The Cranston–Li–Su–Wang–Xu (2026) conjecture is true

Dave Goel*

October 2026
Revised: November 10, 2026

This paper has an expanded version, with further results [3]: https://doi.org/10.5281/zenodo.23605838

hexagon:2611.00365v1

Abstract

Let A be a finite abelian group. A nowhere-zero A-flow of a graph can be changed along a cycle: going around the cycle, add a fixed non-zero element of A to the value of every edge that points forward and subtract it from the value of every edge that points backward. The Cranston–Li–Su–Wang–Xu (2026) conjecture on the reconfiguration of nowhere-zero flows states that if A has at least four elements, then in every graph in which all degrees are even, any two nowhere-zero A-flows are joined by a chain of such changes through nowhere-zero A-flows. It also states that, for every integer $k \geq 4$, the same holds for integer flows whose values are between 1 and $k-1$ in absolute value, where each change adds a non-zero integer. We prove this conjecture. The proof is by induction on the number of edges. If a flow enters a vertex along one edge and leaves it along another edge with the same value, the two edges can be replaced by a single edge; this gives a smaller graph. The main step is to change every flow into a flow with such a pair of edges. For A-flows we do this when all degrees are at most 4; repeatedly replacing a vertex of degree at least 6 by a triangle, following Cranston et al., reduces the general case to this one. For integer flows, we show that, in a graph in which all degrees are even, any two nowhere-zero flows with values in the cyclic group $\mathbb{Z}_k$, $k \geq 4$, are joined by changes that add only 1, -1, 2 or -2, and that such changes lift to integer flows. All theorems of this paper are also proved in the Lean 4 proof assistant.

2020 Mathematics Subject Classification: 05C21 (primary), 05C45 (secondary).
Keywords: nowhere-zero flow, group flow, reconfiguration, flow-connected graph, Eulerian graph.

1 Introduction

This paper proves the Cranston–Li–Su–Wang–Xu (2026) conjecture [1, Conjecture 4.1] on the reconfiguration of nowhere-zero flows in graphs in which every vertex has even degree, that is, on whether every nowhere-zero flow can be turned into any other one by changes along cycles that keep the flow nowhere-zero.

Let G be a finite graph; loops and parallel edges are allowed. For a finite abelian group A (for example the cyclic group $\mathbb{Z}_m$ of the integers modulo m), a nowhere-zero A-flow of G assigns to every edge, in a fixed reference direction, a non-zero element of A, so that at every vertex the sum of the values on the entering edges equals the sum on the leaving edges. Two such flows differ by a *step* with *value* $t \neq 0$ if there is a cycle, traversed in one of its two directions, such that their difference is t

*All theorems of this paper are proved in Lean 4 from Lean's three standard axioms alone, with no other result encoded as an axiom, and checked with the Lean comparator:
https://github.com/david44hello/papers_supporting_materials/tree/cranston-li-su-wang-xu-2026-conjecture-v2.00/cranston-li-su-wang-xu-2026-conjecture

on every edge of the cycle whose reference direction agrees with the traversal, $-t$ on every other edge of the cycle, and 0 on every edge not on the cycle. The graph $\mathcal{F}(G, A)$ has the nowhere-zero A-flows of G as its vertices, two of them being adjacent if they differ by a step, and G is *A-flow-connected* if $\mathcal{F}(G, A)$ is connected. For an integer $k \geq 2$, the graph $\mathcal{F}(G, k)$ is defined in the same way with integers. Its vertices are the *k-flows* of G, that is, the integer flows F of G with $1 \leq |F(e)| \leq k-1$ on every edge e (they are usually called nowhere-zero k-flows), and two of them are adjacent if they differ by a step whose value is a non-zero integer. The graph G is *k-flow-connected* if $\mathcal{F}(G, k)$ is connected. Figure 1 shows a step.

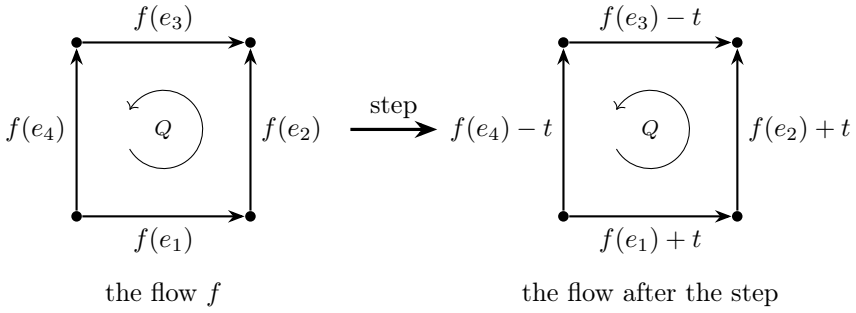


Figure 1: A step with value t along a cycle Q with four edges, traversed counter-clockwise. The arrow on each edge is its reference direction. On the two edges whose reference direction agrees with the traversal, t is added to the value; on the other two, t is subtracted; the edges not on Q (not drawn) keep their values. The flow condition still holds at every vertex.

Esperet, Hendrey, Lagoutte, Marseloo, Norin and Steiner [2] introduced the graphs $\mathcal{F}(G, A)$ and $\mathcal{F}(G, k)$ and asked for which graphs G, groups A and integers k they are connected. Cranston, Li, Su, Wang and Xu [1] call a graph in which all degrees are even Eulerian. Since "Eulerian" often also requires the graph to be connected, we call a graph *even* if every vertex has even degree; an even graph need not be connected.

Theorem 1.1. *Let G be a graph in which every vertex has even degree.*

(a) G is A-flow-connected for every finite abelian group A with $|A| \geq 4$.

(b) G is k-flow-connected for every integer $k \geq 4$.

Theorem 1.1 is [1, Conjecture 4.1]. Part (a) is Theorem 2.3 (Section 5), and part (b) is Theorem 7.1 (Section 7). The table and Figure 2 compare Theorem 1.1 with earlier results.

Best result before this paper	New result
Which graphs are A-flow-connected, and for which finite abelian groups A?	
All even graphs: even $\|A\|$ [1, Theorem 4.3]; $\|A\| \geq 5{,}251{,}339$ [1, Corollary 4.5]; for $\|A\| \geq 5$, reduced to 4-regular graphs [1, Observation 4.6]. All graphs without a bridge: $\|A\| \geq 3024$ [5, Theorem 1.1]; some groups of smaller order, including $\mathbb{Z}_{55}$ [4, Theorem 4.5]. Even loopless planar graphs: all $\|A\| \geq 4$, by [1] and [2] (see below).	All even graphs: all $\|A\| \geq 4$ (Theorem 1.1 (a)).
Which graphs are k-flow-connected, and for which integers k?	
All even graphs: even k [1, Theorem 4.2]. All graphs without a bridge and without loops: some odd k, for example $k = 729$ [4, Theorem 7.4]. Even loopless planar graphs: all $k \geq 4$, by [1] and [2] (see below).	All even graphs: all $k \geq 4$ (Theorem 1.1 (b)).

An even graph has no bridge, so the results in the table for graphs without a bridge, from [4] and [5], apply to even graphs. Such results go back to [2, Theorems 6.15 and 6.23], for the group $\mathbb{Z}_2^8$ and for all groups of sufficiently large order. Neither [4] nor [5] is used here.

