THE HUMPHREYS SUPPORT CONJECTURE

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Abstract. We prove the Humphreys support conjecture for connected reductive groups over algebraically closed fields of characteristic $p > h$.

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1. Introduction

1.1. **The support formula.** Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$, and let G be a connected reductive group over $\Bbbk$. Fix a maximal torus and a Borel subgroup. Write X for the character lattice of the torus, $\Phi \subset X$ for the root system, Φ^+ for the chosen positive roots, and X^+ for the dominant weights. A finite-dimensional G-module is tilting if it has a filtration by Weyl modules and a filtration by dual Weyl modules. The indecomposable tilting module of highest weight $\lambda \in X^+$ is denoted by $T_p(\lambda)$. Put $h = \max h_i$, where the h_i are the Coxeter numbers of the irreducible components of Φ; set $h = 1$ for a torus. Throughout the paper we assume

$$p > h.$$

Let G_1 be the kernel of the relative Frobenius morphism $G \to G^{(1)}$, and let $\mathcal{N}_\Bbbk$ be the nilpotent cone. For a finite-dimensional G-module M, its full cohomological support is

$$V_{G_1}(M) = \operatorname{Supp}_{H^{2\bullet}(G_1,\Bbbk)} \operatorname{Ext}^\bullet_{G_1}(M, M). \tag{1.1}$$

The cohomology ring is the coordinate ring of the Frobenius twist $\mathcal{N}_\Bbbk^{(1)}$; see [4, Lemma 8.1]. Using the Frobenius homeomorphism, we regard the support as a closed G-stable subset of $\mathcal{N}_\Bbbk$. Supports throughout this paper are closed sets.

Let W be the finite Weyl group, let $Q = \mathbb{Z}\Phi$, and put $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$. The affine and extended affine Weyl groups and their scaled dot action are

$$W_{\mathrm{aff}} = Q \rtimes W, \qquad W_{\mathrm{ext}} = X \rtimes W, \qquad (t_\nu w) \bullet_p \lambda = p\nu + w(\lambda + \rho) - \rho.$$

Here t_ν denotes translation by ν before scaling by p. The fundamental alcove is

$$\mathscr{A}_0 = \{\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R} : 0 < \langle \lambda + \rho, \alpha^\vee \rangle < p \text{ for all } \alpha \in \Phi^+\},$$

where $\alpha^\vee$ is the coroot of α. Its central directions are unrestricted. The point 0 is regular because $p > h$. For an alcove described by $m_\alpha p < \langle \lambda + \rho, \alpha^\vee \rangle < (m_\alpha + 1)p$, its *lower closure* is obtained by replacing the left inequalities by weak inequalities. Each weight lies in the lower closure of a unique alcove. At a wall, this is the alcove containing $\lambda + \epsilon\eta$ for sufficiently small $\epsilon > 0$, where η is strictly dominant.

Lusztig's correspondence attaches a nilpotent orbit $\mathcal{O}_w$ to each ordinary affine two-sided cell [36]. If the lower-closure alcove of λ is $w \bullet_p \mathscr{A}_0$, put $\mathcal{O}_\lambda = \mathcal{O}_w$. Length-zero elements and central characters preserve this orbit label. For a torus the label is the zero orbit.

Theorem 1.1. *For every connected reductive group G, every prime $p > h$, and every dominant weight λ,*

$$V_{G_1}(T_p(\lambda)) = \overline{\mathcal{O}_\lambda}.$$

In particular, if w is a minimal representative in $W\backslash W_{\mathrm{ext}}$, then

$$V_{G_1}(T_p(w \bullet_p 0)) = \overline{\mathcal{O}_w}.$$

1.2. **Background and previous results.** The support theory for the first Frobenius kernel is rooted in the cohomology of restricted Lie algebras. Friedlander–Parshall developed support varieties for restricted enveloping algebras in 1986 [23]. For a rational representation of a reductive group, restriction to its first Frobenius kernel places the module in this setting. The resulting support is stable under conjugation, so nilpotent orbits provide its natural geometric constituents.

The orbit label in the theorem comes from Lusztig's study of cells in affine Weyl groups. In *Cells in affine Weyl groups. IV* [36], he constructed the bijection between two-sided cells and unipotent classes in the corresponding complex reductive group. Under the nilpotent–unipotent correspondence, these are nilpotent orbits. He also identified the value of the a-function on a cell with the dimension of the Springer fiber of its orbit. For a nilpotent element x, the Springer fiber $\mathcal{B}_x$ is the variety of Borel subgroups whose Lie algebras contain x; write $\mathcal{B}_\mathcal{O}$ for a fiber

with $x \in \mathcal{O}$. Lusztig's construction uses affine Hecke algebras, their asymptotic based rings, and restriction to finite parabolic Hecke algebras. These finite restrictions will also enter our proof.

In 1997, Humphreys proposed the support formula in [28, §12], in a comparison of modular representations of semisimple groups with representations of their Lie algebras. Humphreys asked whether, for indecomposable tilting modules with regular highest weights, it is exactly the orbit closure determined by the affine-cell correspondence. Translation functors extend the formulation to the lower closures of alcoves. Cooper's 2010 work studied support formulae for singular weights and several infinite families of tilting modules [17].

Ginzburg–Kumar's calculation of small-quantum-group cohomology in 1993 provided the corresponding nilpotent-cone support theory for quantum groups at roots of unity [26]. The quantum tilting support formula was established first. In 1998, Ostrik computed the supports of tilting modules for quantum groups of type A [38]. In 2006, Bezrukavnikov described the cohomology of quantum tilting modules by coherent sheaves on the Springer resolution and the nilpotent cone [12]. His description uses the exotic t-structure and perverse-coherent sheaves, and identifies the relevant orbit through the affine-cell correspondence. It gives the quantum support formula in all types; the passage to full self-extension support is also explained in [4, §8.4]. This quantum formula provides the characteristic-zero comparison used below. In 2026, Coulembier proved naturality of small-quantum-group support under restriction to Levi subalgebras for finite-dimensional type-1 modules extending to Lusztig's quantum group, including tilting modules [18, Theorem 4.2.1].

For algebraic groups in positive characteristic, Hardesty proved the conjecture in type A in 2018 [27]. His proof uses the partition description of affine weight cells to obtain an upper bound, and comparison with quantum tilting modules to obtain the opposite inclusion. The resulting formula holds throughout $p > h$; see [27, Theorem 1.4.1] and [4, §8.2]. In 2019, Achar–Hardesty–Riche proved, for every root system and every $p > h$, the lower bound

$$\overline{\mathcal{O}_\lambda} \subseteq V_{G_1}(T_p(\lambda)).$$

They proved equality when p is sufficiently large, with a bound depending on the root system [4, Theorem 9.3]. Their argument develops integral exotic parity objects and compares their coherent supports under specialization.

Tilting characters in characteristic p are governed by the p-canonical basis. Parity sheaves provide its geometric construction [31], and the character theorem of Achar–Makisumi–Riche–Williamson identifies the antispherical p-canonical basis with modular tilting characters for $p > h$ [6]. Ordinary canonical elements give quantum tilting characters. Consequently a character-preserving quantum lift of a modular indecomposable tilting module can have several indecomposable summands. Finite p-cells have also been studied directly. The subregular and submaximal cells in finite classical Weyl groups are described in the 2026 work of Miemietz–Roth–Tubbenhauer [37, Theorems 3.11 and 4.10]. Our finite constructions use ordinary cell bounds and particular p-independent objects inside the finite Hecke categories.

In their 2024 papers, Achar–Hardesty constructed co-t-structures on coherent derived categories of the nilpotent cone and parabolic Springer resolutions [3]. The nonzero pushforwards of exotic parity objects are indecomposable objects in the corresponding cohearts. Their later construction by nilpotent orbits and silting complexes identifies the support of these objects and proves the relative Humphreys conjecture for $p > h$ [2, Theorem 8.1 and Remark 8.2]. This determines the support of $H^\bullet(G_1, T)$ for the tilting modules with nonzero cohomology. In our proof, a coevaluation summand of $T \otimes T^*$ relates that relative support to the full support (1.1).

The ICM 2026 account of Achar–Riche records these developments [7, §3 and Tables 1–2]. Its Conjecture 3.3 is the traditional support formula for weights in the extended affine orbit of 0, and Remark 3.4 explains its equivalence with the formulation for arbitrary dominant

weights. The passage from relative to full support requires a compatibility with ordinary cells [7, Remark 3.11(2)]. Theorem 1.1 proves the traditional support formula throughout $p > h$.

1.3. **Character and Springer conventions.** In the proof up to the final reduction in Section 9, G is assumed to be semisimple and simply connected. We use the positive Kazhdan–Lusztig normalization $C_s = H_s + v$, where H_s is the standard Hecke generator; thus $C_s(1) = 1 + s$. The standard and ordinary canonical Hecke bases are denoted by H_w and C_w, respectively. Write $\mathcal{H}(W)$ and $\mathcal{H}(W_{\mathrm{aff}})$ for the corresponding Hecke algebras. The antispherical module is

$$\mathcal{M}_{\mathrm{asph}} = \mathrm{sgn} \otimes_{\mathcal{H}(W)} \mathcal{H}(W_{\mathrm{aff}}),$$

where H_s acts on sgn by $-v$. The ordinary and p-canonical bases are denoted by N_w and ${}^pN_w$, indexed by the minimal representatives of $W\backslash W_{\mathrm{aff}}$. Under the regular-block character identification, their specialization at $v = 1$ gives quantum and modular tilting characters, respectively [4, 6]. The ordinary two-sided preorder is oriented so that $x \leq_{LR} w$ implies $\mathcal{O}_x \subset \overline{\mathcal{O}_w}$. A *downset* is a union of nilpotent orbits closed under taking orbit closures.

Let $G_{\mathbb{C}}$ be the connected complex reductive group with the same root datum as G. We use the ordinary Springer convention in which the regular nilpotent orbit corresponds to the trivial representation and the zero orbit to the determinant representation. For a finite parabolic Weyl group H, a standard Springer label F acts in the positive Hecke model as $F^+ = \det_H \otimes F$. We compare orbit labels in characteristic zero and characteristic p by their common good-characteristic parametrization. Write ι for the resulting correspondence on unions of orbit closures; it preserves the closure order [4, §4.1]. We use the same orbit notation over $\mathbb{C}$ and $\Bbbk$; the ambient support variety specifies the field.

1.4. **Organization of the proof.** Let q be a primitive p-th root of unity, and let u_q be the small quantum group for the same root system. For a modular tilting module M, let $Q(M)$ be the quantum tilting module having the same character, with all its indecomposable summands retained. We first prove

$$V_{G_1}(M) = \iota V_{u_q}(Q(M)). \tag{1.2}$$

Here quantum support is defined by self-extensions over u_q. The proof uses a coevaluation summand, the relative support theorem, and specialization of an integral exotic parity object at the specified prime.

Equation (1.2) reduces the theorem to an upper bound on the ordinary quantum summands of $Q(T_p(\lambda))$. For a downset $\mathcal{C}$, let $J^q_{\mathcal{C}}$ be the complex span of quantum tilting characters supported in $\mathcal{C}$, and let $J^+_{\mathcal{C}}$ be the span of modular tilting characters whose quantum lifts are supported there. We prove

$$J^+_{\mathcal{C}} = J^q_{\mathcal{C}}. \tag{1.3}$$

Tensor operations and linkage projections generate a coefficient operator algebra. Its stable character spaces correspond to Weyl-stable ideals in the group algebra $\mathbb{C}[X]$, with an additional finite-center stability condition. The resulting correspondence commutes with completion.

At a semisimple torus point, let $M \subset W$ be its stabilizer. The ordinary coefficient ideal is contained in the sum of the polynomial M-isotypic ideals whose local Springer orbits, viewed in the ambient nilpotent cone, lie in $\mathcal{C}$. A geometric comparison at Hecke parameter 1 proves this containment. Kato's canonical trace filtration shows that the sum is generated by the lowest polynomial copies of the Springer representations for trivial local systems. Finite parity objects and tensor operations supply these generators with the required support bounds.

In classical types the construction is uniform in the rank. For the exceptional groups of rank at most six, finite p-canonical calculations provide the sources. In types E_7 and E_8, selected sources and exact finite-group traces give the local generators. The last E_8/A_7 source is obtained from an integral calculation for a trivector double cover. The completed ideals of these finite

constructions are constant on each root-stabilizer stratum. A boundary argument therefore extends the full-rank calculations to the whole torus. Finally, highest-weight triangularity gives the original label $\mathcal{O}_\lambda$, and translation and central isogenies give the statement for all weights and all connected reductive groups. Appendix A contains the finite tables and their verification rules.

2. The whole quantum lift

Let q be a primitive p-th root of unity, let U_q be Lusztig's quantum group, and let u_q be its small quantum group. For $\lambda \in X^+$, let $T_q(\lambda)$ denote the indecomposable quantum tilting module of highest weight λ. We write $\langle n \rangle$ for an internal grading shift and $[n]$ for a cohomological shift. A modular tilting module M has a unique quantum tilting module $Q(M)$, up to isomorphism, with $\operatorname{ch} Q(M) = \operatorname{ch} M$; see [4, Remark 7.3]. Uniqueness follows from the linear independence of indecomposable quantum tilting characters. In particular, it gives objectwise isomorphisms

$$\begin{aligned} Q(M \oplus N) &\cong Q(M) \oplus Q(N), \\ Q(M \otimes N) &\cong Q(M) \otimes Q(N), \\ Q(M^*) &\cong Q(M)^*. \end{aligned} \tag{2.1}$$

These objectwise identities also preserve every direct-summand decomposition.

2.1. **A coevaluation summand.** Put $R_G = H^{2\bullet}(G_1, \Bbbk)$. For a finite-dimensional rational module M, the annihilator of its self-extension module is the kernel of the central action

$$R_G \longrightarrow \operatorname{Ext}^\bullet_{G_1}(M, M).$$

Denote this ideal by I_M. Indeed, a class annihilating the self-extension module kills its identity in degree zero; conversely, the action on every extension is Yoneda composition with the action on that identity.

Lemma 2.1. *Let T be an indecomposable tilting module. There are an indecomposable tilting summand D of $T \otimes T^*$ and a map $\delta : \Bbbk \to D$ such that $\delta \otimes 1_T : T \to D \otimes T$ has a left inverse. Moreover,*

$$I_T = I_D = \operatorname{Ann}_{R_G} H^\bullet(G_1, D) = \operatorname{Ann}_{R_G}(R_G\delta).$$

Proof. Decompose $T \otimes T^* = \bigoplus_i D_i$, and let $\delta_i : \Bbbk \to D_i$ be the components of coevaluation. Evaluation gives maps $s_i : D_i \otimes T \to T$ with

$$\sum_i s_i(\delta_i \otimes 1_T) = 1_T.$$

The finite-dimensional ring $\operatorname{End}_G(T)$ is local. Its nonunits form an ideal, so at least one summand in this sum is invertible. Replacing the corresponding s_i by an invertible multiple gives $s(\delta \otimes 1_T) = 1_T$.

For $a \in H^{2n}(G_1, \Bbbk)$, write a_T for its central action on T. Tensoring the class $a\delta \in \operatorname{Ext}^{2n}_{G_1}(\Bbbk, D)$ with T gives

$$s[2n] \circ ((a\delta) \otimes 1_T) = a_T,$$

because $s(\delta \otimes 1_T) = 1_T$. Thus $a\delta = 0$ implies $a_T = 0$. Conversely, $a_T = 0$ implies $a_{T \otimes T^*} = a_T \otimes 1_{T^*} = 0$; restriction to the summand D, followed by composition with δ, gives $a\delta = 0$. Hence $I_T = \operatorname{Ann}(R_G\delta)$. The retractions

$$D \mid T \otimes T^*, \qquad T \mid D \otimes T$$

give $I_T = I_D$. Finally,

$$I_D \subseteq \operatorname{Ann} H^\bullet(G_1, D) \subseteq \operatorname{Ann}(R_G\delta) = I_T,$$

which proves all equalities. □

The nonzero map δ puts D in the principal linkage component. Write $D = T_p(d \bullet_p 0)$, where $d \in W_{\mathrm{aff}}$ is minimal in Wd. Since $\mathrm{Hom}_G(\Bbbk, D) \neq 0$, the element d is also minimal in dW, and hence in WdW, by [4, Lemmas 8.7–8.8]. The relative support theorem and its ordinary-cell identification [2, Theorem 8.1 and Remark 8.2], together with Lemma 2.1, give

$$V_{G_1}(T) = V_{G_1}(D) = \iota\overline{\mathcal{O}_d}. \tag{2.2}$$

2.2. **Specialization of the entire parity object.** At the fixed prime $p > h$, choose a Bott–Samelson object containing the exotic parity object of the given principal-block tilting. Its degree-zero endomorphism algebra is finite-dimensional over $\mathbb{F}_p$ and splits over a finite extension. Over such an extension choose an indecomposable summand whose base change is the given parity object. Lift its primitive idempotent over a complete discrete valuation ring $\mathbb{O}$, finite over $\mathbb{Z}_p$, to obtain a summand of the integral Bott–Samelson object. Choose embeddings of its fraction field into $\mathbb{C}$ and its residue field into $\Bbbk$. Extension along the latter recovers an isomorphic geometric object. Its standard-Hom modules are finite free and commute with base change. Simultaneous integral representatives for nilpotent orbits can be chosen over such a ring. These assertions follow from [4, Corollaries 3.4 and 3.6, the proof of Lemma 3.11, Lemma 4.5, and §4.4].

Lemma 2.2. *For the summand D of Lemma 2.1,*

$$\iota V_{u_q}(Q(D)) \subseteq V_{G_1}(D).$$

Proof. Let $\mathcal{E}_{\mathbb{O}}$ be an integral exotic parity lift of the object corresponding to D. Its characteristic-zero fiber is a finite direct sum of indecomposable exotic parity objects, with diagonal grading shifts. For every standard object $\Delta_{\mathbb{O}}(\mu)$, the groups

$$\mathrm{Hom}\big(\Delta_{\mathbb{O}}(\mu), \mathcal{E}_{\mathbb{O}}\langle n\rangle[n]\big)$$

are free and compatible with both fibers. Indeed this holds for the Bott–Samelson object and passes to the image of its lifted idempotent. The degrading equivalences identify the finite sums of these ranks with the corresponding standard-filtration multiplicities. Thus the entire generic fiber has quantum character $\mathrm{ch}\, D$, and its quantum image is $Q(D)$. The principal root-lattice component satisfies the equivariance condition in the quantum comparison [4, Proposition 9.1 and Remark 9.2].

Write $\widetilde{\pi}$ for the Springer map over $\mathbb{O}$. After forgetting equivariance, consider

$$\mathcal{K}_{\mathbb{O}} = R\widetilde{\pi}_* R\mathcal{H}om(\mathcal{E}_{\mathbb{O}}, \mathcal{E}_{\mathbb{O}}).$$

The Springer scheme over $\mathbb{O}$ is regular, so $\mathcal{E}_{\mathbb{O}}$ is perfect. The map is proper and $\mathcal{K}_{\mathbb{O}}$ is a bounded coherent complex. For a simultaneous integral nilpotent representative $x : \mathrm{Spec}\,\mathbb{O} \to \mathcal{N}_{\mathbb{O}}$, its derived pullback is a bounded-above pseudo-coherent complex over $\mathbb{O}$, with finitely generated cohomology in each degree. The integral Springer scheme and nilpotent cone are flat over $\mathbb{O}$, so their scalar-change squares are Tor-independent. Proper base change identifies the generic and special fibers of this pullback with the pullbacks of the respective self-Hom pushforwards. If its special fiber were zero, derived Nakayama over $\mathbb{O}$ would make the pullback zero, and hence its generic fiber zero. Therefore every nonzero generic fiber has nonzero special fiber.

The self-Hom of each generic indecomposable summand is a direct summand of the self-Hom of the whole generic object. Hence the support of every such summand specializes into the special self-extension support. The modular and quantum coherent support comparisons identify these supports with the two sides of the asserted inclusion. □

Theorem 2.3. *For every tilting G-module M,*

$$V_{G_1}(M) = \iota V_{u_q}(Q(M)).$$

Proof. First let $M = T$ be indecomposable, and choose D as above. The top quantum tilting $T_q(d \bullet_p 0)$ occurs in $Q(D)$ with multiplicity one. Quantum support theory gives

$$\overline{\mathcal{O}_d} \subseteq V_{u_q}(Q(D)).$$

Together with Lemma 2.2 and (2.2), this proves $\iota V_{u_q}(Q(D)) = V_{G_1}(T)$. The objectwise identities (2.1) transfer the two retractions in Lemma 2.1 to quantum tilting modules:

$$Q(D) \mid Q(T) \otimes Q(T)^*, \qquad Q(T) \mid Q(D) \otimes Q(T).$$

For quantum modules A, B, the central cohomology action on $A \otimes B$ is obtained by tensoring the action on A with 1_B, so $V_{u_q}(A \otimes B) \subseteq V_{u_q}(A)$. Apply this inclusion and summand monotonicity to the two retractions to obtain $V_{u_q}(Q(T)) = V_{u_q}(Q(D))$. For arbitrary M, take its finite decomposition into indecomposable tiltings and use the direct-sum formula for support. □

Corollary 2.4. *If*

$${}^pN_w = \sum_x m_{x,w}(v) N_x,$$

then

$$V_{G_1}(T_p(w \bullet_p 0)) = \bigcup_{m_{x,w} \neq 0} \iota\overline{\mathcal{O}_x}.$$

In particular, it contains $\iota\overline{\mathcal{O}_w}$.

Proof. The Laurent polynomials $m_{x,w}$ have nonnegative integral coefficients, and $m_{w,w} = 1$. Their values at 1 are the quantum summand multiplicities in $Q(T_p(w \bullet_p 0))$. Apply Theorem 2.3 and the quantum support formula. □

3. Actual character operators and coefficient ideals

All vector spaces and algebras in this section are over $\mathbb{C}$. The characteristic p remains the fixed modular characteristic. Put

$$R = \mathbb{C}[X], \qquad A = R^W, \qquad \Delta = \sum_{w \in W} \det(w) e^{w\rho}, \qquad \psi_p(e^\nu) = e^{p\nu}.$$

Multiplication by Δ identifies the character ring A with the anti-invariants $R^{\det}$. An *actual character operation* is tensoring with a modular tilting module, projection to a linkage component, or a composite of these operations. We take their complex linear span only after they have been applied to actual modules.

3.1. **The large operator algebra.** Let $\mathscr{B}$ be the operator algebra on $R^{\det}$ generated by multiplication by A and by the projections onto W-orbits of residues in X/pQ. The shift by ρ has already been incorporated in the numerator model. These are exactly the available linkage projections, with a projection understood as the sum of categorical block projections when necessary. Since tilting characters form a basis of A, multiplication by any element of A is a linear combination of actual tensor operations.

Set

$$R_0 = \mathbb{C}[pQ], \qquad D_0 = R \rtimes \mathrm{Hom}(X/pQ, \mathbb{C}^\times), \qquad E_0 = D_0^W.$$

Fourier projections onto individual residue classes, followed by multiplication by Laurent monomials, give all matrix units. Consequently

$$D_0 = \mathrm{End}_{R_0}(R). \tag{3.1}$$

The action of E_0 on $R^{\det}$ is faithful. To see this, let $e_{\det} = |W|^{-1} \sum_w \det(w) w$ in $D_0 \rtimes W$. The corner $e_{\det}(D_0 \rtimes W)e_{\det}$ is identified with E_0 by $a \mapsto a e_{\det}$. Distinct automorphisms of the fraction field of R are linearly independent over that field, so the crossed product acts faithfully

on R. If $a \in E_0$ kills $R^{\det}$, then $ae_{\det}$ kills R; its identity-W coefficient gives $a = 0$. In particular, equalities of the stated character operators are equalities in E_0.

Proposition 3.1. *For $p > h$, the actual operator algebra is E_0.*

Proof. Let f be the principal linkage projector. Since 0 lies in the fundamental alcove, the weights $w\rho$ are distinct modulo pQ, and

$$fR = \bigoplus_{w \in W} R_0 e^{w\rho}.$$

This is the left regular module of $R_0 \rtimes W$. It follows that

$$fE_0 f = \mathrm{End}_{R_0 \rtimes W}(fR) \cong (R_0 \rtimes W)^{\mathrm{op}}. \tag{3.2}$$

Contracting pQ to Q, this is the opposite affine group algebra. On anti-invariants the corresponding map is

$$g \longmapsto \Delta^{-1} \sum_{w \in W} \det(w) e^{w\rho} \psi_p(wg), \qquad g \in \mathbb{C}[Q].$$

Under the antispherical identification, right translation by ν acts by multiplication by e^{ν}, while a finite Weyl element acts by its determinant times the inverse polynomial action. The wall-crossing formula [4, §7.2, equation (7.1)] is therefore precisely $\Theta_s = 1 + s$. Subtracting the identity obtains every affine simple reflection, and these generate the affine group algebra. Thus $f\mathscr{B}f = fE_0 f$.

It remains to prove that f is full in $\mathscr{B}$. Represent each linkage class by an integral point μ in the closed fundamental alcove, and let W_J be its finite affine stabilizer. The representative μ need not be dominant. Translation out of the μ-block and back has character

$$T_0^{\mu} T_{\mu}^{0} = |W_J| \mathrm{id}. \tag{3.3}$$

Indeed, for a standard character with strongly minimal double-coset label y, translation out has the standard characters with labels yz, $z \in W_J$, each once; translation back sends each to the original standard character. This is the translation principle [7, §2.5(a),(b), following equation (2.4)]. On characters, translation is multiplication by the character of a finite-dimensional module, preceded and followed by linkage projections. That character belongs to A, which is spanned by tilting characters. Its translation operator therefore lies in $\mathscr{B}$. Hence, writing P_μ for the linkage projector,

$$P_\mu = |W_J|^{-1} T_0^{\mu} f T_{\mu}^{0} \in \mathscr{B} f \mathscr{B}.$$

There are finitely many residue orbits. Summing gives $1 = \sum_i b_i f c_i$, with $b_i, c_i \in \mathscr{B}$. For any $x \in E_0$,

$$x = \sum_{i,j} b_i (f c_i x b_j f) c_j.$$

The middle factor belongs to $fE_0 f = f\mathscr{B}f$. Thus $x \in \mathscr{B}$, as required. □

3.2. **The small determinant corner.** Put $R_p = \mathbb{C}[pX]$ and

$$D = \mathrm{End}_{R_p}(R) = R \rtimes \mathrm{Hom}(X/pX, \mathbb{C}^\times), \qquad \mathscr{S} = D \rtimes W.$$

Let

$$Z = \mathrm{Hom}(X/Q, \mathbb{C}^\times), \qquad \tau_z(e^\lambda) = z(\lambda) e^\lambda \quad (z \in Z).$$

Every prime divisor of $|X/Q|$ is at most h, so p is prime to $|Z|$. The subgroups $\mathrm{Hom}(X/pX, \mathbb{C}^\times)$ and Z of $\mathrm{Hom}(X/pQ, \mathbb{C}^\times)$ have coprime orders and their product has order $|X/pQ|$. Consequently

$$\begin{aligned} \mathrm{Hom}(X/pQ, \mathbb{C}^\times) &= \mathrm{Hom}(X/pX, \mathbb{C}^\times) \times Z, \\ E_0 &= D^W \rtimes Z. \end{aligned} \tag{3.4}$$

For the second equality, first write $D_0 = D \rtimes Z$. The Weyl group acts trivially on X/Q, so taking W-invariants coefficientwise in the Z-basis gives $(D \rtimes Z)^W = D^W \rtimes Z$.

Lemma 3.2. *The weights $w\rho$, $w \in W$, are pairwise distinct modulo pX.*

Proof. If $w\rho - \rho = p\nu$, the extended affine element $t_{-\nu}w$, acting with translation scale p, fixes ρ in the unshifted fundamental alcove. It therefore belongs to the stabilizer Ω of that alcove. The affine wall coordinates of ρ are

$$(p - h + 1, 1, \ldots, 1).$$

The affine coordinate is the unique coordinate greater than 1. An element of Ω fixing ρ must fix the affine wall and its opposite vertex 0. It then has zero translation part; an alcove-preserving element of W is the identity. This proves the assertion in an irreducible root system, and products follow factor by factor. □

Proposition 3.3. *There is an inclusion-preserving bijection between character subspaces $J \subset A$ stable under actual tensor operations and linkage projections, and ideals $I \subset R$ stable under W and all central translations τ_z, $z \in Z$. For each such subspace,*

$$\Delta J = (R\psi_p(I))^{\det}. \tag{3.5}$$

In particular, this construction applies to the modular and quantum support subspaces. It is an equality of modules and ideals, including their completions.

Proof. By Lemma 3.2, the module R contains the $R_p \rtimes W$-stable direct summand

$$\bigoplus_{w \in W} R_p e^{w\rho} \cong R_p \otimes_{\mathbb{C}} \mathbb{C}[W].$$

The right-hand side is the regular module of $R_p \rtimes W$. Moreover R is projective over this skew group ring: it is free over R_p, and a splitting can be averaged over W. It is thus a projective generator, as is its determinant twist. Equivariant matrix Morita equivalence now says that $e_{\det}$ is full in $\mathscr{S}$.

A submodule of the natural $D = \mathrm{End}_{R_p}(R)$-module R has the form RI_p for a unique ideal $I_p \subset R_p$. It is $\mathscr{S}$-stable precisely when I_p is W-stable. Fullness of $e_{\det}$ gives a bijection between these submodules and the submodules of $R^{\det}$ for its corner D^W. Transport I_p through $\psi_p : R \to R_p$. This proves (3.5), its uniqueness, and preservation of inclusions for D^W-submodules. For $z \in Z$, one has

$$\tau_z(\Delta) = z(\rho)\Delta, \qquad \tau_z\psi_p = \psi_p\tau_{z^p}.$$

Multiplication by the nonzero scalar $z(\rho)$ preserves each subspace, and $z \mapsto z^p$ permutes Z. Thus the character submodule is Z-stable precisely when its coefficient ideal is Z-stable. Proposition 3.1 and (3.4) now give the stated correspondence for all actual operators.

All rings are Noetherian, and the modules in question are finite over R_p^W. Hence the module and ideal identities commute with flat completion. □

Remark 3.4. In the aggregate principal component modulo pX, the coefficient map is

$$R \longrightarrow A, \qquad g \longmapsto \Delta^{-1} \sum_{w \in W} \det(w) e^{w\rho} \psi_p(wg).$$

It is the extended antispherical polynomial model

$$\mathrm{sgn} \otimes_{\mathbb{C}[W]} \mathbb{C}[W_{\mathrm{ext}}] \longrightarrow R, \qquad 1 \otimes t_\nu w \longmapsto \det(w) e^{w^{-1}\nu}.$$

The length-zero components are realized by translation equivalences between regular linkage classes. Their character operators belong to $\mathscr{B}$ by the same multiplication and linkage-projection formula.

3.3. **The exact local tensor operation.** Write $\mathbb{T} = \operatorname{Spec} R$, put $Y = \operatorname{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$, and write $y^{\lambda} = e^{\lambda}(y)$. For $y \in \mathbb{T}$, let $M = W_y$ and $S_y = \widehat{\mathcal{O}}_{\mathbb{T},y}$. The centralizer of y in $G_{\mathbb{C}}$ is connected, since the ambient derived group is simply connected. Thus M is the reflection group of the roots α with $\alpha(y) = 1$. Logarithmic coordinates $u \in Y \otimes_{\mathbb{Z}} \mathbb{C}$, defined by

$$e^{\lambda}(y \exp u) = y^{\lambda} \exp\langle \lambda, u \rangle,$$

identify S_y with the power-series ring on this vector space, carrying the natural linear action of M.

If $t^p = y$, then $W_t \subseteq M$. Denote this subgroup by L_t, and denote its linear reflection discriminant by Δ_{L_t}. For two operator-stable character families with coefficient ideals I, J, let $I \star J$ be the coefficient ideal of the operator closure of their actual tensor products.

Proposition 3.5. *In the completed local ring at y,*

$$\widehat{(I \star J)}_y = \sum_{t^p = y \ /M} S_y M \Big[\Delta_{L_t} (\widehat{I}_y : \Delta_{L_t})^{L_t} (\widehat{J}_y : \Delta_{L_t})^{L_t} \Big]. \tag{3.6}$$

The notation on the right means the ideal generated by the M-translates of all the indicated products.

Proof. Let $P \subset A$ be the character span of the tensor products. Both input character spaces are Z-stable by Proposition 3.3, and τ_z is an algebra automorphism of A. Hence P, and therefore ΔP, is Z-stable. Since Z normalizes D^W, the submodule $D^W(\Delta P)$ is also Z-stable. Equation (3.4) gives

$$E_0(\Delta P) = D^W(\Delta P).$$

It therefore suffices to compute the determinant-corner closure.

Complete first over the W-orbit of y in the base torus. The characteristic-zero p-power map is finite etale and splits over the completion into its p-th-root local factors. In the factor at t, an anti-invariant numerator is an L_t-anti-invariant element of $\widehat{I}_y$. The Weyl numerator has the form

$$u_t \Delta_{L_t},$$

where u_t is an invertible L_t-invariant series: the vanishing roots give the discriminant, the other factors are units, and both numerator and discriminant transform by the same determinant character. Thus the local character module is

$$u_t^{-1} (\widehat{I}_y : \Delta_{L_t})^{L_t}.$$

The analogous expression holds for J. Multiplying characters and restoring one Weyl numerator gives the term in (3.6), up to the invariant unit u_t^{-1}. That unit can be absorbed in either invariant colon module.

The matrix algebra $\operatorname{End}_{R_p}(R)$ extracts every coefficient component; after the etale splitting, this is the same as extraction of every root factor. Weyl saturation gives the outer M-span. Completion commutes with these products and images because all modules are finite over the invariant base. Their image submodules are adically closed, proving (3.6). □

Corollary 3.6. *Suppose a source at y contains a copy of an irreducible M-module E. Let $L = W_t$ occur for a p-th root of y. If a second source contains Δ_L and $\operatorname{Res}^M_L E$ contains $\det_L$, their tensor source retains a copy of E. It retains the whole E-isotypic ideal if the first source contains that ideal. Its support is bounded by the intersection of the two source bounds.*

Proof. Choose a nonzero L-anti-invariant vector g in the given copy. It has the form $\Delta_L h$ with $h \in S_y^L$. The L-term of (3.6) contains $\Delta_L h \cdot 1 = g$. Its M-translates give the whole copy. Apply this argument in every degree for an entire isotypic ideal. The support assertion follows from tensor-product support for G_1 and Theorem 2.3. □

3.4. **The two support character spaces.** For a closed nilpotent downset $\mathcal{C}$ in characteristic zero, define

$$J^q_{\mathcal{C}} = \operatorname{span}_{\mathbb{C}}\{\operatorname{ch} T_q(\lambda) : V_{u_q}(T_q(\lambda)) \subseteq \mathcal{C}\},$$
$$J^+_{\mathcal{C}} = \operatorname{span}_{\mathbb{C}}\{\operatorname{ch} T_p(\lambda) : V_{u_q}(Q(T_p(\lambda))) \subseteq \mathcal{C}\}.$$

The whole-lift theorem also describes the second space using modular support in $\iota\mathcal{C}$. Both are coordinate subspaces in their respective tilting bases and are stable under actual operations. Let $I_{\mathcal{C}}$ and $I^+_{\mathcal{C}}$ be the coefficient ideals corresponding to $J^q_{\mathcal{C}}$ and $J^+_{\mathcal{C}}$, respectively. Then

$$I^+_{\mathcal{C}} \subseteq I_{\mathcal{C}}. \tag{3.7}$$

In the extended antispherical model, $I_{\mathcal{C}}$ is the coefficient image at $v = 1$ of the ordinary two-sided based ideal for $\mathcal{C}$. This follows by applying the quantum support theorem to its canonical antispherical basis and the coefficient identification of Remark 3.4. The next section describes this ideal near each semisimple point.

4. Ordinary cell ideals and Springer generators

4.1. **Finite generation at parameter one.** Let J be the asymptotic algebra of the equal-parameter affine Hecke algebra. We use its usual basis t_w. Write $C_xC_y = \sum_z h_{x,y,z}(v)C_z$. Lusztig's a-function is defined by

$$a(z) = \min\{n \geq 0 : v^n h_{x,y,z}(v) \in \mathbb{Z}[v] \text{ for all } x, y\};$$

its common value on a two-sided cell $\mathbf{c}$ is denoted by $a(\mathbf{c})$. For a finite standard parabolic subgroup W_I, let J^I be its finite asymptotic subalgebra. Over $\mathbb{C}$, Xi's finite-parabolic generation theorem [40, Theorem 3.1] gives

$$J_{\mathbb{C}} = \sum_{I:\, W_I \text{ finite}} J_{\mathbb{C}} J^I_{\mathbb{C}} J_{\mathbb{C}}. \tag{4.1}$$

The statement applies to the affine and extended groups used here. The maximal extended algebra is the crossed product of the affine algebra by its length-zero group Ω. Conjugation by Ω permutes the finite standard parabolics. The ideal generated by their affine subalgebras is therefore Ω-stable. Equation (4.1) in the crossed product implies the same equality in its identity homogeneous component. It then implies the equality in every intermediate extended group. The argument also applies to products.

Lemma 4.1. *Let B be a unital algebra with multiplication $*$. Suppose a second associative multiplication μ on its underlying vector space satisfies the two mixed associativity identities*

$$\mu(x * y, z) = x * \mu(y, z), \qquad \mu(x, y * z) = \mu(x, y) * z.$$

Writing e for the $*$-unit and $d = \mu(e, e)$, one has $\mu(x, y) = x * d * y$. Suppose unital subalgebras $S_i \subset B$, with units e_i, generate B as a $*$-ideal, and that $d_i = e_i * d * e_i$ is invertible in S_i. Then the same subspaces generate the whole vector space as a μ-ideal.

Proof. The first assertion follows by inserting e into the mixed identities. Let $u_i = d_i^{-1}$ in S_i. Since $u_i = e_i * u_i = u_i * e_i$,

$$u_i * d * e_i * d * u_i = u_i * d_i * d_i * u_i = e_i.$$

Choose a finite expression $e = \sum_j b_j * e_{i_j} * c_j$. For every $x \in B$,

$$\begin{aligned}\sum_j \mu(x * b_j * u_{i_j}, \mu(e_{i_j}, u_{i_j} * c_j)) \\ &= \sum_j x * b_j * u_{i_j} * d * e_{i_j} * d * u_{i_j} * c_j \\ &= \sum_j x * b_j * e_{i_j} * c_j = x.\end{aligned}$$

This is an expression in the μ-ideal generated by the S_i. □

Proposition 4.2. *Let $\mathscr{U}$ be a downset of ordinary affine two-sided cells. At $v = 1$, its based ideal is generated by its intersections with the finite standard parabolic Hecke algebras:*

$$\mathcal{H}_{\mathscr{U}}(1) = \sum_{I:\, W_I \text{ finite}} \mathbb{C}[W_{\mathrm{ext}}]\ \mathrm{span}_{\mathbb{C}}\{C_x(1) : x \in W_I \cap \mathscr{U}\}\ \mathbb{C}[W_{\mathrm{ext}}]. \tag{4.2}$$

Each finite intersection is a sum of complete matrix blocks of $\mathbb{C}[W_I]$.

Proof. For a two-sided cell $\mathbf{c}$, take its exact ordinary layer $\mathcal{H}_{\leq \mathbf{c}}/\mathcal{H}_{<\mathbf{c}}$. Identify its basis with the asymptotic cell algebra $J_{\mathbf{c}}$. The equal-parameter bimodule identity [40, §2.1(d)] gives $\mu(x * y, z) = x * \mu(y, z)$ on each a-layer. The common anti-involution $t_w \mapsto t_{w^{-1}}$, $C_w \mapsto C_{w^{-1}}$ gives $\mu(x, y * z) = \mu(x, y) * z$. P4 and P11 restrict these identities to the exact cell layer: every output cell is below the input cell, and equality of a-values forces the two cells to coincide. Thus Lemma 4.1 applies to $J_{\mathbf{c}}$.

