alpha'| \le 2v_0 W$, we have $|\nabla_U\psi_t| \le Ca_0 W|U| + c_{\mathrm{kick}}|U|$ and

$$\partial_t\psi_t + C_5|\nabla_U\psi_t|^2 + \mu W^2|U|^2 \le (6v_0 + Ca_0^2 + 2\mu - c_{\mathrm{kick}})W^2|U|^2 + C\big(|\lambda' b|^2 + W^2|\beta(t)|^2\big)$$

for W large. We choose $\mu = c_{\mathrm{kick}}/8$, and a_0 and v_0 so small that $6v_0 + Ca_0^2 \le c_{\mathrm{kick}}/4$; the coefficient of $W^2|U|^2$ is then negative. Inserting this bound into (19.32) and integrating over $[0,\tau]$, we obtain

$$\|e^{\psi_\tau}v^p u(\tau)\| \le \exp\big\{C\big(W^2|\beta_j - \beta_{j-1}|^2 + |\beta_{j-1}|^2 + |\beta_j|^2\big) + C_k/W\big\}\|e^{\psi_0}v^p u(0)\|, \tag{19.33}$$

where we used $W^2\tau = 1/2$, $C_k h^{-2}\tau = C_k/(2W)$ and

$$\int_0^\tau |\lambda'|^2\,\mathrm{d}t = \frac{|\lambda_j - \lambda_{j-1}|^2}{\tau} = 2W^2|\lambda_j - \lambda_{j-1}|^2.$$

At $t = 0$ the weight is that of $\|\cdot\|_{\lambda_{j-1};p,a'_{j-1}}$, and at $t = \tau$ it is the weight of $\|\cdot\|_{\lambda_j;p,a'_j}$ divided by $e^{c_{\mathrm{kick}}|U|^2/2}$. At a shared node the output and input use the same pull $\mathcal{P}_{\lambda_j}$, so their half-Jacobian conventions agree exactly. No comparison of distinct coefficient

metrics is made there. These estimates hold for arbitrary, including nonmonotone, choices of the λ_j.

The onsite factor $w \in \{s_\Gamma w_0, w_0\}$ of S_j makes up this deficit. By Lemma 19.7, which applies since $|b| \le c_1 R_k \le R_k$, multiplication by w and by the cutoff of S_j maps the weight e^{ψ_τ} at $t = \tau$, which is the weight of $\|\cdot\|_{\lambda_j;p,a_j'}$ divided by $e^{c_{\mathrm{kick}}|U|^2/2}$, to the weight of $\|\cdot\|_{\lambda_j;p,a_j'}$, and it changes the norm by at most $\exp\{C_k/W + C_2|\beta_j|^2 + C_k R_k|\beta_j|^2\}$. Multiplying the resulting bounds over j, and using $|\beta_{j-1}|^2 \le 2|\beta_j|^2 + 2|\beta_j - \beta_{j-1}|^2$ and $\sum_j C_k/W \le C_k C_0$, we obtain (19.29).

For a jet step $S^{(\alpha_i)}$, Lemma 19.7 gives the additional factor $C_k v^{d_{\alpha_i}}$. We absorb it by lowering the body weight by d_{α_i} from that step on, and the rest of the argument is unchanged. The input then carries the body weight $p + d$, and the J jet steps contribute the factor C_k^J, which proves (19.30). □

The same energy method with an exponential weight in the normal coordinate gives a Davies–Gaffney estimate [24], which Section 19.7 uses to show that a disturbance created at normal distance of order one from the saddle manifold is exponentially small near it after one block. We state it on the tube of one signature α, as in Section 19.7.

Lemma 19.9. *There exists a universal constant $C > 0$ such that the following holds at fixed rank k, after choosing R_k sufficiently small and then W sufficiently large. Let α be a signature and $p \in \mathbb{R}$. Let L_α^{raw} be the generator of the raw local heat, that is, of the ambient heat in the raw realization on a slightly larger Dirichlet tube of the signature α, with all odd coordinates scaled by h, and let $\mathcal{E}_\alpha^{\mathrm{real}}(u) = \mathcal{E}_{\tan}(u) + \mathcal{E}_{\mathrm{nor}}(u)$ be the real gradient energy of Lemma 18.4. Let d_{n} be a real function of the normal coordinate with Lipschitz constant one whose gradient $\nabla_U d_{\mathrm{n}}$ vanishes outside $\{|U| \le 2R_k\}$, and let $a_{\mathrm{D}} \in \mathbb{R}$. Then there exists $c > 0$ such that, in the raw norm with the body weight v^p, for all u in the form domain and all $t \ge 0$,*

$$\mathrm{Re}\langle u, L_\alpha^{\mathrm{raw}} u\rangle \le -c\,\mathcal{E}_\alpha^{\mathrm{real}}(u) + C_k W\|u\|^2, \qquad \|e^{a_{\mathrm{D}} d_{\mathrm{n}}} e^{tL_\alpha^{\mathrm{raw}}} e^{-a_{\mathrm{D}} d_{\mathrm{n}}}\| \le e^{(C_k W + C a_{\mathrm{D}}^2)t}. \quad (19.34)$$

Moreover, for the onsite factors $w \in \{s_\Gamma w_0, w_0\}$ with $\Gamma \in \mathcal{U}_2$, the local step satisfies

$$\|e^{a_{\mathrm{D}} d_{\mathrm{n}}} w\, e^{\tau_{\mathrm{phys}} L_\alpha^{\mathrm{raw}}} e^{-a_{\mathrm{D}} d_{\mathrm{n}}}\| \le e^{C_k/W + (C_k W + C a_{\mathrm{D}}^2)\tau_{\mathrm{phys}}}.$$

Proof. We first prove the form bound, using the orthogonal normal and horizontal decomposition (18.8). By Lemma 18.4, the ordinary principal part is a uniformly accretive perturbation of the flat normal metric and of the orbit metric of the group chart, with a universal margin in the normal block. A nilpotent ordinary principal coefficient contains two odd coordinates, so that it is $O(h^2)$, and it is absorbed in that margin. In the coordinates scaled by h, one odd normal derivative has norm h^{-1}, and the odd second-order part has norm at most $C_k h^{-2} = C_k W$. By Young's inequality, the remaining finite matrix terms and the terms of the transported frame are bounded by a small multiple of

$\mathcal{E}_\alpha^{\mathrm{real}}(u)$ plus $C_k W\|u\|^2$, and so are the terms that the body weight v^p adds, with constants depending on p. This proves the first bound in (19.34).

For the second bound we use Davies' method of exponential weights [24]. The weight $e^{a_{\mathrm{D}} d_{\mathrm{n}}}$ depends only on the normal coordinate. On the flat normal part, which is $e^{2\mathrm{i}\theta_\sigma}$ times the flat Laplacian on a root block of type σ, the divergence terms cancel exactly, as in the proof of Lemma 19.8, and since $|\nabla_U d_{\mathrm{n}}| \le 1$ we have

$$\mathrm{Re}\langle u, e^{a_{\mathrm{D}} d_{\mathrm{n}}} e^{2\mathrm{i}\theta_\sigma} \Delta e^{-a_{\mathrm{D}} d_{\mathrm{n}}} u\rangle \le -(\cos 2\theta_\sigma - |\sin 2\theta_\sigma|)\|\nabla u\|^2 + 2a_{\mathrm{D}}^2\|u\|^2.$$

In $L_H(U)$, the normal derivatives come only from the normal charges, which are first-order operators in U with coefficients of size $O(|U|)$. In the form of the conjugated operator, their action on $e^{\pm a_{\mathrm{D}} d_{\mathrm{n}}}$ produces terms that carry the factor $a_{\mathrm{D}} \nabla_U d_{\mathrm{n}}$. By hypothesis, this factor vanishes outside $\{|U| \le 2R_k\}$, where the coefficients of the normal charges are at most $C_k R_k$. By Young's inequality, these terms are therefore bounded by a small multiple of $\mathcal{E}_{\tan}(u)$ plus $C_k (2R_k)^2 a_{\mathrm{D}}^2 \|u\|^2$, and $4C_k R_k^2 \le 1$ once R_k is small. Therefore

$$\mathrm{Re}\langle u, e^{a_{\mathrm{D}} d_{\mathrm{n}}} L_\alpha^{\mathrm{raw}} e^{-a_{\mathrm{D}} d_{\mathrm{n}}} u\rangle \le C_k W\|u\|^2 + C a_{\mathrm{D}}^2 \|u\|^2$$

with a universal C, and the energy inequality for $e^{a_{\mathrm{D}} d_{\mathrm{n}}} e^{t L_\alpha^{\mathrm{raw}}} e^{-a_{\mathrm{D}} d_{\mathrm{n}}} u$ gives the second bound in (19.34). Finally, multiplication by $e^{a_{\mathrm{D}} d_{\mathrm{n}}}$ commutes with every onsite multiplier, and by Lemma 19.7 with $b = 0$, multiplication by w has norm at most $e^{C_k/W}$. This implies the bound for the local step. □

We set $c_{\mathrm{D}} = 1/(4C)$, with the universal constant C of Lemma 19.9. It is the rate of the Gaussian decay in the barrier (19.62) below.

Since the energy interval of Theorem 7.6 depends on the coefficient C_1 of Lemma 19.8 through (19.50) below, we now list its sources and check that it does not depend on the rank.

Remark 19.10. Summing (19.33) and (19.27) over the steps, with $|\beta_{j-1}|^2 \le 2|\beta_j|^2 + 2|\beta_j - \beta_{j-1}|^2$, shows that we may take $C_1 = 3C + C_2$, where C is the constant of (19.33) and C_2 that of (19.27). The constant C comes from the time derivative of the weight, with the universal parameters a_0, v_0, and c_{kick}, from the velocity term, through Young's inequality against the flat normal energy, and from the flat normal metric through C_5. The constant C_2 comes from the quadratic part q_2 of $\log w_0$, which is the scalar m_σ times $\mathrm{STr}(U^2)$, and from Young's inequality with the universal c_{kick}.

None of these involves the dimension of the normal space: the flat normal metric and the scalar Hessian act on the Hilbert–Schmidt norm, and the half-divergence in the moving-graph velocity cancels exactly in (18.20). The rank-one inverse bound (18.19) makes the velocity cost in (19.31) universal, with no loss proportional to the total variation of the graph heights. The rank appears only in three ways: through the cubic Taylor remainder r_{T} and the horizontal coefficients at complex points, which come with a factor

R_k and appear as $C_k R_k |\beta_j|^2$; through the smallness of $|b|/R_k$ and R_k needed to absorb the graph terms of the horizontal operator; and through prefactors $e^{C_k/W}$ per step, which include the odd-charge weight commutators and give C_k over a block. The constants c_1, a_0, and v_0 are universal for the same reasons.

Consequently the constant $C_n = 3C_1$ of (19.36) below is universal. The constants C_{geo}, C_*, and c_{D} are dimension-free as well, by Lemma 19.15, by Proposition 19.17, whose FF part rests on Lemma 15.8, and by Lemma 19.9, whose constant comes from the flat normal metric and from the normal charges. The normal charges act on the Davies weight only where its gradient is nonzero, which by the hypothesis of that lemma is inside $\{|U| \le 2R_k\}$, and there they contribute at most $C_k R_k^2 a_{\mathrm{D}}^2 \le a_{\mathrm{D}}^2$, for the parameter a_{D} of that lemma, once R_k is small. Therefore the energy threshold E_{geom} of (19.50) is universal.

We now introduce the terms in which the rest of Part IV states its Gaussian bounds. Let $a \in \mathbb{R}$. We say that a function f on the tube has *Gaussian margin* a if $\|w_a f\| = \|e^{a|Z|^2} f\|$ is finite, in the body weights at hand. By Lemma 19.5, the endpoint factors g_R and g_L have every sufficiently small negative margin $-a$, but no positive margin. We call a sequence $a'_0, \dots, a'_n$ with $a'_0 = -a$, $a'_n = a$, and $|a'_j - a'_{j-1}| \le v_0 h^2$, as in Lemma 19.8, a *signed ramp* of length n, and we call a family *prepared* if it has a positive Gaussian margin a with $0 < a < a_0$. In the symmetric normal gauge, Lemma 19.8 gives the following bound along a signed ramp, which turns a negative margin into a positive one.

Corollary 19.11. *Choose $0 < a < a_0$ and $\varrho > 0$ so that $2a/\varrho < v_0$. For $\varrho W \le n \le C_0 W$,*

$$\|w_a \mathbf{F}_h^n w_a\| + \|w_a A_h^n w_a\| \le C_{k,a,\varrho,C_0},$$

uniformly for $\Gamma \in \mathcal{U}_2$ and in every fixed body weight. Constant Gaussian weights $w_{a'}$ with $|a'| \le a_0$ are likewise stable over $0 \le n \le C_0 W$. If a fixed local source multiplier m_{src} has coefficient growth $C_k v^d \langle Z \rangle^{d'}$, then for $0 < a' < a$ it maps the input norm with Gaussian weight w_a and body weight v^{p+d} to the norm with $w_{a'}$ and body weight v^p, with a norm that does not depend on h. If $d' = 0$, we may take $a' = a$.

Proof. We take $a'_j = -a + 2aj/n$, so that

$$|a'_j - a'_{j-1}| = \frac{2a}{n} \le \frac{2a}{\varrho W} \le v_0 h^2.$$

Applying Lemma 19.8 with $b = 0$, we obtain $\|w_a S_\Gamma^n w_a\|_{\mathrm{raw}} \le C_k$ and the same bound for S_N, in every body weight v^p. By (19.22) and (19.23), the same bounds hold for A_h^n and $\mathbf{F}_h^n$. The assertion for constant weights uses $a'_j = a'$ for all j. Finally, $\sup_Z \langle Z \rangle^{d'} e^{-(a-a')|Z|^2} < \infty$, which proves the multiplier bound, and for $d' = 0$ the multiplier commutes with w_a. $\square$

We now separate the positive margin produced by the signed ramp from the negative Gaussian norms of the endpoint factors g_R and g_L.

Corollary 19.12. *Let a and ϱ be as in Corollary 19.11, and let $\varrho W \le n \le C_0 W$. In the balanced signature,*

$$\|w_a \mathbf{F}_h^n g_R\| + \|w_a \mathbf{F}_h^n g_L\| \le C_{k,a,\varrho,C_0},$$

and at every other signature the same norms are at most $C_{k,a,\varrho,C_0} W^{A_k}$. The same bounds hold in the body weight v^{p_0} of Lemma 19.5.

Proof. Let $\widetilde{\chi}$ be a real cutoff as in Lemma 19.5 that equals one on the support of χ_{inv}, so that $\mathbf{F}_h = \mathbf{F}_h \widetilde{\chi}$. Then

$$\|w_a \mathbf{F}_h^n g\| \le \|w_a \mathbf{F}_h^n w_a\| \, \|e^{-a|Z|^2} \widetilde{\chi} g\|,$$

and Corollary 19.11 and Lemma 19.5 give the bounds. The weight v^{p_0} commutes with $\mathbf{F}_h$. □

We now state the bound for a block of adapted steps on a normal graph in the symmetric normal gauge. Let b be a real normal direction with $|b| \le c_1 R_k$, and let $U_1 = U + \mathrm{i}\psi_{R_k}(U) b$ be the compact normal graph (18.15) with $\bar{r} = R_k$ and $\lambda = 1$. It has the constant height b on the inner tube $\{|U| \le R_k\}$, and it is the identity wherever $\chi_{\mathrm{inv}} \neq 1$. For $a \in \mathbb{R}$, a body weight p, and a function f on the tube that is holomorphic in the normal variables where $\psi_{R_k} \neq 0$, we define the graph norm in the symmetric normal gauge

$$\|f\|_{b;p,a} = \big\| w_a(U_1)\, v^p \rho_1^{1/2}\, f(U_1) \big\|, \tag{19.35}$$

where the norm on the right is the positive Wick metric of Definition 16.6. For $b = 0$, the norm (19.35) is $\|w_a v^p f\|$. By the version of (19.23) on graphs, $\|f\|_{b;p,a} \le C_k \|w_0^{1/2} \gamma_h f\|_{1;p,a}$, with the raw graph norm (19.24).

Proposition 19.13. *There exist universal constants $a, C_n > 0$ and a universal C_0 such that, after choosing the coefficient metric and R_k at fixed rank, the following holds for the adapted kernel. Let $m = \lceil C_0 W \rceil$, let $\Gamma \in \mathcal{U}_2$, let $p \in \mathbb{R}$, and let b be a real normal direction with $|b| \le c_1 R_k$. Then for all g on the real tube,*

$$\|A_h^m g\|_{b;p,a} \le C_k(R_k) \exp\big\{ C_n C_0 W |b|^2 + C_k W R_k |b|^2 \big\} \|w_{-a} v^p g\|, \tag{19.36}$$

with $C_n = 3C_1$. The input has the Gaussian margin $-a$, and the output has the Gaussian margin a, on the real tube ($b = 0$) and on the graph. The estimates are uniform on the nested damping balls. For an input f in the raw realization that is holomorphic in the normal variables where $\psi_{R_k} \neq 0$ and has $\|f\|_{1;p,a} < \infty$, using the same normal graph at every node gives, for every $0 \le j \le C_0 W$, including $j = 0$,

$$\|S_\Gamma^j f\|_{1;p,a} \le C_k \exp\big\{ (C_1 + C_k R_k) j |b|^2 \big\} \|f\|_{1;p,a}.$$

If a prescribed finite number of complete local jets are inserted as jet steps (19.25)*, these bounds hold with the changes of* (19.30)*.*

Proof. We keep every body variable on its real cycle and deform only the normal coordinates of the intermediate nodes, as in the proof of Lemma 19.8, with $\lambda_j = j/m$ for $0 \le j \le m$. The graph vanishes wherever an original real cutoff varies, so the input is real. We use the signed weights $a'_j = -a + 2aj/m$, running from $-a$ to a; this is possible since $m \ge \varrho W$. The heights $\beta_j = \lambda_j b$ satisfy

$$\sum_{j=1}^{m} \bigl(|\beta_j|^2 + W^2|\beta_j - \beta_{j-1}|^2\bigr) \le \Bigl(m + \frac{W^2}{m}\Bigr)|b|^2 \le 3C_0 W|b|^2,$$

where we used $m \le 2C_0 W$ and $W^2/m \le W/C_0 \le C_0 W$, since $C_0 \ge 1$. Similarly,

$$\sum_{j=1}^{m} R_k|\beta_j|^2 \le 2C_0 R_k W|b|^2.$$

Inserting these bounds into (19.29), we obtain the raw form of (19.36), for S_Γ^m applied to $w_0^{1/2}\gamma_h g$. By (19.22), we have $A_h^m g = \gamma_h^{-1} w_0^{-1/2} S_\Gamma^m (w_0^{1/2}\gamma_h g)$. The lower bound in (19.23) bounds the raw input norm by $C_k\|w_{-a}v^p g\|$, and its version on graphs bounds $\|A_h^m g\|_{b;p,a}$ by C_k times the raw output norm on the graph. This proves (19.36) with $C_n = 3C_1$. The source appears only in the onsite bound Lemma 19.7, which holds uniformly for $\Gamma \in \mathcal{U}_2$, and it adds no rank-dependent term proportional to $W|b|^2$. By Remark 19.10, the constants C_1 and $C_n = 3C_1$ are universal.

For the assertion with the same graph at all nodes, we use Lemma 19.8 with $\lambda_j = 1$ for all j, including the input, and with $a'_j = a$ for all j. The velocity terms vanish, and the exponent in (19.29) reduces to $(C_1 + C_k R_k)j|b|^2$, for every $0 \le j \le C_0 W$. A complete local jet enters as a jet step, and in both arguments (19.30) replaces (19.29) for blocks with jet steps. Finally, normal derivatives of order ℓ of the output, obtained by Cauchy's estimates on polydisks of radius c/W in the normal variables, are bounded with an additional factor CW^ℓ, and body derivatives require fixed stronger polynomial weights. This completes the proof. □

The estimates above use the source factor only through (19.27). The following lemma shows that all of them are uniform on the damping balls.

Lemma 19.14. *The power bounds of Theorem 17.1, the body-weight gain of Lemma 17.20 and the signed normal preparation of Corollary 19.11 and Proposition 19.13 hold uniformly for $\Gamma \in \mathcal{U}_2$, in every fixed body weight, with the fixed changes of body weight that each finite jet requires. In particular, a leg of macroscopic length with positive damping, followed by a final block of $C_0 W$ steps, gives a family with a positive Gaussian margin in the family norm* (19.1)*. For the even*

damping family and the finite BB and FF jet components of the original sources, this bound is $O(1)$ *in the balanced signature. For the odd components of these jets, the bound holds with additional fixed powers of W and fixed losses of body weight, and the polynomial factors of the unbalanced signatures are kept until they are suppressed in Section 20. Moreover, multiplication by* s_Γ *is holomorphic in* $\Gamma \in \mathcal{U}_2$ *as a map into the bounded operators on each weighted space, and every finite product of the operations of Definition 19.1 is holomorphic in* $\Gamma \in \mathcal{U}_2$ *as a Banach-space-valued function.*

Proof. For the power bounds there is nothing to prove, since Theorem 17.1 holds uniformly for complex damping in $\mathcal{U}_2$, with the complex frozen kick of (17.2) in its reference contraction; the constant c_Γ of (17.1) is kept out of the kick and gives a scalar term of the four-term bound. The other estimates use the source factor only through the onsite bound (19.27) and through the damping, and we now check that these are uniform on $\mathcal{U}_2$.

By Lemma 19.6, the source factor has the damping $|s_\Gamma|_1 \le e^{C_k\eta - c_0\eta v}$ uniformly on $\mathcal{U}_2$. The fixed derivatives of the exponent $\mathrm{Tr}(Q_B\Gamma_{BB})$ of (19.26) along the orbit are bounded by $C_{k,j}v$, since conjugated unit tangent generators insert bounded commutators and $\|\Gamma\|$ is bounded. The onsite bound (19.27) is thus uniform on the parameter ball, as Lemma 19.7 states, and so are the estimates of Corollary 19.11 and Proposition 19.13, which use the source only through it. The same positive damping proves the body-weight gain of Lemma 17.20, for the step and for its adjoint. A finite jet uses its fixed changes of body weight and Gaussian margin once, and the raw inputs have the bounds of Lemma 19.5.

We next prove that s_Γ is holomorphic. By (4.15), $s_\Gamma = e^{-\mathrm{i}\eta_0\,\mathrm{STr}(S\Gamma)}$, and the exponent is linear in Γ. For a small even matrix δ the quotient $s_{\Gamma+\delta}/s_\Gamma$ is the exponential of a function that is linear in δ. In the normalization (15.6), this function is at most $C\eta|\delta|v$ on the tube, as is the damping exponent, since $\|S_{BB}\| \le Cv$ by (15.3). By Lemma 19.6, we have $|s_\Gamma|_1 \le e^{C_k\eta - c_0\eta v}$, and therefore

$$|s_{\Gamma+\delta} - s_\Gamma|_1 \le |s_\Gamma|_1\, C\eta|\delta|v\, e^{C\eta|\delta|v} \le C|\delta|\, e^{C_k\eta} \sup_{v\ge 1} \eta v\, e^{-(c_0 - C|\delta|)\eta v},$$

which is at most $C'|\delta|$, uniformly in v, for $C|\delta| \le c_0/2$. The same expansion to second order in δ shows that the multiplication by s_Γ is complex differentiable in the operator norm. The heat factors, the half-weights, and the cutoffs do not depend on Γ. It follows that every finite product of the operations of Definition 19.1 is holomorphic in the ordinary damping entries, with values in the weighted spaces. Integrating the uniform bounds over the ball proves the bound for (19.1). All local BB sources remain formal finite derivatives at zero. □

19.5 Canonical coordinates near the saddle manifold

The physical chain lives on the cycle $\mathcal{S}$ and the adapted chain on the tube. To compare them near the saddle manifold, we write each point of the cycle in tubular coordinates. In the FF block, the cycle and the real tube share the compact orbit and differ only by the rotation $e^{-\mathrm{i}\theta_\sigma}$ of the normal axes. In the BB block, the point of the cycle does not lie on the real tube, and a small complex conjugation, which we call the canonical gauge, brings it to a complex point of the tube. The normal height of that complex point must be bounded by $C_{\mathrm{geo}}(|E|R + R^2)$ at normal radius R, with a constant C_{geo} that does not depend on the rank, the dimension, or the boost.

The canonical equation. We use the coordinates of Section 15.3. By (15.26), the BB block of a point of the cycle satisfies $S_B = \mathrm{g}^{-1} R_{\mathrm{g}}(\widetilde{P})\mathrm{g}$, where for a block-diagonal n_B we write

$$R_{\mathrm{g}}(n_B) = R_{*,B} + n_B + \mathrm{th}_{\mathrm{g}}\, n_B.$$

For an off-diagonal matrix T we set $Y_{\mathrm{g}}(T) = \mathrm{ch}_{\mathrm{g}}^{-1}\, T$ and $T^\natural = \mathrm{Ad}_{\mathrm{g}}\, Y_{\mathrm{g}}(T) = T + \mathrm{th}_{\mathrm{g}}\, T$. Since $\mathrm{ad}_{\log \mathrm{g}}$ is selfadjoint for the Hilbert–Schmidt inner product, $\|\,\mathrm{ch}_{\mathrm{g}}^{-1}\,\| \le 1$ and $\|\,\mathrm{th}_{\mathrm{g}}\,\| \le 1$. Hence $\|Y_{\mathrm{g}}(T)\| \le \|T\|$, and $\|T^\natural\| \le \sqrt{2}\|T\|$ since its two block parities are orthogonal. Let Π_{off} and $\Pi_{\natural}$ be the projections onto off-diagonal and block-diagonal matrices. We look for an off-diagonal $T = T_{\mathrm{g}}(n_B)$ such that

$$\Pi_{\mathrm{off}}\big(e^{T^\natural} R_{\mathrm{g}}(n_B) e^{-T^\natural}\big) = 0, \qquad U_{\mathrm{g}}(n_B) = \Pi_{\natural}\big(e^{T^\natural} R_{\mathrm{g}}(n_B) e^{-T^\natural}\big) - R_{*,B}. \tag{19.37}$$

We call $U_{\mathrm{g}}(n_B)$ the canonical normal coordinate of the BB block. Since $e^{T^\natural} = \mathrm{g} e^{Y_{\mathrm{g}}(T)} \mathrm{g}^{-1}$, the BB block of the point of the cycle is then

$$S_B = T_{\mathrm{can}}(R_{*,B} + U_{\mathrm{g}}(n_B)) T_{\mathrm{can}}^{-1}, \qquad T_{\mathrm{can}} = S_{\mathrm{g}}^{-1}\mathrm{g}^{-1}, \quad S_{\mathrm{g}} = e^{Y_{\mathrm{g}}(T_{\mathrm{g}}(n_B))}, \tag{19.38}$$

so that the canonical frame differs from the real frame g^{-1} by the small complex gauge S_{g}^{-1}.

Lemma 19.15. *Let $E_0 \le 1/2$. There exist universal constants $r_{\mathrm{c}}, C > 0$ and C_{geo} such that the following holds for all $|E| \le E_0$.*

(a) (BB block.) *For every Hermitian off-diagonal X and every block-diagonal n_B with $\|n_B\| \le r_{\mathrm{c}}$, there exists a unique off-diagonal $T_{\mathrm{g}}(n_B)$ with $\|T_{\mathrm{g}}(n_B)\| \le s^{-1}\|n_B\|$ that solves* (19.37)*. It is holomorphic in n_B, and*

$$\|U_{\mathrm{g}}(n_B) - n_B\| \le C\|n_B\|^2, \qquad \|S_{\mathrm{g}}^{\pm 1} - \mathrm{Id}\| \le C\|n_B\|.$$

If moreover n_B is Hermitian with $\|n_B\| \le R \le r_{\mathrm{c}}$, then the height of $U_{\mathrm{g}}(n_B)$ above the real tube satisfies $\|w_B\| \le C_{\mathrm{geo}}(|E|R + R^2)$.

(b) (FF block.) *Let M be Hermitian and $p \in \{0, \dots, 2k\}$, and suppose that $M = V(-s\Sigma_F + n_F)V^*$ with V unitary, $\Sigma_F = \operatorname{diag}(\mathrm{Id}_p, -\mathrm{Id}_{2k-p})$ and n_F Hermitian and block diagonal with $\|n_F\| \le R \le r_c$. Then the FF block of the point of the cycle is the point of the complexified tube with the real orbit coordinate V and the normal coordinate $U_F = \mathrm{i}n_F$, and its height above the real tube satisfies $\|w_F\| = |\sin\theta_\sigma|\,\|n_F\| \le C_{\mathrm{geo}}|E|R$, blockwise for the roots σ.*

Consequently, the body of the canonical normal coordinate of a point of the cycle at normal radius $R \le r_c$ has height at most $C_{\mathrm{geo}}(|E|R + R^2)$. The constants do not depend on k, on X or on the matrix dimension.

Part (b) computes the rotation of the FF normal axes, which is of order $|E|R$. In the FF block no complex gauge is needed. Indeed, a Hermitian matrix $M = V(-s\Sigma_F + n_F)V^*$ as in (b), whose eigenvalues are close to $\pm s$, is block diagonalized exactly by the unitary matrix V. The cycle point is therefore a point of the tube with a real orbit coordinate, and its normal coordinate $\mathrm{i}n_F$ lies on the unrotated real form. Its height above the adapted real form $e^{-\mathrm{i}\theta_\sigma}\mathrm{i}\,\mathrm{Herm}$ comes only from the rotation, which is of order $|E|$ since the FF normal Hessian m_σ has argument of order $|E|$.

Proof. We begin with (a), which is a quantitative implicit function theorem in the Hilbert–Schmidt norm; we check that every constant is independent of the dimension and of the boost. Write $T^\natural$ as above and $\tilde n_B = n_B + \mathrm{th}_{\mathrm{g}}\, n_B$, so that $R_{\mathrm{g}}(n_B) = R_{*,B} + \tilde n_B$ and $\|\tilde n_B\| \le \sqrt{2}\|n_B\|$. The matrix $\mathrm{th}_{\mathrm{g}}\, T$ is block diagonal and commutes with Λ, which gives

$$[T^\natural, R_{*,B}] = [T, R_{*,B}] = 2\mathrm{i}s\Lambda T.$$

The linear map $\mathcal{L}T = 2\mathrm{i}s\Lambda T$ preserves the off-diagonal matrices, and $\|\mathcal{L}^{-1}\| = (2s)^{-1}$.

Expanding $e^{T^\natural} R_{\mathrm{g}} e^{-T^\natural} = e^{\mathrm{ad}_{T^\natural}} R_{\mathrm{g}}$, we have

$$\Pi_{\mathrm{off}}\big(e^{\mathrm{ad}_{T^\natural}} R_{\mathrm{g}}(n_B)\big) = \mathcal{L}T + \mathrm{th}_{\mathrm{g}}\, n_B + \mathcal{R}(T, n_B),$$

where

$$\mathcal{R}(T, n_B) = \Pi_{\mathrm{off}}\Big([T^\natural, \tilde n_B] + \sum_{n\ge 2} \frac{\mathrm{ad}^n_{T^\natural} R_{\mathrm{g}}(n_B)}{n!}\Big).$$

Since $\|\mathrm{ad}^n_{T^\natural} R_{*,B}\| \le (2\|T^\natural\|)^{n-1} 2s\|T\|$ and $\|T^\natural\| \le \sqrt{2}\|T\|$, there exists a universal $C_{\mathcal{R}}$ such that for $\|T\|, \|n_B\| \le 1$,

$$\|\mathcal{R}(T, n_B)\| \le C_{\mathcal{R}}(\|T\|\|n_B\| + \|T\|^2),$$
$$\|\mathcal{R}(T, n_B) - \mathcal{R}(T', n_B)\| \le C_{\mathcal{R}}(\|T\| + \|T'\| + \|n_B\|)\|T - T'\|.$$

The equation (19.37) is therefore the fixed point equation

$$T = -\mathcal{L}^{-1}\big(\mathrm{th}_{\mathrm{g}}\, n_B + \mathcal{R}(T, n_B)\big).$$

For $\|n_B\| \le r_c$ with r_c small, depending only on $C_{\mathcal{R}}$ and s_0, the right side maps the ball $\{\|T\| \le s^{-1}\|n_B\|\}$ into itself and is a contraction there. This gives $T_{\mathfrak{g}}(n_B)$, which is holomorphic in n_B as a locally uniform limit of holomorphic iterates.

Next, we compare $U_{\mathfrak{g}}(n_B)$ with n_B. We have $\Pi_{\mathfrak{k}}\tilde{n}_B = n_B$, and $\Pi_{\mathfrak{k}}[T^{\natural}, R_{*,B}] = 0$ since $\mathcal{L}T$ is off-diagonal. Hence

$$U_{\mathfrak{g}}(n_B) - n_B = \Pi_{\mathfrak{k}}\Big([T^{\natural}, \tilde{n}_B] + \sum_{n\ge 2} \frac{\operatorname{ad}^n_{T^{\natural}} R_{\mathfrak{g}}(n_B)}{n!}\Big),$$

whose norm is at most $C(\|T\|\|n_B\| + \|T\|^2) \le C\|n_B\|^2$. The bound on $S_{\mathfrak{g}}^{\pm 1} - \mathrm{Id}$ follows from $\|Y_{\mathfrak{g}}(T)\| \le \|T\| \le s^{-1}\|n_B\|$.

For the height, let n_B be Hermitian. Since $s^2 = 1 - E^2/4$, we have $m_\sigma = 1 - r_\sigma^2 = 2s^2 - \mathrm{i}\sigma E s$ and $c_\sigma^2 = m_\sigma(W^2 + m_\sigma/4)$ by (16.40), so that

$$|\arg c_\sigma| \le \tfrac{1}{2}\arctan\frac{|E|}{2s} + \frac{C|E|}{W^2} \le C|E|, \qquad |\theta_\sigma| \le C|E|.$$

On the root block of type σ, $n_B = e^{-\mathrm{i}\theta_\sigma}(e^{\mathrm{i}\theta_\sigma} n_B)$, and $e^{\mathrm{i}\theta_\sigma} n_B = \cos\theta_\sigma n_B + \mathrm{i}\sin\theta_\sigma n_B$, and the height of n_B is $\sin\theta_\sigma$ times its block of type σ. Therefore the height w_B of $U_{\mathfrak{g}}(n_B)$ satisfies

$$\|w_B\| \le |\sin\theta_\sigma|\|n_B\| + \|U_{\mathfrak{g}}(n_B) - n_B\| \le C|E|\|n_B\| + C\|n_B\|^2,$$

which proves (a). Every constant depends only on s_0 and on the inequality $\|[A,Y]\| \le 2\|A\|_{\mathrm{op}}\|Y\|$, which does not depend on the dimension.

We now prove (b). The FF block of the point of the cycle is $\kappa\mathrm{Id} + \mathrm{i}M = V(\kappa\mathrm{Id} - \mathrm{i}s\Sigma_F + \mathrm{i}n_F)V^*$, and $\kappa\mathrm{Id} - \mathrm{i}s\Sigma_F$ is the FF block $R_{*,F}$ of the root matrix of signature p. Thus the point is $V(R_{*,F} + U_F)V^{-1}$ with V in the real compact orbit group and the block-diagonal normal coordinate $U_F = \mathrm{i}n_F$, which is the tubular form (16.12) of the point; no complex conjugation is needed. The real form of the FF normal coordinates is i times the Hermitian matrices, and on the root block of type σ we have $\mathrm{i}n_F = e^{-\mathrm{i}\theta_\sigma}\big(\cos\theta_\sigma\,\mathrm{i}n_F + \mathrm{i}\sin\theta_\sigma\,\mathrm{i}n_F\big)$. The real part of U_F is $\cos\theta_\sigma\,\mathrm{i}n_F$, and its height is $w_F = \sin\theta_\sigma\,\mathrm{i}n_F$. Thus $\|w_F\| = |\sin\theta_\sigma|\|n_F\| \le C|E|R$, with the universal constant of the bound on θ_σ above. This proves (b).

Finally, the body of the full supermatrix splits into its BB and FF blocks, and the body of the canonical map is the product of the two constructions above. Taking C_{geo} larger than the constants in (a) and (b), and using $\|w\|^2 = \|w_B\|^2 + \|w_F\|^2$, we obtain the last assertion with $\sqrt{2}C_{\mathrm{geo}}$ in place of C_{geo}, and we rename the constant. □

The construction extends to the full supermatrix by induction in the total odd degree. First apply the body frames of Lemma 19.15, so that the transformed supermatrix is $R_* + U_{\mathrm{b}} + N$, where U_{b} is the body normal coordinate, $[\Sigma, U_{\mathrm{b}}] = 0$, $\|U_{\mathrm{b}}\| \le Cr_c$, and N has zero body. We seek an additional even nilpotent conjugation e^Y, with $\Sigma Y = -Y\Sigma$, that

removes all entries between the two root spaces. At each positive degree, its unknown component satisfies a linear equation with the full body Jacobian

$$\mathcal{J}_{\mathrm{b}} Y = [Y, R_* + U_{\mathrm{b}}] = \mathcal{L}_\Sigma Y + [Y, U_{\mathrm{b}}], \qquad \mathcal{L}_\Sigma Y = 2\mathrm{i}s\Sigma Y.$$

This formula applies to the BB, FF, BF and FB entries between the root spaces. In the Hilbert–Schmidt coefficient norm, $\|\mathcal{L}_\Sigma^{-1}\| = (2s)^{-1}$ and $\|\mathcal{J}_{\mathrm{b}} - \mathcal{L}_\Sigma\| \le 2\|U_{\mathrm{b}}\|_{\mathrm{op}} \le C r_{\mathrm{c}}$. After a universal decrease of r_{c}, a Neumann series therefore gives $\|\mathcal{J}_{\mathrm{b}}^{-1}\| \le (2s - C r_{\mathrm{c}})^{-1}$. The right side at each degree is a polynomial in coefficients already determined. There are finitely many degrees at fixed rank, and this constructs every nilpotent normal shift and odd frame entry, with fixed-rank polynomial bounds in the body weight on each smaller tube. The body map is unchanged, so only its heights enter the energy threshold; the nilpotent parts are finite coefficient jets with rank-dependent bounds.

To fix the full frame, let T_{b} be the product of the BB canonical body frame and the FF unitary frame, and let T_{r} be the corresponding real body frame. We include the correction just constructed by setting $\widehat{T}_{\mathrm{can}} = T_{\mathrm{b}} e^{-Y}$ and $\widehat{S} = T_{\mathrm{r}} \widehat{T}_{\mathrm{can}}^{-1}$. Then the original supermatrix is $\widehat{T}_{\mathrm{can}}(R_* + U)\widehat{T}_{\mathrm{can}}^{-1}$ with $[\Sigma, U] = 0$, and $\widehat{T}_{\mathrm{can}} = \widehat{S}^{-1} T_{\mathrm{r}}$. The body of $\widehat{S}$ is S_{g} in the BB block and the identity in the FF block.

We let Can denote the resulting canonical map, which takes a function on the tube, with its half-density, its Gaussian gauge, and its full nilpotent structure, to a function near the saddle manifold on the cycle. By Lemma 19.2(b) and the full frame identity above, for a family generated from the endpoint the composition with Can is the family at the conjugated damping $\widehat{S}\Gamma\widehat{S}^{-1}$ and directions $\widehat{S}\mathbf{V}\widehat{S}^{-1}$, evaluated at the real frame; the pushed frame keeps the normal coordinates unchanged. The body selection uses S_{g} as in (19.38), and the nilpotent corrections are retained through their finite coefficient expansion, as in Lemma 19.2(c).

The physical cutoff. We now fix the real physical cutoff χ_{phys} on the cycle, which equals one near the saddle manifold and vanishes at normal distance r_{phys} from it. Its body is a radial function of the real part of the canonical normal coordinate. Its nilpotent part is obtained by transporting a cutoff on the real tube along a homotopy of body cycles from the cycle to the real tube, which Section 20 also uses to return the cycle of integration to the tube.

Let χ be a smooth function on $[0, \infty)$ with values in $[0, 1]$, equal to one on $[0, r_1]$ and to zero on $[r_{\mathrm{phys}}, \infty)$. For a complex normal coordinate U we write $U = U^{\mathrm{re}} + \mathrm{i}U^{\mathrm{im}}$, where U^{re} and U^{im} lie in the real normal space of the adapted tube, so that U^{im} is the height of Lemma 19.15. On the real tube we write $\chi_{\mathrm{tube}} = \chi(|U|)$.

We parametrize $\mathcal{S}$ near the saddle manifold by $y = (X, \widetilde{P}, M)$ as in Section 15.3, together with the odd entries of the supermatrix, which we call the ambient odd coordinates, and we let $\mathcal{Y}$ be the parameter domain of y. We write $n_B = \widetilde{P}$, we let $T^\natural$ and $U_{\mathrm{g}}(n_B)$

be as in Lemma 19.15(a), and we write $M = V(-s\Sigma_F + n_F)V^*$ as in Lemma 19.15(b). For $0 \le t \le 1$, let $\Phi_t(y)$ be the body point with the blocks

$$\begin{aligned} \Phi_t(y)_{\mathrm{BB}} &= \mathrm{g}^{-1} e^{-tT^\natural} \big(R_{*,B} + U_{\mathrm{g}}(n_B)^{\mathrm{re}} + \mathrm{i}t\, U_{\mathrm{g}}(n_B)^{\mathrm{im}}\big) e^{tT^\natural} \mathrm{g}, \\ \Phi_t(y)_{\mathrm{FF}} &= V\big(R_{*,F} + (\mathrm{i}n_F)^{\mathrm{re}} + \mathrm{i}t\,(\mathrm{i}n_F)^{\mathrm{im}}\big) V^*, \end{aligned} \tag{19.39}$$

and write $\Phi(t,y) = \Phi_t(y)$. We write U_{can} for the canonical normal coordinate of the point y of the cycle, which is $U_{\mathrm{g}}(n_B)$ in the BB block and $\mathrm{i}n_F$ in the FF block, and $U_{\mathrm{can}}^{\mathrm{re}}$ for its real part.