For even planar graphs without loops (the graphs of [2] have no loops), the conjecture already followed from earlier results, applied to each connected component. A connected even plane graph G without loops has no bridge, and its dual is bipartite; so the simple graph underlying the dual is bipartite and planar, hence 3-degenerate by Euler's formula (as noted in [2, Section 5]). By [2, Corollary 5.8], if the simple graph underlying the dual is d-degenerate, then G is A-flow-connected for $|A| \geq d+2$ and k-flow-connected for $k \geq d+2$; here $d = 3$. The cases $|A| = 4$ and $k = 4$ are [1, Theorems 4.3 and 4.2].

Known before this paper

	3	4	5	6	7	8	9	⋯
groups, $\lvert A\rvert =$	no	yes	open	yes	open	yes	open	⋯
integers, $k =$	no	yes	open	yes	open	yes	open	⋯

Known after this paper

	3	4	5	6	7	8	9	⋯
groups, $\lvert A\rvert =$	no	yes	yes	yes	yes	yes	yes	yes
integers, $k =$	no	yes	yes	yes	yes	yes	yes	yes

Figure 2: Is every even graph A-flow-connected (rows "groups", by the order $|A|$) and k-flow-connected (rows "integers", by k)? For 3 the answer is no [1, Theorem 2.1]; see Figure 3. Left: what was known before this paper for the values drawn [1, Theorems 4.2 and 4.3]. Right: what is known after this paper; every "yes" there follows from Theorem 1.1.

The number 4 in Theorem 1.1 cannot be replaced by 3. By [1, Theorem 2.1], for a graph with a 3-flow, the following are equivalent: it is $\mathbb{Z}_3$-flow-connected; it is 3-flow-connected; all its degrees are even and no two of its vertices are joined by more than four edge-disjoint paths. The even graph with two vertices and six parallel edges has a 3-flow (Figure 3), and its two vertices are joined by six edge-disjoint paths. So it is neither $\mathbb{Z}_3$-flow-connected nor 3-flow-connected (for $\mathbb{Z}_3$, this is also noted in [2, Section 5]).

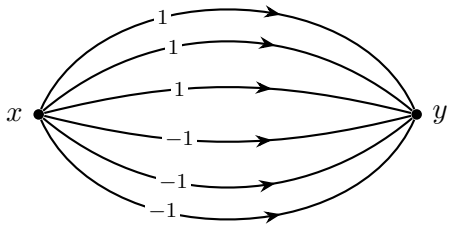


Figure 3: The even graph with two vertices and six parallel edges, with a 3-flow: all edges are directed from x to y; three have the value 1 and three have the value -1.

Two further theorems. For the cyclic groups $\mathbb{Z}_m$ with $m \geq 4$, any two nowhere-zero $\mathbb{Z}_m$-flows of an even graph are joined by a chain of steps whose values are in $\{1, -1, 2, -2\}$ (Theorem 2.4); for even m, $\mathbb{Z}_m$-flow-connectedness, with steps of any value, was known [1, Theorem 4.3]. Part (a) rests on Theorem 2.2: every even graph in which all degrees are at most 4 is A-flow-connected for every finite abelian group A. For $|A| \leq 3$ this is not new: it is trivial for $|A| \leq 2$, as a graph then has at most one nowhere-zero A-flow, and for $A = \mathbb{Z}_3$ it follows from [1, Theorem 2.1], because an even

graph has a 3-flow (the value 1 along an Eulerian orientation) and, when all its degrees are at most 4, no two of its vertices are joined by more than four edge-disjoint paths.

Outline of the proof. The proof is by induction on the number of edges, and its main tool is the balanced pair. A flow has a balanced pair at a vertex v if it enters v along an edge e from a vertex u and leaves v along another edge e' to a vertex w with the same value (Definition 2.1). The two edges can then be replaced by a single edge uw (Figure 4), and the flows with this balanced pair correspond to the flows of the smaller graph (Lemma 3.2). By the induction hypothesis the smaller graph is flow-connected, so any two flows with a balanced pair on the same two edges are joined by a chain of steps.

Flows whose balanced pairs are on different edges are joined through constant flows: a constant flow has the same value $a \neq 0$ on every edge, in the direction of an Eulerian orientation (an orientation in which, at every vertex, as many edges enter as leave; one exists because every degree is even). Any two edges at a vertex of a loopless even graph form a balanced pair in some constant flow with value a, and all constant flows with value a are joined to each other, for every $a \neq 0$ when steps of all values are allowed and for $a = 1$ when only steps with values ± 1 and ± 2 are allowed (Lemma 3.3). So all flows with a balanced pair are joined to each other, and it remains to join every flow to one of them. This last statement is called (LOCAL) in Proposition 3.4, which states the reduction precisely.

Constant flows along Eulerian orientations and splitting off two edges at a vertex both come from [1]. In the proofs of [1, Theorems 4.2 and 4.3] every flow is brought to a constant flow along an Eulerian orientation, and any two Eulerian orientations are joined by reversals of directed cycles [1, Lemma 2.3] (see Lemma 3.3 (a)). The proof of [1, Lemma 2.3] also splits off two edges at a vertex; Lemma 3.2 extends this step from Eulerian orientations to all flows with a balanced pair.

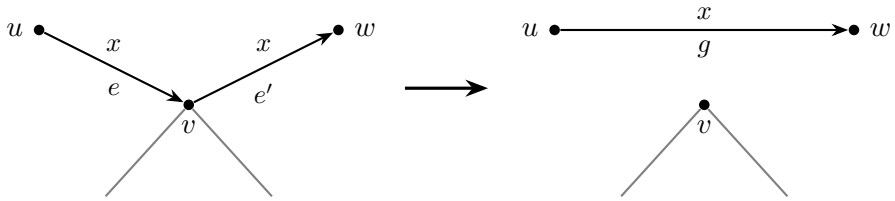


Figure 4: A balanced pair at v: the flow enters v along e, from u, and leaves v along e', to w, with the same value x. The two edges are replaced by one edge $g = uw$ with the value x; the other edges at v (gray) are unchanged. The flows with this balanced pair correspond to the flows of the smaller graph (Lemma 3.2).

Statement (LOCAL) is needed only for loopless graphs in which every degree is 0 or at least 4: loops can be deleted (Lemma 3.1), and at a vertex of degree 2 the two edges form a balanced pair in every flow, by the flow condition. Section 4 proves it when every degree is 0 or 4; this gives Theorem 2.2 (all degrees at most 4). In Section 5 a vertex of larger degree is replaced by a triangle, with two of the edges at the vertex going to each of two vertices of the triangle and the others to the third (Figure 5). This is the reduction of [1, Observation 4.6], which shows for $|A| \geq 5$ that all even graphs are A-flow-connected if and only if all 4-regular graphs are. We need only one direction, and it holds for $|A| \geq 4$: if the graph with the triangle is A-flow-connected, then so is the original graph (Lemma 5.1, a special case of [1, Theorem 5.9(1)], see Remark 5.2; we include its short proof).

Part (b) of Theorem 1.1 does not follow from part (a) by a general principle: whether every $\mathbb{Z}_k$-flow-connected graph is k-flow-connected is, for general k, an open problem [2, Problem 2.6]. The steps of Section 4 can have any value, but the lemmas that turn a step between two nowhere-zero

$\mathbb{Z}_k$-flows into a step between two k-flows with these residues modulo k (Lemmas 7.5 and 7.6) apply only to steps with values ±1 and ±2. For the cyclic groups $\mathbb{Z}_m$ with $m \geq 4$, Section 6 therefore proves (LOCAL) using only steps with values ±1 and ±2, by a counting argument that needs no bound on the degrees. This gives Theorem 2.4, which also proves part (a) for cyclic groups (Remark 6.4 shows where the counting argument fails for other groups). Section 7 lifts these chains to integer flows, using Tutte's lifting theorem [7] in the form of [2, Theorem 2.3]: every nowhere-zero $\mathbb{Z}_k$-flow is the residue modulo k of a k-flow.