Projecting (4.1) to $J_{\mathbf{c}}$ gives generation by

$$S_{I,\mathbf{c}} = \mathrm{span}_{\mathbb{C}}\{t_w : w \in W_I \cap \mathbf{c}\}.$$

Finite-parabolic compatibility of the Kazhdan–Lusztig structure constants and equality of intrinsic and ambient a-values (property P12) identify this space with a direct sum of finite intrinsic cell algebras, with its own asymptotic unit $e_{I,\mathbf{c}}$. At $v = 1$, the ordinary product on the same vector space is the quotient of the two ideals

$$\mathbb{C}[W_I] \cap \mathcal{H}_{\leq \mathbf{c}}(1), \qquad \mathbb{C}[W_I] \cap \mathcal{H}_{<\mathbf{c}}(1).$$

The group algebra $\mathbb{C}[W_I]$ is semisimple. Its indicated ideal quotient therefore has a unit f_I. If its product is $x * d_I * y$, substituting the asymptotic unit in the ordinary unit identities gives

$$f_I * d_I = e_{I,\mathbf{c}} = d_I * f_I.$$

Thus f_I is the required inverse in the specialized asymptotic corner.

Lemma 4.1 now gives generation of each exact ordinary cell layer. Induction through a linear extension of the finite cell order, with lower cells first, gives (4.2). Finally a two-sided ideal of a finite semisimple algebra is a sum of its full matrix blocks. □

4.2. **The orbit of a finite source.** If a finite affine parabolic has projected group $H \subset W$, a finite two-sided family has a unique special representation F_{sp} in the standard Springer normalization. For an irreducible H-module F, let $b(F)$ be the least degree in which F occurs in the polynomial ring on the reflection representation of H, or equivalently in its coinvariant algebra.

Proposition 4.3. *The ambient affine cell $\mathbf{c}$ containing that finite family has orbit*

$$\mathcal{O}(j_H^W(F_{\mathrm{sp}})).$$

Here j is induction truncated at the smallest coinvariant degree. It gives the Springer representation of a trivial local system. The Springer-fiber dimension of its orbit is

$$a(\mathbf{c}) = b(F_{\mathrm{sp}}).$$

Before applying this rule, a family in the positive Hecke convention is twisted by $\det_H$.

Proof. First pass to the adjoint quotient, which has the same affine Coxeter group and finite standard parabolics. In this setting Lusztig defines the orbit by truncated induction in [36, §6.5], and identifies its Springer fiber dimension with the finite a-value. The finite asymptotic module occurs in a simple ambient asymptotic module by [36, §5.4]. The multiplicity persists in its generic Hecke counterpart by [36, §5.6(b)]. Theorem 4.8 of that paper assigns the ambient cell and its a-value to the latter module. Its Lemma 7.2 puts the associated nilpotent element in the closure of the orbit defined in §6.5, and puts it in that orbit when the Springer fiber dimensions agree. P12 supplies that equality. Sections 4.10–4.12 of [36] extend the classification from the adjoint case to the affine and extended groups of the given root datum. The length-zero group acts trivially on affine two-sided cells, and the orbit labels correspond under the central isogeny. The finite Weyl group, Springer fibers, and their trivial-local-system representations are unchanged. Thus the same truncated-induction label applies to the present group. The determinant twist is the positive-normalization convention already fixed above. □

4.3. **The local support containment.** Let $y \in \mathbb{T}$, $M = W_y$, $G_y = Z_{G_{\mathbb{C}}}(y)$, and $\mathfrak{g}_y = \mathrm{Lie}(G_y)$. Let $\mathcal{C}_y = \mathcal{C} \cap \mathcal{N}(\mathfrak{g}_y)$. For $E \in \mathrm{Irr}(M)$, write $\mathcal{O}_M(E)$ for its ordinary Springer orbit. In this notation the local system is part of the label E, although the support condition only uses its orbit. Define the completed Springer downset ideal

$$K_{\mathcal{C},y} = \sum_{\mathcal{O}_M(E) \subseteq \mathcal{C}_y} S_y \, S_y[E],$$

where $S_y[E]$ is the whole natural E-isotypic component. Write $\mathfrak{m}_y$ for the maximal ideal of S_y.

Proposition 4.4. *For every downset* $\mathcal{C}$ *and every* $y \in \mathbb{T}$,

$$\widehat{I}_{\mathcal{C},y} \subseteq K_{\mathcal{C},y}.$$

Moreover, the completion of the ordinary based ideal for $\mathcal{C}$ *is generated by constant matrix blocks in* $\mathrm{Mat}_{[W:M]}(S_y \rtimes M)$. *Every selected block, expressed in the natural Springer convention, has a label* E *with* $\mathcal{O}_M(E) \subseteq \mathcal{C}_y$.

Proof. Xi's theorem [41, Theorem 1.5] identifies the rational ordinary based ideal with the image of equivariant coherent K-theory of the Steinberg correspondence supported over $\mathcal{C}$. Denote this ideal by J and its ambient rational Hecke algebra by $\mathcal{H}$. The complementary Kazhdan–Lusztig basis makes $\mathcal{H}/J$ free over $\mathbb{Q}[v, v^{-1}]$. Hence the inclusion induces an injection

$$J/(v-1)J \hookrightarrow \mathcal{H}/(v-1)\mathcal{H}.$$

By right exactness of specialization, the supported K-theory map onto J specializes to a surjection onto $J/(v-1)J$. Its image in the specialized Hecke algebra is therefore exactly this embedded ideal.

Put $n = [W : M]$. Completing the crossed-product algebra at $[y]$ and splitting the points of Wy gives

$$\widehat{R \rtimes W}_{[y]} \cong \mathrm{Mat}_n(S_y \rtimes M). \tag{4.3}$$

Choose $r_z \in W$ with $r_z z = y$ for each $z \in Wy$. A finite affine parabolic, projected to $H \subset W$, has a rational fixed point $\xi \in X \otimes_{\mathbb{Z}} \mathbb{Q}$; its elements are $t_{\xi - h\xi} h$, with $\xi - h\xi \in X$. Write u for the logarithmic coordinate at y. The matrix entry from the z-factor to the hz-factor is

$$(hz)^{\xi - h\xi} \exp\langle r_{hz}\xi - m r_z \xi, u\rangle \, m, \qquad m = r_{hz} h r_z^{-1} \in M.$$

Let g be the diagonal matrix with entries $g_z = \exp\langle r_z \xi, u\rangle$. Conjugation by g^{-1} on the left and g on the right cancels the exponential, since $m(g_z) = \exp\langle m r_z \xi, u\rangle$. The conjugated finite group therefore lies in $\mathrm{Mat}_n(\mathbb{C}[M])$, with the residual scalars $(hz)^{\xi - h\xi}$.

Let $e_F \in \mathbb{C}[H]$ be the central idempotent of a full finite matrix block. In the semisimple algebra $\mathrm{Mat}_n(\mathbb{C}[M])$, the two-sided ideal generated by its image is the direct sum of the blocks on whose simple modules that image acts nontrivially. Indeed, a nonzero element of a simple matrix algebra generates that algebra as a two-sided ideal, and the central block projections separate the factors. Extending this equality to $\mathrm{Mat}_n(S_y \rtimes M)$ shows that the full completed ideal is generated by the same constant blocks. Inner conjugation by g preserves every two-sided ideal. Proposition 4.2 now proves constant-block generation for the completed ordinary based ideal.

Let $\mathfrak{m}_{S_y^M}$ be the maximal ideal of the invariant power-series ring. The central fiber of (4.3) is

$$\mathrm{Mat}_n((S_y/\mathfrak{m}_{S_y^M} S_y) \rtimes M).$$

The positive-degree coinvariant ideal is nilpotent, and its quotient is $\mathrm{Mat}_n(\mathbb{C}[M])$. This quotient is semisimple, so the coinvariant ideal is the Jacobson radical. Every selected constant block therefore survives in the semisimple quotient of the central fiber.

We next determine a necessary support condition for a selected block. Apply nonabelian completion localization to the Springer resolution and its Steinberg correspondence at $(y, 1)$. Theorem 4.3(a) and Corollary 4.5(b) of [21] are natural for the proper map from the closed supported correspondence. They transport its image to the fixed supported correspondence over $\mathcal{C}_y$. The convolution normalization uses inverse normal Euler classes from the smooth fixed Springer resolution. These are invertible at y and preserve the supported image.

Every component of the fixed flag variety is a flag variety for the connected reductive group G_y [15, Proposition 8.8.7]. The fixed Springer object is thus a sum of $[W : M]$ copies of the ordinary Springer sheaf for G_y, all with the same perverse shift. The total-extension description of the central Hecke fiber [15, Proposition 8.1.5, Lemma 8.6.1 and Theorem 8.6.7] applies at $(y, 1)$; see also [8, Theorem 6.4 and its proof]. It uses the simply connected derived group of the ambient $G_{\mathbb{C}}$. The maps through fixed-point K-theory and Borel–Moore homology commute with these proper supported maps and with restriction to an open subset of the base. Thus the image of the supported correspondence restricts to zero on $\mathcal{N}(\mathfrak{g}_y) \setminus \mathcal{C}_y$.

Write this semisimple perverse sheaf as

$$\mathcal{S}_y = \bigoplus_{(\mathcal{O},\mathscr{L})} V_{\mathcal{O},\mathscr{L}} \otimes \mathrm{IC}(\overline{\mathcal{O}}, \mathscr{L}).$$

On the complement of $\mathcal{C}_y$, exactly the summands with $\mathcal{O} \subseteq \mathcal{C}_y$ vanish. The other summands remain nonzero and pairwise nonisomorphic simple perverse sheaves, because their dense orbits and local systems remain visible. Hence the kernel of restriction on degree-zero endomorphisms is

$$\bigoplus_{\mathcal{O}\subseteq\mathcal{C}_y,\mathscr{L}} \mathrm{End}(V_{\mathcal{O},\mathscr{L}}).$$

Negative extensions vanish; the positive-degree ideal of the finite total Ext algebra is nilpotent. Its semisimple quotient is the degree-zero algebra. The central-fiber isomorphism preserves Jacobson radicals and induces an isomorphism of the two semisimple quotients. Consequently, a selected constant block must map into the displayed restriction kernel. This determines the allowed simple blocks.

In the positive Hecke normalization a constant Hecke type F has the standard Springer type $F \otimes \det_M$; the compatibility is recorded in [9, Remark 4.1, Theorem 8.2 and §12]. The antispherical right polynomial action is $f \cdot w = \det(w) w^{-1} f$. These two twists cancel. The natural polynomial type E can therefore be selected only when $\mathcal{O}_M(E) \subseteq \mathcal{C}_y$. If $e_E \in \mathbb{C}[M]$ is the central projector onto the natural polynomial type E, then

$$(S_y \rtimes M) e_E S_y = S_y S_y[E].$$

Constant-block generation therefore gives $\widehat{I}_{\mathcal{C},y} \subseteq K_{\mathcal{C},y}$. □

4.4. **Lowest Springer copies generate downsets.** Let M now be any Weyl group, with reflection representation V, and put $S = \mathbb{C}[V]$. For an orbit $\mathcal{O}$ in its ordinary nilpotent cone, let $E_0(\mathcal{O})$ be the Springer representation for the trivial local system. Put $d_{\mathcal{O}} = \dim \mathcal{B}_{\mathcal{O}}$. The proof below shows that $E_0(\mathcal{O})$ first occurs once in polynomial degree $d_{\mathcal{O}}$. Denote that copy by $U_{\mathcal{O}}$.

Proposition 4.5. *For every nilpotent downset $\mathscr{D}$,*

$$\sum_{\mathcal{O}_M(E)\subseteq\mathscr{D}} S\,S[E] = \sum_{\mathcal{O}\subseteq\mathscr{D}} S\,U_{\mathcal{O}}. \tag{4.4}$$

This also holds after adding variables on which M acts trivially and after completion at the origin.

Proof. Consider $B = S \rtimes \mathbb{C}[M]$. The natural polynomial module S is the projective B-module with trivial head. The Springer extension-algebra description and canonical trace filtration are given by [32, Theorems 3.1 and 3.5] and [33, Theorem 1.8, Remark 1.9, Corollary 3.12 and Theorem 4.1]. These results apply to the ordinary Springer block and its orbit-closure order.

Choose a total order $\mathcal{O}_1, \dots, \mathcal{O}_m$ refining closure, with all orbits of $\mathscr{D}$ first. Let $e_j \in \mathbb{C}[M]$ be the sum of the central matrix-block idempotents for the Springer pairs on $\mathcal{O}_j$, and set

$$F_0 = 0, \qquad F_j = B(e_1 + \cdots + e_j)S.$$

Since each central projector extracts its full isotypic component, F_j is exactly the sum of $S\,S[E]$ over these first orbits. The proof of [33, Theorem 4.1] constructs the quotients of a projective by its earlier trace submodules. Applied to S, it identifies F_j/F_{j-1} with a direct sum of shifted modules $\widetilde{K}_{\mathcal{O}_j,\mathscr{L}}$. Explicitly this is the actual kernel

$$\ker(S/F_{j-1} \longrightarrow S/F_j) = F_j/F_{j-1}$$

of the consecutive projective trace-quotient maps. Using full central projectors gives the same trace submodule as using one primitive projector for each simple type, since the corresponding matrix units generate one another under multiplication by B. Each $\widetilde{K}_{\mathcal{O}_j,\mathscr{L}}$ is generated in degree zero by its irreducible Springer head and is a quotient of the corresponding graded projective. It is concentrated in nonnegative degrees and is annihilated by $e_1, \dots, e_{j-1}$. The multiplicities in the trace quotient are given by Brauer–Humphreys reciprocity.

The trivial Weyl representation occurs in Springer cohomology exactly once, in $H^0(\mathcal{B}_x)$, and the component group acts trivially on that line [13, §§2.3 and 2.9]. Reciprocity therefore gives

$$F_j/F_{j-1} \cong \widetilde{K}_{\mathcal{O}_j,1}\langle d_{\mathcal{O}_j}\rangle, \tag{4.5}$$

where the shift places the head in polynomial degree $d_{\mathcal{O}_j}$. In particular, that head is a single copy of $E_0(\mathcal{O}_j)$.

To locate its lift inside S, fix $\mathcal{O}$ and choose the order with $\overline{\mathcal{O}}$ first. Every orbit strictly below $\mathcal{O}$ has larger Springer-fiber dimension. Its shifted trace standard therefore contributes no vector in degrees $\le d_{\mathcal{O}}$. Every later trace standard is annihilated by the earlier projector onto $E_0(\mathcal{O})$. Exactness of this projector on each graded piece and (4.5) show that $E_0(\mathcal{O})$ occurs in S with multiplicity zero below degree $d_{\mathcal{O}}$ and with multiplicity one in that degree. This defines $U_{\mathcal{O}}$.

Return to the order with $\mathscr{D}$ first. Semisimplicity of $\mathbb{C}[M]$ lifts the head in (4.5) to $U_{\mathcal{O}_j} \subset F_j$. Since that quotient is generated by its head,

$$F_j = F_{j-1} + S\,U_{\mathcal{O}_j}.$$

Induction through the orbits of $\mathscr{D}$ proves (4.4). Polynomial extension and completion are flat and preserve the finite generating sum. □

Lemma 4.6. *Let $S = \mathbb{C}[[V]]$, with maximal ideal $\mathfrak{m}$, and let K be the completion of the ideal in* (4.4)*. Suppose $J \subseteq K$ is an M-stable ideal. If, for every surviving type $U_{\mathcal{O}}$ in $K/\mathfrak{m}K$, an element of J has nonzero degree-$d_{\mathcal{O}}$ term in that type, then $J = K$.*

Proof. There is a graded equivariant surjection

$$\bigoplus_{\mathcal{O}} U_{\mathcal{O}} \longrightarrow K/\mathfrak{m}K.$$

The types $E_0(\mathcal{O})$ are pairwise nonisomorphic. A surviving type therefore occurs just once, in its single displayed degree. Project the given element of J to that type by a finite group average. Its higher terms are in $\mathfrak{m}K$. The infinite tail is also there, because this submodule is adically closed. Its M-span fills the indicated summand of $K/\mathfrak{m}K$. Thus $J + \mathfrak{m}K = K$; Nakayama applied to the finite module K/J proves the claim. □

For actual source families with quantum support in $\mathcal{C}$, let $J_{\mathrm{source},y}$ be the sum of their completed coefficient ideals. Then

$$J_{\mathrm{source},y} \subseteq \widehat{I}^{+}_{\mathcal{C},y} \subseteq \widehat{I}_{\mathcal{C},y} \subseteq K_{\mathcal{C},y}. \tag{4.6}$$

The first two inclusions are supplied by actual source support; the last is Proposition 4.4. It is therefore enough to obtain a nonzero degree-$d_{\mathcal{O}}$ component in every surviving generator type $U_{\mathcal{O}}$ of $K_{\mathcal{C},y}/\mathfrak{m}_y K_{\mathcal{C},y}$.

5. Finite sources, retaining cuts, and globalization

5.1. **From a finite parity object to an actual source.** Let H be a finite affine parabolic, and suppose a finite canonical element C_x is the character of an indecomposable parity object over the modular field. Thus ${}^{p}C_x = C_x$. Tensor this object with arbitrary Bott–Samelson prefixes and suffixes, and pass to the antispherical category. The resulting characters have nonnegative Laurent coefficients in the p-canonical basis. At $v = 1$ they are therefore characters of actual tilting modules [4, Lemma 5.8 and Theorem 7.2]. Their ordinary characters belong to the based two-sided ideal generated by C_x. Hence every indecomposable summand of their quantum lifts has support in the closure of the ordinary cell of x. Theorem 2.3 gives the same bound for these modular tilting modules and their direct summands. The character action of the fixed parity object lies in the linear span of Bott–Samelson character actions, since $C_s(1) = 1 + s$.

At $v = 1$, the two-sided ideal of $\mathbb{C}[H]$ generated by $C_x(1)$ contains exactly the complete matrix blocks on which $C_x(1)$ acts nontrivially. Every finite group element is a product of $s = C_s(1) - 1$. Hence those matrix blocks are in the complex span of the stated actual prefixes and suffixes, with the same support bound. Left multiplication is performed in the regular Hecke category before passing to the antispherical module. Thus the complete finite blocks belong to the same bounded character span. Its closure under tensor and linkage character operators still has this support bound, so Proposition 3.3 applies to that stable span.

An *actual finite block source* will mean a block obtained in this manner, with its proven ordinary support bound. More than one p-independent canonical vector may be used to obtain it. The exact-family selection in the classical construction is an instance of this definition.

5.2. **The finite trace formula.** Project the finite affine parabolic to $H \subset W$, and choose its rational fixed point ξ. Its elements have the form $t_{\xi - h\xi}h$, where $\xi - h\xi \in X$. Let $y \in \mathbb{T}$ have stabilizer M. For each $z \in Wy$, choose $r_z \in W$ with $r_z z = y$. For $E \in \mathrm{Irr}(M)$, define

$$\chi_{E,y,\xi}(h) = \sum_{\substack{z \in Wy \\ hz = z}} z^{\xi - h\xi}\, \chi_E(r_z h r_z^{-1}). \tag{5.1}$$

Lemma 5.1. *In the standard Springer normalization, a full H-matrix block F acts nontrivially on the simple constant matrix module labelled by E in* (4.3) *if and only if*

$$m(F, E; y, \xi) = \frac{1}{|H|} \sum_{h \in H} \overline{\chi_F(h)}\, \chi_{E,y,\xi}(h) > 0. \tag{5.2}$$

When this condition holds, the coefficient ideal of the source contains the entire completed natural isotypic ideal $S_y S_y[E]$.

Proof. After the determinant twists fixed in Proposition 4.4 cancel, the constant matrix module is a direct sum of copies of E, indexed by Wy. The affine element $t_{\xi - h\xi} h$ sends the copy at z to the copy at hz; its diagonal entry when $hz = z$ is

$$z^{\xi - h\xi}\, r_z h r_z^{-1}.$$

Its trace is therefore (5.1). Changing r_z conjugates the corresponding M-action and preserves this trace. The character inner product (5.2) is the multiplicity of F in this finite-dimensional H-module. Hence the F-block acts nontrivially exactly when that multiplicity is positive.

This is Mackey restriction through the full subgroup $H \cap r_z^{-1} M r_z$, with the displayed scalar character. The constant matrix ideal then contains the E-block by semisimplicity. The matrix argument in Proposition 4.4 extends that inclusion to the completed algebra, and its action on the natural polynomial module gives $S_y S_y[E]$. □

At $y = 1$, formula (5.2) reduces to the multiplicity of E in $\mathrm{Ind}_H^W F$. This is also a normalization check on the determinant twist. The positive Hecke block used as an actual object is $\det_H \otimes F$, while the formula itself uses standard Springer labels.

5.3. **Native discriminant sources.** An ordinary parabolic L of M may occur as the stabilizer of a p-th root of y. The longest element of its corresponding finite affine stabilizer has a smooth parity object. Its standard Springer block is $\det_L$. If that affine parabolic is not standard, start with the conjugate standard longest source and apply the conjugating group elements on its two sides in the character span. Writing $s = C_s(1) - 1$ expresses these as finite combinations of actual Bott–Samelson operations, preserving the ordinary two-sided support bound.

For the simply laced exceptional full-rank points, the residual scalar equals 1. Normalize the invariant form so that every root has squared length 2, and use it to identify the weight and cocharacter spaces in this paragraph. A root sheet is torsion, so write $t = \exp(2\pi i \beta)$ with $\beta \in Y \otimes_{\mathbb{Z}} \mathbb{R}$. Put $Q^\vee = \sum_{\alpha \in \Phi} \mathbb{Z}\alpha^\vee$. Since the ambient group is simply connected, $Y = Q^\vee$; thus $\beta - h\beta \in Q^\vee = Q$ for its stabilizer. Since h is orthogonal,

$$\exp\bigl(2\pi i(\beta, \beta - h\beta)\bigr) = \exp\bigl(\pi i \|\beta - h\beta\|^2\bigr) = 1,$$

because the simply laced root lattice is even. The same equality holds at $y = t^p$. After the exponential gauge is removed, the local coefficient ideal therefore contains Δ_L. Its support bound is

$$\mathcal{C}_L = \overline{\mathcal{O}(j_L^W \det_L)}$$

by Proposition 4.3. Corollary 3.6 permits this bound to cut an existing source while retaining every E-type with $\mathrm{Hom}_L(\det_L, \mathrm{Res}_L^M E) \neq 0$.

Here is the root-sheet existence argument for the exceptional full-rank points used below. Their orders are among $1, 2, 3, 4, 5, 6$, hence prime to p. Choose a p-th root y_0 that is a power of y, so M fixes y_0. For a standard parabolic $L \subset M$, prescribe root pairings zero on its simple roots and one on the other simple roots of M, modulo p. The relevant Cartan and lattice indices have prime divisors at most 7; they are invertible in the E_7, E_8 ranges $p > 18, p > 30$. There is consequently a p-torsion point z with these pairings. For a positive root of M outside L, the sum of the prescribed simple coefficients is between 1 and $h(M) - 1 < p$, so does not vanish modulo p. Thus $W_{y_0 z} = L$. A root fixing $y_0 z$ must already fix its p-th power y, so no root outside M intervenes. Conjugation in M gives every ordinary parabolic needed for the cuts. The classical constructions below give their root sheets directly in signed coordinates, including their nontrivial affine phases.

Proposition 5.2 (Finite source criterion)**.** *Fix $y \in \mathbb{T}$. Suppose that for every local nilpotent orbit $\mathcal{O}$ there is an actual finite block source F, of bound $\mathcal{C}_F$, and finitely many retaining cuts $L_1, \dots, L_r$ such that*

$$m(F, E_0(\mathcal{O}); y, \xi) > 0, \qquad \mathrm{Hom}_{L_j}(\det_{L_j}, E_0(\mathcal{O})) \neq 0,$$

and

$$\mathcal{C}_F \cap \bigcap_{j=1}^{r} \mathcal{C}_{L_j} \subseteq \overline{G_{\mathbb{C}} \cdot \mathcal{O}}. \tag{5.3}$$

Then $\widehat{I}^{+}_{\mathcal{C},y} = \widehat{I}_{\mathcal{C},y}$ for every downset $\mathcal{C}$. It is also enough to construct the nonzero minimum jets in Lemma 4.6 by actual finite tensor source words with these bounds.

Proof. Lemma 5.1 supplies the entire $E_0(\mathcal{O})$-isotypic ideal from F. Each cut retains it by Corollary 3.6, while intersecting the source bounds. Thus every minimum generator for the local downset $\mathcal{C}_y$ lies in $\widehat{I}^{+}_{\mathcal{C},y}$. Use Proposition 4.5 and the containment (4.6). For the minimum-jet formulation, apply Lemma 4.6 to the same sandwich. □

5.4. **Central translation.**

Lemma 5.3. *The coefficient ideals of actual finite source words, at their specified support bounds, are stable under the finite center $Z = \mathrm{Hom}(X/Q, \mathbb{C}^{\times})$. The same is true of the ordinary cell-downset coefficient ideals.*

Proof. Take the actual-operator closure of each finite source word at its proved support bound. Tensoring and linkage projection preserve that bound, so its character space is an E_0-submodule. Proposition 3.3 makes its coefficient ideal invariant under Z. The ordinary cell-downset ideal corresponds to the operator-stable quantum support space $J^{q}_{\mathcal{C}}$, so the same proposition applies. □

5.5. **Constancy and the finite full-rank test.** Stratify $\mathbb{T}$ by the connected components of intersections of root hypertori $\alpha(y) = 1$, with all further root hypertori removed. This is a finite stratification. On a stratum the subgroup $M = W_y$ is constant.

For a fixed finite affine source, and h in a fixed intersection with M, the function

$$y \longmapsto y^{\xi - h\xi}$$

is a regular Laurent monomial. If h has order r, the product of its r translates telescopes to 1. Since h fixes y, this function is an r-th root of unity. It is therefore constant on the connected stratum. The full finite trace formula and its exponential gauge give one fixed completed ideal in logarithmic coordinates throughout that stratum.

This property is preserved by finite sums and finite tensor words. To see the tensor step, pull back to the finite etale p-power cover. For a root of M, its value on a root sheet is in μ_p, hence is locally constant. A root outside M cannot fix a sheet. Thus the multiset of the groups W_t is locally constant; monodromy can only permute the sheets. Formula (3.6) sums over all of them and is consequently constant as an ideal in the common logarithmic coordinate ring. The sum includes every root-sheet stabilizer. The retaining cuts use the ordinary parabolics specified in their construction.

Lemma 5.4. *The closure of each root-stabilizer stratum contains a point whose root stabilizer has full rank.*

Proof. Write the closure as a connected subtorus coset $a\mathbb{T}_0$, and put $\ell = \operatorname{rank} X$. Choose independent roots $\alpha_1, \dots, \alpha_r$ that equal 1 on this coset and span its normal space. Extend them to a full-rank set $\alpha_1, \dots, \alpha_\ell$. The restrictions of $\alpha_{r+1}, \dots, \alpha_\ell$ to $\mathbb{T}_0$ are independent and span its rational character space. They define an isogeny

$$\mathbb{T}_0 \longrightarrow (\mathbb{C}^{\times})^{\ell - r}, \qquad t \longmapsto (\alpha_{r+1}(t), \dots, \alpha_\ell(t)).$$

Choose t mapping to $(\alpha_{r+1}(a)^{-1}, \dots, \alpha_\ell(a)^{-1})$. The point at lies on all ℓ root hypertori, so its root stabilizer has full rank. □

Theorem 5.5. *Suppose finite actual source words satisfy Proposition 5.2 at every full-rank point, up to Weyl conjugacy and central translation. Then*

$$I_{\mathcal{C}}^{+} = I_{\mathcal{C}}$$

for every downset $\mathcal{C}$.

Proof. A full-rank point satisfies a full-rank set of root equations, so it is torsion. Write it as $y = \exp(2\pi i\beta)$, with $\beta \in Y \otimes_{\mathbb{Z}} \mathbb{R}$. Here the exponential has kernel $Y = Q^{\vee}$. Consequently Weyl-conjugacy classes are represented in a closed alcove for $Q^{\vee} \rtimes W$ acting on the cocharacter space $Y \otimes_{\mathbb{Z}} \mathbb{R}$. A full-rank root stabilizer fixes a zero-dimensional face, hence a vertex. There are therefore finitely many such points, and Lemma 5.3 permits central translation among their representatives.

For a fixed $\mathcal{C}$, take the finite sum $L_{\mathcal{C}}$ of the source-word ideals providing all required generators at these points. It has the established bound, so $L_{\mathcal{C}} \subseteq I_{\mathcal{C}}^{+} \subseteq I_{\mathcal{C}}$. The ordinary ideal has a constant formal expression on each stratum by Proposition 4.2 and finite block compression. The preceding constancy argument applies to $L_{\mathcal{C}}$ as well.

The support of the finite module $I_{\mathcal{C}}/L_{\mathcal{C}}$ is closed. On any connected stratum it is either empty or the whole stratum, because completed inclusion is constant and completion is faithfully flat. If it were nonempty, its closure would contain a full-rank point by Lemma 5.4. This contradicts the verified equality there. Thus $L_{\mathcal{C}} = I_{\mathcal{C}}$, which proves the theorem. □

6. Positive sources in the classical types

We construct the local minimum Springer copies required in Proposition 4.5. All coefficient ideals and polynomial representations in this section are over $\mathbb{C}$. Source objects are parity objects with coefficients of characteristic p, and give tilting objects under the categorical realization.

For a finite reflection group K, write ε_K for its determinant character and Δ_K for the product of its positive roots. For a finite reflection subgroup L, set

$$\mathrm{Alt}_L(f) = \frac{1}{|L|} \sum_{w \in L} \det(w) w(f).$$

We use the standard Springer convention, in which the regular nilpotent orbit corresponds to the trivial representation. Thus, if E is a representation on which a positive Hecke element $C_s(1) = 1 + s$ acts, its standard Springer label is $E \otimes \varepsilon_K$. We call the latter the *standard block* of the source.

For disjoint sets of variables I and J, put

$$D_a(I) = \prod_{\substack{i<j \\ i,j \in I}} (t_i^2 - t_j^2), \qquad B_b(J) = \left(\prod_{j \in J} t_j \right) \prod_{\substack{i<j \\ i,j \in J}} (t_i^2 - t_j^2), \qquad |I| = a, \quad |J| = b. \tag{6.1}$$

An empty product is 1. Their degrees are $a(a-1)$ and b^2. For a bipartition (α, β), with column heights $a_j = \alpha'_j$ and $b_j = \beta'_j$, a signed column Specht polynomial is

$$P_{\alpha,\beta} = \prod_j D_{a_j}(I_j) B_{b_j}(J_j), \tag{6.2}$$

where all the column sets are disjoint. In type B, its Weyl-group span is the unique minimum-degree copy of the irreducible representation indexed by (α, β) [19, Proposition 7.6]. Its degree is

$$b(\alpha, \beta) = 2n(\alpha) + 2n(\beta) + |\beta|, \qquad n(\alpha) = \sum_i (i-1)\alpha_i.$$

In type D, restriction identifies the two ordered labels when $\alpha \neq \beta$. For the distinguished labels used below we choose $\alpha_i \geq \beta_i$ for every i. If $\alpha \neq \beta$, then $|\alpha| > |\beta|$; the two type-B extensions have minimum degrees $b(\alpha, \beta)$ and $b(\beta, \alpha)$, with $b(\beta, \alpha) - b(\alpha, \beta) = |\alpha| - |\beta| > 0$. Restriction of the lower-degree polynomial span therefore gives the unique minimum copy. A label with $\alpha = \beta$ is called *balanced*, or *degenerate*; its restriction has two irreducible halves. Their minimum copies are treated separately below.

6.1. **From finite parity objects to complete matrix blocks.** Write $\mathcal{H}_{\mathrm{ext}}$ for the extended Hecke algebra. At $v = 1$, its antispherical quotient map is

$$\pi : \mathcal{H}_{\mathrm{ext}}(1) \longrightarrow \mathrm{sgn} \otimes_{\mathbb{C}[W]} \mathbb{C}[W_{\mathrm{ext}}], \qquad \pi(h) = 1 \otimes h.$$

We also write π for the corresponding quotient functor on Hecke objects.

Lemma 6.1. *Let K be a finite standard affine parabolic subgroup, and let Z be an actual object of its finite Hecke category, with character $z \in \mathcal{H}_K$. Suppose that z belongs to an ordinary based two-sided ideal $\mathcal{H}_{\mathcal{U}}$ of the ambient Hecke algebra and that $z(1)$ acts nontrivially on an irreducible $\mathbb{C}[K]$-module E. The character span of actual sources with ordinary support in $\mathcal{U}$ contains the entire antispherical image*

$$\pi\big(\mathcal{H}_{\mathrm{ext}}(1) e_E \mathcal{H}_{\mathrm{ext}}(1)\big),$$

where e_E is the central block idempotent of E. Consequently it contains the full finite-block coefficient ideal, with all its completed local degrees.

Proof. The nonzero matrix $z(1)|_E$ generates $\mathrm{End}_{\mathbb{C}}(E)$ as a two-sided ideal. Semisimplicity of $\mathbb{C}[K]$ therefore gives

$$e_E \in \mathbb{C}[K] z(1) \mathbb{C}[K].$$

The characters of Bott–Samelson objects span the Hecke algebra: the character of a reduced Bott–Samelson word has its corresponding standard basis element with coefficient 1, and all other terms have smaller length. Include the invertible length-zero objects to obtain the extended algebra. Hence every element of the displayed antispherical image is a finite complex linear combination of characters of objects

$$\pi\big(B_{\underline{x}} \otimes Z \otimes B_{\underline{y}}\big).$$

Here the left factor is applied in the regular category before taking the antispherical quotient. The quotient sends an indecomposable to its antispherical indecomposable or to zero.

Because $\mathcal{H}_{\mathcal{U}}$ is a based two-sided ideal, all ordinary character terms in these products lie in $\mathcal{U}$. The expansions of actual objects in the modular basis, and of the modular basis in the ordinary basis, have nonnegative coefficients. Thus no actual summand can acquire an ordinary term outside $\mathcal{U}$. The coefficient correspondence of Proposition 3.3 now gives the asserted ideal inclusion. The idempotent calculation thus gives complex linear relations among characters of actual objects. □

The finite affine-parabolic label rule will be used throughout this section. If $F = E \otimes \varepsilon_K$ and F_{sp} is the special representation in its ordinary family, the ambient ordinary cell of a source in that family is the Springer orbit of

$$j_K^W(F_{\mathrm{sp}}). \tag{6.3}$$

Here j denotes truncated induction in minimum polynomial degree. This is the finite-parabolic cell theorem of [36, §§5.4–5.6, 6.5 and Lemma 7.2]. For standard Levi factors, truncated induction is compatible with ordinary induction of nilpotent orbits. In the classical partition notation, a symmetric-group longest factor on m coordinates appends the two columns (m, m), followed by the appropriate classical collapse. We will compute the special family before applying (6.3).

Proposition 6.2. *At every odd coefficient characteristic the following standard blocks have actual finite sources in their exact ordinary families:*

$$((1^a),(1^b)) \quad \text{in } W(B_{a+b}), \qquad \{(1^a),(1^b)\} \quad \text{in } W(D_{a+b}).$$

For a balanced type-D block both halves are available, using the two fork parabolics. Arbitrary symmetric-group longest factors may be adjoined, and the complete corresponding matrix-block ideal is then available.

Proof. Let $A = \mathfrak{S}_s$ be the permutation parabolic of B_s, or one of the two spinor parabolics of D_s. The spherical parity theorem for minuscule flag varieties gives the ordinary spherical canonical basis at every good prime, hence at every odd prime in these types [11, Theorem 9.1]. The normalized smooth pullback to the full flag variety sends the spherical basis vector indexed by u to C_{uw_A}. Its character formula is independent of the coefficient field. In characteristic zero it is one IC character; nonnegativity of the modular-to-ordinary transition then implies that the modular pullback is the same single canonical character. Consequently all the full-flag canonical vectors in this spherical module are actual p-independent sources.

Fourier decomposition for the normal sign subgroup gives

$$\operatorname{Ind}_{\mathfrak{S}_s}^{B_s} \mathbf{1} = \bigoplus_{b=0}^{s} E_{((s-b),(b))}.$$

Restriction to D_s identifies complementary nonbalanced labels. For $s = 2b$, the two fork embeddings supply the two balanced halves. After tensoring by the full determinant these are precisely the standard blocks in the statement.

It remains to select a source in the exact family of the desired block. The spherical module $\mathcal{H}C_{w_A}$ and an ordinary based two-sided ideal are coordinate subspaces of the same canonical basis. Their intersection therefore has as a basis exactly the canonical vectors belonging to both. Intersect with the downset of the family of E. Its exact family quotient has a nonzero E-row, since $E^A \neq 0$. At least one canonical vector in that quotient acts nontrivially on E. Vectors in strictly lower families annihilate E, so the chosen vector belongs to the desired family itself. Applying Lemma 6.1 gives the complete block. External products with smooth symmetric-group longest objects preserve all these conclusions. □

Remark 6.3. We use $D_2 = A_1 \times A_1$ and $D_3 = A_3$ with their usual interpretations. A displayed D_1 factor denotes one unused coordinate with the identity object. The identity object also handles an empty block.

6.2. **Orthogonal rank varieties and their integral forms.** The remaining orthogonal mixed pairs use pure two-row blocks. We construct them by computing every graded local intersection form.

Set $\Lambda = \mathbb{Z}[1/2]$. Let $A \geq R \geq 1$, let $n = A + R$, and fix an isotropic R-plane Z in a quadratic space of dimension $2n + 1$. Choose a Witt splitting

$$Z^{\perp} = Z \oplus V, \qquad \dim V = 2A + 1.$$

Inside $\mathrm{OG}(R, Z^{\perp})$, define

$$X_j = \{U : \dim(U \cap Z) \geq j\}, \qquad 0 \leq j \leq R. \tag{6.4}$$

Set $X_{R+1} = \varnothing$. These are Schubert varieties for an isotropic flag containing Z. The stratum $S_k = X_k \setminus X_{k+1}$ is an affine bundle over

$$\mathrm{Gr}(k, R) \times \mathrm{OG}(R - k, 2A + 1).$$

Indeed $U \cap Z$ and the image of U in V determine the base point, and the remaining datum is a linear graph map. In particular the strata are simply connected. If $c = A - R$, then

$$\dim X_0 = 2AR - \frac{R(R-1)}{2}, \qquad \operatorname{codim}_{X_0} S_k = 2ck + \frac{k(3k+1)}{2}. \tag{6.5}$$

For $0 \le r \le R$, let $\mathcal{U}_r$ denote the tautological bundle on $\mathrm{OG}(r, V)$. The incidence morphism

$$Y_r = \mathrm{Gr}_{\mathrm{OG}(r,V)}(R, Z \oplus \mathcal{U}_r) \longrightarrow X_{R-r}, \qquad (U, L) \longmapsto U \tag{6.6}$$

has smooth projective source and is birational onto its image. Over a point whose projection to V has rank j, its fiber is $\mathrm{OG}(r-j, 2(A-j)+1)$.

Lemma 6.4. *Transverse to the projection-rank-j stratum, the resolved slice of* (6.6) *is*

$$\operatorname{Tot}\operatorname{Hom}(\mathbb{C}^{R-j}, \mathcal{U}_{r-j}) \longrightarrow \{B : \mathbb{C}^{R-j} \longrightarrow \mathbb{C}^{2(A-j)+1} : B^{\mathsf{t}} Q B = 0,\ \operatorname{rank} B \le r-j\}.$$

Here Q is a Gram matrix of the residual quadratic space. The fiber over the origin is the zero section.