Lemma 19.16. *Let $0 \le t \le 1$ and $y \in \mathcal{Y}$, and write $U = U_{\mathrm{g}}(n_B)$.*

(a) The BB block of $\Phi_t(y)$ has the frame $S_{\mathrm{g}}^{-t}\mathrm{g}^{-1}$ and the normal coordinate $U^{\mathrm{re}} + \mathrm{i}t\, U^{\mathrm{im}}$, and its FF block has the frame V and the normal coordinate $(\mathrm{i}n_F)^{\mathrm{re}} + \mathrm{i}t\,(\mathrm{i}n_F)^{\mathrm{im}}$.

(b) The point $\Phi_1(y)$ is the body of the point y of $\mathcal{S}$, and Φ_0 is a diffeomorphism from a neighborhood of the saddle manifold in $\mathcal{S}$ onto a neighborhood of the saddle manifold in the real tube.

(c) There exists A_Φ such that $|\partial_t \Phi_t(y)|$ and $\|D\Phi_t(y)\|^{\dim -1}$ are bounded by $C_k v_{\mathcal{S}}^{A_\Phi}$, where $D\Phi_t$ is the derivative in y and dim *is the real dimension of the body of $\mathcal{S}$.*

Proof. Recall that $S_{\mathrm{g}} = e^{Y_{\mathrm{g}}}$, with $Y_{\mathrm{g}} = Y_{\mathrm{g}}(T_{\mathrm{g}}(n_B))$, and that $e^{T^\natural} = \mathrm{g} e^{Y_{\mathrm{g}}} \mathrm{g}^{-1}$. Hence $\mathrm{g}^{-1} e^{-tT^\natural} = S_{\mathrm{g}}^{-t}\mathrm{g}^{-1}$, and (a) follows from (19.39). At $t = 1$, the BB block has the canonical frame

$$T_{\mathrm{can}} = S_{\mathrm{g}}^{-1}\mathrm{g}^{-1} = \mathrm{g}^{-1} e^{-T^\natural}$$

of (19.38) and the normal coordinate $U^{\mathrm{re}} + \mathrm{i}U^{\mathrm{im}} = U$. Therefore (19.38) and Lemma 19.15(b) show that $\Phi_1(y)$ is the body of the point y of $\mathcal{S}$. At $t = 0$, the frames g^{-1} and V and the normal coordinates are real, and Lemma 19.15 implies that Φ_0 is a diffeomorphism onto a neighborhood of the saddle manifold in the real tube. This proves (b).

For (c), recall that Φ_t is built from $\mathrm{g}^{\pm 1}$, whose norms are at most $(3v_{\mathcal{S}})^{1/2}$ by (15.3) and whose logarithmic derivatives are bounded by (15.30), and from $T^\natural$ and $U_{\mathrm{g}}(n_B)$, which are holomorphic in n_B with bounds uniform in the boost by Lemma 19.15. For instance, we have

$$\partial_t \Phi_t(y)_{\mathrm{BB}} = \mathrm{g}^{-1} e^{-tT^\natural} \big(-[T^\natural, R_{*,B} + U^{\mathrm{re}} + \mathrm{i}tU^{\mathrm{im}}] + \mathrm{i}U^{\mathrm{im}}\big) e^{tT^\natural} \mathrm{g}, \qquad \|T^\natural\| \le \sqrt{2}\, s^{-1} \|n_B\|.$$

The other derivatives are products of the same factors, and each factor is bounded by a power of $v_{\mathcal{S}}$. □

We fix the nilpotent part of the physical cutoff by transporting χ_{tube} along the homotopy. In the ambient odd coordinates, the nilpotent coefficients of $\chi_{\mathrm{tube}} = \chi(|U|)$ are

polynomials in the nilpotent part of U with coefficients $\chi^{(j)}(|U|)$, $j \geq 1$. On $[0,1] \times \mathcal{Y}$ we set

$$\chi_\Phi(t,y) = \chi_{\text{tube}}(\Phi_0(y)), \qquad \chi_{\text{phys}}(y) = \chi_\Phi(1,y), \tag{19.40}$$

coefficientwise in the ambient odd coordinates. By Lemma 19.16(a) at $t = 0$, the cutoff χ_{phys} has the body $\chi(|U^{\text{re}}_{\text{can}}|)$. Therefore $\chi_{\text{phys}} = 1$ where $|U_{\text{can}}| \leq r_1$, and χ_{phys}, together with all its nilpotent coefficients, vanishes where $|U^{\text{re}}_{\text{can}}| \geq r_{\text{phys}}$. The nilpotent coefficients of χ_Φ and the derivatives of all its coefficients are supported in the annulus $\{y : r_1 \leq |U^{\text{re}}_{\text{can}}(y)| \leq r_{\text{phys}}\}$, which we call the annulus where χ_{phys} varies.

The strip of the actual cycle. The physical chain on the cycle is compared with the references on the tube through normal graphs of the cycle itself. For a constant small Hermitian block-diagonal matrix D, we shift the physical normal $\widetilde{P}$ to $\widetilde{P} + \mathrm{i}D$, that is, P_0 to $P_0 + \mathrm{i}P_D$ at fixed X, where we write $P_D = \mathrm{ch}_{\mathrm{g}}^{-1} D$. This shift has unit Jacobian, since only the block-diagonal coordinate is translated. For a Hermitian D_F, we also shift the FF block from $\kappa\mathrm{Id} + \mathrm{i}M$ to $\kappa\mathrm{Id} + \mathrm{i}M + D_F$. We write $\mathcal{S}_D$ for the shifted cycle, and we define w_{Γ,D,D_F} to be the onsite weight at the shifted point, as a function of the coordinates (P_0, X, M) and the odd coordinates. Recall the defect $d_X(\Xi)$ of (15.28), and let

$$d(X,X') = \|X - X'\|^2 - \|f_0(X) - f_0(X')\|^2$$

denote the defect of the BB graph between X and X', which is nonnegative by (B.6). The following proposition shows that the partial chains continue to the shifted cycles, with bounds that grow like $e^{C_* j(\|D\|^2 + \|D_F\|^2)}$, and it gives the resulting bound near the saddle manifold in the coordinates of the tube.

Proposition 19.17. *There exist universal constants $r_{\mathrm{s}} \in (0, s_0/4]$, $R_{\mathrm{s}}, C, C_*, c' > 0$ such that the following holds at fixed rank k for all sufficiently large W. Let $\Gamma \in \mathcal{U}_2$, let $p \in \mathbb{R}$, and let D be a block-diagonal Hermitian matrix and D_F a Hermitian matrix with $\|D\| + \|D_F\| \leq r_{\mathrm{s}}$.*

(a) For every block-diagonal Hermitian matrix Ξ,

$$\|DP^*_D \Xi\| \leq C\|D\| d_X(\Xi)^{1/2}, \qquad |\langle DF^*_X \Xi, DP^*_D \Xi\rangle| \leq C\|D\| d_X(\Xi), \tag{19.41}$$

and the ambient heat pulled to $\mathcal{S}_D$ satisfies the bound $\exp[(C_k W + C_{k,p})t]$ of Lemma 15.7 on $\mathcal{H}_p(\mathcal{S})$, for $0 \leq t \leq \tau_{\text{phys}}$.

(b) The shifted onsite weight satisfies

$$|w_{\Gamma,D,D_F}|_{1;h} \leq e^{C_*(\|D\|^2 + \|D_F\|^2) + C_k/W + C_k\eta}\, e^{-c'\vartheta - c'\eta v_{\mathcal{S}}}. \tag{19.42}$$

(c) For every $j \geq 1$, the partial chain $f_j = T^j_\Gamma 1$ extends holomorphically from $\mathcal{S}$ to a neighborhood of $\mathcal{S}$ that contains $\mathcal{S}_D$, and $f_{j+1}|_{\mathcal{S}_D}$ is the image of $f_j|_{\mathcal{S}_D}$ under the shifted transfer operator, the heat on $\mathcal{S}_D$ followed by multiplication by w_{Γ,D,D_F}. For $1 \leq j \leq C_0 W$,

$$\|f_j|_{\mathcal{S}_D}\|_{\mathcal{H}_p(\mathcal{S})} \leq C_k W^{A_k} \exp\big[C_k C_0 + C_* j(\|D\|^2 + \|D_F\|^2)\big]. \tag{19.43}$$

(d) Let $0 < R \le R_{\mathrm{s}}$ and $1 \le j \le C_0 W$. The partial chain f_j, written in the coordinates of the tube through the canonical map, has norm on the set $\{|U| \le R\}$ of the tube at most

$$C_k W^{A_k} \exp\bigl[C_k C_0 + 16 C_{\mathrm{geo}}^2 C_* C_0 W (E^2 R^2 + R^4)\bigr], \tag{19.44}$$

uniformly for Γ in a compact parameter ball strictly inside the parameter ball of Proposition 19.4.

Proof. We first prove (a). Let Ξ be the Fourier variable of P_0, as in Lemma 15.7. Since $\mathrm{ch}_{\mathrm{g}} = \frac{1}{2}(\mathrm{Ad}_{\mathrm{g}} + \mathrm{Ad}_{\mathrm{g}^{-1}})$, its derivative in a direction V is

$$(D_V \mathrm{ch}_{\mathrm{g}})Y = \tfrac{1}{2}\bigl([D_V \mathrm{g}\, \mathrm{g}^{-1}, \mathrm{g} Y \mathrm{g}^{-1}] + [\mathrm{g}^{-1} Y \mathrm{g}, \mathrm{g}^{-1} D_V \mathrm{g}]\bigr).$$

For block-diagonal Y we have $\| \mathrm{Ad}_{\mathrm{g}^{\pm 1}} Y\|^2 = 2\| \mathrm{ch}_{\mathrm{g}}\, Y\|^2 - \|Y\|^2$, and therefore

$$\|(D_V \mathrm{ch}_{\mathrm{g}})P_D\| \le \sqrt{2}\bigl(\|D_V \mathrm{g}\, \mathrm{g}^{-1}\| + \|\mathrm{g}^{-1} D_V \mathrm{g}\|\bigr)\|D\|.$$

Differentiating $\mathrm{ch}_{\mathrm{g}}\, P_D = D$, we obtain $\langle \Xi, D_V P_D\rangle = -\langle \mathrm{ch}_{\mathrm{g}}^{-1}\, \Xi, (D_V \mathrm{ch}_{\mathrm{g}})P_D\rangle$. Using the first line of (15.30) for unit directions V, and the second line for $V = DF_X^* \Xi$, we obtain (19.41). The second, stronger bound is needed for the directions Ξ whose defect $d_X(\Xi)$ is small compared with $\|\Xi\|^2$. In the Fourier variable the shift replaces ϕ_Ξ by $\phi_\Xi - \langle \Xi, P_D(X)\rangle$, which is still real, and (19.41) implies that the shifted defect satisfies

$$\|\Xi\|^2 - \|D(F_X + P_D)^* \Xi\|^2 \ge \bigl(1 - 2C\|D\| - C^2\|D\|^2\bigr) d_X(\Xi).$$

For r_{s} small, the right side is at least $d_X(\Xi)/2 \ge 0$, and the proof of Lemma 15.7 gives the shifted kinetic estimate in (a).

We next prove (b), beginning with the shifted BB onsite factor. Let $\kappa = E/2$ and set

$$L = \mathrm{g}^{-1} \Lambda P_0 \mathrm{g}^{-1}, \qquad L_D = \mathrm{g}^{-1} \Lambda P_D \mathrm{g}^{-1} = \Lambda\, \mathrm{Ad}_{\mathrm{g}}(P_D), \qquad A_\kappa = s\mathrm{Id} + \mathrm{i}(L - \kappa\Lambda).$$

The holomorphic BB phase is, up to its saddle constant, $\widetilde{V}_\kappa(P_0) = \frac{1}{2}\operatorname{Tr} P_0^2 + \kappa \operatorname{Tr} P_0 - \mathrm{i}s \operatorname{Tr} L + \operatorname{Tr} \log A_\kappa$, with real part V_κ. Its derivative in the direction $\mathrm{i}P_D$ has real part $\operatorname{Tr}[(s\mathrm{Id} - \operatorname{Re} A_\kappa^{-1}) L_D]$. Let $A = \mathrm{Id} + L^2$ and $\mathcal{M}_L = A^{-1/2}\{\Lambda, L\}A^{-1/2}$. The inequality $2|\langle \Lambda u, Lu\rangle| \le \|u\|^2 + \|Lu\|^2$ implies that $\|\mathcal{M}_L\|_{\mathrm{op}} \le 1$. With $H_L = L - \kappa\Lambda$, we have $\operatorname{Re} A_\kappa^{-1} = s(s^2 + H_L^2)^{-1}$ and $s^2 + H_L^2 = A^{1/2}(\mathrm{Id} - \kappa\mathcal{M}_L)A^{1/2}$, by $\kappa^2 + s^2 = 1$. Therefore

$$\operatorname{Re} A_\kappa^{-1} = sA^{-1/2}(\mathrm{Id} - \kappa\mathcal{M}_L)^{-1} A^{-1/2}.$$

We also use $\|\mathrm{Id} - A^{-1}\|^2 \le \operatorname{Tr} \log A$ and $\|LA^{-1/2}\|^2 \le \operatorname{Tr} \log A$, which follow from $x/(1+x) \le \log(1+x)$, so that $\|\mathcal{M}_L\| \le 2(\operatorname{Tr} \log A)^{1/2}$. It follows that

$$\begin{aligned} \|s\mathrm{Id} - \operatorname{Re} A_\kappa^{-1}\| &\le s\|\mathrm{Id} - A^{-1}\| + s\bigl\|A^{-1/2}\bigl((\mathrm{Id} - \kappa\mathcal{M}_L)^{-1} - \mathrm{Id}\bigr)A^{-1/2}\bigr\| \\ &\le \Bigl(1 + \frac{2|\kappa|}{1 - |\kappa|}\Bigr)(\operatorname{Tr} \log A)^{1/2}. \end{aligned}$$

By the concavity bound (15.20) for the first relation, and since $L_D = \Lambda(D + \mathrm{th}_{\mathrm{g}}\, D)$ is an orthogonal sum with $\|\mathrm{th}_{\mathrm{g}}\, D\|^2 = \|D\|^2 - \|P_D\|^2$ for the second, we have

$$\operatorname{Tr}\log A = 2\Big(V_0 - \frac{1}{2}\operatorname{Tr} P_0^2\Big) \le 4V_\kappa, \qquad \|L_D\|^2 = 2\|D\|^2 - \|P_D\|^2 \le 2\|D\|^2,$$

and Young's inequality bounds the first variation by $\mu V_\kappa + C_\mu\|D\|^2$. Let $0 \le t \le 1$. Since $\sqrt{2}\|D\| \le s_0/2$ and $\|L_D\|_{\mathrm{op}} \le \|L_D\| \le \sqrt{2}\|D\|$, the Hermitian part $s\mathrm{Id} - tL_D$ of $A_\kappa - tL_D$ is at least $s\mathrm{Id}/2$; in particular the BB block stays invertible on $\mathcal{S}_D$, so the shifted factor has no poles. The second variation of the logarithm is bounded by $C\|L_D\|^2 \le 2C\|D\|^2$, and the Gaussian second variation is $-\|P_D\|^2$. Taylor's formula yields, with dimension-free constants,

$$\operatorname{Re}\widetilde{V}_\kappa(P_0 + \mathrm{i}P_D) \ge (1-\mu)V_\kappa(P_0) - C_\mu\|D\|^2.$$

For the FF block, Lemma 15.8 with the shift D_F bounds the body. For the nonbody coefficients we use, as in the proof of Lemma 15.8, Hadamard's bound by the column norms $y_i^{1/2}$ in the eigenbasis of M, since the joint diagonalization used in Lemma 15.9 is not available for the shifted factor; the damping is treated as in Lemma 15.9. Combining these bounds, we obtain (19.42).

We now prove (c), that is, we identify the partial chains on the shifted cycle with the holomorphic continuation of the partial chains on $\mathcal{S}$. Let f be one of the partial chains $f_j = T_\Gamma^j 1$. For a point S of $\mathcal{S}_D$, we define $f(S)$ by the same iterated integral over $\mathcal{S}$, with the complex output point S. The link Gaussian from the complex output S to a point $S' \in \mathcal{S}$ has, up to the FF and odd parts, the real exponent

$$\|P_0 - P_0'\|^2 - \|P_D\|^2 + d(X, X') + 2\operatorname{Tr}\big(P_D\Lambda(f_0(X) - f_0(X'))\big).$$

The defect and the cross term grow at the same exponential rate as X' moves away from X, and the integral converges because $\|D\|$ is small.

To see this, we write $\mathrm{g} = \mathrm{g}_X$ and $\mathrm{g}' = \mathrm{g}_{X'}$, and we let $\mathrm{g}\mathrm{g}'^{-1} = k_1 e^Y k_2$ be the Cartan decomposition of this element of $U(k,k)$, with k_1 and k_2 block diagonal and unitary, and with Y Hermitian and off-diagonal, with eigenvalues $\pm y_j$ for $1 \le j \le k$. By (B.6), we have

$$d(X, X') = s^2 \operatorname{Tr}\big(\mathrm{g}^2\mathrm{g}'^{-2} + \mathrm{g}^{-2}\mathrm{g}'^2 - 2\mathrm{Id}\big) = 4s^2\sum_{j=1}^{k}(\cosh 2y_j - 1). \tag{19.45}$$

The matrix $f_0(X')$ is the block-diagonal part of $s\mathrm{g}'^2$, by (15.25). Since P_D is block diagonal, $\mathrm{g}\Lambda = \Lambda\mathrm{g}^{-1}$ and $\mathrm{Ad}_{\mathrm{g}}\, P_D = D + \mathrm{th}_{\mathrm{g}}\, D$, we find

$$2\operatorname{Tr}\big(P_D\Lambda(f_0(X) - f_0(X'))\big) = 2s\operatorname{Tr}\big((D + \mathrm{th}_{\mathrm{g}}\, D)(\Lambda - k_1\Lambda e^{-2Y}k_1^{-1})\big).$$

We have $\|D + \mathrm{th}_{\mathrm{g}}\, D\| \le \sqrt{2}\|D\|$, the matrix k_1 commutes with Λ, and

$$\|\Lambda - \Lambda e^{-2Y}\|^2 = 4\sum_{j=1}^{k}\cosh 2y_j\,(\cosh 2y_j - 1) \le \frac{d(X,X')}{s^2} + \frac{d(X,X')^2}{4s^4}.$$

By the Cauchy–Schwarz inequality for the Hilbert–Schmidt inner product, it follows that

$$\left|2\,\mathrm{Tr}\big(P_D\Lambda(f_0(X)-f_0(X'))\big)\right| \le \sqrt{2}\|D\|\big(s^{-1}d(X,X')+2d(X,X')^{1/2}\big),$$

uniformly in the boost. Since $\sqrt{2}\|D\| \le s_0/2$, Young's inequality and $\|P_D\| \le \|D\|$ show that the real exponent is at least $\|P_0 - P_0'\|^2 + d(X,X')/4 - 9\|D\|^2$. In particular, the integral converges locally uniformly, and f is holomorphic near $\mathcal{S}$.

Next, the heat on $\mathcal{S}_D$ applied to the restriction of f is the restriction of $K_\tau f$. Indeed, at fixed (X', M') and fixed odd coordinates, we may shift the P_0'-integral back by Cauchy's theorem, since the integrand is holomorphic in P_0' on the strip $\|\operatorname{Im}\widetilde{P}'\| \le \|D\|$, has no poles there by the invertibility above, and has the Gaussian decay $e^{-\|P_0'\|^2/2+\|P_D\|^2/2}$ uniformly on the strip. The FF shift is treated in the same way in M'. By induction on j, $f_{j+1}|_{\mathcal{S}_D}$ is the shifted onsite weight times $(K_\tau f_j)|_{\mathcal{S}_D}$, which is the image of $f_j|_{\mathcal{S}_D}$ under the shifted transfer operator.

We start from $\|w_{\Gamma,D,D_F}\|_{\mathcal{H}_p(\mathcal{S})} \le C_k\eta^{-A_k}e^{C_*\|D\|^2}$, and we apply the shifted kinetic estimate of (a) and the bound (19.42) at each step. Since $e^{-c'\vartheta} \le 1$ and the factors $e^{C_k/W+C_k\eta}$ multiply to at most $e^{C_kC_0}$ over $j \le C_0W$ steps, we obtain (19.43).

Finally, we prove (d). The canonical gauge and the normal height selected by the canonical map both depend on the output point. We control both selections by Lemma A.5, applied jointly in $\zeta = (\Gamma, D, D_F)$ before the real body is integrated, on a compact parameter ball strictly inside the Hardy ball of Section 19.2, on which the Bergman integral is bounded by the Hardy norm. If the selected heights have norm at most B, we use a ball of radius B in (D, D_F) about the selected heights and a fixed radius in Γ, after a linear rescaling as in Remark A.6. For complex D, the imaginary part of D is a real translation of P_0 with unit Jacobian at fixed X. It acts as a real magnetic potential, which does not change the real part of the form and preserves the norm of $\mathcal{H}_p(\mathcal{S})$, and the genuine imaginary strip is $\operatorname{Re} D$, of norm at most $2B$. For R_{s} small, $2B \le r_{\mathrm{s}}/2$, and therefore (19.43) holds on that ball with $4B^2$ in place of $\|D\|^2 + \|D_F\|^2$. The volume ratio of the submean lemma is a constant that depends on k but not on B or W. Fubini's theorem bounds the norm of the selected values by that constant times $e^{4C_*mB^2}$ times the bound (19.43), where $m \le 2C_0W$. The nilpotent canonical corrections are kept coefficientwise, and they change the bound by fixed powers of W, once, at the output. By Lemma 19.15, we have $B^2 \le 2C_{\mathrm{geo}}^2(E^2R^2 + R^4)$ on the set $\{|U| \le R\}$, and therefore

$$4C_*mB^2 \le 16C_{\mathrm{geo}}^2C_*C_0W(E^2R^2 + R^4).$$

This proves (19.44). □

Curved graphs. Bounds at constant height imply bounds on curved graphs, with a prefactor that depends on k. Let F be Hilbert-space valued and holomorphic in the complex normal coordinate $w \in \mathbb{C}^d$, with all half-densities included, and let $b(x)$ be

measurable with $|b(x)| \le B$. Suppose that $\int \|F(x+\mathrm{i}v)\|^2 \,\mathrm{d}x \le M^2$ for all $|v| \le 2B$. We apply Lemma A.5 with $\zeta = w$, the function $(w,x) \mapsto F(x+w)$, the selection $\zeta_*(x) = \mathrm{i}b(x)$, and $\rho = B$. The ball $B(\mathrm{i}b(x), B)$ lies in $\{|\operatorname{Re} w| < B,\ |\operatorname{Im} w| < 2B\}$, and Fubini's theorem in $y = x + \operatorname{Re} w$ and $v = \operatorname{Im} w$ gives

$$\int \|F(x+\mathrm{i}b(x))\|^2 \,\mathrm{d}x \le \frac{\mathrm{vol}_{\mathbb{R}^d}(B)\,\mathrm{vol}_{\mathbb{R}^d}(2B)}{\mathrm{vol}_{\mathbb{R}^{2d}}(B)} M^2. \tag{19.46}$$

The ratio depends on d but not on B or W, and it enters once, at the output, and not at the nodes. On a chart where the cutoffs equal one the submean balls lie where the real cutoff equals one.

19.6 The energy threshold and the physical residual

We now fix the energy threshold and show that a family of finite prepared norm, continued to the cycle by the canonical map, has an exponentially small residual on the annulus $r_1 \le |U| \le r_{\mathrm{phys}}$, where the physical cutoff varies. We begin with the prepared norm.

The prepared norm. We fix the constant C_k in the exponent of (19.36), and we set

$$C_{\mathrm{prep}} = C_n C_0 + C_k R_k,$$

so that this exponent is $C_{\mathrm{prep}} W |b|^2$. For a family $f_{\Gamma,\mathbf{v}}$ generated from the endpoint that is holomorphic in the normal variables where $\psi_{R_k} \neq 0$, for a parameter ball $\mathcal{U}$ and a body weight p, we define the *prepared norm*

$$\|f\|^2_{\mathrm{prep},\mathcal{U};p} = \sup_{|b| \le c_1 R_k} e^{-2C_{\mathrm{prep}} W |b|^2} \int_{\mathcal{U}} \sum_{\ell} \|f_{\Gamma,\ell}\|^2_{b;p,a} \,\mathrm{d}\Gamma, \tag{19.47}$$

with the components $f_{\Gamma,\ell}$ of (19.1) and the Gaussian margin a of Proposition 19.13. The term with $b = 0$ is the family norm (19.1), so that $\|f\|_{\mathcal{U};p,a} \le \|f\|_{\mathrm{prep},\mathcal{U};p}$. Integrating (19.36) over $\Gamma \in \mathcal{U}_2$ and summing over the components, we find that the output of a block of $m = \lceil C_0 W \rceil$ adapted steps satisfies

$$\|A_h^m g\|_{\mathrm{prep},\mathcal{U}_2;p} \le C_k(R_k) \|g\|_{\mathcal{U}_2;p,-a} \tag{19.48}$$

for every family g generated from the endpoint, where on the right is the family norm (19.1) with the Gaussian margin $-a$. By the last assertion of Proposition 19.13, (19.48) also holds for a block that contains finitely many jet steps, with the changes of (19.30).

The prepared norm controls the values of a family on the complex normal graphs of height $|b| \le c_1 R_k$. A positive Gaussian margin on the real tube, which makes a family prepared, does not bound these values. The residual bounds of this section use them, and they are therefore stated for families of finite prepared norm, such as the outputs of blocks in (19.48).

The energy threshold. The Gaussian weight $e^{a|Z|^2}$ of the prepared norm, evaluated at the complex normal point $U + \mathrm{i}b$, has the real exponent

$$a(|Z|^2 - |Z_b|^2) \geq c_{\mathrm{ann}} W |U|^2 - C_{\mathrm{ann}} W |b|^2,$$

with the universal constants $c_{\mathrm{ann}} = a c_Z$ and $C_{\mathrm{ann}} = a C_Z$. Conversion from the symmetric normal gauge to the physical image under Can also multiplies by γ_h, besides the bounded half-weight and half-density factors. The ordinary body of this Gaussian satisfies

$$|\gamma_h(U + \mathrm{i}b)_\emptyset|^2 = e^{-(|Z|^2 - |Z_b|^2)/2}, \qquad e^{|Z_b|^2/4} \leq e^{C_{\mathrm{gau}} W |b|^2}, \qquad C_{\mathrm{gau}} = C_Z/4. \tag{19.49}$$

Since the ordinary Wick density is $e^{-|Z|^2/2}$, the conversion of a norm on this graph therefore costs $e^{|Z_b|^2/4}$. The odd Gaussian factors are bounded coefficient multipliers; the other changes of half-density and frame require only fixed constants and polynomial body weights. Thus C_{gau} is a separate, universal height cost, while $C_{\mathrm{ann}} = a C_Z$ records only the signed Gaussian weight. Recall the universal rate c_{D} of the Davies–Gaffney estimate Lemma 19.9, which gives the decay term $-c_{\mathrm{D}} W d_{\mathrm{col}}^2 / C_0$ in the exponent of the barrier (19.62) below, and recall that the collar distance of the barrier is $d_{\mathrm{col}} = R_k/8$. We choose $E_{\mathrm{geom}} > 0$ so small that

$$4(C_n C_0 + C_{\mathrm{ann}} + C_{\mathrm{gau}}) C_{\mathrm{geo}}^2 E_{\mathrm{geom}}^2 \leq \frac{c_{\mathrm{ann}}}{4}, \qquad 8192\, C_{\mathrm{geo}}^2 C_* C_0^2 E_{\mathrm{geom}}^2 \leq c_{\mathrm{D}}. \tag{19.50}$$

The first condition is used in the proof of Lemma 19.18. The second makes the term $16 C_{\mathrm{geo}}^2 C_* C_0 W E_{\mathrm{geom}}^2 R_k^2$ of the barrier (19.62) at most an eighth of the decay term $c_{\mathrm{D}} W d_{\mathrm{col}}^2 / C_0 = c_{\mathrm{D}} W R_k^2 / (64 C_0)$ of its exponent. Since c_{ann} is proportional to a, the two conditions amount to

$$E_{\mathrm{geom}} \lesssim C_{\mathrm{geo}}^{-1} \Big(\frac{a}{C_n C_0 + C_{\mathrm{ann}} + C_{\mathrm{gau}}} \Big)^{1/2}, \qquad E_{\mathrm{geom}} \lesssim \frac{1}{C_{\mathrm{geo}} C_0} \Big(\frac{c_{\mathrm{D}}}{C_*} \Big)^{1/2},$$

where $0 < a < a_0$ and $C_0 \geq \varrho > 2a/v_0$. Every constant in this chain is universal and depends only on constants chosen before it in Table 19.1: C_n by Remark 19.10, $C_{\mathrm{gau}} = C_Z/4$ by the standardized normal-coordinate bounds, C_{geo} by Lemma 19.15, C_* by (19.42) and c_{D} by Lemma 19.9. Therefore E_{geom} is universal. We do not give it explicitly, since these universal constants are not computed. We set $c_{\mathrm{sh}} = c_{\mathrm{ann}}/4$.

The shell jets. Let $K_{\mathrm{ad},s}$ denote the adapted heat over time s, and $\mathbf{1}_{\mathrm{ann}}$ the indicator of the annulus $r_1 \leq |U_{\mathrm{can}}^{\mathrm{re}}| \leq r_{\mathrm{phys}}$ where the physical cutoff varies, defined after (19.40). Let $\| \cdot \|_{\mathrm{jets}}$ be the sum of the $\mathcal{H}_p(\mathcal{S})$ norms of the finitely many normal and tangent derivatives used in Section 20, together with the derivatives of order at most one; we call these derivatives, restricted to the annulus, the *shell jets*. Recall the canonical map Can of Section 19.5 and the physical cutoff χ_{phys} of (19.40), and set $\mathrm{Can}_{\mathrm{phys}} = \chi_{\mathrm{phys}} \mathrm{Can}$.

The following proposition, with its two lemmas, shows that a family of finite prepared norm (19.47), continued to the cycle by the canonical map, has an exponentially small heat residual: the physical step applied to its canonical image differs from the canonical image of the adapted step only through the commutator of the physical heat with χ_{phys}, which lives on the annulus, and there the family is exponentially small.

Lemma 19.18. *Let $|E| \le E_{\mathrm{geom}}$, choose the nested radii of Table 19.1 at fixed rank, and let $p \in \mathbb{R}$. There exist $A_{\mathrm{ann}} \ge 0$ and $p_{\mathrm{ann}} \ge p$, depending only on k, p, the orders of the shell jets, and the prescribed local jets of the family, such that the following holds for all sufficiently large W. Let $f = f_{\Gamma,\mathbf{v}}$ be a family generated from the endpoint that is holomorphic in the normal variables where $\psi_{R_k} \neq 0$, and let $\Gamma \in \mathcal{U}_0$. Then*

$$\sup_{0 \le s \le \tau_{\mathrm{phys}}} \big\| \mathbf{1}_{\mathrm{ann}}\, \mathrm{Can}\, K_{\mathrm{ad},s}(\chi_{\mathrm{inv}} f_{\Gamma,\mathbf{v}}) \big\|_{\mathrm{jets}} \le C_k W^{A_{\mathrm{ann}}} e^{-c_{\mathrm{sh}} W r_1^2} \|f\|_{\mathrm{prep},\mathcal{U}_2;p_{\mathrm{ann}}}. \tag{19.51}$$

Proof. We first continue the family. By Lemma 19.15 and (19.38), $S_{\mathrm{g}} - \mathrm{Id}$ and its logarithm Y_{g} are $O(|U|)$ uniformly in the boost. The path $t \mapsto e^{tY_{\mathrm{g}}}$, $t \in [0,1]$, therefore stays in a neighborhood of the identity of size $O(|U|)$. By Lemma 19.2(b), the continuation of the body along this path is the family on the real tube at damping $S_{\mathrm{g}} \Gamma S_{\mathrm{g}}^{-1}$ and directions $S_{\mathrm{g}} V_a S_{\mathrm{g}}^{-1}$. We choose the tube so small that these damping values stay in $\mathcal{U}_1$ for $\Gamma \in \mathcal{U}_0$.

We now evaluate at the canonical point. The parameter and the normal height that the canonical map selects both depend on the output, and we control the two selections together. We cover the annulus by the shells $\Omega_s = \{s \le |U| \le (1+\theta)s\}$, where $(1+\theta)^6 \le 2$. By Lemma 19.15, the canonical height on Ω_s is at most

$$B_s = C_{\mathrm{geo}}\big(|E|(1+\theta)s + (1+\theta)^2 s^2\big).$$

The first condition in (19.50) and $c_Z \le C_Z$ give $C_{\mathrm{geo}}|E| \le 1/4$, and therefore $B_s \le s$ for $s \le r_{\mathrm{phys}}$ once r_{phys} is small. On each shell we apply Lemma A.5 jointly in $\zeta = (\Gamma, w)$, where w is the complex normal coordinate, with the selection $\zeta_*(U) = (S_{\mathrm{g}} \Gamma S_{\mathrm{g}}^{-1}, \mathrm{i}b(U))$, after rescaling w so that the ball has radius θB_s in w and, in Γ, the radius equal to the distance from $\mathcal{U}_1$ to the complement of $\mathcal{U}_2$, as in Remark A.6. The submean balls lie inside the region where $\chi_{\mathrm{inv}} = 1$ and $\psi_{R_k} = 1$. We apply the submean estimate to the physical half-densities, including the Gaussian gauge, so that their ordinary normal measure is Lebesgue measure. As in (19.46), Fubini's theorem reduces the estimate, up to a volume ratio depending only on k, to physical norms at constant heights $|b'| \le (1+\theta)B_s$, restricted to the real points at distance at most θB_s from Ω_s. We compare these norms with the graph norms $\|f_{\Gamma',\ell}\|_{b';p_{\mathrm{ann}},a}$ of (19.35), accounting for (19.49) and the signed Gaussian weight below. By the definition (19.47), the integrals of the squared graph norms over $\mathcal{U}_2$ are at most $e^{2C_{\mathrm{prep}} W |b'|^2} \|f\|^2_{\mathrm{prep},\mathcal{U}_2;p_{\mathrm{ann}}}$.

The derivatives in the shell jets change these bounds by fixed factors. By Cauchy's estimates on polydisks of radius c/W, normal derivatives of order ℓ are bounded with an

additional factor CW^{ℓ}, and the finite odd canonical corrections, which we retain as finite jets, give further fixed powers of W. Derivatives of the Gaussian gauge also cost only fixed powers of W on the fixed tube. By the equivariance (19.3), tangent derivatives become parameter derivatives, and fixed body derivatives require fixed polynomial weights, which the body weight $p_{\mathrm{ann}} \geq p$ provides. These factors enter once, at the output, and not at the nodes. We take A_{ann} to be the sum of the resulting exponents of W and of the exponent of the final partial heat below.

We now collect the exponents on Ω_s. The real points U' of the submean balls satisfy $|U'| \geq (1-\theta)s$, since $B_s \leq s$. At the point $U' + \mathrm{i}b'$ of the inner tube, the Gaussian weight of the prepared norm has modulus $e^{a(|Z'|^2 - |Z_{b'}|^2)}$, where Z' and $Z_{b'}$ are the standardized versions of U' and b', and $|Z'|^2 \geq c_Z W|U'|^2$, $|Z_{b'}|^2 \leq C_Z W|b'|^2$. Removing this weight, converting to the physical norm by (19.49), and using the factor $e^{C_{\mathrm{prep}} W|b'|^2}$ contribute the factors

$$\exp\bigl\{-c_{\mathrm{ann}}(1-\theta)^2 W s^2 + (C_{\mathrm{ann}} + C_{\mathrm{gau}})W(1+\theta)^2 B_s^2\bigr\},$$
$$\exp\bigl\{(C_n C_0 + C_k R_k)W(1+\theta)^2 B_s^2\bigr\},$$

where $c_{\mathrm{ann}} = a c_Z$, $C_{\mathrm{ann}} = a C_Z$, and $C_{\mathrm{gau}} = C_Z/4$. Since $(1+\theta)^6 \leq 2$, we have

$$(1+\theta)^2 B_s^2 \leq 2C_{\mathrm{geo}}^2 (1+\theta)^6 (E^2 s^2 + s^4) \leq 4C_{\mathrm{geo}}^2 (E^2 s^2 + s^4),$$

and the first condition in (19.50) implies that the part of the exponent proportional to $(C_n C_0 + C_{\mathrm{ann}} + C_{\mathrm{gau}})E^2$ is at most $c_{\mathrm{ann}} W s^2/4$. The quartic part and the terms with $C_k R_k$ are at most $c_{\mathrm{ann}} W s^2/8$ once r_{phys} and R_k are small at fixed rank. Since $(1-\theta)^2 \geq 1 - 2\theta$ and $2\theta \leq 3/8$, the exponent is at most

$$-c_{\mathrm{ann}}\Bigl(1 - 2\theta - \frac{1}{4} - \frac{1}{8}\Bigr) W s^2 \leq -\frac{c_{\mathrm{ann}}}{4} W s^2 \leq -c_{\mathrm{sh}} W r_1^2$$

on Ω_s, and the finite number of shells changes only the constant. This bound is also uniform along the homotopy (19.39): its selected heights are $tb(U)$ with $0 \leq t \leq 1$, hence no larger than B_s, and its conjugated dampings remain in $\mathcal{U}_1$.

A final partial heat satisfies the same bound. Indeed, its normalized Gaussian either stays in the region where $\chi_{\mathrm{inv}} = 1$, or makes a jump across a fixed gap, which contributes at most $C_k W^{A_{\mathrm{ann}}} e^{-cW^2 r_1^2}$. The bound is uniform down to zero heat time, since the input is holomorphic in the normal variables on polydisks of a fixed radius about the real points of the annulus. This proves (19.51). Since $W \geq 1$ and the prepared norm increases with the body weight, the bound remains true when A_{ann} and p_{ann} are replaced by larger numbers. □

The exponent and the body weight of the residual. Let L denote the generator of the physical heat on the cycle, as in Section 19.2. By the construction of χ_{phys} in (19.40), the commutator $[L, \chi_{\mathrm{phys}}]$ is a first-order operator whose coefficients are supported on the

annulus where χ_{phys} varies. At fixed rank, these coefficients are bounded by fixed powers of W, from the odd coordinates, and by fixed powers of the body weight, from the boost, and the shell jets contain the derivatives of order at most one. Hence there exist $A_1 \geq 0$ and $q \geq 0$, depending only on k, such that

$$\|[L, \chi_{\mathrm{phys}}]u\|_{\mathcal{H}_p(\mathcal{S})} \leq C_k W^{A_1} \|\mathbf{1}_{\mathrm{ann}} u\|_{\mathrm{jets},p+q}, \tag{19.52}$$

where the norm on the right is that of the shell jets in the body weight $p+q$. Let A_{ann} and p_{ann} be the exponent and the body weight of Lemma 19.18 for the body weight $p+q$. We set

$$A_{\mathrm{sh}} = A_{\mathrm{ann}} + A_1, \qquad p' = p_{\mathrm{ann}}.$$

They depend only on k, p, the orders of the shell jets, and the prescribed local jets of the family, and $p' \geq p+q \geq p$. Since $v \geq 1$ and $q \geq 0$, the norm of the shell jets in the body weight p is at most their norm in the body weight $p+q$. Therefore (19.51) for $p+q$ implies (19.51) for p with A_{sh} and p' in place of A_{ann} and p_{ann}. We use these A_{sh} and p' in the rest of this section.

Lemma 19.19. *Let E, the radii, p, f and Γ be as in Lemma 19.18, let A_{sh} and p' be as defined before this lemma, and suppose that $\|f\|_{\mathrm{prep},\mathcal{U}_2;p'} < \infty$. For $0 \leq s \leq \tau_{\mathrm{phys}}$, let $u_s = \mathrm{Can}\, K_{\mathrm{ad},s}(\chi_{\mathrm{inv}} f)$. Then the physical step T_Γ and the adapted step A_h satisfy the coefficientwise identity*

$$(T_\Gamma \,\mathrm{Can}_{\mathrm{phys}} - \mathrm{Can}_{\mathrm{phys}}\, A_h) f = M_{w_\Gamma} \int_0^{\tau_{\mathrm{phys}}} K_{\tau_{\mathrm{phys}}-s}\,[L, \chi_{\mathrm{phys}}]\, u_s \,\mathrm{d}s, \tag{19.53}$$

where L is the generator of the physical heat and M_{w_Γ} is the multiplication by w_Γ.