An expanded version of this paper, with further results, is posted on Zenodo [3]. It bounds the diameter of $\mathcal{F}(G, A)$ by a polynomial in the number of edges when G is even and $|A| \geq 4$, and it treats graphs with few vertices of odd degree. None of its results is used here.

2 Definitions and results

Graphs. All graphs are finite; loops and parallel edges are allowed. A graph is *even* if every vertex has even degree, where a loop counts twice. A *cycle* is a connected subgraph in which every vertex has degree 2; a loop is a cycle with one edge, and two parallel edges form a cycle with two edges. A *directed cycle* is a cycle together with one of its two directions of traversal. An *Eulerian orientation* of a graph directs every edge so that at every vertex the in-degree equals the out-degree; the directed edges are its *arcs*. Every even graph has an Eulerian orientation.

Flows. Let A be a finite abelian group, and let G be a graph. Fix a reference direction of every edge. An *A-flow* of G is a map $f\colon E(G) \to A$ such that at every vertex the sum of f over the edges that leave the vertex equals the sum over the edges that enter it. The value of f on an edge e is $f(e)$ in the reference direction of e and $-f(e)$ in the other direction. From now on, a *flow* is a nowhere-zero A-flow, that is, $f(e) \neq 0$ for every edge e.

For a directed cycle Q, let $\chi_Q(e)$ be 1 if Q traverses e in its reference direction, -1 if Q traverses e against it, and 0 if e is not on Q. A *step* is a pair of flows f and $f' = f + t\chi_Q$, where Q is a directed cycle and $t \neq 0$ is an element of A, the *value* of the step. The value is determined only up to sign, as reversing Q or exchanging f and f' replaces t by $-t$. The graph $\mathcal{F}(G, A)$ has the flows of G as its vertices, and two flows are adjacent if they form a step. In [1] two different flows are adjacent if they agree outside a cycle Q, and in [2] if the support of their difference is a cycle Q; both give the same graph $\mathcal{F}(G, A)$, because the difference of two such flows is an A-flow that is zero outside Q, and such an A-flow is $t\chi_Q$ for some $t \in A$ and a direction of Q. The graph G is *A-flow-connected* if $\mathcal{F}(G, A)$ is connected; here $\mathcal{F}(G, A)$ counts as connected when it has at most one vertex, in particular when G has no flow.

Let T be a set of non-zero elements of A with $T = -T$. A *T-step* is a step whose value is in T, and two flows are *T-joined* if a chain of T-steps (possibly empty) leads from one to the other.

A *multi-step* with value $t \neq 0$ is a pair of flows f and $f' = f + t(\chi_{Q_1} + \cdots + \chi_{Q_r})$, where $r \geq 1$ and $Q_1, \ldots, Q_r$ are directed cycles, no two of which have a common edge. Every partial sum $f + t(\chi_{Q_1} + \cdots + \chi_{Q_j})$ is then a flow, as it agrees with f or f' on every edge; so a multi-step is a chain of r steps with value t.

Ends of edges. Every edge has two ends, one at each endpoint; a loop at v has both ends at v. The degree of v is the number of ends at v. If v is a vertex of a cycle Q, the *ends of Q at v* are the two ends at v of the edges of Q. For an A-flow f and an end i at v of the edge e, the *outward value* $y_i(f) \in A$ is the value of f on e in the direction that leaves v through i; it is non-zero if f is a flow.

The flow condition at v says

$$\sum_{i \text{ an end at } v} y_i(f) = 0. \tag{1}$$

If a directed cycle Q enters v through the end i and leaves v through the end j, then

$$y_i(f + t\chi_Q) = y_i(f) - t, \qquad y_j(f + t\chi_Q) = y_j(f) + t, \tag{2}$$

and the other ends at v keep their outward values.

Definition 2.1. Let G have no loop. A *balanced pair* of a flow f at a vertex v is a pair of different ends i, j at v with $y_i(f) + y_j(f) = 0$; that is, f enters v through one of them and leaves v through the other with the same value. (Loops are excluded because the two ends of a loop would always form a balanced pair.)

We prove the following three theorems.

Theorem 2.2. *Let A be a finite abelian group. Every even graph in which all degrees are at most 4 is A-flow-connected.*

Theorem 2.3. *Let A be a finite abelian group with $|A| \geq 4$. Every even graph is A-flow-connected.*

Theorem 2.4. *Let $m \geq 4$. In every even graph, any two nowhere-zero $\mathbb{Z}_m$-flows are joined by a chain of steps whose values are in $\{1, -1, 2, -2\}$.*

Sections 5 and 6 prove these theorems, and Section 7 deduces Theorem 1.1 (b) from Theorem 2.4.

3 Reduction to flows with a balanced pair

This section proves Proposition 3.4, the reduction described in the outline. In this section A is a finite abelian group with at least two elements, and T is a set of non-zero elements of A with $T = -T$. Here T is the set of allowed step values ($A \setminus \{0\}$ for Theorems 2.2 and 2.3, and $\{1, -1, 2, -2\}$ for Theorem 2.4). We use two conditions on T:

(T1) any two non-zero elements of A are joined by a sequence of non-zero elements of A in which consecutive elements differ by an element of T;

(T2) there is a non-zero $a \in A$ with $2a = 0$ or $2a \in T$.

Both conditions hold for $T = A \setminus \{0\}$, and also for $A = \mathbb{Z}_m$ with $m \geq 3$ and $T = \{1, -1, 2, -2\}$: the sequence $1, 2, \ldots, m-1$ gives (T1), and $a = 1$ gives (T2). Condition (T1) is used for loops in Lemma 3.1, and (T2) is used in Proposition 3.4 to join the constant flows of Lemma 3.3.

Lemma 3.1 (deleting a loop)**.** *Let T satisfy* (T1)*, and let ℓ be a loop of G. If any two flows of $G - \ell$ are T-joined, then any two flows of G are T-joined.*

Proof. A loop adds the same term to both sides of the flow condition at its vertex. So a flow of G is a flow of $G - \ell$ together with any non-zero value on ℓ. A step of $G - \ell$, with the value on ℓ kept, is a step of G, and a change of the value on ℓ by an element of T is a T-step along the cycle ℓ. Given two flows of G, join their restrictions by T-steps of $G - \ell$, then change the value on ℓ along a sequence as in (T1). □

Splitting off. Figure 4 shows the following operation. Let G have no loop, let v be a vertex, and let $e = vu$ and $e' = vw$ be two different edges at v ($u = w$ is allowed). Let G' be the graph $G - e - e' + g$ with a new edge $g = uw$ (a loop if $u = w$): the first end of g is at u and replaces the end of e at u, and its second end is at w and replaces the end of e' at w. The direction of g *from u to w* is the direction from its first end to its second end; if $u = w$, this fixes one of the two directions of the loop g. For a map $f' \colon E(G') \to A$, let x be the value of f' on g from u to w, and let $Lf' \colon E(G) \to A$ be the map that agrees with f' on the edges other than e, e', g (the common edges of G and G') and has the value x on e in the direction from u to v and on e' in the direction from v to w. Let $B(v; e, e')$ be the set of flows f of G in which the ends of e and e' at v are a balanced pair.