Proof. Choose a basis of the rank-j isotropic image and extend it to a Witt decomposition. On the open chart in which the corresponding rank-j minor is invertible, row and column elimination normalize that minor to the identity. The Gram equations involving one of these j columns solve uniquely for its coordinates in the dual isotropic summand. These solved coordinates and the remaining graph coordinates are smooth parameters. The unsolved columns form a map from $\mathbb{C}^{R-j}$ to the nondegenerate residual space of dimension $2(A-j)+1$, and their only remaining equations are $B^{\mathsf{t}} Q B = 0$.

In the incidence resolution, the isotropic r-plane containing the image descends, after removing the normalized j-plane, to an isotropic $(r-j)$-plane containing the residual image. Over the Grassmannian of such planes the residual map is unrestricted. Thus the resolved slice is the displayed vector bundle. □

Write $k = r - j$, $a = A - j$, and $K = R - j$. The top Chern class of $\operatorname{Hom}(\mathbb{C}^K, \mathcal{U}_k)$ is z^K, where $z = c_k(\mathcal{U}_k)$. The self-intersection formula for the zero section, followed by integral Poincare duality on the smooth Grassmannian, identifies every graded local form with a graded piece of

$$z^K : H^*(\mathrm{OG}(k, 2a+1); \Lambda) \longrightarrow H^*(\mathrm{OG}(k, 2a+1); \Lambda). \tag{6.7}$$

Poincare duality is unimodular over Λ. A submodule over Λ is *saturated* if its quotient is torsion-free.

Lemma 6.5. *For every $a \ge k \ge 1$ and every $K \ge 0$, the image of every graded piece of* (6.7) *is saturated over Λ.*

Proof. Let $x_1, \dots, x_k$ be the Chern roots of $\mathcal{U}_k$, and put $c = a - k$. Write e_i and h_i for the elementary and complete symmetric functions, defined by

$$\sum_{i \ge 0} e_i(x^2) u^i = \prod_{j=1}^{k} (1 + x_j^2 u), \qquad \sum_{i \ge 0} h_i(x^2) u^i = \prod_{j=1}^{k} (1 - x_j^2 u)^{-1}.$$

Thus $e_0 = h_0 = 1$; negative-index complete symmetric functions are zero. The Borel presentation, after inverting the torsion index, is

$$H^*(\mathrm{OG}(k, 2a+1); \Lambda) = \frac{\Lambda[x_1, \dots, x_k]^{\mathfrak{S}_k}}{(h_{c+1}(x^2), \dots, h_{c+k}(x^2))}. \tag{6.8}$$

Only the prime 2 divides the relevant orthogonal torsion indices. For this paragraph let G_0 be a simply connected complex group of type B_m or D_m. Choose a maximal torus $T_0 \subset B_0 \subset P_0$, where B_0 is a Borel subgroup and P_0 is a parabolic subgroup of G_0. Let W_0 be the Weyl group of G_0, and W_P that of the Levi subgroup of P_0 containing T_0. After tensoring with Λ, identify

$X^*(T_0)$ with the orthogonal coordinate lattice, and put $\mathscr{P} = \Lambda[x_1, \dots, x_m]$. A subscript $+$ denotes the positive-degree ideal. Write $\mathrm{cl} : \mathscr{P} \to H^*(G_0/B_0; \Lambda)$ for the characteristic map and $\partial_s(f) = (f - sf)/\alpha_s$ for the divided difference, where α_s is the corresponding simple root. The additive graded form of [14, Theorem 6.7, Remark 6.8, and Corollaries 13.9–13.10], together with the Schubert-cell identification of Chow and singular cohomology over $\mathbb{C}$, gives

$$H^*(G_0/B_0; \Lambda) \cong \mathscr{P}/(\mathscr{P}_+^{W_0})\mathscr{P}$$

and makes $\mathscr{P}$ finite free over $\mathscr{P}^{W_0}$. Every torsion index used here, including that of the semisimple part of the Levi, is a power of 2.

Write ∂_{w_P} for the longest divided-difference operator. The characteristic map for the Levi flag variety supplies a homogeneous polynomial q_P with $\partial_{w_P}(q_P) = 1$: an integral representative whose pushforward is the Levi torsion index can be divided by that index in Λ. Consequently

$$\Pi_P(f) = \partial_{w_P}(q_P f)$$

is a $\mathscr{P}^{W_0}$-linear projection from $\mathscr{P}$ onto $\mathscr{P}^{W_P}$. Indeed ∂_{w_P} takes values in the W_P-invariants, and for invariant f the expression is $f\partial_{w_P}(q_P) = f$. If $\pi : G_0/B_0 \longrightarrow G_0/P_0$ is the flag projection, the geometric divided-difference identity identifies this projection in cohomology with

$$a \longmapsto \pi^*\pi_*(\mathrm{cl}(q_P)a).$$

The Schubert classes of the Levi flag variety lift to G_0/B_0 after inverting its torsion index. Leray–Hirsch therefore makes π^* injective. The displayed operator has image exactly $\pi^*H^*(G_0/P_0; \Lambda)$, since $\pi_* \mathrm{cl}(q_P) = 1$. Its polynomial version, being $\mathscr{P}^{W_0}$-linear, also gives

$$\mathscr{P}^{W_P} \cap (\mathscr{P}_+^{W_0})\mathscr{P} = (\mathscr{P}_+^{W_0})\mathscr{P}^{W_P}.$$

It follows that the partial-flag characteristic presentation is

$$H^*(G_0/P_0; \Lambda) \cong \mathscr{P}^{W_P}/(\mathscr{P}_+^{W_0})\mathscr{P}^{W_P}.$$

For $W_0 = B_a$ and $W_P = \mathfrak{S}_k \times B_{a-k}$, elimination of the residual squared-root elementary functions gives (6.8). For $W_0 = D_a$ and $W_P = \mathfrak{S}_k \times D_{a-k}$, the same elimination retains the residual Euler class; the full Euler relation is $zw = 0$, and the remaining relations are exactly those of (6.11). The case $a - k = 1$ has a residual torus and is included by taking its coordinate as w.

Apply the same argument with $W_0 = D_k$ and $W_P = \mathfrak{S}_k$, for $k \geq 2$. Put

$$\mathscr{P}_k = \Lambda[x_1, \dots, x_k], \qquad \mathscr{P}_k^{D_k} = \Lambda[e_1(x^2), \dots, e_{k-1}(x^2), z].$$

In this case one may take $q_P = x_1^{k-1}x_2^{k-2}\cdots x_{k-1}$, with the sign chosen according to the positive-root convention, so no division by $k!$ is involved. The projection Π_P makes $\mathscr{P}_k^{\mathfrak{S}_k}$ a direct summand of the finite free $\mathscr{P}_k^{D_k}$-module $\mathscr{P}_k$. Thus $\mathscr{P}_k^{\mathfrak{S}_k}$ is projective over $\mathscr{P}_k^{D_k}$. Its quotient by $(\mathscr{P}_k^{D_k})_+$ is the cohomology of one spinor partial flag variety, which has a homogeneous Schubert basis of size

$$[D_k : \mathfrak{S}_k] = 2^{k-1}.$$

Lift this basis homogeneously to $\mathscr{P}_k^{\mathfrak{S}_k}$. Graded Nakayama gives a surjection from the free $\mathscr{P}_k^{D_k}$-module on these lifts to $\mathscr{P}_k^{\mathfrak{S}_k}$. Projectivity makes its kernel a direct summand; reducing the split sequence modulo $(\mathscr{P}_k^{D_k})_+$ shows that this kernel has zero quotient. Graded Nakayama applied again makes the kernel zero. This proves the asserted homogeneous freeness over $\mathscr{P}_k^{D_k}$, with rank 2^{k-1}. For $k = 2$ the same proof uses $D_2 = A_1 \times A_1$.

Set

$$B = \frac{\Lambda[e_1(x^2), \dots, e_k(x^2)]}{(h_{c+1}(x^2), \dots, h_{c+k}(x^2))} \cong H^*(\mathrm{Gr}(k, a); \Lambda),$$

where the grading on the last ring is doubled. The invariant tower $\mathfrak{S}_k \subset D_k \subset B_k$ gives

$$\Lambda[x]^{D_k} = \Lambda[x]^{B_k}[z]/(z^2 - e_k(x^2)), \qquad z = x_1 \cdots x_k.$$

By the freeness just proved, (6.8), as a module over $B[z]/(z^2-e_k)$, is a direct sum of homogeneous shifts of that ring. For $k=1$ this assertion is simply $\Lambda[x]=\Lambda[x^2]\oplus x\Lambda[x^2]$; for $k=2$ it uses $D_2=A_1\times A_1$.

If $K=2m$, multiplication by z^K is e_k^m on each of the two summands B and zB. If $K=2m+1$, its matrix is

$$\begin{pmatrix} 0 & e_k^{m+1} \\ e_k^m & 0 \end{pmatrix}. \tag{6.9}$$

In the integral Schur basis of the $k\times c$ Grassmannian rectangle,

$$e_k^m s_\lambda = \begin{cases} s_{\lambda+(m^k)}, & \lambda+(m^k)\subseteq(c^k), \\ 0, & \text{otherwise.} \end{cases} \tag{6.10}$$

Here s_λ is the Schur polynomial in $x_1^2,\dots,x_k^2$. Multiplication by $e_k(x^2)^m=(x_1\cdots x_k)^{2m}$ adds m to every part of λ; the Grassmannian relations kill the classes outside the rectangle. The matrices of (6.10) and (6.9) are partial permutation matrices in every degree. Their nonzero invariant factors are units, and homogeneous free-module shifts do not change that assertion. □

For even orthogonal spaces, let $\dim V=2A$ in (6.4). We use X_j only for $j\geq 1$, with the resolutions Y_r of (6.6) for $r\leq R-1$. The resolved normal slices are $\mathrm{Hom}(\mathbb{C}^{R-j},\mathcal{U}_{r-j})$ over $\mathrm{OG}(r-j,2(A-j))$. Their stable difference $A-r$ is at least 1, so these orthogonal Grassmannians are nonmaximal and connected.

Lemma 6.6. *Let $a>k\geq 1$ and $z=c_k(\mathcal{U}_k)$. Every graded multiplication map by a positive power of z on $H^*(\mathrm{OG}(k,2a);\Lambda)$ has saturated image.*

Proof. Put $c=a-k$. Remove the same homogeneous free factor of rank 2^{k-1} as in the preceding proof. The remaining Borel algebra is

$$C=\frac{\Lambda[e_1,\dots,e_k,z,w]}{(z^2-e_k,\ w^2-(-1)^c h_c,\ zw,\ h_{c+1},\dots,h_{c+k-1})}. \tag{6.11}$$

Here the e_i,h_i are symmetric functions in squared roots, and w is the Euler class of the residual D_c factor. The relations imply

$$e_k h_c=0, \qquad z h_c=0, \qquad w e_k=0.$$

Modulo the intervening complete symmetric functions, $e_k h_c=0$ is equivalent, up to sign, to $h_{c+k}=0$. Let $B=H^*(\mathrm{Gr}(k,a);\Lambda)$, again with doubled degrees. Reducing words in z,w in (6.11) gives the actual module decomposition

$$C=B\oplus zB/(h_c)\oplus wB/(e_k), \tag{6.12}$$

with

$$B/(h_c)\cong H^*(\mathrm{Gr}(k,a-1);\Lambda), \qquad B/(e_k)\cong H^*(\mathrm{Gr}(k-1,a-1);\Lambda).$$

The three displayed annihilator relations are exactly the relations needed to make (6.12) well-defined.

Multiplication by z sends B to $zB/(h_c)$ by the quotient map, sends $zB/(h_c)$ to B by multiplication by e_k, and kills the w-summand. In Schur bases the quotient deletes the partitions touching the removed boundary of the rectangle, while multiplication by e_k shifts by a full column. Every positive power is therefore a partial permutation matrix. Restoring the homogeneous free factor proves the assertion in all degrees. □

Proposition 6.7. *In odd dimension, $IC(X_0)$ and $IC(X_1)$ have their ordinary characters as parity objects over every field of odd characteristic. In even dimension the same is true of $IC(X_j)$ for every $j\geq 1$. Their normalized full-flag pullbacks are actual p-independent canonical objects.*

Proof. All fibers of the smooth projective resolutions above have free integral cohomology in even degrees. Lemma 6.4 and its even-dimensional version identify all local forms with the Euler maps, whose ranks are unchanged over every odd field by Lemmas 6.5 and 6.6. The local intersection-form decomposition criterion [31, Theorem 3.13], applied successively to the simply connected rank strata, therefore gives the same indecomposable summands and graded characters as in characteristic zero. In odd dimension the first-defect support X_1 occurs already in the resolution of X_0: on its generic normal fiber $\mathrm{OG}(1, 2c+3)$, multiplication by the tautological Euler class has nonzero rank. Alternatively it can be obtained from its own image resolution. In even dimension we use the image resolution of X_1 directly, including when $A = R$. The smooth full-flag projection has even flag-variety fibers, so normalized pullback preserves the character assertion. □

6.3. **The pure two-row blocks.** We identify the family and a nonzero matrix block of each IC summand.

Lemma 6.8. *For the odd orthogonal variety X_0, the total unshifted IC stalk dimension at radical defect k is*

$$f_B(k) = \begin{cases} 1, & k = 0, \\ 2^{k-1}, & k \geq 1. \end{cases}$$

For the even orthogonal variety X_1, it is

$$f_D(k) = \begin{cases} 0, & k = 0, \\ (k+1)2^{k-2}, & k \geq 1. \end{cases}$$

These formulas are independent of the stable difference.

Proof. We derive the formulas from the smooth image resolutions. Let $M_B(a, K, r)$ be the total rank of multiplication by z^K on $H^*(\mathrm{OG}(r, 2a+1))$, with $M_B(a, K, 0) = 1$. For $r \geq 1$, the two-summand calculation above and (6.10) give

$$M_B(a, 2m, r) = 2^r \binom{a-m}{r}, \tag{6.13}$$

$$M_B(a, 2m+1, r) = 2^{r-1} \left(\binom{a-m}{r} + \binom{a-m-1}{r} \right).$$

Here $\binom{n}{r} = 0$ unless $0 \leq r \leq n$. Indeed the number of Schur classes surviving a shift by m full columns is $\binom{a-m}{r}$.

Fix radical dimension K, and let $F_j(A, K)$ be the stalk dimension at Z of the rank-j image X_{K-j}; then $F_0 = 1$. Taking the stalk of each smooth image resolution at Z, and using the local-form multiplicities, gives

$$2^r \binom{A}{r} = \sum_{j=0}^{r} M_B(A-j, K-j, r-j)\, F_j(A, K), \qquad 0 \leq r \leq K. \tag{6.14}$$

Consequently

$$(1+2t)^A = \sum_{j=0}^{K} F_j t^j (1+2t)^{A-\lceil (K+j)/2 \rceil} (1+t)^{(K-j) \bmod 2} \pmod{t^{K+1}}.$$

After division by $(1+2t)^{A-K}$, both sides are polynomials of degree at most K, so this is an exact identity. Set

$$u = \frac{t}{\sqrt{1+2t}}, \qquad q = \sqrt{1+u^2}.$$

Then $\sqrt{1+2t} = q+u$, and the identity becomes

$$(q+u)^K = \sum_{j=0}^{K} F_j u^j q^{(K-j) \bmod 2}, \qquad q^2 = 1+u^2. \tag{6.15}$$

Expanding the left-hand side and reducing powers of q gives

$$F_j = \sum_{\substack{0\le h\le j \\ h\equiv j \ (2)}} \binom{K}{h}\binom{\lfloor (K-h)/2 \rfloor}{(j-h)/2}.$$

For $j = K > 0$ this is the sum of the binomial coefficients of one parity, namely 2^{K-1}. The resolved-slice calculation applies the same formula at every defect, proving f_B.

For even orthogonal Grassmannians and positive K, the total rank is

$$M_D(a,2m,r) = 2^{r-1}\left(\binom{a-m}{r} + \binom{a-m-1}{r}\right), \tag{6.16}$$
$$M_D(a,2m+1,r) = 2^r\binom{a-m-1}{r}.$$

This is $M_B(a,K+1,r)$. The recurrence for the even rank images through $r \le K-1$ is therefore the odd recurrence with domain size $K+1$, through the same ranks. The preceding coefficient formula gives

$$F_{K-1}(A,K+1) = \frac{1}{2}\sum_{\substack{0\le h\le K+1 \\ h\equiv K+1 \ (2)}} (K+1-h)\binom{K+1}{h} = (K+1)2^{K-2}.$$

For $K = 1$ the same formula is $F_0 = 1$. This proves the asserted X_1 stalk formula. Only nonmaximal orthogonal Grassmannians occur. The recurrence is used only through rank $K-1$, and its displayed coefficient is independent of A; the same formula therefore applies when $A = R$. □

Proposition 6.9. *There are actual sources at every odd coefficient characteristic with the following standard blocks and exact special families:*

K	*range*	*standard block*	*special family member*
B_{A+R}	$A \ge R \ge 1$	$(\varnothing;\ 1^A + 1^R)$	$(1^R;\ 1^A)$
D_{A+R}	$A \ge R \ge 2$	$\{1^A + 1^R;\ \varnothing\}$	$\{(2,1^{A-1});\ 1^{R-1}\}$.

Here sums of partitions are rowwise. The corresponding positive blocks are $E_{((A,R),\varnothing)}$ and $E_{\{(A,R),\varnothing\}}$, respectively.

Proof. In odd dimension use the full-flag pullback of $IC(X_0)$. With $c = A-R$, and with the convention that s_0 changes the first coordinate sign, its maximal full-flag label is

$$w = w_0^{B_{A+R}} u, \qquad u = (1,\dots,c,A+1,\dots,A+R,c+1,\dots,A).$$

Set $J = B_A \times \mathfrak{S}_R$, and use the isotropic coordinate flag for which $Z = \langle e_{A+1},\dots,e_{A+R}\rangle$. The corresponding J-minimal representative is

$$w^J = (1,\dots,c,A+1,\dots,A+R;-A,\dots,-(c+1)), \qquad w = w^J w_J.$$

Its last R coordinates select an isotropic plane in the nondegenerate summand of $Z^\perp$, so its partial-flag Borel orbit lies in S_0. The displayed representative is J-minimal: its first A entries are

positive and increasing, and its last R entries are increasing in signed order. The only inversions in u are between its two blocks of size R, so $\ell(u) = R^2$. Hence

$$\begin{aligned}\ell(w^J) &= (A+R)^2 - R^2 - \left(A^2 + \frac{R(R-1)}{2}\right) \\ &= 2AR - \frac{R(R-1)}{2} = \dim X_0.\end{aligned}$$

The irreducibility of X_0 follows also from its birational image resolution. Thus the indicated Borel orbit is dense in X_0, proving that w^J, and therefore w after full-flag pullback, is its actual Schubert label. Row insertion of u gives the two-row shape (A, R): the initial small block and the large block form the first row, and the last middle block bumps exactly R large entries. The special type-A representation is $S^{(A,R)}$, of minimum degree R. Its truncated induction to B_{A+R} is the special bipartition $((A),(R))$: the horizontal Pieri rule exhibits this constituent and it is the unique one of that degree. Ordinary parabolic cell induction, followed by multiplication by w_0, therefore gives the special family member $(1^R; 1^A)$. In particular its a-value is $A^2 + R(R-1)$.

At $v = 1$, map the signed-subset spherical module to the unsigned R-subset module by killing the normal sign subgroup. For an unsigned subset with k radical positions, the condition $U \subset Z^\perp$ forces the signs of those positions; the remaining $R-k$ signs are free. Lemma 6.8 gives coefficient

$$2^{R-k} f_B(k) = 2^{R-1}(1 + \mathbf{1}_{\{k=0\}}).$$

A harmonic polytabloid for $S^{(A,R)}$ is the alternating sum over the 2^R choices of one element from each of R disjoint top-bottom pairs. Choose each top element outside the radical block and each bottom element inside; $A \geq R$ ensures that this is possible. Give a choice with k bottom elements the sign $(-1)^k$. Its pairing with the constant function is $(1-1)^R = 0$, and its pairing with $\mathbf{1}_{\{k=0\}}$ is 1. The block action therefore has nonzero pairing 2^{R-1}.

In even dimension use $IC(X_1)$, with parabolic $J = D_A \times \mathfrak{S}_R$. Put $n = A + R$ and $\delta = (-1)^n$. The full-flag signed permutation is

$$w = (\delta, -2, \ldots, -(c+1), -(A+2), \ldots, -n, A+1, -(c+2), \ldots, -A). \tag{6.17}$$

It is even-signed. To see that it is the required geometric label, the partial-flag Schubert label is

$$((-1)^{R+1}, 2, \ldots, c+1, A+2, \ldots, n;\ -A, \ldots, -(c+2), A+1).$$

Its final R entries select the $R-1$ largest negative nondegenerate coordinates and the first radical coordinate, which are exactly the incidence conditions defining X_1. Multiplication by w_J gives (6.17). For all $A \geq R \geq 2$, denote the displayed partial-flag representative by w^J, so that $w = w^J w_J$. Its first A entries have no D_A-descent, and its last R entries are increasing, so it is J-minimal. Its selected isotropic plane lies in $Z^\perp$ and has exactly one radical coordinate, hence its Borel orbit lies in S_1. In our signed-permutation convention the longest element of D_n is $((-1)^{n-1}, -2, \ldots, -n)$. Consequently, for either parity of n,

$$v = w_0^{D_n} w = (-1, 2, \ldots, c+1, A+2, \ldots, n, -(A+1), c+2, \ldots, A).$$

For an even signed permutation, the length is the number of inversions in signed order plus the number of pairs of entries with negative sum. In this word those two numbers are respectively

$$(R-1)^2 + (R-1) + c + 1, \qquad 1 + c + (R-1) = A.$$

Thus $\ell(v) = R^2 + 2c + 1$, and

$$\begin{aligned}\ell(w^J) &= n(n-1) - (R^2 + 2c + 1) - \left(A(A-1) + \frac{R(R-1)}{2}\right) \\ &= 2A(R-1) - \frac{(R-1)(R-2)}{2}.\end{aligned}$$

The right-hand side is $\dim Y_{R-1} = \dim X_1$: the base $\mathrm{OG}(R-1,2A)$ has dimension $2A(R-1) - \big(3(R-1)^2 + (R-1)\big)/2$, and the ordinary Grassmannian fiber has dimension $R(R-1)$. Since X_1 is irreducible, the indicated Borel orbit is dense in it. This proves the actual Schubert-label assertion, including the boundary case $A = R$.

The doubled word $(-w(n), \dots, -w(1), w(1), \dots, w(n))$ has RSK shape

$$\begin{cases} (4, 2^{\,2R-2}, 1^{\,2c}), & n \text{ even}, \\ (3, 2^{\,2R-2}, 1^{\,2c+1}), & n \text{ odd}. \end{cases} \tag{6.18}$$

To compute its Greene invariants, write its successive blocks as

$$\begin{aligned} &(A, \dots, c+2),\ -(A+1),\ (n, \dots, A+2),\ (c+1, \dots, 2), \\ &-\delta, \delta,\ (-2, \dots, -(c+1)),\ (-(A+2), \dots, -n),\ A+1,\ (-(c+2), \dots, -A). \end{aligned}$$

A decreasing chain avoiding $\pm(A+1)$ uses at most one of the two positive blocks of length $R-1$, at most $2c$ letters from the head blocks, and at most one of the two negative blocks of length $R-1$. It can use at most one of $-1, 1$ when n is even, and both when n is odd. Its length is therefore at most $2A-1$ or $2A$, respectively. A chain containing $-(A+1)$ can use only the first positive block before it and the block $-(A+2), \dots, -n$ after it. A chain containing $A+1$ has the symmetric bound. Such a chain has length at most $2R-1$.

When n is even, the two decreasing chains

$$\begin{aligned} (A, \dots, 2, -1, -2, \dots, -A), \\ (n, \dots, A+2, 1, -(A+2), \dots, -n) \end{aligned}$$

cover all letters except $\pm(A+1)$, and the first has length $2A-1$. The increasing chain $(-(A+1), -1, 1, A+1)$ forces two decreasing chains to omit at least two letters, and three to omit at least one. Adding either special letter as a singleton gives a three-chain cover of $2n-1$ letters; adding both gives a four-chain cover of the word. The maximal sizes of unions of one, two, three and four decreasing chains are therefore $2A-1, 2n-2, 2n-1, 2n$.

When n is odd, the decreasing chains

$$\begin{aligned} (n, \dots, A+2, c+1, \dots, 1, -1, \dots, -A), \\ (A, \dots, c+2, -(A+1), -(A+2), \dots, -n) \end{aligned}$$

have lengths $2A$ and $2R-1$, and cover all letters except $A+1$. The increasing chain $(-(A+1), 1, A+1)$ prevents a two-chain cover of the entire word. Adding $A+1$ as a singleton gives a three-chain cover. The maximal union sizes are $2A, 2n-1, 2n$. Greene's theorem [10, Theorem 2.14] gives conjugate RSK partitions $(2A-1, 2R-1, 1, 1)$ and $(2A, 2R-1, 1)$, respectively, which are equivalent to (6.18).

Apply the type-D symbol construction to this doubled word [10, §3.3, Propositions 3.5 and 3.6]. Pad its RSK partition ν with zeros to length $2A$. In the defect-zero normalization, the union of the two symbol rows is

$$\left\{ \left\lfloor \frac{\nu_i + 2A - i}{2} \right\rfloor : 1 \le i \le 2A \right\}.$$

For either parity this multiset is the union of

$$(1, 2, \dots, A-1, A+1), \qquad (0, 1, \dots, c, c+2, \dots, A).$$

Indeed the first row of ν contributes $A+1$. Its $2R-2$ rows of length two contribute A and $c+1$ once, and each intervening integer twice. The remaining rows contribute 0 once and $1, \dots, c$ twice. Sorting the symbol gives the special label $\{(2, 1^{A-1}); 1^{R-1}\}$, whose a-value is $A(A-1) + (R-1)^2$.

The same sign count gives the unsigned-subset coefficient of the even IC source:

$$2^{R-k} f_D(k) = 2^{R-2}(k + 1 - \mathbf{1}_{\{k=0\}}).$$

For $R \geq 2$, the same harmonic polytabloid kills the linear function $k+1$, since

$$\sum_{k=0}^{R}(-1)^k\binom{R}{k}(k+1) = (1-1)^R - R(1-1)^{R-1} = 0.$$

Its pairing with the indicator is 1, so the block action is $-2^{R-2} \neq 0$. Proposition 6.7 and Lemma 6.1 now prove both assertions. □

6.4. **Column formulas and the common discriminant.** The following elementary column function will organize the construction:

$$c(a,b) = \begin{cases} (2a-1, 2b+1), & a > b, \\ (2b, 2a), & a \leq b. \end{cases} \tag{6.19}$$

Zero columns are omitted. We call the two cases respectively the D-orientation and the B-orientation of a pair; these names describe the formula, not the type of the ambient local factor.

Lemma 6.10. *The trivial-local-system Springer representations and their orbit columns have the following descriptions.*

(1) In type C, a distinguished bipartition satisfies

$$\alpha_i \geq \beta_i - 1, \qquad \beta_i \geq \alpha_{i+1} - 1.$$

If $a_j = \alpha'_j$ and $b_j = \beta'_j$, the conjugate Springer partition is the decreasing concatenation

$$c(a_1, b_1), c(a_2, b_2), \ldots.$$

(2) In type B, the distinguished inequalities are $\alpha_i \geq \beta_i - 2$ and $\beta_i \geq \alpha_{i+1}$. The orbit columns are

$$(2b_1 + 1,\ c(a_1, b_2), c(a_2, b_3), \ldots).$$

The minimum polynomial has the initial factor B_{b_1}.

(3) In type D, orient the unordered distinguished label so that $\alpha_i \geq \beta_i$ and $\beta_i \geq \alpha_{i+1} - 2$. The orbit columns are

$$(2a_1,\ c(a_2, b_1), c(a_3, b_2), \ldots).$$

The minimum polynomial has initial factor D_{a_1}.

Proof. We spell out the column count in the Springer partition formulas [5, §2], [30, §§5.2, 7.2–7.3]. In type C, interleave the even rows $2\alpha_1, 2\beta_1, 2\alpha_2, 2\beta_2, \ldots$. Distinguishedness says that an ascending adjacent pair has the form $(2j-2, 2j)$; replace it by $(2j-1, 2j-1)$. Before replacement the two columns $2j-1, 2j$ both have height $a_j + b_j$. The indicator strings for $\alpha_i \geq j$ and $\beta_i \geq j$ contain a_j and b_j initial ones. Their interleaving has $b_j - a_j$ upward transitions if $b_j \geq a_j$, and $a_j - b_j - 1$ otherwise. Each replacement raises the odd column by one and lowers the even column by one. This gives exactly (6.19). The distinguished inequalities, equivalently $a_j \geq b_{j+1}$ and $b_j \geq a_{j+1} - 1$, show that the concatenation is decreasing.

For type B, remove the first column of β. The pair $(\alpha, (\beta_i - 1)_+)$ satisfies the type-C inequalities. Counting columns in the interleaved rows $(2\alpha_i + 1, 2\beta_i - 1)$, with ascending pairs averaged, adds the initial column $2b_1 + 1$ to the type-C tail. For type D, remove the first column of α instead; the same count in $(2\alpha_i - 1, 2\beta_i + 1)$ adds $2a_1$. The factorizations of (6.2) give the claimed head factors. □

The tensor calculation will always use the same elementary observation.

Lemma 6.11. *Let L be a parabolic subgroup of $M = W_y$, occurring among the p-th-root stabilizers in Proposition 3.5. Suppose that the completed ideals of a finite collection of actual source families contain $\Delta_L h_1, \dots, \Delta_L h_m$, respectively, with every $h_i \in S_y^L$. The completed coefficient ideal of their tensor product and block closure contains*

$$\Delta_L \prod_{i=1}^{m} h_i.$$

Its support bound is the intersection of the bounds of the source families.

Proof. Use the term with $L_t = L$ in formula (3.6). It sends $\Delta_L h_i$ and $\Delta_L h_{i+1}$ to $\Delta_L h_i h_{i+1}$. Iteration proves the first assertion. The corresponding operations are tensor products and block projections of actual objects, so the support assertion follows from the support interface and Theorem 2.3. □

Lemma 6.12. *Let y be a full-rank torus point in type B_d, C_d or D_d, and let $p > h$. Each common parabolic L displayed in Propositions 6.14, 6.17 and 6.19 is W_t for some t with $t^p = y$.*

Proof. The vector coordinates of y are $\epsilon_i \in \{1, -1\}$. The displayed groups consist of symmetric coordinate blocks and, in orthogonal type, retained plus and minus heads of sizes r, s. Assign to each symmetric block a common value z satisfying $z^p = \epsilon_i$. For blocks of the same sign, choose values from distinct pairs $\{z, z^{-1}\}$, excluding $z = \epsilon_i$. There are $(p-1)/2$ such pairs for each sign. Values belonging to different signs are neither equal nor inverse, since their p-th powers differ. On each retained head put $z = \epsilon_i$. Thus permutations preserving the chosen values give exactly the prescribed symmetric factors; coordinate flips occur only in the heads.

For types B_d, C_d, the inequality $p > 2d$ provides at least d pairs, enough for all symmetric blocks. For type D_d, the inequality $p > 2d - 2$ provides at least $d - 1$ pairs. By Lemma 6.10, every nonempty distinguished local D-label has a nonempty head. The tail therefore contains at most $d - 1$ symmetric blocks.

For Sp_{2d}, these vector coordinates already define t, with the required stabilizer. For a Spin torus they have two lifts. Multiplication by the nontrivial element in the kernel of Spin $\to$ SO interchanges their p-th powers because p is odd. Exactly one lift therefore has p-th power y.

To compute its stabilizer, let e_i be the i-th vector coordinate character and use the spin weight $\omega = (e_1 + \cdots + e_d)/2$. A coordinate permutation fixes ω. Flipping coordinates in a set I sends it to $\omega - \sum_{i \in I} e_i$, changing its value by $\prod_{i \in I} z_i^{-1}$. Generic tail values exclude coordinate flips. In the heads this product is $(-1)^{|I_-|}$, where I_- is the set of flipped minus-head coordinates. In odd orthogonal type the lift stabilizer is consequently $B_r \times D_s$ on the heads. In even orthogonal type the global even-sign condition also forces an even number of plus-head flips, giving $D_r \times D_s$. Together with the symmetric factors these are precisely the stated groups L. □

If $L_j = \mathfrak{S}_{a+b}$ permutes the variables in one column pair, the alternation of $D_a(I)B_b(J)$ is nonzero. Indeed its alternant exponents are

$$\{0, 2, \dots, 2a-2\} \ \cup \ \{1, 3, \dots, 2b-1\},$$

which are all distinct. The alternant is divisible by the ordinary Vandermonde Δ_{L_j}, with invariant quotient.

6.5. **Symplectic sources at the full-rank points.** For $G = \mathrm{Sp}_{2d}$, a full-rank torus point has stabilizer $M = B_u \times B_v$, on its $+1$ and -1 coordinate blocks. Choose distinguished local types (α_+, β_+) and (α_-, β_-). For one column position, abbreviate the heights to (a_+, b_+) and (a_-, b_-). Write

$$(X, Y) = c(a_+, b_+) + c(a_-, b_-), \qquad m = (X + Y)/2,$$

where pairs are added componentwise.

Lemma 6.13. *There is an actual minuscule finite affine-parabolic source on these m coordinates whose exact symplectic orbit has columns (X, Y), and whose local coefficient ideal contains the minimum copy of*

$$((1^{a_+}), (1^{b_+})) \boxtimes ((1^{a_-}), (1^{b_-})).$$

Proof. For a standard two-column B-block $(1^a; 1^b)$, the exact family orbit in symplectic type has columns

$$\begin{cases} (2b, 2a), & a \le b, \\ (2a-1, 2a-1), & a = b+1, \\ (2a-2, 2b+2), & a \ge b+2. \end{cases} \tag{6.20}$$

In the first two cases the finite label is special. In the last case its sorted defect-one symbol has special representative $(1^{b+1}; 1^{a-1})$; Lemma 6.10 then gives the third line. A special two-column D-block $\{1^a; 1^b\}$ gives $(2a-1, 2b+1)$ when $a > b$, and $(2a, 2a)$ when $a = b$, by (6.3).

The required choices are

local orientations	standard finite source block
B, B	$B_m:\ (1^{a_+ + a_-};\ 1^{b_+ + b_-})$
D, D	$B_m:\ (1^{a_+ + a_-};\ 1^{b_+ + b_-})$
D, B or B, D	$D_m:\ \{1^{a_+ + b_-};\ 1^{b_+ + a_-}\}$.

For the first line, the first case of (6.20) applies. For the second line the two strict inequalities give $a_+ + a_- \ge b_+ + b_- + 2$, so the third case applies. For the mixed line the two heights are unequal; the larger height is the one associated with the D-oriented local pair. Substitution gives (X, Y) in all cases.

In the B-sources, splitting a vertical partition between the two local coordinate blocks has Littlewood–Richardson coefficient one in each color. In the D-source use the affine subgroup fixing $\xi = (1/2, \ldots, 1/2)$ on its coordinates. At a plus coordinate its finite character is trivial; on the minus coordinates it is the sign-subgroup character, which interchanges the two colors. Thus the global ordered extension with heights $(a_+ + b_-, b_+ + a_-)$, restricted to the whole intersection with M and twisted by that character, contains the required local type. Both ordered extensions restrict to the same nondegenerate D_m-representation, so there is no choice of a degenerate half here. The exact finite Mackey formula and Lemma 6.1 give the whole local isotypic ideal, and hence its minimum copy. If $m = 1$, the mixed pair is $(1, 0)$ and its minimum polynomial is 1; use the identity source. □

Proposition 6.14. *At every full-rank symplectic torus point, each local trivial-local-system minimum Springer copy $U_{\mathcal{O}}$ is contained in the completed ideal of a finite actual source family with support in the ambient orbit closure of $\mathcal{O}$.*

Proof. Let λ be the row union of the two local symplectic partitions. Its columns are the decreasing list

$$(X_1, Y_1, X_2, Y_2, \ldots), \qquad (X_j, Y_j) = c(a_j^+, b_j^+) + c(a_j^-, b_j^-).$$

Put $m_j = (X_j + Y_j)/2$. For each j, use Lemma 6.13 on group j, and a symmetric-group longest source on every other group. By the finite label rule its bound σ_j is the symplectic collapse of the partition with columns

$$(m_1, m_1, \ldots, m_{j-1}, m_{j-1}, X_j, Y_j, m_{j+1}, m_{j+1}, \ldots). \tag{6.21}$$

This list is decreasing. Pair averaging decreases column dominance, so its uncollapsed row partition lies above λ. Since λ is symplectic, it remains below the collapse.

For a partition ν, the rank of the k-th power of a nilpotent with partition ν is

$$r_k(\nu) = |\nu| - \sum_{i=1}^{k} \nu'_i.$$

Every even column prefix in (6.21) equals the corresponding prefix of λ, and the odd prefix ending at X_j equals it for cut j. Collapse can only increase the column prefixes, and cannot cross λ. Therefore

$$\min_j r_k(\sigma_j) = r_k(\lambda) \quad \text{for every } k, \qquad \bigcap_j \overline{\mathcal{O}_{\sigma_j}} = \overline{\mathcal{O}_\lambda}. \tag{6.22}$$

Choose disjoint column coordinate groups and put

$$P_j = D_{a_j^+} B_{b_j^+} D_{a_j^-} B_{b_j^-}, \quad L_j = \mathfrak{S}_{a_j^+ + b_j^+} \times \mathfrak{S}_{a_j^- + b_j^-}, \quad L = \prod_j L_j.$$

Lemma 6.12 realizes L as a p-th-root stabilizer. The exact local block inclusion puts P_j in the active ideal; each passive longest source supplies Δ_{L_i}. Thus the j-th ideal contains

$$g_j = \mathrm{Alt}_{L_j}(P_j) \prod_{i \neq j} \Delta_{L_i} = \Delta_L h_j, \qquad h_j \in S_y^L.$$

The distinct even and odd exponent sets prove every alternant nonzero. Lemma 6.11 produces

$$\Delta_L \prod_j h_j = \prod_j \mathrm{Alt}_{L_j}(P_j) = \mathrm{Alt}_L \left(\prod_j P_j \right) \neq 0.$$

The product before alternation is the signed column Specht polynomial in the unique minimum copy of the desired local type. Its nonzero group-algebra projection remains in that copy, and M-saturation gives all of it. The support bound is (6.22). The construction has one cut for each nonempty pair, and is finite in every rank. In type $B_u \times B_v$ the corresponding minimum Specht ideal is also the full isotypic ideal [19, Proposition 7.6]; only the minimum-copy assertion is needed for the general downset argument. □

6.6. **Orthogonal mixed points and complete intersection groups.** We next treat the simply connected orthogonal groups. Their full-rank centralizers have Weyl groups

$$B_u \times D_v \quad \text{for } \mathrm{Spin}_{2d+1}, \qquad D_u \times D_v \quad \text{for } \mathrm{Spin}_{2d}.$$

Their vector torus coordinates are $+1$ and -1. All lifts to the Spin torus will be retained. The head-and-pair formulas in Lemma 6.10 reduce the construction to local heads and one pair at a time.

We make explicit a point about the finite intersections. An affine parabolic at the half-shift end fixes a rational vector ξ, and its constant character on an element h fixing the torus point is $y^{\xi - h\xi}$. When coordinates are split into two global ends and two local signs, the intersection need not be a product of four even-sign groups. It contains coupled odd sign changes between quadrants. We shall compute restrictions in the product of the four hyperoctahedral overgroups, twist there by the sign character in the half-shift minus quadrant, and only then restrict to the whole intersection. This retains the value on every coupled element.