Proof. We first say what u_s means at the points of the cycle, which are complex points of the tube. Let

$$g = \varpi\, w_0^{1/2} \gamma_h\, \chi_{\mathrm{inv}} f$$

be the raw input on the real tube, let k_s be the kernel of the ambient heat over time s, and let $s > 0$. For a real frame x and a complex normal coordinate U with $|U| \leq r_{\mathrm{sep}}$, we set

$$u_s(x, U) = \int k_s\big(R(x, U), R(y')\big)\, g(y')\, \mathrm{d}y', \tag{19.54}$$

where $y' = (x', U')$ runs over the real tube with the flat measure $\mathrm{d}U'\,\mathrm{d}\nu(x')$. By (18.13), the real part of the exponent of k_s is bounded above by $-c\, s_E^2\, \mathfrak{d}(x, x')/(4s)$ plus $|\operatorname{Im} U|^2/s$, and the odd coordinates contribute only polynomial factors. Therefore the integral converges locally uniformly, and u_s is holomorphic in U. To evaluate u_s at the canonical point of a point of the cycle, we use (19.54) for the family at the conjugated damping $\widehat{S}\Gamma\widehat{S}^{-1}$ and directions $\widehat{S}\mathbf{V}\widehat{S}^{-1}$, including the finite nilpotent expansion, as in the definition

of Can. The partial heat $K_{\mathrm{ad},s}(\chi_{\mathrm{inv}} f)$ is a family generated from the endpoint, and by Lemma 19.2(b) this is its canonical image $\mathrm{Can}\, K_{\mathrm{ad},s}(\chi_{\mathrm{inv}} f)$. By the same lemma, u_s is holomorphic along the complexified orbits. Together with its holomorphy in U, this makes u_s a holomorphic function of the ambient point near the canonical points.

We now show that u_s satisfies the heat equation $\partial_s u_s = \Delta_{\mathrm{amb}} u_s$ near the canonical points, where Δ_{amb} acts on the holomorphic extension just described; the points $R(x,U)$ with x real and U complex do not form an open set. The real orbit directions and the complex normal directions at such a point span the ambient tangent space over $\mathbb{C}$. The holomorphic derivatives of a holomorphic function at the point are therefore determined by its derivatives in these directions, and Δ_{amb} acts as the operator $\varpi^{-1}(\Delta_{\mathrm{nor}} + L_H(U))\varpi$ of (16.29) in the coordinates (x,U). Recall from Section 18.3 that k_s is entire in its first argument and satisfies $\partial_s k_s = \Delta_{\mathrm{amb}} k_s$. We differentiate (19.54) under the integral sign in s, in x and in U, which is justified by (18.13) locally uniformly for $s > 0$, and we obtain $\partial_s u_s = \Delta_{\mathrm{amb}} u_s$ at the points $R(x,U)$ with x real and $|U| \le r_{\mathrm{sep}}$. The function $\partial_s u_s - \Delta_{\mathrm{amb}} u_s$ is holomorphic on the connected open set of points $S^{-1}R(x,U)S$ with x real, U complex and S near the identity, and it vanishes on the real tube, which is a totally real submanifold of maximal dimension. By the identity theorem, it vanishes on this set, which contains the canonical points. The generator L of the physical heat is the restriction of Δ_{amb} to the cycle, so the chain rule gives $\partial_s u_s = L u_s$ on the cycle.

The identity at time $\tau = \tau_{\mathrm{phys}}$ is exact. By (16.38) and (16.33), with the cutoffs of (19.21),

$$A_h f = \chi_{\mathrm{inv}}\, s_\Gamma \gamma_h^{-1} w_0^{1/2} \varpi \cdot (u_\tau \circ R)$$

on the real tube. The canonical map multiplies by the inverse half-density and the inverse gauge $\varpi^{-1} w_0^{1/2} \gamma_h$, every factor other than χ_{inv} is holomorphic in the normal variables, and $\chi_{\mathrm{inv}} = 1$ on the support of χ_{phys}. Since $s_\Gamma w_0 = w_\Gamma$, we obtain

$$\mathrm{Can}_{\mathrm{phys}}\, A_h f = \chi_{\mathrm{phys}}\, w_\Gamma\, u_\tau. \tag{19.55}$$

Both sides are continued to complex orbit points by Lemma 19.2(b), so they agree there by the identity theorem.

We next apply Duhamel's formula on $[\varepsilon, \tau]$, for $0 < \varepsilon < \tau$. The explicit Gaussian form of k_s bounds u_s and its derivatives on the support of χ_{phys}, at fixed W and uniformly for $\varepsilon \le s \le \tau$. Hence $\chi_{\mathrm{phys}} u_s$ lies in the domain of L in $\mathcal{H}_p(\mathcal{S})$, and $s \mapsto \chi_{\mathrm{phys}} u_s$ is continuously differentiable there, with derivative $\chi_{\mathrm{phys}} L u_s$. Since $\partial_s(K_{\tau-s}\chi_{\mathrm{phys}} u_s) = -K_{\tau-s}[L, \chi_{\mathrm{phys}}] u_s$, we obtain

$$K_{\tau-\varepsilon}(\chi_{\mathrm{phys}} u_\varepsilon) - \chi_{\mathrm{phys}} u_\tau = \int_\varepsilon^\tau K_{\tau-s}\, [L, \chi_{\mathrm{phys}}]\, u_s \, \mathrm{d}s. \tag{19.56}$$

It remains to let $\varepsilon \downarrow 0$. We show that

$$\lim_{\varepsilon \downarrow 0} \big\| \chi_{\mathrm{phys}} (u_\varepsilon - \mathrm{Can}\, f) \big\|_{\mathcal{H}_p(\mathcal{S})} = 0. \tag{19.57}$$

Fix a point of the cycle in the support of χ_{phys}, with canonical frame x and canonical normal coordinate $U + ib$. In (19.54), the orbit coordinate of the input is already in the real frame, at the conjugated damping. By Cauchy's theorem in the normal input variables, we move the normal input onto the constant-height graph through the output height b. This deformation takes place where $\chi_{\mathrm{inv}} = 1$ and the graph cutoff equals one, except for inputs at a normal distance at least $d > 0$ from the output, where d depends only on the nested radii. By (18.13), these inputs contribute $O(e^{-cd^2/\varepsilon})$, which tends to zero.

After this deformation, u_ε is the pulled Gaussian evolution (18.16) on the constant-height graph, with constant parameter and with the values of f on the same graph as initial data. As $\varepsilon \downarrow 0$ it tends to f in L^2 of the graph. For smooth f this is the Laplace method in the proof of Lemma 18.6, and for general f it follows by density, since that proof bounds the maps $f \mapsto \mathbf{1}_{\mathcal{T}_{\mathrm{in}}} u(t)$ uniformly in t. The damping and the height of the graph depend on the output. As in the proof of Lemma 19.18, we control this selection by Lemma A.5 in $\zeta = (\Gamma, w)$, which bounds the selected values by an integral of graph norms over a ball in (Γ, w). Each integrand tends to zero as $\varepsilon \downarrow 0$. It is bounded by a constant times the squared graph norm of f, which is integrable over the ball since $\|f\|_{\mathrm{prep},\mathcal{U}_2;p'} < \infty$. By dominated convergence, we obtain (19.57).

We now let $\varepsilon \downarrow 0$ in (19.56). The left side converges by (19.57), together with the uniform bound and the strong continuity of K_t from Lemma 15.7. By (19.52) and Lemma 19.18 in the body weight $p + q$, which applies since $\|f\|_{\mathrm{prep},\mathcal{U}_2;p'} < \infty$ for the body weight $p' = p_{\mathrm{ann}}$ defined before this lemma, we have

$$\int_0^\tau \|[L, \chi_{\mathrm{phys}}] u_s\| \, \mathrm{d}s < \infty,$$

and the right side converges. We obtain

$$K_\tau \operatorname{Can}_{\mathrm{phys}} f - \chi_{\mathrm{phys}} u_\tau = \int_0^\tau K_{\tau - s} \, [L, \chi_{\mathrm{phys}}] \, u_s \, \mathrm{d}s.$$

Multiplying by w_Γ, and using $T_\Gamma = M_{w_\Gamma} K_\tau$ and (19.55), we obtain (19.53). □

Proposition 19.20. *Let $|E| \le E_{\mathrm{geom}}$, choose the nested radii of Table 19.1 at fixed rank, and let $p \in \mathbb{R}$. Let $A_{\mathrm{sh}} \ge 0$ and $p' \ge p$ be the exponent and the body weight defined before Lemma 19.19, which depend only on k, p, the orders of the shell jets, and the prescribed local jets of the family. The following holds for all sufficiently large W. Let $f = f_{\Gamma,\mathbf{v}}$ be a family generated from the endpoint that is holomorphic in the normal variables where $\psi_{R_k} \neq 0$, with $\|f\|_{\mathrm{prep},\mathcal{U}_2;p'} < \infty$, and let $\Gamma \in \mathcal{U}_0$. Then f satisfies the shell bound* (19.51) *with A_{sh} and p' in place of A_{ann} and p_{ann}, the identity* (19.53)*, and, with u_s as in Lemma 19.19,*

$$\begin{aligned} &\Big\| \int_0^{\tau_{\mathrm{phys}}} K_{\tau_{\mathrm{phys}} - s} \, [L, \chi_{\mathrm{phys}}] \, u_s \, \mathrm{d}s \Big\|_{\mathcal{H}_p(\mathcal{S})} + \big\| (T_\Gamma \operatorname{Can}_{\mathrm{phys}} - \operatorname{Can}_{\mathrm{phys}} A_h) f \big\|_{\mathcal{H}_p(\mathcal{S})} \\ &\qquad \le C_k W^{A_{\mathrm{sh}}} e^{-c_{\mathrm{sh}} W r_1^2} \|f\|_{\mathrm{prep},\mathcal{U}_2;p'}. \end{aligned} \tag{19.58}$$

In particular, the heat residual is exponentially small, with the full onsite weight w_Γ, including its nilpotent terms.

Proof. By the choice of A_{sh} and p' before Lemma 19.19, the shell bound (19.51) holds with A_{sh} and p' in place of A_{ann} and p_{ann}, and Lemma 19.19 gives (19.53).

It remains to prove (19.58). By (19.52) and Lemma 19.18 in the body weight $p+q$, we have

$$\sup_{0\le s\le\tau} \|[L,\chi_{\mathrm{phys}}]u_s\|_{\mathcal{H}_p(\mathcal{S})} \le C_k W^{A_1+A_{\mathrm{ann}}} e^{-c_{\mathrm{sh}} W r_1^2} \|f\|_{\mathrm{prep},\mathcal{U}_2;p'}.$$

By Lemma 15.7, for $0 \le s \le \tau$ we have

$$\|K_{\tau-s}\|_{\mathcal{H}_p(\mathcal{S})\to\mathcal{H}_p(\mathcal{S})} \le e^{(C_k W + C_{k,p})\tau} \le C_k,$$

and $\tau \le 1$. Integrating over s, we obtain the bound (19.58) for the first term. By Lemma 15.9, multiplication by w_Γ, including its nilpotent terms, has norm at most $e^{C_k/W + C_k\eta}$ on $\mathcal{H}_p(\mathcal{S})$, and therefore (19.53) gives the bound for the second term. This completes the proof. □

By (19.48), the hypotheses of Proposition 19.20 hold for the output $f = A_h^m g$ of a block of $m = \lceil C_0 W \rceil$ adapted steps, for every family g generated from the endpoint with $\|g\|_{\mathcal{U}_2;p',-a} < \infty$, and then $\|f\|_{\mathrm{prep},\mathcal{U}_2;p'} \le C_k(R_k)\|g\|_{\mathcal{U}_2;p',-a}$. For a block with finitely many jet steps, the same holds with the changes of (19.30). This is how Section 20 uses the proposition. The long partial chains with which these residuals are paired there stay on the real cycle at their original BB damping, and they are bounded there by the $e^{o(W)}$ estimate of Theorem 15.14 in the opposite body weight.

19.7 The early barrier and the attachment of the endpoint

In this section we compare the physical block $f_m = T_\Gamma^m 1$ with local references on the tube, and prove the attachment theorem. For each signature $\alpha \in \{0,\dots,2k\}$ we choose disjoint normal neighborhoods of the corresponding FF orbit, with a common BB normal radius below the root gap. We work first in the raw realization, before the Gaussian gauge. Let ϖ_α be the half-density and $\mathrm{Can}_\alpha^{\mathrm{raw}}$ the canonical pullback with the inverse half-density and frame.

The domains of the comparison. In each of these neighborhoods we fix three nested normal neighborhoods of the orbit, defined through the norm $|U|$ of the normal coordinate. They are the *inner region* $\{|U| \le R_k/2\}$, on which we compare the physical chain with the references; the larger *comparison domain* $\{|U| < 7R_k/8\}$; and the region where the cutoffs χ_{inv} of the raw steps equal one. Recall from Section 19.4 that the last region contains the support of ψ_{R_k}, and hence the neighborhood $\{|U| \le R_k\}$ of the comparison domain.

We also use the graph cutoff $\psi_{\bar r}$ of (18.15) with $\bar r = 15R_k/16$. We have $\psi_{\bar r} = 1$ on the neighborhood $\{|U| \le 15R_k/16\}$ of the comparison domain, and the support of $\psi_{\bar r}$ lies in that of ψ_{R_k}, at distance at least $R_k/16$ from the complement of the latter. We choose an invariant normal class cutoff ξ_α, with all its nilpotent terms, equal to one on a neighborhood of the support of $\psi_{\bar r}$ and supported in the interior of the support of ψ_{R_k}, where $\chi_{\mathrm{inv}} = 1$. Its body takes values in $[0,1]$, and it is fixed in the physical normal variable U. In particular, ξ_α differs from one only at distance at least $R_k/16$ from the comparison domain. We also choose a real normal cutoff ϕ_α, supported in the comparison domain and equal to one on the inner region, whose collar lies in $\{5R_k/8 \le |U| \le 3R_k/4\}$; we call it the *barrier cutoff*. Its collar lies in $\{|U| \le R_k\}$, at normal distance at least $d_{\mathrm{col}} = R_k/8$ from the inner region.

We define the raw references

$$g^{\mathrm{raw}}_{\alpha,0} = \xi_\alpha \omega_\alpha, \qquad g^{\mathrm{raw}}_{\alpha,j} = S^j_{\alpha,\Gamma} g^{\mathrm{raw}}_{\alpha,0},$$

where $S_{\alpha,\Gamma}$ is the adapted step with cutoffs of (19.21) in the signature α, in its onsite-after-heat order. Then $\mathrm{Can}^{\mathrm{raw}}_\alpha g^{\mathrm{raw}}_{\alpha,0} = 1$ throughout the comparison domain, with the physical orientation. In the symmetric normal gauge, these references become $\gamma_h^{-1} w_0^{-1/2} g^{\mathrm{raw}}_{\alpha,j}$; both endpoint factors are transported together. Let $f^{\mathrm{raw}}_{\alpha,j}$ denote the partial chain f_j written in the raw chart of the signature α through the canonical map, as in Proposition 19.17(d). For a body weight p, we write $\|\cdot\|_{\mathrm{comparison},\mathcal{H}_p}$ and $\|\cdot\|_{\mathrm{inner},\mathcal{H}_p}$ for the raw norms in the body weight v^p of the restrictions to the comparison domain and to the inner region, taken coefficientwise.

The main result of this section is the following attachment theorem.

Theorem 19.21. *Let $m = \lceil C_0 W \rceil$, let $\chi^{\mathrm{phys}}_\alpha$ be real physical normal cutoffs supported in the disjoint tubes of radius r_{phys}, and set $\mathrm{Can}^{\mathrm{raw}}_{\alpha,\mathrm{phys}} = \chi^{\mathrm{phys}}_\alpha \mathrm{Can}^{\mathrm{raw}}_\alpha$. Let $p \in \mathbb{R}$, and let a finite family of derivatives in the damping parameter and in the local sources at zero be given. Then*

$$T^m_\Gamma 1 = \sum_{\alpha=0}^{2k} \mathrm{Can}^{\mathrm{raw}}_{\alpha,\mathrm{phys}}\, g^{\mathrm{raw}}_{\alpha,m} + e^R_m, \qquad \|e^R_m\|_{\mathcal{H}_p(\mathcal{S})} \le C_k W^{A_k} e^{-\gamma_k W}, \tag{19.59}$$

and the same bound holds for the derivatives of e^R_m in the given family, where $\gamma_k > 0$, A_k and C_k may depend on p and on the family. The constants are uniform for Γ in the smallest damping ball $\mathcal{U}_0$ of (15.36)*, for $|E| \le E_{\mathrm{geom}}$, and on compact sets of the aspect ratio and of the damping ε.*

We prove Theorem 19.21 at the end of this section. Near the saddle manifold, the physical chain and the references satisfy the same local equation and agree initially, so that their difference is created only on the collar of the barrier cutoff, at normal distance d_{col} from the inner region. The Davies–Gaffney estimate Lemma 19.9 shows that over one block this difference reaches the inner region only with a factor $e^{-c_{\mathrm{D}} W d^2_{\mathrm{col}}/C_0}$. The proof

uses five lemmas: the bound on the references near the collar (Lemma 19.22), the first step (Lemma 19.23), the barrier (Lemma 19.24), the choice of the radii (Lemma 19.25), and the continuation of the error to the canonical graph (Lemma 19.26).

Early annuli. The barrier below uses the raw references on the collar, where they need not be small, and it needs a bound there for every $0 \le j \le C_0 W$, including $j = 0$. We call the finitely many normal and tangent derivatives of a fixed order, restricted to the part of an annulus $|U| \ge r$ that lies in the comparison domain, the *annular jets*, and we write $\|\cdot\|_{\mathrm{raw,jets}}$ for the sum of their raw norms in the body weight v^{p_0} of Lemma 19.5. The annular jets are measured in raw norms of functions on the tube, on the part of $\{|U| \ge r\}$ in the comparison domain, whereas the shell jets of Section 19.6 are measured in $\mathcal{H}_p(\mathcal{S})$, for functions on the cycle, on the annulus where χ_{phys} varies. The comparison domain contains the collar of the barrier cutoff, where the annular jets are used. Let $c_{\mathrm{e}} = \min(v_0, a_0/C_0)$.

Lemma 19.22. *For all $0 \le j \le C_0 W$ and $r > 0$, the raw references satisfy*

$$\big\|\mathbf{1}_{\{|U|\ge r\}}\, g^{\mathrm{raw}}_{\alpha,j}\big\|_{\mathrm{raw,jets}} \le C_k W^{A_k} e^{-c_{\mathrm{e}} c_Z j r^2}. \tag{19.60}$$

For a positive body weight v^p, the same bound holds for $j \ge 1$ with a larger power of W.

Proof. The bound is a corollary of Lemma 19.8. Let $a'_i = c_{\mathrm{e}} i h^2$ for $0 \le i \le j$. Then

$$a'_0 = 0, \qquad 0 \le a'_i \le c_{\mathrm{e}} C_0 W h^2 \le a_0, \qquad |a'_i - a'_{i-1}| = c_{\mathrm{e}} h^2 \le v_0 h^2.$$

We apply Lemma 19.8 with $b = 0$ to the raw steps $S_{\alpha,\Gamma}$, in the body weight v^{p_0}. The input $\xi_\alpha \varpi_\alpha$ is bounded and supported in the tube, and its raw norm is at most $C_k W^{A_k}$, since the tube has volume $O(W^{A_k})$ in the standardized coordinates. Therefore

$$\|w_{a'_j} v^{p_0} g^{\mathrm{raw}}_{\alpha,j}\|_{\mathrm{raw}} \le C_k \|v^{p_0} \xi_\alpha \varpi_\alpha\|_{\mathrm{raw}} \le C_k W^{A_k}.$$

Since $|Z|^2 \ge c_Z W |U|^2$ and $a'_j W = c_{\mathrm{e}} j$, we have $w_{a'_j} \ge e^{c_{\mathrm{e}} c_Z j r^2}$ where $|U| \ge r$, and this proves (19.60) without derivatives. For a positive body weight v^p and $j \ge 1$, the damping $e^{-c\eta v}$ of the first onsite factor bounds the weight v^{p-p_0}, since $v^q e^{-c\eta v} \le C_q \eta^{-q}$ for $q \ge 0$, and η^{-q} is a power of W.

For the normal derivatives we use the same graph at every node, with the graph cutoff $\psi_{\bar r}$, $\bar r = 15R_k/16$, of the domains of the comparison in place of ψ_{R_k}. The proof of Lemma 19.8 applies verbatim to this cutoff, for heights $|b| \le c_1 \bar r$. Indeed, since $R_k/2 \le \bar r \le R_k$, the support of $\psi_{\bar r}$ is contained in that of ψ_{R_k}, where the cutoffs equal one, and the bound on the graph Jacobian satisfies $C|b|/\bar r \le 2C|b|/R_k$. Let b be a constant real normal direction with $|b| \le c/W$, and apply Lemma 19.8 in this form with $\lambda_i = 1$ for all i, including the input, and with the same weights a'_i. This is allowed since $\xi_\alpha \varpi_\alpha$ is

holomorphic in the normal variables on the support of $\psi_{\bar{r}}$, where $\xi_\alpha = 1$. There are no velocity terms, and the exponent in (19.29) is at most

$$(C_1 + C_k R_k) C_0 W |b|^2 \le (C_1 + C_k R_k)\, C_0 c^2 / W = O(1).$$

Where $\psi_{\bar{r}} = 1$, the weight on such a graph satisfies

$$|w_{a'_j}(U_\lambda)| = e^{a'_j(|Z|^2 - |Z_b|^2)}, \qquad a'_j |Z_b|^2 \le a_0 C_Z c^2 / W,$$

and the last quantity is small. For W large, the polydisks of radius c/W about the real points of the annulus in the comparison domain lie where $\psi_{\bar{r}} = 1$, and Cauchy's estimates on them bound the normal derivatives of order ℓ with an additional factor CW^ℓ. Shifting the annulus by c/W changes $e^{-c_e c_Z j r^2}$ only by a bounded factor, since $j \le C_0 W$. The tangent derivatives are derivatives in the parameters, by the equivariance (19.2), since the references are generated from the endpoint. By Cauchy's formula on the larger damping ball, they are bounded in fixed stronger body weights. The comparison with the Dirichlet heat is contained in the proof of Lemma 19.8. □

At small j the bound (19.60) gives no decay, since the raw references have no positive Gaussian margin at early times, and at $j = m$ it is exponentially small on each fixed annulus.

The first step. The partial chain and the reference satisfy the same local heat equation between steps and have the same onsite factor, including its nilpotent terms, and their initial data agree on the comparison domain. In a negative body weight this is literal agreement at time zero. For a positive body weight we start after the first step.

Lemma 19.23. *For every body weight p, coefficientwise, including every nilpotent term of the cutoff,*

$$\| f^{\mathrm{raw}}_{\alpha,1} - g^{\mathrm{raw}}_{\alpha,1} \|_{\mathrm{comparison},\mathcal{H}_p} \le C_k W^{A_k} e^{-cW^2}. \tag{19.61}$$

Proof. The local heat of the constant function is one, and the input cutoff ξ_α differs from one only at a fixed distance from the comparison domain. Since the heat kernel of one link is a Gaussian of variance of order W^{-2}, the part of the input where $\xi_\alpha \ne 1$ contributes at most $C_k W^A e^{-cW^2}$ on the comparison domain. The two chains are then multiplied by the same onsite factor. In a positive body weight v^p, the fresh damping $e^{-c\eta v}$ of this factor bounds the weight, since $v^p e^{-c\eta v} \le C_p \eta^{-p}$, and η^{-p} is a power of W. These two bounds imply (19.61). □

The barrier. We now compare $f^{\mathrm{raw}}_{\alpha,j}$ and $g^{\mathrm{raw}}_{\alpha,j}$ on the inner region. Since the comparison takes place on the real raw chart, the difference is forced only on the collar of the barrier cutoff, through two collar norms. For the partial chain, which is pulled to the chart by

the complex canonical map, the collar norm is bounded by (19.44). For the reference, only its real collar norm enters, and Lemma 19.22 bounds it, including at $j = 0$. Neither bound needs a Gaussian margin, since the decay comes from the distance d_{col}.

Lemma 19.24. *Let $p \in \mathbb{R}$, and suppose that $R_k \le R_{\mathrm{s}}$, with R_{s} as in Proposition 19.17. Then, uniformly for $\Gamma \in \mathcal{U}_0$,*

$$\|f^{\mathrm{raw}}_{\alpha,m} - g^{\mathrm{raw}}_{\alpha,m}\|_{\mathrm{inner},\mathcal{H}_p} \le C_k W^{A_k} \exp\Big[-\frac{c_{\mathrm{D}} W d_{\mathrm{col}}^2}{C_0} + 16 C_{\mathrm{geo}}^2 C_* C_0 W (E^2 R_k^2 + R_k^4)\Big]. \quad (19.62)$$

Proof. We write $\tau = \tau_{\mathrm{phys}}$, $L = L^{\mathrm{raw}}_\alpha$, and $\phi = \phi_\alpha$. Let $f^{\mathrm{raw}}(t)$ and $g^{\mathrm{raw}}(t)$ be the two chains on the heat intervals, so that $f^{\mathrm{raw}}(t) = e^{(t-j\tau)L} f^{\mathrm{raw}}_{\alpha,j}$ for $j\tau \le t < (j+1)\tau$, and similarly for $g^{\mathrm{raw}}(t)$, and at the end of each heat interval both are multiplied by the onsite factor w of the step. By Lemma 18.6, the heat of the chains agrees with the Dirichlet heat e^{tL} on the support of the cutoffs up to an error $C_k W^{C_k} e^{-c R_k^2 W^2}$ times the input norm. The input norms are at most e^{CW} on the tube, and there are $O(W)$ steps, so that these errors are superexponentially small, and we omit them below.

We first derive the Duhamel formula. Let $\delta(t) = f^{\mathrm{raw}}(t) - g^{\mathrm{raw}}(t)$. On each heat interval we have $\partial_t \delta = L\delta$ on the comparison domain, and since ϕ is supported there,

$$\partial_t(\phi\delta) = L(\phi\delta) - [L,\phi]\delta.$$

The commutator $[L,\phi]$ is of first order, and its coefficients vanish outside the collar of ϕ. By Duhamel's formula on the heat interval of step $j+1$, we have

$$\phi\delta\big((j+1)\tau^-\big) = e^{\tau L}\phi\delta(j\tau) - \int_{j\tau}^{(j+1)\tau} e^{((j+1)\tau - t)L}[L,\phi]\delta(t)\,\mathrm{d}t.$$

At the end of the interval, both chains are multiplied by the same onsite factor w, which commutes with ϕ. Let $S_{\mathrm{loc}} = we^{\tau L}$ be the local step. Iterating from $j = 1$ to $j = m-1$, we obtain

$$\phi\,\delta_m = S_{\mathrm{loc}}^{m-1}\phi\,\delta_1 - \sum_{j=1}^{m-1}\int_{j\tau}^{(j+1)\tau} S_{\mathrm{loc}}^{m-j-1}\, w\, e^{((j+1)\tau - t)L}[L,\phi]\delta(t)\,\mathrm{d}t, \quad (19.63)$$

where $\delta_j = f^{\mathrm{raw}}_{\alpha,j} - g^{\mathrm{raw}}_{\alpha,j}$. Since $\phi = 1$ on the inner region, we have $\phi\,\delta_m = \delta_m$ there.

Next, we insert a Davies weight. Let d_{n} be the normal distance to the collar of ϕ, truncated at d_{col}. It has Lipschitz constant one, it vanishes on the collar, and it equals d_{col} on the inner region. Since the collar lies in $\{5R_k/8 \le |U| \le 3R_k/4\}$, the gradient of d_{n} vanishes outside $\{R_k/2 \le |U| \le 7R_k/8\}$, and Lemma 19.9 therefore applies to d_{n}. Let $a_{\mathrm{D}} \ge 0$. A function F supported on the collar satisfies $e^{-a_{\mathrm{D}} d_{\mathrm{n}}} F = F$, and on the inner region $1 \le e^{-a_{\mathrm{D}} d_{\mathrm{col}}} e^{a_{\mathrm{D}} d_{\mathrm{n}}}$. By Lemma 19.9, the conjugated local step satisfies

$$\|e^{a_{\mathrm{D}} d_{\mathrm{n}}} S_{\mathrm{loc}} e^{-a_{\mathrm{D}} d_{\mathrm{n}}}\| \le e^{C_k/W + (C_k W + C a_{\mathrm{D}}^2)\tau},$$

and the conjugated heat satisfies the same bound without the factor $e^{C_k/W}$. Since $m \leq 2C_0W$ and $m\tau \leq C_0/W$, it follows that for $0 \leq s \leq \tau$ and $1 \leq j < m$,

$$\big\| \mathbf{1}_{\text{inner}}\, S_{\text{loc}}^{m-j-1} w\, e^{sL} F \big\| \leq e^{-a_{\text{D}} d_{\text{col}}}\, e^{2C_kC_0 + (C_kW + Ca_{\text{D}}^2)m\tau} \|F\| \leq C_k \exp\Big[-a_{\text{D}} d_{\text{col}} + \frac{CC_0}{W} a_{\text{D}}^2 \Big] \|F\|,$$

where the terms C_kW of Lemma 19.9 contributed the factor $e^{C_kWm\tau} \leq e^{C_kC_0}$.

We now bound the forcing. At each time $\tau \leq t \leq m\tau$, the function $[L, \phi]\delta(t)$ is bounded by $C_kW^{A_k}$ times the collar norms of $f^{\text{raw}}(t)$ and $g^{\text{raw}}(t)$ and of their derivatives of order one. For the partial chain, recall from the choice of the domains at the start of this section that the collar lies in $\{|U| \leq R_k\}$. By the choice of ρ in Table 19.1, the closure of $\mathcal{U}_0$ is a compact subset of the open parameter ball of radius ρ_{out} about Λ of Proposition 19.4, and $R_k \leq R_{\text{s}}$. Hence Proposition 19.17(d) with $R = R_k$ applies uniformly for $\Gamma \in \mathcal{U}_0$. Together with the kinetic bound on the heat intervals, it bounds the collar norms of $f^{\text{raw}}(t)$ and $\nabla f^{\text{raw}}(t)$ by $C_kW^{A_k} \exp[16C_{\text{geo}}^2 C_* C_0 W(E^2R_k^2 + R_k^4)]$. For the reference, let $j\tau \leq t < (j+1)\tau$ with $j \geq 1$. We apply Lemma 19.22 at the step j, the heat-step bound (19.33) with $\lambda_i = 1$ on the interval $[j\tau, t]$, and Cauchy's estimates on polydisks of radius c/W, as in the proof of Lemma 19.22. They bound the collar norms of $g^{\text{raw}}(t)$ and $\nabla g^{\text{raw}}(t)$ by $C_kW^{A_k}$. Therefore

$$\|[L, \phi]\delta(t)\| \leq C_kW^{A_k} \exp\big[16C_{\text{geo}}^2 C_* C_0 W(E^2R_k^2 + R_k^4)\big].$$

For the first term of (19.63), the bound of Lemma 19.9 with $a_{\text{D}} = 0$ and Lemma 19.23 give $\|S_{\text{loc}}^{m-1} \phi\, \delta_1\| \leq C_kW^{A_k} e^{-cW^2}$.

Inserting these bounds into (19.63), and using $m\tau \leq 1$ for the total length of the heat intervals, we obtain

$$\|\delta_m\|_{\text{inner}, \mathcal{H}_p} \leq C_kW^{A_k} e^{-cW^2} + C_kW^{A_k} \exp\Big[-a_{\text{D}} d_{\text{col}} + \frac{CC_0}{W} a_{\text{D}}^2 + 16C_{\text{geo}}^2 C_* C_0 W(E^2R_k^2 + R_k^4) \Big].$$

We now choose the Davies parameter. The function $a_{\text{D}} \mapsto -a_{\text{D}} d_{\text{col}} + CC_0 a_{\text{D}}^2/W$ is minimal at

$$a_{\text{D}} = \frac{Wd_{\text{col}}}{2CC_0}, \qquad \text{with the value} \quad -\frac{Wd_{\text{col}}^2}{4CC_0} = -\frac{c_{\text{D}} Wd_{\text{col}}^2}{C_0},$$

since $c_{\text{D}} = 1/(4C)$. The first term e^{-cW^2} is smaller than the second for W large, and the finite jets and the number of steps change only W^{A_k}. This proves (19.62). □

The radii. The barrier decays at the rate $c_{\text{D}} d_{\text{col}}^2/C_0$, while the canonical heights over the collar produce the growth $16C_{\text{geo}}^2 C_* C_0(E^2R_k^2 + R_k^4)$. The second condition in (19.50) and the choice of R_k make the growth a small fraction of the decay. We set

$$\gamma_k^{\text{bar}} = \frac{c_{\text{D}} R_k^2}{128\, C_0}.$$

Lemma 19.25. *Let $|E| \le E_{\mathrm{geom}}$. If R_k is sufficiently small, then the exponent in* (19.62) *is at most $-3\gamma_k^{\mathrm{bar}} W/2$. This choice of R_k at fixed rank does not change the energy threshold E_{geom}.*

Proof. Since $d_{\mathrm{col}} = R_k/8$, the decay term of (19.62) satisfies $c_{\mathrm{D}} W d_{\mathrm{col}}^2/C_0 = c_{\mathrm{D}} W R_k^2/(64 C_0) = 2\gamma_k^{\mathrm{bar}} W$. By the second condition in (19.50),

$$16 C_{\mathrm{geo}}^2 C_* C_0 W E^2 R_k^2 \le \frac{16\, c_{\mathrm{D}}}{8192\, C_0} W R_k^2 = \frac{c_{\mathrm{D}} W R_k^2}{512\, C_0} = \frac{\gamma_k^{\mathrm{bar}} W}{4},$$

which is an eighth of the decay term. If $R_k^2 \le c_{\mathrm{D}}/(8192\, C_{\mathrm{geo}}^2 C_* C_0^2)$, then the quartic term $16 C_{\mathrm{geo}}^2 C_* C_0 W R_k^4$ is at most $\gamma_k^{\mathrm{bar}} W/4$ as well. Adding these bounds, we see that the exponent is at most $-2\gamma_k^{\mathrm{bar}} W + \gamma_k^{\mathrm{bar}} W/2 = -3\gamma_k^{\mathrm{bar}} W/2$. The energy threshold was chosen in (19.50) before the radii, and the condition on R_k above does not involve it. □

Continuation to the canonical graph. The barrier bounds the error on the real chart. We also need it on the complex canonical graph and at the canonical frame, and we obtain it from the real bound and a rough bound on a polydisk by the two-constants theorem [106].

Lemma 19.26. *Let $|E| \le E_{\mathrm{geom}}$, let $R_k \le R_{\mathrm{s}}$ be as in Lemma 19.25, and let $p \in \mathbb{R}$. If r_{phys}/R_k is sufficiently small at fixed rank, then there exists $\gamma_k' > 0$ such that for every signature α,*

$$\big\| \chi_\alpha^{\mathrm{phys}} \big(f_m - \mathrm{Can}_\alpha^{\mathrm{raw}}\, g_{\alpha,m}^{\mathrm{raw}} \big) \big\|_{\mathcal{H}_p(\mathcal{S})} \le C_k W^{A_k} e^{-\gamma_k' W},$$

uniformly for $\Gamma \in \mathcal{U}_0$.

Proof. On the inner region the output cutoffs equal one, so the physical endpoint and the reference have holomorphic normal continuations to the polydisk of radius ρ_{pd} about each point of the inner region. Here ρ_{pd} is a fixed fraction of R_k, chosen so that the real parts of the points of the polydisk lie at distance less than $3R_k/8$ from the inner region, hence in the comparison domain, so that the polydisk lies in the strip of Proposition 19.17, and so that its heights satisfy $|b| \le c_1 \bar r$, with $\bar r = 15R_k/16$ as in the domains of the comparison. The physical support lies in the inner region, since r_{phys}/R_k is small. Since there are finitely many normal coordinates, $\|D\|^2 + \|D_F\|^2 \le C_k R_k^2$ on the polydisk, and by (19.43) with $j \le m \le 2C_0 W$, the physical endpoint is bounded there by $W^{A_k} e^{B_k W}$, with $B_k \le C_* C_0 C_k R_k^2$. For the reference we use the same constant normal graph at every node, with the graph cutoff $\psi_{\bar r}$, including the initial datum $\xi_\alpha \varpi_\alpha$, which is holomorphic on the support of $\psi_{\bar r}$, where $\xi_\alpha = 1$. Then Lemma 19.8, in the form with the cutoff $\psi_{\bar r}$ of the proof of Lemma 19.22, applied to the raw steps $S_{\alpha,\Gamma}$ with $\lambda_j = 1$ for all j and without Gaussian weight, gives a bound of the same form.

The real comparison (19.62) holds also after fixed finite real normal derivatives, by the same collar equation, and Sobolev embedding on a slightly smaller box makes its Hilbert-valued supremum on the real axis at most $W^{A_k} e^{-\gamma_k^{\mathrm{bar}} W}$, by Lemma 19.25.

We pass from the real box to the canonical graph by the two-constants theorem [106]. In one normal coordinate, we slit a rectangle of size comparable to ρ_{pd} along its inner real interval. The logarithm of the norm of a holomorphic vector-valued function is subharmonic. Suppose that the norm is at most M_0 on the slit and at most M_1 on the rectangle. Then

$$\|f(z)\| \le M_0^{\omega(z)} M_1^{1-\omega(z)},$$

where ω is the harmonic measure of the slit; for vector-valued functions this follows by applying the scalar inequality to every unit dual functional and taking the supremum. Above every smaller real interval we have $1-\omega(z) \le C|\operatorname{Im} z|/\rho_{\mathrm{pd}}$. We apply this successively to the finitely many normal coordinates, shrinking the real interval once each time, with $M_0 = W^{A_k} e^{-\gamma_k^{\mathrm{bar}} W}$ and $M_1 = W^{A_k} e^{B_k W}$. By Lemma 19.15, on the physical support the canonical graph has height at most $C_{\mathrm{geo}}(|E| r_{\mathrm{phys}} + r_{\mathrm{phys}}^2)$, and the product ω of the harmonic measures satisfies $1-\omega \le C_k r_{\mathrm{phys}}/R_k$. Since γ_k^{bar} and B_k are both of order R_k^2, the condition

$$\omega \gamma_k^{\mathrm{bar}} > (1-\omega) B_k$$

holds once r_{phys}/R_k is small enough at fixed rank, and then the continued error is at most $W^{A_k} e^{-\gamma_k' W}$ on that graph, where, since $1-\omega \le C_k r_{\mathrm{phys}}/R_k$,

$$\gamma_k' = \gamma_k^{\mathrm{bar}} - \frac{C_k r_{\mathrm{phys}}}{R_k}\left(\gamma_k^{\mathrm{bar}} + B_k\right) > 0.$$

It remains to pass from the real frame to the canonical frame. The partial chain and the raw reference satisfy the same equivariance (19.2), by Lemma 19.2, and so does their difference. We apply Lemma A.5 with $\zeta = \Gamma$ to the difference, with the selection given by (19.38), as in the proof of Lemma 19.2(c). It keeps the exponential error on the body of the canonical image, and all nilpotent canonical corrections are retained as finite coefficient jets. □

Proof of Theorem 19.21. The exact identity

$$f_m - \sum_\alpha \chi_\alpha^{\mathrm{phys}} \operatorname{Can}_\alpha^{\mathrm{raw}} g_{\alpha,m}^{\mathrm{raw}} = \Big(1 - \sum_\alpha \chi_\alpha^{\mathrm{phys}}\Big) f_m + \sum_\alpha \chi_\alpha^{\mathrm{phys}} \left(f_m - \operatorname{Can}_\alpha^{\mathrm{raw}} g_{\alpha,m}^{\mathrm{raw}}\right)$$

splits the error into two parts. The first term is supported away from the union of the normal cores, and the global tail of f_m after Proposition 19.4 bounds it by $C_k W^{A_k} e^{-cWr_0^2}$, where $r_0 > 0$ is the radius of these cores. By Lemma 19.26, each remaining term is at most $C_k W^{A_k} e^{-\gamma_k' W}$, and their finite number changes only the constant. We obtain (19.59) with $\gamma_k = \min(\gamma_k', cr_0^2)$.

Transporting by $\gamma_h^{-1} w_0^{-1/2}$ once converts the raw references to the symmetric normal gauge without changing the identity; its endpoint factors are those of (16.38). Parameter derivatives follow from Cauchy's formula on the larger parameter ball. Local BB jets, if needed later, are taken as finite coefficients at zero before the return to the contour. □

SECTION 20

Proof of the interface theorem

In this section we prove Theorem 7.6. Part III uses this theorem and nothing else from Part IV, and with its proof the reduction of the band matrix to the sigma model of Section 5 is complete. The heuristics behind the argument are explained in Section 2. Here we combine the a priori bounds of Section 15, the normal tube and the normal oscillator of Section 16, the stability estimates of Section 17, the consistency estimates of Section 18 and the endpoint estimates of Section 19.

Recall from Proposition 4.11 and (4.21) that the source ratio is, up to an explicit factor, a ratio of two chain integrals,

$$T_N^{(k)}(\mathbf{u}, \mathbf{b}) = e^{N\eta_0^2 \,\mathrm{STr}(\Gamma^2)/2} \, \frac{\mathcal{Z}_N(\Gamma)}{\mathcal{Z}_N(\Gamma_0)}, \tag{20.1}$$

where $\Gamma = \Gamma(\mathbf{u}, \mathbf{b})$ is the source supermatrix of Section 4 and $\eta_0 = a_N^{-1}$. Our main task is to show that $\mathcal{Z}_N(\Gamma)$ converges to the partition function $\mathcal{Z}_{X,\varepsilon}^{(k)}(\mathbf{u}, \mathbf{b})$ of Definition 5.16. The chain integral $\mathcal{Z}_N(\Gamma)$ is a chain of N onsite weights and $N-1$ heat links on the product of Schäfer–Wegner cycles. We compare it with the adapted chain of Section 16, which lives on a tube around the saddle manifold, and which converges to the continuum semigroup by the results of Section 18.