Lemma 3.2 (splitting off)**.** *Let G, v, e, e', G', L and $B(v; e, e')$ be as above.*

(a) L is a bijection from the flows of G' onto $B(v; e, e')$.

(b) If f'_2 comes from f'_1 by a multi-step of G' with value t, then Lf'_2 comes from Lf'_1 by a multi-step of G with value t. In particular, if two flows of G' are T-joined, their images under L are T-joined.

Proof. (a) In Lf' the ends of e and e' at v have the outward values $-x$ and x, so the flow condition holds at v; at u the edge e leaves with value x as g did, and at w the edge e' enters with value x as g did. So Lf' is a flow in $B(v; e, e')$. Conversely, for $f \in B(v; e, e')$ the value of f on e from u to v equals its value on e' from v to w; giving g this value from u to w and keeping the other values defines the only map f' with $Lf' = f$; it is a flow, as the ends of g have the outward values of the ends of e, e' that they replace, and the ends of e, e' at v have outward values with sum 0.

(b) Let Q' be a directed cycle of G'. If g is not on Q', let $W = Q'$, a directed cycle of G. If g is on Q', replace g by the two edges e, e', traversed through v in the direction of Q' (first e from u to v and then e' from v to w if Q' traverses g from u to w, and the reverse otherwise); this gives a closed walk W of G that uses every edge at most once. If $u = w$, then $Q' = g$ and W is the cycle formed by e and e'. If $u \neq w$ and Q' does not pass through v, then W is a directed cycle. If $u \neq w$ and Q' passes through v, then W visits v twice and no other vertex twice, and splitting W at v gives two directed cycles with no common edge. In every case let χ_W be the sum of the maps χ of these one or two directed cycles; then $L(f' + t\chi_{Q'}) = Lf' + t\chi_W$. If $Q'_1, \dots, Q'_r$ are directed cycles of G', no two of which have a common edge, then no two of the directed cycles of G obtained from them have a common edge: they use the edges of the Q'_j, with e, e' instead of g. So L maps a multi-step of G' to a multi-step of G with the same value. □

Part (a) of the next lemma restates [1, Lemma 2.3], which is about Eulerian orientations, in the language of flows; we include its short proof.

Lemma 3.3 (constant flows)**.** *Let G be even and let $a \neq 0$ be an element of A. For an Eulerian orientation D of G, let $a\mathbf{1}_D$ be the constant flow with value a along every arc of D.*

(a) For two Eulerian orientations D_1, D_2 of G, either $a\mathbf{1}_{D_2} = a\mathbf{1}_{D_1}$, or $a\mathbf{1}_{D_2}$ comes from $a\mathbf{1}_{D_1}$ by a multi-step with value $-2a$.

(b) In the situation of Lemma 3.2, $B(v; e, e')$ contains $a\mathbf{1}_D$ for some Eulerian orientation D of G.

Proof. (a) The arcs of D_1 that are reversed in D_2 form a directed graph with in-degree equal to out-degree at every vertex, because D_1 and D_2 are both Eulerian orientations; so they are the union of directed cycles $Q_1, \dots, Q_r$ of D_1, no two of which have a common arc. The reversal of an arc

changes its value along D_1 from a to $-a$. So $a\mathbf{1}_{D_2} = a\mathbf{1}_{D_1} - 2a(\chi_{Q_1} + \cdots + \chi_{Q_r})$. If $2a = 0$ or $r = 0$, the two flows are equal; otherwise this is a multi-step with value $-2a$.

(b) The graph G' is even. Take an Eulerian orientation D' of G', and let D agree with D' on the common edges and direct e and e' through v in the direction of g in D' (from u to v to w if D' directs g from u to w, and the reverse otherwise). Then D is an Eulerian orientation of G and $a\mathbf{1}_D = L(a\mathbf{1}_{D'})$. □

Proposition 3.4 (reduction to a balanced pair)**.** *Let $\mathcal{C}$ be a class of even graphs such that $G - \ell \in \mathcal{C}$ for every $G \in \mathcal{C}$ and every loop ℓ of G, and $G' \in \mathcal{C}$ for every $G \in \mathcal{C}$ with no loop and every graph G' obtained from G as in Lemma 3.2. Let T satisfy* (T1) *and* (T2)*. Suppose that the following statement holds.*

(LOCAL) *For every loopless $G \in \mathcal{C}$ in which every vertex has degree 0 or at least 4, and which has at least one edge, every flow of G is T-joined to a flow that has a balanced pair at some vertex.*

Then in every $G \in \mathcal{C}$ any two flows are T-joined.

Proof. We use induction on the number of edges of G. A graph with no edge has exactly one flow. If G has a loop ℓ, use Lemma 3.1 and the induction hypothesis for $G - \ell$. From now on G has no loop.

If some vertex v has degree 2, its two ends belong to two different edges e, e', and by (1) they are a balanced pair in every flow. So every flow of G is in $B(v; e, e')$, and by Lemma 3.2 and the induction hypothesis for G', any two flows of G are T-joined.

Otherwise G has an edge and every vertex has degree 0 or at least 4. Fix a as in (T2). For every vertex v and any two different edges e, e' at v, any two flows in $B(v; e, e')$ are T-joined, by the induction hypothesis for G' and Lemma 3.2, and $B(v; e, e')$ contains a flow $a\mathbf{1}_D$, by Lemma 3.3 (b). By Lemma 3.3 (a) and (T2), all flows $a\mathbf{1}_D$ are T-joined to one another (if $2a \neq 0$, the value $-2a$ is in T, because $2a \in T$ and $T = -T$). A flow with a balanced pair is in some $B(v; e, e')$, because G has no loop. So all flows with a balanced pair are T-joined to one another, and (LOCAL) joins every flow to one of them. □

4 Reaching a balanced pair when all degrees are 0 or 4

This section proves statement (LOCAL) of Proposition 3.4 for graphs in which every degree is 0 or 4 (Proposition 4.4). We fix generators of the cyclic summands of A and call an edge small if its value is one of these generators or its negative (precise definitions follow). First, steps along cycles change a flow into one in which the edges that are not small form a forest (Corollary 4.2); then, as the graph has an edge and every vertex with an edge has degree 4, the small edges contain a cycle, and one more step along that cycle gives a balanced pair (Lemma 4.3).

Let A be a finite abelian group with at least two elements. Write A as a direct sum of cyclic groups,

$$A = \langle s_1 \rangle \oplus \cdots \oplus \langle s_q \rangle, \qquad s_h \neq 0,$$

so that every element c of A can be written in exactly one way as $c = c_1 + \cdots + c_q$ with $c_h \in \langle s_h \rangle$; the c_h are the *parts* of c. Let $S = \{s_1, -s_1, \ldots, s_q, -s_q\}$. The elements of S are called *small*, and the *color* of $s \in S$ is the index h with $s \in \langle s_h \rangle$. We have $S = -S$ and $0 \notin S$, and S generates A. For a flow f, an edge is *small* if its value is small (this does not depend on the direction); the *color* of a small edge is the color of its value, which is the same in both directions. Let $\rho(f)$ be the number of edges that are not small.