Lemma 6.15. *For the active blocks specified below, vertical-column Littlewood–Richardson restriction, with the indicated half-shift character, puts the desired local head-and-pair minimum polynomial in the source ideal. Passive symmetric-group longest factors supply the ordinary column discriminants with trivial cocycle. For a balanced local type-D label, the initial assertion is occurrence of at least one nonzero minimum-half component; the individual halves are treated in Lemma 6.21.*

Proof. The exact finite-block coefficient formula takes the restriction of the global block to the full intersection, twists by $y^{\xi-h\xi}$, and induces to the local stabilizer. A common constituent of those restrictions therefore supplies the entire corresponding local isotypic ideal, by the matrix-block calculation.

On the four hyperoctahedral overgroups, the character is the sign of coordinate flips in the quadrant which lies both at the half-shift end and in the minus block. Tensoring by this character exchanges its two partition colors. The vertical Littlewood–Richardson constituents listed below thus give the desired restrictions before passing to the full intersection, so they also give them after restriction. In particular their character on a coupled pair of odd flips is retained. Each passive symmetric block has constant ξ, hence $\xi - h\xi = 0$, and its determinant restricts to the ordinary determinant of its intersection. Its coefficient image therefore contains the corresponding discriminant. Adjoining the remaining passive column pairs preserves the required full local constituent: in each color the rowwise sum of its vertical column partitions is the desired partition, and the corresponding Cartan Littlewood–Richardson coefficient is one. On an unsigned pair group the determinant occurs in the restriction of the required two-column type, again with coefficient one. Thus ordinary support containment applies to the complete target representation selected by the finite Mackey formula. □

6.7. **Odd orthogonal head-and-pair sources.** Write a plus-side distinguished head and pair as $(r; a, b)$, and a minus-side distinguished head and pair as $(s; c, d)$. Thus

$$F_+ = (1^a;\, 1^r + 1^b), \qquad F_- = \{1^s + 1^c;\, 1^d\}, \qquad r \geq a-1,\ r \geq b,\ s \geq c, d.$$

The local row union has columns

$$H = 2(r+s)+1, \qquad (X, Y) = c(a, b) + c(c, d). \tag{6.23}$$

The affine Coxeter diagram has two finite B-ends. For $P \in \{r, b\}$, $Q \in \{s, c\}$, use the standard blocks

$$E_1 = (1^{a+Q};\, 1^{P+d}), \qquad E_2 = (\varnothing;\, 1^{r+b-P+s+c-Q}) \tag{6.24}$$

on the untwisted and half-shift ends respectively. The first is minuscule and the second is longest. The special representative of E_1 is

$$(1^u; 1^v), \qquad u = \min(a+Q, P+d+1), \quad v = \max(P+d, a+Q-1).$$

Writing $z = r+b-P+s+c-Q$, truncated induction gives $(1^u; 1^v + 1^z)$.

Lemma 6.16. *For every choice of local orientations there is an actual active source whose bound contains the target in* (6.23) *and whose two largest column prefix is $H + X$.*

Proof. For orientations B, B, choose $P = b, Q = c$ in (6.24). Then $z = r+s$, $a+c \leq b+d$, and $r+s \geq b+d$. The induced label has columns $(H, 2(b+d), 2(a+c)) = (H, X, Y)$. For orientations D, D, make the same choice. Now $a+c \geq b+d+2$, so $u = b+d+1$, $v = a+c-1$, and $z = r+s \geq v$. The columns are $(H, 2(a+c)-2, 2(b+d)+2) = (H, X, Y)$.

For orientations D, B, take $P = r, Q = s$. Here $z = b+c$ and both $a+s$ and $r+d+1$ are at least $b+c+1$. Thus $u \geq z+1$ and $v \geq u-1$. Its final column is $2z+1 = Y$, so its first two columns sum to $H + X$. The first column need not itself be H, which will be supplied by a separate head cut.

The remaining orientation B, D uses Proposition 6.9. Put

$$P_0 = r+s, \qquad R_0 = b+c,$$

and use

$$E_1 = (1^a; 1^d), \qquad E_2 = (\varnothing;\, 1^{P_0} + 1^{R_0}) \tag{6.25}$$

on the two ends. The inequalities imply $P_0 \geq R_0$. If $u = \min(a, d+1)$ and $v = \max(d, a-1)$, their special representatives truncate-induce to

$$(1^{R_0} + 1^u;\, 1^{P_0} + 1^v).$$

Since $P_0 \geq R_0 > u, v$ and $v \geq u - 1$, this is distinguished. Its columns are

$$(2P_0 + 1,\ 2R_0 - 1,\ 2v + 1,\ 2u - 1,\ 1),$$

with the last two entries omitted when $u = 0$. The first two entries are H, X, and the remainder has total $Y = 2(a + d) + 1$.

We verify the local occurrence, which also supplies the lower orbit inequality. In (6.24), split the plus beta columns into $P, r + b - P$, and the minus alpha columns into $Q, s + c - Q$. The remaining plus alpha and minus beta columns lie in the first end. Vertical restriction has coefficient one, and the half-shift minus twist turns the second minus beta column into the required alpha column. For (6.25), the required coefficient is the transpose-Cartan coefficient

$$c^{1^{r+s}+1^{b+c}}_{1^r+1^b,\ 1^s+1^c} = 1.$$

The half-shift twist again changes the minus color, and E_1 supplies its remaining two columns. Lemma 6.15 gives the full intersection occurrence in all four cases. The constant matrix-block argument in the proof of Proposition 4.4, applied to these selected local blocks, forces the target orbit to lie in each source bound. □

Proposition 6.17. *At every full-rank torus point for* Spin_{2d+1}*, every local trivial-local-system minimum Springer copy has a finite actual source family with the exact ambient orbit bound.*

Proof. Write F_+, F_- for the full local Springer labels. Use Lemma 6.16 for each tail-pair position j, and symmetric-group longest factors on all other positions. Add a head source: the ordinary B_{r+s} longest object on the combined heads, with the same passive symmetric factors. The common local parabolic is

$$L = B_r \times D_s \times \prod_j (\mathfrak{S}_{a_j+b_j} \times \mathfrak{S}_{c_j+d_j}).$$

This is a p-th-root stabilizer by Lemma 6.12. The head source is ordinary, with fixed point $\xi = 0$, so its character is untwisted and its local image contains Δ_L. If an active minus-side label is balanced, apply the odd-orthogonal half switch described after Lemma 6.21 to that source ideal first. Choose the flipped coordinate in its head, so all passive coordinates stay fixed. Central stability gives both minimum halves at the same bound, and hence the unprojected active minimum polynomial. The j-th active source contains $\Delta_L h_j$, where h_j is obtained by alternating $D_{a_j} B_{b_j}$ and $D_{c_j} B_{d_j}$ within their pair groups, and dividing by those two Vandermondes. The alternants are nonzero by their distinct even and odd exponent sets. Lemma 6.11, including the head factor 1, gives

$$\Delta_L \prod_j h_j = \mathrm{Alt}_L(P_{F_+} P_{F_-}) \neq 0.$$

This is a nonzero vector in the target minimum copy, and M-saturation gives the copy.

Let the target columns be $(H, X_1, Y_1, X_2, Y_2, \ldots)$. The head source gives every odd prefix: its head and its preceding averaged pairs have the same sum as the target prefix. For the even prefix ending at X_j, use the j-th active source; its first two active columns sum to $H + X_j$, and the passive preceding pairs retain their totals. These selected columns are a subset of the appropriate size in the raw source partition, so the largest columns have at least that sum. Orthogonal collapse can only increase it. Conversely the full-intersection occurrence of the target in every source, including the passive factors, and Proposition 4.4, give the opposite inequality. Thus at each prefix one source attains the target exactly. The rank functions consequently have pointwise

minimum equal to that of the target, and the intersection of the source closures is its orbit closure.

If a local D-type is degenerate, an odd coordinate flip in that local block is a global B-Weyl element. It fixes the vector torus values and may change the Spin lift by the central element. The central stability proved in Lemma 6.20 returns to the same Spin point and exchanges the two halves. Their ambient odd orthogonal orbit is the same. Thus both required halves are supplied at the same bound. □

6.8. **Even orthogonal head-and-pair sources.** At a full-rank mixed point of Spin_{2d}, write

$$F_+ = \{1^r + 1^a; 1^b\}, \qquad F_- = \{1^s + 1^c; 1^d\}, \qquad r \geq a, b, \quad s \geq c, d, \quad P = r + s.$$

The target head is $H = 2P$, and its tail pair is $(X, Y) = c(a, b) + c(c, d)$. Use the two D-ends of the affine diagram. A two-column block means a minuscule source from Proposition 6.2; when it is balanced retain the two actual fork sources at this stage.

Lemma 6.18. *The following standard active blocks supply a nonzero target local minimum and a source bound whose first two column prefix is $H + X$:*

orientations	*untwisted end*	*half-shift end*
B, B	$\{1^P; 1^{b+d}\}$	$\{1^a; 1^c\}$
D, B	$\{1^P; 1^b\}$	$\{1^{a+d}; 1^c\}$
B, D	$\{1^P; 1^d\}$	$\{1^a; 1^{b+c}\}$
D, D	$\{1^P + 1^{a+c}; \varnothing\}$	$\{1^d; 1^b\}$.

For a balanced local type this initially supplies at least one nonzero half. The individual required halves are obtained in Lemma 6.21 and Proposition 6.22.

Proof. Only the last row uses a new source. There $R = a + c \geq 2$ and $P \geq R$, so Proposition 6.9 supplies the first block. Its special representative is $\{(2, 1^{P-1}); 1^{R-1}\}$.

For the intersection calculation, split the coordinates into quadrants $11, 12, 21, 22$, with the first index specifying the global end and the second the local sign. In the product of their hyperoctahedral overgroups use the following ordered bipartitions:

	11	12	21	22
B, B	$(1^r; 1^b)$	$(1^s; 1^d)$	$(1^a; \varnothing)$	$(\varnothing; 1^c)$
D, B	$(1^r; 1^b)$	$(1^s; \varnothing)$	$(1^a; \varnothing)$	$(1^d; 1^c)$
B, D	$(1^r; \varnothing)$	$(1^s; 1^d)$	$(1^a; 1^b)$	$(\varnothing; 1^c)$
D, D	$(1^r + 1^a; \varnothing)$	$(1^s + 1^c; \varnothing)$	$(\varnothing; 1^b)$	$(1^d; \varnothing)$.

Vertical-column restriction gives the two global-end representations in each row. In the last row the nontrivial coefficient is

$$c^{1^{r+s}+1^{a+c}}_{1^r+1^a,\ 1^s+1^c} = 1.$$

Exchange the colors in quadrant 22, as required by the half-shift character. The local columns then assemble to F_+ and F_-. Restricting these common overgroup constituents to the whole actual intersection proves the desired occurrence, including the coupled odd flips. When a local unordered type is balanced, take a nonzero projection onto one of its two halves; occurrence of the other half is not inferred merely by projection.

We compute the source bounds from special finite representatives. In the B, B row set $u = \max(a, c)$, $v = \min(a, c)$. The induced distinguished label has alpha columns (P, u) and beta columns $(b+d, v)$. Since $P \geq b+d \geq u \geq v$, its first two orbit columns are $(2P, 2(b+d)) = (H, X)$. For D, B, its alpha columns are $(P, a + d)$ and its beta columns are $(\max(b, c), \min(b, c))$. Since $a+d > \max(b, c)$, its first two orbit columns are $(2P, 2(a+d)-1) = (H, X)$. The B, D calculation gives $(2P, 2(b + c) - 1) = (H, X)$.

For D, D, put $R = a + c$, $u = \max(b, d)$, $v = \min(b, d)$. Induction of the special first label and the second two-column label gives alpha columns $(P, u, 1)$ and beta columns $(R - 1, v)$ when $u > 0$. If $u = v = 0$, the alpha columns are $(P, 1)$. Because $P \geq R \geq b + d + 2$, these labels are distinguished. Their first two orbit columns are $(2P, 2R - 2) = (H, X)$. The remaining columns are $(2u, 2v, 2)$ if $v > 0$, $(2u, 1, 1)$ if $v = 0 < u$, and (2) if $u = 0$. Their total is $Y = 2(b + d) + 2$. The finite cell rule (6.3) therefore gives exactly the required prefix in every row. The local block occurrence and ordinary support containment give the reverse orbit inequality, as in the odd case. □

Proposition 6.19. *For a target whose ambient even orthogonal partition is not very even, the active sources of Lemma 6.18, together with one head source, give a finite actual source family containing every required local minimum copy and having exactly the target orbit bound.*

Proof. Write F_+, F_- for the full local Springer labels. For pair j, use its active row and passive symmetric-group longest factors at all other positions. Take as head source the ordinary D_P longest object, with all pairs passive. Use the common parabolic

$$L = D_r \times D_s \times \prod_j (\mathfrak{S}_{a_j+b_j} \times \mathfrak{S}_{c_j+d_j}).$$

This is a p-th-root stabilizer by Lemma 6.12. The head source has fixed point 0, so its coefficient image contains the untwisted Δ_L. The active sources contain the full active minimum polynomials, as follows. For a full local label $\{\alpha, \beta\}$ with $\alpha_i \geq \beta_i$, its j-th active label $\{1^r + 1^{a_j}; 1^{b_j}\}$ is balanced only if $a_j = \alpha'_{j+1} = 0$ and $b_j = \beta'_j = r = \alpha'_1$. These equalities force $\alpha = \beta = (j^r)$. Thus at most one active local factor is balanced here. For each such source, apply Lemma 6.21 before forming the tensor product, choosing the two flipped coordinates in the heads. It supplies both active halves at the same bound and fixes every passive coordinate. Consequently the active source for pair j contains $\Delta_L h_j$, where h_j is the invariant quotient of the two pair alternants. The distinct-exponent calculation and Lemma 6.11 give

$$\Delta_L \prod_j h_j = \mathrm{Alt}_L(P_{F_+} P_{F_-}) \neq 0.$$

If a local factor is balanced, this vector first gives a nonzero minimum-half component.

The source orbit meet is the exact target: the head source supplies every odd column prefix, and the j-th active source supplies the even prefix ending at its first pair column. The preceding passive averages retain their totals. As in Proposition 6.17, selected columns give one inequality, collapse strengthens it, and the complete local block occurrence gives the reverse inequality by Proposition 4.4. Rank functions now give the asserted meet. There is only one even orthogonal orbit for this non-very-even partition.

If exactly one local factor is balanced, Lemma 6.21 exchanges its two halves within the completed ideal of this final tensor family, at the same point and the same support bound. This supplies each required half. If neither factor is balanced the minimum copy is irreducible. If both nonempty factors were balanced, the ambient partition would be very even, which is excluded here and treated next. □

6.9. **Central homogeneity and the two orthogonal halves.**

Lemma 6.20. *For a simply connected classical orthogonal group at an odd prime p, the coefficient ideal of an actual source family with a fixed ordinary support bound is stable under central translation on the complex torus. The same holds after a finite number of tensor operations and block projections.*

Proof. Let X and Q be the weight and root lattices. The weights of a tilting module $T_p(\lambda)$ are all congruent to λ modulo Q. Thus its character is homogeneous for X/Q. Tensor products

preserve this grading and linkage projections preserve each homogeneous component. The Weyl numerator Δ has central class ρ, since $w\rho - \rho \in Q$.

In the principal coefficient extraction, an exponent is of the form $w\rho + p\nu$. For a homogeneous character of class λ, its coefficient therefore satisfies

$$p\nu \equiv \lambda \pmod{Q}.$$

The center of a Spin group has order a power of 2; multiplication by the odd integer p is consequently invertible on X/Q. Coefficient extraction preserves homogeneous classes, with their indices permuted by this inverse. The coefficient ideal generated by the source family is therefore an X/Q-graded ideal.

Translation by $z \in Z$ multiplies every homogeneous component by its nonzero central scalar, and hence preserves the ideal. This argument applies to the already specified source family at its original bound, including its tensor and block closure.

Equivalently, in a finite affine source the exponents $\xi - h\xi$ belong to Q, so central translation preserves its finite character. In the tensor formula, choose the unique central p-th root of z. It permutes the p-th-root sheets, leaves their root stabilizers unchanged, and changes the true Weyl numerator only by its nonzero ρ-scalar. This also shows directly that the completed tensor formula respects the same symmetry. □

Lemma 6.21. *Let y be a mixed full-rank point for Spin_{2n}, with nonempty plus and minus blocks and stabilizer $D_u \times D_v$. Let J be a Weyl-stable coefficient ideal as in Lemma 6.20. There is an automorphism of $\widehat{J}_y$ whose linear action is an odd coordinate flip in each local factor. If exactly one local irreducible type is degenerate, occurrence of one of its minimum halves in $\widehat{J}_y$ implies occurrence of the other, with the nondegenerate factor unchanged.*

Proof. Use

$$X = \mathbb{Z}^n + \mathbb{Z}\omega, \quad \omega = \tfrac{1}{2}(e_1 + \cdots + e_n), \qquad Q = \{(a_i) \in \mathbb{Z}^n : \textstyle\sum_i a_i \text{ is even}\}.$$

Choose i in the plus block and j in the minus block, and let $r_0 = \sigma_i\sigma_j$ invert these two vector coordinates. Then $r_0 \in W(D_n)$. It fixes all vector coordinate values $y^{e_k} = \pm 1$, but

$$r_0\omega = \omega - e_i - e_j, \qquad (r_0 y)^\omega = -y^\omega.$$

Let c be the central element with $c^{e_k} = 1$ and $c^\omega = -1$. It has order two and $r_0 y = cy$. The composition of r_0 with translation by c therefore fixes y and preserves $\widehat{J}_y$, by Weyl and central stability. In logarithmic coordinates its derivative is exactly r_0; central translation has the identity derivative. Thus it preserves degree and nonzero lowest vectors and induces one odd flip on each local factor.

Clifford theory for $D_m \triangleleft B_m$ shows that odd conjugation fixes a nondegenerate unordered representation $\{\alpha, \beta\}$, $\alpha \neq \beta$, and exchanges the two halves of a balanced label $\{\alpha, \alpha\}$. Applying the automorphism to a nonzero minimum vector gives the other half. The operation acts on the completed ideal of the final tensor family, so the original support intersection remains unchanged. □

An empty factor has no half choice. A formal D_1 coordinate is also not a degenerate representation factor and requires no odd-flip construction. For odd orthogonal groups the same argument uses a single minus-coordinate flip in the global B-group, followed by the same central correction on the Spin torus. It exchanges the two local D-halves and does not change their ambient odd orthogonal orbit.

Proposition 6.22. *For Spin_{2n}, each of the two ambient very-even orbit closures separately has actual sources for all its compatible local trivial-local-system minimum copies.*

Proof. Fix a full-rank point y with $M = W_y = D_u \times D_v$, where $u + v = n$; either factor may be empty. A local distinguished type-D label is balanced exactly when its orbit is very even. This follows also from the row formula in Lemma 6.10: equal paired rows average to even rows, whereas the first strict inequality in an ordered distinguished bipartition leaves an odd row. The ambient row union is very even exactly when each nonempty local factor is balanced. Write the local labels (α_+, α_+) and (α_-, α_-), omitting an empty factor, and put $a_j^{\pm} = (\alpha_\pm)'_j$.

Take the ordinary parabolic

$$K = \prod_j \mathfrak{S}_{2(a_j^+ + a_j^-)} \subset W(D_n)$$

and its smooth longest source. Parabolic truncated induction of its determinant has one of the two global balanced Springer labels with precisely the desired very-even partition. The other fork embedding, obtained by the outer diagram involution, gives the other global orbit. These are two actual longest-element sources.

At y, the relevant intersection is

$$K \cap M = \prod_j (\mathfrak{S}_{2a_j^+} \times \mathfrak{S}_{2a_j^-}).$$

The source is ordinary with $\xi = 0$, so there is no affine character. Its local coefficient ideal contains the discriminant of this intersection. In each local factor its degree is

$$\sum_j \binom{2a_j^\pm}{2} = 4n(\alpha_\pm) + |\alpha_\pm|,$$

which is exactly the minimum degree of the balanced Springer representation. The parabolic truncated-induction rule therefore places a nonzero projection of this discriminant in one local half-pair. Let E be a local half-pair with nonzero projection of this discriminant. That projection is still $(K \cap M)$-anti-invariant, so

$$\mathrm{Hom}_{K\cap M}(\det_{K\cap M}, E) \neq 0.$$

The identity double-coset summand in the finite Mackey formula has trivial cocycle here. Frobenius reciprocity gives $m(\det_K, E; y, 0) > 0$. By Lemma 5.1, the source selects the full constant E-block. Proposition 4.4 therefore assigns this pair to the chosen global orbit; it cannot be assigned to the other very-even orbit, whose closure is incomparable at this partition.

When both local factors are nonempty, denote the half labels of the selected pair by (ϵ_+, ϵ_-). Lemma 6.21 switches both halves simultaneously. This gives the two compatible pairs

$$(\epsilon_+, \epsilon_-), \qquad (-\epsilon_+, -\epsilon_-)$$

inside the same global orbit bound. Indeed switching one local half is induced by an isometry of determinant -1 on one summand, so it exchanges the two global orthogonal orbits; switching both has determinant $+1$ and preserves the global orbit. The two remaining pairs belong to the other fork source. If one factor is empty, the chosen global orbit has only one corresponding local half, already supplied by its fork source. □

6.10. **Type A and the local conclusion.** For a type-A partition λ, take the Young parabolic

$$K_\lambda = \prod_j \mathfrak{S}_{\lambda'_j}.$$

Its longest Schubert variety is a smooth partial flag fiber, so its canonical object is independent of every coefficient characteristic. Its RSK shape is λ, and the standard block S^λ occurs by column antisymmetrization. The source is consequently in the exact ordinary cell of λ. At a central torus point its coefficient image contains the ordinary column Specht polynomial $\prod_j \Delta_{\mathfrak{S}_{\lambda'_j}}$, whose W-span is the unique minimum copy of S^λ. Full-rank stabilizers for the simply connected

type-A torus are central; the ordinary source has trivial cocycle there. This supplies the required minimum copies at all such points.

Theorem 6.23. *Let G be simply connected of classical type and let $p > h$. At each full-rank torus point y, every local nilpotent orbit $\mathcal{O}$ has a finite actual source family whose completed coefficient ideal contains its trivial-local-system minimum Springer copy $U_{\mathcal{O}}$, and whose support is contained in the closure of its ambient nilpotent orbit. In even orthogonal type the two very-even closures are treated separately.*

Proof. Type A is the preceding construction, type C is Proposition 6.14, and odd orthogonal type is Proposition 6.17. For even orthogonal type combine Propositions 6.19 and 6.22, using the central homogeneity and half-switch lemmas.

The finite parity inputs hold at every odd coefficient characteristic, since the local intersection forms are saturated over $\mathbb{Z}[1/2]$. The operator bound in Proposition 3.1 and the root-sheet construction in Lemma 6.12 require $p > h$. Explicitly it is $p > 2d$ for B_d, C_d and $p > 2d - 2$ for D_d; it also makes p prime to the Spin center. □

For any ambient closed orbit downset $\mathcal{C}$, take the finite sums of these source families over its permitted local orbits at each full-rank point. The ordinary inclusion Proposition 4.4 and the downset generation Proposition 4.5 then give the required local equality with $I_{\mathcal{C}}$. The general globalization argument applies to these finite source and tensor words.

7. Finite sources in the exceptional types

We construct the remaining generators in the finite criterion of Proposition 5.2. Throughout this section the nodes of E_r are numbered so that the edges are

$$(1,3),\ (3,4),\ (2,4),\ (4,5),\dots,(r-1,r).$$

A string $i_1 \cdots i_d$ denotes the product $s_{i_1} \cdots s_{i_d}$; juxtaposed lines in a displayed string are concatenated. If J is a subset of the finite simple reflections S, let w_J be its longest element and W^J the set of minimal representatives of right W_J-cosets. All words denoted by x below belong to W^J. We use the normalization

$$(H_s + v)(H_s - v^{-1}) = 0, \qquad C_s = H_s + v.$$

In particular, $C_s(1) = 1 + s$. For $I \subset S$, let $P_I \subset G_{\mathbb{C}}$ be the standard parabolic subgroup corresponding to I.

7.1. **Actual elements and finite blocks.** An *actual ordinary source* is an element w for which ${}^pC_w = C_w$ in the indicated range of characteristics. This equality identifies an indecomposable parity object, and hence an actual tilting source under the character correspondence used above.

Our finite character labels use the ordinary Springer convention: the trivial representation corresponds to the regular orbit. A label $\phi_{d,b}$ denotes an irreducible character of degree d whose first coinvariant occurrence has degree b. A representation F with this label is represented in the positive Hecke normalization by

$$F^+ = \det \otimes F. \tag{7.1}$$

An actual source detects F if $C_w(1)$ acts nontrivially on F^+. Since the finite source group algebra is semisimple, one nonzero matrix in this block generates its full matrix algebra as a two-sided ideal. The source bound is the affine orbit attached to the ordinary cell of w, with the finite-to-affine convention of Proposition 4.3; this bound depends on the special representation of the finite family.

We use the following precise finite inputs. The exhaustive calculations in [22, Appendix, especially §5.4] give ${}^pC_w = C_w$ for every element of every finite Weyl group of rank at most six when $p > 7$. In types F_4 and G_2 the only interesting primes in those calculations are 2 and 3, so

the corresponding whole-basis equality holds for $p > 3$. Here "interesting" means that the two bases differ. In particular the G_2 case includes characteristic 7.

For a spherical (co)minuscule quotient, the ordinary and p-canonical bases agree in good characteristic by [11, Theorem 9.1]. We apply this to the classical minuscule quotients and to E_7/E_6. In type A, the input needed here is smaller: every ordinary cell has an actual parabolic-longest source. Indeed a permutation which reverses consecutive blocks of sizes $\lambda'_1, \lambda'_2, \dots$ has Robinson–Schensted shape λ, and its Schubert variety is a finite flag variety. This gives a source in each cell for every p.

Here is the finite-algebra reason that these inputs supply the blocks used later. If every maximal right W_K-coset element in a finite group H is actual, the cell filtration of $\mathcal{H}(H)C_{w_K}$ supplies an actual element detecting each ordinary constituent of $\mathrm{Ind}_{W_K}^H$ det. To see this, specialize the filtration at $v = 1$ and project to a simple matrix block. Its first nonzero cell quotient is the quotient for the family of that constituent; if every basis element of that quotient acted by zero, the quotient would act by zero. Thus some basis element in the required cell has nonzero action. For products of irreducible components, take products of the sources. For a special representation one may use any positive basis element in its cell: the positive special-module theorem gives nonzero action [34, Propositions 9, 13, 17 and Corollary 16]. We shall use this last statement only for special representations. The nonspecial actions required beyond the spherical inputs are computed explicitly in Section 7.5.

In types E_6, F_4, G_2, every canonical element of every finite standard affine parabolic is actual, with respective bounds $p > 12, p > 12, p > 6$. For G_2 the finite parabolics have types $G_2, A_2, A_1 \times A_1$ or smaller, so the same assertion includes $p = 7$. Proposition 4.2 expresses each ordinary based downset ideal in terms of these finite elements. Lemma 6.1 therefore puts its entire coefficient image in $I_{\mathcal{C}}^+$, proving $I_{\mathcal{C}}^+ = I_{\mathcal{C}}$ in these types. The remaining finite calculation concerns E_7 and E_8.

7.2. **Two propagation lemmas.**

Lemma 7.1 (Partial-Levi inflation)**.** *Let $I, J \subset S$, put $K = I \cap J$, and let $x \in W_I$ be minimal in its right W_J-coset. If*

$$ {}^pC_{xw_K} = C_{xw_K} \quad \text{in } \mathcal{H}(W_I), $$

then ${}^pC_{xw_J} = C_{xw_J}$ in $\mathcal{H}(W)$.

Proof. Let G_I be the standard Levi subgroup of $G_{\mathbb{C}}$ with nodes I. The closed inclusion $G_I/(G_I \cap P_J) \longrightarrow G_{\mathbb{C}}/P_J$ identifies the Schubert variety indexed by x with the same Schubert variety in the ambient partial flag variety; it identifies their Schubert stratifications as well [39, Lemma 4.8]. Normalized smooth pullback along the full-to-partial flag projections takes its indecomposable parity complex to the complexes indexed by xw_K and xw_J, respectively. It takes the characteristic-zero intersection cohomology complex to the corresponding normalized pullback. Equality of the parity and intersection cohomology characters therefore transfers between the two full flag varieties. Equivalently, the two relevant spherical canonical coefficients are the coefficients of the same stratified variety. This proves the assertion for an indecomposable source. □

Lemma 7.2 (Common-coset propagation)**.** *Suppose ${}^pC_z = C_z$, $w = sz > z$, and*

$$ C_sC_z = C_w + \sum_{y \in Y} \mu_y C_y, \qquad \mu_y \in \mathbb{Z}_{>0}. \tag{7.2} $$

For every $y \in Y$, suppose w, y belong to a common left or right coset of a finite standard parabolic W_I. In that coset write the upper element as a minimal coset representative times its parabolic factor, using inversion in the left-coset case. Suppose that this parabolic factor b satisfies ${}^pC_b = C_b$. Then ${}^pC_w = C_w$.

Proof. Positivity of the p-canonical structure constants and of the transition to the ordinary basis implies

$$ {}^pC_w = C_w + \sum_{y\in Y} a_y C_y, \qquad a_y \in \mathbb{Z}_{\geq 0}[v, v^{-1}]. $$

In fact this follows by comparing the ordinary-basis coefficients of the actual product ${}^pC_s\,{}^pC_z = C_sC_z$: the leading summand ${}^pC_w$ occurs with multiplicity one, and all other contributions are nonnegative. Let $h_{y,w}$ and ${}^ph_{y,w}$ be the H_y-coefficients of C_w and ${}^pC_w$, respectively. Jensen's parabolic coefficient theorem [29, Corollary 3.10] and the equality for b give ${}^ph_{y,w} = h_{y,w}$ for each $y \in Y$. Comparing the coefficient of H_y gives

$$ 0 = \sum_{t\in Y} a_t h_{y,t}. $$

All coefficients on the right are nonnegative and $h_{y,y} = 1$; hence $a_y = 0$. This holds for every y. Inversion preserves both canonical bases and proves the left-coset assertion. □

The hypothesis in Lemma 7.2 concerns the parabolic upper element. We verify it directly for the two rank-seven upper factors below. All other common-coset factors have connected components of rank at most six.

7.3. **The two upper sources for the last geometric construction.** We first record these sources explicitly, because both are inputs to the construction in Section 8. Let $J = \{1, 3, 4, 5, 6, 7, 8\}$, of type A_7, and define

$$ x_{509} = \texttt{43154234565423143542}, $$
$$ x_{1221} = \texttt{4315436547658765423143542}. $$

Their lengths are 20 and 25, respectively.

Proposition 7.3. *For $p > 7$ one has*

$$ {}^pC_{x_{509}w_J} = C_{x_{509}w_J}, \qquad {}^pC_{x_{1221}w_J} = C_{x_{1221}w_J}. $$

The first equality uses only the rank-six finite input; the second uses a smooth longest element and rank-two common-coset factors.

Proof. The word x_{509} is supported on $I = \{1, 2, 3, 4, 5, 6\}$, of type E_6, and $I \cap J$ is of type A_5. Apply Lemma 7.1 and the rank-six input.

For the second equality set $z_0 = w_J$. Table 1 defines $z_k = s_{i_k} z_{k-1}$ for $1 \leq k \leq 25$, and gives sets A_k. The words are reduced and $z_{25} = x_{1221}w_J$. The complete ordinary multiplication identities are

$$ C_{s_{i_k}} C_{z_{k-1}} = C_{z_k} + \sum_{j\in A_k} C_{s_j z_{k-1}}. \tag{7.3} $$

There are 23 correction terms in total, all with coefficient one. These are identities in the spherical module $\mathcal{H}(E_8)C_{w_J}$; the multiplication rule (A.2) gives a direct integer Laurent-polynomial verification. More specifically, start with C_{w_J}, use the listed i_k, and impose bar invariance and strict positive-degree triangularity after each multiplication. The complete list of terms to subtract is exactly the last column of Table 1. This is a finite verification with 337 nonzero spherical terms at the last step.

For $j \in A_k$ put $y = s_j z_{k-1}$. The same root-matrix calculation gives $\ell(y) = \ell(z_{k-1}) - 1$ and

$$ z_k = s_{i_k} s_j y. $$

Thus y, z_k belong to the same left coset of $W_{\{i_k,j\}}$, a rank-two finite group. Its ordinary and p-canonical bases agree for $p > 7$. Lemma 7.2, starting from the smooth source w_J, proves all 25 assertions in order. □

Table 1: The complete chain for the upper source $x_{1221}w_J$. Here $z_0 = w_J$, $z_k = s_{i_k} z_{k-1}$, and the last column is the set A_k in (7.3).

k	i_k	Name of z_k	A_k
1	2	2	$\varnothing$
2	4	3	$\{2\}$
3	5	5	$\{4\}$
4	6	8	$\{5\}$
5	7	12	$\{6\}$
6	8	18	$\{7\}$
7	3	26	$\varnothing$
8	4	36	$\{3\}$
9	5	49	$\{4\}$
10	6	66	$\{5\}$
11	7	87	$\{6, 8\}$
12	2	113	$\varnothing$
13	4	145	$\{2\}$
14	5	183	$\{4\}$
15	6	229	$\{7, 5\}$
16	3	284	$\varnothing$
17	4	348	$\{3\}$
18	5	422	$\{6, 4\}$
19	1	507	$\varnothing$
20	3	605	$\{1\}$
21	4	717	$\{3\}$
22	5	834	$\{4\}$
23	6	969	$\{5\}$
24	7	1101	$\{6\}$
25	8	1221	$\{7\}$

7.4. **The additional exceptional elements.** Table 2 specifies the nine elements needed in addition to the uniform finite sources. Table 3 records the blocks that they detect. The ordinary family label determines the source bound.

Table 2: The nine additional source elements $w = xw_J$. Digit strings are reduced words for x.

W	Name	J	x
E_7	$x_{7,43}$	$\{1, 3, 4, 5, 6, 7\}$	5423143542
E_7	$x_{7,259}$	$\{1, 3, 4, 5, 6, 7\}$	31435465765423143542
E_8	$x_{8,1540}$	$\{2, 3, 4, 5, 6, 7\}$	342543654276542314567 8
E_8	$x_{8,191}$	$\{1, 2, 3, 4, 5, 6, 7\}$	765423456787654231435426 54 3176542345678
E_8	$x_{8,232}$	$\{1, 2, 3, 4, 5, 6, 7\}$	24354265431765423456787654 2314354265431765423456 78
E_8	$x_{8,2611}$	$\{1, 3, 4, 5, 6, 7, 8\}$	24565423457654234587654231 43542
E_8	$x_{8,418}$	$\{2, 3, 4, 5, 6, 7, 8\}$	24315423456542314354276543 1

W	Name	J	x
E_8	$x_{8,1356}$	$\{2,3,4,5,6,7,8\}$	24354265437654287654231435 426543765428765431
E_8	$x_{8,2794}$	$\{1,3,4,5,6,7,8\}$	31435465431765423458765423 143542

Type	Source	Ordinary block F	a	Bound
E_7	$x_{7,43}w_{A_6}$	$\phi_{189,22}$	22	A_2+2A_1
E_7	$x_{7,259}w_{A_6}$	$\phi_{15,28}$	25	A_2+A_1
E_8	$x_{8,1540}w_{D_6}$	$\phi_{1400,32}$	32	$D_4(a_1)+A_1$
E_8	$x_{8,191}w_{E_7}$	$\phi_{35,74}$	74	$2A_1$
E_8	$x_{8,232}w_{E_7}$	$\phi_{8,91}$	91	A_1
E_8	$x_{8,2611}w_{A_7}$	$\phi_{1050,34}$	32	$D_4(a_1)+A_1$
E_8	$x_{8,418}w_{D_7}$	$\phi_{560,47}$	47	A_2+2A_1
E_8	$x_{8,1356}w_{D_7}$	$\phi_{210,52},\phi_{50,56}$	52	A_2+A_1
E_8	$x_{8,2794}w_{A_7}$	$\phi_{175,36}$	32	$D_4(a_1)+A_1$

TABLE 3. Additional detected blocks and their source bounds.

Proposition 7.4. *The first eight elements in Table 3 are actual for $p>7$. The last element is actual for $p>30$. They detect the listed blocks and have the listed source bounds.*

Proof. The short word for $x_{7,43}$ is supported on $I=\{1,2,3,4,5\}$, of type D_5. Here $I\cap J$ has type A_4. Lemma 7.1 proves its actuality directly from the finite rank-five input.

All the other assertions of actuality in the first eight rows are proved by the complete common-coset tables in Appendix A.3. We describe the roots of these arguments to make their dependencies explicit. The E_7/A_6 chain for 259 starts at the word `2315436542`, supported on E_6. The E_8/D_6 chain for 1540 starts at `145678`, whose support is A_1+A_5. The E_8/E_7 chains for 191 and 232 start at `345678`, with support A_6. The E_8/A_7 chain for 2611 starts at `65423143542`, supported on E_6. Each seed is actual by Lemma 7.1. Every ordinary product, lower correction, and common-coset relative word is printed in the appendix, so induction with Lemma 7.2 proves the desired equalities.

There are three auxiliary E_7/D_5 elements, named $1104, 1455, 1486$. Their words and complete chains are also given in Appendix A.3. Their common seed is supported on $\{2,3,4,5,6,7\}$, of rank six, and every subsequent correction factor has rank at most six. The E_8/D_7 element 418 is the partial-Levi inflation of 1486. The chain for 1356 starts with the inflation of 1455 and uses 1104 once as a common-coset upper factor. In these two inflations $I=\{1,\dots,7\}$ and $I\cap J=\{2,\dots,7\}$ is D_6; the equality of the full upper element with the certified E_7/D_5 word is checked as an equality of Weyl-group elements. The notation E_7/D_5 for the auxiliary construction records the quotient used to construct that word; the inflations use the intersection D_6 computed here.

The remaining rank-seven upper factor, in the chain for 2611, is the E_7/A_6 element with short word `4315436542`. Its short word is supported on E_6, so it is itself actual by Lemma 7.1. These observations close every leaf of the common-coset construction in a finite rank-at-most-six input. Finally the last row is Theorem 8.1, using Proposition 7.3 for its two upper sources.

For ordinary cell membership use the complete star paths in Table 6. They take each displayed full word to the representative cell of [24, §6] identified in Table 7. Every step is specified by an

adjacent pair and is verified by the integer root matrices (A.1). Left and right star operations preserve two-sided cells. In the order of Table 3, the resulting special characters are

$$\begin{array}{l|l} E_7 & \phi_{189,22},\ \phi_{120,25}, \\ E_8 & \phi_{1400,32},\ \phi_{35,74},\ \phi_{8,91},\ \phi_{1400,32},\ \phi_{560,47},\ \phi_{210,52},\ \phi_{1400,32}. \end{array}$$

Their minimum degrees give the printed a-values. Proposition 4.3, with the Springer tables in [36, 25], gives the affine bounds. The positive special-module theorem proves detection for the special blocks. The four additional nonspecial blocks are verified by the explicit actions in Section 7.5. This proves all the assertions, with the determinant conversion (7.1). □

There is one further source inside the affine D_8 subgroup of E_8. It is the pure positive $(5, 3)$ source of the classical construction. It detects the D_8 character `.22211`, whose family has $a = 24$, with ambient bound $A_4 + 2A_1$. This source is recorded as mask 510, row 28 in Table 15. Its actuality is the classical geometric construction, while its finite family and embedding determine the displayed bound.

7.5. **The nonspecial matrix actions.** The following exact actions prove detection of the four nonspecial blocks used here. Let R be an integral representation of the finite Weyl group, set

$$e_J = \sum_{u \in W_J} R(u), \qquad C_{xw_J} = \sum_{y \in W^J} q_y(v) H_y C_{w_J}.$$

Then its specialization acts as

$$Z_x = \left(\sum_y q_y(1) R(y) \right) e_J. \tag{7.4}$$

The integral simple-reflection matrices, their class character checks, and the vectors used in (7.4) form part of the explicit certificates in Appendix A.4. The auxiliary prime used in these checks identifies the integral representations in characteristic zero.