The main difficulty in this comparison is the one of exponential orders explained in Section 2.5. Every change of integration cycle between the physical chain and the tube produces an error of order $e^{-\gamma W}$ for some $\gamma > 0$, while the physical powers are bounded in operator norm only by e^{CW}, and Theorem 15.14 bounds by $e^{o(W)}$ only the partial chains started from the constant function. We therefore use a one-sided telescoping identity, Lemma 20.4, in which each local error is paired directly with such a partial chain. Every comparison of the physical chain with the adapted chain below is an instance of this lemma. The limit of the adapted chain is then identified by the product approximation of Section 18, after an initial layer in which the normal variables relax.

The section is organized as follows. In Section 20.1, we fix the order in which the constants are chosen and the two energy intervals of Theorem 7.6. In Section 20.2, we write the chain integral in one-sided form. Section 20.3 collects three properties of the adapted chain, the first of which is proved in Section 17.9, and Section 20.4 proves the telescoping lemma. Parts (a), (b), (c) and (d) of Theorem 7.6 are proved in Sections 20.5 to 20.8, respectively. Figure 20.1 shows how the chain is decomposed.

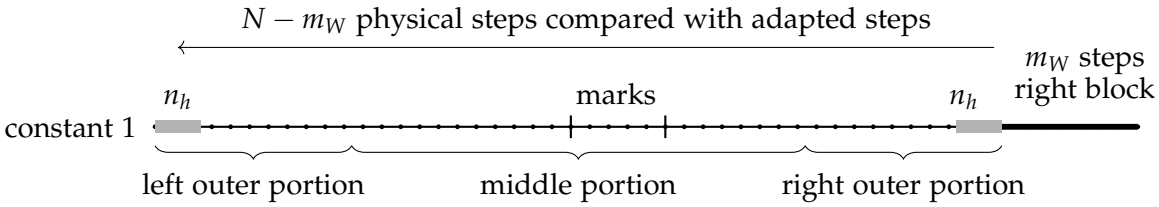


Figure 20.1. The decomposition of the chain in (20.7). The right block is attached to the tube by Theorem 19.21, the remaining $N - m_W$ physical steps are compared with adapted steps by Lemma 20.4, and the left endpoint is the constant function 1. In the balanced adapted chain, the shaded initial layers of n_h steps (Lemma 20.3) move the endpoints to the normal fixed state. The outer portions of length about θN suppress the unbalanced signatures and prepare body weights, and the balanced tangent variables converge to the continuum semigroup. The two ticks show sites that carry the local sources of Theorem 7.6(b), which Section 20.6 calls marks.

20.1 Conventions and the order of choices

Throughout this section, $k \geq 1$ is the rank, and we use the scales

$$h = W^{-1/2}, \qquad \delta_W = h^4 = W^{-2}, \qquad \tau_{\mathrm{phys}} = (2W^2)^{-1}$$

of Part IV. In one link of the chain, the normal variables relax by a factor $1 - ch^2$, while the tangent variables move by an amount of order h^2; their evolution per link has a generator of order δ_W. Let $\eta = \varepsilon\eta_0 = \varepsilon/a_N$ denote the physical imaginary part. The damping ε and the parameter $X_N(E)$ take values in compact subsets of $(0, \infty)$, and the energy E may depend on N and W within the intervals fixed below. Constants $C, c > 0$ may depend on k, on these compact sets, and on the choices listed below, but not on N, W, E, or the sites. Since $X_N(E) = N/((4 - E^2)W^2)$ and $|E| \leq 1/2$ below, we have $N \asymp W^2$.

The constants are chosen in the following order. Each choice may depend on all choices made before it.

(i) The universal constants of Section 19, in the order of Table 19.1. Among them are the block length C_0, the Gaussian margin a of the prepared families, the coefficient C_n in the exponent of the prepared block bound (19.36), the coefficient C_* of the shifted onsite bound in (19.42), the constant C_{geo} of the canonical normal height in Lemma 19.15, the constants $c_{\mathrm{ann}} = ac_Z$ and $C_{\mathrm{ann}} = aC_Z$ of the Gaussian factor on the annulus, and the shell-gain constant $c_{\mathrm{sh}} = c_{\mathrm{ann}}/4$ of (19.51). We add to them the energy E_{sep} and the radius r_{sep} of the Gaussian separation in Lemma 18.5. None of these constants depends on k, and Remark 19.10 explains why, in particular for the coefficient C_n of $|b|^2$.

(ii) The energy threshold $E_{\mathrm{geom}} > 0$ of (19.50). Its two conditions involve the constants of (i) alone.

(iii) The outer energy bound

$$E_{\mathrm{out}} = \min\bigl(\tfrac{1}{2}, E_{\mathrm{geom}}, E_{\mathrm{sep}}\bigr). \tag{20.2}$$

Here $1/2$ is the energy bound of Theorem 15.14 and of Lemma 19.15. The estimates of Sections 16 and 17 are uniform on every compact set of energies, by Theorem 17.1, and we use them on $[-1/2, 1/2]$. Since $E_{\mathrm{out}} \le 1/2$, Lemma 7.2 and Lemma 7.3 apply on $[-E_{\mathrm{out}}, E_{\mathrm{out}}]$.

(iv) The inner energy bound E_*, with $0 < E_* < E_{\mathrm{out}}$.

(v) The choices at rank k, in the order of Table 19.1. These are the radius r_k and the degree-metric parameter $\varkappa_k$ of Lemma 18.4; the degree-metric parameter ω and the cutoff radius $r \le r_k$ of Theorem 17.1; the finitely many graph orders of Section 18; the damping balls $\mathcal{U}_0 \Subset \mathcal{U}_1 \Subset \mathcal{U}_2$ of Section 19.1; the Hardy radii ρ and ρ_{out} of Proposition 19.4; the radius R_k of the root tube, which satisfies

$$16R_k \le r_{\mathrm{out}}, \qquad 4R_k \le r_{\mathrm{in}} \le \tfrac{3}{4} r_{\mathrm{out}}, \qquad 2r_{\mathrm{out}} \le \min(r_k, r_{\mathrm{sep}})$$

for the radii $r_{\mathrm{in}} < r_{\mathrm{out}}$ of Lemma 18.6; the collar distance $d_{\mathrm{col}} = R_k/8$; the polydisk radius ρ_{pd}; the support radius r_{phys} of the physical cutoffs; the inner radius $r_1 < r_{\mathrm{phys}}$ of the annulus on which they vary; and the saturation radius r_{esc} of the escape function. Each of them is constrained only as in Sections 18 and 19.

(vi) The source radius $r_{k,\varepsilon}$ of Section 20.2.

(vii) The lower threshold W_0 on W.

Since E_{out} and E_* are chosen in (iii) and (iv), before the choices at rank k in (v), they do not depend on k. They do not depend on the number m of marks or on the orders of the derivatives in Theorem 7.6(b) either. Indeed, a complete jet changes the block estimates of Section 19 only as (19.30) states, and we show in Lemma 20.11 that the margins and the shell gain therefore hold for the chain with jets with the same universal constants and under the same conditions (19.50). The rank, the number of marks, and the orders change only the constants, the tube, the source radius, W_0, and the rate of convergence, as Theorem 7.6 asserts.

Parts (a) and (b) of Theorem 7.6 are proved for $|E| \le E_{\mathrm{out}}$, and parts (c) and (d) for $|E| \le E_*$. The margin between the two intervals absorbs the change of energy $E \mapsto E + u/a_N(E)$ in parts (c) and (d), through the following lemma.

Lemma 20.1. *Let $u_0 > 0$, let $|E| \le E_*$ and $|u| \le u_0$, and define*

$$E' = E + \frac{u}{a_N(E)}, \qquad \varepsilon' = \varepsilon\, \frac{\rho(E')}{\rho(E)}. \tag{20.3}$$

Then $E + (u + \mathrm{i}\varepsilon)/a_N(E) = E' + \mathrm{i}\varepsilon'/a_N(E')$. *Moreover, there exist* $C > 0$ *and* N_0, *depending only on* u_0, E_*, *and* E_{out}, *such that for all* $N \geq N_0$,

$$|E'| \leq E_{\mathrm{out}}, \qquad \left|\frac{\varepsilon'}{\varepsilon} - 1\right| + \left|\frac{X_N(E')}{X_N(E)} - 1\right| \leq \frac{C}{N}. \tag{20.4}$$

Proof. Recall that $a_N(E) = N\rho(E)$. The identity follows from

$$\frac{\varepsilon'}{a_N(E')} = \frac{\varepsilon}{N\rho(E)} = \frac{\varepsilon}{a_N(E)}.$$

Let $\rho_* > 0$ denote the minimum of ρ on $[-E_{\mathrm{out}}, E_{\mathrm{out}}]$. Then

$$|E' - E| \leq \frac{u_0}{N\rho_*}, \qquad |E'| \leq E_* + \frac{u_0}{N\rho_*} \leq E_{\mathrm{out}} \quad \text{for } N \geq \frac{u_0}{\rho_*(E_{\mathrm{out}} - E_*)}.$$

The functions ρ and $x \mapsto (4 - x^2)^{-1}$ are smooth and positive on $[-E_{\mathrm{out}}, E_{\mathrm{out}}]$, and the second bound in (20.4) follows from the first display and the mean value theorem. □

20.2 The chain in one-sided form

Recall from Section 4 the cycle $\mathcal{S}$, its bilinear pairing $\mathcal{B}_{\mathcal{S}}(f, g) = \int_{\mathcal{S}} fg \, \mathrm{d}S$, the onsite weight w_Γ, the heat factor K_τ, and the transfer operator $T_\Gamma = M_{w_\Gamma} K_{\tau_{\mathrm{phys}}}$, where M_w denotes multiplication by w. Recall also that $K_\tau 1 = 1$, that the analytic transpose of T_Γ for $\mathcal{B}_{\mathcal{S}}$ satisfies $T_\Gamma^{\mathrm{tr}} = K_{\tau_{\mathrm{phys}}} M_{w_\Gamma}$, and that by Proposition 4.11, for all integers $1 \leq m \leq N/2$,

$$\mathcal{Z}_N(\Gamma) = \mathcal{B}_{\mathcal{S}}(\ell_m, T_\Gamma^{N-2m} f_m), \qquad f_m = T_\Gamma^m 1, \qquad \ell_m = (T_\Gamma^{\mathrm{tr}})^m 1 = K_{\tau_{\mathrm{phys}}} f_m. \tag{20.5}$$

We define the block length to be

$$m_W = \lceil C_0 W \rceil, \tag{20.6}$$

with the constant C_0 of Section 19. Taking $m = m_W$ in (20.5) and moving the m_W factors of ℓ_{m_W} to the right by analytic transposition, we obtain

$$\mathcal{Z}_N(\Gamma) = \mathcal{B}_{\mathcal{S}}(1, T_\Gamma^{N-m_W} f_{m_W}). \tag{20.7}$$

In this form the chain is attached to the tube only at its right endpoint (see Figure 20.1). By Theorem 19.21, the block f_{m_W} is close to a sum of reference vectors on the tube, and the remaining $N - m_W$ steps are compared with adapted steps. The left endpoint is the constant function 1. The point of this form is that the left factors produced by the comparison are the partial chains $T_\Gamma^j 1$ and their versions with local sources, which Theorem 15.14 bounds.

We now fix the source radius of (vi). Recall from Section 19 the nested complex balls $\mathcal{U}_0 \Subset \mathcal{U}_1 \Subset \mathcal{U}_2$ of damping parameters around Γ_0, on which the estimates of that section

are uniform. We choose $r_{k,\varepsilon} > 0$ independent of N, W, and E and so small that $b_j \geq \varepsilon/2$ for all j and $\Gamma(\mathbf{u}, \mathbf{b}) \in \mathcal{U}_0$ whenever $\mathbf{u}$ and $\mathbf{b}$ are real, $\sum_j u_j = 0$ and

$$\|\mathbf{u}\| + \|\mathbf{b} - \varepsilon \mathbf{1}\| < r_{k,\varepsilon}. \tag{20.8}$$

This is possible since Γ depends linearly on $(\mathbf{u}, \mathbf{b})$ and $\Gamma(\mathbf{0}, \varepsilon\mathbf{1}) = \Gamma_0$. The radius depends on k and on the compact range of ε. The FF block of Γ is the part that depends on $(\mathbf{u}, \mathbf{b})$. It is traceless, as Theorem 15.14 requires.

20.3 Three properties of the adapted chain

We now prove two properties of the adapted transfer operator A_h of (16.38) that the comparison uses, and we recall a third. The adapted step improves polynomial body weights, by Lemma 17.20, it contracts exponentially in the unbalanced root signatures, and in the balanced signature it moves the right endpoint factor g_R to the normal fixed state Ω_h over a short initial block. Recall the identity (17.38) for the weighted spaces $\mathcal{H}_p^{\mathrm{tube}}$.

We also record how the analytic transpose of the chain reduces to the chain itself. By (16.38) and (18.24), we have $A_h = s_\Gamma T_h$, where the source factor s_Γ is a multiplier and T_h is the unsourced symmetric step. Recall the analytic Berezin pairing $\mathcal{B}_{\mathrm{ad}}(f, g) = \int f g \, \gamma_h^2 \, \mathrm{d}U \, \mathrm{d}\nu$ of the adapted tube, defined in (16.36). The step T_h is symmetric for $\mathcal{B}_{\mathrm{ad}}$, since Δ_{amb} is symmetric for the flat Berezin form $\varpi^2 \, \mathrm{d}U \, \mathrm{d}\nu$ of (16.15), and since ϖ, γ_h, $w_0^{1/2}$, and the cutoffs are multipliers. Let A_h^{tr} denote the transpose of A_h for $\mathcal{B}_{\mathrm{ad}}$. Since multipliers are symmetric, we have $A_h^{\mathrm{tr}} = T_h s_\Gamma$, and, as in Remark 18.13,

$$(A_h^{\mathrm{tr}})^n = T_h \, A_h^{n-1} \, s_\Gamma, \qquad n \geq 1. \tag{20.9}$$

The factor T_h is bounded on every space $\mathcal{H}_p^{\mathrm{tube}}$ by Lemma 18.7, and s_Γ is a bounded multiplier by the proof of Lemma 19.7. Every bound on the powers of A_h below therefore holds for the powers of A_h^{tr}, up to a constant factor, and every limit of the powers of A_h implies the corresponding limit for A_h^{tr}.

Unbalanced signatures

Recall from Section 16 that the saddle manifold has $2k + 1$ components, indexed by the signature $p \in \{0, \dots, 2k\}$, and that the signature $p = k$ is balanced. Theorem 19.21 retains all signatures. We now show that the unbalanced ones are exponentially small after an adapted leg of macroscopic length. We use the following lemma at each place below where an unbalanced signature occurs.

Lemma 20.2. *Let $p \neq k$ be an unbalanced signature, and let $q \in \mathbb{R}$. There exist $C, c > 0$ such that, in this signature,*

$$\|A_h^n\|_{\mathcal{H}_q^{\mathrm{tube}}} \leq C e^{-cnh^2}, \qquad n \geq 0. \tag{20.10}$$

In particular, for all $\theta > 0$ there exists $c' > 0$, depending on θ and on the compact range of $X_N(E)$, such that $\|A_h^n\|_{\mathcal{H}_q^{\mathrm{tube}}} \le Ce^{-c'W}$ for all $n \ge \theta N$.

Proof. In the signature p, the reference contraction C_h of Theorem 17.1 on $\mathcal{H}_0^{\mathrm{tube}}$ carries the full scalar factor λ of Proposition 16.22, including the physical saddle phase $\zeta_p = w_0(R_*)$. The defects depend only on $|\lambda|$, so by (16.54) its right defect satisfies $D_R \ge c_1 h^2 \mathrm{Id}$. On $\mathcal{H}_q^{\mathrm{tube}}$ we use the reference $C_{h,q} = v^{-q} C_h v^q$. Since multiplication by v^q is unitary from $\mathcal{H}_q^{\mathrm{tube}}$ onto $\mathcal{H}_0^{\mathrm{tube}}$, the right defect of $C_{h,q}$ in $\mathcal{H}_q^{\mathrm{tube}}$ satisfies

$$D_{R,q} = v^{-q} D_R v^q \ge c_1 h^2 \mathrm{Id}.$$

Using the conclusion of Theorem 17.1 in $\mathcal{H}_q^{\mathrm{tube}}$ and $\delta_W = h^4$, we find for h small

$$A_h^* A_h \le (1 + B\delta_W/2)\mathrm{Id} - \tfrac{1}{2} D_{R,q} \le \big(1 + Bh^4/2 - \tfrac{1}{2} c_1 h^2\big)\mathrm{Id} \le \big(1 - \tfrac{1}{4} c_1 h^2\big)\mathrm{Id},$$

where the adjoint is taken in $\mathcal{H}_q^{\mathrm{tube}}$, and in the second inequality we used $D_{R,q} \ge c_1 h^2 \mathrm{Id}$. It follows that $\|A_h\|_{\mathcal{H}_q^{\mathrm{tube}}} \le e^{-c_1 h^2/8}$, which implies (20.10) with $c = c_1/8$ and $C = 1$.

We now prove the second assertion. Since $X_N(E)$ is bounded below and $|E| \le 1/2$, there exists $c_X > 0$ such that $N \ge c_X W^2$, and for $n \ge \theta N$ we obtain

$$nh^2 \ge \frac{\theta N}{W} \ge c_X \theta W. \qquad \square$$

In our applications, the other factors of a pairing in an unbalanced signature are bounded by fixed powers of W. These are the endpoint norms of Lemma 19.5, the bounds on the preparation in Lemma 19.14, and the terminal bound of Lemma 20.6. We carry each unbalanced signature, with its phase and orientation, until an adapted leg of length at least θN suppresses it by Lemma 20.2.

The initial layer

Throughout this subsection, we work in the balanced root signature $p = k$. The unbalanced signatures are treated by Lemma 20.2.

Recall the endpoint factors g_R and g_L and the frozen normal transfer $\mathbf{F}_h$ from (16.38). Recall from Lemma 18.3 that the normal fixed state Ω_h is the unique closed vector with normalized Gaussian mean one such that $\mathbf{F}_h \Omega_h = \Omega_h$, that it satisfies the bounds (18.4), that closed vectors relax to multiples of it at rate h^2 per step by (18.5), and that its analytic Berezin pairing in the normal variables satisfies

$$\mathcal{B}_{\mathrm{nor}}(\Omega_h, \Omega_h) = 1 \tag{20.11}$$

by (18.6). By Lemma 16.8, the endpoint factor is a normal class function times the constant function of the tangent variables, and we denote it by $g_R \otimes 1$. Let $c_{\mathrm{lay}} \ge 1$, and define the length of the initial layer to be

$$n_h = \big\lceil c_{\mathrm{lay}} h^{-2} \log(1/h) \big\rceil. \tag{20.12}$$

Lemma 20.3. *In the balanced signature $p = k$, let ξ be either 1 or a fixed smooth invariant normal class cutoff in the physical variable U, with its full nilpotent completion and equal to one near $U = 0$. This includes the cutoff ξ_k used in Theorem 19.21. Let c_{lay} be sufficiently large. Then $n_h \delta_W \to 0$, and*

$$A_h^{n_h}((\xi g_R) \otimes 1) = \Omega_h \otimes 1 + o(1) \tag{20.13}$$

in the base normal norm and a sufficiently negative fixed body weight. The same holds for the analytic transpose of the chain, with ξg_L in place of ξg_R.

We first explain the idea. On the tangent constant, the adapted step without its source factor reduces to the frozen normal transfer $\mathbf{F}_h \otimes \mathrm{Id}$, which relaxes at rate h^2 per step. The source factor changes each step by $O(\delta_W)$, and over n_h steps the source error is $O(n_h \delta_W) = O(h^2 \log(1/h))$, whereas the relaxation error is $O(h^{cc_{\mathrm{lay}}})$.

Proof. We first note that

$$n_h \delta_W \le 2c_{\mathrm{lay}} h^2 \log(1/h) \longrightarrow 0.$$

Set $u_R = \xi g_R$ and $u_L = \xi g_L$. By Lemma 16.8, the endpoint factors are invariant normal class functions. The cutoff ξ has the same invariance, including its nilpotent terms, so u_R and u_L are annihilated by the normal charges and are $\mathfrak{q}_{\mathrm{nor}}$-closed. Since ξ equals one near the origin, their body values at $U = 0$ are one in the balanced signature. Normal localization Lemma 16.13 therefore gives $\ell_0(u_R) = \ell_0(u_L) = 1$. The bounds of Lemma 19.5 also hold for u_R and u_L: the fixed cutoff has bounded coefficients in the standardized metric, and each of its nonbody terms carries the corresponding positive power of h. For an input $f \otimes 1$ with f of this kind, the horizontal part of the decomposition (16.29) of the ambient generator vanishes, since the horizontal derivatives act on a constant and the normal stabilizer fields annihilate the class function f. Therefore the adapted step without its source factor agrees with $\mathbf{F}_h \otimes \mathrm{Id}$ on these inputs.

A larger normal cutoff changes this identity only by a Gaussian error from the separated boundary, which is $O(h^L)$ for all L. Indeed, a Gaussian input with negative margin grows at most like $e^{Ch^{-2}}$ on the tube, while a jump to the separated boundary carries a factor $e^{-ch^{-4}}$.

Next, the source factor differs from one by $O(\delta_W v)$. We absorb the factor v in a sufficiently negative body weight p_0 such that $v \in \mathcal{H}^{\mathrm{tube}}_{p_0}$. Writing $\|f\|_{p,a'} = \|e^{a'|Z|^2} f\|_{\mathcal{H}^{\mathrm{tube}}_p}$, where Z is the standardized normal coordinate, we obtain

$$\big\|(A_h - \mathbf{F}_h \otimes \mathrm{Id})(\mathbf{F}_h^j u_R \otimes 1)\big\|_{p_0, a_{j+1}} \le C\delta_W, \qquad j \ge 0. \tag{20.14}$$

Here (a_j) is the signed Gaussian ramp of Corollary 19.11, which increases from $-a$ to a over the first m_W steps, and $a_j = 0$ for $j > m_W$. The frozen path at step $j+1$ has bounded norm in this signed weight, and the source factor commutes with the weight. This ramp

uses real normal inputs, so its application to u_R requires only the negative-margin bound and evaluates ξ only on the real tube.

We now telescope. We have

$$A_h^{n_h}(u_R \otimes 1) - (\mathbf{F}_h^{n_h} u_R) \otimes 1 = \sum_{j=0}^{n_h-1} A_h^{n_h-1-j}(A_h - \mathbf{F}_h \otimes \mathrm{Id})(\mathbf{F}_h^j u_R \otimes 1).$$

For $j < m_W$, we use the signed weights at all steps. The weighted energy estimate Lemma 19.8, in the form of Corollary 19.11, bounds the powers $A_h^{n_h-1-j}$ from the signed weight at step $j+1$ to the base norm.

For $m_W \le j < n_h$, the inputs $\mathbf{F}_h^j u_R$ are closed normal class functions with bounded base norm, and we use the power bounds of Theorem 17.1. Inserting (20.14) into the telescoping sum, we obtain

$$\left\| A_h^{n_h}(u_R \otimes 1) - (\mathbf{F}_h^{n_h} u_R) \otimes 1 \right\| \le C m_W \delta_W + C n_h \delta_W \le C h^2 \log(1/h).$$

It remains to relax the frozen path. The closed-space Ward identity preserves the Gaussian mean one of u_R. After the signed block, $\mathbf{F}_h^{m_W} u_R$ is therefore closed, has Gaussian mean one and has bounded base norm. Using (18.5), we obtain

$$\|\mathbf{F}_h^{n_h} u_R - \Omega_h\| \le e^{-c(n_h - m_W)h^2} \|\mathbf{F}_h^{m_W} u_R - \Omega_h\| = O(h^{cc_{\mathrm{lay}}}),$$

where we used $(n_h - m_W)h^2 \ge c_{\mathrm{lay}} \log(1/h) - 2C_0$. The three estimates together imply (20.13).

For the analytic transpose we use (20.9), which gives

$$(A_h^{\mathrm{tr}})^{n_h}(u_L \otimes 1) = T_h \, A_h^{n_h-1}(s_\Gamma u_L \otimes 1).$$

The factor u_L is a closed invariant normal class function with Gaussian mean one, like u_R, it has the negative Gaussian margins proved above, and $s_\Gamma - 1 = O(\delta_W v)$. The argument above therefore shows that $A_h^{n_h-1}(s_\Gamma u_L \otimes 1) = \Omega_h \otimes 1 + o(1)$. Finally, T_h is bounded, and $T_h(\Omega_h \otimes 1) = \Omega_h \otimes 1 + O(\delta_W h)$ by (18.37), since $L_{H,0} 1 = 0$. □

20.4 The telescoping comparison

We now compare the physical chain on the cycle with the adapted chain on the tube. Every comparison has the same three steps. We attach the right endpoint by Theorem 19.21, we telescope the chain from the right so that all left factors are physical partial chains started from the constant function, and we return the final pairing with the constant function to the tube. The main result of this section is Lemma 20.4, which carries out the second step; the first and the third enter through its hypothesis (H4). We state it after introducing the objects that it uses, and we prove it after the lemmas that we use to verify its hypotheses.

Recall from Section 19.5 the physical cutoff χ_{phys} of (19.40), with the radial profile χ, the decomposition $U = U^{\mathrm{re}} + \mathrm{i}U^{\mathrm{im}}$, and the cutoff $\chi_{\mathrm{tube}} = \chi(|U|)$ on the real tube, and the map $\mathrm{Can}_{\mathrm{phys}} = \chi_{\mathrm{phys}}\,\mathrm{Can}$ of Section 19.6. We let $g_L^{\mathrm{cut}} = \chi_{\mathrm{tube}}\, g_L$ denote the left endpoint factor multiplied by the cutoff. Recall finally the families generated from g_R of Definition 19.1, their parameter-family norm $\|\cdot\|_{\mathcal{U}_2;p,a}$ of (19.1), with body weight p and Gaussian margin a, and their prepared norm $\|\cdot\|_{\mathrm{prep},\mathcal{U}_2;p}$ of (19.47). The prepared norm is the supremum over the heights $|b| \le c_1 R_k$ of the family norms of the values on the complex normal graphs of height b, with the Gaussian margin a of Proposition 19.13 and the factor $e^{-C_{\mathrm{prep}} W |b|^2}$ of (19.47). Its term with $b = 0$ is the family norm, so that $\|f\|_{\mathcal{U}_2;p,a} \le \|f\|_{\mathrm{prep},\mathcal{U}_2;p}$. The shell bound and the residual bound of Proposition 19.20 are stated in the prepared norm, since their proofs use the values of the family on these graphs.

We now write the difference between a physical step and an adapted step. Let $T_i = M_{w_i} K_{\tau_{\mathrm{phys}}}$ denote a physical step, where w_i is either the onsite weight w_Γ or, at finitely many sites, a derivative of the onsite weight with respect to a local source, as in Section 20.6. The pullback to the tube and the gauge of (16.38) give an adapted step $A_i = M_{w_i,\mathrm{ad}} K_{\mathrm{ad}}$ such that $M_{w_i}\,\mathrm{Can}_{\mathrm{phys}} = \mathrm{Can}_{\mathrm{phys}}\, M_{w_i,\mathrm{ad}}$, where K_{ad} is the gauged heat factor of the adapted tube. Therefore

$$T_i\,\mathrm{Can}_{\mathrm{phys}} - \mathrm{Can}_{\mathrm{phys}}\, A_i = M_{w_i}(K_{\tau_{\mathrm{phys}}}\,\mathrm{Can}_{\mathrm{phys}} - \mathrm{Can}_{\mathrm{phys}}\, K_{\mathrm{ad}}).$$

On the support of the cutoffs, the canonical map intertwines the two heat generators, and the difference of the heat factors comes only from the commutator of the physical heat generator $L_{\mathcal{S}}$ with the cutoff. On the families generated from g_R of finite prepared norm, Duhamel's formula then yields the exact identity (19.53) of Lemma 19.19, which we write as

$$\begin{aligned} T_i\,\mathrm{Can}_{\mathrm{phys}} - \mathrm{Can}_{\mathrm{phys}}\, A_i &= M_{w_i}\mathcal{R}, \\ \mathcal{R}f &= \int_0^{\tau_{\mathrm{phys}}} K_{\tau_{\mathrm{phys}}-s}\,[L_{\mathcal{S}},\chi_{\mathrm{phys}}]\;\mathrm{Can}\,K_{\mathrm{ad},s}(\chi_{\mathrm{inv}} f)\,\mathrm{d}s, \end{aligned} \tag{20.15}$$

where χ_{inv} is the invariant normal input cutoff of the adapted step. We call $\mathcal{R}$ the heat residual. Its Duhamel forcing $[L_{\mathcal{S}},\chi_{\mathrm{phys}}]\,\mathrm{Can}\,K_{\mathrm{ad},s}(\chi_{\mathrm{inv}} f)$ is supported on the annulus where χ_{phys} varies; the outer heat operator propagates this forcing beyond the annulus.

Lemma 20.4. *Let $1 \le n \le N$, let $T_1,\dots,T_n$ and $A_1,\dots,A_n$ be physical and adapted steps as above, let g be a family generated from g_R, and set $g_j = A_j \cdots A_1 g$ for $0 \le j \le n$. Let $f = \mathrm{Can}_{\mathrm{phys}}\, g + e$. Let p be at least the body weight of Theorem 15.14 for the rank k and the total order d of the local derivatives among the steps, and let A_{sh} and p' be as in Proposition 19.20 for p. Suppose that there exist $A \ge 0$ and $C, c > 0$ such that the following hold, with these p and p' in all four hypotheses.*

(H1) Every marked partial chain F of Definition 15.2 of length $1 \le j \le N$ whose on-site weights are the weights of j consecutive given steps, in reversed order, satisfies $\|F\|_{\mathcal{H}_{-p}(\mathcal{S})} + \|K_{\tau_{\mathrm{phys}}} F\|_{\mathcal{H}_{-p}(\mathcal{S})} \le e^{o(W)}$.

(H2) For all $0 \le j \le n$, the state g_j is a family generated from g_R that is holomorphic in the normal variables where $\psi_{R_k} \neq 0$, and $\|g_j\|_{\mathrm{prep},\mathcal{U}_2;p'} \le W^A$.

(H3) For all $0 \le j < n$, we have $\|\mathcal{R} g_j\|_{\mathcal{H}_p(\mathcal{S})} \le C W^{A_{\mathrm{sh}}} e^{-cW} \|g_j\|_{\mathrm{prep},\mathcal{U}_2;p'}$.

(H4) We have $\|e\|_{\mathcal{H}_p(\mathcal{S})} \le W^A e^{-cW}$ and $c_{\mathrm{sh}} r_1^2 \ge c$, and

$$\left|\mathcal{B}_{\mathcal{S}}(1, \mathrm{Can}_{\mathrm{phys}}\, g_n) - \mathcal{B}_{\mathrm{ad}}(g_L^{\mathrm{cut}}, g_n)\right| \le C_k W^{A_{\mathrm{sh}}+4k^2} e^{-c_{\mathrm{sh}} r_1^2 W} \|g_n\|_{\mathrm{prep},\mathcal{U}_2;p'}.$$

Then

$$\left|\mathcal{B}_{\mathcal{S}}(1, T_n \cdots T_1 f) - \mathcal{B}_{\mathrm{ad}}(g_L^{\mathrm{cut}}, g_n)\right| \le C N W^{A_{\mathrm{sh}}+A} e^{-cW+o(W)}. \tag{20.16}$$

Hypothesis (H1) is the a priori bound of Theorem 15.14, which remains true when p is increased, since $v_{\mathcal{S}} \ge 1$. In our applications, we take p so large that Lemma 20.5 below applies, and (H4) then follows from Theorem 19.21 and Lemma 20.6. There we also have $N \asymp W^2$, and the right side of (20.16) tends to zero. The proof of the lemma is elementary. The work is in the verification of (H1)–(H4), which uses Sections 15 and 19. The bound (H3) is available only on families generated from g_R, which is why (H2) asks for this structure and not only for a bound on the norm. We prove Lemma 20.4 at the end of this section, and we begin with the return of the final pairing to the tube, in two steps.

The first step returns the cycle of integration from $\mathcal{S}$ to the real tube, by Stokes' theorem along the homotopy Φ_t of body cycles of (19.39), defined in Section 19.5 on the parameter domain $\mathcal{Y}$ of the coordinates y of $\mathcal{S}$, with the properties stated in Lemma 19.16.

We use the orientation of the real tube fixed before Lemma 16.2. Near each component of the saddle manifold, the domain of Φ_0 is connected, so that Φ_0 either preserves or reverses this orientation there. A reversal would multiply the integral in (20.17) below by -1, for all sources, and with it the term of that signature in each comparison below. Near the balanced component, Φ_0 preserves the orientation, as we show in Section 20.5 by comparing the limit of $\mathcal{Z}_N(\Gamma_0)$ with (4.21). The terms of the unbalanced signatures tend to zero, so that their signs play no role. In these signatures, the orientation of G_0/K_0 fixed before Lemma 16.2 is arbitrary, and by (16.15) the orientation of the real tube reverses with it. We take it such that Φ_0 preserves the orientation.

Recall from (19.40) that $\chi_{\mathrm{phys}} = \chi_\Phi(1, \cdot)$, where $\chi_\Phi(t, y) = \chi_{\mathrm{tube}}(\Phi_0(y))$ coefficient-wise in the ambient odd coordinates, and that the nilpotent coefficients of χ_Φ, and the derivatives of every coefficient of χ_Φ, vanish outside the annulus where χ_{phys} varies. We denote this annulus by

$$\mathcal{A} = \left\{y : r_1 \le |U_{\mathrm{can}}^{\mathrm{re}}(y)| \le r_{\mathrm{phys}}\right\}.$$

We now state the first step.

Lemma 20.5. *Let p be sufficiently large, and let A_{sh} and $p' \geq p$ be the exponent and the body weight of Proposition 19.20 for p. Let $f = f_{\Gamma,\mathbf{V}}$ be a family generated from g_R in a fixed signature that is holomorphic in the normal variables where $\psi_{R_k} \neq 0$, with $\|f\|_{\mathrm{prep},\mathcal{U}_2;p'} < \infty$, and let $\Gamma \in \mathcal{U}_0$. Then*

$$\mathcal{B}_{\mathcal{S}}(1, \mathrm{Can}_{\mathrm{phys}} f) = \int \chi_{\mathrm{tube}}\, \mathrm{Can}\, f\, \varpi^2\, \mathrm{d}U\, \mathrm{d}\nu + O\big(C_k W^{A_{\mathrm{sh}}+4k^2} e^{-c_{\mathrm{sh}} W r_1^2}\big) \|f\|_{\mathrm{prep},\mathcal{U}_2;p'}, \tag{20.17}$$

and the integral on the right converges absolutely.

Proof. The function $\mathrm{Can}\, f$ is holomorphic, but the cutoff is not, and we apply Stokes' theorem to their product along the homotopy Φ. The exterior derivative then falls only on the cutoff, which varies only on the annulus $\mathcal{A}$, and there the bound (19.51) applies along the homotopy.

We begin by reducing to body cycles. We write both integrals in the ambient odd coordinates. On $\mathcal{S}$ these are free generators, and $\mathcal{B}_{\mathcal{S}}(1, \mathrm{Can}_{\mathrm{phys}} f)$ is the integral over the body of $\mathcal{S}$ of the top odd coefficient of $\mathrm{Can}_{\mathrm{phys}} f$. On the real tube we pass from the tubular to the ambient coordinates by Berezin's formula Theorem 3.15, which is exact for integrands with compact support; we apply it only after an exhaustion cutoff has been inserted. Let U_{can} be the canonical normal coordinate of Section 19.5 with its nilpotent part, which is obtained by solving (19.37) in successive odd degree. It depends holomorphically on the body point, and on the real tube it is the tubular coordinate U. Consequently, by Lemma 19.2(b) and the normal holomorphy of f, the coefficients of $F = \mathrm{Can}\, f$ in the ambient odd coordinates are holomorphic functions of the body point z on the region swept by Φ.

We now apply Stokes' theorem with an exhaustion. Let φ be a smooth function on $[0, \infty)$ that equals one on $[0, 1]$ and zero on $[2, \infty)$, and set

$$\varphi_R(y) = \varphi\big(\log v_{\mathcal{S}}(X)/R\big), \qquad R \geq 1.$$

Let $\mathrm{d}z$ be the flat holomorphic volume form of the body of the ambient space, normalized as the body part of $\mathrm{d}S$. Consider the form

$$\omega_R = \varphi_R\, \chi_\Phi\, (F \circ \Phi)\, \Phi^*\, \mathrm{d}z$$

on $[0, 1] \times \mathcal{Y}$, coefficientwise in the ambient odd coordinates. It has compact support, since χ_Φ vanishes where $|U_{\mathrm{can}}^{\mathrm{re}}| \geq r_{\mathrm{phys}}$ and φ_R vanishes where $v_{\mathcal{S}} \geq e^{2R}$. Since F is holomorphic, $\mathrm{d}F \wedge \mathrm{d}z = 0$, and therefore $\mathrm{d}\omega_R = \mathrm{d}(\varphi_R \chi_\Phi) \wedge (F \circ \Phi)\, \Phi^*\, \mathrm{d}z$. Stokes' theorem gives

$$\int_{\{1\}\times\mathcal{Y}} \omega_R - \int_{\{0\}\times\mathcal{Y}} \omega_R = \int_{[0,1]\times\mathcal{Y}} \big(\varphi_R\, \mathrm{d}\chi_\Phi + \chi_\Phi\, \mathrm{d}\varphi_R\big) \wedge (F \circ \Phi)\, \Phi^*\, \mathrm{d}z = I_R + II_R. \tag{20.18}$$

The integrand of the second term on the left has compact support on the real tube. Using the definition (19.40) of χ_{phys} for the first term, and Theorem 3.15 and (16.15) for the second, we obtain

$$\int_{\{1\}\times\mathcal{Y}} \omega_R = \int_{\mathcal{S}} \varphi_R \, \mathrm{Can}_{\mathrm{phys}} f \, \mathrm{d}S, \qquad \int_{\{0\}\times\mathcal{Y}} \omega_R = \int \big(\varphi_R \circ \Phi_0^{-1}\big)\, \chi_{\mathrm{tube}} \, \mathrm{Can}\, f \, \varpi^2 \, \mathrm{d}U \, \mathrm{d}\nu.$$

Since χ_Φ and φ_R do not depend on t, only the component $\mathrm{d}t \wedge \iota_{\partial_t \Phi} \, \mathrm{d}z$ of $\Phi^* \, \mathrm{d}z$ contributes to I_R and $I\!I_R$. The integrand of I_R is supported in $[0,1] \times \mathcal{A}$, and that of $I\!I_R$ where $e^R \le v_{\mathcal{S}} \le e^{2R}$.

We now bound I_R. Let $y \in \mathcal{A}$ and $0 \le t \le 1$. By Lemma 19.16(a) and Lemma 19.2(b), the value $F(\Phi_t(y))$ is the family at the conjugated damping $e^{tY_{\mathrm{g}}} \Gamma e^{-tY_{\mathrm{g}}}$, with the conjugated jet directions, evaluated at the real frame g^{-1} and at the normal coordinate $U^{\mathrm{re}} + \mathrm{i}tU^{\mathrm{im}}$. These dampings lie in $\mathcal{U}_1$ for the tube of Section 19.5, since $\|S_{\mathrm{g}}^{\pm t} - \mathrm{Id}\| \le C\|n_B\|$ by Lemma 19.15, and on the shell Ω_s of the proof of Lemma 19.18 the heights satisfy $t|U^{\mathrm{im}}| \le B_s$. That proof applies Lemma A.5 jointly in the damping and in the complex normal coordinate on each shell, to selections of the damping in $\mathcal{U}_1$ and of heights at most B_s, and it bounds the selected values by the prepared norm. It therefore applies verbatim to the selections $(e^{tY_{\mathrm{g}}} \Gamma e^{-tY_{\mathrm{g}}}, \mathrm{i}tU^{\mathrm{im}})$, which yields, uniformly in t,

$$\int_{\mathcal{A}} v^{2p} \big|F(\Phi_t(y))\big|^2 \, \mathrm{d}y \le C_k W^{2A_{\mathrm{sh}}} e^{-2c_{\mathrm{sh}} W r_1^2} \|f\|^2_{\mathrm{prep},\mathcal{U}_2;p'}, \tag{20.19}$$

where $|\cdot|$ is the coefficient norm and v is the weight of the family at the real frame g^{-1}, which is the weight of Q_X.