Lemma 4.1 (a cycle with no small edge)**.** *Let f be a flow of a graph G and Q a directed cycle of G none of whose edges is small. Then there is a step $f' = f + t\chi_Q$ with $\rho(f') \le \rho(f) - 1$.*

Proof. Let X be the set of values of the edges of Q in the direction of Q. Then $0 \notin X$ and $X \cap S = \emptyset$. We claim that there are $x \in X$ and $s \in S$ with $x - s \notin X$. If not, then $X - S \subseteq X$, that is, $X + S \subseteq X$ as $S = -S$; adding elements of S repeatedly gives $X + \langle S \rangle \subseteq X$, and as S generates A and X is not empty, $X = A$, contradicting $0 \notin X$. Let $t = s - x$. Then $t \neq 0$ because $x \notin S$. A new value $x' + t$ with $x' \in X$ is 0 only if $x' = x - s$, which is not in X. So $f' = f + t\chi_Q$ is a flow. An edge of Q with value x has the new value s and is small in f'; the edges outside Q did not change; and no edge of Q was small in f. So $\rho(f') \le \rho(f) - 1$. □

Corollary 4.2. *Every flow f of a graph G is joined by a chain of at most $\rho(f)$ steps to a flow in which the edges that are not small form a forest.*

Proof. Apply Lemma 4.1 as long as the edges that are not small contain a cycle. Each application is a single step and lowers ρ by at least 1, and $\rho \ge 0$. □

Lemma 4.3 (a cycle of small edges)**.** *Let G be a loopless graph in which every vertex has degree 0 or 4. Let f be a flow of G with no balanced pair, and Q a cycle of G all of whose edges are small. Then there are a direction of Q and an element $t \neq 0$ of A such that $f + t\chi_Q$ is a flow with a balanced pair.*

Proof. Let v be a vertex of Q, to be chosen below. As v lies on Q, its degree is 4; number the four ends at v as $1, 2, 3, 4$, where 1 and 2 are the ends at v of the two edges of Q at v, and write $y_i = y_i(f)$. Direct Q so that it enters v through the end 1 and leaves through the end 2. For $j \in \{3, 4\}$ let $t_j = y_1 + y_j$, and let j' be the other element of $\{3, 4\}$; t_j is not 0 because f has no balanced pair. By (2) and (1), in $f + t_j\chi_Q$ the end 1 has the outward value $y_1 - t_j = -y_j$ and the end 2 has the outward value $y_2 + t_j = -y_{j'}$. Neither is 0, and the ends 1 and j are a balanced pair. An edge of Q that is not at v gets the value 0 exactly if its value in the direction of Q was $-t_j$. Hence the following holds.

(∗) If no edge of Q that is not at v has the value $-t_j$ in the direction of Q, then $f + t_j\chi_Q$ is a flow with a balanced pair.

In particular, (∗) applies if t_j is not small, because then $-t_j$ is not small either ($S = -S$), while all values on Q are small. Below we use the identity $t_3 + t_4 = 2y_1 + y_3 + y_4 = y_1 - y_2$, which holds by (1).

Case 1: two consecutive edges of Q have different colors. Choose v to be a common vertex of these two edges. Then $y_1 = a$ is small, of color h, and $y_2 = b$ is small, of color $h' \neq h$. Suppose that t_3 and t_4 are both small. Their sum $t_3 + t_4 = a - b$ has exactly two non-zero parts, a in $\langle s_h \rangle$ and $-b$ in $\langle s_{h'} \rangle$. If t_3 and t_4 had the same color, both would lie in one cyclic summand of A, and their sum would have at most one non-zero part; so they have different colors. Then t_3 and t_4 are non-zero elements of two different cyclic summands of A, so by the uniqueness of the parts they are the two non-zero parts of $a - b$, that is, $\{t_3, t_4\} = \{a, -b\}$. So $t_3 = a$ or $t_4 = a$; but $t_3 = a$ means $y_3 = 0$, and $t_4 = a$ means $y_4 = 0$. This is impossible. So t_3 or t_4 is not small, and (∗) gives the step.

Case 2: all edges of Q have the same color h. Here v may be any vertex of Q. Then $y_1, y_2 \in \{s_h, -s_h\}$ and $y_1 + y_2 \neq 0$, so $y_1 = y_2 = a$ and $2a \neq 0$. (If $2s_h = 0$ this case does not occur: the ends $1, 2$ would be a balanced pair.) The values of the edges of Q in the direction of Q are a or $-a$. Now $-t_3 = a$ means $y_3 = -2a$ and then $y_4 = -2a - y_3 = 0$; and $-t_3 = -a$ means $y_3 = 0$. Both are impossible, so (∗) applies with $j = 3$ and gives a step with value t_3. □

Proposition 4.4 (reaching a balanced pair when all degrees are 0 or 4)**.** *Let G be a loopless graph in which every vertex has degree 0 or 4, and which has at least one edge. Every flow f of G is joined by a chain of at most $\rho(f)+1$ steps to a flow with a balanced pair.*

Proof. By Corollary 4.2, f is joined by a chain of at most $\rho(f)$ steps to a flow f^* in which the edges that are not small form a forest. Let $n \geq 1$ be the number of vertices of degree 4. Then G has $2n$ edges, all on these n vertices; at most $n-1$ of them are not small in f^* (a forest on n vertices has at most $n-1$ edges), so at least $n+1$ are small, and the small edges contain a cycle Q. If f^* has no balanced pair, Lemma 4.3 gives a further step to a flow with a balanced pair. □

5 Finite abelian groups: proofs of Theorems 2.2 and 2.3

Proof of Theorem 2.2. If $|A| = 1$, then G has at most one flow (none if it has an edge), so G is A-flow-connected. Let $|A| \geq 2$, let $\mathcal{C}$ be the class of even graphs with all degrees at most 4, and let $T = A \setminus \{0\}$; it satisfies (T1) and (T2). Deleting a loop lowers the degree of its vertex by 2. Splitting off as in Lemma 3.2 lowers the degree of v by 2 and keeps the other degrees (a new loop at $u = w$ counts twice). So $\mathcal{C}$ is closed under both operations. A graph in $\mathcal{C}$ with no loop and no vertex of degree 2 has only vertices of degree 0 or 4, so (LOCAL) is Proposition 4.4. Proposition 3.4 gives the theorem. □

For the proof of Theorem 2.3, a vertex of degree at least 6 is replaced by a triangle (Figure 5); Lemma 5.1 shows that if the new graph is A-flow-connected, then so is the original graph.

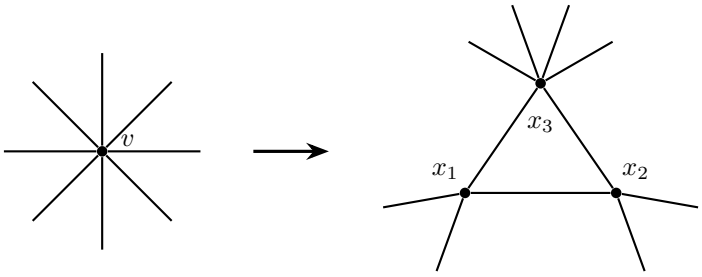


Figure 5: A vertex v of degree $d \geq 6$ (here $d = 8$) is replaced by a triangle $x_1x_2x_3$: two of the edges at v go to x_1, two go to x_2, and the other $d-4$ go to x_3. The new degrees are 4, 4 and $d-2$. This is the replacement used in the proof of [1, Observation 4.6].