For E_7, take the 15-dimensional representation $\phi_{15,7} = \det \otimes \phi_{15,28}$. Its simple-reflection traces are 5, its commutant has dimension one, and $\operatorname{tr}(e_J)/5040 = 1$ for $J = A_6$. The ordinary spherical vector for $x_{7,259}$ has 141 nonzero terms. Direct substitution into (7.4) gives

$$\operatorname{tr}(Z_{259}) = 20160, \qquad \sum_{a,b} (Z_{259})_{ab}^2 = 2649894912. \tag{7.5}$$

In particular this actual source detects the nonspecial ordinary block $\phi_{15,28}$, with bound $A_2 + A_1$.

For each of the three E_8 representations of dimension d, set

$$(v_0)_i = ((37i + 11) \bmod 101) - 50 \quad (1 \le i \le d), \qquad u = e_J v_0,$$

where the remainder lies in $\{0, \dots, 100\}$. Then u is integral and W_J-invariant. The exact nonzero vectors $z = \sum_y q_y(1) R(y) u$ are as follows. The simple-reflection matrices satisfy the Coxeter

Source	Positive representation	z_1	$\sum_i z_i^2$
2611	$\phi_{1050,10}$	28800	1839562447896576
1356	$\phi_{50,8}$	5160960	1165436657860608
2794	$\phi_{175,12}$	−241920	17245228105728

TABLE 4. Exact integral nonzero actions in the three nonspecial E_8 blocks.

relations, and their traces on all 112 conjugacy classes identify the indicated characters. Integer finite-order traces have absolute value at most the dimension. Thus the class checks modulo 1000003, which is larger than twice all three dimensions, determine the integer traces uniquely. The ordinary spherical coefficients in the action formula are obtained from bar invariance and triangularity; there are 897 nonzero terms for 2794. Equation (7.4) and Table 4 prove nonzero action. Actuality of its last row comes separately from Theorem 8.1.

7.6. **The two local constituents detected by the last source.** The two remaining local representations can be verified directly in the same 175-dimensional representation. This also records the multiplicities needed in the final coverage argument. Write $s_0 = s_\theta$, where

$$\theta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8.$$

The following explicit conjugation expresses its matrix in terms of the eight simple generators:

$$s_0 = u s_8 u^{-1}, \qquad u = \texttt{876542314354265431765423456 7}. \tag{7.6}$$

The word for u has length 28, so this is a 57-letter reflection word. The equality follows either by applying it to the eight simple roots, or by verifying $u(\alpha_8) = \theta$.

Lemma 7.5. *In the ordinary Springer convention,*

$$[\mathrm{Res}^{W(E_8)}_{W(E_6)\times W(A_2)}\, \phi_{175,36} : \phi_{10,9} \boxtimes \mathbf{1}] = 1,$$
$$[\mathrm{Res}^{W(E_8)}_{W(A_5)\times W(A_2)\times W(A_1)}\, \phi_{175,36} : S^{(3,3)} \boxtimes \mathbf{1} \boxtimes \mathbf{1}] = 1.$$

The first subgroup is obtained by deleting node 7 *from the extended diagram, and the second by deleting node* 4*. Both local representations have ambient orbit* $D_4(a_1) + A_1$*.*

Proof. Use the positive representation $V = \phi_{175,12}$. For the first subgroup, E_6 has nodes $1, \dots, 6$ and A_2 has nodes $8, 0$. The sign projector

$$P^-_{A_2} = \frac{1}{6} \sum_{a \in W(A_2)} \det(a) R(a)$$

has image of dimension 10. The six E_6 generators commute with this projector, and their action on its image has a one-dimensional commutant. Thus the image is the unique degree-10 irreducible of E_6, namely $\phi_{10,9}$, with multiplicity one. Determinant twist makes the A_2 factor trivial, and $\phi_{10,9}$ is self-associated under determinant twist.

For the second subgroup the factors have nodes

$$\{5, 6, 7, 8, 0\}, \qquad \{1, 3\}, \qquad \{2\},$$

respectively. The product of the A_2 and A_1 sign projectors has image of dimension 5 and the induced A_5 action has one-dimensional commutant. In the order of cycle types

$$6,\ 51,\ 42,\ 411,\ 33,\ 321,\ 3111,\ 222,\ 2211,\ 21111,\ 111111$$

its character is

$$0,\ 0,\ -1,\ 1,\ 2,\ -1,\ -1,\ 3,\ 1,\ -1,\ 5.$$

The Murnaghan–Nakayama rule identifies this with $S^{(2,2,2)}$. Determinant twist gives $S^{(3,3)}$ and makes both smaller factors trivial.

For completeness, the dimensions and commutant assertions are exact rational statements. The matrices of the projectors are rational with denominators dividing 12. Their images and commutants can be computed after reduction modulo 1000003. The computed image dimensions agree with the projector traces, hence with the rational ranks. A commutant of dimension one over this finite field implies rational commutant dimension at most one, since rank cannot increase upon reduction. Scalar matrices give the reverse inequality; semisimplicity over $\mathbb{C}$ then gives irreducibility. These are ordinary characteristic-zero restriction calculations, independent of the characteristic p of the theorem. The embedded weighted Dynkin diagrams of their Springer orbits give $D_4(a_1) + A_1$ in both cases. They appear explicitly in Tables 27 and 32. □

7.7. **Exact local trace and support cuts.** We describe the finite calculation underlying the coverage tables. This description includes the phases at noncentral points.

Let y be a full-rank point of the coefficient torus, $M = W_y$, and $E \in \mathrm{Irr}(M)$. Let H be the projection of a finite affine standard parabolic, with rational fixed weight ξ. Its elements are $t_{\xi - h\xi}h$, $h \in H$. For $z \in Wy$, choose r_z with $r_z z = y$. The trace on the fibre induced from E is

$$\chi_{E,y,\xi}(h) = \sum_{\substack{z \in Wy \\ hz = z}} z^{\xi - h\xi}\, \chi_E(r_z h r_z^{-1}). \tag{7.7}$$

The difference $\xi - h\xi$ is integral, so the phase is a well-defined root of unity. The value is independent of the chosen r_z. For a detected ordinary block F define

$$m(F, E; y, \xi) = \frac{1}{|H|} \sum_{h \in H} \overline{\chi_F(h)}\, \chi_{E,y,\xi}(h). \tag{7.8}$$

This is a nonnegative integer. All finite characters in this formula use the ordinary convention; the determinant in (7.1) has already been incorporated in passing from a positive Hecke source to its ordinary block. The fixed-point sum uses the entire intersection $H \cap W_z$.

When ξ is integral, conjugation by the global monomial of weight ξ removes the cocycle and the matrix of multiplicities is

$$(\mathrm{Ind}_H^W)^{\mathrm{t}}\, \mathrm{Ind}_M^W .$$

In general (7.7) is evaluated in $\mathbb{Z}[\zeta_e]$, where e is the order of y and ζ_e is a primitive e-th root of unity. For every E one also has the degree identity

$$\sum_{F \in \mathrm{Irr}(H)} \dim(F) m(F, E; y, \xi) = [W : M] \dim(E). \tag{7.9}$$

For each ordinary parabolic subgroup L of M, let D_L be the ambient orbit attached to $j_L^W \det_L$. A cut associated with L retains E whenever

$$[\mathrm{Res}_L^M E : \det_L] > 0.$$

By Corollary 3.6 the corresponding allowed tensor operations retain the required generator and impose the upper bound $\overline{D_L}$. Define the finite orbit down-set

$$\mathcal{D}(E) = \bigcap_{\substack{L \text{ ordinary parabolic in } M \\ [\mathrm{Res}_L^M E : \det_L] > 0}} \overline{D_L}. \tag{7.10}$$

Thus each cut is indexed by an ordinary parabolic of the local centralizer M.

Thus an actual source block F with bound B_F supplies the local generator attached to E if

$$m(F, E; y, \xi) > 0, \qquad \overline{B_F} \cap \mathcal{D}(E) \subseteq \overline{\mathcal{O}(E)}. \tag{7.11}$$

Here $\mathcal{O}(E) = G_{\mathbb{C}} \cdot \mathcal{O}_M(E)$ is the ambient nilpotent orbit containing the local Springer orbit. It is computed by embedding the local weighted Dynkin diagram and taking its ambient dominant representative. The constant matrix-block test in the proof of Proposition 4.4 also gives $\mathcal{O}(E) \subseteq \overline{B_F}$ whenever the displayed multiplicity is positive.

7.8. **Coverage of the exceptional local generators.** The full-rank centralizers and the numbers of required generators are recorded in Table 5. The order of y is its order in the torus of the simply connected datum. Central translates are retained for E_7.

Proposition 7.6 (Exceptional finite coverage)**.** *For E_7 and $p > 18$, respectively for E_8 and $p > 30$, every local generator required by Proposition 5.2 satisfies (7.11) for an actual source. There are* 352 *such full-rank rows in E_7 and* 567 *in E_8.*

E_7				E_8			
Vertex	Order	Centralizer	Rows	Vertex	Order	Centralizer	Rows
1	1	E_7	45	1	1	E_8	70
2	2	E_7	45	2	2	D_8	75
3	2	$D_6 + A_1$	62	3	2	$E_7 + A_1$	90
4	2	$D_6 + A_1$	62	4	3	$E_6 + A_2$	63
5	3	$A_5 + A_2$	33	5	3	A_8	30
6	4	$2A_3 + A_1$	50	6	4	$D_5 + A_3$	80
7	4	A_7	22	7	4	$A_7 + A_1$	44
8	6	$A_5 + A_2$	33	8	5	$2A_4$	49
				9	6	$A_5 + A_2 + A_1$	66
		Total	352			Total	567

TABLE 5. The full-rank local generators with trivial local system.

Proof. The selected finite blocks are specified, including their affine node subsets, ordinary character labels and support bounds, in Tables 14 and 15. Their sources are the finite rank and minuscule inputs, the type-A longest elements, Proposition 7.4, and the single classical D_8 source described above. The bounds are computed by taking the special representation of each finite family, applying truncated induction to the ambient finite Weyl group, and using its ordinary affine-cell orbit. The defining truncated constituent has multiplicity one and matching b-value.

Tables 16–23 and 24–32 give one source for every required row. For a row, use its local orbit with trivial local system to determine E by Springer correspondence. Evaluate (7.7) and (7.8) for the specified mask and block. Its multiplicity is positive. For all ordinary parabolic $L \subset M$, compute the sign-restriction multiplicity and retain exactly those L for which it is positive. Intersect the corresponding orbit down-sets with the recorded source down-set. The result is contained in the target down-set printed in the same row. Appendix A.5 gives the complete deterministic rules for these finite calculations; the attached certificates supply integral representation and character data, exact algorithms for the cyclotomic traces, and orbit incidence data.

For the two rows using source 8.G, Lemma 7.5 gives multiplicity one, and the source bound equals the ambient target. All remaining rows are evaluated by the same trace-and-cut rule. The lists comprise every trivial-local-system Springer pair in each centralizer in Table 5; hence the numbers 352 and 567 count all the required generators.

The finite criterion now supplies the corresponding completed Springer ideals. The specialization and globalization result of Theorem 5.5 passes from these full-rank vertices to all semisimple strata. The trivial-local-system generators suffice by the ordinary generation theorem of Proposition 4.5. □

Together with the classical constructions, this proves the source condition in every irreducible root system. Tensor products of sources give the same condition for a product of root systems.

8. Integral geometry of the final E_8 source

We give an integral geometric construction of the remaining source. All varieties in this section are complex varieties. A prime appearing as a coefficient characteristic is the prime of the parity category, and ultimately is the characteristic $p > 30$ of the reductive group. We use Chow degree, so that a class in H^{2d} has degree d. For a vector bundle E, write $e(E) = c_{\operatorname{rank} E}(E)$ for its top Chern class. For a variety of complex dimension d, an IC exponent r at a point denotes the stalk

group in degree $-d+2r$. When these are the only stalk degrees, the IC polynomial records their ranks as the coefficients of q^r, with q a formal variable.

8.1. **The Schubert varieties and their words.** Use the Bourbaki diagram with edges

$$1{-}3{-}4{-}5{-}6{-}7{-}8, \qquad 2{-}4,$$

and put $J=\{1,3,4,5,6,7,8\}$. The Levi factor of P_J has derived group SL_8. If u is a minimal right W_J-coset representative, write $X_u \subset G_{\mathbb{C}}/P_J$ for its Schubert variety and $\mathbf{c}_u$ for its spherical canonical element. Normalized smooth pullback to the full flag variety sends $\mathbf{c}_u$ to C_{uw_J}.

The elements used below are defined by the following reduced words; the entries are multiplied in their displayed order:

$$a=(4,3,1,5,4,2,3,4,5,6,5,4,2,3,1,4,3,5,4,2), \tag{8.1}$$

$$b=(6,5,4,3,1,7,6,5,4,3,8,7,6,5,4,2), \tag{8.2}$$

$$c=(4,3,1,5,4,3,6,5,4,7,6,5,8,7,6,5,4,2,3,1,4,3,5,4,2), \tag{8.3}$$

$$z=(3,1,4,3,5,4,6,5,4,3,1,7,6,5,4,2,\ 3,4,5,8,7,6,5,4,2,3,1,4,3,5,4,2). \tag{8.4}$$

Thus their partial dimensions are $20,16,25,32$, respectively. The source called 2794 in the finite tables is zw_J. The numbers $509,277,1221$ refer to aw_J, bw_J, cw_J. The displayed words specify these objects.

Theorem 8.1. *For every prime $p>30$,*

$$^{p}C_{zw_J}=C_{zw_J}.$$

The inputs from Proposition 7.3 are the p-independence of aw_J and cw_J. We prove the needed statement for bw_J below. The proof of Theorem 8.1 occupies the rest of this section.

8.2. **The decomposable cone.** Let V be an eight-dimensional vector space. We use the finite central covering $\mathrm{GL}(V)\to L$ of the Levi for which the first graded representation is $\bigwedge^3 V$. Its kernel consists of scalar cube roots of unity and acts trivially on $\bigwedge^3 V$ and $\bigwedge^6 V$. Thus the varieties and incidence maps below descend to L. The opposite big cell at the base point of $G_{\mathbb{C}}/P_J$ is the negative unipotent radical. Exponential coordinates are equivariant for the Levi, and its first two graded pieces are

$$\bigwedge^3 V, \qquad \bigwedge^6 V.$$

The third graded piece has dimension eight. The constructions below have zero coordinate in that piece.

The opposite chart of $X_{s_2}\cong \mathbf{P}^1$ is the root line of $-\alpha_2$. In $\bigwedge^3 V$, it is a decomposable line. Its Levi saturation is therefore

$$\{\, v_1\wedge v_2\wedge v_3 : v_i\in V \,\},$$

the affine cone over $\mathrm{Gr}(3,V)$. This Levi saturation is also the relevant parabolic saturation. Indeed, for a positive root $\gamma\neq\alpha_2$, the simply laced root string shows that conjugating its root group by $\exp(tf_{\alpha_2})$ keeps it in P_J. It fixes the affine root line. The root group of α_2 preserves the whole projective line. Thus the radical of P_J preserves X_{s_2}, and its Levi saturation has P_J-stable closure.

For the Weyl-group identification, begin with s_2w_J and apply left Demazure multiplication successively at

$$(4,3,1,5,4,3,6,5,4,7,6,5,8,7,6).$$

Each indicated multiplication increases length, and the result is bw_J, of length 44. To verify the assertion, use

$$s_i(\alpha_j)=\alpha_j-a_{ij}\alpha_i$$

for the E_8 Cartan matrix and count positive roots sent negative. These formulas also verify the word equality. It follows that the displayed cone is the opposite chart of X_b. The nonzero cone is a single Levi orbit, and the α_2-subgroup moves the other point of the root projective line into its nonzero affine part. Hence the projective variety has only the smooth nonvertex orbit and the vertex for the purpose of the local calculation.

We record the integral calculation for this cone, and also for a smaller cone used later. For $\mathrm{Gr}(3,n)$, let $\mathcal{P}_d(3,n-3)$ be the partitions of d in the 3 by $n-3$ rectangle. In the Schubert basis, multiplication by the positive Plucker class has matrix

$$P_d(\mu,\lambda)=\begin{cases}1, & \mu/\lambda \text{ consists of one box,}\\ 0, & \text{otherwise,}\end{cases}\qquad \lambda\in\mathcal{P}_d(3,n-3),\ \mu\in\mathcal{P}_{d+1}(3,n-3). \tag{8.5}$$

For $n=8$, the ranks by degree are

$$1,1,2,3,4,5,6,6,6,6,5,4,3,2,1,1.$$

The nonzero Smith factors of every P_d are 1, with the following exceptions:

d	nonzero Smith factors
6	$(1,1,1,1,1,3)$
7	$(1,1,1,1,1,2)$
8	$(1,1,1,1,1,3)$.

These small integer matrices are determined completely by (8.5); row and column elimination over $\mathbb{Z}$ gives the displayed diagonal forms. In particular every kernel and cokernel is free after inverting 6.

The punctured affine cone is the complement of the zero section in $\mathcal{O}_{\mathrm{Gr}}(-1)$. Its Gysin sequence identifies the relevant link cohomology with the kernels and cokernels of the adjacent Plucker maps. Truncation at the middle degree gives the stalk of the cone IC, and Verdier duality gives its costalk. It follows that the IC over $\mathbb{Z}[1/6]$ has free even stalk and costalk, compatible with reduction modulo every prime greater than 3. Thus

$${}^pC_{bw_J}=C_{bw_J}\qquad (p>3). \tag{8.6}$$

The Gysin sequence uses precisely the adjacent maps in (8.5).

8.3. **The contact double cover and its induction.** Fix a six-plane $W\subset V$. The subgroup on nodes $\{1,\dots,6\}$ is of type E_6; its partial flag variety at node 2 is the adjoint contact variety, of dimension 21. Its Schubert divisor has word a and dimension 20.

Here is the equation of its opposite chart. For $\omega\in\bigwedge^3 W$, define

$$K_\omega : W^*\longrightarrow W^*\otimes\det W,\qquad K_\omega(\xi)=(\iota_\xi\omega)\wedge\omega,$$

using $\bigwedge^5 W\cong W^*\otimes\det W$. The invariant quartic $Q_W(\omega)\in(\det W)^2$ is, up to a fixed nonzero scalar, $\frac{1}{6}\operatorname{tr}(K_\omega^2)$. With a compatible normalization, the chart of the divisor is

$$Y_W=\{(\omega,\eta)\in\bigwedge^3 W\oplus\det W : \eta^2+Q_W(\omega)=0\}. \tag{8.7}$$

This equation also follows directly from the contact grading. Let β be the highest root of this E_6, and choose root vectors e_β, f_β in its $\mathfrak{sl}_2$-triple. In the lowest-root coordinate of $\exp(\mathrm{ad}(\omega+\eta f_\beta))e_\beta$, the possible terms have degrees $\eta^2,\eta\omega^2$, and ω^4. The first is nonzero by the root $\mathfrak{sl}_2$, the middle term vanishes because the exterior-cube pairing is alternating, and the last is the nonzero contact quartic. The latter contact identity is also given in [1, §1.2]. The lowest-root coordinate defines a Borel-stable hyperplane section. The adjoint highest weight is the fundamental weight at node 2, so this hyperplane section represents the positive generator of the Picard group. It is therefore the unique Schubert divisor, with multiplicity one. This proves (8.7). Rescaling its nonzero complex coefficients gives an isomorphic complex variety.

Let Q_L be the parabolic in the Levi L stabilizing W. The equation and its projective closure are Q_L-stable. The associated variety is projective over $L/Q_L = \mathrm{Gr}(6, V)$, with projective fiber X_a. Consequently it has a proper morphism

$$\pi : L \times_{Q_L} X_a \longrightarrow G_{\mathbb{C}}/P_J. \tag{8.8}$$

On the opposite chart it is the closed incidence projection

$$\left\{ \begin{array}{l} W \in \mathrm{Gr}(6, V), \quad \omega \in \bigwedge^3 W, \quad \eta \in \det W, \\ \eta^2 + Q_W(\omega) = 0 \end{array} \right\} \longrightarrow \bigwedge^3 V \oplus \bigwedge^6 V.$$

A trivector of support six determines W uniquely, and that support is recovered regularly on opens defined by nonzero support minors. Thus π is an isomorphism on its support-six locus, including the quartic branch. It is birational onto its image.

Let U be the unipotent radical of P_J. The morphism is P_J-equivariant. To check the radical, consider a positive root outside the indicated E_6. Its expansion retains a positive coefficient at node 7 or 8 under conjugation by the E_6 subgroup; every resulting root is positive. This root group fixes the embedded E_6 contact variety pointwise. The remaining radical root groups lie in the E_6 Borel and preserve its Schubert divisor. This gives the action on $P_J \times_{U \rtimes Q_L} X_a$, whose underlying associated variety is the domain of (8.8).

Its image is the Schubert variety X_z. Explicitly, begin with aw_J, of length 48, and apply left Demazure multiplication in the order

$$(7, 6, 5, 4, 3, 1, 8, 7, 6, 5, 4, 3). \tag{8.9}$$

All twelve steps increase length. The resulting root matrix is that of zw_J, of length 60, by the simple-reflection formula above. Parabolic saturation therefore identifies the reduced image with X_z.

The geometry is a double cover of the support-six trivector locus. For a generic $\omega = a_1 + a_2$, with a_i volume forms of complementary three-spaces, the two choices of η are proportional to $\pm a_1 \wedge a_2$. Interchanging the three-spaces interchanges the sheets. This monodromy determines the Fourier summand below.

Fix a basis $e_1, \ldots, e_8$ of V and write $e_{ijk} = e_i \wedge e_j \wedge e_k$. The orbit types needed for the fibers are as follows:

support	representative or condition	partial orbit dimension	$\pi^{-1}(\omega, \eta)$
6	$Q_W(\omega) \neq 0$	32	pt
6	$Q_W(\omega) = 0$, ω of support 6	31	pt
5	$e_{123} + e_{145}$	25	$\mathbf{P}^2$
3	e_{123}	16	$\mathrm{Gr}(3, 5)$
0	0	0	$\mathrm{Gr}(6, 8)$.

The two support-six entries are understood in the cover. The classification follows from the normal forms for trivectors supported on a six-space; see [20, Table I]. For support at most five, $\eta = 0$, and the fiber consists of six-planes containing the support. This proves the last column directly. These are all the positive-dimensional fiber types. The three lower stabilizers are connected. In support three, their reductive part is $\mathrm{SL}_3 \times \mathrm{GL}_5$; in support five it is $\mathrm{GSp}_4 \times \mathrm{GL}_3$. The remaining factors are block-triangular unipotent groups, and the scalar on the distinguished line in support five is fixed by the symplectic similitude. Thus every irreducible equivariant local system on these lower strata is trivial.

These chart computations cover the projective target. Every Schubert closure contains the base point, and the opposite chart is an open neighborhood of that point. Its intersection with a Schubert closure is a nonempty open subset, hence meets the dense Schubert cell. Since the

Borel is contained in P_J, every P_J-orbit meets the chart. Equivariance therefore transports each local computation to the entire corresponding projective orbit.

8.4. **The characteristic-zero decomposition.** Let $K = \{1, 3, 4, 5, 6, 8\} \subset J$, of type $A_5 + A_1$. There are 28 minimal right cosets W_J/W_K, and their largest length is 12. Associated induction has the spherical character

$$\left(\sum_{u \in W_J^K} v^{12-\ell(u)} H_u \right) \mathbf{c}_a = \mathbf{c}_z + (v + v^{-1})\mathbf{c}_c + (v^2 + 1 + v^{-2})\mathbf{c}_b + (v^2 + 1 + v^{-2})\mathbf{c}_1. \tag{8.10}$$

Here 1 denotes the zero-dimensional partial Schubert variety, and W_J^K consists of the elements with no right descent in K. Formula (8.10) is a finite canonical-basis identity. Enumerate those 28 cosets using the root matrices. In the ambient spherical module, put $m_x = H_x C_{w_J}$ for $x \in W^J$. The ordinary Hecke rule is

$$H_s m_x = \begin{cases} m_{sx}, & \ell(sx) > \ell(x),\ sx \in W^J, \\ m_{sx} + (v^{-1} - v)m_x, & \ell(sx) < \ell(x), \\ v^{-1} m_x, & sxW_J = xW_J. \end{cases}$$

Construct $\mathbf{c}_a$ by bar-triangular recursion, apply the 28 operators H_u in (8.10), and eliminate leading terms using the unique bar-invariant vectors $\mathbf{c}_x \in m_x + \sum_{x' < x} v\mathbb{Z}[v]m_{x'}$. The four resulting coefficient polynomials are exactly those in (8.10); the words (8.1)–(8.4) and the displayed recursion specify every term of this calculation.

This decomposition specifies which integral multiplicity forms must be checked. Its characteristic-zero point Euler identity is

$$28 \cdot 4 = 43 + 2 \cdot 24 + 3 \cdot 6 + 3.$$

The following calculations determine the integral multiplicity forms for the three lower supports.

8.5. **The support-five slice.** Write $V = W_5 \oplus K_3$, where $W_5 = \mathbb{C}e_0 \oplus U_4$, and fix $\omega_0 = e_0 \wedge \sigma$, with σ symplectic on U_4. Contraction gives an injection

$$W_5^* \longrightarrow \bigwedge^2 W_5, \qquad \text{im} = \mathbb{C}\sigma \oplus e_0 \wedge U_4.$$

The complementary summand is the five-dimensional primitive space

$$\bigwedge_0^2 U_4 = \{\tau \in \bigwedge^2 U_4 : \sigma \wedge \tau = 0\}.$$

Fix the $\bigwedge^3 W_5$ coordinate to be ω_0, and set to zero the projection of the $K_3 \otimes \bigwedge^2 W_5$ coordinate to $K_3 \otimes (\mathbb{C}\sigma \oplus e_0 \wedge U_4)$. These are $10 + 15 = 25$ independent equations. They remove exactly the tangent directions of the support-five orbit, so they define a transverse slice.

For an incident six-plane W, its projection onto W_5 is surjective, since the projected trivector has support W_5. Its kernel is a line $\ell \subset K_3$, and W is a graph of a map $q : W_5 \to K_3/\ell$. The coordinate with one K_3/ℓ entry is precisely the contraction of ω_0 by q. The slice equations and the injection above force $q = 0$. Choose a local generator l of ℓ; the residual coordinate is the tensor $l \otimes \tau \in \ell \otimes \bigwedge_0^2 U_4$. Then

$$W = W_5 + \ell, \qquad \omega = e_0 \wedge \sigma + l \wedge \tau, \quad \tau \in \bigwedge_0^2 U_4, \qquad \eta \in \ell \otimes \det W_5.$$

Conversely every displayed pair satisfies the incidence and slice equations, so these coordinates describe the inverse-image slice.

The quartic restricts, up to a nonzero scalar, to the binary Pfaffian discriminant

$$(\sigma \wedge \tau)^2 - (\sigma \wedge \sigma)(\tau \wedge \tau).$$

The first term vanishes for primitive τ, and the second is a nondegenerate quadratic form on its five coordinates. Together with η^2, the equation is a nondegenerate quadric in six coordinates, all with the same ℓ-twist. Let $Q^4 \subset \mathbf{P}^5$ be the smooth quadric defined by this equation. The target slice is therefore the cone over the Segre embedding

$$\mathbf{P}^2 \times Q^4, \tag{8.11}$$

and the inverse-image slice is the corresponding quadric cone family in $\mathcal{O}_{\mathbf{P}^2}(-1) \otimes \mathbb{C}^6$.

Here is the integral link calculation. On the quadric Q^4, the ranks by degree are $1, 1, 2, 1, 1$. In compatible Schubert bases its adjacent hyperplane maps are

$$(1), \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad (1 \;\; 1), \qquad (1).$$

Let h be the hyperplane class on $\mathbf{P}^2$. On $\mathbf{P}^2 \times Q^4$, multiplication by the Segre hyperplane is the sum of this operator and the unit shift on $1, h, h^2$. The resulting nonzero Smith factors are units except for one factor 2 in each of the degree 2 and degree 3 maps. This follows by forming the tensor-product bases and adding the two displayed shift matrices. Its IC polynomial is $1 + q + 2q^2$; the quadric cone itself has IC polynomial $1 + q^2$. The Gysin and truncation argument used for (8.6) shows compatibility with all odd coefficient fields.

The direct image over this slice is parity: its only positive-dimensional fiber is $\mathbf{P}^2$, and its source IC restricts to a free even constant complex. Its top parity summand is the IC of (8.11). Comparison of characters therefore leaves the same two shifted point summands as in characteristic zero. This proves constancy of the support-five multiplicity ranks over every odd coefficient field.

8.6. **Fourier transform and the double-cover IC.** The remaining forms are computed by proper vector-bundle maps. In the Fourier and integral-lattice calculations below, write $\Lambda = \mathbb{Z}[1/2]$. The functor F is the perversely normalized Fourier–Sato transform of conic constructible complexes. Let W now have dimension six, and put

$$B_0 = \mathrm{Gr}(3, W), \quad U \subset W \otimes \mathcal{O}_{B_0}, \quad L = \bigwedge^3 U, \quad V_0 = \bigwedge^3 W.$$

The map $\pi_L : \mathrm{Tot}(L) \to V_0$ resolves the cone of decomposable trivectors. Its Fourier transform is the proper annihilator-bundle map

$$\pi_{L^\perp} : \mathrm{Tot}(L^\perp) \longrightarrow V_0^*.$$

With the perverse normalization, Fourier base change gives

$$\mathsf{F}(\pi_{L*}\Lambda_{\mathrm{Tot}(L)}[10]) \cong \pi_{L^\perp *}\Lambda_{\mathrm{Tot}(L^\perp)}[28]. \tag{8.12}$$

This is the usual Fourier identity for a linear subbundle, applied relatively and followed by proper pushforward. It holds over a coefficient ring and commutes with its derived change of coefficients.

For $\mathrm{Gr}(3, 6)$, the ranks by degree are

$$1, 1, 2, 3, 3, 3, 3, 2, 1, 1.$$

The adjacent Pieri maps have unit nonzero Smith factors, except for the middle map, with factors $1, 1, 2$. For example, with the degree-four partitions $(3, 1), (2, 2), (2, 1, 1)$ and degree-five partitions $(3, 2), (3, 1, 1), (2, 2, 1)$, its matrix is

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}. \tag{8.13}$$

All the other diagonal factors follow at once from (8.5) and complementary-partition duality. Thus the line-resolution pushforward splits over $\mathbb{Z}[1/2]$ into the cone IC and point summands,

using actual inclusion and retraction maps. The cone IC has free stalks and costalks and commutes with reduction.

In characteristic zero its Fourier transform is the full-support sign IC on V_0^*; see [35, Theorem 3.9 and §§3.3–3.4] in the (A_5, ω_3) case. The same identification holds over every odd coefficient field. Indeed, Fourier is a perverse exact equivalence, so it takes the simple cone IC to a simple perverse sheaf. Its generic Euler characteristic is unchanged by reduction, by (8.12) and the integral splitting. It is nonzero and of absolute value one. Hence the Fourier image has full support and generic rank one. The stabilizer of $e_{123} + e_{456}$ for the actual GL_6-action has connected part $\mathrm{SL}_3 \times \mathrm{SL}_3$ and component group $\mathbb{Z}/2$, generated by interchange of the two three-spaces. The rank-one local system cannot be trivial, since the inverse Fourier transform of the constant IC of the full vector space is supported at the origin. It is therefore the sign local system.

The finite map $Y_W \to V_0$ of (8.7) is small. For odd coefficients its IC pushforward splits under the two deck idempotents. The invariant summand is the constant IC of V_0, and the anti-invariant summand is the sign IC just described. Over the zero and decomposable trivectors, the finite map has a unique preimage, with $\eta = 0$. Proper base change and its Verdier-dual form therefore identify the canonical costalk-to-stalk maps of $\mathrm{IC}(Y_W)$ with those of its finite pushforward at these points. The same identification holds fiberwise in the associated families. The two deck projectors split these maps into their invariant and anti-invariant parts. Thus the Fourier calculation gives the anti-invariant part of the contact-cover forms. An isomorphism of the corresponding simple self-dual objects changes a self-duality by a nonzero scalar. It therefore preserves the ranks of all local maps over each coefficient field.

These identifications extend to a varying six-plane. The vector-bundle maps are GL_6-equivariant, and the integral projectors constructed below from (8.22) are equivariant as well. On the open orbit, both Fourier and double-cover constructions carry the sign representation of the stabilizer component group. Intermediate extension therefore identifies them as GL_6-equivariant perverse sheaves. For a rank-six bundle, perform the construction on its frame bundle and descend by GL_6. Relative Fourier base change commutes with the transition functions, so the resulting identification is an identification of the associated families. It identifies their canonical costalk-to-stalk maps. Under this identification the pairings from the chosen self-dualities differ by a nonzero scalar, constant on the connected base. Proper pushforward preserves this identification, so the integrated multiplicity forms have the same ranks over each coefficient field.

In the Smith calculations below, the integral lattice is the direct summand of the Fourier bundle model. The equivariant identification just established transfers the ranks of its reductions to the contact IC multiplicity forms, both before and after proper pushforward.

There is a dual-bundle convention in this argument. To obtain a sheaf on $\bigwedge^3 W$, we apply Fourier to the decomposable cone in $\bigwedge^3 W^*$. The pairing of two exterior cubes has values in $\det W$, so the dual bundle in this Fourier transform is kept throughout the relative construction. For explicit arithmetic we first give the universal formulas using the Chern roots of W. Replacing all roots by their negatives implements the required dual input. The origin classes below have even degrees $10, 12$, and are unchanged. The rank-three matrix has degrees $4, 5, 6$, and is conjugated by $\mathrm{diag}(1, -1)$, up to an overall orientation sign. All subsequent Smith factors are unchanged.

8.7. **The two projectors and the origin forms.** We first record how relative pairings give the multiplicity forms of the proper map.

Lemma 8.2. *Let $f : Y \to X$ be proper, let $B = f^{-1}(x)$ be smooth and projective, and write $i : B \hookrightarrow Y$. Let R be a coefficient ring such that $H^*(B; R)$ is finite free and concentrated in even degrees. Fix a self-duality of $\mathscr{F}$, and suppose that $i^*\mathscr{F}$ is a finite direct sum of constant free sheaves shifted by even integers. Choose homogeneous generators e_k for the dual sum $i^!\mathscr{F}$, and their duals for $i^*\mathscr{F}$. Express the canonical map $i^!\mathscr{F} \to i^*\mathscr{F}$ by homogeneous classes*

$C_{kl} \in H^*(B;R)$. *Then its local multiplicity pairing on the direct image is*

$$(ae_k, be_l) = \int_B abC_{kl}.$$

Proof. Proper base change identifies the stalk and costalk of $f_*\mathscr{F}$ at x with the hypercohomology of $i^*\mathscr{F}$ and $i^!\mathscr{F}$, respectively. Self-duality and Verdier duality on B identify these two modules as duals. Under this identification, the pairing is the canonical map followed by the trace $H^*(B;R) \to R[-2\dim B]$. Verdier duality on the complex manifold B shifts by $2\dim B$, so the dual summands also have even shifts. All cup factors and matrix classes therefore have even degree. The projection formula makes the value on ae_k, be_l equal to the trace of abC_{kl}, which is the displayed integral. The same argument commutes with derived change of coefficients. □

Let A_0 be the cone-IC projector on the stalk of the line-resolution pushforward and let $A_0^\dagger$ be its projector on the costalk, adjoint under Grassmannian Poincare duality. The zero-section map for the line bundle is multiplication by $e(L)$. Naturality gives

$$A_0 e(L) = e(L) A_0^\dagger, \qquad e(V_0/L) A_0 = A_0^\dagger e(V_0/L). \tag{8.14}$$

For the second identity, localize equivariantly, use $e(L)e(V_0/L) = e(V_0)$, and then descend: equivariant Grassmannian cohomology is free. Fourier exchanges the stalk and costalk projectors; (8.14) gives their compatibility with the two Euler-class maps.

The cone-IC primitive stalk classes can be represented by

$$1, \quad c_2(U), \quad c_3(U).$$

The class 1 belongs wholly to the IC summand: in the shifted line-resolution stalk it has degree -10, whereas point summands start in degree at least -8. Let h be the positive Plucker class and σ_λ the Schubert class indexed by λ. In cohomological degree four,

$$h^2 = \sigma_2 + \sigma_{11}, \qquad c_2(U) = \sigma_{11}.$$

Thus $c_2(U)$ is an integral primitive with coefficient one modulo the Plucker image. Its point-summand part pairs to zero with 1 in the Fourier form, by (8.14).

The three cone primitives have Chow degrees $0, 2, 3$. Poincare duality on the nine-dimensional B_0, followed by the shift [28] in (8.12), places them at IC exponents $5, 3, 2$, respectively, in the twenty-dimensional contact cover. Its invariant summand adds exponent zero. An origin pairing between exponents r, s has polynomial degree $20 - r - s$. Over $\mathrm{Gr}(6,8)$, of dimension 12, only the exponents $3, 5$ contribute, with block

$$\begin{pmatrix} 0 & \alpha_{12} \\ \alpha_{12} & \beta_{10} \end{pmatrix}, \tag{8.15}$$

where

$$\beta_{10} = \int_{\mathrm{Gr}(3,W)} e\left(\textstyle\bigwedge^3 W / \bigwedge^3 U\right), \tag{8.16}$$

$$\alpha_{12} = \int_{\mathrm{Gr}(3,W)} e\left(\textstyle\bigwedge^3 W / \bigwedge^3 U\right) c_2(U). \tag{8.17}$$

Changing the equivariant primitive lift changes α_{12} by β_{10} times an integral degree-two class on the base. This is exactly an integral change of basis for (8.15).

By Lemma 8.2, their integrals over $\mathrm{Gr}(6,8)$ give the origin forms in the Fourier model. The identified contact family has the same graded ranks over each odd coefficient field. We give explicit finite formulas for these integrals. Choose distinct weights $t_1, \dots, t_8$. For $I \subset J_6 \subset \{1, \dots, 8\}$,

with $|I|=3, |J_6|=6$, set

$$D_{J_6}=\prod_{\substack{i\in J_6\\ j\notin J_6}}(t_j-t_i),\qquad D_{I,J_6}=\prod_{\substack{i\in I\\ j\in J_6\setminus I}}(t_j-t_i),\qquad E_{I,J_6}=\prod_{\substack{K\subset J_6,\ |K|=3\\ K\neq I}}\left(\sum_{k\in K}t_k\right).$$

For a symmetric insertion F of degree two, localization on $\mathrm{Flag}(3,6;8)$, with tautological bundles $S_3\subset S_6$, reads

$$\int_{\mathrm{Flag}(3,6;8)} e(\textstyle\bigwedge^3 S_6/\bigwedge^3 S_3)F=\sum_{\substack{|J_6|=6\\ I\subset J_6,\ |I|=3}}\frac{E_{I,J_6}\,F(t_I,t_{J_6})}{D_{J_6}D_{I,J_6}}. \tag{8.18}$$

The 560 summands give

F	$c_1(S_6)^2$	$c_2(S_6)$	$c_2(S_3)$	$c_1(S_3)^2$	$c_1(S_3)c_1(S_6)$
$\int e(\bigwedge^3 S_6/\bigwedge^3 S_3)F$	18	12	6	0	0.

(8.19)

For example, taking $t_i=i$ in (8.18) and adding rational numbers gives the displayed integers. The integrand has total degree $19+2=21$, equal to the flag dimension. Its equivariant integral has degree zero. Thus this evaluation computes its exact constant value. Equation (8.18), with elementary symmetric functions for the insertions, is also a direct algorithm over $\mathbb{Q}$ for every entry of (8.19).