The remaining factors in I_R are polynomial in $v_{\mathcal{S}}$. Indeed, Lemma 19.16(c) bounds $|\partial_t \Phi_t|$ and $\|D\Phi_t\|^{\dim -1}$ by $C_k v_{\mathcal{S}}^{A_\Phi}$. The coefficients of $\mathrm{d}\chi_\Phi$ are bounded at fixed rank, and the passage from the top coefficient in the ambient odd coordinates to the coefficient norm loses at most a factor $C_k W^{4k^2}$, as in Lemma 15.1. Finally, $v_{\mathcal{S}} = 1 + 2s(v(Q_X) - 1)$ by Section 15, so that $v_{\mathcal{S}}$ and v are comparable. By the Cauchy–Schwarz inequality, with the weight $v_{\mathcal{S}}^{-p}$ on the first factor and p so large that $v_{\mathcal{S}}^{2A_\Phi - 2p}$ is integrable over $\mathcal{A}$, and by (20.19), we obtain

$$|I_R| \le C_k W^{A_{\mathrm{sh}} + 4k^2} e^{-c_{\mathrm{sh}} W r_1^2} \|f\|_{\mathrm{prep},\mathcal{U}_2;p'},$$

uniformly in R.

We next show that $I\!I_R \to 0$ as $R \to \infty$. The same selection argument, without the smallness of the annulus and with heights at most $B_{\max} = C_{\mathrm{geo}}(|E| r_{\mathrm{phys}} + r_{\mathrm{phys}}^2)$ on the support of χ_Φ, shows that $v^p \, F \circ \Phi_t$ is square integrable there, uniformly in t; here a bound that depends on W suffices. Since $|\,\mathrm{d}\varphi_R| \le C/R$ and the integrand of $I\!I_R$ is supported where $v_{\mathcal{S}} \ge e^R$, the Cauchy–Schwarz inequality and dominated convergence give $I\!I_R \to 0$.

It remains to let $R \to \infty$ on the left of (20.18). By the case $t = 1$ of the selection argument, $\mathrm{Can}_{\mathrm{phys}} f \in \mathcal{H}_p(\mathcal{S})$. We pair it with a compact normal cutoff rather than require

the constant function to belong to the opposite weighted space. Indeed, Lemma 19.15 on the chosen small canonical chart bounds n_B and n_F uniformly on the support of χ_{phys}. Since $P_0 = \mathrm{ch}_{\mathrm{g}}^{-1} n_B$ with $\| \mathrm{ch}_{\mathrm{g}}^{-1} \| \le 1$, and $M = V(-s\Sigma_F + n_F)V^*$ with V unitary, the physical normal coordinates (P_0, M) lie in a fixed compact set, independently of X. The nilpotent coefficients of χ_{phys} have the same body support. Choose a smooth scalar cutoff $\Theta(P_0, M)$, compactly supported in the physical normal variables and equal to one on a neighborhood of this set. Then $\Theta \operatorname{Can}_{\mathrm{phys}} f = \operatorname{Can}_{\mathrm{phys}} f$, and $\Theta \in \mathcal{H}_{-p}(\mathcal{S})$ for p sufficiently large: its normal fiber volume is bounded, while $v_{\mathcal{S}}(X) \ge c(1 + |X|)$ and the real dimension of the X space is $2k^2$.

Dominated convergence gives $(\varphi_R - 1) \operatorname{Can}_{\mathrm{phys}} f \to 0$ in $\mathcal{H}_p(\mathcal{S})$. Applying Lemma 15.1 with Θ therefore gives

$$\int_{\mathcal{S}} \varphi_R \operatorname{Can}_{\mathrm{phys}} f \, \mathrm{d}S \longrightarrow \mathcal{B}_{\mathcal{S}}(1, \operatorname{Can}_{\mathrm{phys}} f), \qquad R \to \infty.$$

In the tubular coordinates, $\varphi_R \circ \Phi_0^{-1} - 1$ and its nilpotent coefficients are bounded and supported where $v \ge e^R / C$, and the integrand $\chi_{\mathrm{tube}} \operatorname{Can} f \varpi^2$ is integrable. Indeed, on the adapted tube we have $\operatorname{Can} f = \varpi^{-1} w_0^{1/2} \gamma_h f$, and $g_L = \varpi w_0^{1/2} \gamma_h^{-1}$ by (16.38), so that

$$\chi_{\mathrm{tube}} \operatorname{Can} f \, \varpi^2 = g_L^{\mathrm{cut}} f \, \gamma_h^2. \tag{20.20}$$

The right side is integrable by the Cauchy–Schwarz inequality in the opposite body weights $-p$ and p and the bound of Lemma 19.5 on g_L. Letting $R \to \infty$ in (20.18), we obtain (20.17). □

We now identify the returned integral with the analytic Berezin pairing.

Lemma 20.6. *Let p, A_{sh}, p', f, and Γ be as in Lemma 20.5. There exists C such that*

$$\begin{aligned} \mathcal{B}_{\mathcal{S}}(1, \operatorname{Can}_{\mathrm{phys}} f) &= \mathcal{B}_{\mathrm{ad}}(g_L^{\mathrm{cut}}, f) + O\big(C_k W^{A_{\mathrm{sh}} + 4k^2} e^{-c_{\mathrm{sh}} W r_1^2}\big) \|f\|_{\mathrm{prep}, \mathcal{U}_2; p'}, \\ |\mathcal{B}_{\mathrm{ad}}(g_L^{\mathrm{cut}}, f)| &\le C \|f\|_{\mathcal{U}_2; p, a}. \end{aligned} \tag{20.21}$$

In the balanced signature C is independent of W, and in an unbalanced signature $C \le C_k W^{A_k}$.

Proof. By Lemma 20.5, it remains to identify the returned integral and to bound it. Integrating (20.20), we obtain

$$\int \chi_{\mathrm{tube}} \operatorname{Can} f \, \varpi^2 \, \mathrm{d}U \, \mathrm{d}\nu = \int g_L^{\mathrm{cut}} f \, \gamma_h^2 \, \mathrm{d}U \, \mathrm{d}\nu = \mathcal{B}_{\mathrm{ad}}(g_L^{\mathrm{cut}}, f).$$

Together with (20.17), this gives the identity in (20.21).

It remains to bound $\mathcal{B}_{\mathrm{ad}}(g_L^{\mathrm{cut}}, f)$. The analytic Berezin pairing is represented, relative to the positive coefficient metric, by a bounded signed matrix on the exterior coefficients. The Cauchy–Schwarz inequality therefore implies that

$$|\mathcal{B}_{\mathrm{ad}}(g_L^{\mathrm{cut}}, f)| \le C \, \| e^{-a|Z|^2} v^{-p} \widetilde{\chi} g_L \| \, \| e^{a|Z|^2} v^p f_\Gamma \|,$$

where $\widetilde{\chi}$ is a real invariant normal cutoff as in Lemma 19.5 that equals one on the support of χ_{tube}.

Once p is so large that $-p \le p_0$, Lemma 19.5 bounds the first factor by a constant in the balanced signature and by $C_k W^{A_k}$ otherwise. By Lemma A.5, applied with $\zeta = \Gamma$ and the constant selection $\zeta_* = \Gamma \in \mathcal{U}_0$, the second factor is bounded by $C\|f\|_{\mathcal{U}_2;p,a}$, where C depends on the distance from $\mathcal{U}_0$ to the complement of $\mathcal{U}_2$. □

The hypotheses (H2) and (H3) of Lemma 20.4 bound the states of the adapted chain in the prepared norm, since Proposition 19.20 bounds the heat residual by it. Every state that we use is the output of a block of m_W adapted steps, and (19.48) bounds the prepared norm of such an output in the body weight of the input. The input of the first block is $(\xi_\mu g_R) \otimes 1$, with the invariant cutoff of Theorem 19.21; Lemma 19.5 bounds it only in a negative body weight. The following lemma shows that a block also raises the body weight, at a cost polynomial in W, through the damping of its last source factor.

Lemma 20.7. *Let $p \in \mathbb{R}$ and $q \ge 0$. There exists C, depending on k, p, and q, such that for every family g generated from the endpoint,*

$$\|A_h^{m_W} g\|_{\mathrm{prep},\mathcal{U}_2;p+q} \le C W^{2q} \|g\|_{\mathcal{U}_2;p,-a}. \tag{20.22}$$

Proof. The source factor of the adapted step carries the BB damping $|s_\Gamma|_1 \le e^{C_k\eta - c_0\eta v}$ of Lemma 19.6, for some $c_0 > 0$. As in the proof of Lemma 17.20, let $\mathbf{S}_\flat = e^{-c\delta_W v/2}$, where $c > 0$ is so small that $c\delta_W \le c_0\eta/2$. The last of the m_W steps is $A_h = s_\Gamma T_h$, with the multiplier s_Γ on the left, and we write

$$A_h^{m_W} = \mathbf{S}_\flat^2 B^\flat, \qquad B^\flat = (\mathbf{S}_\flat^{-2} s_\Gamma)\, T_h\, A_h^{m_W - 1}.$$

The block $B^\flat$ differs from $A_h^{m_W}$ only in the source factor of its last step, and $\mathbf{S}_\flat^{-2} s_\Gamma = e^{c\delta_W v} s_\Gamma$ keeps at least half of the BB damping of s_Γ. The proof of Proposition 19.13 uses the source factor only through Lemma 19.7. The proof of Lemma 19.7 shows that (19.27) also holds with s_Γ replaced by $e^{c'\eta v} s_\Gamma$, for every $0 \le c' \le c_0/2$. Applied with $c' = c\delta_W/\eta$, which is at most $c_0/2$ by our choice of c, this gives (19.27) for the source factor $\mathbf{S}_\flat^{-2} s_\Gamma$. Hence (19.48) holds for $B^\flat$,

$$\|B^\flat g\|_{\mathrm{prep},\mathcal{U}_2;p} \le C_k(R_k) \|g\|_{\mathcal{U}_2;p,-a}.$$

The multiplier $v^q \mathbf{S}_\flat^2$ depends only on the tangent variables, which the normal graphs of (19.35) leave unchanged, and, as in the derivation of (17.42),

$$\sup_{x \ge 1} x^q e^{-c\delta_W x} \le C_q \delta_W^{-q} = C_q W^{2q}.$$

Therefore $\|\mathbf{S}_\flat^2 F\|_{b;p+q,a} \le C_q W^{2q} \|F\|_{b;p,a}$ for all $|b| \le c_1 R_k$ and all F, and (20.22) follows from the bound for $B^\flat$. □

Proof of Lemma 20.4. By (20.15), we have the telescoping identity

$$T_n \cdots T_1 \operatorname{Can}_{\mathrm{phys}} g - \operatorname{Can}_{\mathrm{phys}} g_n = \sum_{j=1}^{n} T_n \cdots T_{j+1} M_{w_j} \mathcal{R} g_{j-1}.$$

We pair it with 1 and move all physical factors to the left by analytic transposition. Since $T_i^{\mathrm{tr}} = K_{\tau_{\mathrm{phys}}} M_{w_i}$ and multiplication is symmetric for the pairing, we obtain

$$\mathcal{B}_{\mathcal{S}}(1, T_n \cdots T_{j+1} M_{w_j} \mathcal{R} g_{j-1}) = \mathcal{B}_{\mathcal{S}}(F_j, \mathcal{R} g_{j-1}), \qquad F_j = M_{w_j} K_{\tau_{\mathrm{phys}}} M_{w_{j+1}} \cdots K_{\tau_{\mathrm{phys}}} M_{w_n} 1.$$

Note that each F_j is a marked partial chain as in (H1), with $n - j + 1$ onsite factors in reversed chronological order (this is where the one-sided form of the pairing is used). Without local sources, (20.5) gives

$$F_j = M_{w_\Gamma} (T_\Gamma^{\mathrm{tr}})^{n-j} 1 = M_{w_\Gamma} K_{\tau_{\mathrm{phys}}} T_\Gamma^{n-j} 1 = T_\Gamma^{n-j+1} 1.$$

In particular, the last term also uses a partial chain with one onsite factor. Similarly, $\mathcal{B}_{\mathcal{S}}(1, T_n \cdots T_1 e) = \mathcal{B}_{\mathcal{S}}(K_{\tau_{\mathrm{phys}}} F_1, e)$.

Lemma 15.1 bounds the pairing $\mathcal{B}_{\mathcal{S}}$ on $\mathcal{H}_{-p}(\mathcal{S}) \times \mathcal{H}_p(\mathcal{S})$ by $C_k W^{4k^2}$, and this factor is absorbed in $e^{o(W)}$. Hypotheses (H1), (H2) and (H3) bound the n residual terms, and (H1) and (H4) bound the term with e. We find

$$\big|\mathcal{B}_{\mathcal{S}}(1, T_n \cdots T_1 f) - \mathcal{B}_{\mathcal{S}}(1, \operatorname{Can}_{\mathrm{phys}} g_n)\big| \le e^{o(W)} \big(n W^{A_{\mathrm{sh}} + A} e^{-cW} + W^A e^{-cW} \big).$$

Finally, (H4) and (H2) imply that

$$\begin{aligned} |\mathcal{B}_{\mathcal{S}}(1, \operatorname{Can}_{\mathrm{phys}} g_n) - \mathcal{B}_{\mathrm{ad}}(g_L^{\mathrm{cut}}, g_n)| &\le C_k W^{A_{\mathrm{sh}} + 4k^2} e^{-c_{\mathrm{sh}} r_1^2 W} \|g_n\|_{\mathrm{prep}, \mathcal{U}_2; p'} \\ &\le C_k W^{4k^2} W^{A_{\mathrm{sh}} + A} e^{-cW}, \end{aligned}$$

where in the last inequality we used $c_{\mathrm{sh}} r_1^2 \ge c$. As above, the factor W^{4k^2} is absorbed in $e^{o(W)}$. Combining the last two displays, we obtain (20.16). □

20.5 The source limit: proof of part (a)

We prove Theorem 7.6(a) in two steps. The first step replaces $\mathcal{Z}_N(\Gamma)$ by a sum of adapted pairings, one for each signature, and the second identifies the limit of these pairings. In this section **u** and **b** are real, $\sum_j u_j = 0$, and (20.8) holds.

Replacement of the physical chain

Recall from Theorem 19.21 that the right block has the decomposition

$$f_{m_W} = \sum_{\mu=0}^{2k} \operatorname{Can}_{\mu,\mathrm{phys}}\, g_\mu + e_{m_W}, \qquad \|e_{m_W}\|_{\mathcal{H}_p(\mathcal{S})} \le C_k W^{A_k} e^{-\gamma_k W}, \tag{20.23}$$

for all required body weights p. Here μ runs over the signatures, $\mathrm{Can}_{\mu,\mathrm{phys}}$ is the canonical map of the component μ with its physical cutoff, and g_μ is the reference endpoint of that component, transported to the gauge (16.38). Let $A_{h,\mu}$ denote the adapted step in the signature μ, and $g^{\mathrm{cut}}_{L,\mu}$ the corresponding left endpoint factor. The raw input in Section 19.7 is $\xi_\mu \varpi_\mu$, so the exact similarity (19.22) gives

$$g_\mu = A_{h,\mu}^{m_W}((\xi_\mu g_R) \otimes 1).$$

The invariant normal cutoff ξ_μ is retained in this input throughout the comparison and the initial layer. In particular, g_μ is a family generated from g_R by Definition 19.1.

Proposition 20.8. *Uniformly on compact subsets of the set* (20.8)*,*

$$\mathcal{Z}_N(\Gamma) = \sum_{\mu=0}^{2k} \mathcal{B}_{\mathrm{ad}}(g^{\mathrm{cut}}_{L,\mu}, A_{h,\mu}^{N-m_W} g_\mu) + o(1). \tag{20.24}$$

Proof. By (20.7) and (20.23), we have

$$\mathcal{Z}_N(\Gamma) = \sum_\mu \mathcal{B}_{\mathcal{S}}\big(1, T_\Gamma^{N-m_W}(\mathrm{Can}_{\mu,\mathrm{phys}}\, g_\mu + e_\mu)\big),$$

where we split e_{m_W} into $2k+1$ pieces e_μ with the same bound. For each μ, we apply Lemma 20.4 without local sources, with $T_i = T_\Gamma$, $A_i = A_{h,\mu}$, $n = N - m_W$, $g = g_\mu$, and $e = e_\mu$. We now verify its hypotheses.

Hypothesis (H1) is Theorem 15.14 without local sources. Its hypotheses hold, since $N \asymp W^2$, $|E| \le E_{\mathrm{out}} \le 1/2$, $\eta = \varepsilon/a_N \asymp W^{-2}$, and the FF block of the source is traceless. The factor $(1+\eta^{-1})^{A'}$ in that theorem, where A' denotes its exponent, is therefore polynomial in W.

For (H2), we set $g_{-m_W} = (\xi_\mu g_R) \otimes 1$, with the invariant cutoff used to define g_μ, and $g_j = A_{h,\mu}^{m_W+j} g_{-m_W}$ for $-m_W < j \le n$. Then $g_0 = g_\mu$ exactly, the states of Lemma 20.4 are the g_j with $0 \le j \le n$, and

$$g_j = A_{h,\mu}^{m_W}\, g_{j-m_W}, \qquad 0 \le j \le n.$$

These states are families generated from g_R, since intermediate powers, invariant normal cutoffs, and partial pure heats are among the operations of Definition 19.1. For $j \ge 0$ they are outputs of adapted steps, and hence holomorphic in the normal variables where $\psi_{R_k} \neq 0$. Let p_0 be the body weight of Lemma 19.5. For $-m_W \le j < 0$, Lemma 19.5 and the constant Gaussian weights of Corollary 19.11 bound $\|g_j\|_{\mathcal{U}_2;p_0,-a}$ by a constant in the balanced signature and by $C_k W^{A_k}$ otherwise, uniformly on $\mathcal{U}_2$ by Lemma 19.14. Applying Lemma 20.7 with the body weight p_0 and $q = p' - p_0$, we obtain

$$\|g_j\|_{\mathrm{prep},\mathcal{U}_2;p'} \le C W^{2(p'-p_0)} \|g_{j-m_W}\|_{\mathcal{U}_2;p_0,-a}, \qquad 0 \le j \le n.$$

For $j < m_W$, the right side is bounded by a power of W. For $j \geq m_W$, the margin $-a$ is weaker than the base norm, and the power bounds of Theorem 17.1, which are uniform on $\mathcal{U}_2$ by Lemma 19.14, bound the base norm of $g_{j-m_W} = A_{h,\mu}^{j-m_W} g_0$ in the body weight p_0 by a constant times $\|g_0\|_{\mathrm{prep},\mathcal{U}_2;p'}$. Hence (H2) holds for all $0 \leq j \leq n$.

For (H3), Proposition 19.20 applies to every g_j with $0 \leq j < n$, since $\Gamma \in \mathcal{U}_0$ and g_j has finite prepared norm by (H2). Its residual bound (19.58) implies (H3) with the rate $c = c_{\mathrm{sh}} r_1^2$. The continuation of the families to the canonical body coordinates is made inside Proposition 19.20, by Lemma 19.2. Only the right states are continued analytically; the left factors are evaluated on the real cycle at the original real sources.

Hypothesis (H4) is (20.23) together with Lemma 20.6, which applies by (H2). Applying Lemma 20.4 with $c = \min(c_{\mathrm{sh}} r_1^2, \gamma_k)$ in each signature and summing over μ, we obtain

$$\Big| \mathcal{Z}_N(\Gamma) - \sum_{\mu=0}^{2k} \mathcal{B}_{\mathrm{ad}}(g_{L,\mu}^{\mathrm{cut}}, A_{h,\mu}^{N-m_W} g_\mu) \Big| \leq C(2k+1) N W^{A_{\mathrm{sh}}+A} e^{-cW+o(W)},$$

and the right side is $o(1)$ since $N \asymp W^2$. The bounds are uniform on compact parameter sets by Lemma 19.14. □

The limit of the adapted chain

In the balanced signature, recall from Lemma 18.8 that $N\delta_W L_h^{\tan} = -\mathsf{G}_{X_N(E),\Gamma}$, where $L_h^{\tan} = \frac{1}{2} L_{H,0} - V_0$ is the tangent generator of one adapted step and $\mathsf{G}_{X,\Gamma}$ is written as in (18.29). Thus j steps of this balanced chain correspond to the macroscopic time j/N of the semigroup $e^{-t\mathsf{G}_{X_N(E),\Gamma}}$.

Proposition 20.9. *Uniformly on compact subsets of the set* (20.8)*,*

$$\sum_{\mu=0}^{2k} \mathcal{B}_{\mathrm{ad}}(g_{L,\mu}^{\mathrm{cut}}, A_{h,\mu}^{N-m_W} g_\mu) \longrightarrow \mathcal{Z}_{X,\varepsilon}^{(k)}(\mathbf{u}, \mathbf{b}). \tag{20.25}$$

Proof. The terms with $\mu \neq k$ are $O(W^A e^{-cW})$, by Lemma 20.2 and the polynomial bounds of Lemma 20.6 and Lemma 19.5. We consider the balanced term and drop the index $\mu = k$. Fix $0 < \theta < 1/4$ (the limit will not depend on θ), and split the $N - m_W$ adapted steps into a right outer portion, a middle portion, and a left outer portion, of lengths $n_R = \lfloor \theta N \rfloor$, n_M, and $n_L = \lfloor \theta N \rfloor$, so that $n_M = (1-2\theta)N + O(W)$. We treat the left portion in the analytic transpose order, so that

$$\mathcal{B}_{\mathrm{ad}}(g_L^{\mathrm{cut}}, A_h^{N-m_W} g) = \mathcal{B}_{\mathrm{ad}}\big((A_h^{\mathrm{tr}})^{n_L} g_L^{\mathrm{cut}}, A_h^{n_M} A_h^{n_R} g\big). \tag{20.26}$$

We begin with the right state. Since $n_h > m_W$ for W large, Lemma 20.3, applied with $\xi = \xi_k$, implies that

$$A_h^{n_h - m_W} g = A_h^{n_h}((\xi_k g_R) \otimes 1) = \Omega_h \otimes 1 + o(1)$$

in the base normal norm and a sufficiently negative body weight. Its tangent time n_h/N tends to zero. A base-norm error suffices, since the power bounds of Theorem 17.1 propagate base-norm errors.

Next, we use the product limit Theorem 18.14 (the Trotter–Kurtz theorem [82, 131], in the form of Lemma 18.1), whose hypotheses are the power bounds of Theorem 17.1 and the consistency relation of Proposition 18.12. It implies that

$$\sup_{n\delta_W \le T} \left\| A_h^n \iota_h f - \iota_h e^{n\delta_W L_h^{\tan}} f \right\| \longrightarrow 0$$

for all $T > 0$ and all f in the base space, uniformly on compact parameter sets. Recall that $e^{j\delta_W L_h^{\tan}} = e^{-(j/N)\mathsf{G}_{X_N(E),\Gamma}}$, by Lemma 18.8. Since $X_N(E) \to X$, the generators $\mathsf{G}_{X_N(E),\Gamma}$ converge to $\mathsf{G}_{X,\Gamma}$ on the common core, and the parameter version Corollary 18.2 of the product limit gives

$$\sup_{0 \le j \le TN} \left\| A_h^j (\Omega_h \otimes f) - \Omega_h \otimes e^{-(j/N)\mathsf{G}_{X,\Gamma}} f \right\| \longrightarrow 0. \tag{20.27}$$

The tangent constant 1 belongs to the base space once its negative body weight is strong enough.

We now combine the two steps. Using the power bounds once more to propagate the error of the initial layer, together with (20.27) on the right outer portion and then on the middle portion, we find

$$A_h^{n_R} g = \Omega_h \otimes e^{-\theta \mathsf{G}_{X,\Gamma}} 1 + o(1), \qquad A_h^{n_M} A_h^{n_R} g = \Omega_h \otimes e^{-(1-\theta)\mathsf{G}_{X,\Gamma}} 1 + o(1),$$

in a negative body weight, where we used the semigroup property and $(n_M + n_R)/N \to 1 - \theta$. Since $n_R \ge \theta N/2$, Lemma 17.20 bounds $A_h^{n_R} g$ and $A_h^{n_M} A_h^{n_R} g$ in all positive body weights, and interpolation between the weighted L^2 norms promotes both convergences to every body weight.

For the left state, we first replace g_L^{cut} by g_L. The two differ only where $|U| \ge r_1$. There the right state $A_h^{N-m_W} g$ is exponentially small, by the positive Gaussian margin of Proposition 19.13, and this replacement changes the pairing by $O(e^{-cWr_1^2})$. We next reduce the transposed chain to the chain itself. By (20.9),

$$(A_h^{\mathrm{tr}})^{n_L} g_L = T_h \, A_h^{n_L - 1} (s_\Gamma g_L).$$

The factor g_L has the structure of g_R that the initial layer uses, with a constant tangent part, and $s_\Gamma - 1 = O(\delta_W v)$. The argument for the right state, with Lemma 20.3 for g_L, therefore shows that $A_h^{n_L - 1}(s_\Gamma g_L) = \Omega_h \otimes e^{-\theta \mathsf{G}_{X,\Gamma}} 1 + o(1)$ in a negative body weight. The final factor T_h is bounded by Lemma 18.7, and by (18.37) it changes $\Omega_h \otimes F$ only by $O(\tau_{\mathrm{phys}})$ for F in the operator core of Proposition 5.15. Hence

$$(A_h^{\mathrm{tr}})^{n_L} g_L = \Omega_h \otimes e^{-\theta \mathsf{G}_{X,\Gamma}} 1 + o(1)$$

in a negative body weight, and, by Lemma 17.20 applied to $A_h^{n_L-1}$, in every body weight.

In the balanced signature, (16.37) with $u = u' = \Omega_h$, which is even, factors the adapted pairing of two vectors $\Omega_h \otimes f$ and $\Omega_h \otimes f'$ into the normal pairing $\mathcal{B}_{\mathrm{nor}}(\Omega_h, \Omega_h)$ and $\int ff' \, \mathrm{d}\nu$. In the tubular measure of Section 16.5, the tangent measure $\mathrm{d}\nu$ is the invariant measure of $\mathcal{M}_k$ with the orientation of $\mathcal{B}_k$, so that $\int ff' \, \mathrm{d}\nu = \mathcal{B}_k(f, f')$. By (20.26), the convergence just proved in opposite body weights, and (20.11), we conclude that

$$\mathcal{B}_{\mathrm{ad}}(g_L^{\mathrm{cut}}, A_h^{N-m_W} g) \longrightarrow \mathcal{B}_{\mathrm{nor}}(\Omega_h, \Omega_h)\, \mathcal{B}_k\big(e^{-\theta \mathsf{G}_{X,\Gamma}} 1, e^{-(1-\theta)\mathsf{G}_{X,\Gamma}} 1\big) = \mathcal{Z}_{X,\varepsilon}^{(k)}(\mathbf{u}, \mathbf{b}),$$

where the last equality is Definition 5.16, and the pairing is finite and independent of θ by Proposition 5.15(d). Since every estimate above holds uniformly on compact parameter sets, so does the convergence. □

Normalization

By Corollary 5.20, we have $\mathcal{Z}_{X,\varepsilon}^{(k)}(\mathbf{0}, \varepsilon \mathbf{1}) = 1$, and Propositions 20.8 and 20.9, applied at $\Gamma = \Gamma_0$, give

$$\mathcal{Z}_N(\Gamma_0) \longrightarrow 1. \tag{20.28}$$

This agrees with the exact identity $\mathcal{Z}_N(\Gamma_0) = 1$ of (4.21). The agreement also shows that Φ_0 preserves the orientation of the real tube near the balanced component, as stated before Lemma 20.5. To see this, suppose that Φ_0 reversed it there. Then the proofs of Lemma 20.5, Proposition 20.8 and Proposition 20.9 would apply with the balanced term multiplied by -1, while the unbalanced terms would still tend to zero, and we would obtain $\mathcal{Z}_N(\Gamma_0) \to -1$. The proof of Theorem 7.6 uses only that the limit is nonzero. Indeed, the source ratio (20.1), and its versions with local sources in Section 20.6, are chain integrals divided by $\mathcal{Z}_N(\Gamma_0)$. A constant factor $\lambda \neq 0$ in the normalizations of the adapted chain would multiply the limits of the numerator and of $\mathcal{Z}_N(\Gamma_0)$ alike and would cancel in every ratio. Such a factor would not depend on the sources, since the orientations, the half-weight $w_0^{1/2}$, the gauge γ_h, and the half-density ϖ are built from the source-free weight.

Proof of Theorem 7.6(a). By Proposition 20.8 and Proposition 20.9, we have $\mathcal{Z}_N(\Gamma) \to \mathcal{Z}_{X,\varepsilon}^{(k)}(\mathbf{u}, \mathbf{b})$, uniformly on compact subsets of the set (20.8), and in particular (20.28) holds. The source supermatrix Γ is bounded on this set, and since

$$N\eta_0^2 = \frac{1}{N\rho(E)^2} \longrightarrow 0,$$

the prefactor in (20.1) tends to one uniformly. Inserting these limits into (20.1), we obtain

$$T_N^{(k)}(\mathbf{u}, \mathbf{b}) \longrightarrow \mathcal{Z}_{X,\varepsilon}^{(k)}(\mathbf{u}, \mathbf{b}),$$

uniformly on compact subsets of the set (20.8), for $|E| \le E_{\text{out}}$. This is part (a), with the radius $r_{k,\varepsilon}$ of (20.8). □

20.6 Local sources: proof of part (b)

We now add finitely many local sources to the chain. Let $m \ge 1$, $\delta \in (0, 1/2)$, sites $i_1 < \cdots < i_m$ with $\delta \le i_a/N \le 1-\delta$, local sources $\mathbf{Y} = (Y_1, \dots, Y_m)$ and multi-indices $\alpha = (\alpha_1, \dots, \alpha_m)$ be as in Theorem 7.6(b). We call the sites i_a the marks. We label the sites so that the right block of (20.7) consists of the sites $N - m_W + 1, \dots, N$, and the j-th step after it carries the onsite weight of the site $N - m_W + 1 - j$.

Recall from Section 7.5 that the local source function of Definition 7.5 is the function $T_N^{(k)}(\Gamma; \mathbf{Y})$ of Lemma 4.14, in which the local sources are added to $E + \mathrm{i}\eta_0\Gamma$. By (4.30), the local source Y_a multiplies the onsite weight at the site i_a by

$$\mathrm{SDet}\big(\mathrm{Id} + (E-S)^{-1}Y_a\big)^{-1},$$

where S is the chain variable at that site. We define the complete local jet at the mark i_a and the differentiated chain integral by

$$\begin{aligned} w_\Gamma^{(\alpha_a)} &= \partial_{Y_a}^{\alpha_a} w_{\Gamma, Y_a}\big|_{Y_a=0} = \partial_{Y_a}^{\alpha_a}\big(w_\Gamma \,\mathrm{SDet}(\mathrm{Id} + (E-S)^{-1}Y_a)^{-1}\big)\big|_{Y_a=0}, \\ \mathcal{Z}_N(\Gamma;\alpha) &= \mathcal{Z}_N\Big(\Gamma; \prod_{a=1}^m \frac{w_\Gamma^{(\alpha_a)}(S_{i_a})}{w_\Gamma(S_{i_a})}\Big), \end{aligned} \tag{20.29}$$

in the notation (4.17). In words, $\mathcal{Z}_N(\Gamma;\alpha)$ is the chain integral with the jets $w_\Gamma^{(\alpha_a)}$ in place of the onsite weights at the marks, and the jet $w_\Gamma^{(\alpha_a)}$ contains all the derivatives at the site i_a. By (4.20) and (4.21), we have

$$\partial_{\mathbf{Y}}^{\alpha} T_N^{(k)}(\mathbf{u}, \mathbf{b}; \mathbf{Y})\big|_{\mathbf{Y}=0} = e^{N\eta_0^2 \,\mathrm{STr}(\Gamma^2)/2}\, \frac{\mathcal{Z}_N(\Gamma;\alpha)}{\mathcal{Z}_N(\Gamma_0)}. \tag{20.30}$$

We obtain the orbit value of a jet from the expansion of $(E-S)^{-1}$ on the tube. Recall from (16.12) the tubular coordinates $S = T(x)(R_* + U)T(x)^{-1}$, where $R_* = \kappa_E\mathrm{Id} - \mathrm{i}s_E\Sigma$, and set $Q = T(x)\Sigma T(x)^{-1}$, so that $\kappa_E\mathrm{Id} - \mathrm{i}s_E Q$ runs over the saddle manifold as x varies. Recall also the orbit resolvent $\mathcal{G}(Q) = -\kappa_E\mathrm{Id} + \mathrm{i}s_E Q$ of (7.16) and the orbit multipliers $\mathrm{j}^{(\alpha_a)}$ of (7.17).

Proposition 20.10. *Let k, ε, and the local sources be as above.*

(a) On a fixed root-separated tube, there exist C and M_0 such that

$$(E-S)^{-1} = -\mathcal{G}(Q) + \mathcal{E}_h(S), \qquad \|v^{-1}\mathcal{E}_h\|_{\text{coeff}} \le Ch\langle Z\rangle^{M_0}, \tag{20.31}$$

as a multiplier from a space with finitely many stronger normal moments and body weights to the required space. Consequently, the adapted jet $w^{(\alpha_a)}_{\Gamma,\mathrm{ad}}/w_{\Gamma,\mathrm{ad}}$ *and the orbit multiplier* $\mathfrak{j}^{(\alpha_a)}$ *differ by* $O(hv^{d_a}\langle Z\rangle^{M_a})$ *between such spaces, where* $d_a=|\alpha_a|$.

(b) In each signature, the chain with the jets satisfies the hypotheses (H1)–(H4) of Lemma 20.4, and

$$\mathcal{Z}_N(\Gamma;\alpha)=\mathcal{B}_{\mathrm{ad}}\big(g_L^{\mathrm{cut}},A_{N-m_W}\cdots A_1 g\big)+o(1),$$

where the right side is the term of the balanced signature, g *is its reference endpoint from* (20.23)*, and* A_j *is the adapted step of the j-th step after the right block, with the adapted jet at the marks.*

(c) As $N\to\infty$ *with* $i_a/N\to x_a$*, the adapted pairing in (b) converges to*

$$\mathcal{B}_k\big(1,\,e^{-x_1\mathsf{G}_{X,\Gamma}}\mathfrak{j}^{(\alpha_1)}e^{-(x_2-x_1)\mathsf{G}_{X,\Gamma}}\mathfrak{j}^{(\alpha_2)}\cdots\mathfrak{j}^{(\alpha_m)}e^{-(1-x_m)\mathsf{G}_{X,\Gamma}}1\big),$$

uniformly on compact parameter sets. This holds also when some of the differences $x_{a+1}-x_a$ *vanish.*

We prove the three parts in turn. The order of the two replacements matters. The physical comparison in (b) retains the exact jets, and the algebraic $O(h)$ error of (a) is used only afterward, inside the stable adapted chain. In this way an algebraic error is never paired with a physical partial chain, whose norm is only $e^{o(W)}$.

Proof of Proposition 20.10(a). Recall that S is the shifted chain variable of Section 4. In particular, the inverse $(E-S)^{-1}$ contains no η. We first compute it on the saddle manifold. Using $E=2\kappa_E$, $\Sigma^2=\mathrm{Id}$, and $\kappa_E^2+s_E^2=1$, we find

$$(E-R_*)(\kappa_E\mathrm{Id}-\mathrm{i}s_E\Sigma)=(\kappa_E\mathrm{Id}+\mathrm{i}s_E\Sigma)(\kappa_E\mathrm{Id}-\mathrm{i}s_E\Sigma)=(\kappa_E^2+s_E^2)\mathrm{Id}=\mathrm{Id},$$

so that $(E-R_*)^{-1}=-\mathcal{G}(\Sigma)$. Conjugation by $T(x)$ gives the leading term $-\mathcal{G}(Q)$ in (20.31).

We now bound the remainder. We expand the uniformly invertible matrix $E-R_*-U$ on a smaller root-separated tube. Every term of the difference $(E-R_*-U)^{-1}-(E-R_*)^{-1}$ contains $U=hZ$, and the finite nilpotent part of the inverse retains the full normal and tangent exterior algebra.

Conjugation by the odd part of $T(x)$ is bounded in the degree metric, conjugation by the BB part of $T(x)$ is bounded by Cv, and the FF part of $T(x)$ is unitary. The normal Gaussian margin bounds $\|\langle Z\rangle^{M_0}f\|$ uniformly on the prepared families, and only normal multiplication moments are needed. We conclude that (20.31) holds.

The jet $w^{(\alpha_a)}_{\Gamma,\mathrm{ad}}/w_{\Gamma,\mathrm{ad}}$ is the derivative $\partial^{\alpha_a}_{Y_a}\mathrm{SDet}(\mathrm{Id}+(E-S)^{-1}Y_a)^{-1}|_{Y_a=0}$, a polynomial of degree d_a in the entries of $(E-S)^{-1}$ formed before this expansion, so that the terms that couple its several derivatives are retained. Its orbit value is

$$\partial^{\alpha_a}_{Y_a}\mathrm{SDet}\big(\mathrm{Id}-\mathcal{G}(Q)Y_a\big)^{-1}\big|_{Y_a=0}=\mathfrak{j}^{(\alpha_a)}(Q),$$

by (7.17). The bound on the difference follows from (20.31) and the product rule. □

The chain with jets

We now prove Proposition 20.10(b). Recall that the physical comparison retains the exact jets. At the mark i_a we set $T_{i_a} = M_{w_\Gamma^{(\alpha_a)}} K_{\tau_{\mathrm{phys}}}$, and the pullback and the gauge give an adapted step $A_{i_a} = M_{w_{\Gamma,\mathrm{ad}}^{(\alpha_a)}} K_{\mathrm{ad}}$ such that $M_{w_\Gamma^{(\alpha_a)}} \mathrm{Can}_{\mathrm{phys}} = \mathrm{Can}_{\mathrm{phys}} M_{w_{\Gamma,\mathrm{ad}}^{(\alpha_a)}}$. Since the similarity of the normal metric is conjugated into $w_{\Gamma,\mathrm{ad}}^{(\alpha_a)}$, a step at a mark has the same heat residual $\mathcal{R}$ as in (20.15), and the jet becomes part of the left factor F_j in the proof of Lemma 20.4. Hypothesis (H1) is Theorem 15.14 with the given local derivatives, and (H4) is verified as in the proof of Proposition 20.8. It remains to verify (H2) and (H3) for the states of the adapted chain with the jets.

The block estimates of Section 19 hold for blocks that contain jets, with the changes stated in (19.30), and every state of the chain with jets has finite prepared norm (19.47), as in the chain without marks, whatever the distance between the marks. We define the adapted jet ratio at the mark i_a by

$$J_{\alpha_a} = \frac{w_{\Gamma,\mathrm{ad}}^{(\alpha_a)}}{w_{\Gamma,\mathrm{ad}}} = \partial_{Y_a}^{\alpha_a} \mathrm{SDet}\big(\mathrm{Id} + (E-S)^{-1} Y_a\big)^{-1}\big|_{Y_a=0}. \tag{20.32}$$

It does not depend on Γ, and in the raw realization the step at the mark is the jet step $S^{(\alpha_a)}$ of (19.25). Recall that $d_a = |\alpha_a|$, and let $d = d_1 + \cdots + d_m$ denote the total order. By Lemma 19.7, we have $|J_{\alpha_a}|_1 \le C_k v^{d_a}$ at the complex points of the tube, with no growth in $\langle Z \rangle$.

Lemma 20.11. *Let $p \in \mathbb{R}$, and let B be the product of m_W consecutive adapted steps of the chain with jets, in one signature, some of which may be steps at marks.*

(a) The bounds of Corollary 19.11, (19.36)*, and* (19.48) *hold for B in place of $A_h^{m_W}$, with the same universal constants a, C_n, and C_0, with an additional factor C_k^m, and with the body weight of the input raised by d. In particular, B maps a family with the Gaussian margin $-a$ to a family with the Gaussian margin a, and for every family g generated from the endpoint,*

$$\|Bg\|_{\mathrm{prep},\mathcal{U}_2;p} \le C_k^{1+m} \|g\|_{\mathcal{U}_2;p+d,-a}. \tag{20.33}$$

(b) For all $q \ge 0$, the bound (20.22) *holds for B in place of $A_h^{m_W}$, with an additional factor C_k^m and with the body weight of the input raised by d.*

Proof. In the raw realization of B, the steps at the marks are the jet steps $S^{(\alpha_a)}$ of (19.25), and there are at most m of them. Therefore (19.30) bounds B with the exponent of (19.29), up to a factor C_k^m, and with the body weight of the input raised by the total degree of these jet steps. Since $v \ge 1$, we may raise it by d instead. The proofs of Corollary 19.11

and Proposition 19.13 use the steps only through (19.29), the similarity (19.22), and the conversion (19.23), and their universal constants a, C_n, and C_0 are those of (19.29). For (19.36) and (19.48), this is the last assertion of Proposition 19.13, in the graph norm (19.35), as stated after (19.48). This proves (a).

For (b), the last step of B has the multiplier s_Γ on the left, also when it is a step at a mark, whose onsite factor is $s_\Gamma w_0 J_{\alpha_a}$. The proof of Lemma 20.7 therefore applies, with (a) in place of (19.48). □

Let g_j, $0 \le j \le n$, be the states of Lemma 20.4 for the chain with jets, so that $g_0 = g_\mu$ is the image of $(\xi_\mu g_R) \otimes 1$ under the m_W adapted steps of the right block, and set $g_{-m_W} = (\xi_\mu g_R) \otimes 1$ and $g_j = A_h^{m_W+j} g_{-m_W}$ for $-m_W < j < 0$, as in the proof of Proposition 20.8. Then, for all $0 \le j \le n$,

$$g_j = A_j A_{j-1} \cdots A_{j-m_W+1}\, g_{j-m_W}, \tag{20.34}$$

where A_i for $i \le 0$ are the steps of the right block. Some of the m_W steps in (20.34) may be steps at marks. The right block contains no mark, since the marks are interior; in part (c) we place it at the endpoint farther from the mark.