Lemma 5.1 (contracting a triangle)**.** *Let $|A| \geq 4$. Let G^+ be a graph and Δ a triangle of G^+ with vertices x_1, x_2, x_3, such that the three edges of Δ are the only edges of G^+ whose two ends are both at vertices of Δ. Let $G = G^+/\Delta$ be the graph obtained by deleting the three edges of Δ and identifying x_1, x_2, x_3 to one vertex x. For a flow f of G^+, let pf be its restriction to $E(G) = E(G^+) \setminus E(\Delta)$.*

(a) pf is a flow of G, and every flow of G is pf for some flow f of G^+.

(b) If f_2 comes from f_1 by a multi-step with value t, then $pf_2 = pf_1$ or pf_2 comes from pf_1 by a multi-step with value t.

So if G^+ is A-flow-connected, then G is A-flow-connected.

Proof. (a) The flow condition of pf at x is the sum of the flow conditions of f at x_1, x_2, x_3, in which the edges of Δ cancel. Let φ be a flow of G. For $i = 1, 2, 3$, let β_i be the sum of the outward values of φ over the ends at x that are ends at x_i in G^+; then $\beta_1 + \beta_2 + \beta_3 = 0$. Give the edges x_1x_2, x_2x_3, x_3x_1 of Δ, in these directions, the values z, $z - \beta_2$, $z + \beta_1$ (the value z on x_1x_2 is free, and the flow conditions at x_2 and x_1 then fix the other two values). The flow conditions at x_1, x_2, x_3 are $\beta_1 + z - (z + \beta_1) = 0$, $\beta_2 + (z - \beta_2) - z = 0$ and $\beta_3 + (z + \beta_1) - (z - \beta_2) = 0$; they hold for every z. The three values are not 0 if $z \notin \{0, \beta_2, -\beta_1\}$, and $|A| \geq 4$ leaves such a z.

(b) Let $f_2 = f_1 + t(\chi_{Q_1} + \cdots + \chi_{Q_r})$ with directed cycles of G^+, no two of which have a common edge. Let H be the directed graph on $V(G)$ whose arcs are the edges of $Q_1, \ldots, Q_r$ outside Δ, each with the direction of its cycle. At every vertex other than x, the in-degree of H equals its out-degree, because this holds for the union of the cycles. It also holds at x: in the union of the cycles, the total in-degree of x_1, x_2, x_3 equals their total out-degree, and every arc of the cycles inside Δ adds one to the total in-degree and one to the total out-degree. So H is the union of directed cycles of G, no two of which have a common arc, and $pf_2 - pf_1$ is t times the sum of their maps χ. If H has no arc, $pf_2 = pf_1$; otherwise this is a multi-step of G with value t.

The last statement follows, because a multi-step is a chain of steps and p is onto by (a). □

Remark 5.2. Lemma 5.1 is the special case of [1, Theorem 5.9(1)] for a triangle, which is A-connected in the sense of [6] when $|A| \geq 4$ (this notion is different from A-flow-connectedness).

Proof of Theorem 2.3. By Lemma 3.1 (with $T = A \setminus \{0\}$), deleting the loops one by one, we may assume that G has no loop. Let $\Phi(G) = \sum_v \max(\deg v - 4, 0)$. We use induction on $\Phi(G)$. If $\Phi(G) = 0$, this is Theorem 2.2. Otherwise let v be a vertex of degree $d \geq 6$. Let G^+ be the graph obtained from G by replacing v with a triangle Δ with vertices x_1, x_2, x_3: two of the ends at v become ends at x_1, two become ends at x_2, and the other $d - 4$ become ends at x_3 (Figure 5). The new degrees are 4, 4 and $d - 2$, so G^+ is even and $\Phi(G^+) = \Phi(G) - 2$. The graph G^+ has no loop, and the edges of Δ are the only edges whose two ends are both at vertices of Δ, because G has no loop at v. Also $G = G^+/\Delta$. By the induction hypothesis G^+ is A-flow-connected, and Lemma 5.1 gives the same for G. □

6 Cyclic groups: steps with values ± 1 and ± 2

This section proves Theorem 2.4. Let $A = \mathbb{Z}_m$ with $m \geq 4$ and $T = \{1, -1, 2, -2\}$. An edge is small for a flow f if its value is 1 or -1 (as in Section 4 with $S = \{1, -1\}$), and $\rho(f)$ is the number of edges that are not small. The value of an edge that is not small is, in each direction, the residue of an integer in $\{2, \ldots, m - 2\}$.

Lemma 6.1 (a cycle with no small edge, in $\mathbb{Z}_m$)**.** *Let f be a flow of a graph G and Q a cycle none of whose edges is small. Then f is T-joined, by steps along Q with value -1, to a flow f' with $\rho(f') \leq \rho(f) - 1$.*

Proof. Fix a direction of Q. For an edge e of Q, let $x_e \in \{2, \ldots, m - 2\}$ be the integer whose residue is the value of e in this direction, and let x be the least of the x_e. For $j = 1, \ldots, x - 1$ the map $f - j\chi_Q$ has on e the residue of $x_e - j$, where $1 \leq x_e - j \leq m - 3$; so it is a flow. In $f' = f - (x - 1)\chi_Q$, an edge with $x_e = x$ has the value 1; the edges outside Q did not change, and no edge of Q was small in f. So $\rho(f') \leq \rho(f) - 1$. □

Lemma 6.2 (a cycle of small edges, in $\mathbb{Z}_m$)**.** *Let G have no loop, let f be a flow with no balanced pair, Q a cycle all of whose edges are small, and v a vertex of Q.*

(a) The two ends of Q at v have the same outward value $a \in \{1, -1\}$.

(b) If an end at v that is not an end of Q has an outward value $c \neq -2a$, then f is T-joined, by steps along Q, to a flow with a balanced pair at v.

Proof. (a) Both values are in $\{1, -1\}$ and their sum is not 0.

(b) Call the two ends of Q at v the end 1 and the end 2, direct Q so that it enters v through the end 1 and leaves through the end 2, and let $t = a + c$. Then $t \neq 0$ (f has no balanced pair), $t \neq a$ ($c \neq 0$), and $t \neq -a$ ($c \neq -2a$). So t is the residue of an integer τ with $2 \leq \tau \leq m - 2$. The values of the edges of Q in the direction of Q are 1 or -1. For $j = 2, 3, \ldots, \tau$ the map $f + j\chi_Q$ has on the edges of Q the residues of $1 + j$ or $-1 + j$, integers between 1 and $m - 1$; so it is a flow. The chain f, $f + 2\chi_Q$, $f + 3\chi_Q$, $\ldots$, $f + \tau\chi_Q$ consists of one step with value 2 and $\tau - 2$ steps with value 1. (The chain does not start with a step with value 1: by (a), one of the two edges of Q at v has the value -1 in the direction of Q, and $f + \chi_Q$ is 0 on that edge.) In $f + \tau\chi_Q = f + t\chi_Q$ the end 1 has the outward value $a - t = -c$ by (2), and the end with the outward value c did not change; so these two ends are a balanced pair at v. □

Proposition 6.3 (reaching a balanced pair in $\mathbb{Z}_m$)**.** *Let $m \geq 4$, and let G be a loopless even graph in which every vertex has degree 0 or at least 4, and which has at least one edge. Every flow of G is T-joined to a flow with a balanced pair.*