The two classes $c_1(S_6)^2, c_2(S_6)$ form a unimodular basis in degree two on $\mathrm{Gr}(6,8)$. The off-center origin maps in this model therefore have nonzero Smith factor $\gcd(18,12,6)=6$, whereas the central map has nonzero Smith factor 18. The primitive ambiguity in α adds a combination of 18 and 12 to 6, leaving that joint gcd unchanged. Hence the three forms in the Fourier model have nonzero Smith factors

$$6,\quad 18,\quad 6 \tag{8.20}$$

up to signs and the units inverted in constructing the projectors. Their reductions, and consequently the contact multiplicity forms, have their characteristic-zero ranks over every coefficient field of characteristic greater than 3.

8.8. **The integral support lattice at a decomposable vector.** We next identify a transverse slice of the global induction map at a support-three point. Fix $V=A\oplus Q$, where $\dim A=3$ and $\dim Q=5$, and fix a volume vector $\omega_A\in\bigwedge^3 A$. In the opposite chart set the $\bigwedge^3 A$-coordinate of ω equal to ω_A, and set its $\bigwedge^2 A\otimes Q$-coordinate equal to zero. These are $1+15=16$ independent equations. The tangent space to the decomposable orbit at $(\omega_A,0)$ is $\bigwedge^3 A\oplus(\bigwedge^2 A\otimes Q)$, so these equations define a transverse slice.

An incident six-plane W projects surjectively onto A, since the projected trivector is ω_A. Put $U=W\cap Q$. Then $\dim U=3$, and W is the graph, modulo U, of a map $q:A\to Q/U$. The component with two entries in A and one in Q/U is the contraction of ω_A by q. Contraction identifies A^* with $\bigwedge^2 A$; hence the slice equations give $q=0$ and $W=A\oplus U$. The remaining $\bigwedge^2 A\otimes U$-coordinate vanishes by the same equations. Thus the inverse-image slice is the family over $\mathrm{Gr}(3,Q)$ given by

$$\omega=\omega_A+Y+\delta,\qquad Y\in A\otimes\textstyle\bigwedge^2 U,$$
$$\delta\in\det U,\quad \eta\in\det A\otimes\det U,\qquad \eta^2+Q_W(\omega)=0.$$

The chosen volume identifies $\det A$ with $\mathbb{C}$. Using $\bigwedge^2 U\cong U^*\otimes\det U$, regard Y as a map $U\to A$ with values twisted by $\det U$. Its determinant then belongs to $\det A\otimes(\det U)^2\cong(\det U)^2$. To

compute the quartic, choose bases a_i of A and b_i of U, with volume vectors ω_A and ω_U. On the dense open set $\det Y \neq 0$, an $\mathrm{SL}(A)$-change of basis puts

$$Y = \sum_{i=1}^{3} y_i a_i \wedge \iota_{b_i^*}\omega_U, \qquad \delta = d\omega_U.$$

The definition of K_ω then gives, on $\langle a_i^*, b_i^* \rangle$, the matrix

$$\begin{pmatrix} d & -2y_jy_k \\ -2y_i & -d \end{pmatrix}, \qquad \{i,j,k\} = \{1,2,3\},$$

after trivializing $\det W$ by $\omega_A \wedge \omega_U$. Its square is $(d^2 + 4y_1y_2y_3)I$. Both sides are polynomial in the coordinates, so this proves $Q_W(\omega) = \delta^2 + 4\det Y$ everywhere in the compatible normalization of (8.7). Taking two linear combinations of η, δ and rescaling puts the equation in the form

$$Y \in A \otimes \bigwedge^2 U, \qquad u, v \in \det U, \qquad uv = \det Y.$$

Both sides take values in $(\det U)^2$. This identifies the entire inverse-image slice, including its determinant twists. We compute its pairing by the relative Fourier bundle model over $\mathrm{Gr}(3, Q)$.

The fiber of the annihilator-bundle map at the decomposable covector is the Plucker hyperplane $H \subset B_0 = \mathrm{Gr}(3,6)$. Its complement is the big cell $\mathbb{A}^9$, and H has a paving by nineteen even-dimensional real cells. Under the equivariant Gysin map, its Borel–Moore homology is identified, by Poincare duality on the smooth B_0, with the kernel of restriction to the complementary fixed point. This realizes the pairing on an integral Borel–Moore support lattice.

Let $\mathcal{P}$ be the twenty partitions in a 3 by 3 rectangle. Write U_3 for the tautological rank-three bundle on B_0. Put

$$f_0 = 1, \quad f_2 = c_2(U_3), \quad f_3 = c_3(U_3), \qquad h = c_1(U_3) = e(\bigwedge^3 U_3),$$

and let $g_1, \ldots, g_{17}$ be the Schur classes $s_\lambda(U_3)$ for

$$\lambda \in \mathcal{P} \setminus \{(3,3,3), (3,2,2), (2,2,2)\}.$$

The excluded classes are, up to sign, the Poincare duals of the three primitives. Define the equivariant matrices

$$M_{ij} = \int_{B_0} h g_i g_j, \qquad B_{ki} = \int_{B_0} f_k g_i. \tag{8.21}$$

The determinant of M has degree zero, since

$$\sum_i \deg g_i = 68, \qquad 2 \cdot 68 + 17 - 9 \cdot 17 = 0.$$

At zero equivariant parameters, one-box Pieri and complementary-partition duality give

$$\mathrm{SNF}(M) = (1^{16}, 2). \tag{8.22}$$

One can see the single factor 2 in (8.13); every other degree block eliminates using unit pivots. Thus $\det M = \pm 2$ for all equivariant parameters, and M^{-1} is defined over the equivariant ring with 2 inverted. It splits the seventeen actual point summands of the line resolution.

In the Fourier transform their support vectors are hg_i. The full origin Fourier pairing on the three primitives is, up to the fixed orientation sign,

$$F_{kl} = \int_{B_0} e(V_0/L) f_k f_l - e(V_0)(BM^{-1}B^{\mathsf{t}})_{kl}. \tag{8.23}$$

Indeed the point-summand Gram matrix is $e(V_0)M$, and the cross matrix is $e(V_0)B$. Taking the Schur complement is an operation on actual integral direct summands.

At the decomposable covector, denote the complementary fixed point by A. The classes

$$z_2 = f_2 - f_2(A), \qquad z_3 = f_3 - f_3(A)$$

are integral support classes. Together with the seventeen hg_i, they generate the support lattice after inverting 2. Here is a check of saturation. At zero parameters, express the nineteen columns

$$(hg_1, \dots, hg_{17}, z_2, z_3)$$

in the nineteen positive-degree Schur classes. Their Smith form is

$$(1^{18}, 2). \tag{8.24}$$

In degrees two and three, z_2, z_3 supply exactly the two primitive complements to the Plucker image. All remaining degree blocks are the unit blocks of the Pieri maps, except for (8.13). This proves (8.24). For a maximal torus T of the stabilizer of the chosen decomposable covector, one has $h(A) = 0$. The twenty Schur classes form a free basis over $H_T^\bullet(\mathrm{pt}; \mathbb{Z})$. Since restriction sends 1 to 1, its kernel has the free basis

$$s_\lambda(U_3) - s_\lambda(A) \qquad (\lambda \in \mathcal{P},\ \lambda \neq \varnothing).$$

Express the support columns in this basis. The sum of column degrees is $68 + 17 + 2 + 3 = 90$, equal to the sum of positive Schur degrees. Hence its equivariant determinant also has degree zero and is ± 2. Thus the rank-two residual Fourier lattice is saturated over $\mathbb{Z}[1/2]$.

8.9. **The transverse Borel–Moore pairing.** Choose a torus in $\mathrm{SL}(A) \times \mathrm{GL}(U)$. Write the weights on A as a_1, a_2, a_3, with $a_1 + a_2 + a_3 = 0$, and those on U as b_1, b_2, b_3. Let $x = (a_1, a_2, a_3, b_1, b_2, b_3)$. For a three-element subset $I \subset \{1, \dots, 6\}$, put

$$\lambda_I = \sum_{i \in I} x_i, \qquad d_I = \prod_{\substack{i \in I \\ j \notin I}} (x_j - x_i).$$

The excluded fixed point is $A = \{1, 2, 3\}$; $\lambda_A = 0$. We may assume all other λ_I and all denominators nonzero. The tangent to the decomposable orbit consists of its scalar line and the nine weights with $|I \cap A| = 2$. Its ten-dimensional normal representation N therefore has Euler class

$$e(N) = \prod_{\substack{|I|=3 \\ |I \cap A| \leq 1}} \lambda_I. \tag{8.25}$$

Lemma 8.3. *After splitting the seventeen constant summands, the local form of the residual Fourier summand on the support classes z_2, z_3 is*

$$C_{kl} = e(N) \left(\sum_{I \neq A} \frac{z_k(I) z_l(I)}{\lambda_I d_I} - (B_z M^{-1} B_z^{\mathrm{t}})_{kl} \right), \qquad k, l \in \{2, 3\}, \tag{8.26}$$

where

$$(B_z)_{ki} = \sum_{|I|=3} \frac{z_k(I) g_i(I)}{d_I}.$$

Over every odd coefficient field this form identifies with the contact-IC form, up to the nonzero scalar comparing the chosen self-dualities. Its entries have degrees $4, 5, 6$, in the order

$$C = \begin{pmatrix} \gamma_4 & \beta_5 \\ \beta_5 & \alpha_6 \end{pmatrix}.$$

Proof. The annihilator-bundle total space is smooth. Its equivariant action maps onto every tangent direction of the decomposable orbit. Hence inverse image of a transverse slice is smooth near the fiber. At a surviving fixed plane I, its tangent K-class is

$$T_I B_0 + N - L_I$$

with the dual-bundle orientation convention fixed as in Section 8.6. Its Euler class is therefore $d_I e(N)/\lambda_I$, up to one common orientation sign.

A Borel–Moore support class whose Gysin image in B_0 is z has localized coefficient $z(I)/d_I$ at I. Intersecting two such classes in this smooth transverse source consequently gives

$$e(N)\sum_{I\neq A}\frac{z_k(I)z_l(I)}{\lambda_I d_I}.$$

The excluded fixed point contributes nothing because the support classes vanish there. This proves the unprojected term directly from the local geometry.

For the actual constant summands represented by hg_i, use $h(I)=\lambda_I$. Their transverse Gram matrix is $e(N)M$, and their cross matrix with the z_k is $e(N)B_z$. Splitting them by M^{-1}, which is integral with 2 inverted, gives exactly (8.26). By (8.24) the remaining two vectors are an integral basis of the residual Fourier summand over that ring. Finally, the tangent Euler class and the degrees of z_2, z_3 give entry degrees $4, 5, 6$. This also proves that every determinant twist in (8.25) is retained. □

8.10. **Integration of the support-three forms.** Let $B_1=\mathrm{Gr}(3,\mathbb{C}^5)$, with tautological bundle U and $h_U=c_1(U)$. The global slice just constructed identifies the relative contact family with the inverse-image slice of π. Formula (8.26) gives relative pairing classes for the associated Fourier family on B_1, with the fixed SL_3-equivariance on A. Lemma 8.2 computes its multiplicity forms by integrating these classes. The fieldwise identification of the two families gives the same graded ranks for the support-three contact multiplicity forms. Exact integration gives

$$\int_{B_1}\alpha_6=1,\qquad \int_{B_1}\beta_5 h_U=-3,\qquad \int_{B_1}\gamma_4 h_U^2=9,\qquad \int_{B_1}\gamma_4 c_2(U)=6. \tag{8.27}$$

The following rational-arithmetic formula computes these four integrals. Choose

$$(a_1,a_2,a_3)=(-7,2,5),\qquad (t_1,\dots,t_5)=(11,17,23,31,41).$$

For each three-subset $J_3\subset\{1,\dots,5\}$, use $b=(t_j)_{j\in J_3}$ in (8.21) and (8.26). Schur evaluations are given by

$$s_\lambda(u_1,u_2,u_3)=\frac{\det(u_i^{\lambda_j+3-j})_{i,j=1}^3}{\det(u_i^{3-j})_{i,j=1}^3}. \tag{8.28}$$

In particular every entry of the 17 by 17 matrix M is

$$M_{ij}=\sum_{|I|=3}\frac{\lambda_I s_{\nu_i}(x_I)s_{\nu_j}(x_I)}{d_I},$$

where the ν_i are the seventeen partitions specified above. Form B_z, invert M over $\mathbb{Q}$, and use (8.26). Finally sum the four quantities

$$\frac{1}{\prod_{i\in J_3,j\notin J_3}(t_j-t_i)}\left(C_{33},\ C_{23}\sum_{j\in J_3}t_j,\ C_{22}\left(\sum_{j\in J_3}t_j\right)^2,\ C_{22}\sum_{\substack{i<j\\ i,j\in J_3}}t_it_j\right)$$

over the ten subsets J_3. The sum is $(1,-3,9,6)$. This rule involves only determinants of size three, a matrix with determinant ±2, and finite sums of rational numbers. It specifies all entries of (8.27), including signs. Each inserted class has total degree six, the dimension of B_1; its equivariant pushforward is a constant integer. Thus one admissible numerical weight specialization evaluates the integral exactly.

In the displayed convention, the central matrix in the Fourier model is

$$\begin{pmatrix}1&-3\\-3&9\end{pmatrix}.$$

It has rank one and nonzero Smith factor 1. The two off-center maps are dual rank-one rectangular forms with coordinate row $(-3,9,6)$, and hence nonzero Smith factor 3. Dualizing the Fourier

input changes the off-diagonal signs by the unimodular conjugation already described, so the three nonzero Smith factors are

$$3, \quad 1, \quad 3. \tag{8.29}$$

The invariant summand of the finite double cover has internal IC exponent zero. Its associated degrees in these forms are $7, 8, 10$, so its contributions vanish on the six-dimensional B_1. The reductions of the forms in (8.29), and hence the contact multiplicity forms, have their characteristic-zero ranks in every coefficient characteristic greater than 3.

8.11. **Completion of the geometric proof.**

Proof of Theorem 8.1. The input X_a is p-independent in the required range by Proposition 7.3. Its associated IC on the domain of (8.8), normalized in dimension 32, is parity. The stabilizer of a point in each of the three lower strata acts transitively on its Grassmannian fiber. Equivariance therefore makes the cohomology sheaves of the restricted IC locally constant. These fibers are simply connected, so the local systems are constant; their cohomology groups are free and even. The restricted IC has cohomology in even degrees. The extensions between its shifted constant cohomology sheaves lie in odd cohomology groups of the Grassmannian and hence vanish. Thus it is a direct sum of constant free sheaves in even shifts. Proper base change gives even stalks of the direct image, and self-duality gives even costalks. The direct image is therefore parity.

On the entire support-six locus the map is an isomorphism. The support-five slice calculation is compatible with every odd coefficient characteristic. At support three, Lemma 8.3 computes the local forms on the saturated Fourier lattice. The equivariant fieldwise identification with the contact family, Lemma 8.2, and (8.29) give the actual multiplicity forms their characteristic-zero ranks for $p > 3$. At the vertex the same statement follows from (8.20). These computations cover every projective orbit by the equivariance argument following (8.8).

The lower indecomposable IC objects are also p-independent: X_c by Proposition 7.3, X_b by (8.6), and the point trivially. For a parity complex $\mathscr{F}$ and the inclusion i of a closed stratum, the ranks of the canonical maps $i^!\mathscr{F} \to i^*\mathscr{F}$ determine the summands supported there by [31, Proposition 2.26]. Apply this criterion to the parity direct image along the dense Schubert cell of each lower support, after removing the boundary of that cell. The cell is then closed and contractible, so its star and shriek restrictions are direct sums of shifted constant local systems. Restriction to this open union of Schubert strata preserves nonzero indecomposable parity complexes by [31, Proposition 2.11]; hence these local multiplicities are the global multiplicities. All its ranks are the characteristic-zero ranks. Therefore the lower direct summands occur with the multiplicities in (8.10) over every field of characteristic $p > 30$. The remaining top indecomposable parity summand has character $\mathbf{c}_z$. Normalized smooth pullback to the full flag variety gives ${}^pC_{zw_J} = C_{zw_J}$, as asserted. □

9. The support formula for all highest weights

9.1. **Equality of character spaces.** We first assume that the root system is irreducible and that G is simply connected. Theorem 6.23 provides the finite local generators in every classical type. For types E_7, E_8 they are provided by Proposition 7.6, with the last geometric source supplied by Theorem 8.1. The finite criterion includes every full-rank stabilizer and every trivial-local-system Springer generator at that stabilizer. In particular the exceptional argument uses only the characteristic bounds stated there.

For these types, apply Theorem 5.5 to each orbit downset $\mathcal{C}$. In types E_6, F_4, G_2, the finite generation argument at the start of Section 7 gives the same conclusion, including $p = 7$ for G_2. Thus in every type,

$$I_{\mathcal{C}}^+ = I_{\mathcal{C}}.$$

The coefficient correspondence of Proposition 3.3 now gives the equality

$$J_{\mathcal{C}}^{+} = J_{\mathcal{C}}^{q}. \tag{9.1}$$

9.2. **Recovering the original label.** Let λ be a regular dominant weight and put $\mathcal{C} = \overline{\mathcal{O}_\lambda}$. The quantum support theorem places $\operatorname{ch} T_q(\lambda)$ in $J_{\mathcal{C}}^{q}$. By (9.1), there is a finite expression

$$\operatorname{ch} T_q(\lambda) = \sum_{\mu \in F} a_\mu \operatorname{ch} T_p(\mu), \qquad a_\mu \in \mathbb{C}, \tag{9.2}$$

where every $\mu \in F$ satisfies

$$V_{u_q}(Q(T_p(\mu))) \subseteq \mathcal{C}.$$

Discard the zero coefficients. Every weight in a highest-weight module of highest weight ν is at most ν, and its highest-weight multiplicity is one. If a weight μ in F were not at most λ, choose one maximal for the dominance order. The coefficient of e^μ in the right-hand side of (9.2) would be $a_\mu \neq 0$, whereas it is zero in the left-hand side. This is impossible. Thus every $\mu \in F$ is at most λ. Comparison of the coefficient of e^λ then gives $a_\lambda = 1$. Consequently

$$V_{u_q}(Q(T_p(\lambda))) \subseteq \overline{\mathcal{O}_\lambda}. \tag{9.3}$$

The quantum tilting module $T_q(\lambda)$ occurs with multiplicity one in $Q(T_p(\lambda))$: this follows again from the unique highest weight and its multiplicity. The quantum support theorem gives the reverse inclusion in (9.3). Finally Theorem 2.3 yields

$$V_{G_1}(T_p(\lambda)) = \overline{\mathcal{O}_\lambda}.$$

9.3. **Singular weights.** Let $\lambda \in X^+$ be arbitrary, and let w label its lower-closure alcove as in the introduction. The translation principle for tilting tensor ideals gives

$$V_{G_1}(T_p(\lambda)) = V_{G_1}(T_p(w \bullet_p 0));$$

see [4, Lemma 7.6]. We recall the mechanism relevant to this equality. Translation to the face containing λ and translation out of that face are linkage summands of tensoring with tilting modules. The lower-closure convention selects the tilting summand in the regular block, and the two tiltings belong to each other's tensor ideals. Taking a summand can only decrease support, and tensoring intersects supports. The two inclusions therefore give equality. The regular weight $w \bullet_p 0$ has the orbit label $\mathcal{O}_w = \mathcal{O}_\lambda$, so the established regular formula proves the theorem for λ as well.

9.4. **Products, central quotients, and the connected center.** For a product of simply connected simple groups, indecomposable tilting modules are exterior tensor products. The cohomological Künneth isomorphism identifies their supports with products of the supports in the factors. Ordinary affine cells and their nilpotent orbits have the same product decomposition. The theorem therefore holds for every semisimple simply connected group.

Now let G be connected reductive, let $G_{\mathrm{der}}^{\mathrm{sc}}$ be the simply connected cover of its derived group, and let Z° be its connected center. The structure theorem for reductive groups [16, Corollary 5.3.3 and Proposition 5.3.4] gives a central isogeny

$$f : G_{\mathrm{der}}^{\mathrm{sc}} \times Z^\circ \longrightarrow G. \tag{9.4}$$

Its kernel injects into the center of $G_{\mathrm{der}}^{\mathrm{sc}}$. The primes dividing the order of that center are at most the Coxeter number of one of its simple factors. Since $p > h$, the kernel of f is finite etale. Frobenius is an isomorphism on this kernel, and the induced map on first Frobenius kernels is an isomorphism

$$(G_{\mathrm{der}}^{\mathrm{sc}})_1 \times (Z^\circ)_1 \xrightarrow{\sim} G_1.$$

The differential of f identifies the nilpotent cones in the derived Lie algebras and preserves their orbit labels.

Pullback under the surjective map f is fully faithful. It takes Weyl and dual Weyl modules to the corresponding modules for the covering root datum, so it takes an indecomposable tilting module to the indecomposable tilting module with the pulled-back highest weight. Such a module is a tilting module for $G_{\mathrm{der}}^{\mathrm{sc}}$ tensored with a character of Z°. The infinitesimal torus $(Z^\circ)_1$ is diagonalizable and linearly reductive; its positive-degree cohomology vanishes. Thus this torus factor contributes no nonzero direction to the support. Applying the simply connected formula and the isomorphism of Frobenius kernels proves Theorem 1.1 for G. If G itself is a torus, both its nilpotent cone and every nonzero module support are the single point 0, giving the same conclusion.

Corollary 9.1. *If the quantum tilting module $T_q(\mu)$ occurs as a summand of the whole quantum lift $Q(T_p(\lambda))$, then*

$$\mathcal{O}_\mu \subseteq \overline{\mathcal{O}_\lambda}.$$

Proof. The support of a direct summand is contained in the support of the whole module. Apply the quantum support theorem, Theorem 2.3, and Theorem 1.1. □

Appendix A. Explicit finite certificates

This appendix specifies the finite computations used in Section 7. The certificate consists of reduced words, integer Laurent polynomials, integral representation matrices, exact character calculations in cyclotomic fields, and finite orbit down-sets. The rules below determine the mathematical meaning and the verification of every record. The accompanying electronic files give the larger integer arrays in machine-readable form and implement these rules.

A.1. **Roots, words, and spherical canonical vectors.** For E_r, let $A = (a_{ij})$ be the Cartan matrix with diagonal 2 and off-diagonal entries -1 on the edges listed at the start of Section 7. On the column basis $e_1, \dots, e_r$ set

$$S_i = I - e_i(a_{i1}, \dots, a_{ir}). \tag{A.1}$$

Starting with $\{e_1, \dots, e_r\}$ and closing under the S_i gives the roots. The positive roots are those with nonnegative coordinates; there are 63 and 120 for $r = 7$ and 8. For a word $a = i_1 \cdots i_d$ write $S(a) = S_{i_1} \cdots S_{i_d}$. Its element has length

$$\ell(S(a)) = \#\{\alpha > 0 : S(a)\alpha < 0\}.$$

Thus reducedness is an integer matrix test. Right W_J-minimality is the test $S(a)e_j > 0$ for every $j \in J$. Repeated removal of right descents in J gives the unique minimal coset representative and the accompanying parabolic factor. Repeated addition of right ascents in J constructs w_J.

Put $m_x = H_x C_{w_J}$ for $x \in W^J$. If s is simple, let x' be the minimal element of sxW_J. The elementary spherical rule is

$$C_s m_x = \begin{cases} m_{x'} + v m_x, & \ell(x') = \ell(x) + 1, \\ m_{x'} + v^{-1} m_x, & \ell(x') = \ell(x) - 1, \\ (v + v^{-1}) m_x, & x' = x. \end{cases} \tag{A.2}$$

This rule and the integer root matrices suffice for all the spherical polynomial calculations in the paper.

For each x, the vector $\mathbf{c}_x = C_{xw_J}$ is uniquely specified by

$$\overline{\mathbf{c}_x} = \mathbf{c}_x, \qquad \mathbf{c}_x = m_x + \sum_{y<x} q_{y,x}(v) m_y, \qquad q_{y,x}(v) \in v\mathbb{Z}[v]. \tag{A.3}$$

The bar operation is computed by $\bar{v} = v^{-1}$, $\overline{C_{w_J}} = C_{w_J}$, and $\overline{H_s} = H_s + v - v^{-1}$. To construct $\mathbf{c}_x$, choose a left descent s of x, construct $\mathbf{c}_{sx}$, and multiply by C_s. In decreasing Bruhat order remove the unique bar-invariant multiple of each already constructed $\mathbf{c}_y$ which eliminates its nonpositive-degree coefficient. Termination follows from the strictly decreasing leading index. This is also an independent way to verify a supplied vector: check its diagonal coefficient, the strict triangular condition, and bar invariance using (A.2). Uniqueness in (A.3) then proves that it is the ordinary canonical vector. The calculation uses integer coefficients throughout.

A.2. **Ordinary two-sided cell certificates.** Let s, t be adjacent simple reflections. Set

$$D_R(s,t) = \{w : \#\{a \in \{s,t\} : \ell(wa) < \ell(w)\} = 1\}.$$

For $w \in D_R(s,t)$, its right star is the unique member of $\{ws, wt\}$ that belongs to $D_R(s,t)$. Define the left star by inverting, applying the right star, and inverting again. These operations preserve ordinary two-sided cells [24, §6]. The length test using (A.1) determines every step without a choice of reduced expression.

The initial element in each row of Table 6 is the full word $x_{r,i}w_J$ of Table 2. Apply its right-star row and then its left-star row; an empty row means the identity. Each two-digit entry specifies the two simple nodes. All 204 steps satisfy the defining descent test. Table 7 gives the resulting

reduced words and their labels in the representative-cell data of [24, §6]. These paths prove the ordinary-cell assignments used in Proposition 7.4.

Table 6: Complete star paths. Apply the right row first and then the left row. Each two-digit entry is the pair (s, t).

Source	Side	Pairs in order
$x_{7,43}w_J$	R	42 43 31 54 76
$x_{7,43}w_J$	L	65 54 42 31 42 54 31 65 43 42 65 42 43 31
$x_{7,259}w_J$	R	42 43 31 54 76
$x_{7,259}w_J$	L	$\varnothing$
$x_{8,1540}w_J$	R	31 43 42 65 42 43 31 87 65 54 42 76 65 54 43 42
		87 76 54 43 31 65 54 31 42 54 65 87 65 54 42 31
		54 31 42 65 54 31 42 54 31 65 42 76 65 42 87 76
$x_{8,1540}w_J$	L	31 43 42 65 42 43 31 87 65 54 42 76 65 54 43 42
		87 76 54 43 31 65 54 31 42 54 65 87 65 54 42 31
		54 31 42 65 54 31 42 54 31 65 42 76 65 42 87 76
$x_{8,191}w_J$	R	$\varnothing$
$x_{8,191}w_J$	L	31 43 42 65 42 43 31 65 54 42 31 42 54 65 76 87
$x_{8,232}w_J$	R	$\varnothing$
$x_{8,232}w_J$	L	31 43 54 65 76 87
$x_{8,2611}w_J$	R	42 43 31 54 76
$x_{8,2611}w_J$	L	43 54 65 76 87
$x_{8,418}w_J$	R	31 43 42 65 42 43 31 65 54 42 76 65 42 87
$x_{8,418}w_J$	L	$\varnothing$
$x_{8,1356}w_J$	R	31 43 42 65 42 43 31 65 54 42 76 65 42 87
$x_{8,1356}w_J$	L	31 43 42 65 42 43 31 65 54 42 76 65 42 87
$x_{8,2794}w_J$	R	42 43 31 54 76
$x_{8,2794}w_J$	L	42 43 31 54 76

Table 7: Endpoints in the representative cells of [24, §6]. The cell index is the zero-based index of the distributed representative-cell list; all generator digits use the numbering of this paper.

Source	Cell	Special character	Endpoint word
$x_{7,43}w_J$	27	$\phi_{189,22}$	234234542314565423143546576542314356
$x_{7,259}w_J$	19	$\phi_{120,25}$	131431542316542314357654231435426543176543
$x_{8,1540}w_J$	91	$\phi_{1400,32}$	231423542346542314354265431765423143542654317654234568765423143542654317654234568765423143562654317654234568765423143562654317654234568765423143562

Source	Cell	Special character	Endpoint word
$x_{8,232}w_J$	104	$\phi_{8,91}$	123142314354231435426542314354265431765423456787654 231435426542654317654234565678 765423143542654317654234567876542 3143542654317654234567
$x_{8,2611}w_J$	95	$\phi_{1400,32}$	234234542345654231435676542314354265476587654231435426543765876
$x_{8,418}w_J$	96	$\phi_{560,47}$	123142314354231435426542314354265431765423143542654317654287654231435426543765876
$x_{8,1356}w_J$	101	$\phi_{210,52}$	231425423654231435426543176542314354265431765423456876542314354265431765423456787654231435426543765423143542654317654234567876542314354265431765423456787654231435426543765423143542654317654234567876542314354265431765423456787654231435426543765423143542654317654234567

as above. The two seeds of the E_8/D_7 table have upper factors 1455 and 1486 from the same E_7/D_5 table. Hence every dependency is either in an earlier table or in a finite rank-at-most-six source.

The unmerged source chains have 220 nodes and 210 multiplication identities. Merging their shared prefixes leaves 157 nodes, 150 distinct multiplication identities, and 169 lower corrections in the tables below. These counts are checks on the size of the certificate; the defining data are the words and identities themselves. The independent 25-step chain for the upper source 1221 is already printed in Table 1, with all 23 of its lower corrections.

E_7, $J = \{2,3,4,5,6\}$, *targets 1104,1455,1486.* The seed named 124 is xw_J, where

$$\texttt{7654234567} \quad \text{is the word for } x, \qquad I = \{2,3,4,5,6,7\}.$$

The connected components of I have rank at most six.

Table 8: Exact multiplication and common-coset data in E_7, $J = \{2,3,4,5,6\}$.

Name w	Predecessor z	s	Correction descriptors (δ, r)
158	124	1	$\varnothing$
197	158	3	$(L, \texttt{31})$
242	197	4	$(L, \texttt{43})$
292	242	5	$(L, \texttt{54})$
346	292	2	$\varnothing$
406	346	4	$(L, \texttt{42})$, $(L, \texttt{45})$
471	406	3	$(L, \texttt{34})$
538	471	1	$(L, \texttt{13})$
609	538	6	$\varnothing$
681	609	5	$(L, \texttt{56})$
755	681	4	$(L, \texttt{45})$
827	755	7	$\varnothing$
830	755	3	$(L, \texttt{31})$, $(L, \texttt{34})$
901	827	3	$(L, \texttt{31})$, $(L, \texttt{34})$
902	830	2	$\varnothing$
971	901	2	$\varnothing$
972	902	4	$(L, \texttt{42})$, $(L, \texttt{43})$
1039	971	4	$(L, \texttt{42})$, $(L, \texttt{43})$
1040	972	5	$(L, \texttt{54})$
1104	1039	5	$(L, \texttt{54})$, $(L, \texttt{54234576})$
1105	1040	6	$(L, \texttt{65})$
1164	1104	6	$\varnothing$
1165	1105	7	$\varnothing$
1219	1164	5	$(L, \texttt{5676})$, $(L, \texttt{56})$
1220	1165	6	$(L, \texttt{67})$
1269	1219	4	$(L, \texttt{45})$
1270	1220	5	$(L, \texttt{56})$
1312	1269	2	$(L, \texttt{24})$, $(L, \texttt{24565437})$
1315	1270	4	$(L, \texttt{45})$
1351	1312	3	$\varnothing$
1354	1315	3	$(L, \texttt{34})$
1385	1351	4	$(L, \texttt{42})$, $(L, \texttt{43})$
1387	1354	2	$\varnothing$
1413	1385	5	$(L, \texttt{54})$, $(L, \texttt{56})$
1416	1387	4	$(L, \texttt{42})$, $(L, \texttt{43})$

Name w	Predecessor z	s	Correction descriptors (δ, r)
1438	1413	1	$(R, \texttt{13454231436542})$
1439	1416	5	$(L, \texttt{54})$
1455	1439	6	$(L, \texttt{65})$, $(L, \texttt{67})$
1458	1438	3	$(L, \texttt{31})$
1474	1458	4	$(L, \texttt{45})$, $(L, \texttt{43})$
1486	1474	2	$(L, \texttt{24})$

E_7, $J = \{1, 3, 4, 5, 6, 7\}$, *targets 259.* The seed named 45 is xw_J, where

$\texttt{2315436542}$ is the word for x, $\quad I = \{1, 2, 3, 4, 5, 6\}$.

The connected components of I have rank at most six.

Table 9: Exact multiplication and common-coset data in E_7, $J = \{1, 3, 4, 5, 6, 7\}$.

Name w	Predecessor z	s	Correction descriptors (δ, r)
59	45	7	$\varnothing$
76	59	6	$(L, \texttt{65})$, $(L, \texttt{67})$
95	76	4	$\varnothing$
117	95	5	$(L, \texttt{54})$, $(L, \texttt{5423})$
139	117	3	$(L, \texttt{234542})$, $(L, \texttt{235426})$
163	139	4	$(L, \texttt{43})$, $(L, \texttt{45})$, $(L, \texttt{4565})$, $(L, \texttt{43542316})$
184	163	6	$(L, \texttt{65})$
210	184	1	$\varnothing$
239	210	3	$(L, \texttt{31})$, $(L, \texttt{34})$, $(L, \texttt{3142346542})$
259	239	7	$(L, \texttt{76})$

E_8, $J = \{1, 3, 4, 5, 6, 7, 8\}$, *targets 2611.* The seed named 67 is xw_J, where

$\texttt{65423143542}$ is the word for x, $\quad I = \{1, 2, 3, 4, 5, 6\}$.

The connected components of I have rank at most six.

Table 10: Exact multiplication and common-coset data in E_8, $J = \{1, 3, 4, 5, 6, 7, 8\}$.

Name w	Predecessor z	s	Correction descriptors (δ, r)
89	67	7	$(L, \texttt{76})$
115	89	5	$\varnothing$
148	115	4	$(L, \texttt{45})$, $(L, \texttt{4576})$
187	148	3	$(L, \texttt{34})$
232	187	2	$\varnothing$
288	232	4	$(L, \texttt{42})$, $(L, \texttt{43})$
354	288	5	$(L, \texttt{5676})$, $(L, \texttt{54})$, $(L, \texttt{54234576})$
431	354	8	$(L, \texttt{87})$
521	431	6	$\varnothing$
624	521	7	$(L, \texttt{76})$, $(L, \texttt{78})$
743	624	5	$\varnothing$
878	743	4	$(L, \texttt{45})$, $(L, \texttt{4576})$
1030	878	3	$(L, \texttt{34})$
1195	1030	2	$\varnothing$

Name w	Predecessor z	s	Correction descriptors (δ, r)
1383	1195	4	$(L, \mathtt{42})$, $(L, \mathtt{43})$
1589	1383	5	$(L, \mathtt{5676})$, $(L, \mathtt{54})$
			$(L, \mathtt{54234576})$, $(R, \mathtt{1234231543654765})^{\dagger}$
1815	1589	6	$(L, \mathtt{67})$
2061	1815	5	$(L, \mathtt{56})$
2327	2061	4	$(L, \mathtt{45})$
2611	2327	2	$(L, \mathtt{24})$

E_8, $J = \{2, 3, 4, 5, 6, 7\}$, *targets 1540.* The seed named 34 is xw_J, where

$\mathtt{145678}$ is the word for x, $I = \{1, 4, 5, 6, 7, 8\}$.

The connected components of I have rank at most six.

Table 11: Exact multiplication and common-coset data in E_8, $J = \{2, 3, 4, 5, 6, 7\}$.

Name w	Predecessor z	s	Correction descriptors (δ, r)
50	34	3	$\varnothing$
70	50	2	$(L, \mathtt{2343})$, $(L, \mathtt{2314})$
97	70	4	$(L, \mathtt{43})$
130	97	5	$(L, \mathtt{54})$, $(L, \mathtt{154231})$
172	130	6	$(L, \mathtt{65})$
221	172	2	$(L, \mathtt{1234565431})$
284	221	7	$(L, \mathtt{76})$
357	284	4	$(L, \mathtt{42})$
443	357	5	$(L, \mathtt{54})$
546	443	6	$(L, \mathtt{65})$, $(L, \mathtt{67})$
656	546	3	$\varnothing$
789	656	4	$(L, \mathtt{43})$
947	789	5	$(L, \mathtt{54})$, $(L, \mathtt{56})$
1114	947	2	$\varnothing$
1315	1114	4	$(L, \mathtt{42})$, $(L, \mathtt{45})$
1540	1315	3	$(L, \mathtt{34})$

E_8, $J = \{1, 2, 3, 4, 5, 6, 7\}$, *targets 191,232.* The seed named 8 is xw_J, where

$\mathtt{345678}$ is the word for x, $I = \{3, 4, 5, 6, 7, 8\}$.

The connected components of I have rank at most six.

Table 12: Exact multiplication and common-coset data in E_8, $J = \{1, 2, 3, 4, 5, 6, 7\}$.

Name w	Predecessor z	s	Correction descriptors (δ, r)
9	8	2	$\varnothing$
12	9	4	$(L, \mathtt{42})$, $(L, \mathtt{43})$
14	12	5	$(L, \mathtt{54})$
17	14	6	$(L, \mathtt{65})$
20	17	7	$(L, \mathtt{76})$
24	20	8	$(L, \mathtt{87})$

Name w	Predecessor z	s	Correction descriptors (δ, r)
28	24	1	$\varnothing$
32	28	3	$(L, \mathtt{31})$
36	32	4	$(L, \mathtt{43})$
41	36	5	$(L, \mathtt{54})$
46	41	6	$(L, \mathtt{65})$
52	46	7	$(L, \mathtt{76}),\quad (L, \mathtt{78})$
58	52	2	$\varnothing$
64	58	4	$(L, \mathtt{42})$
70	64	5	$(L, \mathtt{54})$
76	70	3	$\varnothing$
81	76	1	$(L, \mathtt{13}),\quad (L, \mathtt{1354}),\quad (L, \mathtt{135476})$
88	81	4	$(L, \mathtt{45}),\quad (L, \mathtt{4576})$
95	88	3	$(L, \mathtt{31}),\quad (L, \mathtt{34})$
101	95	2	$\varnothing$
108	101	4	$(L, \mathtt{42}),\quad (L, \mathtt{43})$
115	108	5	$(L, \mathtt{54}),\quad (L, \mathtt{54234576})$
123	115	6	$\varnothing$
130	123	7	$(L, \mathtt{5765}),\quad (L, \mathtt{76})$
136	130	8	$(L, \mathtt{87})$
143	136	5	$(L, \mathtt{56})$
150	143	4	$(L, \mathtt{45})$
158	150	3	$(L, \mathtt{34})$
164	158	2	$\varnothing$
172	164	4	$(L, \mathtt{42}),\quad (L, \mathtt{43})$
178	172	5	$(L, \mathtt{54})$
185	178	6	$(L, \mathtt{65})$
191	185	7	$(L, \mathtt{78}),\quad (L, \mathtt{76})$
197	191	1	$\varnothing$
202	197	3	$(L, \mathtt{31})$
207	202	4	$(L, \mathtt{43})$
211	207	5	$(L, \mathtt{54})$
215	211	6	$(L, \mathtt{65}),\quad (L, \mathtt{67})$
219	215	2	$\varnothing$
223	219	4	$(L, \mathtt{42})$
226	223	5	$(L, \mathtt{54}),\quad (L, \mathtt{56})$
228	226	3	$\varnothing$
230	228	4	$(L, \mathtt{43}),\quad (L, \mathtt{45})$
232	230	2	$(L, \mathtt{24})$

E_8, $J = \{2,3,4,5,6,7,8\}$, targets 418,1356. The seed named 328 is xw_J, where

$\mathtt{654234567654231435426543 1}$ is the word for x, $\quad I = \{1,2,3,4,5,6,7\}$.

Its Levi upper element $xw_{I\cap J}$ is the already certified E_7/D_5 element named 1455. The seed named 418 is xw_J, where

$\mathtt{2431542345654231435427654 31}$ is the word for x, $\quad I = \{1,2,3,4,5,6,7\}$.

Its Levi upper element $xw_{I\cap J}$ is the already certified E_7/D_5 element named 1486.