Proof of Proposition 20.10(b). It remains to verify (H2) and (H3). Every g_j with $0 \le j \le n$ is a family generated from g_R, since Definition 19.1 includes complete local jets, and it is the output of an adapted step, hence holomorphic in the normal variables where $\psi_{R_k} \neq 0$. Let p_0 be the body weight of Lemma 19.5. By (20.34) and Lemma 20.11(b), applied with the body weight $p_0 - d$ and $q = p' - p_0 + d$, we have

$$\|g_j\|_{\mathrm{prep},\mathcal{U}_2;p'} \le C\, C_k^{1+m} W^{2(p'-p_0+d)} \|g_{j-m_W}\|_{\mathcal{U}_2;p_0,-a}, \qquad 0 \le j \le n,$$

uniformly on $\mathcal{U}_2$ by Lemma 19.14. For $j < m_W$, the input $g_{j-m_W} = A_h^j g_{-m_W}$ carries no jet, since the right block contains no mark, and Lemma 19.5 and the constant Gaussian weights of Corollary 19.11 bound its norm on the right by a power of W. For $j \ge m_W$, the margin $-a$ is weaker than the base norm. The adapted legs are bounded in every body weight by Theorem 17.1, and each jet maps $\mathcal{H}^{\mathrm{tube}}_{q+d_a}$ to $\mathcal{H}^{\mathrm{tube}}_q$ with norm at most C_k, by Lemma 19.7. Hence the base norm of g_{j-m_W} in the body weight p_0 is at most $C\, C_k^m$ times that of g_0 in the body weight $p_0 + d$. This is at most $\|g_0\|_{\mathrm{prep},\mathcal{U}_2;p_0+d}$, which Lemma 20.7 bounds by a power of W. This proves (H2).

For (H3), Proposition 19.20 applies to every g_j with $0 \le j < n$, since $\Gamma \in \mathcal{U}_0$ and g_j has finite prepared norm by (H2). Its residual bound (19.58), with the same shell gain c_{sh} and the original conditions (19.50), implies (H3) with the rate $c_{\mathrm{sh}} r_1^2$, for all states, including those immediately after a mark and those between marks that are fewer than m_W steps apart. The constants depend on the number m of marks and on the orders d_a only through the factor C_k^{1+m}, the exponent A_{sh}, and the body weights.

We now apply Lemma 20.4 in each signature. By Theorem 15.14 with local derivatives, the bound $e^{o(W)}$ of (H1) takes the form $W^{A'}(1+\eta^{-1})^{A'}e^{\mathcal{E}_W(\eta)}$, where A' denotes the exponent of that theorem. Then (20.16) bounds the total replacement error by $CNW^{A_{\mathrm{sh}}+A}e^{-\gamma W+o(W)} = o(1)$, with $\gamma = \min(c_{\mathrm{sh}}r_1^2, \gamma_k)$. The unbalanced signatures are $O(W^A e^{-cW})$ by Lemma 20.2, since the leg before the first mark has length at least $\delta N/2$. □

The same argument bounds the Gaussian moments of the states at macroscopic times independently of W. We will use this in the proof of Proposition 20.10(c).

Lemma 20.12. *Let g_j, $0 \le j \le N - m_W$, be the states of the chain with jets in the balanced signature, and let $\delta' > 0$ and $p \in \mathbb{R}$. There exists $C_{p,a,\delta'}$, which does not depend on W and depends on the marks only through their number m and the orders d_a, such that*

$$\sup_{\delta' N \le j \le N - m_W} \left\| e^{a|Z|^2} g_j \right\|_{\mathcal{H}^{\mathrm{tube}}_p} \le C_{p,a,\delta'}. \tag{20.35}$$

The same bound holds for the states of the chain with jets in the analytic transpose order.

Proof. Let W be so large that $m_W \le \delta' N/2$, and let $\delta' N \le j \le N - m_W$. By (20.34) and Lemma 20.11(a), and since $e^{-a|Z|^2} \le 1$, we have

$$\left\| e^{a|Z|^2} g_j \right\|_{\mathcal{H}^{\mathrm{tube}}_p} \le C_k^{1+m} \left\| e^{-a|Z|^2} g_{j-m_W} \right\|_{\mathcal{H}^{\mathrm{tube}}_{p+d}} \le C_k^{1+m} \| g_{j-m_W} \|_{\mathcal{H}^{\mathrm{tube}}_{p+d}},$$

with $j - m_W \ge \delta' N/2$. It suffices to bound the base norms $\|g_i\|_{\mathcal{H}^{\mathrm{tube}}_{p+d}}$ for $i \ge \delta' N/2$ independently of W.

Let p_0 be the body weight of Lemma 19.5. In the balanced signature, Lemma 19.5 and Corollary 19.11, combined as in the proof of Corollary 19.12, bound $\|e^{a|Z|^2} g_0\|_{\mathcal{H}^{\mathrm{tube}}_{p_0}}$ independently of W, and the power bounds of Theorem 17.1 keep the base norm of $A_h^i g_0$ in $\mathcal{H}^{\mathrm{tube}}_{p_0}$ bounded for $i \le N$. The leg before the first mark has length $n_0 \ge \delta N/2$. For $\min(\delta, \delta')N/2 \le n \le N$, we have $n\delta_W \ge c$ for some $c > 0$, since $N \asymp W^2$, and Lemma 17.20 bounds A_h^n from $\mathcal{H}^{\mathrm{tube}}_{p_0}$ to $\mathcal{H}^{\mathrm{tube}}_{p+2d}$ independently of W. This bounds $\|g_i\|_{\mathcal{H}^{\mathrm{tube}}_{p+2d}}$ for $\delta' N/2 \le i \le n_0$. After the first mark, the jets map $\mathcal{H}^{\mathrm{tube}}_{q+d_a}$ to $\mathcal{H}^{\mathrm{tube}}_q$ with norm at most C_k, by the bound on $|J_{\alpha_a}|_1$, and the adapted legs between them are bounded in every fixed body weight by Theorem 17.1. The total loss of body weight is at most d, which bounds $\|g_i\|_{\mathcal{H}^{\mathrm{tube}}_{p+d}}$ for $i > n_0$.

For the analytic transpose order, recall that the multipliers commute and that T_h is symmetric for $\mathcal{B}_{\mathrm{ad}}$. As in (20.9), the transposed chain with jets started from g_L is therefore the chain with the jets in the reversed order, started from $s_\Gamma g_L$ and followed by one step T_h. The same argument applies, with Lemma 19.5 for g_L and Lemma 18.7 for T_h. □

Proof of Proposition 20.10(c). We work with the balanced pairing in (b). Recall that we use (20.31) only after the exact physical replacement of (b), inside the stable adapted chain. Lemma 20.12 supplies all the stronger body weights and normal moments with $O(1)$ norms for the finitely many $O(h)$ errors of Proposition 20.10(a). Observe that the orbit multiplier $\mathsf{j}^{(\alpha_a)}$ acts as the identity on the normal factor, so that it preserves Ω_h.

The terms that a jet creates in the normal complement of Ω_h have real Hilbert norm $O(h)$, and they remain $O(h)$ under the power bounds of Theorem 17.1. We may therefore replace each jet by its orbit multiplier with an error $o(1)$.

It remains to identify the limit of the adapted chain with the orbit multipliers inserted. After the right initial layer, the state has the form $\Omega_h \otimes 1 + o(1)$ in the base norm, by Lemma 20.3 with $\xi = \xi_k$. On a segment between two marks, (20.27) holds, and we extend it to the inputs produced by the preceding segments. Suppose that $f_N \to f$ in a weighted base space. The power bounds imply that

$$\big\| A_h^j(\Omega_h \otimes f_N) - \Omega_h \otimes e^{-(j/N)\mathsf{G}_{X,\Gamma}} f \big\| \le C\|f_N - f\| + \big\| A_h^j(\Omega_h \otimes f) - \Omega_h \otimes e^{-(j/N)\mathsf{G}_{X,\Gamma}} f \big\|,$$

and both terms tend to zero uniformly for $j \le TN$.

An orbit multiplier of degree d_a is bounded from $\mathcal{H}^{\mathrm{tube}}_{q+d_a}$ to $\mathcal{H}^{\mathrm{tube}}_{q}$, and there are finitely many marks. We start from convergence in a sufficiently strong body weight, which the leg before the first mark supplies by Lemma 17.20, and we lose d_a weights at the mark i_a. A segment of length $o(N)$, including length one, acts as the identity in the limit, since $e^{-t\mathsf{G}_{X,\Gamma}} f \to f$ as $t \downarrow 0$.

Distinct sites remain separate jets, even when they are adjacent, while the complete jet at one site also contains the terms that couple its several derivatives, such as the term $\mathcal{G}^{\mathrm{BB}}_{\mathrm{RA}}\mathcal{G}^{\mathrm{BB}}_{\mathrm{AR}}$ of Remark 7.9. Pairing with the left state as in the proof of Proposition 20.9, and using (20.11) and the symmetry of $e^{-t\mathsf{G}_{X,\Gamma}}$ for $\mathcal{B}_k$, we obtain the limit stated in Proposition 20.10(c). □

Proof of Theorem 7.6(b). By Proposition 20.10(b) and (c), the numerator $\mathcal{Z}_N(\Gamma;\alpha)$ converges to the pairing of semigroups and multipliers in Proposition 20.10(c). The denominator converges to one by (20.28), and the prefactor tends to one, as in the proof of part (a). Inserting these limits into (20.30), we obtain (7.19) for $|E| \le E_{\mathrm{out}}$, uniformly for $(\mathbf{u}, \mathbf{b})$ in compact subsets of (20.8). □

20.7 All-site moments: proof of part (c)

We now bound the moments of the diagonal resolvent entries at all sites, including the sites near the endpoints of the chain. There is a single complete jet, at a site i that may be anywhere in $\{1, \dots, N\}$, and the argument uses only the boundedness of the adapted pairing, not its limit.

Proof of Theorem 7.6(c). We first take $u = 0$ and $|E| \le E_{\mathrm{out}}$, and we fix $p \ge 1$. We take $k = 1$, $\mathbf{u} = 0$, and $\mathbf{b} = \varepsilon$, so that $Z_{\mathrm{F}} = Z_{\mathrm{B}}$ in Definition 7.5, and the local source $Y = \mathrm{diag}(Y^{\mathrm{B}} \mid 0)$ at the site i, with $Y^{\mathrm{B}} = \mathrm{diag}(y, y')$. Using

$$\det(z - H + y e_i e_i^*) = \det(z - H)\big(1 - y G_{ii}(z)\big), \qquad G_{ii}(\bar z) = \overline{G_{ii}(z)},$$

we obtain

$$\begin{aligned} \partial_y^p \partial_{y'}^p T_N^{(1)}(0, \varepsilon; Y)\big|_{y=y'=0} &= \partial_y^p \partial_{y'}^p \mathbb{E}\big[(1 - y G_{ii}(z_+^{\mathrm{B}}))^{-1}(1 - y' G_{ii}(z_-^{\mathrm{B}}))^{-1}\big]\big|_{y=y'=0} \\ &= (p!)^2\, \mathbb{E}|G_{ii}(E + \mathrm{i}\eta)|^{2p}. \end{aligned} \tag{20.36}$$

We now bound the left side of (20.36). By the reflection $i \mapsto N + 1 - i$ of the chain, we may assume that $i \le (N+1)/2$, and we write the chain in the one-sided form (20.7) with its right block at the farther endpoint. The leg from the right block to the site i then carries no local source. Its length is at least $(N-1)/2 - m_W$. By Lemma 19.14, the signed preparation, the BB damping on this leg of macroscopic length and one final block of m_W steps put the state arriving at i in all prescribed positive body weights with a positive Gaussian margin, with an $O(1)$ norm, uniformly on $\mathcal{U}_2$. After this leg, the unbalanced signatures are $O(W^A e^{-cW})$, by Lemma 20.2.

The complete jet of (20.36) has the multiplier bound $|J_\alpha|_1 \le C_p v^{2p}$ of Lemma 19.7, and the incoming state supplies these body weights. After the jet, adapted propagation over an arbitrary number of steps has bounded base norms, by Theorem 17.1. Every state after the jet, including the state immediately after it, is the image of the m_W adapted steps that precede it, one of which may be the jet step, as in (20.34). Lemma 20.11 therefore implies that it has the positive Gaussian margin a, with a norm bounded independently of W.

By Proposition 20.10(b), with (H1)–(H4) verified as there for this single jet, the differentiated chain integral differs from $\mathcal{B}_{\mathrm{ad}}(g_L^{\mathrm{cut}}, g_{\mathrm{out}})$ by $o(1)$, where g_{out} denotes the adapted state after the last step. Lemma 20.6 implies that

$$|\mathcal{B}_{\mathrm{ad}}(g_L^{\mathrm{cut}}, g_{\mathrm{out}})| \le C \|g_{\mathrm{out}}\|_{\mathcal{U}_2;\bar p,a} \le C',$$

where $\bar p$ denotes the body weight of Lemma 20.6, since p is the order of the moment here. For $i = 1$ there is no step after the jet, and the left factor of the last residual term in the proof of Lemma 20.4 is the jet times the constant function. This is a partial chain of length one with a local derivative, which Theorem 15.14 covers.

By (20.30), the left side of (20.36) is the differentiated chain integral divided by $\mathcal{Z}_N(\Gamma_0)$, at $\Gamma = \Gamma_0$, where the prefactor is one. Since $\mathcal{Z}_N(\Gamma_0) \to 1$ by (20.28), we conclude that $(p!)^2 \mathbb{E}|G_{ii}(E + \mathrm{i}\eta)|^{2p} = O(1)$, uniformly in i. This is part (c) for $u = 0$, for all $|E| \le E_{\mathrm{out}}$.

The bound is uniform in the critical regime of Definition 1.2, with E_{out} in place of E_*. Fix $0 < X_- \le X_+ < \infty$ and ε in a compact subset of $(0, \infty)$. If the bound failed uniformly over all (N, W, E) with $|E| \le E_{\mathrm{out}}$ and $X_- \le X_N(E) \le X_+$, there would be sequences of

such parameters, of dampings, and of sites i along which the left side of (20.36) tends to infinity. Along a subsequence, $X_N(E)$ and ε converge, and $N \asymp W^2$. The argument above applies along this subsequence, with the energy depending on N as Theorem 7.6 allows, and shows that the left side stays bounded, which is a contradiction.

Now let $u_0 > 0$ be the bound on $|u|$ in Theorem 7.6(c), let $|u| \le u_0$ and $|E| \le E_*$, and let E' and ε' be as in (20.3). By Lemma 20.1, we have

$$G_{ii}\big(E + (u + \mathrm{i}\varepsilon)/a_N(E)\big) = G_{ii}\big(E' + \mathrm{i}\varepsilon'/a_N(E')\big),$$

and for $N \ge N_0$ we have $|E'| \le E_{\mathrm{out}}$, while ε' and $X_N(E')$ lie in slightly larger compact ranges, by (20.4). The case $u = 0$ at the energy E' with the damping ε', which holds uniformly over these ranges, therefore implies (7.20), uniformly in the critical regime. □

20.8 One-point normalization: proof of part (d)

The last part of Theorem 7.6 identifies the mean of the diagonal resolvent. It is the case of one site and one first derivative of part (b), evaluated by the localization lemma Lemma 5.19.

Proof of Theorem 7.6(d). We first take $u = 0$ and $|E| \le E_{\mathrm{out}}$. We take $k = 1$, $\mathbf{u} = 0$, and $\mathbf{b} = \varepsilon$, and the local source $Y = \mathrm{diag}(0 \mid tE_{\mathrm{RR}})$ at the site i, where E_{RR} is the matrix unit of the retarded flavor. By (7.15), we have

$$-\partial_t T_N^{(1)}(0, \varepsilon; Y)\big|_{t=0} = \mathbb{E} G_{ii}(E + \mathrm{i}\eta).$$

The supermatrix $\mathrm{Id} - \mathcal{G}Y$ is block upper triangular with FF block $\mathrm{Id} - t\mathcal{G}^{\mathrm{FF}}E_{\mathrm{RR}}$, so that $\mathrm{SDet}(\mathrm{Id} - \mathcal{G}Y)^{-1} = \det(\mathrm{Id} - t\mathcal{G}^{\mathrm{FF}}E_{\mathrm{RR}})$, and the corresponding orbit multiplier satisfies

$$-\partial_t \,\mathrm{SDet}\big(\mathrm{Id} - \mathcal{G}(Q)Y\big)^{-1}\big|_{t=0} = \mathcal{G}(Q)^{\mathrm{FF}}_{\mathrm{RR}} = -\kappa_E + \mathrm{i}s_E Q^{\mathrm{FF}}_{\mathrm{RR}}.$$

Let $i/N \to x \in [\delta, 1-\delta]$, and write $\mathsf{G} = \mathsf{G}_{X,\Gamma_0}$ at rank one. By part (b) with $m = 1$, applied to $-\partial_t$, we have

$$\mathbb{E} G_{ii}(E + \mathrm{i}\eta) = -\kappa_E \,\mathcal{B}_1\big(1, e^{-\mathsf{G}}1\big) + \mathrm{i}s_E \,\mathcal{B}_1\big(1, e^{-x\mathsf{G}} Q^{\mathrm{FF}}_{\mathrm{RR}} e^{-(1-x)\mathsf{G}}1\big) + o(1).$$

By Corollary 5.20 for the first pairing, and by the symmetry of $e^{-x\mathsf{G}}$ for $\mathcal{B}_1$ in Proposition 5.15(c) for the second, we have

$$\mathcal{B}_1\big(1, e^{-\mathsf{G}}1\big) = \mathcal{Z}^{(1)}_{X,\varepsilon}(0, \varepsilon) = 1,$$
$$\mathcal{B}_1\big(1, e^{-x\mathsf{G}} Q^{\mathrm{FF}}_{\mathrm{RR}} e^{-(1-x)\mathsf{G}}1\big) = \mathcal{B}_1\big(e^{-x\mathsf{G}}1, Q^{\mathrm{FF}}_{\mathrm{RR}} e^{-(1-x)\mathsf{G}}1\big).$$

We now evaluate the second pairing by localization. The two factors $e^{-x\mathsf{G}}1$ and $e^{-(1-x)\mathsf{G}}1$ are invariant under $\mathfrak{k}$, by Corollary 5.20, and Q_{RR} is advanced-invariant, by

(5.6). The integrand $(e^{-x\mathsf{G}}1)\, Q_{\mathrm{RR}}^{\mathrm{FF}}\, (e^{-(1-x)\mathsf{G}}1)$ is therefore invariant under $\mathfrak{k}_{\mathrm{A}}$, hence $\mathfrak{q}_{\mathrm{A}}$-closed by Lemma 5.18, and its coefficients are polynomially integrable by Proposition 5.15. By Lemma 5.19, its integral is the value of its body at $Q = \Lambda$, where $Q_{\mathrm{RR}}^{\mathrm{FF}} = 1$. The two factors also have the body value one at Λ. Indeed, for $t \in \{x, 1-x\}$, the superfunction $e^{-t\mathsf{G}}1$ is invariant under $\mathfrak{k}$, hence $\mathfrak{q}_{\mathrm{A}}$-closed, and has polynomially integrable coefficients, by the same results. Lemma 5.19, applied with $A = e^{-t\mathsf{G}}1$, and Corollary 5.20, applied with $z = 2\pi\varepsilon$ and $T = t$, since $\mathsf{G} = \mathsf{G}_{X,2\pi\varepsilon}$, therefore give

$$\big(e^{-t\mathsf{G}}1\big)_{\varnothing}(\Lambda) = \int e^{-t\mathsf{G}}1 = 1.$$

Therefore the second pairing is one, and

$$\mathbb{E}G_{ii}(E + \mathrm{i}\eta) - (-\kappa_E + \mathrm{i}s_E) \longrightarrow 0.$$

This convergence is uniform for $\delta \le i/N \le 1-\delta$, for ε in compact subsets of $(0,\infty)$, and over all (N, W, E) with $|E| \le E_{\mathrm{out}}$ and $X_- \le X_N(E) \le X_+$, for every $0 < X_- \le X_+ < \infty$. Indeed, otherwise there would exist $c > 0$ and sequences of sites and parameters along which the difference stays above c. Along a subsequence of these, i/N, ε, and $X_N(E)$ converge, and part (b) applies along it, since the energy may depend on N there. This contradicts the convergence just proved.

Now let $u_0 > 0$ be the bound on $|u|$ in Theorem 7.6(d), let $|u| \le u_0$ and $|E| \le E_*$, and let E' and ε' be as in (20.3). By Lemma 20.1 and the case $u = 0$ at the energy E', we obtain

$$\mathbb{E}G_{ii}\big(E + (u + \mathrm{i}\varepsilon)/a_N\big) = \mathbb{E}G_{ii}\big(E' + \mathrm{i}\varepsilon'/a_N(E')\big) = -\kappa_{E'} + \mathrm{i}s_{E'} + o(1),$$

uniformly for $\delta \le i/N \le 1-\delta$ and $|u| \le u_0$, and uniformly in the critical regime of Definition 1.2, since $|E'| \le E_{\mathrm{out}}$ and, by (20.4), $X_N(E')$ and ε' lie in slightly larger compact ranges. Since $|E' - E| \le C/N$, we have $-\kappa_{E'} + \mathrm{i}s_{E'} = -\kappa_E + \mathrm{i}s_E + o(1)$, which is the first limit in (7.21).

Recall that $\ell_{i,N}^{\varepsilon}(u) = s_E^{-1} \operatorname{Im} G_{ii}(E + (u + \mathrm{i}\varepsilon)/a_N)$. Taking imaginary parts, we conclude that $\mathbb{E}\ell_{i,N}^{\varepsilon}(u) \to 1$. This is part (d), and the proof of Theorem 7.6 is complete. □

APPENDIX A

General tools

This appendix collects four general facts that are used in several sections. The first is a stability estimate for an operator that differs from a contraction by a perturbation that is small relative to the dissipation of the contraction; it is used in Sections 16 and 17. The second is a Dirichlet-form bound for integral operators whose kernels take values in operators on a Hilbert space, and is used in Section 17. The third concerns Poisson transforms of positive measures on the line, and is used in Part III to identify limiting pair and occupation measures. The fourth is a submean inequality for holomorphic families of Hilbert-space-valued functions, which is used in Sections 19 and 20. All four are elementary, and we give complete proofs. The letters used in this appendix are local to it, except for the defects D_R and D_L of (A.1), which Sections 16 and 17 also use, and for P_a and $\|\cdot\|_{\mathrm{loc}}$ in Appendix A.3.

A.1 Perturbations of a contraction

Let $\mathcal{H}$ be a Hilbert space, and let C be a contraction on $\mathcal{H}$ (a bounded operator with $\|C\| \le 1$). We define the *right defect* and the *left defect* of C to be

$$D_R = \mathrm{Id} - C^*C, \qquad D_L = \mathrm{Id} - CC^*. \tag{A.1}$$

These are positive operators of norm at most one, and their square roots are the defect operators of C and C^* in the sense of Sz.-Nagy and Foiaş [130]. For all $f \in \mathcal{H}$, we have

$$\|Cf\|^2 = \|f\|^2 - \|D_R^{1/2} f\|^2.$$

Thus D_R measures how much C dissipates the norm of f, and D_L does the same for C^*.

In our applications, C is an explicit model for one step of a transfer chain, and T is the actual step. The difference $T - C$ is not small in norm. However, it is small relative to the defects of C, up to an error of the order of the step size δ_0. The following lemma shows that T then keeps half of the dissipation of C, up to a loss of order δ_0 per step, so that the powers T^j with $j\delta_0 \le t$ stay bounded.

Lemma A.1. *Let C be a contraction on a Hilbert space $\mathcal{H}$, with defects D_R and D_L as in* (A.1).

(a) Let $0 < \delta_0 \le 1$, $A \ge 1$, and $0 < \theta \le 1/8$ be such that $A\delta_0 \le 1$, and let T be a bounded operator on $\mathcal{H}$ such that

$$|\langle g, (T - C) f\rangle| \le \theta \, \|(D_L + A\delta_0)^{1/2} g\| \, \|(D_R + A\delta_0)^{1/2} f\| \qquad \text{for all } f, g \in \mathcal{H}. \tag{A.2}$$

Then

$$T^*T \le (1+4\theta A\delta_0)\mathrm{Id} - \tfrac{1}{2}D_R, \qquad TT^* \le (1+4\theta A\delta_0)\mathrm{Id} - \tfrac{1}{2}D_L. \tag{A.3}$$

(b) Let $0 < \delta_0 \le 1$ and $A_1 \ge 0$, and let T be a bounded operator on $\mathcal{H}$ such that

$$T^*T \le (1+A_1\delta_0)\mathrm{Id} - \tfrac{1}{2}D_R, \qquad TT^* \le (1+A_1\delta_0)\mathrm{Id} - \tfrac{1}{2}D_L. \tag{A.4}$$

Let $n \ge 1$ and $t > 0$ satisfy $n\delta_0 \le t$. Then $\|T^j\| \le e^{A_1 t/2}$ for $0 \le j \le n$, and for all $f, g \in \mathcal{H}$,

$$\sum_{j=0}^{n-1} \|D_R^{1/2} T^j f\|^2 + \sum_{j=0}^{n-1} \|D_L^{1/2} (T^*)^j g\|^2 \le 2\big(1 + A_1 t e^{A_1 t}\big)\big(\|f\|^2 + \|g\|^2\big). \tag{A.5}$$

In particular, under the hypotheses of part (a), the conclusions of part (b) hold with $A_1 = 4\theta A$ for all n and t with $n\delta_0 \le t$.

Proof. We begin with part (a). For $f \in \mathcal{H}$, set

$$q(f) = \|(D_R + A\delta_0)^{1/2} f\|^2 = \|D_R^{1/2} f\|^2 + A\delta_0 \|f\|^2.$$

Expanding $\|Cf + (T-C)f\|^2$ and using $\|Cf\|^2 = \|f\|^2 - \|D_R^{1/2} f\|^2$, we have

$$\|Tf\|^2 = \|f\|^2 - \|D_R^{1/2} f\|^2 + 2\,\mathrm{Re}\langle Cf, (T-C)f\rangle + \|(T-C)f\|^2. \tag{A.6}$$

We will bound the last two terms by multiples of $q(f)$.

To bound the cross term, we first note that

$$C^*C(\mathrm{Id} - C^*C) \le \mathrm{Id} - C^*C,$$

since $0 \le C^*C \le \mathrm{Id}$. It follows that

$$\|D_L^{1/2} Cf\|^2 = \langle Cf, (\mathrm{Id} - CC^*)Cf\rangle = \langle f, C^*C(\mathrm{Id} - C^*C)f\rangle \le \|D_R^{1/2} f\|^2.$$

Together with $\|Cf\| \le \|f\|$, this implies

$$\|(D_L + A\delta_0)^{1/2} Cf\|^2 = \|D_L^{1/2} Cf\|^2 + A\delta_0 \|Cf\|^2 \le q(f).$$

Taking $g = Cf$ in (A.2), we obtain

$$|\langle Cf, (T-C)f\rangle| \le \theta\, q(f). \tag{A.7}$$

For the last term, we take the supremum in (A.2) over all g with $\|g\| = 1$. Using $\|D_L\| \le 1$ and $A\delta_0 \le 1$, we find

$$\|(T-C)f\|^2 \le 2\theta^2 q(f). \tag{A.8}$$

Inserting (A.7) and (A.8) into (A.6), we find

$$\|Tf\|^2 \le \|f\|^2 - \|D_R^{1/2} f\|^2 + (2\theta + 2\theta^2) q(f) \le (1 + 4\theta A\delta_0)\|f\|^2 - (1 - 4\theta)\|D_R^{1/2} f\|^2,$$

where we used $2\theta^2 \le 2\theta$ in the second inequality. By the assumption that $\theta \le 1/8$, we have $1 - 4\theta \ge 1/2$, and the first inequality in (A.3) follows.

For the second inequality, observe that

$$|\langle f, (T^* - C^*) g\rangle| = |\langle g, (T - C) f\rangle| \qquad \text{for all } f, g \in \mathcal{H},$$

and that the right and left defects of the contraction C^* are D_L and D_R. This shows that the pair (T^*, C^*) satisfies (A.2), with the roles of f and g exchanged. The first inequality in (A.3) for (T^*, C^*) is the second inequality for (T, C).

We now prove part (b). Let $f \in \mathcal{H}$, and set $x_j = \|T^j f\|^2$. Applying the first inequality in (A.4) to $T^j f$, we obtain

$$x_{j+1} \le (1 + A_1\delta_0) x_j - \tfrac{1}{2}\|D_R^{1/2} T^j f\|^2. \tag{A.9}$$

In particular, for $0 \le j \le n$, we have

$$x_j \le (1 + A_1\delta_0)^j x_0 \le e^{A_1\delta_0 j}\|f\|^2 \le e^{A_1 t}\|f\|^2,$$

which is the bound $\|T^j\| \le e^{A_1 t/2}$.

Next, we bound the defect sums. Summing (A.9) over $0 \le j \le n-1$, we find

$$\frac{1}{2}\sum_{j=0}^{n-1} \|D_R^{1/2} T^j f\|^2 \le x_0 - x_n + A_1\delta_0 \sum_{j=0}^{n-1} x_j \le \bigl(1 + A_1\delta_0 n e^{A_1 t}\bigr)\|f\|^2 \le \bigl(1 + A_1 t e^{A_1 t}\bigr)\|f\|^2.$$

The same argument for T^*, with the second inequality in (A.4), bounds the second sum in (A.5) by $2(1 + A_1 t e^{A_1 t})\|g\|^2$. The final claim of the lemma follows, since (A.3) is (A.4) with $A_1 = 4\theta A$. □

Most applications verify (A.2) through a bound with four terms: a term with a defect on each side, two terms that carry one defect and a factor $\sqrt{\delta_0}$, and a term of order δ_0. The following corollary turns such a bound into the hypothesis (A.4) of Lemma A.1(b), with explicit constants.

Corollary A.2. *Let C be a contraction on a Hilbert space $\mathcal{H}$, with defects D_R and D_L as in* (A.1). *Let $0 \le a \le 1/16$, $A_0 \ge 0$, and $0 < \delta_0 \le 1$, and let T be a bounded operator on $\mathcal{H}$ such that for all $f, g \in \mathcal{H}$,*

$$\begin{aligned}|\langle g, (T - C) f\rangle| \le{}& a\|D_L^{1/2} g\|\,\|D_R^{1/2} f\| \\ &+ A_0\sqrt{\delta_0}\bigl(\|g\|\,\|D_R^{1/2} f\| + \|D_L^{1/2} g\|\,\|f\|\bigr) + A_0\delta_0\|g\|\,\|f\|. \end{aligned}\tag{A.10}$$

Set

$$A = \max\{1, 2304A_0^2\}, \qquad A_1 = A\big((1 + a + 3A_0)^2 + 1\big). \tag{A.11}$$

Then T satisfies (A.4) *with this A_1. If moreover $A\delta_0 \le 1$, then T satisfies* (A.4) *with $A/2$ in place of A_1. In particular, the conclusions of Lemma A.1(b) hold for all n and t with $n\delta_0 \le t$, with $A/2$ in place of A_1 when $A\delta_0 \le 1$.*

The bound (A.10) holds, for example, if

$$T - C = D_L^{1/2} Y_0 D_R^{1/2} + \sqrt{\delta_0}\big(Y_1 D_R^{1/2} + D_L^{1/2} Y_2\big) + \delta_0 Y_3$$

for bounded operators $Y_0, \dots, Y_3$ with $\|Y_0\| \le a$ and $\|Y_1\|, \|Y_2\|, \|Y_3\| \le A_0$.

Proof. Suppose first that $A\delta_0 \le 1$. Let $f, g \in \mathcal{H}$, and set

$$q_R = \|(D_R + A\delta_0)^{1/2} f\|, \qquad q_L = \|(D_L + A\delta_0)^{1/2} g\|.$$

Then

$$\|D_R^{1/2} f\| \le q_R, \qquad \sqrt{\delta_0}\,\|f\| \le A^{-1/2} q_R, \qquad \|D_L^{1/2} g\| \le q_L, \qquad \sqrt{\delta_0}\,\|g\| \le A^{-1/2} q_L.$$

Inserting these bounds into (A.10), we obtain

$$|\langle g, (T - C) f\rangle| \le \big(a + 2A_0 A^{-1/2} + A_0 A^{-1}\big) q_L q_R \le \big(a + 3A_0 A^{-1/2}\big) q_L q_R \le \tfrac{1}{8}\, q_L q_R,$$

where we used $A \ge 1$ in the second inequality, and $a \le 1/16$ and $A^{1/2} \ge 48A_0$, which follows from $2304 = 48^2$, in the third. Thus (A.2) holds with $\theta = 1/8$, and Lemma A.1(a) implies (A.4) with $4\theta A = A/2$ in place of A_1. Since $A/2 \le A_1$, the bound (A.4) also holds with A_1 itself.

Suppose next that $A\delta_0 > 1$. We take the supremum in (A.10) over unit vectors f and g. Since the defects have norm at most one and $\delta_0 \le 1$, we find

$$\|T - C\| \le a + 3A_0, \qquad \|T\| \le 1 + a + 3A_0.$$

The second bound implies that

$$T^*T \le (1 + a + 3A_0)^2 \mathrm{Id} \le \big(1 + A\delta_0((1 + a + 3A_0)^2 + 1)\big)\mathrm{Id} - \tfrac{1}{2} D_R,$$

where we used $A\delta_0 > 1$ and $D_R \le \mathrm{Id}$ in the second inequality. The same bound holds for TT^*, with D_L in place of D_R, and (A.4) follows. The final claim is now a consequence of Lemma A.1(b). □

A.2 An operator-valued Dirichlet-form bound

We will need the following bound for a symmetric integral operator whose kernel takes values in bounded operators on a Hilbert space. In Section 17, these values are coefficient operators that need not commute with one another. The lemma bounds the difference between the integral operator and the multiplication by its marginal by the Dirichlet form of the underlying scalar kernel, uniformly in the operator values.

Throughout this section, $(\mathcal{Y}, \mu)$ is a σ-finite measure space and $\mathcal{K}$ is a separable Hilbert space. Let $L^2(\mathcal{Y}; \mathcal{K})$ denote the space of square-integrable measurable functions from $\mathcal{Y}$ to $\mathcal{K}$. We call a measurable function $p\colon \mathcal{Y} \times \mathcal{Y} \to [0, \infty)$ a *symmetric Markov kernel* if $p(x, y) = p(y, x)$ for all $x, y \in \mathcal{Y}$ and $\int p(x, y)\, \mathrm{d}\mu(y) = 1$ for all $x \in \mathcal{Y}$. For such a kernel, we define

$$\Pi f(x) = \int p(x, y) f(y)\, \mathrm{d}\mu(y). \tag{A.12}$$

Using the symmetry and the normalization of p, the Schur test shows that Π is a self-adjoint operator on $L^2(\mathcal{Y}; \mathcal{K})$ with $\|\Pi\| \leq 1$. We call a family $(Y(x, y))_{x,y \in \mathcal{Y}}$ of bounded operators on $\mathcal{K}$ *strongly measurable* if $(x, y) \mapsto Y(x, y)e$ is measurable for all $e \in \mathcal{K}$.

Lemma A.3. *Let p be a symmetric Markov kernel on $\mathcal{Y}$, and let $(R(x, y))_{x,y \in \mathcal{Y}}$ be a strongly measurable family of bounded operators on $\mathcal{K}$. Let $c_R \geq 0$ be such that*

$$\|R(x, y)\| \leq c_R \qquad \text{for all } x, y \in \mathcal{Y}. \tag{A.13}$$

Define the operator T on $L^2(\mathcal{Y}; \mathcal{K})$ and the bounded operators $R_{\mathrm{marg}}(x)$ on $\mathcal{K}$ by

$$Tf(x) = \int p(x, y) R(x, y) f(y)\, \mathrm{d}\mu(y), \qquad R_{\mathrm{marg}}(x) = \int p(x, y) R(x, y)\, \mathrm{d}\mu(y), \tag{A.14}$$

where the second integral is taken strongly. We call R_{marg} the marginal *of the kernel, and we let R_{marg} also denote the multiplication by $R_{\mathrm{marg}}(x)$ on $L^2(\mathcal{Y}; \mathcal{K})$.*

(a) Suppose that $R(x, y) = R(y, x)$ for all $x, y \in \mathcal{Y}$. Then for all $f, g \in L^2(\mathcal{Y}; \mathcal{K})$,

$$\langle g, (T - R_{\mathrm{marg}}) f \rangle = -\frac{1}{2} \int_{\mathcal{Y}^2} p(x, y) \big\langle g(x) - g(y), R(x, y)\big(f(x) - f(y)\big)\big\rangle\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y) \tag{A.15}$$

and

$$|\langle g, (T - R_{\mathrm{marg}}) f \rangle| \leq c_R \langle g, (\mathrm{Id} - \Pi) g \rangle^{1/2} \langle f, (\mathrm{Id} - \Pi) f \rangle^{1/2}. \tag{A.16}$$

If moreover $\Pi \geq 0$, then for all $f \in L^2(\mathcal{Y}; \mathcal{K})$,

$$\langle f, (\mathrm{Id} - \Pi) f \rangle \leq \|(\mathrm{Id} - \Pi^2)^{1/2} f\|^2. \tag{A.17}$$

(b) Transported kernels. *Let $(\tau_{xy})_{x,y \in \mathcal{Y}}$ be a strongly measurable family of unitary operators on $\mathcal{K}$ such that $\tau_{yx} = \tau_{xy}^*$ for all x, y, and suppose that $R(y, x) = \tau_{yx} R(x, y) \tau_{xy}$ for all x, y.*

Define T_τ and Π_τ by replacing $f(y)$ with $\tau_{xy}f(y)$ in the definitions of T and Π. Then part (a) holds for T_τ and Π_τ in place of T and Π, with $f(x) - f(y)$ and $g(x) - g(y)$ replaced by $f(x) - \tau_{xy}f(y)$ and $g(x) - \tau_{xy}g(y)$ in (A.15).

Suppose that the operators $R(x,y)$ act on the fibers $\mathcal{K}_x$ of a Hilbert bundle over $\mathcal{Y}$, with unitary transports from $\mathcal{K}_y$ to $\mathcal{K}_x$, and that the fibers are identified with $\mathcal{K}$ by a measurable choice of unitary frames. Then the transports become operators τ_{xy} as in part (b). This is the form in which part (b) is applied.

Proof. We first check that all integrals converge absolutely. By (A.13), we have

$$\int p(x,y)\|R(x,y)\|\,\mathrm{d}\mu(y) \le c_R \qquad \text{for all } x \in \mathcal{Y}, \tag{A.18}$$

and the same bound holds for the integral in x, since p is symmetric. By the Schur test, we have $\|T\| \le c_R$, and (A.18) also implies that $\|R_{\mathrm{marg}}(x)\| \le c_R$ for all x. Moreover,

$$\int_{\mathcal{Y}^2} p(x,y)\|R(x,y)\|\,\|g(x)\|\big(\|f(x)\| + \|f(y)\|\big)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \le 2c_R\|g\|\,\|f\|,$$

which justifies the uses of Fubini's theorem below. Since the operators τ_{xy} in part (b) are unitary, the same bounds hold for the kernel $p(x,y)R(x,y)\tau_{xy}$ of T_τ.

We now prove part (a). By the definitions of T and R_{marg}, we have

$$\langle g, (T - R_{\mathrm{marg}})f\rangle = \int_{\mathcal{Y}^2} p(x,y)\big\langle g(x), R(x,y)\big(f(y) - f(x)\big)\big\rangle\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y).$$

We exchange the names of x and y. Using $p(x,y) = p(y,x)$ and $R(x,y) = R(y,x)$, we see that the same integral equals

$$\int_{\mathcal{Y}^2} p(x,y)\big\langle g(y), R(x,y)\big(f(x) - f(y)\big)\big\rangle\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y).$$

Averaging the two expressions, we obtain (A.15).

Next, we prove (A.16). For $f \in L^2(\mathcal{Y};\mathcal{K})$, define

$$\mathcal{E}_p(f) = \frac{1}{2}\int_{\mathcal{Y}^2} p(x,y)\|f(x) - f(y)\|^2\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y). \tag{A.19}$$

Expanding the square and using the symmetry and the normalization of p and the self-adjointness of Π, we find

$$\mathcal{E}_p(f) = \|f\|^2 - \mathrm{Re}\langle f, \Pi f\rangle = \langle f, (\mathrm{Id} - \Pi)f\rangle. \tag{A.20}$$

By the Cauchy–Schwarz inequality, (A.15) implies

$$\begin{aligned}|\langle g, (T - R_{\mathrm{marg}})f\rangle| &\le \frac{c_R}{2}\int_{\mathcal{Y}^2} p(x,y)\|g(x) - g(y)\|\,\|f(x) - f(y)\|\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\\ &\le c_R\,\mathcal{E}_p(g)^{1/2}\mathcal{E}_p(f)^{1/2},\end{aligned}$$

which is (A.16). If $\Pi \geq 0$, then $0 \leq \Pi \leq \mathrm{Id}$, since $\|\Pi\| \leq 1$. The spectral theorem then yields $\Pi^2 \leq \Pi$, which is (A.17).