Proof. Let $n \geq 1$ be the number of vertices of degree at least 4; all edges are on these vertices, and $|E(G)| \geq 2n$. By Lemma 6.1, used as long as the edges that are not small contain a cycle, a given flow is T-joined to a flow f^* in which they form a forest; so at most $n - 1$ edges are not small, as they lie on the n vertices. Suppose that f^* has no balanced pair (otherwise there is nothing to prove), and that the hypothesis of Lemma 6.2 (b) fails for every cycle Q of small edges and every vertex v of Q. Then at every vertex v of such a cycle, all ends at v that are not ends of the cycle have the outward value $-2a$, where $a \in \{1, -1\}$ is the common outward value at v of the two ends of the cycle (Lemma 6.2 (a)). As $m \geq 4$, $-2a$ is neither 1 nor -1: $-2a = a$ would mean $3 \equiv 0$, and $-2a = -a$ would mean $1 \equiv 0$, modulo m. So the edges of these ends are not small, and every vertex of a cycle of small edges lies on exactly two small edges, the two edges of the cycle. Hence every cycle of small edges is a whole component of the graph formed by the small edges on the n vertices. So every component of this graph is a tree or a cycle (a component that contains a cycle is that cycle), and there are at most n small edges. Then $|E(G)| \leq n + (n - 1) < 2n$, a contradiction. So, for some cycle Q of small edges and some vertex v of Q, Lemma 6.2 (b) joins f^* to a flow with a balanced pair. □

Proof of Theorem 2.4. Apply Proposition 3.4 to the class $\mathcal{C}$ of all even graphs and $T = \{1, -1, 2, -2\}$: (T1) and (T2) hold, as noted at the start of Section 3, and (LOCAL) is Proposition 6.3. □

Remark 6.4. Theorem 2.4 proves Theorem 2.3 again for cyclic groups, without Lemma 5.1. The counting argument of Proposition 6.3 uses that the value $-2a$ in Lemma 6.2 (b) is not small. This holds in $\mathbb{Z}_m$ with $m \geq 4$, but fails, for example, in $\mathbb{Z}_3 \times \mathbb{Z}_3$ with S as in Section 4, where $-2a = a$: then a vertex of a cycle of small edges can lie on more than two small edges, and the bound of at most n small edges in that argument no longer follows. For example, a vertex of degree 6 can have six ends with the same small outward value a: the flow condition holds, as $6a = 0$; there is no balanced pair, as $2a \neq 0$; and all six edges are small.

7 Integer flows: proof of Theorem 1.1 (b)

This section turns a chain of nowhere-zero $\mathbb{Z}_k$-flows from Theorem 2.4 into a walk in $\mathcal{F}(G,k)$: every $\mathbb{Z}_k$-flow of the chain is replaced by a path of k-flows whose residues modulo k are that $\mathbb{Z}_k$-flow. Throughout this section, a *$\mathbb{Z}_k$-flow* means a nowhere-zero $\mathbb{Z}_k$-flow (a flow in the sense of Section 2 for $A = \mathbb{Z}_k$); we name the group to tell these flows apart from the k-flows, whose values are integers.

Definitions. An *integer flow* of a graph G is a map $F\colon E(G) \to \mathbb{Z}$ that satisfies the flow condition at every vertex. For a map x from $E(G)$ to $\mathbb{Z}$ or to the real numbers, the *boundary* $(\partial x)(v)$ at a vertex v is the sum of x over the edges that enter v minus the sum over the edges that leave v (a loop at v is in both sums). So an integer map x is an integer flow if and only if $\partial x = 0$; also $\partial \chi_Q = 0$ for every directed cycle Q.

Let $k \geq 2$. The k-flows, the graph $\mathcal{F}(G,k)$ and k-flow-connectedness are defined in Section 1; in the notation of Section 2, two k-flows are adjacent if they differ by $\lambda\chi_Q$, where Q is a directed cycle and $\lambda \neq 0$ is an integer. As for $\mathcal{F}(G,A)$, the graph $\mathcal{F}(G,k)$ counts as connected when it has at most one vertex, and it is the same as the graph $\mathcal{F}(G,k)$ of $[1,2]$. An *integer lift* of a $\mathbb{Z}_k$-flow f is a k-flow F with $F(e) \equiv f(e)$ modulo k for every edge e. The residues modulo k of a k-flow form a $\mathbb{Z}_k$-flow, because no value of a k-flow is a multiple of k.

Theorem 7.1. *Let $k \geq 4$. Every even graph is k-flow-connected.*

This is part (b) of Theorem 1.1. The proof lifts a chain from Theorem 2.4 step by step. Lemma 7.3 is Tutte's lifting theorem in the form of [2, Theorem 2.3], and Lemma 7.4 is contained in the proof of [2, Lemma 5.5]. We include their short proofs to keep the paper self-contained; the rounding lemma (Lemma 7.2) gives Lemma 7.3 and is used again in the proof of Lemma 7.6.

Lemma 7.2 (rounding)**.** *Let G be a graph and x a map from $E(G)$ to the real interval $[0,1]$ such that $(\partial x)(v)$ is an integer for every vertex v. Then there is a map $z\colon E(G) \to \{0,1\}$ with $\partial z = \partial x$ and with $z(e) = x(e)$ for every edge e with $x(e) \in \{0,1\}$.*

Proof. Call an edge e *fractional* if $0 < x(e) < 1$. If no edge is fractional, take $z = x$. Otherwise the fractional edges contain a cycle Q. If some loop is fractional, it is such a cycle. If no loop is fractional, then no vertex v meets exactly one fractional edge, because the other edges at v have integer values and $(\partial x)(v)$ is an integer; and a graph with no loop, at least one edge and no vertex of degree 1 contains a cycle. Give Q a direction, and let $\delta > 0$ be the least number such that $x + \delta\chi_Q$ has the value 0 or 1 on some edge of Q. The map $x + \delta\chi_Q$ takes values in $[0,1]$, has the same boundary as x, agrees with x on the edges that are not fractional, and has fewer fractional edges than x. After at most $|E(G)|$ such changes no edge is fractional; take z to be the final map. □

For a $\mathbb{Z}_k$-flow f and an edge e, let $r_f(e) \in \{1,\dots,k-1\}$ be the integer whose residue is the value of f on e in the reference direction. Then $\partial r_f \equiv 0$ modulo k at every vertex, and an integer lift of f has on every edge e the value $r_f(e)$ or $r_f(e) - k$.

Lemma 7.3 (integer lifts)**.** *Let $k \geq 2$. Every $\mathbb{Z}_k$-flow f of a graph G has an integer lift.*

Proof. The map $x = r_f/k$ takes values in $(0,1)$ and has integer boundary. Let z be as in Lemma 7.2, and $F = r_f - kz$. Then $\partial F = \partial r_f - k\,\partial z = \partial r_f - k\,\partial x = 0$, $F \equiv f$ modulo k, and $F(e)$ is $r_f(e) \in [1, k-1]$ or $r_f(e) - k \in [-(k-1), -1]$. □

Lemma 7.4 (integer lifts of a given flow)**.** *Let $k \geq 2$ and let f be a $\mathbb{Z}_k$-flow of a graph G. Any two integer lifts of f are joined by a path in $\mathcal{F}(G,k)$ all of whose vertices are integer lifts of f.*