Table 13: Exact multiplication and common-coset data in E_8, $J = \{2, 3, 4, 5, 6, 7, 8\}$.

Name w	Predecessor z	s	Correction descriptors (δ, r)
368	328	8	$\varnothing$
411	368	7	$(L, 78)$
453	411	1	$\varnothing$
498	453	3	$(L, 31)$
546	498	4	$(L, 43)$
596	546	5	$(L, 54)$, $(L, 5676)$
649	596	6	$(L, 67)$
704	649	5	$(L, 56)$
762	704	4	$(L, 45)$
821	762	3	$(L, 34)$
880	821	2	$\varnothing$
941	880	4	$(L, 42)$
1003	941	5	$(L, 54)$
1066	1003	6	$(L, 65)$, $(L, 346543)$
1130	1066	7	$(L, 76)$
1181	1130	3	$\varnothing$
1239	1181	4	$(L, 43)$, $(L, 45)$
1296	1239	2	$(L, 24)$, $(L, 2431542345765431)^{\ddagger}$
1356	1296	8	$(L, 87)$

A.4. **Representation and nonzero-action certificates.** An integral representation record contains its dimension d and the r matrices R_i for the simple reflections. A sparse matrix is a list of triples (a, b, c), meaning entry c in row a and column b, with unlisted entries zero; repeated triples are added. Let $m_{ij} = 3$ when nodes i, j are adjacent in the Dynkin diagram and $m_{ij} = 2$ otherwise. The required tests are

$$R_i^2 = I, \qquad (R_i R_j)^{m_{ij}} = I \quad (1 \le i < j \le r).$$

Thus a record defines a representation of the finite Weyl group. For every conjugacy class it also contains a representative word and the prescribed ordinary character value. Multiplying the corresponding R_i and taking the trace verifies the character label. The E_8 records contain all 112 class checks for each of dimensions $50, 175, 1050$. The input arrays are `E7_phi15_source259_receipt.json`, `E8_phi50_positive_rep.json`, `E8_phi175_positive_rep.toml`, and `E8_phi1050_positive_rep.json` in the accompanying certificate data. The three E_8 character-check records have the respective names `E8_phi50_class_checks.json`, `E8_phi175_class_checks.toml`, and `E8_phi1050_class_checks.json`.

For E_7 the 15-dimensional matrices have scalar commutant and trace 5 on a simple reflection. The finite character table has two degree-15 irreducibles, distinguished by that sign, so this identifies $\phi_{15,7}$. Its full integral action is recorded in (7.5). In the E_8 checks one may reduce modulo 1000003: since $1000003 > 2d$, the integer trace of a finite-order matrix is the unique integer in $[-d, d]$ with its computed residue. This makes the class-trace check exact.

To construct e_J, choose a chain of standard parabolics

$$1 = W_{J_0} \subset W_{J_1} \subset \cdots \subset W_{J_t} = W_J$$

and a set D_a of minimal coset representatives at each step. The products of the D_a give each element of W_J once, so

$$e_J = \prod_{a=t}^{1} \left(\sum_{d \in D_a} R(d) \right)$$

with the order chosen to match that coset decomposition. The certificate lists the factors as words. Their distinctness, minimality, and cardinalities are verified by the root matrices. Starting with the seed vector v_0 defined in Section 7.5, this computes the integer vector $u = e_J v_0$. One checks $R_j u = u$ for all $j \in J$. Finally, specialize the independently defined spherical vector in (A.3) at $v = 1$ and evaluate (7.4). The resulting exact coordinates give Table 4.

The largest conservative intermediate integer bounds in these three vector calculations are, respectively, 133497216, 123604992, and 34384896. Alternatively arbitrary precision integer arithmetic gives the same calculation without an a priori bound. A nonzero coordinate already proves the required block detection; the displayed sum of squares supplies an additional exact check on the entire vector.

For the two restriction calculations, use (7.6) to construct R_0. The sign projectors are averages over groups of orders 6 and 12. Their images have dimensions 10 and 5, respectively. On a basis of each image, form the linear equations

$$XR_i - R_i X = 0$$

for the remaining simple generators. The coefficient matrices have nullities one modulo 1000003, hence also over $\mathbb{Q}$. The character identifications and determinant conversion are given in Lemma 7.5. This verifies both multiplicities directly from the integral matrices.

A.5. **The finite trace-and-cut verification rules.** We now specify the objects occurring in the source and witness tables. The affine node 0 is represented in the finite Weyl group by reflection in the highest root. If b is a mask in rank r, write

$$b = \sum_{j=1}^{r} \epsilon_j 2^{j-1} + \epsilon_0 2^r, \qquad \epsilon_j \in \{0, 1\}.$$

Then H_b is generated by s_j with $\epsilon_j = 1$ and by s_θ when $\epsilon_0 = 1$. Only proper affine node subsets are used. Thus the mask specifies the embedding of the subgroup. For example mask 255 is the finite E_8, and mask 510 is its affine D_8 subgroup.

For a mask not using node 0, take $\xi = 0$. Otherwise let k be the least missing finite node and set

$$\langle \xi, \alpha_i^\vee \rangle = \delta_{ik}/m_k, \tag{A.5}$$

where $\theta^\vee = \sum_i m_i \alpha_i^\vee$. This is a fixed point for the corresponding finite affine parabolic. Its lift is exactly $t_{\xi - h\xi} h$. One checks $\xi - h\xi$ is integral for every generator and hence every h. This fixes all phases in (7.7).

Here are explicit conventions for the vertex representatives. Let $\omega_i^\vee$ be the fundamental coweights and put $y_i = \exp(2\pi \mathrm{i}\, \omega_i^\vee / m_i)$ for $i > 0$, $y_0 = 1$. The highest-root coefficient lists are

$$(m_i)_{E_7} = (2, 2, 3, 4, 3, 2, 1), \qquad (m_i)_{E_8} = (2, 3, 4, 6, 5, 4, 3, 2).$$

In simple-coroot coordinates, the representative is the ith column of A^{-1} divided by m_i, taken modulo $\mathbb{Z}^r$. The tables list the vertices in the following order; the entries are the deleted-node indices:

$$\begin{aligned} E_7 &: \quad (0, 7, 6, 1, 3, 4, 2, 5), \\ E_8 &: \quad (0, 1, 8, 7, 2, 6, 3, 5, 4). \end{aligned} \tag{A.6}$$

Deleting that node from the extended diagram gives its centralizer. The representatives retain all central translates in the simply connected datum. The full root systems of the centralizers

can also be recovered without diagram recognition: take exactly the roots α for which $\alpha(y) = 1$ and their reflections.

In the product-orbit notation of the witness tables, the ordered factors are as follows:

E_7 vertices	$E_7, E_7, D_6A_1, D_6A_1, A_2A_5, A_3A_1A_3, A_7, A_5A_2$
E_8 vertices	$E_8, D_8, E_7A_1, E_6A_2, A_8, D_5A_3, A_1A_7, A_4A_4, A_2A_1A_5$.

Components are ordered by their least finite node; the affine node is placed last for this purpose. Within a component use its standard based root system. Positive and negative very-even labels distinguish the two orbits in the usual type-D Springer tables. A string of single digits denotes a partition; a part larger than nine is written in a parenthesized list. Commas between factors denote products. The label 1 in an exceptional orbit column denotes the zero orbit.

The finite input for a subgroup is its ordinary character table, class representative words and centralizer orders, the Springer pairs, the a- and b-values, and the orbit closure incidence relation. These are the standard finite Weyl-group and Springer data in [36, 25, 24]. Character row labels in the source tables are standard $\phi_{d,b}$ labels in exceptional type, partitions in type A, and bipartitions separated by a dot in type D; signs distinguish a degenerate pair. Product characters are written as lists of their factor labels. The additional row number fixes the ordering of the accompanying character array, but the character label and embedded subgroup specify the representation independently of that number.

For each row of a witness table perform the following finite steps.

1. Construct its vertex point y, its centralizer M, and the irreducible E corresponding to its printed local nilpotent orbit with trivial local system. Embed its weighted Dynkin diagram in the ambient root space, make it dominant, and check that it gives the printed ambient orbit.
2. Read the indicated source ID in Table 14 or 15. Construct H_b and ξ by (A.5). For each conjugacy-class representative h of H_b, enumerate the points $z \in Wy$ fixed by h, and evaluate (7.7). Combine these values with the ordinary character of F using class sizes to obtain (7.8). Reduce cyclotomic polynomials modulo the e-th cyclotomic polynomial Φ_e, with e the order of y. The resulting value is a strictly positive integer. Check nonnegativity and integrality for the full multiplicity column and the degree identity (7.9).
3. For every ordinary parabolic L of M, compute $[\mathrm{Res}^M_L E : \det_L]$ by character inner product. Keep exactly the positive cases. Compute $j^W_L \det_L$: among the constituents of $\mathrm{Ind}^W_L \det_L$, it is the unique one with $b = b(\det_L)$, and its multiplicity is one. Read its Springer orbit D_L.
4. In the finite orbit incidence relation intersect the down-set of the source bound with the down-sets of all retained D_L. Every element of this intersection lies in the down-set of the printed target. Check also that the target itself lies in the source down-set. These are precisely the two assertions in (7.11).

Parabolic subgroups in the third step may be taken up to conjugacy in M, because both the restriction multiplicity and the ambient orbit are constant on such a class. The identity mask and the trivial subgroup are included. The group-intersection step in the trace calculation uses all fixed points and all conjugated elements $r_z h r_z^{-1}$ in the full intersection.

For source ID 8.G, Step 2 can instead use the explicit multiplicity-one calculation of Lemma 7.5; Step 4 then holds because its source bound is already the target.

The electronic implementation separates three kinds of input. The integral word and representation records are verified by matrix multiplication and Laurent-polynomial identities. The ordinary character and Springer arrays are finite classification data; their row and column identifications, degrees, class orthogonality, induction dimensions, and closure down-sets fix the conventions. The cyclotomic trace and cut routines evaluate the four steps just given from these inputs. The published exhaustive p-canonical calculations supply the precise rank bound stated

in Section 7. The propagation and geometric arguments establish actuality of the remaining sources.

A.6. **Selected actual finite blocks.** Tables 14 and 15 list the distinct ordinary blocks used by the coverage calculation. A source ID has the form $r.k$ in type E_r and identifies a row of this table. For each ordinary block F, its bound is obtained from the special representation of its family by truncated induction from the embedded H_b to $W(E_r)$. The exceptional actuality certificates are Proposition 7.4 and Theorem 8.1; the uniform blocks are supplied by the finite, minuscule, and classical constructions described in Section 7.

Table 14: Selected actual finite blocks in E_7. The mask defines H; F is in the ordinary Springer convention.

ID	Mask	Row	F	Bound B_F
7.001	0	1	`Id`	E_7
7.002	13	1	`1111`	$D_6(a_1)$
7.003	15	1	`11111`	$D_5(a_1)$
7.004	19	1	`11,11,11`	$E_7(a_3)$
7.005	23	1	`111,11,11`	$E_7(a_4)$
7.006	27	1	`11,1111`	$E_7(a_5)$
7.007	30	3	`1.111`	$E_7(a_5)$
7.008	30	4	`.1111`	A_5''
7.009	31	2	`1.1111`	A_4
7.010	31	3	`.11111`	$(A_3+A_1)''$
7.011	47	1	`11111,11`	A_4+A_1
7.012	53	1	`111,111`	D_5+A_1
7.013	55	1	`111,11,111`	$E_7(a_5)$
7.014	61	1	`111111`	$D_4(a_1)+A_1$
7.015	63	2	`phi1,36`	$3A_1''$
7.016	63	5	`phi6,25`	A_2+A_1
7.017	63	8	`phi15,17`	$D_4(a_1)+A_1$
7.018	83	1	`11,11,11,11`	$E_6(a_1)$
7.019	87	1	`111,11,11,11`	A_6
7.020	90	1	`1111,11`	D_5
7.021	91	1	`11,1111,11`	$E_6(a_3)$
7.022	94	7	`.1111,11`	A_4
7.023	95	3	`1.1111,11`	A_3+A_2
7.024	95	5	`.11111,11`	$2A_2$
7.025	109	1	`1111,111`	$D_5(a_1)+A_1$
7.026	111	1	`11111,111`	$A_3+A_2+A_1$
7.027	119	1	`111,11,1111`	A_4+A_2
7.028	122	1	`111111`	D_4
7.029	123	1	`11,111111`	$D_4(a_1)$
7.030	125	1	`1111111`	A_2+3A_1
7.031	126	3	`11.1111`	$D_4(a_1)$
7.032	126	4	`1.11111`	A_3
7.033	126	5	`.111111`	A_2
7.034	126	18	`.222`	$D_5(a_1)$
7.035	127	2	`phi1,63`	1
7.036	127	3	`phi7,46`	A_1
7.037	127	5	`phi15,28`	A_2+A_1
7.038	127	12	`phi27,37`	$2A_1$
7.039	127	35	`phi189,22`	A_2+2A_1

ID	Mask	Row	F	Bound B_F
7.040	212	1	`11,11,11,11`	D_6
7.041	213	1	`1111,11,11`	$D_6(a_2)$
7.042	215	1	`1111,11,11,11`	A_5'
7.043	221	1	`111111,11`	$A_3 + 2A_1$
7.044	222	9	`1.111,11,11`	$D_5(a_1) + A_1$
7.045	222	13	`.1111,11,11`	$A_3 + A_2$
7.046	223	5	`11.1111,11`	$(A_3 + A_1)'$
7.047	223	7	`1.11111,11`	$A_2 + 2A_1$
7.048	223	9	`.111111,11`	$3A_1'$
7.049	223	35	`.222,11`	$A_4 + A_1$
7.050	235	1	`111,111,111`	$A_5 + A_1$
7.051	239	1	`111111,111`	$2A_2 + A_1$
7.052	245	1	`1111,1111`	$D_4 + A_1$
7.053	247	1	`1111,11,1111`	$A_3 + A_2 + A_1$
7.054	253	1	`11111111`	$4A_1$

Table 15: Selected actual finite blocks in E_8. The mask defines H; F is in the ordinary Springer convention.

ID	Mask	Row	F	Bound B_F
8.001	0	1	`Id`	E_8
8.002	13	1	`1111`	$E_7(a_1)$
8.003	15	1	`11111`	$E_7(a_3)$
8.004	19	1	`11,11,11`	$E_8(a_3)$
8.005	23	1	`111,11,11`	$E_8(b_4)$
8.006	27	1	`11,1111`	$E_8(b_5)$
8.007	30	3	`1.111`	$E_8(b_5)$
8.008	30	4	`.1111`	E_6
8.009	31	2	`1.1111`	$E_6(a_1)$
8.010	31	3	`.11111`	D_5
8.011	47	1	`11111,11`	$E_6(a_1) + A_1$
8.012	55	1	`111,11,111`	$E_8(b_5)$
8.013	61	1	`111111`	$D_6(a_1)$
8.014	63	2	`phi1,36`	D_4
8.015	63	5	`phi6,25`	$D_5(a_1)$
8.016	63	8	`phi15,17`	$D_6(a_1)$
8.017	83	1	`11,11,11,11`	$E_8(a_4)$
8.018	87	1	`111,11,11,11`	$E_8(a_5)$
8.019	91	1	`11,1111,11`	$E_8(a_6)$
8.020	94	5	`1.111,11`	$E_8(a_6)$
8.021	94	7	`.1111,11`	$E_6(a_1)$
8.022	95	3	`1.1111,11`	$E_7(a_4)$
8.023	95	5	`.11111,11`	$E_6(a_3)$
8.024	109	1	`1111,111`	$D_7(a_1)$
8.025	111	1	`11111,111`	$D_5 + A_2$
8.026	119	1	`111,11,1111`	$E_8(b_6)$
8.027	123	1	`11,111111`	$E_8(a_7)$
8.028	125	1	`1111111`	$D_4 + A_2$
8.029	126	3	`11.1111`	$E_8(a_7)$
8.030	126	4	`1.11111`	$E_6(a_3)$

ID	Mask	Row	F	Bound B_F
8.031	126	5	`.111111`	A_4
8.032	126	18	`.222`	$E_7(a_3)$
8.033	127	2	`phi1,63`	A_2
8.034	127	3	`phi7,46`	A_3
8.035	127	5	`phi15,28`	$D_5(a_1)$
8.036	127	12	`phi27,37`	$D_4(a_1)$
8.037	127	35	`phi189,22`	$D_5(a_1) + A_1$
8.038	175	1	`11111,11,11`	$D_7(a_2)$
8.039	183	1	`111,11,111,11`	$E_8(a_6)$
8.040	191	3	`phi1,36,11`	$D_4(a_1)$
8.041	191	9	`phi6,25,11`	$A_4 + A_1$
8.042	191	15	`phi15,17,11`	$E_8(a_7)$
8.043	222	7	`1.111,111`	$E_8(b_6)$
8.044	222	10	`.1111,111`	A_6
8.045	223	4	`1.1111,111`	$E_8(a_7)$
8.046	223	7	`.11111,111`	$A_4 + A_2$
8.047	237	1	`1111,1111`	$D_7(a_2)$
8.048	247	1	`111,11,11111`	$A_6 + A_1$
8.049	251	1	`11,1111111`	$A_4 + A_2 + A_1$
8.050	253	1	`11111111`	$D_4(a_1) + A_2$
8.051	254	2	`11.11111`	$A_4 + 2A_1$
8.052	254	3	`1.111111`	$A_3 + A_2$
8.053	254	4	`.1111111`	$2A_2$
8.054	255	2	`phi1,120`	1
8.055	255	6	`phi35,74`	$2A_1$
8.056	255	9	`phi50,56`	$A_2 + A_1$
8.057	255	16	`phi210,52`	$A_2 + A_1$
8.058	255	38	`phi1050,34`	$D_4(a_1) + A_1$
8.059	255	44	`phi1400,32`	$D_4(a_1) + A_1$
8.060	255	69	`phi8,91`	A_1
8.061	255	82	`phi560,47`	$A_2 + 2A_1$
8.062	338	1	`11,11,11,11`	E_7
8.063	343	1	`111,11,11,11,11`	D_7
8.064	346	1	`1111,11,11`	$E_7(a_2)$
8.065	347	1	`11,1111,11,11`	$D_7(a_1)$
8.066	350	9	`1.111,11,11`	$D_7(a_1)$
8.067	350	13	`.1111,11,11`	$E_7(a_4)$
8.068	351	5	`1.1111,11,11`	A_6
8.069	351	9	`.11111,11,11`	A_5
8.070	375	1	`111,11,1111,11`	A_7
8.071	378	1	`111111,11`	$D_5 + A_1$
8.072	379	1	`11,111111,11`	$E_7(a_5)$
8.073	382	5	`11.1111,11`	$E_7(a_5)$
8.074	382	7	`1.11111,11`	$D_5(a_1) + A_1$
8.075	382	9	`.111111,11`	$A_3 + A_2$
8.076	382	10	`.111111,2`	A_4
8.077	382	35	`.222,11`	$E_6(a_1) + A_1$
8.078	383	3	`phi1,63,11`	$3A_1$
8.079	383	4	`phi1,63,2`	A_2
8.080	383	5	`phi7,46,11`	$A_2 + 2A_1$
8.081	383	6	`phi7,46,2`	A_3

ID	Mask	Row	F	Bound B_F
8.082	383	9	`phi15,28,11`	$A_4 + A_1$
8.083	383	17	`phi21,36,11`	$D_4(a_1)$
8.084	383	23	`phi27,37,11`	$A_3 + A_1$
8.085	383	69	`phi189,22,11`	$A_4 + A_2$
8.086	437	1	`111,111,111`	$E_6 + A_1$
8.087	445	1	`111111,111`	$E_6(a_3) + A_1$
8.088	447	4	`phi1,36,111`	$2A_2 + A_1$
8.089	447	13	`phi6,25,111`	$D_4(a_1) + A_2$
8.090	447	14	`phi6,25,21`	$A_4 + A_1$
8.091	447	22	`phi15,17,111`	$E_6(a_3) + A_1$
8.092	447	28	`phi15,16,111`	$E_6(a_3) + A_1$
8.093	447	29	`phi15,16,21`	$E_8(a_7)$
8.094	447	35	`phi20,20,21`	$E_6(a_3)$
8.095	474	1	`1111,1111`	D_6
8.096	475	1	`11,1111,1111`	$D_5 + A_2$
8.097	478	16	`.1111,1111`	$D_6(a_2)$
8.098	479	1	`11.111,1111`	$E_8(a_7)$
8.099	479	6	`1.1111,1111`	$D_5(a_1) + A_2$
8.100	479	11	`.11111,1111`	$2A_3$
8.101	479	13	`.11111,22`	A_5
8.102	495	1	`11111,11111`	$A_4 + A_3$
8.103	503	1	`111,11,111111`	$A_5 + A_1$
8.104	503	23	`21,11,111111`	$E_7(a_5)$
8.105	506	1	`11111111`	$D_4 + A_1$
8.106	507	1	`11,11111111`	$A_3 + A_2 + A_1$
8.107	507	2	`11,2111111`	$A_4 + A_2 + A_1$
8.108	509	1	`111111111`	$2A_2 + 2A_1$
8.109	509	2	`21111111`	$D_4(a_1) + A_2$
8.110	510	3	`111.11111`	$A_3 + A_2 + A_1$
8.111	510	4	`11.111111`	$A_3 + 2A_1$
8.112	510	5	`1.1111111`	$A_2 + 3A_1$
8.113	510	6	`.11111111`	$4A_1$
8.114	510	28	`.22211`	$A_4 + 2A_1$
8.G	255	14	$\phi_{175,36}$	$D_4(a_1) + A_1$

A.7. **All required local witnesses in type E_7.** For a local orbit in the following tables, the local system is always trivial. The Springer correspondence therefore determines the local irreducible. The row number identifies the same entry in the full ordinary character array; gaps in those row numbers are the omitted nontrivial-local-system entries. Each displayed source satisfies the positive trace and cut-intersection conditions of (7.11).

Table 16: E_7, vertex 1, $M = E_7$: all 45 trivial-local-system generators.

Row	Local nilpotent orbit	Ambient orbit	Source ID
1	E_7	E_7	7.001
2	1	1	7.001
3	A_1	A_1	7.036
4	$E_7(a_1)$	$E_7(a_1)$	7.001
5	$4A_1$	$4A_1$	7.054

Row	Local nilpotent orbit	Ambient orbit	Source ID
9	$3A_1''$	$3A_1''$	7.001
10	E_6	E_6	7.001
11	$E_7(a_2)$	$E_7(a_2)$	7.001
12	$2A_1$	$2A_1$	7.001
15	D_6	D_6	7.040
16	$3A_1'$	$3A_1'$	7.048
17	A_2	A_2	7.001
18	$E_7(a_3)$	$E_7(a_3)$	7.001
19	$2A_2 + A_1$	$2A_2 + A_1$	7.051
20	$A_5 + A_1$	$A_5 + A_1$	7.050
21	$D_4 + A_1$	$D_4 + A_1$	7.052
25	A_6	A_6	7.001
26	$A_2 + 3A_1$	$A_2 + 3A_1$	7.001
27	A_5''	A_5''	7.001
28	D_4	D_4	7.001
29	$E_6(a_1)$	$E_6(a_1)$	7.001
30	$A_2 + A_1$	$A_2 + A_1$	7.016
31	$D_5 + A_1$	$D_5 + A_1$	7.001
32	$2A_2$	$2A_2$	7.001
35	$A_2 + 2A_1$	$A_2 + 2A_1$	7.001
36	$E_7(a_4)$	$E_7(a_4)$	7.001
37	$(A_3 + A_1)''$	$(A_3 + A_1)''$	7.001
38	D_5	D_5	7.001
39	$D_6(a_1)$	$D_6(a_1)$	7.001
40	A_3	A_3	7.032
41	$A_4 + A_2$	$A_4 + A_2$	7.001
42	$A_3 + A_2 + A_1$	$A_3 + A_2 + A_1$	7.001
43	$A_3 + 2A_1$	$A_3 + 2A_1$	7.043
44	A_5'	A_5'	7.042
47	$D_6(a_2)$	$D_6(a_2)$	7.041
48	$(A_3 + A_1)'$	$(A_3 + A_1)'$	7.043
49	$D_4(a_1)$	$D_4(a_1)$	7.001
50	$E_7(a_5)$	$E_7(a_5)$	7.001
53	$A_3 + A_2$	$A_3 + A_2$	7.001
54	$D_5(a_1) + A_1$	$D_5(a_1) + A_1$	7.001
55	$E_6(a_3)$	$E_6(a_3)$	7.001
56	$D_4(a_1) + A_1$	$D_4(a_1) + A_1$	7.001
57	$D_5(a_1)$	$D_5(a_1)$	7.001
58	A_4	A_4	7.001
60	$A_4 + A_1$	$A_4 + A_1$	7.001

Table 17: E_7, vertex 2, $M = E_7$: all 45 trivial-local-system generators.

Row	Local nilpotent orbit	Ambient orbit	Source ID
1	E_7	E_7	7.001
2	1	1	7.001
3	A_1	A_1	7.036
4	$E_7(a_1)$	$E_7(a_1)$	7.001
5	$4A_1$	$4A_1$	7.054

Row	Local nilpotent orbit	Ambient orbit	Source ID
9	$3A_1''$	$3A_1''$	7.001
10	E_6	E_6	7.001
11	$E_7(a_2)$	$E_7(a_2)$	7.001
12	$2A_1$	$2A_1$	7.001
15	D_6	D_6	7.040
16	$3A_1'$	$3A_1'$	7.048
17	A_2	A_2	7.001
18	$E_7(a_3)$	$E_7(a_3)$	7.001
19	$2A_2 + A_1$	$2A_2 + A_1$	7.051
20	$A_5 + A_1$	$A_5 + A_1$	7.050
21	$D_4 + A_1$	$D_4 + A_1$	7.052
25	A_6	A_6	7.001
26	$A_2 + 3A_1$	$A_2 + 3A_1$	7.001
27	A_5''	A_5''	7.001
28	D_4	D_4	7.001
29	$E_6(a_1)$	$E_6(a_1)$	7.001
30	$A_2 + A_1$	$A_2 + A_1$	7.016
31	$D_5 + A_1$	$D_5 + A_1$	7.001
32	$2A_2$	$2A_2$	7.001
35	$A_2 + 2A_1$	$A_2 + 2A_1$	7.001
36	$E_7(a_4)$	$E_7(a_4)$	7.001
37	$(A_3 + A_1)''$	$(A_3 + A_1)''$	7.001
38	D_5	D_5	7.001
39	$D_6(a_1)$	$D_6(a_1)$	7.001
40	A_3	A_3	7.032
41	$A_4 + A_2$	$A_4 + A_2$	7.001
42	$A_3 + A_2 + A_1$	$A_3 + A_2 + A_1$	7.001
43	$A_3 + 2A_1$	$A_3 + 2A_1$	7.043
44	A_5'	A_5'	7.042
47	$D_6(a_2)$	$D_6(a_2)$	7.041
48	$(A_3 + A_1)'$	$(A_3 + A_1)'$	7.043
49	$D_4(a_1)$	$D_4(a_1)$	7.001
50	$E_7(a_5)$	$E_7(a_5)$	7.001
53	$A_3 + A_2$	$A_3 + A_2$	7.001
54	$D_5(a_1) + A_1$	$D_5(a_1) + A_1$	7.001
55	$E_6(a_3)$	$E_6(a_3)$	7.001
56	$D_4(a_1) + A_1$	$D_4(a_1) + A_1$	7.001
57	$D_5(a_1)$	$D_5(a_1)$	7.001
58	A_4	A_4	7.001
60	$A_4 + A_1$	$A_4 + A_1$	7.001

Table 18: E_7, vertex 3, $M = D_6 + A_1$: all 62 trivial-local-system generators.

Row	Local nilpotent orbit	Ambient orbit	Source ID
1	$222222+, 11$	$3A_1'$	7.048
2	$222222+, 2$	A_2	7.033
3	$222222-, 11$	$3A_1''$	7.015
4	$222222-, 2$	$4A_1$	7.054
5	$22221111, 11$	$2A_1$	7.038

Row	Local nilpotent orbit	Ambient orbit	Source ID
6	$22221111, 2$	$3A_1'$	7.048
7	$2211111111, 11$	A_1	7.036
8	$2211111111, 2$	$2A_1$	7.038
9	$111111111111, 11$	1	7.035
10	$111111111111, 2$	A_1	7.036
15	$332211, 11$	$A_2 + A_1$	7.016
16	$332211, 2$	$A_2 + 2A_1$	7.039
17	$33111111, 11$	A_2	7.033
18	$33111111, 2$	$A_2 + A_1$	7.016
19	$3111111111, 11$	$2A_1$	7.038
20	$3111111111, 2$	$3A_1''$	7.015
21	$3333, 11$	$2A_2$	7.024
22	$3333, 2$	$2A_2 + A_1$	7.051
23	$4422+, 11$	$(A_3 + A_1)'$	7.046
24	$4422+, 2$	$D_4(a_1)$	7.029
25	$4422-, 11$	$(A_3 + A_1)''$	7.010
26	$4422-, 2$	$A_3 + 2A_1$	7.043
27	$333111, 11$	$A_2 + 2A_1$	7.039
28	$333111, 2$	$A_2 + 3A_1$	7.030
29	$441111, 11$	A_3	7.032
30	$441111, 2$	$(A_3 + A_1)'$	7.046
31	$32211111, 11$	$3A_1'$	7.048
32	$32211111, 2$	$4A_1$	7.054
33	$4431, 11$	$A_3 + 2A_1$	7.043
34	$4431, 2$	$D_4(a_1) + A_1$	7.014
35	$322221, 11$	$4A_1$	7.054
36	$322221, 2$	$A_2 + A_1$	7.037
39	$5322, 11$	$D_4(a_1) + A_1$	7.014
40	$5322, 2$	$A_3 + A_2$	7.023
41	$531111, 11$	$D_4(a_1)$	7.029
42	$531111, 2$	$D_4(a_1) + A_1$	7.014
43	$51111111, 11$	A_3	7.032
44	$51111111, 2$	$(A_3 + A_1)''$	7.010
47	$5331, 11$	$A_3 + A_2$	7.023
48	$5331, 2$	$A_3 + A_2 + A_1$	7.026
49	$5511, 11$	A_4	7.009
50	$5511, 2$	$A_4 + A_1$	7.011
51	$522111, 11$	$(A_3 + A_1)'$	7.046
52	$522111, 2$	$A_3 + 2A_1$	7.043
53	$66+, 11$	A_5'	7.042
54	$66+, 2$	$E_6(a_3)$	7.021
55	$66-, 11$	A_5''	7.008
56	$66-, 2$	$A_5 + A_1$	7.050
61	$7311, 11$	$D_5(a_1)$	7.003
62	$7311, 2$	$D_5(a_1) + A_1$	7.025
63	$711111, 11$	D_4	7.028
64	$711111, 2$	$D_4 + A_1$	7.052
65	$75, 11$	$D_6(a_2)$	7.041
66	$75, 2$	$E_7(a_5)$	7.006
67	$7221, 11$	$D_4 + A_1$	7.052
68	$7221, 2$	$D_5(a_1)$	7.034

Row	Local nilpotent orbit	Ambient orbit	Source ID
69	$93, 11$	$D_6(a_1)$	7.002
70	$93, 2$	$E_7(a_4)$	7.005
71	$9111, 11$	D_5	7.020
72	$9111, 2$	$D_5 + A_1$	7.012
73	$(11, 1), 11$	D_6	7.040
74	$(11, 1), 2$	$E_7(a_3)$	7.004

Table 19: E_7, vertex 4, $M = D_6 + A_1$: all 62 trivial-local-system generators.

Row	Local nilpotent orbit	Ambient orbit	Source ID
1	$222222+, 11$	$3A_1''$	7.015
2	$222222+, 2$	$4A_1$	7.054
3	$222222-, 11$	$3A_1'$	7.048
4	$222222-, 2$	A_2	7.033
5	$22221111, 11$	$2A_1$	7.038
6	$22221111, 2$	$3A_1'$	7.048
7	$2211111111, 11$	A_1	7.036
8	$2211111111, 2$	$2A_1$	7.038
9	$111111111111, 11$	1	7.035
10	$111111111111, 2$	A_1	7.036
15	$332211, 11$	$A_2 + A_1$	7.016
16	$332211, 2$	$A_2 + 2A_1$	7.039
17	$33111111, 11$	A_2	7.033
18	$33111111, 2$	$A_2 + A_1$	7.016
19	$3111111111, 11$	$2A_1$	7.038
20	$3111111111, 2$	$3A_1''$	7.015
21	$3333, 11$	$2A_2$	7.024
22	$3333, 2$	$2A_2 + A_1$	7.051
23	$4422+, 11$	$(A_3 + A_1)''$	7.010
24	$4422+, 2$	$A_3 + 2A_1$	7.043
25	$4422-, 11$	$(A_3 + A_1)'$	7.046
26	$4422-, 2$	$D_4(a_1)$	7.029
27	$333111, 11$	$A_2 + 2A_1$	7.039
28	$333111, 2$	$A_2 + 3A_1$	7.030
29	$441111, 11$	A_3	7.032
30	$441111, 2$	$(A_3 + A_1)'$	7.046
31	$32211111, 11$	$3A_1'$	7.048
32	$32211111, 2$	$4A_1$	7.054
33	$4431, 11$	$A_3 + 2A_1$	7.043
34	$4431, 2$	$D_4(a_1) + A_1$	7.014
35	$322221, 11$	$4A_1$	7.054
36	$322221, 2$	$A_2 + A_1$	7.037
39	$5322, 11$	$D_4(a_1) + A_1$	7.014
40	$5322, 2$	$A_3 + A_2$	7.023
41	$531111, 11$	$D_4(a_1)$	7.029
42	$531111, 2$	$D_4(a_1) + A_1$	7.014
43	$51111111, 11$	A_3	7.032
44	$51111111, 2$	$(A_3 + A_1)''$	7.010
47	$5331, 11$	$A_3 + A_2$	7.023

Row	Local nilpotent orbit	Ambient orbit	Source ID
48	$5331, 2$	$A_3 + A_2 + A_1$	7.026
49	$5511, 11$	A_4	7.009
50	$5511, 2$	$A_4 + A_1$	7.011
51	$522111, 11$	$(A_3 + A_1)'$	7.046
52	$522111, 2$	$A_3 + 2A_1$	7.043
53	$66+, 11$	A_5''	7.008
54	$66+, 2$	$A_5 + A_1$	7.050
55	$66-, 11$	A_5'	7.042
56	$66-, 2$	$E_6(a_3)$	7.021
61	$7311, 11$	$D_5(a_1)$	7.003
62	$7311, 2$	$D_5(a_1) + A_1$	7.025
63	$711111, 11$	D_4	7.028
64	$711111, 2$	$D_4 + A_1$	7.052
65	$75, 11$	$D_6(a_2)$	7.041
66	$75, 2$	$E_7(a_5)$	7.006
67	$7221, 11$	$D_4 + A_1$	7.052
68	$7221, 2$	$D_5(a_1)$	7.034
69	$93, 11$	$D_6(a_1)$	7.002
70	$93, 2$	$E_7(a_4)$	7.005
71	$9111, 11$	D_5	7.020
72	$9111, 2$	$D_5 + A_1$	7.012
73	$(11, 1), 11$	D_6	7.040
74	$(11, 1), 2$	$E_7(a_3)$	7.004

Table 20: E_7, vertex 5, $M = A_5 + A_2$: all 33 trivial-local-system generators.

Row	Local nilpotent orbit	Ambient orbit	Source ID
1	$111, 111111$	1	7.035
2	$111, 21111$	A_1	7.036
3	$111, 2211$	$2A_1$	7.038
4	$111, 222$	$3A_1''$	7.015
5	$111, 3111$	A_2	7.033
6	$111, 321$	$A_2 + A_1$	7.016
7	$111, 33$	$2A_2$	7.024
8	$111, 411$	A_3	7.032
9	$111, 42$	$(A_3 + A_1)''$	7.010
10	$111, 51$	A_4	7.009
11	$111, 6$	A_5''	7.008
12	$21, 111111$	A_1	7.036
13	$21, 21111$	$2A_1$	7.038
14	$21, 2211$	$3A_1'$	7.048
15	$21, 222$	$4A_1$	7.054
16	$21, 3111$	$A_2 + A_1$	7.016
17	$21, 321$	$A_2 + 2A_1$	7.039
18	$21, 33$	$2A_2 + A_1$	7.051
19	$21, 411$	$(A_3 + A_1)'$	7.046
20	$21, 42$	$A_3 + 2A_1$	7.043
21	$21, 51$	$A_4 + A_1$	7.011
22	$21, 6$	$A_5 + A_1$	7.050

Row	Local nilpotent orbit	Ambient orbit	Source ID
23	$3, 111111$	A_2	7.033
24	$3, 21111$	$A_2 + A_1$	7.016
25	$3, 2211$	$A_2 + 2A_1$	7.039
26	$3, 222$	$A_2 + 3A_1$	7.030
27	$3, 3111$	$2A_2$	7.024
28	$3, 321$	$2A_2 + A_1$	7.051
29	$3, 33$	$D_4(a_1)$	7.029
30	$3, 411$	$A_3 + A_2$	7.023
31	$3, 42$	$A_3 + A_2 + A_1$	7.026
32	$3, 51$	$A_4 + A_2$	7.027
33	$3, 6$	$E_7(a_5)$	7.013

Table 21: E_7, vertex 6, $M = 2A_3 + A_1$: all 50 trivial-local-system generators.

Row	Local nilpotent orbit	Ambient orbit	Source ID
1	$1111, 11, 1111$	1	7.035
2	$1111, 11, 211$	A_1	7.036
3	$1111, 11, 22$	$2A_1$	7.038
4	$1111, 11, 31$	A_2	7.033
5	$1111, 11, 4$	A_3	7.032
6	$1111, 2, 1111$	A_1	7.036
7	$1111, 2, 211$	$2A_1$	7.038
8	$1111, 2, 22$	$3A_1''$	7.015
9	$1111, 2, 31$	$A_2 + A_1$	7.016
10	$1111, 2, 4$	$(A_3 + A_1)''$	7.010
11	$211, 11, 1111$	A_1	7.036
12	$211, 11, 211$	$2A_1$	7.038
13	$211, 11, 22$	$3A_1'$	7.048
14	$211, 11, 31$	$A_2 + A_1$	7.016
15	$211, 11, 4$	$(A_3 + A_1)'$	7.046
16	$211, 2, 1111$	$2A_1$	7.038
17	$211, 2, 211$	$3A_1'$	7.048
18	$211, 2, 22$	$4A_1$	7.054
19	$211, 2, 31$	$A_2 + 2A_1$	7.039
20	$211, 2, 4$	$A_3 + 2A_1$	7.043
21	$22, 11, 1111$	$2A_1$	7.038
22	$22, 11, 211$	$3A_1'$	7.048
23	$22, 11, 22$	A_2	7.033
24	$22, 11, 31$	$A_2 + 2A_1$	7.039
25	$22, 11, 4$	$D_4(a_1)$	7.029
26	$22, 2, 1111$	$3A_1''$	7.015
27	$22, 2, 211$	$4A_1$	7.054
28	$22, 2, 22$	$A_2 + A_1$	7.037
29	$22, 2, 31$	$A_2 + 3A_1$	7.030
30	$22, 2, 4$	$D_4(a_1) + A_1$	7.014
31	$31, 11, 1111$	A_2	7.033
32	$31, 11, 211$	$A_2 + A_1$	7.016
33	$31, 11, 22$	$A_2 + 2A_1$	7.039
34	$31, 11, 31$	$2A_2$	7.024

Row	Local nilpotent orbit	Ambient orbit	Source ID
35	$31, 11, 4$	$A_3 + A_2$	7.023
36	$31, 2, 1111$	$A_2 + A_1$	7.016
37	$31, 2, 211$	$A_2 + 2A_1$	7.039
38	$31, 2, 22$	$A_2 + 3A_1$	7.030
39	$31, 2, 31$	$2A_2 + A_1$	7.051
40	$31, 2, 4$	$A_3 + A_2 + A_1$	7.026
41	$4, 11, 1111$	A_3	7.032
42	$4, 11, 211$	$(A_3 + A_1)'$	7.046
43	$4, 11, 22$	$D_4(a_1)$	7.029
44	$4, 11, 31$	$A_3 + A_2$	7.023
45	$4, 11, 4$	A_4	7.022
46	$4, 2, 1111$	$(A_3 + A_1)''$	7.010
47	$4, 2, 211$	$A_3 + 2A_1$	7.043
48	$4, 2, 22$	$D_4(a_1) + A_1$	7.014
49	$4, 2, 31$	$A_3 + A_2 + A_1$	7.026
50	$4, 2, 4$	$A_4 + A_1$	7.049

Table 22: E_7, vertex 7, $M = A_7$: all 22 trivial-local-system generators.