We now prove part (b). We have

$$\langle g, (T_\tau - R_{\mathrm{marg}})f \rangle = \int_{\mathcal{Y}^2} p(x,y) \big\langle g(x), R(x,y)\big(\tau_{xy}f(y) - f(x)\big)\big\rangle \, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y).$$

We again exchange the names of x and y. Recall that $\tau_{yx} = \tau_{xy}^*$, so that $\tau_{yx} = \tau_{xy}^{-1}$. Using also $R(y,x) = \tau_{yx} R(x,y) \tau_{xy}$, we find

$$\big\langle g(y), R(y,x)\big(\tau_{yx}f(x) - f(y)\big)\big\rangle = \big\langle \tau_{xy}g(y), R(x,y)\big(f(x) - \tau_{xy}f(y)\big)\big\rangle.$$

Averaging the two expressions gives (A.15) with the transported differences.

To prove the transported versions of (A.16) and (A.17), let $\mathcal{E}_{p,\tau}(f)$ be defined by (A.19), with $f(x) - \tau_{xy}f(y)$ in place of $f(x) - f(y)$. The operator Π_τ is self-adjoint, since p is symmetric and $\tau_{yx} = \tau_{xy}^*$, and the Schur test gives $\|\Pi_\tau\| \leq 1$. The expansion that gave (A.20), together with $\|\tau_{xy}f(y)\| = \|f(y)\|$, yields

$$\mathcal{E}_{p,\tau}(f) = \langle f, (\mathrm{Id} - \Pi_\tau) f \rangle,$$

and the rest of the proof is unchanged. □

A.3 Poisson transforms of translation-bounded measures

For $a > 0$, let $P_a(u) = a/(\pi(u^2 + a^2))$ denote the Poisson kernel; for $a = \varepsilon$, this is the kernel P_ε of Section 1. For a positive Radon measure ν on $\mathbb{R}$, we set

$$\|\nu\|_{\mathrm{loc}} = \sup_{s \in \mathbb{R}} \nu([s, s+1]), \tag{A.21}$$

and when $\|\nu\|_{\mathrm{loc}} < \infty$, we define the *Poisson transform* $P_a * \nu$ of ν by

$$(P_a * \nu)(u) = \int P_a(u - r)\, \nu(\mathrm{d}r), \qquad u \in \mathbb{R}.$$

This integral is finite by part (a) of the following lemma, which is used to identify limiting pair and occupation measures from their Poisson transforms.

Lemma A.4. *Let ν be a positive Radon measure on $\mathbb{R}$ with $\|\nu\|_{\mathrm{loc}} < \infty$.*

(a) For all $a > 0$, $R \geq 1$, and $u \in \mathbb{R}$ with $|u| \leq R/2$,

$$\int_{|r| > R} P_a(u - r)\, \nu(\mathrm{d}r) \leq \frac{16\, a\, \|\nu\|_{\mathrm{loc}}}{\pi R}.$$

(b) For all $f \in C_c(\mathbb{R})$,

$$\lim_{a \downarrow 0} \int f(u)(P_a * \nu)(u)\,\mathrm{d}u = \int f\,\mathrm{d}\nu.$$

In particular, ν is determined by the functions $P_a * \nu$ along every sequence $a \downarrow 0$.

(c) For all $u \in \mathbb{R}$,

$$\nu(\{u\}) = \lim_{a \downarrow 0} \pi a\,(P_a * \nu)(u).$$

(d) Let $(\nu_n)_{n\geq 1}$ be positive Radon measures on $\mathbb{R}$ such that $\sup_n \|\nu_n\|_{\mathrm{loc}} \leq C_0$, and suppose that there exists a function ϑ on $(0,\infty) \times \mathbb{R}$ such that $(P_a * \nu_n)(u) \to \vartheta(a,u)$ for all $a > 0$ and $u \in \mathbb{R}$. Then ν_n converges vaguely to a positive Radon measure ν with $\|\nu\|_{\mathrm{loc}} \leq 2C_0$ and $P_a * \nu = \vartheta(a,\cdot)$ for all $a > 0$.

Proof. We begin with part (a). Let $|r| > R$. Since $R \geq 2|u|$, we have

$$|u - r| \geq |r| - |u| \geq |r|/2,$$

and therefore

$$P_a(u - r) \leq \frac{a}{\pi(u-r)^2} \leq \frac{4a}{\pi r^2}. \tag{A.22}$$

Covering (R,∞) by the intervals $[R+k, R+k+1]$ with $k \geq 0$, we have

$$\int_{r>R} \frac{\nu(\,\mathrm{d}r)}{r^2} \leq \|\nu\|_{\mathrm{loc}} \sum_{k=0}^{\infty} \frac{1}{(R+k)^2} \leq \|\nu\|_{\mathrm{loc}} \Big(\frac{1}{R^2} + \frac{1}{R}\Big) \leq \frac{2\|\nu\|_{\mathrm{loc}}}{R}, \tag{A.23}$$

where we used $R \geq 1$ in the last step. The same bound holds for the integral over $r < -R$. Combining (A.22) and (A.23) proves part (a).

Next, we prove part (b). Let $\operatorname{supp} f \subset [-R_0, R_0]$, and set $R_1 = 2R_0 + 2$. If $|r| \geq R_1$ and $|u| \leq R_0$, then $|r - u| \geq |r|/2$, which implies that

$$|(P_a * f)(r)| \leq \|f\|_{L^1} \sup_{|u|\leq R_0} P_a(r-u) \leq \frac{4a\|f\|_{L^1}}{\pi r^2}. \tag{A.24}$$

The bound (A.23) shows that the right side of (A.24), with $|f|$ in place of f, is integrable against ν over $|r| > R_1$. Therefore Fubini's theorem applies, and we obtain

$$\int f(u)(P_a * \nu)(u)\,\mathrm{d}u = \int (P_a * f)\,\mathrm{d}\nu. \tag{A.25}$$

We now let $a \downarrow 0$ in (A.25). As $a \downarrow 0$, we have $P_a * f \to f$ uniformly on $\mathbb{R}$, since f is bounded and uniformly continuous and $(P_a)_{a>0}$ is an approximate identity. Since $\nu([-R_1, R_1]) < \infty$, this implies

$$\lim_{a \downarrow 0} \int_{[-R_1,R_1]} (P_a * f - f)\,\mathrm{d}\nu = 0.$$

Outside $[-R_1, R_1]$, the function f vanishes. By (A.24) and by (A.23) with $R = R_1$, applied on both sides of the origin, we have

$$\Big| \int_{|r|>R_1} (P_a * f) \, \mathrm{d}\nu \Big| \le \frac{4a\|f\|_{L^1}}{\pi} \int_{|r|>R_1} \frac{\nu(\,\mathrm{d}r)}{r^2} \le \frac{16a\|f\|_{L^1}\|\nu\|_{\mathrm{loc}}}{\pi R_1},$$

which tends to zero as $a \downarrow 0$. This proves the limit in part (b). The second claim of part (b) follows, since a positive Radon measure is determined by its integrals against $C_c(\mathbb{R})$.

We now prove part (c). The functions

$$\pi a P_a(u-r) = \frac{a^2}{a^2 + (u-r)^2}$$

are bounded by one and decrease to $\mathbf{1}_{\{r=u\}}$ as $a \downarrow 0$. Dominated convergence on $[u-1, u+1]$ shows that

$$\lim_{a \downarrow 0} \int_{[u-1,u+1]} \pi a P_a(u-r) \, \nu(\,\mathrm{d}r) = \nu(\{u\}).$$

Outside this interval, the covering argument of part (a) gives

$$\int_{|r-u|>1} \pi a P_a(u-r) \, \nu(\,\mathrm{d}r) \le a^2 \int_{|r-u|>1} \frac{\nu(\,\mathrm{d}r)}{(u-r)^2} \le 4a^2 \|\nu\|_{\mathrm{loc}},$$

which tends to zero.

Finally, we prove part (d). Note that $\nu_n([-m, m]) \le 2mC_0$ for all $m \ge 1$ and n. By Helly's selection theorem and a diagonal argument, every subsequence of (ν_n) has a further subsequence that converges vaguely to a positive Radon measure. Let ν be such a limit along a subsequence $(\nu_{n'})$. For every open interval G, we have

$$\nu(G) \le \liminf_{n' \to \infty} \nu_{n'}(G).$$

Applying this bound with $G = (s - \gamma, s + 1 + \gamma)$, where $s \in \mathbb{R}$ and $0 < \gamma < 1/2$, we find

$$\nu([s, s+1]) \le \nu((s-\gamma, s+1+\gamma)) \le \liminf_{n'} \nu_{n'}([s-\gamma, s+1+\gamma]) \le 2C_0,$$

so that $\|\nu\|_{\mathrm{loc}} \le 2C_0$.

It remains to identify the Poisson transform of ν. Fix $a > 0$ and $u \in \mathbb{R}$, and let $R \ge \max(1, 2|u|)$. Let $\chi \in C_c(\mathbb{R})$ satisfy $0 \le \chi \le 1$, $\chi = 1$ on $[-R, R]$ and $\operatorname{supp} \chi \subset [-R-1, R+1]$. By vague convergence, we have

$$\lim_{n' \to \infty} \int P_a(u-r)\chi(r) \, \nu_{n'}(\,\mathrm{d}r) = \int P_a(u-r)\chi(r) \, \nu(\,\mathrm{d}r).$$

By part (a), with $\|\nu_{n'}\|_{\mathrm{loc}} \le C_0$ and $\|\nu\|_{\mathrm{loc}} \le 2C_0$, we have

$$\int P_a(u-r)(1-\chi(r))\,\nu_{n'}(\,\mathrm{d}r) \le \frac{16aC_0}{\pi R}, \qquad \int P_a(u-r)(1-\chi(r))\,\nu(\,\mathrm{d}r) \le \frac{32aC_0}{\pi R}.$$

Letting $n' \to \infty$ and then $R \to \infty$, we conclude that $(P_a * \nu)(u) = \vartheta(a,u)$. By part (b), all subsequential vague limits therefore coincide, and the full sequence converges vaguely to ν. □

A.4 A submean inequality for holomorphic families

We say that a function f from an open set $\mathcal{U} \subset \mathbb{C}^n$ to a Hilbert space $\mathcal{H}$ is *holomorphic* if $\zeta \mapsto \lambda(f(\zeta))$ is holomorphic on $\mathcal{U}$ for every continuous complex-linear functional λ on $\mathcal{H}$. With the inner product linear in its first argument, this is equivalent to requiring $\zeta \mapsto \langle f(\zeta), e\rangle$ to be holomorphic for all $e \in \mathcal{H}$. Let $\mathrm{d}^{2n}\zeta$ denote Lebesgue measure on $\mathbb{C}^n = \mathbb{R}^{2n}$, let $|S|$ denote the Lebesgue measure of a set $S \subset \mathbb{C}^n$, and let $B(\zeta, r)$ denote the open Euclidean ball of radius r about ζ. Recall that $|B(\zeta,r)| = \pi^n r^{2n}/n!$.

Lemma A.5. *Let $\mathcal{U} \subset \mathbb{C}^n$ be open, let $\mathcal{H}$ be a separable Hilbert space, and let $(\mathcal{Y}, m)$ be a σ-finite measure space. Let $f\colon \mathcal{U}\times\mathcal{Y} \to \mathcal{H}$ be measurable and such that $\zeta \mapsto f(\zeta,y)$ is holomorphic on $\mathcal{U}$ for all $y \in \mathcal{Y}$. Let $\zeta_*\colon \mathcal{Y} \to \mathcal{U}$ be measurable, and let $r > 0$ be such that $B(\zeta_*(y), r) \subset \mathcal{U}$ for all $y \in \mathcal{Y}$. Then*

$$\int_{\mathcal{Y}} \|f(\zeta_*(y),y)\|^2\, m(\,\mathrm{d}y) \le \frac{n!}{\pi^n r^{2n}} \int_{\mathcal{Y}} \int_{B(\zeta_*(y),r)} \|f(\zeta,y)\|^2\, \mathrm{d}^{2n}\zeta\, m(\,\mathrm{d}y). \tag{A.26}$$

Proof. Fix $y \in \mathcal{Y}$ and an orthonormal basis $(e_k)_{k\ge1}$ of $\mathcal{H}$, and define the holomorphic functions

$$f_k(\zeta) = \langle f(\zeta,y), e_k\rangle, \qquad \zeta \in \mathcal{U}.$$

Their real and imaginary parts are harmonic on $\mathcal{U} \subset \mathbb{R}^{2n}$. Let $0 < r' < r$, and write $B' = B(\zeta_*(y), r')$ and $B = B(\zeta_*(y), r)$, so that the closure of B' lies in $\mathcal{U}$. The mean value property of harmonic functions and the Cauchy–Schwarz inequality give

$$|f_k(\zeta_*(y))|^2 = \Big|\frac{1}{|B'|}\int_{B'} f_k(\zeta)\,\mathrm{d}^{2n}\zeta\Big|^2 \le \frac{1}{|B'|}\int_{B'} |f_k(\zeta)|^2\,\mathrm{d}^{2n}\zeta \le \frac{1}{|B'|}\int_{B} |f_k(\zeta)|^2\,\mathrm{d}^{2n}\zeta.$$

Summing over k and using the monotone convergence theorem, we obtain

$$\|f(\zeta_*(y),y)\|^2 \le \frac{1}{|B'|}\int_B \|f(\zeta,y)\|^2\,\mathrm{d}^{2n}\zeta.$$

Letting $r' \uparrow r$ gives (A.26) without the integral over y.

It remains to integrate in y, and for this we check that the two sides are measurable functions of y. The left side is the composition of f with the measurable map $y \mapsto$

$(\zeta_*(y), y)$. After the change of variables $\zeta = \zeta_*(y) + w$, the right side is the integral over $w \in B(0,r)$ of the jointly measurable function $(w,y) \mapsto \|f(\zeta_*(y)+w, y)\|^2$, and Tonelli's theorem shows that it is measurable in y. Integrating in y completes the proof. □

Remark A.6. The ball in Lemma A.5 may be replaced by an ellipsoid. Let A be an invertible complex-linear map of $\mathbb{C}^n$, and suppose that $\zeta_*(y) + AB(0,r) \subset \mathcal{U}$ for all y. We apply Lemma A.5 to the function $(w,y) \mapsto f(Aw, y)$ on $A^{-1}\mathcal{U}$, with the center $A^{-1}\zeta_*(y)$ and the same r. The change of variables $\zeta = Aw$ multiplies Lebesgue measure by $|\det A|^2$, which is also the ratio of $|AB(0,r)|$ to $|B(0,r)|$. We obtain

$$\int_{\mathcal{Y}} \|f(\zeta_*(y), y)\|^2 \, m(\,\mathrm{d}y) \le \frac{1}{|AB(0,r)|} \int_{\mathcal{Y}} \int_{\zeta_*(y)+AB(0,r)} \|f(\zeta, y)\|^2 \, \mathrm{d}^{2n}\zeta \, m(\,\mathrm{d}y). \tag{A.27}$$

The same argument applies when $A = A(y)$ depends measurably on y, since the proof of Lemma A.5 is pointwise in y before the final integration.

APPENDIX B

The product contour

In this appendix, we prove the contour statements used in Section 4. These are the Gaussian integral (4.7) on the flat cycle, the deformation of Proposition 4.8 onto the translated product of Schäfer–Wegner cycles, the facts about the heat kernel in Lemma 4.10, and the deformation used in the proof of Proposition 4.13. In Appendix B.8, we also bound the source factor of the onsite weight on the saddle manifold, which Part IV uses. As explained after Proposition 4.8, the earlier results on the Schäfer–Wegner domain concern a single matrix variable, and the decay in the noncompact directions comes there from the coupling to the vector variables [139] or from the condition $As > 0$ on the source [60, 96]. Here we need N matrix variables coupled by a nearest-neighbor Laplacian, with a source of rank one at each site, and the decay must come from the damping, which translates the cycle. The new ingredients are the identity (B.6) for the link terms, which shows that the map $X \mapsto f_0(X)$ is a contraction, and the bound of Lemma B.4, which is uniform along a deformation that ends on the translated cycle. All statements concern integrals of fixed finite dimension, and the constants may depend on all the parameters. The only delicate point is the integrability along the deformations, since on the untranslated cycle the Gaussian weight does not decay in the noncompact directions; the rest of the argument is elementary.

We use the notation of Section 4, in particular $f_0(X)$ and $\mathfrak{g}_X$ of (4.11) and their properties in Lemma 4.7. We let $\mathcal{S}^{\mathrm{B}}$ denote the set of matrices $P - \mathrm{i}\Lambda_{\mathrm{B}}(f_0(X) + X)$ with $P \in \mathrm{Herm}_{\mathrm{d}}(2k)$ and $X \in \mathrm{Herm}_{\mathrm{o}}(2k)$, which is the set of BB blocks of the cycle $\mathcal{S}$.

B.1 A Stokes lemma for real cycles

The deformations of this appendix move the BB and FF blocks of the integration variables in complex space and leave the odd variables unchanged. Each coefficient of the integrand, in the odd variables, is then an ordinary integral of a holomorphic function over a real cycle of half dimension. We will use the following form of Cauchy's theorem.

Lemma B.1. *Let $d \geq 1$, let F be an entire function on $\mathbb{C}^d$, and let $(t, x) \mapsto \mathbf{z}_t(x)$ be a continuously differentiable map from $[0,1] \times \mathbb{R}^d$ to $\mathbb{C}^d$. Let $D\mathbf{z}_t(x)$ denote the complex $d \times d$ matrix of partial derivatives of $\mathbf{z}_t$ in x, and set $j_t(x) = \det D\mathbf{z}_t(x)$. Suppose that there exists*

$F^\sharp \in L^1(\mathbb{R}^d)$ *such that, for all* $t \in [0,1]$ *and* $x \in \mathbb{R}^d$,

$$|F(\mathbf{z}_t(x))|\big(|j_t(x)| + |\partial_t \mathbf{z}_t(x)|\,\|D\mathbf{z}_t(x)\|_{\mathrm{op}}^{d-1}\big) \le F^\sharp(x). \tag{B.1}$$

Then $\int_{\mathbb{R}^d} F(\mathbf{z}_t(x)) j_t(x)\,\mathrm{d}x$ *does not depend on* $t \in [0,1]$.

Proof. The holomorphic d-form $F(z)\,\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_d$ on $\mathbb{C}^d$ is closed, and so is its pullback by $(t,x) \mapsto \mathbf{z}_t(x)$ to $[0,1] \times \mathbb{R}^d$. We apply Stokes' theorem to this pullback on $[0,1] \times \{|x| \le r\}$, where $r > 0$, and we obtain

$$\int_{|x|\le r} F(\mathbf{z}_1) j_1\,\mathrm{d}x - \int_{|x|\le r} F(\mathbf{z}_0) j_0\,\mathrm{d}x = \pm \int_{[0,1]\times\{|x|=r\}} \mathbf{z}^*\big(F\,\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_d\big).$$

Restricted to $[0,1] \times \{|x| = r\}$, the pulled-back form is the product of $F(\mathbf{z}_t)$ and a sum of $d \times d$ minors of the derivative of $(t,x) \mapsto \mathbf{z}_t(x)$ that contain the column $\partial_t \mathbf{z}_t$. The density of this form with respect to $\mathrm{d}t$ and the surface measure is therefore bounded by

$$C_d |F(\mathbf{z}_t)|\,|\partial_t \mathbf{z}_t|\,\|D\mathbf{z}_t\|_{\mathrm{op}}^{d-1} \le C_d F^\sharp.$$

Let σ_r denote the surface measure on the sphere $\{|x| = r\}$. Since

$$\int_0^\infty \int_{\{|x|=r\}} F^\sharp\,\mathrm{d}\sigma_r\,\mathrm{d}r = \int_{\mathbb{R}^d} F^\sharp\,\mathrm{d}x < \infty,$$

there exist $r_j \to \infty$ such that the integral of $F^\sharp$ over the sphere $\{|x| = r_j\}$ tends to zero. Letting $r = r_j \to \infty$ and using dominated convergence for the two integrals on the left, we find the claim. □

We apply Lemma B.1 to the BB and FF blocks of N supermatrices, with $d = 8k^2N$. We use the real coordinates of Definition 4.5 on $\mathbb{R}^d$. On $\mathbb{C}^d$, we use at each site the complex coordinates introduced after Definition 4.6, in which a BB block takes the form $P^{\mathbb{C}} - \mathrm{i}\Lambda_{\mathrm{B}} X^{\mathbb{C}}$, with $P^{\mathbb{C}}$ complex block diagonal and $X^{\mathbb{C}}$ complex block off-diagonal, and an FF block the form $\mathrm{i}M^{\mathbb{C}}$. These complex coordinates restrict to the real coordinates on the flat cycle $\mathcal{S}_\flat$, so that $j_t = 1$ there. A superfunction whose coefficients are entire functions of the BB and FF entries is integrated over a deformed cycle coefficientwise, and the integral of Definition 4.6 is the case in which $\mathbf{z}_t$ parametrizes $\mathcal{S}$.

B.2 Gaussian integrals on the flat cycle

We first prove (4.7). This is the Gaussian Fourier transform on flat superspace, and we include the short proof because Proposition 4.11 and Proposition 4.13 use the fact that its constant equals one. For families $R = (R_i)_{i\in[N]}$ and $B = (B_i)_{i\in[N]}$ of even $(2k|2k)$ supermatrices and a real symmetric $N \times N$ matrix $\mathbf{M}$, we set

$$\langle R, \mathbf{M}B\rangle = \sum_{i,j=1}^{N} \mathbf{M}_{ij}\,\mathrm{STr}(R_i B_j).$$

This form is symmetric, since $\mathrm{STr}(R_iB_j) = \mathrm{STr}(B_jR_i)$ for even supermatrices by Lemma 3.11(c).

Lemma B.2. *Let* $\mathbf{M}$ *be a positive definite real symmetric* $N \times N$ *matrix, and let* $B_1, \dots, B_N$ *be even supermatrices whose entries lie in auxiliary generators, with complex bodies. Then*

$$\int_{\mathcal{S}_\flat^N} e^{-\langle R, \mathbf{M}R\rangle/2 - \mathrm{i}\langle R, B\rangle} \, \mathrm{d}R = e^{-\langle B, \mathbf{M}^{-1}B\rangle/2}.$$

The same identity holds with $\mathcal{S}_\flat^N$ *replaced by its translate by any family* $(C_i)_{i\in[N]}$ *of supermatrices with complex entries and vanishing odd blocks.*

Proof. We first take $B = 0$. For $\mathbf{M} = \mathrm{Id}$, the identity is the N-fold product of (4.6).

For general $\mathbf{M}$, we substitute

$$R = (\mathbf{M}^{-1/2} \otimes \mathrm{Id})R',$$

which is a real linear change of the even coordinates and a linear change of the odd coordinates, given in each case by the matrix $\mathbf{M}^{-1/2}$ acting on the site index. This substitution turns $\langle R, \mathbf{M}R\rangle$ into $\langle R', R'\rangle$. Since each site carries $8k^2$ even and $8k^2$ odd coordinates, the Lebesgue integral over the even coordinates gains the factor $\det(\mathbf{M})^{-4k^2}$, and by Lemma 3.6(c) the Berezin integral over the odd coordinates gains the factor $\det(\mathbf{M})^{4k^2}$. The two factors cancel, as in the direct computation after (4.7).

For general B, we complete the square,

$$-\frac{1}{2}\langle R, \mathbf{M}R\rangle - \mathrm{i}\langle R, B\rangle = -\frac{1}{2}\big\langle R + \mathrm{i}\mathbf{M}^{-1}B, \mathbf{M}(R + \mathrm{i}\mathbf{M}^{-1}B)\big\rangle - \frac{1}{2}\langle B, \mathbf{M}^{-1}B\rangle.$$

The shift $R \mapsto R + \mathrm{i}\mathbf{M}^{-1}B$ adds odd elements to the odd entries, even nilpotent elements to the even entries, and complex numbers to the even entries. Recall that on $\mathcal{S}_\flat$,

$$\langle R, \mathbf{M}R\rangle_\varnothing = \sum_{i,j} \mathbf{M}_{ij}\big(\mathrm{Tr}(R_i^{\mathrm{BB}}R_j^{\mathrm{BB}}) + \mathrm{Tr}(M_iM_j)\big),$$

so that the coefficients of the Gaussian are polynomials times $e^{-\langle R, \mathbf{M}R\rangle_\varnothing/2}$, and all their derivatives are integrable. By Lemma 3.6(d) and (e), the first two shifts do not change the integral.

For the shift by a family c of complex constants, we apply Lemma B.1 with $\mathbf{z}_t(x) = x + tc$, coefficientwise. Here $j_t = 1$ and $|\partial_t \mathbf{z}_t| = |c|$. In the notation $|\cdot|_\varnothing$ of the proof of Proposition 4.11 for the modulus of the body, we have

$$\left|e^{-\langle R, \mathbf{M}R\rangle/2}\right|_\varnothing \le e^{-\lambda_{\min}(\mathbf{M})|x|^2/2 + C|x| + C} \qquad \text{at } R = x + tc,$$

uniformly in $t \in [0,1]$. The other coefficients are bounded by polynomials times the right side, and (B.1) holds. Together with the case $B = 0$, this implies the identity. The last assertion follows in the same way from the shift by C. □

B.3 Real parts along the deformation

We now compute the real parts of the Gaussian terms along the deformations. For $c \geq 0$, $t \in [0,1]$, $P \in \mathrm{Herm_d}(2k)$, and $X \in \mathrm{Herm_o}(2k)$, we set

$$R_t(P,X) = P - \mathrm{i}\Lambda_{\mathrm{B}}\big((1-t)(f_0(X)-c)+X\big). \tag{B.2}$$

Thus $R_1(P,X) = P - \mathrm{i}\Lambda_{\mathrm{B}}X$ is Hermitian, and $R_0(P,X)$ is the point of $\mathcal{S}^{\mathrm{B}} + \mathrm{i}c\Lambda_{\mathrm{B}}$ with coordinates (P,X). With $c = \eta$, the matrices $R_t(P,X)$ are the BB blocks of the cycle $\mathcal{S}_{\Gamma,t}$ of Section 4.4.

Lemma B.3. *Let $c \geq 0$ and $\eta' \geq 0$, let $\zeta \in \mathbb{C}^{2k}$, and let $t \in [0,1]$.*

(a) For $m \geq 0$ and $R = R_t(P,X)$,

$$\begin{aligned}\mathrm{Re}\Big[-\frac{m}{2}\operatorname{Tr}R^2 - \mathrm{i}\operatorname{Tr}\big(R(\zeta\zeta^*+\eta')\Lambda_{\mathrm{B}}\big)\Big] \\ = -\frac{m}{2}\Big(\|P\|^2 + \|X\|^2 - (1-t)^2\|f_0(X)-c\|^2\Big) \\ - \zeta^*\big((1-t)(f_0(X)-c)+X\big)\zeta - (1-t)\eta'\big(\operatorname{Tr}f_0(X) - 2kc\big).\end{aligned} \tag{B.3}$$

If $m > 0$ and $\bar\eta = mc + \eta' > 0$, the right side is at most

$$-\frac{m}{2}\|P\|^2 - \frac{\bar\eta}{4}\operatorname{Tr}f_0(X) + mk(s_E^2+c^2) + 2k\eta'c + (s_E+c)|\zeta|^2 + \frac{|\zeta|^4}{m} + \frac{k\bar\eta s_E}{2} + \frac{k\bar\eta^2}{6m}. \tag{B.4}$$

(b) For $w \geq 0$, $R = R_t(P,X)$, and $R' = R_t(P',X')$,

$$\mathrm{Re}\Big[-\frac{w}{2}\operatorname{Tr}(R-R')^2\Big] = -\frac{w}{2}\Big(\|P-P'\|^2 + \|X-X'\|^2 - (1-t)^2\|f_0(X)-f_0(X')\|^2\Big), \tag{B.5}$$

and

$$\|X-X'\|^2 - \|f_0(X)-f_0(X')\|^2 = s_E^2\operatorname{Tr}\big(\mathrm{g}_X^2\mathrm{g}_{X'}^{-2} + \mathrm{g}_X^{-2}\mathrm{g}_{X'}^2 - 2\mathrm{Id}\big) \geq 0. \tag{B.6}$$

In particular, the right side of (B.5) *is at most $-\frac{w}{2}\|P-P'\|^2$, and $\|f_0(X)-f_0(X')\| \leq \|X-X'\|$.*

(c) Let $c = 0$, $w \geq 0$, and $R_* = P_* - \mathrm{i}\Lambda_{\mathrm{B}}(f_0(X_*)+X_*) \in \mathcal{S}^{\mathrm{B}}$. Then for $R = R_t(P,X)$,

$$\mathrm{Re}\Big[-\frac{w}{2}\operatorname{Tr}(R-R_*)^2\Big] \leq -\frac{w}{2}\|P-P_*\|^2 - \frac{ws_E}{2\|\mathrm{g}_{X_*}^2\|_{\mathrm{op}}}\operatorname{Tr}f_0(X) + 2wks_E^2 + \frac{w}{2}\|f_0(X_*)\|^2.$$

Proof. We begin with (B.3). We set

$$Z = (1-t)(f_0(X)-c)+X,$$

which is Hermitian, so that $R = P - \mathrm{i}\Lambda_{\mathrm{B}} Z$. As in the proof of Lemma 4.7(c), the trace $\operatorname{Tr}(P\Lambda_{\mathrm{B}} Z)$ is real, and

$$\Lambda_{\mathrm{B}} Z \Lambda_{\mathrm{B}} = (1-t)(f_0(X) - c) - X,$$

since $f_0(X)$ is block diagonal and X block off-diagonal. We obtain

$$\operatorname{Re}\operatorname{Tr} R^2 = \|P\|^2 - \operatorname{Tr}(\Lambda_{\mathrm{B}} Z \Lambda_{\mathrm{B}} Z) = \|P\|^2 + \|X\|^2 - (1-t)^2 \|f_0(X) - c\|^2.$$

Moreover, $\operatorname{Tr}(P(\zeta\zeta^* + \eta')\Lambda_{\mathrm{B}})$ is real, since $\Lambda_{\mathrm{B}} P$ and $\zeta\zeta^* + \eta'$ are Hermitian, and

$$\operatorname{Tr}\left(\Lambda_{\mathrm{B}} Z(\zeta\zeta^* + \eta')\Lambda_{\mathrm{B}}\right) = \zeta^* Z \zeta + \eta' \operatorname{Tr} Z, \qquad \operatorname{Tr} Z = (1-t)(\operatorname{Tr} f_0(X) - 2kc).$$

Combining the last two displays yields (B.3).

We next prove (B.4), and we begin with the Gaussian term. Since $f_0(X)^2 = s_E^2 + X^2$, we have

$$\|f_0(X) - c\|^2 = 2ks_E^2 + \|X\|^2 - 2c \operatorname{Tr} f_0(X) + 2kc^2,$$

and the Gaussian term in (B.3) becomes

$$-\frac{m}{2}\|P\|^2 - \frac{m}{2} t(2-t)\|X\|^2 + (1-t)^2 mk(s_E^2 + c^2) - (1-t)^2 mc \operatorname{Tr} f_0(X).$$

We now bound the source term. We write

$$Z = \big(f_0(X) + X\big) - t f_0(X) - (1-t)c.$$

Using $f_0(X) + X > 0$, $\|f_0(X)\|_{\mathrm{op}} \le s_E + \|X\|$, and Young's inequality, we find

$$-\zeta^* Z \zeta \le t(s_E + \|X\|)|\zeta|^2 + c|\zeta|^2 \le (s_E + c)|\zeta|^2 + \frac{mt}{4}\|X\|^2 + \frac{|\zeta|^4}{m}.$$

We now combine the two bounds. Since $t \le t(2-t)$, the sum of the terms with $\|X\|^2$ and $\operatorname{Tr} f_0(X)$ is bounded by

$$-\frac{m}{4} t(2-t)\|X\|^2 - \big((1-t)^2 mc + (1-t)\eta'\big) \operatorname{Tr} f_0(X).$$

When $t \le 1/2$, the second term is bounded by $-\frac{\bar\eta}{4} \operatorname{Tr} f_0(X)$. When $t \ge 1/2$, we have $t(2-t) \ge 3/4$. Using $\operatorname{Tr} f_0(X) \le 2ks_E + \sqrt{2k}\|X\|$ and maximizing over $\|X\|$, we find

$$\frac{\bar\eta}{4} \operatorname{Tr} f_0(X) - \frac{3m}{16}\|X\|^2 \le \frac{k\bar\eta s_E}{2} + \frac{k\bar\eta^2}{6m}.$$

In each case, the last display but one is bounded by

$$-\frac{\bar\eta}{4} \operatorname{Tr} f_0(X) + \frac{k\bar\eta s_E}{2} + \frac{k\bar\eta^2}{6m}.$$

Collecting the terms gives (B.4).

We now prove (b). The constant c cancels in $R - R'$. The computation of (a) with $m = w$, $\zeta = 0$, and $\eta' = 0$, and with $f_0(X) - f_0(X')$ and $X - X'$ in place of $f_0(X) - c$ and X, gives (B.5).

For (B.6), we use $s_E \mathrm{g}_X^2 = f_0(X) + X$ and $s_E \mathrm{g}_X^{-2} = f_0(X) - X$ to compute

$$s_E^2 \operatorname{Tr}\left(\mathrm{g}_X^2 \mathrm{g}_{X'}^{-2} + \mathrm{g}_X^{-2} \mathrm{g}_{X'}^2\right) = 2 \operatorname{Tr}\left(f_0(X) f_0(X')\right) - 2 \operatorname{Tr}(XX').$$

On the other hand, since $\|f_0(X)\|^2 = 2ks_E^2 + \|X\|^2$, we have

$$\|X - X'\|^2 - \|f_0(X) - f_0(X')\|^2 = 2 \operatorname{Tr}\left(f_0(X) f_0(X')\right) - 2 \operatorname{Tr}(XX') - 4ks_E^2,$$

and the identity follows. To see that its right side is nonnegative, note that for positive definite matrices A and A',

$$\operatorname{Tr}\left(AA'^{-1} + A^{-1}A' - 2\mathrm{Id}\right) = \sum_\lambda \left(\lambda + \lambda^{-1} - 2\right) \geq 0,$$

where λ runs over the eigenvalues of $A'^{-1/2} A A'^{-1/2}$. Finally, the right side of (B.5) is nonincreasing in t, and (B.6) bounds its value at $t = 0$ by $-\frac{w}{2}\|P - P'\|^2$.

Finally, we prove (c). As in the computation of (b), we have

$$\operatorname{Re}\Big[-\frac{w}{2} \operatorname{Tr}(R - R_*)^2 \Big] = -\frac{w}{2}\Big(\|P - P_*\|^2 + \|X - X_*\|^2 - \|(1-t) f_0(X) - f_0(X_*)\|^2 \Big).$$

Expanding the last norm, we see that this exceeds its value at $t = 0$ by

$$\begin{aligned} \frac{w}{2}\Big(-t(2-t)\|f_0(X)\|^2 + 2t \operatorname{Tr}\left(f_0(X) f_0(X_*)\right)\Big) &\leq \frac{w}{2} \sup_{y \geq 0} \left(-ty^2 + 2ty \|f_0(X_*)\| \right) \\ &= \frac{w}{2} t \|f_0(X_*)\|^2. \end{aligned}$$

The identity (B.6) shows that the value at $t = 0$ is

$$\begin{aligned} -\frac{w}{2}\|P - P_*\|^2 - \frac{w s_E^2}{2} \operatorname{Tr}\left(\mathrm{g}_X^2 \mathrm{g}_{X_*}^{-2} + \mathrm{g}_X^{-2} \mathrm{g}_{X_*}^2 - 2\mathrm{Id}\right) \\ \leq -\frac{w}{2}\|P - P_*\|^2 - \frac{w s_E^2}{2} \operatorname{Tr}\left(\mathrm{g}_X^2 \mathrm{g}_{X_*}^{-2}\right) + 2wks_E^2, \end{aligned}$$

and

$$\operatorname{Tr}\left(\mathrm{g}_X^2 \mathrm{g}_{X_*}^{-2}\right) \geq \|\mathrm{g}_{X_*}^2\|_{\mathrm{op}}^{-1} \operatorname{Tr} \mathrm{g}_X^2 = \left(s_E \|\mathrm{g}_{X_*}^2\|_{\mathrm{op}}\right)^{-1} \operatorname{Tr} f_0(X),$$

where we used $\operatorname{Tr} X = 0$. □

The main point of this appendix is the identity (B.3) at $t = 0$, which explains the role of the translation discussed after Lemma 4.7. With $c = \eta' = 0$, its right side does not decay in X at $t = 0$. On the untranslated cycle $\mathcal{S}^{\mathrm{B}}$, the Gaussian is flat in the noncompact directions, and the source $\zeta\zeta^*$ of rank one produces decay in only some of them. Either a translation $c > 0$ or a linear damping $\eta' > 0$ restores the decay, through the term $-\frac{\bar{\eta}}{4} \operatorname{Tr} f_0(X)$ in (B.4). In the open chain, the damping enters as the translation $c = \eta$, and in the anchored chain as the linear term with $\eta' = \eta$.

B.4 The deformation lemma

We now combine these computations for a finite graph. Let $n \geq 1$ and let $m_1, \dots, m_n \geq 0$ be masses. We fix edge weights $w_{ij} = w_{ji} \geq 0$ for $i \neq j$, and weights $w_{i*}, w_{i**} \geq 0$ of links to two fixed points $R_*, R_{**} \in \mathcal{S}^{\mathrm{B}}$. Fix also $c, \eta' \geq 0$ and $\zeta_1, \dots, \zeta_n \in \mathbb{C}^{2k}$. For $R = (R_1, \dots, R_n) \in (\mathbb{C}^{2k\times 2k})^n$, we set

$$\begin{aligned}\mathcal{E}(R) = &-\sum_i \frac{m_i}{2} \operatorname{Tr} R_i^2 - \sum_{i<j} \frac{w_{ij}}{2} \operatorname{Tr}(R_i - R_j)^2 \\ &- \sum_i \Big(\frac{w_{i*}}{2} \operatorname{Tr}(R_i - R_*)^2 + \frac{w_{i**}}{2} \operatorname{Tr}(R_i - R_{**})^2 \Big) \\ &- \mathrm{i} \sum_i \operatorname{Tr}\big(R_i(\zeta_i\zeta_i^* + \eta')\Lambda_{\mathrm{B}}\big).\end{aligned} \tag{B.7}$$

Lemma B.4. *Suppose that one of the following holds.*

(i) We have $m_i > 0$ for all i, $c + \eta' > 0$, and $c = 0$ if some w_{i*} or w_{i**} is positive.

(ii) We have $n = 1$, $m_1 = 0$, $\zeta_1 = 0$, $c = \eta' = 0$, and $w_{1*} > 0$.

Then there exist $C, c' > 0$, depending on all the data, such that for all $t \in [0,1]$ and all $(P_i, X_i)_{i\in[n]}$,

$$\operatorname{Re} \mathcal{E}\big((R_t(P_i, X_i))_{i\in[n]}\big) \leq C - c' \sum_{i=1}^n \big(\|P_i\|^2 + \operatorname{Tr} f_0(X_i) \big). \tag{B.8}$$

Consequently, for every polynomial p in the entries of R, the integral $\int p(R_t) e^{\mathcal{E}(R_t)} \,\mathrm{d}P \,\mathrm{d}X$ over $(\mathrm{Herm_d}(2k) \times \mathrm{Herm_o}(2k))^n$, with $R_t = (R_t(P_i, X_i))_i$, converges absolutely and does not depend on $t \in [0,1]$.

Proof. We begin with case (i). We bound the vertex terms with (B.4), where $\bar\eta_i = m_i c + \eta' > 0$, the edge terms with Lemma B.3(b), and the links to R_* and R_{**}, if present, with Lemma B.3(c), keeping only the constants. Summing these bounds, we obtain (B.8) with $c' = \min_i \min(m_i/2, \bar\eta_i/4)$.

In case (ii), the vertex term vanishes. We use Lemma B.3(c) for the link to R_*, together with

$$\|P - P_*\|^2 \geq \frac{1}{2}\|P\|^2 - \|P_*\|^2,$$

and we note that the link to R_{**} contributes at most a constant.