Proof. Let F, F' be integer lifts of f. On every edge both have the value $r_f(e)$ or $r_f(e) - k$, so $\eta = (F' - F)/k$ is an integer flow with values in $\{-1, 0, 1\}$. Direct every edge e with $\eta(e) \neq 0$ so that η is 1 in this direction. By the flow condition these arcs form a directed graph with in-degree equal to out-degree at every vertex, so they are the union of directed cycles $Q_1, \dots, Q_j$, no two of which have a common arc, and $\eta = \chi_{Q_1} + \cdots + \chi_{Q_j}$. Let $F_i = F + k(\chi_{Q_1} + \cdots + \chi_{Q_i})$. On every edge F_i has the value of F or the value of F', so F_i is an integer lift of f; and $F_i - F_{i-1} = k\chi_{Q_i}$. So $F = F_0, F_1, \dots, F_j = F'$ is a path in $\mathcal{F}(G, k)$ of integer lifts of f. □

Lemma 7.5 (a step with value 1)**.** *Let $k \geq 2$, let Q be a directed cycle of a graph G, and let f and $f + \chi_Q$ be $\mathbb{Z}_k$-flows of G. Then for every integer lift F of f, the map $F + \chi_Q$ is an integer lift of $f + \chi_Q$.*

Proof. Choose the reference directions so that Q traverses all its edges in the reference direction. (The statement does not depend on this choice: a change of the reference direction of an edge changes the signs of $F(e)$, of $f(e)$ and of $\chi_Q(e)$.) The map $F + \chi_Q$ is an integer flow with the residues of $f + \chi_Q$. Let e be an edge of Q. Then $r_f(e) \neq k - 1$, because $f + \chi_Q$ is not 0 on e. The value $F(e)$ is $r_f(e)$ or $r_f(e) - k$, so $F(e) + 1$ is in $[2, k - 1]$ or in $[2 - k, -1]$. □

Lemma 7.6 (a step with value 2)**.** *Let $k \geq 3$, let Q be a directed cycle of a graph G, and let f and $f + 2\chi_Q$ be $\mathbb{Z}_k$-flows of G. Then there is an integer lift F of f such that $F + 2\chi_Q$ is an integer lift of $f + 2\chi_Q$.*

Proof. Lemma 7.5 cannot simply be used twice, because $f + \chi_Q$ can have zeros. As in the proof of Lemma 7.5, choose the reference directions so that Q traverses all its edges in the reference direction. No edge of Q has $r_f(e) = k - 2$, because $f + 2\chi_Q$ is not 0 there. Let E_1 be the set of edges e of Q with $r_f(e) = k - 1$, and let

$$x = (r_f + \chi_Q)/k.$$

All values of x are in $(0, 1]$, and $x(e) = 1$ exactly for $e \in E_1$. (The term χ_Q is there for the edges of E_1: it forces $z(e) = 1$ below, so that the lift has the value -1 on these edges and the step leads to the value 1, not to $k + 1$.) As $\partial\chi_Q = 0$, the boundary $\partial x = (\partial r_f)/k$ is an integer at every vertex. By Lemma 7.2 there is $z\colon E(G) \to \{0, 1\}$ with $\partial z = \partial x$ and with $z(e) = 1$ for every $e \in E_1$. Let $F = r_f - kz$. As in the proof of Lemma 7.3, F is an integer lift of f.

Now $F + 2\chi_Q$ is an integer flow with the residues of $f + 2\chi_Q$, and outside Q it equals F. If $e \in E_1$, then $z(e) = 1$, so $F(e) = -1$ and the new value is 1. If e is an edge of Q not in E_1, then $r_f(e) \leq k - 3$; hence $3 \leq F(e) + 2 \leq k - 1$ if $F(e) = r_f(e)$, and $3 - k \leq F(e) + 2 \leq -1$ if $F(e) = r_f(e) - k$. So $F + 2\chi_Q$ is a k-flow. □

Proposition 7.7 (from $\mathbb{Z}_k$-flows to k-flows)**.** *Let $k \geq 3$ and let G be a graph. If any two $\mathbb{Z}_k$-flows of G are joined by a chain of steps whose values are in $\{1, -1, 2, -2\}$, then G is k-flow-connected.*

Proof. Let F and F' be k-flows of G, and let f and f' be the $\mathbb{Z}_k$-flows formed by their residues modulo k. If $f = f'$, then F and F' are two integer lifts of f, and they are joined in $\mathcal{F}(G, k)$ by Lemma 7.4. Otherwise let $f = f_0, f_1, \dots, f_s = f'$ with $s \geq 1$ be a chain of $\mathbb{Z}_k$-flows with $f_i = f_{i-1} + t_i\chi_{Q_i}$ and $t_i \in \{1, -1, 2, -2\}$. If t_i is neither 1 nor 2, replace Q_i by the same cycle with the other direction and t_i by $-t_i$; so we may assume that $t_i \in \{1, 2\}$. For every i from 1 to s, Lemmas 7.3 and 7.5 (if $t_i = 1$) or Lemma 7.6 (if $t_i = 2$) give an integer lift F_{i-1}^+ of f_{i-1} such that $F_i^- = F_{i-1}^+ + t_i\chi_{Q_i}$ is an integer lift of f_i; these two k-flows are adjacent in $\mathcal{F}(G, k)$. By Lemma 7.4, two integer lifts of the same $\mathbb{Z}_k$-flow are joined in $\mathcal{F}(G, k)$: F is joined to F_0^+ (both are integer lifts of f_0), F_i^- to F_i^+ for $0 < i < s$ (of f_i), and F_s^- to F' (of f_s). So F and F' are joined in $\mathcal{F}(G, k)$. □

Proof of Theorem 7.1. By Theorem 2.4 with $m = k$, any two $\mathbb{Z}_k$-flows of an even graph are joined by a chain of steps with values in $\{1, -1, 2, -2\}$; so Proposition 7.7 shows that the graph is k-flow-connected. □

Acknowledgments

This work was done overwhelmingly by AI models and supervised by the author. The models were Anthropic's Claude and OpenAI's ChatGPT. The author takes full responsibility for the paper.

References

[1] D. W. Cranston, J. Li, B. Su, Z. Wang, N. Xu, *Reconfiguration of nowhere-zero flows*, arXiv:2606.24685v1 (June 23, 2026). Statement numbers refer to this version.

[2] L. Esperet, K. Hendrey, A. Lagoutte, M. Marseloo, S. Norin, R. Steiner, *Nowhere-zero flow reconfiguration*, arXiv:2512.17342v4 (July 3, 2026). Statement numbers refer to this version.

[3] D. Goel, *The Cranston–Li–Su–Wang–Xu (2026) conjecture is true, with extensions to graphs with few vertices of odd degree*, Zenodo (2026), doi:10.5281/zenodo.23605838

[4] D. Goel, *Flow reconfiguration with one modulus for all graphs, and Cereceda's conjecture for bipartite* 2*-degenerate graphs*, Zenodo, version 2.30 (October 23, 2026). Statement numbers refer to this version. doi:10.5281/zenodo.23603238

[5] D. Goel, *Nowhere-zero flow reconfiguration in every finite abelian group of order at least* 3024, Zenodo, version 1.20 (October 23, 2026). Statement numbers refer to this version. doi:10.5281/zenodo.23605837

[6] F. Jaeger, N. Linial, C. Payan, M. Tarsi, *Group connectivity of graphs—a nonhomogeneous analogue of nowhere-zero flow properties*, J. Combin. Theory Ser. B **56** (1992), no. 2, 165–182.

[7] W. T. Tutte, *A contribution to the theory of chromatic polynomials*, Canad. J. Math. **6** (1954), 80–91.