Row	Local nilpotent orbit	Ambient orbit	Source ID
1	11111111	1	7.035
2	2111111	A_1	7.036
3	221111	$2A_1$	7.038
4	22211	$3A_1'$	7.048
5	2222	A_2	7.033
6	311111	A_2	7.033
7	32111	$A_2 + A_1$	7.016
8	3221	$A_2 + 2A_1$	7.039
9	3311	$2A_2$	7.024
10	332	$2A_2 + A_1$	7.051
11	41111	A_3	7.032
12	4211	$(A_3 + A_1)'$	7.046
13	422	$D_4(a_1)$	7.029
14	431	$A_3 + A_2$	7.023
15	44	A_4	7.022
16	5111	A_4	7.009
17	521	$A_4 + A_1$	7.011
18	53	$A_4 + A_2$	7.027
19	611	A_5'	7.042
20	62	$E_6(a_3)$	7.021
21	71	A_6	7.019
22	8	$E_6(a_1)$	7.018

Table 23: E_7, vertex 8, $M = A_5 + A_2$: all 33 trivial-local-system generators.

Row	Local nilpotent orbit	Ambient orbit	Source ID
1	$111111, 111$	1	7.035
2	$111111, 21$	A_1	7.036
3	$111111, 3$	A_2	7.033
4	$21111, 111$	A_1	7.036
5	$21111, 21$	$2A_1$	7.038
6	$21111, 3$	$A_2 + A_1$	7.016
7	$2211, 111$	$2A_1$	7.038
8	$2211, 21$	$3A_1'$	7.048
9	$2211, 3$	$A_2 + 2A_1$	7.039
10	$222, 111$	$3A_1''$	7.015
11	$222, 21$	$4A_1$	7.054
12	$222, 3$	$A_2 + 3A_1$	7.030
13	$3111, 111$	A_2	7.033
14	$3111, 21$	$A_2 + A_1$	7.016
15	$3111, 3$	$2A_2$	7.024
16	$321, 111$	$A_2 + A_1$	7.016
17	$321, 21$	$A_2 + 2A_1$	7.039
18	$321, 3$	$2A_2 + A_1$	7.051
19	$33, 111$	$2A_2$	7.024
20	$33, 21$	$2A_2 + A_1$	7.051
21	$33, 3$	$D_4(a_1)$	7.029
22	$411, 111$	A_3	7.032
23	$411, 21$	$(A_3 + A_1)'$	7.046
24	$411, 3$	$A_3 + A_2$	7.023
25	$42, 111$	$(A_3 + A_1)''$	7.010
26	$42, 21$	$A_3 + 2A_1$	7.043
27	$42, 3$	$A_3 + A_2 + A_1$	7.026
28	$51, 111$	A_4	7.009
29	$51, 21$	$A_4 + A_1$	7.011
30	$51, 3$	$A_4 + A_2$	7.027
31	$6, 111$	A_5''	7.008
32	$6, 21$	$A_5 + A_1$	7.050
33	$6, 3$	$E_7(a_5)$	7.013

A.8. **All required local witnesses in type E_8.** The conventions are the same as for E_7. In particular the two entries with source 8.G are the two explicit multiplicity-one restrictions in Lemma 7.5.

Table 24: E_8, vertex 1, $M = E_8$: all 70 trivial-local-system generators.

Row	Local nilpotent orbit	Ambient orbit	Source ID
1	E_8	E_8	8.001
2	1	1	8.001
5	$E_8(a_2)$	$E_8(a_2)$	8.001
6	$2A_1$	$2A_1$	8.055
9	$4A_1$	$4A_1$	8.113
10	E_7	E_7	8.062

Row	Local nilpotent orbit	Ambient orbit	Source ID
11	$3A_1$	$3A_1$	8.078
14	$2A_2 + 2A_1$	$2A_2 + 2A_1$	8.108
15	$E_8(a_4)$	$E_8(a_4)$	8.001
16	$A_2 + A_1$	$A_2 + A_1$	8.057
17	$A_4 + A_3$	$A_4 + A_3$	8.102
22	E_6	E_6	8.001
23	D_4	D_4	8.001
24	$E_7(a_1)$	$E_7(a_1)$	8.001
25	A_3	A_3	8.034
28	$D_4 + A_1$	$D_4 + A_1$	8.105
29	$E_8(a_5)$	$E_8(a_5)$	8.001
30	$2A_2$	$2A_2$	8.001
33	$2A_3$	$2A_3$	8.100
35	D_6	D_6	8.095
38	$A_3 + 2A_1$	$A_3 + 2A_1$	8.111
39	D_5	D_5	8.001
40	$E_7(a_2)$	$E_7(a_2)$	8.064
41	$A_3 + A_1$	$A_3 + A_1$	8.084
43	$E_8(a_6)$	$E_8(a_6)$	8.001
44	$D_4(a_1) + A_1$	$D_4(a_1) + A_1$	8.059
47	$E_6(a_3) + A_1$	$E_6(a_3) + A_1$	8.087
50	$D_6(a_2)$	$D_6(a_2)$	8.097
51	$E_8(b_6)$	$E_8(b_6)$	8.001
52	$D_4(a_1) + A_2$	$D_4(a_1) + A_2$	8.001
53	$E_8(a_7)$	$E_8(a_7)$	8.001
54	$E_7(a_3)$	$E_7(a_3)$	8.001
55	A_4	A_4	8.001
57	$A_6 + A_1$	$A_6 + A_1$	8.001
58	$A_4 + A_2 + A_1$	$A_4 + A_2 + A_1$	8.001
60	$D_5 + A_1$	$D_5 + A_1$	8.071
61	A_5	A_5	8.069
63	$A_4 + A_1$	$A_4 + A_1$	8.015
64	$D_7(a_2)$	$D_7(a_2)$	8.001
65	$A_4 + 2A_1$	$A_4 + 2A_1$	8.051
66	$E_7(a_4)$	$E_7(a_4)$	8.001
67	$D_5(a_1) + A_1$	$D_5(a_1) + A_1$	8.001
68	$E_8(a_1)$	$E_8(a_1)$	8.001
69	A_1	A_1	8.060
72	$E_8(a_3)$	$E_8(a_3)$	8.001
73	A_2	A_2	8.001
77	D_7	D_7	8.063
78	$A_2 + 3A_1$	$A_2 + 3A_1$	8.112
79	$E_6 + A_1$	$E_6 + A_1$	8.086
80	$2A_2 + A_1$	$2A_2 + A_1$	8.088
81	$E_8(b_4)$	$E_8(b_4)$	8.001
82	$A_2 + 2A_1$	$A_2 + 2A_1$	8.061
83	$D_5(a_1) + A_2$	$D_5(a_1) + A_2$	8.099
88	$A_5 + A_1$	$A_5 + A_1$	8.103
91	A_7	A_7	8.070
92	$A_3 + A_2 + A_1$	$A_3 + A_2 + A_1$	8.105
93	$E_8(b_5)$	$E_8(b_5)$	8.001

Row	Local nilpotent orbit	Ambient orbit	Source ID
94	$D_4(a_1)$	$D_4(a_1)$	8.001
97	$E_6(a_1)$	$E_6(a_1)$	8.001
98	$D_5(a_1)$	$D_5(a_1)$	8.015
100	$D_7(a_1)$	$D_7(a_1)$	8.001
101	$A_3 + A_2$	$A_3 + A_2$	8.001
104	$E_7(a_5)$	$E_7(a_5)$	8.071
105	$E_6(a_1) + A_1$	$E_6(a_1) + A_1$	8.001
107	A_6	A_6	8.001
108	$D_4 + A_2$	$D_4 + A_2$	8.001
109	$D_5 + A_2$	$D_5 + A_2$	8.001
110	$A_4 + A_2$	$A_4 + A_2$	8.001
111	$D_6(a_1)$	$D_6(a_1)$	8.001
112	$E_6(a_3)$	$E_6(a_3)$	8.001

Table 25: E_8, vertex 2, $M = D_8$: all 75 trivial-local-system generators.

Row	Local nilpotent orbit	Ambient orbit	Source ID
1	22222222−	$4A_1$	8.113
2	22222222+	A_2	8.033
3	2222221111	$3A_1$	8.078
4	222211111111	$2A_1$	8.055
5	22111111111111	A_1	8.060
6	1111111111111111	1	8.054
10	33222211	$A_2 + 2A_1$	8.061
11	3322111111	$A_2 + A_1$	8.057
12	331111111111	A_2	8.033
13	31111111111111	$2A_1$	8.055
15	333322	$2A_2 + A_1$	8.088
16	442222−	$A_3 + 2A_1$	8.111
17	442222+	$D_4(a_1)$	8.036
18	33331111	$2A_2$	8.053
19	44221111	$A_3 + A_1$	8.084
20	3331111111	$A_2 + 2A_1$	8.061
21	4411111111	A_3	8.034
22	322111111111	$3A_1$	8.078
24	333331	$2A_2 + 2A_1$	8.108
25	443311	$A_3 + A_2$	8.052
26	33322111	$A_2 + 3A_1$	8.112
27	44311111	$A_3 + 2A_1$	8.111
28	3222211111	$4A_1$	8.113
29	4444−	$2A_3$	8.100
30	4444+	A_4	8.031
31	443221	$D_4(a_1) + A_1$	8.058
32	32222221	$A_2 + A_1$	8.056
35	532222	$A_3 + A_2$	8.052
36	53221111	$D_4(a_1) + A_1$	8.059
37	5311111111	$D_4(a_1)$	8.036
38	511111111111	A_3	8.034
42	533311	$D_4(a_1) + A_2$	8.050

Row	Local nilpotent orbit	Ambient orbit	Source ID
43	552211	$A_4 + A_1$	8.041
44	53311111	$A_3 + A_2$	8.052
45	55111111	A_4	8.031
46	5221111111	$A_3 + A_1$	8.084
48	5533	$A_4 + A_2$	8.046
49	533221	$A_3 + A_2 + A_1$	8.110
51	553111	$A_4 + 2A_1$	8.051
52	52222111	$A_3 + 2A_1$	8.111
53	5443	$A_4 + 2A_1$	8.114
54	6622−	$A_5 + A_1$	8.103
55	6622+	$E_6(a_3)$	8.023
56	544111	$2A_3$	8.100
57	661111	A_5	8.069
59	5551	$A_4 + A_3$	8.102
60	6631	$E_6(a_3) + A_1$	8.087
64	732211	$D_5(a_1) + A_1$	8.037
65	73111111	$D_5(a_1)$	8.015
66	7111111111	D_4	8.014
68	7333	$D_5(a_1) + A_2$	8.099
69	7522	$E_7(a_5)$	8.072
70	733111	$D_4 + A_2$	8.028
71	751111	$D_6(a_2)$	8.097
72	72211111	$D_4 + A_1$	8.105
74	7531	$E_8(a_7)$	8.027
75	722221	$D_5(a_1)$	8.035
77	7441	$D_6(a_2)$	8.097
78	7711	A_6	8.044
80	88−	A_7	8.070
81	88+	$E_6(a_1)$	8.021
84	9322	$E_7(a_4)$	8.022
85	931111	$D_6(a_1)$	8.013
86	91111111	D_5	8.010
88	9331	$D_5 + A_2$	8.025
89	9511	$D_7(a_2)$	8.038
90	922111	$D_5 + A_1$	8.071
91	97	$E_8(b_6)$	8.026
94	$(11, 3, 1, 1)$	$D_7(a_1)$	8.024
95	$(11, 1, 1, 1, 1, 1)$	D_6	8.095
96	$(11, 5)$	$E_8(a_6)$	8.019
97	$(11, 2, 2, 1)$	$E_7(a_3)$	8.032
98	$(13, 3)$	$E_8(a_5)$	8.018
99	$(13, 1, 1, 1)$	D_7	8.063
100	$(15, 1)$	$E_8(a_4)$	8.017

Table 26: E_8, vertex 3, $M = E_7 + A_1$: all 90 trivial-local-system generators.

Row	Local nilpotent orbit	Ambient orbit	Source ID
1	$E_7, 11$	E_7	8.062
2	$E_7, 2$	$E_8(a_3)$	8.004

Row	Local nilpotent orbit	Ambient orbit	Source ID
3	$1, 11$	1	8.054
4	$1, 2$	A_1	8.060
5	$A_1, 11$	A_1	8.060
6	$A_1, 2$	$2A_1$	8.055
7	$E_7(a_1), 11$	$E_7(a_1)$	8.002
8	$E_7(a_1), 2$	$E_8(b_4)$	8.005
9	$4A_1, 11$	$4A_1$	8.113
10	$4A_1, 2$	$A_2 + A_1$	8.056
17	$3A_1'', 11$	$3A_1$	8.078
18	$3A_1'', 2$	A_2	8.033
19	$E_6, 11$	E_6	8.008
20	$E_6, 2$	$E_6 + A_1$	8.086
21	$E_7(a_2), 11$	$E_7(a_2)$	8.064
22	$E_7(a_2), 2$	$E_8(b_5)$	8.006
23	$2A_1, 11$	$2A_1$	8.055
24	$2A_1, 2$	$3A_1$	8.078
29	$D_6, 11$	D_6	8.095
30	$D_6, 2$	$E_7(a_3)$	8.032
31	$3A_1', 11$	$3A_1$	8.078
32	$3A_1', 2$	$4A_1$	8.113
33	$A_2, 11$	A_2	8.033
34	$A_2, 2$	$A_2 + A_1$	8.057
35	$E_7(a_3), 11$	$E_7(a_3)$	8.003
36	$E_7(a_3), 2$	$D_7(a_1)$	8.024
37	$2A_2 + A_1, 11$	$2A_2 + A_1$	8.088
38	$2A_2 + A_1, 2$	$2A_2 + 2A_1$	8.108
39	$A_5 + A_1, 11$	$A_5 + A_1$	8.103
40	$A_5 + A_1, 2$	$E_6(a_3) + A_1$	8.087
41	$D_4 + A_1, 11$	$D_4 + A_1$	8.105
42	$D_4 + A_1, 2$	$D_5(a_1)$	8.035
49	$A_6, 11$	A_6	8.044
50	$A_6, 2$	$A_6 + A_1$	8.048
51	$A_2 + 3A_1, 11$	$A_2 + 3A_1$	8.112
52	$A_2 + 3A_1, 2$	$2A_2$	8.053
53	$A_5'', 11$	A_5	8.069
54	$A_5'', 2$	$E_6(a_3)$	8.023
55	$D_4, 11$	D_4	8.014
56	$D_4, 2$	$D_4 + A_1$	8.105
57	$E_6(a_1), 11$	$E_6(a_1)$	8.009
58	$E_6(a_1), 2$	$E_6(a_1) + A_1$	8.011
59	$A_2 + A_1, 11$	$A_2 + A_1$	8.057
60	$A_2 + A_1, 2$	$A_2 + 2A_1$	8.061
61	$D_5 + A_1, 11$	$D_5 + A_1$	8.071
62	$D_5 + A_1, 2$	$D_6(a_1)$	8.013
63	$2A_2, 11$	$2A_2$	8.053
64	$2A_2, 2$	$2A_2 + A_1$	8.088
69	$A_2 + 2A_1, 11$	$A_2 + 2A_1$	8.061
70	$A_2 + 2A_1, 2$	$A_2 + 3A_1$	8.112
71	$E_7(a_4), 11$	$E_7(a_4)$	8.022
72	$E_7(a_4), 2$	$D_5 + A_2$	8.025
73	$(A_3 + A_1)'', 11$	$A_3 + A_1$	8.084

Row	Local nilpotent orbit	Ambient orbit	Source ID
74	$(A_3+A_1)'', 2$	$D_4(a_1)$	8.036
75	$D_5, 11$	D_5	8.010
76	$D_5, 2$	D_5+A_1	8.071
77	$D_6(a_1), 11$	$D_6(a_1)$	8.013
78	$D_6(a_1), 2$	$E_7(a_4)$	8.022
79	$A_3, 11$	A_3	8.034
80	$A_3, 2$	A_3+A_1	8.084
81	$A_4+A_2, 11$	A_4+A_2	8.046
82	$A_4+A_2, 2$	$A_4+A_2+A_1$	8.049
83	$A_3+A_2+A_1, 11$	$A_3+A_2+A_1$	8.106
84	$A_3+A_2+A_1, 2$	$D_4(a_1)+A_2$	8.050
85	$A_3+2A_1, 11$	A_3+2A_1	8.111
86	$A_3+2A_1, 2$	$D_4(a_1)+A_1$	8.058
87	$A_5', 11$	A_5	8.069
88	$A_5', 2$	A_5+A_1	8.103
93	$D_6(a_2), 11$	$D_6(a_2)$	8.097
94	$D_6(a_2), 2$	$E_7(a_5)$	8.072
95	$(A_3+A_1)', 11$	A_3+A_1	8.084
96	$(A_3+A_1)', 2$	A_3+2A_1	8.111
97	$D_4(a_1), 11$	$D_4(a_1)$	8.036
98	$D_4(a_1), 2$	$D_4(a_1)+A_1$	8.059
99	$E_7(a_5), 11$	$E_7(a_5)$	8.072
100	$E_7(a_5), 2$	$E_8(a_7)$	8.027
105	$A_3+A_2, 11$	A_3+A_2	8.052
106	$A_3+A_2, 2$	$A_3+A_2+A_1$	8.110
107	$D_5(a_1)+A_1, 11$	$D_5(a_1)+A_1$	8.037
108	$D_5(a_1)+A_1, 2$	D_4+A_2	8.028
109	$E_6(a_3), 11$	$E_6(a_3)$	8.023
110	$E_6(a_3), 2$	$E_6(a_3)+A_1$	8.087
111	$D_4(a_1)+A_1, 11$	$D_4(a_1)+A_1$	8.059
112	$D_4(a_1)+A_1, 2$	A_3+A_2	8.052
113	$D_5(a_1), 11$	$D_5(a_1)$	8.015
114	$D_5(a_1), 2$	$D_5(a_1)+A_1$	8.037
115	$A_4, 11$	A_4	8.031
116	$A_4, 2$	A_4+A_1	8.041
119	$A_4+A_1, 11$	A_4+A_1	8.041
120	$A_4+A_1, 2$	A_4+2A_1	8.051

Table 27: E_8, vertex 4, $M = E_6+A_2$: all 63 trivial-local-system generators.

Row	Local nilpotent orbit	Ambient orbit	Source ID
1	$E_6, 111$	E_6	8.008
2	$E_6, 21$	E_6+A_1	8.086
3	$E_6, 3$	$E_8(b_5)$	8.012
4	$1, 111$	1	8.054
5	$1, 21$	A_1	8.060
6	$1, 3$	A_2	8.033
7	$2A_2+A_1, 111$	$2A_2+A_1$	8.088
8	$2A_2+A_1, 21$	$2A_2+2A_1$	8.108

Row	Local nilpotent orbit	Ambient orbit	Source ID
9	$2A_2 + A_1, 3$	$D_4(a_1) + A_1$	8.G
10	$E_6(a_1), 111$	$E_6(a_1)$	8.009
11	$E_6(a_1), 21$	$E_6(a_1) + A_1$	8.011
12	$E_6(a_1), 3$	$E_8(b_6)$	8.026
13	$A_1, 111$	A_1	8.060
14	$A_1, 21$	$2A_1$	8.055
15	$A_1, 3$	$A_2 + A_1$	8.057
25	$A_5, 111$	A_5	8.069
26	$A_5, 21$	$A_5 + A_1$	8.103
27	$A_5, 3$	$E_7(a_5)$	8.072
28	$3A_1, 111$	$3A_1$	8.078
29	$3A_1, 21$	$4A_1$	8.113
30	$3A_1, 3$	$A_2 + 3A_1$	8.112
31	$D_5, 111$	D_5	8.010
32	$D_5, 21$	$D_5 + A_1$	8.071
33	$D_5, 3$	$D_5 + A_2$	8.025
34	$2A_1, 111$	$2A_1$	8.055
35	$2A_1, 21$	$3A_1$	8.078
36	$2A_1, 3$	$A_2 + 2A_1$	8.061
37	$D_4, 111$	D_4	8.014
38	$D_4, 21$	$D_4 + A_1$	8.105
39	$D_4, 3$	$D_4 + A_2$	8.028
40	$2A_2, 111$	$2A_2$	8.053
41	$2A_2, 21$	$2A_2 + A_1$	8.088
42	$2A_2, 3$	$D_4(a_1)$	8.040
43	$E_6(a_3), 111$	$E_6(a_3)$	8.023
44	$E_6(a_3), 21$	$E_6(a_3) + A_1$	8.087
45	$E_6(a_3), 3$	$E_8(a_7)$	8.027
46	$A_2, 111$	A_2	8.033
47	$A_2, 21$	$A_2 + A_1$	8.057
48	$A_2, 3$	$2A_2$	8.053
49	$A_3 + A_1, 111$	$A_3 + A_1$	8.084
50	$A_3 + A_1, 21$	$A_3 + 2A_1$	8.111
51	$A_3 + A_1, 3$	$A_3 + A_2 + A_1$	8.106
52	$D_4(a_1), 111$	$D_4(a_1)$	8.036
53	$D_4(a_1), 21$	$D_4(a_1) + A_1$	8.059
54	$D_4(a_1), 3$	$D_4(a_1) + A_2$	8.050
58	$A_4 + A_1, 111$	$A_4 + A_1$	8.041
59	$A_4 + A_1, 21$	$A_4 + 2A_1$	8.051
60	$A_4 + A_1, 3$	$A_4 + A_2 + A_1$	8.049
61	$A_2 + 2A_1, 111$	$A_2 + 2A_1$	8.061
62	$A_2 + 2A_1, 21$	$A_2 + 3A_1$	8.112
63	$A_2 + 2A_1, 3$	$2A_2 + 2A_1$	8.108
64	$D_5(a_1), 111$	$D_5(a_1)$	8.015
65	$D_5(a_1), 21$	$D_5(a_1) + A_1$	8.037
66	$D_5(a_1), 3$	$D_5(a_1) + A_2$	8.099
67	$A_2 + A_1, 111$	$A_2 + A_1$	8.057
68	$A_2 + A_1, 21$	$A_2 + 2A_1$	8.061
69	$A_2 + A_1, 3$	$2A_2 + A_1$	8.088
70	$A_4, 111$	A_4	8.031
71	$A_4, 21$	$A_4 + A_1$	8.041

Row	Local nilpotent orbit	Ambient orbit	Source ID
72	$A_4, 3$	$A_4 + A_2$	8.046
73	$A_3, 111$	A_3	8.034
74	$A_3, 21$	$A_3 + A_1$	8.084
75	$A_3, 3$	$A_3 + A_2$	8.052

Table 28: E_8, vertex 5, $M = A_8$: all 30 trivial-local-system generators.

Row	Local nilpotent orbit	Ambient orbit	Source ID
1	111111111	1	8.054
2	21111111	A_1	8.060
3	2211111	$2A_1$	8.055
4	222111	$3A_1$	8.078
5	22221	$4A_1$	8.113
6	3111111	A_2	8.033
7	321111	$A_2 + A_1$	8.057
8	32211	$A_2 + 2A_1$	8.061
9	3222	$A_2 + 3A_1$	8.112
10	33111	$2A_2$	8.053
11	3321	$2A_2 + A_1$	8.088
12	333	$D_4(a_1)$	8.040
13	411111	A_3	8.034
14	42111	$A_3 + A_1$	8.084
15	4221	$A_3 + 2A_1$	8.111
16	4311	$A_3 + A_2$	8.052
17	432	$A_3 + A_2 + A_1$	8.106
18	441	$2A_3$	8.100
19	51111	A_4	8.031
20	5211	$A_4 + A_1$	8.041
21	522	$A_4 + 2A_1$	8.051
22	531	$A_4 + A_2$	8.046
23	54	$A_4 + A_3$	8.102
24	6111	A_5	8.069
25	621	$A_5 + A_1$	8.103
26	63	$E_7(a_5)$	8.072
27	711	A_6	8.044
28	72	$A_6 + A_1$	8.048
29	81	A_7	8.070
30	9	$E_8(a_6)$	8.039

Table 29: E_8, vertex 6, $M = D_5 + A_3$: all 80 trivial-local-system generators.

Row	Local nilpotent orbit	Ambient orbit	Source ID
1	$222211, 1111$	$2A_1$	8.055
2	$222211, 211$	$3A_1$	8.078
3	$222211, 22$	$4A_1$	8.113
4	$222211, 31$	$A_2 + 2A_1$	8.061

Row	Local nilpotent orbit	Ambient orbit	Source ID
5	$222211, 4$	$A_3 + 2A_1$	8.111
6	$22111111, 1111$	A_1	8.060
7	$22111111, 211$	$2A_1$	8.055
8	$22111111, 22$	$3A_1$	8.078
9	$22111111, 31$	$A_2 + A_1$	8.057
10	$22111111, 4$	$A_3 + A_1$	8.084
11	$1111111111, 1111$	1	8.054
12	$1111111111, 211$	A_1	8.060
13	$1111111111, 22$	$2A_1$	8.055
14	$1111111111, 31$	A_2	8.033
15	$1111111111, 4$	A_3	8.034
21	$3322, 1111$	$A_2 + A_1$	8.057
22	$3322, 211$	$A_2 + 2A_1$	8.061
23	$3322, 22$	$A_2 + 3A_1$	8.112
24	$3322, 31$	$2A_2 + A_1$	8.088
25	$3322, 4$	$A_3 + A_2 + A_1$	8.106
26	$331111, 1111$	A_2	8.033
27	$331111, 211$	$A_2 + A_1$	8.057
28	$331111, 22$	$A_2 + 2A_1$	8.061
29	$331111, 31$	$2A_2$	8.053
30	$331111, 4$	$A_3 + A_2$	8.052
31	$31111111, 1111$	$2A_1$	8.055
32	$31111111, 211$	$3A_1$	8.078
33	$31111111, 22$	A_2	8.033
34	$31111111, 31$	$A_2 + 2A_1$	8.061
35	$31111111, 4$	$D_4(a_1)$	8.036
36	$3331, 1111$	$A_2 + 2A_1$	8.061
37	$3331, 211$	$A_2 + 3A_1$	8.112
38	$3331, 22$	$2A_2$	8.053
39	$3331, 31$	$2A_2 + 2A_1$	8.108
40	$3331, 4$	$D_4(a_1) + A_2$	8.050
41	$4411, 1111$	A_3	8.034
42	$4411, 211$	$A_3 + A_1$	8.084
43	$4411, 22$	$A_3 + 2A_1$	8.111
44	$4411, 31$	$A_3 + A_2$	8.052
45	$4411, 4$	$2A_3$	8.100
46	$322111, 1111$	$3A_1$	8.078
47	$322111, 211$	$4A_1$	8.113
48	$322111, 22$	$A_2 + A_1$	8.056
49	$322111, 31$	$A_2 + 3A_1$	8.112
50	$322111, 4$	$D_4(a_1) + A_1$	8.058
56	$5311, 1111$	$D_4(a_1)$	8.036
57	$5311, 211$	$D_4(a_1) + A_1$	8.059
58	$5311, 22$	$A_3 + A_2$	8.052
59	$5311, 31$	$D_4(a_1) + A_2$	8.050
60	$5311, 4$	$A_4 + 2A_1$	8.114
61	$511111, 1111$	A_3	8.034
62	$511111, 211$	$A_3 + A_1$	8.084
63	$511111, 22$	$D_4(a_1)$	8.036
64	$511111, 31$	$A_3 + A_2$	8.052
65	$511111, 4$	A_4	8.031

Row	Local nilpotent orbit	Ambient orbit	Source ID
66	$55, 1111$	A_4	8.031
67	$55, 211$	$A_4 + A_1$	8.041
68	$55, 22$	$A_4 + 2A_1$	8.051
69	$55, 31$	$A_4 + A_2$	8.046
70	$55, 4$	$A_4 + A_3$	8.102
71	$5221, 1111$	$A_3 + A_1$	8.084
72	$5221, 211$	$A_3 + 2A_1$	8.111
73	$5221, 22$	$D_4(a_1) + A_1$	8.058
74	$5221, 31$	$A_3 + A_2 + A_1$	8.106
75	$5221, 4$	$A_4 + A_1$	8.082
76	$73, 1111$	$D_5(a_1)$	8.015
77	$73, 211$	$D_5(a_1) + A_1$	8.037
78	$73, 22$	$D_4 + A_2$	8.028
79	$73, 31$	$D_5(a_1) + A_2$	8.099
80	$73, 4$	$E_8(a_7)$	8.045
81	$7111, 1111$	D_4	8.014
82	$7111, 211$	$D_4 + A_1$	8.105
83	$7111, 22$	$D_5(a_1)$	8.035
84	$7111, 31$	$D_4 + A_2$	8.028
85	$7111, 4$	$D_6(a_2)$	8.097
86	$91, 1111$	D_5	8.010
87	$91, 211$	$D_5 + A_1$	8.071
88	$91, 22$	$D_6(a_1)$	8.013
89	$91, 31$	$D_5 + A_2$	8.025
90	$91, 4$	$D_7(a_2)$	8.047

Table 30: E_8, vertex 7, $M = A_7 + A_1$: all 44 trivial-local-system generators.

Row	Local nilpotent orbit	Ambient orbit	Source ID
1	$11, 11111111$	1	8.054
2	$11, 2111111$	A_1	8.060
3	$11, 221111$	$2A_1$	8.055
4	$11, 22211$	$3A_1$	8.078
5	$11, 2222$	A_2	8.033
6	$11, 311111$	A_2	8.033
7	$11, 32111$	$A_2 + A_1$	8.057
8	$11, 3221$	$A_2 + 2A_1$	8.061
9	$11, 3311$	$2A_2$	8.053
10	$11, 332$	$2A_2 + A_1$	8.088
11	$11, 41111$	A_3	8.034
12	$11, 4211$	$A_3 + A_1$	8.084
13	$11, 422$	$D_4(a_1)$	8.036
14	$11, 431$	$A_3 + A_2$	8.052
15	$11, 44$	A_4	8.031
16	$11, 5111$	A_4	8.031
17	$11, 521$	$A_4 + A_1$	8.041
18	$11, 53$	$A_4 + A_2$	8.046
19	$11, 611$	A_5	8.069
20	$11, 62$	$E_6(a_3)$	8.023

Row	Local nilpotent orbit	Ambient orbit	Source ID
21	$11, 71$	A_6	8.044
22	$11, 8$	$E_6(a_1)$	8.021
23	$2, 11111111$	A_1	8.060
24	$2, 2111111$	$2A_1$	8.055
25	$2, 221111$	$3A_1$	8.078
26	$2, 22211$	$4A_1$	8.113
27	$2, 2222$	$A_2 + A_1$	8.056
28	$2, 311111$	$A_2 + A_1$	8.057
29	$2, 32111$	$A_2 + 2A_1$	8.061
30	$2, 3221$	$A_2 + 3A_1$	8.112
31	$2, 3311$	$2A_2 + A_1$	8.088
32	$2, 332$	$2A_2 + 2A_1$	8.108
33	$2, 41111$	$A_3 + A_1$	8.084
34	$2, 4211$	$A_3 + 2A_1$	8.111
35	$2, 422$	$D_4(a_1) + A_1$	8.058
36	$2, 431$	$A_3 + A_2 + A_1$	8.106
37	$2, 44$	$A_4 + A_1$	8.082
38	$2, 5111$	$A_4 + A_1$	8.041
39	$2, 521$	$A_4 + 2A_1$	8.051
40	$2, 53$	$A_4 + A_2 + A_1$	8.049
41	$2, 611$	$A_5 + A_1$	8.103
42	$2, 62$	$E_6(a_3) + A_1$	8.087
43	$2, 71$	$A_6 + A_1$	8.048
44	$2, 8$	$E_6(a_1) + A_1$	8.077

Table 31: E_8, vertex 8, $M = 2A_4$: all 49 trivial-local-system generators.

Row	Local nilpotent orbit	Ambient orbit	Source ID
1	$11111, 11111$	1	8.054
2	$11111, 2111$	A_1	8.060
3	$11111, 221$	$2A_1$	8.055
4	$11111, 311$	A_2	8.079
5	$11111, 32$	$A_2 + A_1$	8.057
6	$11111, 41$	A_3	8.081
7	$11111, 5$	A_4	8.076
8	$2111, 11111$	A_1	8.060
9	$2111, 2111$	$2A_1$	8.055
10	$2111, 221$	$3A_1$	8.078
11	$2111, 311$	$A_2 + A_1$	8.057
12	$2111, 32$	$A_2 + 2A_1$	8.061
13	$2111, 41$	$A_3 + A_1$	8.084
14	$2111, 5$	$A_4 + A_1$	8.090
15	$221, 11111$	$2A_1$	8.055
16	$221, 2111$	$3A_1$	8.078
17	$221, 221$	$4A_1$	8.113
18	$221, 311$	$A_2 + 2A_1$	8.061
19	$221, 32$	$A_2 + 3A_1$	8.112
20	$221, 41$	$A_3 + 2A_1$	8.111
21	$221, 5$	$A_4 + 2A_1$	8.051

Row	Local nilpotent orbit	Ambient orbit	Source ID
22	$311, 11111$	A_2	8.079
23	$311, 2111$	$A_2 + A_1$	8.057
24	$311, 221$	$A_2 + 2A_1$	8.061
25	$311, 311$	$2A_2$	8.053
26	$311, 32$	$2A_2 + A_1$	8.088
27	$311, 41$	$A_3 + A_2$	8.052
28	$311, 5$	$A_4 + A_2$	8.085
29	$32, 11111$	$A_2 + A_1$	8.057
30	$32, 2111$	$A_2 + 2A_1$	8.061
31	$32, 221$	$A_2 + 3A_1$	8.112
32	$32, 311$	$2A_2 + A_1$	8.088
33	$32, 32$	$2A_2 + 2A_1$	8.108
34	$32, 41$	$A_3 + A_2 + A_1$	8.106
35	$32, 5$	$A_4 + A_2 + A_1$	8.107
36	$41, 11111$	A_3	8.081
37	$41, 2111$	$A_3 + A_1$	8.084
38	$41, 221$	$A_3 + 2A_1$	8.111
39	$41, 311$	$A_3 + A_2$	8.052
40	$41, 32$	$A_3 + A_2 + A_1$	8.106
41	$41, 41$	$2A_3$	8.100
42	$41, 5$	$A_4 + A_3$	8.102
43	$5, 11111$	A_4	8.076
44	$5, 2111$	$A_4 + A_1$	8.090
45	$5, 221$	$A_4 + 2A_1$	8.051
46	$5, 311$	$A_4 + A_2$	8.085
47	$5, 32$	$A_4 + A_2 + A_1$	8.107
48	$5, 41$	$A_4 + A_3$	8.102
49	$5, 5$	$E_8(a_7)$	8.098

Table 32: E_8, vertex 9, $M = A_5+A_2+A_1$: all 66 trivial-local-system generators.

Row	Local nilpotent orbit	Ambient orbit	Source ID
1	$111, 11, 111111$	1	8.054
2	$111, 11, 21111$	A_1	8.060
3	$111, 11, 2211$	$2A_1$	8.055
4	$111, 11, 222$	$3A_1$	8.078
5	$111, 11, 3111$	A_2	8.079
6	$111, 11, 321$	$A_2 + A_1$	8.057
7	$111, 11, 33$	$2A_2$	8.053
8	$111, 11, 411$	A_3	8.081
9	$111, 11, 42$	$A_3 + A_1$	8.084
10	$111, 11, 51$	A_4	8.076
11	$111, 11, 6$	A_5	8.101
12	$111, 2, 111111$	A_1	8.060
13	$111, 2, 21111$	$2A_1$	8.055
14	$111, 2, 2211$	$3A_1$	8.078
15	$111, 2, 222$	A_2	8.079
16	$111, 2, 3111$	$A_2 + A_1$	8.057
17	$111, 2, 321$	$A_2 + 2A_1$	8.061

Row	Local nilpotent orbit	Ambient orbit	Source ID
18	$111, 2, 33$	$2A_2 + A_1$	8.088
19	$111, 2, 411$	$A_3 + A_1$	8.084
20	$111, 2, 42$	$D_4(a_1)$	8.083
21	$111, 2, 51$	$A_4 + A_1$	8.090
22	$111, 2, 6$	$E_6(a_3)$	8.094
23	$21, 11, 111111$	A_1	8.060
24	$21, 11, 21111$	$2A_1$	8.055
25	$21, 11, 2211$	$3A_1$	8.078
26	$21, 11, 222$	$4A_1$	8.113
27	$21, 11, 3111$	$A_2 + A_1$	8.057
28	$21, 11, 321$	$A_2 + 2A_1$	8.061
29	$21, 11, 33$	$2A_2 + A_1$	8.088
30	$21, 11, 411$	$A_3 + A_1$	8.084
31	$21, 11, 42$	$A_3 + 2A_1$	8.111
32	$21, 11, 51$	$A_4 + A_1$	8.090
33	$21, 11, 6$	$A_5 + A_1$	8.103
34	$21, 2, 111111$	$2A_1$	8.055
35	$21, 2, 21111$	$3A_1$	8.078
36	$21, 2, 2211$	$4A_1$	8.113
37	$21, 2, 222$	$A_2 + A_1$	8.056
38	$21, 2, 3111$	$A_2 + 2A_1$	8.061
39	$21, 2, 321$	$A_2 + 3A_1$	8.112
40	$21, 2, 33$	$2A_2 + 2A_1$	8.108
41	$21, 2, 411$	$A_3 + 2A_1$	8.111
42	$21, 2, 42$	$D_4(a_1) + A_1$	8.058
43	$21, 2, 51$	$A_4 + 2A_1$	8.051
44	$21, 2, 6$	$E_6(a_3) + A_1$	8.092
45	$3, 11, 111111$	A_2	8.079
46	$3, 11, 21111$	$A_2 + A_1$	8.057
47	$3, 11, 2211$	$A_2 + 2A_1$	8.061
48	$3, 11, 222$	$A_2 + 3A_1$	8.112
49	$3, 11, 3111$	$2A_2$	8.053
50	$3, 11, 321$	$2A_2 + A_1$	8.088
51	$3, 11, 33$	$D_4(a_1)$	8.083
52	$3, 11, 411$	$A_3 + A_2$	8.052
53	$3, 11, 42$	$A_3 + A_2 + A_1$	8.106
54	$3, 11, 51$	$A_4 + A_2$	8.085
55	$3, 11, 6$	$E_7(a_5)$	8.104
56	$3, 2, 111111$	$A_2 + A_1$	8.057
57	$3, 2, 21111$	$A_2 + 2A_1$	8.061
58	$3, 2, 2211$	$A_2 + 3A_1$	8.112
59	$3, 2, 222$	$2A_2$	8.053
60	$3, 2, 3111$	$2A_2 + A_1$	8.088
61	$3, 2, 321$	$2A_2 + 2A_1$	8.108
62	$3, 2, 33$	$D_4(a_1) + A_1$	8.G
63	$3, 2, 411$	$A_3 + A_2 + A_1$	8.106
64	$3, 2, 42$	$D_4(a_1) + A_2$	8.109
65	$3, 2, 51$	$A_4 + A_2 + A_1$	8.107
66	$3, 2, 6$	$E_8(a_7)$	8.093

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