For the second assertion, we apply Lemma B.1 with $\mathbf{z}_t(P, X) = R_t$ and with the entire function $F = p\, e^{\mathcal{E}}$. In the complex coordinates of Appendix B.1, the map $\mathbf{z}_t$ is

$$(P, X) \mapsto \big(P - \mathrm{i}(1-t)\Lambda_{\mathrm{B}}(f_0(X) - c),\ X\big),$$

at each site. The Jacobian matrix of $\mathbf{z}_t$ is block triangular with identity blocks on the diagonal, and therefore $j_t = 1$. The bound $\|f_0(X) - f_0(X')\| \le \|X - X'\|$ of Lemma B.3(b) implies that $\|D\mathbf{z}_t\|_{\mathrm{op}} \le 2$. Moreover, there exist C and K such that for all $t \in [0,1]$,

$$|\partial_t \mathbf{z}_t| = \Big(\sum_i \|f_0(X_i) - c\|^2\Big)^{1/2} \le C\Big(1 + \sum_i \|X_i\|\Big),$$
$$|p(R_t)| \le C\Big(1 + \sum_i \big(\|P_i\| + \|X_i\|\big)\Big)^K.$$

Using (B.8) and $\mathrm{Tr}\, f_0(X) \ge \|X\|$, we conclude that (B.1) holds with an integrable $F^\sharp$. □

B.5 Proof of Proposition 4.8

Recall the cycles $\mathcal{S}_{\Gamma,t}$ of Section 4.4 and the integrand

$$F_\psi(R) = \exp\Big(-\frac{1}{2}\langle R, J^{-1}R\rangle - \mathrm{i}\sum_{i=1}^N \mathrm{STr}\,\big(R_i\psi_i\psi_i^*\Lambda\big)\Big),$$

and fix $\zeta_1, \dots, \zeta_N \in \mathbb{C}^{2k}$. We expand pF_ψ in the odd entries of the R_i and the odd coordinates χ_i, $\bar\chi_i$. By (4.4), we have $\mathrm{STr}(R_i\psi_i\psi_i^*\Lambda) = \psi_i^*\Lambda R_i\psi_i$, whose body $\zeta_i^*\Lambda_{\mathrm{B}}R_i^{\mathrm{BB}}\zeta_i$ involves only the BB block, and the odd entries appear in the exponent only through nilpotent terms. Each coefficient of pF_ψ is therefore a polynomial in the BB and FF entries of the R_i, times

$$e^{\mathcal{E}(R^{\mathrm{BB}})}\exp\Big(\frac{1}{2}\sum_{i,j}(J^{-1})_{ij}\,\mathrm{Tr}\,\big(R_i^{\mathrm{FF}}R_j^{\mathrm{FF}}\big)\Big).$$

Here $\mathcal{E}$ is (B.7) with $n = N$, $m_i = 1$, $w_{i,i+1} = W^2$, all other weights zero, and $\eta' = 0$, with the given ζ_i. We used (4.8) and the identity $\zeta_i^*\Lambda_{\mathrm{B}}R_i^{\mathrm{BB}}\zeta_i = \mathrm{Tr}(R_i^{\mathrm{BB}}\zeta_i\zeta_i^*\Lambda_{\mathrm{B}})$, and the plus sign in the FF factor comes from the minus sign of STr on the FF block.

The BB blocks of $\mathcal{S}_{\Gamma,t}$ are the matrices $R_t(P,X)$ of (B.2) with $c = \eta$, so case (i) of Lemma B.4 applies, with $\bar\eta_i = \eta$. Summing the bound (B.4) over the sites and the bound of Lemma B.3(b) over the links yields (4.14), with any constant $C_{k,\eta}$ such that

$$C_{k,\eta}\big(1 + |\zeta_i|^4\big) \ge k(s_E^2 + \eta^2) + (s_E + \eta)|\zeta_i|^2 + |\zeta_i|^4 + \frac{k\eta s_E}{2} + \frac{k\eta^2}{6}.$$

Since $s_E \le 1$, we may choose $C_{k,\eta}$ depending only on k and η. This completes the proof of Proposition 4.8(b).

Recall that the FF blocks of $\mathcal{S}_{\Gamma,t}$ have the form $\mathrm{i}M_i + (1-t)c_{\mathrm{F}}$, where $c_{\mathrm{F}} = \kappa_E\mathrm{Id} + \mathrm{i}\eta_0\Gamma_{\mathrm{F}}$. Since $J^{-1} \ge \mathrm{Id}$, the real part of the exponent of the FF factor is at most $-\frac{1}{2}\sum_i \|M_i\|^2 + C\sum_i\|M_i\| + C$ on $\mathcal{S}_{\Gamma,t}$, uniformly in t. We now apply Lemma B.1 to each coefficient, with $d = 8k^2N$ and the joint deformation $\mathbf{z}_t$ of the BB and FF blocks that parametrizes $\mathcal{S}_{\Gamma,t}^N$.

The Jacobian of $\mathbf{z}_t$ is one, as in the proof of Lemma B.4, since the FF blocks are translated by a constant. The derivative $\partial_t \mathbf{z}_t$ and the polynomial factors grow at most polynomially. By (B.8) and the Gaussian bound on the FF factor, (B.1) holds with an integrable $F^\sharp$. We conclude that each coefficient of $\int_{\mathcal{S}^N_{\Gamma,t}} pF_\psi \, \mathrm{d}R$ converges absolutely and is independent of t, which is Proposition 4.8(a). □

B.6 The heat kernel on the cycle

We now prove Lemma 4.10. Let $t, t' > 0$ and $S, S'' \in \mathcal{S}$. To avoid a clash with the heat time, we write $\tau \in [0,1]$ for the deformation parameter in (B.2), with $c = 0$.

The symmetry $k_t(S, S') = k_t(S', S)$ is clear. Let (P, X, M) and (P', X', M') be the coordinates of S and S'. Since $S - S'$ has the FF block $\mathrm{i}(M - M')$, the body of $\mathrm{STr}(S - S')^2$ has real part

$$\|P - P'\|^2 + \|X - X'\|^2 - \|f_0(X) - f_0(X')\|^2 + \|M - M'\|^2 \geq \|P - P'\|^2 + \|M - M'\|^2,$$

where we used (B.5) with $\tau = 0$ and (B.6). This implies the bound on the body of $k_t(S, S')$.

For the first identity, consider the coefficients of $k_t(S, S')$ in the odd entries of S and S' as functions of S'^{BB} and S'^{FF}. They are polynomials times $e^{\mathcal{E}(S'^{\mathrm{BB}})} e^{\mathrm{Tr}(S'^{\mathrm{FF}} - S^{\mathrm{FF}})^2/(4t)}$, where $\mathcal{E}$ is (B.7) with $n = 1$, $m_1 = 0$, $\zeta_1 = 0$, $\eta' = 0$, $w_{1*} = (2t)^{-1}$, $w_{1**} = 0$, and $R_* = S^{\mathrm{BB}}$. The FF factor is a Gaussian in M', since $\mathrm{Tr}(\mathrm{i}(M' - M))^2 = -\|M' - M\|^2$. Case (ii) of Lemma B.4 shows that $k_t(S, \cdot)$ is integrable on $\mathcal{S}$. Together with Lemma B.1, applied as in Appendix B.5, it also shows that the integral does not change if the BB block of S' is deformed from $\mathcal{S}^{\mathrm{B}}$ to the Hermitian matrices. The resulting cycle is the translate of $\mathcal{S}_\flat$ by $\mathrm{diag}(0 \mid \kappa_E \mathrm{Id})$, and on it we write

$$-\frac{1}{4t} \mathrm{STr}(S' - S)^2 = -\frac{1}{4t} \mathrm{STr}\, S'^2 - \mathrm{i}\, \mathrm{STr}\left(S' \cdot \frac{\mathrm{i}S}{2t}\right) - \frac{1}{4t} \mathrm{STr}\, S^2.$$

We apply Lemma B.2 with $N = 1$, $\mathbf{M} = (2t)^{-1}$, and $B = \mathrm{i}S/(2t)$, and we obtain

$$\int_{\mathcal{S}} k_t(S, S') \, \mathrm{d}S' = e^{\mathrm{STr}(S^2)/(4t)} e^{-\mathrm{STr}(S^2)/(4t)} = 1.$$

For the second identity, we argue in the same way with $w_{1*} = (2t)^{-1}$, $w_{1**} = (2t')^{-1}$, $R_* = S^{\mathrm{BB}}$, and $R_{**} = S''^{\mathrm{BB}}$. We then apply Lemma B.2 on the flat cycle, with $\mathbf{M} = (t + t')/(2tt')$ and $B = \mathrm{i}(S/(2t) + S''/(2t'))$, and we find

$$\int_{\mathcal{S}} k_t(S, S') k_{t'}(S', S'') \, \mathrm{d}S' = \exp\Big(\frac{tt'}{t + t'} \mathrm{STr}\left(\frac{S}{2t} + \frac{S''}{2t'}\right)^2 - \frac{\mathrm{STr}\, S^2}{4t} - \frac{\mathrm{STr}\, S''^2}{4t'}\Big).$$

Expanding the square and using $\mathrm{STr}(SS'') = \mathrm{STr}(S''S)$, we find

$$\frac{tt'}{t + t'} \mathrm{STr}\left(\frac{S}{2t} + \frac{S''}{2t'}\right)^2 - \frac{\mathrm{STr}\, S^2}{4t} - \frac{\mathrm{STr}\, S''^2}{4t'} = -\frac{\mathrm{STr}(S - S'')^2}{4(t + t')}.$$

This completes the proof of Lemma 4.10. □

B.7 The deformation in the anchored chain

In the proof of Proposition 4.13, we deform the variables $S_1, \dots, S_n$ of the anchored chain from $\mathcal{S}$ to the cycle with Hermitian BB blocks, for fixed ψ. We now explain why Lemma B.4 applies. The BB part of the exponent of $\ell_n(S;\psi)$ is (B.7) with unit masses, $w_{i,i+1} = W^2$, the link $w_{1*} = W^2$ to $R_* = S^{\mathrm{BB}}$, the given ζ_i, and $\eta' = \eta$. Indeed, since the BB block of Γ is $\varepsilon\Lambda_{\mathrm{B}}$, the BB part of $-\mathrm{i}\eta_0 \operatorname{STr}(S_i\Gamma)$ is the linear term $-\mathrm{i}\eta \operatorname{Tr}(S_i^{\mathrm{BB}}\Lambda_{\mathrm{B}})$. The cycle $\mathcal{S}$ is the case $\tau = 0$ of (B.2) with $c = 0$, so case (i) applies. The FF blocks stay on $\kappa_E\mathrm{Id} + \mathrm{iHerm}(2k)$, where the FF part of the exponent is a Gaussian in the M_i, and the odd variables appear polynomially. As in Appendix B.5, each coefficient of the integrand is integrable on the deformed cycles, uniformly in τ, and its integral is independent of τ. At $\tau = 1$, the cycle is the translate of $\mathcal{S}_\flat^n$ by $\operatorname{diag}(0 \mid \kappa_E\mathrm{Id})$ at each site, and applying Lemma B.2 with $\mathbf{M} = J_n^{-1}$ and $B_i = \mathrm{i}b_i$, we obtain the Gaussian integral used there.

For the marked anchored chains, the extra factor $\prod_i p_{\alpha_i}(\psi_i)$ involves only the ψ_i, and the same argument applies. The deformation of this section, with the inverse superdeterminants written as ψ-integrals and the order of the integrals exchanged, proves Proposition 4.13 directly in the direction from the chain to the Gaussian matrix, without an auxiliary regularization. The damping of the onsite weights supplies the decay in the noncompact directions, through the term $-\frac{\bar\eta}{4}\operatorname{Tr} f_0(X)$ of (B.4) with $\bar\eta = \eta' = \eta$.

B.8 The source factor on the saddle manifold

We now bound the BB part of the source factor $w_\Gamma / w_0 = e^{-\mathrm{i}\eta_0 \operatorname{STr}(S\Gamma)}$ of Section 4.7, together with its derivatives, at the points S with $S^{\mathrm{BB}} = \kappa_E - \mathrm{i}s_E Q$ and $Q \in H_k$. These bounds are used in Part IV, and they are stated in the geometry of Section 5. Recall the space H_k, the weight v of (5.3), and the Levi-Civita connection ∇ of Section 5.

Lemma B.5. *Let Γ be a source supermatrix, and for $Q \in H_k$ set $\mathbf{S}(Q) = e^{-\mathrm{i}\eta_0 \operatorname{Tr}((\kappa_E - \mathrm{i}s_E Q)\varepsilon\Lambda_{\mathrm{B}})}$, which is the BB part of the source factor at a point S with $S^{\mathrm{BB}} = \kappa_E - \mathrm{i}s_E Q$. Then*

$$\mathbf{S}(Q) = e^{-2\eta s_E (v(Q)-1)} \in \big(0, e^{-2k\eta s_E}\big]. \tag{B.9}$$

In particular $0 < \mathbf{S} \le 1$. Moreover, for $m \in \{1,2\}$ there exists C, depending only on k and s_E, such that $|\nabla^m \mathbf{S}| \le C\eta v e^{-s_E\eta v}$ on H_k whenever $\eta \le 1$. For a point $S \in \mathcal{S}$ with coordinates $(\kappa_E\mathrm{Id}, X, M)$, we have $\mathbf{S}(Q_X) = e^{-\eta \operatorname{Tr} f_0(X)}$, where Q_X is as in Lemma 4.7(d).

Proof. Using $\operatorname{Tr}\Lambda_{\mathrm{B}} = 0$, we compute the exponent,

$$-\mathrm{i}\eta \operatorname{Tr}\left(\kappa_E\Lambda_{\mathrm{B}} - \mathrm{i}s_E Q\Lambda_{\mathrm{B}}\right) = -\eta s_E \operatorname{Tr}(\Lambda_{\mathrm{B}} Q),$$

and the definition (5.3) of v gives (B.9). The bound follows from $\frac{1}{2}\operatorname{Tr}(\Lambda_{\mathrm{B}} Q) \ge k$, which is Lemma 5.1(a).

For the derivatives, we write $\mathbf{S} = e^{-2\eta s_E(v-1)}$, so that

$$\nabla \mathbf{S} = -2\eta s_E (\nabla v)\mathbf{S}, \qquad \nabla^2 \mathbf{S} = \big(4\eta^2 s_E^2\, \nabla v \otimes \nabla v - 2\eta s_E \nabla^2 v\big)\mathbf{S}.$$

Recall from Lemma 5.1(b) that

$$|\nabla v| + |\nabla^2 v| \le Cv.$$

For the term with $\eta^2 v^2$, we use

$$\eta^2 v^2 e^{-2\eta s_E v} \le \big(\eta v e^{-\eta s_E v}\big) \sup_{x \ge 0} x e^{-s_E x} = (e s_E)^{-1} \eta v e^{-\eta s_E v},$$

and together with $e^{2\eta s_E} \le e^2$, this implies the bound.

Finally, Lemma 4.7(a) implies that $\Lambda_{\mathrm{B}} \mathrm{g}_X^{-1} \Lambda_{\mathrm{B}} = \mathrm{g}_X$, and therefore

$$s_E \operatorname{Tr}(\Lambda_{\mathrm{B}} Q_X) = s_E \operatorname{Tr}\big(\Lambda_{\mathrm{B}} \mathrm{g}_X^{-1} \Lambda_{\mathrm{B}} \mathrm{g}_X\big) = s_E \operatorname{Tr} \mathrm{g}_X^2 = \operatorname{Tr}\big(f_0(X) + X\big) = \operatorname{Tr} f_0(X),$$

where we used $\operatorname{Tr} X = 0$ in the last step. Inserting this into the exponent above, we find $\mathbf{S}(Q_X) = e^{-\eta \operatorname{Tr} f_0(X)}$. □

In the critical regime, $\eta = \varepsilon/(N\rho(E))$ is of order $\delta_W = W^{-2}$, so that the bound of Lemma B.5 reads $|\nabla^m \mathbf{S}| \le C\delta_W v e^{-c\delta_W v}$, with constants depending on ε, E, and the range of $X_N(E)$.

APPENDIX C

Proof of the regular realization

In this appendix, we prove Proposition 5.15, the description (5.25) of the domains, and Lemma 5.17, which are stated in Section 5.3. We use the notation of Section 5, and the proofs use only the material of Section 5 that precedes these statements. The operator G is a second-order elliptic system on the bundle of superfunctions over $\mathcal{M}_k^0$, with a potential that grows like v. We realize it by Kato's form method, and we follow the standard arguments for elliptic operators on manifolds of bounded geometry [124]. Two features need care. The principal part of C_k is not a multiple of the identity on the odd coefficients, and the potential is not real. Both problems are solved by the choice of a small weight ω in the coefficient norms, since in the norm $|\cdot|_{2;\omega}$ the nilpotent parts of the principal symbol and of the potential become small.

Throughout the proof, constants may depend on k, X, ε, R_{F}, p, and ω, and they are bounded for (X,ε) in compact subsets of $(0,\infty)^2$. We write $\|\cdot\| = \|\cdot\|_0$ and $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_0$.

C.1 Local structure

By Lemma 5.8(c) and the homogeneity of Section 5.2, in every translated chart the generator has the form

$$\mathsf{G} = -X\Delta_{\mathcal{M}_k^0} + X\sum_{\mu,\nu} n^{\mu\nu}\partial_\mu\partial_\nu + \sum_\mu \tilde{b}^\mu\partial_\mu + \tilde{c} + \pi\,\mathrm{STr}(Q\Gamma), \tag{C.1}$$

where the coefficients $n^{\mu\nu}$, $\tilde{b}^\mu$, and $\tilde{c}$ are linear maps of $\bigwedge_{4k^2}$ with smooth coefficients, the same in all translated charts. Each $n^{\mu\nu}$ raises the degree by at least two. In the norm $|\cdot|_{2;\omega}$, a derivative in χ_a costs a factor ω^{-1} and a factor χ_a gains a factor ω, by Lemma 5.13. Therefore

$$\|n^{\mu\nu}\|_{|\cdot|_{2;\omega}\to|\cdot|_{2;\omega}} \le C\omega^2, \qquad \|\tilde{b}^\mu\| + \|\tilde{c}\| \le C_\omega, \tag{C.2}$$

where C is uniform in $\omega \in (0,1]$, and C_ω may depend on ω. For the potential, (5.22), Lemma 5.14, and the bound $|\mathrm{Tr}(Q^{\mathrm{FF}}M)| \le 2k\|M\|_{\mathrm{op}}$ for the body imply

$$\begin{aligned} \mathrm{Re}\,\big(\pi\,\mathrm{STr}(Q\Gamma)\big)_\varnothing &\ge 2\pi\varepsilon(v-1-k) - CR_{\mathrm{F}}, \\ \big|\pi\,\mathrm{STr}(Q\Gamma) - (\pi\,\mathrm{STr}(Q\Gamma))_\varnothing\big|_{1;\omega} &\le C\omega^2(\varepsilon v + R_{\mathrm{F}}). \end{aligned} \tag{C.3}$$

Here we used that the soul of Q^{FF} has degree at least two, since odd powers of Θ have vanishing diagonal blocks, and that the imaginary part of the body comes only from the bounded fermionic term.

C.2 The form

For compactly supported smooth superfunctions f and f', we set $\mathfrak{t}[f,f'] = \langle \mathsf{G} f, f' \rangle$ and $\|f\|_V^2 = \|\nabla f\|^2 + \|v^{1/2} f\|^2 + \|f\|^2$. We claim that there exist $c > 0$ and C such that

$$\operatorname{Re} \mathfrak{t}[f,f] + C\|f\|^2 \geq c\|f\|_V^2, \qquad |\mathfrak{t}[f,f']| \leq C\|f\|_V \|f'\|_V, \tag{C.4}$$

for all compactly supported smooth f and f', provided ω is small enough.

To prove (C.4), we fix a partition of unity $1 = \sum_\ell \phi_\ell^2$ on $\mathcal{M}_k^0$ subordinate to a cover by the domains of translated charts, with uniformly bounded multiplicity, such that each ϕ_ℓ is the transport of one of finitely many fixed functions by the action of G_0. Such a cover exists since $\mathcal{M}_k^0$ is homogeneous. In each chart, we integrate the second-order terms by parts once. The term $-X\Delta_{\mathcal{M}_k^0}$ gives $X\|\nabla f\|^2$, up to terms bounded by $C\|\nabla f\|\|f\|$ that come from the derivatives of the ϕ_ℓ and from the change of the odd frame between charts. Using (C.2), we bound the terms with $n^{\mu\nu}$ by $C\omega^2\|\nabla f\|^2 + C_\omega\|\nabla f\|\|f\|$, and the first-order and zeroth-order terms by $C_\omega\|\nabla f\|\|f\| + C_\omega\|f\|^2$.

For the potential, (C.3) implies that

$$\operatorname{Re}\langle \pi \operatorname{STr}(Q\Gamma) f, f\rangle \geq 2\pi\varepsilon\|v^{1/2} f\|^2 - C\omega^2(\varepsilon + R_{\mathrm{F}})\|v^{1/2} f\|^2 - C\|f\|^2,$$

where we used Lemma 5.13(a) for the soul. We now choose ω so small that $C\omega^2 \leq X/2$ in the bound for the principal part and $C\omega^2(\varepsilon + R_{\mathrm{F}}) \leq \pi\varepsilon$ in the bound for the potential. Absorbing the products $\|\nabla f\|\|f\|$ by Young's inequality then yields the first inequality in (C.4). These two conditions are the only requirements on ω in the proof, and they hold uniformly for (X,ε) in compact subsets of $(0,\infty)^2$. The second inequality follows from the same integration by parts and the bound $|\pi \operatorname{STr}(Q\Gamma)|_{1;\omega} \leq Cv$.

By (C.4), the form $\mathfrak{t} + C$ is sectorial and closable, and the domain of its closure is the completion of the compactly supported smooth superfunctions in $\|\cdot\|_V$. This completion is the space V_0 of part (a), since a superfunction with $\|f\|_V < \infty$ is the limit in $\|\cdot\|_V$ of $\psi_R f$, with the cutoffs ψ_R of (5.21), followed by mollification in the charts. By Kato's first representation theorem [72, Theorem VI.2.1], there exists an m-sectorial operator $\mathsf{G}_{(0)} + C$ such that $f \in D(\mathsf{G}_{(0)})$ if and only if $f \in V_0$ and there exists $g \in L_0^2$ with $\bar{\mathfrak{t}}[f,f'] = \langle g, f'\rangle$ for all $f' \in V_0$, and then $\mathsf{G}_{(0)} f = g$. Since the compactly supported smooth superfunctions are dense in V_0, it suffices to test with these f', and an integration by parts yields the characterization

$$D(\mathsf{G}_{(0)}) = \{f \in V_0 : \mathsf{G} f \in L_0^2\}, \tag{C.5}$$

where $\mathsf{G}f$ is taken in the sense of distributions, and $\mathsf{G}_{(0)}f = \mathsf{G}f$. In particular, $\mathsf{G}_{(0)}$ extends G on compactly supported smooth superfunctions. Since $\mathsf{G}_{(0)} + C$ is m-sectorial, the operator $-\mathsf{G}_{(0)}$ generates a holomorphic semigroup with $\|e^{-t\mathsf{G}_{(0)}}\| \le e^{Ct}$ [72].

C.3 The core

We now show that the compactly supported smooth superfunctions are a core for $\mathsf{G}_{(0)}$, and we first record the local regularity of the elements of $D(\mathsf{G}_{(0)})$. Let $f \in D(\mathsf{G}_{(0)})$. By (C.1), the principal part of G in a chart is triangular with respect to the degree in the χ_a, with diagonal part $-X\Delta_{\mathcal{M}_k^0}$, since each $n^{\mu\nu}$ raises the degree. Since $f \in V_0$ and $\mathsf{G}f \in L^2_0$, the component f_j of f of degree j satisfies $-X\Delta_{\mathcal{M}_k^0} f_j = g_j$ in each chart, where g_j is square integrable once the components of degree less than j have square integrable second derivatives. By induction on j and interior regularity for scalar elliptic equations [62], f has locally square integrable second derivatives.

We claim that $\psi_R f \to f$ in the graph norm of $\mathsf{G}_{(0)}$. We have $\psi_R f \in V_0$ and

$$\mathsf{G}(\psi_R f) = \psi_R \mathsf{G} f + [\mathsf{G}, \psi_R] f.$$

The potential commutes with ψ_R. The commutator is therefore a first-order operator whose coefficients are derivatives of ψ_R of order one and two, multiplied by bounded coefficients of G, and

$$\|[\mathsf{G}, \psi_R] f\| \le C\big(\|\nabla \psi_R\|_\infty + \|\nabla^2 \psi_R\|_\infty\big)\big(\|\nabla f\| + \|f\|\big) \le C R^{-1} \|f\|_V \longrightarrow 0.$$

Moreover $\psi_R f \in D(\mathsf{G}_{(0)})$ by (C.5), and $\psi_R f \to f$ and $\psi_R \mathsf{G} f \to \mathsf{G}_{(0)} f$ in L^2_0, which proves the claim. Finally, a compactly supported element of $D(\mathsf{G}_{(0)})$ is the limit in the graph norm of its mollifications in finitely many charts, since its second derivatives are square integrable. We have proved part (a) for $p = 0$.

C.4 Weights

Let $p \in \mathbb{R}$. Multiplication by v^p is a unitary map from L^2_p onto L^2_0 that preserves the compactly supported smooth superfunctions. We have

$$v^p \mathsf{G} v^{-p} = \mathsf{G} + \mathcal{R}_p,$$

where $\mathcal{R}_p$ is a first-order operator whose coefficients are polynomials in the derivatives of $\log v$ of order one and two, multiplied by coefficients of the differential part of G. In particular, conjugating the first-order term contributes the bounded zeroth-order term $-p \sum_\mu \tilde{b}^\mu \partial_\mu \log v$. These coefficients are bounded, by Lemma 5.1(b) and (C.2), and (C.4) and the arguments of Appendices C.2 and C.3 hold for $\mathsf{G} + \mathcal{R}_p$ on L^2_0. Transporting back

by multiplication by v^{-p}, we obtain part (a) for all p, with $V_p = v^{-p}V_0$, which is the space displayed there, and the description (5.25) of the domain of $\mathsf{G}_{(p)}$.

We now prove the consistency in part (b). Let $q \geq p$. Then $V_q \subset V_p$, since $v^p f = v^{p-q}(v^q f)$, and multiplication by $v^{p-q} \leq 1$ preserves V_0 by Lemma 5.1(b). Let λ be larger than the shifts of $\mathsf{G}_{(p)}$ and $\mathsf{G}_{(q)}$, let $f \in L^2_q$, and let $g = (\mathsf{G}_{(q)} + \lambda)^{-1} f$. Then $g \in V_q \subset V_p$, and $\mathsf{G}g = f - \lambda g \in L^2_p$ in the sense of distributions. Using (5.25), we find that $g \in D(\mathsf{G}_{(p)})$ and $(\mathsf{G}_{(p)} + \lambda)g = f$. The resolvents of $\mathsf{G}_{(p)}$ and $\mathsf{G}_{(q)}$ agree on L^2_q, and so do the semigroups, by the exponential formula $e^{-t\mathsf{G}} = \lim_{n\to\infty}(\mathrm{Id} + t\mathsf{G}/n)^{-n}$ [72].

Next, we prove the smoothing estimate. Let $f \in L^2_0$ and $g(t) = e^{-t\mathsf{G}_{(0)}} f$. Since the semigroup is holomorphic, $\|\mathsf{G}_{(0)} g(t)\| \leq Ct^{-1}\|f\|$ for $0 < t \leq T$, and (C.4) implies that

$$c\|v^{1/2} g(t)\|^2 \leq \mathrm{Re}\langle \mathsf{G}_{(0)} g(t), g(t)\rangle + C\|g(t)\|^2 \leq C(t^{-1}+1)\|f\|^2.$$

This is the case $p = 0$, $m = 1$ of the estimate, and the same argument for $\mathsf{G} + \mathcal{R}_p$ yields the case $m = 1$ for all p. For general m, we write $e^{-t\mathsf{G}} = (e^{-t\mathsf{G}/m})^m$, which is legitimate by consistency, and we apply the case $m = 1$ with the weights $p, p + \frac{1}{2}, \dots, p + \frac{m-1}{2}$.

It remains to show that $e^{-t\mathsf{G}} f$ is rapidly decreasing for $t > 0$ and $f \in L^2_p$. Recall the domains $D(\mathsf{G}^j_{(p)})$ of the powers of G from Section 5.3. By the smoothing estimate, consistency, and holomorphy, $g = e^{-t\mathsf{G}} f$ belongs to $D(\mathsf{G}^n_{(p')})$ for all p' and all n. We use interior elliptic estimates in the translated charts. Let $x \in \mathcal{M}^0_k$, and let $B_1 \subset B_2$ denote the images of the balls of radii one and two centered at Λ in the translated chart centered at x. We claim that for every integer $j \geq 1$ and every $p \in \mathbb{R}$, there exists C, independent of x, such that

$$\sum_{i=0}^{2j} \|\nabla^i g\|_{L^2(B_1)} \leq C\, v(x)^j \sum_{l=0}^{j} \|\mathsf{G}^l g\|_{L^2(B_2)}, \qquad g \in D(\mathsf{G}^j_{(p)}), \tag{C.6}$$

where the norms are those of the coefficients in the chart.

To prove (C.6), we first note that the local form of the coercivity estimate (C.4), with a cutoff function, bounds the first derivatives of g on a ball between B_1 and B_2. The part of degree s of the equation $\mathsf{G}g = h$ is then an equation $-X\Delta_{\mathcal{M}^0_k} g_s = h_s - (\dots)$, in which the omitted terms contain second derivatives only of the parts of g of degree at most $s-2$, since each $n^{\mu\nu}$ in (C.1) raises the degree by at least two, and first derivatives of all parts. By induction on s and interior regularity for $-X\Delta_{\mathcal{M}^0_k}$ [62], we obtain the case $j = 1$, and the general case follows by differentiating the equation. The constant is uniform in x, since the coefficients of $\mathsf{G} - \pi\,\mathrm{STr}(Q\Gamma)$ are the same in all translated charts, the potential is a multiplier of degree one by Lemma 5.14, and v varies by at most a bounded factor on B_2, since $\log v$ is Lipschitz.

Multiplying (C.6) for $g = e^{-t\mathsf{G}} f$ by $v(x)^{p'}$, summing the squares over a cover of bounded multiplicity, and using the Sobolev embedding theorem, we find that g is smooth and that $\|\nabla^j g\|_{p'} < \infty$ for all j and p'. This completes the proof of part (b).

C.5 Symmetry, the split pairing, and small damping

We begin with part (c). Recall from (5.11) that $\mathcal{B}_k(f,f') = \int_{\mathcal{M}_k^0} \int \rho f f' \,\mathrm{d}\chi \,\mathrm{dvol}$. The inner Berezin integral is the top coefficient of $\rho f f'$, and

$$\Big| \int \rho f f' \,\mathrm{d}\chi \Big| \le \omega^{-4k^2} |\rho f f'|_{1;\omega} \le C |f|_{2;\omega} |f'|_{2;\omega},$$

where we used Lemma 5.13(a) and the Cauchy–Schwarz inequality over the 2^{4k^2} monomials. Writing $1 = v^p v^{-p}$ and applying the Cauchy–Schwarz inequality on $\mathcal{M}_k^0$, we obtain $|\mathcal{B}_k(f,f')| \le C \|f\|_p \|f'\|_{-p}$.

By Lemma 5.8(b), we have $\mathcal{B}_k(\mathsf{G}f, f') = \mathcal{B}_k(f, \mathsf{G}f')$ for compactly supported smooth f and f', since the potential acts by multiplication by an even superfunction. These superfunctions are cores for $\mathsf{G}_{(p)}$ and $\mathsf{G}_{(-p)}$. Since $\mathcal{B}_k$ is bounded on $L^2_p \times L^2_{-p}$, the identity extends to $f \in D(\mathsf{G}_{(p)})$ and $f' \in D(\mathsf{G}_{(-p)})$. For such f and f', the function $s \mapsto \mathcal{B}_k(e^{-s\mathsf{G}} f, e^{-(t-s)\mathsf{G}} f')$ on $[0,t]$ is differentiable, with derivative

$$-\mathcal{B}_k(\mathsf{G}e^{-s\mathsf{G}} f, e^{-(t-s)\mathsf{G}} f') + \mathcal{B}_k(e^{-s\mathsf{G}} f, \mathsf{G}e^{-(t-s)\mathsf{G}} f') = 0.$$

Comparing $s = 0$ and $s = t$, and using that the domains are dense, we obtain part (c).

We now prove part (d). By the layer cake formula and Lemma 5.1(c), we have $\|1\|^2_{-q} = \int v^{-2q} \,\mathrm{dvol} < \infty$ for $2q > q_1$, and part (b) shows that $e^{-t\mathsf{G}}1$ is rapidly decreasing for $t > 0$. Applying part (c) with $p = -q$, $f = 1$, $t = a$, and $f' = e^{-(T-a)\mathsf{G}}1 \in L^2_q$, and using consistency, we find

$$\mathcal{B}_k(e^{-a\mathsf{G}}1, e^{-(T-a)\mathsf{G}}1) = \mathcal{B}_k(1, e^{-T\mathsf{G}}1) = \int e^{-T\mathsf{G}}1.$$

Finally, we prove part (e), in which the main point is that the soul of the damping is controlled by its body, uniformly as $z \downarrow 0$. Let $0 \le z \le z_0$. Using Lemma 5.14(b) and (5.10), we find

$$|d_k - (d_k)_\varnothing|_{1;\omega} \le C\omega^2 v \le C\omega^2 \big((d_k)_\varnothing + 1 + k\big).$$

Choose $\omega_0 > 0$ small enough for the principal-part estimate of Appendix C.2 and for $C\omega_0^2 \le \frac{1}{2}$, and fix $\omega \in (0, \omega_0]$ for the rest of the argument. Then half of the nonnegative body $z(d_k)_\varnothing$ absorbs the soul of $z d_k$, up to the constant $(1+k)z_0$, and the argument of Appendix C.2 gives, for compactly supported smooth f,

$$\mathrm{Re}\langle \mathsf{G}_{X,z} f, f\rangle + C_\omega \|f\|^2 \ge c\|\nabla f\|^2 + \frac{z}{2}\langle (d_k)_\varnothing f, f\rangle \ge c\|\nabla f\|^2,$$

$$|\,\mathrm{Im}\langle \mathsf{G}_{X,z} f, f\rangle| \le C_\omega\big(\|\nabla f\|^2 + \|f\|^2\big) + \frac{z}{2}\langle (d_k)_\varnothing f, f\rangle,$$

with c and C_ω uniform in z. The dependence on ω comes from the first-order and zeroth-order coefficient bounds C_ω in (C.2); Young's inequality retains this dependence

in the form shift. The same holds with the weight v^{2p}, by the argument of Appendix C.4. For $z = 0$, these bounds show, as in Appendices C.2 to C.4, that $X\mathsf{C}_k$ is closable in L^2_p, and that its closure has the form domain H^1_p and the domain $\{f \in H^1_p : \mathsf{C}_k f \in L^2_p\}$; the potential plays no role in these arguments. The numerical range of $\mathsf{G}_{X,z} + C_\omega$ lies in a fixed sector for all $z \in [0, z_0]$, and for $0 < s \le T$ and $j \ge 0$,

$$\|e^{-s\mathsf{G}_{X,z}}\|_{L^2_p \to L^2_p} \le e^{C_\omega s}, \qquad \|\mathsf{G}^j_{X,z} e^{-s\mathsf{G}_{X,z}}\|_{L^2_p \to L^2_p} \le C_{j,\omega} s^{-j},$$

uniformly in z, with ω fixed; the first bound holds for all $s \ge 0$. For X in a compact subset of $(0, \infty)$, the threshold ω_0 and, at fixed admissible ω, these constants can be chosen uniformly in X.

The bound on the coefficients of the superfunctions $e^{-s_n \mathsf{G}_{X,z}} m_n \cdots e^{-s_1 \mathsf{G}_{X,z}} m_1 1$ in (e) is part (b) of Lemma 5.17, whose proof in Appendix C.6 below uses only the parts of Proposition 5.15 proved so far. This completes the proof of Proposition 5.15.

C.6 Semigroup products at all times

We now prove Lemma 5.17, which gives bounds on the semigroup products (5.26) that are uniform in the times.

Proof of Lemma 5.17. We write $\mathsf{G} = \mathsf{G}_{X,z}$, and we fix ω so small that Proposition 5.15 holds for the finitely many weights used below, uniformly in $z \in [0, z_0]$. We use three facts.

(A) Interior estimates. Recall the interior estimate (C.6) from the proof of Proposition 5.15(b), for the balls $B_1 \subset B_2$ of the translated chart centered at x. Its proof uses only that $g \in D(\mathsf{G}^j_{(p)})$, and it applies to $\mathsf{G} = \mathsf{G}_{X,z}$ for $z \in [0, z_0]$, with the realization of Proposition 5.15(e) if $z = 0$. The constant C in (C.6) is then uniform in $z \le z_0$, since $z d_k$ is z times a multiplier of degree one.

Multiplying (C.6) by $v(x)^{p-j}$, squaring, and summing over the centers of a cover of bounded multiplicity, as in Appendix C.2, we obtain

$$\sum_{i=0}^{2j} \|\nabla^i g\|_{p-j} \le C \sum_{l=0}^{j} \|\mathsf{G}^l g\|_p, \qquad g \in D(\mathsf{G}^j_{(p)}). \tag{C.7}$$

(B) Multipliers. Let m be a multiplier of degree s, let $j \ge 1$, and let u be a smooth superfunction with $\|\nabla^i u\|_p \le C$ for $i \le 2j + 1$. In a translated chart, $\mathsf{G}^l(mu)$ is a sum of terms $a_\beta \partial^\beta u$ with $|\beta| \le 2l$, whose coefficients are products of derivatives of m, of the coefficients of C_k, and of $z d_k$. The coefficients of C_k are bounded with all their derivatives in the translated charts, and m and $z d_k$ are multipliers of degrees s and one, so that

$$|a_\beta|_{1;1} \le C v^{s+l}, \qquad |\beta| \le 2l,$$

with all derivatives, uniformly in $z \le z_0$. Let $p' = p - s - j - 1$. Then $\mathsf{G}^l(mu) \in V_{p'}$ for $l < j$, with $H^1_{p'}$ in place of $V_{p'}$ if $z = 0$, and $\mathsf{G}^j(mu) \in L^2_{p'}$, with norms bounded by a constant. We apply (5.25), or its analogue for $z = 0$ in the proof of Proposition 5.15(e), in turn to $\mathsf{G}^{j-1}(mu), \mathsf{G}^{j-2}(mu), \dots, mu$, and we obtain

$$mu \in D(\mathsf{G}^j_{(p')}), \qquad \sum_{l=0}^{j} \|\mathsf{G}^l(mu)\|_{p'} \le C.$$

(C) The semigroup. For $u \in D(\mathsf{G}^j_{(p)})$ and $s \ge 0$, we have $e^{-s\mathsf{G}}u \in D(\mathsf{G}^j_{(p)})$ and $\mathsf{G}^l e^{-s\mathsf{G}}u = e^{-s\mathsf{G}}\mathsf{G}^l u$ for $l \le j$, and the first bound of Proposition 5.15(e) implies that

$$\|\mathsf{G}^l e^{-s\mathsf{G}}u\|_p \le e^{Cs}\|\mathsf{G}^l u\|_p, \qquad l \le j,\ s \in [0,T].$$

We now prove (a) by induction over the construction of the semigroup product. Let $q > q_1/2$, so that $1 \in L^2_{-q}$ by Proposition 5.15(d). The derivatives of the constant 1 vanish in the translated charts, and (B) with $m = 1$ and $u = 1$ shows that $1 \in D(\mathsf{G}^J_{(p_J)})$ for every J, with $p_J = -q - J - 1$. If u satisfies the bounds of (a) for some $J \ge 2$, then (A) bounds $\|\nabla^i u\|_{p-J}$ for $i \le 2J$, and (B) shows that mu satisfies them with $J - 1$ in place of J and a smaller weight. By (C), the bounds are preserved by a semigroup factor. Starting with $J = j + n + 1$, we obtain $f \in D(\mathsf{G}^{j+1}_{(p)})$ for some p, with bounds uniform in z and in the times, and (A) then implies (5.27), after lowering p. For (b), let K be compact. On a compact neighborhood of K, the weight v is bounded above and below, and (a) with j large and the Sobolev embedding theorem in the translated charts bound the coefficients of f and their derivatives of order at most j on K.

For (c), fix z and the times. By (a), for every j there exists p with $f \in D(\mathsf{G}^{j+1}_{(p)})$. The map $t \mapsto e^{-t\mathsf{G}}f$ is then continuously differentiable on $[0,\infty)$ in the graph norm of $\mathsf{G}^j_{(p)}$, with derivative $-\mathsf{G}e^{-t\mathsf{G}}f = -e^{-t\mathsf{G}}\mathsf{G}f$. Iterating, we find that $t \mapsto e^{-t\mathsf{G}}f$ is l times continuously differentiable in the graph norm of $\mathsf{G}^{j-l}_{(p)}$, for $l \le j$, with

$$\partial_t^l e^{-t\mathsf{G}}f = (-1)^l e^{-t\mathsf{G}}\mathsf{G}^l f.$$

By (C.7) and the Sobolev embedding theorem, the coefficients of $\partial_t^l e^{-t\mathsf{G}}f$ have continuous derivatives in the body variables up to an order that tends to infinity with $j - l$, jointly continuous in $t \in [0,\infty)$. Since j is arbitrary, the coefficients are smooth up to $t = 0$, and $\partial_t u = -\mathsf{G}u$ holds classically. The bound on $\|\nabla^i e^{-t\mathsf{G}}f\|_p$ follows from (C) and (C.7).

For (d), let f be rapidly decreasing. Then (B) with $m = 1$ and $u = f$ shows that $f \in D(\mathsf{G}^j_{(p)})$ for all j and p, and the proof of (c) applies with every p.

Finally, (e) follows from the proofs above and those of Proposition 5.15, in which there are then no odd coordinates, the terms $n^{\mu\nu}$ are absent, and the potential $z(d_k)_\emptyset$ has no soul.

In this case the condition on ω disappears, and the form $f \mapsto X\|\nabla f\|^2 + z\langle (d_k)_\varnothing f, f\rangle$ of $A_{X,z}$ on L^2_0 is real and nonnegative. For $z > 0$, its form domain is V_0, and the positive-damping realization and smoothing proofs apply. At $z = 0$, its form domain is H^1_0 and the corresponding weighted realization has form domain H^1_p. The small-damping estimates and the semigroup-product arguments (A)–(C) apply throughout $0 \le z \le z_0$, using this H^1_p domain at zero, as in (B). They give the asserted weighted, compact, time-regularity, and rapid-data bounds also at $z = 0$. □

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